Problem 1

a) The probability that at least four visitors arrive during one minute.

With $\mu = \lambda t = 3 \cdot 1$ we get $P(X = x) = \frac{3^x}{x!} e^{-3}$ which gives

$$P(X = 3) = \frac{3^3}{3!} e^{-3} = \frac{27}{6} \cdot 0.049787 = 0.2242415$$

$$P(X = 2) = \frac{3^2}{2!} e^{-3} = \frac{9}{2} \cdot 0.049787 = 0.2242415$$

$$P(X = 1) = \frac{3^1}{1!} e^{-3} = \frac{3}{1} \cdot 0.049787 = 0.149361$$

$$P(X = 0) = \frac{3^0}{0!} e^{-3} = \frac{1}{1} \cdot 0.049787 = 0.049787$$

$$P(X \geq 4) = 1 - P(X < 4) =$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)] =$$

$$= 1 - 0.049787 - 0.149361 - 0.2242415 - 0.2242415 = 0.35276$$

The probability that a visitor spends more than 5 minutes on the website.

From the lecture we have that the pdf of gamma distribution is:

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, \qquad x \geq 0$$

$$f(x) = \frac{1}{3^2 \Gamma(2)} x^{2-1} e^{-\frac{x}{3}};$$

$$P(X > 5) = \int_5^\infty \frac{1}{9} x e^{-\frac{x}{3}} = -\frac{1}{3} e^{-\frac{x}{3}} (x + 3) = 0.5036$$

We can make a simulation algorithm following the next steps:

1) Using Poisson distribution we can make a vector of length N, where Ni will denote the number of visitors arriving during 1 minute.
2) Then using Gamma distribution we can make a vector X where Xi will denote the time visitor i spend on the website
3) To get Vt we calculate the sum of all times in vector X

To estimate $P(V_t > a)$ we can run the simulation algorithm a large number of times and check the amount of times when V_t > a. Then divide that number by the number of simulations.

Problem 2

a) First, we need to calculate the cdf which is:

$$F(x) = \int_0^x f(t)dt = \int_0^x \alpha\beta t^{\beta-1}e^{-\alpha t^\beta}dt = [-e^{\alpha(-t^\beta)}]_0^x = 1 - e^{\alpha(-x^\beta)}$$

Inverting $u = F(x)$:

$$u = 1 - e^{\alpha(-x^\beta)}$$

$$e^{\alpha(-x^\beta)} = 1 - u$$

$$\alpha(-x^\beta) = \ln(1-u)$$

$$-x^\beta = \frac{\ln(1-u)}{\alpha}$$

$$x = (-\frac{\ln(1-u)}{\alpha})^{\frac{1}{\beta}}$$

The algorithm will be:

1. Generate $u = Uniform$ [0,1]
2. Let $x = (-\frac{\ln(1-u)}{\alpha})^{\frac{1}{\beta}}$

c) A special case of the Weibull distribution when $\beta = 1$ is the Exponential distribution: $f(x) = \alpha e^{-\alpha x}$

The probability that one pump fails before 1 year (12 months) period is:

$$P(X < 12) = \int_0^{12} f(x)dx = \int_0^{12} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} =$$

$$= \int_0^{12} 0.08e^{-0.08x} = [-e^{-0.08x}]_0^{12} = 0.62$$

For $i = 1, \dots, 10$ let

$$I_j = \begin{cases} 0, & with\, probability\, 1 - p_j \\ 1, & with\, probability\, p_j \end{cases}$$

where I_j = 1 means that pump failed. Then

1) $P(I_1 = 1 \cap I_2 = 1 \cap I_3 = 1)$ = $P(I_1 = 1)P(I_2 = 1)P(I_3 = 1) = p1p2p3 =$
$= 0.23$

2) P = 0.92

3) P = 0.68

The algorithm to estimate the expected time until failure for scenario *i)* can be stated as following:

1. First, we generate the distribution of failure times X using Weibull distribution for every of the three pumps.
2. Then we compare X*i (time until failure)* for every pump and chose a value depending on the scenario:
   i) maximum value
   ii) minimum value
   iii) second maximum value
3. Find the mean of those values.

e) The expectation formula for Weibull distribution is $E(x) = \alpha^{-\frac{1}{\beta}}\Gamma(1 + \frac{1}{\beta})$

In case where we have $\beta = 1$ the expectation formula will be $E(x) = \frac{1}{\alpha}$

hence $\alpha = \frac{1}{E(x)}$. Using this formula and triangle distribution we can implement an algorithm to find the distribution of the probabilities that the system fails within one year for each of the three scenarios.

1. Simulate the time until failure with triangle distribution given the intervals as min = 10, max = 50, most likely = 20
2. For every value in the triangle distribution find parameter value α using formula $\alpha = \frac{1}{E(x)}$
3. For every parameter value α we calculate the probabilities that the system fails within one year for each of the three scenarios.

Problem 3

a) We can make a simulation algorithm following the next steps:

1) Start round 1.
2) Draw *n* numbers uniformly among the integers 1 to 6.
3) Reduce *n* by the amount of numbers that were equal to the number of the round.
4) Sum the numbers equal to the number of the round throughout 1-6 rounds.
5) Make 2 more draws.
6) Do rounds 2,3,4,5,6.

To estimate the probability of the score to be at least 42 we can simulate a large number of outcomes of the upper section score. Count the number of outcomes that are equal or larger then 42 points, and then divide that number by the amount of total simulations.

We need to make $n > \frac{1}{0.01^2} = 10\ 000$ simulations for an error of at most 0.01