$$\vec{E}(\vec{r},t)=\vec{E}(x,y)e^{ik_z z-i\omega y},\ \vec{B}(\vec{r},t)=\vec{B}(x,y)e^{ik_z z-i\omega t}$$

$$\nabla_\perp(E_z(x,y)e^{ik_z z})-\frac{\partial}{\partial z}(\vec{E}_\perp(x,y)e^{ik_z z})=\frac{i\omega}{c}[\vec{e_z}\times\vec{B}_\perp(x,y)e^{ik_z z}]$$

$$1)\quad \nabla_\perp E_z(x,y)-ik_z\vec{E}_\perp(x,y)=\frac{i\omega}{c}[\vec{e_z}\times\vec{B}_\perp(x,y)]$$

$\mathrm{rot}\vec{H}=\dfrac{i\omega\varepsilon(\omega)}{c}\vec{E}\to$ домножим на $\mu(\omega)\Rightarrow\mathrm{rot}\vec{B}=-\dfrac{i\omega}{c}\varepsilon(\omega)\mu(\omega)\vec{E}$, далее аналогично домножаем векторно на $\vec{e_z}$ и выделяем $\perp$ составляющую:

$$2)\quad \nabla_\perp B_z(x,y)-ik_z\vec{B}_\perp(x,y)=-\frac{i\omega}{c}\varepsilon\mu[\vec{e_z}\times\vec{E}_\perp(x,y)]$$

$$\nabla_\perp E_z(x,y)-ik_z\vec{E}_\perp(x,y)=\frac{i\omega}{c}\left[\vec{e_z}\times\left(\frac{\nabla_\perp B_z(x,y)+\frac{i\omega\varepsilon\mu}{c}[\vec{e_z}\times\vec{E}_\perp]}{ik_z}\right)\right];$$

$$[\vec{e_z}\times[\vec{e_z}\times\vec{E}_\perp]]=\vec{e_z}\overset{0}{\cancel{(\vec{e_z},\vec{E}_\perp)}}-\vec{E}_\perp$$

$$ik_z\nabla_\perp E_z(x,y)+k_z^2+\vec{E}_\perp(x,y)=\frac{i\omega}{c}[\vec{e_z}\times\nabla_\perp B_z(x,y)]+\frac{\omega^2\varepsilon\mu}{c^2}\vec{E}_\perp(x,y);\quad æ^2=\frac{\varepsilon(\omega)\mu(\omega)\omega^2}{c^2}-k_z^2$$

$$\vec{E}_\perp(x,y)=\frac{ik_z}{æ^2}\nabla_\perp E_z(x,y)-\frac{i\omega}{cæ^2}[\vec{e_z}\times\nabla_\perp B_z(x,y)]$$

$$\vec{B}_\perp(x,y)=\frac{ik_z}{æ^2}\nabla_\perp B_z(x,y)+\frac{i\omega\varepsilon\mu}{cæ^2}[\vec{e_z}\times\nabla_\perp E_z(x,y)]$$

Уравнения на $E_z(x,y)$ и $B_z(x,y)$

$$\begin{cases}\mathrm{rot}(\mathrm{rot}\vec{E})=\dfrac{i\omega}{c}\mathrm{rot}(\mu\vec{H})=\dfrac{i\omega}{c}\mu\left(-\dfrac{i\omega\varepsilon}{c}\right)\vec{E}\\ \mathrm{rot}(\mathrm{rot}\vec{E})=\overset{0}{\cancel{\nabla\mathrm{div}\vec{E}}}-\Delta\vec{E},\quad \mathrm{div}\vec{D}=0=\mathrm{div}-\varepsilon\vec{E}=\varepsilon\mathrm{div}\vec{E}=0\end{cases}\Rightarrow$$

$$\Rightarrow\Delta\vec{E}(x,y,z)+\frac{\omega^2\varepsilon\mu}{c^2}\vec{E}(x,y,z)=0$$

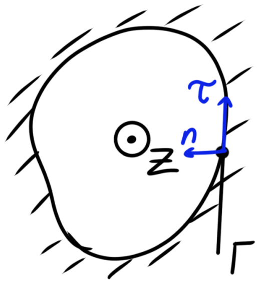
$$\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\frac{\partial}{\partial z}\right)E_z(x,y)e^{ik_z z}+\frac{\omega^2}{c^2}\varepsilon\mu E_z(x,y)e^{ik_z z}=0$$

$$\Delta_\perp E_z(x,y)e^{ik_z z}-k_z^2E_z(x,y)e^{ik_z z}+\frac{\omega^2\varepsilon\mu}{c^2}E_z(x,y)e^{ik_z z}=0$$

$$\Delta_\perp E_z(x,y)+æ^2E_z(x,y)=0-\text{ двумерное волновое уравнение}$$

$$\text{Аналогично: }\nabla_\perp B_z(x,y)+æ^2B_z(x,y)=0$$

Граничные условия:




$$E_\tau|_\Gamma = 0 \Rightarrow (\vec{E}, \vec{r}) = 0, \quad (\tau - \text{ вдоль поверхности}), \quad B_n|_\Gamma = 0 \Rightarrow (\vec{n}, \vec{B})|_\Gamma = 0$$

$$\text{Пусть } \vec{\tau} \parallel \vec{e_z} \Rightarrow E_z(x,y)e^{ik_z z - i\omega t}|_\Gamma = 0 \Rightarrow E_z(x,y)|_\Gamma = 0$$

$$\text{Пусть } \vec{\tau} \nparallel \vec{e_z} \Rightarrow E_\tau = (\vec{\tau}, (\vec{e_z}E_z + \vec{E}_\perp))|_\Gamma = \underbrace{(\vec{\tau}, \vec{e_z})\cancelto{0}{E_z}|_\Gamma}_{0} + (\vec{\tau}, \vec{E}_\perp)$$

$$= \left(\vec{\tau}, \left\{\frac{ik_z}{æ^2}\nabla_\perp E_z(x,y) - \frac{i\omega}{cæ^2}[\vec{e_z} \times \nabla_\perp B_z(x,y)]\right\}\right)\Bigg|_\Gamma = \frac{ik_z}{æ^2}\frac{\partial E_z}{\partial \tau}\Bigg|_\Gamma - \frac{i\omega}{cæ^2}(\vec{\tau}, [\vec{e_z} \times \nabla_\perp B_z(x,y)])\Bigg|_\Gamma =$$

$$\pm\frac{i\omega}{cæ^2}(\nabla B_z, \vec{n})\Bigg|_\Gamma = \pm\frac{i\omega}{cæ^2}\frac{\partial B_z(x,y)}{\partial n}\Bigg|_\Gamma = 0 \Rightarrow E_z(x,y)\Bigg|_\Gamma = 0, \frac{\partial B_z}{\partial n}\Bigg|_\Gamma = 0$$

Второе граничное условие:

$$(\vec{n}, \vec{B})\Bigg|_\Gamma = 0 \Rightarrow \left(\vec{n}, \left\{\vec{e_z}B_z(x,y) + \frac{ik_z}{æ^2}\nabla_\perp B_z(x,y) + \frac{i\omega\varepsilon\mu}{cæ^2}[\vec{e_z} \times \nabla_\perp E_z(x,y)]\right\}\right)\Bigg|_\Gamma =$$

$$\frac{ik_z}{æ^2}\frac{\partial B_z(x,y)}{\partial n}\Bigg|_\Gamma + \frac{i\omega\varepsilon\mu}{cæ^2}(\vec{n}, [\vec{e_z} \times \nabla_\perp E_z(x,y)])\Bigg|_\Gamma = \frac{i\omega\varepsilon\mu}{cæ^2}\frac{\partial E_z(x,y)}{\partial n}\Bigg|_\Gamma = 0$$

Для упрощения анализа разделим общее решение на два:

1-ый тип: $E$ - волна (TM - волна) $E_z(x,y) \neq 0, B_z(x,y) = 0$ внутри волновода;

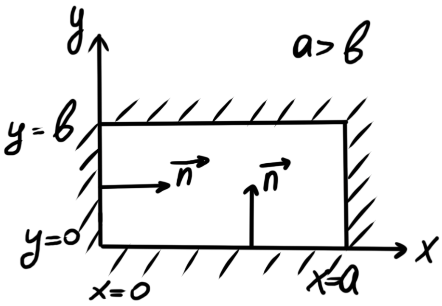
2-ой тип: $H$ - волна (TE - волна) $E_z(x,y) = 0, B_z(x,y) \neq 0\ldots$

Для $E$ - волны: $\Delta_\perp E_z(x,y) + æ^2 E_z(x,y) = 0 +$ Г.У $E_z|_\Gamma = 0$

- задача Штурмана-Лиувилля, из решения которой находятся собственные значения $æ_n, n = ....$ и собственные функции $E_z^{(n)}(x,y)$

Для $H$ - волны: $\Delta_\perp B_z(x,y) + æ^2 B_z(x,y) = 0 +$ Г.У. $\dfrac{\partial B_z}{\partial n}|_\Gamma = 0$

Пример 1: $H$ - волна в прямоугольном волноводе




$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)B_z(x,y)+æ^2B_z(x,y)=0,\ +\text{ Г.У. }\frac{\partial B_z}{\partial n}|_\Gamma=0\Rightarrow\frac{\partial B_z}{\partial x}|_{x=0,a}=0,\ \frac{\partial B_z}{\partial y}|_{y=0,b}=0$$

Пусть $B_z(x,y)=B_1(x)B_2(y)$

$$B_2(y)B_1''(x)+B_1(x)B_2''(y)+æ^2B_1(x)B_2(y)=0$$

$$\Rightarrow\ \underbrace{\frac{B_1''(x)}{B_1(x)}}_{=\text{const}=-k_x^2}+\underbrace{\frac{B_2''(y)}{B(y)}}_{=\text{const}=-k_y^2}+æ^2=0\Rightarrow\begin{cases}B_z(x,y)=B_0\sim(k_xx\alpha_x)\sin(k_yy\alpha_y)\\ æ^2=k_x^2+k_y^2\end{cases}$$

Граничные условия:

$$\left.\frac{\partial B_z}{\partial x}\right|_{x=0}=k_xB_0\cos\alpha_x\sin(k_yy+\alpha_y)=0\text{ при }\forall y\Rightarrow\alpha_x=\frac{\pi}{2}$$

$$\left.\frac{\partial B_z}{\partial x}\right|_{x=a}=k_xB_0\cos\left(k_xa+\frac{\pi}{2}\right)\sin(k_yy+\alpha_y)=0\Rightarrow k_xa=n_x\pi,\ k_x=\frac{n_x\pi}{a},\ n_x\in\mathbb{Z}$$

Аналогично: $\alpha_y=\frac{\pi}{2},\ k_y=\frac{n_y\pi}{b},\ n_y\in\mathbb{Z}\Rightarrow æ_{n_x,n_y}=\sqrt{\left(\frac{n_x\pi}{a}\right)^2+\left(\frac{n_y\pi}{b}\right)^2}$ –собственные числа

$$B_z(x,y)=B_0\cos(k_xx)\cos(k_yy),\ B_z(\vec r,t)=B_0\cos(k_xx)\cos(k_yy)e^{ik_zz-i\omega t}$$

$$E_\perp(x,y)=-\frac{i\omega}{cæ^2_{n_x,n_y}}[\vec{e_z}^2\times\nabla_\perp B_z(x,y)],\ B_\perp(x,y)=\frac{ik_z}{æ^2}\nabla_\perp B_z(x,y)$$

$$\nabla_\perp B_z(x,y)=\left(\vec{e_x}\frac{\partial}{\partial x}+\vec{e_y}\frac{\partial}{\partial y}\right)B_z(x,y)=-B_0\left\{\vec{e_x}k_x\sin(k_xx)\cos(k_yy)+\vec{e_y}k_y\cos(k_xx)\sin(k_yy)\right\}$$

Какие $n_x,n_y$ допустимы? Пусть $n_x=0,n_y=0$

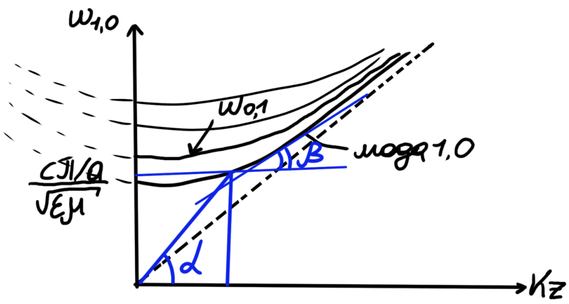
$B_z(\vec{r},t) = B_0 e^{ik_z z - i\omega t},\ \vec{B}_\perp(\vec{r},t) = \dfrac{ik_z}{0^2}0-$ проверить из уравнений Максвелла, что $B_\perp = 0$

$$\mathrm{div}\vec{B} = 0 = \frac{\partial B_z}{\partial z} = ik_z B_0 e^{ik_z z - i\omega t}(k_z \neq 0) \Rightarrow B_0 \text{ (такой моды нет)}$$

Пусть: $n_x = 1, n_y = 0 \Rightarrow æ_{1,0} = \dfrac{\pi}{a},\ \dfrac{\omega_{1,0}^2 \varepsilon\mu}{c^2} = \left(\dfrac{\pi}{a}\right)^2 + k_z^2$

$$B_z(x,y) = B_0\cos(k_x x),\ B_\perp(x,y) = \frac{ik_z}{æ_{1,0}^2} B_0(-k_x)\sin(k_x x)\vec{e_x},\ \text{далее } \rightarrow \begin{cases}\varepsilon(\omega) = \text{const}\\ \mu(\omega) = \text{const}\end{cases}$$

$$\vec{E}_\perp(x,y) = -\frac{i\omega}{cæ^2}\vec{e_y} B_0(-k_x)\sin(k_x x)$$




- мода$_{1,0}$ - основная мода волновода.

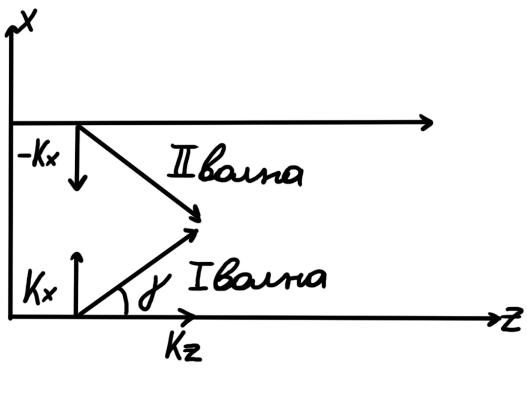
$$v_\Phi = \frac{\omega}{k_z},\ v_g = \frac{d\omega}{dk_z}$$

$$v_\Phi v_g = \frac{c_2}{\varepsilon\mu}$$

$$v_\Phi = tg\alpha,\ v_g = tg\beta$$

Представление в виде плоских волн:

$$B_z(\vec{r},t) = B_0\left(\frac{e^{ik_x x} + e^{-ik_x x}}{2}\right)e^{ik_z z - i\omega t} = \frac{B_0}{2}e^{ik_x x + ik_z z - i\omega t} + \frac{B_0}{2}e^{-ik_x x - ik_z z - i\omega t}$$



$$v_\Phi = \frac{c}{\sqrt{\varepsilon\mu}\cos\gamma}, \quad v_g = \frac{c}{\sqrt{\varepsilon\mu}}\cos\gamma, \cos\gamma = \frac{k_x}{\sqrt{k_x^2 + k_z^2}}$$