CS2215 - Computer Graphics

Lecture 2 - Transformations

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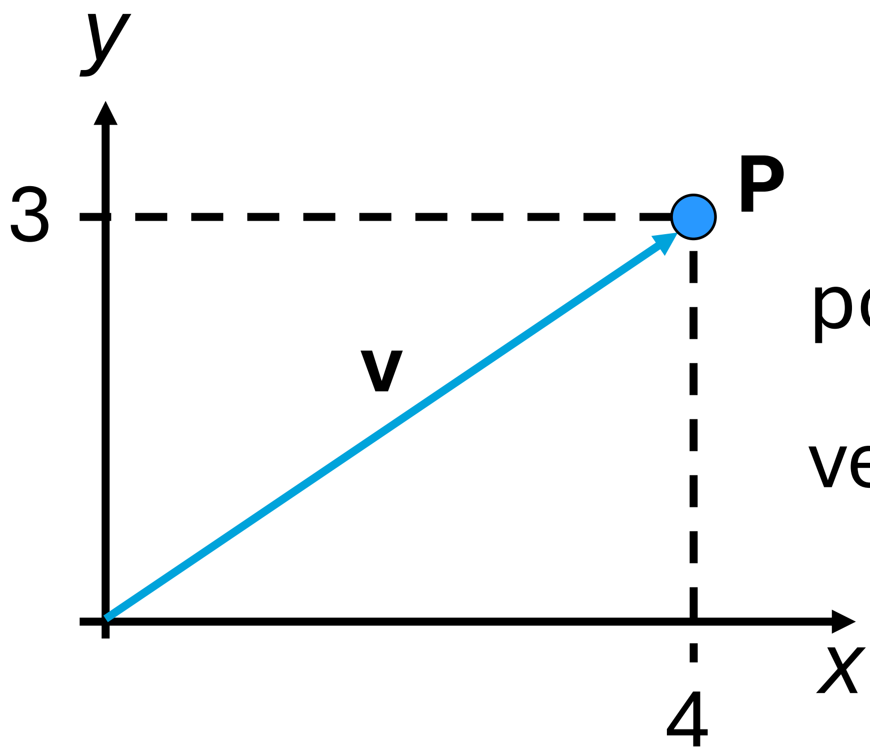
Points and vectors



point **P (4,3)**

vector **v (4,3)**

- same notation
- same operations (math)
- defined by context, e.g.:
  - positions → points (vertices)
  - directions → vectors

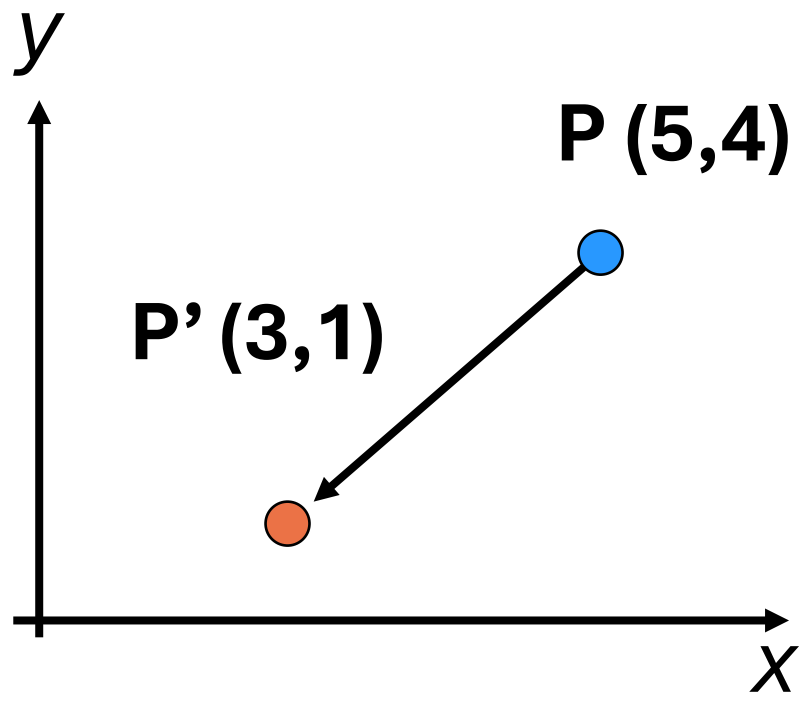
Matrices

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- matrices represent transformations!
- operates on
  - points, vectors and other transformations (matrices)

Translation



- move point P

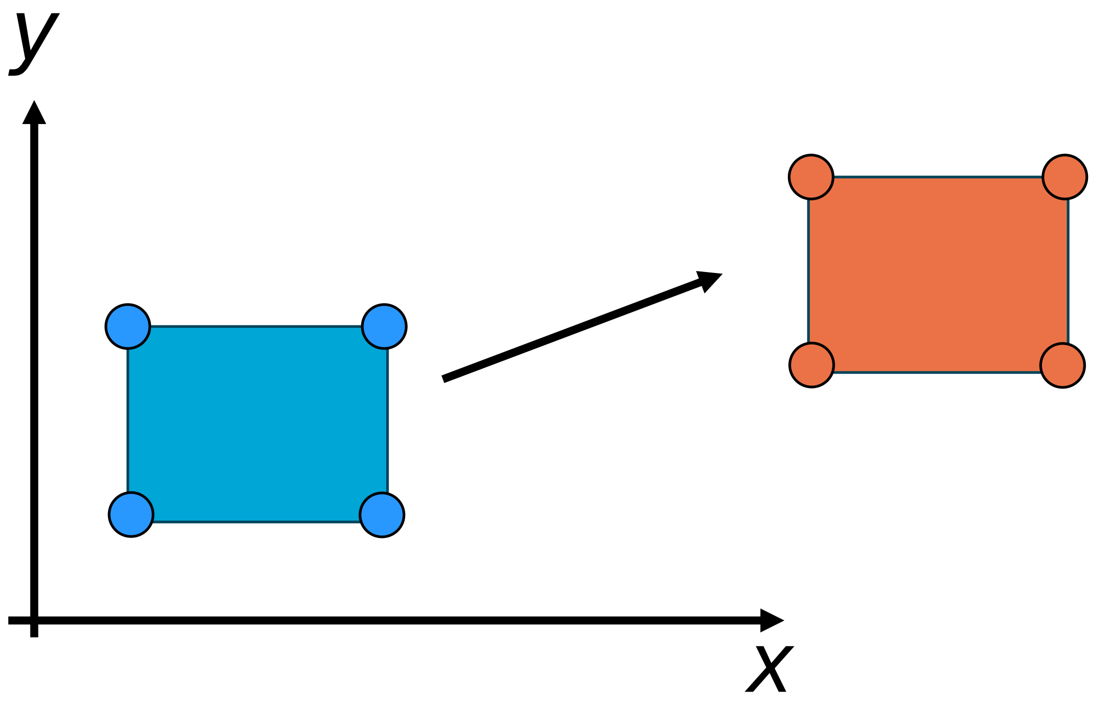
$$\begin{bmatrix}5\\4\end{bmatrix}+\begin{bmatrix}?\\?\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$$

$$\begin{bmatrix}5\\4\end{bmatrix}+\begin{bmatrix}-2\\-3\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$$

- Translation:

$$\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}t_x\\t_y\end{bmatrix}=\begin{bmatrix}x+t_x\\y+t_y\end{bmatrix}$$

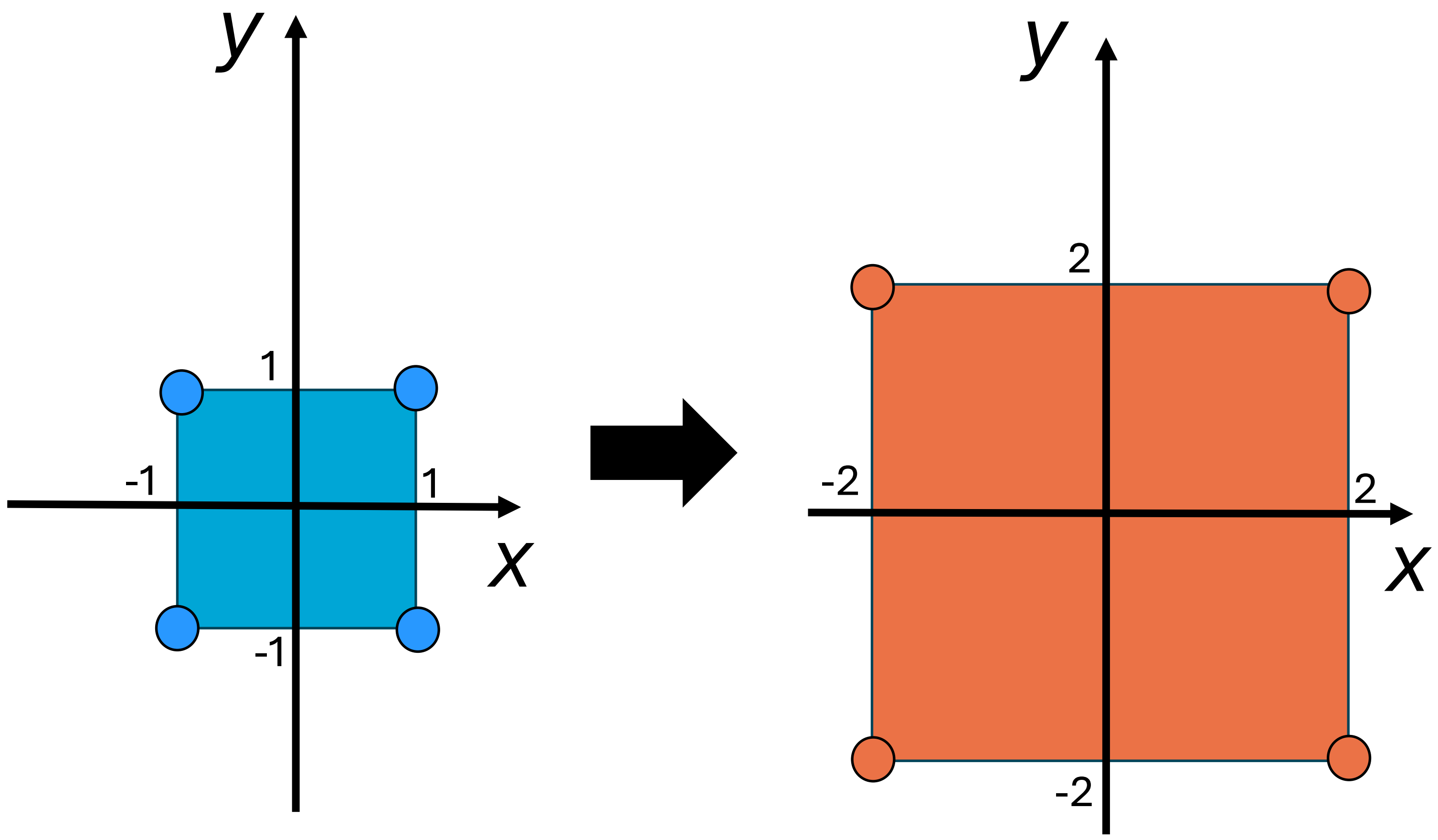
Translation



- move an object (mesh)
- move every point (vertex) by same amount

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

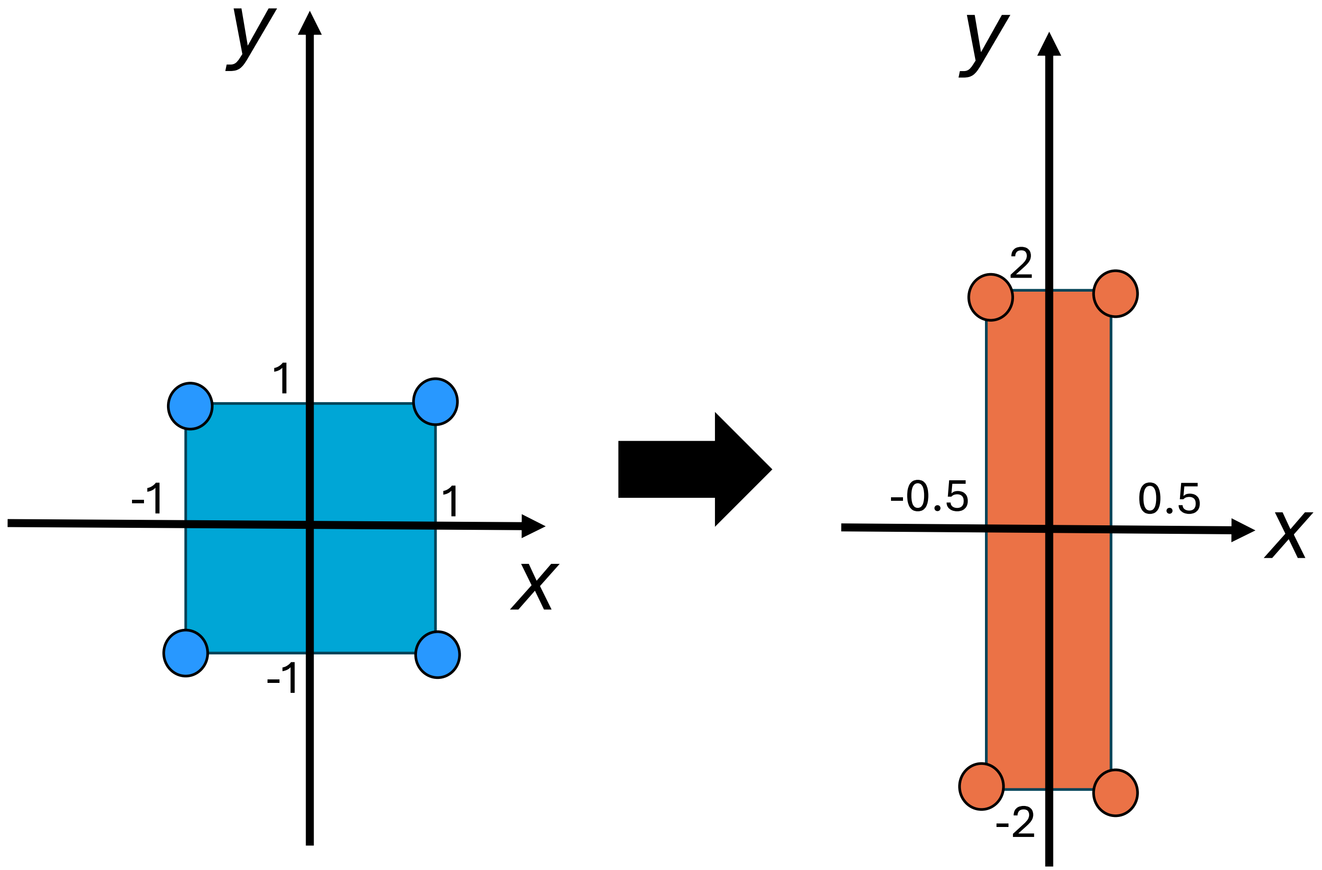
Scale



- scale by 2
- multiply coordinates by 2
  - (1, 1) → (2, 2)
  - (-1, 1) → (-2, 2)
  - (1,-1) → (2,-2)
  - (-1,-1) → (-2,-2)

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} 2x_i \\ 2y_i \end{bmatrix}$$

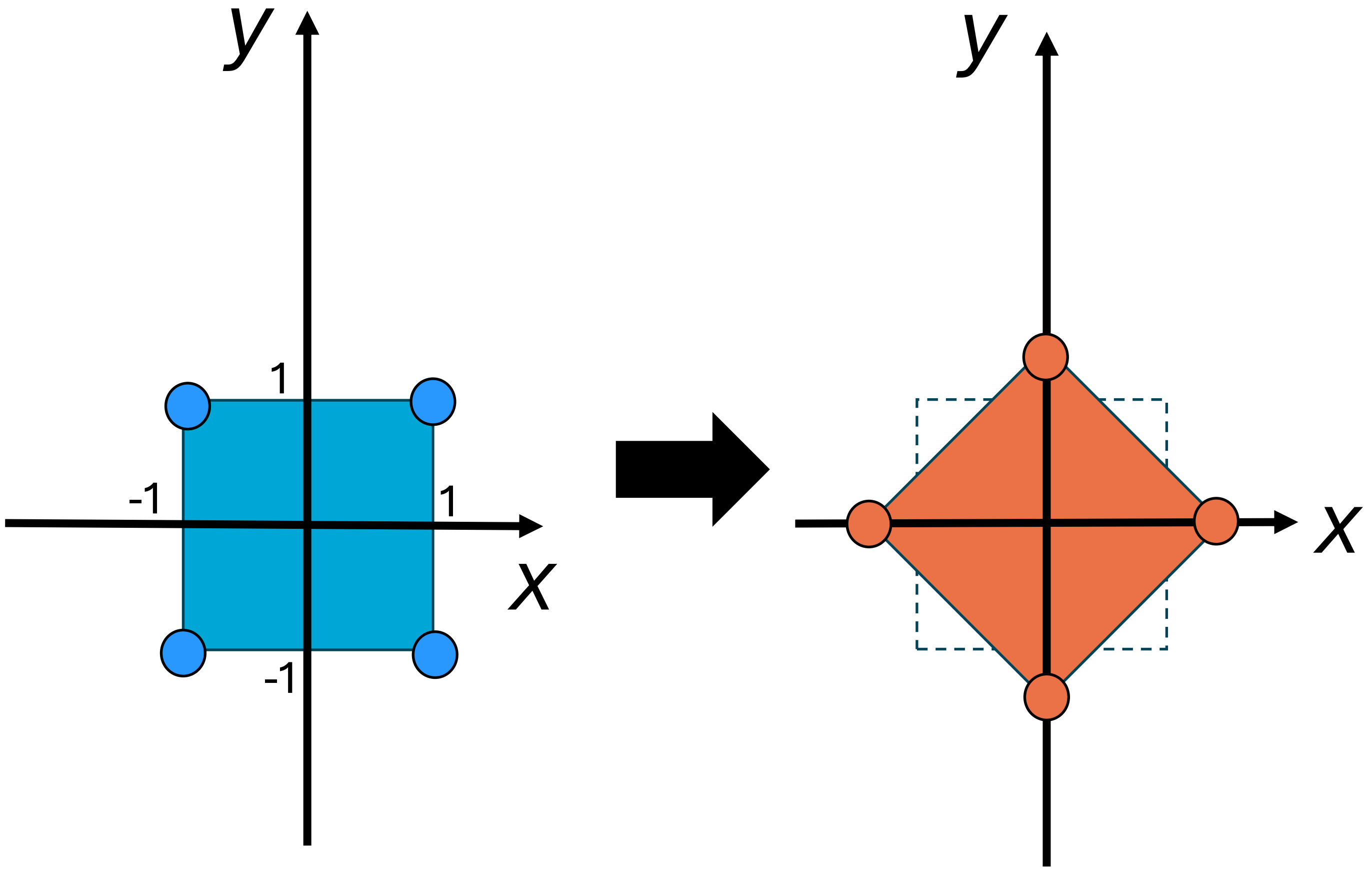
Scale



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = ?$$

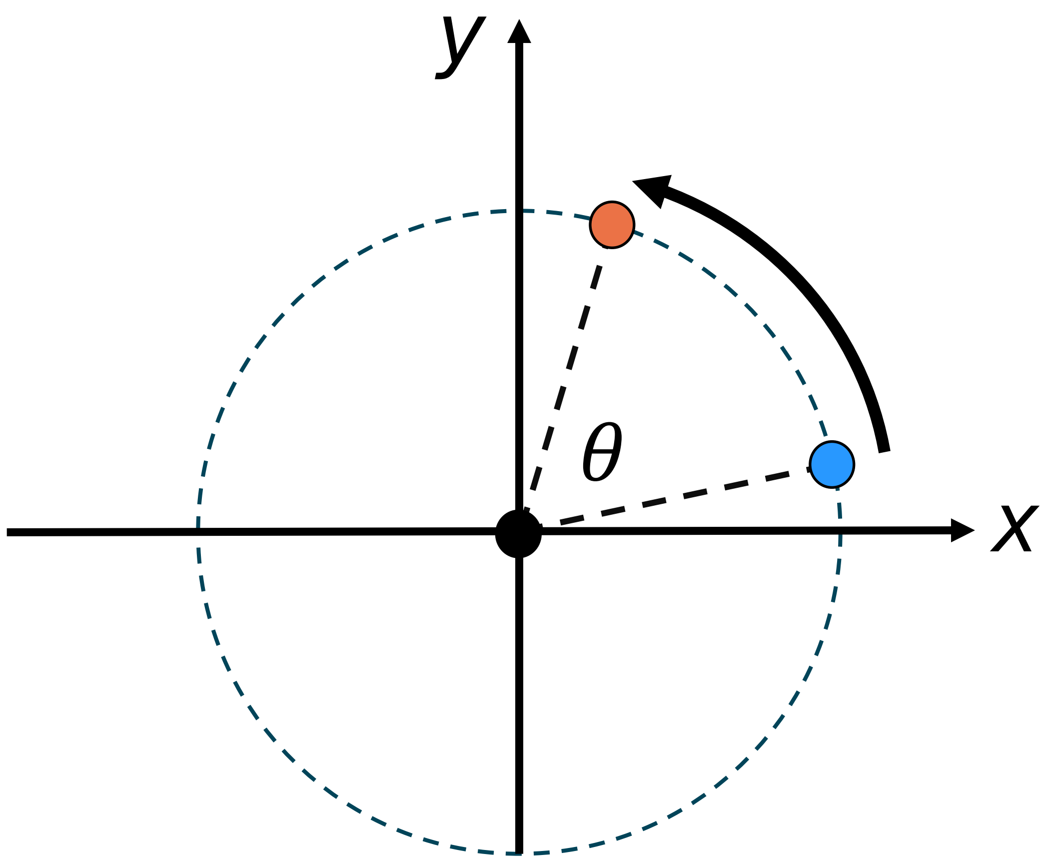
- (1, 1) → (0.5, 2)
- (-1, 1) → (-0.5, 2)
- (1,-1) → (0.5,-2)
- (-1,-1) → (-0.5,-2)

Rotate



- rotate 45° counterclockwise
- (1, 1) → (?, ?)
- (1, 1) → $(0, \sqrt{2})$

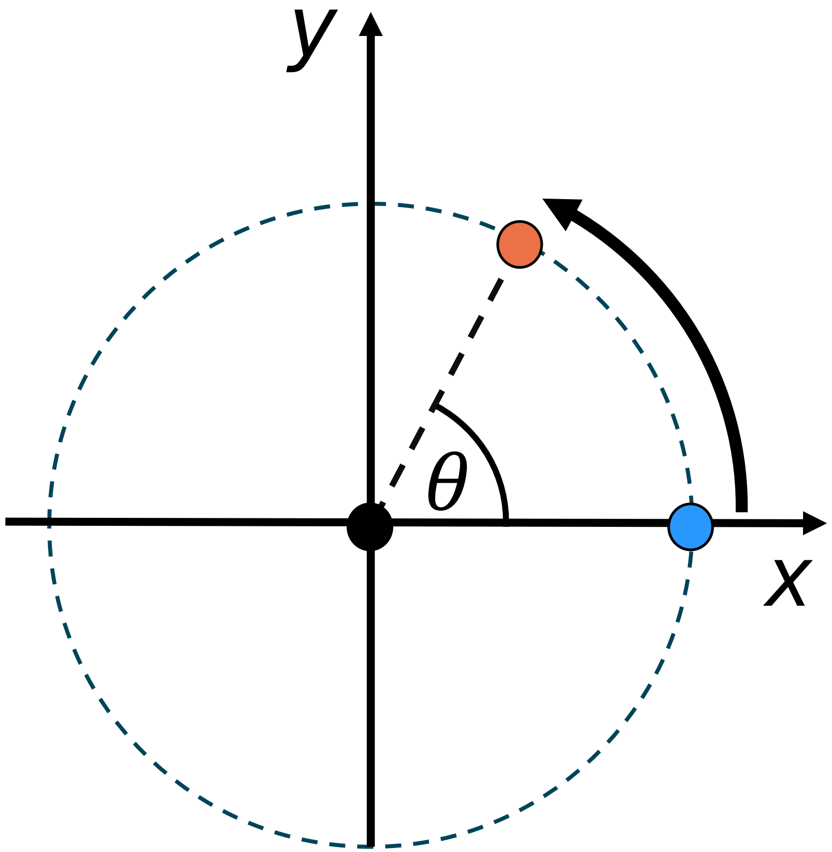
Rotation

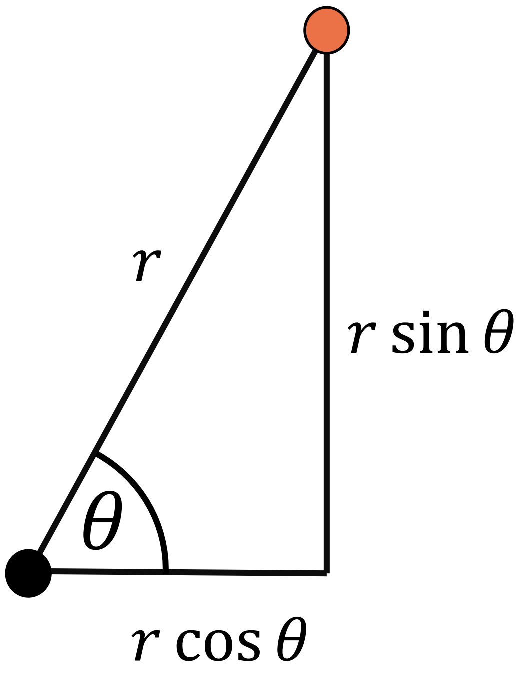


- rotation is always around origin
- distance to origin is preserved
- rotates θ degrees (or radians)

Rotation

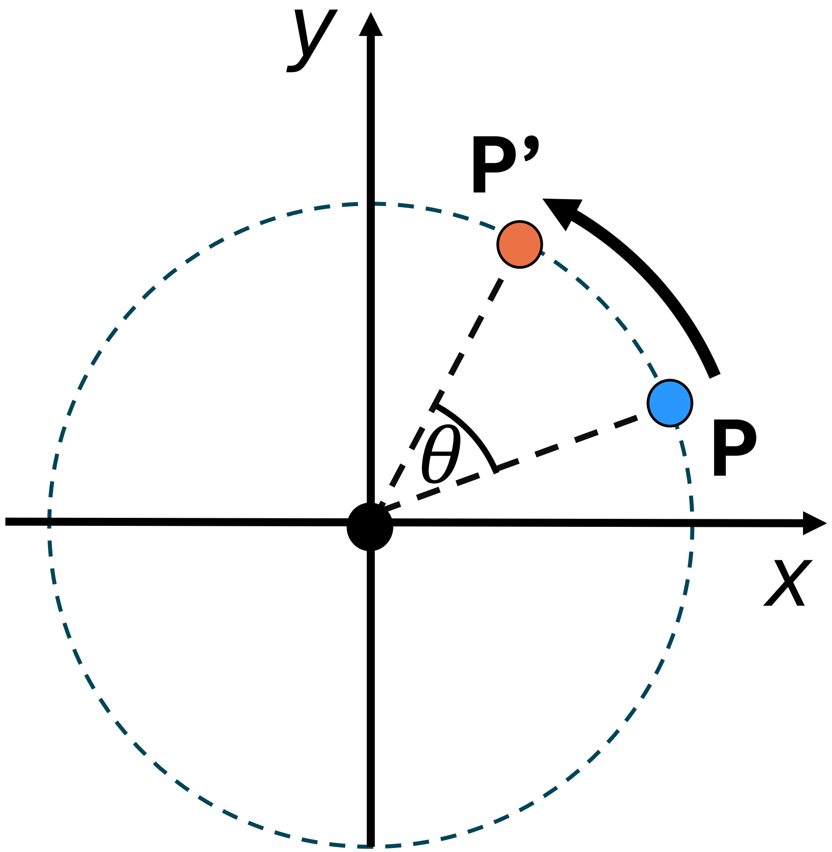
- rotate point (r,0) by θ degrees





$$x' = r\cos\theta$$
$$y' = r\sin\theta$$

Rotation



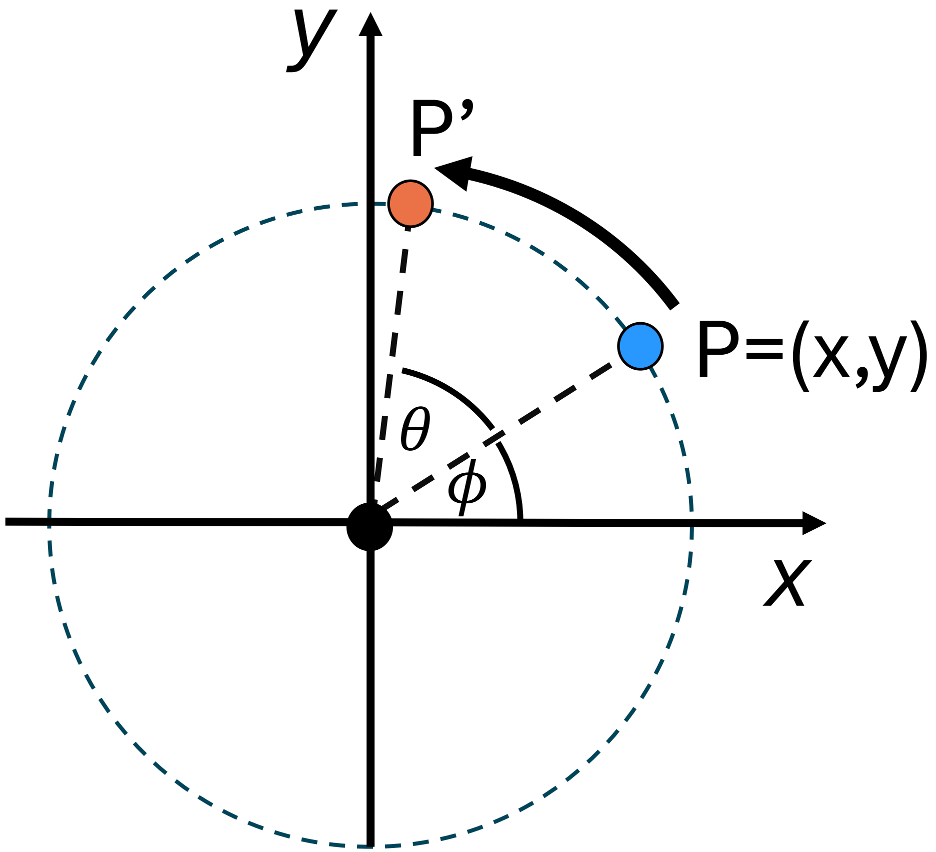
- rotate point (x , y) around origin by θ degrees

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x' = x\cos\theta \ - \mathrm{y}\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

exercise: derive the rotation matrix



- rotate point (x , y) around origin by θ degrees

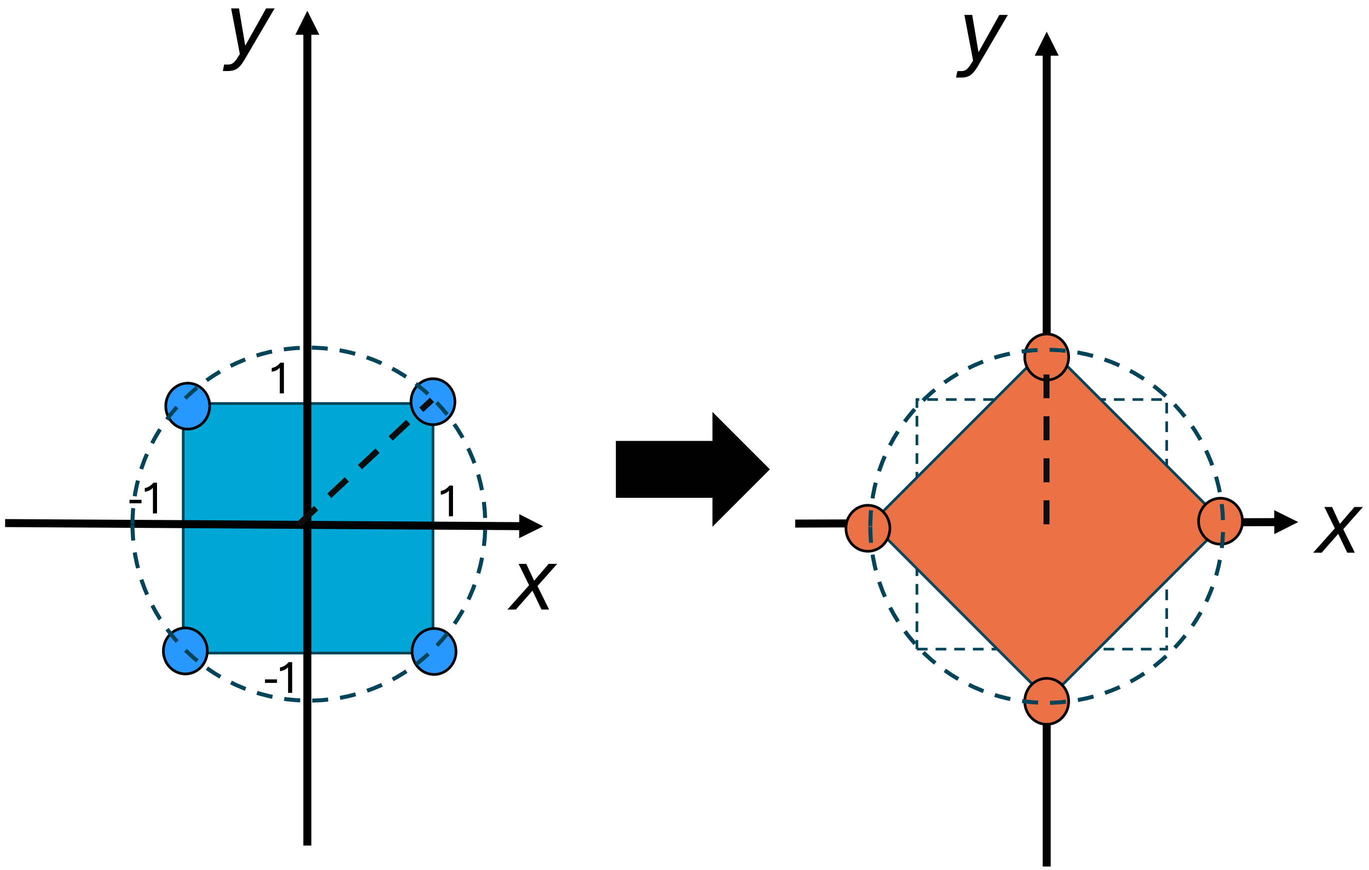
$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x'\\ y'\end{bmatrix}$$

$$x' = x\cos\theta \ - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

tip: lookup $\cos(\phi+\theta)$ and $\sin(\phi+\theta)$

Rotation



- rotate 45° counterclockwise

$$\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

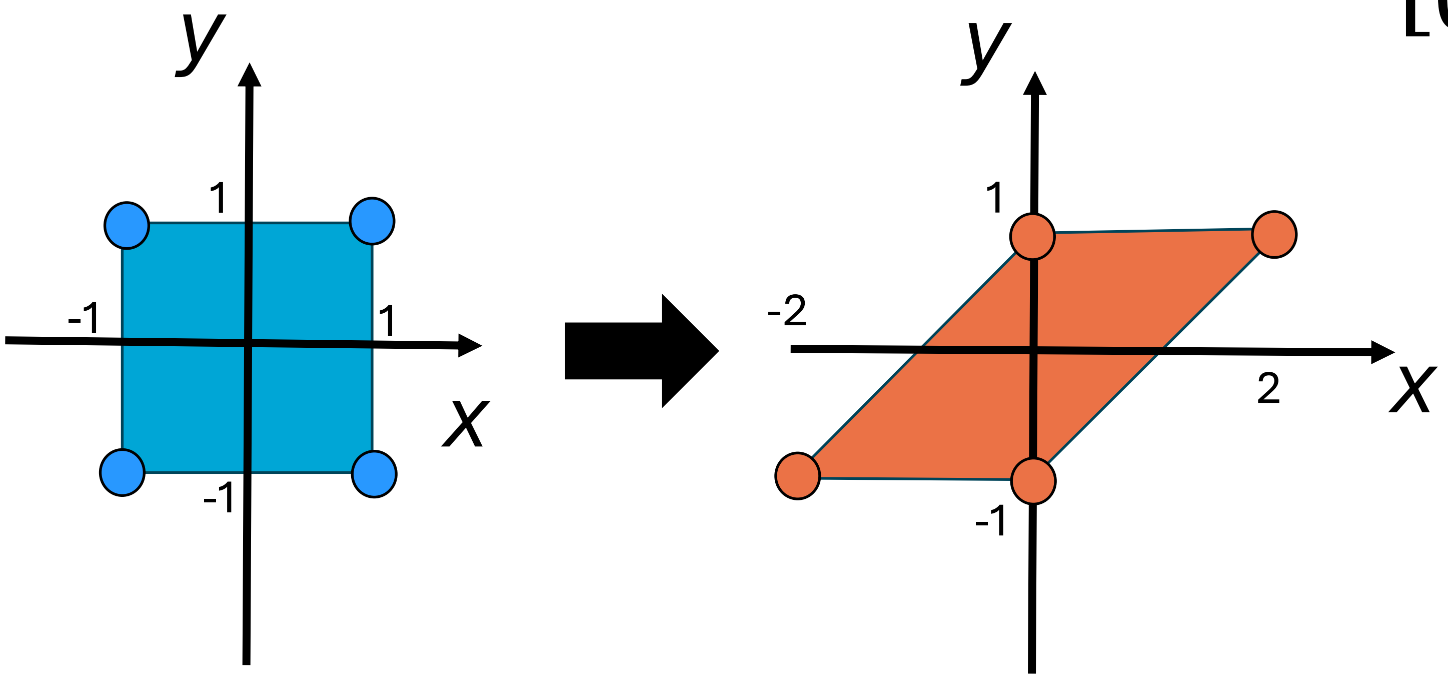
$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

Shear

- what is the result of this transformation?

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x_i\\y_i\end{bmatrix} = ?$$



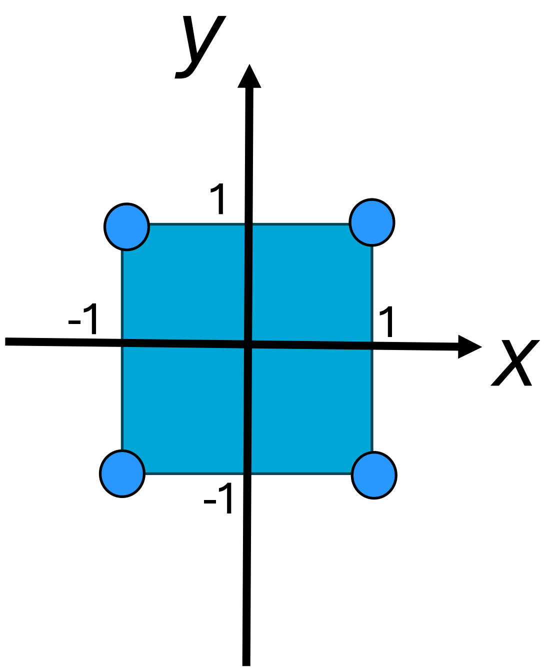
$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}-2\\-1\end{bmatrix}$$

Linear Transformations



- 2D linear transformation
  - 2x2 matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

this course: mostly rotation and scale

- special cases

rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

scale

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

shear

$$\begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

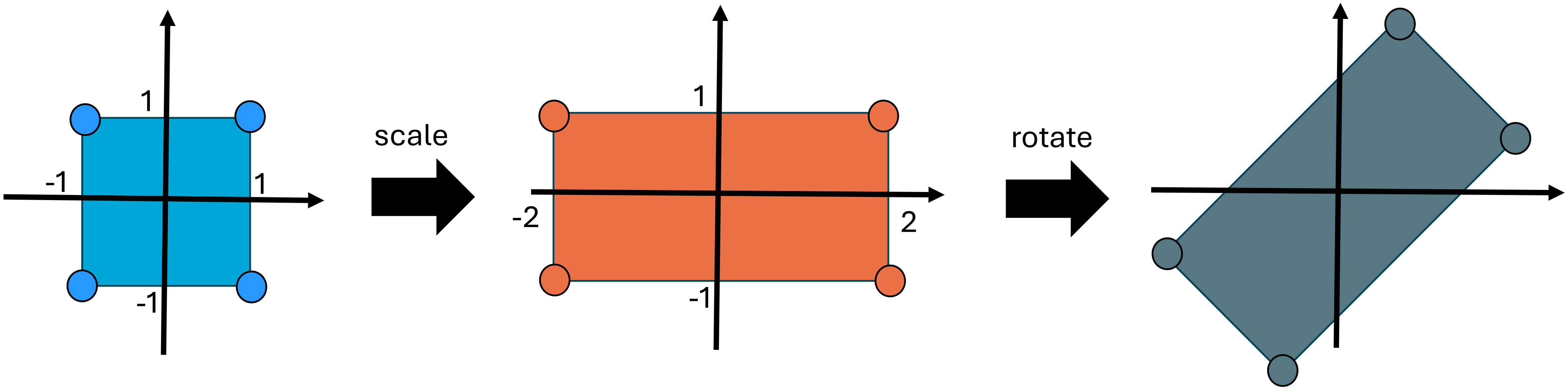
Concatenating Linear Transformations

example 1:

$$\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{array}{cc} \text{2nd} & \text{1st} \\ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} & \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \end{bmatrix}$$

order: right to left



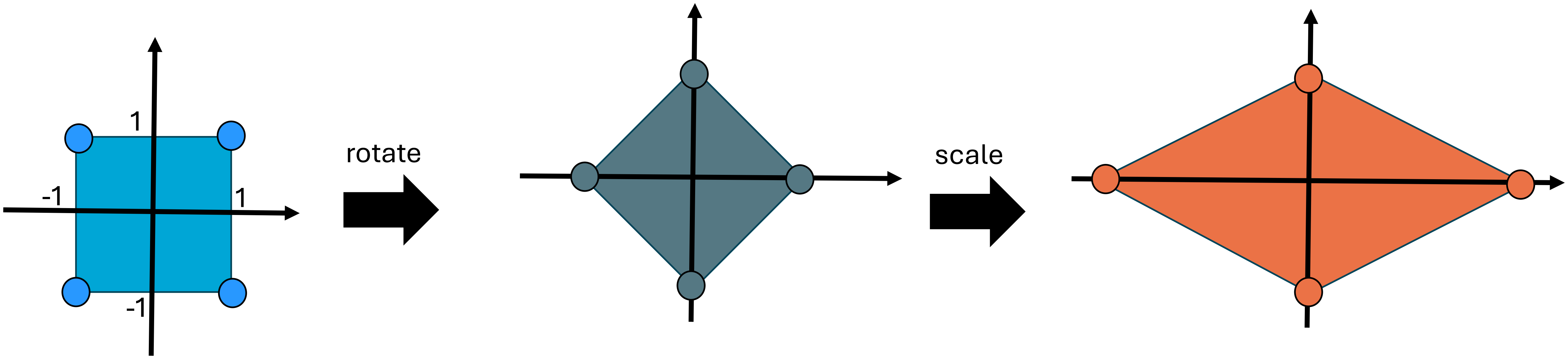
Concatenating Linear Transformations

example 2:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{array}{cc} 2^{nd} & 1^{st} \\ \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} & \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \end{bmatrix}$$

order: right to left



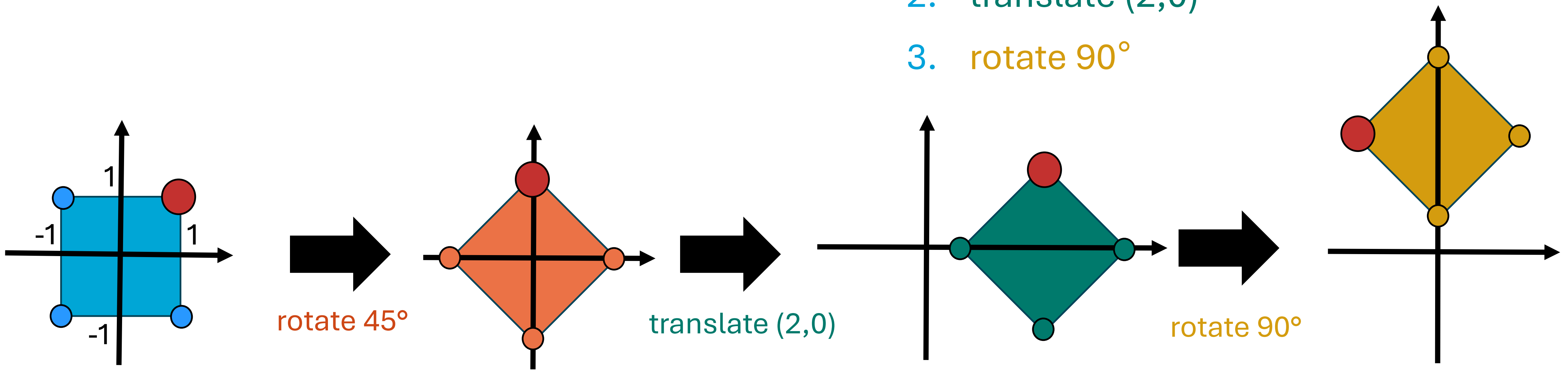
exercise: try the two combinations below

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

Concatenating translation

- sequence:
  1. rotate 45°
  2. translate (2,0)
  3. rotate 90°



operations for point (1,1)

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix}$$

issue: cannot concatenate linear transformation and translation into one operation (matrix)

Issue with translation

- not a linear transformation
  - moves the origin
  - cannot be represented as a matrix (e.g., 2x2 matrix in 2D)
- the order still matters
  - cannot always translate first or last
  - switch between matrix and vector operations (not efficient)
- question: can we turn translation into a matrix?

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}$$

Issue with translation

- question: can we turn translation into a matrix?
  - e.g.: translate by $(0, \sqrt{2})$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}$$

- does not work when applied to other points
  - object's vertices will move differently

$$\begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

Homogeneous coordinates

- projective geometry
  - solution is to go one dimension higher
- represent 2D point in 3D homogeneous space
  - last coordinate is w

$$\text{point in } \mathcal{R}^2 \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

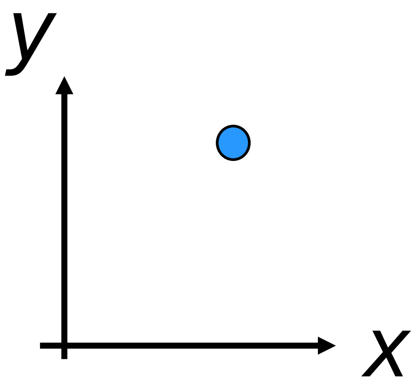
$$\text{point in 2D homogenous space } P^2 \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

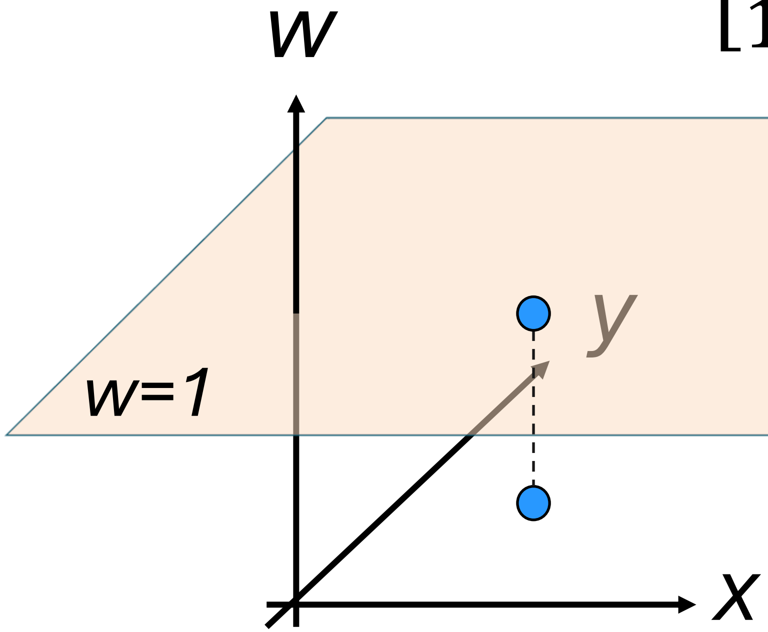
Homogeneous coordinates

- affine plane (w=1): all points (x,y,1)

$$P^2 \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

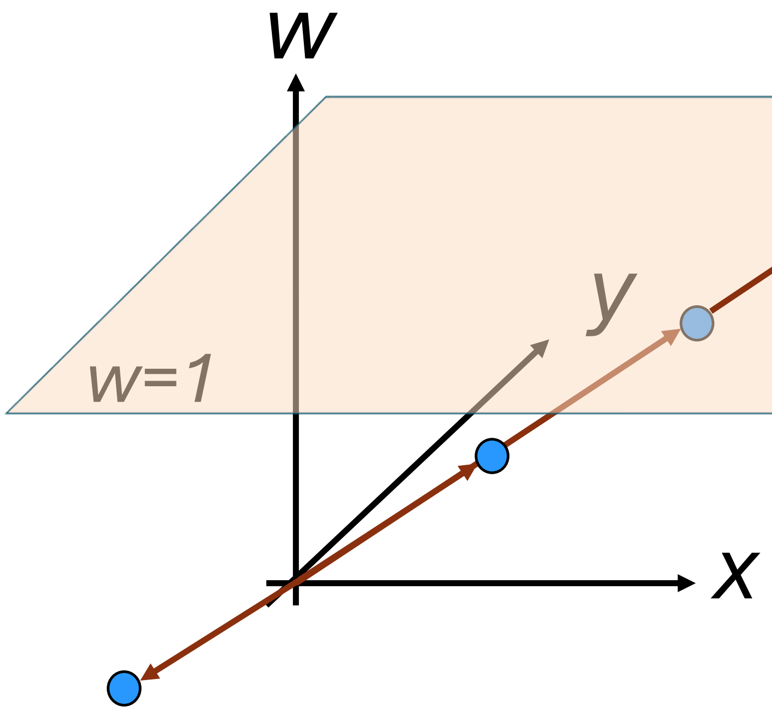
$$\mathcal{R}^2 \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$





Homogeneous points

- points in P^2
- all points on line that passes through origin are equal

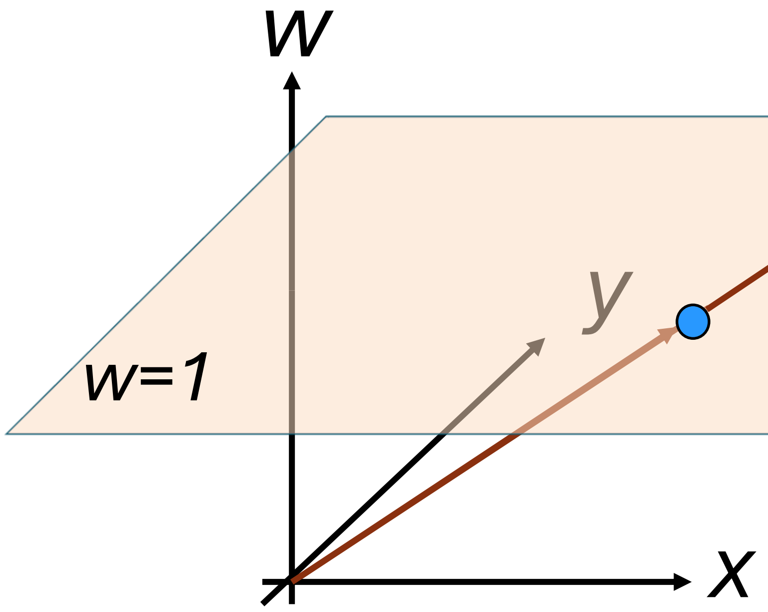


$$P^2 \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathrm{k} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathrm{k} \neq 0$$

$$P^2 \longrightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1 \\ -0.5 \end{bmatrix}$$

Homogeneous points

- if $\mathrm{w} \neq 0$, we can go back to affine plane by dividing by w
- attention: for $\mathrm{w} = 0$ cannot go back!

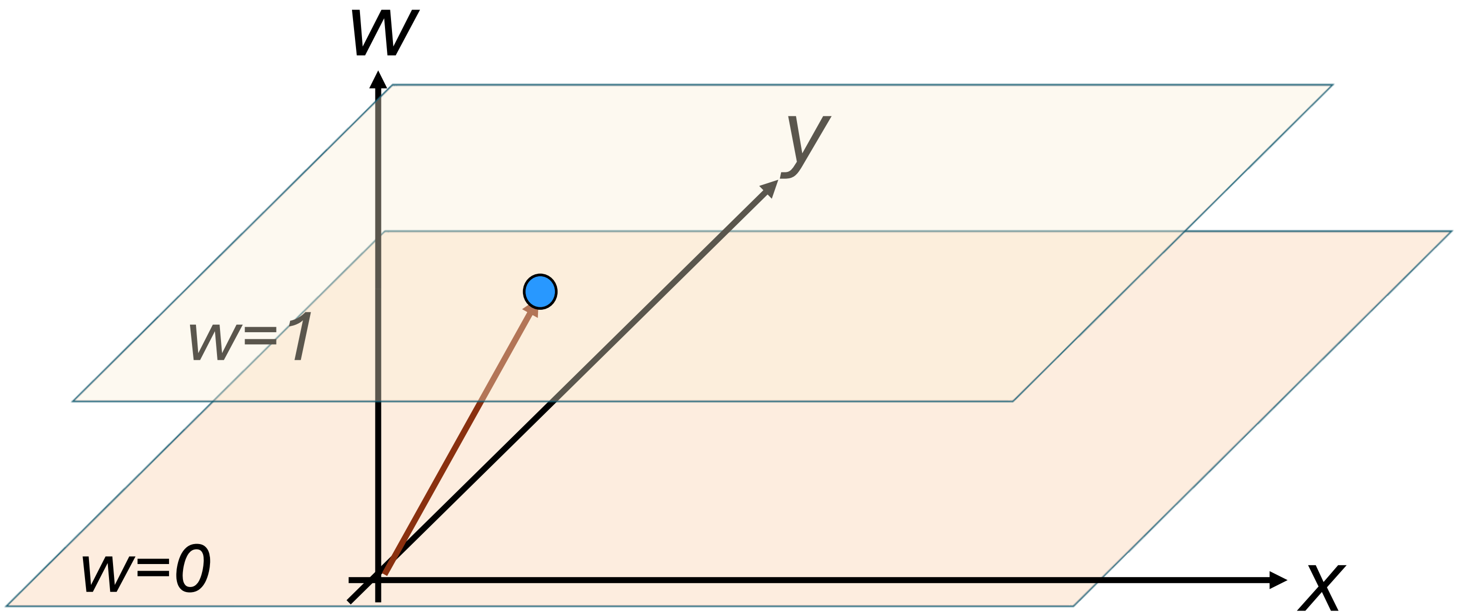


$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \\ \frac{w}{w} \\ \frac{w}{w} \end{bmatrix} = \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{6}{2} \\ \frac{4}{2} \\ \frac{2}{2} \\ \frac{2}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Homogeneous points

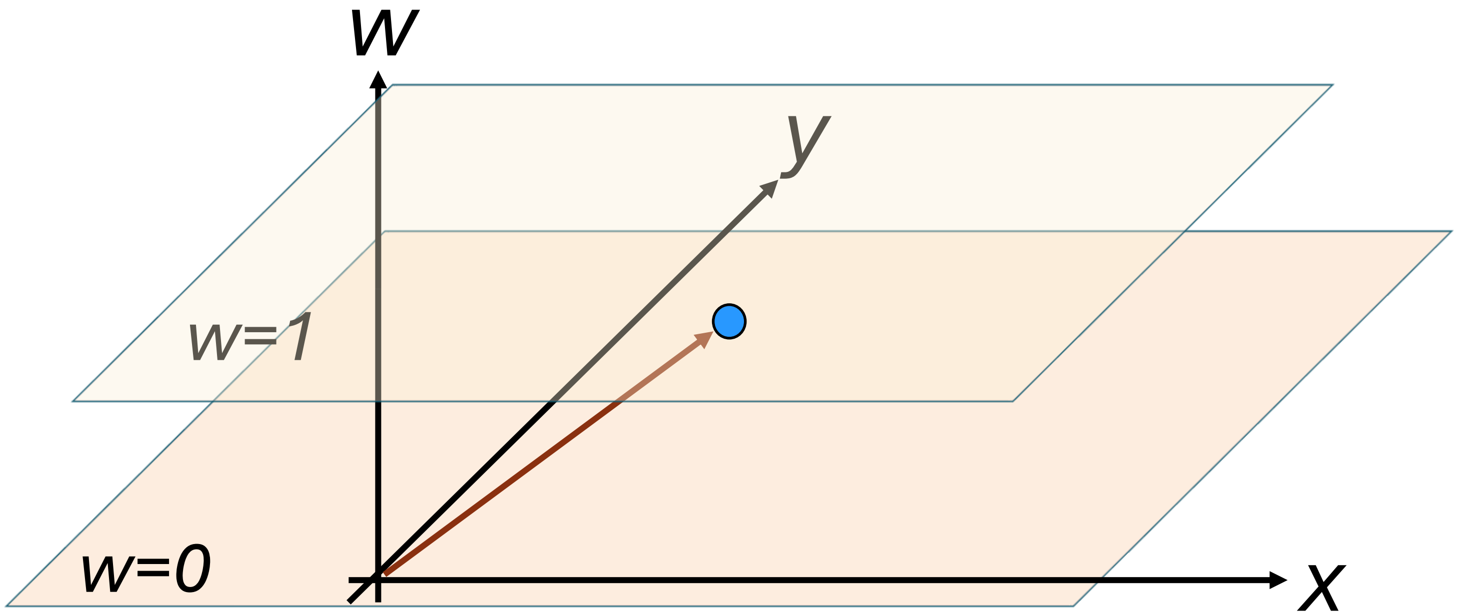
- points with $\mathrm{w} = 0$ are *special*
- observe what happens when we decrease w for the same point



$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous points

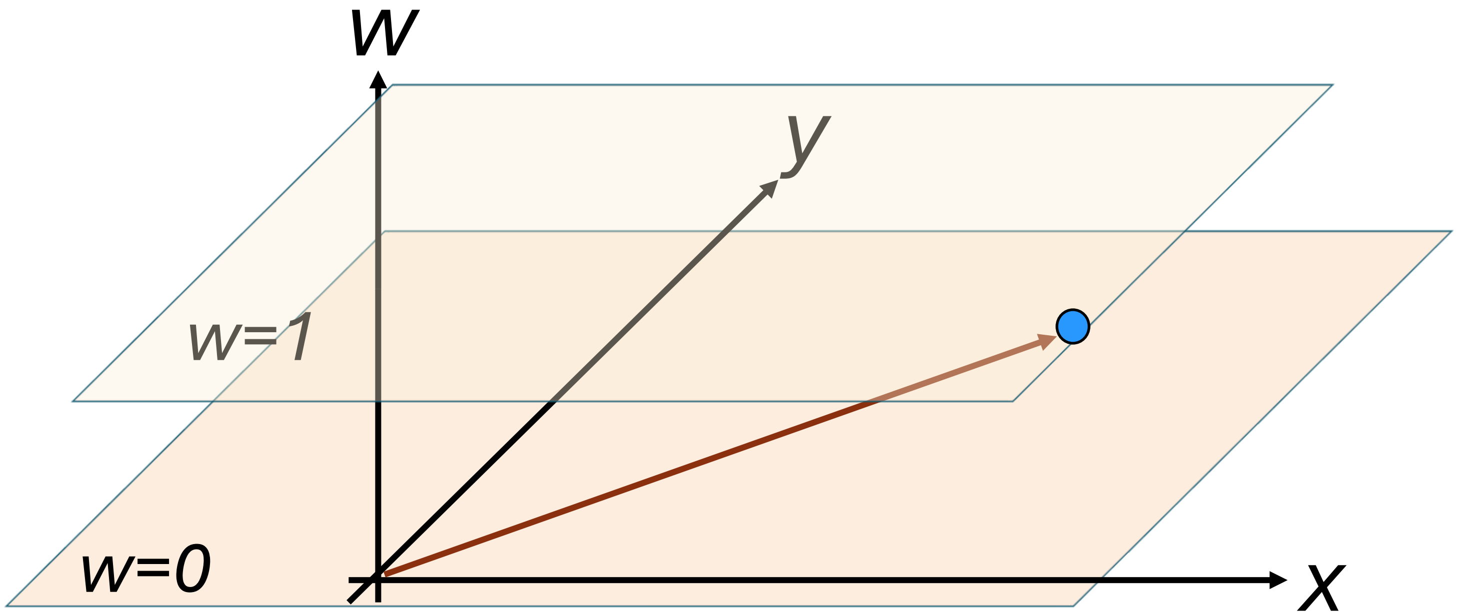
- points with $\mathrm{w} = 0$ are *special*
- observe what happens when we decrease w for the same point



$$\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous points

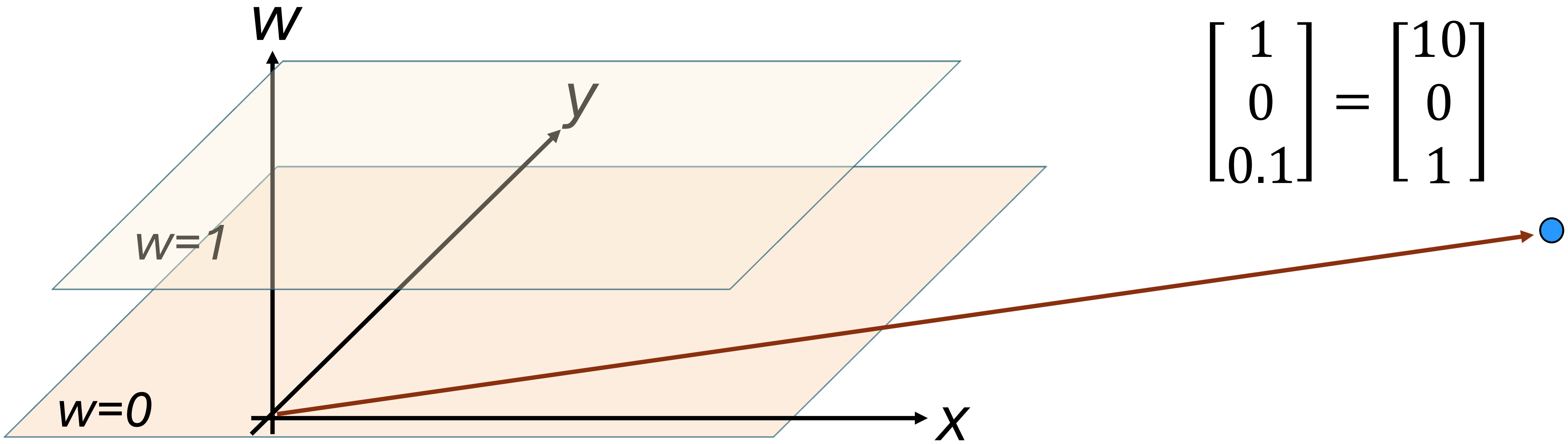
- points with $\mathrm{w} = 0$ are *special*
- observe what happens when we decrease w for the same point



$$\begin{bmatrix} 1 \\ 0 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous points

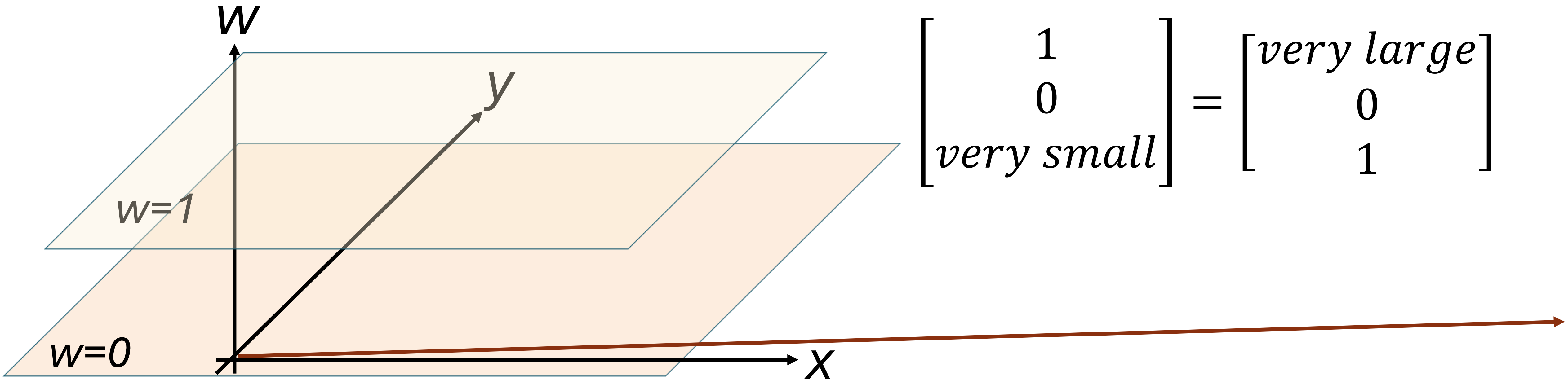
- points with $\mathrm{w} = 0$ are *special*
- observe what happens when we decrease w for the same point



$$\begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous points

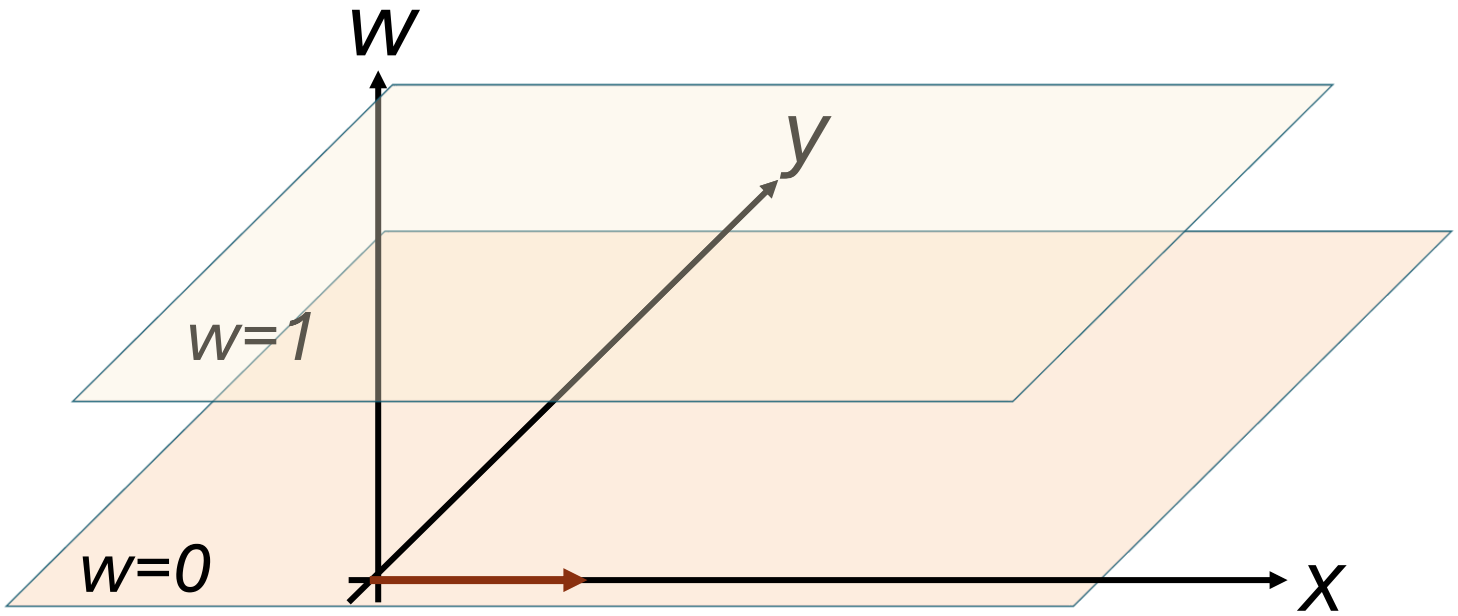
- points with $\mathrm{w} = 0$ are *special*
- observe what happens when we decrease w for the same point



$$\begin{bmatrix} 1 \\ 0 \\ very\ small \end{bmatrix} = \begin{bmatrix} very\ large \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous points

- points with $\mathrm{w} = 0$ are *special*
- when $\mathrm{w} \longrightarrow 0$, point goes to infinity



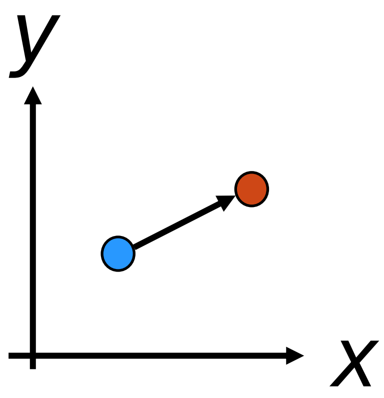
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$w = 0$ are points at infinity!

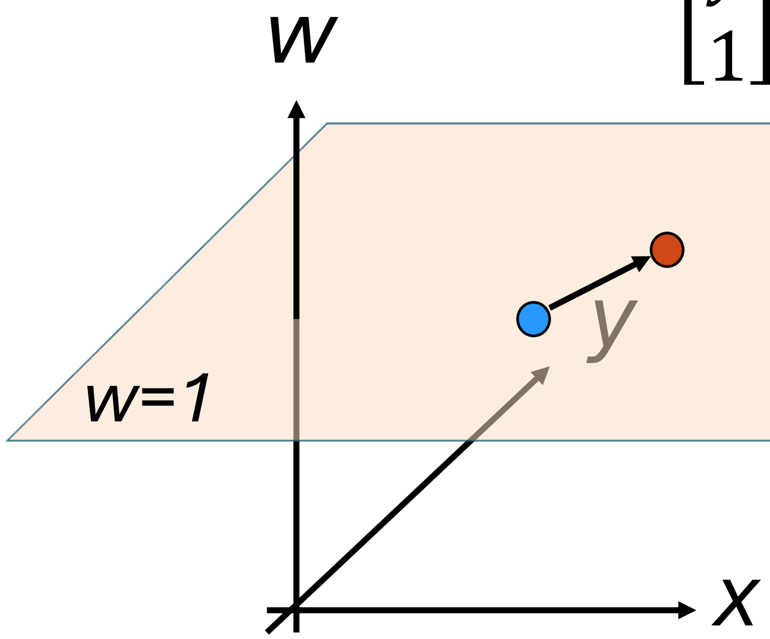
we will come back to this concept later!

Translations in homogeneous coordinates

$$\mathcal{R}^2 \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$



$$P^2 \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



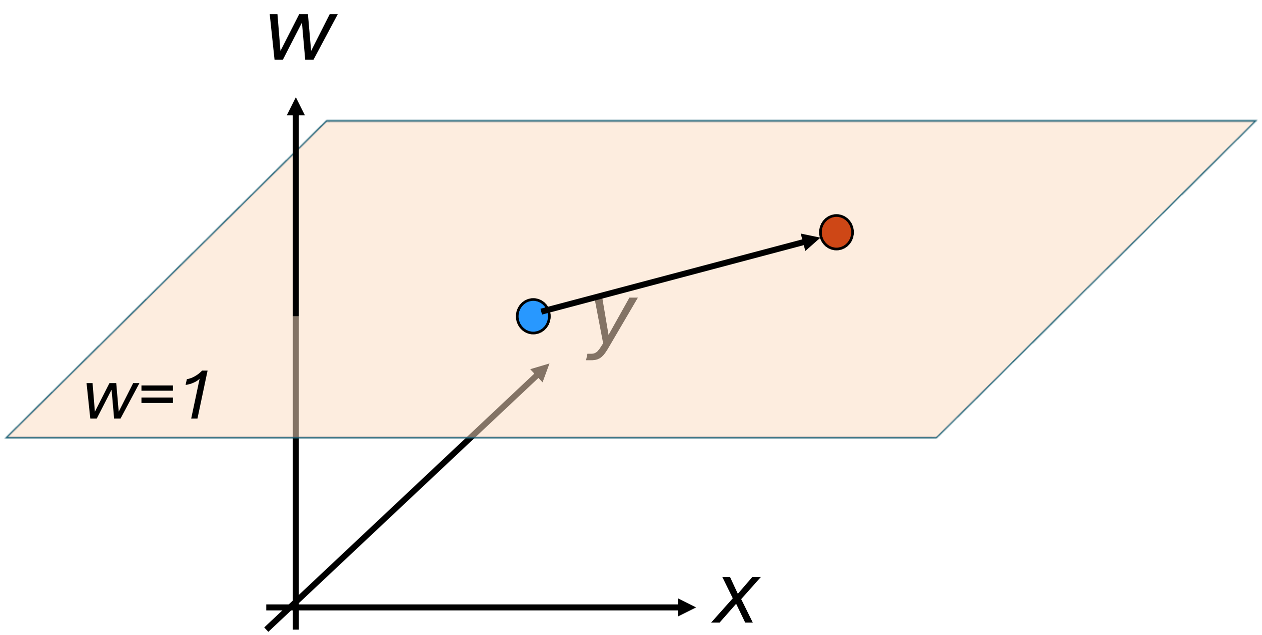
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Translations in homogeneous coordinates

- example: translate by (2,1)

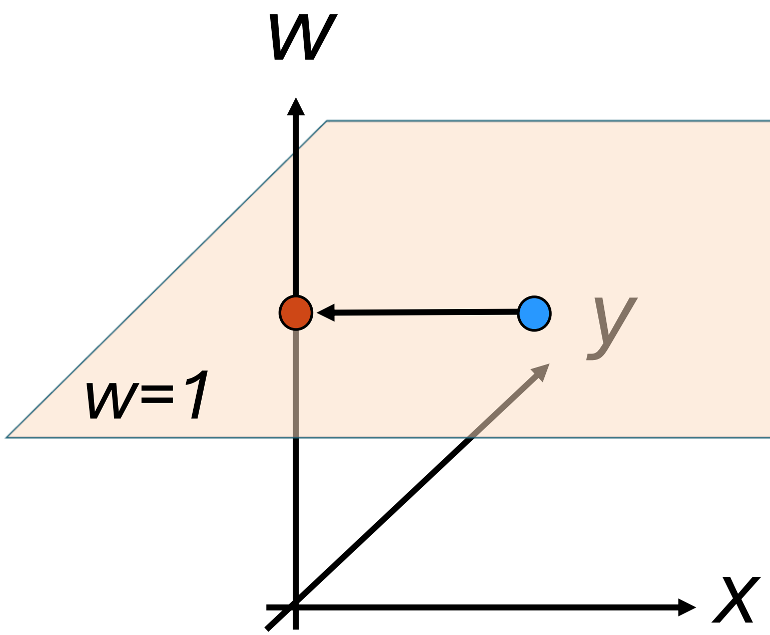
$$\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}1+2\\ 1+1\\ 1\end{bmatrix} = \begin{bmatrix}3\\ 2\\ 1\end{bmatrix}$$



Translations in homogeneous coordinates

- does it work for any point on the line?
- example: translate by (-1,0)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

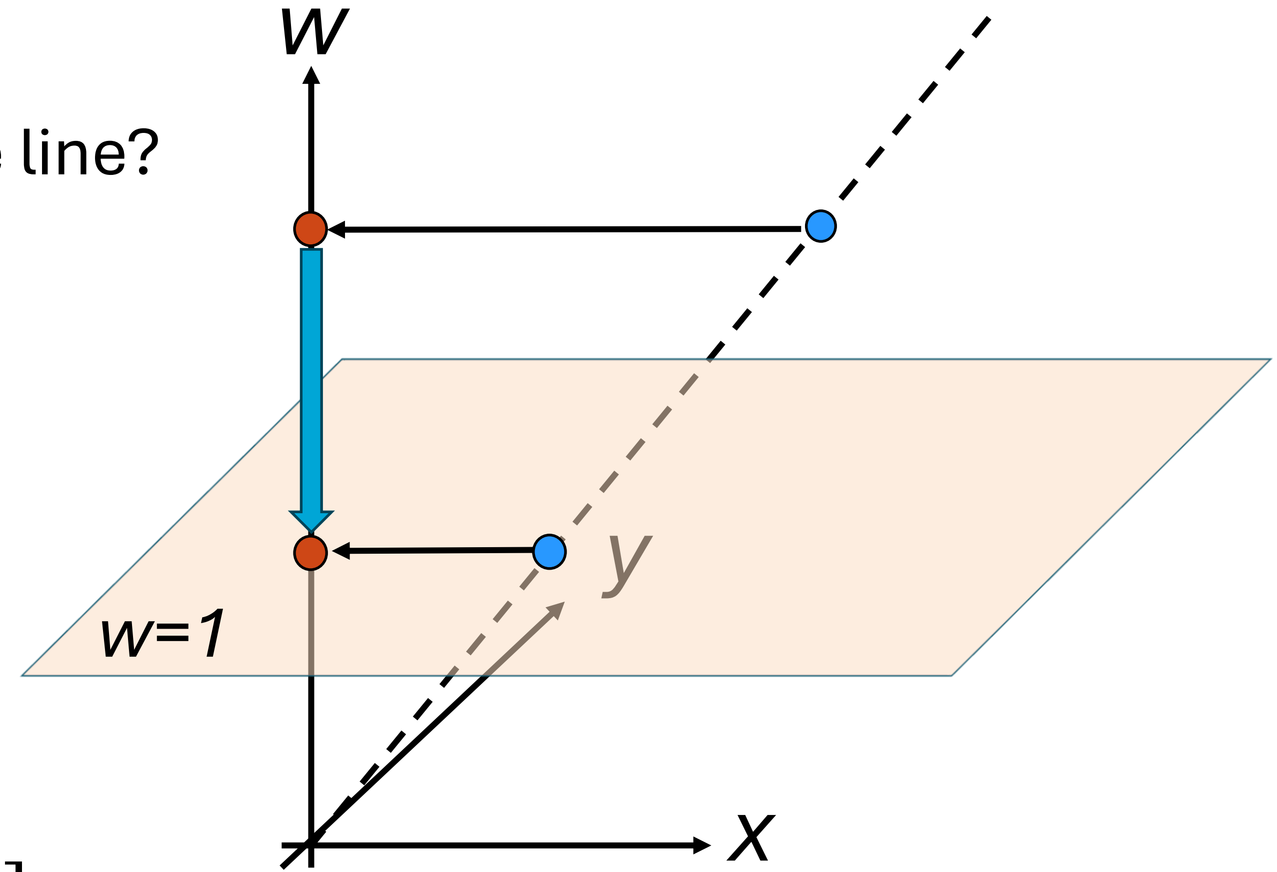


Translations in homogeneous coordinates

- does it work for any point on the line?
- example: translate by (-1,0)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



2D Transformations with homogeneous coordinates

- linear + translation in one 3x3 matrix!!!

2D linear transformations (rotation, scale, shear ..)

2D translation

$$\begin{bmatrix} \boxed{\begin{matrix} a & c \\ b & d \end{matrix}} & \boxed{\begin{matrix} t_x \\ t_y \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations with homogeneous coordinates

- concatenate many transformations, result is a 3x3 matrix $\begin{bmatrix} a & c & t_x \\ b & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- example:
  1. rotation R
  2. translation T
  3. scale S

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$S\ T\ R\ p$$

exercise: try the two combinations below

- attention! order matters
- RT vs TR

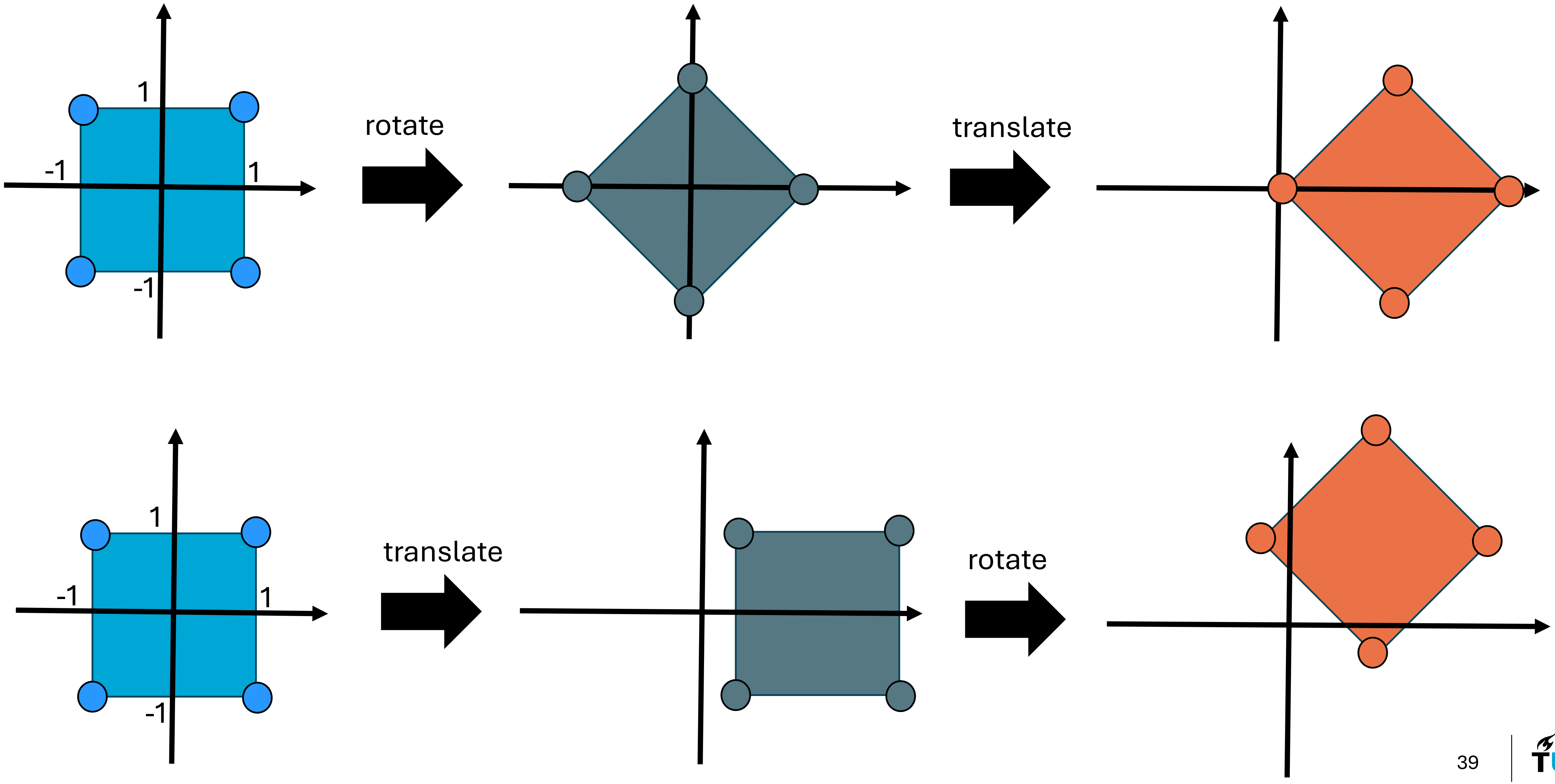
$$T\ R\ p$$

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

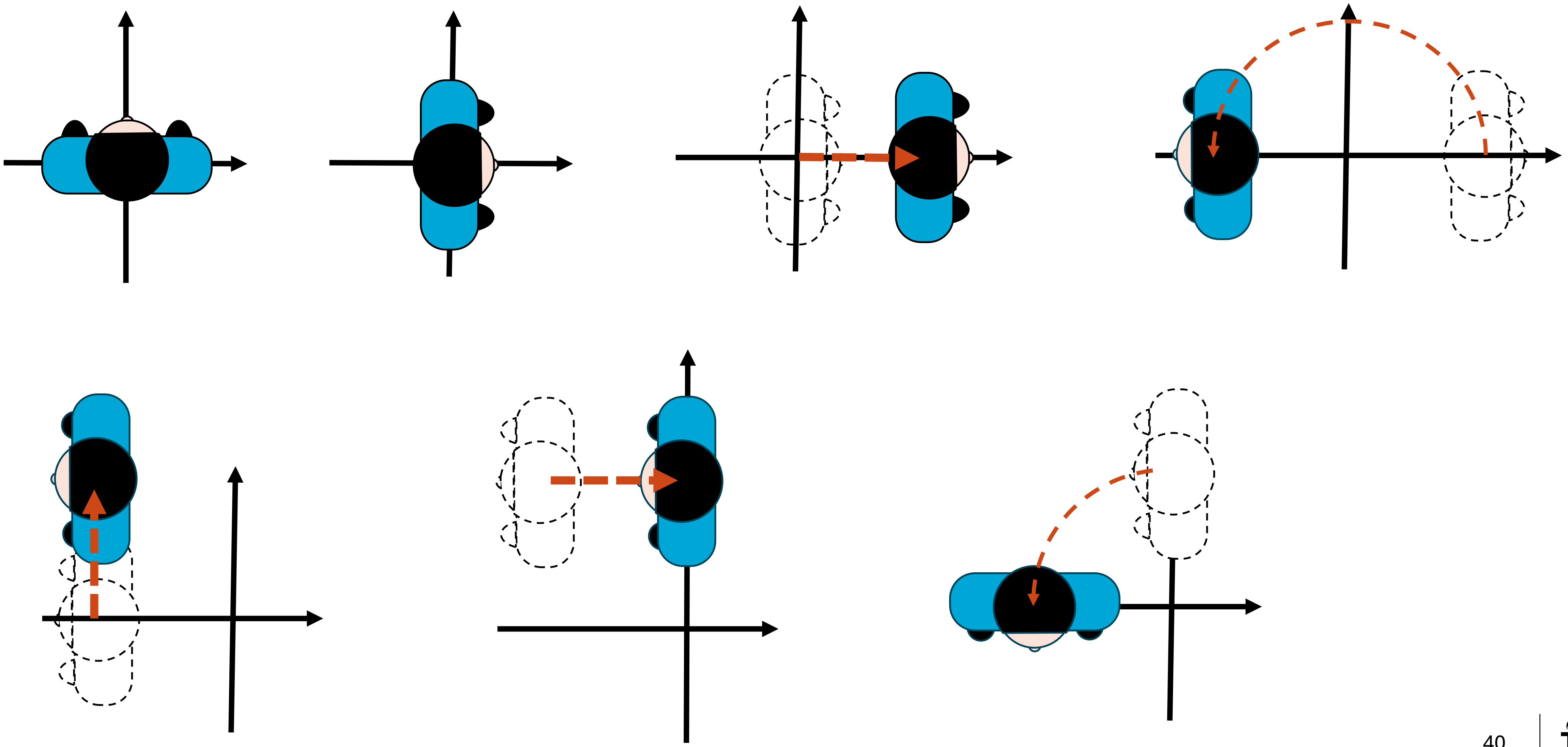
$$R\ T\ p$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

RT vs TR



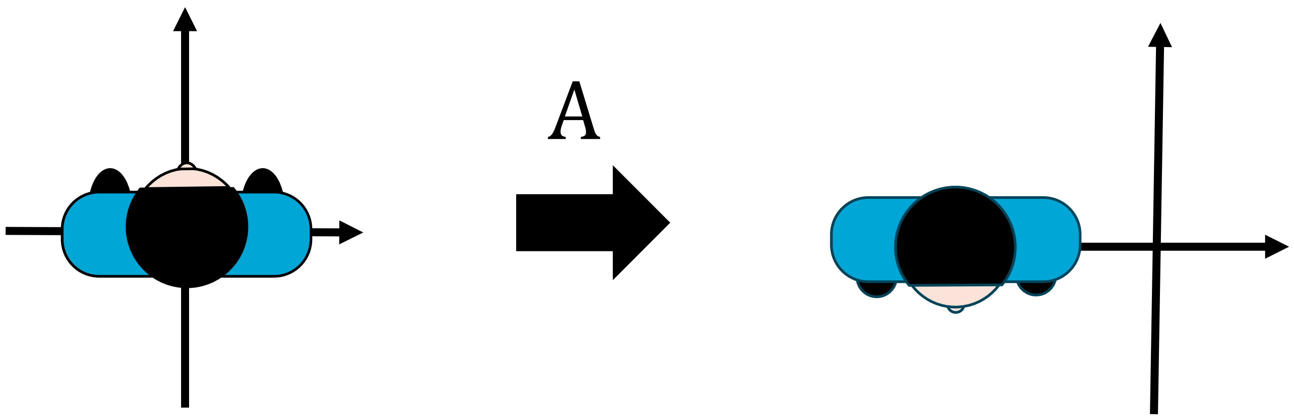
$$T = R(90^o)T(\begin{bmatrix}1\\0\end{bmatrix})T(\begin{bmatrix}0\\1\end{bmatrix})R(180^o)T(\begin{bmatrix}1\\0\end{bmatrix})R(-90^o)$$



exercise: write the matrix A

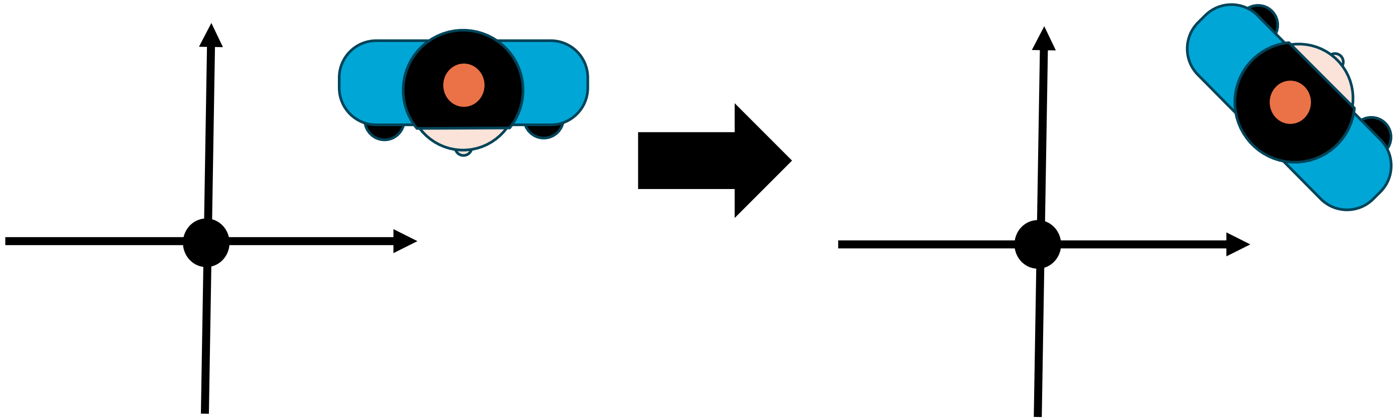
- Matrix **A** performs the entire sequence in one step

$$A = R(90^o)T(\begin{bmatrix}1\\0\end{bmatrix})T(\begin{bmatrix}0\\1\end{bmatrix})R(180^o)T(\begin{bmatrix}1\\0\end{bmatrix})R(-90^o)$$



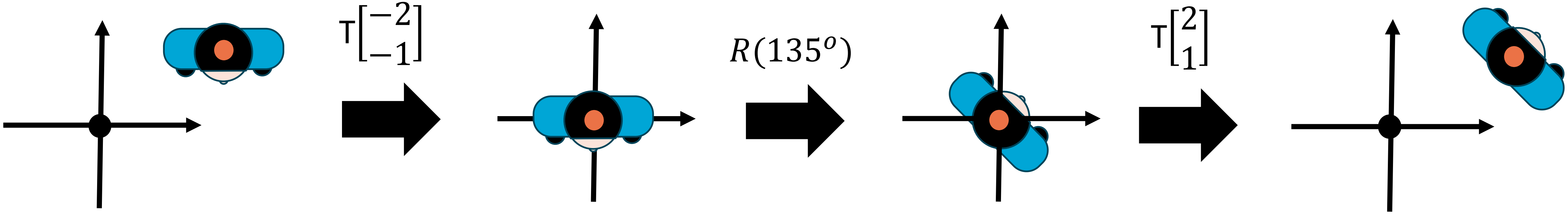
Rotation around arbitrary point

rotate 135° around orange point



center of rotation

 $= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



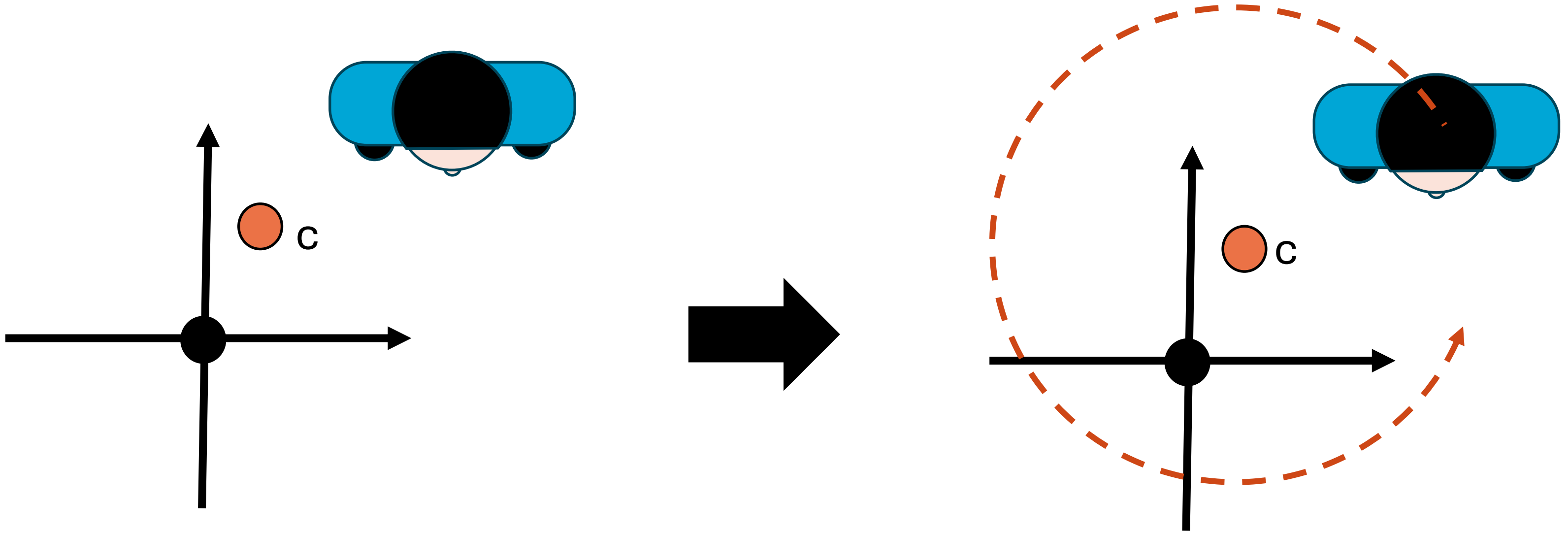
$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} R(135^o) T\begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Rotation around arbitrary point

center of rotation

$$c=\begin{bmatrix} x \\ y \end{bmatrix}$$

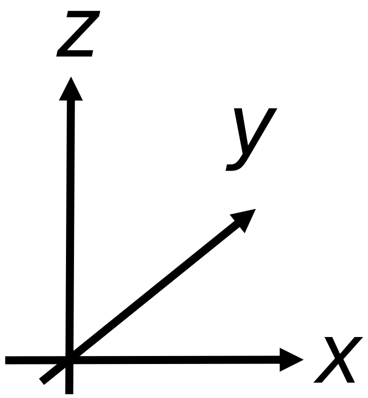
rotate θ around point



$$\mathrm{T}\begin{bmatrix} x \\ y \end{bmatrix} R(\theta)\,\mathrm{T}\begin{bmatrix} -x \\ -y \end{bmatrix}$$

1. translate by –c (c goes to origin)
2. rotate by θ
3. translate back by +c

3D Transformations



- 3D linear transformation
  - 3x3 matrix

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- special cases

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

$$R_y = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Sh}=\begin{bmatrix} 1 & Sh_{xy} & Sh_{xz} \\ Sh_{yx} & 1 & Sh_{yz} \\ Sh_{zx} & Sh_{zy} & 1 \end{bmatrix}$$

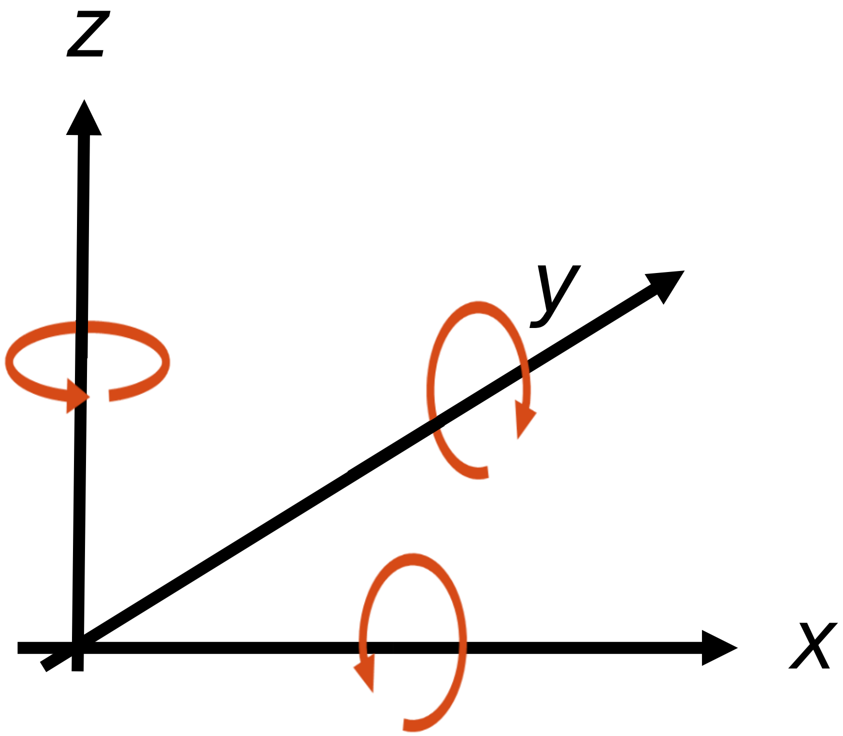
$$\text{S}=\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

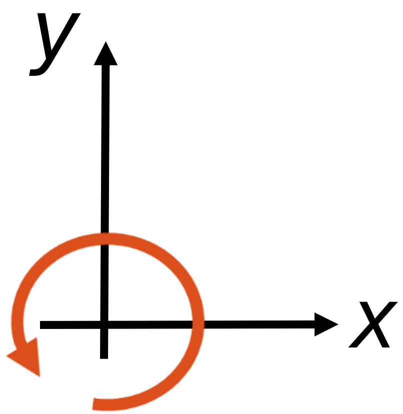
3D Transformations - rotations

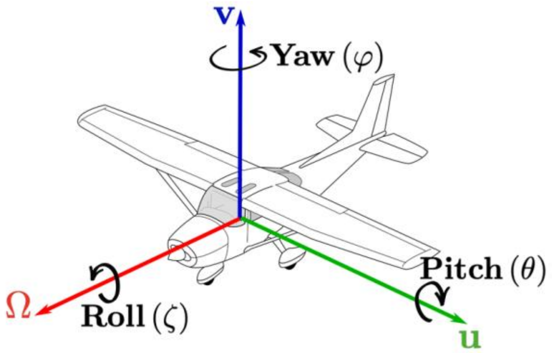
$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

$$R_y = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$







https://arxiv.org/abs/2101.10864

3D Transformations - rotations

- Order matters

$$R_x R_y R_z \neq R_z R_y R_x$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

$$R_y = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

exercise: try the two combinations below

$$R_z(\phi)R_x(\theta) = \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} = ?$$

$$R_x(\theta)R_z(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = ?$$

3D Transformations with homogeneous coordinates

- linear + translation in one 4x4 matrix!!!

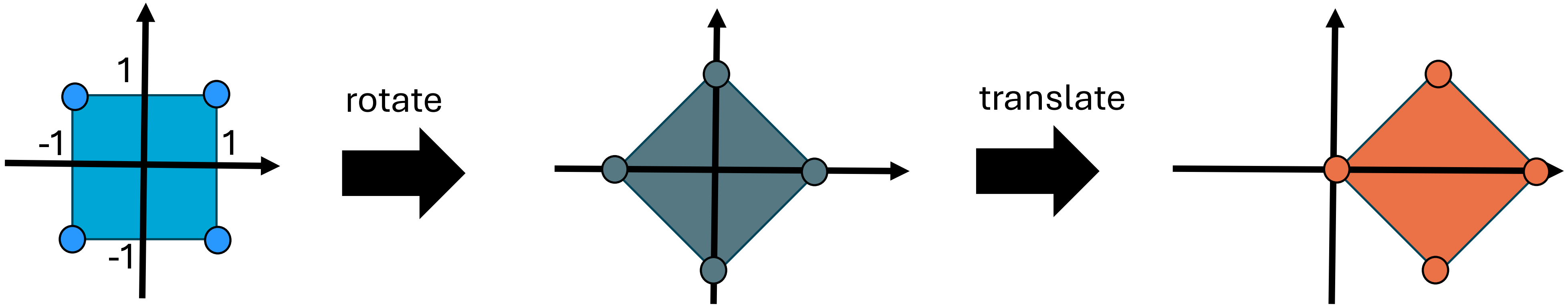
3D linear transformations (rotation, scale, shear ..)

3D translation

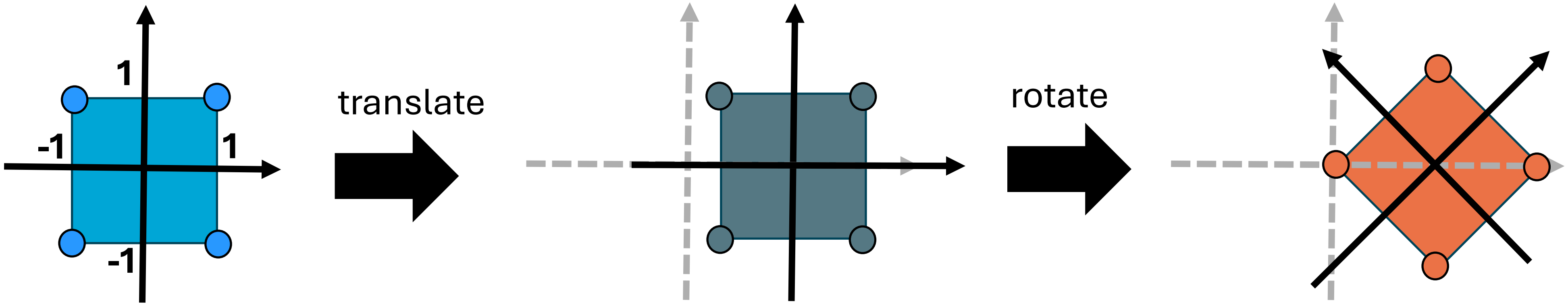
$$\begin{bmatrix} \boxed{\begin{matrix} a & d & g \\ b & e & h \\ c & f & i \end{matrix}} & \boxed{\begin{matrix} t_x \\ t_y \\ t_z \end{matrix}} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Another view on transformations order - TR

- TR right to left:
  - first rotate, then translate. Coordinate system stays fixed

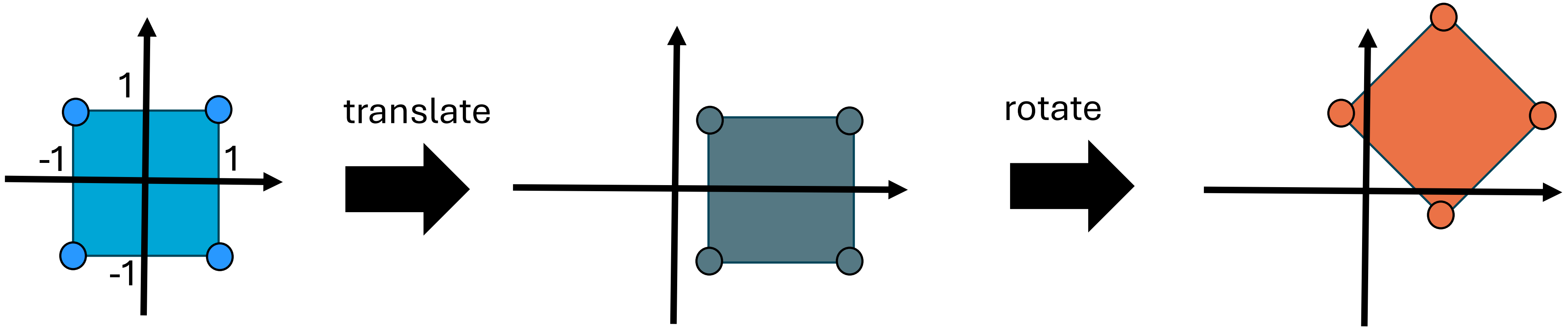


- TR left to right:
  - first translate, then rotate. Coordinate follows transformations

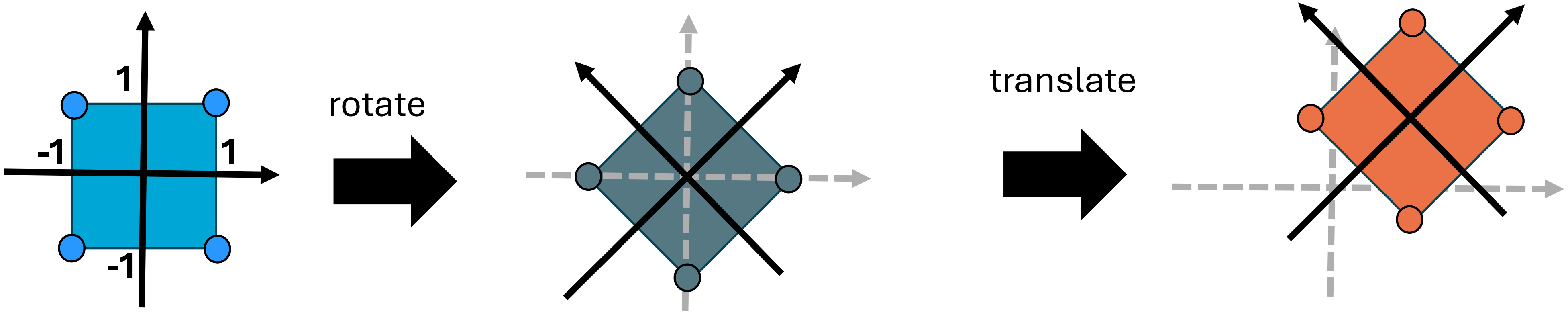


Another view on transformations order - RT

- RT right to left:
  - first translate, then rotate. Coordinate system stays fixed

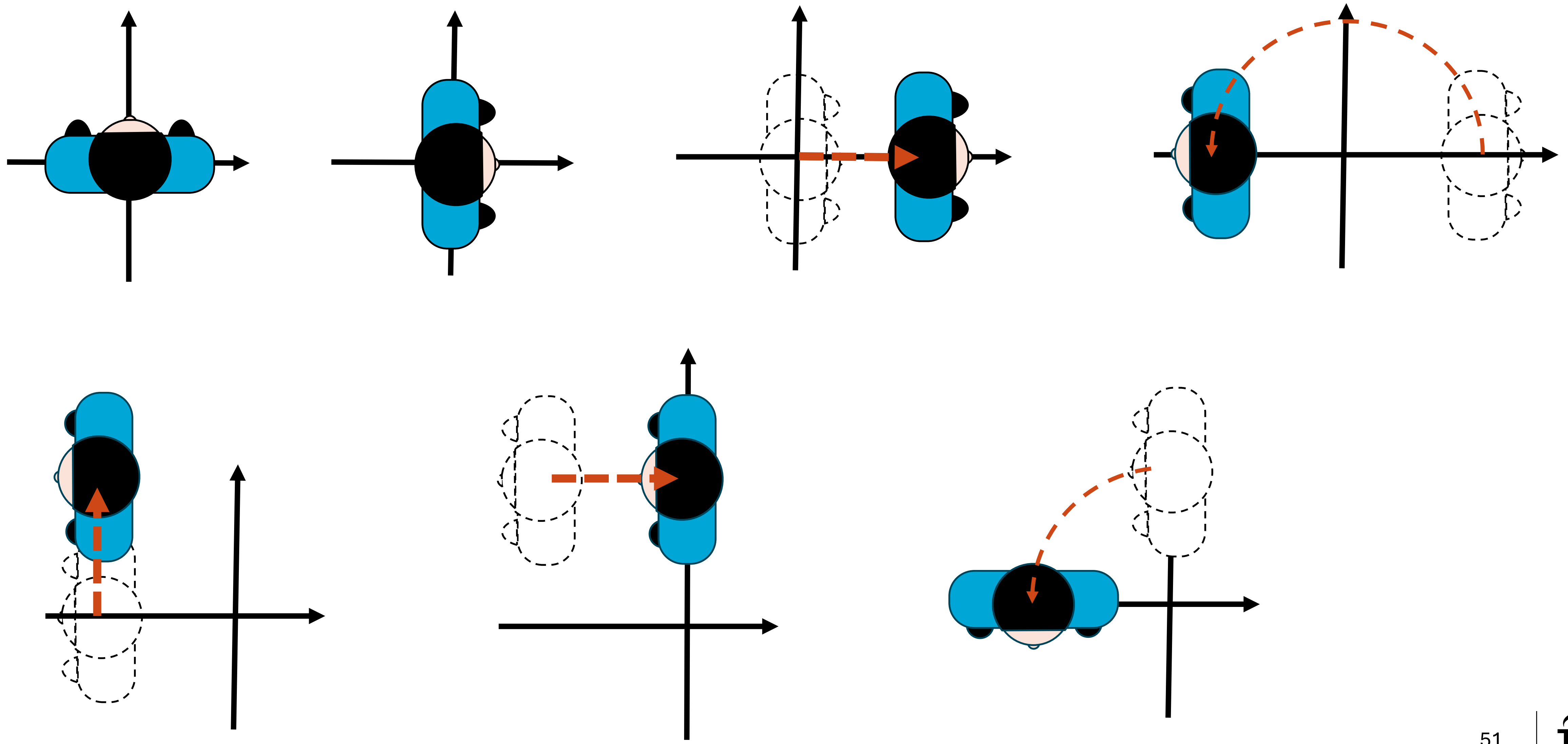


- RT left to right:
  - first rotate, then translate. Coordinate follows transformations



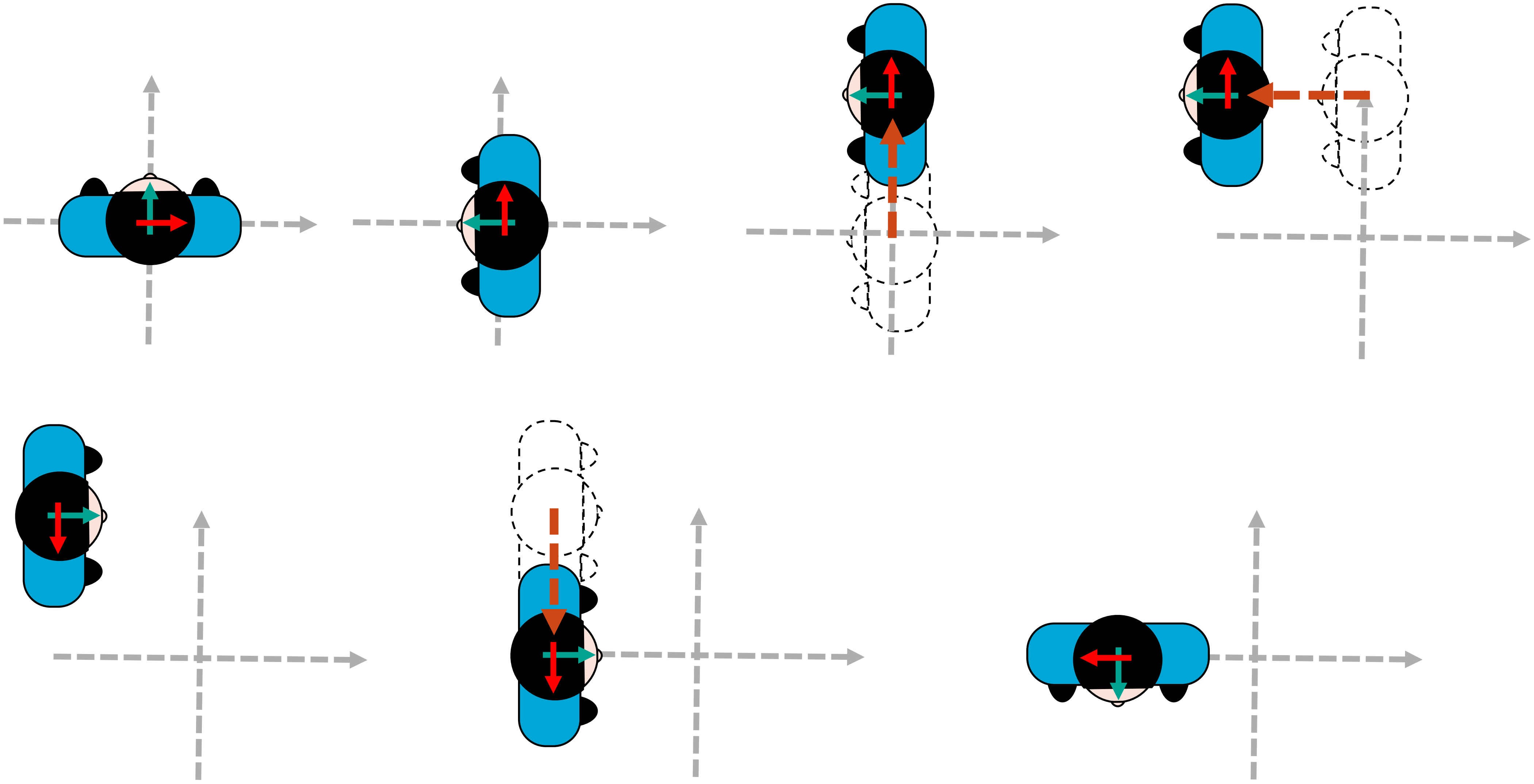
Right to Left

$$T = R(90^o)T(\begin{bmatrix}1\\0\end{bmatrix})T(\begin{bmatrix}0\\1\end{bmatrix})R(180^o)T(\begin{bmatrix}1\\0\end{bmatrix})R(-90^o)$$



Left to Right

$$T = R(90^o)T(\begin{bmatrix}1\\0\end{bmatrix})T(\begin{bmatrix}0\\1\end{bmatrix})R(180^o)T(\begin{bmatrix}1\\0\end{bmatrix})R(-90^o)$$



Today we saw ...

- 2D Linear Transformations
- Homogeneous Coordinates
- 2D Linear Transformation + Translation (Affine)
- 3D Transformations
- Right to Left vs Left to Right

Thank you for your attention!