The Theory of Implication.

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The purpose of the present article is to set forth the first chapter of the deduction of pure mathematics from its logical foundations. This first chapter is necessarily concerned with deduction itself, *i. e.* with the principles by which conclusions are inferred from premisses. If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. Symbolic logic is often regarded as consisting of two coordinate parts, the theory of classes and the theory of propositions. But from our point of view these two parts are not coordinate; for in the theory of classes we deduce one proposition from another by means of principles belonging to the theory of propositions, whereas in the theory of propositions we nowhere require the theory of classes. Hence, in a deductive system, the theory of propositions necessarily precedes the theory of classes.

But the subject to be treated in what follows is not quite properly described as the theory of *propositions*. It is in fact the theory of how one proposition can be inferred from another. Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p, we say that p *implies* q. Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. In the present article, certain propositions concerning implication will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all *necessary*, and it is probable that the number of them might be diminished. All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that I do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate.

The symbolism adopted in what follows is that of Peano, with certain additions and changes. I have adopted his symbol for implication ($\supset$), and his use of dots instead of brackets; also his plan of numbering propositions with an integral and a decimal part.* But although the symbolism is in the main Peano's, the ideas are more those of Frege.† Frege's work, probably owing to the inconvenience of his symbols, has received far less recognition than it deserves. I shall not refer to him in detail in what follows, but whoever will consult his work will see how much I owe to him. Especially I have adopted from him the assertion-sign (cf. ★1·1 *infra*), the interpretation of "p implies q" (cf ★1·2 *infra*), and the distinction between asserting a proposition for *all* values of the variable or variables, and asserting it for any values (cf. ★7 *infra*). The plan of taking implication and negation as our primitive ideas is also his.

★1. *Primitive Ideas.*

Since all definitions of terms are effected by means of other terms every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems the simpler and easier. I know no way of proving that such-and-such a system of undefined ideas contains as few as will give such-and-such results.‡ Hence we can only say that such and such

* For an explanation of Peano's symbolism, cf. Whitehead, "On Cardinal Numbers," *American Journal of Mathematics*, Vol. XXIV, No. 4.

† Cf. his "Grundgesetze der Arithmetik," Vol. I, Jena, 1893; Vol. II, 1903.

‡ The recognized methods of proving independence are not applicable, without reserve, to fundamentals. Cf. my "Principles of Mathematics", §17. What is there said concerning primitive propositions applies with even greater force to primitive ideas.

ideas are undefined in such and such a system, not that they are indefinable. Following Peano, I shall call the undefined ideas and the undemonstrated propositions *primitive* ideas and propositions respectively. These ideas are *explained* by means of descriptions intended to point out to the reader what is meant; but the explanations do not constitute definitions, because they really involve the ideas they explain.

★1·1. *Assertion.*—Any proposition may be either asserted or merely considered. If I say "Caesar died", I assert the proposition "Caesar died" if I say "'Caesar died' is a proposition", I make a different assertion, and "Caesar died" is no longer asserted, but merely considered. Similarly in a hypothetical proposition, *e. g.* "if $a = b$, then $b = a$", we have two unasserted propositions, namely "$a = b$" and "$b = a$" while what *is* asserted is that the first of these implies the second. In language, we indicate when a proposition is merely considered by "*if* so-and-so" or "*that* so-and-so" or merely by inverted commas. In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by

$$\text{"}\vdash . p\text{"}$$

The sign "$\vdash$" is called the assertion-sign; it may be read "it is true that" (although philosophically this is not what it means). The dots after the assertion-sign indicate its range; that is to say, everything following is asserted until we reach either an equal number of dots or the end of the sentence. Thus "$\vdash : p . \supset . q$" means "it is true that p implies q", whereas "$\vdash . p . \supset \vdash . q$" means "$p$ is true; therefore q is true". The first of these does not necessarily involve the truth either of p or of q, while the second involves the truth of both.

★1·2. *Implication.*—The meaning to be given to implication in what follows may at first sight appear somewhat artificial; but although there are other legitimate meanings, the one here adopted is, if I am not mistaken, very much more convenient that any of its rivals. The essential property that we require of implication is this: "What is implied by a true proposition is true". It is in virtue of this property that implication yields proofs. But this property by no means determines whether anything, and if so what, is implied by a false proposition, or by something which is not a proposition at all. What it does determine is that, if p

implies q, then it cannot be the case that p is true and q is not true. The most convenient interpretation of implication is to say, conversely, that unless p is true and q is not true, "p implies q" is to be true. Hence, "p implies q" will be a relation which holds between any two entities p and q unless p is true and q is not true, *i. e.* whenever either p is not true or q is true.* The proposition "p implies q" is equivalent to "if p is true, then q is true", *i. e.* "'p is true' implies 'q is true'"; it is also equivalent to "if q is false, p is false". When p is in fact true, "implies" may be replaced by "therefore", *i. e.* in place of "p implies q" we may say "p is true; therefore q is true". For "implies" we use the symbol "$\supset$", thus

"$p \supset q$" means "p implies q"

"$p . \supset . q \supset r$" means "$p$ implies that q implies r", etc.

The chief advantage of the above interpretation is that it avoids hypotheses which are otherwise necessary. We wish, for example, to assert "$p \supset p$". If implication can only hold between propositions, it is necessary to preface "$p \supset p$" by the hypothesis "p is a proposition"; and whenever we want to use "$p \supset p$" in a particular case, we shall have to prove first that what we are applying it to is a proposition.† But this is highly inconvenient. Again, paradoxes result from restricting the meaning of implication. For example, it will be admitted that "if p and q are true, then r is true" is equivalent to "if p is true, then if q is true, r is true", *i. e.* to "if p is true, then q implies r". Also it will be admitted that if p and q are true, then p is true. Hence, by the above admission, if p is true, then q implies p; and here q is not subject to any limitation; for, even if q is not a proposition, it must be admitted that if p and q *were* both true, p *would be* true. Hence, unless a true proposition p is to be implied by *every* entity q, one at least of the above obvious propositions will have to be denied.

★1·3. *The Variable.*—A single letter, unless specially defined to have a certain constant meaning, will always stand for an independent variable. The

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* Cf. ★5·53 and ★5·54 below.

† In my "Principles of Mathematics" I adopted the interpretation "p and q are propositions, and p is *false* or q is true", instead of, as here, "p is *not true* or q is true". Thus "$p \supset q$" was false if p was not a proposition, and "$p \supset p$" was equivalent to "p is a proposition". The advantages of the present interpretation may be seen by comparing the primitive propositions of ★2 below with those of §18 of the above-mentioned work.

possible values of an independent variable are always to include all entities absolutely. The reason for this is as follows: If we affirm some statement about x, where x is restricted by some condition, we must mention the condition to make our statement accurate; but then we are really affirming that the truth of the condition implies the truth of our original statement about x; and this, in virtue of our interpretation of implication, will hold equally when the condition is not fulfilled. The "universe of discourse", as it has been called, must be replaced by a general hypothesis concerning the variable, and then our formulæ are true whether the hypothesis is verified or not, because an implication holds whenever its hypothesis is not true. The old theory of the "universe" had the defect of introducing tacit hypotheses, thus making all enunciations incomplete, since a hypothesis does not cease to be an essential part of a proposition merely because we do not take the trouble to state it.

In such a formula as (say) "$x . = . p \supset q$", the x is still an *independent* variable, because we are not only concerned with the value of x that makes the formula *true*, but also with all the other values that make it false. On the other hand, "$p \supset q$" is a *dependent* variable.

★1·4. *Propositional Functions.*—Any statement about a variable x will be expressed by

$$(C \between x)$$

or by $(A \between x)$, $(B \between x)$, etc. Similarly any statement about two variables x and y will be expressed by

$$(C \between x, y)$$

or by $(A \between x, y)$, $(B \between x, y)$ etc., and so on for any number of variables. Such expressions are functions whose values are propositions; hence we call them *propositional functions*. Thus "$p \supset q$" is a propositional function of p and q; "$p \supset p$" is a propositional function of p. There are other kinds of functions in mathematics, but they are not required for the theory of implication.

When an expression of the form $(C \between x)$ is asserted, what is meant is that the expression in question is true for *any* value of the variable x. Thus, *e. g.*

$$\text{"} \vdash . p \supset p \text{"}$$

means: "For *any* value of p, p implies p". But such an expression as

$(C \between p \supset q)$ does not assert that $(C \between x)$ holds for *any* argument, but only for such as are of the form $p \supset q$; *e. g.* if $(C \between x)$ is "x is a proposition", then

$$\text{“}\vdash . (C \between x)\text{” is false, but}$$
$$\text{“}\vdash . (C \between p \supset q)\text{” is true.}$$

Generally, "$\vdash . (C \between (A \between y))$" means: "For any value of y, $(C \between (A \between y))$ is true", or "For any value of y, $(A \between y)$ is a value of x for which $(C \between x)$ is true". The C in $(C \between x)$ has no meaning by itself; all that its use assumes is that $(C \between x)$ and $(C \between y)$ are to have similarity of *form*, although this form is not always capable of separate specification apart from any argument x.

★1·5. *Negation.*—The proposition "p is not true" is expressed by

$$\sim p.$$

Thus "$\sim p$" will be true if p is not a proposition, and if p is a false proposition; it will only be false if p is a true proposition.

One other primitive idea will be introduced in ★7. Two others, or perhaps three, complete what is required for the whole of pure mathematics. Thus, six are necessary for the theory of implication, and eight or nine for all pure mathematics.

★2. *Primitive Propositions.*

·1 Anything implied by a true proposition is true. Pp.* This proposition is used in every proof without exception; hence it will not be referred to in giving proofs. The very meaning of *proof* is demonstration of truth by the fact of being implied by a true proposition. It must be understood that the principle is not the same as: "*If* p is true, then *if* p implies q, q is true". This is a true proposition, but it holds equally when p is not true, and when p does not imply q. It does not, like the principle we are concerned with, enable us to assert q simply, without any hypothesis. We cannot express the principle symbolically, partly because any symbolism in which p is variable only gives the *hypothesis* that p is true, not the *fact* that it is true.†

* The letters "Pp" stand for "primitive proposition" as with Peano.

† For further remarks on this principle, cf. my "Principles of Mathematics", §38.

When this principle is used, we write

$$\text{“}\vdash . p . \supset \vdash . q\text{”}$$

meaning "p, which is true, implies q, which is therefore true", or "p is true; therefore q is true". *Therefore* is distinguished from *implies* by being only applicable to implications between *true* propositions.

★ 2·2. If a propositional function $(C \between y)$ is true for *any* value of y, it is true for such-and-such a value. Pp.

This principle will be called the "principle of substitution", and will be referred to as "Subst." It allows us, for example, from "For any value of y, if y is a man, y is a mortal", to infer "if Socrates is a man, Socrates is a mortal". The value substituted is supposed to be a definite constant value, such as Socrates, or the fourth proposition of Euclid. This principle, like its predecessor, cannot be symbolized, because any possible symbolic expression would deal with *any* constant, and we should need the principle itself to deduce, from such a statement, that it holds also concerning Socrates or some other definite constant. "Any constant" is, in fact, not distinguishable from a variable: it is like what is called a parameter, *i. e.* a variable which we choose to regard as a constant.

★ 2·3. If a propositional function $(C \between y)$ is true for any value of y, then $(C \between (A \between z))$ is true for any value of z. Pp.

This may be called the "principle of the substitution of a dependent for an independent variable". It may be referred to as "Dep.", where "Dep." is short for "dependent variable". It is obvious that, since all our variables have an unrestricted range, the various values of $(A \between z)$ are among the values of y, since all entities are among the values of y. Hence what holds for any value of y holds also for any value of $(A \between z)$. [This is not a proof, but merely an elucidation.]

To symbolize this principle, we should need a symbol for the supposition, as opposed to the assertion, that $(C \between y)$ is true for *any* value of y. Such a symbol is not afterwards needed, since our principle will only be applied when $(C \between y)$ is *asserted* for any value of y. In any such case the principle becomes

$$\vdash . (C \between y) . \supset \vdash . (C \between (A \between z)).$$

The use of this principle is constant. For example, it is needed for the inference that, since "$p \supset p$" is true for any value of p, it is true if we substitute $p \supset q$ for p, so that "$p \supset q . \supset . p \supset q$" is true for any values of p and q. (The principle is to be extended to two or more variables.) When this principle is used, the substitution effected will be indicated by

$$(C \between y) \frac{(A \between z)}{y}$$

meaning that, in $(C \between y)$, $(A \between z)$ is to be substituted for y. Generally the substitution is to be effected in a proposition referred to by its name or number. Thus, *e. g.* "$\vdash . p \supset p$" will be $\star$ 2·5; thus,

$$\text{“}\star 2{\cdot}5 \frac{p \supset q\text{”}}{p}$$

means what $\star$ 2·5 becomes when "$p \supset q$" is substituted for p. Thus we write

$$\vdash . \star 2{\cdot}5 \frac{p \supset q}{p} . \supset \vdash : p \supset q . \supset . p \supset q$$

in which we use our present principle. In like manner,

$$(C \between x , y) \frac{(A \between x , y), (B \between x , y)}{x , y}$$

means the result of substituting $(A \between x , y)$, $(B \between x , y)$ for x , y in $(C \between x , y)$. Thus $\star$ 2·6 will be "$\vdash : p . \supset . q \supset p$", *i. e.* "If p is true, then, if q is true, p is true". Hence

$$\text{“}\star 2{\cdot}6 \frac{p \supset q . \supset . q , q \supset p\text{”}}{p , q}$$

denotes the result of substituting $p \supset q . \supset . q$ for p, and $q \supset p$ for q: thus we may write

$$\vdash . \star 2{\cdot}6 \frac{p \supset q . \supset . q , q \supset p}{p , q} .$$

$$\supset \vdash :: p \supset q . \supset . q : \supset : . q \supset p . \supset : p \supset q . \supset . q$$

[In reading expressions of this kind, the first thing to notice is the arrangement of the dots, which gives the structure of the sentence. Thus the second line of the above reads: "If 'p implies q' implies q, then, if q implies p, it follows that 'p implies q' implies q".]

In like manner, when a constant, say Socrates, is to be substituted to obtain an inference by ★2·2, we may indicate the inference by

$$(C \wr y) \frac{\text{Socrates}}{y}.$$

Thus, $\vdash . \bigstar 2\cdot 6 \dfrac{\text{Socrates}}{q} . \supset \vdash : p . \supset . \text{Socrates} \supset p$.

But this is less often necessary, since the substitution of a constant is generally easily recognizable.

The remaining principles of deduction are simpler. They are as follows:

★2·5 $\vdash . p \supset p$ Pp. ["Id"]*

This principle asserts that anything implies itself. We decided that "$p \supset q$" was to be equivalent to "p is not true or q is true"; hence "$p \supset p$" is equivalent to "p is not true or p is true", *i. e.* to the law of excluded middle. This principle will be referred to as "Id", which is short for "identity"; for the principle is a form of the law of identity.

★2·6 $\vdash : p . \supset . q \supset p$ Pp. ["Simp"]

This principle asserts that if p is true, then if q is true, p is true; it is equivalent to "if p and q are true, then p is true". On this account it is called (following Peano) the "principle of simplification", which is abbreviated into "Simp." This principle asserts that a true proposition is implied by anything, for it may be read: "If p is true, then q implies p".

★2·7 $\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$ Pp. ["Syll"]

This principle asserts: "If p implies q, then, if q implies r, it follows that p implies r". This principle is called the "principle of the syllogism" (shortened to "Syll"), because it is the source of the syllogism in Barbara.†

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* A name in inverted commas, enclosed in square brackets, is the name of the proposition after which it occurs. I have given names to a few of the more important principles, for convenience of reference. When there are no inverted commas, the enclosure in square brackets refers to the proposition or propositions by which the one in question is proved.

† This occurs, in outline, as follows:

"All a is b" is equivalent to "for any value of x, x is an $a . \supset . x$ is a b".

"All b is c" is equivalent to "for any value of x, x is a $b . \supset . x$ is a c".

By the above principle, if these both hold, it follows that for any value of x, x is an $a . \supset . x$ is a c, *i. e.* "All a is c". The complete proof is more elaborate, but the above is only intended to justify the name.

★2·8 $\vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$ Pp. ["Comm"]

i. e., given that, if p is true, then, if q is true, r is true, it follows that, if q is true, then, if p is true, r is true. Roughly, if r follows from p and q, then r follows from q and p. This is called "the commutative principle" (shortened to "Comm").

·9 $\vdash . \sim(\sim p) \supset p$ Pp. ["Neg"]

i. e., if it is false that p is false, then p is true. This is the "principle of double negation" (shortened to "Neg").

·91 $\vdash : p \supset \sim p . \supset . \sim p$ Pp. ["Abs"]

i. e., whatever implies its own untruth is untrue. This principle is very useful in philosophy, where many widely-held positions sin against it. *E. g.* the principle "no truth is quite true" implies that itself is quite true, and therefore that some truth is quite true. Hence it implies its own falsehood, and is therefore false. It is called the "principle of the reductio ad absurdum" (shortened into "Abs").*

★2·92 $\vdash : p \supset \sim q . \supset . q \supset \sim p$ Pp. ["Transp"]

i. e., if the truth of p implies the untruth of q, then the truth of q implies the untruth of p. In other words, if p is incompatible with q, then q is incompatible with p. This is called the "principle of transposition" (shortened to "Transp").

In ★7, three more primitive propositions will be given. For the present the above completes the list. In what follows, a few of the more important propositions will be given names, but the rest will be referred to by their numbers.

★3. *Elementary Properties of Implication and Negation.*

The propositions that follow are all such as are actually needed in deducing mathematics from our primitive propositions. I shall omit proofs of some of the less important, and shall sometimes only briefly indicate the proofs where they are very obvious. But in most cases I shall give the proofs in full,† because the importance of the present subject lies, not in the propositions themselves, but (1) in the fact that they follow from the primitive propositions, (2) in the fact that it is the

* There is an interesting historical article on this principle by Vailati, "A proposito d' un passo del Teeteto e di una dimostrazione di Euclide", Rivista di Filosofia e scienze affine, 1904.

† Certain abbreviating processes will, however, be gradually introduced.

easiest, simplest, and most elementary example of the symbolic method of dealing with the principles of mathematics generally. Later portions —the theories of classes, relations, cardinal numbers, series, ordinal numbers, geometry, etc.—all employ the same method, but with an increasing complexity in the entities and functions considered.

★3·1 $\vdash :. p . \supset : p \supset q . \supset . q$

Dem.

$$\vdash . \mathrm{Id}\, \frac{p \supset q}{p} . \supset \vdash : p \supset q . \supset . p \supset q \qquad (1)$$

$$\vdash . \mathrm{Comm}\, \frac{p \supset q, p, q}{p, q, r} . \supset \vdash :: p \supset q . \supset . p \supset q : \supset :. p . \supset : p \supset q . \supset . q \qquad (2)$$

$\vdash . (1) . \supset : \vdash . (2) . \supset \vdash .$ Prop.

In the above, *Dem.* is short for *demonstration.* The proposition proved may be stated: "If p is true, then, if p implies q, q is true". This differs from ★ 2·1 by the fact that it is not only concerned with the case when p actually is true, and does not end by asserting that q actually is true. The symbol "Id $\frac{p \supset q}{p}$" has been explained in connection with ★ 2·3. The passage from (1) and (2) to the conclusion is effected in accordance with ★ 2·1; (1) asserts the hypothesis of (2), and the apodosis in (2) is the proposition to be proved. In future, instead of "$\vdash . (1) . \supset : \vdash . (2) . \supset \vdash .$ Prop.", I shall write "$\vdash . (1) . (2) . \supset \vdash .$ Prop.". The numbers (1), (2), etc. refer to everything to the right of the last assertion-sign in the line in which they occur. "Prop." always means the proposition to be proved; thus every proof ends with "$\supset \vdash .$ Prop.", which takes the place of "Q. E. D."

★3·11 $\vdash : q . \supset . p \supset p$

Dem.

$$\vdash . \mathrm{Comm}\, \frac{p}{r} . \supset \vdash :. p . \supset . q \supset p : \supset : q . \supset . p \supset p \qquad (1)$$

$\vdash . (1) . \mathrm{Simp.} \supset \vdash .$ Prop.

·12 $\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$

Dem.

$$\vdash . \mathrm{Comm}\, \frac{p \supset q, q \supset r, p \supset r}{p, q, r} . \supset$$

$$\vdash :: p \supset q . \supset : q \supset r . \supset . p \supset r :. \supset :. q \supset r . \supset : p \supset q . \supset . p \supset r \qquad (1)$$

$\vdash . (1) . \mathrm{Syll.} \supset \vdash .$ Prop.

★3·2 $\vdash . q \supset \sim(\sim q)$

Dem.

$\vdash . \text{Transp} \dfrac{\sim q}{p} . \supset \vdash : \sim q \supset \sim q . \supset . q \supset \sim(\sim q)$ (1)

$\vdash . (1) . \text{Id.} \quad \supset \vdash . \text{Prop.}$

The above is the converse of the principle of double negation (★2·9); it asserts that if p is true, the negation of p is false.

·21 $\vdash : \sim p \supset q . \supset . \sim q \supset p$

Dem.

$\vdash . ★3{\cdot}12 \dfrac{\sim p, \sim(\sim q)}{p, r} . \supset \vdash :. q \supset \sim(\sim q) . \supset : \sim p \supset q . \supset . \sim p \supset \sim(\sim q)$ (1)

$\vdash . ★3{\cdot}2 \dfrac{q}{p} \quad . \supset \vdash . q \supset \sim(\sim q)$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \sim p \supset q . \supset . \sim p \supset \sim(\sim q)$ (3)

$\vdash . ★2{\cdot}92 \dfrac{\sim p, \sim q}{p, q} \quad . \supset \vdash : \sim p \supset \sim(\sim q) . \supset . \sim q \supset \sim(\sim p)$ (4)

$\vdash . ★3{\cdot}12 \dfrac{\sim q, \sim(\sim p), p}{p, q, r} . \supset \vdash :. \sim(\sim p) \supset p . \supset : \sim q \supset \sim(\sim p) . \supset . \sim q \supset p$ (5)

$\vdash . (5) . ★2{\cdot}9 . \quad \supset \vdash : \sim q \supset \sim(\sim p) . \supset . \sim q \supset p$ (6)

$\vdash . (3) . (4) . (6) . \text{Syll.} \quad \supset \vdash . \text{Prop.}$

Note. In the last line of the above proof, "(3) . (4) . (6) . Syll." is an abbreviation for a process which occurs constantly, and which is tedious to write out. It will be observed that (3) is of the form $\vdash . a \supset b$, (4) is of the form $\vdash . b \supset c$, and (6) is of the form $\vdash . c \supset d$, while the proposition to be proved is $a \supset d$. The process of proof, in full, is as follows:

$\vdash . \text{Syll.} \quad \supset \vdash :. a \supset b . \supset : b \supset c . \supset . a \supset c$ (7)

$\vdash . (3) . (7) . \quad \supset \vdash : b \supset c . \supset . a \supset c$ (8)

$\vdash . (4) . (8) . \quad \supset \vdash . a \supset c$ (9)

$\vdash . \text{Syll.} \quad \supset \vdash :. a \supset c . \supset : c \supset d . \supset . a \supset d$ (10)

$\vdash . (9) . (10) . \supset \vdash : c \supset d . \supset . a \supset d$ (11)

$\vdash . (6) . (11) . \supset \vdash . a \supset d$

and "$a \supset d$" was the proposition to be proved. [In the above, a is $\sim p \supset q$, b is $\sim p \supset \sim(\sim q)$, c is $\sim q \supset \sim(\sim p)$, d is $\sim q \supset p$.]

The above proposition is a second form of "Transposition"; the two following ($\star 3\cdot 22$ and $\star 3\cdot 23$) are two further forms. All four may sometimes be referred to as "Transp".

$\star 3\cdot 22$ $\vdash : p \supset q . \supset . \sim q \supset \sim p$

Dem.

$$\vdash . \star 3\cdot 12 \frac{\sim(\sim q)}{r} . \supset \vdash :. q \supset \sim(\sim q) . \supset : p \supset q . \supset . p \supset \sim(\sim q) \qquad (1)$$

$$\vdash . (1) . \star 3\cdot 2 . \quad \supset \vdash : p \supset q . \supset . p \supset \sim(\sim q) \qquad (2)$$

$$\vdash . \star 2\cdot 92 \frac{\sim q}{q} . \quad \supset \vdash : p \supset \sim(\sim q) . \supset . \sim q \supset \sim p \qquad (3)$$

$\vdash . (2) . (3) . \text{Syll.} \supset \vdash . \text{Prop.}$

$\cdot 23$ $\vdash : \sim q \supset \sim p . \supset . p \supset q$

Dem.

$$\vdash . \star 2\cdot 92 \frac{\sim q , p}{p , q} . \quad \supset \vdash : \sim q \supset \sim p . \supset . p \supset \sim(\sim q) \qquad (1)$$

$$\vdash . \star 3\cdot 12 \frac{\sim(\sim q) , q}{q , r} . \supset \vdash :. \sim(\sim q) \supset q . \supset : p \supset \sim(\sim q) . \supset . p \supset q \qquad (2)$$

$$\vdash . (2) . \star 2\cdot 9 . \quad \supset \vdash : p \supset \sim(\sim q) . \supset . p \supset q \qquad (3)$$

$\vdash . (1) . (3) . \text{Syll.} \quad \supset \vdash . \text{Prop.}$

$\star 3\cdot 3$ $\vdash : p . \supset . \sim p \supset q$

i. e., if p is true, then $\sim p$ implies q, whatever q may be.

Dem.

$$\vdash . \text{Simp} \frac{\sim q}{q} . \quad \supset \vdash : p . \supset . \sim q \supset p \qquad (1)$$

$$\vdash . \star 3\cdot 21 \frac{q , p}{p , q} . \quad \supset \vdash : \sim q \supset p . \supset . \sim p \supset q \qquad (2)$$

$\vdash . (1) . (2) . \text{Syll.} \supset \vdash . \text{Prop.}$

$\cdot 31$ $\vdash : \sim p . \supset . p \supset q$

i. e., if p is not true, p implies q, whatever q may be.

Dem.

$$\vdash . \text{Comm.} \frac{\sim p , q}{q , r} . \supset \vdash :. p . \supset . \sim p \supset q : \supset : \sim p . \supset . p \supset q \qquad (1)$$

$\vdash . (1) . \star 3\cdot 3 . \quad \supset \vdash . \text{Prop.}$

$\cdot 32$ $\vdash : \sim(p \supset q) . \supset . p$ $[\star 3\cdot 31\cdot 21]$

Note. Here $\star 3\cdot 31\cdot 21$ is short for $\star 3\cdot 31 \cdot \star 3\cdot 21$. When a proof is very obvious, as in the present case, I shall merely indicate the propositions employed in it by enclosing the references in square brackets, as above.

·33 $\vdash : \sim(p \supset q) . \supset . \sim q$ [Simp. ★3·22]

★3·34 $\vdash : \sim(p \supset q) . \supset . p \supset \sim q$

Dem.

$\vdash . \text{Simp} \frac{\sim q, p}{p, q} . \supset \vdash : \sim q . \supset . p \supset \sim q$ (1)

$\vdash . (1) . ★3·33 . \text{Syll.} \supset \vdash$ Prop.

·35 $\vdash : \sim(p \supset q) . \supset . \sim p \supset q$ [★3·32·3]

·36 $\vdash : \sim(p \supset \sim q) . \supset . p$ $\left[★3·32 \frac{\sim q}{q}\right]$

·37 $\vdash : \sim(p \supset \sim q) . \supset . q$ $\left[★3·33 \frac{\sim q}{q} . ★2·9\right]$

·38 $\vdash :. p . \supset : q . \supset . \sim(p \supset \sim q)$

Dem.

$\vdash . ★3·1 \frac{\sim q}{q} . \supset \vdash :. p . \supset : p \supset \sim q . \supset . \sim q$ (1)

$\vdash . ★2·92 \frac{p \supset \sim q}{p} . \supset \vdash :. p \supset \sim q . \supset . \sim q : \supset : q . \supset . \sim(p \supset \sim q)$ (2)

$\vdash . (1) . (2) . \text{Syll.} \supset \vdash$ Prop.

Note. It follows from ★3·36·37 that when $\sim(p \supset \sim q)$ is true, p and q are true, and it follows from ★3·38 that when p and q are true, $\sim(p \supset \sim q)$ is true. This justifies the definition below (★4·1) of the joint assertion or propositional product of p and q.

★3·4 $\vdash :. p \supset q . \supset . r : \supset : p \supset q . \supset . p \supset r$

Dem.

$\vdash . ★3·1 \frac{p \supset q, r}{p, q} . \supset \vdash :: p \supset q . \supset :. p \supset q . \supset . r : \supset . r$ (1)

$\vdash . ★3·12 \frac{p \supset q . \supset . r, r, p \supset r}{p, q, r} . \supset$

$\vdash ::. r . \supset . p \supset r : \supset :: p \supset q . \supset . r : \supset . r :. \supset :. p \supset q . \supset . r : \supset . p \supset r$ (2)

$\vdash . (2) . \text{Simp.} \supset \vdash :: p \supset q . \supset . r : \supset . r :. \supset :. p \supset q . \supset . r : \supset . p \supset r$ (3)

$\vdash . (1) . (3) . \text{Syll.} \supset \vdash :: p \supset q . \supset :. p \supset q . \supset . r : \supset . p \supset r$ (4)

$\vdash . \text{Comm} \frac{p \supset q, p \supset q . \supset . r, p \supset r}{p, \quad q, \quad r} . \supset$

$\vdash ::. p \supset q . \supset :. p \supset q . \supset . r : \supset . p \supset r :: \supset :: p \supset q . \supset . r : \supset :$
$p \supset q . \supset . p \supset r$ (5)

$\vdash . (4) . (5) . \supset \vdash .$ Prop.

★3·41 $\vdash : \sim p \supset p \,.\, \supset \,.\, p$

Dem.

$\vdash \,.\, ★3{\cdot}12 \dfrac{\sim p,\, p,\, \sim(\sim p)}{p,\, q,\, r} \,.\, \supset \vdash :. \, p \supset \sim(\sim p) \,.\, \supset : \sim p \supset p \,.$
$\supset \,.\, \sim p \supset \sim(\sim p)$ (1)

$\vdash \,.\, (1) \,.\, ★3{\cdot}2 \,.\; \supset \vdash : \sim p \supset p \,.\, \supset \,.\, \sim p \supset \sim(\sim p)$ (2)

$\vdash \,.\, ★2{\cdot}91 \dfrac{\sim p}{p} \,.\, \supset \vdash : \sim p \supset \sim(\sim p) \,.\, \supset \,.\, \sim(\sim p)$ (3)

$\vdash \,.\, ★2{\cdot}9 \,.\; \supset \vdash : \sim(\sim p) \,.\, \supset \,.\, p$ (4)

$\vdash \,.\, (2) \,.\, (3) \,.\, (4) \,.\, \mathrm{Syll.} \;\; \supset \vdash \,.\, \mathrm{Prop.}$

The above is the first use of ★2·91. It asserts that a proposition must be true if it can be deduced from the supposition that it is false.

★3·42 $\vdash :. \, p \supset q \,.\, \supset : \sim p \supset q \,.\, \supset \,.\, q$

Dem.

$\vdash \,.\, ★3{\cdot}22 \,.\; \supset \vdash :. \, p \supset q \,.\, \supset : \sim q \supset \sim p$ (1)

$\vdash \,.\, \mathrm{Syll} \; \dfrac{\sim q,\, \sim p,\, q}{p,\, q,\, r} \,.\, \supset \vdash :. \sim q \supset \sim p \,.\, \supset : \sim p \supset q \,.\, \supset \,.\, \sim q \supset q$ (2)

$\vdash \,.\, ★3{\cdot}12 \dfrac{\sim p \supset q,\, \sim q \supset q,\, q}{p,\qquad q,\qquad r} \,.\, \supset$

$\vdash :: \sim q \supset q \,.\, \supset \,.\, q : \supset :. \sim p \supset q \,.\, \supset \,.\, \sim q \supset q : \supset : \sim p \supset q \,.\, \supset \,.\, q$ (3)

$\vdash \,.\, (3) \,.\, ★3{\cdot}41 \,.\; \supset \vdash :. \sim p \supset q \,.\, \supset \,.\, \sim q \supset q : \supset : \sim p \supset q \,.\, \supset \,.\, q$ (4)

$\vdash \,.\, (1) \,.\, (2) \,.\, (4) \,.\, \mathrm{Syll.} \;\; \supset \vdash \,.\, \mathrm{Prop.}$

·43 $\vdash :. \sim p \supset q \,.\, \supset : p \supset q \,.\, \supset \,.\, q$ [★3·42 . Comm.]

★3·44 $\vdash :. \, p \supset q \,.\, \supset : p \supset \sim q \,.\, \supset \,.\, \sim p$

Dem.

$\vdash \,.\, ★3{\cdot}22 \,.\; \supset \vdash :. \, p \supset q \,.\, \supset : \sim q \supset \sim p$ (1)

$\vdash \,.\, ★3{\cdot}12 \dfrac{\sim q,\, \sim p}{q,\, r} \,.\, \supset \vdash :. \sim q \supset \sim p \,.\, \supset : p \supset \sim q \,.\, \supset \,.\, p \supset \sim p$ (2)

$\vdash \,.\, ★3{\cdot}12 \dfrac{p \supset \sim q,\, p \supset \sim p,\, \sim p}{p,\qquad q,\qquad r} \,.\, \supset$

$\vdash :: p \supset \sim p \,.\, \supset \,.\, \sim p : \supset :. \, p \supset \sim q \,.\, \supset \,.\, p \supset \sim p : \supset : p \supset \sim q \,.\, \supset \,.\, \sim p$ (3)

$\vdash \,.\, (3) \,.\, ★2{\cdot}91 \,.\; \supset \vdash :. \, p \supset \sim q \,.\, \supset \,.\, p \supset \sim p : \supset : p \supset \sim q \,.\, \supset \,.\, \sim p$ (4)

$\vdash \,.\, (1) \,.\, (2) \,.\, (4) \,.\, \mathrm{Syll.} \;\; \supset \vdash \,.\, \mathrm{Prop.}$

·45 $\vdash :. \, p \supset q \,.\, \supset \,.\, q : \supset \,.\, \sim p \supset q$

Dem.

$\vdash \,.\, \mathrm{Syll} \; \dfrac{\sim p,\, p \supset q,\, q}{p,\qquad q,\qquad r} \,.\, \supset \vdash :: \sim p \,.\, \supset \,.\, p \supset q : \supset :. \, p \supset q \,.\, \supset \,.\, q : \supset \,.\, \sim p \supset q$ (1)

$\vdash \,.\, (1) \,.\, ★3{\cdot}31 \,.\; \supset \vdash \,.\, \mathrm{Prop.}$

$\star 3{\cdot}46 \quad \vdash :. \sim p \supset q . \supset . q : \supset . p \supset q \quad \left[\star 3{\cdot}45 \frac{\sim p}{p}\right]$

Note. In all formulæ concerned only with implications, p may be substituted for $\sim(\sim p)$, in virtue of $\star 2{\cdot}9$ and $\star 3{\cdot}2$. In future, this substitution will be made (as it has been in proving $\star 3{\cdot}46$) tacitly; it would be justified, in each case, by exactly similar steps, and it is not worth while to repeat them explicitly on each occasion.

$\cdot 47 \quad \vdash :. p \supset q . \supset . q : \supset : q \supset p . \supset . p$

Dem.

$\vdash . \star 3{\cdot}45. \quad \supset \vdash :. p \supset q . \supset . q : \supset . \sim p \supset q \qquad (1)$

$\vdash . \star 3{\cdot}21. \quad \supset \vdash : \sim p \supset q . \supset . \sim q \supset p \qquad (2)$

$\vdash . \star 3{\cdot}43 \frac{q, p}{p, q} . \supset \vdash :. \sim q \supset p . \supset : q \supset p . \supset . p \qquad (3)$

$\vdash . (1) . (2) . (3) .$ Syll. $\supset \vdash .$ Prop.

$\star 3{\cdot}5 \quad \vdash :. p . \supset . p \supset q : \supset . p \supset q$

Dem.

$\vdash . \star 3{\cdot}43 \frac{p \supset q}{q} . \supset \vdash :: \sim p . \supset . p \supset q : \supset :. p . \supset . p \supset q : \supset . p \supset q \qquad (1)$

$\vdash . (1) . \star 3{\cdot}31. \quad \supset \vdash .$ Prop.

$\cdot 51 \quad \vdash :. p \supset q . \supset . p : \supset . p \quad \left[\star 3{\cdot}5 \cdot \star 3{\cdot}47 \frac{p \supset q}{q}\right]$

$\star 3{\cdot}6 \quad \vdash :. p . \supset . q \supset r : \supset : p \supset q . \supset . p \supset r$

Dem.

$\vdash .$ Comm. $\quad \supset \vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r \qquad (1)$

$\vdash . \star 3{\cdot}12 \frac{p \supset r}{r} . \supset \vdash :: q . \supset . p \supset r : \supset :. p \supset q . \supset : p . \supset . p \supset r \qquad (2)$

$\vdash . \star 3{\cdot}12 \frac{p \supset q ,\, p . \supset . p \supset r ,\, p \supset r}{p ,\quad q ,\quad r} . \supset$

$\vdash ::. p . \supset . p \supset r : \supset . p \supset r :. \supset :: p \supset q . \supset : p . \supset . p \supset r :.$
$\supset :. p \supset q . \supset . p \supset r \qquad (3)$

$\vdash . 3{\cdot}5 \frac{r}{q} . \supset \vdash :. p . \supset . p \supset r : \supset . p \supset r \qquad (4)$

$\vdash . (3) . (4) . \supset \vdash :: p \supset q . \supset : p . \supset . p \supset r :. \supset :. p \supset q . \supset . p \supset r \qquad (5)$

$\vdash . (1) . (2) . (5) .$ Syll. $\supset \vdash .$ Prop.

$\star 3{\cdot}61$ $\vdash :: p \,.\supset.\, q \supset r : \supset :. \, p \,.\supset.\, r \supset s : \supset : p \,.\supset.\, q \supset s$

Dem.

$\vdash .\, \star 3{\cdot}12 \, \dfrac{q \supset r,\; r \supset s \,.\supset.\, q \supset s}{q, \qquad r} \,.\supset$

$\vdash ::. \, q \supset r \,.\supset: r \supset s \,.\supset.\, q \supset s :. \supset :: p \,.\supset.\, q \supset r : \supset :. \, p \,. \supset : r \supset s \,.\supset.\, q \supset s$ (1)

$\vdash .\, (1) \,.\, \star 3{\cdot}12 \, \dfrac{q, r, s}{p, q, r} \,.\supset \vdash :: p \,.\supset.\, q \supset r : \supset :. \, p \,.\supset: r \supset s \,.\supset.\, q \supset s$ (2)

$\vdash .\, \star 3{\cdot}6 \, \dfrac{r \supset s,\, q \supset s}{q, r} \,. \quad \supset \vdash :: p \,.\supset: r \supset s \,.\supset.\, q \supset s :. \supset :. \, p \,.\supset.\, r \supset s : \supset : p \,.\supset.\, q \supset s$ (3)

$\vdash .\, (2) \,.\, (3) \,.$ Syll. $\quad \supset \vdash .$ Prop.

$\star 3{\cdot}62$ $\vdash :. \, p \supset q \,.\supset.\, p \supset r : \supset : p \,.\supset.\, q \supset r$

Dem.

$\vdash .$ Simp. $\quad \supset \vdash : q \,.\supset.\, p \supset q$ (1)

$\vdash .\, \star 3{\cdot}1 \, \dfrac{p \supset q,\, p \supset r}{p, q} \,.\supset \vdash :: p \supset q \,.\supset :. \, p \supset q \,.\supset.\, p \supset r : \supset .\, p \supset r$ (2)

$\vdash .\, (1) \,.\, (2) \,.$ Syll. $\quad \supset \vdash :: q \,.\supset :. \, p \supset q \,.\supset.\, p \supset r : \supset .\, p \supset r$ (3)

$\vdash .$ Comm $\quad \dfrac{q,\; p \supset q \,.\supset.\, p \supset r,\; p \supset r}{p, \qquad q, \qquad r} \,.\supset$

$\vdash ::. \, q \,.\supset :. \, p \supset q \,.\supset.\, p \supset r : \supset .\, p \supset r :: \supset :: p \supset q \,.\supset.\, p \supset r : \supset : q \,.\supset.\, p \supset r$ (4)

$\vdash .\, (3) \,.\, (4) \,.\supset \vdash .$ Prop.

$\star 4.$ *Multiplication and Addition.*

In this section we shall be concerned with the fundamental properties of the *propositional product* and the *propositional sum* of two entities p and q. The *propositional product* is practically "p and q are both true". But this, as it stands, would have to be a new primitive idea, and it is desirable to avoid primitive ideas whenever we can. Now we have seen ($\star 3{\cdot}38$, note) that $\sim (p \supset \sim q)$ is true when and only when p and q are both true. Hence we may take $\sim (p \supset \sim q)$ as the *propositional product* of p and q, since this implies, or is implied by, anything that "p and q are both true" implies, or is implied by.

The *propositional sum* of p and q is practically "either p is true or q is true". We avoid a new primitive idea by taking as the propositional sum $\sim p \supset q$, *i. e.*, "if p is not true, then q is true", which is obviously equivalent to "either p or q is true". There are some advantages in taking the propositional sum as a primitive idea instead of "$p \supset q$"; we then define "$p \supset q$" as the propositional sum of $\sim p + q$. The choice is a matter of taste.

Both the propositional product and the propositional sum of p and q are propositional functions of p and q. They are *significant* for all values of p and q, *i. e.*, even for values which are not propositions; but they are not *important* unless p and q are propositions.

In a definition, we write the term, or combination of terms, to be defined to the left of the sign of equality, and the defining combination of terms to the right; at the end we write "Df." The sign of equality and the "Df." are to be considered as together forming one symbol, meaning "is defined to mean". A definition is not properly part of the subject, being concerned with the symbols, not with what they symbolize; nevertheless, *practically*, the definitions are usually the most important part in a study of principles.

The sign of equality *not* followed by "Df." will be used to express identity, and must not be confounded with the sign of equality followed by "Df."

★4·1 $p \,.\, q \,. \quad = . \sim (p \supset \sim q)$ Df.

·11 $p \vee q \,. \quad = . \sim p \supset q$ Df.

·12 $p \equiv q \,. \quad = . \, p \supset q \,.\, q \supset p$ Df.

·13 $p \supset q \supset r \,. = . \, p \supset q \,.\, q \supset r$ Df.

·14 $p \equiv q \equiv r \,. = . \, p \equiv q \,.\, q \equiv r$ Df.

Definitions have no assertion-signs, because they are not expressions of propositions, but of volitions. Of the above definitions, ★4·1·11 have been already explained. ★4·13·14 are merely convenient abbreviations. ★4·12 defines *equivalence*: "$p \equiv q$" is read "p is equivalent to q". This holds when and only when p and q are either both true or both not true. In the theory of implication it plays a part analogous to that of equality in ordinary mathematics.

When it is necessary, the propositional product of two terms will be expressed by placing several dots between them. Thus, *e. g.*,

$$p \supset q \,.\, \supset . \, q : q \supset p$$

will express the propositional product of $p \supset q \,.\, \supset . \, q$ and $q \supset p$;

$$p \,.\supset :q\supset r \,.\supset .\, s:.\, q \,.\, s \,.\supset :p:p\supset q \,.\supset .\, r$$

will express the propositional product of

$$p \,.\supset :q\supset r \,.\supset .\, s \text{ and } q \,.\, s \,.\supset :p:p\supset q \,.\supset .\, r$$

where $q \,.\, s \,.\supset :p:p\supset q \,.\supset .\, r$ expresses that the propositional product of q and s implies the propositional product of p and $p\supset q \,.\supset .\, r$.

★4·2 $\vdash :.\, p \,.\supset :q \,.\supset .\, p \,.\, q$ [★3·38]

·21 $\vdash :.\, q \,.\supset :p \,.\supset .\, p \,.\, q$ [★4·2 . Comm.]

·22 $\vdash :p \,.\, q \,.\supset .\, q \,.\, p$

Dem.

$$\vdash .\, ★2{\cdot}92 \frac{q,p}{p,q} .\supset \vdash :q\supset \sim p \,.\supset .\, p\supset \sim q \qquad (1)$$

$$\vdash .\, ★3{\cdot}22 \frac{q\supset \sim p,\, p\supset \sim q}{p,q} .\supset$$

$$\vdash :.\, q\supset \sim p \,.\supset .\, p\supset \sim q :\supset : \sim (p\supset \sim q) \,.\supset .\, \sim (q\supset \sim p) \qquad (2)$$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .$ Prop.

·23 $\vdash :p\vee q \,.\supset .\, q\vee p$ [★3·21]

·24 $\vdash .\, \sim (p \,.\sim p)$

Dem.

$\vdash .$ ★3·2 .	$\supset \vdash .\, p\supset \sim (\sim p) \,.$	(1)
$\vdash .$ ★3·2 .	$\supset \vdash :p\supset \sim (\sim p) \,.\supset .\, \sim [\sim \{p\supset \sim (\sim p)\}]$	(2)
$\vdash .$ Id. (★4·1).	$\supset \vdash : \sim [\sim \{p\supset \sim (\sim p)\}] \,.\supset .\, \sim (p \,.\sim p)$	(3)
$\vdash .\, (2) \,.\, (3) \,.$ Syll.	$\supset \vdash :p\supset \sim (\sim p) \,.\supset .\, \sim (p \,.\sim p)$	(4)
$\vdash .\, (1) \,.\, (4)$.	$\supset \vdash$ Prop.	

The above is the law of contradiction, in the form "nothing is both true and not true". When a definition is referred to, as in the third line of the above proof, it is adduced in round brackets, because it is not *logically* relevant, but only *symbolically*.

Proofs may be made both shorter and easier by the following plan. Suppose we know some proposition $(C)(p,q)$, and suppose that some previously proved proposition (a) gives

$$(C)(p,q) \,.\supset .\, (D)(p,q).$$

We may then write the proof of $(D)(p,q)$ in the form

$$\vdash .\, (C)(p,q) \,.$$

$[(a)]$ $\qquad \supset \vdash .\, (D)(p,q)$,

where the proposition by which the transition is made is indicated in square brackets on the left. Thus in $\star 4 \cdot 24$ we should write

$$\begin{array}{ll} \vdash . \star 3 \cdot 2 . & \supset \vdash . p \supset \sim (\sim p) . \\ [\star 3 \cdot 2] & \supset \vdash . \sim [\sim \{ p \supset \sim (\sim p) \}] \\ [\mathrm{Id.} (\star 4 \cdot 1)] & \supset \vdash . \sim (p . \sim p) \end{array}$$

Another similar abbreviation is got by the use of $\star 4 \cdot 13$. Suppose we have

$$\begin{array}{ll} \vdash . (a) . & \supset \vdash : (A \between p, q) . \supset . (B \between p, q) \\ \vdash . (b) . & \supset \vdash : (B \between p, q) . \supset . (C \between p, q) \\ \vdash . (c) . & \supset \vdash : (C \between p, q) . \supset . (D \between p, q) \end{array}$$

and we wish to prove $(A \between p, q) . \supset . (D \between p, q)$. We write

$$\begin{array}{lll} \vdash . (a) . \supset \vdash : (A \between p, q) . \supset . (B \between p, q) . & & \\ [(b)] & \supset . (C \between p, q) . & \\ [(c)] & \supset . (D \between p, q) & (1) \end{array}$$

where "(1)" means "$(A \between p, q) . \supset . (D \between p, q)$", which follows from the above by means of *Syll.* This abbreviation is of very great convenience.

$\star 4 \cdot 25$ $\vdash . p \vee \sim p \left[\mathrm{Id} \dfrac{\sim p}{p} \right]$

This is the law of excluded middle, in the form "everything is either true or not true".

$\cdot 26$ $\vdash : p . q . \supset . p$ $[\star 3 \cdot 36]$

$\cdot 27$ $\vdash : p . q . \supset . q$ $[\star 3 \cdot 37]$

$\cdot 28$ $\vdash : p . \supset . p \vee q$ $[\star 3 \cdot 3]$

$\cdot 29$ $\vdash : q . \supset . p \vee q$ $\left[\mathrm{Simp} \dfrac{q, \sim p}{p, q} \right]$

$\star 4 \cdot 3$ $\vdash :. p . q . \supset . r : \supset : p . \supset . q \supset r$ ["Exp"]

This is an important principle of inference, called by Peano "exportation", because the q is exported out of the hypothesis. It will be adduced as "Exp".

Dem.

$$\begin{array}{ll} \vdash . \mathrm{Id.} (\star 4 \cdot 1) . \supset \vdash :. p . q . \supset . r : \supset : \sim (p \supset \sim q) . \supset . r : & \\ [\star 3 \cdot 21] & \supset : \sim r . \supset . p \supset \sim q : \\ [\mathrm{Comm}] & \supset : p . \supset . \sim r \supset \sim q : \\ [\star 3 \cdot 23 \cdot 12] & \supset : p . \supset . q \supset r :. \supset \vdash . \mathrm{Prop.} \end{array}$$

In this proof we use the second of the abbreviations mentioned in ★4·24 *note.* The fact that there are *two* dots, not *three*, at the ends of the lines (except the last line), indicates that it is not the whole of the first line, but only its consequent, that implies the second line, and so with the transition from the second line to the third and from the third to the fourth. Thus, by *Syll,* the hypothesis of the first line implies each successive subsequent line. The three dots before "$\supset \vdash .$ Prop." indicate that now the hypothesis of the first line is again to be taken into account.

★4·31 $\vdash :. p . \supset . q \supset r : \supset : p . q . \supset . r$ ["Imp"]

This principle is of equal importance with ★4·3, and is called by Peano "importation", because q is imported into the hypothesis. It will be adduced as "Imp".

Dem.

$\vdash . ★3·22·12 · \supset \vdash :. p . \supset . q \supset r : \supset : p . \supset . \sim r \supset \sim q :$

[Comm] $\quad \supset : \sim r \supset . p \supset \sim q :$

[★3·21] $\quad \supset : \sim (p \supset \sim q) . \supset . r :$

[(★4·1)] $\quad \supset : p . q . \supset . r :. \supset \vdash .$ Prop.

★4·32 $\vdash :. p . q . \supset . r : \equiv : p . \supset . q \supset r : \equiv : q . \supset . p \supset r : \equiv : q . p . \supset . r$

Dem.

$\vdash . ★4·2 \dfrac{\text{Exp}, \text{Imp}}{p , q} . \quad \supset \vdash :. \text{Exp.} \supset : \text{Imp.} \supset . \text{Exp. Imp.} \qquad (1)$

$\vdash . (1) . \text{Exp.} \quad \supset \vdash : \text{Imp.} \supset . \text{Exp. Imp.} \qquad (2)$

$\vdash . (2) . \text{Imp.} \quad \supset \vdash : \text{Exp. Imp.} :$

$[\text{Id.} (★4·12)] \quad \supset \vdash :. p . q . \supset . r : \equiv : p . \supset . q \supset r \qquad (3)$

$\vdash . \text{Comm.} \quad \supset \vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r \qquad (4)$

$\vdash . \text{Comm} \dfrac{q , p}{p , q} . \quad \supset \vdash :. q . \supset . p \supset r : \supset : p . \supset . q \supset r \qquad (5)$

$\vdash . \{(4) . (5)\} . ★4·2 . \quad \supset \vdash :. p . \supset . q \supset r : \equiv : q . \supset . p \supset r \qquad (6)$

$\vdash . \{\text{Imp. Exp.}\} . ★4·2 . \supset \vdash :. q . \supset . p \supset r : \equiv : q . p . \supset . r \qquad (7)$

$\vdash . \{(3) . (6)\} . (7) . \quad \supset \vdash .$ Prop.

Note. The use of $\star 4 \cdot 2$, which is given at length above in proving $\vdash .$ Exp. Imp., will in future be assumed, as it is above in passing from (4) and (5) to (6), and from *Imp* and *Exp* to (7). By $\star 4 \cdot 2$, when two propositions have been proved, it follows that their propositional product is true; but it would be unnecessarily lengthy to go through the steps each time this principle is used. Observe that (1), in the above proof, is obtained from $\star 4 \cdot 2$ by the use of $\star 2 \cdot 2$, not of $\star 2 \cdot 3$.

$\star 4 \cdot 33$ $\vdash : p \supset q . q \supset r . \supset . p \supset r$ [Syll. Imp.]

$\cdot 34$ $\vdash : q \supset r . p \supset q . \supset . p \supset r$ [$\star 3 \cdot 12$. Imp.]

These two propositions will hereafter be referred to as "Syll"; they are usually more convenient than either $\star 2 \cdot 7$ or $\star 3 \cdot 12$.

$\cdot 35$ $\vdash : p . p \supset q . \supset . q$ ["Ass"]

[$\star 3 \cdot 1$. Imp.]

This is an important principle of inference, which I shall call the "principle of assertion", and refer to as "Ass".

$\cdot 36$ $\vdash : \sim q . p \supset q . \supset . \sim p$ [$\star 3 \cdot 22$. Comm. Imp.]

$\cdot 37$ $\vdash :. p . q . \supset . r : \supset : p . \sim r . \supset . \sim q$

Dem.

$\vdash . \star 3 \cdot 22 . \supset \vdash : q \supset r . \supset . \sim r \supset \sim q :$

$[\star 3 \cdot 12] \quad \supset \vdash :. p . \supset . q \supset r : \supset : p . \supset . \sim r \supset \sim q$ (1)

$\vdash .$ Exp. $\quad \supset \vdash :. p . q . \supset . r : \supset : p . \supset . q \supset r$ (2)

$\vdash .$ Imp. $\quad \supset \vdash :. p . \supset . \sim r \supset \sim q : \supset : p . \sim r . \supset . \sim q$ (3)

$\vdash . (2) . (1) . (3) .$ Syll. $\supset \vdash .$ Prop.

This is another form of transposition, and is a principle constantly employed.

$\star 4 \cdot 4$ $\vdash : p . q / \supset . p \supset q$ $\left[\star 3 \cdot 34 \frac{\sim q}{q} \right]$

$\cdot 41$ $\vdash :. p \supset r . \supset : p . q . \supset . r$ [$\star 4 \cdot 26$. Syll.]

$\cdot 42$ $\vdash :. q \supset r . \supset : p . q . \supset . r$ [$\star 4 \cdot 27$. Syll.]

$\cdot 43$ $\vdash :. p \supset q . p \supset r . \supset : p . \supset . q . r$ ["Comp"]

This is called by Peano the "principle of composition"; it asserts that if p implies each of two propositions, it implies their propositional product. It will be referred to as "Comp".

Dem.

$\vdash . \star 4\cdot2 . \supset \vdash :: q . \supset :. r . \supset . q . r :.$

$[\star 3\cdot12] \quad \supset :. p \supset r . \supset : p . \supset . q . r :.$

$[\text{Comm}] \quad \supset :. p . \supset : p \supset r . \supset . q . r$ (1)

$\vdash . (1) . \star 3\cdot12 . \supset \vdash ::. p \supset q . \supset :: p . \supset :. p . \supset : p \supset r . \supset . q . r ::$

$[\star 3\cdot5] \quad \supset :: p . \supset : p \supset r . \supset . q . r ::$

$[\text{Comm}] \quad \supset :: p \supset r . \supset : p . \supset . q . r$ (2)

$\vdash . (2) . \text{Imp.} \supset \vdash . \text{Prop.}$

$\star 4\cdot44 \quad \vdash :. q \supset p . r \supset p . \supset : q \vee r . \supset . p$ ["Alt."]

This principle is analogous to $\star 4\cdot43$; I shall call it the "principle of the alternative" and refer to it as "Alt". The analogy between $\star 4\cdot43$ and $\star 4\cdot44$ is of a sort which generally subsists between formulæ concerning products and formulæ concerning sums.

Dem.

$\vdash . \text{Syll.} \supset \vdash :. \sim q \supset r . r \supset p . \supset : \sim q \supset p :$

$[\star 3\cdot43] \quad \supset : q \supset p . \supset . p$ (1)

$\vdash . (1) . \text{Exp.} \supset \vdash :: \sim q \supset r . \supset :. r \supset p . \supset : q \supset p . \supset . p :.$

$[\text{Comm. Imp}] \quad \supset :. q \supset p . r \supset p . \supset . p$ (2)

$\vdash . (2) . \text{Comm.} \supset \vdash :. q \supset p . r \supset p . \supset : \sim q \supset r . \supset . p :.$

$[(\star 4\cdot11)] \quad \supset \vdash . \text{Prop.}$

$\star 4\cdot45 \quad \vdash :. p \supset q . \supset : p . r . \supset . q . r$ ["Fact."]

This principle shows that we may multiply both sides of an implication by a common factor; hence it is called by Peano the "principle of the factor". I shall refer to it as "Fact".

Dem.

$\vdash . \text{Syll} \frac{\sim r}{r} . \supset \vdash :. p \supset q . \supset : q \supset \sim r . \supset . p \supset \sim r :$

$[\star 3\cdot22] \quad \supset : \sim (p \supset \sim r) . \supset . \sim (q \supset \sim r) :.$

$[(\star 4\cdot1)] \quad \supset \vdash . \text{Prop.}$

★4·46 $\vdash :. p \supset q . \supset : p \vee r . \supset . q \vee r$ ["Sum."]

This principle is analogous to ★4·45; it may be called the "principle of the summand". I shall refer to it as "Sum".

Dem.

$\vdash . ★3·22 . \supset \vdash :. p \supset q . \supset : \sim q \supset \sim p :$

[Syll] $\supset : \sim p \supset r . \supset . \sim q \supset r :.$

[(★4·11)] $\supset \vdash$. Prop.

★4·47 $\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . r . s$

This proposition, or rather its analogue for classes, was proved by Leibniz, and evidently pleased him, since he calls it "præclarum theorema".*

Dem.

$\vdash . ★4·26 . \supset \vdash :. p \supset r . q \supset s . \supset : p \supset r :$

[Fact] $\supset : p . q . \supset . r . q :$

[★4·22] $\supset : p . q . \supset . q . r$ (1)

$\vdash . ★4·27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$

[Fact] $\supset : q . r . \supset . s . r :$

[★4·22] $\supset : q . r . \supset . r . s$ (2)

$\vdash . (1) . (2) . ★4·2 . \supset \vdash :: p \supset r . q \supset s . \supset : p . q . \supset . q . r :. p \supset r . q \supset s .$
$\supset : q . r . \supset . r . s ::$

[Comp] $\supset \vdash :: p \supset r . q \supset s . \supset :. p . q . \supset . q . r : q . r . \supset . r . s :.$

[Syll] $\supset :. p . q . \supset . r . s :: \supset \vdash$. Prop.

★4·48 $\vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . r \vee s$

This theorem is the analogue of ★4·47.

Dem.

$\vdash . ★4·26 . \supset \vdash :. p \supset r . q \supset s . \supset : p \supset r :$

[Sum] $\supset : p \vee q . \supset . r \vee q :$

[★4·23] $\supset : p \vee q . \supset . q \vee r$ (1)

$\vdash . ★4·27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$

[Sum] $\supset : q \vee r . \supset . s \vee r :$

[★4·23] $\supset : q \vee r . \supset . r \vee s$ (2)

$\vdash . (1) . (2) . ★4·2 . \text{Comp} . \supset$
$\vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . q \vee r : q \vee r . \supset . r \vee s :$

[Syll] $\supset : p \vee q . \supset . r \vee s :. \supset \vdash$. Prop.

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*Philosophical works, Gerhardt's edition, Vol. VII, p. 223.

★4·5 $\vdash :. p \vee q . \equiv : p \supset q . \supset . q$ [★3·43·45]

·51 $\vdash :. p \supset q . \supset . q : \equiv : q \supset p . \supset . p$ $\left[★3·47. \quad ★3·47 \frac{q, p}{p, q} \right]$

·6 $\vdash :. p \equiv q . \supset : p \supset r . \equiv . q \supset r$

Dem.

$\vdash$. ★4·22. $\supset \vdash : p \equiv q . \supset . q \supset p . p \supset q$ (1)

$\vdash$. Syll. $\supset \vdash :. q \supset p . \supset : p \supset r . \supset . q \supset r$ (2)

$\vdash$. Syll. $\supset \vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$ (3)

$\vdash$. (2) . (3). ★4·47 . $\supset$

$\vdash :. q \supset p . p \supset q . \supset : p \supset r . \equiv . q \supset r$ (4)

$\vdash$. (1) . (4) . Syll. $\supset \vdash$. Prop.

·61 $\vdash :. p \equiv q . \supset : r \supset p . \equiv . r \supset q$ [Similar proof.]

★5 *Formal rules.*

In this section, we shall be concerned with rules analogous, more or less, to those of ordinary algebra. It is from these rules that the usual "calculus of formal logic" starts. Treated as a "calculus", the rules of deduction are capable of many other interpretations. But all other interpretations depend upon the one here considered, since in all of them we deduce consequences from our rules, and thus presuppose the theory of deduction. One very simple interpretation of the "calculus" is as follows: The entities considered are to be numbers which are all either 0 or 1; "$p \supset q$" is to have the value 0 if p is 1 and q is 0; otherwise it is to have the value 1; $\sim p$ is to be 1 if p is 0, and 0 if p is 1; $p . q$ is to be 1 if p and q are both 1, and is to be 0 in any other case; $p \vee q$ is to be 0 if p and q are both 0, and is to be 1 in any other case; and the assertion-sign is to mean that what follows has the value 1.

To show that our primitive propositions are sufficient for the propositional calculus, we may compare our propositions with the first set of postulates in Huntington's "Sets of Independent Postulates for the Algebra of Logic".* In making this comparison, observe that his class K is replaced by the class of all the entities, so that postulates concerning membership of K are necessarily satisfied by everything. His two rules of combination are our addition and multiplication; his $\wedge$ is our $(s) . s$ [see ★7], and his V is our $\sim (s) . s$. Then his postulates Ia and Ib are

*Trans. Amer. Math. Soc., July, 1904, p. 292.

satisfied because the class K is the class of all entities; his equality is replaced by our equivalence, so that IIa is our ★7·3, and IIb is ★7·31, IIIa and IIIb are respectively ★5·31 and ★5·3. IVa is ★5·41, and IVb is ★5·4; V results from ★7·34·35, and VI results from ★7·26. Thus our material (together with the additions to be made in ★7) is sufficient to found the propositional calculus. Symbolic logic considered as a calculus has undoubtedly much interest on its own account; but in my opinion this aspect has hitherto been too much emphasized, at the expense of the aspect in which symbolic logic is merely the most elementary part of mathematics, and the logical prerequisite of all the rest. For this reason, I shall only deal briefly with what is required for the algebra of symbolic logic.

★5·1 $\vdash : p \supset q . \equiv . \sim q \supset \sim p$ [★3·22·23]

·11 $\vdash : p \equiv q . \equiv . \sim p \equiv \sim q$ [★3·22·23. ★4·47·22]

·12 $\vdash : p \equiv \sim q . \equiv . q \equiv \sim p$ [★2·92. ★3·21]

·13 $\vdash . p \equiv \sim (\sim p)$ [★2·9. ★3·2]

·14 $\vdash :. p . q . \supset . r : \equiv : p . \sim r . \supset . \sim q$ [★4·37. ★5·13]

Observe that if $p \equiv q$, q may be substituted for p, or vice versa, in any formula involving no primitive ideas except implication and negation, without altering the truth or falsehood of the formula. This can be proved for each separate case, but not generally, because we have no means of specifying (with our apparatus of primitive ideas) that a complex $(C \wr p, q)$ is to be one that can be built up out of implication and negation alone.

·15 $\vdash :. p . q . \supset . \sim r : \equiv : q . r . \supset . \sim p$ [★4·22. ★5·13·14]

★5·2 $\vdash . p \equiv p$ [Id. ★4·2]

·21 $\vdash : p \equiv q . \equiv . q \equiv p$ [★4·22]

·22 $\vdash : p \equiv q . q \equiv r . \supset . p \equiv r$

Dem.

$\vdash .$ ★4·26.	$\supset \vdash : p \equiv q . q \equiv r . \supset . p \equiv q .$	
[★4·26]	$\supset . p \supset q$	(1)
$\vdash .$ ★4·27.	$\supset \vdash : p \equiv q . q \equiv r . \supset . q \equiv r .$	
[★4·26]	$\supset . q \supset r$	(2)
$\vdash . (1) . (2) .$ Comp.	$\supset \vdash : p \equiv q . q \equiv r . \supset . p \supset q . q \supset r .$	
[Syll]	$\supset . p \supset r$	(3)

$\vdash . \star 4.27. \qquad \supset \vdash : p \equiv q . q \equiv r : \supset . q \equiv r .$

$[\star 4\cdot 27] \qquad \supset . r \supset q \qquad (4)$

$\vdash . \star 4\cdot 26. \qquad \supset \vdash : p \equiv q . q \equiv r . \supset . p \equiv q .$

$[\star 4\cdot 27] \qquad \supset . q \supset p \qquad (5)$

$\vdash . (4) . (5) . \text{Comp.} \quad \supset \vdash : p \equiv q . q \equiv r . \supset . r \supset q . q \supset p .$

$[\text{Syll}] \qquad \supset . r \supset p \qquad (6)$

$\vdash . (3) . (6) . \text{Comp.} \quad \supset \vdash . \text{Prop.}$

Note. The above three propositions show that the relation of equivalence is reflexive ($\star 5\cdot 2$), symmetrical ($\star 5\cdot 21$), and transitive ($\star 5\cdot 22$). Implication is reflexive and transitive, but not symmetrical. The properties of being symmetrical, transitive, and (at least within a certain field) reflexive are essential to any relation which is to have the formal characters of equality.

$\star 5\cdot 24 \quad \vdash : p . \equiv . p . p$

Dem.

$\vdash . \star 4\cdot 26 . \quad \supset \vdash : p . p . \supset . p \qquad (1)$

$\vdash . \star 4\cdot 2 . \quad \supset \vdash : . p . \supset : p . \supset . p . p . :$

$[\star 3\cdot 5] \quad \supset \vdash : p . \supset . p . p \qquad (2)$

$\vdash . (1) . (2) . \star 4\cdot 21. \quad \supset \vdash . \text{Prop.}$

$\cdot 25 \quad \vdash : p . \equiv . p \lor p$

Dem.

$\vdash . \star 4\cdot 28. \qquad \supset \vdash : p . \supset . p \lor p \qquad (1)$

$\vdash . \text{Id.} (\star 4\cdot 11). \quad \supset \vdash : p \lor p . \supset . \sim p \supset p .$

$[\star 3\cdot 41] \qquad \supset . p \qquad (2)$

$\vdash . (1) . (2). \quad \star 4\cdot 2. \quad \supset \vdash . \text{Prop.}$

Note. $\star 5\cdot 24\cdot 25$ are the two forms of the *law of tautology*, which is what chiefly distinguishes the algebra of symbolic logic from ordinary algebra.

$\star 5\cdot 3 \quad \vdash : p . q . \equiv . q . p \qquad [\star 4\cdot 22]$

Note. Whenever we have, whatever values p and q may have

$$(C \between p , q) . \supset . (C \between q , p) ,$$

we have also

$$(C \between p , q) . \equiv . (C \between q , p) .$$

For $\{(C \between p, q) . \supset . (C \between q ; p)\} \frac{q, p}{p, q} . \supset : (C \between q, p) . \supset . (C \between p, q)$.

·31 $\vdash : p \vee q . \equiv . q \vee p$ [$\star$3·21]

$\star$5·32 $\vdash :(p . q) . r . \equiv . p . (q . r)$

Dem.

$\vdash . \star 5{\cdot}15 . \quad \supset \vdash :. p . q . \supset . \sim r : \equiv : q . r . \supset . \sim p :$

[$\star$5·12] $\quad \equiv : p . \supset . \sim (q . r)$ (1)

$\vdash . (1) . \star 5{\cdot}11 . \quad \supset \vdash : \sim (p . q . \supset . \sim r) . \equiv . \sim (p . \supset . \sim (q . r)) :$

[($\star$4·1)] $\supset \vdash$. Prop.

Note. Here "(1)" stands for "$\vdash :. p . q . \supset . \sim r : \equiv : p . \supset . \sim (q . r)$", which is obtained from the above steps by $\star$5·22. The use of $\star$5·22 will often be tacit, as above. The principle is the same as that explained in respect of implication in $\star$4·24 *note.*

·33 $\vdash : (p \vee q) \vee r . \equiv . p \vee (q \vee r)$

Dem.

$\vdash . \star 5{\cdot}2 . (\star 4{\cdot}11) . \supset \vdash :. (p \vee q) \vee r . \equiv : \sim (\sim p \supset q) . \supset . r :$

[$\star$5·1] $\quad \equiv : \sim r . \supset . \sim p \supset q :$

[$\star$4·32] $\quad \equiv : \sim p . \supset . \sim r \supset q :$

[$\star$5·1 · $\star$4·61] $\quad \equiv : \sim p . \supset . \sim q \supset r :$

[$\star$5·2 · ($\star$4·11)] $\quad \equiv : p \vee (q \vee r) :. \supset \vdash$. Prop.

The above are the associative laws for multiplication and addition. To avoid brackets, we introduce the following definitions:

·34 $p . q . r . = . (p . q) . r$ Df.

·35 $p \vee q \vee r . = . (p \vee q) \vee r$ Df.

$\star$5.36 $\vdash :. p \equiv q . \supset : p . r . \equiv . q . r$ [Fact. $\star$4·47]

·37 $\vdash :. p \equiv q . \supset : p \vee r . \equiv . q \vee r$ [Sum. $\star$4·47]

·38 $\vdash :. p \equiv r . q \equiv s . \supset : p . q . \equiv . r . s$ [$\star$4·47 . $\star$5·32 . $\star$4·22]

·39 $\vdash :. p \equiv r . q \equiv s . \supset : p \vee q . \equiv . r \vee s$ [$\star$4·48·47 . $\star$5·32 . $\star$4·22]

$\star 5{\cdot}4$ $\vdash :. p . q \vee r . \equiv : p . q . \vee . p . r$

This is the first form of the distributive law.

Dem.

$\vdash . \star 4{\cdot}2 . \quad \supset \vdash :: p . \supset :. q . \supset . p . q :. p . \supset : r . \supset . p . r ::$

$[\text{Comp.}] \quad \supset \vdash :: p . \supset :. q . \supset . p . q : r . \supset . p . r :.$

$[\star 4{\cdot}48] \quad \supset :. q \vee r . \supset : p . q . \vee . p . r \quad (1)$

$\vdash . (1) . \text{Imp.} \quad \supset \vdash :. p . q \vee r . \supset : p . q . \vee . p . r \quad (2)$

$\vdash . \star 4{\cdot}26 . \quad \supset \vdash :. p . q . \supset . p : p . r . \supset . p :.$

$[\text{Alt}] \quad \supset \vdash :. p . q . \vee . p . r : \supset . p \quad (3)$

$\vdash . \star 4{\cdot}27 . \quad \supset \vdash :. p . q . \supset . q : p . r . \supset . r :.$

$[\star 4{\cdot}48] \quad \supset \vdash :. p . q . \vee . p . r : \supset . q \vee r \quad (4)$

$\vdash . (3) . (4) . \text{Comp.} \quad \supset \vdash :. p . q . \vee . p . r : \supset . p . q \vee r \quad (5)$

$\vdash . (2) . (5) . \quad \supset \vdash . \text{Prop.}$

$\star 5{\cdot}41$ $\vdash :. p . \vee . q . r : \equiv . p \vee q . p \vee r$

This is the second form of the distributive law—a form to which there is nothing analogous in ordinary algebra. By the conventions as to dots, "$p . \vee . q . r$" means "$p \vee (p . r)$".

Dem.

$\vdash . \star 4{\cdot}26 . \text{Sum.} \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee q \quad (1)$

$\vdash . \star 4{\cdot}27 . \text{Sum.} \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee r \quad (2)$

$\vdash . (1) . (2) . \text{Comp.} \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee q . p \vee r \quad (3)$

$\vdash . \text{Id.} (\star 4{\cdot}11) . \quad \supset \vdash :. p \vee q . p \vee r . \supset : \sim p \supset q . \sim p \supset r :$

$[\text{Comp}] \quad \supset : \sim p . \supset . q . r :$

$[\text{Id.} (\star 4{\cdot}11)] \quad \supset : p . \vee . q . r \quad (4)$

$\vdash . (3) . (4) . \supset \vdash . \text{Prop.}$

$\cdot 42$ $\vdash :. p . \equiv : p . q . \vee . p . \sim q$

Dem.

$\vdash . \star 4{\cdot}21 . \supset \vdash :. q \vee \sim q . \supset : p . \supset . p . q \vee \sim q :.$

$[\star 4{\cdot}25] \quad \supset \vdash : p . \supset . p . q \vee \sim q \quad (1)$

$\vdash . \star 4{\cdot}26 . \supset \vdash : p . q \vee \sim q . \supset . p \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :. p . \equiv : p . q \vee \sim q :$

$[\star 5{\cdot}4] \quad \equiv : p . q . \vee . p . \sim q :. \supset \vdash . \text{Prop.}$

★5·43 $\vdash :. p . \equiv : p \vee q . p \vee \sim q$

Dem.

$\vdash . \star 4{\cdot}28 . \quad \supset \vdash : p . \supset . p \vee q : p . \supset . p \vee \sim q :$

[Comp] $\quad \supset \vdash : p . \supset . p \vee q . p \vee \sim q$ (1)

$\vdash . \star 3{\cdot}44 \frac{\sim p}{p} . \supset \vdash :. \sim p \supset q . \supset : \sim p \supset \sim q . \supset . p :.$

[Imp] $\quad \supset \vdash :. \sim p \supset q . \sim p \supset \sim q . \supset . p :.$

[(★4·11)] $\quad \supset \vdash :. p \vee q . p \vee \sim q . \supset . p$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash .$ Prop.

·44 $\vdash :. p . \equiv : p . \vee . p . q$

Dem.

$\vdash . \star 4{\cdot}28 . \quad \supset \vdash :. p . \supset : p . \vee . p . q$ (1)

$\vdash .$ Id. $\star 4{\cdot}26 . \supset \vdash :. p \supset p : p . q . \supset . p :.$

[★4·44] $\quad \supset \vdash :. p . \vee . p . q : \supset . p$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash .$ Prop.

·45 $\vdash : p . \equiv . p . p \vee q$ [★4·26·28]

The following propositions are important. They show how to transform implications into sums or into denials of products, and vice versa. Compare ★1·2.

★5·5	$\vdash :$	$\sim (p . q) . \equiv . p \supset \sim q$	[★5·12.(★4·1)]
·51	$\vdash :$	$\sim p \vee \sim q . \equiv . p \supset \sim q$	[★5·13.(★4·11)]
·52	$\vdash :$	$\sim (\sim p . \sim q) . \equiv . \sim p \supset q$	[★5·13]
·53	$\vdash :$	$p \vee q . \equiv . \sim p \supset q$	[★5·2]
·54	$\vdash :$	$\sim (p . \sim q) . \equiv . p \supset q$	[★5·5·13]
·55	$\vdash :$	$\sim p \vee q . \equiv . p \supset q$	[★5·13]
·56	$\vdash :$	$\sim (\sim p . q) . \equiv . \sim p \supset \sim q$	$\left[\star 5{\cdot}5 \frac{\sim p}{p}\right]$
·57	$\vdash :$	$p \vee \sim q . \equiv . \sim p \supset \sim q$	$\left[\star 5{\cdot}53 \frac{\sim q}{q}\right]$

From the above we obtain De Morgan's formulæ, as follows:

·6	$\vdash :$	$\sim (p . q) . \equiv . \sim p \vee \sim q$	[★5·5·51]
·61	$\vdash :$	$p . q . \equiv . \sim (\sim p \vee \sim q)$	[★5·6·12]
·62	$\vdash :$	$\sim (\sim p . \sim q) . \equiv . p \vee q$	[★5·52·53]
·63	$\vdash :$	$\sim p . \sim q . \equiv . \sim (p \vee q)$	[★5·62·12]
·64	$\vdash :$	$\sim (p . \sim q) . \equiv . \sim p \vee q$	[★5·54·55]
·65	$\vdash :$	$p . \sim q . \equiv . \sim (\sim p \vee q)$	[★5·64·12]
·66	$\vdash :$	$\sim (\sim p . q) . \equiv . p \vee \sim q$	[★5·56·57]
·67	$\vdash :$	$\sim p . q . \equiv . \sim (p \vee \sim q)$	[★5·66·12]

$\star 5 \cdot 7 \quad \vdash :. p \supset q . \equiv : p . \supset . p . q$

Dem.

$\vdash . \star 4 \cdot 26 . \star 3 \cdot 12 . \supset \vdash :. p . \supset . p . q : \supset . p \supset q \qquad (1)$

$\vdash .$ Comp. $\quad \supset \vdash :. p \supset p . p \supset q . \supset : p . \supset . p . q :.$

[Exp] $\quad \supset \vdash :: p \supset p . \supset :. p \supset q . \supset : p . \supset . p . q ::$

[Id] $\quad \supset \vdash :. p \supset q . \supset : p . \supset . p . q \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash .$ Prop.

$\cdot 71 \quad \vdash :. p \supset q . \equiv : p . \equiv . p . q$

Dem.

$\vdash . \star 4 \cdot 21 . \quad \supset \vdash :: p . q . \supset . p : \supset :. p . \supset . p . q : \supset : p . \equiv . p . q ::$

$[\star 4 \cdot 26] \quad \supset \vdash :. p . \supset . p . q : \supset : p . \equiv . p . q \qquad (1)$

$\vdash . \star 4 \cdot 26 . \quad \supset \vdash :. p . \equiv . p . q : \supset : p . \supset . p . q \qquad (2)$

$\vdash . (1) . (2) . \quad \supset \vdash :. p . \supset . p . q : \equiv : p . \equiv . p . q \qquad (3)$

$\vdash . (3) . \star 5 \cdot 7 . \quad \supset \vdash .$ Prop.

The above proposition enables us to transform every implication into an equivalence, which is an advantage if we wish to assimilate symbolic logic as far as possible with ordinary algebra. But when symbolic logic is regarded as an instrument of proof, we need implications, and it is merely inconvenient to substitute equivalences.

$\star 5 \cdot 72 \quad \vdash :. p \supset q . \equiv : q . \equiv . p \vee q$

Dem.

$\vdash . \star 5 \cdot 1 . \quad \supset \vdash :. p \supset q . \equiv : \sim q \supset \sim p :$

$\left[\star 5 \cdot 71 \dfrac{\sim q , \sim p}{p , q} \right] \quad \equiv : \sim q . \equiv . \sim q . \sim p :$

$[\star 5 \cdot 12] \quad \equiv : q . \equiv . \sim (\sim q . \sim p) :$

$[\star 5 \cdot 62] \quad \equiv : q . \equiv . q \vee p :$

$[\star 5 \cdot 31] \quad \equiv : q . \equiv . p \vee q :. \supset \vdash .$ Prop.

$\cdot 73 \quad \vdash :. q . \supset : p . \equiv . p . q \qquad$ [Simp. $\star 5 \cdot 71$]

This proposition is very useful, since it shows that a true factor may be omitted from a product without altering its truth or falsehood, just as a true hypothesis may be omitted from an implication.

$\cdot 74 \quad \vdash :. \sim p . \supset : q . \equiv . p \vee q \qquad [\star 3 \cdot 31 . \star 5 \cdot 72]$

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$\star 5{\cdot}76$ $\vdash :. p \supset q . p \supset r . \equiv : p . \supset . q . r$ $\left[\star 5{\cdot}41 \frac{\sim p}{p} . \star 5{\cdot}55\right]$

$\cdot 77$ $\vdash :. q \supset p . r \supset p . \equiv : q \vee r . \supset . p$ [Alt. $\star 4{\cdot}28{\cdot}29$]

$\cdot 78$ $\vdash :. p \supset q . \vee . p \supset r : \equiv : p . \supset . q \vee r$

Dem.

$\vdash . \star 5{\cdot}55{\cdot}39 . \supset \vdash :. p \supset q . \vee . p \supset r : \equiv : \sim p \vee q . \vee . \sim p \vee r :$

$[\star 5{\cdot}33]$ $\equiv . \sim p . \vee . q \vee \sim p \vee r :$

$[\star 5{\cdot}31{\cdot}37]$ $\equiv : \sim p . \vee . \sim p \vee q \vee r :$

$[\star 5{\cdot}33]$ $\equiv : \sim p \vee \sim p . \vee . q \vee r :$

$[\star 5{\cdot}25]$ $\equiv : \sim p . \vee . q \vee r :$

$[\star 5{\cdot}55]$ $\equiv : p . \supset . q \vee r :. \supset \vdash .$ Prop.

$\cdot 79$ $\vdash :. q \supset p . \vee . r \supset p : \equiv : q . r . \supset . p$

Dem.

$\vdash . \star 5{\cdot}1{\cdot}39 . \supset \vdash :. q \supset p . \vee . r \supset p : \equiv : \sim p \supset \sim q . \vee . \sim p \supset \sim r :$

$[\star 5{\cdot}78]$ $\equiv : \sim p . \supset . \sim q \vee \sim r :$

$[\star 5{\cdot}6 . \star 4{\cdot}61]$ $\equiv : \sim p . \supset . \sim (q . r) :$

$[\star 5{\cdot}1]$ $\equiv : q . r . \supset . p :. \supset \vdash .$ Prop.

Note. The analogues, for classes, of $\star 5{\cdot}78{\cdot}79$ are false. Take, *e.g.*, $\star 5{\cdot}78$, and put $p =$ English people, $q =$ men, $r =$ women. Then p is contained in q or r, but is not contained in q and is not contained in r.

$\star 5{\cdot}8$ $\vdash : p \supset \sim p . \equiv . \sim p$ [$\star 2{\cdot}91$. Simp.]

$\cdot 81$ $\vdash : \sim p \supset p . \equiv . p$ [$\star 3{\cdot}41$. Simp.]

$\cdot 82$ $\vdash : p \supset q . p \supset \sim q . \equiv . \sim p$ [$\star 3{\cdot}44$. Imp. $\star 3{\cdot}31$. Comp.]

$\cdot 83$ $\vdash : p \supset q . \sim p \supset q . \equiv . q$ [$\star 3{\cdot}42$. Imp. Simp. Comp.]

Note. $\star 5{\cdot}82{\cdot}83$ may also be obtained from $\star 5{\cdot}43$, of which they are virtually other forms.

$\star 6{\cdot}$ *Miscellaneous propositions.*

The following are mainly propositions inserted on account of their subsequent utility. They are all easy to prove, and I shall therefore either wholly omit the proofs, or merely indicate the propositions used in the proofs.

·1 $\vdash : p \,.\, \vee \,.\, p \supset q$ [$\star$3·31]

·11 $\vdash : p \supset q \,.\, \vee \,.\, {\sim} p \supset q$ [$\star$3·35]

·12 $\vdash : p \supset q \,.\, \vee \,.\, p \supset {\sim} q$ [$\star$3·34]

·13 $\vdash : p \supset q \,.\, \vee \,.\, q \supset p$ [$\star$3·32 . Simp.]

·14 $\vdash : p \supset q \,.\, \vee \,.\, q \supset r$ [$\star$3·33·31]

·15 $\vdash : p \equiv q \,.\, \vee \,.\, p \equiv {\sim} q$

·16 $\vdash \,.\, {\sim}(p \equiv q \,.\, p \equiv {\sim} q)$

·17 $\vdash : p \vee q \,.\, {\sim}(p \,.\, q) \,.\, \equiv \,.\, p \equiv {\sim} q$

·18 $\vdash : p \equiv q \,.\, \equiv \,.\, {\sim}(p \equiv {\sim} q)$ [$\star$6·15·16·17]

·19 $\vdash : {\sim}(p \equiv {\sim} p)$ $\left[\star 6{\cdot}18 \frac{p}{q} \,.\, \star 5{\cdot}2\right]$

$\star$6·2 $\vdash : p \,.\, q \,.\, \supset \,.\, p \equiv q$ [$\star$4·4·22]

·21 $\vdash : {\sim} p \,.\, {\sim} q \,.\, \supset \,.\, p \equiv q$ [$\star$6·2 . $\star$5·11]

·22 $\vdash :. {\sim}(p \equiv q) \,.\, \equiv : p \,.\, {\sim} q \,.\, \vee \,.\, {\sim} p \,.\, q$ [$\star$5·6·54]

·23 $\vdash :. \quad p \equiv q \,.\, \equiv : p \,.\, q \,.\, \vee \,.\, {\sim} p \,.\, {\sim} q$ $\left[\star 6{\cdot}18 \,.\, \star 6{\cdot}22 \frac{{\sim} q}{q}\right]$

·24 $\vdash :. {\sim}(p \,.\, q \,.\, \vee \,.\, {\sim} p \,.\, {\sim} q) \,.\, \equiv : p \,.\, {\sim} q \,.\, \vee \,.\, {\sim} p \,.\, q$ [$\star$6·22·23]

·3 $\vdash :. p \,.\, q \,.\, \supset \,.\, r : \equiv : p \,.\, q \,.\, \supset \,.\, p \,.\, r$

·31 $\vdash :. r \,.\, p \supset q : \supset : p \,.\, \supset \,.\, q \,.\, r$

·32 $\vdash :. p \,.\, \supset \,.\, q \equiv r : \equiv : p \,.\, q \,.\, \equiv \,.\, p \,.\, r$ [$\star$5·76 . $\star$4·32 . $\star$6·3]

This proposition is constantly required in subsequent proofs.

·33 $\vdash :. p \,.\, q \supset r \,.\, \equiv : p : p \,.\, q \,.\, \supset \,.\, r$

·34 $\vdash :. p \supset q \,.\, \supset : p \,.\, q \supset r \,.\, \supset \,.\, r$

·35 $\vdash :. p \supset q \,.\, p \supset r \,.\, \supset : p \,.\, \supset \,.\, q \equiv r$

·36 $\vdash : p \,.\, p \equiv q \,.\, \equiv \,.\, q \,.\, p \equiv q$

$\star$6·4 $\vdash :. p \,.\, \supset \,.\, p \supset q : \equiv \,.\, p \supset q$ [Simp. $\star$3·5]

·41 $\vdash :. p \supset q \,.\, \supset \,.\, p \supset r : \equiv : p \,.\, \supset \,.\, q \supset r$ [$\star$3·62·6]

·42 $\vdash :: p \,.\, \supset \,.\, q \supset r : \equiv :. p \,.\, \supset : q \,.\, \supset \,.\, p \,.\, r$ [$\star$6·3 . Exp.]

·43 $\vdash :: p \supset s :. p \,.\, \supset : q \,.\, \equiv \,.\, r \,.\, s :. \supset : p \,.\, \supset \,.\, q \equiv r$

·44 $\vdash :: p \supset q \,.\, \supset :. p \supset r \,.\, \equiv : p \,.\, \supset \,.\, q \,.\, r$

·5 $\vdash :. p . \supset : p \supset q . \equiv . q$

·51 $\vdash :. p \supset q . \supset : p . \supset . q \vee r$

·52 $\vdash :. q . \vee . p \supset q : \equiv . p \supset q$

·53 $\vdash :. p \vee q \vee r . \supset . s : \equiv : p \supset s . q \supset s . r \supset s$ [★5·77]

·54 $\vdash :. p . q . \equiv . p : \vee : p . q . \equiv . q$

·55 $\vdash :. p \vee q . \equiv . p : \vee : p \vee q . \equiv . q$

·6 $\vdash :. p . \sim q . \supset . r : \equiv : p . \supset . q \vee r$

·61 $\vdash : p \vee q . \sim q . \equiv . p . \sim q$

·62 $\vdash :. p . q . \vee . \sim q : \equiv . p \vee \sim q$ $\left[\bigstar 5{\cdot}7 \frac{q, p}{p, q} \right]$

·7 $\vdash :. p \vee r . \equiv . q \vee r : \equiv : r . \vee . p \equiv q$

·71 $\vdash ::. r \supset p . \supset :: s . \sim p . \vee . q : \supset . r :. \equiv . s \supset p . q \supset r$

★7· *Propositions Concerning All Values of the Variables.*

In this section, we have to introduce a new primitive idea, namely, the idea "$(C \between x)$ is true for all values of x." Our formulæ hitherto have concerned *any* value of the variable, not *all* values, and what we may call the *range* of the variable has always been the whole of the sentence (*i. e.*, the whole of one asserted proposition, or of as many asserted propositions as enter into a single chain of deductions). The point of the new primitive idea may be illustrated as follows: Consider ★3·31, *i.e.*, $\vdash : \sim p . \supset . p \supset q$. This means: "For any values of p and q, not-p implies that $p \supset q$". Here although q only occurs at the end, its "range" is the whole proposition. But suppose we wish to say: "Not-p implies that, for any value of q, p implies q". Here q has as its range only "p implies q", not the whole proposition. We may proceed a stage further, and say: "Not-p implies that p implies that, for any value of q, q is true." Here the range of q is confined to q itself. When the range of (say) q is not the whole of an asserted proposition, we indicate the fact by putting (q) before the beginning of the range, followed by as many dots as occur at the end of the range, or (if the end of the range is the end of the sentence) by one more dot than occurs anywhere between the beginning of the range and the end of the sentence. Thus "$\vdash : \sim p . \supset . (q) . p \supset q$" means: "For any value of p, not-p implies that, for all values of q, p implies q". Similarly, "$\vdash :. \sim p . \supset : p . \supset . (q) . q$" means: "For any value of p, not-p implies that p implies that, for all values of q, q is true". When the assertion-sign is absent, any complex $(C \between x)$, containing x, is a function of x, which in general has different values for different values of x. But the proposition "$(C \between x)$ is true for all values of x" is not a function of x; the result, so far as x is concerned, is a constant. [The case is more or less like that of the x in $\int_a^b f(x)\,dx$.] Hence we may, following Peano, call the x only an *apparent* variable in such a case. A variable which is not apparent is called *real*. [The sense of *real* which is opposed to *complex* cannot well occur in a context in which the above sense might be meant.] We denote by

$$(x) . (C \between x) \text{ or } (x) : (C \between x)$$

(or by a larger number of dots, if necessary) the proposition "$(C \between x)$ is true for *all* values of x". Thus the "range" of an apparent variable becomes that proposition (whatever it is) whose truth for all values of the variable is being considered.

The utility of the above idea and notation arises when the proposition $(x) . (C \between x)$ is not itself asserted, but is part of an asserted proposition. Thus in the above case, in

$$\vdash :. \sim p . \supset : p . \supset . (q) . q,$$

$(q) . q$ is not itself asserted, but is a constituent of an asserted proposition. Note that $(q) . q$ is an absolute constant, meaning "everything is true". Thus the formula

$$\vdash :. \sim p . \supset : p . \supset . (q) . q$$

may be read: "If p is false, then p implies that everything is true."

In $(C \between x)$, where the x is undetermined, the x is supposed to have some value, but not this or that definite value. Thus, when we assert $(C \between x)$, we assert an ambiguous proposition; this assertion is true if all the values of $(C \between x)$ are true, but not otherwise. This is what is asserted in ★7·1·11 below. For the sake of definiteness, we may say that

"$\vdash . (C \between x)$" means "$(C \between x)$ is true, where x may be anything".

"$\vdash . (x) . (C \between x)$" means "$(C \between x)$ is true for all values of x".

The necessity for this rather subtle distinction arises from certain difficulties concerning deduction with variables, which need not be here considered, as well as from the necessity of a notation for the case where the "range" of a variable is less than the whole of one asserted proposition.*

We may now proceed to the formal development, which will make the above points clearer.

★7·0. *Primitive idea:* "$(x) \,.\, (C \between x)$" means "the truth of $(C \between x)$ for all values of x."

·01 $(x, y) \,.\, (C \between x, y) \,.=:\, (x) : (y) \,.\, (C \between x, y)$ Df.

i.e., "$(x, y) \,.\, (C \between x, y)$" means "the truth for all values of x, of the truth, for all values of y, of $(C \between x, y)$"; *i.e.,* practically, "the truth of $(C \between x, y)$ for all values of x and y". A similar form will be used, when required, for more than two variables.

·02 $(A \between x) \,.\supset_x.\, (B \between x) :=: (x) : (A \between x) \,.\supset.\, (B \between x)$ Df.

Peano employs exclusively the notation $(A \between x) \,.\supset_x.\, (B \between x)$, which expresses what I call a "formal implication". He has no means of expressing $(x) \,.\, (C \between x)$ except when $(C \between x)$ is of the form $(A \between x) \,.\supset.\, (B \between x)$. In this respect Frege is preferable. But the above subscript notation is shorter, and often convenient, where implications are concerned; it may therefore be retained with advantage for use on suitable occasions.

·03 $(A \between x, y) \,.\supset_{x, y}.\, (B \between x, y) :=: (x, y) : (A \between x, y) \,.\supset.\, (B \between x, y)$ Df.

A similar form may be used for more than two variables.

★7·04 $\sim (x) \,.\, (C \between x) \,.=.\, \sim \{ (x) \,.\, (C \between x) \}$ Df.

This definition serves only for the avoidance of brackets. A similar notation will be used for several variables.

·1 $\vdash : (x) \,.\, (C \between x) \,.\supset.\, (C \between y)$ Pp.

This proposition may be read: "What is true of all is true of any."

·11 What is true of any is true of all. Pp.

*The possibility of taking "all values" of a variable, and the nature of the total set of values concerned, are subject to some rather complicated restrictions, not here considered. This results from the paradox of the Cretan who said that all Cretans are liars, from Burali-Forti's paradox, ["Una questione sui numeri transfiniti", Rendiconti del Circolo Matematico di Palermo, Tomo XI, 1897], from the paradox considered in my "Principles of Mathematics", chap. X, and from other analogous paradoxes. As an example of the limitations, "all values of C" in a statement about $(C \between x)$ would be meaningless.

This proposition is the converse of ★7·1. It cannot be symbolized without inventing a new symbol for the purpose, which seems not worth while, as this symbol would not be afterwards required. It might be supposed at first that what we mean could be expressed by

$$\text{“}\vdash : (C \between y) \,.\supset.\, (x) \,.\, (C \between x)\text{”}.$$

But this would mean: "$(C \between y)$ implies that $(C \between x)$ is true for all values of x, where y may be anything", which is not what we mean, and is not in general true. What we mean is: "If $(C \between y)$ is true whatever y may be, then $(C \between x)$ is true for all values of x". When $(C \between y)$ is asserted, we have

$$\vdash . (C \between y) \,.\supset \vdash .\, (x) \,.\, (C \between x)$$

but we cannot adopt this as our formula, because we must only put the assertion-sign before what really is true, and here $(C \between y)$ is to be only a hypothesis, which may or may not be true.

★7·12 $\vdash :.\, (x) \,.\, p \supset (C \between x) \,.\supset : p \,.\supset.\, (x) \,.\, (C \between x)$ Pp.

i. e., "If it is true, for all values of x, that p implies $(C \between x)$, then p implies that $(C \between x)$ is true for all values of x".

We now have all the apparatus necessary for deductions such as occur in the ordinary syllogism, *e. g.*, for proving

"If all Greeks are men, and all men are mortals, then all Greeks are mortals".

The symbolic statement of "all Greeks are men" is

"$(x) : x$ is a Greek. $\supset . \, x$ is a man".

Thus the general form of Barbara is

$$\vdash :.\, (x) \,.\, (A \between x) \supset (B \between x) : (x) \,.\, (B \between x) \supset (C \between x) : \supset : (x) : (A \between x) \supset (C \between x) .$$

This can be deduced from the above premisses; but for the present I shall confine myself to cases in which there are no undetermined complexes, such as $(A \between x)$, $(B \between x)$, $(C \between x)$. The additional apparatus now available leads to certain further propositions, of great importance, in the theory of implication, and it is only these that concern us now.

★7·13. If a is any constant, $\{(x) \,.\, (C \between x)\}$ implies $(C \between a)$.

This is not a primitive proposition, but a consequence of ★2·2 and ★7·1. On the impossibility of expressing it symbolically, the same remarks apply as were made concerning ★2·2.

$\star$7·14. $\vdash : (x) . (C \between x) . \supset . (C \between (A \between x))$ [$\star$7·1·$\star$2·3]

Although $\star$2·3 could not be symbolized with our symbols, $\star$7·14 can be, because we are now able to indicate the limited *range* of the x.

·15 $\vdash :. (x) . p \supset (C \between x) . \equiv : p . \supset . (x) . (C \between x)$

Dem.

$\vdash . \star 7{\cdot}12 . \quad \supset \vdash :. (x) . p \supset (C \between x) . \supset : p . \supset . (x) . (C \between x)$ (1)

$\vdash . \star 3{\cdot}12 . \quad \supset \vdash :: (x) . (C \between x) . \supset . (C \between y) : \supset :. p . \supset . (x) . (C \between x) :$
$\supset : p . \supset . (C \between y)$ (2)

$\vdash . (2) . \star 7{\cdot}1 . \supset \vdash :. p . \supset . (x) . (C \between x) : \supset : p . \supset . (C \between y)$

[$\star$7·11] $\supset \vdash :. (x) :. p . \supset . (x) . (C \between x) : \supset : p . \supset . (C \between x) :.$

[$\star$7·12] $\supset \vdash :. p . \supset . (x) . (C \between x) : \supset : (x) : p . \supset . (C \between x)$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash .$ Prop.

Note that in the last two lines of the above proof two different x's occur. In the last line but one, the x which has come down from the previous line has $(C \between x)$ for the whole of its range; the other x replaces the y of the previous line, in accordance with $\star$7·11, and has the whole proposition for its range. The practice of using two different x's in one proposition is not in general commendable, for in *practice* it is liable to cause confusion; but the system of dots makes confusions *theoretically* impossible, *i. e.*, the meaning is never ambiguous, tho' it may be a little difficult to discover. I shall, however, in future avoid using the same letter for different apparent variables, except where doing so conduces to clearness. Different *real* variables must never be represented by the same letter.

The effect of the above primitive propositions may be stated in certain rules.

1. Given any *asserted* proposition containing a real variable x, we may put an x in brackets at the beginning, with a sufficient number of dots to cover the whole proposition. *E.g.*, given $\vdash : x \equiv y . y \equiv z . \supset . x \equiv z$, we may infer

$$\vdash : (x) : x \equiv y . y \equiv z . \supset . x \equiv z$$

2. Given any proposition beginning by an x in brackets, followed by a sufficient number of dots to cover the whole proposition, we may infer the proposition which results from omitting the x in brackets.

3. Given an implication which begins with an x in brackets, followed by a sufficient number of dots to cover the whole proposition, then, if the hypothesis does not contain x, we may remove the x in brackets to the beginning of the apodosis.

4. Given an *asserted* implication, containing x in the apodosis, but not in the hypothesis, we may insert at the beginning of the apodosis an x in brackets, followed by a sufficient number of dots to cover the whole of the apodosis.

$\star 7{\cdot}2$ $\vdash :. p . \equiv : (s) . s \supset p$

This proposition states that p implies and is implied by the proposition "for all values of s, s implies p"; *i. e.*, p implies and is implied by the proposition "everything implies p".

Dem.

$\vdash . \text{Simp}. \quad \supset \vdash :. p . \supset . s \supset p$ (1)

$\vdash . (1) . \star 7{\cdot}11 . \quad \supset \vdash :. (s) : p . \supset . s \supset p$ (2)

$\vdash . \star 7{\cdot}12 . \quad \supset \vdash :: (s) : p . \supset . s \supset p : \supset :. p . \supset : (s) . s \supset p$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. p . \supset : (s) . s \supset p$ (4)

$\vdash . \star 7{\cdot}14 . \quad \supset \vdash :. (s) . s \supset p . \supset . \sim p \supset p .$

$[\star 3{\cdot}41] \quad \supset . p$ (5)

$\vdash . (4) . (5) . \quad \supset \vdash . \text{Prop.}$

$\star 7{\cdot}21$ $\vdash : \sim p . \equiv . (s) . p \supset s$

i. e., the falsehood of p implies and is implied by the proposition "for all values of s, p implies s".

Dem.

$\vdash . \star 3{\cdot}31 . \quad \supset \vdash : \sim p . \supset . p \supset s$ (1)

$\vdash . (1) . \star 7{\cdot}11 . \quad \supset \vdash : (s) : \sim p . \supset . p \supset s :$

$[\star 7{\cdot}12] \quad \supset \vdash : \sim p . \supset . (s) . p \supset s$ (2)

$\vdash . \star 7{\cdot}14 . \quad \supset \vdash : (s) . p \supset s . \supset . p \supset \sim p .$

$[\star 2{\cdot}91] \quad \supset . \sim p$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash . \text{Prop.}$

Note. $\star 7{\cdot}11$ and $\star 7{\cdot}12$ are constantly being used together. They enable us, from any *asserted* proposition of the form

$\vdash : (C \wr x) . \supset . (A \wr x, y)$ [x and y being undetermined]

to infer $\vdash : (C \wr x) . \supset . (y) . (A \wr x, y)$.

In order to apply $\star 7{\cdot}11$, it is necessary, in such a case, to take the *whole* proposition $(C \between x) . \supset . (A \between x, y)$, not only $(A \between x, y)$. For $(A \between x, y)$ is unasserted, and therefore does not necessarily imply $(y) . (A \between x, y)$. For this reason it would be a fallacy to write

$$\begin{aligned} &\vdash : (C \between x) . \quad \supset . (A \between x, y) . \\ &[\star 7{\cdot}11{\cdot}12] \supset . (y) . (A \between x, y) \end{aligned}$$

although our conclusion, namely $(C \between x) . \supset . (y) . (A \between x, y)$, would be true.

$\star 7{\cdot}22$ $\vdash :. \sim p . \equiv : p . \supset . (s) . s$

Dem.

$\vdash . \star 7{\cdot}15 . \quad \supset \vdash :. (s) . p \supset s . \equiv : p . \supset . (s) . s$ (1)

$\vdash . (1) . \star 7{\cdot}21 . \supset \vdash .$ Prop.

Thus p is false when, and only when, p implies the proposition "everything is true".

$\cdot 23$ $\vdash :. \sim p . \equiv : p . \equiv . (s) . s$

Dem.

$\vdash . \star 7{\cdot}14 . \quad \supset \vdash : (s) . s . \supset . p :$

$[\star 5{\cdot}73] \quad \supset \vdash :. p . \supset . (s) . s : \equiv : p . \equiv . (s) . s$ (1)

$\vdash . (1) . \star 7{\cdot}22 . \supset \vdash .$ Prop.

$\cdot 24$ $\vdash :. p . \equiv : p . \equiv . \sim (s) . s$

Dem.

$\vdash . \star 7{\cdot}23 \frac{\sim p}{p} . \supset \vdash :. p . \equiv : \sim p . \equiv . (s) . s :$

$[\star 5{\cdot}12] \quad \equiv : p . \equiv . \sim (s) . s :. \supset \vdash .$ Prop.

Thus both p and not-p may be replaced, in implicational formulæ, by equivalences. The above propositions $\star 7{\cdot}23{\cdot}24$ may be read as follows: To say that p is false is equivalent to saying that p is equivalent to "everything is true"; and to say that p is true is equivalent to saying that p is equivalent to "not everything is true".

★7·25 $\vdash . \sim (s) . s$

i.e., "not everything is true".*

Dem.

$\vdash . \text{Id.} \supset \vdash : (s) . s . \supset . (s) . s :$

$\left[★7{\cdot}22 \frac{(s) . s}{p} \right] \supset \vdash : \sim (s) . s : \supset \vdash . \text{Prop.}$

·26 $\vdash . \sim \{ (s) . s . \equiv . \sim (s) . s \}$ $\left[★6{\cdot}19 \frac{(s) . s}{p} \right]$

Note that ★2·2 is used here, not ★2·3.

·27 $\vdash : (s) . s . \supset . p$ [★7·1]

·28 $\vdash : p . \supset . \sim (s) . s$ [★7·25 . Simp.]

Note. $(s) . s$ and $\sim (s) . s$ are absolute constants, which play a part analogous to that of 0 and 1 in ordinary algebra. The analogy is illustrated by ★7·3·31·33 below; but ★7·32 makes $\sim (s) . s$ analogous rather to ∞ than to 1.

★7·3 $\vdash :. \ p . \equiv : p . \vee . (s) . s$ [★5·74 . ★7·25]

·31 $\vdash :. \ p . \equiv : p : \sim (s) . s$ [★5·73 . ★7·25]

·32 $\vdash :. \sim (s) . s . \equiv : p . \vee . \sim (s) . s$ [★5·72 . ★7·28]

·33 $\vdash :. \ (s) . s . \equiv : p : (s) . s$ [★5·71 . ★7·27]

·34 $\vdash : \ p . \sim p . \equiv . (s) . s$ $\left[★4{\cdot}24 . ★7{\cdot}23 \frac{p . \sim p}{p} \right]$

·35 $\vdash : \ p \vee \sim p . \equiv . \sim (s) . s$ $\left[★4{\cdot}25 . ★7{\cdot}24 \frac{p \vee \sim p}{p} \right]$

★7·4 $\vdash :. p . \equiv : p \supset s . \supset_s . s$

Dem.

$\vdash . ★3{\cdot}1 . \quad \supset \vdash :. p . \supset : p \supset s . \supset . s :.$

$[★7{\cdot}11] \quad \supset \vdash :. (s) :. p . \supset : p \supset s . \supset . s :.$

$[★7{\cdot}12] \quad \supset \vdash :. p . \supset : p \supset s . \supset_s . s$ (1)

$\vdash . ★7{\cdot}14 . \quad \supset \vdash :. p \supset s . \supset_s . s : \supset : p \supset p . \supset . p :$

$\left[★3{\cdot}51 \frac{p}{q} \right] \quad \supset : p$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop.}$

The above proposition may be stated in words as follows: p is true when and only when everything that p implies is true.

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* "Everything is not true" would be "$(s) . \sim s$".

★7·41 $\vdash :: p . q . \equiv :. p . \supset . q \supset s : \supset_s . s$

Dem.

$\vdash$. Imp. $\supset \vdash :: p . \supset . q \supset s : \supset : p . q . \supset . s ::$

[Comm] $\supset \vdash :: p . q . \supset :. p . \supset . q \supset s : \supset . s ::$

[★7·11] $\supset \vdash :: (s) :: p . q . \supset :. p . \supset . q \supset s : \supset . s ::$

[★7·12] $\supset \vdash :: p . q . \supset :: p . \supset . q \supset s : \supset_s . s$ (1)

$\vdash$. ★7·14 . $\supset \vdash ::. p . \supset . q \supset s : \supset_s . s : . \supset :: p . \supset . q . \supset . p . q : . \supset . p . q ::$

[★4·2] $\supset :: p . q$ (2)

$\vdash$. (1) . (2) . $\supset \vdash$. Prop.

★7·42 $\vdash :: p . q . \equiv :. p \supset s . \vee . q \supset s : \supset_s . s$

Dem.

$\vdash$. ★5·79 . $\supset \vdash :: p \supset s . \vee . q \supset s : \supset : p . q . \supset . s ::$

[Comm] $\supset \vdash :: p . q . \supset :. p \supset s . \vee . q \supset s : \supset . s ::$

[★7·11·12] $\supset \vdash :: p . q . \supset :. p \supset s . \vee . q \supset s : \supset_s . s$ (1)

$\vdash$. ★7·14 . $\supset \vdash :: p \supset s . \vee . q \supset s : \supset_s . s : . \supset$

$:. p . \supset . p . q : \vee : q . \supset . p . q : \supset . p . q$ (2)

$\vdash$. ★6·54 . $\supset \vdash : p . \supset . p . q : \vee : q . \supset . p . q$ (3)

$\vdash$. (2) . (3) . $\supset \vdash :: p \supset s . \vee . q \supset s : \supset_s . s : . \supset . p . q$ (4)

$\vdash$. (1) . (4) . $\supset \vdash$. Prop.

·43 $\vdash :. p . q . r . \equiv :. p \supset s . \vee . q \supset s . \vee . r \supset s : \supset_s . s$ [★7·42]

·5 $\vdash :. p \vee q . \equiv : p \supset s . q \supset s . \supset_s . s$ [★5·77]

The proof proceeds on the same lines as the proof of ★7·42.

·51 $\vdash :. p \vee q \vee r . \equiv : p \supset s . q \supset s . r \supset s . \supset_s . s$ [★7·5]

Note. Instead of taking negation as a primitive idea, as was done in ★1, it is possible to regard the property stated in ★7·22 as giving the *definition* of negation, *i. e.*, we may put

$$\sim p . = : p . \supset . (s) . s \qquad \text{Df.}$$

This requires that the primitive idea $(x) . (C \between x)$ should be introduced in ★2, and not postponed till ★7. Instead of the definitions in ★4·1·11 it is now simpler to take ★7·41 and ★4·5 as the definitions of $p\,.\,q$ and $p \vee q$ respectively, putting

$$p\,.\,q\,.=:.\,p\,.\supset.\,q\supset s:\supset_s.\,s \qquad \text{Df.}$$

$$p \vee q\,.=:p\supset q\,.\supset.\,q \qquad \text{Df.}$$

Instead of ★2·9·91·92, we can now substitute a single primitive proposition, namely

$$\vdash:.\,p\supset q\,.\supset.\,p:\supset.\,p \qquad \text{Pp.*}$$

(which is ★3·51 in the above).

This method is more artificial and much more difficult than the method adopted above, but these disadvantages are more than outweighed by the fact that we have one less primitive idea and two less primitive propositions.

My reason for not adopting this method is not its artificiality or its difficulty, but the fact that it never enables us to know that anything whatever is *false*. It enables us to prove the *truth* of whatever can be proved true by the method adopted above, and it does not enable us to prove the truth of anything which in fact is false. It even enables us to prove, concerning all the propositions which can be proved false by the above method, that, if they are true, then everything is true; but if any man is so credulous as to believe that everything is true, then the method in question is powerless to refute him. For example, we get the law of contradiction in the form

$$\vdash:p\,.\sim p\,.\supset.\,(s)\,.\,s;$$

but this does not show that $p\,.\sim p$ is false, unless we assume that $(s)\,.\,s$ is false. Now in the system considered, *falsehood* is not among the ideas that occur in our apparatus; hence we cannot assume that $(s)\,.\,s$ is false without introducing a new primitive idea. But when once we have introduced this new idea, it is economical to make all possible use of it; and this leads us to the method adopted above.

The above propositions give what is most important in the theory of *material* implication, *i. e.*, of propositions of the form

$$p\supset q\,.$$

* This is the principle called "reduction" in my "Principles of Mathematics", §18, (10), p. 17.

The subject which comes next in logical order is the theory of *formal* implication, *i. e.*, of propositions of the form

$$(x) : (A \between x) . \supset . (B \between x) .$$

These have been touched on in ★7, but are not there considered on their own account. It is only after these have been considered that we can advance to the theory of classes, and thence to the theory of relations. The necessity of avoiding the contradiction of Burali–Forti concerning the greatest ordinal, and a class of contradictions of which the simplest is discussed in my "Principles of Mathematics", Chap. X, necessitate certain distinctions which render the subsequent development somewhat less simple than it would otherwise be. By distinguishing different *types* of variables, and confining the notion of "all values of the variable" to all values within the type concerned, these contradictions can, I believe, be all satisfactorily avoided.