On expected occupation time of Brownian bridge

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Abstract

Consider a Brownian bridge from 0 to $c > 0$. It is known that the density of the expected occupation time by the Brownian bridge is constant in $[0, c]$. We give a simple elementary proof for this result based on a direct examination of the corresponding integral. The expected occupation time plays an important role in the analysis of animal movement.

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1. Introduction

Let us start from a simple calculus observation. Consider the following function of x:

$$\int_0^1 \frac{1}{\sqrt{t(1-t)}} \exp\left\{ -\frac{(x-t)^2}{2t(1-t)} \right\} \, \mathrm{d}t. \tag{1}$$

This function is constant when x is between 0 and 1. This fact can be easily checked via a numerical integration. But how to prove it? Almost immediately one wants to take derivative with respect to x and show that it is 0. More specifically, one can represent the derivative as a sum of two integrals

$$-\int_0^{k(x)} \frac{(x-t)}{[t(1-t)]^{3/2}} \exp\left\{ -\frac{(x-t)^2}{2t(1-t)} \right\} \, \mathrm{d}t$$

and

$$-\int_{k(x)}^1 \frac{(x-t)}{[t(1-t)]^{3/2}} \exp\left\{ -\frac{(x-t)^2}{2t(1-t)} \right\} \, \mathrm{d}t.$$

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Then we can try to choose an appropriate $k(x)$ that would allow us to transform (via substitution) one integral into another with an opposite sign. Unfortunately, we could not follow this path. Our proof is based on a more direct examination of integral (1).

Before we go to our proof let us explain a practical context in which integral (1) appears. It is related to analysis of animal movement via Brownian motion. Here we give just a brief discussion. We refer readers more detailed exposition to Horne *et al.* (2007) where this modeling idea was first introduced to the ecological community.

Assume that movement of an animal can be modeled by a driftless rotation invariant two-dimensional Brownian motion with diffusion coefficient σ^2. Suppose that we know the animal locations at times 0 and 1, and those are $(0,0)$ and (c, b). The possible paths between these two locations are then modeled by Brownian bridge. In particular, the probability density function of the animal location at time $t \in (0, 1)$ is given by the following well-known formula:

$$f_t(x, y) = \frac{1}{2\pi\sigma^2 t(1-t)} \exp\left\{-\frac{(x - ct)^2 + (y - bt)^2}{2\sigma^2 t(1-t)}\right\}. \tag{2}$$

The *expected occupation time* plays an important role in animal movement modeling. By definition, it is the expected fraction of time the animal spends in area A (a Borel set) on its move from $(0,0)$ to (c, b). More specifically, if we denote the Brownian bridge process as B_t ($0 \leq t \leq 1$), then the expected occupation time is given by

$$\mathrm{E} \int_0^1 \mathbf{1}\{B_t \in A\}\mathrm{d}t,$$

where $\mathbf{1}\{\cdot\}$ is an indicator function. One can show that the expected occupation time, as a function of A, is an absolutely continuous probability measure on the plane equipped with Borel σ-field. Moreover, by Fubini Theorem we have

$$\mathrm{E} \int_0^1 \mathbf{1}\{B_t \in A\}\,\mathrm{d}t = \int_0^1 \mathrm{P}\{B_t \in A\}\,\mathrm{d}t$$

$$= \int_0^1 \left[\iint_A f_t(x, y)\,\mathrm{d}x\,\mathrm{d}y\right]\mathrm{d}t$$

$$= \int_A \left[\int_0^1 f_t(x, y)\,\mathrm{d}t\right]\mathrm{d}x\,\mathrm{d}y.$$

Therefore, the density function of the expected occupation time is given by

$$h(x, y) = \int_0^1 f_t(x, y)\,\mathrm{d}t. \tag{3}$$

The plot of the function for $c = b = 1$ and $\sigma^2 = 1$ is presented in Figure 1. It is very intuitive. As expected, the animal on average spends more time

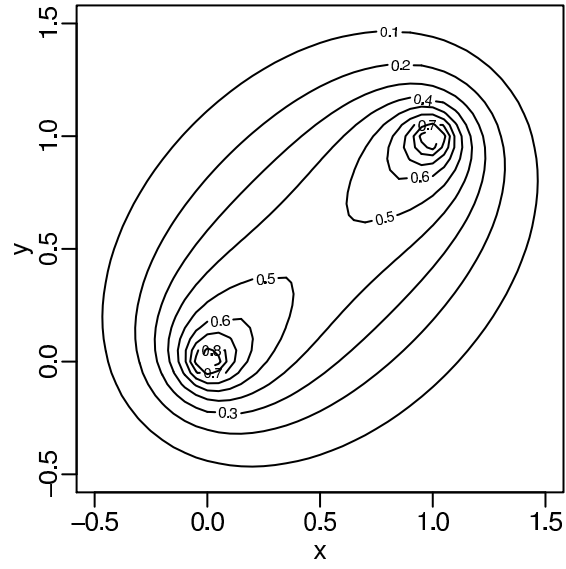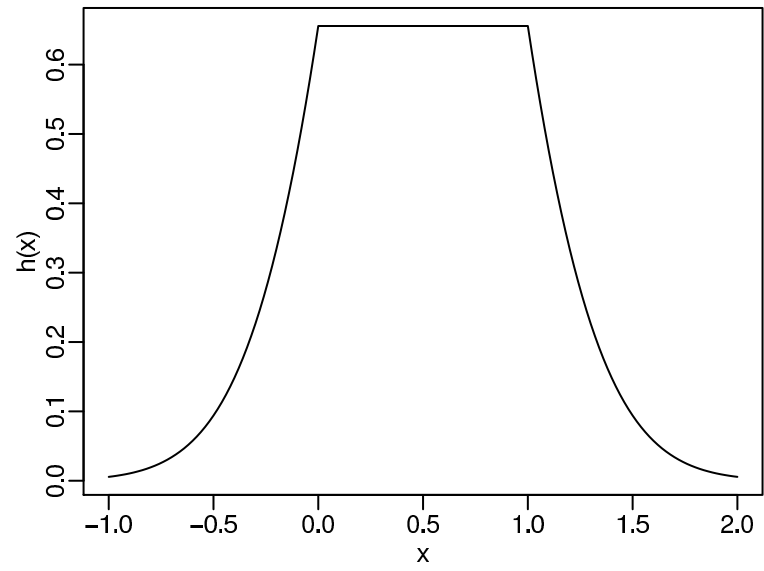Figure 1: Contour plot of $h(x, y)$

Figure 2: Plot of $h(x)$

around the starting and end points, and the density is high over the line that connect these two locations.

Now if we are only interested in the x-expected occupation time then the corresponding density is given by

$$h(x) = \int_{-\infty}^{\infty} h(x, y)\, \mathrm{d}y = \int_0^1 \frac{1}{\sqrt{2\pi\sigma^2 t(1-t)}} \exp\left\{-\frac{(x-ct)^2}{2\sigma^2 t(1-t)}\right\} \mathrm{d}t. \quad (4)$$

The plot of $h(x)$ (for $c = 1$ and $\sigma = 1$) is in Figure 2. One surprising observation is that the function is constant when $0 \le x \le c$! This was noted but left unexplained in an example in Pozdnyakov et al (2014) on the density of occupation time for a Brownian motion with measurement error, which reduces to Brownian motion when the variance of the measurement error is zero.

In fact, because of the rotational symmetry of two-dimensional Brownian motion, if we take any two parallel lines that lie between $(0, 0)$ and (c, b), and integrate density $h(x, y)$ over these two lines, the results will be the same! From the contour plot of $h(x, y)$ in Figure 1, this is not obvious at all.

The distributions of the occupation time for Brownian processes are important in applied probability as well. Often the distribution of occupation time can be viewed as a limit distribution of functionals of queueing or branching processes. They were studied extensively, for instance, see Hooghiemstra (2002), Howard and Zumbrun (1999) and Takács (1999). The most closely related result to our Proposition 1 can be found in Howard and Zumbrun (1999), which actually implies our result. However, their proof relies on the strong Markov property of the Brownian bridge. In contrast, the proof presented here is direct and elementary.

2. Main Result

Here is our main finding. It gives an elementary proof that the density of the one-dimensional expected occupation time $h(x)$ is constant for $0 \le x \le c$, and the constant is expressed in terms of normal distribution.

Proposition 1. *Let $c > 0$ and $\sigma > 0$. Define $h(x)$ by (4). Then for $x \in (0, c)$,*

$$h(x) = \mathrm{E}\left[\frac{c}{c^2 + \xi^2}\right], \quad \xi \sim N(0, \sigma^2). \quad (5)$$

Remark 1. Howard and Zumbrun (1999) established certain translation invariance of the occupation time distribution but did not consider its density. However, from their result, one can get

$$h(x) = h(0) = \int_0^1 \frac{1}{\sqrt{2\pi\sigma^2 t(1-t)}} \exp\left\{-\frac{c^2 t}{2\sigma^2(1-t)}\right\} \mathrm{d}t.$$

By change of variable $u = c\sqrt{t/(1-t)}$, the last integral is the same as the right hand side of (5). On the other hand, our proof has completely no use of the result or argument in Howard and Zumbrun (1999).

Remark 2. Consider the integral of $h(x,y)$ in (3) over a straight line lying between $(0,0)$ and (c,b). Suppose the distances between the points and the line are d_1 and d_2, respectively. Then by rotational symmetry and Proposition 1, it can be seen that the integral is equal to $\mathrm{E}[d/(d^2 + \xi^2)]$, where $d = d_1 + d_2$.

Proof. Let $T \sim \mathrm{unif}(0,1)$ be independent of ξ. By (4), $h(x)$ is the density of $\sqrt{T(1-T)}\xi + cT$. Denote the right hand side of (5) by K. Then (5) is equivalent to

$$\mathrm{P}\{\sqrt{T(1-T)}\xi + cT \in \mathrm{d}x\} = K\mathrm{d}x, \quad x \in (0,c). \tag{6}$$

Put $x = cs$, $Z = \xi/c$. Denote

$$\varphi(s) = \mathrm{P}\{0 \le \sqrt{T(1-T)}Z + T \le s\}.$$

Then (6) is equivalent to

$$\varphi(s) = cKs, \quad s \in (0,1). \tag{7}$$

Since $Z \sim -Z$ and $T \sim 1 - T$, and Z and T are independent,

$$\begin{aligned} \varphi(s) &= \mathrm{P}\{0 \le -\sqrt{T(1-T)}Z + 1 - T \le s\} \\ &= \mathrm{P}\{1 - s \le \sqrt{T(1-T)}Z + T \le 1\} \\ &= \varphi(1) - \varphi(1-s). \end{aligned}$$

Consequently, to show (7), it suffices to consider $s \in (0, 1/2]$ only.

To calculate $\varphi(s)$, we condition on Z. Since $\mathrm{P}\{Z \ne 0\} = 1$, we only need to consider nonzero values of Z. Given $z \ne 0$,

$$\mathrm{P}\{0 \le \sqrt{T(1-T)}z + T \le s \,|\, Z = z\} = |\{t \in (0,1) : 0 \le \sqrt{t(1-t)}z + t \le s\}|,$$

where $|\cdot|$ stands for the Lebesgue measure. Put $g = z^2$. First assume $z > 0$. Then the set on the right hand side consists of the solutions to

$$\sqrt{t(1-t)}z \le s - t, \quad 0 \le t \le 1. \tag{8}$$

Note that as $z > 0$, the first inequality implies $t \le s$. Thus (8) is equivalent to

$$q(t) := (t-s)^2 - t(1-t)g \ge 0, \quad 0 \le t \le s.$$

Since $q(t)$ is a quadratic function with $q(s) < 0$, $q(0) > 0$, $q(1) > 0$, and $q(t) \to \infty$ as $t \to \pm\infty$, the set of t satisfying the above inequalities is $[0, t_-]$, where $t_- = t_-(g) \in (0, s)$ is the smaller solution to $q(t) = 0$, the other one being $t_+ = t_+(g) \in (s, 1)$. By elementary algebra,

$$t_{\pm} = \frac{2s + g \pm \sqrt{(2s+g)^2 - 4s^2(1+g)}}{2(1+g)}. \tag{9}$$

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One can directly check that $t_\pm$ are real by noting $2s + g > 2s\sqrt{1+g}$, which follows from $g - 2s(\sqrt{1+g} - 1) \geq g - (\sqrt{1+g} - 1) > 0$. As a result,

$$P\{0 \leq \sqrt{T(1-T)}z + T \leq s \,|\, Z = z\} = t_-(z^2), \quad z > 0. \tag{10}$$

Next assume $z < 0$. Then we need to solve

$$0 \leq t - \sqrt{t(1-t)}|z| \leq s, \quad 0 < t < 1. \tag{11}$$

The inequality on the left end of (11) is equivalent to $t \geq g/(1+g)$. The second one is equivalent to

$$t - s \leq \sqrt{t(1-t)}|z|.$$

Clearly, any $t \in [0, s]$ is a solution to the above inequality. On the other hand, if $t > s$, then the inequality holds if and only if $t \in (s, t_+]$, where t_+ is as in (9). Thus the set of solutions is $[0, t_+]$. By $t_+ > t_+ - s = \sqrt{t_+(1-t_+)}|z|$, $t_+ > g/(1+g)$. As a result, the set of solutions to (11) is $[g/(1+g), t_+]$ and

$$P\{0 \leq \sqrt{T(1-T)}z + T \leq s \,|\, Z = z\} = t_+(z^2) - \frac{z^2}{1+z^2}, \quad z < 0. \tag{12}$$

Combining (10) and (12) and taking expectation then yield

$$\varphi(s) = \mathrm{E}[t_-(Z^2)\mathbf{1}\{Z > 0\}] + \mathrm{E}\left[\left(t_+(Z^2) - \frac{Z^2}{1+Z^2}\right)\mathbf{1}\{Z < 0\}\right].$$

Then by symmetry and $t_-(Z^2) + t_+(Z^2) = (2s + Z^2)/(1+Z^2)$,

$$
\begin{aligned}
\varphi(s) &= \frac{1}{2}\mathrm{E}\left[t_-(Z^2) + t_+(Z^2) - \frac{Z^2}{1+Z^2}\right] \\
&= s\mathrm{E}\left[\frac{1}{1+Z^2}\right] \\
&= s\mathrm{E}\left[\frac{1}{1+\xi^2/c^2}\right],
\end{aligned}
$$

which is (7). $\qquad\square$

3. Acknowledgements

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