# Algebră liniară
## Spații vectoriale

### Seminar 10

**3.1.31** Să se arate că $\mathbb{R}^*_+ = (0, \infty)$ este un $\mathbb{R}$-spațiu vectorial în raport cu adunarea vectorilor

$$\boxplus : \mathbb{R}^*_+ \times \mathbb{R}^*_+ \to \mathbb{R}^*_+ , \quad x \boxplus y = xy , \quad \forall\, x,y \in \mathbb{R}^*_+$$

și cu înmulțirea cu scalari

$$\boxdot : \mathbb{R} \times \mathbb{R}^*_+ \to \mathbb{R}^*_+ , \quad \alpha \boxdot x = x^\alpha , \quad \forall\, x \in \mathbb{R}^*_+ , \alpha \in \mathbb{R}$$

Soluție

I $(\mathbb{R}^*_+, \boxplus)$ grup abelian?

① parte stabilă

$\forall\, x \in \mathbb{R}^*_+$

$\forall\, y \in \mathbb{R}^*_+$

$\xrightarrow{\boxplus}$

$$x \boxplus y = \underbrace{x}_{\in \mathbb{R}^*_+} \cdot \underbrace{y}_{\in \mathbb{R}^*_+} \Rightarrow x \boxplus y \in \mathbb{R}^*_+$$

② asociativitate

$$\forall\, x,y,z \in \mathbb{R}^*_+ , \quad (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$$

$$(x \boxplus y) \boxplus z = (x \cdot y) \boxplus z = x \cdot y \cdot z = x \cdot (y \boxplus z) = x \boxplus (y \boxplus z)$$

③ comutativitate

$$\forall\, x,y \in \mathbb{R}^*_+ , \quad x \boxplus y = y \boxplus x$$

$$x \boxplus y = x \cdot y = y \cdot x = y \boxplus x$$

④ element neutru

$$\exists\, e \in \mathbb{R}^*_+ , \forall\, x \in \mathbb{R}^*_+ , \quad \underbrace{x \boxplus e = e \boxplus x}_{\text{comutativă}} = x$$

$$x \boxplus e = x \iff x \cdot e = x \quad | : x$$

$$e = 1 \in \mathbb{R}^*_+ .$$

⑤ elemente simetrizabile

$\forall x \in \mathbb{R}_+^*, \exists x' \in \mathbb{R}_+^*$ a.î. $\underbrace{x \boxplus x' = x' \boxplus x}_{\text{comutativ}} = e$

$x \boxplus x' = e$

$x \cdot x' = 1$

$x' = \frac{1}{x} \quad \forall x \in \mathbb{R}_+^* \Rightarrow x' \in \mathbb{R}_+^*$

$1, 2, 3, 4, 5 \Rightarrow (\mathbb{R}_+^*, \boxplus)$ grup abelian

II Verificăm axiomele spațiului vectorial

1) $\alpha \boxdot (x \boxplus y) = \alpha \boxdot x \boxplus \alpha \boxdot y$

$\alpha \boxdot (x \cdot y) = (x \cdot y)^\alpha = x^\alpha \cdot y^\alpha = (\alpha \boxdot x) \cdot (\alpha \boxdot y) = \alpha \boxdot x \boxplus \alpha \boxdot y$

2) $(\alpha + \beta) \boxdot x = \alpha \boxdot x \boxplus \beta \boxdot x$ , adunare normală $\alpha + \beta$

$(\alpha + \beta) \boxdot x = (\alpha + \beta) \boxdot x = x^{\alpha + \beta} = x^\alpha \cdot x^\beta = x^\alpha \boxplus x^\beta =$

$= (x \boxdot \alpha) \boxplus (\beta \boxdot x)$

3) $(\alpha \cdot \beta) \boxdot x = \alpha \boxdot (\beta \boxdot x)$ , $\alpha \cdot \beta$ - înmulțire normală de 2 scalari

$\cancel{(\alpha \cdot \beta) \boxdot x = x^{\alpha \cdot \beta} = (x^\alpha)^\beta = (\alpha \boxdot x)^\beta = (\alpha \boxdot x) \boxdot \beta}$

$(\alpha \cdot \beta) \boxdot x = x^{\alpha \cdot \beta} = (x^\beta)^\alpha = (\beta \boxdot x)^\alpha = \alpha \boxdot (\beta \boxdot x)$

4) $1 \boxdot x = x \quad , \forall x \in \mathbb{R}_+^*$

$1 \boxdot x = x^1 = x$

I, II $\Rightarrow \mathbb{R}_+^*$ este un $\mathbb{R}$-spațiu vectorial împreună cu $\boxplus$ și op. externă $\boxdot$

3.1.32. Să se verifice că operațiile

$\boxplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $x \boxplus y = \sqrt[5]{x^5 + y^5}$, $\forall\, x, y \in \mathbb{R}$

$\boxdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\alpha \boxdot x = \sqrt[5]{\alpha}\, x$, $\forall\, \alpha, x \in \mathbb{R}$

definesc o structură $\mathbb{R}$-spațiu vectorial pe $\mathbb{R}$

Soluție

I Verificăm $(\mathbb{R}, \boxplus)$ grup abelian

1) parte stabilă

$\forall\, x \in \mathbb{R}$

$\forall\, y \in \mathbb{R}$

$\xrightarrow{\boxplus}$

$x \boxplus y = \sqrt[5]{\underbrace{x^5}_{\in \mathbb{R}} + \underbrace{y^5}_{\in \mathbb{R}}} \in \mathbb{R}$

$\Rightarrow x \boxplus y \in \mathbb{R}$

2) asociativitatea

$\forall\, x, y, z \in \mathbb{R} \Rightarrow (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$

$(x \boxplus y) \boxplus z = \sqrt[5]{x^5 + y^5} \boxplus z = \sqrt[5]{\left(\sqrt[5]{x^5 + y^5}\right)^5 + z^5} = \sqrt[5]{x^5 + y^5 + z^5}$

$= \sqrt[5]{x^5 + \left(\sqrt[5]{y^5 + z^5}\right)^5} = \sqrt[5]{x^5 + (y \boxplus z)^5} = x \boxplus (y \boxplus z)$

3) comutativitatea

$\forall\, x, y \in \mathbb{R}$, $x \boxplus y = y \boxplus x$

$x \boxplus y = \sqrt[5]{x^5 + y^5} = \sqrt[5]{y^5 + x^5} = y \boxplus x$

4) element neutru

$\forall\, x \in \mathbb{R}$, $\exists\, e \in \mathbb{R}$ a.î. $\underbrace{x \boxplus e = e \boxplus x}_{\text{comutativă}} = x$

$x \boxplus e = x$

$\sqrt[5]{x^5 + e^5} = x \quad |(\;)^5$

$x^5 + e^5 = x^5 \quad | - x^5$

$e^5 = 0 \Rightarrow e = 0 \in \mathbb{R}$

5) elemente simetrizabile

$\forall x \in \mathbb{R}, \exists x' \in \mathbb{R}$ a.i. $\underbrace{x \boxplus x' = x' \boxplus x}_{\text{comutativa}} = e$

$x \boxplus x' = 0 \iff \sqrt[5]{x^5 + x'^5} = 0 \mid (\,)^5$

$x^5 + x'^5 = 0$

$x'^5 = -x^5$

$x' = -x \quad \forall x \in \mathbb{R}$

$1, 2, 3, 4, 5 \Rightarrow (\mathbb{R}, \boxplus)$ grup abelian

II Verificăm axiomele spațiului vectorial

1) $\alpha \boxdot (x \boxplus y) = (\alpha \boxdot x) \boxplus (\alpha \boxdot y), \forall x, y, \alpha \in \mathbb{R}$

$\alpha \boxdot x = \sqrt[5]{\alpha} \cdot x$

$x \boxplus y = \sqrt[5]{x^5 + y^5}$

$\alpha \boxdot (x \boxplus y) = \alpha \boxdot (\sqrt[5]{x^5 + y^5}) = \sqrt[5]{\alpha} \cdot (\sqrt[5]{x^5 + y^5}) = \sqrt[5]{\alpha (x^5 + y^5)}$

$= \sqrt[5]{\alpha x^5 + \alpha y^5}$ (a)

$(\alpha \boxdot x) \boxplus (\alpha \boxdot y) = (\sqrt[5]{\alpha} \cdot x) \boxplus (\sqrt[5]{\alpha} \cdot y) = \sqrt[5]{(\sqrt[5]{\alpha}\, x)^5 + (\sqrt[5]{\alpha} \cdot y)^5}$

$= \sqrt[5]{\alpha x^5 + \alpha \cdot y^5}$ (b)

$a = b$

2) $(\alpha + \beta) \boxdot x = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

$(\alpha + \beta) \boxdot x = \sqrt[5]{\alpha + \beta} \cdot x$ (a)

$(\alpha \boxdot x) \boxplus (\beta \boxdot x) = (\sqrt[5]{\alpha}\, x) \boxplus (\sqrt[5]{\beta}\, x) = \sqrt{(\sqrt[5]{\alpha}\, x)^5 + (\sqrt[5]{\beta}\, x)^5}$

$= \sqrt[5]{\alpha x^5 + \beta x^5} = \sqrt[5]{x^5 (\alpha \cdot \beta)} = \sqrt[5]{\alpha \cdot \beta} \cdot x$ (b)

$a = b$

3) $\alpha \boxdot (\beta \boxdot x) = (\alpha \cdot \beta)\, x$

$\alpha \boxdot (\beta \boxdot x) = \alpha \boxdot (\sqrt[5]{\beta}\, x) = \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x$ (a)

$(\alpha \cdot \beta) \cdot x = \sqrt[5]{\alpha \cdot \beta} \cdot x = \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x$ (b)

$a = b$

4) $1 \boxdot x = x$

$1 \boxdot x = \sqrt[5]{1} \cdot x = x$

I, II $\Rightarrow \mathbb{R}$ este un $\mathbb{R}$-spațiu vectorial împreună cu op internă $\boxplus$ și $\boxdot$ operația externă

3.1.33 Care dintre următoarele submulțimi ale mulțimii $\mathbb{R}^3$ sunt $\mathbb{R}$-subspații?

$\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$

$+ : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$,

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3 y_3)$

Teorema 3.1.6

$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$\alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3)$

$0_3 = (0,0,0)$ elementul neutru în $\mathbb{R}^3$

a) $A = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0\}$

i) $0_3 = (0,0,0) \in A$

ii) fie $x, y \in A \overset{?}{\Rightarrow} x + y \in A$

$x = [x_1, x_2, x_3]$, $y = [y_1, y_2, y_3]$

$x + y = [x_1, x_2, x_3] + [y_1, y_2, y_3] = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$

$2(x_1 + y_1) + (x_2 + y_2) - (x_3 + y_3) =$

$2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3 =$

$\underbrace{2x_1 + x_2 + x_3}_{0} + \underbrace{2y_1 + y_2 - y_3}_{0} = 0 + 0 = 0 \Rightarrow x + y \in A$

iii) fie $\alpha \in \mathbb{R}$

$x = [x_1, x_2, x_3] \in \mathbb{R}^3$, $x \in A$

$\alpha x = [\alpha x_1, \alpha x_2, \alpha x_3]$

$2\alpha x_1 + \alpha x_2 + \alpha x_3 = \alpha(2x_1 + x_2 - x_3) = \alpha \cdot 0 = 0 \Rightarrow \alpha x \in A$

$\Longrightarrow A \leq_K \mathbb{R}^3$

b) $B = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 1\}$

i) $O_3 = (0,0,0) \overset{?}{\in} B$

$2 \cdot 0 + 0 - 0 = 0 + 0 - 0 = 0 \neq 1 \Rightarrow B \not\leq_{\mathbb{R}} \mathbb{R}^3$

c) $C = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$

i) $O_3 = (0,0,0) \overset{?}{\in} C$

$0 = 0 = 0 \Rightarrow O_3 \in C$

ii) fie $x, y \in C$, $x + y \overset{?}{\in} C$

$x = [x_1, x_2, x_3]$, $y = [y_1, y_2, y_3]$

$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$

$x_1 + y_1 = x_2 + y_2 = x_3 + y_3$

$\Rightarrow x + y \in C$

iii) $x \in C$, $\alpha \in \mathbb{R}$, $\alpha x \overset{?}{\in} C$

$x = [x_1, x_2, x_3]$, $x_1 = x_2 = x_3$

$\alpha x = [\alpha x_1, \alpha x_2, \alpha x_3]$, $\alpha x_1 = \alpha x_2 = \alpha x_3$

$\Rightarrow \alpha x \in C$

$\Longrightarrow C \leq_K \mathbb{R}^3$

d) $D = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1^2 + x_2 = 0\}$

i) $O_2 \overset{?}{\in} D$

$O_2 = (0,0)$

$0^2 + 0 = 0 \Rightarrow O_2 \in D$

ii) fie $x, y \in D$, $x + y \overset{?}{\in} D$

$x = [x_1, x_2, x_3]$, $x_1^2 + x_2 = 0$

$y = [y_1, y_2, y_3]$, $y_1^2 + y_2 = 0$

$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$

$(x_1 + y_1)^2 + (x_2 + y_2) =$

$\underline{x_1^2} + 2x_1y_1 + y_1^2 + \underline{x_2} + y_2 =$

$0 + 0 + 2x_1y_1 = 2x_1y_1 \neq 0 \quad \forall x_1, y_1 \in \mathbb{R}$

$\Rightarrow D \not\leq_K \mathbb{R}$

$(a+b)^2 = a^2 + 2ab + b^2$

Ex parte cu contraexemplu

e) $E = \mathbb{R}^3 \setminus A$

$E = \mathbb{R}^3 \setminus \{[x_1, x_2, x_3] \in \mathbb{R} \mid 2x_1 + x_2 - x_3 = 0\}$

$0_3 \in A$ așa că $0_3 \notin \mathbb{R}^3 \setminus A$

$\Rightarrow E \not\leq_K \mathbb{R}^3$

f) $F = (\mathbb{R}^3 \setminus A\} \cup \{0\}$

i) $0_3 \in F$

ii) fie $x, y \in F$, $x+y \overset{?}{\in} F$

$x = [x_1, x_2, x_3]$, $2x_1 + x_2 - x_3 \neq 0$

$y = [y_1, y_2, y_3]$, $2y_1 + y_2 - y_3 \neq 0$

$x+y = [x_1+y_1, x_2+y_2, x_3+y_3]$

$2(x_1+y_1) + x_2 + y_2 - x_3 - y_3 =$

$\underbrace{2x_1 + x_2 - x_3}_{\neq 0} + \underbrace{2y_1 + y_2 - y_3}_{\neq 0} = \quad \Rightarrow \neq 0$

$x = (0, 1, 2) \quad 2 \cdot 0 + 1 - 2 = -1$

$y = (0, -1, -2) \quad 2 \cdot 0 - 1 + 2 = +1$

$x+y = -1 + 1 = 0$

$\Rightarrow F \not\leq_K \mathbb{R}$

! când sunt combinații liniare egale cu 0 sunt de obicei sub spații vectoriale:

$ax_1 + bx_2 + cx_3 = 0$ etc.

**3.1.35** Să se găsească ecuațiile care determină vectorii din (ecuațiile acestor subsp) următoarele subspații:

a) $S = \langle [1,2,-1] \rangle$

b) $T = \langle [1,2,1], [-2,1,-3] \rangle$ ale lui $\mathbb{R}^3$

a) $x \in S \Leftrightarrow \exists\, t \in \mathbb{R}$ a.î. $x = t \cdot [1,2,-1]$

$x = [x_1, x_2, x_3] \Leftrightarrow x = [t, 2t, -t]$

$$\begin{cases} x_1 = t \\ x_2 = 2t \\ x_3 = -t \end{cases} \Leftrightarrow \begin{cases} x_2 = 2x_1 \\ x_3 = -x_1 \end{cases}$$

$S = \{ x \in \mathbb{R}^3 \mid x_2 = 2x_1,\ x_3 = -x_1 \}$

a) fie $[x_1, x_2, x_3] \in T$

$\Updownarrow$

$\exists\, t, s \in \mathbb{R}$ a.î. $[x_1, x_2, x_3] = t[1,2,1] + s[-2,1,-3]$

$$\begin{cases} x_1 = t + (-2)s \\ x_2 = 2t + s \\ x_3 = t + (-3)s \end{cases} \Leftrightarrow \begin{cases} t - 2s = x_1 & \Rightarrow t = 2s + x_1 \\ 2t + s = x_2 & \Rightarrow s = x_2 - 2t \\ t - 3s = x_3 & \Rightarrow t = x_3 + 3s \end{cases}$$

$$\begin{cases} x_2 = 2(2s + x_1) \\ 2s + x_1 = x_3 + 3s \end{cases} \Leftrightarrow \begin{cases} x_2 = 4s + x_1 \\ x_3 = x_1 - s \end{cases}$$

$\cancel{T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2}$

$\cancel{x_1 = x_3 + 3s - 2s}$

$\cancel{x_1 = x_3 + s}$

$\cancel{x_1 = x_3 + x_2 - 2(2s + x_1)}$

$\cancel{x_1 = x_3 + x_2 - 4s - 2x_1}$

$\cancel{3x_1 = x_3 + x_2}$

scriem $x_2, x_3$ în funcție de $x_1$

$x_2 = 2t + s$

~~$x_2 = 2t + x_2 = 2t$~~

$\begin{cases} t = 2s + x_1 \\ t = x_3 + 3s \end{cases}$

$2s + x_1 = x_3 + 3s$

$x_1 - x_3 = s$

$x_2 = 2t + s$

$x_2 = 2t + x_1 - x_3$

$x_2 = 2(2s + x_1) + x_1 - x_3$

$x_2 = 4(x_1 - x_3) + 2x_1 + x_1 - x_3$

$x_2 = 4x_1 - 4x_3 + 2x_1 + x_1 - x_3$

$x_2 = 7x_1 - 5x_3$

$\Rightarrow 7x_1 - x_2 - 5x_3 = 0$

$T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 7x_1 - x_2 - 5x_3 = 0\}$

**3.136** Să se scrie subspațiile $S, T$ ale lui $\mathbb{R}^3$ ca subspații generate (cu număr minimal de generatori)

a) $S = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 - x_2 - x_3 = 0\}$

b) $T = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 - x_2 = x_2 - x_3 = x_3 - x_1\}$

a) notăm $x_3 = t$, $x_2 = s \Rightarrow x_1 = t + s$

$[x_1, x_2, x_3] = [t+s, s, t]$

$\Updownarrow$

$[x_1, x_2, x_3] = [t, 0, t] + [s, s, 0] \in \langle [1,0,1], [1,1,0] \rangle$

$[x_1, x_2, x_3] = t[1,0,1] + s[1,1,0]$

$S = \langle [1,0,1], [1,1,0] \rangle$

b) $x_1 - x_2 = x_2 - x_3 = x_3 - x_1$

$\begin{cases} x_1 - x_2 = x_2 - x_3 \\ x_2 - x_3 = x_3 - x_1 \end{cases} \Leftrightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 2x_3 + x_1 = 0 \end{cases} (-)$

$-3x_2 + 3x_3 = 0$

$$-3x_2 + 3x_3 = 0 \quad | \cdot (-1)$$

$$3x_2 - 3x_3 = 0 \quad | : 3$$

$$x_2 - x_3 = 0$$

$$x_2 = x_3$$

fie $x_1 = t$, $x_2 = t$, $x_3 = t$

$$[x_1, x_2, x_3] = [t, t, t] = t \cdot [1, 1, 1]$$

$$T = \langle [1, 1, 1] \rangle$$