# On a sufficient condition for Eurodollar futures convexity bias

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Based on joint work with J. M. Steele

## Outline

- Model
- LIBOR, Forwards and Futures
- The Futures Rate-Forward Rate Inequality
- An Empirical Paradox?
- Suggested Empirical Analysis

## Model: Instantaneous Forward Rate

Consider the $n$-dimensional HJM model directly under the equivalent martingale measure $P$.

That is, the dynamics of the instantaneous forward rate is given by

$$df(t,T) = -\boldsymbol{\sigma}(t,T)^{\top}\boldsymbol{a}(t,T)\,dt + \boldsymbol{\sigma}(u,T)^{\top}d\boldsymbol{B}_t, \tag{1}$$

where

$$\boldsymbol{a}(t,T) = -\int_t^T \boldsymbol{\sigma}(t,u)\,du, \tag{2}$$

and $\boldsymbol{B}_t$ is the $n$-dimensional Brownian motion under $P$ with the standard filtration $\{\mathcal{F}_t\}$.

This relationship between volatility and drift of the instantaneous forward rate is needed to guarantee the absence of arbitrage between zero-coupon bonds of different maturities.

## Model: Bond

The $t$ price of a bond that pays one dollar at the maturity date $T$ is given by

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right) \qquad 0 \le t \le T \le \tau. \tag{3}$$

One can show that $P(t,T)$ satisfy the SDE

$$dP(t,T) = P(t,T)[\,r(t)\,dt + \boldsymbol{a}(t,T)^\top d\boldsymbol{B}_t\,], \tag{4}$$

where $r(t) = f(t,t)$ is the spot rate.

## $\lambda$-LIBOR Rate

The "$\lambda$-LIBOR rate" that is offered at time $t$ for a Eurodollar deposit for a maturity of $\lambda 360$ days. This rate is also called the *spot $\lambda$-LIBOR rate* when one needs to emphasize its distinction from the corresponding forward or futures rates and it is denoted by $L_\lambda(t)$. In terms of a corresponding zero coupon bond the spot $\lambda$-LIBOR rate is given by

$$L_\lambda(t) = \frac{1}{\lambda}\left(\frac{1}{P(t, t+\lambda)} - 1\right) \qquad 0 < t < \tau. \tag{5}$$

## Forward Rate $L_\lambda(t,T)$

The *forward rate* $L_\lambda(t,T)$ reflects the (add-on) interest rate available at time $t$ for a riskless loan that begins at date $T$ and which is paid back at time $T+\lambda$, and for this rate one has the representation

$$L_\lambda(t,T) = \frac{1}{\lambda}\left(\frac{P(t,T)}{P(t,T+\lambda)} - 1\right) \qquad 0 < t < T < \tau. \tag{6}$$

## $\lambda$-**LIBOR Futures Rate**

The $\lambda$-*LIBOR futures rate* $F_\lambda(t,T)$ is given by

$$F_\lambda(t,T) = E\left[\frac{1}{\lambda}\left(\frac{1}{P(T,T+\lambda)} - 1\right)\middle|\mathcal{F}_t\right] = E\left[L_\lambda(T)|\mathcal{F}_t\right] \quad 0 < t < T < \tau. \tag{7}$$

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## The Futures Rate-Forward Rate Inequality

**Theorem 1 (Futures Rate-Forward Rate Inequality)** *For all* $0 < t < T < \tau$*, the futures rate* $F_\lambda(t,T)$ *and the forward rate* $L_\lambda(t,T)$ *satisfy the inequality*

$$F_\lambda(t,T) \geq L_\lambda(t,T) \tag{8}$$

*with probability one, provided that the underlying HJM family of bond prices* $\{P(t,T) : 0 \leq t \leq T \leq \tau\}$ *satisfies the* inner product condition *which asserts that for any* $0 < t < U < T$

$$-\boldsymbol{a}(s,U)^\top[\boldsymbol{a}(s,T) - \boldsymbol{a}(s,U)] = |\boldsymbol{a}(s,U)|^2 - \boldsymbol{a}(s,U)^\top \boldsymbol{a}(s,T) \leq 0 \quad \textit{a.s.} \tag{9}$$

## An Empirical Paradox?

The relationship (8) is well known, and it was discussed in the financial literature, for example, see Burghardt and Hoskins (1995), Grinblatt and Jegadeesh (1996), and Pozdnyakov and Steele (2002). However, as it was demonstrated in the mentioned above articles the futures-forward rate inequality is quite often violated in practice when the time to maturity $T - t$ is relatively small – less that a year.

One possible explanation is that the inner product condition does not hold in practice. So, the primary goal of our project is to see whether we have evidence in the market data in support of (9).

## Spot LIBOR Rate Dynamics

First note that the process $P(\cdot, \cdot + \lambda)$ is also a diffusion that satisfies the SDE:

$$dP(t, t+\lambda) = P(t, t+\lambda)[(r(t) - f(t, t+\lambda))dt + \boldsymbol{a}(t, t+\lambda)^\top d\boldsymbol{B}_t].$$

Therefore,

$$dL_\lambda(t) = \frac{1}{\lambda P(t, t+\lambda)}[(f(t, t+\lambda) - r(t) + |\boldsymbol{a}(t, t+\lambda)|^2)dt - \boldsymbol{a}(t, t+\lambda)^\top d\boldsymbol{B}_t]. \qquad (10)$$

Every day we observe twelve spot LIBOR rates with maturities $\lambda$ that varies from one to twelve months. Let us try to use this data to verify the inner product condition.

## Empirical Analysis: Simplifying Assumption

**An Assumption.** *Assume that for a relatively short period of time (30 days or 1/12) the bond volatility* $\boldsymbol{a}(t, t+\lambda)$ *is (conditionally) deterministic and depends only on* $\lambda$, *i.e.* $\boldsymbol{a}(t, t+\lambda) \equiv \boldsymbol{a}(\lambda)$.

## Empirical Analysis: Two Expressions for Quadratic Variation

If we observe the LIBOR rate $L_\lambda(t)$ during $K$ business days, that is, at times $0 < t_1 < t_2 < \ldots < t_K < 1/12$, then using two different representations for the quadratic variation we obtain

$$\langle L_\lambda(\cdot), L_\lambda(\cdot)\rangle_t \approx \sum_{i=1}^{K-1}[L_\lambda(t_{i+1}) - L_\lambda(t_i)]^2 \approx \frac{|\boldsymbol{a}(\lambda)|^2}{\lambda^2}\sum_{i=1}^{K-1}\frac{t_{i+1}-t_i}{P(t_i, t_i+\lambda)^2}.$$

Therefore, taking into account that $P(t, t+\lambda) = (1+\lambda L_\lambda(t))^{-1}$ we get

$$|\boldsymbol{a}(\lambda)|^2 \approx \lambda^2 \frac{\sum_{i=1}^{K-1}[L_\lambda(t_{i+1}) - L_\lambda(t_i)]^2}{\sum_{i=1}^{K-1}(t_{i+1}-t_i)(1+\lambda L_\lambda(t_i))^2},$$

and all the quantities on the right side are observable.

## Empirical Analysis: Two Expressions for Quadratic Cross-Variation

Similarly,

$$\begin{aligned}\langle L_{\lambda_2}(\cdot), L_{\lambda_1}(\cdot)\rangle_t &\approx \sum_{i=1}^{K-1}[L_{\lambda_2}(t_{i+1}) - L_{\lambda_2}(t_i)][L_{\lambda_1}(t_{i+1}) - L_{\lambda_1}(t_i)] \\ &\approx \frac{\boldsymbol{a}(\lambda_2)^\top \boldsymbol{a}(\lambda_1)}{\lambda_2\lambda_1}\sum_{i=1}^{K-1}\frac{t_{i+1}-t_i}{P(t_i, t_i+\lambda_2)P(t_i, t_i+\lambda_1)},\end{aligned}$$

and, as a consequence,

$$\boldsymbol{a}(\lambda_2)^\top \boldsymbol{a}(\lambda_1) \approx \lambda_2\lambda_1 \frac{\sum_{i=1}^{K-1}[L_{\lambda_2}(t_{i+1}) - L_{\lambda_2}(t_i)][L_{\lambda_1}(t_{i+1}) - L_{\lambda_1}(t_i)]}{\sum_{i=1}^{K-1}(t_{i+1}-t_i)(1+\lambda_2 L_{\lambda_2}(t_i))(1+\lambda_1 L_{\lambda_1}(t_i))}.$$

These two approximations alow us to check the inner product condition (9). If we want to work with the convexity bias in the pricing Eurodollar futures and associated forward LIBOR rates, then we need to check the "no name condition" only when $\lambda_2 - \lambda_1$ is equal to 90 days, or 1/4 on the 360-day financial year scale.

## Empirical Analysis: Inner Product Condition

| Maturities $(\lambda_1, \lambda_2)$ | Estimated difference $\lvert \boldsymbol{a}(\lambda_1)\rvert^2 - \boldsymbol{a}(\lambda_2)^\top \boldsymbol{a}(\lambda_1)$ Negative | Zero | Positive |
|---|---|---|---|
| (1/12, 4/12) | 56 | 3 | 4 |
| (2/12, 5/12) | 61 | 0 | 2 |
| (3/12, 6/12) | 62 | 0 | 1 |
| (4/12, 7/12) | 61 | 0 | 2 |
| (5/12, 8/12) | 62 | 0 | 1 |
| (6/12, 9/12) | 63 | 0 | 0 |
| (7/12, 10/12) | 62 | 0 | 1 |
| (8/12, 11/12) | 63 | 0 | 0 |
| (9/12, 12/12) | 63 | 0 | 0 |

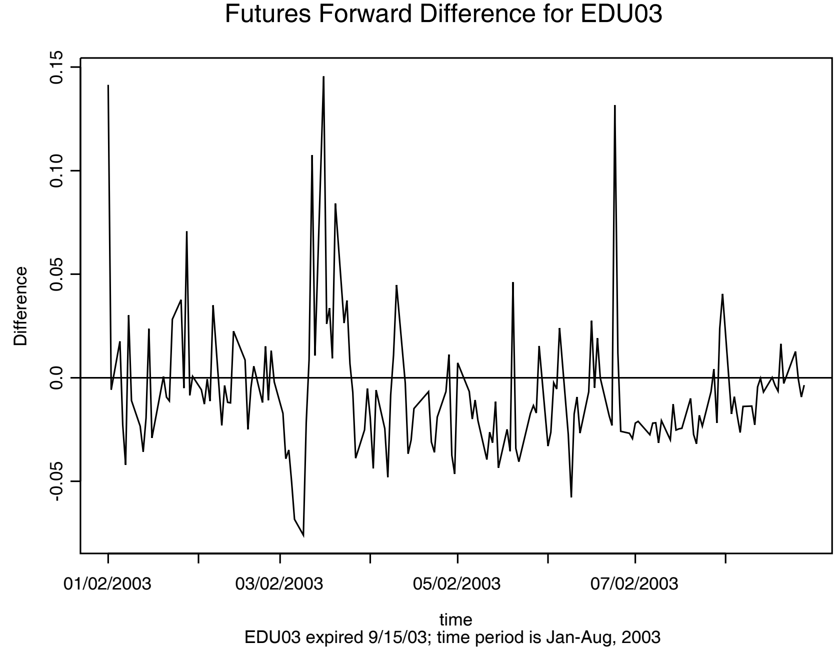
63 time periods from Jan 2000 to Dec 2004, each period is 20 business days

## Empirical Analysis: Futures-Forward Difference

| Contract | Obs | %V | Mean Δ | Std Δ | Median Δ | MAD Δ |
|---|---|---|---|---|---|---|
| EDZ00 | 166 | 18.07 | .0422 | .0667 | .0304 | .0418 |
| EDH01 | 164 | 42.07 | .0084 | .0419 | .0045 | .0350 |
| EDM01 | 164 | 42.07 | .0086 | .0418 | .0054 | .0357 |
| EDU01 | 166 | 63.86 | -.0071 | .0403 | -.0091 | .0349 |
| EDZ01 | 165 | 34.55 | .0255 | .0615 | .0180 | .0514 |
| EDH02 | 163 | 37.42 | .0175 | .0465 | .0095 | .0282 |
| EDM02 | 163 | 34.97 | .0257 | .0501 | .0118 | .0378 |
| EDU02 | 162 | 47.53 | .0083 | .0410 | .0029 | .0338 |
| EDZ02 | 163 | 46.63 | .0056 | .0483 | .0041 | .0383 |
| EDH03 | 164 | 58.54 | .0066 | .0459 | -.0098 | .0264 |
| EDM03 | 162 | 65.43 | -.0048 | .0406 | -.0109 | .0299 |
| EDU03 | 164 | 73.78 | -.0066 | .0323 | -.0112 | .0201 |
| EDZ03 | 164 | 69.51 | -.0050 | .0284 | -.0117 | .0198 |
| EDH04 | 163 | 61.34 | .0009 | .0298 | -.0032 | .0150 |
| EDM04 | 163 | 52.76 | .0090 | .0352 | -.0030 | .0223 |
| EDU04 | 164 | 54.27 | .0038 | .0403 | -.0058 | .0334 |
| EDZ04 | 163 | 33.13 | .0202 | .0501 | .0126 | .0297 |

Percentage of observations that violate the Futures Rate-Forward Rate Inequality

# Empirical Analysis: EDU03




Futures Rate/Forward Rate difference $\Delta$ for EDU03
EDU03 expired 9/15/03; time period is Jan-Aug, 2003

# THANK YOU