1. Продолжение. Спектр свертки двух функций

$$F(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} d\tau \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega\tau}d\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\omega')e^{i\omega'(t-\tau)}d\omega' =$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \hat{f}(\omega)\hat{g}(\omega')e^{-\omega' t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{d\tau e^{-i(\omega-\omega')\tau}}_{=2\pi\delta(\omega-\omega')} = \int_{-\infty}^{\infty} d\omega \hat{f}(\omega) \int_{-\infty}^{\infty} d\omega' \hat{g}(\omega')\delta(\omega-\omega')e^{-i\omega' t} =$$

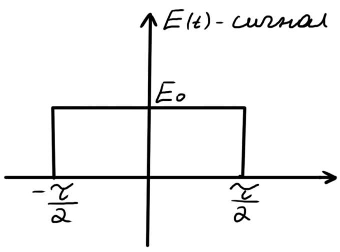
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t}d\omega \Rightarrow F(t) \risingdotseq \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$$

2. Соотношение неопределенностей

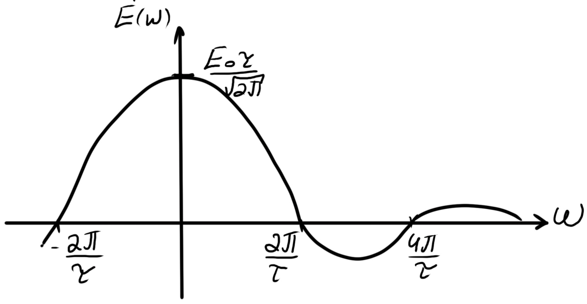
Определение 1. *Определенная связь между длительностью сигнала и шириной его спектра называется соотношение неопределенностей.*

Покажем эту связь на примерах:

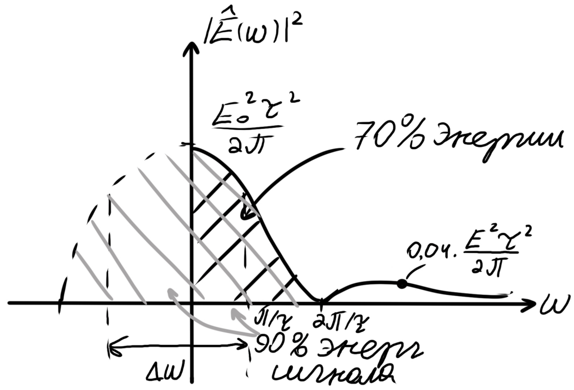
1) Спектр прямоугольного сигнала $E_1(t) = \begin{cases} E_0, |t| \le \frac{\tau}{2} \\ 0, |t| > \frac{\tau}{2} \end{cases}$ E(t) - сигнал.



$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\tau}{2}}^{-\frac{\tau}{2}} E_0 E^{+i\omega t}dt = \frac{E_0\tau}{\sqrt{2\pi}} \frac{e^{i\omega\frac{\tau}{2}} - e^{-i\omega\frac{\tau}{2}}}{2i\omega\frac{\tau}{2}} = \frac{E_0\tau}{\sqrt{2\pi}}\text{sinc}(\frac{\omega\tau}{2})$$



Спектральная плотность энергии $= |\hat{E}(\omega)|^2$

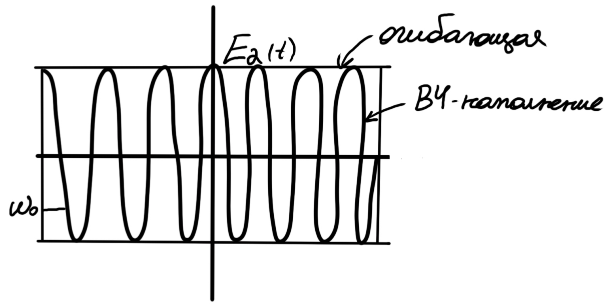


$$\Delta\omega \sim \frac{2\pi}{\tau} \Rightarrow \Delta\omega\tau \sim \pi - \text{ соотношение неопределенности}$$

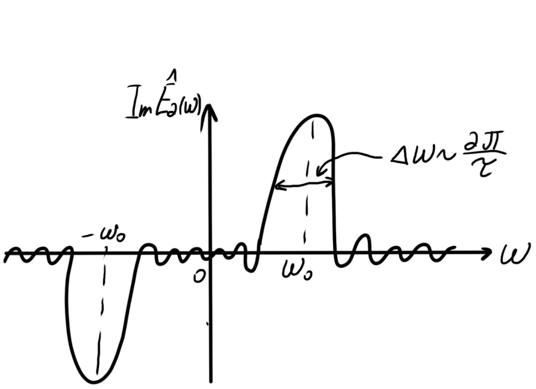
$$\tau \to \infty \Rightarrow \hat{E} \sim \delta(\omega)$$

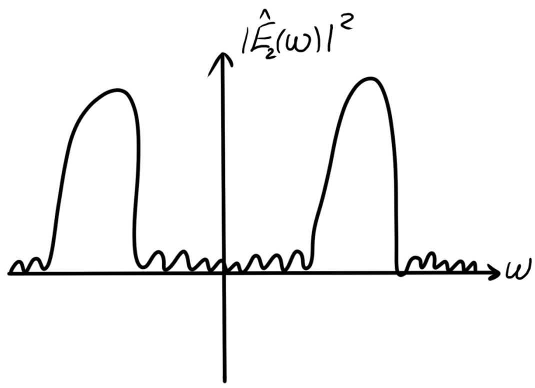
2)Спектр синусоидальной волны:

$$E_2(t) = \begin{cases} E_0 \sin\omega_0 t, |t| \le \frac{\tau}{2} \\ 0, |t| > \frac{\tau}{2} \end{cases} \quad ,\omega_0 = \frac{2\pi}{\tau} \quad \text{, пусть } \tau = NT, N(\text{целое}) \gg 1$$



$$\hat{E}_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} e^{i\omega t} dt = \frac{1}{2i} \left\{ \frac{E_0\tau}{\sqrt{2\pi}} \text{sinc}\left[(\omega+\omega_0)\frac{\tau}{2}\right] - \frac{E_0\tau}{\sqrt{2\pi}} \text{sinc}\left[(\omega-\omega_0)\frac{\tau}{2}\right] \right\}$$



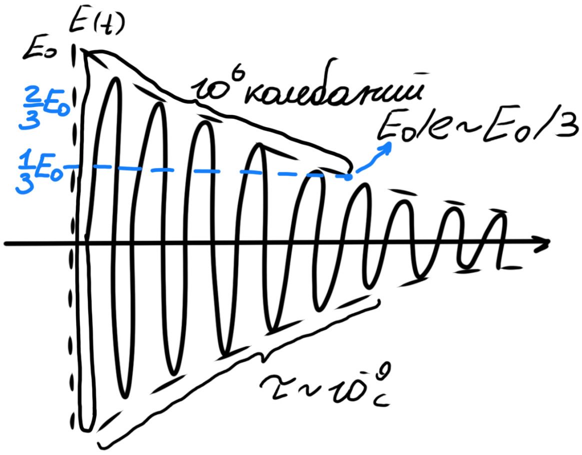


Если $\frac{\Delta\omega}{\omega_0} \ll 1$, то такая волна - квазимонохроматическая.
3) Спектр радиационно затухающего осциллятора:
Механистическая модель атома:

$$E(t) = \begin{cases} E_0 e^{\gamma t}\cos\omega_0 t, t > 0 \\ 0, t < 0 \end{cases}$$

$$\gamma = \frac{e^2\omega_0^2}{3mc^2} \sim 10^9 c^{-1} \quad \omega_0 \approx 2\cdot 10^{16}\frac{rad}{c} \Rightarrow f \sim 3\cdot 10^8 c^{-1}$$

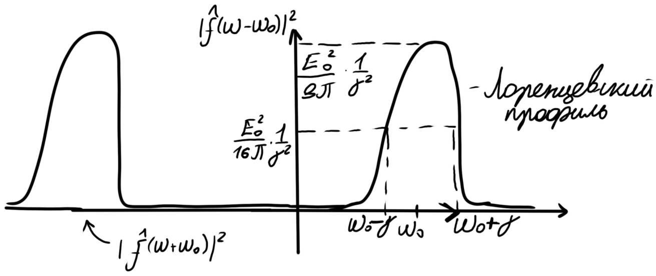
$$e^{-\gamma t} = e^{-\frac{t}{\tau}} \Rightarrow \tau \sim \frac{1}{\gamma}$$



$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty E_0 e^{-\gamma t}\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} e^{+i\omega t} dt$$

$$\hat{E}(\omega) = \frac{E_0}{2\sqrt{2\pi}}\left\{-\frac{1}{-\gamma + i(\omega+\omega_0)} - \frac{1}{-\gamma + i(\omega-\omega_0)}\right\} = \hat{f}(\omega+\omega_0)\hat{f}(\omega-\omega_0)$$

$$|\hat{f}(\omega-\omega_0)|^2 = \frac{E_0^2}{8\pi}$$



$$\Delta\omega \sim 2\gamma - \text{ширина спектра}$$

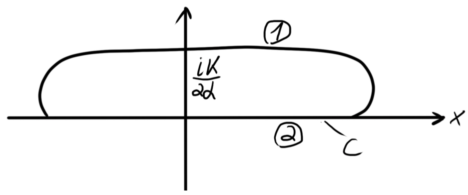
$$\Delta\omega \underbrace{\Delta t}_{\sim\tau} \sim 2\gamma\frac{1}{\gamma} \sim 2$$

$$|\hat{E}(\omega)|^2 = |\hat{f}(\omega+\omega_0)|^2 + |\hat{f}(\omega-\omega_0)|^2 + \cancel{\text{поправка}}$$

Поправка мала, если $10^9c^{-1} \sim \Delta\omega \ll \omega_0 \sim 2\cdot 10^{16}\frac{rad}{c}$

4) Спектр гауссовой функции : $f(x) = E_0 e^{-\alpha x^2}$ $\quad \hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx =$

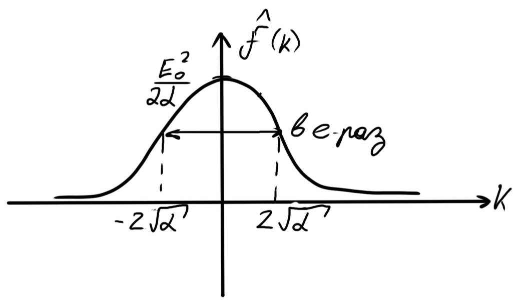
$$\frac{1}{\sqrt{2\pi}}E_0\int_{-\infty}^{\infty} e^{-\alpha x^2 - ikx}dx; -\alpha x^2 - ikx = -\alpha\left(x^2 + 2x\frac{ik}{2\alpha} - \frac{k^2}{4\alpha^2}\right) - \frac{k^2}{4\alpha} = -\alpha\left(x + \frac{ik}{2\alpha}\right)^2 - \frac{k^2}{4\alpha}$$

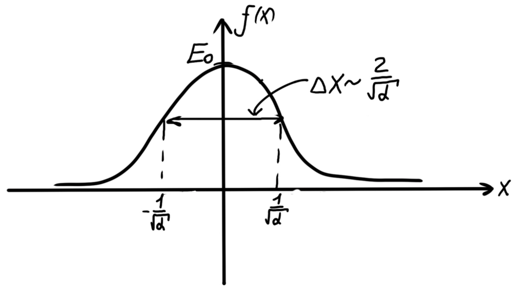


$$\int_C e^{-\alpha z^2}dz = 0 = \int_1 + \int_2 \Rightarrow \int_1 = \int_{-2}$$

$$\int_{-\infty}^{\infty} e^{-\alpha z^2}dz = \int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\hat{f}(k) = \frac{E_0}{\sqrt{2\pi}}\sqrt{\frac{\pi}{\alpha}}e^{-\frac{k^2}{4\alpha}}$$





$$\begin{array}{l} \Delta k \sim 4\sqrt{\alpha} \\ \Delta x \sim \frac{2}{\sqrt{\alpha}} \end{array} \Rightarrow \Delta k \Delta x \sim 8 \sim \pi$$

5) Модулированный гауссиан: $E(x) = E_0 e^{-\alpha x^2}\cos k_0 x$

3. Преобразование Фурье функции четырех переменных (x,y,z,t). Уравнения Максвелла в Фурье преобразованиия

$$f(x,y,z,t) = \frac{1}{\sqrt{2\pi}^4}\iiiint \hat{f}(k_x,k_y,k_z,k_t)e^{ik_xx}e^{ik_yy}e^{ik_zz}e^{-i\omega t}dk_xdk_ydk_zdk_t$$

$$f(\vec{r},t) = \frac{1}{(2\pi)^2}\iiiint \hat{f}(\vec{k},\omega)e^{i(\vec{k},\vec{r})-i\omega t}d^3kd\omega$$

$$\frac{\partial f(\vec{r},t)}{\partial t} = \frac{1}{(2\pi)^2}\iiiint \hat{f}(\vec{k},\omega)(-i\omega)e^{i\vec{k}\vec{r}-i\omega t}d^3kd\omega \quad \frac{\partial f}{\partial t} \risingdotseq -i\omega\hat{f}(\vec{k},\omega)$$

$$\frac{\partial f(\vec{r},t)}{\partial x} \risingdotseq ik_x\hat{f}(\vec{k},\omega)$$

$$\mathrm{div}\hat{\vec{D}}(\vec{r},t) = \frac{\partial\hat{\vec{D}}_x}{\partial x}+\frac{\partial\hat{\vec{D}}_y}{\partial y}+\frac{\partial\hat{\vec{D}}_z}{\partial z} \risingdotseq ik_x\hat{D}_x(\vec{k},\omega)+ik_y\hat{D}_y(\vec{k},\omega)+ik_z\hat{D}_z(\vec{k},\omega) = i(\vec{k},\hat{\vec{D}}(\vec{r},\omega))$$

$$\mathrm{rot}\hat{\vec{E}} = [\nabla\times\hat{\vec{E}}] = \frac{1}{(2\pi)^2}\iiiint \underbrace{\left[(\nabla\times\hat{\vec{E}}(\vec{k},\omega)e^{i(\vec{k}\vec{r})})\right]}_{\nabla e^{i(\vec{k},\vec{r})}\times\vec{E}(\vec{k},\omega)} e^{-i\omega t}d^3d\omega =$$

$$\text{Где } \nabla e^{i(\vec{k},\vec{r})} = \vec{e_x}ik_xe^{(i(\vec{k},\vec{r}))}+\vec{e_y}ik_ye^{(i(\vec{k},\vec{r}))}+\vec{e_z}ik_ze^{(i(\vec{k},\vec{r}))} = i\vec{k}e^{i(\vec{k},\vec{r})}$$

$$= \frac{1}{(2\pi)^2}\iiiint \left[i\vec{k}\times\hat{\vec{E}}(\vec{k},\omega)\right]e^{i(\vec{k},\vec{r})-i\omega t}d^3kd\omega$$

$$\mathrm{rot}\vec{E} \risingdotseq i\left[\vec{k}\times\hat{\vec{E}}(\vec{k},\omega)\right]$$

$$\begin{cases}\mathrm{rot}\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \risingdotseq i\left[\vec{k}\times\hat{\vec{E}}(\vec{k},\omega)\right] = \frac{i\omega}{c}\hat{\vec{B}}(\vec{k},\omega)\\ \mathrm{rot}\vec{H} = \frac{4\pi}{c}\vec{j}+\frac{1}{c}\frac{\partial\vec{D}}{\partial t} \risingdotseq i\left[\vec{k}\times\hat{\vec{H}}(\vec{k},\omega)\right] = \frac{4\pi}{c}\hat{\vec{j}}(\vec{k},\omega)-\frac{i\omega}{c}\hat{\vec{D}}(\vec{k},\omega)\\ \mathrm{div}\vec{D} = 4\pi\rho \risingdotseq i(\vec{k},\hat{\vec{D}}(\vec{k},\omega)) = 4\pi\hat{\rho}(\vec{k},\omega)\\ \mathrm{div}\vec{B} = 0 \Rightarrow (\vec{k},\hat{\vec{B}}(\vec{k},\omega)) = 0\end{cases}$$

В системе слева это уравнения Максвелла, а справа преобразование Фурье уравнений Максвелла.

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\vec{j} = 0 \risingdotseq -i\omega\hat{\rho}(\vec{k},\omega)+i(\vec{k},\hat{\vec{j}}(\vec{k},\omega)) = 0$$

Если $\vec{B} = \mu\vec{H}, \vec{D} = \varepsilon\vec{E}, \varepsilon, \mu - \mathrm{const}$

$$\underset{a}{\vec{k}}\times\left[\underset{b}{\vec{k}}\times\underset{c}{\vec{E}}\right] = \frac{\omega\mu}{c}\left(-\frac{\omega}{c}\varepsilon\hat{\vec{E}}\right)$$

$$||$$

$$\vec{k}\underbrace{(\vec{k},\hat{\vec{E}})}_{=0}-\hat{\vec{E}}k^2 = -\frac{\varepsilon\mu}{c^2}\omega^2\hat{\vec{E}} \Rightarrow k^2 = \frac{\omega^2}{\left(\frac{c}{\sqrt{\varepsilon\mu}}\right)^2} = \frac{\omega^2}{v_{\text{в}}^2}$$