4) Так как выборка — $X_1, \dots, X_n$, это необходимо учесть.

Выпишем функцию правдоподобия:

$$L(\theta) = \prod_{i=1}^{n} P(X=0)P(X=1)$$

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{1+\theta}{2}\right)^{X_i} \left(\frac{1-\theta}{2}\right)^{1-X_i}$$

Натуральный логарифм:

$$\ln L(\theta) = \ln\left(\prod_{i=1}^{n} \left(\frac{1+\theta}{2}\right)^{X_i} \left(\frac{1-\theta}{2}\right)^{1-X_i}\right)$$

$$\ln L(\theta) = \sum_{i=1}^{n} \left(X_i \ln\left(\frac{1+\theta}{2}\right) + (1-X_i)\ln\left(\frac{1-\theta}{2}\right)\right)$$

$$\ln L(\theta) = \sum_{i=1}^{n} X_i \ln(1+\theta) + \sum_{i=1}^{n} (1-X_i)\ln(1-\theta) - n\ln 2$$

$$(\ln L(\theta))' = \sum_{i=1}^{n} \left(\frac{X_i}{1+\theta} - \frac{1-X_i}{1-\theta}\right)$$

$$\sum_{i=1}^{n} \left(\frac{X_i}{1+\theta} - \frac{1-X_i}{1-\theta}\right) = 0$$

Введём число единиц в нашей выборке $X_1, \dots, X_n$ как $S = \sum_{i=1}^{n} X_i$ и подставим в уравнение.

$$\frac{S}{1+\theta} - \frac{n-S}{1-\theta} = 0 \quad \Big| \cdot (1+\theta)(1-\theta)$$

$$S(1-\theta) - (n-S)(1+\theta)$$

$$S - \cancel{S\theta} - n - n\theta + S + \cancel{S\theta} = 0$$

$$2S - n - n\theta = 0$$

$$n\theta = -n + 2S$$

$$\theta = \frac{2S}{n} - \frac{n}{n} \Rightarrow \theta = \frac{2S}{n} - 1$$

1 k орлов подряд.

5) **МО:**

$$E_1(x) = \int_0^\theta x \cdot \frac{1}{\theta}\, dx = \frac{1}{\theta}\int_0^\theta x\, dx = \frac{x^2}{2}\Big|_0^\theta = \frac{\theta^2}{2}$$

$$E_2(x) = \int_\theta^1 x \cdot \frac{1}{1-\theta}\, dx = \frac{1+\theta}{2}$$

$$E(x) = p \cdot \frac{\theta}{2} + (1-p) \cdot \frac{1+\theta}{2} = \frac{1}{2}\left(p\theta + (1-p)(1+\theta)\right)$$

$$E(x) = \frac{1}{2}(p\theta + 1 - p + \theta - p\theta)$$

$$E(x) = \frac{1}{2}(1 - p + \theta)$$

$$\bar{x} \approx E(x)$$

$$\bar{x} = \frac{1}{2}(1 - p + \theta) \Rightarrow p = 1 + \theta - 2\bar{x} \quad \left(E(x) \approx \frac{\theta + 1 - p}{2}\right)$$

Дисперсия:

$$E_1(x^2) = \int_0^\theta x^2 \cdot \frac{1}{\theta}\, dx = \frac{\theta^2}{3}$$

$$D_1(x) = \frac{\theta^2}{3} - \left(\frac{\theta}{2}\right)^2 = \frac{\theta^2}{12}$$

$$E_2(x^2) = \int_\theta^1 x^2 \cdot \frac{1}{1-\theta}\, dx = \frac{1}{1-\theta}\int_\theta^1 x^2 dx = \frac{1-\theta^3}{3}$$

$$\Rightarrow D_2(x) = \frac{(1+\theta)^2}{4}$$

$D(x)$ [illegible]

(5) $E(X^2) = p\cdot\frac{\theta^2}{3} + (1-p)\cdot\frac{1-\theta^3}{3(1-\theta)}$

$$\begin{cases}\bar{X} \approx E(X)\\ S^2 \approx D(X)\end{cases}$$

$$\begin{cases}\bar{X} \approx \frac{\theta+1-p}{2} \Rightarrow 2\bar{X} = \theta+1-p \Rightarrow p = \theta+1-2\bar{X} \\ S^2 \approx \overbrace{\frac{p\cdot\theta^2}{3}+\frac{(1-p)(1-\theta^3)}{3(1-\theta)}}^{E(X^2)} - \left(E(X)\right)^2\end{cases}$$

$$\cancel{S^2 = \frac{(2\bar{X}-\theta-1)\theta^2}{3} + \frac{(1-2\bar{X}+\theta+1)(1-\theta^3)}{3(1-\theta)}} =$$

$$= \cancel{\frac{2\theta^2+2+3\theta-2X-2\theta X}{3}}$$

$$E(X^2) = \frac{(\theta+1-2\bar{x})\cdot\theta^2}{3} + \frac{(1-\theta-1+2\bar{x})(1-\theta^3)}{3(1-\theta)}$$

$$= \frac{-\theta+2\bar{x}+2\theta\bar{x}}{3}$$

$$\Rightarrow D(x) = E(x^2) - (E(x))^2$$

$$(E(x))^2 = \left(\frac{\theta+1-p}{2}\right)^2 = \left(\frac{\theta+1-\theta-1+2\bar{x}}{2}\right)^2 = \bar{x}^2$$

$$D(x) = \frac{2\bar{x}-\theta+2\theta\bar{x}}{3} - \bar{x}^2$$

$$\Rightarrow \begin{cases} \overset{(=\bar{x})}{E(x)} = \dfrac{\theta+1-p}{2} \Rightarrow p = \theta+1-2\bar{x} \\ D(x) = \dfrac{2\bar{x}-\theta+2\theta\bar{x}}{3} - \bar{x}^2 \end{cases}$$

$$\Rightarrow S^2 = \frac{\theta(2\bar{x}-1)+2\bar{x}}{3} - \bar{x}^2$$

$$\theta(2\bar{x}-1)+2\bar{x} = 3\bar{x}^2+3S^2$$

$$\theta = \frac{3\bar{x}^2+3S^2-2\bar{x}}{2\bar{x}-1}$$

Ответ:

$$\Rightarrow \left[\begin{array}{l} p = \dfrac{3S^2+2\bar{x}-\bar{x}^2-1}{2\bar{x}-1} \\ \theta = \dfrac{3\bar{x}^2+3S^2-2\bar{x}}{2\bar{x}-1} \end{array}\right.$$