A BOUND ON LIBOR FUTURES PRICES
FOR HJM YIELD CURVE MODELS

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ABSTRACT. We prove that for a large class of widely used term structure models there is a simple theoretical upper bound for value of LIBOR futures prices. When this bound is compared to observed futures prices, one nevertheless finds that the theoretical bound is sometimes violated by market prices. The main consequence of this observation is that virtually all of the important fixed income models have theoretical implications that are sometimes at odds with market realities, at least when they are applied to futures markets.

1. INTRODUCTION

The main purpose of this article is to prove that for a large class of stochastic models for the term structure of interest rates, one has the *futures price inequality:*

$$(1) \qquad F_\lambda(t,T) \leq 100\left(\frac{1}{\lambda}+1\right) - \left(\frac{100}{\lambda}\right)\frac{1+(T-t+\lambda)L_{T-t+\lambda}(t)}{1+(T-t)L_{T-t}(t)},$$

where $F_\lambda(t,T)$ denotes the LIBOR futures price at date t for settlement at date T and where $L_\lambda(t)$ denotes the London interbank offered rate (LIBOR) at time t for a term deposit with a term of λ years. An intriguing feature of this inequality is that it holds for essentially all stochastic yield curve models that are currently used, and, even so, one finds that the inequality is sometimes violated by market prices. This article focuses almost entirely on the mathematical consequences of the yield curve models, and the resolution of the resulting empirical paradox is left as an open problem.

2. BACKGROUND ON THE HJM MODEL

Since its introduction almost a decade ago, the term structure model of Heath, Jarrow, and Morton (1992) has become an almost universal standard for the theoretical analysis of fixed income securities and their associated derivatives (see e.g., Baxter and Rennie (1996), Duffie (1996), or Musiela and Rutkowski (1997)). The HJM model also has the honor of including essentially all earlier bond models as special cases, and, in particular, the HJM model contains both the widely used model of Ho and Lee (1986) and the seminal model of Vasicek (1977).

Still, the most fundamental benefit of the HJM model is that it provides a broad class of natural conditions under which one can rule out the possibility of risk-free arbitrage between bonds of different maturities, and freedom from arbitrage

is arguably the most fundamental condition of economic realism. For all these reasons, the HJM models deserve the careful attention of anyone interested the analytic understanding of fluctuating interest rates.

One of the basic insights of Heath, Jarrow, and Morton (1992) is that the price $P(t, T)$ at time t of a bond that pays one dollar at the maturity date $T \leq \tau$ is best viewed in terms of an integral representation:

$$(2) \qquad P(t, T) = \exp \left(- \int_t^T f(t, u) \, du \right) \qquad 0 \leq t \leq T \leq \tau.$$

A simple (but useful) consequence of such a representation is that it immediately guarantees $P(T, T) = 1$ and $P(t, T) > 0$, two essential properties of any bond price model.

Nevertheless, this first step only defers the modeling of the ensemble of processes $\{P(t, T) : 0 \leq t \leq T \leq \tau\}$ to the modeling of the corresponding ensemble of kernel processes $\{f(t, T) : 0 \leq t \leq T \leq \tau\}$. For the integral representation of $P(t, T)$ to be genuinely useful, one must be able to specify models for $f(t, T)$ that are both financially feasible and mathematically tractable.

The HJM model addresses these aims by focusing on random kernels $f(t, T)$ that have a representation as an Itô integral of the form

$$(3) \qquad f(t, T) = f(0, T) + \int_0^t \alpha(u, T) \, du + \int_0^t \boldsymbol{\sigma}(u, T)^\perp d\mathbf{B}_u,$$

where $\mathbf{B}_t$ denotes an n-dimensional Brownian motion and where

$$\{\alpha(u, T) : 0 \leq u \leq T \leq \tau\} \quad \text{and} \quad \{\boldsymbol{\sigma}(u, T) : 0 \leq u \leq T \leq \tau\}$$

are respectively $\mathbb{R}$ and $\mathbb{R}^n$ valued processes that are adapted to the standard filtration $\mathcal{F}_t$ of $\{\mathbf{B}_t\}$. The transpose symbol $\perp$ in the second integral of (3) serves to remind us that $\mathbf{B}_t$ and $\boldsymbol{\sigma}(u, T)$ are both viewed as column vectors.

The intuitive idea behind the kernel process $f(t, T)$ is that it should represent an interest rate for which one can contract at time t for a riskless loan that begins at date T and which is paid back "an instant" later. Traditionally, such quantities are called *forward rates*, and Heath, Jarrow, and Morton (1992) showed that many important properties of the bond prices $P(t, T)$ can be expressed most easily in terms of a model for $f(t, T)$. In particular, they showed that by imposing a natural restriction on the SDE model for $f(t, T)$ one can guarantee that the price processes $\{P(t, T)\}$ will not offer any opportunity for the construction of a riskless arbitrage between bonds of differing maturities.

3. The Forward Rate Drift Restriction

If the coefficient processes $\alpha(t, T)$ and $\boldsymbol{\sigma}(t, T)$ of the SDE for $f(t, T)$ have the property that $\alpha(t, T)$ may be written as

$$(4) \qquad \alpha(t, T) = \boldsymbol{\sigma}(t, T)^\perp \left[\boldsymbol{\gamma}(t) + \int_t^T \boldsymbol{\sigma}(t, u) \, du \right]$$

where $\boldsymbol{\gamma}(t)$ is an adapted n-dimensional process such that

$$(5) \qquad E \exp \left(- \int_0^\tau \boldsymbol{\gamma}(u)^\perp d\mathbf{B}_u - \frac{1}{2} \int_0^\tau |\boldsymbol{\gamma}(u)|^2 \, du \right) = 1,$$

then we say that $f(t,T)$ satisfies the *forward rate drift restriction*, and the process $\boldsymbol{\gamma}(t)$ that appears in this condition is called the *market price for risk*. The importance of these notions for the theory of bond price processes $P(t,T)$ is that when the forward rate drift restriction applies, one can show that the probability measure $\widetilde{P}$ defined on $\mathcal{F}_\tau$ by

$$(6) \qquad \widetilde{P}(A) = \mathbf{E}\left[1_A \exp\left(-\int_0^\tau \boldsymbol{\gamma}(u)^\perp d\mathbf{B}_u - \frac{1}{2}\int_0^\tau |\boldsymbol{\gamma}(u)|^2\, du\right)\right]$$

has several useful properties. In particular, if we let

$$(7) \qquad r(t) = f(t,t) \quad \text{and} \quad \beta(t) = \exp\left(\int_0^t r(u)\, du\right),$$

then for each T the discounted process $\{P(t,T)/\beta(t) : 0 \leq t \leq T\}$ is a $\widetilde{P}$-martingale with respect to the filtration $\mathcal{F}_t$.

The process $r(t) = f(t,t)$ is called the *spot rate* $r(t)$ and $\beta(t)$ is called the *accumulation factor* (or *discount factor*). The intuition behind $\beta(t)$ is that it should represent the amount of money that one has in a money market account that begins with a balance of one dollar at time zero and that accrues interest at the (random) spot rate $r(u)$ for $0 \leq u \leq t$. We will follow tradition and refer to $\widetilde{P}$ as the *equivalent martingale measure*, reflecting the fact that $\widetilde{P}$ has the same null sets as P and that the process $\{P(t,T)/\beta(t) : 0 \leq t \leq T\}$ is a $\widetilde{P}$-martingale for each $T \in [0,\tau]$. Another key property of $\widetilde{P}$ that follows directly from the definition (6) and Girsanov's theorem is that the vector process defined by

$$(8) \qquad \widetilde{\mathbf{B}}_t = \mathbf{B}_t + \int_0^t \boldsymbol{\gamma}(u)\, du$$

is a standard $\widetilde{P}$-Brownian motion. The measure $\widetilde{P}$ and the associated expectation $\widetilde{E}$ will be used without comment throughout the rest of our analysis.

One of the basic deductions in Heath, Jarrow, and Morton (1992) is that the representation of the bond price processes $\{P(t,T)\}$ in terms of the forward rate processes given by (2) and the SDE for $\{f(t,T)\}$ may be used to show that the processes $\{P(t,T)\}$ satisfy the SDEs

$$(9) \qquad dP(t,T) = P(t,T)[\, r(t)\, dt + \mathbf{a}(t,T)^\perp d\widetilde{\mathbf{B}}_t\,],$$

where $\mathbf{a}(t,T)$ is the n-dimensional column vector of integrated volatilities defined by

$$(10) \qquad \mathbf{a}(t,T) = -\int_t^T \boldsymbol{\sigma}(t,u)\, du.$$

From the SDE (9) and the definition of the discount factor $\beta(t)$, one can also show that given the *initial yield curve* $P(0,T)$ one has a very useful representation for $P(t,T)$ for all $0 \leq t \leq T$:

$$(11) \qquad P(t,T) = P(0,T)\beta(t)\exp\left[\int_0^t \mathbf{a}(s,T)^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_0^t |\mathbf{a}(s,T)|^2\, ds\right].$$

Finally, if the price processes $\{P(t,T)\}$ satisfy equations (2) through (11) we will say that $\{P(t,T)\}$ follows the HJM model and we will refer to $\{P(t,T)\}$ as an *HJM family of bond prices*.

To be sure, this quick review of the HJM model does little justice to its full complexity and richness. Still, this summary does at least provides us with all of the facts and notation that we need in our development of an HJM model for LIBOR quotes and the associated upper bound for LIBOR futures prices.

4. Connecting LIBOR Quotes with Bond Prices

The stochastic integral representation for $P(t, T)$ given by (11) turns out to be one of the keys to the LIBOR inequality (1), but before any information about the HJM model can be put to use, we have to work out an elementary bookkeeping link between LIBOR quotes and the prices of zero coupon bonds. By tradition, LIBOR quotes are stated as *add-on yields*, and, accordingly, some accounting is needed to place such yields in the context of the HJM model for zero-coupon bonds.

The defining convention for LIBOR quotes is that if L is the current quote for a 90 day deposit, the investor who commits M dollars now will receive after 90 days the return of the principle and an interest payment of $M \cdot L \cdot (90/360)$ dollars. Because of this convention, it is useful for us to scale time so that for any two times s and t the number of calendar days between these times is $360(t - s)$. We will also let $L_\lambda(t)$ denote the LIBOR quote at time t for a deposit of 360λ days, so $\lambda = 1/4$ corresponds to a 90-day term of deposit.

Now, with these conventions, if we let $P(t, T)$ denote the value of a zero coupon bond at time t that pays one dollar at time T and that matches the return of a deposit for a period of 360λ calendar days that is made at LIBOR, then we have

$$(12) \qquad L_\lambda(t) = \frac{1}{\lambda} \left(\frac{1}{P(t, t + \lambda)} - 1 \right),$$

or, equivalently,

$$(13) \qquad P(t, t + \lambda) = \frac{1}{1 + \lambda L_\lambda(t)}.$$

From the last two relations, we see that any model for the price $P(t, T)$ of a zero coupon bonds has an immediate translation into a model for LIBOR deposits. Our next step is to examine how LIBOR quotes relate to LIBOR futures.

5. Connecting LIBOR Quotes with Futures Prices

We now let $F_\lambda(t, T)$ denote a stochastic process which we will design to model the LIBOR futures price at time t for futures contracts based on a deposit commitment of 360λ days and with a settlement date T. By conventions of the trading exchanges, the value of the LIBOR futures price at the settlement time T is equal to $100(1 - L_\lambda(T))$; or, in other words, $F_\lambda(t, T)$ satisfies the terminal condition:

$$(14) \qquad F_\lambda(T, T) = 100(1 - L_\lambda(T)).$$

For the actual traded futures, the settlement value $L_\lambda(T)$ is determined at time T by a survey of banks that is contractually specified by the trading exchange that supports the futures contract. Descriptions of this survey process may be found in Amin and Morton (1994, p.148), Kuprianov (1993, p.197), and the website of the British Bank Association (www.bba.org.uk). For us, the details of the survey process for the determination of the settlement price of LIBOR futures contracts are inessential.

Here the critical point is that any sensible model for the futures price $F_\lambda(t,T)$ must satisfy (14), and, when this observation is teamed up with the bookkeeping relation (13), we find that there is an inescapable link between LIBOR futures prices and *any* model that one might pose for the prices $P(t,T)$ of a family of zero coupon bonds. The problem we now face is that of forging a connection between the HJM model and the futures price $F_\lambda(t,T)$ for general values of $t \in [0,T]$.

The missing link may be provided by *assuming* that $F_\lambda(t,T)$ is a $\widetilde{P}$-martingale. One can motivate this assumption by examining the structure of the cash flows that are associated with the futures contract and by exploiting the fact that one can enter into a futures contract on either the long or short side for an initial cost of zero. Nevertheless, even if one is just interested the detailed motivation behind this assumption, there are some technical issues that must be faced. In particular, the martingale property for interest rate futures turns out to require a gentler touch than one typically takes for commodity futures, and, in the interest of clarity, it seems best to defer both the structural motivation and the technical support for the martingale property until a full treatment can be given in Section 10. For the moment, we simply take the martingale property $F_\lambda(t,T)$ as if it were given to us *ex cathedra.*

6. Two Roles for Reciprocal Price

Since the value of $F_\lambda(t,T)$ at time $t = T$ is $100(1 - L_\lambda(T))$, our fundamental assumption that $\{F_\lambda(t,T): \quad 0 \le t \le T\}$ is a martingale tells us that we have the representation

$$(15) \qquad\qquad F_\lambda(t,T) = \widetilde{E}(100(1 - L_\lambda(T))|\mathcal{F}_t),$$

and when we expand this identity by the bookkeeping relation (12), we come our most basic representation for LIBOR futures prices:

$$(16) \qquad F_\lambda(t,T) = 100\left(\frac{1}{\lambda} + 1\right) - \frac{100}{\lambda}\widetilde{E}\left[\frac{1}{P(T,T+\lambda)}|\mathcal{F}_t\right].$$

This identity is our guide to the proof of the futures price inequality. Still, before it can be put to use, we need to find an appropriate representation for the last expectation.

Because the futures price given by equation (16) is a linear function of the reciprocal of the bond price, any further progress will depend on finding a good representation for expressions like $1/P(T,T+\lambda)$. The next proposition provides such a representation by showing that for any $t' \in [t,T]$ the reciprocal $1/P(t',T)$ may be written as the product of a positive random variable that is $\mathcal{F}_t$ measurable and two other terms that have simple bounds when conditioned on $\mathcal{F}_t$.

Proposition 1 (Product Representation for the Reciprocal Bond Price)**.** *For any family of HJM bond prices* $\{P(t,T)\}$ *and for all* $0 \le t \le t' \le T$, *we have the product representation*

$$\frac{1}{P(t',T)} = \frac{P(t,t')}{P(t,T)} \times \exp\left[-\int_t^{t'} \mathbf{a}(s,T)^\perp[\mathbf{a}(s,t') - \mathbf{a}(s,T)]\,ds\right]$$

$$\times \exp\left[\int_t^{t'} [\mathbf{a}(s,t') - \mathbf{a}(s,T)]^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_t^{t'} \left|\mathbf{a}(s,t') - \mathbf{a}(s,T)\right|^2 ds\right].$$

Proof. Using the basic bond price representation (11) we find that for all $0 \leq u \leq t' \leq T$

$$
\frac{P(u, t')}{P(u, T)} = \frac{P(0, t')\beta(u) \exp\left[\int_0^u \mathbf{a}(s, t')^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_0^u |\mathbf{a}(s, t')|^2\, ds\right]}{P(0, T)\beta(u) \exp\left[\int_0^u \mathbf{a}(s, T)^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_0^u |\mathbf{a}(s, T)|^2\, ds\right]}
$$

$$
= \frac{P(0, t')}{P(0, T)} \times
$$

$$
\times \exp\left[\int_0^u [\mathbf{a}(s, t') - \mathbf{a}(s, T)]^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_0^u [|\mathbf{a}(s, t')|^2 - |\mathbf{a}(s, T)|^2]\, ds\right]
$$

$$
= \frac{P(0, t')}{P(0, T)}\eta(u)\xi(u),
$$

where

$$
\eta(u) = \exp\left[\int_0^u [\mathbf{a}(s, t') - \mathbf{a}(s, T)]^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_0^u |\mathbf{a}(s, t') - \mathbf{a}(s, T)|^2\, ds\right],
$$

and

$$
\xi(u) = \exp\left[-\int_0^u \mathbf{a}(s, T)^\perp [\mathbf{a}(s, t') - \mathbf{a}(s, T)]\, ds\right].
$$

Using the representation

$$
\frac{P(u, t')}{P(u, T)} = \frac{P(0, t')}{P(0, T)}\eta(u)\xi(u)
$$

two times for $u = t'$ and $u = t$ we find

$$
\frac{1}{P(t', T)} = \frac{P(t', t')}{P(t', T)} = \frac{P(0, t')}{P(0, T)}\eta(t')\xi(t')
$$

$$
= \frac{P(0, t')}{P(0, T)}\eta(t)\xi(t)\frac{\eta(t')}{\eta(t)}\frac{\xi(t')}{\xi(t)}
$$

$$
= \frac{P(t, t')}{P(t, T)}\frac{\eta(t')}{\eta(t)}\frac{\xi(t')}{\xi(t)}.
$$

This is exactly our target identity, so the proof of the proposition is complete. $\square$

7. The Futures Price Inequality

We now have all the tools we need to prove the advertised upper bound for the value of LIBOR futures prices. The formal statement just requires two modest technical assumptions on the underlying HJM model and two structural assumptions that link the LIBOR quote $L_\lambda(t)$ to the HJM family of bond prices.

Theorem 1 (Futures Price Inequality). *Let* $\{P(t, T) : 0 \leq t \leq T \leq \tau\}$ *denote an HJM family of bond prices for which*

(17)   *all components of the volatility coefficient* $\boldsymbol{\sigma}(t, T)$ *are nonnegative*

and for which the integrated volatility $\mathbf{a}(s, \tau)$ *satisfies*

(18) $\widetilde{E} \exp\left[\frac{1}{2}\int_0^\tau |\mathbf{a}(s, \tau)|^2\, ds\right] < \infty.$

If the LIBOR quote $L_\lambda(t)$ is linked to the HJM model by the bookkeeping relation

$$(19) \qquad L_\lambda(t) = \frac{1}{\lambda}\left(\frac{1}{P(t, t+\lambda)} - 1\right),$$

and the LIBOR futures price process $F_\lambda(t, T)$ satisfies the martingale identity

$$(20) \qquad F_\lambda(t, T) = \widetilde{E}(100(1 - L_\lambda(T))|\mathcal{F}_t),$$

then it also satisfies the inequality

$$(21) \qquad F_\lambda(t, T) \leq 100\left(\frac{1}{\lambda} + 1\right) - \left(\frac{100}{\lambda}\right)\frac{1 + (T - t + \lambda)L_{T-t+\lambda}(t)}{1 + (T - t)L_{T-t}(t)}.$$

Proof. Naturally, we want to exploit Proposition 1, and our first step will be to show that under the hypotheses of the theorem we have the general inequality:

$$(22) \qquad \widetilde{E}\left[\frac{1}{P(t', T)}|\mathcal{F}_t\right] \geq \frac{P(t, t')}{P(t, T)} \quad \text{for all} \quad 0 \leq t \leq t' \leq T \leq \tau.$$

Since $\mathbf{a}(s, T) = -\int_s^T \boldsymbol{\sigma}(s, u)\, du$ for any $s < T$, the nonnegativity of $\boldsymbol{\sigma}(s, u)$ tells us that for all $0 \leq s \leq t' \leq T$ the components a_i of $\mathbf{a}$ satisfy

$$(23) \qquad a_i(s, T) \leq a_i(s, t') \leq 0 \quad \text{for all } 1 \leq i \leq n,$$

and from these coordinatewise bounds we find

$$(24) \qquad -\int_t^{t'} \mathbf{a}^\perp(s, T)[\mathbf{a}(s, t') - \mathbf{a}(s, T)]\, ds \geq 0.$$

Now, if we define the process $\eta(t)$ by setting

$$(25) \qquad \eta(t) = \exp\left[\int_0^t [\mathbf{a}(s, t') - \mathbf{a}(s, T)]^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_0^t |\mathbf{a}(s, t') - \mathbf{a}(s, T)|^2\, ds\right],$$

then Proposition 1 and the nonnegativity relation (24) give us the inequality

$$(26) \qquad \frac{1}{P(t', T)} \geq \frac{P(t, t')}{P(t, T)}\frac{\eta(t')}{\eta(t)} \quad \text{for all } 0 \leq t \leq t'.$$

Now, for any $0 \leq s \leq t' \leq T \leq \tau$ the monotonicity property (23) of the a_i tells us that $|\mathbf{a}(s, t') - \mathbf{a}(s, T)|^2 \leq |\mathbf{a}(s, \tau)|^2$, so the integrability hypothesis (18) implies

$$\widetilde{E}\exp\left[\frac{1}{2}\int_0^{t'} |\mathbf{a}(s, t') - \mathbf{a}(s, T)|^2\, ds\right] \leq \widetilde{E}\exp\left[\frac{1}{2}\int_0^\tau |\mathbf{a}(s, \tau)|^2\, ds\right] < \infty.$$

The validity of this Novikov condition is precisely what we need to be able to say that the exponential process $\eta(t)$ is actually a $\widetilde{P}$-martingale, and, in particular, we have

$$\widetilde{E}\left(\exp\left[\int_t^{t'} [\mathbf{a}(s, t') - \mathbf{a}(s, T)]^\perp d\widetilde{\mathbf{B}}_s - \frac{1}{2}\int_t^{t'} |\mathbf{a}(s, t') - \mathbf{a}(s, T)|^2\, ds\right]\bigg|\mathcal{F}_t\right) = 1,$$

Now, when we take the conditional expectation of the inequality (26) and apply the preceding identity, we complete the proof of the basic inequality (22).

We have already observed that the bookkeeping relation (19) and the martingale representation (20) give us

$$(27) \qquad F_\lambda(t, T) = 100\left(\frac{1}{\lambda} + 1\right) - \frac{100}{\lambda}\widetilde{E}\left[\frac{1}{P(T, T+\lambda)}|\mathcal{F}_t\right],$$

and now we can use (22) with the substitutions $t' \leftarrow T$ and $T \leftarrow T + \lambda$ in order to bound the the expectation in (27) to obtain

$$(28) \qquad F_\lambda(t, T) \leq 100 \left(\frac{1}{\lambda} + 1 \right) - \left(\frac{100}{\lambda} \right) \frac{P(t, T)}{P(t, T + \lambda)}.$$

Finally, to complete the proof of the futures price inequality we just need to apply the bookkeeping relation (19) twice more to express the ratio $P(t, T)/P(t, T + \lambda)$ in terms of LIBOR quotes. $\qquad\square$

Before leaving this theorem, we should note that the nonnegativity hypothesis (17) and the integrability hypothesis (18) are met by almost any sensible HJM model, so these two assumptions only serve to strain out the most pathological possibilities. All of the examples in Heath, Jarrow, and Morton (1992) satisfy these conditions, and, for the models of greatest importance, the verification is almost automatic. For example, in the continuous Ho–Lee model we have $\sigma(\omega, t, T) = \sigma$ where $\sigma > 0$ is constant, and in the Vasicek model we have $\sigma(\omega, t, T) = \sigma \exp(-\delta(T - t)) > 0$; so in both of these cases the required checks are trivial. In the more complex "Square Root Model" of Amin and Morton (1994) we have $\sigma(\omega, t, T) = \sigma |f(t, T)|^{1/2}$, and here nonnegativity is again trivial, but the integrability hypothesis (18) does entail a modest restriction on the growth of the underlying forward rates $\{f(t, T)\}$. Finally, a bigger challenge to the nonnegativity hypothesis is given by the "Proportional model" of Heath, Jarrow, and Morton (1992) where $\sigma(\omega, t, T) = \sigma \min(f(t, T), \delta)$ with $\sigma > 0$ and $\delta > 0$. The nonnegativity of $\sigma(\omega, t, T)$ is not at all obvious for this model, yet Heath, Jarrow, and Morton (1992) provide a clever argument that again establishes nonnegativity.

8. Upper Bound or Theoretical Value?

One of the most intriguing features of the futures price inequality of Theorem 1 is that there is a sense in which the upper bound

$$(29) \qquad U_\lambda(t, T) \equiv 100 \left(\frac{1}{\lambda} + 1 \right) - \frac{100}{\lambda} \frac{1 + (T - t + \lambda) L_{T-t+\lambda}(t)}{1 + (T - t) L_{T-t}(t)}$$

may be interpreted as the *theoretical value* for the futures price. The hypothesis that supports this interpretation is that the spot rate is a deterministic process. To be sure, this hypothesis is a strong one; there simply would be no interest rate futures markets if the spot rate was always exactly deterministic. Nevertheless, the hypothesis of a deterministic spot rate is sensible if it is viewed as either temporary or approximate. Alternatively, one can interpret the hypothesis purely as a theoretical limiting case.

Our first step is to show that if the spot rate is deterministic, then there is a simple formula for the bond price process in terms of the accumulation factor. This result is essentially contained in the argument of Musiela (1995, pp. 228–229), but it seems important enough to deserve explicit treatment. For the purpose of the lemma, a market is said to be *frictionless* provided there are no transaction costs and all instruments can be bought or sold short at the same price.

Lemma 1. *In a frictionless, arbitrage-free market where the spot rate $r(t)$ is a deterministic function, the accumulation factor $\beta(t)$ and the bond price*

processes $P(t,T)$ are also deterministic. Moreover, these processes are related by the formula:

$$P(t,T) = \frac{\beta(t)}{\beta(T)}.$$
(30)

Proof. Since $\beta(t)$ is a function of an integral (7) of $r(\cdot)$ over $[0,t]$, it is trivial that $\beta(t)$ is deterministic. Now, an investment of one dollar at time t in the T-maturity bond made will return $1/P(t,T)$ dollars at time T, and the value of this ratio is known at time t. Also, a one-dollar investment at the spot rate $r(\cdot)$ held over the same period would return

$$\exp\left(\int_t^T r(u)\,du\right) = \beta(T)/\beta(t),$$

and this value is also be known at time t under our hypothesis that $r(\cdot)$ is deterministic.

Since the returns for these two one dollar investments are both determined at time t, these returns must be equal, or else there would be an arbitrage opportunity. Specifically, we could go short one dollar in the instrument with the lesser return and go long one dollar in the instrument with the greater return. This would provide a zero-cost investment over the period $[t,T]$ that is guaranteed to provide a positive return, a circumstance that has been ruled out by our hypothesis that the market is arbitrage-free. This arbitrage argument proves that $P(t,T)$ has the ratio representation (30), and, as a consequence of this representation, we see that $P(t,T)$ must also be deterministic. $\square$

Now, under the conditions of the preceding lemma and the assumption that the martingale representation (20) for $F_\lambda(t,T)$ continues to hold, we are now ready to show that that $U_\lambda(t,T)$ is the unique arbitrage-free value for the futures price. To see why this is so, we first take $t \le t' \le T$ and note that

$$\frac{1}{P(t',T)} = \frac{\beta(T)}{\beta(t')} = \frac{\beta(T)}{\beta(t)}\frac{\beta(t)}{\beta(t')} = \frac{P(t,t')}{P(t,T)}.$$
(31)

Next, we take the martingale representation (20) for $F_\lambda(t,T)$, and note that when the integrand is no longer random, the $\widetilde{E}$-expectation is just the expectation of a constant. Consequently the "martingale representation" simply reduces to the identity,

$$F_\lambda(t,T) = 100(1 - L_\lambda(T)) \quad \text{for all } 0 \le t \le T.$$
(32)

From this identity and the representation (31), we find by two of applications of the bookkeeping relation (19) that we have

$$F_\lambda(t,T) = 100\left(\frac{1}{\lambda} + 1\right) - \frac{100}{\lambda}\frac{1}{P(T,T+\lambda)} = 100\left(\frac{1}{\lambda} + 1\right) - \frac{100}{\lambda}\frac{P(t,T)}{P(t,T+\lambda)}$$

$$= 100\left(\frac{1}{\lambda} + 1\right) - \frac{100}{\lambda}\frac{1 + (T - t + \lambda)L_{T-t+\lambda}(t)}{1 + (T - t)L_{T-t}(t)} = U_\lambda(t,T).$$

The bottom line here is that when assume that we have a deterministic term structure, we are brought quickly to the rather remarkable identity:

$$F_\lambda(t,T) = U_\lambda(t,T).$$
(33)

To be sure, this identity requires the strong assumption that the spot rate is deterministic, and in such a world the interest rate futures markets would dry up very quickly. Perhaps the soundest way to interpret this assumption is to view it as a theoretical limiting case. With this interpretation, our calculations simply say that when one moves from a world with deterministic spot rates to a world that allows for spot rates to fluctuate randomly, the value of the futures price can only decrease.

Now we face an interesting possibility. Is it reasonable in the real world for the random fluctuations of the spot rate to be so small that the theoretical upper bound $U_\lambda(t, T)$ might still serve as a practical proxy for the LIBOR futures price? We provide a partial answer to this question in the next section.

9. Calculation of the $U_\lambda(t, T)$–$F_\lambda(t, T)$ Gap

Under the general HJM model, there does not seem to be any natural way to calculate the gap between the futures price and its theoretical upper bound, but in the important special case where where the integrated volatility coefficients are deterministic, the next proposition shows that there is a formula for $F_\lambda(t, T)$ that leads quickly to a practical formula for the gap between $U_\lambda(t, T)$ and $F_\lambda(t, T)$. The propostion is quite close to some well-known results, such as those summarized in Musiela and Rukowski (1997, Proposition 15.2.1); but, to draw out the explicit connection with LIBOR futures, we include a simple explicit proof.

One should note that here the assumption that the integrated volatility coefficients are deterministic is relatively modest; so, in particular, the Vasicek and Ho–Lee models both satisfy this condition. The proposition therefore gives us just the tool we need to start to assess economic significance of the size of the gap between the futures price and its theoretical upper bound.

Proposition 2. *Suppose that $\{P(t, T) : 0 \leq t \leq T \leq \tau\}$ is an HJM family of bond prices for which the integrated volatility coefficient $\mathbf{a}(t, T)$ is a deterministic function that satisfies the integrability condition*

$$(34) \qquad \int_0^T |\mathbf{a}(s, T)|^2 \, ds < \infty \quad \text{for all } \leq T \leq \tau.$$

If the LIBOR quotes are linked to the bond model by the usual bookkeeping identity, and the LIBOR futures price process $F_\lambda(t, T)$ satisfies the martingale identity

$$F_\lambda(t, T) = \widetilde{E}(100(1 - L_\lambda(T))|\mathcal{F}_t),$$

then $F_\lambda(t, T)$ also satisfies

$$(35) \qquad F_\lambda(t, T) = 100\left(\frac{1}{\lambda} + 1\right) - \frac{100}{\lambda}\frac{1 + (T - t + \lambda)L_{T-t+\lambda}(t)}{1 + (T - t)L_{T-t}(t)}G(t, T, \lambda)$$

where

$$(36) \qquad G(t, T, \lambda) = \exp\left[-\int_t^T \mathbf{a}^\perp(s, T + \lambda)[\mathbf{a}(s, T) - \mathbf{a}(s, T + \lambda)] \, ds\right].$$

Proof. The natural idea is to follow the plan of Theorem 1 and to find the right spot to exploit our strengthened hypotheses. We first note that if we take the

product representation of Proposition 1 and make the substitutions $t' \leftarrow T$ and $T \leftarrow T + \lambda$ then the representation may be written as

$$(37) \qquad \frac{1}{P(T, T+\lambda)} = \frac{P(t,T)}{P(t,T+\lambda)} \, G(t,T,\lambda) \, \frac{\eta(T)}{\eta(t)}$$

where $\eta(t)$ is the exponential process defined by equation (25). Since $\mathbf{a}(\cdot, \cdot)$ is assumed to be deterministic, the integrability condition (34) implies the Novikov condition for $\eta(\cdot)$, so $\eta(\cdot)$ is in fact an honest martingale.

Because the first term of the factorization (37) is $\mathcal{F}_t$–measurable and the second factor is deterministic, we can take the conditional expectation and use the fact that $E(\eta(T)/\eta(t)|\mathcal{F}_t) = 1$ to deduce that

$$(38) \qquad E\left(\frac{1}{P(T,T+\lambda)} \, | \, \mathcal{F}_t \right) = \frac{P(t,T)}{P(t,T+\lambda)} G(t,T,\lambda).$$

Now, if we insert this identity into the representation (27) for the futures price as a conditional expectation, we find

$$(39) \qquad F_\lambda(t,T) = 100 \left(\frac{1}{\lambda} + 1 \right) - \left(\frac{100}{\lambda} \right) \frac{P(t,T)}{P(t,T+\lambda)} G(t,T,\lambda),$$

and if we use the bookkeeping relation to rewrite the bond prices in the last identity in terms of LIBOR quotes, we come at last to the identity claimed by the proposition. □

Here we should note how this calculation exploits the assumption that the integrated volatility $\mathbf{a}(t,T)$ is deterministic. The main observation is that when we take the conditional expectation in the factorization (37), the first two factors can be brought outside the expectation. It is this step that lets us extract the full benefit from the fact that $\eta(t)$ is an exponential martingale. On the other hand, in the proof of Theorem 1 the stochastic dependence of $G(t,T,\lambda)$ and $\eta(T)/\eta(t)$ made the two factors inseparable, so we were forced to argue more indirectly that the expectation of $G(t,T,\lambda)\eta(T)/\eta(t)$ could be bounded below by one.

Further, we should note that the decomposition given by Proposition 2 has an economic interpretation. The factor $G(t,T,\lambda) \geq 1$ provides a price deflator that is readily calculated in terms of the parameters of the underlying model, and, moreover, the explicit formula (36) for $G(t,T,\lambda)$ tells us in essence that models with larger volatilities have larger gaps between the futures price and the theoretical upper bound. The fact that price gap increases with volatility seems to point to a basic qualitative phenomenon that may hold in considerable generality, although, admittedly, there are several ways one might measure the size of the volatility.

Finally, to see how the futures price identity (35) of Proposition 2 works in a concrete example, we do well to consider the important case of the 90-day LIBOR Eurodollar futures, the world's most actively traded interest rate futures contracts. Also, as our bond model, we take the continuous-time Ho–Lee model.

For the Ho–Lee model, the forward rate volatility coefficient $\sigma(\omega, t, T)$ is just a constant that we may denote by σ, and for loans with a 90-day maturity we have $\lambda = 1/4$. The formula (36) for the model deflator $G(t,T,\lambda)$ and trivial integrations therefore yield

$$\ln G(t,T,1/4) = \frac{\sigma^2}{8}(T-t)(T-t+\frac{1}{2}),$$

so the gap $U_{1/4}(t,T) - F_{1/4}(t,T)$ between the upper bound and the futures price process boils down to just

$$400\frac{1 + (T - t + \frac{1}{4})L_{T-t+\frac{1}{4}}(t)}{1 + (T - t)L_{T-t}(t)}\left\{ -1 + \exp\left(\frac{\sigma^2}{8}(T - t)(T - t + \frac{1}{2})\right)\right\}.$$

Now, if we consider the typical situation where $T - t \leq 1$, $\sigma \leq .05$, and the two the quotes $L_{T-t+1/4}(t)$ and $L_{T-t}(t)$ are less than 0.08, then easy estimates show that the price gap for the 90-day Eurodollar futures under the Ho–Lee model has the simple approximation

(40) $$U_{1/4}(t,T) - F_{1/4}(t,T) \approx (400 \pm 40)\frac{\sigma^2}{8}(T - t)(T - t + \frac{1}{2}).$$

One nice feature of this formula is that by accepting a restriction on $L_{T-t}(t)$ and $L_{T-t+1/4}(t)$, we have a deterministic estimate of the price gap. The formula also makes it clear that the price gap will be small when either σ or $T - t$ is small, but the approximation is of most economic interest when σ and $T - t$ take on moderate values. In that case, the gap approximation (40) may be used to argue that the difference between the futures price process and its theoretical upper bound may indeed be economically significant.

In their empirical investigations of bond prices under the Ho–Lee model, Amin and Morton (1994) and Flesaker (1993) found that σ usually ranges between 0.01 and 0.03. If we take the typical values $\sigma = 0.02$ and $T - t = 1$ in our approximation for the futures price gap (40), then we find that our estimate for the gap is about 0.030. To get a feeling for the magnitude of this gap, we should recall that for the LIBOR futures the tick (or minimal price change) is equal to 0.0025. The estimated gap in this example is about therefore about 12 ticks, and, for traders, 12 ticks is likely to be of economic significance. A second way to understand the magnitude of the gap is to note that the median of the absolute daily change of the futures prices in January 1998 for the contract that expired in September 1998 was equal to 0.050; so, for this reasonably typical contract, the observed MAD price change is about the same size as the the estimated gap between the general theoretical upper bound and the theoretical arbitrage price under the Ho–Lee model. Individual views may vary, but from most perspectives, a gap that is comparable to a typical day's price change is also too large to be seen as economically negligible.

10. The Martingale Representation of $F_\lambda(t,T)$

In all the preceding sections, we relied profoundly on the link between the bond model and the futures price that is given by our assumption that $F_\lambda(t,T)$ satisfies the martingale identity

(41) $$F_\lambda(t,T) = \widetilde{E}(B|\mathcal{F}_t),$$

where $B = 100(1 - L_\lambda(T))$ is the contractually guaranteed settlement price at the terminal time T. Representations analogous to (41) are known to hold in a large class of models for commodity futures price processes, say for gold or pork bellies, where one may assume that the spot rate $r(\cdot)$ is particularly well-behaved. Specifically, Duffie (1996, p.170) provides such a martingale representation for a price model where one may assume that spot rate process $r(\cdot)$ satisfies the condition

(42) $$P(A_1 \leq \inf_{0 \leq t \leq T} r(t) \leq \sup_{0 \leq t \leq T} r(t) \leq A_2) = 1$$

for some constants $-\infty < A_1 \leq A_2 < \infty$, and Karatzas and Shreve (1998, p.45) provide a martingale representation under the slightly weaker assumption that

$$(43) \qquad P(\alpha_1 \leq \inf_{0 \leq t \leq T} \beta(t) \leq \sup_{0 \leq t \leq T} \beta(t) \leq \alpha_2) = 1$$

for some constants $0 < \alpha_1 \leq \alpha_2 < \infty$. These assumptions may be perfectly reasonable for many types of futures prices, and committedly practical individuals may feel that such assumptions are always appropriate. Nevertheless, when one is specifically concerned with the theoretical analysis of interest rate futures, then these assumptions are just too strong to be used.

To see why this is so, we only need to consider the special case of the HJM model that corresponds to the continuous-time Ho–Lee model. This is pehaps the most tractable of nontrivial the HJM models, and one can easily show, say as in Heath, Jarrow, and Morton (1992, p.90), that the spot rate $r(t) = f(t, t)$ has the representation

$$(44) \qquad r(t) = f(0, t) + \sigma^2 t^2/2 + \sigma \widetilde{B}_t,$$

where $\{\widetilde{B}_t\}$ is a $\widetilde{P}$-Brownian motion and the initial forward rate curve $f(0, t)$ is only assumed to be bounded. In the nontrivial case when $\sigma \neq 0$, the simple fact that Brownian motion is Gaussian immediately implies that the spot rate boundedness condition (42) does not hold, and a modest computation shows that the accumulation factor boundness condition (43) also fails. Finally, since any generally applicable theory of interest rate futures should at least be able accommodate the widely used continuous Ho–Lee model, we are forced to look for a way to relax the assumption that the spot rate satisfies the boundedness condition.

The main aim of this section is to show how the formalization of general futures price processes given by Karatzas and Shreve (1998) may be modified to provide a more appropriate formalization for interest rate futures. Specifically, we will show how one may replace the accumulation factor condition (43) by an alternative assumption that is easily checked for the Ho–Lee model (and many others). The required modification is small and technical, but nevertheless it seems useful. Perhaps it even rescues the theory of interest rate futures from the horns of a dilemma.

In Karatzas and Shreve (1998), a *cumulative income process* is nothing more or less than a semimartingale, and our first step here is to consider a semimartingale $\{\phi(t) : 0 \leq t \leq T\}$, which we view informally as the cumulative income received by the holder of an interest rate futures contact during the time interval $[0, t]$. Next, we write $\phi(\cdot)$ in terms of the usual semimartingale decomposition

$$(45) \qquad \phi(t) = \phi(0) + \phi^{fv}(t) + \phi^{lm}(t), \qquad 0 \leq t \leq T,$$

where $\phi^{fv}(\cdot)$ is a càdlàg process with finite variation and $\phi^{lm}(\cdot)$ is a $\widetilde{P}$-local martingale. Here, $\widetilde{P}$ continues to refer to the equivalent martingale measure of the underlying HJM model $\mathcal{M}$, and the decomposition is unique $\widetilde{P}$-a.s. provided that we take standardized initial values $\phi^{fv}(0) = 0$ and $\phi^{lm}(0) = 0$. To be completely precise, we should call ϕ a cumulative cash flow process associated with the HJM model $\mathcal{M}$.

We continue in parallel with Karatzas and Shreve (1998, p.18) and say that a cumulative income process $\phi(\cdot)$ associated with an HJM model $\mathcal{M}$ is *integrable* if

provided that it satisfies the two conditions

$$(46) \qquad \widetilde{E} \int_0^T \beta(t)^{-1} d\widehat{\phi}^{fv}(t) < \infty \text{ and } \widetilde{E} \int_0^T \beta(t)^{-2} \, d\langle\phi^{lm}\rangle(s) < \infty,$$

where $\widehat{\phi}^{fv}(t)$ denotes the total variation of $\phi^{fv}(\cdot)$ on $[0, t]$ and $\langle\phi^{lm}\rangle(t)$ denotes the quadratic variation of $\phi^{lm}(\cdot)$ on $[0, t]$. We then take another step with Karatzas and Shreve (1998, p.39) and call the integrable cumulative income process $\phi(\cdot)$ a *European contingent claim associated with the HJM model* $\mathcal{M}$.

From the original paper Heath, Jarrow, and Morton (1992), we know that the HJM bond model is complete, and, therefore, by the general theory of arbitrage prices — or, more specifically by Proposition 2.3 of Karatzas and Shreve (1998, p.41)) — we know that the unique arbitrage-free price at time $t \in [0, T]$ of the European contingent claim $\phi(\cdot)$ is given by the classic pricing formula

$$(47) \qquad \beta(t)\widetilde{E}\left[\int_t^T \beta(s)^{-1} \, d\phi(s)|\mathcal{F}_t\right].$$

Now, we have everything we need to give a definition of a futures price process associated with an HJM model. We only need to take the definition of a futures prices process given by Karatzas and Shreve (1998, p.45) and provide a translation into the HJM context.

Definition 1 (Futures Price Process Associated with an HJM Model)**.**
   If $\{\phi(t) : 0 \le t \le T\}$ is a European contingent claim associated with the HJM model $\mathcal{M}$, then $\phi(\cdot)$ is called a futures price process *with terminal value $B \in \mathcal{F}_T$ provided that $\phi(T) = B$ and the arbitrage-free price (47) of $\phi(\cdot)$ is equal to zero $\widetilde{P}$-a.s. for all $t \in [0, T]$.*

The motivation that drives this abstract notion of a futures price process is the fact that at any time one can enter into a futures contract on either the long or short side at zero cost. Therefore, if $\phi(t)$ represents the cash flow of a futures contract over the period $[0, t]$, then arbitrage free price of that cash flow must also equal zero, or else we would have an arbitrage possibility.

The next proposition is the main result of this section. It provides a simple condition for the futures price process to have the martingale representation property that we took as axiomatic in all of the earlier sections. The statement and proof require only small twists in the formulation and proof Theorem 3.7 of Karatzas and Shreve (1998, p.45), yet these twists let us escape from the draconian assumption (43) of a bounded accumulation factor.

Proposition 3 (Representation of Futures Prices)**.** *Let B be an $\mathcal{F}_T$-measurable random variable such that*
$$\widetilde{E}[B^2] < \infty.$$
If a futures price process $\{\phi(t) : 0 \le t \le T\}$ associated with an HJM model $\mathcal{M}$ has terminal value B and satisfies

$$(48) \qquad \widetilde{E}\langle\phi^{lm}\rangle(T) < \infty,$$

where $\langle\phi^{lm}\rangle(\cdot)$ denotes the quadratic variation of $\phi^{lm}(\cdot)$, then the process $\phi(\cdot)$ is a $\widetilde{P}$-martingale on $[0, T]$, and we have the representation

$$(49) \qquad \phi(t) = \widetilde{E}[B|\mathcal{F}_t] \quad \text{for all} \quad 0 \le t \le T.$$

Conversely, if the martingale $\psi(\cdot)$ defined by

(50) $$\psi(t) = \widetilde{E}[B|\mathcal{F}_t] \quad for \quad 0 \le t \le T$$

satisfies

(51) $$\widetilde{E} \int_0^T \beta(t)^{-2} d\langle\psi\rangle(t) < \infty,$$

where $\langle\psi\rangle(\cdot)$ denotes the quadratic variation of $\psi(\cdot)$, then the process $\psi(\cdot)$ is a futures price process for $\mathcal{M}$ in the sense of definition (1).

Proof. For any HJM model, we have $P(\beta(t) = 0) = 0$ for all $t \in [0, T]$, so, from the zero-price constraint in the definition of a futures price process, we see that ϕ satisfies

(52) $$\widetilde{E}\left[\int_t^T \beta(u)^{-1} d\phi(u) \,|\, \mathcal{F}_t\right] = 0 \quad \widetilde{P}\text{-}a.s. \quad \text{ for all } t \in [0, T].$$

If we define a new process $I(\cdot)$ by the stochastic integral

$$I(t) = \int_0^t \beta(u)^{-1} d\phi(u) \qquad 0 \le t \le T,$$

then (52) tells us that for all $t \in [0, T]$ the process $I(\cdot)$ satisfies

$$\widetilde{E}[I(T) \,|\, \mathcal{F}_t] = \int_0^t \beta(u)^{-1} d\phi(u) + \widetilde{E}\left[\int_t^T \beta(u)^{-1} d\phi(u) \,|\, \mathcal{F}_t\right] = I(t),$$

and from this identity we see that $I(\cdot)$ is a $\widetilde{P}$-martingale. Now, by its construction, $I(\cdot)$ has the stochastic differential

$$dI(t) = \beta(t)^{-1} d\phi(t),$$

and, if multiply this equation by $\beta(t)$ and integrate, we find that $\phi(\cdot)$ has the a simple representation

$$\phi(t) - \phi(0) = \int_0^t \beta(u) \, dI(u).$$

This representation tells us that the process $\phi(\cdot)$ is a $\widetilde{P}$-local martingale, so, in terms of its canonical decomposition as a semimartingale we have $\phi^{fv}(\cdot) \equiv 0$ and $\phi^{lm}(\cdot) = \phi(\cdot) - \phi(0)$. These relations provide the required link to our hypothesis (48) on the quadratic variation of $\phi^{lm}(\cdot)$. Specifically, it is well known, say from Karatzas and Shreve (1997, p. 38), that any local martingale with an integrable quadratic variation is in fact an honest square-integrable martingale. This observation tells us $\phi(\cdot)$ is a martingale, and nothing more is needed to complete the proof of the direct half of the proposition.

The proof of the converse half is even easier; the terminal condition is trivial, so we just need to check that the process $\psi(\cdot)$ defined by (50) is a European contingent claim that satisfies the zero price condition of required by the definition of a futures price process. First, we note that the square integrability hypothesis on B and Jensen's inequality imply that the process $\psi(\cdot)$ is a square integrable $\widetilde{P}$-martingale, so the canonical decomposition of $\psi(\cdot)$ as a semimartingale is trivially given by $\psi^{fv}(\cdot) \equiv 0$ and $\psi^{lm}(\cdot) = \psi(\cdot) - \psi(0)$. By our key assumption (51) on $\psi(\cdot)$ we have the first of the two integrability conditions (46). For $\psi(\cdot)$ the second condition is vacuous, so $\psi(\cdot)$ meets all the requirements of a European contingent claim.

Now, if we define a new process $J(\cdot)$ by the stochastic integral

$$J(t) = \int_0^t \beta(u)^{-1}\, d\psi(u) \qquad t \in [0, T],$$

then, by our hypothesis (51) on the quadratic variation of $\psi(\cdot)$ and the "well known fact" used just a few lines ago, we see that that the process $J(\cdot)$ is a square-integrable $\widetilde{P}$-martingale. From the martingale property of $J(\cdot)$, we trivially find

$$\beta(t)\widetilde{E}[J(T) - J(t)|\mathcal{F}_t] = 0, \quad \text{for all } t \in [0, T].$$

Since this is precisely the zero-price condition for the European contingent claim $\psi(\cdot)$ that is required by the definition of a futures price process and since the uniqueness assertion is immediated from the first part of the proposition, the converse is also complete. $\qquad\square$

As we mentioned earlier, Proposition 3 is a modest refinement of Theorem 3.7 of Karatzas and Shreve (1998, p.45), and the following corollary show how the "two halves" of the accumulation factor condition (43) of Karatzas and Shreve relate to the rather different looking conditions of Proposition 3.

Corollary 1. *Let B be an $\mathcal{F}_T$-measurable random variable such that*

$$\widetilde{E}[B^2] < \infty.$$

If the accumulation factor $\beta(\cdot)$ of the HJM model $\mathcal{M}$ satisfies the two conditions

$$(53) \quad (a) \quad P\left(\alpha_1 \leq \inf_{0 \leq t \leq T} \beta(t)\right) = 1 \quad and \quad (b) \quad P\left(\sup_{0 \leq t \leq T} \beta(t) \leq \alpha_2\right) = 1$$

for some constants $0 < \alpha_1 \leq \alpha_2 < \infty$, then there exist a unique futures price process associated with the HJM and the terminal value B; moreover, it is given by

$$(54) \qquad\qquad \psi(t) = \widetilde{E}[B|\mathcal{F}_t] \quad for\ all \qquad t \in [0, T].$$

Proof. By Proposition 3, we know that the process $\psi(\cdot)$ defined by (54) will be an honest futures price process provided that it satisfies the integrability condition (51). Next, by the part (a) of our hypothesis (53) on the accumulation factor, we find

$$(55) \qquad \widetilde{E}\int_0^T \beta(s)^{-2}\, d\langle\psi\rangle(s) \leq \frac{1}{\alpha_1{}^2}\widetilde{E}\int_0^T d\langle\psi\rangle(s) = \frac{1}{\alpha_1{}^2}\widetilde{\mathrm{Var}}(B) < \infty$$

so $\psi(\cdot)$ satisfies (51) — just as required.

Now, to prove that $\psi(\cdot)$ is the unique HJM futures price process with terminal value B, we first note under part (b) of our hypothesis (53) on the accumulation factor, *any* futures price process $\phi(\cdot)$ satisfies

$$\widetilde{E}\langle\phi^{lm}\rangle(T) = \widetilde{E}\int_0^T \beta(s)^2\beta(s)^{-2}\, d\langle\phi^{lm}\rangle(s) \leq \alpha_2^2\widetilde{E}\int_0^T \beta(s)^{-2}\, d\langle\phi^{lm}\rangle(s) < \infty$$

where the last inequality comes from the fact that a futures price process is an integrable cumulative price process. This inequality confirms the hypothesis (48) of the direct part of Proposition 3 and therefore establishes that $\phi(\cdot)$ is a martingale. Since $\phi(T) = B$, the martingale identity tells us

$$\phi(t) = \widetilde{E}(B|\mathcal{F}_t) = \psi(t) \quad \widetilde{P}\text{-a.s.} \quad \text{for all } t \in [0, T],$$

and this is the detailed assertion of the required uniqueness. $\qquad\square$

Our final task here is to show how Proposition 3 may be used to construct LIBOR futures price processes in concrete HJM models, and, specifically, we want to show how Proposition 3 may be used to overcome the difficulties with the Ho-Lee model that were discussed at the beginning of the section. First of all, the converse part of Proposition 3 tells us that the process $F_\lambda(t,T)$ defined by

$$(56) \qquad F_\lambda(t,T) = \widetilde{E}\left[100(1 - L_\lambda(T)) \mid \mathcal{F}_t\right] \qquad 0 \le t \le T,$$

is a futures price process with the terminal value $B = 100(1 - L_\lambda(T))$ provided that we can check two basic conditions. To get the ball rolling, we need to show

$$(57) \qquad \widetilde{E}[B^2] < \infty,$$

and, more delicately, we need to show that the quadratic variation of the process $F_\lambda(t,T)$ defined by (56) satisfies the integrability condition

$$(58) \qquad \widetilde{E}\int_0^T \beta(s)^{-2} d\langle F_\lambda(\cdot,T)\rangle(s) < \infty,$$

The verification of these inequalities inevitably calls for a detailed understanding of the underlying HJM model, and the case of the Ho-Lee model provides a good illustration of the required computations.

Proposition 4. *Conditions (57) and (58) both hold under the Ho-Lee model, and consequently the process $F_\lambda(t,T)$ given by (56) in a HJM futures price process in the sense of Definition 1. Moreover, $F_\lambda(t,T)$ is the unique HJM futures price process with terminal value B.*

Proof. The bookkeeping identity for the LIBOR quotes tells us that

$$L_\lambda(T) = \frac{1}{\lambda}\left(\frac{1}{P(T,T+\lambda)} - 1\right),$$

so, to prove (57), it suffices to show that

$$(59) \qquad \widetilde{E}\left(P(T,T+\lambda)^{-2}\right) < \infty.$$

The most direct approach to this inequality begins with the important representation (11) from Section 2. If we use that representation for both $P(t,T)$ and $P(t,t) = 1$, we can take the ratio and eliminate the accumulation factor $\beta(t)$, and if we specialize the resulting formula to the Ho-Lee model where $\sigma(t,T) \equiv \sigma$, a short calculation will then show

$$(60) \qquad P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[-\frac{\sigma^2}{2}Tt(T-t) - \sigma(T-t)\widetilde{B}_t\right].$$

From this formula, we see that $P(T,T+\lambda)^{-2}$ is a product of a deterministic function and and a random variable with a lognormal distribution under $\widetilde{P}$ so it has a finite $\widetilde{P}$-expectation, just as required by (59).

In order to prove the quadratic variation bound (58), we first recall from Proposition 2 of Section 9 we have that

$$(61) \qquad F_\lambda(t,T) = 100\left(\frac{1}{\lambda}+1\right) - \left(\frac{100}{\lambda}\right)\frac{P(t,T)}{P(t,T+\lambda)}G(t,T,\lambda),$$

where $G(t,T,\lambda) = \exp\left[(\sigma^2/8)(T-t)(T-t+1/2)\right]$. With the help of the bond price representation (60) we can use this formula to get a completely explicit formula

for the quadratic variation of $F_\lambda(t, T)$. If we use (58) to eliminate the bond prices from (61), we find

$$F_\lambda(t, T) = 100 \left(\frac{1}{\lambda} + 1 \right) - D(t, T, \lambda) \exp(\sigma \lambda \widetilde{B}_t),$$

where

$$D(t, T, \lambda) = \frac{100}{\lambda} \frac{P(0, T)}{P(0, T + \lambda)} \exp \left[\frac{\sigma^2}{2}(T + \lambda)t(T + \lambda - t) - \frac{\sigma^2}{2} Tt(T - t) \right] G(t, T, \lambda)$$

A thoughtless application of Itô's formula would be messy, but, if we first note that the definition (56) of $F_\lambda(t, T)$ tells us $F_\lambda(t, T)$ is a $\widetilde{P}$-martingale, then we know that the drift term of $dF_\lambda(t, T)$ must be zero, so Itô's formula gives us simply

$$dF_\lambda(t, T) = -\sigma \lambda D(t, T, \lambda) \exp[\sigma \lambda \widetilde{B}] \, d\widetilde{B}_t.$$

We then find that the quadratic variation $\langle F_\lambda(\cdot, T) \rangle(\cdot)$ of the process $F_\lambda(\cdot, T)$ satisfies the SDE

$$d\langle F_\lambda(\cdot, T) \rangle(t) = \sigma^2 \lambda^2 D(t, T, \lambda)^2 \exp[2\sigma \lambda \widetilde{B}_t] \, dt.$$

Now, since the spot rate for the Ho-Lee model is given by (44), the accumulation factor is simply

$$\beta(t) = \exp \left[\int_0^t (f(0, s) + \sigma^2 s^2 / 2 + \sigma \widetilde{B}_s) \, ds \right],$$

and, at last, we come to an explicit representation of the critical integral (58):

$$(62) \quad \int_0^T \beta(t)^{-2} d\langle F_\lambda(\cdot, T) \rangle(t) = \int_0^T D_1(t, T, \lambda) \exp \left[-\int_0^t 2\sigma \widetilde{B}_s \, ds + 2\sigma \lambda \widetilde{B}_t \right] dt,$$

where

$$D_1(t, T, \lambda) = \sigma^2 \lambda^2 D(t, T, \lambda)^2 \exp \left[-\int_0^t (2f(0, s) + \sigma^2 s^2) \, ds \right].$$

By a short calculation with Itô's formula and Itô's isometry, we find that

$$-\int_0^t 2\sigma \widetilde{B}_s \, ds + 2\sigma \lambda \widetilde{B}_t = 2\sigma \int_0^t (s - \lambda - t) \, dB_s$$
$$\stackrel{d}{=} N[0, 4\sigma^2 \{ \lambda^3 / 3 - (\lambda - t)^3 / 3 \}].$$

This tells us that the inside integrand of the critical integral (62) has the lognormal distribution, so, by the boundedness of the deterministic function $D_1(t, T, \lambda)$, one obtains the finiteness of the expectation of (62) after routine estimates.   □

Proposition 4 completes the program that began with the observation that the futures price formalizations of Duffie (1996) and Karatzas and Shreve (1998) could not accommodate the Ho-Lee model. We now see that by shifting the focus away from the accumulation factor toward the quadratic variation criterion (48), one obtains a formalization of the theory of futures price processes which is much more comfortable for modelers of interest rate futures.

11. Empirical Observations

The LIBOR futures price theoretical upper bound holds *uniformly* for a very large class of models for the term structure of interest rates, and the natural concern provoked by such generality is possibility that the bound might be unrealistically large. It was therefore quite surprising to discover that the theoretical upper bound is frequently violated by observed market prices, and, moreover, the violations are often large enough to be economically significant. Even though the empirical results reported here are not intended to definitive, they strongly suggests that there are deep conflicts between market realities and the mathematical consequences of an important class of yield curve models.

Before we describe the evidence for our findings, we need to give a brief review of the data sources that were used. To make any use of the futures price upper bound, one needs data on LIBOR quotes, and an important public source of such data is the website of the British Bankers' Association `www.bba.org.uk`. Also, in order to evaluate the performance of the futures price upper bound, one needs market price data for the LIBOR futures. Here we relied upon the data on the Eurodollar LIBOR contracts that are sponsored by Chicago Mercantile Exchange. This price data is publicly available from the website `www.barchart.com/cme/cmedta.htm`, as well as other locations. The great benefit of these data sources is their easy public access, but their use does impose some limitations on our analysis.

There are also some computational details that need to be discussed. As an illustration, we will compute the value of the upper bound as of January 2nd, 1998 for the 90-day LIBOR Eurodollar quotes with settlement in September of 1998. For the quotes on 90-day deposits, we have $\lambda = 1/4$, and the upper bound $U_{1/4}(t, T)$ may be written as

$$(63) \qquad U_{1/4}(t, T) = 500 - 400 \frac{1 + (T - t + 1/4)L_{T-t+1/4}(t)}{1 + (T - t)L_{T-t}(t)},$$

where one needs to keep in mind that λ, t, and T are all scaled so that one unit equals 360 days. By tradition, the settlement date for each contract is the second London bank business day before the third Wednesday of the contract month. For the contract expiring in September 1998, we find that the settlement date T is 14th of September 1998, which is 255 days after the "current date" t, January 2nd, so we find that $T - t = 255/360$. A small point worth noting here is that since we are always concerned with differences between dates like $T - t$, the origin of our time line can be picked arbitrarily.

Finally, to compute $U_{1/4}(t, T)$ for the current date t, January 2nd, and the settlement date T, September 14th, we need LIBOR quotes at time t with maturities of 255 days and $255 + 90$ days, or in our normalized time scale, we need LIBOR quotes with maturities $\lambda = 255/360$ and $\lambda = 345/360$. Such quotes are not publicly available, and, if we are to rely on the data available from the British Bankers' Association, an interpolation is needed. Here we decided to use the cubic spline interpolation given by the standard S-Plus function `spline`, where the interpolating spline was calculated by using the available BBA data for LIBOR quotes with scaled maturities λ given by $7/360, 30/360, 60/360 \ldots, 360/360$ corresponding to loan commitments of one week and k months where $1 \le k \le 12$. For the example under consideration, cubic spline interpolation provided interpolated LIBOR quotes for $\lambda = 255/360$ and $\lambda = 345/360$ that were equal to 5.882092% and 5.929697%

respectively. With these values in hand, all the inputs required by the formula for $U_\lambda(t, T)$ are available, and, one finds that as of January 2, 1998, the theoretical upper bound for the futures price for the contract with September settlement is equal to 94.178. Here we should acknowledge that interpolation is never a comfortable experience, but $L_\lambda(T)$ does depend smoothly on λ, and when other interpolation methods were tested for comparison, there was no significant impact on the overall analysis.

The theoretical upper bound $U_{1/4}(t, T)$ was computed for all t corresponding to the twenty trading days of January 1998 and the fixed settlement date T equal to September 14, 1998. A time series line plots these values is given by the solid line of Figure 1, and, for comparison, the corresponding values of the futures prices $F_{1/4}(t, T)$ from the Chicago Mercantile Exchange were plotted as a dotted line.
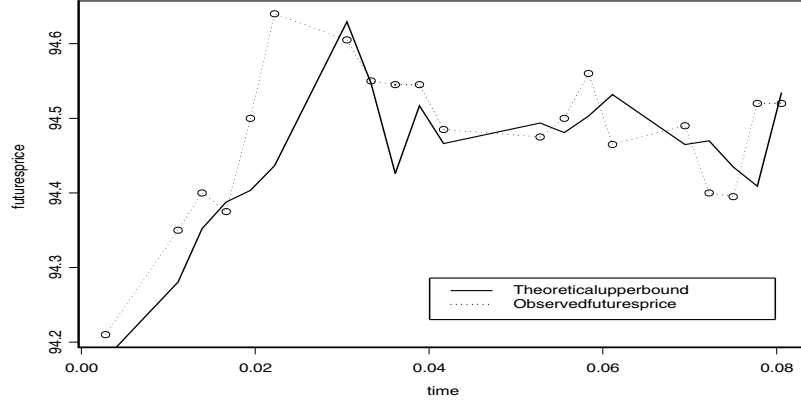

FIGURE 1. 90-Day LIBOR Futures Prices and Theoretical Upper Bound for the Twenty Trading Days During January 1998 for the Contract with Settlement in September 1998.

The most striking feature of this plot is that the solid and dotted lines cross several times, and the solid line does not seem to have any of the features one expects from an upper bound. This behavior was definitely unanticipated, and it is surprising for several reasons. First, the theoretical upper bound does not depend on any unknown parameters, so parameter estimation error is not an issue. Also, the bound applies simultaneously for essentially all of the interest rate models that have been used in practice, so model selection is not an issue. In fact, the derivation of the futures price upper bound depended on little more than the economic assumption of no-arbitrage between bonds of differing maturities and the (relatively modest) mathematical assumptions that come from placing a continuous time model in the a the broad framework of stochastic calculus.

Naturally, there is the possibility that the behavior one sees in in Figure 1 is somehow due to unique features of January 1998, so a parallel analysis was done for 60 additional contracts from January to December 1998. The results of this analysis are summarized in Table 1 where the main message is found in the column

labeled $\%V$ which gives the percentages of times that the market price of the futures was larger than the theoretical upper bound. The bottom line is that the paradoxical behavior one sees in the time series plot of Figure 1 is to be found virtually anywhere one looks.

Month	Cts	TDs	Obs	%V	Mean Δ	Std Δ	Median Δ	MAD Δ
Jan	4	20	80	66.25	0.0260	0.0513	0.0150	0.0307
Feb	4	19	76	28.95	-0.0134	0.0339	-0.0060	0.0143
Mar	6	22	132	47.73	0.0000	0.0252	-0.0016	0.0161
Apr	6	19	114	42.11	-0.0020	0.0414	-0.0036	0.0193
May	6	19	114	57.89	0.0052	0.0352	0.0070	0.0252
Jun	6	22	132	60.60	0.0005	0.0386	0.0081	0.0182
Jul	6	22	132	77.27	0.0090	0.0223	0.0136	0.0123
Aug	6	20	120	85.00	0.0287	0.0354	0.0216	0.0160
Sep	5	20	100	93.00	0.0787	0.0645	0.0705	0.0386
Oct	6	21	126	88.89	0.0903	0.0627	0.0888	0.0427
Nov	0	—	—	—	—	—	—	—
Dec	6	21	126	80.95	0.0455	0.0585	0.0442	0.0399

TABLE 1. PERCENTAGE OF OBSERVATIONS THAT VIOLATE THE THEORETICAL FUTURES PRICE UPPER BOUND (ALL QUALIFYING 90-DAY LIBOR CONTRACTS FOR 1998)

Table 1 contains the results of the analysis of all 90-day LIBOR contracts for 1998 that met two modest restrictions driven by the available data. First of all, we restricted our analysis to just those contracts for which the futures prices and associated LIBOR quotes were available for all of the trading days of the month under consideration. Because the BBA data for November 1998 inexplicably contains only six days of data, no contracts were analysed for that month. Also, since the CME introduced two new 90-day contracts in the middle of February 1998, we are able to study six contracts for most of the months from March onward, but we only study four contracts in February itself.

For a more technical reason, we also studied only those contracts for which the time to settlement $T - t$ was less than $3/4$ years for all the days in the study month. The reason for this restriction comes from the structure of the formula for the theoretical upper bound, or, specifically, from the fact that the formula for $U_{1/4}(t, T)$ requires the value of LIBOR quote $L_{T-t+1/4}(t)$. A fundamental limitation of the BBA data is that it only provides the values of LIBOR quotes with $\lambda \leq 1$; that is, the BBA data is limited to quotes on deposits of duration no greater than one year. In order to avoid the extrapolation of the observed LIBOR quoted to maturities beyond the range of the available data, we simply limited our analysis those futures contracts for which $T - t + 1/4 \leq 1$.

For each month of 1998, the table gives the number of contracts that we analysed (Cts) and the number of trading days for the month (TDs). The product of these numbers gives the number of observations (Obs) of the value of the futures price defect

$$(64) \qquad \Delta = F_\lambda(t, T) - U_\lambda(t, T),$$

and the %V column gives the percentage of the observations for which $\Delta > 0$, signaling a violation of the theoretical upper bound property. The last four columns give the mean, standard deviation, median, and mean absolute deviation of the observed values of the defect Δ.

The most striking months in the table are surely September and October where the theoretical upper bound was violated by an extraordinary 90% of the observations. To be sure, excuses can be made; this was a period of considerable turmoil for all the fixed income markets. Russia halted interest payments on its foreign debt in late August. The spread between US treasury and corporate obligations was at an historical high; the yield curve was inverted, and the solvency of some highly leverage hedge funds was in question. Any one of these circumstances can stress a fixed income model, but a failure rate of 90% for a theoretical upper bound that is almost model-free still seems remarkable. Moreover, the exceptional circumstances of the Fall of 1998 provide no excuse for the performance of the bound during February, March, and April; even during these more placid times, one finds that the upper bound is violated by more than a third of the observations. Finally, one should note that the reported violations cannot be explained away as unimportant technical violations where Δ is practically zero. For example, in October one finds that the *average* size of the violations is equal to 0.0903. In traders' terms this is a violation of 36 ticks, an amount with clear economic significance.

The bottom line here that the theoretical futures price upper bound provided by Theorem 1 is profoundly at odds with the empirical behavior one finds in the prices for the 90-day LIBOR Eurodollar Futures traded on the Chicago Mercantile Exchange. The ultimate meaning of this divergence between markets and models is still unresolved, but the next section suggests that there may be interesting consequences for both.

12. Concluding Remarks

The theoretical upper bound for the price of LIBOR futures that is given by

$$(65) \qquad U_\lambda(t, T) \equiv 100 \left(\frac{1}{\lambda} + 1 \right) - \frac{100}{\lambda} \frac{1 + (T - t + \lambda) L_{T-t+\lambda}(t)}{1 + (T - t) L_{T-t}(t)}$$

has several appealing feature, but one of the nicest is that it does not depend on *any* of the defining parameters of the HJM model. This freedom from model parameters suggests that one may be able to derive the upper bound with out any reference to the HJM model. There is even some support for this possibility in Section 10 where we found that $U_\lambda(t, T)$ emerges naturally in the limiting case when the spot rate is deterministic. This observation suggests that the futures price inequality might follow from an understanding of the deterministic case and some sort of stochastic convexity (or Jensen inequality) argument. Such a derivation would account for the generality of the form of $U_\lambda(t, T)$ and would add substantially to the general understanding of yield curve models. Still, such a proof seems far away, and until some alternative derivation of $U_\lambda(t, T)$ is obtained, we have to regard the futures price upper bound as a feature of the HJM model.

The major issue that we leave open is the resolution of apparent discrepancy between observed futures prices and the theoretical upper bound. If the observed discrepancy cannot be attributed to the inadequacy of the data that were available for our limited exploration, then the generality of the upper bound essentially implies that no HJM yield curve model is adequate for the analysis of LIBOR futures.

Since LIBOR futures are a relatively simple financial derivative, this suggests that one may have reason for concern about the application of HJM models to other instruments such as forward contracts, swaps, and options.

Certainly HJM models have been used in all of these contexts by both market participants and theoreticians. We do not suggest that these applications were inappropriate or that their implications were inconsistent with market realities. We can only record the simple observation that where one inconsistency has been found, there may well be others. For the moment, we can only say that considerable work remains before one can provide conditions under which we can be sure that there is an HJM model that is appropriate for the intended application.

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