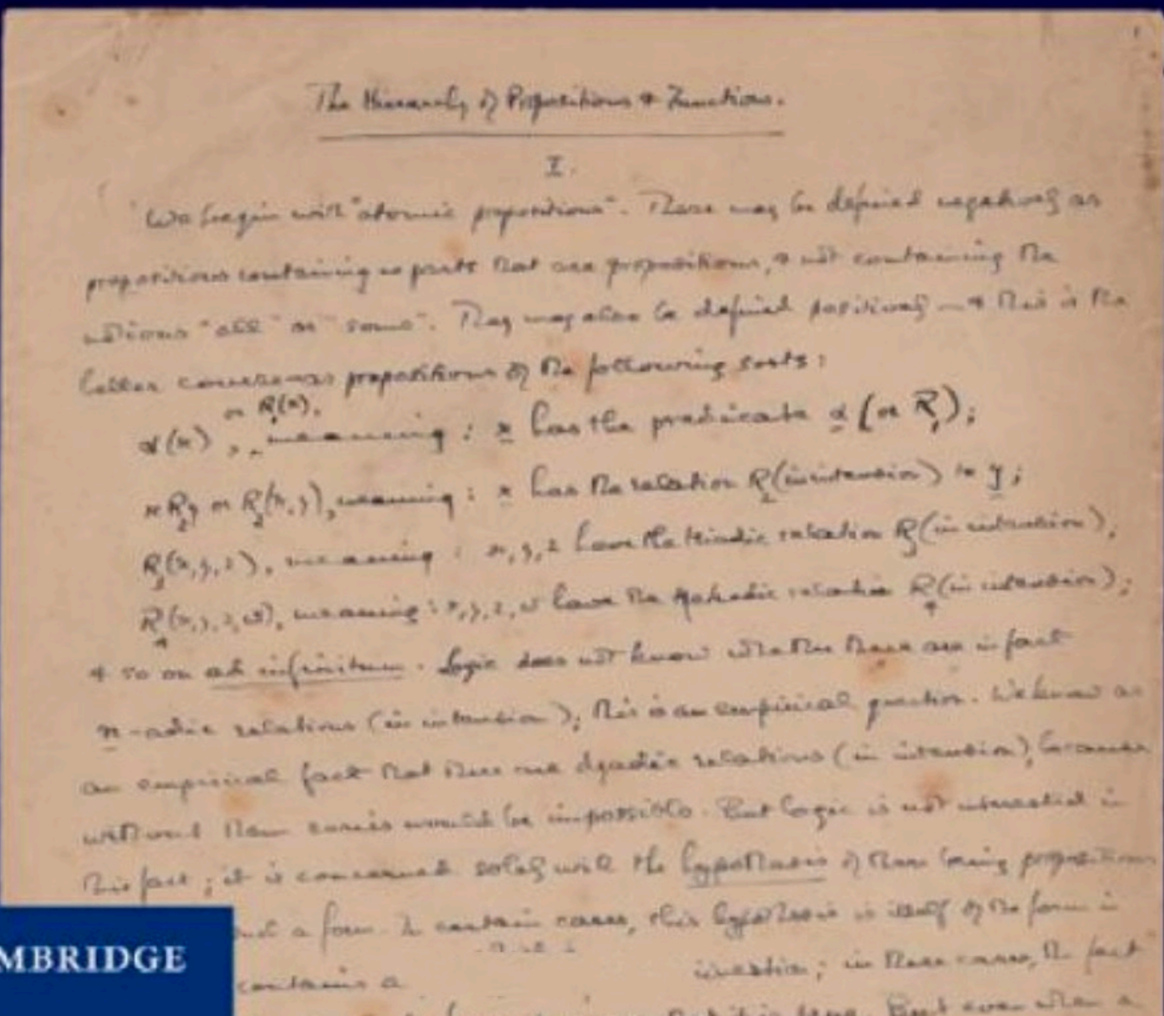
# THE EVOLUTION OF PRINCIPIA MATHEMATICA

## Bertrand Russell's Manuscripts and Notes for the Second Edition

BERNARD LINSKY

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# THE EVOLUTION OF PRINCIPIA MATHEMATICA

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Originally published in 1910, *Principia Mathematica* led to the development of mathematical logic and computers and thus to information sciences. It became a model for modern analytic philosophy and remains an important work.

In the late 1960s, The Bertrand Russell Archives at McMaster University in Canada obtained Russell's papers, letters and library. These archives contained the manuscripts for the new Introduction and three appendices that Russell added to the second edition in 1925. Also included was another manuscript, "The Hierarchy of Propositions and Functions", which was divided up and reused to create the final changes for the second edition. These documents provide fascinating insight, including Russell's attempts to work out the theorems in the flawed Appendix B, "On Induction". An extensive introduction describes the stages of the manuscript material on the way to print, and analyzes the proposed changes in the context of the development of symbolic logic after 1910.

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# Contents

# 1

# Introduction

The second edition of Alfred N. Whitehead and Bertrand Russell's *Principia Mathematica* was published by Cambridge University Press in three volumes between 1925 and 1927. It consists of a reprint of the first edition, which appeared between 1910 and 1913, with the addition of a new Introduction and three Appendices (A, B, and C) written by Russell alone, and a List of Definitions.[1] The new material takes up only 66 pages, yet it proposed radical changes to the system of *Principia Mathematica*, some of which require fundamental rethinking of the nature of logic.

What Russell oddly introduces as the "... most definite improvement resulting from work in mathematical logic during the past fourteen years ..." is the proposal to replace the familiar fundamental logical connectives "or" and "not" with the single "Sheffer stroke", "not-both." This technically trivial change is in fact carried out in a rigorous fashion and does not require any rewriting of the body of *Principia Mathematica* in order to be properly implemented. A second and genuinely fundamental change is the adoption of "extensionality" in the second edition. This requires that all propositional connectives are to be truth-functional, and that co-extensive propositional functions, true of the same arguments, are identified. Russell characterizes this doctrine as the result of two theses, that:

> ... functions of propositions are always truth-functions, and that a function can only occur in a proposition through its values. (*PM*, p.$xiv$)

What this change amounts to, and how it fits with the details of the various traces of the non-extensional system of the first edition that are unaltered, such as the definition of identity, will be discussed below. The third major change in the second edition is the proposal to abandon the axiom of reducibility in the development of mathematics that is the project of *Principia Mathematica*. The

[1] The new material was all added to the first volume as pages numbered $xiii$ to $xlvi$, at the beginning, following the Preface from the first edition, and the appendices at the end, pages 635 to 666 at the end of the volume. For complete bibliographical information on the editions of *PM*, see Blackwell & Ruja (1994, pp.19–25).

*logical types* of propositional functions, the subject matter of the higher order logic of *PM*, are distinguished in what came to be called the "ramified" theory of types. Not only is there the distinction between individuals, functions of individuals, and functions of such functions, and so on, which is familiar from Frege's logic, but there is a further division of *orders* of functions required in order to observe the "vicious circle principle" that Russell decided was the key to resolving the paradoxes of both set theory and logic. To use Russell's example, the property of "having all the qualities that make a great general" is represented by a propositional function of a higher order than the lowest order, or "predicative" functions in terms of which it is defined. The axiom of reducibility asserts that there is a predicative function which is nevertheless co-extensive with any given function of a higher order. Russell adopted the axiom of reducibility reluctantly in the first edition of *PM* and it was the target of many criticisms by contemporary logicians almost immediately after 1910. In the second edition, in Appendix B, Russell proposes a proof of the principle of induction in the new system of *Principia Mathematica*, which does not rely on the axiom of reducibility. This proof was criticized by Kurt Gödel in his famous essay "Russell's mathematical logic" (1944), and the project of deriving the principle of induction in certain systems of extensional ramified theory of types without an axiom of reducibility was shown to be impossible. How Russell could have made the elementary mistake in Appendix B that he does, and what was the precise nature of the revised theory of types in which the proof was to proceed, however, have been unresolved puzzles. All three of these significant changes in the second edition have thus, in various ways, been the source of uncertainty about what exactly Russell was intending. The publication of this manuscript material provides some additional clues to resolve these technical questions.

The Bertrand Russell Archives at McMaster University in Canada possess the original manuscripts of all that new material, as well as an additional 53 unused or revised leaves left from a first draft, and 62 leaves of notes in symbolic notation consisting of rejected attempts at the formal results in Appendix B. Only two and one half pages of the manuscript material for the first edition have survived, and no material so close to a final version.[2] The last six chapters of this book contain transcriptions of all of that material using the fonts of the word processing system LaTeX. The first seven chapters include a discussion of the history of the writing of the second edition, and present some additional, previously unpublished material from the Bertrand Russell Archives and the Carnap Archives. Further chapters summarize the notation and content of the technical portions of the first edition that are

[2] See Linsky & Blackwell (2006).

relevant to the new material. The Introduction to the second edition and the three appendices are summarized, with special attention to the improvements that Russell thought should be made to the first edition. A significant technical issue arises from the criticisms made by Kurt Gödel and John Myhill against the argument in Appendix B. The publication of the leaves of notes for Appendix B helps to resolve those technical questions, although some serious mysteries remain for others to resolve with the careful study of this material.

This work has two purposes: specifically, to make this archival material available in print, and more generally, by that means, to restore the reputation of the second edition of *Principia Mathematica* as a serious contribution to logic. Soon after its publication in 1927 the second edition was superseded by other developments in the field of mathematical logic. Although the new material in the second edition, with the exception of Appendix B, is reprinted in the most widely available version of the work, the paperback *Principia Mathematica to ∗56* from 1962, that new material for the second edition is not studied carefully, and is indeed seen as an unfortunately unsuccessful attempt of Russell's to keep up with a subject that had passed him by. Study of the published and archival material, however, shows that the second edition reveals deep issues about the move from the intensional logic of propositional functions in the "ramified theory of types" of the first edition, to the altered theory of types in an extensional logic that Russell saw as an improvement. The failure of his technical proposal about the derivation of the principle of mathematical induction in Appendix B, and its criticism by Gödel, reveal the nature of the move from ramified type theory to set theory formulated in first order, extensional logic that became the new preferred foundation for mathematics. The second edition of *Principia Mathematica* marks the end of logicism as the leading program in the foundations of mathematics, and the rise of the mathematical logic of Gödel and Tarski as its replacement, and so it is important for understanding the history of logicism.

*Principia Mathematica* is arguably the most important work in the subject of symbolic logic which emerged at the beginning of the twentieth century. In its three volumes there are laid out a cumulative series of definitions and formal proofs by which much of the elementary portions of arithmetic, set theory and the theory of real numbers is deduced in a rigorous fashion. Logic conducted in this way soon became a branch of mathematics, to be called "mathematical logic" and so *Principia Mathematica* suffered the fate of many important works in the history of mathematics. Improvements were made in notation and definitions, and better and more rigorous systems of proof were developed. By the 1930s, Kurt Gödel and others had isolated interest in a fragment of the system, the "first order", extensional quantificational logic which is now the subject of countless textbooks and taught as a part of the undergraduate curriculum in philosophy, and as part of elementary courses in computing science in universities around the world.[3]

Gödel's work realized the notion of formal "meta-logic" that had been first proposed by Hilbert and others, namely the creation of a mathematical theory of sentences and symbolic formulas which allowed the derivation of theorems about the strengths and limitations of symbolic logic. He proved that standard systems of first order, extensional logic were complete, in the sense that every "valid" sentence, in a precisely definable sense, is in fact provable, with a precise formal definition of the notion of proof. He further refined the notion of a mechanically verifiable, rigorous, proof, and so led the way to the isolation of the notion of computable algorithm that became the theory that led to the invention of programmable computers. Using the new notion of a provable sentence, he showed that no first order theory of mathematics could be complete in the sense of being capable of proving all truths expressible in that theory. His model for such a theory is presented in the title of his famous paper, "On formally undecidable propositions of *Principia Mathematica* and related systems I" (Gödel, 1931, p.596). The project of *PM* of reducing mathematics to logic thus seemed impossible. Some other approach to mathematics, beginning again with precise, first order axiomatic formal theories of arithmetic and set theory, seemed to be the proper way to investigate the foundations of mathematics. Even in the classic history of logic, *The Development of Logic* by Kneale & Kneale (1962), this result was still considered to show that the logicist project of *Principia Mathematica* was necessarily a failure.[4]

Also in the 1930s Alfred Tarski developed the foundations of the new field of formal semantics, most classically with his "semantic conception of truth", which enabled a mathematical investigation of the definability of the concept of truth.[5] With the subsequent development of a precise concept of mathematical models, the notion of truth in a model, and so of logical consequence via truth in certain models, and the definition of validity as truth in all models, the way was open for even elementary logic to turn from the construction of proofs to the study of classes of sentences in terms of the models in which they are true. With the notion of a tautology of propositional logic clearly defined, the details of the actual proofs of theorems, which is the subject matter of $*1$ to $*9$ of *Principia Mathematica*, was seen as a merely tedious exercise. The rise of set theory, as the language in which the theory of models and the notion of truth were developed, turned out to provide a much simpler development of the elementary parts of mathematics that had taken three full volumes in *PM*.

3 See Goldfarb (1979), which traces the rise of extensional first order logic, with its model theoretic semantics.
4 See the discussion of the view that Whitehead and Russell's logicism was a failure in Linsky & Zalta (2006, p.6).
5 See Tarski (1936).

As a work in mathematics, *Principia Mathematica* soon became obsolete. Symbolic logic is also a field of philosophy, and so the study of major works from the past has a special role unlike that of the study of the history of mathematics. Certainly there were criticisms of *Principia Mathematica* from the point of view of the philosophy of logic. The intensional nature of the logic of *PM*, when it is seen as based on propositions and propositional functions rather than sentences and predicates, and the potential distinction between co-extensive functions, was alien to the extensional account of logic that grew to supplant *PM*. To W.V. Quine, otherwise a champion of *PM*, this intensionality was seen to be a result of an unfortunate, elementary, confusion of use and mention:[6]

> It is regretable that Frege's own scrupulous observance of this distinction between an expression and its name, between use and mention, was so little heeded by Whitehead, Russell, and their critics. (Quine, 1951a, p.142)

The charge of "use–mention" confusions in *PM* goes back to Frege's first reactions upon reading the work. In a letter to Jourdain of 28 January 1914, Frege had complained that:

> I find it very difficult to read Russell's *Principia*; I stumble over almost every sentence. (Frege, 1980, p.81)

A draft of the letter explains the difficulty, "I never know for sure whether he is speaking of a sign or of its content." (Frege, 1980, p.78). Following the criticisms in Wittgenstein's *Tractatus Logico-Philosophicus*, *Principia Mathematica* took on a role in analytic philosophy of logic as a starting point from which progress was made by first correcting its many errors, errors which in many cases were avoided by Frege earlier. In that way *Principia Mathematica* is often seen as a wrong turn in a progression from Frege's *Grundgesetze der Arithmetik* through the *Tractatus* and on into the philosophy and logic of Carnap, Gödel and Tarski.

For all its technical crudities to our eyes, the two editions of *Principia Mathematica* represent the development of certain important ideas in logic. The notions of type theory, extensionality, truth-functionality, the definability of identity and the primitive notions of set theory, and most importantly, the idea of the reduction of mathematics to logic, all change between the two editions of *PM*. The fundamental idea of logic as a theory of intensional propositional functions, instead of a formal system with a certain class of allowable models as interpretations, was abandoned as backward. Studying the history of *Principia Mathematica* reveals important knowledge about the history and philosophy of logic in the first part of

[6] The back cover of the paperback 1962 edition of *Principia Mathematica to ∗56* includes the quote from Quine: "This is the book that has meant the most to me."

the twentieth century. It is even possible to find in the unsuccessful attempts in the new work a defense of the ideas behind the original ramified theory of types as a viable intensional logic, worthy of study today.

My first task, however, is to motivate interest in the second edition of *PM* in the face of the generally dismal assessment, including that of defenders of the original edition. In his book *The Search for Mathematical Roots: 1870–1940*, in which *Principia Mathematica* is the centerpiece of the story, Ivor Grattan-Guinness dismisses the second edition as "hardly a philosophical advance upon the first." (Grattan-Guinness, 2000, p. 443). Ray Monk (2000), in the second volume of his biography, *Bertrand Russell: The Ghost of Madness: 1921–1970* condemns the work as an inadequate attempt to appreciate Wittgenstein's thought which, at the same time, betrayed Whitehead's co-authorship of the first edition:

> First, Russell had not read most of the recent technical literature on the subject, and had neither the time nor the inclination to master it. Second, a complete acceptance of Wittgenstein's work would require, not just changes to the system of *Principia*, but its complete abandonment. Third, as *Principia Mathematica* was co-written with Whitehead, this new edition would also have to be published under both their names, and Whitehead was deeply unsympathetic to Wittgenstein's work and thus to the general lines on which Russell sought to 'improve' their joint undertaking.
>
> In the face of these difficulties, what Russell produced was a piece of work unsatisfactory in almost every respect, one that failed to realise any of its aims, made no significant technical advances to the subject and was disliked by both Wittgenstein and Whitehead. (Monk, 2000, p.44)

Russell himself only devotes part of one sentence in his *Autobiography* to his work on the second edition.[7] Very little has been written about the second edition, and most of it has Monk's critical tone. The evidence presented in what follows, from the unused notes, and from a careful rereading of the published version, suggests that a more charitable assessment is more accurate.

Chapter 2 presents the history of the writing of the new material for the second edition. The work was all Russell's, and Whitehead seems to have been content with both the process and the results. Russell began the work in late 1923, with what was intended to be a single long addition to the original Introduction. That first manuscript, identified here as "Hierarchy of propositions and functions", was then split up into the final configuration of a separate Introduction to the second edition and the three appendices. Frank Ramsey read and made some comments on that first manuscript in early 1924, and then read the

[7] The whole sentence is: "There was a new edition of *Principia Mathematica* in 1925, to which I made various additions; and in 1927 I published *The Analysis of Matter* (*AMa*), which is in some sense a companion volume to *The Analysis of Mind* (*AMi*), begun in prison and published in 1921." (*Auto.*, p.214). There is, however, a longer discussion of the second edition in *My Philosophical Development* (*MPD*, pp.120–23).

proofs of portions of the final version in the fall of that year, but all the rest of the work was Russell's.

Chapter 3 describes the development of Russell's ideas about logic since the first edition, both among his own intimates around Cambridge, and in the growing literature in the field. At Cambridge after his dismissal from Trinity College, Russell had a circle of associates working on logic; Henry Sheffer, Jean Nicod and Dorothy Wrinch all wrote on logic and discussed it with Russell. Beginning in 1911, just as the first edition was in press, Ludwig Wittgenstein arrived in Cambridge and remained close to Russell until his departure for Norway in 1913 and his subsequent return to Austria when the war began. After the war Russell assisted Wittgenstein with arranging the publication of the *Tractatus Logico-Philosophicus*, to which Russell contributed an introduction. The evidence suggests that Russell developed his ideas between the editions of *PM* over the whole period, entering into intense discussions with Wittgenstein, but also others, and then presenting his evolving views in the Philosophy of Logical Atomism lectures (PLA) in 1918, the *Introduction to Mathematical Philosophy*, 1919, and then his introduction to Wittgenstein's *Tractatus* in 1921. The actual composition of the second edition, during 1923 and 1924, found Russell working alone, with Frank Ramsey reporting to Wittgenstein about what he was reading of it.

Chapter 3 is organized around a discussion of the list of "contributions to mathematical logic since the publication of the first edition of *Principia Mathematica*" which concludes the new introduction. An examination of each of these works in turn contradicts Monk's claim that Russell had neither read the technical literature in his subject nor mastered it. While Russell was aware of developments in mathematical logic, they were developing logic in radically different directions, and so he took from that literature just what was relevant to his alterations to the system of the first edition. The picture that emerges is of new ideas and schools of thought, the Intuitionism of Brouwer, the work of the Hilbert Program, Wittgenstein's *Tractatus*, and others, emerging in reaction to *Principia Mathematica*. Russell was aware of, but did not follow in, these other directions.

Chapter 4 provides a review of the symbolism and content of the first edition of *Principia Mathematica*, in order of their presentation by "numbers", indicated with the asterisk, as in "$*14$" and "$*20$", so that the chapter can be used as a glossary of definitions and theorems for reference in what follows. All of the symbolism and references in the new material for the second edition should be presented in some form earlier in this text, although the reader will want to have a copy of at least the paperback edition *Principia Mathematica to* $*56$ ready to hand.

Chapter 5 describes the proposed changes to *PM* that constitute the system of the second edition. These changes are listed in the Introduction to the second edition, which describes the role of the Sheffer stroke as a preferred primitive connective, and the subsequent theorems needed for that revision, proved in Appendix A. The adoption of extensionality, and accompanying abandonment of the axiom of reducibility, are the main alterations to the system of the first edition. A revision of the theory of types in the first edition is suggested by the account of types in the introductory material, and will be the subject of careful interpretation in the texts.

Chapter 6 studies in detail the content of Appendix B, On induction. The appendix consists of a technical proof that even without the axiom of reducibility, a limited form of the principle of mathematical induction can be derived. This proof was found to contain a technical flaw by Gödel (1944), "Russell's mathematical logic". Later John Myhill (1974) followed this up with a proof showing that the project of Appendix B is impossible in principle. The manuscript material presented here allows for a better understanding of these issues. The "mistake" can be traced through the earlier drafts of the appendix, and a sense of the intended project of the appendix can be seen in the manuscript notes "Amended list of propositions." It appears that the published material in Appendix B was not slipshod or casual, as Myhill suggests, but rather the result of intense efforts, although in the end, the details of the proposed alterations to the logic of the first edition are not clear. A puzzle remains for the reader to solve by careful study of the notes published here.

Chapter 7 is a survey of some of the more prominent reviews of the second edition. It begins with an overview of Frank Ramsey (1926), "The foundations of mathematics" which can be viewed as Ramsey's response to the issues in *PM* about the axiom of reducibility, the nature of quantification and propositional functions which came from his study of Wittgestein's *Tractatus*, and the second edition material which he had read the year before. The reviews of the second edition by Alonzo Church, C. I. Lewis, and others, help to place *Principia Mathematica* in the logical scene in the mid 1920s, although they are directed more at the general program of logicism (identified with Frege and Russell) than the technical issues that are the focus in this work.

There is much still to be learned about the history of logic in this period, and about the nature of intensional logic which was lost in the move towards extensionality that dominated from the 1920s until the great development of interest in intensional logic and the explosion of work on modal logic and possible worlds semantics following Saul Kripke's papers in the early 1960s. Perhaps type theory has yet to enjoy quite such a revival of interest. It is now acknowledged that Gödel's incompleteness theorem, with its attention to systems with recursively enumerable axioms, does not show that "full" second order logic is incomplete, when the predicate quantifiers are interpreted as ranging over all sets of objects in the domain. It will then also not apply to the intensional theory of types of *Principia Mathematica*. The contemporary logical scene may be ready for an investigation of the ramified theory of types as an intensional logic.

## 1.1 The manuscripts

The first manuscript transcribed below is the list of the principal definitions and theorems in *Principia Mathematica* that Russell wrote out by hand for Rudolf Carnap in 1922. It is published here with the permission of the Archive of Scientific Philosophy at the University of Pittsburgh, which holds Carnap's papers.

The other manuscript material, transcribed beginning after Chapter 8, came to the Bertrand Russell Archives as part of the original purchase of Russell's papers by McMaster University in 1968. Some smaller items, including Russell's personal copy of the first edition of *Principia Mathematica* with some important material relevant to possible revisions stuck between the pages, arrived with the so-called "Second Archive" after Russell's death in 1970.[8] The manuscripts here called "Hierarchy of propositions and functions" and "Amended list of propositions" were catalogued under those names, and kept in files with the manuscript of the introduction and appendices. The *Collected Papers of Bertrand Russell* series, being conducted by the Bertrand Russell Research Centre at McMaster University, is dedicated to publishing papers and some additional manuscript materials, but the plan was that the material relevant to Russell's books would be used in critical editions produced by their publishers. While a scholarly critical edition of the whole of *Principia Mathematica* would be a massive editorial project, this volume may be considered a first attempt at such a presentation of the second edition of *Principia Mathematica*, and so an opportunity to publish further holdings of the Archives.

Russell's manuscripts were clearly his final versions, sent to Cambridge University Press for typesetting, and returned to him for proofreading, and thus surviving among his papers. The few differences between these versions and the printed edition presumably occurred at that proof stage, and are indicated in the transcriptions. The manuscript leaves are written in an easily readable handwriting, as they were intended for the use of typesetters. Those typesetters were put to work on four or five leaf sections of manuscripts, and the individual names of some of them are written in the upper left hand corners by someone else, presumably a chief printer who delegated the work. Russell numbered the manuscript leaves and this numbering can be used to reconstruct the original "Hierarchy of propositions and functions" manuscript. Russell moved material, simply striking out old numbering and adding the new.

The "Amended list of propositions" manuscript is very different. It consists of single leaves of notes in symbolic notation, with no numbering, and it is named after the heading of the first leaf of notes, which is almost the only writing in the whole

[8] See Blackwell & Spadoni (1992).

manuscript not in *Principia* symbolic notation. While this material was identified as connected with *PM* and kept with the manuscripts, its nature as repeated attempts at finding proofs for lemmas in Appendix B has not been previously identified in print.

The manuscripts are transcribed here in such a way that the content of each original can be reconstructed. They are identified with Russell's numbering ("foliation") in italics in the upper right hand corner, with a break between the content of leaves. For the introduction and appendices, the page number in the printed edition is inserted into the text where the new page begins. Russell struck out some material with a single line (reproduced here in the text as ~~deletions~~) and sometimes added some material above the line, and that is indicated as well. These insertions, usually above the line with a caret to indicate their position, are inserted in the line with '⌊' and '⌋' around the inserted material, as follows: ⌊insertions⌋.The editorial conventions are explained at the beginning of Chapter 12. For the introduction and appendices the "house style" of capitalization, italics, and spacing is imposed on the manuscript so that the reader can more easily identify the changes between the final manuscript and the published version. These corrections were likely made by Russell in the proof stage. The unpublished manuscripts "Hierarchy of propositions and functions" and "Amended list of propositions" are reproduced literally so that the reader can see the layout of text and symbols on the page.

## 1.2 Acknowledgements

Kenneth Blackwell and Nicholas Griffin have guided me throughout these efforts. This work is, however, not an activity of the Bertrand Russell Research Centre, or McMaster, and Blackwell, the Honorary Archivist, and Griffin, Director of the BRRC, are not responsible for lapses from the high scholarly standards and absolute accuracy of all of their work. Blackwell has nevertheless assisted me at every stage of my work. I came to the Archives as a fledgling Russell scholar for the first time in October 2003, asking if there was any unpublished material "in symbols" that I might see. Ken and Nicholas Griffin together presented me with the box of material on the second edition. During the summer of 2010 Blackwell provided suggestions for the editing of the manuscript material and Griffin read the final manuscript correcting numerous errors. Before I even knew of the manuscripts, between January and June of 1997, I had the opportunity to study Appendix B with Allen Hazen at the University of Melbourne. Together we discussed Gregory Landini's then recently published paper (1996), and worked through the appendix, losing and then finding again the "mistake" in line 3 of the proof of $*89{\cdot}16$. Hazen, Landini and I had occasion to discuss Appendix B further at the conference "One

Hundred Years of Russell's Paradox" organized by Godehard Link at the Ludwig Maximillian University in Munich in June of 2001.[9] With the understanding of these issues that I had gained from Landini and Hazen, identifying the "Amended list of propositions" manuscript was easy.

Ken Blackwell assisted me in finding the letters relevant to the second edition, as well as the offprints and other materials in Russell's library that are described and used in what follows. He instructed me about working with manuscripts and gave advice about transcriptions. My father, Leonard Linsky, has provided both parental encouragement and expert assistance from the first telephone call from Hamilton to Chicago on the day I found the manuscripts. In the 1960s I had met John Myhill in our family home in Urbana, Illinois and so my father's knowledge of the book was long and personal. I have relied on him both for guidance about the reception of *PM* and for technical details that he recalled from the course of his academic career. Overcoming his doubts about the significance of the second edition of *Principia Mathematica* has added a personal motivation that has maintained my enthusiasm through the long months of typing LaTeXformulas.

The work of checking the transcriptions against photocopies of the manuscript originals has been carried out by several graduate students in the Department of Philosophy at the University of Alberta, with some compensation for their work provided through a grant from the Social Sciences and Humanities Research Council of Canada. Ayodele Adejumobi and Shaheen Islam worked on the "Amended list of propositions" manuscript and compared the two editions, line by line, to find the alterations. Seyed Mousavian figured out the fate of the "Hierarchy of propositions and functions" manuscript starting from a list made by Martin K. Smith at McMaster University, dated 8 July 1985. Vladan Djordevic checked the lists of corrections sent by Behmann and Boskovitz. His frustrating work on a list of corrections apparently from Ramsey led to their proper identification as notes from Whitehead dating back to the first edition. Seyed Mousavian, Octavian Ion, and Patrick Gamez read early drafts of the manuscript. Richard Zach saved me from numerous embarrassing errors, gave advice on putting *PM* notation into LaTeX, and made me aware of the influence of the second edition on Carnap's *Abriss*. Katalin Bimbo, Kevin Klement, Hassan Masoud, Graham Sullivan, and Dustin Tucker caught more mistakes in the final version.

Various people have made useful suggestions and caught errors from reading the manuscript or hearing talks based on this material. I would like to thank Allen Hazen, Ed Mares, Bob Meyer and audiences at the University of Oklahoma and the Society for Exact Philosophy for this sort of discussion of the project. Edward N. Zalta discussed the the notation in *PM* and helped me to write the article

[9] Papers from the conference were published in Link (2004).

(Linsky 2009d) on that topic for the *Stanford Encyclopedia of Philosophy* which is the basis for Chapter 4.

The work of the graduate students on this project was supported by grants from the Faculty of Arts and the Vice-President (Research) at the University of Alberta, and two Standard Research Grants from the Social Sciences and Humanities Research Council of Canada (SSHRC).

I would like to dedicate this work to my wife, Elizabeth Millar, encouraging me to press ahead with the sometimes tedious work, and permitting me to take evenings and weekends away from my family duties to pursue my obsession to finish quickly what might well have been a longer, measured process. Betty has always cheerfully excused me from our collective life as I have gone off to my study to work on “my book”.

# 2

# Writing the second edition

The abundance of manuscript material and correspondence relating to the second edition of *Principia Mathematica* in the Bertrand Russell Archives makes possible a reconstruction of the process of its composition. It is clear that the second edition is solely the work of Russell, and what is new is confined to the Introduction to the second edition and three appendices, with the rest of the first edition reset with corrections or simply reprinted. Frank Ramsey was involved with the new edition, reading manuscript material for the new material, and then the printer's proofs late in 1924.

From the manuscript material we can also learn more of the details of this story. The new material for the second edition was originally planned to take the form of a revised chapter of the *Introduction* to the first, and it survives as a manuscript of seventy odd leaves, called "The Hierarchy of Propositions and Functions." ("HPF" in what follows.) This was the first draft, with its leaves dispersed and renumbered, later filled out with the rest of the material for the Introduction to the second edition and Appendices A and B. Appendix C appears to have been then added, with only the final draft surviving.

Appendix B, which is not reprinted in the paperback *Principia Mathematica to ∗56*, with the technical problems identified by Gödel (1944) and confirmed by Myhill's (1974) proof, was the subject of the most preliminary work. The forty some leaves of discarded proofs, called "Amended List of Propositions" after the title of the list on the first leaf, seems to consist of various attempts to prove theorems for inclusion in Appendix B. The discarded material repeats the error that Gödel identified, as well as the numerous logical peculiarities in the appendix which Myhill refers to. Contrary to Myhill's rather negative assessment of the appendix, the wealth of manuscript material suggests that Russell was careful, and quite thoughtful about what he included in it. There are still mysteries around what Russell's intentions were, but we have every reason to take the appendix seriously, intended, as Ramsey wrote to Russell after reading the proof sheets, as "an awfully good theory".

What follows is an account of the preparation of the second edition of *Principia Mathematica*, which Russell started to think about seriously in 1922 after a request for a copy from Rudolf Carnap, who could not obtain a copy. This chapter traces the process of preparing the second edition as revealed in the manuscript materials and letters in the Archives.

## 2.1 The list of definitions for Carnap

Russell had paid careful attention to *Principia Mathematica* before he began to work on a new edition. Rudolf Carnap wrote to Russell on 17 November 1921, to introduce himself, and send a work on space.[1] Carnap assumed that Russell was still in China, but in fact he and Dora Black had returned in September 1921.[2] From 13 June 1922 we have a letter from Carnap asking if Russell can supply a copy of the first volume of *PM* at some author's price. He reports that he was a student of Frege in Jena and that he had then made an excerpt of *PM* from a library copy. A letter from 29 July 1922 thanks Russell for the proposal to write out the definitions and main theorems of volume I. A letter dated "29. Sept", presumably also 1922, thanks Russell for what he sent ("Ihre Sendung"), for the effort that went into it, and promises to pass it along to others who will be interested, including "Gerhards, Aachen; Behmann, Göttingen; Wilke, Weissasser; Reichenbach, Stuttgart".

Carnap kept the list of definitions and theorems that Russell sent. He writes in the "Intellectual Autobiography" in Schilpp (1963, p.14) of his study of Russell's works:

> I also continued to occupy myself with symbolic logic. Since the *Principia Mathematica* was not easily accessible, I began work on a textbook in symbolic logic. There was no copy of the *Principia* in the University Library in Freiburg. The price of a new copy was out of reach because of the inflation in Germany. Since my efforts to find a secondhand copy in England were unsuccessful, I asked Russell whether he could help me in finding one. Instead, he sent me a long list containing all the most important definitions of *Principia*, handwritten by himself, on 35 pages, which I still cherish as a priceless possession. In 1924 I wrote the first version of the later book, *Abriss der Logistik* [1929]. It was based on *Principia*.

The list survives in the Archives of Scientific Philosophy with Carnap's papers and is transcribed and included below with the other manuscript material related

[1] Carnap says "Raum". Presumably this was Carnap's doctoral thesis, *Der Raum*, Carnap (1921).

[2] Russell writes in the *Autobiography* that from 1921 to 1927 he split his time "about equally" between Cornwall and London. (*Auto.*, vol. II, p.212)

to the second edition. There are 35 consecutively foliated leaves of notes, 11 on Volume I, (i.e. up to $*100$), another 14 up to $*250$ in Vol II, and the remaining 11 covering Volume III, ending with $*375$, in other words a uniform, though very selective, summary of the whole of the work. All of the material "to $*56$", for example, is covered in the first six and a half pages. There are very few comments with the verbal material taken from the text of *PM*. One notable item is a remark to $*12$ (on the axiom of reducibility) that "A predicative function can be an apparent variable; a general function cannot." A typo in $*73{\cdot}01$, which figures in the various lists of corrections for the second edition, is just silently corrected in Russell's list for Carnap.

Carnap writes that it would be too expensive to acquire a copy of *PM* because of the inflation in Germany. Russell is well known to have sent copies of books to correspondents, but in this case he may have been limited by the unavailability of copies.[3] This sign of interest in Germany in the work may well have been part of the motivation to produce a new edition. In any case, the interest from Carnap and his colleagues in Germany could have persuaded Russell that there was enough interest in *PM* to justify printing a new edition.

When Russell did decide to produce a new edition, he had recently experienced reading through the whole of the three volumes, and realizing that only material in Volume I was affected by any of the new ideas about logic he had been working on since 1910. So the list of definitions can now serve as an overview of the contents of *PM* for the purposes of this book, just as it had served as a review for Russell in 1922.

## 2.2 A. N. Whitehead and the second edition

The first edition of *Principia Mathematica*, and hence the body of the second, is justifiably viewed as a thoroughly collaborative work, yet it is proper to attribute the additions for the second edition to Russell alone. In fact Whitehead wrote a letter to *Mind* in 1925, declaring that the new material added to the second edition was all the work of Russell (Whitehead, 1926). Some commentators react to a perceived tone in this brief note. Grattan-Guinness calls the letter "testy" (Grattan-Guinness, 2000, p. 441). Monk suggests that Whitehead was quite unsympathetic with the new material, and did not want to be associated with it, although Russell can, in the end, be "cleared of any impropriety in his dealings with Whitehead over the new edition; he kept his co-author informed and allowed him a chance to object to the new material." (Monk, 2000, p.45). The letters surrounding the new

[3] The 750 copies of the first volume and 500 of II and III were probably long sold out. (Grattan-Guiness, 2000, pp. 385–6).

edition, however, suggest a more benign attitude from Whitehead than these authors find.

A letter from Whitehead to Russell, dated 24 May 1923, gives a picture of Whitehead's attitude towards the second edition that seems to differ from Monk's:

> . . . Yes – I thoroughly agree that we cannot undertake a reorganization of the text. But I will send you my ideas for appendices to the various 'Parts' of the book, mostly in the form of press corrections to the proofs, e.g. on the first two sheets you will find a series on the various meanings of 'function', entirely due to the infernal niggling criticisms of Johnson. I think a short note on the various meanings citing some of the occasions of their use might be useful. *A priori*, I should have thought that the text is plain enough. I shall settle down to the last volume in the summer, and send you the result. At present, I am about 3 days behindhand in my various jobs. I don't think that 'types' are quite right. They are 'tending towards the truth', as the Hindoo said of his fifth lie on the same subject. But for heaven's sake, don't alter them in the text. I will send you 3 and 4 at the end of this week, to arrive on Monday . . .

The "infernal niggling criticisms of Johnson" are presumably based on the views of W. E. Johnson (1921) about the nature of predication, in particular the distinction between the subject/propositional function analysis of the logic of *PM* and the subject/copula/predicate analysis of traditional logic.

The reference to "types" is unclear. Monk suggests that this letter shows that Whitehead disapproves of the modifications to the theory of types suggested by the adoption of extensionality and the abandoning of the axiom of reducibility, the very modifications that Ramsey was to refine into his well-known proposal to adopt the simple theory of types to handle the set theoretic paradoxes, and handle the other, intensional or "epistemic" paradoxes with a theory of meaning and language. Whitehead refers to reading "the first two sheets", and to adding material about functions. If these were proof "sheets", and if Whitehead was reading proofs of the reset Introduction, this would have taken him through most of the discussion of Chapter II of the Introduction, "The Theory of Logical Types" to Section V of Chapter, "The Hierarchy of Functions and Propositions". Revisions at that point were the germ of the "Hierarchy of Propositions and Functions" manuscript.

The letter shows Whitehead somewhat disengaged from Russell's work on the second edition, suggesting that instead he will complete "the last volume" over the summer. The projected Volume IV of *Principia Mathematica*, on geometry, was never completed. Whitehead seems content with whatever alterations Russell was planning, wishing simply to add some notes of his own. No mention of Whitehead appears in the manuscript material collected here, or in correspondence during the rest of the completion of the second edition. Whitehead, then, most likely later wrote to *Mind* simply to decline credit for writing the work, rather than to disown it. What's more, he wanted to identify some portions of the initial edition as primarily

Russell's work, and so to break with Russell's way of referring to all of the work as jointly authored.

The letter to the editor of *Mind*, addressed from Harvard University, 5 November 1925, asks that the following statement be inserted in the journal.

> The great labour of supervising the second edition of the *Principia Mathematica* has been solely undertaken by Mr. Bertrand Russell. All the new material in that edition is due to him, unless it shall be otherwise expressly stated. It is also convenient to take this opportunity of stating that the portions in the first edition – also reprinted in the second edition – which correspond to this new matter were due to Mr. Russell, my own share in those parts being confined to discussion and final concurrence. The only minor exception is in respect to $*10$, which preceded the corresponding articles. I had been under the impression that a general statement to this effect was to appear in the first volume of the second edition.

This is not the clearest of attributions. It is evident, however, that Russell is given credit for all of the new material in the second edition. It also seems to indicate that portions of the first edition from volume II, those that "correspond to this new matter" were also mostly Russell's work. That would mean certainly the material on induction covered by Appendix B, but also, as one might suspect, the Introduction to the first edition, and much of the more philosophical material in Volume I. This letter also seems to say that the two treatments of logic of the quantifiers, $*9$ and $*10$, are just what they seem, a compromise presenting the preferred presentations of the two co-authors, with Whitehead as the author of $*10$. The general statement "to this effect" that Whitehead expected could be the simpler statement that the additions in the second edition are due to Russell, but more likely a statement referring to all of the material preceding it in the letter, namely, giving sole credit to Russell for significant material in the first edition. The tone of this letter is even harder to discern, but it looks like a generous statement setting straight the fact that *PM* was not an absolutely equal collaboration in all parts, rather than an attempt to disown the new material introduced by Russell.

In a "Memorandum" dated 23 January 1927, Whitehead wrote in support of an appointment for Henry Sheffer at Harvard University, Whitehead quotes the last paragraph of the first section of the "Introduction to the second edition" beginning: "It should be noted that a new and very powerful method in mathematical logic has been invented by Dr. H. M. Sheffer . . . ".[4] Whitehead introduces the quotation with: "In this connection on p.*xv* of the 2nd edition of Vol. I of *Principia Mathematica* (Russell and Whitehead) we write: . . . " and the quotation is followed by: "These sentences were in fact penned by Mr. Bertrand Russell, and represent his independent opinion with which I entirely concur." (Henle *et al.*, 1951, p. *x*). While the

[4] The "Memorandum" is quoted in Felix Frankfurter's forward to the Festschrift for Sheffer (Henle *et al.*, 1951, pp.*ix*–*x*). This work by Sheffer, which was never published, will be described below in the discussion of Sheffer's manuscript "The General Theory of Notational Relativity".

context is an endorsement of Sheffer, there would be opportunities for Whitehead to distance himself from the second edition in these sentences. This also suggests the sort of reason Whitehead might have had behind his letter to *Mind*, making public that the new second edition material was "penned" by Russell.

This matter is not clear, but the simpler issue of Russell's responsibility for the Introduction and three appendices is settled, and is confirmed by the fact that all the manuscript material and the equal number of leaves of unused notes are all in Russell's hand. Only two and one half pages of the manuscript of the first edition survive, so the fact that all the final manuscripts are by Russell is not by itself conclusive.[5] There are also none of the other sorts of possible evidence of collaboration on the second edition that describe the work on the first edition with drafts going back and forth, then lists of corrections, all accompanied by correspondence and discussions. The new material in the second edition of *Principia Mathematica* is Russell's work alone. In what follows Russell is identified as "the" author of the second edition material, using the definite description, while the first edition material and views will be distinguished as the work of "Whitehead and Russell".

Russell started the actual writing some time in 1923. In a letter from Russell to Jean Nicod, of 13 September 1923, there is a mention of the second edition:

> *Principia Mathematica* is being re-printed and I am writing a new introduction, abolishing the axiom of reducibility, and assuming that functions of props are always truth-functions, and functions of functions only occur through values of the functions, and are always extensional. I don't know if these assumptions are true, but it seems worthwhile to work out their consequences.

Russell says that it is he who is undertaking the revision, which at this point is to consist of a new or revised introduction. The main ideas behind the changes: "abolishing" the axiom of reducibility, and the thesis that functions only "occur through their values" and the adoption of extensionality, are already formulated.

A letter of 21 October 1923 to S. C. Roberts at Cambridge University Press suggests that the setting of the new edition is well under way, with Volume I perhaps already completed, but that work on the new Introduction, much less the Appendices, has barely begun:

> I think it will be quite possible to proceed with the composition of Vol. II as soon as that of Vol. I is completed (except the Introduction). In addition to the Introduction, we wish to add a few notes on single chapters. If you like, these can come immediately after the introduction, though they would come more naturally at the end of the volume. But they cannot well be ready sooner than the introduction.

[5] See Linsky & Blackwell (2006) for an account of the manuscript of the first edition.

While I am in America (Jan.–April) it will be necessary to get Dr. Whitehead to attend to the proofs. He also has certain ideas about the prefatory matter, but I think he and I can arrange that without troubling you. The corrections for Vol. II will certainly be very slight; in fact I doubt if there need be any.

I am very grateful to you for allowing the introduction to stand over till next summer. It will not be long, but involves reading all that has been done in recent years on the subject, most of which, unfortunately, is by Polish mathematicians, whose language is unknown to me.

While it is not obvious just how seriously we are to take this letter, at face value it presents Whitehead as still involved in the project as of October of 1923, at least as one to whom proofs might be sent, the Introduction not yet even begun, as notes to individual chapters were still being considered.[6] On the other hand it does seem that the "Hierarchy of Propositions and Functions" manuscript (HPF), was intended initially as a revision of the original Introduction, and so it may predate this letter. HPF may have begun as a "note" as well, and quickly developed into a draft of "the Introduction". The letter does not, then, allow a precise assessment of the state of the work in October of 1923. The reference to work by "Polish mathematicians" can only be an allusion to the papers sent by Leon Chwistek which included "The theory of constructive types" (1924). While one of Chwistek's published papers from this time is in Polish, the rest are in English or German. Russell probably intended to study Chwistek's work more carefully, and used the joking remark about the Polish language to suggest that the work would take longer than originally expected.

It is most likely that the work on the second edition was carried out between the second half of 1923 and the end of 1924. Russell's lecture tour of the United States extended through April and May of 1924, and so work would have halted during that time. The whole project from conception to printing was thus carried out in less than a year and a half, with the bulk done in the summer of 1924.

## 2.3 Hierarchy of propositions and functions

The largest part of the material added to the second edition, the Introduction to the second edition and both Appendices A and B, started off as parts of a single draft. It is foliated beginning with a first leaf titled "The Hierarchy of Propositions and Functions" (HPF). That title will serve for the entire manuscript. The title in fact suggests that Russell originally planned simply to revise Chapter II, part V of the

[6] In (*Auto.* vol. II, p.97), commenting on a letter from Whitehead of 8 January 1917, Russell says that "Before the war started, Whitehead had made some notes on our knowledge of the external world and I had written a book on the subject in which I made use with due acknowledgement of ideas that Whitehead had passed on to me. The above letter shows that this had vexed him. In fact, it put an end to our collaboration." This supports the conclusion that there was no collaboration later, although, as we see, Whitehead and Russell were still on friendly terms personally, and at least considering further collaboration.

introduction to the first edition, "The Hierarchy of Functions and Propositions". Chapter II is devoted to "The Theory of Logical Types", with part V beginning the explanation of first order functions of individuals and then the construction of the ramified theory of types from them. Part VI introduces the axiom of reducibility, and VII, reasons for adopting the axiom. The chapter concludes with part VIII on the various logical paradoxes and how they are resolved by the theory of types. Thus it seems from the title of HPF that Russell thought that his amendments could take the form of a revision to the last part of Chapter II of the Introduction to the first edition, beginning with part V.

It seems that after completing the HPF manuscript it became clear to Russell that the revisions would require a separate introduction for this edition, and several appendices. The material moved out of the HPF manuscript then was divided into the Introduction to the second edition, Appendix A on the Sheffer stroke, and Appendix B on induction. Corrections to the final manuscript of the new introduction suggest that even later in the process it was still not determined that there would be more than one appendix. Indeed, the very first leaf of the introduction manuscript says simply that "$*9$ is replaced by a new chapter, $*8$, given in the Appendix to this Volume". Only in the final printed version does that become "... given in Appendix A to this Volume".

The HPF manuscript survives in the Archives as a file containing fifty three leaves numbered ("foliated") from *1* to *78*, with underlining, in the upper right hand corner, Russell's standard practice with manuscripts. Many leaves are missing, as indicated by gaps in the numbering and abrupt changes in the material. They were reused, with a new number, and show up in the manuscripts for the published material. The reused leaves can be traced, allowing for an almost total reconstruction of the original draft. The revisions to that earlier draft seem to be mostly improvements in exposition rather than remains of significant changes of mind. The renumbered pages are often only renumbered once, with only a handful of cases where multiple intermediate revisions must be hypothesized, which suggests that HPF was a first draft, and the final material often just the second. As was common with much of Russell's prose composition, while there are corrections, both strikeouts and additions, on the manuscript pages, not much remains of distinct versions. Russell may have discarded intermediate drafts, but even then we would expect some pages to survive more than two drafts, with consequently more than two numberings, but in the prose sections at least, and with only one fatal exception in the technical material in Appendix B, no other leaf has three numberings.[7] As well, there are two leaves with the number *52* and two numbered *74* left over in HPF, but if this

[7] The exception is page *9* of the manuscript, which appears to have begun as *71* in the HPF manuscript, then being renumbered as *8*, and then its final number, *9*. This is the page that contains the erroneous proof of theorem $*89\cdot16$ identified by Gödel, to be discussed much more below. Part of the story of the error is that Russell seems

collection is all the unused material, it is clear that there was not even one complete revision of the document.

## 2.4 Working papers for Appendix C

The manuscript of Appendix C does not make use of any previously numbered pages, and, as well, does not contain material which might be a revision of any leaves remaining in HPF. It thus appears to have been added late in the writing, after the initial manuscript was revised as the Introduction to the second edition and Appendices A and B. Two papers, published in *Papers 9* (24 and 25), appear to be precursors to Appendix C. One of them, What is meant by "A believes *p*?" (*Papers 9*, p.25) consists of two leaves, foliated 1–2. The leaves were found in Russell's copy of Wittgenstein's *Tractatus Logico-Philosophicus* when it arrived in the Bertrand Russell Archives in 1978.[8] This paper mentions Wittgenstein's remark about belief at *TLP* 5.44, and also Russell's Introduction to the *Tractatus*, but the focus of the short discussion is on working out the successor to his earlier "multiple relation" theory of belief. According to this new account, several beliefs may form an equivalence class, if they correspond in the same way to a fact that makes them true or false, and so be said to be beliefs in the same *proposition*. The other paper, "Truth-functions and meaning-functions" (*Papers 9*, p.24) is more directly related to the issues discussed in Appendix C.[9] It concentrates on the issue of how to account for propositions which seem to describe propositional attitudes when it is assumed that all occurrences of propositions in larger contexts are truth-functional. Using the example of "Plato is Greek", the manuscript analyzes the notion of a *constituent* both of the proposition and of the fact that Plato is Greek. It gives the same account of a belief as the previous manuscript, by which a belief is a fact which is directed to a certain proposition, with those propositions themselves corresponding to another fact that makes the proposition either true or false. Appendix C, after repeating the slogan from Appendix A that a propositional function only occurs in a proposition "through its values", turns to giving an account of the occurrence of propositions in beliefs and assertions that is compatible with the thesis of truth-functionality. In Appendix C beliefs are facts which are constituents or parts of believing subjects, propositions are then equivalence classes of such facts, but all mention of correspondence, or of facts making propositions true or false, has now been replaced by the notion of simply having a truth value. The two preliminary

to have made the mistake early, simply reusing the same leaf in later versions, never going back over the proof to revise it, and so not spotting the mistake. The mistake was not deliberately repeated through any revisions of that one leaf.

[8] Yellowed edges suggest that it had been there for a while. It is catalogued as RA 220.148001g.

[9] This paper, RA 230.03140, five leaves foliated 1 to 5, was among the material in the original archive. Both are dated by the editor as having been written in 1923.

papers can be seen as progressively refining the proposal about propositions derived from equivalence classes of facts that constitute propositional attitudes such as beliefs. The purpose, then, is to justify the thesis that all propositional contexts are truth-functional which plays such a large role in the Introduction to the second edition. Seen perhaps originally as a philosophical discussion in the metaphysical language of logical atomism, the resulting Appendix C is limited to an account of the logical status of propositions and propositional attitudes.

## 2.5 Amended list of propositions

Russell preserved a remarkable collection of leaves of technical work, transcribed here and named "Amended list of propositions" (ALP in what follows) after a title on a list on one of the leaves. ALP consists of 62 leaves, with only one numbered.[10] No material is continued from one page to the next. In addition to some miscellaneous pages intended to become various parts of Appendix B, most of the pages contain repeated and obviously failed attempts to prove theorems in the latter half of the appendix. Much of the material is directed towards the generalization of the main result about the derivation of induction without the axiom of reducibility. In particular, Russell tries to generalize the result to "generations" of arbitrary relations, beyond the well behaved but limited case of the one to one functions represented by the successor of natural numbers.

The mere existence of this intense amount of work, some 60 plus leaves full of nothing but symbolic notation, is very revealing about Russell's attitudes towards the content of Appendix B. For one thing, it contains a repetition of the erroneous lemma that has been the focus of most study of Appendix B. More generally, the sheer amount of work shows that Appendix B was the result of serious effort, and was not casual. The appendix shows some confusion about types, and may be unclear in its notation, but this was not the result of having been dashed off or written carelessly. Rather, whatever conceptual errors there are were not the result of simple mistakes (with at least the one notorious exception). Russell worked hard and long on Appendix B, despite what must have been enormous frustration with giving up on intended proofs always after a whole page of symbolism. While Russell's prose work may have consisted of one or two drafts, with some corrections, the logical work was the end result of repeated butting at the problem, and much deletion of work that was not needed or did not get the desired results.

While it is certainly fascinating to look at the discarded notes of the logician at work, publishing these manuscript notes is not a voyeuristic search through the

[10] That exception is ALP [14r] which is foliated '16'. From the content this page is seen to be a late version of the very last theorem of Appendix B, with this version containing a different number as the order needed to prove induction, namely *4* instead of the *5* in the final version.

wastebasket of a great thinker. Russell kept this material with his manuscripts of the second edition. These were ideas that did not just fizzle out until each was worked out to the end of a whole page, and so as records of blind alleys not to pursue again Russell felt they were worth keeping in his records.

When Appendix B was finally worked out, the rest of the second edition material seems to have fallen into place fairly easily. Even the technical material on the Sheffer stroke in Appendix A seems to have undergone only one revision from its first appearance in HPF to its final version in the manuscript sent for publication.

## 2.6 Corrections to the first edition

Aside from the new introduction and appendices, the rest of the second edition of *PM* consists of a faithful reprinting of the original material. The first two volumes had to be reset, although Cambridge University Press was able to reproduce Volume III by a photographic process. The opening sentences of the Introduction to the second edition are as follows:

> In preparing this new edition of *Principia Mathematica*, the authors have thought it best to leave the text unchanged, except as regards misprints and minor errors, even where they were aware of possible improvements. The chief reason for this decision is that any alteration of the propositions would have entailed alterations of the references, which would have meant a very great labour.
>
> *(PM, p.*xiii*)*

The changes to the first edition material are indeed just a relatively small number, corrections of misprints and slight corrections, of the sort that were included on errata sheets even during the printing of the first edition. Not one theorem is added or deleted from that material. Russell clearly did pay careful attention to the citation and use of theorems in later sections, constructing elaborate lists of "Props. used" at some stage in the proofreading process.[11]

What follows is an attempt to list every known correction that Russell made to the first edition. While the list in no way constitutes the sort necessary for a genuine "critical edition" of the second edition of *Principia Mathematica*, it does confirm the statement that the authors tried to leave the text unchanged to the extent possible, and also gives some sense of the likely real number of changes and of the range of their sources. Serious effort was devoted to this task, although nothing like the work that currently goes into the revision of logic textbooks for new editions. Russell corrected errors that had come to his attention from various sources, but did not seek them out with a systematic proofreading of the whole first edition. These

[11] The recovered manuscript pages of the first edition were found on the verso of such lists. See Linsky & Blackwell (2006).

are almost all trivial typographical errors, as can be appreciated even without first reading the explanation of the symbolic notation which follows in Chapter 4 below.

### *2.6.1 Errata*

Some errors in the first edition of *PM* were caught after the printing process and listed as “Errata” on a page just before the Introduction.

ERRATA

p.14, line 2, *for* “states” *read* “allows us to infer.”
p.14, line 7, *after* “$*3\cdot03$” *insert* “$*1\cdot7$, $*1\cdot71$, and $*1\cdot72$.”
p.15, last line but one, *for* “function of $\phi\hat{x}$” *read* “function $\phi\hat{x}$.”
p.34, line 15, *for* “$x$” *read* “$R$.”
p.68, line 20, *for* “classes” *read* “classes of classes.”
p.86, line 2, *after* “must” *insert* “neither be nor.”
p.91, line 8, *delete* “and in $3\cdot03$.”
p.103, line 7, *for* “assumption” *read* “assertion.”
p.103, line 25, at end of line, *for* “$q$” *read* “$r$.”
p.218, last line but one, *for* “$\Lambda$” *read* “$\dot{\Lambda}$” [ owing to brittleness of the type, the same error is liable to occur elsewhere].
p.382, last line but one, *delete* “in the theory of selections ($*83\cdot92$) and.”
p.487, line 13, *for* “$*95$” *read* “$*94$.”
p.503, line 14, *for* “$*88\cdot38$” *read* “$*88\cdot36$.”

The proof sheet of “Additional Errata to Volume I”, from Volume II, survives in the Bertrand Russell Archives, complete with corrections in Russell’s hand.[12] Volume II reports 39 errata for Vols. I and II. A second “Additional Erratum to Volume I”

p.574, line 8, *for* “ $p$‘Ɑ“$(S|R)$ ” *read* “ $p$‘Ɑ“Pot‘$(S|R)$ ”[13]

appears on page $x$ of Volume III, in between 34 errata for Volume III, and 21 errata for Volume II. These changes were all added to the second edition.

What direct evidence we have of corrections genuinely new to the second edition comes from some manuscript leaves that were saved in Russell’s own copy of *PM*, and removed and catalogued when his library came to the Bertrand Russell Archives.[14]

[12] Kenneth Blackwell (1984) reproduces the sheet, catalogued as BRA 210.1457501a, with Russell’s corrections.
[13] At p.547, line 6 from bottom.
[14] This material included, unnoticed, the three half pages of manuscript of the first edition described in Linsky & Blackwell (2006).

### 2.6.2 Changes suggested by Boskovitz

The largest single source of corrections was in a letter from Alfred Boskovitz, a student of mathematics at the University of Göttingen, dated 3 July 1923. It was apparently forwarded to Russell by Heinrich Behmann in a letter of 19 September 1923 (Mancosu, 1999 p.322 n.2). In the notes Boskovitz writes that the twenty odd corrections up to $*72$ date from 1920, with the rest added in 1922 and 1923. He informs Russell that he is a student of Paul Bernays, working on the development of set theory without use of the axiom of reducibility. Some remarks on the twelve pages of notes are in the handwriting of Behmann.[15] The second footnote in the Introduction to the second edition thanks Behmann and Boskovitz "especially" for those corrections. Indeed they seem to be the only known source of large numbers of suggested changes.

Boskovitz made over 162 suggestions covering the whole of Volume I. 44 of these appear as corrections in the second edition.[16] Five of these changes are purely matters of notation, such as correcting the number of dots used for punctuation, or in one case, the number of a theorem. The largest number of changes (fourteen) are corrections of the simplest of typographical mistakes, i.e. those that can be corrected by changing one or two single letters or symbols. In two places a new expression is inserted in the second edition. The other changes involve the annotation of proofs. Four involve adding a theorem number or formula as an additional premise. Another series of five changes involves omitting premises from proofs. In five numbers one or more premises are changed in a proof. Finally, at $*96{\cdot}492$, a line already described as "not needed" and "merely stated" in the text, is deleted from the second edition.

$*1{\cdot}72$. axioms $*1{\cdot}7{\cdot}71{\cdot}72$ will generally replaces axioms will generally
$*11{\cdot}07$. $y$ interchanged except in "$\phi(x, y)$" replaces $y$ interchanged.
$*43{\cdot}491$. $(\beta \upharpoonleft R)$ replaces $(\alpha \upharpoonleft R)$
$*64{\cdot}2$. $\vdash : \dot{\exists}\,!$ replaces $*64{\cdot}2$. $\vdash . \dot{\exists}\,!$
$*70{\cdot}51$. *Dem*. l. 6, $*23{\cdot}34 . * 51{\cdot}236 . \supset .$ replaces $*23{\cdot}34 . \supset .$
$*74{\cdot}511$. $(\beta \upharpoonleft \breve{P})$ replaces $(\breve{P} \upharpoonleft \beta)$
$*74{\cdot}84$. *Dem*. l. 12, $\upharpoonleft R``\kappa \subset 1$ replaces $\upharpoonleft R``\kappa \epsilon 1$
$*82{\cdot}23$. and $*82{\cdot}231$. $Q \upharpoonleft \lambda \epsilon 1 \to 1$ replaces $Q \upharpoonleft \lambda \epsilon \mathrm{Cls} \to 1$
$*83{\cdot}74$. *Dem*. l. 3, $\mu, \nu \epsilon \mathrm{D}``\epsilon_\Delta`\kappa$ replaces $\mu, \nu \epsilon \epsilon_\Delta`\kappa$
$*84{\cdot}21$. $1_{\mathrm{Cls}}$ replaces $1(\mathrm{Cls})$ throughout
$*84{\cdot}422$. $\mathrm{D}``\epsilon_\Delta`\kappa$ replaces $\mathrm{D}``\epsilon_\Delta`\alpha$

[15] For an account of the interest in Russell at Göttingen, see Mancosu (1999) and (2003).
[16] The work of identifying the changes in Boskovitz' list that were made was done by Seyed Mousavian.

$*85{\cdot}13$. In the above proposition, the hypothesis required as to $\overrightarrow{Q}$ by $*82{\cdot}231$ is only $\overrightarrow{Q} \upharpoonright \lambda \in \text{Cls} \to 1$; but since $\overrightarrow{Q} \in 1 \to \text{Cls}$, $\overrightarrow{Q} \upharpoonright \lambda \in \text{Cls} \to 1 . \equiv . \overrightarrow{Q} \upharpoonright \lambda \in 1 \to 1$. deleted from before The above ...

$*90{\cdot}41$. *Dem*. l. 3, $P_*$“$\alpha \cup \breve{P}$“$\alpha$ replaces $P_*$“$\alpha \cap \breve{P}$“$\alpha$

$*91{\cdot}242$. *Dem*. l. 3, $Q|$“$\overrightarrow{R}_{\text{ts}}$‘$P$ replaces $Q|$“$R_{\text{ts}}$‘$P$

$*91{\cdot}43$. $[*37{\cdot}1 . *43{\cdot}1]$ replaces $[*37{\cdot}1 . *43{\cdot}101]$

$*91{\cdot}521$. *Dem*. l. 10, $*91{\cdot}1{\cdot}33$ replaces $*91{\cdot}15$

$*91{\cdot}62$. compared with $*90{\cdot}1$, replaces compared with $*90{\cdot}11$,

$*91{\cdot}73$. *Dem*. l. 10 , $*91{\cdot}27$ replaces $*91{\cdot}504$

$*92{\cdot}144$. $\text{Ɑ}$‘$R \subset \text{D}$‘$R$ replaces $R \in 1 \to \text{Cls} . \text{Ɑ}$‘$R \subset \text{D}$‘$R$

$*92{\cdot}145$. $\text{D}$‘$R \subset \text{Ɑ}$‘$R$ replaces $R \in \text{Cls} \to 1 . \text{D}$‘$R \subset \text{Ɑ}$‘$R$

$*92{\cdot}146$. $\text{Ɑ}$‘$R \subset \text{D}$‘$R$ replaces $R \in 1 \to \text{Cls} . \text{Ɑ}$‘$R \subset \text{D}$‘$R$

$*92{\cdot}147$. $\text{D}$‘$R \subset \text{Ɑ}$‘$R$ replaces $R \in \text{Cls} \to 1 . \text{D}$‘$R \subset \text{Ɑ}$‘$R$

$*92{\cdot}32$. *Dem*. l. 4, $R \mathrel{\dot\subset} R_* \mathrel{\dot\cup} \breve{R}_*$ replaces $R \mathrel{\dot\subset} R_* \cup \breve{R}_*$

$*95{\cdot}1$. *Dem*. l. 3, $[*90{\cdot}11]$ replaces $[*90{\cdot}111]$

$*95{\cdot}37$. $\overrightarrow{B}$‘$Q . \dot{\exists}! R$ replaces $\overrightarrow{B}$‘$Q$ .

$*95{\cdot}382$. *Dem*. l. 4, $*93{\cdot}12$. replaces $*93{\cdot}101$.

$*95{\cdot}41$. $\text{D}$‘$R \subset C$‘$P . \text{Ɑ}$‘$R \subset C$‘$Q . \supset$ : replaces Hp $*95{\cdot}41 . \supset$ :

$*95{\cdot}42$. and $*95{\cdot}43$. Hp $*95{\cdot}411$ replaces Hp $*95{\cdot}41$ $*95{\cdot}411$. Hp

$*95{\cdot}46$. $\overrightarrow{B}$‘$Q$ replaces $\overrightarrow{B}$‘$\breve{Q}$

$*95{\cdot}62$. $s$‘gen‘$\breve{P}$ replaces $s$‘gen‘$P$

$*95{\cdot}7$. *Dem*. l. 8, $\text{D}$‘$S - \text{Ɑ}$‘$R$ replaces $\text{Ɑ}$‘$S - \text{Ɑ}$‘$R$

$*96{\cdot}131$. $\vdash : x \in \text{D}$‘$R$ replaces $*96{\cdot}131$. $\vdash . x \in \text{D}$‘$R$

$*96{\cdot}15$. $\vdash$ . D‘ replaces $*96{\cdot}15$. $\vdash$ : D‘

$*96{\cdot}31$. replaces $*95{\cdot}31$. (in $*96$!)

$*96{\cdot}49$. $R \in \text{Cls} \to 1$ replaces $R \in \text{Cls} \to 1 . x \in \text{D}$‘$R$

$*96{\cdot}492$. l.13, deleted $\vdash : R \in \text{Cls} \to 1 . \supset . (\exists S) . S \in \text{Pot}$‘$R . (I_R$‘$x) \upharpoonleft R_{\text{po}} \mathrel{\dot\subset} S$

$*97{\cdot}23$. *Dem*. l. 8, $\supset$ :: replaces $\supset$ :.

$*97{\cdot}242$. $\text{D}$“$\epsilon_\Delta$‘$\overleftarrow{R}_*$“$\overrightarrow{B}$‘$R$ replaces $\text{D}$“$\overleftarrow{R}_*$“$\overrightarrow{B}$‘$R$

$*97{\cdot}402$. $R \in \text{Cls} \to 1 . x \in \text{D}$‘$R : (\exists S) . S \in \text{Pot}$‘$R$ replaces
$R \in \text{Cls} \to 1 : (\exists S) . S \in \text{Pot}$‘$R$

$*97{\cdot}58$. *Dem*. l. 3, $*93{\cdot}33 . *40{\cdot}13{\cdot}38{\cdot}43$ replaces $*93{\cdot}33 . *40{\cdot}8$

$*97{\cdot}58$. *Dem*. l. 6,7, $s$‘$\overset{\leftrightarrow}{R}_*$ replaces $\overset{\leftrightarrow}{R}_*$ twice

Russell rejected around three quarters of Boskovitz’ corrections, including several that were justified and would not have required renumbering. For example, it appears that the second set of three dots (:.) should indeed be replaced by two (:) in $*20{\cdot}112$, although the use of more dots than necessary might be considered to be allowable optionally. $*21{\cdot}702$ seems to be a simple error, $(R, x)$ must be replaced by $(x, R)$ as Boskovitz suggests, but these corrections were not made. This particular error is especially puzzling because it is on the following list in Russell's hand, also saved in his copy of *PM*. The fate of items on this list is complicated.

*Corrections in PM from Mr Boscovitch.*
$*11{\cdot}13$ and $*11{\cdot}131$ are identical.
$*20{\cdot}03$ ought to come after $*20{\cdot}08$
$*21{\cdot}702$. For $g!(R, x)$ read $g!(x, R)$
$*24{\cdot}492$ note, for "$*93{\cdot}273\ldots$" read "$*73{\cdot}841$".
$*43{\cdot}491$ for "$\alpha$" read "$\beta$".

The first observation, that $*11{\cdot}13$ and $*11 \cdot 311$ are the same (and not $*11 \cdot 131$ as Russell writes) is correct, but no change is made. Russell may have lost track of the error, but most likely the change would have required renumbering in proofs later on.

The second suggestion also seems justified. Boskovitz points out that the use of expressions for functions true of classes of classes must be defined, as in $*20{\cdot}08$, before they are used as in $*20{\cdot}03$. Russell's failure to mark the necessary distinction between classes of classes and classes of functions is a genuine, though remediable, weakness in the presentation of the "no-classes" theory.[17] That this suggestion might make it to a "short list" but still not to the point of corrections is understandable given its particular, vexed, content. The failure to make the third correction is clearly an error on Russell's part, and one without the subtle excuses that can be given for the last. The fourth item on the list is also correct. $*24{\cdot}492$ is indeed first used in $*73{\cdot}841$ rather than $*93{\cdot}273$ as both editions say. Yet this list cannot simply be read as a list of intended corrections that were not made for various reasons, for the last was indeed made in the second edition.

The "Errata" pages and Boskovitz' long letter are the only identified sources of corrections to the first edition. From the record of actual changes made it appears that Russell carefully examined the suggestions, accepting those small ones that would not require serious changes in the text, most importantly, no renumbering of theorems. He seems to have missed some of the suggestions, and declined to make others which do not seem to have required really large corrections. On the other hand, he does not seem to have ignored serious errors, nor indeed to have had any brought to his attention. All of the evidence thus bears out the exact wording in the opening sentence of the new edition. Only misprints and minor errors were changed.

[17] See Linsky (2004b).

### *2.6.3 Other changes to the first edition*

What other changes were made? This is not a simple question, since no normal method of collating editions will work for *PM*. Because the first two volumes were reset, no sort of visual collation would work to discover changes. It is possible to make a collation that looks for the number of lines in a text. For *PM* that would determine if any “starred numbers”, i.e., theorems or definitions, were added or deleted. No such changes were made. A more careful check would require digitalizing the whole of both editions and performing a mechanical check. The production of a perfect “critical edition” of the second edition, therefore, is still waiting. It would seem, however, that the scope and nature of the changes accepted from Boskovitz gives a good indication of the totality of changes to the first edition.

The following changes have come to light in the process of a visual inspection, reading the two editions side by side, looking for changes.[18]

$*2{\cdot}1$. [$*2{\cdot}08$.($*1{\cdot}01$)] replaces [Id.($*1{\cdot}01$)]
$*14{\cdot}241$. *Dem*. l.6, $y = x$ . replaces $y = x$
$*14{\cdot}26$. *Dem*. l.5, $\psi b$ . replaces $\psi b$
$*14{\cdot}27$. *Dem*. l.4, $(\psi x)$ : replaces $(\psi x)$
$*20{\cdot}633$. (The corresponding exception is to be understood in $11{\cdot}07$.)
deleted from before This is the analogue of $11{\cdot}07$
$*33{\cdot}28$. and $*33{\cdot}29$., $\dot{\mathrm{V}}$ replaces V and $\dot{\Lambda}$ replaces $\Lambda$
$*33{\cdot}45$. to be first eliminated. replaces to have the larger scope.
$*34{\cdot}301$. $P|Q = \dot{\Lambda}$ replaces $P|Q = \Lambda$
$*34{\cdot}302$. *Dem*. l.4, $P|Q = \dot{\Lambda}$ replaces $P|Q = \Lambda$
$*34{\cdot}32$. $P = \dot{\Lambda}$ and $P|Q = \dot{\Lambda}$ replace $P = \Lambda$ and $P|Q = \Lambda$
$*35{\cdot}832$. $\vdash . \dot{-}$ replaces $\dot{-}$
$*35{\cdot}89$. $(\alpha \uparrow \gamma)$ : replaces $(\beta \uparrow \gamma)$ :
$*37{\cdot}264$. E! $\beta$ replaces $\exists ! \, \beta$
$*41{\cdot}5$. $\dot{p}\text{‘}s\text{‘}\lambda \;,\!\mid\!,\text{“}\mu$ replaces $\dot{p}\text{‘}(s\text{‘}\lambda \;,\!\mid\!,\text{“}\mu)$
$*55{\cdot}1$. $(\iota\text{‘}x) \uparrow (\iota\text{‘}y)$ replaces $\iota\text{‘}x \uparrow \iota\text{‘}y$
$*55{\cdot}35$. and $*56{\cdot}261$., $\dot{\Lambda}$ replaces $\Lambda$
$*73{\cdot}01$. $(1 \rightarrow 1)$ replaces $1 \rightarrow 1$
$*73{\cdot}241$. $\alpha \; \overline{\mathrm{sm}} \; (\breve{R}\text{“}\alpha)$ replaces $\alpha \; \overline{\mathrm{sm}} \; \breve{R}\text{“}\alpha$
$*81{\cdot}23$. and in *Dem*. l. 2, $\iota\text{‘}R\text{‘}y$ replaces $\iota\text{‘}\overrightarrow{R}\text{‘}y$
p.547, l.17, in this number ($*95$) may replaces in this number may
$*92{\cdot}161$. $T \in \mathrm{Potid}\text{‘}\breve{R} : Q|P$ replaces $T \in \mathrm{Potid}\text{‘}R : \breve{Q}|P$
$*93{\cdot}132$. and *Dem*. l.1 and 2, $= C\text{‘}P \cap \text{Ɑ}\text{‘}T$ replaces $= \text{Ɑ}\text{‘}T$
$*93{\cdot}31$. $\text{Ɑ}\text{‘}(T|P)$ replaces $\text{‘}(T|P)$

[18] Shaheen Islam did this from January to September of 2006.

The original sources of these various corrections are unknown. One, to $*73\cdot01$, was copied out by Russell in his list for Carnap which includes the correction of $1 \to 1$ to $(1 \to 1)$. Almost all the rest of these are simple typographical corrections which may have been made during the printing process by Russell himself.

Two insertions in Volume II reflect issues raised in the new Introduction. At page *ix* in the Prefatory statement (*viii* in the first edition), the remark:

> But identity between a function and a class does not have the usual properties of identity; in fact, though every function is identical with some class, and vice versa, the number of functions is likely to be greater than the number of classes. (*PM* II, p.*viii*)

becomes

> But if we adopt the theory of $*20$, as opposed to that suggested in the Introduction to the second edition, identity between a function and a class does not have the usual properties of identity; in fact, though every function is identical with some class, and vice versa, the number of functions is likely to be greater than the number of classes.
>
> (*PM* II, p.ix)

Chapter $*20$ is titled "General Theory of Classes", which does not assume the extensionality principle by which co-extensive propositional functions are identical.[19]

The second addition is a comment to Cantor's Theorem, $*117\cdot661$, where Russell adds, in parentheses: "See, however, the Introduction to the second edition". This is a reference to the passage preceding the conclusion "Consequently the proof of $2^n > n$ collapses when the axiom of reducibility is not assumed." (*PM*, p.*xliii*). These two brief additions were added as a consequence of the two fundamental changes in the logic of *PM* from the first to second editions, the adoption of extensionality, by which co-extensive propositional functions are identified, and the effort to derive theorems without using the axiom of reducibility. Such fundamental changes, which are the main topic of the new Introduction, only show up as parenthetical remarks in the body of *PM*.

Other differences between the first and second editions of Volume II seem to all be the result of the typesetting process. Thus in $*102\cdot84$ the order of two conjoined antecedents is switched. On pages 108, 120, and 272 of the first edition, some parentheses, '(' and ')' , and some braces, '{ ' and '}', are changed or deleted. At 165 and 316 symbols in the first edition are deleted, clearly by an error of the typesetter. All other changes are already identified on errata sheets appearing with the first edition.

The corrections to the material of the first edition of *Principia Mathematica* do indeed seem to be limited to "misprints and minor errors", the largest block

[19] See Linsky (2004b) for a discussion of identity between functions and classes.

of which can indeed be credited to Boskovitz and Behmann, all as Russell says. In addition to confirming those introductory words in the new Introduction, this listing of corrections should serve to display the variety of kinds of correction that were made and give a rough sense of their total number and distribution through the work. Russell seems to have made a serious effort to correct those errors that he could find, which would at least not require reordering of theorems or other large changes. It is common practice for logic teachers to work through a text, finding errors which are collected at the end of the term and sent off to the author, perhaps for correction in the next edition. However widely read and influential among logicians in the years from 1910 to 1924, *PM* does not appear to have been studied by groups in that fashion.

For the reprint of the second edition in 1950 some further corrections were made. A letter to Russell from "C.F.E." at Cambridge University Press from 16 September 1949 accompanies some pages from Volume II that were to be photographed and lists six corrections.[20] From the limited range of these particular examples, all from Volume II, one can surmise that there were more to be found throughout the three volumes.

### 2.7 Frank Ramsey and the second edition

Frank Ramsey had heard from Ludwig Wittgenstein that Russell was preparing a new edition of *PM* and discussed this in a letter from September of 1923 (Monk, 2000, p.45). C. K. Ogden introduced Ramsey to Russell on 2 February 1924. Ramsey's diary for the next day reports that Russell was "... good against W's identity, poor on types; doubts Dedekindian section as he can't prove it, though he can Math Induction without Axiom of Reducibility." (Monk, 2000, p.46) Ramsey here seems to be referring to Wittgenstein's theory of identity, as stated in *TLP*:

> 5.53 Identity of the object I express by identity of the sign, and not by means of a sign of identity. Difference of the objects by difference of the signs.

Wittgenstein thinks that with a proper notation, using distinct variables to stand for distinct objects, it would not be necessary, or even possible, to express the notion of identity.[21] The problem with Dedekind cuts, which define a real by a bounded set of rational numbers, is that this way of constructing the real numbers still requires the axiom of reducibility if a bounded set of reals is to have as its bound another real number, for the "cut" will be of a higher order than the reals that comprise it. Russell expresses concern about the theory of real numbers without the axiom

[20] Corrections are to page 338 "two-thirds down page", $*205{\cdot}254$, $*210{\cdot}232{\cdot}233$, $*215{\cdot}01$, and $*234{\cdot}521$.
[21] Thus *TLP* 5.533: The identity sign is therefore not an essential constituent of conceptual notation.

of reducibility in the Introduction to the second edition. The remark that Russell can prove "Math Induction without Axiom of Reducibility" shows that Russell had already reached the main conclusions of Appendix B by February 1924. It is thus likely that Russell did all of the work reproduced in this volume in two bursts of activity, between the letter from Whitehead in 1923 and the reading of manuscripts and proofs by Ramsey in 1924.

Ramsey wrote a widely quoted letter on 20 February 1924 to Ludwig Wittgenstein:

> I can't write about work, it is such an effort when my ideas are so vague, and I'm going to see you soon. Anyhow I have done little except, I think, made out the proper solution rather in detail of some of the contradictions which made Russell's Theory of Types unnecessarily complicated, and made him put in the axiom of reducibility. I went to see Russell a few weeks ago, and am reading the manuscript of the new stuff he is putting into the *Principia*. You are quite right that it is of no importance; all it really amounts to is a clever proof of mathematical induction without using the axiom of reducibility. There are no fundamental changes, identity is just as it used to be. I felt he was too old: he seemed to understand and say "yes" to each separate thing, but it made no impression so that 3 minutes afterwards he talked on his old lines. Of all your work he seems now to accept only this: that it is nonsense to put an adjective where a substantive ought to be which helps in his theory of types. (Wittgenstein, 1995, p.197)

The discussion "a few weeks ago" probably is this same meeting of 2 February 1924. The "clever proof" of induction is referred to again. The main theorems of Appendix B had been completed at this point. He indicates in the next letter that he has been reading "Sheffer's thing", and so also presumably the material that ended up as Appendix A dealing with the Sheffer stroke. Ramsey says that, for Russell, "identity is just as it used to be." This seems somewhat different from his remark in the diary that Russell was "good against W's identity". Perhaps he had changed his mind as the force of Russell's criticisms had faded in the intervening weeks. It is interesting to note that Ramsey must have already been formulating new ideas about logical types, such as the proposal to simplify the theory of types with which he became identified.

In a letter to Russell, dated that same day, 20 February 1924, from Trinity College, Cambridge, Ramsey says:

> I am returning Sheffer's thing, but hope I may keep yours a little longer, as I have only just started on induction; and having various other things to do and being quite unused to following symbolic deductions I get on rather slowly.
>
> I wish Sheffer's book were out so that I could see how he used this very abstract stuff which in itself I find rather dull. The abstracts of papers to which he refers give one no idea. Many thanks for lending it to me. I found a few slips in your manuscript ($y$ for $x$ sort of thing); and there is a thing which recurs in the 26 cases of the prim prop and puzzles me very much. It

is that you change the order of apparent variables whenever it seems convenient, for which I can't see the justification. Sometimes it is covered by one of your primitive propositions that $(\exists x){:}(y).\phi x|\phi y$ can be inferred from $(y){:}(\exists x).\ \phi x|\phi y$; but ~~sometimes~~ generally the matrix is not of the right form for this prim prop to be applicable, at least immediately. I think it would be a good idea to give an explanation of the process, which may well puzzle other people besides me.

I'm putting in with this a list of all the recent literature which I have heard of, ~~to~~ which you could possibly want to look at or refer to. I'm afraid it is a poor lot which makes me suspect there is some better stuff I haven't heard about. If I hear of any more I will let you know of it.

Unless you want your manuscript back at once (I expect to have finished with it by the end of next week) please don't bother to answer this now, but wait till I send your manuscript with comments.

Ramsey is well into his reading of a manuscript at this point. "Sheffer's thing" is probably the peculiar typescript titled "The general theory of notational relativity" (Sheffer, 1921), apparently loaned by Russell and returned, as a copy is in the Archives. The reference to a primitive rule "... that $(\exists x){:}(y).\ \phi x|\phi y$ can be inferred from $(y){:}(\exists x).\ \phi x|\phi y$" identifies this as the "Hierarchy of propositions and functions" manuscript. This rule is the version of $*8{\cdot}13$ on the unfoliated leaf between *14* and *20* of HPF. This is one of two primitive propositions that do not appear in the same form in HPF and in the final version. That Ramsey was reading the HPF manuscript is compatible with the reference to the "26 cases of the prim prop" which are revised in the final version. The first six of the theorems ($*8{\cdot}32{\cdot}321{\cdot}322\ {\cdot}323{\cdot}324{\cdot}325$) and then two more, $*8{\cdot}333$ and $*8{\cdot}342$, are left in the HPF manuscript, and were revised for Appendix A. The new proofs are of a much clearer, and more standardized form, and so may well have been revised to meet Ramsey's objections.

The letter also refers to a list of "recent literature" that is probably the origin of the list at the end of the Introduction to which several commentators have referred. This evidence supports the view that Ramsey came up with the bulk of the list, and most probably the names with which Ramsey had just become familiar, in his travels on the continent, which included a visit to Göttingen before Vienna.

Ramsey's letter reports that he has "only just started on induction", which is compatible with his working on the later leaves of the "Hierarchy" manuscript, at (HPF, 60), in February of 1924. If HPF was the only preliminary manuscript, however, then it must have been what Ramsey was reading in February.

Russell may then have sent the material to be typeset before going to the U.S., or at least all but Appendix C, yet the the following suggests that the material wasn't ready for the printer until the fall. In a letter from Ramsey to Russell, dated 22 September 1924 from Mahlerstrasse in Vienna, we find:

I shall be very pleased to read your proofs, of course. You are quite right that I could not read the M-S for some time, as I shall be very busy, so that it would be much best to send it to the printer at once. By the time it is printed I hope I shall have more spare time.

That Ramsey refers to "the M-S" without suggesting that he has already seen parts of it is also compatible with the view that what he had seen before was an earlier draft of the HPF manuscipt.

Ramsey read the printer's proofs of the new material. This undated letter, which includes several pages of corrections, is in the Archives:

Dear Russell,

I have read the Introduction and verified nearly all the references. But I haven't yet been sent proofs of more than a few pages of the appendices. Here are a few corrections and comments on the Introduction.

Yours fraternally,

Frank Ramsey[22]

pXIII line 4 from bottom of text. for $\vdash .(fx).\phi x$ read $\vdash .(\exists x).\phi x$
pXIV line 21 for Chevistek read Chwistek
line 24 for Wittge*r*stein read Wittgenstein
line 35 for Dadekindian read Dedekindian
pXV I think the last paragraph might be improved by a little alteration.

Surely it is not "proofs" in which crude fact is required, for proofs are independent of fact. Also it is not quite clear what view you are taking; what about false atomic propositions which are quite as important as true ones (if p is false ∼p is true)? Do you take Wittgenstein's view that the objects and so the totality of atomic props are given, so that all we want to know is *which of them* are true, and which are false? In this case the false ones are given residually by giving the true ones. But if we do not know what atomic props there are we need to be told both the true ones and the false ones.

*Bibliography*

It seems to me a pity to say that some of these items are "more important" than anything whatever. It might be worth saying that this is only a bibliography of the ultimate foundations of mathematics; there is a good deal of (dull) stuff on sets of axioms for the Mengenlehre etc which is omitted, but is clearly within the scope of *Principia Mathematica*.

Why not put Wittgenstein into the bibliography?

Additions

H. Weyl. Randbemerkungen zu Hauptproblemen der Mathematik. *Mathematische Zeitschrift* vol 20

L. E. J. Brouwer. Intuitionistische Mengenlehre. *Jahresberichten der deutschen Mathematiker-Vereinigung* vol 28

[22] Undated letter, on King's College Cambridge letterhead, no. 111547. The corrections are listed on three pages.

M. Schönwinkel. Ueber die Bausteine der mathematischen Logik.
*Mathematische Annalen, Vol.92.*

Corrections.
*should read*
L. E. J. Brouwer. Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. *Verhandelingen d.K. Akademie v. Wetenschappen.* Amsterdam 1918, 1919.

A. Tajtelbaum-Tarski.
It is not "Tome" but "Tom" apparently a Polish word. But elsewhere you say "Vol."

H. M. Sheffer not H. M. Shaffer.
It would be better to put after *"Bulletin of the American Math Soc"* "Vol XVI" instead of "1910" as the vols do not coincide with years exactly.

Of the additions the only interesting one is Schönwinkel which is curious and I think instructive.[23]

I have not checked Lewis, or Chwistek in the *Annales de la Societé Mathématique de Pologne* which is not in Cambridge.

These fragments of corrections are full of bits of information. Firstly, they are clearly corrections to printer's proof sheets. The paragraph on "proofs" will be discussed separately, when considering the Introduction, to which it pertains. Next, Russell accepted Ramsey's suggestion for describing his list, and so the final version does not include the remark that the entries are "more important" contributions, although this does occur in the earlier manuscript.

Ramsey comments that "It might be worth saying that this is only a bibliography of the ultimate foundations of mathematics; there is a good deal of (dull) stuff on sets of axioms for the Mengenlehre etc. which is omitted, but is clearly within the scope of *Principia Mathematica*." It is clear that this remark shows the intended scope of the list, as including just foundations of mathematics rather than all developments in symbolic logic. The list is not intended to cover all such developments, although it is important to consider what is included as evidence of what Russell knew of the subject at that time. The list does originate with Ramsey, and besides the discussions of Sheffer and Chwistek, it is not clear how much those works influenced Russell. The reason for that will be discussed below when it is seen what these particular papers present, namely alternatives to logicism as foundational projects.

That Ramsey still subscribed to the logicist project, and had not gone over to the view that axiomatic set theory is the proper foundation for mathematics, is seen in the sarcasm of "there is a good deal of (dull) stuff on sets of axioms for the Mengenlehre etc. which is omitted, but is clearly within the scope of *Principia Mathematica*." That the question "Why not put Wittgenstein into the bibliography?"

[23] In both cases in this letter it is "Schönwinkel" rather than "Schönfinkel".

even arose is peculiar. The Introduction openly discusses Wittgenstein's views, and certainly no one would place the *Tractatus* among the "stuff on axioms for the Mengenlehre" which does not belong in a list of works on foundations. Perhaps it didn't seem appropriate to Russell to include it in a list including Hilbert, Weyl and Brouwer who had well worked out technical proproposals. Perhaps the list was intended to just include alternative foundational projects. Or was this just an oversight, as Russell would have cited the *Tractatus* already in the Introduction to the second edition, and so seen no need for a second bibliographical entry?

While most of the corrections show that Ramsey was looking at the originals as he worked, "M. Schönwinkel" rather than "M. Schönfinkel" is one obvious exception, and the fact that the Chwistek article was "not in Cambridge" suggests that Ramsey had not read the article, although Russell had, or at least had received the offprint from Chwistek. Russell, however, would have seen Schönfinkel mentioned in a letter from Behmann to Russell of 20 May 1923, where Schönfinkel is described as generalizing the Sheffer stroke.[24]

The following letter is dated "3rd Dec", presumably of 1924, and is on King's College letterhead:[25]

Dear Russell,
I have read appendices B and C very carefully, but not Appendix A, which does not seem very different from the M-S I read. As I am very busy I hope you will not mind my not reading Appendix A.

I haven't found anything wrong about induction except for a few misprints or slips of which I enclose a list. It seems to me an awfully good theory.

Yours fraternally
F.P. Ramsey

A fragmentary document appears to be the first page, at least, of whatever errors Ramsey found in Appendix B:[26]

$*89{\cdot}1$. I think the punctuation is wrong in the third and fourth lines of the proof.
$*89{\cdot}105$ In (5) insert after "Hyp" "$z \in p\text{‘}\breve{R}\text{“‘}\kappa$"
p.653 last line but one before $*89{\cdot}12$
for "$\eta \cup \iota\text{‘}y - \mu_2$" read "$\eta \cup \iota\text{‘}y = \mu_2$"
$*89{\cdot}13$ For "[$*89{\cdot}11{\cdot}111{\cdot}113$]" read "[$*89{\cdot}11{\cdot}111{\cdot}112{\cdot}113$]"
$*89{\cdot}132$. In (7) insert after "Hp(2)" "$\sim (yR|R_* y)$"
$*89{\cdot}16$ In the enunciation for "$\alpha\ \nu\ \epsilon$" read "$\alpha\ \sim \epsilon$"
$*89{\cdot}18$ in (1) for "$*89{\cdot}14{\cdot}17$" read "$*89{\cdot}12{\cdot}14{\cdot}17$"
$*89{\cdot}19$ in (1) for "$*89{\cdot}15{\cdot}17$" read "$*89{\cdot}12{\cdot}15{\cdot}17$"
$*89{\cdot}2$ in (1) for "$\supset_{\alpha,\mu}$" read "$\supset_{\alpha,u}$"
and for "$R(x \longmapsto\!\dashv y) \in \mu \,.\supset.\, R(x \longmapsto\!\dashv y) \in \mu$"
read "$R(z \longmapsto\!\dashv y) \in \mu \,.\supset.\, R(x \longmapsto\!\dashv y) \in u$"

[24] This letter is quoted in Mancosu (2003, n.46). [25] RA 710.111548. [26] RA 710.111549.

So, by December 1924 the final proofs were ready to correct, and Ramsey read them all, except for Appendix A, which did not seem very different from the manuscript he had seen. If that manuscript was in fact HPF, then he did not carefully check the new proofs of the twenty six theorems about the instances of the primitive proposition formulated with the Sheffer stroke. Ramsey would have seen at a glance that they had been revised and tidied up as he had suggested in his comments on the first manuscript. It is clear from this letter that Ramsey also read the printer's proofs of Appendix B, and had done so carefully enough to find typographical errors in the symbolism. He even found mistakes near, and even in, the two numbers $*89\cdot12$ and $*89\cdot16$ that are difficult to interpret and will be discussed in a chapter below. He did not, however, identify any peculiarity in identifying classes of different types, or notice the error in $*89\cdot16$ which Gödel found many years later.

## 2.8 Printing the second edition

The new Introduction, Appendices A, B, and C, and an eight page "List of definitions" were added to Volume I and the old material was reset . In the first edition Volume I contains a preface, table of contents and other front material from $i$ to $xvi$ and then pages 1 to 666 of numbered material, making a total of 682. In the second edition, the material in those numbered pages takes up $i$ to $xii$ and 1 to 634, for a reduction to 646. The addition of a new introduction (*xiii* to *xlvi*) and Appendices A (pages 635 to 649), B (pages 650 to 658), C (pages 659 to 666), and the List of definitions (pages 668 to 674) make a total of 720 numbered pages for Volume I in the second edition.[27]

In the first edition Volume II begins with a "Prefatory statement of symbolic conventions" on pages *ix* to *xxxiv* and has pages numbered 1 to 772 for a total of 806. That same material is printed on *vii* to *xxxi* and 1 to 742 in the second edition, for a total of 773 printed pages, a reduction of 33, saving two sixteen page "signatures".

Inspection of Volume III shows that it was simply reprinted from the first edition by a photographic process. Volume III has identical pages $v$ to *viii* of preface and numbered pages 1 to 491 for a total of 500 in both editions.

The material repeated in the second edition was undoubtedly set directly from the first. Symbolic material is reset line by line, with the same number of lines of symbols (from 32 to 38) in both editions. The difference in pagination comes from the text, which is much condensed in the second edition. The Introduction runs to

[27] Full bibliographical information is presented in Blackwell & Ruja (1994, A9). The volume number will not be indicated for Volume I, thus (*PM*, p.164) is in Volume I, (*PM* II, p.83) is in Volume II.

88 pages, but the material is set in 84 in the second edition. Text is set with around 41 or 42 lines per page, so the economy comes from setting more characters in a line in the second edition.

It appears that the new material was typeset by a pair working side by side, with each typesetter taking a batch of manuscript of several pages. Their names, "Symonds" and "Dawson", are written in the same manner (at an angle at the upper left of the material, often underlined, and in one hand). The names are on regularly spaced pages, in order, starting with the Introduction to the second edition.[28] Another name, "Rackham", appears in just the same spot on the top of a manuscript leaf from the first edition that has recently emerged in the Bertrand Russell Archives. This little bit of the typesetting process at Cambridge University Press was unchanged between the editions, although, of course, Russell kept the final manuscripts of the second edition, while those of the first have almost completely disappeared.[29]

The second edition has remained in print since the first volumes appeared in 1925 (Volume I) and 1927 (Volumes II and III). Volume I was separately reprinted in 1935. Reprints are listed from 1950, 1957, 1960, 1963, 1968, 1973, 1978 and more, as the book is still in print. This large, hardcover three volume version is the only version of the full work that is available from the publisher. Most readers encounter *Principia Mathematica* in the form of a paperback edition titled *Principia Mathematica to ∗56*, first published in 1962, now in the Cambridge Mathematical Library series and cover. The *Principia Mathematica to ∗56* edition of 1962 is a reprint of the second edition to page 383, and then Appendices A and C are reprinted with numbers 385 to 399 and 401 to 408. The List of abbreviations is on 409 and 410, ending now with a listing for $*56{\cdot}03$. In this work page references for Appendices A and C will use the numbering in *Principia Mathematica to ∗56*. References to Appendix B will use the only page numbers it has ever had, 650–658.

The first paperback edition, with its cover art of *PM* symbolism and endorsements on the back, was widely used for forty years as a textbook. It is this paperback addition that has the well known blurb from Willard Van Orman Quine, "This is the book that has mean the most to me." Some correspondence in the Archives gives the background of this edition. A letter of 25 May 1961 from R. A. Becher, listed on the letterhead of University Press Cambridge as an "Assistant Secretary" with two others, is apparently a second approach to Russell, describing in more

[28] The names occur on pages as follows: Symonds (1), Dawson (5), Symonds (9), Dawson (14), Symonds (17), Dawson (21), Symonds (25), Dawson (29), Symonds (33) , Dawson (37), Symonds (41), Dawson (53), Dawson (57), Symonds (61). Appendix A has fewer of these marks; Dawson (12), Symonds (16), Dawson (20), Symonds (24). Appendix B was set by Dawson (6), Symonds on the page with the error in $*89{\cdot}16$, folio (9). Appendix C has Dawson (1) and Symonds (2).

[29] See Linsky & Blackwell (2006) for an account of these surviving bits of manuscript, and of what can be deduced from them about the printing of the first edition.

detail a proposal from the Press. Becher says that the Press envisages a paperback edition suitable for use by undergraduates in Britain and America, comprising both introductions and Part I (which would run to $*43$). Russell responded to this letter on 27 May 1961. He is agreeable with the project but remarks that:

> I see that what you suggest might be useful to students who are not going too deeply into mathematical logic. At the same time, I am still rather worried by the fact that such a summary would, in effect, put an end to the study of whatever it omitted. I feel this in particular as regards part IV.

Russell goes on to express concern that the proposed edition may cut into sales of *Principia* and also that he is occupied with other matters, and so would be unable to make any alterations whatever.

The result extends the material included to Section A of Part II, "Unit classes and couples", and adds Appendices A and C, but not B. While concerns about the soundness of Appendix B might have influenced the decision not to reprint it, the more likely reason is that it presupposes material after $*56$, indeed it is properly placed with its final numbering as $*89$.

# 3

# Logic since the first edition

At the end of the manuscript of the Introduction to the second edition, Russell introduces a list of works in mathematical logic with this:

> The following are among the more important contributions to mathematical logic since the publication of the first edition of *Principia Mathematica*.[1]

The list has become an issue of discussion among commentators on the second edition because of what it reveals about Russell's knowledge of logical developments in 1924.

Ray Monk proposes a list of difficulties that Russell faced when beginning to work on a genuinely revised second edition, including this damning charge:

> First, Russell had not read most of the recent technical literature on the subject, and had neither the time nor the inclination required to master it.
>
> *(Monk, 2000, p.44)*

In 1929 Russell wrote a letter of reference which recommends Leon Chwistek over Alfred Tarski for a position at the University of Lwow. It raises the question of what Russell knew of Tarski's work in 1924. In their biography of Tarski, Feferman & Feferman (2004) endorse the view that it was "Ramsey and others in Russell's circle" who supplied the references for the list.[2]

In his essay "Logic in the twenties", Warren Goldfarb remarks that it is because Russell, like Frege, was unable to raise meta-systematic questions at all that he did not appreciate the significance of much of the work in logic that led up to model theory in the 1920s:

[1] The phrase "among the more important contributions" becomes "among the contributions" in print, following Ramsey's suggestion.

[2] Feferman & Feferman (2004, p.396). The evidence about Ramsey's role in making the list is discussed in the preceding chapter. See however, later in this chapter in the discussion of Tarski and Chwistek for the view both that Tarski's published work to 1924 is properly represented and that Russell had reason to express a preference for Chwistek.

> Thus Russell can say with a straight face in the Introduction to the second edition of *Principia Mathematica* [1925] that the most important advance in mathematical logic since the first edition was the Sheffer stroke – a purely internal simplification.
>
> *(Goldfarb, 1979, p.353)*

In fact Russell introduces the Sheffer stroke as "the most definite improvement resulting from work in mathematical logic during the past fourteen years ..." (*PM*, p.*xiii*) It is clear that this is intended to be the most definite improvement *to* the first edition of *Principia Mathematica*, not the most definite improvement in mathematical logic in general. That seems to come from confusing Russell's introduction of the Sheffer stroke with the phrase "among the contributions to mathematical logic since the publication of the first edition ..." which introduces the list on page *xlv*. Saul Kripke (2005) makes this point, but goes on, nevertheless, to suggest that Russell had not read Weyl's *Das Kontinuum*, summing up this way:

> The proper conclusion is that although he *is* aware (probably contrary to the impression that Goldfarb has) that significant other developments in logic had taken place, he probably has not kept up with them, even when they are highly relevant to his project in *Principia*.
>
> *(Kripke, 2005, p.1033 n.60)*

There was an explosion of developments in mathematical logic in the early 1920s, leading to the identification of first order logic and understanding of the nature of quantification, the formalization of rules of syntax and inference and the beginnings of proof theory with the attention to consistency proofs, and the notion of decidability, from the Hilbert school in Göttingen, and the investigations of propositional logic and its semantics in the Lwow–Warsaw school in Poland. As well, the "crisis" of the foundations of mathematics, which pitted Brouwer against Hilbert, had broken out in the early 1920s.[3] Before 1925 the work of German logicians was not widely known outside of Germany in the immediate aftermath of the First World War, and both Post and Skolem had done their work in relative isolation. In any case, little indication of any of these developments shows up in the second edition of *Principia Mathematica*.

In what follows I will first describe the study of logic at Cambridge, including Dorothy Wrinch, Henry Sheffer, Jean Nicod, and, most prominently, Ludwig Wittgenstein, who had arrived in Cambridge just as the first edition of *PM* had appeared, and stayed in correspondence, at least, through Russell's Introduction to the *Tractatus Logico-Philosophicus*. I will then go through the works on the list at the end of the Introduction, with an eye to seeing what influence they may have had on Russell's thinking, and to gather the evidence which suggests that Russell read them or at least knew of their contents.

[3] See Hesseling (2003) for an account of the controversy.

## 3.1 Cambridge and Dr Williams' Library

While the most systematic study of logic between 1910 and 1925 was going on in Hilbert's school at Göttingen, the instant fame of *Principia Mathematica* did inspire logical studies around Russell at Cambridge, and these developments played the largest role in the alterations made in the second edition of *PM*.

In the fall and winter of 1910 to 1911 Henry Sheffer visited Cambridge and attended Russell's lectures on mathematical logic from October to December of 1910.[4] He then made a tour of logicians in Europe, which included visits to Peano, Frege, and the Göttingen school, most likely at Russell's suggestion. Ludwig Wittgenstein arrived in Cambridge in October of 1911 and stayed until the end of 1913, when he left for Norway. Their ideas were to play a central role in the Introduction to the second edition and Appendices A and C, as discussed below.

Russell was engaged with the world of logic in other ways. He organized the philosophy section of the Fifth International Congress of Mathematicians, which was held at Cambridge on 22–28 August 1912.[5] In the program the list of participants includes Maxime Bôcher, L. E. J. Brouwer, Paul Carus (editor of *The Monist*), M. Fréchet, G. H. Hardy, E. V. Huntington, Dénes König (Julius König's son), P. E. B. Jourdain, J. E. Littlewood, E. H. Moore, A. R. Padoa, Giuseppe Peano, S. Zaremba (Leon Chwistek's teacher) and Ernst Zermelo.[6] Russell had also invited Frege, who declined with his cryptic remark that "I see that there are weighty reasons for my going to Cambridge, and yet I feel that there is something like an insuperable obstacle."[7] In August of 1912, at least, Cambridge was the center of activity in mathematical logic, and Russell was in a position to hear of all developments in the field.

The most important influence on Russell between the editions of *PM* was of course Ludwig Wittgenstein, who stayed in Cambridge from 1911 to 1913, and then corresponded with Russell after the war about the publication of the *Tractatus Logico-Philosophicus*. Russell reports that Wittgenstein would have nothing more to do with him after 1922, and this estrangement continued until 1929, when Wittgenstein returned to Cambridge.[8] Wittgenstein dictated his "Notes on Logic" in October of 1913, just before leaving for Norway, and then dictated further notes to G.E. Moore while in Norway in April 1914.[9] Russell and Wittgenstein did

[4] Sheffer's notes from these lectures are in Harvard University Library with Sheffer's papers.
[5] See Grattan-Guinness (2000, p.416).
[6] See the *Proceedings*, Hobson & Love, eds. (1913). Russell's introductory remarks are preceded by letters to Ottoline Morrell describing his impressions of the various logicians in *Papers 6*, p.444–9.
[7] Letter of 9 June 1912. Translation in Frege (1980).
[8] See Blackwell (1981) for the interaction between Wittgenstein and Russell in the period between the editions.
[9] The Notes are in Wittgenstein (1979). The biographical details are in (Monk, 1990, §5).

not meet again until after the war. Their communication for those years was by correspondence after that first visit in the Hague.

During the period between the editions, Russell's main concerns were with issues in philosophical logic and less with formal issues. His main concerns were with the nature of logical form, and the accompanying issue of replacing the multiple relation theory of judgement, and the emerging metaphysics of "logical atomism".

The discussion of logical atomism spanned the whole period between the editions of *Principia Mathematica*. His first use of the term in print comes from his paper "La réalisme analytique", from a lecture given in March of 1911, and published soon thereafter.[10] He says "You will observe that this philosophy is the philosophy of logical atomism" (*Papers 6*, p.135). The lecture series "The philosophy of logical atomism" (PLA) delivered in Dr. Williams' Library in 1918 and the essay "Logical atomism" (LA), written in 1923, span the entire time between the two editions.

Russell's move to accepting extensionality in the second edition can be seen to be an evolution of these issues about logical form in this "logical atomism" period. These concerns about logical form, in turn, arise from the developments out of the "multiple relation theory of judgement". This view appears in the Introduction to the first edition of *Principia Mathematica*. Russell introduces the notion of *fact*, in his terminology, a *complex* such as in:

> . . . a complex object composed of two parts $a$ and $b$ standing to each other in the relation $R$. The complex object "$a$-in-the-relation-$R$-to-$b$" may be capable of being *perceived*; when perceived, it is perceived as one object.
>
> *(PM, p.43)*

Thus in perception the perceiver is related to a fact by a two-place relation of perception. The analogous account of belief does not hold. Belief is not a two place relation between a believer and a proposition:

> It will be seen that, according to the above account, a judgment does not have a single object, namely the proposition, but has several interrelated objects. That is to say, the relation which constitutes judgment is not a relation of two terms, namely the judging mind and the proposition, but is a relation of several terms, namely the mind and what are called the constituents of the proposition. That is, when we judge (say) "this is red", what occurs is a relation of three terms, the mind, and "this", and red. On the other hand, when we *perceive* "the redness of this", there is a relation of two terms, namely the mind and the complex object "the redness of this". When a judgement occurs, there is a certain complex entity, composed of the mind and the various objects of the judgment.
>
> *(PM, p.43–44)*

It is because of this relation, which can have various numbers of arguments depending on the number of constituents of the believed "proposition", that this is called

[10] It is translated into English as "analytic realism" (*Papers 6*, pp.132–46).

the "multiple relation theory of judgment". Symbolically, then, the judgment of $x$ that $aRb$ will be symbolized as a complex atomic proposition:

$$J(x, a, R, b).$$

In 1913 there was an exchange between Wittgenstein and Russell about this very multiple relation theory.[11] This included a famous point to which Wittgenstein alludes in a letter to Russell on 27 July 1913, saying that he was "... very sorry to hear that my objection to your theory of judgment paralyzes you. I think it can only be removed by a correct theory of propositions." Wittgenstein's objection was a version of the so-called "direction problem", which he presents as the following challenge for any multiple relation theory: "Every right theory of judgment must make it impossible for me to judge that this table penholders the book. Russell's theory does not satisfy this requirement." (Wittgenstein, 1914–16, p.103). This objection is analyzed in the "Notes" as "The proper theory of judgment must make it impossible to judge non-sense." (Wittgenstein, 1914–16, p.95). These very words are echoed in *Tractatus Logico-Philosophicus* 5.5422.[12] Presumably, the judgment that "this table penholders the book" would have a form

$$J(x, a, b, c)$$

with $a$, $b$, and $c$ representing individuals, the table, the penholder, and the book. This purported judgment is nonsense, but does not seem to be precluded by the theory as presented by Russell.

While Russell did express doubts about the multiple relation theory in his "Philosophy of logical atomism" lectures, "I hope you will forgive the fact that so much of what I say today is tentative and consists of pointing out difficulties" (PLA, p.199), one can read him as not finally abandoning the view until some time between 1918 and the second edition. The account of belief in Appendix C in the second edition presents a belief as a relation between a believer and an entity, a class of sentences, and so returns to the earlier analysis of belief as a two place relation. Crucially, the object of belief is not a proposition, but rather a class of sentences. What remains in the move between accounts of belief, however, is the general nature of the facts, or complexes that constitute belief. On both accounts beliefs relate an individual to entities that are deemed individually acceptable, but not together having the unity of propositions on their own. Already in the multiple relation theory of judgement Russell had accepted that the only facts that exist are those involving relations and the objects they relate, although some of those entities

[11] See Griffin (1985).

[12] (*TLP*, 5.5422) "The correct explanation of the form of the proposition, "$A$ judges $p$", must show that it is impossible to judge a nonsense. (Russell's theory does not satisfy this condition.)"

might look very much like logical constituents of propositions, including "logical forms". It is easy to see how Russell might simply drop the sort of compound fact that is included in other facts, as is described in an account of perception in the first edition of *PM*, and move to an ontology of various atomic and compound facts involving logical forms. All of these facts are atomic in the sense that they do not include other facts as constituents, only objects and relations. Wittgenstein starts with such an ontology and then sees the logical form of complex propositions as simply asserting that certain combinations of states of affairs obtain and others do not. Thus all propositions can be expressed with only the truth functional connections of atomic propositions. In short, then, the multiple relation theory of judgment is just a small step away from the fully extensional account of belief and judgment expressed in Appendix C of the second edition. The latter transition was anticipated in the *Tractatus*, with Russell working his way to it through the evolving multiple relation theory of judgement.[13]

Russell delivered the lectures that became *Introduction to Mathematical Philosophy* at Dr. Williams' Library in Gordon Square, London between October and December of 1917. He presented the "Philosophy of logical atomism" lectures between January and March of 1918. Russell was tried for his statements about the war in early February of 1918 and then served his sentence in Brixton Prison from 1 May 1918 to 14 September 1918. Dorothy Wrinch corresponded with Russell throughout this period and visited him in prison in the role of secretary.

A letter from Wrinch to "Dear Bertie", on 18 July 1919, describing arrangements for rooms for a series of lectures and other information, says that "I proposed starting with a room of the same size as Billy's."[14] The letter includes this remarkable passage (with a footnote):

> I am very excited to hear that Wittgenstein's book has come. It was very strange – I dreamt that he had arrived in England last night and we* were all rushing to sit at his feet. Many thanks for remarks on judgment article. (*This does not include you. You were on a throne or chair, W on his chair and the rest of us on the floor !! )

Wrinch's correspondence with Russell includes some discussions of philosophical issues, including longish letters about her developing ideas about the nature of propositions. They appeared in an article in *Mind*, "On the nature of judgment" (Wrinch, 1919). Wrinch's purpose in the paper is to extend the multiple relation theory of judgement from atomic propositions of the form "*a* loves *b*" to molecular propositions and quantified propositions. To begin with, Wrinch answers the

[13] Godden & Griffin (2009) trace the whole history of Russell's views on propositions and the multiple relation theory of judgement through Appendix C and to the end of his career.

[14] A letter to Russell of 25 September 1919 contains "Billy's Library can only be taken for a course of lectures on Mondays, Tuesdays, or Fridays. Shall I arrange for Mondays?" These are presumably for the lectures of 13 October to 2 December 1919 at Dr Williams' Library on "The analysis of mind". (*Papers 9*, p.*xxxi*)

"criticism [which] is sometimes advanced that on this theory 'I believe that *a* loves *b*' cannot be distinguished from 'I believe that *b* loves *a*' ", namely one of the so-called "order problems" with the multiple relation theory.[15] Wrinch says that if one symbolizes "*a* loves *b*" as:

$$\phi(ab)$$

then "I believe that *a* loves *b*" becomes

$$J(I, \phi, a, b).$$

In its most general statement of the theory the judgments will be of the form

$$J(I, \phi, a_1, a_2, a_3, \ldots, a_n).$$

That "the arguments cannot be interchanged freely" conclusively shows that this order problem does not affect the theory.

The proposal that Wrinch makes, following suggestions that Russell works out in the Theory of knowledge manuscript, is that entities called "logical forms" should be added as constituents of the judgments. A conditional, "If he comes, I will go", which might be symbolized as

$$\phi a \supset \psi b,$$

will involve a form represented by $fx \supset gy$. The "evaluation" $E$ of a form $fx \supset gy$ with the constants $\phi$, $\psi$, $a$, $b$ will be expressed by the complex form

$$J(I, E_{\substack{x=a\ f=\phi \\ y=b\ g=\psi}}, fx \supset gy, \phi, a, \psi, b).$$

Wrinch repeatedly admits that the theory is complicated, but charges that the forms are no more complicated than the range of judgments that it captures. A "generalising" proposition, such as $(x).\phi x$ will be represented with a generalising operator $G$ and $(\exists x).\phi x$, with a "particularising" operator $P$. The logical form of my judgment that something is $\phi$ will then be

$$J(I, P_x E_{f=\phi}, fx).$$

Wrinch's account, then, has logical forms as constituents of judgements, as well as the "evaluation" operators that make sure that the forms are related to the individuals in the judgment in the right way.

Wrinch defends herself against the objection that logical forms will be complex unities of just the same sort as the propositions that the theory was formulated to avoid. She replies that:

15 See Griffin (1985) for an account of the various "order" problems.

I can only suggest that a form is a very colourless thing indeed. It is a few blank spaces with a bare logical structure uniting them: and I feel that the kind of way in which it is a unity does not in the least imply any propositional unity. All that is implied is that it is so constructed that if we operate on it, we shall not get nonsense; the existence of types belonging to each space will make that impossible. And this is an interesting point because it has been advanced as a criticism that on this theory it is possible to judge nonsense. Of course it is essential for any theory of judgment that such a thing should be impossible.

*(Wrinch, 1919, pp.324–5)*

Wrinch seems to be offering Russell a response to Wittgenstein's objections, and although there is no record of a response from Russell (beyond the thanks for "remarks" above) his close association with Wrinch suggests that he at least encouraged her to publish her thoughts in defense of his position even if he did not accept them in full. On the other hand, he did not ever mention the multiple relation theory again, and as we will see, Appendix C presents a very different theory.

Jean Nicod was in Cambridge during the war and returned to France after it, until his death in 1924. He was part of a circle which met weekly in London with Dorothy Wrinch and Victor Lenzen in a study group run by Russell.[16] Correspondence between Nicod and Russell survives, including discussion of a theory that will "get around the axiom of reducibility" in a letter of Nicod of 9 April 1917, through to the letter of 13 September 1923, which describes the plan of the second edition.

The largest influence on the logical doctrines in the second edition of *Principia Mathematica* was certainly Wittgenstein's *Tractatus Logico-Philosophicus*.[17] It mentions almost all of the ideas new in the second edition. The principle that propositions are truth-functional combinations of atomic propositions appears as

4.4 A proposition is the expression of agreement and disagreement with the truth-possibilities of elementary propositions.

The proposal to analyze all molecular propositions with the Sheffer stroke echoes this generalization of the stroke to an operation on arbitrary collections of propositions:

5.5 Every truth-function is a result of the successive application of the operation '$(- - -T)(\xi, \ldots)$' to elementary propositions.

This operation denies all the propositions in the right-hand bracket, and I call it the negation of these propositions.

Wittgenstein challenges the axiom of reducibility as certainly not a principle of logic:

16 According to Grattan-Guinness (2000, p.435).

17 Gregory Landini (2007) presents a different picture of the relationship between Wittgenstein and Russell and of the nature and origin of these doctrines.

6.1233 We can imagine a world in which the axiom of reducibility is not valid. But it is clear that logic has nothing to do with the question whether our world is really of this kind or not.

As such, of course, it would have no place among the principles of logic from which the truths of logic are to be derived in *PM*.

Besides this statement of views later adopted in the second edition, there is a more direct connection between Russell's Introduction to the *Tractatus* and the proposal to analyze belief contexts in Appendix C. This will be discussed below.

What this survey shows, I propose, is that there is an adequate explanation for Russell's interest in the issues that result in the innovations in the second edition in discussions that went on in his own circle involving Sheffer, Nicod, Wrinch, and, of the first importance, Wittgenstein. In the remainder of this chapter a look at the list of works at the end of the Introduction to the second edition will show the extent of Russell's awareness of developments in logic which were taking place outside of Cambridge. It appears that Russell was in touch with those developments, although they may not have influenced him much. Russell was aware of the position of his own version of logicism in the alternative accounts of the foundations of mathematics that arose in the early 1920s, and was also aware of the disputes about the nature of logic and extensionality and the theory of types in the same period. However, it is possible to read his lack of response to these other views as the result of a decision to work out his own position, rather than from some simple lack of understanding on his part.

## 3.2 Contributions since the first edition

While it is clear from the correspondence that Ramsey did send Russell a list at an early stage in the composition of the second edition, there is no evidence of what items Russell may have chosen to use, or which he may have added to the proposed list. Some items, from Chwistek, Nicod, and Sheffer, for example, were likely added by Russell.

The list is organized roughly by schools, with the work of Hilbert and his associates together, and Russell's own "school" of Lewis, Sheffer, Nicod, and Wittgenstein gathered at the end.[18] The position of the lone entry for "Schönwinkel" (as it is spelled in the manuscript and Ramsey's letters) at the end results from it being added to the manuscript late, upon Ramsey's suggestion. The other two additions, Weyl (1924) and Brouwer (1919) were probably added to the lists for

[18] The citations are copied from *PM* and corrected in the bibliography, which also lists English translations when they are available. See Blackwell (2005).

those authors at the proof stage, with "Schönwinkel" added on at the end of the last page.

The examination of the listed items below establishes that Russell almost certainly knew all the works, except perhaps for Ramsey's very late additions. Most of the items explicitly mention *PM*, and correspondence from later shows that Russell does know of the positions of the authors on foundational questions. Russell's views had traditionally been identified (with those of Frege) as a distinctive "logicist" program in the foundations of mathematics. George Boolos (1994) points out that in fact *PM* does not contain any specific assertion of the thesis that mathematics is logic, and so is not committed to the axiom of infinity being a logical truth. Wittgenstein expressed the view that the axioms of infinity and choice are not tautologous and so not logical truths, in correspondence with Russell, and then later in the *Tractatus*. Any notion that *PM* was not really a "logicist" work, however, could only have become possible with Ramsey's (1926) explicit formulation of Wittgenstein's view that logical truths are just tautologies, and so the axiom of reducibility is not a logical truth. In the early 1920s, then, "logicism" was identified with the views of Frege, Whitehead, and Russell, and the view that logic is not simply another branch of mathematics, like Boolean algebra, and that set theory does not provide the foundations of mathematics.

Foundational and technical work in Hilbert's school and Brouwer's intuitionist school, while relevant to the program of *Principia Mathematica*, did not necessarily have to influence or change its direction. It seems clear from what follows that Russell was aware of what he needed to do to revise *PM*, and also had an interest in other developments, even if he may not have had the inclination to master them, as Monk (2000) says. That Russell did not change his mind and start working in logic in the ways that were coming to dominate the field does not show that he had lost touch, or that we should not pay careful attention to those alterations that he did propose for the second edition.

Items below are as listed in the manuscript of the Introduction to the second edition, with the corrected version that occurred in print after proofs noted where it differs.

### *3.2.1 D. Hilbert*

D. Hilbert. (Axiomatisches Denken, *Mathematische Annalen*, **78**. Die logischen Grundlagen der Mathematik, *Mathematische Annalen* **88**. Neue Begründung der Mathematik, *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, 1922).[19]

[19] These are Hilbert (1918), (1923), and (1922).

These three papers present the main points of what is now known as “Hilbert’s program” and with it the emphasis on the axiomatic method and interest in proving the independence and, more importantly, consistency of the axioms of different theories. They include the notion of a formal and precisely defined syntax, including not only the well formed formulas of a system, but also the inference rules and, very importantly, axioms. The papers introduce Hilbert’s “proof theory”, based on the distinctions between mathematical theories and the “metamathematics” with which they are studied. Hilbert distinguishes a special “contentful” and “finite” part of mathematics (finitary mathematics) in which the proofs of metamathematics are to be carried out, and the infinitary mathematics of formal theories introduced by quantification over infinite domains.

None of this is reflected in the second edition of *PM*. It is these papers that might make the strongest case that Russell did not respond to important developments in logic in the second edition. Presenting *PM* with a formal syntax would have required rewriting the formal system, and expanding at least the initial proofs, adding a great deal more detail of the inferential steps. Adopting the metamathematical approach to logic would require first of all presenting the syntax and proof theory of logic as a formal system and so as an object of mathematical study, whatever the conceptual limitations of Russell’s view of logic as “universal” as has been prominently discussed.[20]

Yet Hilbert and Russell were not in different intellectual worlds. They corresponded, although not in a very substantive way about logical issues.[21] Hilbert does mention Russell favorably in the first paper (Hilbert, 1918):

> But since the examination of consistency is a task that cannot be avoided, it appears necessary to axiomatize logic itself and to prove that number theory and set theory are only parts of logic.
>
> This method was prepared long ago (not least by Frege’s profound investigations); it has been most successfully explained by the acute mathematican and logician Russell. One could regard the completion of this magnificent Russellian enterprise of the *axiomatization of logic* as the crowning achievement of the work of axiomatization as a whole.
>
> *(Hilbert, 1918, p.1113)*

Russell does not seem to be interested in Hilbert’s investigations of axiomatic systems as applied to logic, which may have seemed irrelevant to him, given the reduction of the axioms of propositional logic to a single one by Nicod. However, while there is no issue of independence when there is a single axiom, one still might wonder if it is the “shortest” possible axiom, and there is still the issue of

[20] As famously first argued by van Heijenoort (1967b) and then discussed in Hylton (1980).
[21] See the discussion in Sieg (1999) for details of their relationship and for the influence of Russell on Hilbert’s thinking about logic. Zach (2007) describes the development of Hilbert’s views during this time.

the consistency of the system. As well, the extension of the new system with the Sheffer stroke to include quantifiers, as carried out in part in Appendix A, revives Hilbert's sorts of questions.

One aspect of the second edition which might have been influenced by Hilbert is Russell's decision to abandon any distinction between bound and free variables in assertions, by treating theorems with free variables as tacitly bound by initial quantifiers.[22]

While this would have been general practice in logic by 1923, in Hilbert's papers one sees the importance of the equivalence of quantified formulas with certain quantifier free substitutes. Russell makes no mention of Hilbert in his discussions in the new introduction, however.

### *3.2.2 P. Bernays*

P. Bernays (Über Hilberts Gedanken zur Grundlegung der Arithmetik, *Jahresbericht der deutschen Mathematiker-Vereinigung*, **31**).

This paper was delivered in Jena, in September of 1921.[23] It presents "Hilbert's program" in the foundations of mathematics. The consistency of arithmetic is the central problem for the program, with the axiomatic method as it came to be seen in the Hilbert school, as the meta-mathematical study of mathematical systems as formalized precisely. The consistency problem is thus to be solved through the mathematical study of the notion of proof, formalized as "a figure with determinate concrete properties" in the contemporary manner. A proof is a sequence of formulas which are axioms, previously proved formulas, or formulas that follow from preceding lines by substitution or *modus ponens*, described formally as a sequence of the form $A$, $A \rightarrow B$, $B$. A proof of the consistency of arithmetic will amount to a demonstration that a statement $\Omega$ which expresses a contradiction does not appear as the last line of a proof. This *meta-mathematics*, as it is now called, is not subordinate to logic. Logic should be formalized and can be studied with meta-mathematical methods, just as mathematics can be studied from the point of view of logic, with mathematical forms of reasoning seen as the use of axioms with logical relations between them.

However, Bernays says:

The great advantage of Hilbert's procedure rests precisely on the fact that the problems and difficulties that present themselves in the grounding of mathematics are transferred from the epistemologico-philosophical domain into the domain of what is properly mathematical.

[22] See Hilbert (1922, §47), where quantifier-free expressions are treated as equivalent to their universal generalizations. Given the use of sentence letters '$p$', '$q$', etc in $*1$ to $*5$, this seemingly contradicts the assertion that propositions will not be in the range of quantifiers in *PM* made at page 185 (immediately) preceding the occurrence of a bound propositional variable at $*14{\cdot}3$. The suggestion that those letters are *schematic* adds a third class to *PM*'s distinction of *real* and *apparent* variables. It is simplest to assume that the system of *PM* allows, but avoids, the use of bound variables that range over propositions.

[23] This is Bernays (1922). There is no indication that Frege, who had retired in 1918, was present for this talk.

> Mathematics here creates a court of arbitration for itself, before which all fundamental questions can be settled in a specifically mathematical way, without having to rack one's brain about subtle logical dilemmas [Gewissensfragen] such as whether judgements of a certain form have a meaning or not.
>
> *(Bernays, 1922, pp.121–222)*

Bernays is more explicit about the difference between Hilbert's metamathematical program and the logicism of Frege and Russell. The program of "the logical grounding of mathematics" is judged a failure for reasons that have since become familar (Bernays, 1922, p.216). First, the logicist approach was not able to avoid the paradoxes of "naive set theory". The resolution of the paradoxes in *Principia Mathematica* is found wanting because of the necessity of an axiom of infinity and the axiom of reducibility. The objection to these two is not the now familiar view that they constitute existence claims and hence cannot be a part of logic. Rather it is simply that they must be adopted as *axioms*, and hence the logicist program requires the axiomatic approach of the Hilbert program. The justification of existential axioms relies on the replacement of existential statements with explicit constructions where possible. Here Bernays sketches what is now identified as the distinctively *finitist* component of Hilbert's program as Hilbert's concession to the constructivist program of Brouwer and Weyl.

> . . . two types of complete induction are to be distinguished: the narrower form of induction, which relates only to something completely and concretely given, and the wider form of induction, which uses either the general concept of whole number or the operating with variables in an essential manner.
>
> *(Bernays, 1922, p.221)*

We see here then all of the components of the Hilbert program which was later challenged by Gödel's theorem on consistency proofs (the "second incompleteness theorem") which has been discussed since as the source of a notion of finitistic meta-mathematical reasoning.

Though it does not distinguish approaches to the foundations of mathematics as was later done into the rival schools of logicism, intuitionism, and formalism/finitism, Bernays' paper does clearly indicate how Hilbert's program differs from that of Whitehead and Russell. Russell certainly must have been aware of the criticisms of the axiom of reducibility in the Hilbert School (and read the criticism that it is non-constructive explicitly in Chwistek's papers). His own doubts about reducibility had been with him from the beginning, and were reinforced from many directions.

There is no clear effect, then, of Bernays' paper, as with those of Hilbert, in the revisions of the second edition of *PM*. On the other hand Russell cannot be criticized for failing to take them into account, as Bernays (and Hilbert) explicitly present their program as an alternative to the logicism of Frege and Russell. That

Russell might consider working out what restricted portion of mathematics could be developed without the axiom of reducibility, however, does seem quite in keeping with the constructivist program of Hilbert and Brouwer that rivalled logicism.

Russell knew of Bernays' work, and they had a significant correspondence earlier. In a letter of 19 March 1921, Bernays describes the proofs of the independence of each of the primitive propositions of propositional logic: $*1{\cdot}2$ (Taut), $*1{\cdot}3$ (Add), $*1{\cdot}4$ (Perm), and $*1{\cdot}6$ (Sum), from the others in the group consisting of those four, $*1{\cdot}5$ (Assoc), and the theorem $*2{\cdot}08$ (Id). The letter begins with a reference to earlier correspondence of 4 August 1920, which is lost. In the letter of 19 March 1921 Bernays describes how in that previous letter he had proved the redundancy of $*1{\cdot}5$, i.e. that it can in fact be proved from the other four primitive propositions of *PM* in his thesis (1918).[24] These theorems were only published in Bernays (1926) after the appearance of the second edition, but they had already been known to Russell for several years. Russell says that in the new edition he did not make any improvements of which he had become aware because they would require "alteration of the references, which would have meant a very great labour" (*PM*, p.*xiii*). That is perhaps why the proof that $*1{\cdot}5$ is redundant was not worked into the new edition. In any case, as the new formulation of propositional logic with the Sheffer stroke and Nicod's axioms would allow for the proof of *all* of the original "primitive propositions", Bernays' result was obsolete.

Still, it is odd that there is no mention of Bernays' result. Russell did think it conceivable that such a meta-logical result might be proved. The introductory material of $*1$ of *PM* cites the argument in §17 of *Principles of Mathematics* that the independence of "indemonstrables" (i.e. axioms) cannot be proved by any known method, for ". . . the method of supposing an axiom false, and deducing the consequences of this assumption, which has been found admirable is such cases as the axiom of parallels, is here not universally available." (*PM*, p.91). It surely would be worth remarking that a different method of proving independence had been discovered.

### *3.2.3 H. Behmann*

H. Behmann (Beiträge zur Algebra der Logik. *Mathematische Annalen*, **86**).

Heinrich Behmann was a student of Hilbert, writing a dissertation in 1918 titled "Die Antinomie der transfiniten Zahl und ihre Auflösung durch die

[24] See Zach (1999) for an account of Bernays' thesis. Russell's correspondence with Bernays is discussed in Mancosu (2003). Portions of the letter of 19 March 1921 and all of a letter of 8 April 1921 letter are published in Mancosu (2003 n.43 n.44).

Theorie von Russell und Whitehead."[25] The work cited in *PM*, "Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem", Behmann (1922), is his *Habilitationschrift*, from four years later.[26] The topic is the "decision problem." Behmann succeeds in solving the decision problem of determining validity for the cases of propositional logic and the logic of monadic predicates, introducing along the way the notions of *normal forms* and of logic of the "ersten Stufe" or "first order" in the contemporary sense as predicate logic with quantification over individuals. Ramsey cites the article as having solved the problem for "formulas involving only functions of one variable" in his famous paper "On a problem of formal logic", Ramsey (1930), which proves "Ramsey's theorem". That theorem, which spawned a field of combinatorial mathematics, "Ramsey theory", arose from consideration of the decision problem for the first order theory of relations.[27]

While it is clearly Ramsey who shows more interest in this paper than Russell, who does not mention the notions from it in the second edition, Russell did have a continuing relationship with Behmann, and certainly knew his work when composing the list of logic "since the first edition". A letter from Behmann to Russell dated 8 August 1922 describes his interest in *PM* and the topic of his thesis as arising from Hilbert's suggestions.[28] The letter from Alfred Boskovitz, written from Göttingen, 3 July 1923, which contains Boskovitz' corrections to *PM*, also includes marginal comments from Behmann. Boskovitz does not identify Behmann by name in his cover letter, but rather only Bernays, who was his supervisor. Russell wrote a letter assessing Behmann's thesis to the library of Göttingen University, dated 2 July 1924.[29] Russell maintained his relationship with Behmann, later helping him to win a Rockefeller Foundation fellowship. In a letter from Rome, dated 23 February 1927, Behmann reports on his fellowship work, thanks Russell for his assistance in obtaining the support, and goes on to discuss various logical issues in a way that clearly assumes that Russell is already familiar with them. Behmann first suggests that Russell may have been too hasty to give up on "intensional propositional functions" of propositions, such as "*A* believes *p*", and then goes on to discuss work on the decision problem. It may well be that Russell, knowing of Behmann's unpublished dissertation on *Principia Mathematica*, chose this representative of Behmann's work for the list on Ramsey's suggestion. While there is no

[25] "The Antinomy of transfinite numbers and its solution in the theory of Russell and Whitehead." Mancosu (1999) includes a detailed summary of the dissertation, as well as a discussion of the wider issue of discussions of *Principia Mathematica* in Hilbert's school.

[26] "Contributions to the algebra of logic, in particular the decision problem".

[27] Ramsey (1931, p.92). Behmann thought he had solved the general decision problem in the paper. Kneale and Kneale (1962, p.726) describe Behmann (1922) as one of several independent proofs of the decidability of monadic logic from the early part of the century.

[28] See Mancosu (1999, p.308).

[29] See Mancosu (1999, p.209).

evidence that it influenced Russell's work on the second edition, there is evidence that by 1927 Russell knew of the logical issues it discussed. It just didn't excite his interest.

### *3.2.4 L. Chwistek*

L. Chwistek (Über die Antinomien der Prinzipien der Mathematik, *Mathematische Zeitschrift*, **14**.[30] The theory of constructive types. *Annales de la Société Mathematique de Pologne*, 1923.[31])

Leon Chwistek studied logic at Göttingen briefly during 1908 and 1909, then at Krakow under Stanislaw Zaremba, after which he taught in a secondary school in Krakow. Chwistek was also a painter in the Polish "Formist" school of expressionism and a figure in the artistic scene of Poland between the world wars, but not a member of the "Lwow–Warsaw" school of logicians.[32] Beginning in 1929 Chwistek was a Professor of Logic at the University of Lwow in a position for which Alfred Tarski had also applied. Russell wrote a letter about both candidates, on 23 December 1929, to "Prof. Żylinski, Dean of Faculty of Mathematics – Lwów":

> . . . I know the work of Dr Chwistek and think very highly of it. The work of Mr. Tarski I do not at the moment remember, nor have I access to it at present. In these circumstances, I can only say that in choosing Dr Chwistek you will be choosing a man who will do you credit, but I am not in a position to compare his merits with those of Mr. Tarski.
>
> *(Jadacki, 1986, p.243)*

On the basis of the works by each cited in the second edition, one can understand Russell's remark, for Tarski's work on logic was limited at this point, and his most important work by 1929 was in set theory.[33] Chwistek's two papers, however, were directly related to the project of *Principia Mathematica*.

"Über die Antinomien der Prinzipien der Mathematik" presents an argument against the axiom of reducibility. It begins with a formula defining a function $f$ of an argument $x$ such that $x$ is in the domain of a functional relation $R$, which maps $x$ onto a unique $\alpha$ to which $x$ does not belong (Chwistek, 1922, p.239). Then, consider the set of all such functions $f$ (i.e. for the various instances of $R$) which are definable in some finite number of symbols (or syllables). Since there are only countably many definable expressions, we can consider the many–one relation $S$ between numbers and definable functions of the kind of $f$. When we

[30] "On the antinomies of the principles of mathematics", Chwistek (1922).

[31] Published in two parts, Chwistek (1924) and (1925).

[32] See Wolenski (1989) and McCall (1967).

[33] Jan Wolenski has suggested, in conversation, that perhaps Russell didn't recognize the candidate "Tarski" as the same as the "Tajtelbaum-Tarski" he knew from the articles.

let $S$ be substituted for $R$ above, we define the function $\Phi\hat{x}$. We have just defined $\Phi\hat{x}$, so $S$ maps some number $n$ onto it. By a familiar argument, it is easy to show that $\Phi n \equiv \sim \Phi n$, hence a contradiction. This is essentially a version of the Berry paradox of the "least number not nameable in fewer than eighteen syllables". The ramified theory of types blocks this paradox, as it was indeed designed to do, by introducing *orders* for the various functions $\alpha$ and $\Phi$ to show how the latter predicate which is defined in terms of the bound variable $\alpha$ will have to be of a higher order.

Chwistek's argument against the axiom of reducibility begins at this point. The axiom of reducibility guarantees that we can find a $\Phi$ which is of the same type as $f$, and so the paradox is regained. Chwistek himself saw this as a *reductio ad absurdum* of the existence claim involved in the axiom of reducibility, an existence claim not backed up by an actual construction.

Chwistek's argument eventually led to a better understanding the theory of types, but only after it was revived by Irving M. Copi (1951). Copi gives full credit to Chwistek, and indeed includes a substantial discussion of Chwistek's argument. The progress which resulted from this revival of Chwistek's argument came in the form of two well known papers, Alonzo Church's "Comparison of Russell's resolution of the semantical antinomies with that of Tarski", Church (1976), and John Myhill's "A refutation of an unjustified attack on the axiom of reducibility", Myhill (1979).

Chwistek's correspondence with Russell in Jadacki (1986), reveals that Chwistek had sent manuscripts of the material in these papers before their publication. An offprint of "The theory of constructive types" has been found among Russell's papers, signed by the author, and dated 18 October 1923, at just the time when Russell was working on the second edition.[34]

The introductory section "B" of Chwistek's "The theory of constructive types" makes several points that have since become part of discussions of the technical aspects of *PM*. One was later described by Rudolf Carnap in *Meaning and Necessity*, but not with a reference to Chwistek.[35] The problem is that the no-classes theory in $*20$ of *PM* allows for ambiguities of the scope of class terms. The sentence '$\phi\hat{x} \neq \hat{x}\phi x$' asserts that the function $\phi\hat{x}$ is not identical to the class of things $x$ which are $\phi$. However, given that class expressions are eliminated from a context, and that inequality is defined as the negation of identity, there is a familiar sort of scope ambiguity which emerges. The formula has one reading on which it is

[34] This offprint, together with the Schönfinkel article mentioned below, were in a pile of printed material found in a desk in Dora Russell's house upon her death in 1986, and only identified in the Bertrand Russell Archives by Kenneth Blackwell in 2007.

[35] Or to Gödel (1944, p.126), who also alludes to the problem in passing.

true, as $\phi$ is not identical with some function co-extensive with $\phi$, yet it is false on the reading on which it means that some function co-extensive with $\phi$ is not self-identical. Chwistek proposes an adequate solution in the form of explicit scope indicators suggested at *PM* (p.80) and modeled on the scope indicators for definite descriptions which are presented in $*14$ of *PM*. The expression '$\phi(\hat{x}\psi x)$' which says that the class of $\psi$s has the property $\phi$ should be replaced, he argues, by one which explicitly indicates that '$\phi$' is the scope of the class term: '$[\hat{x}\phi x].\ \phi(\hat{x}\psi x)$'.[36]

Chwistek's paper is of particular interest for this study because it shows that Russell was informed of rather basic, but correctable, errors in the system of the first edition of *PM* just as he was working on the second edition. Why did Russell ignore these points in Chwistek's paper? That these points were not mentioned by Russell shows something about Russell's attitude towards the principle of extensionality, and the accompanying loss of the need for a no-classes theory of classes, corrected or not. It appears that Russell had already decided to adopt the principle of extensionality by 1923 and was not just tentatively working out its consequences. The possible distinction of truth value of different scopes disappears for both (proper) definite descriptions and classes if there are no intensional contexts. As well, in an extensional logic propositional functions are identical if co-extensive, and so can simply be identified with classes. There is no need for the "no-classes theory". The lack of recognition of the possible ambiguity of the abstraction notation for functions is not so easy to understand, as it applies to extensional contexts as well.

Chwistek's plan for the technical body of his "Theory of constructive types" is to "... take directly from *Principia* all that remains true, if the axiom of reducibility is false and if functions of a given type are used as variables instead of matrices" (Chwistek, 1924, p.19). He calls this "... *the pure theory of types*, or the *theory of constructive types* ..."(Chwistek, 1924, p.13). The reason for the term "constructive" is Chwistek's view that the axiom of reducibility asserts the existence of a predicative function equivalent to a given function of higher type yet doesn't guarantee that the predicative function is definable. The axiom only asserts the existence of such a function without producing it, and so leads to a system which is not "constructive" in Chwistek's intended sense.

[36] In addition to the problem about scope in the theory of classes, Chwistek makes two other very good points. One is a passing reference to scope problems with Whitehead and Russell's device for indicating propositional functions, specifically, the use of a caret over a variable as in $f\hat{z}$. This notation is also subject to scope ambiguities, and so Chwistek proposes the notation such as $\hat{z}f\hat{z}$ and $\hat{x}\hat{y}R\hat{y}\hat{x}$, which is close to the later lambda notation: $\lambda x f x$ and $\lambda x \lambda y R y x$. Secondly, Chwistek appreciates that Russell's notation for functions of classes $f\hat{\alpha}$, following the use of the Greek $\alpha$ to range over classes, must nevertheless in fact be seen as functions of functions of propositions about classes defined by those functions, as now would be expressed by $\lambda\phi f\{x : \phi x\}$. See Linsky (2004a) for a later rediscovery of this point. Linsky (2009b) discusses Chwistek's relationship with Russell.

Chwistek identifies classes with propositional functions, and he introduces a defined notion of identity, which amounts to co-extensiveness, in such a way that he can also identify extensional functions within his system. What Russell would have to prove about classes, using his no-class theory and the axiom of reducibility, Chwistek is able to prove about the extensional functions within his type theory without the axiom of reducibility. Russell is able to define '$x \epsilon \phi\hat{x}$' and still prove the "axiom" of extensionality as a theorem: classes are the same if they have the same members. Chwistek needs to restrict the membership relation $\epsilon$ to extensional functions.

There is no indication that Russell proposes any modification of $*13{\cdot}01$, his own definition of identity, and, as will be discussed below, it is not clear just what force the identity assertion has. Perhaps Russell was relying on Chwistek's discussion, which he mentions later in the Introduction to the second edition. Russell may have realized that adding a principle of extensionality to the system of ramified intensional functions in the first edition was not so straightforward as he suggests.

### *3.2.5 H. Weyl*

H. Weyl (*Das Kontinuum*. Veit, 1918. Über die neue Grundlagenkrise der Mathematik, *Mathematische Zeitschrift*, **10**).[37]

*Das Kontinuum*, Weyl (1918), is a monograph pamphlet of 83 pages in which Weyl presents a version of predicative analysis. He begins with his own somewhat idiosyncratic version of the syntax of first order formulas of arithmetic, and then, assuming the theory of the natural numbers, including the rule of induction, argues that the theory of rational and then real numbers must be developed without invoking "vicious circle" fallacies. This theory is now clearly recognizable as what is now called a "predicative" analyis.[38] Formally the theory is based on quantification over first order (predicatively) defined classes of natural numbers and relations, so as to define rational numbers and reals as Dedekind cuts in the class of rationals. Without the axiom of reducibility, and without the use of bound variables ranging over properties (or classes) of natural numbers, one is only able to define certain real numbers, and so definable functions of those real numbers are in turn limited. Weyl proves a number of theorems in his version of analysis, some of which are standard, and others not. Then he presents and proves his version of "Cauchy's convergence principle", that if a function converges, then it has a limit, but he is

[37] Weyl (1918) and (1921). "H. Weyl, Randbemerkungen zu Hauptproblemen der Mathematik. *Mathematische Zeitschrift*, **20**" (Remarks around the main problems of mathematics), Weyl (1924), was added at the proof stage following Ramsey's suggestion in his list of three additions.

[38] See Feferman (1988).

unable to prove the standard theorem that a bounded set of real numbers has a unique least upper bound and a unique greatest lower bound (Weyl, 1918, p.77). The latter is said to fail. As well, Weyl is able to prove the standard theorem that a function that is continuous on the unit interval is uniformly continuous there (Weyl, 1918, p.82).

Did Russell read Weyl's book or papers carefully? In the manuscript of page $xiv$ of the introduction to the second edition Russell writes that if the axiom of reducibility is dropped, and extensionality added, then

> . . . the theory of inductive cardinals and ordinals survives; but ⌊it seems that⌋ the theory of infinite Dedekindian and well-ordered series largely collapses, so that irrationals ⌊and real numbers generally,⌋ can no longer be adequately dealt with ~~and it is even doubtful whether we can prove that $\sqrt{2}$ is irrational.~~

Weyl's system takes the natural numbers as given, as well as the principle of induction. In the passage above, Russell alludes to his own proof of induction in Appendix B. So far Russell and Weyl agree, although Russell thinks he is able to prove induction, while Weyl assumes it. As for the rest of the theory of real numbers, Russell says that ". . . irrationals and real numbers generally, can no longer be adequately dealt with . . . ". In fact there is an extended discussion of just why this is so, later in the Introduction to the second edition, (*PM*, pp.*xliv–xlv*). Kripke takes this as evidence that Russell had not read Weyl, but this assumes that Weyl has "adequately" dealt with the theory of real numbers.[39] This assumes that the predicative analysis that has since been seen to arise from *Das Kontinuum* is in fact a successful version of analysis. In his "The foundations of mathematics" essay from 1925, however, Frank Ramsey describes the first two Weyl pieces as showing that without the axiom of reducibility "ordinary analysis crumbles into dust".[40] (Ramsey, 1926, p. 192). So Ramsey links Russell's remark about the connection between the axiom of reducibility and the loss of analysis with Weyl. Both Russell and Ramsey failed to forsee the possibility of a predicative analysis that did not require reducibility.

It is worth noting that Russell writes and then strikes out the remark "it is even doubtful whether we can prove that $\sqrt{2}$ is irrational." The proof that $\sqrt{2}$ is irrational is in fact an elementary proof that can be used to illustrate the method of

[39] See Kripke (2005, p.1033, n.60).

[40] The context of this remark is in full: "The principal mathematical methods which appear to require the Axiom of Reducibility are mathematical induction and Dedekindian section, the essential foundations of arithmetic and analysis respectively. Mr Russell has succeeded in dispensing with the axiom in the first case, but holds out no hope of a similar success in the second. Dedekindian section is thus left as an essentially unsound method, as has often been emphasized by Weyl, and ordinary analysis crumbles into dust."

proof by *reductio ad absurdum*.[41] Although Russell thought better of this remark, it could be that here he is confusing Weyl's *Das Kontinuum* with Weyl's later turn towards the intuitionism of Brouwer, which would not allow indirect proofs by contradiction, namely those that prove some sentence $A$ by assuming $\sim A$ and deriving a contradiction from that, since they rely on the unsound law of the excluded middle. The proof that $\sqrt{2}$ is irrational is in fact of the intuitionistically valid form which derives $\sim A$ from assuming $A$ (that $\sqrt{2}$ is rational) and deriving a contradiction. Russell's realizing that would explain his striking the remark out, but also why he considered adding it in the first place.

One final, *ad hominem*, argument suggests that Russell had read *Das Kontinuum*, however much he may have misunderstood its positive proposal. Weyl explicitly mentions and rejects Russell's definition of numbers and the theory of types with the axiom of reducibility. "... Russell and I are separated by a veritable abyss" (Weyl, 1918, p.47). (In fact both Weyl and Russell adopt the vicious circle principle.) Surely Russell would have looked at a paper that proposed to develop the theory of real numbers without the axiom of reducibility while keeping a theory of types, or at least a ban on higher order functions.

We know that Russell read later works of Weyl, one chapter of *The Analysis of Matter* (*AMa*) from 1927, Chapter X, "Weyl's Theory" is devoted to Weyl's views on relativity theory. Much later, in the Introduction to the second edition of *The Principles of Mathematics*, in 1938, Russell refers to Weyl's views on foundations, but as an ally of Brouwer, without mentioning Weyl's own original constructivism.

The next two papers, "Über die neue Grundlagenkrise der Mathematik", from 1921, and "Randbemerkungen zu Hauptproblemen der Mathematik", the latter added to the list on Ramsey's suggestion in 1924 as it had just appeared in print, mark Weyl's movement over to Brouwer's intuitionist camp. The first paper introduced the phrase "foundational crisis" (Grundlagenkrise), and briefly reviewed the system of *Das Kontinuum* before going on to avow Brouwer's intuitionism in contrast with Hilbert's alleged formalism, complete with the analogy of mathematics on Hilbert's view with the game of chess. Weyl expresses some doubts about aspects of Brouwer's views, including the suggestion that Brouwer's distinction of the false and that which is unproven might be collapsed under a wider notion of the false, thus saving the law of the excluded middle. Weyl's real attraction is to the constructivist element of intuitionism. These last two papers place Weyl intellectually with Brouwer, whose works are cited next.

[41] One needs as a lemma that if $n^2$ is even then $n$ is even, something easy to prove by induction. Suppose that $\sqrt{2} = n/m$ for $n$ and $m$ with no common factors. Then $n^2/m^2 = 2$, i.e., $n^2 = 2m^2$ and so $n^2$ is even. But then $n$ is even. Similarly $m^2$ is even, and so $m$ is even, but then $n$ and $m$ do have a common factor, namely, 2. This is a contradiction.

### *3.2.6 L. E. J. Brouwer*

L. E. J. Brouwer (Begründung der Mengenlehre unabhängig vom Gesetz des ausgeschlossenen Dritten. *Verhandlungen d. K. Akademie der Wetenschappen*, Amsterdam, 1918).[42]

Ramsey added the article "Intuitionistische Mengenlehre" (Intuitionist set theory) to the proofs of the Introduction. It is a German version of a brief note, of four to eight pages in length in various versions, which itself was originally a Dutch report on the two long papers listed in the first version, "Begründung der Mengenlehre". The short summary states Brouwer's negative theses, beginning with a rejection of the principle of comprehension, that "all things with a certain property are joined in a set", and the principle of the excluded middle, which is associated with Hilbert's notion that all mathematical problems can be decided. But this paper was added to the list by Ramsey in 1924 in his letter about the proofs, and most likely was not known to Russell. What Russell did list in the manuscript of the Introduction to the second edition was the first of the two long "Begründung der Mengenlehre" articles. It is purely positive, though foundational, concentrating on presenting the definitions and basic notions of the constructive theories of *species* (sets governed by a rule), real numbers and ordinals. Brouwer's paper does not seem to have had any impact on the second edition.

Russell would have met Brouwer in person at the Fifth International Congress of Mathematicians, held in Cambridge in August of 1912. As well, Russell expressed views about Brouwer within a few years after the publication of the second edition. In 1929 Trinity College gave Wittgenstein a grant, and in May of 1930 Russell wrote a required report on his progress. He says of Wittgenstein's new work on the foundations of mathematics that "What he says about infinity tends, obviously against his will, to have a certain resemblance to what has been said by Brouwer." (*Auto.* vol.1, p.288). Ramsey's "Foundations" paper begins with the now standard classification of foundational programs into those of Hilbert, described as "formalist", Brouwer and Weyl, characterized as "intuitionists or finitists", and finally Frege, Whitehead, and Russell, the "logicians"(Ramsey, 1926, p.1). Russell must have understood his list of works since the first edition to be representative of this range of views. Russell might have missed the developing resurgence of the algebraic tradition of Schröder (1890) and Skolem, that was about to rise to prominence with Tarski, but many others did as well in the early 1920s.

[42] In print this becomes: "L. E. J. Brouwer. Begründung der Mengenlehre unabhängig vom logischen Satz des ausgeschlossenen Dritten". (Foundations of set theory independent of the logical principle of the excluded middle) *Verhandlungen d. K. Akademie v. Wetenschappen*, 1918, 1919", on Ramsey's correction to the proofs. Ramsey also adds the full citation: Intuitionistische Mengenlehre, *Jahresbericht der deutschen Matematiker-Vereinigung*, **28**. They are Brouwer (1918) and 1919).

### 3.2.7 A. Tarski

A. Tajtelbaum-Tarski (Sur le terme primitif de la logistique, *Fundamenta Mathematicae*, **4**. Sur les "truth-functions" au sens de MM. Russell et Whitehead, *Fundamenta Mathematicae*, **5**. Sur quelques théorèmes qui équivalent à l'axiome du choix, *Fundamenta Mathematicae*, **5**.)[43]

The first of these papers shows how all the connectives can be defined in terms of material equivalence, '≡'. Tarski cites *Principles of Mathematics*, (*PoM*, p.16) which presents in words what would later come to be symbolized as

$$p \,.\, q \ =_{df} \ p \supset . \, q \supset r \ : \supset r.$$

Tarski considers the expression "$=_{df}$" to be another undefined term, and so prefers the following theorem as his way of "defining" conjunction:[44]

$$[p, q] :: p \,.\, q \,. \equiv :. \, [f] :. \, p \equiv : [r]. \, p \equiv f(r) \,. \equiv . \, [r]. \, q \equiv f(r).$$

This had no impact on Russell, who retained his notion of '$=_{df}$'. Whitehead and Russell officially avoid quantifiers over propositions, except in the controversial theorem $*14{\cdot}3$ (see Chapter 4) which they describe as showing that the scope of a definite description is irrelevant in a truth-functional context.

A copy of "Sur les 'truth-functions' au sens de MM. Russell et Whitehead" was found in Russell's copy of the first edition of *PM* when it arrived at the Bertrand Russell Archives. The offprint appears to have come out of its envelope, but was then simply readdressed and stamped by post offices on the first page as it was forwarded from 31 Sydney St., SW3 (Russell's London home), to Sir Frederick Black in Surrey (Dora Black's father) and on to Russell in Cornwall, where he was spending the summer.[45] Postmarks indicate that the offprint left Zakopane in southeastern Poland, where Tarski often took holidays, and, as registered mail, passed through Aldershot and Kensington around 23–25 August 1923.[46] Tarksi's return address is given as "Alfred Tajtelbaum-Tarski, Pologne-Polska, Varsovie-Warszawa, Kosykowa 51-14", his family home. On the top of the first page, Tarski writes "Hommages respectueux de l'auteur". Although there are no markings or marginalia in the body of the offprint, it is most likely that Russell was aware of its contents, especially as he tucked it into his copy of *PM* with the letter from Bernays and corrections from Boskovitz that would both be relevant to writing a new edition.

[43] "On the primitive terms of logistic" (1923), " On 'truth-functions', in the sense of Russell and Whitehead" (1924a), "On several theorems equivalent to the axiom of choice" (1924b).

[44] The square brackets indicate universal quantifiers ranging over propositions.

[45] See (*Papers 9*, p.*xxxiv*), for Russell's addresses during this time.

[46] These visits to Zakopane began in 1923, while Tarski was still a student. See Feferman & Feferman (2004, p.57).

The second paper, on the way to a shorter definition of conjunction, proposes a definition of the notion that "$f$ is a *truth-function*":

$$[p, q] : p \equiv q \;.\; f(p) \supset f(q)$$

and the subsequent "law of substitution", to the effect that every function $f$ is a truth-function:

$$[p, q, f] : p \equiv q \;.\; f(p) \supset f(q).$$

In the discussion of $*14{\cdot}3$, Russell had already rejected this way of expressing the notion of truth-functionality. Seeing it used explicitly, rather than rejected, might have encouraged thought about how to express the thesis of truth-functionality properly. If variables for propositions such as '$p$' and '$q$' could not occur as real (bound) variables, instead they would have to represent the occurrence of functions "through their values" that is, through propositions. If expressions with propositional variables are replaced by expressions for functions, and the result is said to be true of all individuals, then we arrive at the thesis of extensionality, in the sense that co-extensive functions share all properties $f$ and so, by definition, are identical. This move from the truth-functionality of connectives to the extensionality of all functions is the subject matter of much of the Introduction to the second edition. Having these expressions right before him in Tarski's papers may well have influenced the composition of the second edition, although no copy of this article can be found in the Russell Archives.

It is understandable, at least, that given acquaintance with only these two of his papers, the only two in logic before the letter of reference for the chair at Lwow in Logic, that Russell would have remembered the work of Chwistek and not that of Tarski.

### *3.2.8 F. Bernstein*

F. Bernstein (Die Mengenlehre Georg Cantors und der Finitismus, *Jahresbericht der deutschen Mathematiker-Vereinigung*, **28**).[47]

Felix Bernstein was a senior member of the Göttingen school, having received his Habilitation under Hilbert in 1901, and then staying as a teacher from 1907 until he was dismissed by the Nazis in 1933.[48] This paper from 1919 contrasts the finitist view, which Bernstein identifies collectively with Kronecker, Poincaré, Weyl, Brouwer, and others unnamed, with the tradition of Cantor, Dedekind, Hilbert, and Russell. In this essay aimed at making the internal arguments of mathematicians accessible to a wider audience, Bernstein traces the origins of finitism. He

[47] Bernstein (1919), "Georg Cantor's set theory and finitism".

[48] See Mancosu (2003, p.62).

finds them in reactions to Zeno's paradox of Achilles and the tortoise, and others, in which there seemed to be a contradiction in thinking of a completed infinite collection, in the case of Achilles, of all the steps by which Achilles catches up with the tortoise. A different response, starting with Eudoxus, was the development of the notion of limit, and other mathematical tools for dealing with the infinite, with which Bernstein suggests that the infinite can be tamed. As with the case of non-Euclidean geometry, these were steps beyond the familiar portions of mathematics, and although justified, are still subject to Hilbert's demand for a proofs that they do not introduce inconsistency. Bernstein sees the finitist reaction to the paradoxes of set theory (in particular the Richard paradox) as similar to the reactions of a child who first stumbles onto dividing by zero. The proper response, as exemplified by Zermelo with the new axiomatic set theory, is to start with a limited sphere in constructing a consistent, mathematical theory to address the problematic area. Bernstein presents Russell's theory of types as extending to other, logical, paradoxes the sort of solution that Zermelo presents for the theory of infinite sets. Bernstein does not seem to see axiomatic set theory as a quite different approach to the paradoxes of "naive set theory" as was later the standard view. On this view the very distinction between a system with a limited axiom of separation in which only properties picking out elements of a pre-existing set are postulated, on the one hand, and a restricted comprehension principle, with the restrictions of type theory on the predicates that have sets as extensions, on the other.

As before, Russell would have seen in this paper a confirmation of his logicism as an alternative to the Hilbert program, sharing many of the same assumptions, and not as a challenge which must be met with a revision of his own logic. Weyl's *Das Kontinuum* (1918) is also discussed in the paper, with Bernstein disagreeing with the constructivist account of real numbers that Weyl presents which leaves him unable to deal with least upper bounds. Again, reading Bernstein would leave Russell content with exploring the consequences for the theory of the natural and real numbers of abandoning the axiom of reducibility, much in the spirit of considering rival, non-Euclidean geometries, not as an incomplete move towards accepting the constructivist objections to the axiom.

### 3.2.9 J. König

J. König (*Neue Grundlagen der Logik, Arithmetik und Mengenlehre*. Veit, 1914).[49]

This book of 290 some pages was published posthumously by the Hungarian mathematician Julius König's son Dénes König.[50] Julius is known best for a

[49] "New foundations of logic, arithmetic, and set theory".

[50] Dénes König, a more prominent mathematician than his father, is the source of "König's lemma", that every finitely branching tree which is infinite contains an infinite path.

notorious, unsound, claim at a congress in Heidelberg in 1904 that $2^{\aleph_0}$ is not an "aleph", which is equivalent to the claim that the real numbers cannot be well ordered, and the subsequent, sound, "König's inequality" establishing certain bounds on where $2^{\aleph_0}$ occurs in the aleph scale.[51] König quickly replaced his faulty argument with a new one in his 1905 paper, aimed directly at the well-ordering principle. Irving Copi (1971, p.10) reconstructs the argument in König (1905) as follows. There are only $\aleph_0$ definable real numbers, so if the reals can be well ordered, then every subset of the reals, such as those which are not definable, has a least member. Hence there will be a "least indefinable real number", which has, however, just been defined. This argument, assuming the truth of the well-ordering principle, will lead to a paradox, and so is item 6 in the list in section VI, "The contradictions" in the Introduction to the first edition of *PM*, (*PM*, p.61), and the reader is directed to König's paper (1905).[52] Berry's paradox from 1906 shows that this phenomenon has nothing to do with well ordering, for with just finite integers one can ask about "the least number not nameable in fewer than eighteen syllables."[53]

König's book provides an introduction to logic and in particular the basics of set theory through such results as the Schröder–Bernstein theorem and the arithmetic of infinite cardinal and ordinal numbers. The emphasis is on the philosophical justification of existence claims about sets, rather than the purely axiomatic approach of Hilbert which König rejects. There is frequent mention of Russell's paradox, although the theory of types is criticized as overly stringent. (König, 1914, p.155). Dénes König had attended the International Congress of Mathematicians in Cambridge in 1912, and most likely sent the copy of *Neue Grundlagen* that is to be found in Russell's offprint file. While Russell does not mention the book elsewhere in the second edition, it is certain, then, that he knew of its contents, and he may have included it mainly because it was one of the few primarily philosophical works on "mathematical logic" at the time.[54]

*Neue Grundlagen der Logik, Arithmetik und Mengenlehre* survives in the literature of set theory for the proof that by coding integers as series of 0s and 1s, since the "binary" decimals are uncountable, it follows that $2^{\aleph_0} = \aleph_c$, the cardinality of the continuum.[55]

[51] See Hallett, (1984, p.83), Dauben (1979, pp.247–50), Fraenkel (1968, p.98), and Ebbinghaus (2007, pp.50–3) for accounts of the 1904 argument.

[52] In the introductory material to the translation of König (1905), van Heijenoort (1967a) reports that this is a new argument.

[53] Berry's paradox is the fifth in Whitehead and Russell's list of seven "contradictions" on pages 60 to 61 of the Introduction to *PM*.

[54] See Franchella (2000) for a re-evaluation of König (1914).

[55] See Fraenkel (1968, p.116).

### 3.2.10 C. I. Lewis

C. I. Lewis (*Types of Logical Theory*, University of California, 1921).[56]

This is Lewis' *A Survey of Symbolic Logic*, a copy of which remains in Russell's library, and about which we have a letter dated 15 May 1919, from Battersea:

> Allow me to thank you warmly for your *"Survey of Symbolic Logic"*, for which I am most grateful. So far, I have only had time to turn over the leaves and look at some crucial passages. But I see that it is a fine work, and capable of being very useful. I am glad to see your chapter on "Strict Implication." I have never felt that there was any very vital difference between you and me on this subject, since I fully recognize that there is such a thing as "strict implication", and have only doubted its practical importance in logistic.
>
> Your last chapter, on the nature of mathematics and mathematical method, raises very large and difficult questions, but whatever the result may be I feel that your discussion is so clear that it is bound to be a help. . .

Lewis' book starts with a chapter on the history of symbolic logic, beginnning with Leibniz, followed by two chapters on the "Boole–Schröder algebra" which was the form of "logistic" studied by the school of Josiah Royce, Lewis' teacher. Logistic was essentially applied symbolic logic, using symbols for both logical notions, and non-logical constants for relations and operations particular to a specific application. Russell's letter later refers to a classification of approaches to logistic which distinguished Royce's approach from that of Russell and Peano. Lewis presents Peano as using symbolic logic to formalize specific axiomatic systems, and so allow all proofs to be seen to be deductive logical inferences of the same universally applicable kind. Russell's logistic is said to proceed by further defining the primitive notions of a system of mathematics, thus reducing mathematics to logic. This is the logicist program in which Russell did indeed see himself as engaging. Finally, Royce (and Kempe) are presented as using symbolic logic to formulate a general theory of "order", the abstract structure of certain mathematical systems. Logic is seen as just one of many mathematical systems, in particular, as one in which there are certain strings of symbols (axioms and definitions) from which others (theorems) are derived by certain rules (of substitution and other inference rules). Lewis sketches how, using the notions of "triads", or ternary relations, both the structure of geometry, using an interpretation of the relation of "between" for points, and the notion of sequence needed for developing a formal theory of syntax, can be seen as simply two different applications of the logistic method.

[56] That this citation is incorrect, though Russell had received a copy of the book from Lewis, shows that Russell corrected proofs, and likely also even wrote this part of the Introduction to the second edition while in Cornwall, away from his library. The "Schönwinkel" entry in the list also suggests this.

It is easy to see in this work an interest in formal syntax, via the notion of sequences of symbols, that was behind Quine's investigations in his 1932 Harvard thesis. This preceded the independent development of formal syntax, and then semantics, in the work of Tarski and Gödel, which Quine came to learn of following his connections with the Vienna Circle in the late 1920s. Russell was certainly aware of the "algebraic" approach to logic of Boole and then Schröder which was brought into the Hilbert school's investigations to create the ideas of meta-mathematics and model theory. In Lewis, again, this approach is presented as an alternative to that of *Principia Mathematica*, and so, again, the failure of Russell to work these developments into the second edition of *PM* does not show that he was necessarily out of touch with developments in the field, or unable to keep up with them.[57]

The lack of a response to Lewis' introduction of systems of strict implication, which were to lead to the development of modal logic over the next decades, is frustrating, however.[58] It is not helpful to hear that Russell never doubted the existence of strict implication. The suggestion may be that material implication is better suited to this particular application of logistic in the logicist reduction of mathematics. While to our eyes the combination of only truth-functional propositional connectives with an intensional logic containing propositional functions might seem to be an inconsistent combination, in fact it was precisely the step of moving to an extensional view of propositional functions for the logic of mathematics that was the central concern of the second edition. Moving back, as Russell would have seen it, to a theory of intensional propositional connectives was simply contrary to the movement from the first edition of *Principia Mathematica*, and so, understandably, not something that Russell needed to address in the new material in the second edition.

### *3.2.11 H. M. Sheffer*

H. M. Sheffer (Total determinations of deductive systems with special reference to the Algebra of Logic. *Bulletin of the American Mathematical Society*, 1910. A set of five independent postulates for Boolean algebras, with applications to logical constants, *Trans. Amer. Math. Soc.* **14**, pp. 481–8. *The general theory of notational relativity*. Cambridge, Mass.: Harvard University 1921).

Henry M. Sheffer (1882–1964) had been a graduate student at Harvard under Josiah Royce, completing his PhD in 1908.[59] In 1910, after a fellowship,

[57] In *Analysis of Matter* from 1927 we find Russell himself using the notion of "... what professor G. I. Lewis calls 'strict implication'." (*AMa*, pp.199–200) The 'G' is an error, repeated in the index.

[58] There is a brief mention and dismissal at (*IMP*, p.153–4) of the proposal to include "formal deducibility" among the primitive notions of logic.

[59] For a biography, and an assessment of Sheffer's place in logic in the United States, see Scanlan (2000) and (1991).

Sheffer began a year in Europe, beginning with a term at Cambridge where he met Russell and attended his lectures on "Mathematical Philosophy" for Michaelmas term (October to December). In later correspondence Sheffer addressed Russell in a joking fashion, suggesting a certain familiarity. In a letter of 2 March 1911, Sheffer writes to Russell asking for a letter of reference to J. Royce or R. B. Perry for a second appointment as a fellow at Harvard. He mentions a lecture Russell is to give on the 19th of March, and looks forward to more on the "mysterious adventures of 'E. Shriek, R. Hook & Co.' ". Sheffer had been talking to logicians but found Poincaré and Couturat disappointing, and looked forward to the "Peanists" in Italy. Sheffer wrote to Russell from Göttingen on 28 July 1911, reporting that he had just read a paper by Schönflies (1911) that criticises Russell, using the slogan "Für den Cantorismus, aber gegen den Russellismus!" (for Cantorism but against Russellism), and remarked that Schönflies would do away with the distinction between $\epsilon$ and $\subset$ (the set membership and subset relations).[60] The same, for $\supset$ and $\subset$, was repeated in Italy, where Sheffer talked with Peano, Burali-Forti, and Padoa. "I saw Frege too" he reported. "He insists, as you know, on the sharp distinction between Gegenstände and Begriffe."[61]. Sheffer says, however, that Frege's "polemic against Hilbert–Korselt makes amusing reading", and concludes that his application to Harvard being unsuccessful, he is off to Seattle, Washington, for a year on an instructorship. This remarkable tour was clearly taken on Russell's advice. All of the figures in logic in 1911 were included, in France, Italy, and Germany. Even Hilbert is mentioned, though only from Frege's side of their dispute. As the letter is addressed from Göttingen, Sheffer must have learned something of the views on logic there as well, although Hilbert's interest in logic didn't revive until around 1918.[62]

In the Introduction to the second edition, Russell lists Sheffer's abstract (1910), paper (1913), and the typescript *The general theory of notational relativity* dated 1921. The abstract reports a talk, which may have contained the announcement of the "Sheffer stroke", but only speaks of finding "the basal relations (and operations) for that algebra [of logic]" and asserts that logical addition, multiplication, and inclusion are "isolated cases" of that "total set". This may well have been the result that the other operations could be defined in terms of the one connective, but this is only stated in clear terms in the paper of 1913. There Sheffer presents an axiomatization of "Boolean algebras" using the stroke " | ", when the algebra in question is that of the logic of propositions, in which case the stroke means

[60] An offprint of Schönflies (1921), unmarked and with the pages uncut, was found in Russell's offprint file, now in the Archives.

[61] That is, objects and concepts

[62] See Sieg (1999) for an account of Hilbert's lectures on logic in 1917/1918 which were the basis for Hilbert & Ackermann (1928).

"neither-nor". The standard reading of the Sheffer stroke as "not-both" or "nand", which is the connective used by Nicod and then by Russell in the second edition, is introduced in a footnote indicating that the two are dual connectives in Boolean algebras. The paper with the stroke was Sheffer's only published contribution to logic. His reputation relied rather on his personality, which manifested itself throughout his long teaching career at Harvard University, and which clearly must have impressed Russell in 1910.

*The general theory of notational relativity* was never published, for good reason. It seems to be a preliminary attempt to abstract a number of algebras from logic, and is full of definitions, and 25 pages of "grafs" or matrices of values like truth tables, but quite devoid of theorems and general results. There is an unmarked copy of this 60 page typescript in Russell's offprint file. It is most likely this work, however, that Russell alludes to in his remark at the end of the first section of the Introduction to the second edition, where he is describing the "improvements" to be made:

> It should be stated that a new and very powerful method in mathematical logic has been invented by Dr H. M. Sheffer. This method, however, would demand a complete rewriting of *Principia Mathematica*. We recommend this task to Dr. Sheffer, since what has so far been published by him is scarcely sufficient to enable others to undertake the necessary reconstruction.
>
> (*PM,* p.*xi*)

Russell is being generous, as the *The general theory of notational relativity* manuscript gives even less than "scarcely sufficient" indications of how to reformulate a logical system.[63] This passage certainly does not refer to the writings on the "stroke", as Russell himself is able to accomplish the necessary rewriting to take the Sheffer stroke as the basic connective, and provides the detail in the Introduction to the second edition, and Appendix A.

A letter to Russell dated 18 February 1926, from Emerson Hall at Harvard where Sheffer finally had a position, proposes a connective for quantificational logic to parallel the stroke, which is to mean "not for every $x$", symbolized as $\overline{x}$. The universal quantifier is thus symbolized, using negation as $\sim \overline{x}$. This is a mere notational variant of the standard notation of universal quantifier and sentential negation, but indicates the origin of Sheffer's idea that some sort of algebraic study of logic can be carried out, when one looks at his assertions such as: "$(\exists x).(y). \stackrel{df}{=} . \overline{x} \sim \sim \overline{y} = \overline{x}\ \overline{y}$".

[63] This passage is quoted by Whitehead in the "Memorandum" in support of Sheffer of 23 January 1927 (Henle *et al.*, 1951, pp.*ix*–*x*). Whitehead refers to the Sheffer stroke, but then goes on to write of Sheffer's study of Logic as the "general theory of structure." Whitehead writes of Sheffer that "He has not only enunciated the principles, but achieved the far more difficult task of making substantial progress in constructing the groundwork of the science."

Despite this fruitless investigation of the algebra of logic, Sheffer was not out of touch with other developments, and discusses them in the same letter. He says that "Wittgenstein has generalized the notion of the 'stroke' ".[64] Despite his own unclear proposal, Sheffer is aware of Schönfinkel's use of a single connective $\phi x \mid^x \psi x$, and correctly describes it as defined "to mean $(x).\ \phi x \mid \psi x$." Thus, at least by early 1926, Russell was having Schönfinkel's results cited to him as though they were already familiar.

### *3.2.12 J. Nicod*

J. G. P. Nicod (A reduction in the number of the primitive propositions of logic. *Proc. Camb. Phil. Soc.* **19**).

Nicod's paper was read to the Cambridge Philosophical Society on 30 October 1916, but was published in 1920. It is famous for showing how, using the Sheffer stroke alone, all of propositional logic can be derived with only one axiom and one rule of inference.[65] The rule is:

$$\text{from } p \mid (r \mid q) \text{ and } p, \text{ infer } q.$$

Before stating the single axiom, Nicod presents the two axioms which Russell actually uses in his derivations in the second edition:

$$p \mid (p \mid p)$$

and

$$p \mid (q \mid q) \ .\supset .\ (s \mid q) \supset (p \mid s).$$

Nicod's single axiom is mentioned by Russell, but not used in any of the formal development:

$$p \mid (q \mid r)\ .\mid .\ [t \mid (t \mid t)\ .\mid .\ \{(s \mid q) \mid ((p \mid s) \mid (p \mid s))\}].$$

Nicod goes on to derive the primitive propositions of *PM* from the two axiom version, and then some additional results. He incidentally remarks that using a variant of Whitehead and Russell's primitive proposition $q\ .\supset .\ p \vee q$ (Add), namely $q\ .\supset .\ q \vee p$ (Add*), the primitive proposition $p \vee q\ .\supset .\ q \vee p$ (Perm) can be proved from the others; Sum, Taut, and Assoc and the new Add*. Bernays (1926) includes this same result with a much simpler proof, not citing Nicod as apparently it was arrived at independently. Much of Bernays (1926) started in his

64 Although he continues "but it does not seem to have occurred to him to make use of a new general operator, as here suggested."

65 Two, if the rule of substitution is made explicit.

thesis at Göttingen from 1918, while Nicod's paper was read in 1916, with the two in countries on opposite sides in the war, so the results were probably discovered independently.

In a letter to Russell of 18 July 1919, after discussing another reference in Russell's works, Dorothy Wrinch writes "I remember how delighted Nicod was when you talked about his paper on $p|q$ in your lectures (Autumn 1917)!" The lectures at Dr Williams' Library became (*IMP*). The discussion of Nicod is in Chapter XIV. Although Russell did write about the notion of "incompatibility" as a basic logical notion, any consideration of this more philosophical notion in the logical atomism period of (*IMP*) and (PLA) is likely as much due to Russell's own thinking, and the influence of Wittgenstein, as from any ideas directly from Nicod. Certainly, however, the influence on the second edition just came from this paper. A letter of 28 September 1919 tells Russell that "I have definitely arranged to write a thesis on the external world."[66] Nicod was indeed a member of Russell's circle, and carried on a correspondence with Russell after leaving Cambridge, up to his early death in February 1924.[67]

### *3.2.13 L. Wittgenstein*

L. Wittgenstein (*Tractatus Logico-Philosophicus*, Kegan Paul, 1922).

In the Introduction to the second edition, after describing Leon Chwistek's proposed "constructive" type theory that would reconstruct *PM* after simply dropping the axiom of reducibility, Russell credits Wittgenstein with what are the two principal "improvements" in the second edition:

> There is another course, recommended by Wittgenstein [BR's footnote: *Tractatus Logico-Philosophicus*, $*5\cdot54$ ff.] for philosophical reasons. This is to assume that functions of propositions are always truth functions, and that a function can only occur in a proposition through its values. There are difficulties in the way of this view, but perhaps they are not insurmountable. It involves the consequence that all functions of functions are extensional. It requires us to maintain that "A believes $p$" is not a function of $p$. How this is possible is shown in *Tractatus Logico-Philosophicus* (*loc.cit.* and pp. 19–21). We are not prepared to assert that this theory is certainly right, but it has seemed worth while to work out its consequences in the following pages.

These themes can be traced from Russell's introduction to *Tractatus* from 1922 through the "working notes" discussed above, and from there to the Introduction to the second edition. Those working notes show that the account of "propositions as facts", which is credited to Wittgenstein, also has antecedents in Russell's

[66] A copy of the thesis, without markings, is in Russell's offprint file.
[67] This correspondence is published in Volume II of the *Autobiography*.

own "multiple relation theory" of judgement. On that theory, the occurrence of a propositional attitude, while seemingly the relation of a subject to a unitary proposition, is in reality a complex fact involving the individuals and universals that are constituents of the purported proposition. A belief is true when that complex belief fact corresponds with a fact in the world in the right way. Russell can be seen as having found in the *Tractatus* account of logical form the last piece he needed for the theory. Correspondence is, namely, sameness of form. The doctrine that "... a function can only occur in a proposition through its values ..." looks like the view that Russell credits to Wittgenstein in the logical atomism lectures, at (PLA, p.182). This was before its tacit appearance in *Tractatus* in which all states of affairs are atomic, involving individuals and universals, and not involving higher order functions applying to lower order functions (or monadic universals true of other universals). Yet the very existence of the debate about Wittgenstein's putative nominalism shows that this view does not stand out prominently in the *Tractatus*. Something in Wittgenstein's view struck Russell as amounting to the thought that a function could not appear as an argument of another function, but only through its values. Whatever it might be, it is likely the very doctrine that Ramsey describes to Wittgenstein in his letter of 20 February 1924:

> Of all your work he seems now to accept only this: that it is nonsense to put an adjective where a substantive ought to be which helps with his theory of types.

This is surely what Ramsey made of some talk of Russell about a function only "occurring through its values".

Russell's crediting of Wittgenstein thus seems correct, neither overgenerous nor insincere. What then of Ramsey's need to suggest adding Wittgenstein's name with his remark "What about Wittgenstein?" Perhaps Russell considered the *Tractatus* to be a work of philosophical rather than "mathematical" logic. Yet it is not the only non-technical work in the list. It must be that Russell considered this as a list of additional works, which, with the exception of Sheffer and Nicod, is not discussed in the body of the Introduction to the second edition, and that he had already cited the *Tractatus* in the body of the Introduction.

### *3.2.14 M. Schönfinkel*

M. Schönwinkel (Über die Bausteine der mathematische Logick. *Math. Annalen*, **92**).

This is one of the three items added by Ramsey with the results of his proof reading, an understandably last minute addition, as it was presented by Schönfinkel in 1920, but only published in the March issue of 1924 (appearing in September), edited by Heinrich Behmann.[68] Ramsey's letter spells the name as "Schönwinkel", and that spelling makes it into the published version.[69] Schönfinkel was a member of the Göttingen school, and Behmann's paper credits both Boskowitz and Bernays for points relating to the material, yet Schönfinkel himself was apparently unknown to Ramsey and Russell. The paper begins with presenting a generalization of the Sheffer stroke from propositional to a quantified connective '$|^x$' such that $fx \mid^x gx$ amounts to $(x)(\sim Fx \vee \sim Gx)$. To achieve the further goal of eliminating variables, Schönfinkel proceeds to develop quantificational logic in a general theory of operators on formulas. The operator '$|^x$' is thus represented as an operator $U$ on functions, so that $Ufg = fx \mid^x gx$. The operators are $I$, $C$, $T$, $Z$, and $S$, defined by $Ix = x$, $Cxy = x$, $(T\phi)xy = \phi yx$, $Z\phi\chi x = \phi(\chi x)$, and $S\phi\chi x = (\phi x)(\chi x)$. They are quickly reduced to three, $C$, $S$, and $U$. All of the functions can be viewed as monadic, so that, for example, $S\phi\chi x$ would be the function resulting from the application of $S$ to $\phi$ itself applied to $\chi$ and the result in turn applied to $x$, as represented better with parentheses: $((S\phi)\chi)x$. The system is intended to represent functions which can apply to themselves, and so replaces the logic of propositional functions and relations with a typeless theory of mathematical functions.

Although revising logic with Schönfinkel's connective would have required further massive changes to the new Introduction and Appendix A, it is still surprising that the paper is not at least mentioned. This is a case of work that was clearly relevant to Russell's project, which both he and Ramsey only came across late in the work on the second edition.

[68] See the account in the preface to Schönfinkel (1924) by W. V. Quine in van Heijenoort (1967a). Ramsey returned to Cambridge for the new term in October.

[69] Despite the reference to "Schönfinkel", with the right spelling, in the February 1926 letter from Sheffer to Russell, and the fact that an offprint of the paper found its way into Russell's possession among the stack recently identified in Dora Russell's materials in the Archives.

# 4

# Notation and logic

This chapter presents the notation and logical system of *Principia Mathematica*, taking as illustrations many formulas that will appear in the second edition and the manuscripts, and so will prepare the reader who is new to the work. While the syntax of *PM* is not presented in that work explicitly, and is not up to contemporary standards in logic in certain respects, it is easy to distinguish primitive from defined expressions, and follow the introduction of defined symbols into the system.[1] Symbols are then used in a uniform way after they are introduced. Russell's notation evolved from 1900 to 1910, starting with his adoption of Peano's symbolism, and then developed through his collaboration with Whitehead. The second edition of *PM* uses the notation of the first, with the exception of the new Sheffer stroke, and so the changes are primarily matters of logical theory. This chapter will also review the distinctive logical doctrines of the first edition of *PM*, particularly the theory of types, the theory of definite descriptions, and the "no-class theory of classes", as necessary prerequisites for understanding the additions of the second edition.

## 4.1 Primitive symbols and punctuation

The basic symbols include the following. Some are primitive and some are defined:

$*$ Pronounced "star"; indicates a chapter or section ("number") and one of three kinds of sentence, either an axiom, a theorem, or a definition. "$*20$ General Theory of Classes" is a number, with $*20{\cdot}01$ its first definition, and $*20{\cdot}1$ its first theorem.

$\cdot$ The centered dot (the British decimal point); indicates a numbered sentence in the order by first digit (all the 0s preceding all the 1s etc.), then

[1] Thus Gödel (1944, p.126): "It is to be regretted that this first comprehensive and thorough going presentation of a mathematical logic and the derivation of Mathematics from it is so greatly lacking in formal precision in the foundations (contained in $*1 - *21$ of *Principia*), that it presents in this respect a considerable step backwards as compared with Frege."

second digit, and so on. The first definitions and propositions of $*8$ illustrate this "lexicographical" ordering: $*8{\cdot}01$, $*8{\cdot}011$, $*8{\cdot}012$, $*8{\cdot}013$, $*8{\cdot}1$, $*8{\cdot}11$, $*8{\cdot}12$, $*8{\cdot}13$, $*8{\cdot}2$, $*8{\cdot}21$, and so on. The ordering generally indicates the relative importance of theorems, but was used during writing to allow adding or deleting particular theorems without changing the numbers of others.

$\vdash$ The assertion-sign; indicates an *assertion*, that is, either an axiom (i.e., a *primitive proposition*, which is annotated "Pp") or a logical truth proved as a theorem.

Df This follows a definition, and should be considered part of one symbol with the '$=$' sign that accompanies it, much as '$=_{df}$' is now used. Definitions are considered to be merely "symbolic", i.e., as establishing the rules for replacing one expression by another. Theorems, proofs, and even some axioms are written using defined expressions.[2]

$. , : , :. , ::$ , etc. The dots used for delimiting punctuation with varying scope, along with the (, ), (parentheses) [, ], (brackets) and {, }, (braces) of contemporary logic. They also indicate conjunction. See below.

$p$, $q$, $r$, etc. These are propositional variables. Given the apparent commitment to the "multiple relation theory" in the Introduction, and the accompanying view that there are no propositions as unities in some sense, many prefer to read these as schematic letters allowing sentences of propositional logic as substitution instances.[3] In what follows, however, these will be read as "real", or free variables.

$\equiv$ , $\supset$, $.$ , $\vee$ , and $\sim$ These are the familiar sentential connectives, corresponding to 'if and only if', ' if-then', 'and', 'or', and 'not'. *PM* takes $\vee$ and $\sim$ as primitives and the others are defined.[4] (The dual use of dots, both as punctuation and as the connective 'and', will be explained below.)

$|$ The "Sheffer stroke" is added in the second edition of *PM* as a new primitive connective, with the others defined in terms of it. '$\cdots | —$ ' means 'not both $\cdots$ and — '.

$x$, $y$, $z$, etc. The individual variables, which are to be read with "typical ambiguity", i.e., as schematic for variables which have specified logical types (see below).

$a$, $b$, $c$, etc. Individual constants which denote for individuals (of the lowest type). These occur only in the introductions to *PM*, and not in the body of the work.[5]

2 This is without an explicit proof that rules of inference apply to defined expressions, as is pointed out in Gödel (1944). This becomes important when treating the issue of scope in the theories of definite descriptions and classes, given that identity is also a defined notion. See previous note.

3 See Landini (1998, p.258–9).

4 The definitions are: $*1{\cdot}01.\ \ p \supset q\ .=.\ \sim p \vee q$ Df, $*3{\cdot}01.\ \ p\,.\,q\ .=\ .\sim(\sim p \vee \sim q)$ Df, discussed below, and $*4{\cdot}01.\ \ p \equiv q\ .=.\ p \supset q\ .\ q \supset p$ Df.

5 "Constants do not appear in logic, that is to say, the $a$, $b$, $c$ which we have been supposing constant are to be regarded as obtained by extra-logical assignment of values to variables." (*PM*, p.$xxx$). They are more like parameters in contemporary logic.

$\phi$, $\psi$, $\chi$, etc., and $f$, $g$, etc. Variables which range over propositional functions, no matter whether those functions are simple or complex.

$\phi x$, $\psi x$, $\phi(x, y)$, etc. These expressions are consequently to be interpreted as open atomic formulas in which $x$, $y$, etc. are free variables. Given that '$\phi$', etc. are being interpreted as free variables, rather than as schematic letters, it follows that '$\phi x$', '$\psi x$', etc. are also free variables ranging over propositional functions, differing only notationally from '$\phi$', '$\psi$', etc. by indicating explicitly occurrences of free variables that may be bound by quantifiers. Occurrences of function variables in parentheses, to indicate universal quantifiers, '$(\phi)$', '$(\psi)$', etc. always occur without variables.

$R$, $S$, $T$, etc. Italic, "Latin", capital letters indicate (universal) relations in the introductions and early material, but are reserved for "relations in extension" after $*21$ (*PM*, p.201). Universals, as constituents of judgements according to the "multiple relation theory", must be distinguished from propositional functions, which are not ultimate constituents of propositions. No terms for universals appear in the body of *PM*, i.e. outside of the Introduction, but the distinction between propositional functions and universals is important for the metaphysical interpretation of the logic.[6]

ˆ The circumflex; when placed over a variable in an open formula (as in $\phi\hat{x}$) results in a term for a propositional function. When the circumflected variable precedes a complex variable, the result indicates a class, as in $\hat{x}\phi x$.[7] Contemporary notation might express the circumflex over the variable indicating propositional function notation, '$\phi\hat{x}$', with one variable binding operator, say '$\lambda x\ \phi x$' and the other, class expression '$\hat{x}\phi x$' with the familiar '$\{x : \phi x\}$', which of course should be seen as the result of a term forming operator binding the variable '$x$'.

$\exists$ and parentheses around variables; $(x)$, $(\phi)$, etc. The quantifiers "there exists" and "for all" ("every"), respectively. For example, when $\phi x$ is an open formula, $(\exists x).\phi x$ says that there exists an $x$ such that $\phi x$, and $(\exists\phi).\phi x$ that there exists a propositional function $\phi$ such that $\phi x$. $(x).\phi x$ says that every $x$ is such that $\phi x$, and $(\phi).\phi x$ that every propositional function $\phi$ is such that $\phi x$.

$\supset_x$ , $\equiv_x$ The subscripted variables are used to indicate universally quantified variables. They are defined in $*10$, as described below.

! The shriek after a function variable, as in $\phi!x$ or $f!\hat{x}$, indicates that the function is predicative. A predicative function $\phi!x$ is one which is of "... the lowest

6 See Linsky (1999, §2).

7 Landini (1998, p.265) claims that the circumflex in expressions for propositional functions is not a term forming operator, but is instead a syntactic, meta-linguistic device that only occurs in the introductory material of *PM* and not in the system proper. Interpreting propositional function expressions such as '$\phi$', etc. as genuine free variables, as will be done here, however, requires treating the circumflex as a term forming operator. Landini is certainly correct, however, in pointing out that expressions such as '$\phi\hat{x}$' do not occur in the body of *PM*.

order compatible with its having that argument". (*PM*, p.53). The notion is central to the ramified theory of types and plays a crucial role in the changes in the second edition. See below for an account of the ramified theory of types.

$\imath$ The inverted iota, read as 'the', is the "description operator" and is used to form definite descriptions such as $(\imath x)\phi x$. The contextual definition of these expressions in $*14$ is the heart of the theory of definite descriptions.

$[(\imath x)\phi x]$ The occurrence of a definite description in brackets indicates the "scope" of the enclosed definite description, to be discussed below.

$\iota$ The upright iota, arising from Peano's original notation, is a function that yields the class containing just the argument, i.e. $\iota x$ is the singleton containing $x$. (The inverted iota is thus etymologically seen as its converse.)

$E!$ This expression is defined at $*14{\cdot}02$, in the context $E!(\imath x)\phi x$, to mean that $(\imath x)\phi x$ is "proper", that there is exactly one $\phi$.

$\exists!$ This combination is defined at $*24{\cdot}03$, in the context $\exists!\alpha$, to mean that the class $\alpha$ is non-empty.

One initial obstacle to readers of *Principia Mathematica* is the unfamiliar use of patterns of dots, rather than the more common parentheses and brackets now used for punctuation. The system is, however, quite precise, and can be mastered with practice, rather than by demanding an algorithm for converting dots to parentheses.[8] The use of dots for punctuation is not unique to *PM*. Originating with Peano, it was later used in works by Alonzo Church, W. V. O. Quine, and others, but it has now largely disappeared. What follows is an explanation as presented at *PM* (pp. 9–10), followed by a number of examples which illustrate each of its clauses.

*The use of dots*. Dots on the line of the symbols have two uses, one to bracket off propositions, the other to indicate the logical product of two propositions. Dots immediately preceded or followed by "$\vee$" or "$\supset$" or "$\equiv$" or "$\vdash$", or by "$(x)$", "$(x, y)$", "$(x, y, z)$" ... or "$(\exists x)$","$(\exists x, y)$", "$(\exists x, y, z)$" ... or "$[(\imath x)\phi x]$" or "$[R\text{‘}y]$" or analogous expressions, serve to bracket off a proposition; dots occurring otherwise serve to mark a logical product. The general principle is that a larger number of dots indicates an outside bracket, a smaller number indicates an inside bracket. The exact rule as to the scope of the bracket indicated by dots is arrived at by dividing the occurrences of dots into three groups which we will name I, II, and III. Group I consists of dots adjoining a sign of implication ($\supset$) or equivalence ($\equiv$) or of disjunction ($\vee$) or of equality by definition ($=$ Df). Group II consists of dots following brackets indicative of an apparent variable, such as $(x)$ or $(x, y)$ or $(\exists x)$ or $(\exists x, y)$ or $[(\imath x)\phi x]$ or analogous expressions. Group III consists of dots which stand between propositions in order to indicate a logical product. Group I is of greater force than Group II, and Group II than Group III. The scope of the bracket indicated by any collection

[8] Turing (1942) presents such an algorithm for Church's modification of the *PM* notation. Turing's proofs of the readability of this notation include arguments based on exhaustive lists of symbols that can be paired one after the other, the sort of reasoning that was being used by Turing and others to automate code breaking during the war. While Gödel justifiably compares the treatment of formal syntax in *PM* negatively to that of Frege (as noted above), the demand for an algorithm to convert dots to parentheses seems a few years premature.

of dots extends backwards or forwards beyond any smaller number of dots, or any equal number from a group of less force, until we reach either the end of the asserted proposition or a greater number of dots or an equal number belonging to a group of equal or superior force. Dots indicating a logical product have a scope which works both backwards and forwards; other dots only work away from the adjacent sign of disjunction, implication, or equivalence, or forward from the adjacent symbol of one of the other kinds enumerated in Group II. Some examples will serve to illustrate the use of dots.

The following series of examples takes propositions in *PM* to illustrate how to translate them step by step into modern notation.[9]

The one explicit rule of inference in the first chapters is stated as a "primitive proposition":

$\ast 1{\cdot}1$ Anything implied by a true elementary proposition is true. (Pp)

This is one of the occasions for commentators to assert that there is not sufficient attention to the distinction between use and mention in the system. If one takes "true propositions" to be those that are provable, and "$q$ is implied by $p$" to mean that the implication $p \supset q$ is provable, then this is simply a statement of the rule of *modus ponens*:[10]

$$\text{If } \vdash p \text{ and } \vdash p \supset q \text{ then } \vdash q.$$

The rule is stated to hold for "elementary" (atomic) propositions, but as complex instances are justified by this rule from the beginning of PM, it is clear that there is also tacit use being made of a rule of substitution, which is not made explicit.

The first axiom is

$$\ast 1{\cdot}2 \quad \vdash : p \vee p \,.\supset.\, p \quad \text{Pp} \quad \text{(Taut).}$$

The numbering indicates that this is the second assertion of "star" 1. It is in fact an axiom or *primitive proposition* as indicated by the 'Pp'. (It also has the name "Taut", for "Tautology"). That this is an assertion (axiom or theorem) and not a definition is indicated by the use of '$\vdash$'. From the passage above we know that " . . . a larger number of dots indicates an outside bracket, a smaller number indicates an inside bracket . . . ," so the colon here represents an outside bracket. So, a first step in translating $\ast 1{\cdot}2$ into contemporary notation is

$$\vdash [\, p \vee p \,.\supset.\, p \,].$$

[9] In this section symbols are sometimes used as names for themselves, thus avoiding some otherwise needed quotation marks. Russell is often suspected of confusing use and mention, but any confusion here is mine, and deliberate, in aid of the exposition.

[10] This is discussed at (*PM*, p.9) in the Introduction, where it is said that "The process of the inference cannot be put into symbols".

As the brackets '[' and ']' represent the colon in $*1{\cdot}2$, the scope of the colon is thus seen to extend past any smaller number of dots (i.e., one dot) to the end of the formula. Next, the dots around the '$\supset$' are represented in modern notation by the parenthesis around the antecedent and consequent. In the above passage, we find "... dots only work away from the adjacent sign of disjunction, implication, or equivalence... ". The next step in the translation process is thus the formula

$$\vdash [\,(p \vee p\,) \supset (p)\,].$$

Finally, standard modern conventions allow us to delete the outer brackets and the parentheses around single letters, yielding the familiar looking

$$\vdash (p \vee p) \;\supset\; p.$$

The other primitive propositions of propositional logic are

$$*1{\cdot}3 \quad \vdash : q \;.\supset .\; p \vee q \quad \text{Pp} \quad \text{(Add)},$$

$$*1{\cdot}4 \quad \vdash : p \vee q \;.\supset .\; q \vee p \quad \text{Pp} \quad \text{(Perm)},$$

$$*1{\cdot}5 \quad \vdash :\; p \vee (q \vee r) \;.\supset .\; q \vee (p \vee r) \quad \text{Pp} \quad \text{(Assoc)},$$

$$*1{\cdot}6 \quad \vdash :.\, q \supset r \;.\; \supset :\; p \vee q \;.\supset .\; p \vee r \quad \text{Pp} \quad \text{(Sum)}.$$

Primitive proposition $*1{\cdot}5$ was shown to be provable from the other propositions by Bernays in his 1918 thesis, with the result first published in Bernays (1926). To get a sense of how theorems of propositional logic are proved in *PM*, consider this sketch of Bernays' proof that Assoc is redundant. This sketch mixes parentheses and dots, as Whitehead and Russell themselves do on occasion, to make the structure of a formula more obvious. Starting from instances of Add and Sum, prove

$$q \vee r \;.\supset : q \vee .\; p \vee r$$

which, using Sum, yields

$$p \vee .\; q \vee r \;.\supset : p \vee .\,(q \vee .\; p \vee r).$$

Perm then yields the first lemma:

$$p \vee .\; q \vee r \;.\supset : (\,q \vee .\; p \vee r\,) \,.\vee p.$$

Starting again, Add and Perm yield

$$p \;.\supset .\; p \vee r.$$

From this, with the following instance of Add:

$$p \vee r \;.\supset : q \vee .\; p \vee r,$$

derive

$$p\ .\supset : q \vee .\ p \vee r.$$

To conclude the proof by putting these lemmas together, we use a very long instance of Sum:

$$\begin{aligned} &p\ .\supset : (q \vee .\ p \vee r). \\ &\qquad\qquad \supset .\ [(q \vee .\ p \vee r). \vee p\ ] . \supset .\ [(q \vee .\ p \vee r) . \vee . (q \vee .\ p \vee r)]. \end{aligned}$$

The last two sentences, with Taut, yield together

$$(q \vee .\ p \vee r) . \vee p\ . \supset . (q \vee .\ p \vee r).$$

This, together with the first lemma, gives the result, Assoc:

$$p \vee . q \vee r\ . \supset : q \vee .\ p \vee r.$$

To return to the development of the use of dots as punctuation, the following theorem is “Comm”, and has three levels of dots, like the primitive proposition “Sum” above:

$$*2{\cdot}04 \quad \vdash :.\ p\ . \supset . q \supset r : \supset : q\ . \supset . p \supset r.$$

The first single dot “works away” from the adjacent connective ‘$\supset$’, to the left, the second, to the right until it meets with the double dots. The double dots both “work away” from the conditional that they flank, revealing it to be the main connective of the formula. The next two dots reveal the scope of the conditional in the consequent, yielding as modern equivalent

$$\vdash \ [p \supset (q \supset r)] \supset [q \supset (p \supset r)].$$

The next example includes the use of dots to indicate both “logical product” (conjunction), and as delimiting brackets:

$$*3{\cdot}01.\ \ p\ .\ q\ . = \ . \sim(\sim p \vee \sim q) \ \ \mathrm{Df}.$$

(Notice that parentheses and braces are frequently used for punctuation in the early numbers of *PM*. There is no mention of this in the explanation of the scope of dots.) As a first step in translating $*3{\cdot}01$ into modern notation, replace the first dot by an ampersand (and its corresponding scope delimiters) and replace ‘$. = .$’ followed by ‘Df’ at the end of the formula by the more familiar ‘$=_{df}$’, and introduce brackets to yield

$$(p \ \&\ q) =_{df} [\sim(\sim p \vee \sim q)].$$

The above step clearly illustrates how a “dot indicating a logical product has a scope which works both backwards and forwards”. Finally, our modern conventions allow us to eliminate the outer parentheses from the definiendum and the brackets '[' and ']' from the definiens, yielding a familiar definition of conjunction in terms of negation and disjunction:

$$p \,\&\, q =_{df} \sim(\sim p \vee \sim q).$$

The notion of the scope of dots as distinguished by "groups" is illustrated with

$$*9{\cdot}01 \sim\{(x).\phi x\}. = .(\exists x). \sim\phi x \quad \text{Df.}$$

If we apply the rule that "dots only work away from the adjacent sign of disjunction, implication, or equivalence, or forward from the adjacent symbol of one of the other kinds enumerated in Group II" (where Group II includes '$(\exists x)$'), then the modern equivalent would be

$$\sim\forall x\, \phi x \;=_{df}\; \exists x \sim\phi x.$$

The ranking of connectives in terms of relative scope, rather than by this notion of "force", is the standard convention regarding punctuation in contemporary logic. It is used to avoid parentheses as punctuation, rather than to interpret the dot punctuation as it is in *PM*. If there are no explicit parentheses to indicate the scope of a connective those which have least precedence in the ranking are presumed to be the principal connective, and so on for subformulas. Thus, instead of formulating DeMorgan's law as the cumbersome $[(\sim p)\vee(\sim q)] \equiv [\sim(p \,\&\, q)]$, we write it simply as $\sim p \vee \sim q \equiv \sim(p \,\&\, q)$.[11] This simpler formulation is natural because $\vee$ and & take precedence over $\equiv$ (have narrower "scope" than it), and $\sim$ takes precedence over them all, with the narrowest scope. Indeed, parentheses are often unneeded around $\equiv$, given a further convention by which $\supset$ takes precedence over $\equiv$. Thus, the formula '$p \supset q \equiv \sim p \vee q$' becomes unambiguous. We might represent these conventions by listing the connectives in groups with those with widest scope at the top:

$$\begin{array}{l} \equiv \\ \supset \\ \&, \vee \\ \sim \end{array}$$

For Whitehead and Russell, however, the symbols $\supset$, $\equiv$, $\vee$ and $\ldots = \ldots$ Df, all in Group I, are of equal force. Group II consists of the variable binding expressions, quantifiers and scope indicators for definite descriptions, and Group III consists of conjunctions. Negation is below all of these. So the ranking in

[11] In *PM* this is proved as: $*4{\cdot}51.\ \vdash : \sim(p\,.\,q). \equiv . \sim p \vee \sim q.$

*PM* would be:

$\supset, \equiv, \vee$ and $\ldots = \ldots$ Df
$(x), (x, y) \ldots (\exists x), (\exists x, y) \ldots [(\iota\, x)\phi x]$
$p\ .\ q$ (dot as conjunction)
$\sim$

This is what Whitehead and Russell mean by "Group I is of greater force than Group II, and Group II than Group III."

Numbers $*9$ and $*10$ cover the same material, presenting alternative developments of the "theory of apparent variables", i.e., the logic of quantifiers.[12] In $*9$ the use of propositional connectives applied to quantified expressions is defined in terms of formulas in prenex form, in which all the quantifers come at the beginning. Another definition of this type is

$$*9{\cdot}04\ \ p\ .\vee .\ (x)\ .\ \phi x\ :=.\ (x)\ .\ p \vee \phi x \quad \text{Df},$$

which would now be written as

$$p \vee \forall x\, \phi x =_{df}\ \forall x\, (p \vee \phi x).$$

The axioms, or primitive propositions, include

$$*9{\cdot}1\quad \vdash : \phi x\ .\supset .\ (\exists z).\ \phi z\ \ \text{Pp},$$

which is:

$$\vdash \phi x \supset \exists z\, \phi z,$$

and

$$*9{\cdot}11\quad \vdash : \phi x \vee \phi y .\supset .\ (\exists z).\ \phi z\ \ \text{Pp},$$

which is

$$\vdash \phi x \vee \phi y \supset \exists z\, \phi z.$$

These include the *real*, or free, variables $x$ and $y$, in addition to the *apparent*, or bound, variable $z$. This axiom must be taken as typically ambiguous, with $\phi$ as a free variable ranging over propositional functions of each type appropriate to $x$, $y$, (and $z$) as an argument.

As rules of inference we have a repetition of $*1{\cdot}1$, described as an "analogue", intended to apply to propositions involving individual variables:

$*9{\cdot}12$ What is implied by a true premiss is true Pp

[12] As was discussed above, Whitehead's letter to the editor of *Mind* (Whitehead, 1926) suggests that he was the author of $*10$.

and a rule of universal generalization, like *modus ponens*, not to be "reduced to symbols" (*PM*, p.9):

$*9\cdot13$ In any assertion containing a real variable, this real variable may be turned into an apparent variable of which all possible values are asserted to satisfy the function in question. Pp.

With these definitions and primitive propositions, the rest of $*9$ consists of proofs of basic theorems of quantificational logic, such as

$$*9\cdot21 \quad \vdash :. \, (x).\, \phi x \;\supset\; \psi x \,.\supset : (x).\, \phi x \,.\supset .(x).\, \psi x.$$

Note the asymmetrical use of dots around the main connective in what is now written as

$$\vdash \;\; \forall x\, (\phi x \supset \psi x) \;\supset\; (\forall x\, \phi x \;\supset \forall x\, \psi x).$$

The next chapter, $*10$, is an alternative development of quantificational logic, which allows for the propositional connectives between quantified formulas as primitive notation. This chapter has different primitive propositions and defines the subscripted variable notation that comes from Peano:

$$*10\cdot02 \;\; \phi x \supset_x \psi x \,.=.\, (x).\, \phi x \supset \psi x \quad \mathrm{Df}$$

and

$$*10\cdot03 \;\; \phi x \equiv_x \psi x \,.=.\, (x).\, \phi x \equiv \psi x \quad \mathrm{Df}.$$

The subscripted variables are not used in contemporary notation, but could be added with simple definitions:

$$\phi x \supset_x \psi x \;=_{df}\; \forall x\, (\phi x \supset \psi x)$$

and

$$\phi x \equiv_x \psi x \;=_{df}\; \forall x\, (\phi x \equiv \psi x).$$

The five primitive propositions in $*10$ are theorems in $*9$, while $*9\cdot1$ and $*9\cdot11$ of the previous section can now be proved as theorems. The primitive propositions of $*10$ include also the axiomatic equivalent of a rule of universal instantiation:

$$*10\cdot1 \quad \vdash : (x).\, \phi x. \supset .\, \phi\, y$$

or

$$\vdash \forall x\, \phi x \supset \phi y.$$

This Whitehead and Russell gloss as

*I.e.* what is true in all cases is true in any one case.

The distinction between "all" and "any" plays an important role in the theory of types, and is also modified in the second edition. The rule of inference is universal generalization, expressed as

$*10{\cdot}11$ If $\phi y$ is true whatever possible argument $y$ may be, then $(x).\phi x$ is true.

## 4.2 Propositional functions and logical types

It is important to distinguish the two kinds of function in the logic of *Principia Mathematica*. Expressions for *propositional functions*, such as '$\hat{x}$ is a natural number' are not to be confused with the expressions for the familiar, *mathematical functions*, such as 'sin $x$' (the sine function). The latter are called "descriptive functions" (*PM*, p. 31). Descriptive functions are defined using relations and the theory of definite descriptions. For a relation $R$ the descriptive function $R`y$, read as "the $R$ of $y$", is to stand for the $x$ such that $xRy$.

Thus in our notation, "sin $y$" would be written "sin`$y$", and "sin" would stand for the relation which sin`$y$ has to $y$.

*(PM, p. 31)*

This notion of "descriptive functions" illustrates the role of the theory of definite descriptions in the technical work of *PM* of reducing mathematics to logic. In contrast with Frege, who took the mathematical notion of function as a primitive in his logic, Whitehead and Russell reduce it to the notions of definite descriptions and relations, or two-place *propositional functions*.[13]

Whitehead and Russell distinguish between expressions with a free variable (such as '$x$ is hurt') and names of propositional functions (such as '$\hat{x}$ is hurt'). (*PM*, pp. 14–15). The propositions which result from the formula by assigning allowable values to the free variable '$x$' are said to be the various "ambiguous values" of the function. Expressions using the circumflex notation, such as $\phi\hat{x}$, only occur in the material introductory to the technical sections of *PM* and not in the technical sections themselves (with the exception of the sections on the theory of classes), prompting Gregory Landini to argue that such expressions do not really occur in the formal system of *PM* as terms.[14] Are they "term-forming operators" which turn an open formula into a name for a function, or simply a syntactic device, a placeholder, for indicating the variable for which a substitution

[13] See Linsky (2009a) and (2009c).

[14] See Landini (1998, p.265). Landini finds this to be an important feature of *PM*, namely that abstraction is *not* a term forming operation. Definite descriptions are not "terms" because they are incomplete symbols which can be eliminated by a contextual definition. No definition is provided for a function expression such as '$\phi\hat{x}$'. Landini does not think that such expressions occur in the formal language of *PM*, and they do only appear in the introductory material.

can made in an open formula? If they are to be treated as term-forming operators, the modern notation for $\phi\hat{x}$ would be '$\lambda x \phi x$'. The $\lambda$-notation has the advantage of clearly revealing that the variable $x$ is *bound* by the term-forming operator $\lambda$, which takes a predicate $\phi$ and yields a term '$\lambda x \phi x$'. Unlike $\lambda$-notation, the *PM* notation using the circumflex cannot indicate scope. Lambda operators can bind variables in lambda terms, producing iterated expressions such as '$\lambda x \lambda z \phi x z$' which names the function which, for argument $x$, yields as value the function $\lambda z \phi x z$. The function expression '$\phi(\hat{x}, \hat{z})$' is ambiguous between '$\lambda x \lambda z \phi x z$' and '$\lambda z \lambda x \phi x z$', without some further convention.[15] The ambiguity is brought out most clearly by using $\lambda$ notation: the first denotes the relation of being an $x$ and $z$ such that $\phi x z$ and the second denotes the converse relation of being a $z$ and $x$ such that $\phi x z$.

Except for some notation for "relative" types in the "Prefatory statement of symbolic conventions" that begins Volume II (*PM* II, pp. *i–xxxi*) there are notoriously no symbols for types in *Principia Mathematica*. The formulas of *PM* are to be taken as "typically ambiguous", and so as schematic, standing for expressions of each well-formed expression of whatever types can coherently be assigned. In keeping with Russell's view that logic consists of completely general statements, it may have seemed best to use typically ambiguous statements as part of the most general expression of logic. No constants, either for individuals or functions, appear in *PM*, and so no specific type assignments will be relevant to the development of mathematics in the system.

It is, however, useful to introduce a system of notation for logical types as part of a discussion of *PM* even if it is not needed for the application of the system that Whitehead and Russell intended. The system of "$r$-type", for "ramified type" symbols developed by Alonzo Church (1976) is consistent with the explicit remarks about types in *PM*, and will be used in what follows. First, however, it is useful to construct a distinct system for indicating "simple" types.

For the purposes of exposition, the full, or *ramified* theory of types in *PM* is best seen as a development of a theory of *simple* types, whether or not that is the order in which they presented themselves to Russell in the period leading up to *PM*.[16] Church presents the theory first as a definition of type symbols, to which is then added an account of the types which they represent. Here the symbols, and the types they represent, will be introduced together.

[15] Whitehead and Russell seem to have specified this convention for "relations in extension", namely, on page 200 of the introductory material of $*21$, in terms of the order of the variables, but don't have a convention for interpreting multiple abstracts in propositional function expressions.

[16] It appears that Russell briefly considers a simple theory of types, at least for classes, in Appendix B to *Principles of Mathematics*, but abandons it because of the need for a distinction between types of proposition as a consequence of the "paradox of propositions" in the end of that appendix.

- $\iota$ is the simple type of individuals.
- Where $\tau_1, \ldots, \tau_m$ are any simple types, then $(\tau_1, \ldots, \tau_m)$ is the simple type of a propositional function whose arguments are of simple types $\tau_1, \ldots, \tau_m$, respectively.
- ( ) is the simple type of propositions, i.e. 0-place propositional functions.[17]

If we consider '**Socrates**' to be a constant for an individual, which is of (simple type) $\iota$, the proposition **Socrates is mortal** of simple type ( ) will be expressed by that constant concatenated with a constant for the function $\hat{x}$ **is mortal**. The function will be of simple type $(\iota)$. The relation of a father to his child, if it is seen as relating individuals, might be represented as the function $\hat{x}$ **is father of** $\hat{y}$, and would be of simple type $(\iota, \iota)$.

Propositional functions of types $(\iota)$, $(\iota, \iota)$, etc., are often called "first order", and hence the name "first order logic" for the familiar logic with variables that range only over the individuals that are arguments of those first order functions. One-place functions of those functions, of simple type $((\iota))$, are quite appropriately called "second order" functions. Binary, and all $n$-ary relations, will have arguments of distinct types. A relation between an individual $x$ and a proposition $p$, as in $\hat{x}$ **believes that** $\hat{p}$, will be of simple type $(\iota, (\,))$. Simple types are usually used in conjunction with a theory of classes, or of functions which do not include propositions. In that case there is no use for the type ( ), while an extensional type theory for logic will have ( ) as the symbol for the type of the truth values T and F.

The type symbols of the full "ramified" theory of types of *Principia Mathematica* keep track not only of the arguments of each function, but also the quantifiers used in the definition of that function. The need for distinguishing functions in this way, as a recognition of the "vicious circle principle", is explained in the introduction to the first edition of *PM*, Chapter II (*PM*, pp. 37–65), and will be discussed further below. Although there is no discussion of a notation for types in *PM*, there have been numerous proposals for such a notation made since.[18] The system of "$r$-types" presented in Church (1976) is the most fine-grained of these various proposals, making more distinctions of type than the others. A system of symbolization for types is to some extent conventional. Although certain functions must be of different types, as a consequence of the "vicious circle principle", and in order to avoid paradox, it is not necessary to distinguish further aspects of their logical form. Conjunctions of functions of a given type are thus assigned the same $r$-type, despite the difference in logical form between the conjunction and its

[17] This is treated by Church as the case of $\tau_1, \ldots, \tau_m$ where $m = 0$.

[18] See Linsky (1999, §4.3), for a comparison of those of Copi (1971), Hatcher (1968), Chihara (1973), and Myhill (1979). Urquhart (2003) follows Church.

conjuncts. Differences of $r$-types will be crucial in two parts of the logic. One such case where a function must be assigned a particular $r$-type is in the restrictions on what counts as a well-formed expression, for a function will take only arguments of a given $r$-type, and will itself only be an argument consistently with its $r$-type. The second is in the formulation of the comprehension principle. In keeping with the formulation of the "vicious circle principle" by which "whatever involves *all* of a collection must not be one of the collection" (*PM*, p. 37), when a function is defined in terms of quantification over other functions of a given $r$-type, or with respect to free occurrences of variables for propositional functions of a given $r$-type, that defined function must be of another, "higher", $r$-type. This can be represented as a restriction on $r$-types in an explicitly stated comprehension principle, as appears in Church (1976), or perhaps in some other fashion. Church accomplishes this by introducing a new notion of "level" in addition to specified notions of "order" and his new term, "$r$-type". The ramified theory of types will be presented here following this terminology.

To use Whitehead and Russell's example from the first Introduction:

$\hat{x}$ had all the qualities that make a great general

is a function true of individuals of type $\iota$, in particular, Napoleon, we shall suppose. Whitehead and Russell propose that the function of "having all the qualities that make a great general" in fact will be defined in terms of a universal quantification over functions such as $\hat{x}$ **is brave** and $\hat{x}$ **is decisive**, etc. If the latter are functions of individuals of the lowest *level*, in Church's notation, they will be of "$r$-type" $(\iota)/1$, where the '1' indicates that lowest level.[19] The predicate for 'has all the qualities that make a great general', because of the quantification over functions of type $(\iota)/1$ in its definition, will need a higher level to indicate that fact, and so is of type $(\iota)/2$.

The resulting system of "$r$-types", and "type-symbols" for them, as Church calls them, can represent these differences in the type of functions of a given argument.

- $\iota$ is the $r$-type for an individual.
- If $\tau_1, \ldots, \tau_m$ are any $r$-types then $(\tau_1, \ldots, \tau_m)/n$ is the $r$-type of a propositional function of *level* $n$, whose arguments are of types $\tau_1, \ldots, \tau_m$, respectively.
- $(\ )/n$ is the $r$-type of a proposition of *level* $n$.

The notion of the *order* of a function is different from its standard use which comes from the simple theory of types. The order of an $r$-type can be defined inductively.

[19] Hazen (2004) calls this the "badness" of a function rather than its "level". See Hazen (1983).

- The order of an individual of type $\iota$ is 0.
- The order of a function $(\tau_1, \ldots, \tau_m)/n$ is $k + n$ where $k$ is the greatest of the orders of $\tau_1, \ldots, \tau_m$.

On this account the simplest functions of individuals, of $r$-type $(\iota)/1$, will be of order 1, functions which have those functions (of lowest level), and are thus of $r$-type $((\iota)/1)/1$, will be of order 2, etc. For these functions, and functions of functions, the language of "first order", "second order", etc., coincides with contemporary usage, and terminology of the simple theory of types. It is when a function of individuals is defined in terms of higher type functions that one can find more functions of individuals of order two, and so get a divergence from contemporary usage. The function expressed by "has all the qualities that make for a great general" will be one of these functions of order two which takes individuals as arguments.

Both the requirements that a function must be of a higher order than its arguments, and that a function defined in terms of a totality must be of a higher order than that totality are captured in Church's formulation in restrictions on a single comprehension principle ('$A$' is a schematic letter for a formula):[20]

$$(\exists\phi)(\forall x_1)(\forall x_2)\ldots(\forall x_m)\ [\phi(x_1, x_2 \ldots x_m) \equiv\ A],$$

where $\phi$ is a functional variable of r-type $(\beta_1, \beta_2, \ldots, \beta_m)/n$ and $x_1, x_2, \ldots, x_m$ are variables of $r$-types $\beta_1, \beta_2, \ldots, \beta_m$ and the bound variables of $A$ are all of order less than the order of $\phi$, and the free variables of $A$ and the constants occurring in $A$ are all of order not greater than the order of $\phi$ (and $\phi$ does not occur free in $A$).

The function expressed by "has all the qualities that make for a great general" will be of a higher order than those qualities over which the definition quantifies. Church could represent this definition as an instance of such a comprehension principle which guarantees the existence of a function true of all and only individuals who have the "qualities that make for a great general". A fully intensional formulation of the ramified theory of types, which one would want to assert the existence of that very function, "$\hat{x}$ has all the qualities that make for a great general", would need some way of expressing the identity of functions, such as a sign for identity of functions, or some sort of term-forming operator for functions.[21] This comprehension principle does, however, illustrate the consequences of observing the vicious circle principle in attributing orders to propositional functions. The other role of type restrictions, to correlate the types of functions and their

[20] This formula is altered slightly from the original at Church (1976, p. 748).
[21] See Anderson (1989) and Linsky (1999, pp. 58–60) for a discussion of these issues.

arguments, can be implemented by restrictions on what counts as a well-formed formula.[22]

## 4.3 Real and apparent variables

There are no individual or predicate constants occurring as terms in the formal system of *PM*, only individual variables. The upper case "Roman" letters '$R$', '$S$', etc. are used in the introductory material to represent constants for atomic predicates, then after $*20$ as variables for relations in extension. *PM* makes special use of the distinction between *real*, or free, variables and *apparent*, or bound, variables. Consequently, in the introductory material and before $*20$, '$xRy$' will be an atomic formula, with '$x$'and '$y$' real variables. When such formulas are combined with the propositional connectives $\sim$, $\vee$, etc., the result is a *matrix*. Thus '$xRy \mathrel{.\vee.} yRx$' is a matrix. There are also variables which range over functions: '$\phi$, $\psi$,..., $f$, $g$', etc. The expression '$\phi x$' contains two concatenated variables which together stand for a proposition, namely, the result of applying the propositional function $\phi$ to the individual $x$.

Theorems are stated with real variables, which gives them a special significance with regard to the theory. For example, $*10{\cdot}1$, $(x).\ \phi x \mathrel{.\supset.} \phi y$, is a fundamental axiom of the quantificational theory of *PM*. In this primitive proposition the variables '$\phi$' and '$y$' are real (free), and the '$x$' is apparent (bound). As there are no constants in the system, this is the closest that *PM* comes to a rule of universal instantiation. Whitehead and Russell interpret '$(x).\ \phi x$' as "the proposition which asserts *all* the values for $\phi\hat{x}$" (*PM*, p.41). The use of the word 'all' has special significance within the theory of types. They present the vicious circle principle, which underlies the theory of types, as asserting that

> . . . generally, given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total. By saying that the set has "no total", we mean, primarily, that no significant statement can be made about "all its members".
>
> *(PM, p.37)*

Specifically, since a quantified expression talks about *all* the members of a totality, it must range over a specific logical type in order to observe the vicious circle principle. Thus, when interpreting a bound variable, we must assume that it ranges over a specific type of entity and so types must be assigned to the other entities represented by expressions in the formula, in accordance with the theory of types.

[22] This appears above in the condition that if $\phi$ is a functional variable of $r$-type $(\beta_1, \beta_2, \ldots, \beta_m)/n$ then the variables to which it applies are of $r$-types $\beta_1, \beta_2, \ldots, \beta_m$.

The statements of primitive propositions and theorems in *PM* such as $*10\cdot1$ are taken to be *typically ambiguous*, that is, ambiguous with respect to type. These statements are actually schematic and represent all the possible specific assertions which can be derived from them by interpreting types appropriately. But if statements like $*10\cdot1$ are schemata and yet have bound variables, how do we assign types to the entities over which the bound variables range? The answer is that we first decide the type of thing over which the free variables in the statement range. For example, assuming that the variable $y$ in $*10\cdot1$ ranges over individuals (of type $\iota$), then the variable $\phi$ must range over functions of type $(\iota)/n$, for some $n$. Then the bound variable $x$ will also range over individuals. If, however, we assume that the variable $y$ in $*10\cdot1$ ranges over functions of type $(\iota)/1$, then the variable $\phi$ must range over functions of type $((\iota)/1)/m$, for some $m$. In this case, the bound variable $x$ will range over functions of the same type as $y$, namely, $(\iota)/1$. Whitehead and Russell frequently express this by saying that real variables are taken to ambiguously denote "any" of their instances, while bound variables ambiguously denote "all" of their instances (within a legitimate totality, i.e., type).

The seemingly minor technical question of whether proposition letters '$p$', '$q$', etc., can appear as apparent, or bound, variables in fact raises deep issues about the logic of *PM*.[23] In the Introduction to the first edition, the authors describe the "multiple relation theory of judgement" and conclude:

> Owing to the plurality of the objects of a single judgement, it follows that what we call a "proposition" (in the sense in which it is distinguished from the phrase expressing it) is not a single entity at all. That is to say, the phrase which expresses a proposition is what we call an "incomplete" symbol; it does not have meaning in itself, but requires some supplementation in order to acquire a complete meaning.
>
> *(PM, p.44)*

Landini (1998) finds the absence of propositional variables to be significant, in accordance with his view that propositional functions are to be derived from propositions via the "substitutional theory" that preceded *PM*. Yet the remarks that propositional variables will not be used as bound variables all suggest that this is not a matter of principle, but simply a report that it won't be necessary to settle the deeper issue in what follows. Thus:

> Since we never have occasion, in practice, to consider propositions as apparent variables, it follows that the hierarchy of propositions (as opposed to the hierarchy of functions) will never be relevant in practice after the present number.
>
> *(PM, p.129)*

[23] The connection between this issue and the question of the ontological status of propositional functions and universals is discussed in Stevens (2005, pp. 81–9). For an alternative view which defends quantification over propositions and a realist, as opposed to nominalist, view of propositional functions, see Linsky (1999).

The "present number" of the quote above is $*9$, so what the authors say is not strictly true. Propositions do appear as apparent variables, in $*14{\cdot}3$, which is intended to show that the scope of (proper) definite descriptions is irrelevant in extensional contexts. That a context $f$ is extensional is expressed with propositional variables:

$$p \equiv q \,.\supset_{p,q} .\, f(p) \equiv f(q).$$

Immediately after, however, we have this qualification:

In this proposition, however, the use of propositions as apparent variables involves an apparatus not required elsewhere, and we have therefore not used this proposition in subsequent proofs.

*(PM, p.185)*

There are at least three other uses of bound propositional variables in *PM*. One is in the discussion of the "epistemic paradoxes" in the Introduction, where, for example, the liar sentence, "I am lying" is represented as "There is a proposition which I am affirming and which is false." (*PM*, p.62) There is a second occurrence of a bound propositional variable on page 129 in the discussion of the move from propositional to quantificational logic. The third is on the first page of the Introduction to the second edition (*PM*, p.*viii*), where Russell explains the new treatment of apparent variables in propositions as equivalent to bound variables. None is in the formal development of the system, but rather in introductory material. For the purposes of *PM* both in the development of propositional logic in $*1$ to $*5$ and the later logic of propositional functions no such uses of propositional variables need be considered. These remarks are all part of what would now be called "meta-logic", and so perhaps Whitehead and Russell were becoming aware of the difference between what can be said *about* their system and what can be said *in* it.

In Church's notation, which I am following in this presentation of *PM*, a proposition will have a type $(\,)/n$ of level $n$. Functions, on the other hand, will have some type symbol $\alpha$ representing an argument in their types symbols $(\alpha)/n$. So, following this notation, propositional variables could occur as bound, "apparent", variables, but they will not, except when necessary to represent a function or proposition which does not actually occur in the work.

One of the "improvements" proposed in the Introduction to the second edition of *PM* is that the distinction between "any" and "all", represented with the use of real and apparent variables, be abandoned. One consequence of this is that the sentence letters in numbers $*1$ to $*5$ now must be tacitly read as bound variables ranging over propositions, and so going back on the view expressed in the first edition that there is no "occasion" for such real variables. Given the result in Appendix C, however, there *are* propositions in the second edition of *PM*, and so this is to be expected.

## 4.4 The axiom of reducibility: $*12$ – Identity: $*13$

The exclamation mark '!' following a variable for a function and preceding the argument, as in '$f!x$', '$\phi!x$' and '$\phi!\hat{x}$' indicates that the function is *predicative*, that is, of the lowest order which can apply to its arguments. In Church's notation, this means that predicative functions are all of the first level, with types of the form $(\ldots)/1$. As a result, predicative functions will be of order one more than the highest order of any of their arguments. This definition is presented in the Introduction to *PM*.

> We will define a function of one variable as predicative when it is of the next order above that of its argument, i.e., of the lowest order compatible with its having that argument.
>
> *(PM, p.53)*

Unfortunately this does not appear to the be same definition found in the summary of $*12$, where we find

> A predicative function is one which contains no apparent variables, i.e., is a matrix."
>
> *(PM, p.167)*

On this definition, by which predicative functions are those that are quantifier free, so that $\exists x . \phi x \hat{y}$ is no longer counted as predicative, but only the quantifier free "matrix", $\phi \hat{x} \hat{y}$, the predicative functions would be described by the simple theory of types.

Whitehead and Russell themselves recognize the difference, saying in the Preface to the first edition that:

> The explanation of the hierarchy of types in the Introduction differs slightly from that given in $*12$ of the body of the work. The latter explanation is stricter and is that which is assumed throughout the rest of the book.
>
> *(PM, p.vii)*

This would hardly seem to be a "slight" difference if one were to frequently encounter functions with arguments defined using quantifiers, of the $r$-type of the form $((\alpha)/2)/1$, for example, as one does when discussing the contradictions and "vicious circle fallacies" in the Introduction. However, the mathematical material in the body of *PM* involves definitions of extensional functions of classes and relations in extension, which are all defined using predicative functions and relations via the "no-classes" theory, to be presented below. The "explanation" of predicative functions in $*12$ is "stricter" than that of the Introduction in that matrices, the predicative functions of $*12$, are only some of the predicative functions countenanced in the Introduction. While satisfactory for all the needs of the "rest" of *PM*, the claim that there is a predicative function that is equivalent to any arbitrary, higher level, function, in other words, the axiom of reducibility, becomes stronger on the account in $*12$. There are no other technical consequences of the difference, and so, the account of the Introduction, by which a predicative function is simply of the lowest order compatible with its arguments, will be followed below.[24]

The account of predicative functions as matrices is repeated in the Introduction to the second edition, however, and will figure in interpretive problems discussed in the next chapter. Indeed in the new Introduction to the second edition at *xxvii*, we find: "Thus $\phi!x$ is a function of two variables, $x$ and $\phi!\hat{z}$. It is a matrix, since it contains no apparent variable and has elementary propositions for its values." Russell does not suggest that the meaning of "!" has changed, in fact he does use the term "predicative function" in the new Introduction.

The axiom of reducibility states that for any function there is an equivalent function (i.e., one true of all the same arguments) which is predicative:

$$*12{\cdot}1 \quad \vdash : (\exists f) : \phi x \, . \equiv_x . \, f!x \quad \text{Pp.}$$

Whitehead and Russell express doubts about the axiom of reducibility in the first edition of *PM*, and one of the major "improvements" proposed for the second edition is to do away with the axiom. Yet the mathematics developed in *PM*, which includes the elements of analysis, requires frequent use of impredicative definitions of classes. For example, the least upper bound of a class $X$ of reals of order $n$ is defined as a real number which is the least which is greater than or equal to each of *those* very numbers in $X$:

$x$ is the l.u.b. of $X$ iff

$$(y)(y \in X \rightarrow y \leq x) \wedge [(z)((y)(y \in X \rightarrow y \leq z) \rightarrow x \leq z].$$

Because $X$ occurs as a free variable in the definition above, the least upper bound must be of at least of the order of $X$, that is at least $n+1$ . (If a variable of the type of $X$ occurred as a bound variable, the order of the defined notion would

[24] Church (1976, n.1) says that the account of the Introduction is carried over from Russell (1908) and that the account of $*12$ is repeated in the Introduction to the second edition. His own account is a modification of the account in the Introduction. Michael Potter (2000, §10.9, §5.10, §7.2) notes the two accounts and carries them through his discussion of both editions. He describes the Introduction explanation as a later addition after the body of *PM* had been written, and the resulting co-existence of two distinct accounts of the hierarchy of types as a "bizarre" oversight on Russell's part (Potter, 2000, p.144). Landini (1998), (2007), and elsewhere asserts that the identification of predicative functions with matrices is the official doctrine of *PM*. A semantic account of predicative functions is given in Mares (2007).

also have to be at least $n + 1$.) Yet much of the theory of real numbers relies on the fact that the least upper bound of a class of numbers not only can potentially be, but often is, a member of that very class. If $X$ is the closed interval [0, 1], for example, the l.u.b. of $X$, which is 1, is obviously a member of $X$. Some numbers will only be identifiable as a least upper bound of a given set, and so, as far as the theory of types is concerned, of a different type from the members of that set. While nothing in mathematics hinges on which of several co-extensive definitions of specific classes of rational numbers is identified with a given real number, the ramified theory of types will distinguish those numbers by their definitions. The axiom of reducibility guarantees that numbers and sets defined with the aid of quantification over a higher order are nevertheless of the lowest order possible.

In other places the axiom of reducibility is needed to define notions which would otherwise have to violate the theory of types by referring to "all" types in a way that would require an illegitimate totality. Consider the definition of identity offered in *PM*:

$$*13{\cdot}01 \quad x = y \;.=:\; (\phi) : \phi!x \;.\supset .\; \phi!y \quad \text{Df},$$

that is, $x$ is identical with $y$ if and only if $y$ has every predicative function $\phi$ which is possessed by $x$. (Of course the second occurrence of '=' indicates a definition, and does not independently have meaning. It is the first occurrence, relating individuals $x$ and $y$, which is defined.) The more familiar definition of identity on which objects are identical iff they share *all* properties, is not even statable in the theory of types, where quantifiers can range only over "legitimate" totalities, and not all properties of whatever types. There is no totality of *all* functions which take arguments of a given $r$-type $\alpha$, although there will be a totality of functions of the $r$-type $(\alpha)/n$ for a specific $n$. An informal argument, however, shows that $*13{\cdot}01$ has the force of the intuitive, Leibnizian, notion of identity as the sharing of all properties, as follows. Suppose that some $x$ and $y$ differ on some property (propositional function) $\phi\hat{x}$ of some arbitrary type. Suppose that $x$ has $\phi$, but $y$ does not. Now the axiom of reducibility guarantees that there will be a predicative function $f!\hat{x}$, which is co-extensive with $\phi\hat{x}$ such that $fx$ but not $fy$. So, by $*13{\cdot}01$, they are not identical, exactly the result that we want the definition to yield.

Given this intimate association between the definition of identity and the axiom of reducibility, it is especially problematic that in the second edition of *PM*, in which the axiom of reducibility is abandoned, the definition of identity of $*13{\cdot}01$ is untouched.[25] This issue will be pursued further in the discussion of Appendix B in Chapter 6, below.

[25] See the reference to 13·01 at the beginning of Appendix C (*PM*, p.401).

## 4.5 Definite descriptions: $*14$

The inverted Greek letter iota, '$\iota$', followed by a variable, is used to begin a definite description.[26] The complete description '$(\iota x)\phi x$' is to be read as "the $x$ such that $x$ is $\phi$", or more simply, as "the $\phi$". Such expressions occur in subject position as would singular terms, as in '$\psi(\iota x)\phi x$', read as "the $\phi$ is $\psi$". The formal part of Russell's famous "theory of definite descriptions" consists of a definition which applies to each context $\psi$ in which the description '$(\iota x)\phi x$' occurs. To distinguish the portion $\psi$ from the rest of a larger sentence in which the expression '$(\iota x)\phi x$' occurs, its "context", the scope of the description is indicated by repeating the definite description within brackets:

$$[(\iota x)\phi x] \,.\, \psi(\iota x)\phi x.$$

The notion of scope is meant to explain a distinction which Russell first discusses in "On denoting" (OD). The notion of scope has become central in recent discussions of the theory of definite descriptions in the philosophy of language. In *PM*, however, scope does not loom large after section $*14$, because only extensional notions such as classes and relations in extension are considered thereafter. Russell's theory of descriptions is relevant to the second edition of *PM* only tangentially, and then only because of the role of scope in the "no-classes" theory that Leon Chwistek pointed out in his "Theory of constructive types" (1924).

In "On denoting", Russell says that the sentence 'The present king of France is not bald' is ambiguous between two readings: (1) the reading where it says of the present king of France that he is not bald, and (2) the reading on which it denies that the present king of France is bald. The former reading requires that there be a unique king of France on the list of things that are not bald, whereas the latter simply says that there is not a unique king of France that appears on the list of bald things. Russell says only the latter, but not the former, can be true in a circumstance in which there is no king of France. Russell analyzes this difference as a matter of the scope of the definite description.

In *PM* Whitehead and Russell introduce a method for indicating the scope of the definite description. The fundamental definition of the theory of descriptions is

$$*14{\cdot}01.\;\; [(\iota x)\phi x] \,.\, \psi(\iota x)\phi x \;.=:\; (\exists b) : \phi x \,.\equiv_x .\, x = b : \psi b \;\; \text{Df.}$$

This has come to be called a "contextual" definition, to be contrasted with explicit definitions. An explicit definition of the definite description would be something

[26] It was created by rotating the piece of type for the iota, $\iota$. Several of Peano's distinctive contributions to logical symbolism, such as the '$\exists$' and '$\supset$', started out this way. '$\supset$' began as a capital 'C' which was rotated, and then later simplified.

like the following:

$$(\imath x)\phi x \ = \ldots \ \text{Df},$$

which would allow the definite description to be replaced in any context by whichever defining expression fills in the ellipsis. By contrast, ∗14·01 shows how a sentence in which there is an occurrence of a description $(\imath x)\phi x$ in a context $\psi$ $\ldots$, can be replaced by some other sentence (involving $\phi$ and $\psi$) to which it is equivalent. To develop an instance of this definition, consider the following famous example from "On denoting":

The present king of France is bald.

Using '$Kx$' to represent the propositional function of being a present king of France and $B$ to represent the propositional function of being bald, Whitehead and Russell would represent the above claim as

$$[(\imath x)Kx] \,.\, B(\imath x)Kx,$$

which by ∗14·01 is defined as

$$(\exists b) : Kx \,.\, \equiv_x \,.\, x = b : Bb.$$

In words (mixing in a few symbols), there is a $b$ such that anything $x$ is king of France if and only if it is that $b$, and $b$ is bald, or, there is one and only one thing which is a present king of France and it is bald.

Now consider a different example, which shows how the scope of the description makes a difference:

The present king of France is not bald.

There are two options for representing this sentence:

$$[(\imath x)Kx] \,.\, \sim B(\imath x)Kx$$

and

$$\sim [(\imath x)Kx] \,.\, B(\imath x)Kx.$$

In the first, the description has "wide" scope, and in the second, the description has "narrow" scope. (Russell says that the description has "primary occurrence" in the former, and "secondary occurrence" in the latter.) Given the definition ∗14·01, the two formulas immediately above become expanded into primitive notation as

$$(\exists b) : Kx \,.\, \equiv_x \,.\, x = b : \sim Bb$$

and

$$\sim(\exists b) : Kx \,.\equiv_x .\, x = b : Bb.$$

The former says that there is one and only one object which is a present king of France and it is not bald; i.e., there is exactly one present king of France and he is not bald. This reading is false, given that there is no present king of France. The latter says it is not the case that there is exactly one present king of France and it is bald. This reading is true.

In fact, in extensional contexts, it is only in cases of such "improper" descriptions, which do not denote a unique object, that the scope of the description makes a difference. Whitehead and Russell prove assertion $*14{\cdot}3$, which they describe as showing that the scope of a description $(\imath x)\phi x$ does not "matter" to the truth value of a proposition in which it occurs, when that description is proper, and so "occurs in what we may call a 'truth function', i.e. a function whose truth or falsehood depends only upon the truth or falsehood of its argument or arguments." (*PM*, p.184)

This proposition is unique in *PM*, both for introducing the notion of a truth-functional context explicitly, and for using *real* (bound) variables for propositions, which, after the initial five "numbers" of propositional logic, otherwise disappear from the symbolism of *PM*, whether free or bound:

$$*14{\cdot}3.\ \vdash :.\ p \equiv q \,.\supset_{p,q} .\, f(p) \equiv f(q) :\ \mathrm{E!}(\imath x)(\phi x) : \supset :$$
$$f\{[(\imath x)(\phi x)] \,.\, \chi(\imath x)(\phi x)\} \,.\equiv .\, [(\imath x)(\phi x)].\ f\{\chi(\imath x)(\phi x)\}.$$

This proposition is anomolous not only for using bound propositional variables $p$ and $q$ and as the only occasion on which the argument to a function, $f$, is a proposition expressed by a complex formula, $\chi(\imath x)(\phi x)$, rather than a single variable. It is also one of the few clear cases of meta-logical theorems in the work.[27] Theorem $*14{\cdot}3$ is also significant for understanding changes in the new second edition, in two ways. First, it only makes sense if there is no general guarantee that the contexts in which functions will occur are always truth-functions. This makes clear the intensional nature of the logic of the first edition, and so what substance there is to the explicit adoption of extensionality in the second edition. Secondly, Theorem $*14{\cdot}3$ shows the importance of the notion of scope in the logic of *PM*, and so makes more surprising the absence of any explicit consideration of scope in connection with the theory of classes to be introduced below. It was this oversight that was identified by Chwistek in his (1924) "Theory of constructive types".

[27] See Kripke (2005) for a discussion of these issues.

## 4.6 Classes: $*20$

The circumflex ‘ ^ ’ over a variable preceding a formula is used to indicate a class, thus $\hat{x}\psi x$ is the class of things $x$ which are such that $\psi x$. In modern notation we represent this class as $\{x : \psi x\}$.

Recall that ‘$\phi\hat{x}$’, with the circumflex over a variable after the predicate variable, expresses the propositional function of being an $x$ such that $\phi x$. In the type theory of *PM*, the class $\hat{x}\phi x$ has the same logical type as the function $\phi\hat{x}$. This makes it appropriate to use the following contextual definition, which allows one to eliminate the class term $\hat{x}\psi x$ from occurrences in the context $f\{\ldots\}$:

$$*20{\cdot}01 \quad f\{\hat{z}(\psi z)\} \,.=:\, (\exists\phi) : \phi\,!\,x \,.\equiv_x .\, \psi x : f\{\phi\,!\,\hat{z}\} \quad \text{Df.}$$

In contemporary notation this would be

$$f(\{x : \psi x\}) =_{df} \exists\phi\forall x(\phi x \equiv \psi x \;\&\; f((\lambda x)\phi x)),$$

$$[\text{ for}(\lambda x)\phi x \text{ a predicative function of} x].$$

Note that $f$ has to be interpreted as a higher-order function which is predicated of the function $\phi!\hat{x}$. In the modern notation used above, the language has to be a typed language in which expressions such as ‘$\phi$’ are allowed in argument position.

As was subsequently pointed out, by Chwistek (1924) and Carnap (1947), there are formulas that are ambiguous unless one introduces scope indicators for class expressions just as there are for definite descriptions.[28] Chwistek, and then later Carnap, use the example of the pair of formulas:

(1) $\hat{x}\psi x \neq \psi\hat{x}$

and

(2) $\hat{x}\psi x = \psi\hat{x}$

which can both be true, reading them, respectively, as:

(3) $(\exists\phi) : \phi\,!\,x \,.\equiv_x .\, \psi x : \phi\hat{x} \neq \psi\hat{x}$

and

(4) $(\exists\phi) : \phi\,!\,x \,.\equiv_x .\, \psi x : \phi\hat{x} = \psi\hat{x}$.

Chwistek, following a suggestion in (*PM*, p.80), proposed imitating the notation for definite descriptions, thus replacing the definition $*20{\cdot}01$ by

$$[\hat{z}(\psi z)] \,.\, f\{\hat{z}(\psi z)\} \,.=:\, (\exists\phi) : \phi\,!\,x \,.\equiv_x .\, \psi x : f\{\phi\,!\,\hat{z}\}.$$ [29]

[28] Gödel uses almost the same formula, $\phi!\hat{u} = \hat{u}\phi!u$, to show that the order of elimination of defined expressions is “not indifferent”. (Gödel, 1944, p.126)

[29] The proposal to indicate scope for class expressions is also included in the revival of discussion of scope in descriptions in Smullyan (1948).

With indicators of scope for class expressions, the seeming contradiction between (1) and (2) can be handled as was the example of "the present king of France is not bald". The first sentence, (1) $\hat{x}\psi x \neq \psi\hat{x}$ , is seen as ambiguous between two readings, namely that given in (3), with wide scope for the class term corresponding to the wide scope for the description as in "the present king of France is not bald", and a second, narrow, scope for the class term:

$$(5) \sim (\exists\phi): \phi\,!\,x \,.\equiv_x .\, \psi x : \phi x = \psi x$$

corresponding to the narrow scope for the description in "It is not the case that the present king of France is bald." This last expression (5) is false when (4) is true, and vice versa. The reading of (1) as (5) is the true contradictory of (2). Contemporary formalizations of set theory make use of something like these contextual definitions, when they require an "existence" theorem of the form $\exists x\forall y(y \in x \equiv \phi x)$, in order to justify the introduction of a singular term $\{x : \phi x\}$ . (From the axiom of extensionality, it follows from $\exists x\forall y(y \in x \equiv \phi x)$ that there is a unique such set.)

The relation of membership in classes, $\epsilon$, is defined in *PM* by first defining a similar relationship between objects and propositional functions:

$$*20{\cdot}02 \quad x \,\epsilon\, (\phi\,!\,\hat{z}) \,.=.\, \phi\,!\,x \quad \text{Df.}$$

$*20{\cdot}01$ and $*20{\cdot}02$ together can be used to define the more familiar notion of membership in a class. The formal expression '$y \,\epsilon\, \hat{z}(\phi!z)$' can now be interpreted as being constructed from a context in which the class term occurs and is then eliminated by the contextual definition $*20{\cdot}01$, yielding the theorem

$$*20{\cdot}3 \quad \vdash : x \,\epsilon\, \hat{z}(\psi z) \,.\equiv.\, \psi x.$$

This is an unrestricted "abstraction" principle, limited only by the restrictions imposed by the theory of types on $\psi$ and $x$, and so avoiding the paradox.

*PM* also uses Greek letters as variables for classes: $\alpha$, $\beta$, $\gamma$, etc. These will appear both as "apparent" and as "real" variables and in the expressions for propositional functions true of classes, as in $\phi\hat{\alpha}$.

$$*20{\cdot}07 \quad (\alpha) \,.\, f\alpha \,.=.\, (\phi) \,.\, f\{\hat{z}(\phi\,!\,z)\} \quad \text{Df.}$$

Thus universally quantified class variables are defined in terms of quantifiers ranging over predicative functions. Likewise for existential quantification:

$$*20{\cdot}071 \quad (\exists\alpha) \,.\, f\alpha \,.=.\, (\exists\phi) \,.\, f\{\hat{z}(\phi\,!\,z)\} \quad \text{Df.}$$

Expressions with a free Greek variable to the left of $\epsilon$ are defined by

$$*20{\cdot}081 \quad \alpha \,\epsilon\, (\psi\,!\,\hat{\alpha}) \,.=.\, \psi\,!\,\alpha \quad \text{Df.}$$

These definitions do not cover all possible occurrences of Greek variables. In the Introduction to *PM*, further definitions of $f\alpha$ and $f\hat{\alpha}$ are proposed, but it is remarked that the definitions are in some way peculiar and they do not appear in the body of the work. The definition considered for $f\hat{\alpha}$ is

$$f\hat{\alpha} \; . = . \; (\exists\psi) \; . \; \hat{\phi}!x \equiv_x \psi!x \; . \; f\{\psi \,!\, \hat{z}\}$$

or, in a mixed modern and original notation, when restricting $\phi$ and $\psi$ to predicative functions,

$$f\hat{\alpha} =_{df} \lambda\phi \, [\, \exists\psi\forall x(\phi x \equiv \psi x) \;\&\; f\{\lambda z(\psi z\}]$$

or, in even more thoroughly modern notation, not using the "no-classes" theory, but instead explicitly using class terms,

$$\lambda\alpha f\alpha =_{df} \lambda\phi f\{x : \phi x\}.$$

That is, '$f\hat{\alpha}$' is an expression naming the function which takes a function $\phi$ to a proposition which asserts $f$ is of the class of $\phi$s. The versions in modern notation show that in the proposed definition of '$f\hat{\alpha}$' in *PM* notation, we should not expect '$\alpha$' to appear in the definiens, since it is really a bound variable in '$f\hat{\alpha}$'; similarly, we shouldn't expect '$\phi$' in the definiendum because it is a bound variable in the definiens. Whitehead and Russell suggest that this feature of the definition is a "peculiarity", and so the definition does not appear in the body of the text. The qualms about the formal correctness of the definition, however, are not justified. Still, expressions with the circumflex (abstracts) appear only rarely in the introductory material in the body of *PM*, and not at all in the technical sections. One might also expect definitions like ∗20·07 and ∗20·071 to hold for cases in which the Roman letter '$z$' is replaced by a Greek letter, and other such expressions mixing class and propositional function variables in various combinations. The definitions in *PM* are thus not complete, but it is possible to guess at how they would be extended to cover all occurrences of Greek letters.[30] This would complete the project of the "no-classes" theory of classes by showing how all talk of classes can be reduced to the theory of propositional functions.

Determining the power of the theory of classes in *PM* is a subtle technical question. It is likely close to Zermelo's (1908a) original axiomatization of set theory (which did not include the axiom of replacement). While some of the axioms of set theory, such as infinity and replacement, have a different force in the context of a typed language, others become easily proved as theorems. The most striking example of such a theorem is the basic set theoretical "axiom" of

[30] See Linsky (2004b).

extensionality, which in *PM* is a theorem:

$$*20{\cdot}15 \ \vdash :. \ \psi x \ . \equiv_x . \ \chi x : \equiv . \ \hat{z}(\psi z) = \hat{z}(\chi z).$$

## 4.7 General theory of relations: $*21$

Although students of philosophy usually read no further than $*20$ in *PM*, and indeed the changes in the second edition mostly concern these earlier numbers, this is in fact the point where the "construction" of mathematics really starts. $*21$ presents the "General theory of relations", the theory of relations in extension; which in contemporary set theory are treated as sets of sequences, ordered pairs, triples, and so on. Rather than being treated as sets of ordered sequences, in *PM* relations (in extension) are just the counterpart of classes, only for two and more place relations:

$$*21{\cdot}01 \quad f\{\hat{x}\hat{y}\ \psi(x, y)\} \ . = \\ : (\exists\phi) : \phi\,!\,(x, y) \ . \equiv_{x,y} . \ \psi(x, y) : f\{\phi\,!\,(\hat{u}, \hat{v})\} \quad \text{Df.}$$

From $*21$ on "capital Latin letters", i.e. '$R$', '$S$', '$T$', etc., are reserved for these relations in extension. They are variables, replaced by such expressions as $\hat{x}\hat{y}\ \psi\,!\,(x, y)$, the authors say "just as we used Greek letters for variable expressions of the form $\hat{z}(\phi\,!\,z)$." (*PM*, p.201) These new symbols for relations in extension are written between variables, such as $xRy$, $uSv$, or $u\ \hat{x}\hat{y}\psi!(x, y)\ v$, where as propositional function letter would precede the variables, as in $\psi(u, v)$. It is not clear how this "infix" notation for relations in extension would be extended to three or four place relations. As well, this seems to give a new use to the capitals such as '$R$', which elsewhere expresses a relational universal, as when in the Introduction the "multiple relation" theory of judgement is presented.

> The universe consists of various objects having various qualities and standing in various relations. Some of the objects which occur in the universe are complex. When an object is complex it consists of related parts. Let us consider a complex object composed of two parts *a* and *b* standing to each other in the relation *R*. The complex object "a-in-the-relation-*R*-to-b" may be capable of being *perceived*; when perceived, it is perceived as one object.
>
> *(PM, p.43)*

Here a relation expressed by '$R$' is much more clearly a universal, the sort of universal that is a constitutent of a perceived fact, rather than a relation in extension. In the mathematical portions of *PM*, and all of the new material in the second edition, however, the relations symbolized by '$R$', '$S$', and '$T$' are always relations in extension, as defined in $*21$.

"$*22$ Calculus of classes" presents the elementary theory of intersections, unions and the empty class, which is often all the notions of the theory of classes needed for elementary mathematics of other sorts. The relation of inclusion (subset) between classes is defined as in contemporary set theory, although generally symbolized as '$\subseteq$', with the symbol which is used in *PM* now being reserved for *proper* subsets:

$$*22{\cdot}01 \quad \alpha \subset \beta . \; = : x \epsilon \alpha . \supset_x . x \epsilon \beta \quad \text{Df.}$$

The *logical product* of two classes is their intersection:

$$*22{\cdot}02 \quad \alpha \cap \beta \; = \hat{x}(x \epsilon \alpha . x \epsilon \beta) \quad \text{Df.}$$

The *logical sum* of two classes is their union:

$$*22{\cdot}03 \quad \alpha \cup \beta \; = \hat{x}(x \epsilon \alpha . \vee . x \epsilon \beta) \quad \text{Df.}$$

The *negation* of a class will be its "complement". In set theory the complement of a set is not a set, but rather is called a "proper class".[31] In type theory the complement will always exist, being the complement *within* a type:

$$*22{\cdot}04 \quad -\alpha = \hat{x}(x \sim\epsilon \alpha) \quad \text{Df.}$$

By the "no-classes" theory of ∗20 a class variable $\alpha$ is to be eliminated in favor of a propositional function variable. The "negation" of a class $\alpha$ will consist of all those $x$ which are of the type appropriate to that function, but to which it does not apply. Inadvertent definitions of proper classes are thus avoided by the theory of types, though replaced by the danger of violating the theory of types when expressing a function.

The notation for subtraction of classes appears frequently, though not defined as it might be in contemporary set theory. It is defined as

$$*22{\cdot}05 \quad \alpha - \beta = \alpha \cap -\beta \quad \text{Df.}$$

Because $-\beta$ will be a "proper class", $\alpha - \beta$ is defined as $\{x : x \in \alpha \;\&\; x \notin \beta\}$ in axiomatic set theory. This set is shown to exist by the axiom of separation if $\alpha$ is a set.

Because classes will be restricted to members from a given simple type, there will be, for each type, a universal class:

$$*24{\cdot}01 \quad \mathrm{V} = \hat{x}(x = x) \quad \text{Df.}$$

There is no set of $x$ which are self-identical, $\{x : x = x\}$. Again, we have only a "proper class", and not a set.

[31] There is no set of all things, and the union of a set and its complement would include everything as an element.

The empty class is defined as the complement of the universal class, and so there will be a distinct empty class for each type:

$$*24\cdot02 \quad \Lambda = -\mathrm{V} \quad \mathrm{Df.}$$

## 4.8 Logic of relations: $*30$–$*40$ and beyond

The mathematics that is the target of the reductions of the logicist program includes the natural numbers, rationals, and real numbers, as well as the objects of geometry, points, lines, and figures; and mappings or functions between them. Frege's logic took the notions of concept and object as fundamental, treating concepts as functions from objects to truth values. The fundamental notions for Frege, then, were object and function. Russell's logic was based on his notion of propositional function and proposition, which in turn are to be given a metaphysical account in terms of objects and various universals. Propositional functions of one argument do not differ in kind from binary and ternary relations, and are quite appropriately described of as "one-place relations" in presentations of predicate logic.

Although he never remarks on this in his discussions of Frege's philosophy, Russell would seem to have seen Frege's logic as still containing primitive mathematical notions.[32] Frege's account of functions is clearly in mind in the "summary" at the beginning of number $*30$. This passage, with its analysis of the mathematical notion of function, also marks an important waypoint in the development of *PM*. After this point all relations are "relations in extension" and the functions are mathematical rather than propositional functions.

> The functions hitherto considered, with the exception of a few particular functions such as $\alpha \cap \beta$, have been propositional, i.e. have had propositions for their values. But the ordinary functions of mathematics, such as $x^2$, $\sin x$, $\log x$, are not propositional. Functions of this kind always mean "the term having such and such a relation to $x$." For this reason they may be called *descriptive* functions, because they *describe* a certain term by means of its relation to their argument. Thus "$\sin \pi/2$" describes the number 1; yet propositions in which $\sin \pi/2$ occurs are not the same as they would be if 1 were substituted for $\sin \pi/2$. This appears, e.g. from the proposition "$\sin \pi/2 = 1$", which conveys valuable information, whereas "$1 = 1$" is trivial. Descriptive functions, like descriptions in general, have no meaning by themselves, but only as constituents of propositions.
>
> *(PM, p. 231)*

While Russell's theory of definite descriptions from "On denoting" (OD) in 1905 has come to be associated with the philosophy of natural language, its use in the

[32] Peter Hylton speaks of Frege as having "mathematicized" logic with his use of function as a primitive logical notion. Whitehead and Russell can then, by comparison, be seen as having "logicized" the mathematical notion of function. See Hylton (1990, p.261 n.19).

logic of *Principia Mathematica* comes out here. Frege's notion of function and the whole realm of *Bedeutung*, referents, is supplemented by the notion of *Sinn* or sense, precisely to explain such examples of the "information" or cognitive significance which distinguishes informative identities from those that are trivial.[33] After ∗30, definite descriptions do no appear explicitly in *PM*, and all the further development is in terms of relations in extension and descriptive functions. Descriptive functions are defined as follows:

$$*30{\cdot}01 \quad R`y = (\imath x)(xRy) \quad \text{Df.}$$

Thus "$R`y$" is to be read as "the term $x$ which has the relation $R$ to $y$" [34] (*PM*, p.232).

While Whitehead and Russell thus have the apparatus to reduce a more familiar development of mathematics in terms of classes and functions to the logical basis of one-place propositional functions and individuals, in fact the remainder of the three volumes makes use of formulations in the logic of relations in extension.[35] The development of rational numbers follows the ancient Greek geometric interpretation of rational numbers as relations between relations. Ratios are defined at ∗303 (in Volume III) as relations between "vectors". In the case of mass, for example:

> The "vector quantity" $R$, which relates a quantity $m_1$ with a quantity $m_2$, is the relation arising from the existence of some definite physical process of addition by which a body of mass $m_1$ will be tranformed into another body of mass $m_2$. Thus $\sigma$ such steps, symbolized by $R^\sigma$, represents the addition of the mass $\sigma(m_1 - m_2)$.
>
> *(PM* III*, pp.261–2)*

If one expresses the composition of a relation $R$ with itself as $R^2$ (i.e. the relation between $x$ and $z$ when there is some $y$ such that $xRy$ and $yRz$), then 2 is the "exponent" of that relation. Correspondingly, $R^3$ holds between entities two steps away by the relation $R$, and so on. The ratio between two quantities is defined in terms of the exponents of relations that represent them. Thus, for the case of distances, where the relations are spatial, rather than with mass, as above:

> If we call two such distances $R$ and $S$, we may say that they have the ratio $\mu/\nu$ if, starting from some point $x$, $\nu$ repetitions of $R$ take us to the same point $y$ as we reach by $\mu$ repetitions of $S$, i.e. if $xR^\nu y \,.\, xS^\mu y$.
>
> *(PM* III*, p.260)*

[33] Indeed, Frege's paper "On sense and reference" ("Über Sinn und Bedeutung") (1892) begins, famously, with an account of the significance of identity sentences.

[34] In practice it is easiest to read as "the $R$ of $y$". Whitehead and Russell themselves say that the apostrophe is to be read as "of". (*PM*, p.232).

[35] See Linsky (2009a) and (2009c) for an account of "descriptive functions" and the different modern conception of functions as sets of ordered pairs.

Real numbers are constructed as classes of ratios of vectors (at $*310$ in Volume III). The principal definitions of that theory are contained in the list Russell made for Carnap, included as Chapter 8 below. The whole construction of real numbers is carried out in terms of classes of relations, rather than as is standardly done in set theory, with sets of sets, which will include ordered pairs and thus functions as yet more sets. Although these are relations in extension, and so like sets in that regard, real numbers are still not individual sets of sets, but inherently relational in having first and second arguments, etc.

In the discussions over the foundations of mathematics, beginning in the 1920s, it came to be the received view that logic, via the logic of relations and the "no-classes" theory which eliminated classes in favor of propositional functions, was in competition with the view that sets should be taken as primitive mathematical objects, to be codified with the emerging axiomic set theory, then just becoming standardized with the Zermelo–Fraenkel (ZF) axioms. In the second edition of *PM* Russell was adapting to some elements of this discussion. He became convinced to add the principle of extensionality to his logic. One of the distinct features of the sets of ZF is the axiom of extensionality; that sets $x$ and $y$ are identical if all and only members $z$ of $x$ are members of $y$:

$$\forall x \forall y \forall z[(z \in x \equiv z \in y) \supset x = y],$$

which Whitehead and Russell were already able to prove as a theorem ($*20\cdot 15$) of the "no-classes theory". The second edition was to extend this to propositional functions. The deep differences between *PM* and ZF as alternative foundations for mathematics leave traces in the introduction of extensionality to *PM*. It is the complications due to the ramified theory of types for propositional functions and their intensional nature that give rise to the technical complications, and are responsible for the ultimate failure of the "improvements" in the second edition. These distinctive features of *PM* arise from its basis in the logic of relations.

The *converse* of a relation $P$ is defined as

$$*31\cdot 02 \quad \breve{P} = \hat{x}\hat{y}(yPx) \quad \text{Df.}$$

As the "capital Latin letters" such as '$P$' are reserved for relations in extension, this means that the converse of $P$ will be the relation (in extension) which holds between $x$ and $y$ when $P$ holds between $y$ and $x$. Many of these notions of the logic of relations have survived and make an appearance in axiomatic set theory, although the notation is largely obsolete.

In preparation for developing the sequence of natural numbers as the successors of 0, the notions of the "predecessors" and "successors" with respect to a relation is developed in general:

the class of terms which have a relation $R$ to a given term $y$ are called the *referents* of $y$, and the class of terms to which a given term $x$ has the relation $R$ are called the *relata* of $x$. *(PM, p.242)*

The predecessor relation is defined by

$$*32\cdot01 \quad \overrightarrow{R} = \hat{\alpha}\hat{y}\{\alpha = \hat{x}(xRy)\} \quad \text{Df.}$$

As a result, then, $\overrightarrow{R}\text{‘}y$ will be the class of $x$ such that $xRy$, the *referents* of $y$:

$$*32\cdot11 \ \vdash . \ \hat{x}(xRy) = \overrightarrow{R}\text{‘}y.$$

The $R$-successors of $x$ are similarly defined by

$$*32\cdot02 \quad \overleftarrow{R} = \hat{\beta}\hat{x}\{\beta = \hat{y}(xRy)\} \quad \text{Df,}$$

and so

$$*32\cdot111 \ \vdash . \ \hat{y}(xRy) = \overleftarrow{R}\text{‘}x.$$

The familiar notions of *domain* D, “range” or *converse domain* Ɑ, and *field* $C$, of a relation are defined in ∗33. They are introduced as higher order relations between individuals and relations, with the more familiar interpretations using quantification introduced by theorems. The definitions are these:

$$*33\cdot01 \ \ \mathrm{D} = \hat{\alpha}\hat{R}\,[\alpha = \hat{x}\{(\exists y) . xRy\}] \quad \text{Df,}$$

$$*33\cdot02 \ \ \text{Ɑ} = \hat{\beta}\hat{R}\,[\beta = \hat{y}\{(\exists x) . xRy\}] \quad \text{Df,}$$

$$*33\cdot03 \ \ C = \hat{\gamma}\hat{R}\,[\gamma = \hat{x}\{(\exists y) : xRy . \vee . yRx\}] \quad \text{Df.}$$

The theorems specify the meaning of these notions as applied to a given $R$:

$$*33\cdot11 \ \vdash . \ \mathrm{D}\text{‘}R = \hat{x}\{(\exists y) . xRy\},$$

$$*33\cdot111 \vdash . \text{Ɑ‘}R = \hat{y}\{(\exists x) . xRy\},$$

$$*33\cdot112 \ \vdash . \ C\text{‘}R = \hat{x}\{(\exists y) : xRy . \vee . yRx\}.$$

Thus the *domain* of $R$ is the class of $x$ such that for some $y$, $xRy$. The *converse domain* or *range* of $R$ is the class of $y$ such that for some $x$, $xRy$. Finally the *field* (or Latin “*campus*”) of $R$ is the class of $x$ such that there is a $y$ such that either $xRy$ or $yRx$, i.e. the union of the domain and converse domains. Several familiar notions from the logic of relations are defined directly in terms of classes, and have been kept in axiomatic set theory, though often with a different notation.

The *relative product* of $R$ and $S$ is defined as:

$$*34\cdot01 \quad R|S = \hat{x}\hat{z}\,\{(\exists y). \, xRy \, . \, ySz\} \quad \text{Df.}$$

In axiomatic set theory this is generally expressed as the *composition* operation on relations or functions: $R \circ S$ or $f \circ g$.

The *restriction* of $R$ to a class $\beta$ is

$$*35{\cdot}02 \quad R \upharpoonright \beta = \hat{x}\hat{y}(xRy \,.\, y \in \beta) \quad \text{Df.}$$

This notation is still in use. The *Cartesian product* of two classes $\alpha$ and $\beta$, now written as $\alpha \times \beta$, is defined in PM by

$$*35{\cdot}04 \quad \alpha \uparrow \beta = \hat{x}\hat{y}(x \in \alpha \,.\, y \in \beta) \quad \text{Df.}$$

Plural descriptive functions are introduced in $*37$. Written as $R``\beta$ and which might be read as "the $R$s of $\beta$s" (by analogy with $R`x$, "the $R$ of $x$"), they are defined by

$$*37{\cdot}01 \quad R``\beta = \hat{x}\{(\exists y).\, y \in \beta \,.\, xRy\} \quad \text{Df.}$$

This would now be identified as the *projection* of $\beta$ by $R$. The relation of this class $R``\beta$ to $\beta$ has its own name, "$R_\epsilon$":

$$*37{\cdot}02 \quad R_\epsilon = \hat{\alpha}\hat{\beta}(\alpha = R``\beta) \quad \text{Df.}$$

The notions of the product of a pair of classes and of the sum of a pair of classes are generalized to products and sums of arbitrary classes of classes, just as intersections and unions are generalized in contemporary set theory:

$$*40{\cdot}01 \quad p`\kappa = \hat{x}(\alpha \in \kappa \,.\, \supset_\alpha .\, x \in \alpha) \quad \text{Df,}$$

which in contemporary notation is

$$\cap\kappa =_{df} \{x : \forall\alpha(\alpha \in \kappa \supset x \in \alpha)\};$$

and

$$*40{\cdot}02 \quad s`\kappa = \hat{x}\{(\exists\alpha).\, \alpha \in \kappa \,.\, x \in \alpha\} \quad \text{Df,}$$

in contemporary notation:

$$\cup\kappa =_{df} \{x : \exists\alpha(\alpha \in \kappa \;\&\; x \in \alpha)\}.$$

Part I of *Principia Mathematica* is called "Mathematical logic", and ends with the notions of the logic of relations described above. These notions are quite complicated if compared with a presentation of the analogous notions in set theory, where relations and functions are represented as sets of ordered pairs. Indeed Quine seems to present this as a defect in the notation of *PM*. As he says in the concluding remark of his introduction to "Mathematical logic as based on the theory of types", in the collection *From Frege to Gödel*:

There is a needless proliferation of notation: thus Russell's $\overrightarrow{R}$‘$x$, $\overleftarrow{R}$‘$x$, D‘$R$, and Ɑ‘$R$ could be rendered quite briefly enough, using others of his notations, as $R$“$\iota$‘$x$, $\breve{R}$“$\iota$‘$x$, $R$“$V$, and $\breve{R}$“$V$. In *Principia Mathematica*, where the ideas of this essay are reproduced and developed in great detail, the price of this notational excess is evident; scores of theorems serve merely to link up the various ways of writing things.

*(van Heijenoort, 1967a, p.152)*

Quine's wording suggests that *PM* is poorly conceived. There is indeed some "notational excess" which is perhaps simply a stylistic defect. Yet the suspicion remains that *PM* is syntactically faulty, and perhaps not even capable of being reformulated with a proper formal syntax. In many cases the apparent duplication comes from the special role that relations play in *PM*. Thus the converse of a relation $\breve{R}$ is defined in terms of the relation Cnv between a relation and its converse. Consequently $\breve{R}$ is also expressible as "the converse of $R$", i.e. Cnv‘$R$ in theorem $*31 \cdot 12$, $\vdash . \ \breve{P} = \mathrm{Cnv}$‘$P$. Another important reason for these cases of distinct constructions which express the "same" relations or classes is that the definitions express classes and relations in extension which are nevertheless provably co-extensive with each other. Quine's deep seated objections to intensional logic, including his argument that intensionality originates in the sin of confusing use and mention lead him to interpret intensional phenomena as arising from failures of syntax.[36] Quine describes the existence of distinct but co-extensive formulations of certain classes and relations as a "notational excess", thus suggesting its origins in a lack of attention to precise notation and syntax. Instead the multiple "definitions" of a single notion present in a striking fashion the intensional nature of propositional functions, and relations (in intension), in contrast with the extensional logic based on mathematical functions in Frege. Interestingly, Alonzo Church (1974) devoted a separate review to Quine's introductory material to the essay in the van Heijenoort volume, though focussing on the charge of use–mention confusions rather than notational excess. He suggests that Quine's criticism of the axiom of reducibility as undoing the ramification of the theory of types is founded on a misunderstanding of the role of intensionality in *PM*, which is also a theme in Church (1976), and is the basis for what might be called his "intensional interpretation of *PM*".[37]

[36] See Quine (1976, p.177).
[37] See Linsky (1999, p.99) for this view and a defense of the axiom of reducibility against Quine's objections which is based on it.

# 5

# Improvements in the new edition

Russell begins the Introduction to the second edition by writing that the body of the text has not been changed for the new edition because any new propositions would disrupt the elaborate system of numbered theorems. Changes, he says, will be proposed in the Introduction, but not worked into the system. Appendix A does have a number ($*8$), and numbered theorems, but the other changes are not formulated so technically. Thus even the definitions of the standard connectives in terms of the Sheffer stroke remain unnumbered. More serious for the reader, however, is the lack of precision in the discussion of extensionality and the abandonment of the axiom of reducibility. The informal approach to the discussion of the changes leaves it unclear which are minor, and which central, and for all, exactly what is being proposed. Following the opening excuse for not modifying the technical body of the work, Russell goes on to present a seemingly disjointed list of changes which are "desirable", beginning with what he describes as "the most definite improvement resulting from work in mathematical logic during the past fourteen years ... ". This improvement turns out to be the reduction of the propositional connectives to one, the Sheffer stroke "$p|q$" for "not both $p$ and $q$." It is true that the foundation for the logic circuits of contemporary computer chips is the thousands of connections executing operations on current that model this very stroke, now called the "nand" connective, so, in a way, this use of the Sheffer stroke in "logic" is the greatest legacy of the second edition of *PM*. Yet, given our hindsight of the developments in logic following the second edition of *PM*, in particular Gödel's proof of the completeness of first order logic and the incompleteness of theories of arithmetic, the stroke seems trivial. It is conventional now to see the growing work on the semantics of quantifiers and the special role of first order logic, as well as issues that would be seen as leading to model theory, which are all present in the the work of Löwenheim and Skolem, as providing the most important possible improvements to *Principia Mathematica*.[1] Warren Goldfarb feels obliged to rationalize Russell's enthusiasm about the Sheffer stroke as a consequence of the "universalist" conception of logic that Russell shared with Frege. According to this view there could be no meta-logical study of logic as a formal system for Russell, for all logic consists solely in proving results within the system. Recall Goldfarb's remark that "Thus Russell can say with a straight face in the Introduction to the second edition of *Principia Mathematica* [1925] that the most important advance in mathematical logic since the first edition was the Sheffer stroke – a purely internal simplification." (Goldfarb, 1979, p.353). However, Russell does not say that this is the most important improvement in logic, but rather only the most definite improvement that could be made to *PM*.[2]

In fact, however, the improvements in the second edition are not simply more small technical points that have since been surpassed in the development of mathematical logic. To start, they must be seen as what Russell viewed as the most important improvements that he wished to make to the *Principia* system. The introduction of the Sheffer stroke, which is clearly a truth-functional connective, and the fact that all the connectives of *PM* can be defined in terms of it, point towards what is in fact the most significant logical innovation of the second edition, namely Russell's attempt to adopt the principle of extensionality into the ramified type theory of the first edition. Extensionality runs against much of the underpinnings of the logic of *PM* from the first edition. Explaining that troubled imposition of extensionality on the logic of *PM* is, consequently, the focus of the following account of the new material in the Introduction and appendices of the second edition. The adoption of extensionality and abandonment of the axiom of reducibility are certainly more significant changes than the introduction of the Sheffer stroke. Both proposed changes, however, are tentative. Only a portion of mathematics can be recovered without the axiom of reducibility, namely the results in Appendix B. Russell singles out the introduction of the Sheffer stroke as the most "definite improvement" also, perhaps, to contrast it with the other changes with less certain or "definite" results.[3]

Seeing the novelty of the second edition in this way requires rethinking an established view of the first edition of *PM*, one which still manifests itself in discussions of logicism as a foundation of mathematics. Quine (1963, pp.249–58) raised the issue as part of his discussion of the axiom of reducibility. If one sees the logic of *PM* as a logic for classes, then the axiom of reducibility seems to

[1] Löwenheim (1910) and (1915) and Skolem (1920).
[2] This is noted by Kripke in (2005). For a recent discussion, see Proops (2007), who discusses the "universalist" conception of logic as it appears in Russell's work.
[3] Patrick Gamez directed my attention to the use of the word "definite" and its possible significance.

cancel out the ramification of the types that is the most distinctive feature of the logic. For, on this view, if the whole problem is to introduce various sets necessary to develop mathematics, then any sort of care about impredicative definitions marked by type distinctions is simply undone by the axiom of reducibility. That this is Quine's view of the extensional nature of the logic of *PM* is seen in this remark:

> The foundational portion of Whitehead and Russell's *Principia Mathematica*, taking up barely the first two hundred pages, contrasts markedly with the main body of that work. Propositional functions are in evidence only in the foundational part; thereafter the work proceeds in terms of classes and relations-in-extension.
>
> *(Quine, 1963, p.251)*

Now, it is true that in the material related to mathematical induction, for example, we find variables for classes and the "capital Latin" letters such as $R$, $S$, and $T$, standing for relations-in-extension, following the convention expressed in $*21$ (*PM*, p.201). Still, the first edition was intended to allow the formalization of intensional language, including the sort of example, "*A* believes Socrates is Greek", which in Appendix C of the second edition is tentatively given a new, extensional, analysis. While the part of the logic of *PM* used to found mathematics is an account of extensional classes and relations-in-extension, the logic is still that of the "foundational part", and is intensional. Dropping that for a thoroughgoing extensional logic was a big change for Russell, and one, we shall see, that was not completely accomplished. In Appendix B, at least, Russell's understanding of the effect of the addition of extensionality on the ramification of types is not complete. As was noted by Gödel in (1944), and by others since, it appears that in the second edition the ramification is indeed partly undone, although in a way other than that Quine charges, in that functions are allowed to take arguments of a wider range of types than in the first edition. This apparent modification of the theory of types is not explicitly noted by Russell, and indeed, must be attributed without explicit textual evidence to an attempt to make sense of apparently anomalous formulas in Appendix B. Still, the imposition of extensionality on *PM* is not easy. The details of this aspect of Appendix B will be the subject of the next chapter.

After listing the Sheffer stroke as the most "definite" improvement, the rest of the list seems to be a collection of points of varying importance. Russell first cites Jean Nicod's reduction of the logic of propositions to one rule of inference and one axiom stated using the stroke.[4] Following that is the seemingly unrelated claim that there is "no need of the distinction between real and apparent variables", so that when we have "an asserted proposition of the form '$\vdash .\ fx$' or '$\vdash .\ fp$,' this

[4] Again there is no mention of a rule of substitution, although appeals to substitution instances of previous theorems are still used in proofs. Russell had noted the need for a rule of substitution at (*IMP*, p.151 n.).

is to be taken as meaning '$\vdash .(x).fx$' or '$\vdash .(p).fp$'.[5] Then come three seemingly minor technical issues about the difference between Sections A and B of Part I, in which propositional and then quantificational logic are developed. The first of these three is that all use of propositional variables in A is to replaced in B by expressions quantifying over propositional functions and individuals, thus "in place of '$\vdash . (p). \ fp$' we have '$\vdash . (\phi, x). \ f(\phi x)$' ". Secondly, the existential quantifier, as in "$(\exists x).\phi x$", is introduced as a primitive notion. Third "we introduce in Section B general propositions which are molecular constituents of other propositions; thus "$(x). \ \phi x \ .\vee. \ p$" is to mean "$(x). \ \phi x \vee p$".

This summary ends with a single paragraph stating the most important two improvements of all. Russell begins with the seemingly casual remark that "One point in regard to which improvement is obviously desirable is the axiom of reducibility". Two alternatives are then presented, neither of which is totally satisfactory. One is to retain the theory of types of the first edition, but abandon the axiom of reducibility, thus following the "heroic course" of Leon Chwistek in his "Theory of constructive types". The other is to adopt the view that logic is extensional, which Russell credits to Ludwig Wittgenstein as coming from "philosophical reasons". Russell alludes to the results of Appendix B by saying that he is "not prepared to assert that this theory is certainly right, but it has seemed worth while to work out its consequences in the following pages". The hesitation comes from the fact that "the theory of infinite Dedekindian and well-ordered series largely collapses, so that irrationals, and real numbers generally, can no longer be adequately dealt with".

The introductory material concludes with a reference to a "very powerful method in mathematical logic" which has been "invented by Dr H. M. Sheffer". Since use of this method would require rewriting *PM*, and Sheffer has not given more than hints of how this is to be done, the authors "recommend the task to Dr Sheffer".

This seems like an odd list, beginning with minor technical points, continuing with a dilemma between accepting the axiom of reducibility and accepting the thesis of extensionality, of which would seem to result in abandoning the very program that motivates *PM*, and finishing with the bizarre and pointless proposal of rewriting *PM* using Sheffer's method. There is, however, a clear focus to the improvements which will be sketched in the rest of this chapter. The last worry can be quickly dismissed. The recommended reformulation of logic that Russell cites is in fact a general method of presenting axiomatic systems, which Sheffer only hints at in various short published abstracts.[6] It comes out of his general theory of axiomatic

[5] This is one of the occasions where bound propositional variables occur in *PM*. See the discussion above in Section 4.3.

[6] The first words of Sheffer's manuscript are "The following pages present, in outline, a new *method* in mathematical logic." (1921, p.2) The introduction echoes these words.

systems, following the so-called Harvard "postulationist" approach.[7] It is certainly not a proposal that someone rewrite all of *PM* using the Sheffer stroke. Russell in fact completes the task of integrating the Sheffer stroke into *PM* in a proper meta-logical manner. He relies on previous results of Sheffer and Nicod, which show how the primitive connectives of *PM* can be defined in terms of the Sheffer stroke, and then how an axiomatic formulation of propositional logic using one rule and one axiom is capable of deriving all of the axioms, or primitive propositions, of $*2$ to $*6$.[8] The extension of this result to quantificational logic is presented in detail in the Introduction and Appendix A. Definitions of the standard connectives are provided, and then, in Appendix A, theorems are proved justifying the use of propositional inferences with quantificational formulas. The original system of *PM* then can be seen as derived, with its primitive notions defined with the Sheffer stroke, and its primitive propositions now proved from Nicod's axioms. Nothing further needs rewriting for this reduction. On the other hand, the new attitude towards quantifiers, the axiom of reducibility and extensionality are connected in not well understood ways, as will be discussed below. Compared with this uncertainty, it is clear that adoption of the Sheffer stroke as the single primitive connective is indeed the most "definite" improvement to *PM* of all those that Russell proposes.

## 5.1 Occur through its values

Russell proposes that the new system of the second edition is to be extensional. This he expresses with a formula,

$$\phi x \equiv_x \psi x \ . \supset . \ f(\phi \hat{z}) \equiv f(\psi \hat{z}).$$

Yet one would expect this to be presented as a new axiom, to be added to the system. It is not given a number, however, such as the new theorems of Appendix A, as part of the new $*8$. Why is there no proposed "axiom of extensionality"? That appears to be so because Russell thinks he has an *argument* for extensionality, based on both the use of the Sheffer stroke as the sole connective, and an assumption about higher order functions, i.e. functions of functions. The assumption is discussed in the Introduction to the second edition, but since it can't be expressed in the object language of *Principia Mathematica*, the argument for the principle of extensionality couldn't be presented with a "star number", as either an axiom or as a theorem.

The key to the argument can be found in a formula that Russell repeats several times, namely that:[9]

[7] See the discussion above of Sheffer's (1921) "Theory of notational relativity" in Section 3.2.11 above.

[8] $*1$ is a statement of the rule of *modus ponens* as a "primitive proposition".

[9] For example, in a letter to Nicod dated 13 September 1923 we have "*Principia Mathematica* is being reprinted, and I am writing a new introduction, abolishing axiom of reducibility, and assuming that functions of props are

. . . functions of propositions are always truth-functions, and that a function can only occur in a proposition through its values.

*(PM,* p.*xiv)*

My proposal is about the nature of the connection between *truth-functionality* as a feature of the combination of atomic sentences and Russell's cryptic formula that "a function can only occur in a proposition through its values" and the *extensionality* of functions, in the sense that co-extensive functions, true of the same objects, are identified.

Russell can be seen as accepting the following argument for the thesis of extensionality about functions. Consider some predicative second order function $\Gamma\hat{\theta}$ true of some first order functions $\phi\hat{x}$ of individuals. The thesis that a function "occurs in a proposition through its values" means that $\Gamma$ is in fact some truth function of the propositions that are the values of a propositional function $\phi\hat{x}$, for under these conditions the principle of extensionality will indeed hold. Consider some function $\psi\hat{x}$ which is co-extensive with $\phi\hat{x}$, i.e., $\phi x \equiv_x \psi x$. For any argument $x$ the two functions will contribute a *value* or proposition, with the same truth value. Since $\Gamma$ will be a truth function of propositions, it will yield the same value for each argument $x$. As a result $\Gamma\phi\hat{x}$ and $\Gamma\psi\hat{x}$ will have the same value for each argument $x$. Now, by the definition of identity from $*13{\cdot}01$, namely

$$x = y \ . = : (\phi) : \phi!x \ . \supset . \ \phi!y,$$

taking the free variables as typically ambiguous, and so by putting '$\phi\hat{x}$' for '$x$', '$\psi\hat{x}$' for '$y$' and '$\Gamma$' for '$\phi!$' ( assuming that $\Gamma$ is predicative),

$$\phi\hat{x} = \psi\hat{x} \ . = : (\Gamma) : \Gamma!\phi \ . \supset . \ \Gamma!\psi$$

and so we derive the conclusion that $\phi\hat{x} = \psi\hat{x}$.

In essence the argument is that if all functions occur in larger contexts through their values, which are propositions, and those larger contexts are truth functional, then complex functions will be extensional.

This argument indeed seems to occur in the first lines of this important passage in the Introduction:

According to our present theory, all functions of functions are extensional, i.e.

$$\phi x \equiv_x \psi x \ . \supset . \ f(\phi\hat{z}) \equiv f(\psi\hat{z})$$

This is obvious, since $\phi$ can only occur in $f(\phi\hat{z})$ by the substitution of values of $\phi$ for $p, q, r, \ldots$ in a stroke-function, and, if $\phi x \equiv \psi x$, the substitution of $\phi x$ for $p$ in a

always truth-functions, and functions of functions only occur through values of the functions and are always extensional. I don't know if these assumptions are true, but it seems worth while to work out their consequences." (*Auto.* II, p.236). At (*PM*, p.$xxix$) we find ". . . a function can only appear in a matrix through its values."

stroke-function gives the same truth-value to the truth-function as the substitution of $\psi x$. Consequently there is no longer any reason to distinguish between functions and classes, for we have, in virtue of the above,

$$\phi x \equiv_x \psi x \,.\supset.\, \phi \hat{x} = \psi \hat{x}$$

We shall continue to use the notation $\hat{x}(\phi x)$, which is often more convenient than $\phi \hat{x}$; but there will no longer be any difference between the meanings of the two symbols. Thus classes, as distinct from functions, lose even that shadowy being which they retain in $*20$.

*(PM,* p.*xxxix)*

The phrase "According to our present theory, all functions of functions are extensional ..." should be seen as presenting a consequence of "our present theory", rather than leading up to a precise statement of it. If this is right, Russell argues for extensionality rather than simply postulating it. As we shall see, however, the argument is not valid without a further premise, a premise which raises difficulties for interpreting the Introduction to the second edition, and which suggests that a new system of types is tacitly being supposed.

Instead of reading Russell as adopting an axiom of extensionality, or proving it as a theorem, might we rather consider him to have been giving new "identity conditions" for propositional functions, and so simply re-defining identity as co-extensiveness? As Russell knew from his reading of Frege's *Foundations of Arithmetic* (1884, §65), there are limitations to how one can choose to "define" identity in a formal system, in that the rules that allow one to substitute identicals in larger formulas must in fact be sound. (In the above it is the use of the definition reading from left to right.) In a theory of types, this will depend on the types of formulas in which the substitution of '$x$' for '$y$' will be allowed. In the theory of types this depends on the type of $x$ (and $y$ since they must be the same). In Appendix B one finds, however, identifications of functions (or classes) of differing orders. Making sense of this use of identity will be a key to interpreting the argument in the appendix and the use of identity there. Quine (1936) and Church (1951) both observe a "partial extensionality principle" in the second edition, as it is only co-extensive functions of the same (ramified) type that are identified, not arbitrary co-extensive functions themselves true of entities of a given type.[10] Indeed, in the next paragraph after the statement of the extensionality principle with the symbol '$=$', (*PM*, p.*xxxix*), Russell adds that "On the other hand, we now have to distinguish classes of different orders composed of members of the same order". This does not seem to require any change in the definition of identity. Now the distinction of types in the second edition is not as strict as in the first edition,

[10] "Russell adopts a partial extensionality principle for propositional functions, to the extent of identifying functions which are formally equivalent and of the same order." (Quine, 1936, p.498)

or at least that is the hypothesis that seems to explain the troubles in Appendix B. Russell (as well as Quine and Church) do not mention the dependency of the definition of identity at $*13{\cdot}01$ on the axiom of reducibility, now no longer in effect.[11] So, while it is clear that co-extensive functions $\phi$ and $\psi$ of the same order are now to be identified, no one remarks what that identification means according to $*13{\cdot}01$, namely that $\phi$ and $\psi$ share all their predicative properties.

## 5.2 No logical matrix of the form $f!(\phi!\hat{z})$

The argument for extensionality presented above is not valid as it stands, however. It relies crucially on the claim that a function will only "occur through its values", as does the purported consequence that a function $\Gamma$ of another function $\phi\hat{x}$ will really be a truth-functional combination of values of $\phi\hat{x}$. The argument would be successful if indeed every function does indeed only occur in some truth-functional combination. Any attempt to formulate the argument as an inductive argument about formulas of increasing complexity, however, will stall with the proof of the basis case. For surely there could be simply some atomic predication $\Gamma(\phi\hat{x})$. Then, if that $\Gamma$ is not truth-functional, it might distinguish values of $\phi\hat{x}$ with the same truth value.[12] That there can not, however, be just such a formula seems to be explicitly asserted in the following:

> There is no logical matrix of the form $f!(\phi!\hat{z})$. The only matrices in which $\phi\hat{z}$ is the only argument are those containing $\phi!a$, $\phi!b$, $\phi!c$, ..., where $a$, $b$, $c$, ... are constants; but these are not logical matrices, being derived from the logical matrix $\phi!x$.
>
> *(PM,* p.*xxxi)*

This doctrine seems to be presented as fundamental to the Introduction to the second edition, yet its interpretation is problematic. The assertion that "there is no logical matrix of the form $f!(\phi!\hat{z})$" seems manifestly incompatible with the reading of the formulas of *PM* as "typically ambiguous". While usually stated with the "individual variables" such as $x$, $y$, etc., and using the predicate variables $\phi$, $\psi$, etc. for functions of individuals, these formulas are intended to apply to all types, with the types of the rest appropriately adjusted. As there are no type symbols in *PM* it is not possible to indicate the typical ambiguity of the theorems, but they must be so understood. On that understanding of typical ambiguity, however, we immediately confront a conflict when considering the "Theory of propositions

[11] Indeed, in his review (Church, 1932, p.355) of the posthumously published papers of Ramsey (1931), Church contrasts Ramsey's new account of identity with the *PM* view "that $x$ and $y$ are identical (or equal) when every propositional function satisfied by $x$ is also satisfied by $y$". He does not say anything about a restriction on the types of those functions, and in particular nothing about the definition of identity in terms of shared predicative functions at $*13{\cdot}01$.

[12] For example, let $\Gamma$ represent "is Napoleon's favorite quality".

containing one apparent variable" that constitutes $*10$ of *PM*. How is an assertion containing the expression '$\phi x$' to be interpreted as typically ambiguous? Can one substitute a variable for a predicative function for $x$, and one for an appropriately typed second order function for $\phi$? That would seem to involve exactly the sort of "logical matrix" of the form $f!(\phi!\hat{z})$ which Russell denies.

That Russell intends to introduce some very definite change with asserting that "there is no logical matrix of the form $f!(\phi!\hat{z})$" is seen in the contrast with the following from $*12$ on the axiom of reducibility in the first edition:

> It should be observed that, in virtue of the manner in which our hierarchy of functions was generated, non-predicative functions always result from such as are predicative by means of generalization. Hence it is unnecessary to introduce a special notation for non-predicative functions of a given order and taking arguments of a given order. For example, second-order functions of an individual $x$ are always derived by generalization from a matrix
>
> $$f!(\phi!\hat{z}, \psi!\hat{z}, \ldots, x, y, z, \ldots),$$
>
> where the functions $f$, $\phi$, $\psi$, ... are predicative. It is possible, therefore, without loss of generality, to use no apparent variables except such as are predicative.[13]
>
> *(PM,* p.*165)*

In this passage Whitehead and Russell have described the formula

$$f!(\phi!\hat{z}, \psi!\hat{z}, \ldots x, y, z, \ldots)$$

as a "matrix" and, if it is a matrix, this would seem to require that $f!(\phi!\hat{z})$ is a matrix. Russell is clearly abandoning some view that occurs in the first edition, and that view is other than the view that higher order functions are derived from predicative matrices by generalization.[14]

One way to interpret the ban on matrices with functions as sole arguments might be to see it as limited to complex functions of functions and not atomic matrices. The argument for extensionality imputed above would thus only hold for logically complex contexts in which a function occurs, while the thesis of extensionality is

[13] The passage continues as follows. "We require, however, a means of symbolizing a function whose order is not assigned. We shall use "$\phi x$" or "$f(\chi!\hat{z})$" or etc. to express a function ($\phi$ or $f$) whose order, relative to its argument, is not given. Such a function cannot be made into an apparent variable, unless we suppose its order previously fixed. As the only purpose of the notation is to avoid the necessity of fixing the order, such a function will not be used as an apparent variable; the only functions which will be so used will be predicative functions, because, as we have just seen, this restriction involves no loss of generality."

[14] This passage bears on an interpretive question about the logic of the first edition. Gregory Landini cites the passage up to almost the end to support his view that all bound functional variables in *PM* range over predicative functions, and that consequently "All predicate variables are predicative." (Landini, 1998, p.263). The last phrase, "the only functions which will be so used will be predicative functions, because, as we have just seen, this restriction involves no loss of generality", however suggests that this is simply an assertion that no non-predicative functions are called upon in the constructions of mathematics that follow in *PM*, not that among the interpretations of the "typically ambiguous" variables such as '$\phi$' will not also be some non-predicative functions. Indeed, on the very next page (*PM*, p.166) there is talk of cases of "all properties" which does not range only over predicative functions.

intended to hold for all functions. Suppose that $\phi x \equiv_x \psi x$, where $\phi$ and $\psi$ are (co-extensive) atomic functions. The thesis of extensionality can be formulated as the claim that in this case $\phi\hat{x} = \psi\hat{x}$. Indeed Russell formulates it just this way at (*PM*, p.$xxxix$). Yet identity is defined at $*13{\cdot}01$ in terms of indiscernibility, thus $\phi\hat{x}$ will be identical with $\psi\hat{x}$ when $\phi$ and $\psi$ share all the same (second order), predicative, properties $\Gamma$. There is no restriction in $*13{\cdot}01$ that would require $\Gamma$ to be logically complex. Surely there will be logically atomic higher order predicates?

That Russell intends to deny the existence of just such higher order functions is the view taken by Nino Cocchiarella (1989), in his study of the logic of the second edition. To understand his interpretation, consider an atomic sentence '*a* is blue', which in a language with predicate and individual constants could be symbolized as '$B(a)$'. If 'blue is a color' is an example of an atomic second order predication, symbolized as '$C(B\hat{x})$', that would seem to require the existence of a "logical matrix of the form $f!(\phi!\hat{z})$", of just the sort that is denied. Cocchiarella asserts that Russell is (inadvertently) restricting the logic to a version of ramified second order logic. Such a logic is inadequate to the task of logicism that is the job of *PM*, but, on Cocchiarella's view, Russell did not realize this in his enthusiasm for the new doctrine.

Cocchiarella's interpretation is reinforced by remarks that Russell makes in the Philosophy of logical atomism lectures, (PLA, p.182) when he attributes to Wittgenstein a new doctrine that "a predicate can never occur except as a predicate".[15]

The conclusion that the logic of the second edition is at most second order can be supported by the following line of reasoning. Suppose that one allows the use of variables for higher order predicates, but still insists that functions involving such predications must be definable in terms of lower type functions. Consider, then, a formula

$$---\ldots\phi\ldots---$$

which we might want to generalize to derive:

$$(\exists f)---f(\phi)---.$$

It seems that $\phi$ occurs in a third order matrix $f(\phi)$. However, if $\phi$ can only "occur as a predicate", the context ... will in fact be some propositional context, and not an irreducibly higher order function. Likewise with $---$. The variable $f$, though of third order, will only occur in theorems which are equivalent to propositions

[15] This sounds like Frege's view that a concept can never be named as the subject of a sentence, thus leading to the problem with "the concept horse". Yet this issue is not mentioned by Russell, who considers this to be a new point.

of second order logic, one might conjecture. However, this still doesn't mean that the logic in which these theorems hold is a fragment of second order logic. If any formula $\phi!x$ occurs in the language, then there will be a second order function $(\exists\psi!)\ \psi!\hat{z}$, and a third order function $(\exists\theta)\ \theta\hat{z}$, etc., whatever the equivalence of higher order theorems with second order theorems.

A different consideration which initially suggests Cocchiarella's interpretation of the ban on matrices of the form $f!(\phi!\hat{z})$ comes from how Russell uses the '!' notation in the second edition:

> As we shall have occasion later to consider functions whose values are not elementary propositions, we will distinguish those that have elementary propositions for their values by a note of exclamation between the letter denoting the function and the letter denoting the argument. Thus "$\phi!x$" is a function of two variables, $x$ and $\phi!\hat{z}$. It is a matrix, since it contains no apparent variable and has elementary propositions for its values. We shall henceforth write "$\phi!x$" where we have hitherto written $\phi x$.
>
> *(PM,* p.*xxviii)*

As the symbol '!' is used to identify predicative functions, Russell here seems to be following the account of $*12$, by which predicative functions are matrices, rather than the presentation in the Introduction by which predicative functions are of the lowest order compatible with their arguments. My presentation of type theory in Church (1976) and above follows the latter.[16] The proposal, then, is that the predicative functions of the second edition are only the elementary functions made from truth functional combinations of matrices of individuals, but none of the predicative functions of higher types (i.e. type $(\alpha)/1$ for $\alpha$ higher than simply functions of individuals $(\iota)$).[17]

Russell's statement of Wittgenstein's doctrine about the necessity of predicates occurring as predicates in the "Philosophy of logical atomism" lectures is followed by an illustrative example:

> A relation can never occur except as a relation, never as a subject. You will always have to put it in hypothetical terms, if not real ones, such as "If I say that $x$ is before $y$, I assert a relation between $x$ and $y$." It is in this way that you will have to expand such a statement as "'Before' is a relation" in order to get its meaning.
>
> *(PLA p.182)*

This new idea, which Russell attributes to Wittgenstein, is the same as that which was somehow to "help" with the theory of types as Ramsey reported, yet it seems to require abandoning the higher reaches of the theory of types. It is but a short

[16] Landini (2007, §6) discusses the issue of the apparently distinct theories of types in the Introduction to the first edition and in $*12$, and finds it central to his account of the type theory in *PM*.

[17] Potter (2000, pp.198–9) proposes that this is indeed the basis of the theory of types in the second edition.

step from Russell's revision of "'before' is a relation" as "if I say that $x$ is before $y$, I assert a relation between $x$ and $y$" to the nominalist revision of "blue is a color" as "all blue things are colored," where 'colored' represents a first order property of individuals (of $r$-type $(\iota)/1$) rather than a second order property of properties (of $r$-type $((\iota)/1)/1$).

This line of thought is not the only one that can be made out of these remarks, however. One can also take the indefinite article 'a' in 'I assert a relation between $x$ and $y$' as involving quantification over relations. On this interpretation, it should be symbolized as $(x)(y)\ x$ before $y \supset (\exists R)\ xRy$, i.e., if $x$ is before $y$, then there is a relation $R$ that holds between $x$ and $y$.

Indeed, in another passage in the Philosophy of logical atomism lectures we find an occurrence of a color term in precisely the sort of subject position that the new doctrine would seem to forbid:

> You can never place a particular in the sort of place where a universal ought to be, and vice versa. If I say "*a* is not *b*", or if I say "*a* is *b*", that implies that *a* and *b* are of the same logical type. When I say of a universal that it exists, I should be meaning it in a different sense from that in which one says that particulars exist. E.g., you might say "Colours exist in the spectrum between blue and yellow." That would be a perfectly respectable statement, the colours being taken as universals. You mean simply that the propositional function "*x* is a colour between blue and yellow" is one which is capable of truth. But the *x* which occurs there is not a particular, it is a universal.
>
> *(PLA, p.225)*

The manuscripts and notes for the second edition do not provide a resolution for these interpretive issues. In general Russell seems to state these themes from the start with the early draft of HPF, rearranging them in the final version. As with other interpretative issues, what this shows is that the views were serious, and not casual, as one might suspect if they appeared for the first time in one place in the final version; and that they were not easy for Russell, as he did seem to struggle with their expression. This section of the Introduction to the second edition was a troublesome one for Russell, with the HPF manuscript containing several new beginnings of the same account of what "second order" functions of an individual there are.[18] In context, it seems that Russell was wrestling with the distinction between propositions of logic, which will be completely general, and applied propositions, which might include constants.

One passage in the HPF manuscript does suggest one possible resolution of this puzzle. The notion of a "logical matrix" is defined at (*PM*, p.*xxxi*) as a "matrix that can occur explicitly in logic":

[18] In particular, manuscript page *49*, the two revised versions of *52*, and *67*.

A logical matrix is one that contains no constants. Thus $p|q$ is a logical matrix; so is $\phi!x$, where $\phi$ and $x$ are both variables. Taking any elementary proposition, we shall obtain a logical matrix if we replace all its components and constituents by variables.

A paragraph at (*PM*, p.*xxxii*) hints at yet another interpretation of the ban on logical matrices of the form $f!(\phi!\hat{z})$. It originates as follows in HPF, *49* (and then is revised as *37* of the "Introduction" manuscript):

A matrix containing only first-order functions and individuals does not remain a logical matrix when values are assigned to the functions. This, however, is partly arbitrary. If we were to use in logic, as we might do, the symbols

$$R_1(x),\ R_2(x, y),\ R_3(x, y, z), \text{ etc.}$$

where $R_1$ represents a variable predicate, $R_2$ a variable dyadic relation, $R_3$ a variable triadic relation, and so on, we should have logical matrices not containing variable functions. This, however, involves the introduction of variable predicates and relations, which is rendered unnecessary by the use of variable functions.

*(HPF, p.49)*

This suggests that Russell considers the possibility of matrices involving variables over universals rather than functions, but has decided that they would in fact be "unnecessary". Such variables over universals would allow for the expression of propositions that would not be allowed in *PM* as it stands. These might be the propositions which we would be tempted to express with a function appearing in a "logical matrix". Our example $C(B)$ would be turned into a matrix by replacing $C$ with a variable for a universal true of universals (and thus a third order variable) while $B$ would be replaced by a variable for universals, such as $R_1$, a second order variable. Russell, however, says that this is "rendered unnecessary by the use of variable functions". Perhaps, then, he does not see pure logic as dealing with formulas such as '$C(B)$', and so there is no need to consider logical matrices of which that would be an instance.

Unfortunately this also will not work, for surely applied logic should prove:

$$C(B) \supset C(B)$$

for example, as an instance of what surely should be a theorem of pure logic:

$$(f):(\phi).\ f(\phi\hat{z}) \supset f(\phi\hat{z}).$$

Instead, Russell seems to say that it would be an instance of a matrix that uses variables for universals and so one that does not need to show up in *PM* as it stands, without the new variables:

$$C(R_1) \supset C(R_1).$$

But this interpretation does not allow us to prove an obvious logical truth of applied logic using the system of *PM*. What's more, the suggested additional notation which would allow the proof is somehow "rendered unnecessary by the use of variable functions."

To summarize these convolutions: the ban on logical matrices of the form $f!(\phi!\hat{z})$ has led Cocchiarella to propose that the logic of the second edition is a restricted form of second order logic, which allows quantifiers over first order functions, but no second order functions of functions. There is, however, no other evidence that Russell proposed such a weakened system. The suggestion that these sorts of higher order propositions could be asserted in an extended language that included constants for universals is also problematic. For one thing, it will not be possible to exclude such sentences from logic, as the example "if blue is a color then blue is a color" seems to show. What's more, Russell's remark that such an extended language is unnecessary is simply opaque.

Russell seems to have been groping towards the thesis that the only way that atomic formulas could be combined is with quantification and the sentential connectives. In doing so he did not find it necessary to specify that this would be repeated at each type. Rather, he seems to have been concentrating on the issue of how all possible formulas with a free individual variable might be constructed. It looks as though this view of the construction of formulas made him suspect that the axiom of reducibility could be circumvented. The optimism of the early sections of the HPF manuscript is replaced in the last pages by the view that at least the principle of mathematical induction could be proved without the axiom of reducibility. This view of how "logical matrices" could be constructed may thus be a relic of enthusiasm about the possibility of avoiding the axiom of reducibility altogether by showing that the higher level functions encountered in logic are all definable by logical means. This view does not work, and it seems that Russell saw that as work on the second edition developed.

Despite the enormous exegetical difficulties in working out the details of this section of the Introduction to the second edition, it does seem clear that the commitment to extensionality arises from the two doctrines that "functions of propositions are always truth-functions", and that "a function can only occur in a proposition through its values". Russell credits Wittgenstein in the *Tractatus Logico-Philosophicus* for both doctrines. They can, however, also be seen as arising from independent developments in Russell's own logical views. That all complex expressions of both propositional and quantifier logic can be based on the Sheffer stroke gives reason to see all logically complex expressions as somehow truth-functional, for the Sheffer stroke is clearly a truth-functional connective. While the logic of the first edition of *PM* was based on more than one connective, however obviously truth-functional those connectives (negation and disjunction) may be, the possibility yet remained that there might be some logically primitive combination of propositions that isn't truth-functional. Added to this thesis of truth-functionality, it was yet a further extension of atomism that yielded the second doctrine that functions only "occur in propositions through their values".

Russell cites Wittgenstein's *Tractatus* $*5{\cdot}54$ for the view that "$A$ believes $p$" is not a function of $p$. Russell himself had expressed doubts about the existence of false propositions from 1910 at least with his concerns with the "multiple relation" theory of belief. Until the Philosophy of logical atomism lectures he had considered the possibility that perception involves the relation of an individual to a fact. That "$A$ sees $aRb$" might be a compound fact involving $A$ and the fact that $aRb$ still has a proposition occurring other than in a truth function. "$A$ sees ... " is still not a truth functional context, even if it always requires a true complement to be true as a whole. It is merely *factive* like "$A$ knows that ... " It was only when Russell was able to go the final step, and see that facts could not be constituents of other facts, that it became possible to see the only way of building up complex propositions to be with truth functions of their components. That this might require going a further step and denying that there could be atomic facts involving higher order universals may not have been clear to Russell. Yet that does seem to be a consequence of the denial that there are matrices of the form of $f!(\phi!\hat{z})$. The discussion of the possibility of adding variables for universals and relations seems to indicate that Russell would allow for those higher order facts, but that somehow, they wouldn't appear in logic. Yet the technical implementation of these philosophical ideas is obscure. It appears, thus, that Russell's own logical atomism, and enthusiasm for the Sheffer stroke, brought him close to the logical underpinnings of the doctrine of extensionality. How exactly that was to work, and what consequences it has for the rest of the theory of types, is not so clear.

## 5.3 Variables and quantifiers

The thorough-going atomism of the second edition also supports the new views about quantifiers and the "assertion" of propositional functions that are among the "improvements" of the new edition. As recently as the lectures on logical atomism (PLA) in 1918, Russell seems to have acknowledged both negative and universal facts in addition to the atomic facts which correspond with true atomic propositions. It would seem that the statements about quantifiers in the second edition show, however, that universal statements, expressed with quantifiers, do not add any constitutent to what is asserted over and above the other constituents of the proposition. In the first edition of *PM* Russell had distinguished between propositions about "any" and those about "all". A genuine quantification about "all" individuals or functions of a certain sort would treat those as a totality, and since expressions involving that quantification would be of a higher type, they would contribute some constituent to make the difference of type. The many expressions with free variables, which we are tempted to interpret as schematic, can be "typically ambiguous" and so range over instances that don't collectively form a type. One use of the distinction between real and apparent variables, then, is to mark the distinction between typically ambiguous assertions that range over types, and those that involve a determinate, genuine, totality. The move to banish the distinction, and see all "assertions" of functions as equivalent to the assertion of their universal closure is part of this move.

One difficulty with this new doctrine is that one will still need to make use of free variables and open formulas in the system of the second edition. As Landini (2005) points out, on at least one occasion Russell must allow for open formulas. Thus, in an explanation of the nature of quantifiers, we find this passage (*PM*, p.*xxiv*):

> Existence-theorems are very often obtained from the above primitive propositions in the following manner. Suppose we know a proposition
>
> $$\vdash . f(x, x).$$
>
> Since $\phi x \,.\supset .\, (\exists y).\ \phi y$, we can infer
>
> $$\vdash . (\exists y).\ f(x, y),$$
>
> i.e. $$\vdash : (x) : (\exists y).\ f(x, y).$$

This inference would not be allowed if the requirement that the assertion of a formula be interpreted as the assertion of its quantification were to be taken more strongly as a claim that an asserted formula and its universal quantification are to be interchangeable in all circumstances. If $\vdash . f(x, x)$ is to be replaced by $\vdash . (x). f(x, x)$ and the assertion of $\phi x \,.\supset .\, (\exists y).\ \phi y$ by $\vdash : (x) : \phi x \,.\supset .\, (\exists y).\ \phi y$, then it no longer follows that $\vdash : (x) : (\exists y).\ f(x, y)$. The variable function $\phi x$ must be interpreted as instantiated by $f(x, x)$ for the inference to be of the right form, but then the inference would not be licensed by these steps. For the inference to be of the form described, then, it must be the case that a distinction between free and bound variables remains in the statement of the inference rules.

It seems best to interpret this so that what is denied is a *semantic* distinction of the sort that would see universally quantified propositions as something more than a summary or infinite conjunction of their instances. That the proof theory and syntax will still have a use for a notion of open formula cannot be denied.[19] The notion that

[19] Landini's (2005) reconstruction of Appendix A takes another route. He assumes that the proposal to no longer distinguish free and bound variables amounts to that later developed by Quine (1954). This requires amending Russell's list of primitive propositions to adjust to the new rule of supplying the missing quantifiers binding all apparently free variables. Landini's revision has the advantage of meeting contemporary standards of rigor in the formation and derivation rules, and also reveals the short step from the system of Appendix A to the logic for empty domains that Quine develops, which allows models with an empty domain as still validating the rules of quantifier logic. While Russell did see any ontological commitment, such as in the axiom of infinity, as a sign of lack of logical character at (*IMP* p.203 n.), it is unlikely that he had a clear intention that the logic of

a universal quantification is like an infinite conjunction also connects with the view that functions only appear though their values as propositions and truth-functional combinations of those values.[20] A universal quantification, $(x)\phi(x, y)$, if it is seen as a sort of conjunction of its various atomic instances which are propositions of the form $\phi(x, y)$, can also be seen as almost a truth-functional combination of those values. The association of universal generalizations with conjunctions, then, may be of a piece with the view that all propositions occur only in truth-functional combinations.

## 5.4 The Sheffer stroke

The project of revising *PM* using the Sheffer stroke as the sole primitive connective is completed in the Introduction to the second edition and Appendix A, and, as was noted above, is not the work "left" to Sheffer in that Introduction. The technical project is to define the primitive connectives of *PM* with the stroke, and then show that all of the *primitive propositions* are in fact provable from axioms proposed by Nicod, formulated with the stroke. Appendix A then shows that the sentential logic based on the Sheffer stroke can be used with sentences of quantificational logic by showing that the primitive propositions apply to quantified formulas as well as atomic propositions.

Defining the connectives in terms of the Sheffer stroke is simple, and conducted in the new introduction without the usual formulation of definitions with "star" numbers:

$$\begin{array}{rl} \sim p \,.=.\, p|p & \text{Df} \\ p \supset q \,.=.\, p|\sim q & \text{Df} \\ p \vee q \,.=.\, \sim p|\sim q & \text{Df} \\ p \,.\, q \,.=.\, \sim(p|q) & \text{Df} \end{array}$$

Thus all the usual truth-functions can be constructed by means of the stroke. (*PM*, p.*xvi*)[21]

The next step is to present the rule of inference, which is a version of *modus ponens* for the stroke. Russell notes that $p \supset q \,.=.\, p|(q|q)$ by definition and so $p. \supset .q.r$ is equivalent to $p|(q|r)$. As a result, he remarks, $p \supset q$ is "a degenerate case of a function of *three* propositions". The rule replacing *modus ponens* then is presented for the Sheffer stroke as the primitive proposition:

If $p, q, r$ are elementary propositions, given $p$ and $p|(q|r)$, we can infer $r$.

*(PM,* p.*xvii)*

*PM* should hold even in an empty universe. Landini's formulation of the logic of ∗8 does make Quine's logic tantalizingly close, however.

[20] This interpretation of quantifiers is discussed by Ramsey (1926).

[21] Notice that Russell says all "the usual" truth-functions, and not "all the truth functions" for he did not have the notion of truth-functional completeness of Post (1921).

"Elementary" propositions comprise the atomic propositions and all molecular propositions built up from atomic propositions with the stroke function. The propositional logic of *PM* envisages quantifiers ranging over propositions, but now that Russell's plan is to treat free variables as implicitly bound by universal quantifiers, he sees a need for a more general statement of the appropriate rule as a primitive proposition of logic. As logic deals with wholly universal propositions, it is "helpless with atomic propositions, because their truth or falsehood can only be known empirically"(*PM*, p.*xvii*). Consequently the fundamental primitive rule of inference is stated finally in a form that more clearly involves free variables ranging over atomic propositions (*PM*, p.*xviii*):

> The rule of inference, in the form given above, is never required in logic, but only when logic is applied. Within logic, the rule required is different. In the logic of propositions, which is what concerns us at present, the rule used is:
>
> Given, whatever elementary propositions $p, q, r$ may be, both
>
> "$\vdash . F(p, q, r, \ldots)$" and "$\vdash . F(p, q, r, \ldots)|\{G(p, q, r, \ldots)|H(p, q, r, \ldots)\}$"
> we can infer "$\vdash . H(p, q, r, \ldots)$".

Despite the unclarity about the syntactic formation rules of *PM*, the intention is clear. One could use the simpler rule, namely "given $p$ and $p|(q|r)$, we can infer $r$", with allowable substitutions for the free variables $p$, $q$, and $r$ as quantifier free functions of elementary propositions. Again it is clear that the new view of free variables cannot be interpreted as requiring that universal quantifiers be read into every occurrence of variables. A syntactic distinction between free and bound variables will still be maintained, whatever the manner of their semantic interpretation.

Russell attributes the developments thus far to Sheffer, and then goes on to discuss the axiomatization of logic of Jean Nicod. Nicod is best known for the single axiom, which together with the rule above (and substitution), is adequate for proving all of the primitive propositions derivable with $*1 - *6$, (with the *PM* connectives suitably defined using the stroke):

$$\text{Nicod's axiom: } \{p|(q|r)\} \mid [\{r|(r|r)\} \mid \{(s|q)|((p|s)|(p|s))\}].$$

Russell presents a translation of this axiom into the language of negation and the conditional, which reveals it to be a simple conjunction of two others with which he prefers to work: $\vdash . p|(p|p)$
and, secondly, $\vdash : p \supset q . \supset . s|q \supset p|s$,
which he describes as essentially amounting to $p \supset p$, and a principle of the syllogism for propositional logic:[22] $p \supset q . \supset : q \supset \sim s . \supset . p \supset \sim s$. The system

[22] Although these axioms formulated with $\sim$ and $\supset$ rather than $|$ would not form a complete axiomatization with *modus ponens* for $\supset$ as the rule. ($\sim\sim p \supset p$ is unprovable, for example.) The three axioms $p \supset (q \supset p)$,

of propositional logic, then, consists of the single rule of inference above, and these two axioms. It is this system which Nicod has shown to be capable, with the definitions above, of proving all of the theorems of propositional logic of the first sections of *PM*.

It is most likely because Russell takes this to be the work of Nicod that he does not give the primitive propositions "star" numbers as part of the system of *PM*, rather than being merely sketched or tentative in some way.[23] What remains is Russell's own work, the extension of this system of propositional logic to quantified expressions. That work is carried out in Appendix A and the definitions and theorems are given numbers as part of $*8$, to be seen as preceding $*9$ and $*10$, the two presentations of quantifier logic in the first edition. As can be inferred from Whitehead's (1926) letter in *Mind*, the development of quantificational logic in $*9$ was Russell's work, and Russell maintained that approach in the new $*8$ as Appendix A.

The goal of $*8$ is to present the rules for quantificational logic based on the Sheffer stroke (and the new interpretation of free variables). The main bulk of the work, from $*8{\cdot}3$ through $*8{\cdot}367$, shows that the second of the two primitive propositions obtains when the substituends for $p$, $q$, and $r$ are propositions involving quantification over individuals. The rest of the theorems consist of lemmas needed for these proofs, and the theorems needed to show that the first axiom and rule of inference hold for quantified expressions.

Russell's plan is to reduce all compound propositions which include quantified expressions to a prenex form, that is, an equivalent formula in which all the quantifiers precede a complex formula built up from matrices with the stroke function. This translation is accomplished using definitions, so that the primitive propositions need only be formulated as above with matrices and molecular propositions including free variables. Simple primitive propositions then allow one to deal with the prenexed quantifiers. This begins with four definitions:

$$
\begin{array}{ll}
*8{\cdot}01 & \{(x).\,\phi x\}|q\ .=.\ (\exists x).\,(\phi x|q)\quad \mathrm{Df} \\
*8{\cdot}011 & \{(\exists x).\,\phi x\}|q\ .=.\ (x).\,(\phi x|q)\quad \mathrm{Df} \\
*8{\cdot}012 & p|\{(y).\,\psi y\}\ .=.\ (\exists y).\,(p|\psi y)\quad \mathrm{Df} \\
*8{\cdot}013 & p|\{(\exists y).\,\psi y\}\ .=.\ (y).\,(p|\psi y)\quad \mathrm{Df}
\end{array}
$$

Russell remarks that:

These definitions define the meaning of the stroke when it occurs between two propositions of which one is elementary while the other is of the first order. . . . When the stroke occurs

$[p \supset (q \supset s)] \supset [(p \supset q) \supset (p \supset s)]$ and $(\sim p \supset \sim q) \supset (q \supset p)$ are complete with *modus ponens* as the rule of inference. See Thomason (1970, §V).

23 And what numbers could they get? The contemporary device of labeling introductory material as "Chapter 0" was not a common practice at the time.

between two propositions which are both of the first order, we shall adopt the rule that the above definitions are to be applied first to the one on the left, treating the one on the right as if it were elementary, and are then to be applied to the one on the right.

There are four primitive propositions:

$*8{\cdot}1$ $\vdash . (\exists x, y). \phi a|(\phi x|\phi y)$ Pp
$*8{\cdot}11$ $\vdash . (\exists x). \phi x|(\phi a|\phi b)$ Pp
$*8{\cdot}12$ From "$(x). \phi x$" and "$(x). \phi x \supset \psi x$" we can infer "$(x). \psi x$", even when $\phi$ and $\psi$ are not elementary. Pp
$*8{\cdot}13$ If all occurrences of $x$ are separated from all occurrences of $y$ by a certain stroke, we can change the order of $x$ and $y$ in the prefix, i.e. we can replace "$(y):(\exists x). \phi x|\phi y$" by "$(\exists x):(y). \phi x|\phi y$" and vice versa. Pp.[24]

The remainder of Appendix A is devoted to first proving some necessary lemmas, and then the long list of cases by which it is shown that the primitive propositions of propositional logic hold as theorems when quantified expressions replace the elementary $p$s, $q$s, and $r$s, in all combinations. For the first axiom, $p|(p|p)$, Russell proves the two cases for the version as $p \supset p$:

$$*8{\cdot}3 \ \vdash : (x). \phi x \ . \supset . \ (x). \phi x$$

and

$$*8{\cdot}31 \ \vdash : (\exists x). \phi x \ . \supset . \ (\exists x). \phi x.$$

The final 26 cases cover all the possible combinations of elementary and quantified expressions in $p \supset q \ . \supset . \ s|q \supset p|s$, from

$$*8{\cdot}32 \ \vdash : (x). \phi x . \supset . \ q : \supset : s|q \supset \{(x). \phi x\}|s$$

where $p$ is the universally quantified $(x). \phi x$ and $q$ and $s$ are elementary, too, twenty five theorems later:

$$*8{\cdot}367 \vdash : (\exists x). \phi x \ . \supset . \ (\exists x). \psi x : \supset :$$
$$\{(\exists x). \chi x\}|\{(\exists x). \psi x\} \supset \{(\exists x). \phi x\}|\{(\exists x). \chi x\}$$

where $p, q$, and $s$ are all existentially quantified.[25]

As a way of developing quantificational logic from the new formulation of propositional logic due to Sheffer and Nicod, Appendix A is formally adequate.[26]

[24] The last of these rules appears to have been revised in response to Ramsey's questions. (See Section 2 above.) What is now simply an example of a more general rule was the whole of $*8{\cdot}13$ in the HPF manuscript. (On the unfoliated page of "$*8$. List of props.")

[25] Appendix A ends with a list of theorems such as one proving the equivalence of $\sim (x).\phi x$ and $(\exists x). \sim\phi x$. ($*8{\cdot}4$)

[26] Landini (2005) interprets the rules of $*8$ as not complete, but proposes a remedy.

Though perhaps only an elementary result of proof theory, the reduction of the theorems of propositional logic in $*1$ to $*5$ to a single connective governed by one rule and one axiom is similiar in spirit to the semantic reduction of propositional theorems to the tautologies of truth tables. Both are attempts to capture the common part of propositional logic that is preserved in different axiomatizations and is independent of the particular choice of connectives. When thus added to a smooth development of quantificational logic as continuous with propositional logic, the whole being clearly both truth-functional and extensional, it becomes far less of a problem to keep a straight face when reading that the Sheffer stroke is "the most definite improvement resulting from work in mathematical logic during the past fourteen years ... ". While not the most important discovery in the field, it is the source of the "most definite improvement" to *Principia Mathematica*. Russell wanted to adopt extensionality on philosophical grounds, and so adapt his logic to the new thesis. That was the improvement, and it was as definite as a theorem.

## 5.5 Reducibility and extensionality

The second major change proposed for the new edition, to accompany the adoption of extensionality, is to avoid use of the axiom of reducibility whenever possible. This project is introduced somewhat casually: "One point in regard to which improvement is obviously desirable is the axiom of reducibility ($*12\cdot1\cdot11$)" (*PM*, p.$xiv$). This improvement is not completely worked out, however, as no replacement is suggested, and no revision of the rest without the axiom is proposed. Instead, it seems best to consider the abandonment in the spirit of the letter to Nicod of 13 September 1923 quoted in a note above; Russell was trying to work out the consequences of "abolishing" the axiom of reducibility, to see more clearly what exactly depends on it, much as he had done in the first edition with Whitehead for the axioms of infinity and choice. This may have been influenced by the resistance to the axiom of reducibility from Wittgenstein, beginning in 1913, and later by figures such as Weyl and Hilbert, but also is a natural consequence of Russell's own qualms about the axiom in the first edition.[27] It is Russell's attempt to replace certain results which rely on the axiom in the first edition of *PM*, however, that leads to the most interesting technical questions about the theory of types in the second edition.

In the Introduction to the second edition Russell introduces a subscript notation for what we would consider the level (and hence order) of functions of individuals. (Recall that the '!' notation here indicates a matrix.)

[27] Wittgenstein's attacks on the axiom begin in correspondence with Russell from Norway (1995, p.35) and appear in the *Tractatus* (*TLP* 6.1232 and 6.1233). See the sources cited in Section 3 above for the views of Weyl and Hilbert about the axiom.

We can, if we like, introduce a new variable, to denote not only functions such as $\phi!\hat{x}$, but also such as

$$(y).\ \phi!(\hat{x}, y), (y, z).\ \phi!(\hat{x}, y, z), \ldots (\exists y).\ \phi!(\hat{x}, y), \ldots;$$

in a word, all such functions of one variable as can be derived by generalization from matrices containing only individual variables. Let us denote any such functions by $\phi_1 x$, or $\psi_1 x$, or $\chi_1 x$, or etc. Here the suffix 1 is intended to indicate that the values of the functions may be first-order propositions resulting from generalization in respect of individuals. In virtue of $*8$, no harm can come from including such functions along with matrices as values of simple variables.

*(PM,* p.*xxxiii)*

There follows an account of how the results in $*8$ show this harmless addition to be justified. This familar appeal to the equivalence of universal generalizations with infinite conjunctions and existential quantifications with infinite disjunctions appears to contemporary readers as a convenient shorthand for a semantic argument. Russell, in the 1920s, did not have the semantic ideas with which to work, even if they had been open to him.

The notation is extended in the standard way.

We denote by $\phi_2 x$ a function of $x$ in which $\phi_1$ is an apparent variable, but there is no variable of higher order. Similarly $\phi_3 x$ will contain $\phi_2$ as apparent variable, and so on.

*(PM,* p.*xxxiv)*

As soon as these distinctions among higher order functions are defined, it is argued that they don't amount to really new classes of functions. In the HPF manuscript there then follows at page *56*:[28]

Put next $\phi_1 x \ .=.\ (\exists y).\ \phi!(x, y)$.

Then $\qquad (\phi_1 a) \mid (\phi_1 x | \phi_1 b) \ .=:\ (y):(\exists z, w).\ \phi!(a, y) \mid \{\phi!(x, z) | \phi!(b, w)\}$

In this case we merely have to put $z = w = y$ and the result follows. The method will be the same in any other case. Hence generally

$$(\phi_1).\ f!(\phi_1 \hat{z}, x) \ .\equiv.\ (\phi).\ f!(\phi!\hat{z}, x).$$

Although the above arguments do not amount to formal proofs, they suffice to make it clear that, in fact, such variables as $\phi_1$, $f_1$ do not introduce any substantially new propositions. We may therefore dispense with them in such general logical propositions as we have been considering.

*(HPF,* p.*56)*

The view that functions $\phi_1$ involving quantification over individuals do not "introduce substantially new propositions" is easy for us to accept, for the view that

[28] The material in HPF p.*56* was revised and the new version appears in the Introduction manuscript as p.*47*.

quantification over individuals does not raise the *order* of a function of individuals is embedded in Church's system of $r$-types. In fact, the hierarchy of functions $\phi_1$, $\phi_2$, ... is one of levels of functions, and not of functions of functions, as in the example of 'blue is a colour' discussed above. The claim that these newly defined functions don't "introduce" new functions is relevant only to avoiding the axiom of reducibility, not to whether the logic of the second edition is a full type theory rather than a version of second order logic. It does, however, as described above, fit well with the view that all occurrences of propositional functions in complex contexts in fact are occurring as truth functional combinations of their propositional values.

The following is asserted explicitly in the Introduction to the second edition, but somewhat separated from the discussion.

> Although the above arguments do not amount to formal proofs, they suffice to make it clear that, in fact, any general propositions about $\phi!\hat{z}$ are also true about $\phi_1\hat{z}$. This gives us, so far as such functions are concerned, all that could have been got from the axiom of reducibility.
> *(PM,* p.*xxxvii)*

Russell's goal is to show in which cases the axiom of reducibility can be avoided by showing which theorems holding of all matrices follow from theorems about all functions of a more limited range, the functions $\phi_1\hat{z}$. Does this apply further? Do the $\phi_2$ functions add nothing to the $\phi_1$ functions? It would seem that if that is so all order distinctions would collapse and the axiom of reducibility could easily be abandoned as redundant, for it is already true of the new system of types!

The best resolution of these interpretive problems is likely to come from looking at the new theorems that are proved as a result of the revised views about higher order functions. They aim at the goal of showing which parts of the logical project of *PM* rely on the axiom of reducibility, and which do not. While Russell split the discussion of induction from the HPF manuscript and set it into an appendix of its own, the corresponding discussion of identity remains in the Introduction to the second edition.

At the conclusion of section V of the Introduction to the second edition, the results of the preceding work are summarized. Here, it seems, Russell acknowledges that the second order functions which can be proved to exist will be derivable from stroke functions of first order propositions, but the hope of avoiding the axiom of reducibility for wide classes of formulas must be given up. The proof of mathematical induction without the axiom seems to be one of the cases that survived the more ambitious initial program. Going beyond that, to the parts of real analysis that use higher order quantifiers to define least upper bounds, for instance, provides cases where new functions are introduced at the higher orders.

It is those initial hopes, however, that explain Russell's apparent rejection of the need for higher order functions of functions in matrices that we considered above. Consider the case of identity which is not put off to an appendix.

We want to discover whether, or under what circumstances, we have

$$(\phi).\ g!(\phi!\hat{z}, x)\ .\supset .\ g!(\phi_2\hat{z}, x). \qquad \text{(A)}[29]$$

Let us begin with an important particular case. Put

$$g!(\phi!\hat{z}, x)\ . = .\ \phi!a \supset \phi!x.$$

Then $(\phi).\ g!(\phi!\hat{z}, x)\ . = .\ x = a$, according to $*13{\cdot}1$.

*(PM,* p.*xxxvii)*

Proposition A asserts that if a given function $g$ holds of all matrices, and an individual $x$, then it holds of all the $\phi_2$ functions of $x$. One very important instance of such a function is sharing all the predicative functions of $a$, in other words, being identical to $a$. What we wish to prove is that, as the section concludes, "$(\phi).\ \phi!a \supset \phi!x\ .\supset .\ (\phi_2).\ \phi_2 a \supset \phi_2 x$ without the need of any axiom of reducibility". (*PM*, p.*xxxviii*). Here is the argument from that page.

We want to prove

$$(\phi).\ \phi!a \supset \phi!x\ .\supset .\ \phi_2 a \supset \phi_2 x$$

i.e.

$$(\phi).\ \phi!a \supset \phi!x\ .\supset :\ (\phi).f!(\phi!\hat{z}, a)\ .\supset .\ (\phi).\ f!(\phi!\hat{z}, x) :$$
$$(\exists\phi).f!(\phi!\hat{z}, a)\ .\supset .\ (\exists\phi).\ f!(\phi!\hat{z}, x)\text{[30]}$$

Now $f!(\phi!\hat{z}, x)$ must be derived from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values $\phi!x$, $\phi!b$, $\phi!c, \ldots$ where $b, c, \ldots$ are constants. As soon as $\phi$ is assigned, this is of the form $\psi!x$. Hence

$$(\phi).\ \phi!a \supset \phi!x\ .\supset :\ (\phi) :\ f!(\phi!\hat{z}, a)\ .\supset .\ f!(\phi!\hat{z}, x) :$$
$$\supset :\ (\phi).\ f!(\phi!\hat{z}, a)\ .\supset .\ (\phi).\ f!(\phi!\hat{z}, x) :$$
$$(\exists\phi).\ f!(\phi!\hat{z}, a)\ .\supset .\ (\exists\phi).\ f!(\phi!\hat{z}, x).$$

Thus generally $(\phi).\ \phi!a \supset \phi!x\ .\supset .\ (\phi_2).\ \phi_2 a \supset \phi_2 x$ without the need of any axiom of reducibility.

Russell's argument proceeds as follows. Suppose that

$$(\phi).\ \phi!a \supset \phi!x.$$

[29] On page *57* of HPF this is proved as $(\phi).\ f!(\phi!\hat{z}, x)\ .\supset .\ f!(\phi_2\hat{z}, x)$.

[30] Russell makes this inference because the $\phi_2$ functions are those involving either universal or existential generalizations of matrices.

Since any function $f$ of propositions $\phi!a$, etc., will be a truth-functional combination of predicative functions, we have

$$(\phi) : f!(\phi!\hat{z}, a) \,.\supset .\, f!(\phi!\hat{z}, x).$$

By an elementary inference of quantificational logic it follows both that

$$(\phi).\ f!(\phi!\hat{z}, a) \,.\supset .\, (\phi).\ f!(\phi!\hat{z}, x)$$

and that

$$(\exists\phi).\ f!(\phi!\hat{z}, a) \,.\supset .\, (\exists\phi).\ f!(\phi!\hat{z}, x).$$

Therefore, as Russell says, we have

$$(\phi).\ \phi!a \supset \phi!x \,.\supset .\, (\phi_2).\ \phi_2 a \supset \phi_2 x$$

"without the need of the axiom of reducibility".

We have then an initial limited result. Russell concludes that there is no need to use the axiom of reducibility to extend the sharing of predicative functions, which defines identity, to the sharing of functions of the level of the $\phi_2$ functions.

It is likely these passages that lead Cocchiarella to assert that the logic of the second edition is a "fragment of second-order predicate logic", and so not up to the tasks of the original logicist project (Cocchiarella, 1989, p.42). That second order logic is not adequate can be seen from the models of type theory used by Myhill (1974) in his discussion of Appendix B. Myhill shows that at each higher level of the ramification of functions more classes can be defined. As Russell defines the "orders" such as $\phi_2, \phi_3, \ldots$, however, only some of the new sets available at higher orders will be introduced, namely those definable with universal and existential quantification over those of the lower order.

While certainly independent of the trivial technical mistake in Appendix B, to be discussed below, this result also seems to depend on a different "new" conception of the nature of the ramification of the theory of types than the latter proof. This "result" supposes that higher order functions are in some way truth-functional combinations of functions of lower order. The mistake in Appendix B relies on an apparent easing of the restrictions of well-formedness of functions so that functions of different orders can be identified. There are thus two ways in which the theory of types seems altered in the second edition, both, it seems, mistaken.

Now Russell himself does not seem to take his result to mean that individuals sharing all predicative functions share all second order functions, but only those of the higher level shown in the definitions of $\phi_2$, i.e., only those that are second order because they involve quantifiers over matrices (and not because they are functions of matrices). Yet even this limited sort of result he does not think to be always possible.

It must not, however, be assumed that (A) is always true. The procedure is as follows: $f!(\phi!\hat{z}, x)$ results from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values $\phi!x$, $\phi!a$, $\phi!b, \ldots$ ( $a, b, \ldots$ being constants). We assume that, e.g.,

$$\phi_2 x \,.=.\, (\phi).\ f!(\phi!\hat{z}, x).$$

Thus $$\phi_2 x \,.=.\, (\phi).\ F(\phi!x, \phi!a, \phi!b, \ldots).\ \text{(B)}$$

What we want to discover is whether

$$(\phi).\ g!(\phi!\hat{z}, x) \,.\supset.\, g!(\phi_2\hat{z}, x).$$

Now $g!(\phi!\hat{z}, x)$ will be derived from a stroke-function

$$G(p, q, r, \ldots)$$

by substituting $\phi!x$, $\phi!a'$, $\phi!b', \ldots$ for some of $p, q, r, \ldots$ To obtain $g!(\phi_2\hat{z}, x)$, we have to put $\phi_2 x$, $\phi_2 a'$, $\phi_2 b', \ldots$ in $G(p, q, r, \ldots)$, instead of $\phi!x$, $\phi!a'$, $\phi!b', \ldots$ We shall thus obtain a new matrix.

If $(\phi).\ g!(\phi!\hat{z}, x)$ is known to be true because $G(p, q, r, \ldots)$ is always true, then $g!(\phi_2\hat{z}, x)$ is true in virtue of $*8$, because it is obtained from $G(p, q, r, \ldots)$ by substituting for some of $p, q, r, \ldots$ the propositions $\phi_2 x$, $\phi_2 a'$, $\phi_2 b'$ which contain apparent variables. Thus in this case the inference is warranted.

We have thus the following important proposition.

Whenever $(\phi).\ g!(\phi!\hat{z}, x)$ is known to be true because $g!(\phi_2\hat{z}, x)$ is always a value of a stroke-function

$$G(p, q, r, \ldots)$$

which is true for all values of $p, q, r, \ldots$, then $g!(\phi_2\hat{z}, x)$ is also true, and so ~~is~~ (of course) is $(\phi_2).\ g!(\phi_2\hat{z}, x)$.

This, however, does not cover the case where $(\phi).\ g!(\phi!\hat{z}, x)$ is not a truth of logic, but a hypothesis, which may be true for some values of $x$ and false for others. When this is the case, the inference to $g!(\phi_2\hat{z}, x)$ is sometimes legitimate and sometimes not; the various cases must be investigated separately. We shall have an important illustration of the failure of the inference in conjunction with mathematical induction.

*(PM,* p.*xxxviii)*

The restriction that this result does not hold when "$(\phi).\ g!(\phi!\hat{z}, x)$ is not a truth of logic, but a hypothesis" shows the limited nature of all of the discussion of higher order functions in the Introduction to the second edition. It seems that the result is that when it is a theorem, or logical truth, that $x$ and $y$ share all predicative functions, then it will follow that they share all second order properties.[31] Each of

[31] Or it may be that if $x$ and $y$ happen to share all predicative functions, then they will in fact share all the functions definable from them with quantifications over predicative functions, those in the limited class that Russell wants to capture with the expression $\phi_2$. In some way, however, there is a weakening of the claim.

the chain of passages which raise the interpretive problems for the claim that "there is no logical matrix of the form $f!(\phi!\hat{z})$" seems to state certain limited results about how use of the axiom of reducibility can be avoided in particular cases.

This seems to be the way in which the doctrine for which Russell credits Wittgenstein actually "helps" with the theory of types. The help seems to come though this process by which various functions are seen as resulting from truth-functional combinations of their values, so that in each case use of the axiom of reducibility can be avoided.

## 5.6 Appendix C

Appendix C, "Truth-functions and others" defends the doctrine that all functions of propositions are truth-functions.[32] The thesis that a function can only enter into a proposition through its values is restated in the first paragraph. Russell then acknowledges that he might avoid the issue by asserting that mathematics is extensional, and so avoid having to defend the thesis as universally true, but proceeds with a defense of the general hypothesis, though in a tentative fashion. The remainder of Appendix C is an account of how it is that examples such as "$A$ believes $p$" and "$p$ is about $A$" are not, despite appearances, functions of $p$. As a result they are not potential counter-examples to the thesis that all real occurrences of a proposition are truth-functional.

The solution comes by distinguishing "between a proposition as a fact and a proposition as a vehicle of truth or falsehood" (*PM*, p.402). The notion of a proposition occurring as a fact comes from "Wittgenstein's principle, that a logical symbol must, in certain formal respects, resemble what it symbolizes" (*PM*, p.406). Russell says that when a proposition is used as a vehicle of truth or falsehood, that is, when it is asserted, the symbols which express it are used "transparently", that is, to talk about what they refer to. When used as facts, the symbols refer to something else, as it is the string of symbols, or rather sequences of equivalence classes of the symbols themselves, that is the subject matter.[33] Any function of propositions used as propositions, then, will be truth-functional. It is the other use, "as facts", that characterizes propositional attitudes, and so the apparently non-truth-functional occurrences can be explained.

An assertion of "Socrates is Greek" consists of the utterance of a sequence of words, each of which is a member of an equivalence class of expressions which all

[32] Godden & Griffin (2009) place the Appendix C theory in the context of the life-long evolution of Russell's views on psychologism and his various accounts of propositions. They disagree with the continuity that I see between the multiple relation theory and the account in the appendix.

[33] The reference, on page 665 (14 of the manuscript) to "transparent" propositions is cited by Quine (1953, n.3) as the inspiration for his now well known terminology of "opaque" contexts.

"mean" the same thing. To say that someone asserted the proposition that Socrates is Greek, then, is to say that they asserted some string of words, which consisted of a member of the "Socrates" equivalence class, followed by a member of the "is" equivalence class, followed by a member of the "Greek" equivalence class. Belief will be similar. Russell points out that some have held that thought is carried out in language, "Some people maintain that a proposition must be expressed in words before we can believe it; if that were so, there would not, from our point of view, be any vital difference between believing and asserting" (*PM*, p.662). In that case the fact which constitutes the belief is composed of a series of mental entities, "thoughts", which in turn *mean*, Socrates, is, and Greek. A propositional attitude, as reported in "*A* believes 'Socrates is Greek'" is no longer a complex fact involving *A*, the belief relation, Socrates, being, and Greek, as it was in the multiple relation theory of belief that Russell described as problematic in the lectures on Logical atomism (PLA) and other places in the years before 1923, but for which he had no replacement. Now there is a successor positive theory. Now belief is a complex fact involving certain mental particulars, "thoughts", that mean what used to be the constituents. That such a fact is a "belief that *p*" means that the elements, in order, are parts of the relevant equivalence classes. Appendix C provides a new account of the semantics of propositional attitude sentences, but the focus is on providing an account of the fact that the sentences report, and, most importantly, the semantics of the content clause, the "*p*", which occurs in the report. The metaphysics of the account, however, is still that of the "multiple relation theory of judgement" and the fact ontology of the logical atomism lectures. On the new account, as before, a true belief will consist of a correspondence between two facts, the fact which constitutes the belief, and the fact that the belief is about. Russell takes from Wittgenstein the idea that what makes that correspondence possible is some similarity in structure of the two facts.

The other seemingly non-truth-functional context that Russell considers is "*p* is about *A*", as in " 'Socrates is Greek' is about Socrates". Here we have a sentence which seems to involve an expression for a proposition in a non-truth-functional context. Some other "*q*" with the same truth value might not be "about" *A*. "*p* is about *A*" seems to say that an object is a constituent of a proposition. How does it do that?

Consider the fact that makes the assertion " 'Socrates is Greek' is about Socrates" true. Russell does not want to say that there are propositions in the world that are constituents of larger facts. This much is in complete accord with Russell's "multiple relation" theory of judgement which also denied that propositions are in the world. There will be many facts involving Socrates, such as that Socrates is Greek, Socrates is mortal, etc. These bear what Russell calls "particular-resemblance" to each other. Atomic facts with one individual and one universal as "constitutents" will bear distinct patterns of "particular-resemblance", "predicate-resemblance" to different classes of facts. Relational facts will bear various "relation-resemblances" to each other. To say, then, that Socrates is a constituent of the fact that Socrates is Greek is really to say that the fact that Socrates is Greek is a member of a given equivalence class of facts all of which bear the right relation of particular resemblance to each other. "'Socrates is Greek' is about Socrates" can have a different truth value from "'Plato is Greek' is about Socrates" because the real logical form of the sentences involves relations between distinct classes of facts and the individual Socrates.

"Socrates is Greek" again occurs as a "fact" in " 'Socrates is Greek' is about Socrates" rather than as a proposition. At the same time it is not really a sentence that "occurs" either, so Appendix C does not present what might be called a thoroughgoing version of a sentential account of propositional attitude sentences. It is an account that still distinguishes between propositions and their verbal expression, or mental representation, but which does not allow for an occurrence of a proposition in a non-truth-functional context.

Appendix C is Russell's way of working out the proposal just hinted at in the lines leading up to *TLP* 5.542:

> 5.542 It is clear that "*A* believes that *p*", "*A* thinks *p*", and "*A* says *p*", are of the form "'*p*' says p": and here we have no co-ordination of a fact and an object, but a co-ordination of facts by means of a co-ordination of their objects.

Wittgenstein seems to insist that any account of this co-ordination must be something that can only be "shown" but not said. In keeping with his views in the Introduction to the *Tractatus*, Russell goes on to say just what that co-ordination amounts to within a theory involving relational structures. He has an account of both sorts of fact, on either side of the co-ordination, both one fact which expresses or represents a proposition, and another fact that makes it true. The account of what a proposition is "about", namely its constituents, also provides the materials to say what co-ordination makes a belief or assertion true. For the belief that Socrates is Greek, or the assertion that Socrates is Greek, the particular that the proposition is about, Socrates, must have the property which follows as the universal constituent, i.e. Socrates must be Greek.[34]

Appendix C thus goes further than simply providing an explanation of apparent counter-examples to the thesis that all propositions occur only in truth-functional

[34] In the papers leading up to Appendix C, Russell uses a symbol "T" for the relation that a proposition $\alpha$ (which occurs as fact) must have to a fact $f$ that makes it true. See "What is meant by '*A* believes *p*'?" (*Papers 9*, p.25). In the next paper, "Truth-functions and meaning-functions", (*Papers 9*, p.25), he uses '*A*' (for 'assertion'). An asserted sentence or other such fact will bear the truth-making relation to a fact $f$, that is, it will be in the class $\overrightarrow{T}`f$ or $\overrightarrow{A}`f$.

contexts. It gives an account of the logical form of such sentences which also explains what content they have, i.e. what proposition is asserted or believed, for example, and also provides the material for an account of when that proposition is true or false. It is the last stage of the line of thought beginning with the "multiple relation theory" which had first appeared in around 1910, and which is described in the Introduction to the first edition of *Principia Mathematica*. Because of its content, then, we can understand why Appendix C developed as a manuscript independently of the Introduction to the Second Edition, and why, despite its nominal role as support for a premise of the new introduction, remains, with justification, as an Appendix on its own.

# 6

# Induction and types in Appendix B

## 6.1 Induction and inductive classes

The topic of Appendix B is the principle of mathematical induction which, together with the definition of numbers as classes of equinumerous classes, forms the heart of the logicist account of arithmetic. Throughout the earlier history of mathematics, induction had been recognized as a method of proof distinctive of arithmetic. The great achievement of the logicists, in particular, of Frege, was to show how induction could be shown to follow from logical truths and definitions alone.

The principle of induction is used to show that a property holds of all numbers by first showing that it holds of 0 and then that if it holds for some arbitrary $n$ then it holds for $n + 1$. Proofs "by induction" thus have two parts. The first, "basis step" for the induction, is to prove that the given property holds of 0. The second, "induction step", proceeds by assuming that the property holds of an arbitrary number $n$, the so-called "inductive hypothesis", and then showing that it holds for $n + 1$. These two together suffice for proving that the property holds of all numbers.

Induction is not at the heart of the development of mathematics in *Principia Mathematica*, nor even that of Frege's earlier *Grundgesetze der Arithmetik* (1893). By the time of *Introduction to Mathematical Philosophy* (*IMP*) in 1919, however, Russell had come to present it as a central part of what is now the standard account of how mathematics emerges from logic. Using the techniques developed in the nineteenth century, Russell tells us:

> All traditional pure mathematics, including analytic geometry, may be regarded as consisting wholly of propositions about the natural numbers.
>
> (IMP, *p.4)*

Negative and rational numbers can then be "constructed" as pairs of natural numbers. Real numbers are in turn constructed as sets of rational numbers, sets of all rationals with a given upper or lower, bound, or "Dedekind cuts". Functions defined on real numbers, the very material of analysis, are defined as functions of these constructed entities. These constructions are familiar in set theory, where functions, natural numbers and Dedekind cuts are all identified with various sets. In *Prinicipia Mathematica*, however, the constructions are carried out within the logic of relations, with classes ultimately defined in terms of relations by the no-classes theory. It is rather the logic of relations, but including as a part the account of natural numbers, which leads to the analysis of all "pure mathematics".

In his 1919 (*IMP*) account Russell next includes a description of the formulation of the theory of natural numbers by Guiseppe Peano. It requires three primitive notions and five axioms. The primitive notions are 0, Number and Successor.[1] Russell asserts that, using logic alone, these notions can be defined and Peano's axioms can be proved. The axioms are presented as follows (*IMP*, pp.5–6).

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.
(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

Individual natural numbers such as 0 and 1 are defined as classes of equinumerous ("similar") classes, and the addition of 1 to $n$ is defined as the class of classes equinumerous to a member of $n$ with an additional, new, individual included. The natural numbers are then defined as those numbers to which 0 bears the "ancestral" of the relation that holds between a number $x$ and the result of adding 1 to $x$. The principle of induction follows immediately, as the "induction step" guarantees that the given property is passed along the series. Russell learned of this construction from Frege's *Grundgesetze* and then Whitehead and Russell built it into their (consistent) theory of classes based on the theory of types in *Principia Mathematica*.[2]

The particular way that *PM* carries out the constructions is as follows.

Part II of *Principia Mathematica*, called "Prolegomena to cardinal arithmetic", begins with $*50$, "Identity and diversity as relations". These relations $I$ and $J$ are defined, $I$ as:

$$*50{\cdot}01 \quad I = \hat{x}\hat{y}(x = y) \quad \text{Df}$$

1 Russell does not mention the definitions of addition and multiplication in *IMP*. Peano had presented the familiar recursive equations simply as definitions. Dedekind (1888) and following him, Frege (1893), had seen the necessity of justifying such definitions in the sense of proving that given recursion equations are satisfied by a unique function. Russell defines (cardinal) addition $+_c$ directly at $*110{\cdot}02$ and earlier had defined the number 1 (at $*52{\cdot}01$) and so the successor of $n$ would be defined as $n +_c 1$.

2 See Linsky (2005, p.137–8) for the evidence that Russell got the idea of the definition of the ancestral of a relation from Frege. Russell also would have seen the very indirect way that Peano's axioms show up in *Grundgesetze der Arithmetik*.

with $J$ as its complementary relation:

$$*50\cdot02 \quad J = \ \dot{-}I \quad \text{Df.}$$

The relation between an object and the things with which it is identical is named '$\iota$':

$$*51\cdot01 \quad \iota = \overrightarrow{I} \quad \text{Df.}$$

These definitions lead to a theorem which more directly conveys the meaning of the iota:

$$*51\cdot11 \vdash . \ \iota`x = \hat{y}(y = x).$$

Thus $\iota`x$ is the singleton class containing $x$ alone as a member.

Whitehead and Russell directly define more than just the primitive 0 of the Peano axioms. The numbers 0, 1, and 2 are each defined separately and not in terms of 0 and a successor function:

$$*52\cdot01 \ 1 = \hat{\alpha}\{(\exists x). \ \alpha = \iota`x\} \quad \text{Df.}$$

Thus 1 is identified with the class of the singleton classes defined in $*51$:

$$*54\cdot01 \ 0 = \iota`\Lambda \quad \text{Df.}$$

0 is the singleton class containing just the empty class $\Lambda$:

$$*54\cdot02. \ 2 = \hat{\alpha}\{(\exists x, y). \ x \neq y \ . \ \alpha = \iota`x \cup \iota`y\} \quad \text{Df.}$$

The number 2 is defined directly as the class of pairs, i.e. of classes containing some distinct individuals $x$ and $y$.

The notion of the ordered pair of $x$ and $y$, $x \downarrow y$, is defined using the relation $\alpha \uparrow \beta$ between elements of $\alpha$ and elements of $\beta$ in extension, defined earlier at $*35\cdot04$:

$$*55\cdot01 \quad x \downarrow y = \iota`x \uparrow \iota`y \quad \text{Df.}$$

Thus $x \downarrow y$ holds if $x$ stands in the relation in extension to $y$ that holds just in the case $x \in \{x\}$ and $y \in \{y\}$. This definition should be contrasted with that which Norbert Wiener (1914) proposed, by which the pair is defined as $\iota`(\iota`\iota`x \cup \iota`\Lambda) \cup \iota`\iota`\iota`y$. In modern notation this is written as: $\{\{\{x\}, \Lambda\}, \{\{y\}\}\}$. (The definition from Kuratowski (1921) is used more commonly now to define ordered pairs: $\{\{x, y\}, x\}$.) One-place functions are also considered to be sets of ordered pairs (no two of which have the same second member), thus making this definition an important step in the reduction of functions and analysis to set theory. Wiener is reported as remarking that his definition "excited no particular approval on the part of Russell".[3] While Wiener's definition is properly seen as an important step in the reduction of the theory of relations to set theory, it ran counter to the direction of *PM* where ordered pairs and functions are defined in terms of relations. Wiener's definition also makes the ordered pair of $x$ and $y$ of a higher type than the unordered class $\{x, y\}$. Using this sort of construction to produce an infinite sequence, as is done in the axiom of infinity in Zermelo–Fraenkel set theory, would make even an omega series of the order type of the natural numbers go beyond any finite level in the simple theory of types. The axiom of infinity in Zermelo–Fraenkel set theory is presented in Krivine (1971) as

$$\exists x\{\exists y(y \in x) \wedge \forall z \neg(z \in y) \\ \wedge \forall y[y \in x \rightarrow \exists z(z \in x \wedge \forall w(w \in z \equiv w \in y \vee w = y))]\}.$$

In other words, there is a set $x$ (called '$\omega$') which contains the empty set, and if some $y$ is in $\omega$, then so is $\{\{y\} \cup y\}$. The numbers of the set $\omega$ so defined would run through all of the levels of simple type theory, and so is illegitimate in the "no-classes" theory of *PM*. The version of the axiom of infinity in *PM*, "Infin ax", has to be defined differently, as will be seen below.

Using his version of classes, "courses of values", Frege constructed each new number as the class of classes equinumerous with the sum of its predecessors. Thus 1 contains classes equinumerous with 0 (i.e. only the empty class) and 2 contains classes equinumerous with {0, 1}, i.e. the same class of pairs that defines 2 for Whitehead and Russell. In *Principia Mathematica*, however, a class, or rather the function which defines it, will be of a higher order than its members. Hence the totality of all of Frege's numbers would run through all types, and be an "illegitimate totality" banned by the theory of types. Whitehead and Russell's response is to construct the numbers with individuals alone, with their axiom of infinity guaranteeing that there are enough individuals so that every number has a successor. Infinite series could then be defined using relations that have a countable infinite of entities of a single type, as their domain.

*PM* defines a general notion of cardinal number, $Nc$, with the famous "Frege–Russell" definition as classes of similar classes. A theorem gives what is easily seen to be the Frege–Russell notion:

$$*73{\cdot}1 \quad \vdash : \alpha \ \mathrm{sm}\ \beta \ . \equiv . \ (\exists R). \ R \in 1 \rightarrow 1 \ . \ \alpha = \mathrm{D}‘R \ . \ \beta = \mathrm{Ꭷ}‘R.$$

Following Frege's definition, a class $\alpha$ is similar to $\beta$ just in the case when there is a one to one relation $R$ with domain $\alpha$ and range $\beta$.

[3] In the preface to the reprinting of Wiener (1914) in van Heijenoort (1967a, p.224). See Linsky (2009a) and (2009c) for an account of Wiener's reduction of ordered pairs to classes.

The *ancestral* $R_*$ of a relation $R$ is defined by Whitehead and Russell following the original definition of Frege. The definition yields the following theorem, which is the easiest form in which to present the ancestral:

$$*90{\cdot}1 \vdash :. \, xR_*y \, . \equiv : x \in C`R : \breve{R}``\mu \subset \mu \, . \, x \in \mu \, . \supset_\mu . \, y \in \mu.$$

Call a class $\mu$ *R-hereditary* if the things to which the members of $\mu$ bear the relation $R$ are also in $\mu$, or, following the notation more literally, if the things which bear the converse of the relation $R$ to something in $\mu$ are themselves in $\mu$, that is, $\breve{R}``\mu \subset \mu$. The *field* of $R$, namely $C`R$, is everything that is either in the domain or the range of $R$. Then $x$ bears the ancestral of the relation $R$ to $y$ just in case $x$ is in the field of $R$ and $y$ is in any $R$-hereditary class that contains $x$.[4]

The principle of induction itself first appears in a general form, for use with an arbitrary ancestral:

$$*90{\cdot}112 \vdash :. \, xR_*y : \phi z \, . \, zRw \, . \supset_{z,w} . \, \phi w : \phi x : \supset . \, \phi y.$$

If $x$ bears the ancestral of the $R$ relation to $y$ and $x$ possesses any $R$-hereditary property $\phi$, then so does $y$. The recipe for a "proof by induction" is obvious. To prove that $y$ has a property, one must show that $x$ does, that $x$ bears the ancestral of the $R$ relation to $y$, and that the property is $R$-hereditary.

Number $*90$ is the first number of Section E, "Inductive relations", which extends to $*97$ and the conclusion of Volume I of *Principia Mathematica*. Appendix B was numbered, after several changes, as $*89$, to indicate, it would seem, that this new way of developing these notions is preferred to the earlier account starting at $*90$. The issues discussed below, namely "the mistake" in Appendix B and the apparently new system of types, arise from an attempt to redo Section E, only without using the axiom of reducibility, and assuming extensionality. Indeed, most of the material we find in the Amended list of propositions manuscript and the latter portions of the Hierarchy of functions and propositions manuscript, and indeed, the bulk of Appendix B, is devoted to material in Section E, all of which precedes the introduction of the natural numbers in the second volume. This order is required because Whitehead and Russell develop the general theory of induction based on the ancestral of arbitrary relations. The relation of a number to its successor, though

[4] There is no doubt that Russell learned of this definition from Frege. The introduction to Section E ends with the following note: "The present section is based on the work of Frege, who first defined the ancestral relation. See his *Begriffschrift* (1879, Part III., pp. 55–87). Cf. also his *Grundgesetze der Arithmetik*, Vol. I. (1893) §§45, p.59). In this work the ancestral relation is used to prove the properties of finite cardinals and $\aleph_0$." (*PM*, p.548). Russell discovered this during his reading of Frege in 1902. In his notes on *Begriffschrift* we find this comment on the ancestral: "It seems to be a non-numerical definition of $R^N$, and very ingenious: it is better than Peano's mathematical induction." See Linsky (2005, p.137). It is "better" in that Peano's principle of induction need not be taken as an axiom, but can be reduced to "non-numerical", logical, notions.

basic to Peano arithmetic, is just a special case in *PM*, which always focusses on the general logic of relations.

Much of the notation used in the notes originates in this section. One theorem shows a common pattern of notation:

$$*90{\cdot}163 \vdash .\ \breve{R}``\overleftarrow{R_*}`x \subset \overleftarrow{R_*}`x.$$

The things to which the descendents of $x$ bear $R$ are themselves descendants of $x$. Whitehead and Russell comment that: "This proposition is important, since it proves that $\overleftarrow{R_*}`x$ is a hereditary class." (*PM*, p.552)

The next theorem demonstrates a fundamental property of hereditary classes:

$$*90{\cdot}164 \vdash .\ \breve{R}``\breve{R}_*``\alpha \subset \breve{R}_*``\alpha.$$

We can express this using additional individual variables: something $x$ in a member of the class $\alpha$ of classes bears the ancestral of the $R$ relation to some $y$ and $y$ bears $R$ to $z$, then something in a member of $\alpha$ bears the ancestral of $R$ to $z$. The comment on this is "this proposition shows that $\breve{R}_*``\alpha$ is a hereditary class". (*PM*, p.553)

A number of the theorems of $*90$ have to be proved anew in Appendix B so as to avoid use of the axiom of reducibility. The first of these is

$$*90{\cdot}17 \vdash .\ R_*^2 = R_*.$$

A comment states that: "Note that $R_*^2$ means $(R_*)^2$, not $(R^2)_*$" (*PM*, p.553). Thus the relative product of the ancestral of $R$ with itself is just the ancestral of $R$. The comment reminds us that the formula should not be read as asserting that of the ancestral of the product of $R$ with itself.

Volume II of *Principia Mathematica* finally begins the account of cardinal numbers. The "Frege–Russell" definition of cardinal number is expressed in this corollary:

$$*100{\cdot}1 \vdash .\ \mathrm{Nc}`\alpha = \hat{\beta}(\beta\ sm\ \alpha) = \hat{\beta}(\alpha\ sm\ \beta).$$

The cardinal number of $\alpha$ ($\mathrm{Nc}`\alpha$) is the class of $\beta$ which are similar to $\alpha$. (This is the same as the class of $\beta$ to which $\alpha$ is similar, *sm* being an equivalence relation.)

The *arithmetical sum* of two classes, $\alpha + \beta$, is defined at

$$*110{\cdot}01\ \alpha + \beta = \downarrow(\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha)\downarrow``\iota``\beta \quad \mathrm{Df}.$$

In modern symbolism: $\{<\Lambda, y> : y \epsilon \alpha\} \cup \{<\{x\}, \Lambda> : x \epsilon \beta\}$. If finite classes overlap in membership their union will not have as many members as the sum of their individual cardinal numbers. The trick used in this definition is to take the union of $\alpha$ and $\beta$, only with the elements of $\alpha$ and $\beta$ coded so as not to have any duplication, the elements of $\alpha$ as the second element of pairs with $\Lambda$ and the elements of $\beta$, in unit classes, paired with $\Lambda$ as a second element.

Cardinal addition, $+_c$, is defined as the the sum of cardinal numbers:

$$*110{\cdot}02\ \mu +_c \nu = \hat{\xi}\{(\exists\alpha,\beta).\ \mu = N_0c`\alpha\ .\ \nu = N_0c`\beta\ .\ \xi\ sm(\alpha+\beta)\}\ \ \text{Df.}$$

The cardinal sum of two cardinal numbers $\alpha$ and $\beta$ is the class of classes similar to the simple sum of the two, $\alpha + \beta$, as defined in $*110{\cdot}01$.

The symbols '$Nc$' and '$sm$' are typically ambiguous, and almost never appear with any specification of type. Apparently the long prefatory remarks on types at the beginning of Volume II make Whitehead and Russell sensitive to the notation here (*PM* II, p.*vii–xxxi*). Later the subscripts disappear.

Notoriously, it is only at $*110$, on page 83 of Volume II, that we finally have

$$*110{\cdot}643\ \vdash .\ 1 +_c 1 = 2.$$

It is remarked that "The above proposition is occasionally useful. It is used at least three times, in $*113{\cdot}66$, and $*120{\cdot}123{\cdot}472$". This joke emphasizes Russell's thesis that the logical order of presentation of mathematics is different from the order in which it is learned, as well as the epistemological order of what is known with more certainty. While basic in some epistemological sense and in terms of how mathematics is learned by children, '$1 + 1 = 2$' does not play any role in the development of mathematics using symbolic logic in *Principia Mathematica*.

We get the inductive cardinals, NC induct, by starting with 0 and repeatedly adding 1:

$$*120{\cdot}01\ \text{NC induct} = \hat{\alpha}\{\alpha(+_c 1)_* 0\}\ \ \text{Df.}$$

The "inductive", or finite, cardinals are those classes that bear the ancestral of the cardinal addition of 1 relation to 0. They are those classes that are obtained by successively adding 1, starting with the class which is 0.

For each type $\xi$ for the class 0 we will get a different notion of inductive cardinal:

$$*120{\cdot}011\ \text{N}_\xi\text{C induct} = \hat{\alpha}\{\alpha(+_c 1)_* 0_\xi\}\ \ \text{Df.}$$

We are finally able to define the notion of inductive class, Cls induct, one of the two ways of thinking of finite classes:[5]

$$*120{\cdot}02\ \text{Cls induct} = s`\text{NC induct}\ \ \text{Df.}$$

Defined this way, the inductive cardinals are equinumerous classes of individuals produced by adding one thing at a time to the empty class. The *sum* or union of all those cardinals will contain all the finite classes.

[5] The other is defined as one that cannot be mapped one to one into a proper subset of itself, i.e. one that is not "Dedekind infinite". See Boolos (1994).

With 0 defined as above, and "natural number" defined as $\mathrm{N}_\xi\mathrm{C}$ induct (that is, not a property of being a number but rather the class of numbers) and the successor relation as $+_c 1$, Whitehead and Russell are finally able to define and prove the Peano axioms as theorems of their system.

The principle of induction for natural numbers, (5) of Peano's axioms for arithmetic above, follows from this as a special case of induction on arbitrary ancestrals, $*90{\cdot}12$:

$$*120{\cdot}11\ \vdash :. \alpha \in \mathrm{N}_\eta\mathrm{C}\ \text{induct} : \phi\xi \,.\supset_\xi .\, \phi(\xi +_c 1) : \phi 0_\eta : \supset .\, \phi\alpha.$$

Indexed by its type $\eta$, an inductive number $\alpha$ will have any property $\phi$ possessed by the 0 of its type and hereditary over the relation of adding 1.

Peano's first axiom, (1) above, could have been symbolized as Nc'0, i.e. as predicating the property of being a number of 0. Instead it is represented as asserting that 0 is a member of the class of inductive cardinals, NC induct:

$$*120{\cdot}12\ \vdash .\, 0 \in \mathrm{NC}.$$

Peano's axiom (4), that 0 is not the successor of any number, is proved as

$$*120{\cdot}124\ \vdash .\, \alpha +_c 1 \neq 0.$$

This is a more general statement than Peano's version, as it says that 0 is not the successor of any class whatever (not only the particular classes that are the natural numbers).

The last two axioms are (2), that the successor of any number is a number, and (3), that no two numbers have the same successor. These both involve the axiom of infinity.

The axiom of infinity is defined by[6]

$$*120{\cdot}03\ \text{Infin ax} \,.=:\, \alpha \in \mathrm{NC}\ \text{induct} \,.\supset_\alpha .\, \exists!\,\alpha \quad \mathrm{Df}.$$

Without the axiom of infinity, the closest we can come to Peano axiom (2) is

$$*120{\cdot}151\ \vdash :\ \alpha \in \mathrm{NC}\ \text{induct} \,.\, \exists!\,\alpha\ \supset .\, \alpha +_c 1 \in \mathrm{NC}\ \text{induct},$$

that is, if $\alpha$ is a *non-empty* inductive cardinal, then the successor of $\alpha$ is an inductive cardinal. Assuming the axiom of infinity, which confirms that every inductive cardinal is non-empty, $*120{\cdot}151$ guarantees that the successor of every inductive cardinal is an inductive cardinal. The last Peano axiom, (3), that no two numbers

6 It is not an axiom in the sense of a primitive proposition, "Pp", but is always used as the hypothesis in conditional theorems. It is thus introduced by a definition, "Df".

have the same successor, is approximated by

$$*120{\cdot}311 \ \vdash : \exists!\, \alpha +_c 1 \,.\, \alpha +_c 1 \ = \beta +_c 1 \,.\, \supset .\, \alpha = sm``\beta \,.\, \exists!\, \alpha.$$

The contrapositive of Peano's axiom (3) is that if a cardinal $\alpha$ is the same as $\beta$, then they have the same successor. $*120{\cdot}311$ is a generalized version of (3) in two ways. First, it it is true of arbitrary classes and not just inductive cardinals (as $+_c 1$, cardinal addition of 1, is defined for arbitrary classes). The consequent says only that $\alpha$ is identical with the class of classes similar to $\beta$. If we are talking about cardinal numbers $\alpha$ and $\beta$ this amounts to saying that $\alpha = \beta$. Secondly, $*120{\cdot}311$ is independent of the axiom of infinity, for it includes the hypothesis that $\alpha +_c 1$ is in fact non-empty (and so "exists").

No remark is made anywhere in $*120$ that Peano's axioms are finally being proved, and there is no discussion of the way in which they are in fact generalized for this presentation. This is because, unlike Frege's *Grundgesetze* (1893), which is devoted exclusively to the deduction of arithmetic, *Principia Mathematica* is, from the start, devoted to finding the most general formulation of the mathematical structures that it studies. It is almost as if it is left to the reader to find the very specific structure of the natural numbers in the extremely general results of *PM*.

## 6.2 The project of Appendix B

The project of Appendix B is summarized after its last theorem:

$$*89{\cdot}34 \ \vdash : yR_{*5}x \,.\, x \in \lambda \,.\, R``\lambda \subset \lambda \,.\, \supset .\, y \in \lambda. \qquad [*89{\cdot}33]$$

Here $\lambda$ is supposed to be of any order, however high. Hence, so far as mathematical induction is concerned, all proofs remain valid without the axiom of reducibility provided "$R_*$" is understood to mean "$R_{*5}$."

(*PM,* p.*658)*

This is intended to be the most general result of Appendix B, namely that if $y$ inherits all the level 5 $R$-hereditary properties of $x$, then it inherits *any* $R$-hereditary properties of $x$ of whatever level.[7]

As will be familiar by now from the treatment of natural numbers and their arithmetic in *PM*, the organization of Appendix B is based on fully generalized notions. The relation of a natural number to its successor, the ancestral of which generates the finite cardinals from 0, is just a special case as far as *PM* is concerned.

7 Of course it doesn't quite say that, having $yR_{*5}x$ in the antecedent where it should have $xR_{*5}y$. This error occurs in the manuscript as well. In mitigation, it should be noted that the most common expression for the successor of an individual $x$ with respect to a relation $R$ in the appendix and notes is '$\breve{R}$'$x$', reversing the position of the '$x$'. The frequency of occurrence of this pattern must have made it automatic, and explains why it was not detected.

As a result of this striving for generality, the most important case of the result of Appendix B, which shows that any induction on the natural numbers can be carried out with respect to properties of a fixed order, is tucked away in the middle of a series of theorems.

$$*89\cdot24.\ \vdash : R \in \mathrm{Cls}\rightarrow 1\ .\ \breve{R}\text{“}\lambda \subset \lambda\ .\ x \in \lambda\ .\ \supset .\ \overleftarrow{R}_{*3}\text{‘}x\ \subset\ \lambda.$$

Hence if $\lambda$ is an inductive class, it can be used in an induction no matter what its order may be, if $R \in \mathrm{Cls}\rightarrow 1$.

The condition $R \in \mathrm{Cls}\rightarrow 1$ is that $R$ is a many-one, or functional, relation. The successor relation on natural numbers is many-one, and so this theorem applies to induction on the natural numbers. This application to arithmetic is not even noted.

Working backwards from $*89\cdot24$ one finds that the lemmas needed for its proof are results about the level of intervals and of subsets of inductive classes. Indeed, the most powerful, and most problematic result, $*89\cdot12$, is proved in the explanatory text with a single line of argument rather than with the standard format of demonstration (*PM*, p.653).[8]

> We have $(\exists\mu_2).\ \Lambda = \mu_2 : (\exists\mu_2).\ \eta = \mu_2\ .\ \supset .\ (\exists\mu_2).\ \eta \cup \iota\text{‘}y = \mu_2$.
> Now $(\exists\mu_2).\ \eta = \mu_2$ is a third-order property. Hence
> $*89\cdot12\quad \vdash : \rho \in \mathrm{Cls\ induct}_3.\ \supset .\ (\exists\mu_2).\ \rho = \mu_2$.
> This proposition is fundamental.

Theorem $*89\cdot12$ is important for the rest of the appendix, but also problematic, as it seems to violate the theory of types. On the surface it asserts that every inductive or finite class of order 3 is identical with some class of order 2. The three-line proof suggests that this holds because of the level of the operation of adding one individual $y$ to a class $\eta$, yielding $\eta \cup \iota\text{‘}y$, or $M\text{‘}\eta$. The definition does not require functions of more than order 2. The argument seems to be that since each finite class can be seen as built up from one individual by a finite number of uses of this operation, that class must itself be definable by a function of order 2. That itself can be seen to be fallacious, using the models later developed by Myhill, but what's more, the claim of *identity* of these classes of differing types rather than mere co-extensiveness raises issues about what notion of types is in force in the Appendix. These issues arise from the technical objections to Appendix B by Gödel and Myhill, to be considered below.

Next in order in the appendix is the result about intervals that is a principal lemma:

$$*89\cdot21\ \vdash : R \in \mathrm{Cls}\rightarrow 1\ .\ \supset .\ R_3(x \longmapsto\!\!\dashv y) \in \mathrm{Cls\ induct}_3.$$

[8] A more detailed proof occurs at HPF, page *70*. See the discussion below in Section 6.6.

This follows fairly straightforwardly from the fact that the interval from $x$ to $y$ is defined in terms of the descendants of $x$ and the ancestors of $y$:

$$*121{\cdot}013\ \ (x \longmapsto\!\!\dashv y) = \overleftarrow{R}_{*}\text{'}x \cap \overrightarrow{R_*}\text{'}y \quad \text{Df.}$$

Despite the straightforward nature of this result, the notion of interval occurs constantly in the ALP notes. Russell apparently struggled with it for some time.

The other key lemma for $*89{\cdot}24$ is $*89{\cdot}17$, which asserts that every subset of a level 3 inductive class is itself of level 3:

$$89{\cdot}17 \quad \vdash :\ \gamma \in \text{Cls induct}_3\ .\ \alpha \subset \gamma .\supset .\ \alpha \in \text{Cls induct}_3.$$

This is an immediate consequence of the fateful lemma $*89{\cdot}16$ containing the error identified first by Gödel, and it will be discussed in detail below. Given $*89{\cdot}12$, $*89{\cdot}17$ and $*89{\cdot}21$, the proof of $*89{\cdot}24$ is a simple argument by contradiction, as follows. Theorem $*89{\cdot}24$ states that if $R$ is many-one and $\lambda$ is an inductive class to which $x$ belongs, then all of the $R_{*3}$ descendents of $x$ are in $\lambda$. This must hold no matter what the order of $\lambda$, the inductive property. To prove that $\overleftarrow{R}_{*3}\text{'}x \subset \lambda$, suppose that $\overleftarrow{R}_{*3}\text{'}x$ is not a subset of $\lambda$. Then there is some $y$ that belongs to $\overleftarrow{R}_{*3}\text{'}x - \lambda$. Consider an arbitrary $z$ which is an element of both $\lambda$ and $R_3(x \longmapsto\!\!\dashv y)$. What $z$ bears the $R$ relation to will also be in that set, i.e. $\breve{R}\text{'}z \in \lambda \cap R_3(x \longmapsto\!\!\dashv y)$. In other words, $\lambda \cap R_3(x \longmapsto\!\!\dashv y)$ is $R$-hereditary. By lemma $*89{\cdot}21$, then, $R_3(x \longmapsto\!\!\dashv y) \in \text{Cls induct}_3$. By lemma $*89{\cdot}17$ , $\lambda \cap R_3(x \longmapsto\!\!\dashv y)$ will be in Cls $\text{induct}_3$, because it is in a subset of $R_3(x \longmapsto\!\!\dashv y)$. But then, by $*89{\cdot}12$, $\lambda \cap R_3(x \longmapsto\!\!\dashv y)$ is identical with some $\mu_2$, i.e., some second order class. So every member of $\overleftarrow{R}_{*3}\text{'}x$ is in $\lambda \cap R_3(x \longmapsto\!\!\dashv y)$, but that includes $y$, contradicting the assumption. Hence the theorem follows by *reductio*.

The crucial lemmas needed for this most important theorem of Appendix B are $*89{\cdot}12$ and $*89{\cdot}17$. The rest, despite the great amount of complication, and the difficulty of the notation, seems to follow directly. It is clearly a sign of the degree to which Gödel and Myhill studied and understood the content of Appendix B that it is precisely those two theorems that drew their attention. As we shall see, the first requires that there be a new use of types in Appendix B, and the second is simply mistaken.

## 6.3 Myhill's challenge to $*89{\cdot}24$

John Myhill (1974), "The undefinability of the set of natural numbers in the ramified *Principia Mathematica*", argues that the proofs in Appendix B could not have succeeded. Myhill cites a later result, a generalization of $*89{\cdot}24$ which applies to one-many relations as well as to many-one relations, as the key result:

$$*89{\cdot}29\ \ \vdash :\ R \in (1 \to \text{Cls}) \cup (\text{Cls} \to 1)\ .\supset .\ R_{*(3+m)} = R_{*3}.$$

Myhill proves, using a model theoretic argument, that there are instances of induction of a level higher than any given level $k$ which does not follow for properties of levels less than $k$. Thus the particular level 3 which Russell identifies in $*89{\cdot}29$ will not be adequate, for no limit will do. Myhill's argument uses the familiar "non-standard models" of arithmetic which introduce some "non-standard number" distinct from all the elements of the intended model of arithmetic. That construction uses the compactness theorem.

The compactness theorem for a logic states that if every finite subset $\Delta$ of a possibly infinite set $\Gamma$ of sentences in a language of that logic is satisfiable, or has a model, then $\Gamma$ itself is satisfiable. Consider the infinite set of sentences including axioms for arithmetic such as Peano's, and an infinite set containing $u \neq 0, u \neq 1, u \neq 2, \ldots, u \neq m, \ldots$, running through all the numerals for integers. Any finite subset of this set will have a model. (Just pick a number to be $u$ which is larger than any of the numerals in the set.) By the compactness theorem, the whole set has a model, one in which '$u$' is assigned a "non-standard" number which also satisfies the given axioms for arithmetic.[9]

This result is at the heart of a number of phenomena related to first order theories of arithmetic. The existence of non-standard models of theories of arithmetic shows that those theories cannot be *categorical*, that is, that not all models of the theory are isomorphic. The standard or intended model of arithmetic is a structure with an initial element, "0", and a series of successors, an "omega series". The incompleteness of axiomatizable theories of arithmetic proved in Gödel's first incompleteness theorem is a consequence of the inevitability of the existence of non-standard models of any such theory. The omega structure simply is not *definable*, in a precise sense. Second order arithmetic is a theory in a logic in which quantification over properties is allowed. The principle of induction is stated as asserting that any property true of 0 and true of the $n + 1$ if true of $n$ is true of all numbers. Second order arithmetic is categorical, as one can specify the omega structure or intended model of arithmetic with explicit axioms. In this way the incompleteness theorem of Gödel plus the basic notions of model theory such as compactness, which were developed starting with Tarski's work in the 1930s, provide ways of studying the relative power of different theories of arithmetic, and in particular the strength of different formulations of induction, the topic of Appendix B.

Myhill's proof proceeds by using this "non-standard" model for the numbers and arithmetic of each order up to $k$ and the standard, or intended model for orders greater than $k$. It follows that some instance of induction will hold for some level greater than $k$ which fails for one lower, since it describes some class

[9] See Boolos, Burgess, & Jeffrey (2002, p.150) where this proof is given as an exercise.

involving a non-standard number. Hence something can be proved in the higher level arithmetic that cannot be proved in the lower levels. Consequently induction restricted to orders less than $k$ does not have the power of the levels of induction in all orders. This result is relatively straightforward, once the models for type theory are worked out, though of course it makes use of logical techniques not yet developed in 1924.

The "non-standard" models of arithmetic used in Myhill's proof should also cast doubt on Russell's belief, as evidenced in $*89 \cdot 16$, that sets of finite numbers can always be defined using limited logical resources. A propositional function picking out a finite class might be true of some non-standard numbers as well. Non-standard models of the numbers can be thought of as consisting of the "standard" numbers 0, 1, 2, 3, ... followed by yet more, non-standard numbers. Non-standard models of finite classes (or propositional functions) will involve some standard numbers, as well as perhaps some of those non-standard numbers. This will form what might be thought of as "non-standard finite classes". Such a class would not be "defined" in the formal sense of having only isomorphic extensions in each model. Russell had run up against deep problems concerning the logical power of theories that capture the structure of the natural numbers. It is no accident that Gödel looked for some error in Appendix B. It was the very notions of models of arithmetic and the study of the power of various theories of arithmetic that Gödel himself developed in the 1930s that finally allowed a full understanding of the logical force of induction.

Reflection on Gödel's theorems about systems of arithmetic led Hao Wang to assert that the project of Appendix B was impossible already in 1963, well before Myhill's paper:

> In Russell 1925, an attempt is made to derive mathematical induction from his ramified type theory. An error in the derivation is pointed out in Gödel 1944. Indeed, it is easy to give a proof that what Russell tried to do cannot be done since induction with regard to higher order sets can characterize lower order sets completely and therefore no finite number of orders can give the induction principle for all others.
>
> *(Wang, 1962, p.642)*

It is not immediately clear what the argument would be, as types are presumably cumulative and so classes of a lower order are also already of higher orders as well. Thus it is not clear what is added by the notion of "characterizing" lower order sets. A better hint at what Wang had in mind occurs in his 1965 paper "Russell and his logic":

> In Appendix B, Russell offered a proof that the class of integers of order 5 is no larger than the class of integers of order $n$, for any $n > 5$. It would follow that the class of integers can be defined as the intersection of all inductive classes of order 5. This would seem to contradict the fact that one can, using higher order induction, prove the consistency of the

system with lower order induction and, therefore, eliminate more non-standard numbers. As Gödel pointed out, the proof of $*89 \cdot 16$ is not conclusive.

*(Wang, 1965, pp.20–1)*[10]

Wang's suggestion is developed in a 1969 doctoral dissertation by James Royse.[11] Royse first shows how a truth predicate can be defined for a system of predicative arithmetic of a lower order within a system of higher order, following the model of Tarski (1956b, pp.241–65) in which a system of a higher simple type is shown to add the necessary "richness" to define truth in lower simple type. With truth defined, it is then possible to prove that all theorems of the system are true, and thus to prove the consistency of the system in the lower order. Since, as Gödel had shown, no such system can prove its own consistency, this shows that the higher order arithmetic is stronger. This difference in strength can be traced to the induction principles each theory contains. After carrying out this development of Wang's suggested proof, Royse also includes a sketch of the model theoretic version of the argument that Myhill was to publish five years later, apparently arrived at independently.

Myhill's proof is straightforward and does not seem to involve disputed assumptions, other, perhaps, than the fundamental assumptions behind the "non-standard" semantics for higher order logic which he uses in his model theoretic argument. In a stronger system of type theory, i.e., one with fewer models, it might be that one can justify induction with respect to higher order properties on an ancestral relation defined only at some lower order, which is the project of Appendix B. Indeed, the search for a system of types in which the statement of $*89 \cdot 12$ is coherent inevitably involves the investigation of the possibility of systems in which the proofs of $*89 \cdot 29$ and others are valid. Myhill's argument, however, used an obvious and natural semantic interpretation. It is understandable, then, that it became common knowledge that Appendix B had been vitiated. But the reputation of Appendix B had already suffered worse.

## 6.4 The error in Appendix B

As was pointed out by Hao Wang in the quoted passage above, Kurt Gödel found an error in the proofs of Appendix B in his "Russell's mathematical logic" (1944, pp.145–6). In a short passage devoted to the second edition of *PM*, Gödel first describes the project of the Appendix.

As to the question of how far mathematics can be built up on this basis (without any assumptions about the data – i.e., about the primitive predicates and individuals – except,

[10] This is reprinted in Wang (1974, p.116).

[11] In his unpublished PhD dissertation "Some investigations into ramified set theory" at the University of Chicago, Royse (1969).

as far as necessary, the axiom of infinity), it is clear that the theory of real numbers in its present form cannot be obtained. As to the theory of integers, it is contended in the second edition of *Principia* that it can be obtained. The difficulty to be overcome is that in the definition of the integers as "those cardinals which belong to every class containing 0 and containing $x + 1$ if containing $x$", the phrase "every class" must refer to a given order. So one obtains integers of different orders, and complete induction can be applied to integers of order $n$ only for properties of order $n$; whereas it frequently happens that the notion of integer itself occurs in the property to which induction is applied. This notion, however, is of order $n + 1$ for the integers of order $n$. Now, in Appendix B of the second edition of *Principia*, a proof is offered that the integers of any order higher than 5 are the same as those of order 5, which of course would settle all difficulties. The proof as it stands, however, is certainly not conclusive.

Then comes the statement of the mistake at line (3) of $*89{\cdot}16$.

In the proof of the main lemma $*89 \cdot 16$, which says that every subset $\alpha$ (of arbitrarily high order)[38] of an inductive class $\beta$ of order 3 is itself an inductive class of order 3, induction is applied to a property of $\beta$ involving $\alpha$ [namely $\alpha - \beta \neq \Lambda$, which, however, should read $\alpha - \beta \sim\epsilon$ Induct$_2$, because (3) is evidently false]. This property, however, is of order $> 3$ if $\alpha$ is of an order $> 3$. So the question whether (or to what extent) the theory of integers can be obtained on the basis of the ramified hierarchy must be considered as unsolved at the present time.

The footnote numbered 38 above is:

38. That the variable $\alpha$ is intended to be of undetermined order is seen from the later applications of $*89{\cdot}17$ and from the note to $*89{\cdot}17$. The main application is in line (2) of the proof of $*89{\cdot}24$, where the lemma under consideration is needed for $\alpha$s of arbitrarily high orders.

Gödel took this objection, and it appears that he wrote to Russell about it. In the only known letter from Gödel to Russell, which survives in draft form among Gödel's papers but was likely typed and then sent, he expresses the hope that Russell's decision not to reply to his article is not due to the mistaken impression that nothing in it would be controversial.[12] He writes:

I am advocating in some respects the exact opposite of the development inaugurated by Wittgenstein and therefore suspect that many passages will contradict directly your present opinion. Furthermore I am criticizing the vicious circle principle and the appendix B of Principia, which I believe contains formal mistakes that make the proof invalid. The reader would probably find it very strange if there is no reply.

There is no record of a response from Russell. Indeed, in *My Philosophical Development* (*MPD*, p.89), written in 1959, Russell still refers to the second edition of *PM* as successful in showing that the axiom of reducibility is not "indispensible . . . in

[12] Letter of 28 September 1943. In *Kurt Gödel*: *Collected Works* Vol. V: (Gödel 2003, p.208).

all uses of mathematical induction". So there is no published reply from Russell.

John Myhill wrote his paper devoted to disproving the main result of Appendix B, but has little to say about the actual text of the appendix. These are the explicit references to Appendix B in Myhill (1974, pp.19–27).

Specifically, let us define

$$N_{k+1} = \{x \mid (\forall\alpha_k)(0 \in \alpha \wedge (\forall y)(y \in \alpha \rightarrow y+1 \epsilon \alpha) \rightarrow x \in \alpha)\}.$$

Then, in Appendix B, Russell claims to prove, *without using the axiom of reducibility*, a general proposition ($*89{\cdot}29$), having as a special case the result[1] that

$$N_k = N_6 \qquad (k > 6).$$

Suppose he had in fact done this, then we would define $N$ as simply $N_6$ and then we could prove

$$\varphi(0) \wedge (\forall y)(\varphi(y) \rightarrow \varphi(y+1)) \rightarrow (\forall x \in N)\varphi(x)$$

for all $\varphi$ of no matter how high an order.

In his paper in the Schilpp Russell volume, Gödel observed that Russell's proof of $*89 \cdot 29$ is defective, and enquired whether a correct proof could be given. The main purpose of this paper is to answer this question in the negative.[13]

Later, at (1974, p.25) there is another footnote which concludes with this:

... but a more serious error occurs in line 3 of the proof of $*89{\cdot}12$ which reads (without hypotheses)

$$\exists!\, \alpha - \beta \,.\, \alpha \subset \beta \cup \iota y \,.\, \supset .\, \alpha = \beta \cup \iota y$$

which is false on the face of it. Whether it can be replaced by something which is not false so as to give a correct proof of $*89{\cdot}16$ I don't know. At any rate the main result of the present paper shows that not *all* of his errors can be corrected.

It appears, then, that despite the allegations of errors throughout Appendix B, and the proof by Myhill that the main result does not hold, there is really only one definite example of an error in a proof, namely, line (3) in the proof of $*89{\cdot}16$.

[13] (Myhill, 1974, p.21). Footnote 1 in the passage above is as follows. "The reader may wonder how I arrived at the curious number 6, since $*89{\cdot}29$ mentions the number 3. The reason is notational; $*89{\cdot}29$ says that the third-order ancestral $R_{*3}$ of any one-many or many-one relation (e.g. successor) is the same as its $k^{th}$-order ancestral for any $k > 3$. In particular something which has all the third-order hereditary properties of 0, has *all* hereditary properties of 0 (cf. $*89{\cdot}01$). But Russell counts the orders of properties relative to the order of the things that have the properties, not absolutely as we do (*Principia*, p.58), so that his third-order properties are our fifth-order properties. The set of all things that have all fifth-order hereditary properties of 0 is a sixth-order set denoted, by us, as $N_6$."

That error was first identified by Gödel, then Myhill singled it out as well as one of only two examples of what he suggests are many.[14]

### 6.5 The proof of $*89{\cdot}16$

$*89{\cdot}16$ of Appendix B is a lemma for:

$$89{\cdot}17 \quad \vdash :\ \gamma \in \text{Cls induct}_3 \,.\, \alpha \subset \gamma . \supset . \,\alpha \in \text{Cls induct}_3.$$

If $\gamma$ is a level 3 inductive class (member of Cls $\text{induct}_3$), and $\alpha$ is a subset of $\gamma$, then $\alpha$ is also a level 3 inductive class. Since inductive classes are all finite, it would seem that $89{\cdot}17$ must be true, for any subset of a finite set $\gamma$, however defined, should be definable with a formula of a very low order, for it will consist of a finite number of exclusions from $\gamma$. This does not always follow, however, since while $\gamma$ itself may be a class inheriting all the level 3 properties of $\Lambda$, its subset $\alpha$ might be missed, i.e. not inherit all such properties.

Theorem $89{\cdot}17$ is derived from $89{\cdot}16$ by a simple manipulation of propositional logic, proving a conditional of the form $p \,.\, q \,.\supset .\, r$ from something equivalent to $\sim r \,.\, p \,.\supset .\ \sim q$. (The annotation is simply "[$*89{\cdot}$ 16.Transp]".)

The real heart of this passage in the argument of Appendix B is thus:

$$*89{\cdot}16 \quad \vdash :\ \alpha\ \sim\in \text{Cls induct}_3 \,.\, \gamma \in \text{Cls induct}_3 \,.\supset .\, \exists!\, \alpha - \gamma.$$

(If $\alpha$ is not a level 3 inductive class but $\gamma$ is, then there is something in $\alpha$ which is not in $\gamma$.) The demonstration consists of seven lines, the fourth of which, (3), contains the fatal error. The first line of the proof is (1):

$$\text{Hp} \,.\supset : (\exists \mu_3) : \Lambda \in \mu_3 : \beta \in \mu_3 . \supset_{\beta,y} .\, \beta \cup \iota\text{`}y \in \mu_3\ : \gamma \in \mu_3 \,.\, \alpha \sim\in \mu_3. \quad (1)$$

The hypothesis ("Hp") is that while $\gamma$ is in the class of level 3 inductive classes, $\alpha$ is not. (1) fills in the definition of that claim and so states what it means that there is some class $\mu_3$ of level 3 to which $\gamma$ belongs, but to which $\alpha$ does not.

What then follows is a proof of the theorem by induction. Step (2) begins by taking an instance of the quantified variable $\mu_3$ in step (1), and drawing the conclusion that $\alpha \neq \Lambda \,.\, \Lambda \in \mu_3$. Next, if $\Lambda \in \mu_3$, $\gamma \in \mu_3$, and $\alpha \sim\in \mu_3$, it also follows that $\alpha \neq \Lambda \,.\, \Lambda \in \mu_3$:

$$\Lambda \in \mu_3 : \beta \in \mu_3 . \supset_{\beta,y} .\, \beta \cup \iota\text{`}y \in \mu_3 : \gamma \in \mu_3 . \,\alpha \sim\in \mu_3 : \supset : \alpha \neq \Lambda \,.\, \Lambda \in \mu_3$$

and so it follows from the antecedent that there is something in $\alpha$ and not in $\Lambda$, i.e.

$$\supset : \exists!\, \alpha - \Lambda \,.\, \Lambda \in \mu_3 \quad (2)$$

[14] Although Myhill also identifies the unexpected use of type subscripts in $*89{\cdot}12$.

The proof is by induction with respect to $\beta$. The inductive property proved true of all $\beta$ is $\beta \in \mu_3 \,.\, \exists!\, \alpha - \beta$. We have proved the basis step of the induction for $\Lambda$. Now suppose that $\beta \in \mu_3 \,.\, \alpha \sim\in \mu_3 \,.\, \exists!\, \alpha - \beta$. Then we show that $\beta \cup \iota\text{‘}y \in \mu_3$.

Suppose, for proof by contradiction, that $\sim \exists!\, \alpha - \beta \cup \iota\text{‘}y$, in other words, $\alpha \subset \beta \cup \iota\text{‘}y$. Then, if the following, false, lemma (3) were to hold:

$$\exists\,!\, \alpha - \beta \,.\, \alpha \subset \beta \cup \iota\text{‘}y \,.\; \supset .\, \alpha = \beta \cup \iota\text{‘}y, \tag{3}$$

we would have $\alpha = \beta \cup \iota\text{‘}y$. But then $\alpha \in \mu_3$, negating the assumption that it is not in $\mu_3$ above, hence yielding a contradiction.

By *reductio* then, $\exists!\, \alpha - \beta \cup \iota\text{‘}y$. But then, by induction, $\exists!\, \alpha - \beta$ for all $\beta \in$ Cls induct$_3$ and $\alpha \sim\in$ Cls induct$_3$.

This completion of the inductive argument is the substance of the last four lines of the proof:

$(3). \supset :.\, \text{Hp}(2) \,.\supset : \beta \in \mu_3 \,.\, \alpha \sim\in \mu_3 \,.\, \exists\,!\, \alpha - \beta \,.\supset .$
$\qquad \beta \cup \iota\text{‘}y \in \mu_3 \,.\, \alpha \neq \beta \cup \iota\text{‘}y \,.\, \exists\,!\, \alpha - (\beta \cup \iota\text{‘}y) \qquad (4)$
$(4). \supset :.\, \text{Hp}(2) \,.\supset : \beta \in \mu_3 \,.\, \exists\,!\, \alpha - \beta \,.\supset .\, \beta \cup \iota\text{‘}y \in \mu_3 \,.\, \exists\,!\, \alpha - (\beta \cup \iota\text{‘}y) \qquad (5)$
$(2).(5). \supset \vdash :.\, \text{Hp}(2) \,.\supset : \beta \in \text{Cls induct}_3 \,.\supset .\, \beta \in \mu_3 \,.\, \exists\,!\, \alpha - \beta \qquad (6)$
$(1).(6). \supset \vdash .\, \text{Prop}$

Gödel says that (3) is "evidently false". Myhill, as we have seen, seems to follow Gödel, and quotes (3) in full, presenting it as one of the two examples he actually gives of the "many superficial" errors that "can be corrected in various ways". While Myhill says that he gave up on Appendix B because of these many mistakes, Gödel responds differently. He was interested enough in the argument to still seek some explanation from Russell for what he describes as "formal mistakes".

What is the mistake? Line (3) of the proof of ∗89·16 is stated without justification, and does not follow from an earlier line in the proof. It is apparently thought to be an elementary fact of the theory of classes:

$$\exists!\, \alpha - \beta \,.\, \alpha \subset \beta \cup \iota\text{‘}y \,.\; \supset .\, \alpha = \beta \cup \iota\text{‘}y.$$

At first glance it may seem correct. That the difference between $\alpha$ and $\beta$ is non-empty, $\exists!\, \alpha - \beta$, means that there is something in $\alpha$ but not in $\beta$, in other words $\alpha$ is not a subset of $\beta$, i.e. $\alpha \not\subset \beta$. If furthermore, $\alpha \subset \beta \cup \iota\text{‘}y$, then that $\alpha$, on the other hand, is a subset of $\beta$ with only $y$ added, so it should follow that $\alpha = \beta \cup \iota\text{‘}y$, that is, $\alpha$ is identical with $\beta$ with only $y$ added. For, it would seem, the only thing that could be in $\alpha$ but not $\beta$ to begin with must be that very $y$.

However, (3) is in fact only true in particular cases. It is easy to find counter-examples to (3). Suppose that $y$ is in $\alpha$ but not in $\beta$, (so $\exists!\, \alpha - \beta$), and that $\alpha \subset \beta \cup \iota\text{‘}y$. It can still happen that $\alpha \neq \beta \cup \iota\text{‘}y$. All we need is that $y$ is not a member of $\beta$, and that $\alpha$ is a proper subset of $\beta \cup \iota\text{‘}y$. (3) is false in all such cases. Consider the case where $\beta = \{b_1, b_2, b_3\}$ and $\alpha = \{b_1, b_2, y\}$. Then surely $\exists!\, \alpha - \beta$, for $y$ is in $\alpha$ but not in $\beta$. As well, $\alpha \subset \beta \cup \iota\text{‘}y$ for $y$ is already in $\alpha$ and other members of $\alpha$, namely $b_1$ and $b_2$, are in $\beta$. But $\alpha \neq \beta \cup \iota\text{‘}y$, since $b_3$ is not a member of $\alpha$ but it is a member of $\beta$ and hence of $\beta \cup \iota\text{‘}y$.

Where does the initial appearance of truth come from? (3) will be true if $\beta \subset \alpha$, and we seem to be tacitly assuming that in the proof. It is easy to make the mistake if one slides from $\exists!\, \alpha - \beta$, i.e. $\alpha \not\subset \beta$, to the conclusion that $\beta \subset \alpha$. If one is perhaps misled by the analogy between a true "trichotomy" principle, which holds for ordinals, for example, and the false general case, "$\alpha \subset \beta$ or $\alpha = \beta$ or $\beta \subset \alpha$" for arbitrary $\alpha$ and $\beta$, the error is natural. So (3) is evidently false, but understandable. It is a slip, however fatal the consequences for Russell's proof and the appendix.

The history of the Appendix B manuscript is consistent with this assessment of the mistake as being a simple oversight. $*89{\cdot}16$ is on page *9* of the manuscript of Appendix B, but began numbered as *71* in the HPF manuscript, was then renumbered as *8*, finally changed to *9*. It was preserved through more revisions than any other page of the manuscripts. From the deleted numbers we can tell that theorem $*89{\cdot}16$ was first numbered as $*120{\cdot}123$, when the revisions to the theory of induction were to be put in $*120$ and $*121$ in Volume II. It was then renumbered as $*98{\cdot}16$, which would place it at the end of Volume I. It appears that Russell chanced upon the proof and kept it through many later revisions, not seeing the error. There are no traces of any preliminary sketches of the proof in the many working notes of ALP. The final theorem, however, appears on the initial list of "Amended propositions", (ALP, p.[1r]), as $*120{\cdot}123$ and on the next page (ALP, p.[2r]) as $*120{\cdot}123$. So, Russell stumbled onto the incorrect proof early on in the process, and kept it through many renumberings, happy to have that result in hand, and not careful enough to check it again.

Is this mistake easily corrected? As (3) constitutes the inductive step in a proof by induction, it is crucial to the argument. Could a different inductive property be used? In a passage quoted above, Gödel says that in the proof of $*89{\cdot}16$:

> ... induction is applied to a property of $\beta$ involving $\alpha$ [namely $\alpha - \beta \neq \Lambda$, which, however, should read $\alpha - \beta \sim\!\epsilon$ Induct$_2$, because (3) is evidently false],
>
> *(1944, pp.145–146)*

suggesting that a different inductive property should be used.[15] But he rejects that alternative, reasoning that

> This property, however, is of order $> 3$ if $\alpha$ is of an order $> 3$.

[15] Although it seems that this should read Induct$_3$ rather than Induct$_2$.

Gödel has pointed out that in any case Theorems $*89{\cdot}16$ and $*89{\cdot}17$ are simply false. The subset $\alpha$ may be of an order higher than the order 3 of the containing class $\beta$.

The proof of $*89{\cdot}12$ relies on an argument that assumes that finite numbers are definable with simple, low order, expressions. $*89{\cdot}16$ is similarly about finite subsets of simply defined classes. Myhill showed that the existence of "non-standard" models of arithmetic for certain orders blocks any attempt to prove induction, and hence to capture the structure of the numbers, with only low order formulas.[16] Russell seems to have thought that finite classes of numbers could be defined with simple formulas. Instead, we know that because of the existence of such non-standard finite classes this is not possible. Line (3), $\exists !\, \alpha - \beta \,.\, \alpha \subset \beta \cup \iota\text{`}y \,.\, \supset .\, \alpha = \beta \cup \iota\text{`}y$, fails in any case where we cannot reason from $\exists !\, \alpha - \beta$, i.e. $\alpha \not\subset \beta$, to the conclusion that $\beta \subset \alpha$. But when reasoning about inductive, finite, classes, constructing $\alpha$ by adding one element, $y$, to $\beta$, this inference does hold. The situation where (3) fails, where $\beta$ is not a subset of $\alpha$, is exactly one that might be overlooked if one does not think of non-standard numbers. So, if $\beta$ is some finite class distinct from $\alpha$, but also ordered as a series $b_1, b_2, \ldots, b_n$, and, furthermore, $\alpha$ is also a structure such that $\alpha$ is a subset of $b_1, b_2, \ldots, b_n, y$, then surely $\alpha$ just is $b_1, b_2, \ldots, b_n, y$. The counter-example where $\beta$ is not a subset of $\alpha$ might be pictured as one where an object $z$ is outside of this structure, say way beyond the end of the "standard numbers". Thus if $\beta$ is $b_1, b_2, \ldots, b_n, \ldots, z$ and $\alpha$ is $b_1, b_2, \ldots, b_n, y$, it will be true that $\alpha$ is a subset of the result of adding $y$ to $\beta$, $\alpha$ will not be identical to the result of adding $y$ to $\beta$, because it will have missed $z$. It is tempting to hypothesize that since Gödel realized that the project of Appendix B was impossible, precisely because of the impossiblity of constructing a categorical theory of arithmetic with only limited orders of quantifiers, in other words, that something like Myhill's result would be provable. It would be no accident, then, that Gödel found the mistake at line (3) in $*89{\cdot}16$. There would have to be a mistake somewhere, and someone with the notion of non-standard numbers in mind would have the counter-example to (3) ready to mind.

Russell, on the other hand, working before Gödel's results, assumed that he could capture the structure of the numbers with limited resources. One can imagine him checking the validity of (3) with a model of initial segments of the numbers, and finding it to be an obvious truth. Line (3) would be correct of the case where it is applied if the goal of the project of Appendix B was true. Seen this way, the proof of $*89{\cdot}16$ fails because it begs the question. Gödel was not trying to get Russell

[16] See, for example Boolos, Burgess, & Jeffrey (2002, pp.302–5) for an introduction to the study of the structure of non-standard models of arithmetic.

to confess to having made an error of elementary set theory when he wrote asking Russell to respond to his paper. Instead he pointed to an exact point in a technical proof around which to crystalize the larger question of whether Russell's project was bound to fail.

## 6.6 Types in Appendix B

The discussion above has identified an elementary mistake at line (3) of the demonstration of $*89{\cdot}16$. It is clear both what the mistake is, and how Russell might have come to make it, and even, perhaps, how Gödel may have come to find it. But the other problem of identifying classes of different types, highlighted by the example of $*89{\cdot}12$, is not so easy to resolve. Myhill presented it as an example of the "slipshod" notation in Appendix B.[17] Gödel, despite the obvious mistake at line (3) of $*89{\cdot}16$, says only that "... the question whether (or to what extent) the theory of integers can be obtained on the basis of the ramified hierarchy must be considered as unsolved at this time" (Gödel, 1944, p.146).[18] He saw that there was more to the arguments in Appendix B that needed investigation.

The key to resolving the puzzle about $*89{\cdot}12$, and the other apparent confusions about types that Myhill identified, is to see the second edition as introducing a new theory of types. There has long been unclarity and even confusion about exactly what the system of types is, or is intended to be, in the second edition. Even the simple account of the second edition as based on the ramified theory of types, with extensionality, but without the axiom of reducibility, is not universal in the secondary literature. The *Dictionary of Symbols of Mathematical Logic* from 1969 makes this claim about basis of ramified types, the axiom of reducibility and the "no-classes" theory.

> In the second edition of *Principia Mathematica*, this whole approach was dropped in favor of the simple theory of types, and classes were identified with propositional functions, but the necessary changes in the system were only roughly sketched. One disadvantage of this revised standpoint is that the simple theory of types provides in itself no solution of the semantical paradoxes, such as Richard's Paradox ....
>
> *(Feys & Fitch, 1969, p.92)*

This view reports as accomplished in *PM* what is really a proposal which Ramsey (1925b) suggests in his discussion of *PM*, namely to adopt the simple theory of types to handle the set theoretic paradoxes and relegate the others, which might

[17] See Myhill (1974, p.25).
[18] Indeed Landini (1996) proposes that $*89{\cdot}16$ and the rest of Appendix B can be saved with modifications to the theory of types which avoid Myhill's argument. This will be discussed below.

have been thought to require the ramification, to some other theory of semantics, a theory of language and meaning.

Perhaps this confusion about the nature of the theory of types derives from reading Gödel's condensed remark about a new theory of types in the rest of his discussion of the second edition in "Russell's mathematical logic".[19]

In the second edition of *Principia*, however, it is stated in the Introduction (pages XI and XII) that "in a limited sense" also functions of a higher order than the predicate itself (therefore also functions defined in terms of the predicate as, e.g. in $p`\kappa \epsilon \kappa$) can appear as arguments of a predicate of functions; and in Appendix B such things occur constantly. This means that the vicious circle principle for propositional functions is virtually dropped. This change is connected with the new axiom that functions can occur in propositions only "through their values," i.e., extensionally, which has the consequence that any propositional function can take as an argument any function of appropriate type, whose extension is defined (no matter what order of quantifiers is used in the definition of this extension).

*(Gödel, 1944, p.134)*

Here Gödel sees a new theory of types in the logic of Appendix B, namely one in which "any propositional function can take as an argument any function of appropriate type, whose extension is defined (no matter what order of quantifiers is used in the definition of this extension)." This is not simple type theory, but also not the original "ramified" theory of types.

This new system of types has been identified independently in Allen Hazen (1993) and (2004), and Landini (1996). Hazen calls it BMT (from "Appendix B Modified theory of types") to indicate that it is the system of ramified types to be found in Appendix B.[20] BMT is to be distinguished by allowing that, in Gödel's words, "any propositional function can take as an argument any function of appropriate type ... ", regardless of the quantifiers used in the definition of that argument function. Gödel speaks of the arguments of any function whose "extension is defined" and of the "order of quantifiers ... used in the definition of this extension" but since in the second edition classes (or extensions) are simply identified with functions, what this must be is a proposal about the system of types applying to functions. The proposal Gödel makes then, is that the system of the second edition is one in which functions have orders, which reflect the use of quantifiers in their definitions, but in which the arguments of a given function can be of any order, so long as the argument is of a simple type appropriate to that given function. Hazen has made the proposal precise as follows.

There is a BMT type $i$ of individuals. For every non-individual type $t$ ... and every positive integer $n$, there is a BMT-type $t/n$ of propositional functions (or propositions) of *level n*

[19] It is this passage to which A. Hazen refers, crediting J. Davoren with appreciating its significance.

[20] Landini calls the system *Principia*$^W$ to indicate his view that Russell developed it "to evaluate Wittgenstein's ideas" (2007, p.208). See Hazen and Davoren (2000).

and *simple type t* . . . The simple type *corresponding* to the BMT-type $i$ is $i$, and the simple type corresponding to a BMT-type $(\ldots)/n$ is $(\ldots)$.

*(Hazen, 2004, pp.466–7)*

The condition on well formed formulas then is that

if $F$ is a term of some BMT-type $(t_1, \ldots, t_n)/k$ and $a_1, \ldots, a_n$ are terms with corresponding BMT-types $t_1, \ldots, t_n$, then $F[a_1, \ldots, a_n]$ is a formula.

*(Hazen, 2004, p.466–7)*

There is more to the purported system of Appendix B than new type symbols and, thus, new rules for which formulas are well formed expressions. A theory of types must also include a comprehension principle which somehow embodies the notion that a function defined in terms of quantification over functions of a given type must itself be of a yet higher type. Types can enter into the logic in further ways. Hazen's full BMT includes expressions for class abstracts, rules about substitution of those terms for bound variables, and so on. Landini proposes a principle of extensionality which has major consequences for the logic attributed to the appendix. The system of types, however, in particular, the rule allowing for functions to apply to arguments of all levels, is common to both proposals, and seems to be clearly proposed in Gödel's remarks.

What grounds are there for attributing a new logic to the Appendix? Gödel may be right that there may be a new system of types, but the evidence which he cites does not support that conclusion. The difficulty of the task of finding evidence is shown by the fact that Gödel's example '$p`\kappa \in \kappa$' does not in fact occur "constantly", as he says. In fact, it does not ever occur in the formal development of Appendix B, but only in the introductory material. What Russell actually says in the Introduction to the second edition is the following.

There is a kind of proof invented by Zermelo, of which the simplest example is his second proof of the Schröder–Bernstein theorem (given in ∗73). This kind of proof consists in defining a certain class of classes $\kappa$, and then showing that $p`\kappa \in \kappa$. On the face of it, "$p`\kappa \in \kappa$" is impossible, since $p`\kappa$ is not of the same order as members of $\kappa$. This, however, is not all that is to be said. A class of classes $\kappa$ is always defined by some function of the form

$$(x_1, x_2, \ldots) : (\exists y_1, y_2, \ldots) .\ F(x_1 \in \alpha,\ x_2 \in \alpha, \ldots y_1 \in \alpha,\ y_2 \in \alpha, \ldots),$$

where $F$ is a stroke-function, and "$\alpha \in \kappa$" means that the above function is true. It may well happen that the above function is true when $p`\kappa$ is substituted for $\alpha$, and the result is interpreted by ∗8. Does this justify us in asserting $p`\kappa \in \kappa$?

Let us take an illustration which is important in connection with mathematical induction. Put

$$\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha \ .\ a \in \alpha).$$

Then $\qquad \breve{R}``p`\kappa \subset p`\kappa \ .\ a \in p`\kappa$ (see ∗40·81) so that, in a sense $p‘\kappa \in \kappa$. That is to say, if we substitute $p‘\kappa$ for $\alpha$ in the defining function of $\kappa$, and apply $*8$, we obtain a true proposition.

*(PM,* pp.*xxxix–xl)*

What Russell actually asserts is that the effect of the banished expressions ‘$p‘\kappa \in \kappa$’ can still be achieved by consideration of the defining expressions for both classes. The supposed example of a higher order class being a member of one of lower order turns out to be merely something that occurs “in a sense”. There is, however, a direct example of such a higher order class being a member of a lower order class in a deleted formula from page *9* of the Appendix B manuscript, immediately after the proof of $*89{\cdot}17$:

$$\cdot 825 \quad \vdash : \ \exists!\ \text{NC induct}_3 - \text{NC induct}_4 \ . \supset . \ \text{NC induct}_4 \in \text{Cls induct}_3 .$$

The class of inductive numbers, NC induct, does not occur in the printed version of Appendix B, and so there is no definition of the notion relativized to orders, as in NC $\text{induct}_4$. It is clear, however, that the definition of “NC $\text{induct}_4$” will involve quantifiers over variables of order 4, while that of Cls $\text{induct}_3$ involves variables of order 3. Here we have a higher order class as a member of one of lower order. This is a genuine example of the phenomenon that Gödel says “occurs constantly”. The example is, however, not in the final printed version. What might Gödel have seen “constantly” that might have led him, and should lead us, to see there being a new theory of types in the published version of Appendix B?

There is a different sort of evidence of a new theory of types in Appendix B, but it is not found directly in the passages that Gödel identifies. Here is a footnote in Myhill’s discussion, following his identification of the error in line (3) of the proof of $*89{\cdot}16$.

The reader may be curious as to what Russell’s actual mistake was, in Appendix B. The Appendix is a thorny text indeed, full of slipshod notations which it taxed the patience of this reader to correct, and I am not sure that I got to the heart of his error since there are so many superficial ones which can be corrected in various ways (and one does not know which is the right correction, i.e. which one represents just what Russell had in mind). To give the reader some idea of what we are up against, at the very beginning he defines

$$\text{Cls induct}_m \equiv \{\rho|(\forall\mu_m)(\Lambda \in \mu \wedge (\forall\eta y)(\eta \in \mu \rightarrow (\eta \cup (y) \in \mu \rightarrow \rho \in \mu))\}$$[21]

and proves $(*89 \cdot 12)$ $\rho \in \text{Cls induct}_3 \rightarrow (\exists\mu_2)(\rho = \mu_2)$. The definition of Cls $\text{induct}_m$ is, however, defective; the order of $\rho$ ought to be signified, and it is not. If this order is signified as $m-1$, then if $\rho$ has order 2 $*89{\cdot}12$ reduces to the tautology $(\exists\mu_2)(\rho_2 = \mu_2)$; and if $\rho$

[21] Myhill’s punctuation is incorrect. It would be more accurate, but less easily readable, as

$$\{\rho|(\forall\mu_m)((\Lambda \in \mu \wedge ((\forall\eta y)(\eta \in \mu \rightarrow \eta \cup (y)) \in \mu) \rightarrow \rho \in \mu))\}.$$

has any other order $*89{\cdot}12$ is not even well formed. One can correct this mistake by always writing $\mathrm{Cls}_m(\rho)$ instead of $\rho \in \mathrm{Cls\ induct}_m$ (I think!) ...

*(Myhill, 1974, p.25 n.1)*

It is indeed $*89{\cdot}12$, which appears earlier in the Appendix, which requires that there must be a new notion of types in Appendix B:

$$*89{\cdot}12 \quad \vdash : \rho \in \mathrm{Cls\ induct}_3 . \supset . (\exists \mu_2). \rho = \mu_2.$$

Russell remarks that "this proposition is fundamental", and it is fundamental to the appendix, being cited explicitly in $*89{\cdot}22$ and $*89{\cdot}24$, and used in the theorems those two depend on.

Myhill charges that $*89{\cdot}12$ is "defective", because the order of the variable $\rho$ is not specified. He sees this as an example of slipshod notation, and not one that can be coherently amended. If the proposal that there is a new theory of types in Appendix B is right, however, then '$\rho$' should be treated as "typically ambiguous", and so of some order compatible with belonging to $\mathrm{Cls\ induct}_3$, and so coherent after all.

In the HPF manuscript (HPF, p.*70*), where $*89{\cdot}12$ is numbered as $*120{\cdot}81$, we find the following.

The proof of $\rho \in \mathrm{Cls\ induct}_3 . \supset . (\exists \mu_2). \rho = \mu_2$ is very simple. We have $\Lambda = \hat{x}\{(\exists \phi). \phi!x . \sim\phi!x\}$, which is a second-order function. We have $\iota\text{'}y = \hat{x}(\phi!y \supset_\phi \phi!x)$, which is again a second-order function. Moreover the sum of two second-order functions is a second order function. Hence if $\eta$ is defined by a second order function, so is $\eta \cup \iota\text{'}y$. Thus

$$(\exists \mu_2). \rho = \mu_2$$

is an inductive property. It is of the third order. Hence it belongs to all inductive classes of the third order, i.e.

$$*120{\cdot}81 \quad \vdash : \ \rho \in \mathrm{Cls\ induct}_3 . \supset . (\exists \mu_2). \rho = \mu_2.$$

Since $\mathrm{Cls\ induct}_{3+m} \subset \mathrm{Cls\ induct}_3$, this shows that inductive classes of the third and higher orders can all be defined by second-order functions. This proposition enables us, after some trouble, to secure results which otherwise would require the axiom of reducibility.

This explicit evidence shows that $*89{\cdot}12$ was deliberately written with the order indices that it has. Contrary to Myhill's claims, it is not a sign of the "slipshod" use of notation. Theorem $*120{\cdot}81$ is repeated on [1r], [2r] of ALP. It is not casual, or a slip. It is intended, and its intended meaning is that any inductive class of third (or higher) order can be defined by a second order function.

In $*120{\cdot}81$ classes of different orders are identified, and hence ruling out Myhill's one alternative that $*89{\cdot}12$ is simply trivial because members of $\mathrm{Cls\ induct}_3$ will be of order 2. This conclusion is supported by some other formulas found in the manuscripts. Indeed, right after $*89{\cdot}17$ are four theorems involving the notions of NC induct$_3$ and NC induct$_4$, inductive cardinals limited to a particular order. These formulas are deleted, as the notion of orders of inductive cardinals is not needed in Appendix B. The deleted theorems, however, $*89{\cdot}825$, $*89{\cdot}826$, $*89 \cdot 827$, and $*89 \cdot 828$, all include the identification of classes of different orders. Thus consider the deleted

$$\cdot 826 \quad \vdash :\ \exists\ !\ \mathrm{NC\ induct}_3 - \mathrm{NC\ induct}_4\ .\supset .\ (\exists \mu_2).\ \mathrm{NC\ induct}_4 = \mu_2.$$

Its consequent contains an identification of an order 4 class and one of level 2. This formula is repeated in (ALP, p.[2r]), where the number is $*120{\cdot}826$. Thus it is only in material that was deleted from the final version of Appendix B that this issue is settled conclusively.

## 6.7 Identity in Appendix B

Theorem $*89{\cdot}12$ does not in fact assert that an inductive class of third order can be defined by a second order function. What it says is that the third order class is *identical* with a second order function. We are led back to the issue about identity in the second edition of *PM* that was raised in the discussion of the Introduction to the second edition. It is not clear precisely how identity and extensionality should be formulated in the BMT theory of types. In the first edition of *PM* only functions of the same $r$-type can be identical, and they are identical only when they share the same predicative functions. If functions can have arguments of a wide range of types, and functions of differing types can be arguments of the same functions, then these definitions need to be reformulated. It is differences over these notions are the source of the disagreement between Hazen and Landini about the system of Appendix B.

Recall the description of the doctrine of extensionality in the Introduction to the second edition

According to our present theory, all functions of functions are extensional, i.e.

$$\phi x \equiv_x \psi x\ .\supset .\ f(\phi \hat{z}) \equiv f(\psi \hat{z}).$$

This is obvious, since $\phi$ can only occur in $f(\phi \hat{z})$ by the substitution of values of $\phi$ for $p, q, r, \ldots$ in a stroke-function, and, if $\phi x \equiv \psi x$, the substitution of $\phi x$ for $p$ in a stroke-function gives the same truth-value to the truth-function as the substitution of $\psi x$. Consequently there is no longer any reason to distinguish between functions and classes, for we have, in virtue of the above,

$$\phi x \equiv_x \psi x\ .\supset .\ \phi \hat{x} = \psi \hat{x}.$$

We shall continue to use the notation $\hat{x}(\phi x)$, which is often more convenient than $\phi \hat{x}$; but there will no longer be any difference between the meanings of the two symbols.

Thus classes, as distinct from functions, lose even that shadowy being which they retain in $*20$.

*(PM,* p.*xxxix)*

If that is how we are to understand expressions of identity between classes, then $*89\cdot 12$ does indeed say that a third order class will be identical with some second order class. It seems that Myhill is treating as a mere confusion about order, exactly what Gödel more charitably interprets as a loosening of the restrictions of type theory, saying that the "vicious circle principle for propositional functions is virtually dropped". For how could a function of one order be "identical" with one of another order? Russell does not notice this issue because he is thinking of a single class as defined by functions of two different orders. That would be unproblematic in the context of the "no-classes" theory of classes in conjunction with the axiom of reducibility in the first edition. According to that theory, classes are identical if some predicative function co-extensive with the defining function of the first is also co-extensive with the defining function of the second. That can occur even though the functions are of different orders. But in the second edition we are to simply identify functions with classes, and the no-classes theory of $*20$ is presumably to be abandoned as unnecessary. (The no-classes theory of the first edition would not have the power needed, even if it were retained, because in the second edition the axiom of reducibility is abandoned.) The no-classes theory will not be necessary, for due to of the principle of extensionality, no distinction is made between co-extensive functions.

There is a puzzle here. What does it mean to say that functions are "identical" in the absence of the axiom of reducibility? Hazen simply finds the expression $\phi x \equiv_x \psi x . \supset . \phi\hat{x} = \psi\hat{x}$ to be "deeply problematic" (Hazen, 2004, p.469) for it is "... not quite certain that he meant to endorse it at all types". If functions being true of the same values of a lower type implies that they share functions of all higher types, this in effect reinstates an axiom of reducibility. Landini (1996), on the other hand, thinks that the principle of extensionality must be stated with an explicit axiom having just that effect, and chooses as an axiom the claim that if functions are co-extensive, then they will be intersubstitutable in all contexts. Given these problems, perhaps it is necessary to start over again. Must we conclude that the use of identity in $*89\cdot 12$ really does require a new theory of types? Perhaps all that Russell means in $*89\cdot 12$ is that $\rho$ can be defined by a second order function $\mu_2$ and so simply that they are co-extensive and not identical.

In Leon Chwistek's "Theory of constructive types" (1924), which Russell read before composing the second edition of *PM*, there is a proposal to distinguish between two kinds of identity.[22] One kind is what he calls the "Leibnizian" notion

[22] The offprint that Chwistek sent has recently been identified in the Russell Archives. See Section 3.2.4 above.

of identity as indiscernibility, represented as "$x =_L y$", and defined as true if $x$ and $y$ share all properties (of the appropriate type). The other sort of identity is defined for classes, and amounts to $x$ and $y$ having the same members. Perhaps Russell absorbed this point from Chwistek, and assumed that identity was now to be simply co-extensiveness for propositional functions, and thought that nothing more needed to be done about the underlying theory of identity.

Chwistek's "Leibnizian" notion of identity as indiscernibility is represented in the first edition of *PM* using the definition of identity:

$$*13{\cdot}01 \quad x = y \,.=: (\phi) : \phi!x \,.\supset .\, \phi!y \quad \text{Df.}$$

Thus $x$ is identical with $y$ if and only if every predicative function true of $x$ is also true of $y$. It is not possible to state that identity is sharing of *all* properties, since there is no "totality" of all properties to be the subject of a quantifier. Instead the axiom of reducibility guarantees that if any objects $x$ and $y$ differ in some property $f$ of any order, they will differ with respect to the predicative function $\phi$ which is co-extensive with $f$ that the axiom guarantees will exist. Although Russell abandons the axiom of reducibility for the second edition, at least for the purposes of Appendix B, he does not explictly abandon or even change the definition of identity. Thus, in Appendix C, in a discussion of extensionality for propositions, Russell invokes $*13{\cdot}01$ as still in force at (*PM*, p.659). Even in a new theory of types, such as BMT, as long as $*13{\cdot}01$ is retained as the definition of identity, it is still not possible to identify functions of different (BMT) types, even if they are of the same *simple* type. It is thus conceivable that Russell is treating identity, without remark, as co-extensiveness for the newly identified functions/classes. If that is so, this would only constitute an argument for the existence of a new theory of types in Appendix B if there are cases where identified classes are substituted for each other. That in turn would only require a new theory of types if those identified functions/classes are exchanged for one another in contexts where it would seem that their orders would make a difference. If the functions/classes are equally well formed as arguments to functions with BMT types that take functions of the same (corresponding) simple type as arguments, then we will have an account of these substitutions, as showing that indeed there is a new theory of types in Appendix B. This seems to be precisely what happens when $*89{\cdot}12$ is invoked. A class of some higher order, namely 3, is first shown to exist, then $*89{\cdot}12$ is used to argue that that very class is of the lower order 2. This phenomenon is what Gödel says occurs "constantly" in the new material. The resolution of this issue about types, then, comes from paying special attention to the role of types in the proofs of Appendix B, especially those which involve identities between functions (now, indifferently, classes) of differing $r$-types. The inferences may only be expressible on the assumption that there are indeed new kinds of types at work, namely BMT.

## 6.8 The "rectification" of induction

Gödel's paper was published in 1944, Myhill's in 1974. It then took until 1991 for the next discussion of the proof in Appendix B to appear in print. This was in Davoren and Hazen (1991), an abstract of a talk to the Association for Symbolic Logic.

> Gödel, the first commentator to remark on the details of Appendix B, hints in his 1944 at a liberalization of RTT which allows propositional functions to hold of arguments of appropriate (simple) type but arbitrary order, while still maintaining restrictions on the orders of quantified variables in the definition of a propositional function. . . . All of Appendix B can be formalized in the liberalized system and it is consonant with some of Russell's remarks in his Introduction to the second edition. However, there are still interpretative problems, and proofs of crucial lemmas remain, as Gödel pointed out, invalid.
> *(Davoren & Hazen, 1991, p.1109)*

In his (1996) and (2007), Gregory Landini also proposes that the logic of the second edition is modified in this way to allow for the BMT notion of types and consequently relaxes the conditions on well-formed formulas. He adds to this a certain axiom of extensionality, and so produces a proof of $*89{\cdot}17$, thus finally "rectifying induction", as he describes Russell's project.

The modification that Landini proposes is to allow that functions of possibly different types which are co-extensive, with regard to a particular argument of one type, are intersubstitutable in all of the wide range expressions in which they are well formed:

(EXT)
$(\forall v^{t/m})(P^{(t)/a}(v^{t/m}) \equiv Q^{(t)/b}(v^{t/m})). \supset . A(P^{(t)/a}) \supset A[Q^{(t)/b}|P^{(t)/a}]$ where $Q^{(t)/b}$ replaces some (not necessarily all) free occurrences of $P^{(t)/a}$ in $A$, $v^{t/m}$ is an individual variable or predicate variable, and $Q^{(t)/b}$ and $P^{(t)/a}$ are predicate terms (variables or lambda abstracts).[23]

One immediate consequence of this principle, which will be of importance in what follows, is that functions which are co-extensive with respect to arguments of one type, $t/m$, will be co-extensive over all functions of types $t/n$ for which they apply. To demonstrate this, just let $A(P^{(t)/a})$ in the formula above be $P^{(t)/a}(v^{(t)/n})$. This can be analyzed, as the result of a principle of extensionality

[23] I have modified Landini's type notation above. He presents this formula as $({}^{m}v^{t})({}^{a}P^{(t)}({}^{m}v^{t}) \equiv^{b} Q^{(t)}({}^{m}v^{(t)})). \supset . A[{}^{a}P^{(t)}] \supset A[{}^{b}Q^{(t)}|{}^{a}P^{(t)}]$ to keep the order and type indicators apart (Landini, 2007, p.209). See Landini (1996, p.604) where EXT is identified as a primitive proposition "Extensionality$_1$".

$\forall v[P(v) \equiv Q(v) \supset P = Q]$ and some appropriate new definition which has as a consequence $P = Q \supset A(P) \equiv A(Q)$.

With this principle of extensionality, Landini (1996) is able to give a proof of $*89 \cdot 17$. The proof makes use of three lemmas (in his adaptation of *PM* notation):

(1) $\vdash \Gamma \epsilon \overleftarrow{M}_{*3}`\Lambda : \supset : \mathrm{Cl}`\Gamma = \overrightarrow{M}_{*3}`\Gamma$
(2) $\vdash \Gamma \epsilon \overleftarrow{M}_{*3}`\Lambda \,.\, \& \,.\, \alpha \epsilon \overrightarrow{M}_{*3}`\Gamma : \supset : (\exists \mu_2)(\alpha = \mu_2)$
(3) $\vdash \Gamma \epsilon \overleftarrow{M}_{*3}`\Lambda \,.\supset .\, (\alpha_2)(\alpha_2 \leq \Gamma . \supset \alpha_2 \epsilon \overleftarrow{M}_{*3}`\Lambda)$.

Theorem $*89 \cdot 17$ is then proved as $\vdash \Gamma \epsilon \overleftarrow{M}_{*3}`\Lambda \,.\, \& \,.\, \alpha \leq \Gamma . \supset \alpha \epsilon \overleftarrow{M}_{*3}`\Lambda$. The proof of lemma (1) makes use of the new extensionality principle to identify the two classes $\mathrm{Cl}`\Gamma$ (the power set of $\Gamma$), and $\overrightarrow{M}_{*3}`\Gamma$ (the class of level 3 successors of $\Gamma$) on the basis of the co-extensiveness of these functions at just one level. They are then substituted for each other later in the proof. Note that lemma (2) has the same force as the problematic $*89 \cdot 12$.

Hazen & Davoren (2000) challenge Landini's reconstruction of Appendix B on the grounds that the proposed axiom of extensionality is too strong. They provide a direct argument that it must be incorrect, namely that the proposed axiom allows a proof of the axiom of reducibility.[24] Here is the argument. Let $P$ be a function of predicative functions which is true of $\phi!$ just in the case when $\phi!$ is co-extensive with some predicative function $\psi$:

$$P(\phi!) \;\equiv\; \exists \psi ! \forall x (\psi ! x \equiv \phi ! x).$$

Let $Q$ be a function of predicative functions which is true of $\phi!$ just in the case when $\phi!$ is co-extensive with itself:

$$Q(\phi!) \;\equiv\; \forall x (\phi ! x \equiv \phi ! x).$$

Since $P$ and $Q$ are true of the same predicative functions (namely all of them), then by the proposed axiom of extensionality they are identical. But then $P$ and $Q$ will be true of the same functions $\chi$ of any corresponding BMT type, including each case where $\chi$ is not predicative:

$$\forall \chi \; [\exists \psi ! \forall x (\psi ! x \equiv \chi x) \equiv \forall x (\chi x \equiv \chi x)].$$

One side of the resulting biconditional says that $\chi$ is co-extensive with itself, which will be a theorem. So, discharging the other side of the biconditional we derive

$$\exists \psi ! \forall x (\psi ! x \equiv \chi x),$$

which is just an instance of the axiom of reducibility for the type of $\chi$.

24 This formulation of the argument is close to Landini's own (2007, pp.212–13).

Landini is indeed able to "rectify" induction by proving it without using the axiom of reducibility, and with an axiom of extensionality, in the new theory of types that he, Gödel, and Hazen each found in Appendix B. Unfortunately in that system of types the new axiom of extensionality Landini proposes is strong enough to prove the axiom of reducibility. Acknowledging this (2007, p.212), however, Landini suggests that EXT is exactly what Russell was looking for when he says in the Introduction to the second edition that

> Perhaps some further axiom, less objectionable than the axiom of reducibility, might give these results, but we have not succeeded in finding such an axiom.
>
> *(PM,* p.*xiv)*

Yet a principle stronger than the axiom of reducibility, which so directly entails the axiom, would hardly seem "less objectionable", however well it is justified. This does not seem to be a genuine "rectification" of induction in the system of the second edition.

## 6.9 Summary of results about Appendix B

Gödel was right that there is a mistake at line (3) in the proof of $*89{\cdot}16$. Russell did not notice the mistake, which survived several revisions of the appendix intact. The error was to cite without proof an assertion of elementary set theory which is not true in general, but which would hold in the situation of the hypothesis of the argument if the theorem were correct. Russell's error is to beg the question in his proof. Myhill was right to focus on $*89{\cdot}12$ as problematic. He says that it is either ill formed, or trivial. It is indeed intended in the sense that he finds ill formed, as identifying a class of one order with a class of another order. Other theorems, and a line of comment, in the manuscripts, but not the published version, verify that $*89{\cdot}12$ is also intended and not a result of a slip or inattention. The explanation is as Gödel suggested, there must be a new theory of types in Appendix B, one which makes $*89{\cdot}12$ well formed.

Myhill's result about the system of ramified types in the first edition of *PM*, with his proposed semantic interpretation, is quite correct. It is impossible to derive a principle of induction using a principle of extensionality but not an axiom of reducibility, in the system of the first edition. It is an open question whether some other plausible extensional system of ramified type theory (or so-called "predicative analysis") will have such a result.[25] If one assumes that there is a new theory of types in the second edition, then both $*89{\cdot}12$ and the talk of the principle of extensionality in the Introduction to the second edition lead to questions about what it may mean

[25] See Hazen (2004) for the negative conjecture.

to identify two functions. One might, following Chwistek's proposal, simply mean by this that the functions are co-extensive. However, that would not, by itself, allow the substitution of a function $\phi$ for $\psi$ in any but predicative functions. That assumption, then, would not yield some of the substitutions in Appendix B that make use of $*89\cdot12$. So Gödel is right to say that the issue concerning induction is still open, and that Appendix B as it stands cannot be easily repaired. In particular, the difficulties extend beyond the elementary mistake in line 3 of the demonstration of $*89\cdot16$.

This overall assessment is supported by some evidence in the manuscripts. The bulk of the ALP manuscripts show that the work on Appendix B was deliberate and extensive. The history of the manuscript of $*89\cdot16$ helps to explain the nature of the error. Finally, the manuscript evidence urges us to persist with an attempt to sort out Appendix B despite what appeared to Myhill as its being "... full of slipshod notations ...". Rather, it seems that Gödel's more charitable suggestion is more accurate: "The proof as it stands, however, is certainly not conclusive" (Gödel, 1944, p.145). Gödel was justified as well in his request of Russell for some reply about Appendix B. There is not, however, enough in Russell's papers to provide a reply. Landini's hypothesis that Russell saw something special in logical truths has echoes in the discussion of "logical" matrices in the Introduction to the second edition. Perhaps the new notion of identity was somehow only applicable in logical truths.

The errors in Appendix B were serious, and not easily remediable, but not because they were based on superficial confusions and slipshod notations. Rather, it seems, they represented the state of logic in 1924. The notions needed to understand the logical resources necessary for defining the structure of the natural numbers were not yet developed. It was the notions of model theory and the study of the power of deductive theories of arithmetic in the 1930s that would finally allow a better understanding of these issues. As well, it seem clear that Russell had new ideas about the hierarchy of types, ideas that were not fully worked out. The ideas of extensionality and the more thorough-going atomism from his own earlier "Philosophy of logical atomism", and Wittgenstein's *Tractatus*, had influenced Russell's ideas of the structure of the hierarchy of types.

# 7

# The reception of the second edition

## 7.1 "The foundations of mathematics"

Frank Ramsey's paper "The foundations of mathematics" (Ramsey, 1926) was presented to the London Mathematical Society on November 12, 1925, and so was likely written during the fall of 1924 after Ramsey had read the proofs for the second edition of *Principia Mathematica*. In the paper Ramsey makes repeated references to *PM*, citing the first volume of the new edition, and many points in the essay are best read as responses to the second edition. Indeed, the point for which the paper is most well known, the separation of the "mathematical" paradoxes from those due to semantics or "linguistics", and the proposal to solve the former within the simple theory of types, misses the fact that Ramsey also proposes a new ramified theory of types as part of the solution of the semantic paradoxes. This modification of the type theory of *PM* turns attention to the very issues that concerned Russell in Appendix B. It is actually Ramsey's theory that should be given the joking name of the "ramsified theory of types" and which is here named "Ramsey's ramified simple types" or "RS-types".[1]

Ramsey introduces the paper as follows.

> The object of this paper is to give a satisfactory account of the Foundations of Mathematics in accordance with the general method of Frege, Whitehead and Russell. Following these authorities, I hold that mathematics is part of logic and so belongs to what may be called the logical school as opposed to the formalist and intuitionist schools. I have therefore taken *Principia Mathematica* as a basis for discussion and amendment; and believe myself to have discovered how, by using the work of Mr Ludwig Wittgenstein, it can be rendered free from the serious objections which have caused its rejection by the majority of German authorities, who have deserted altogether its line of approach.
>
> *(Ramsey, 1926, p.1)*

[1] Quine refers to the ". . . 'simple' (or in Sheffer's quip, 'ramsified') theory of types" (1963, p.256) thus attributing the joke to Henry Sheffer who was in the Department of Philosophy at Harvard during Quine's time as a student.

Ramsey takes himself to be defending the "logical school" of foundations by defending the idea that mathematics consists of tautologies in a sense developed from Wittgenstein. In the *Tractatus* Wittgenstein does not himself take this route, but instead suggests that mathematics consists primarily of equations expressing the result of the iteration of operations. He holds that they are pseudo-propositions, and like the tautologies of logic, mathematical statements do not describe states of affairs in the world but rather show the logical form of the world (*TLP* 6.22). Ramsey, however, argues that with a suitable extension of the notion of a truth-functional tautology (very much like the yet to be formulated notion of a sentence true in all interpretations) it can be claimed that mathematical truth, like the logic of quantification, consists of tautologies. The first step of this argument is to describe the notion of a truth function in a semantic fashion, with contemporary truth tables for sentential connectives. Ramsey then suggests that a quantified proposition should be thought of as an infinite truth function (like conjunction) of all its instances. Truths of the logic of quantifiers are then seen to be tautologies in the extended sense, true in all of a likely infinite number of atomic cases. Mathematical propositions are also to be seen as tautologies in this extended sense, as true in all models or interpretations.

With this idea of tautology, inspired by Wittgenstein, Ramsey proposes to supply the account of the nature of logical truths missing from Russell's earlier accounts of logic as simply completely general truths. Russell expresses doubts about his own account of logical truths in the *Introduction to Mathematical Philosophy*, suggesting that perhaps Wittgenstein's notion of tautology is the right account.[2] Ramsey proposes to provide just such an account in the "Foundations" paper.

Ramsey's remark that the "German authorities" have "deserted" the logicist approach to foundations reflects the summary that anyone would naturally reach by reading the list of contributions to mathematical logic since the first edition, discussed in Chapter 3 above. Ramsey interprets the Germans (Hilbert and Weyl) as abandoning the view that mathematics can be founded on truths of logic in the way proposed in *Principia Mathematica*. Indeed, he sees himself as defending logicism as perhaps a comfortable, bourgeois, position by alluding to the "Bolshevik menace" of Weyl and Brouwer's intuitionism.[3]

Ramsey agrees with rejecting the axiom of reducibility, on the ground that it is not a logical truth, and because it can be circumvented in practice:

[2] See (*IMP*, p. 205, n.1): "The importance of 'tautology' for a definition of mathematics was pointed out to me by my former pupil Ludwig Wittgenstein, who was working on the problem. I do not know whether he has solved it, or even whether he is alive or dead." (Russell wrote *IMP* in prison during the summer of 1918.) In the Introduction to the *Tractatus* Russell simply says: " All the propositions of logic, he maintains, are tautologies, such, for example, as '*p* or not-*p*'." (*TLP*, p.*xvi*).

[3] "Only so can we preserve it [mathematics] from the Bolshevik menace of Brouwer and Weyl." (1926, p.56)

The principal mathematical methods which appear to require the Axiom of Reducibility are mathematical induction and Dedekindian section, the essential foundations of arithmetic and analysis respectively. Mr Russell has succeeded in dispensing with the axiom in the first case; but holds out no hope of a similar success in the second.

*(Ramsey, 1926, p.29)*

Ramsey here cites the proof in Appendix B, which he had read and praised in his letter to Russell earlier in the year. One of the goals of the "Foundations" paper is to show how, with the proper understanding of propositional functions, it will be seen that the axiom of reducibility is both not needed for the reduction of mathematics to logic, and is indeed not a logical truth. But Ramsey's description suggests that Russell's project in Appendix B is incomplete, successful in showing that "induction" can be justified without the axiom of reducibility, but not successful in so far as it leaves "Dededkindian section", and so analysis, not yet properly founded. Russell is not presented as proposing an intuitionist style "Bolshevik" revolution of mathematics, but rather an incomplete investigation of how to found all of classical mathematics on logic alone, and hence without the axiom of reducibility.

In the formulation of his own technical proposals for revising *PM*, Ramsey begins with the famous distinction between types of paradoxes.[4] The paradox of the set of all sets that are not members of themselves, and others, such as the Burali–Forti paradox of the order type of the class of all ordinals, can be solved by invoking the simple theory of types. Assertions about classes are to be "eliminated" with the no-classes theory, in favor of assertions about propositional functions, with these paradoxes solved by invoking a simple theory of types for functions. The solution to the paradoxes in the first edition of *PM* only uses that aspect of the ramified theory of types which bans functions from applying to functions of their own simple type. The other class of functions is attributed to "linguistic" problems, and Ramsey isolates an issue of "meaning" or what we would term semantics, in those paradoxes. The paradox of "heterological", the predicate that applies to predicates which do not apply to themselves, is here credited to Weyl rather than Grelling (as the paradox appears in Weyl's *Das Kontinuum* (1918), another item on the list discussed in Chapter 3 above.) When formulated carefully, Ramsey says, the paradox involves predicates which are logically individuals, but uses the notion of a predicate *meaning* a propositional function, a notion which will be distinguished by the orders of functions.

[4] Ramsey cites Peano (1906) who distinguished semantic from mathematical paradoxes, at Ramsey (1926, p.21 n.1), but not Chwistek (1924), who had made the same distinction. This further suggests that Ramsey did not read Chwistek's "Theory of constructive types", which he had said in the proof reading corrections was "not in Cambridge". There Chwistek argues that a "simplified" theory of types would solve the set theoretic paradoxes, but not all, including Richard's, and so would not be acceptable.

Ramsey's paper is best known for pointing out that the set theoretic paradoxes can be solved with only the simple theory of types. It is not as often noted that he also proposes a theory of orders for predicates, which accomplishes some of the goals of the full ramified theory of types.[5]

According to this new sort of ramification, a predicate will have a (simple) *type* which is determined by its arguments, individuals, or functions of individuals, or functions of functions of individuals, etc. A predicate will also have an *order* which is determined by the highest type of any function variable quantified in its definition. The order of a function will not be absolute, but only relative to a definition, and so a "linguistic" or semantic matter. With the understanding that it is a theory of types and orders for predicates, rather than functions, it is possible to express this theory of types with type symbols similar to those used for $r$-types and BMT types. Let us call these "ramsified" types, as Sheffer called them, "$rs$-types". A predicate expressing a function will have an an $rs$-type which, for the case of monadic functions, can be defined in terms of the notions of simple types of functions expressed as integers, individuals as 0, functions of individuals 1, functions of things of simple type $n$ as of type $n + 1$, and so on. Here is a definition of such an $rs$-type.

The $rs$-type of a predicate 'F' is a pair $(n, m)$, where $n$ is the simple type of the function expressed by 'F', and $m$ is the highest simple type of any predicate variable in the range of a quantifier in the definition of 'F'.

$n$ is the *type* of the function, and $m$ its *order*, relative to a definition of a predicate that expresses the function.

The liar paradox arises from a sentence "I am lying", which would be represented as at Ramsey (1926, p.48):

$$(\exists``p", p) : \text{ I am saying } ``p". \ ``p" \text{ means } p. \ \sim p.$$

The notion "means" will be limited by the order of the proposition $p$ which is asserted (or rather the order of the sentence "$p$" which expresses it). Consequently that very sentence "I am lying", so analyzed as '($\exists$ "$p$", $p$), I am saying "$p$" ...', is not in the range of the quantifier it contains, and so the contradiction is blocked.

[5] Quine (1936) and, more recently, Ramharter & Rieckh (2006) are exceptions, and note that this is an intermediate and ramified theory of types. In his discussion of the axiom of reducibility, Quine sees predicative functions as having only simple types, and the orders of other functions as a matter of their definitions. He proposes a new, extensional, system in which functions are in simple types, but there are expressions for both predicative and higher order functions. The axiom of reducibility is then replaced by a statement about expressions for functions, asserting that for every higher order expression there is a predicative expression which expresses the same function. Quine (1936) starts with a reference to Ramsey's article, and can be seen as a friendly reformulation of Ramsey's views about the nature of the ramification of the theory of types.

The notions of order and type for functions combined into what I have called "*rs*-types" look very much like the types of BMT attributed by Hazen to Appendix B. In both cases functions are seen to have as arguments all functions of a given simple type, but functions with a given type of argument can still be classified into different orders by the quantifiers used in their definitions. Ramsey sharply distinguishes the type of arguments of functions, which fall into "simple types", from the orders which are defined for the predicates as linguistic items which can sometimes differ yet "mean" the same function. BMT types, on the other hand, apply to both the arguments of functions and the functions themselves. Nevertheless, the basic notion of Ramsey's theory, that functions can be true of extensions directly, independently of how they are defined though still distinguished by the use of quantification in their definitions, is common to both revisions of the original ramified theory of types. It is easy to look at "*rs*-types" as having arisen from struggling with the issues that Ramsey had seen in Russell's Introduction and appendices to the second edition of *PM*. Ramsey presents these revisions as his proposed improvements to *Principia Mathematica*. Perhaps they should best be seen as his improvements to the *first* edition of *PM*, in order to accomplish the project of adopting extensionality and abandoning the axiom of reducibility. They are Ramsey's alternative to the second edition as Russell prepared it.

The purpose of Ramsey's distinction of kinds of paradoxes, which, as we have seen, amounts to a distinction of types for functions and orders for predicates, is in preparation for a discussion of the axiom of reducibility. The axiom of reducibility has two important roles in *PM*, according to Ramsey. One is in conjunction with the "no-classes" theory of classes. A sentence about a class $\hat{x}\phi x$, according to $*20$, is really about a predicative function co-extensive with $\phi$. Without the axiom of reducibility, it would seem, we cannot guarantee that there always is a predicative function of the required sort. As is pointed out in the argument for adopting the axiom of reducibility, in the theory of "Dedekindian" sections, we often use a higher order union of certain classes of rational numbers (cuts) to define a real number which is another Dedekind cut. Without the axiom of reducibility, we do not know that a predicative function exists to define that class. Ramsey responds that in a proper account of predicative functions, the existence of such functions will be known without requiring an axiom of reducibility. Propositional functions can be taken in two ways. Taken "subjectively", as defined by a predicate, as in *PM*, they are incomplete entities that provide a proposition when supplied with an argument. On the other hand there are also propositional functions "in extension", arbitrary mappings from objects to propositions which do not have to be expressible, or to have any rule relating objects to propositions. Thus there could be a propositional function $\phi$ such that "... $\phi$(Socrates) may be Queen Anne is dead, $\phi$(Plato) may be Einstein is a great man; $\phi\hat{x}$ being simply an arbitrary association of propositions

$\phi x$ to individuals $x$". (Ramsey, 1926, p.52). A statement about "all functions" will be a generalization, an infinite truth function in Ramsey's sense of functions taken in extension, whether or not there are actual concepts or predicates of which they are extensions. It is in this sense, in extension, that Ramsey thinks that there obviously exist the "choice functions" postulated by the "multiplicative axiom" Mult ax, mapping each subset of a given set onto some individual in that subset. For Ramsey, then, the axiom of choice is clearly true, as it merely asserts the existence of some relation, in extension, which provides the selection function. The suggestion that it should be provable if it is indeed a truth of logic, rather than taken as a primitive proposition, or axiom, leads Ramsey to a very Leibnizian view, that a proposition might be a tautology only on the basis of some infinite analysis in terms of atomic propositions, and so a truth of logic but not finitely provable, just as Leibniz suggested that seemingly contingent propositions might be found to be necessary upon an infinite analysis only available to an omniscient God.

Ramsey also gives a revised analysis of what a "predicative" function is. This account makes use of his notion of possibly infinite tautologies. Starting with the notion of atomic propositions involving objects and universals connected like links in a chain, which he takes from Wittgenstein, Ramsey describes all of the propositions that can be developed that are truth functions of some number of atomic propositions. A predicative function is one which results from replacing some name in that compound proposition by a variable. Predicative functions are thus not functions in extension, but are also not necessarily finitely expressible.

Ramsey argues that with this notion of predicative function, the axiom of reducibility is not a logical truth. Here he points out the role of the axiom in connection with the definition of identity discussed above. At $*14{\cdot}01$, *PM* defines identity so that $x$ and $y$ are identical if they share all predicative functions. Recall that the axiom of reducibility guarantees that if two objects differ on a property of some higher order they will differ on predicative properties, thus justifying the definition by showing that identity coincides with sharing all properties, i.e., indiscernibility. Ramsey says there is no reason to believe this to be a necessary truth about identical individuals. Consider the version in terms of classes. Objects differing with regard to a predicative function will be members of different classes. Why should we think that any two objects are distinguished by membership in different classes?

Ramsey thinks that while the axiom of reducibility is not a logical truth it is, fortunately, not needed to reduce mathematics to logic. One function of the axiom was to guarantee the existence of propositional functions with the right extensions to go proxy for classes in the no-classes theory. Ramsey is able to simply assert that every function will have a class as its extension. The definitions of some functions may make them appear to be of a higher order, but definition is a feature of language, not of the extensions themselves, and so there is no problem of comparing extensions of functions of different orders.

After giving his new definition of "predicative function" Ramsey says, echoing Russell's formula, that:

> it is clear that a function only occurs in a predicative function through its values.
>
> *(Ramsey, 1926, p.40)*

For Ramsey, this is because a predicative function is derived from a possibly infinite truth function of propositions, and the function must take an individual as argument to produce one of those propositions. That proposition is the "value" of the function. Later Ramsey suggests that he is explaining what Russell must mean by this slogan discussed above in Chapter 5.

> It is, I think, predicative functions which Mr Russell in the Introduction to the second edition of *Principia* tries to describe as functions into which functions enter only through their values. But this is clearly an insufficient description, because $\phi\hat{x}$ enters into $f(\phi\hat{x}) =$ 'I believe $\phi a$' through its values $\phi a$, this is certainly not a function of the kind meant, for it is not extensional...
>
> *(Ramsey, 1926, p.55)*

This is just the conclusion that was reached in Chapter 6 above, where it was suggested that in Russell's mind the slogan of functions "occurring through their values" had something to do with extensionality. But more than this is needed to reach a proper account of extensionality.

The second use that *PM* has for the axiom of reducibility, namely in the logic of identity, is replaced with a more dramatic move. Ramsey cites:

> ... Wittgenstein's discovery that the sign of identity is not a necessary constituent of logical notation, but can be replaced by the convention that different signs must have different meanings.
>
> *(Ramsey, 1931, p.31)*

(The reference is to *TLP* 5.53.) This must have been the issue that Ramsey and Russell discussed when they met in February of 1924, at which time Ramsey noted that Russell was "good against W. on identity", but which Ramsey then reported to Wittgenstein that Russell did not really understand.[6] Ramsey reported that there was "nothing new on identity" in the new edition. Ramsey's view is that the notion of identity, which is used in the definition of identity in *PM*, is inadequate. We can understand the possibility that two objects, i.e. not numerically the same, can have all the same predicative properties. The relation between these objects, however, is not predicative, but rather "made up of two predicative functions":

[6] In Russell's personal copy of Ramsey (1931), in the Bertrand Russell Archives, the word 'discovery' is underlined, and a question mark '?' sits in the margin next to it.

(1) For $x \neq y$

'$x = y$' may be taken to be $(\exists\phi).\ \phi x\ .\sim\phi x$ :

$(\exists\phi).\ \phi y\ .\sim\phi y$, i.e. a contradiction.

(2) For $x = y$

'$x = y$' may be taken to be $(\phi) :.\ \phi x\ .\vee .\sim\phi x$ :

$\phi y\ .\vee .\sim\phi y$, i.e. a tautology.

But '$x = y$' is not itself predicative. (Ramsey, 1926, pp.51–2)

Thus if $x \neq y$, this is a function true of no pair of individuals, if $x = y$ it is (tautologically) true of all individuals, but the relation which is expressed is not definable with a single expression for a "predicative" relation.

In general, questions about numbers of things, including the axiom of infinity, are neither tautologies nor contingent truths expressed with predicative functions, but rather issues about the logical form of the world, to be "shown" rather than "said", following Wittgenstein's suggestions in the *Tractatus*. Whatever this view may otherwise involve, it does have the consequence of making an axiom of reducibility no longer necessary for the revised logic which, Ramsey claims, will provide the proper, logicist, foundation for mathematics.

From the brief summary above it should be clear that Ramsey's essay was primarily a response to the second edition of *Principia Mathematica*. Ramsey's main concern is how to avoid the need for the axiom of reducibility in a properly formulated theory of types that would be based on a proper understanding of predicative functions. While Ramsey does argue for the use of the simple theory of types, that is not the whole of his proposal. There is an account of the order of functions, described above as "$rs$-types", which in fact is similar to the types of BMT that seem to be behind Appendix B of the second edition. The account of predicative functions that Ramsey gives in making sense of the notion of functions "occurring through their values", as he himself says, looks very much like Russell's own new account of the hierarchy of functions in the Introduction to the second edition. That Ramsey had just completed a review of the literature on mathematical logic is reflected in his references to Hilbert, Weyl, and Chwistek in the paper, and even in little details like the attribution of the Grelling paradox to Weyl's paper. Ramsey's concern with identity, which is obviously inspired by Wittgenstein's views, also echoes the issues about identity and the axiom of reducibility that underly the argument in Appendix B. Thus, while marking Ramsey's own original line of thinking, and also inspired primarily by ideas of Wittgenstein, as he himself says, the paper is still clearly the work of someone struggling with the issues addressed by the second edition of *Principia Mathematica*.

Russell reviewed the collection of Ramsey's papers twice (Russell (1931) and (1932)). In the *Mind* review (1931), he finds three main objections to *Principia*;

the first, “supposing that all classes and relations in extension are definable by finite propositional functions”, the second, of “a failure to distinguish two kinds among the contradictions, of which only one requires the theory of types, which can accordingly be much simplified”. The third objection that Russell considers is based on the new “the treatment of identity”. Russell immediately admits to the first two defects, but does not say any more by way of details of how this would alter *PM*, or what would be done differently. The second review also describes Ramsey’s notion of extensional functions, but expresses some qualms.

> If a valid objection exists – as to which I feel uncertain – it must be derived from enquiry into the meaning of “correlation”. A correlation, interpreted in a purely extensional manner, means a collection of ordered pairs. Now such a collection exists if somebody collects it, or if something logical or empirical brings it about. But, if not, in what sense is there such a collection?
>
> *(Russell, 1931, p.116)*

The grip of the intuition that infinite correlations, or functions, such as required by the axiom of choice, must be definable or determined by properties of the related objects is still strong on Russell.[7] Similarly, the notion of an extension as the extension of some propositional function is hard for Russell to shake off, however much we may see it as a natural part of the new “extensional” view of logic.

In both reviews Russell criticizes Ramsey’s view that identity is independent of indiscernibility, arguing that the very hypothesis that *two* objects might share all their properties somehow presupposes the notion of identity that it attacks. Russell remains steadfast in his rejection of both Wittgenstein’s and Ramsey’s views on identity, however much his own presentation of extensionality and arguments in Appendix B seem to require a rethinking of the account in the first edition of *PM* at $*14$.

Russell says nothing about the two kinds of paradox or the simple versus ramified theory of types in the second review, and so it is impossible to see what he took to be the substance of accepting Ramsey’s distinction, and, more importantly for our concerns, whether he saw Ramsey’s criticisms as specific to the new views in the second edition.

## 7.2 Carnap’s *Abriss der Logistik*

Carnap’s plan to write a textbook or “summary” of symbolic logic, his *Abriss der Arithmetik* (1929), began soon after he received the notes on *PM* from Russell in

[7] It continues into *MPD* in 1959 (pp.92–3) where Russell repeats the worry about the axiom of choice applying to an infinite set of pairs of socks.

1922 which are transcribed in the next chapter.[8] The *Abriss* used the notation of *Principia Mathematica* with an eye to the application of the theory of relations in philosophical projects, and it was so used in the Vienna Circle. The logic of the *Abriss* is the simple theory of types, expounded formally for the first time, although, as we have seen, the idea had been expressed in Chwistek (1924) and Ramsey (1926). Carnap certainly is the first to describe the simple theory of types in any detail. His presentation includes a system of notation for types. Types of individuals, classes of individuals, classes of them, etc. are designated by integers; 0, 1, 2, etc. A relation between entities of type $n$ and type $m$ will be of type $(nm)$. Classes of such relations will be of type $((nm))$, and so on. The details of the theory of relations are presented in *PM* notation with relation constants added for the applications to the "Konstitutionssystems" construction (or, more literally, "*constitution*") systems of space and time with a reference to the *Der Logische Aufbau der Welt* (1928).

Carnap had adopted the simple theory of types, and extensionality, prior to his reading of the second edition of *Principia*. He does refer to the second edition, however, and to Ramsey.

> According to a more recent conception (Russell, *PM* $\mathrm{I}^2$, S.XIV; Ramsey [Found.] 275ff.), the ramified theory of types was constructed in the belief that certain antinomies of a special kind could not otherwise be avoided. ... Closer inspection seems to show, however, that these antinomies are not of a logical, but of a semantic kind: i.e., they only occur because of a defect in ordinary language.... Then again, the problem of the antinomies has not been solved completely. It is tied up with the problem of extensionality.
>
> *(Carnap, 1929, p.21).*[9]

Carnap, using the German term "verzweigte" (branching), here introduces a name for the original "ramified" theory of types which distinguishes it from the new "simple" theory. Chwistek had referred to such a "simplified" theory of types (Chwistek, 1924, p.12) but dismissed it as not up to the demand on logic to be able to solve what were in fact logical paradoxes. Ramsey had in fact also described the theory of types without distinguishing orders of functions. His simple theory was motivated as a proper account of how complex propositional functions depend on elementary propositions, however, and thus as a reformation of Whitehead and Russell's own account. In Carnap's remark we find both the first proposal of the simple theory of types as an alternative to the ramified theory of types, and

[8] See Reck (2004) for an account of Carnap's correspondence with Russell, and of the origins and influence of the *Abriss* itself.

[9] Translation from Reck (2004, p.165). The reference to *PM*, "S.XIV", is to page *xiv* in the Introduction to the second edition.

the origin of the term “ramified” for the latter theory.[10] It is notable that while Carnap shares Ramsey’s view that the antinomies have to do with semantics, at this point he viewed them as due to a “defect” in natural language, not as to be better solved in a separate, improved, theory of semantics. More relevant to the reception of the second edition of *PM*, however, is that Carnap clearly relates the distinction between the ramified and simple theory of types with “the problem of extensionality”. The view that the ramification of the theory of types would be obviated by adopting extensionality seems to have been, unintentionaly, the source of the popular misinterpretation of the second edition of *PM* as proposing just that combination.

## 7.3 Hilbert and Ackermann

Hilbert and Ackermann’s *Grundzüge der Theoretischen Logik*, though first published in 1928, was based on Hilbert’s lectures between 1917 and 1922, and makes no explicit reference to the second edition of *PM*. Indeed, the second edition, which had appeared between 1925 and 1927, is not even mentioned in the bibliography. Still, the book concludes with a section on the theory of types (pages 98 to 115) and two pages devoted to “Concluding remarks on theory of types”.[11] After a presentation of paradoxes of the function that does not apply to itself, the Liar, and of the predicate that “is not defined at least once during the twentieth century”, and of how they would be resolved in the ramified theory of types, the authors go on to present the need for the axiom of reducibility with examples of Cantor’s theorem and the definition of the least upper bound of a set of real numbers. The concluding pages state that if “predicates are only distinguished when their associated sets are different” and consequently the theory of types is given a set theoretic interpretation, the axiom of reducibility will be true, and the first paradox will be solved by observing the simple theory of types. The expressions about assertions needed to formulate the Liar, and the notion of definition needed for the third paradox, will not be expressible, as they do not have a “logical character”.[12] These sections (5 to 9) on the theory of types were deleted from the second edition of Hilbert & Ackermann in 1938. They are replaced by two new sections, “ 5. The predicated calculus of order $\omega$” and “ 6. Applications of the calculus of order $\omega$” which present the simple theory of types and the construction of real numbers suggested in the conclusion to the first edition. Whether or not Hilbert and Ackermann deleted the

[10] Remember that Ramsey had not read Chwistek (1924), as can be seen from his letter to Russell about not being able to find it in Cambridge when checking the citations. See the discussion of Chwistek above in Section 3.2.4.

[11] “Schlussbermerkungen zum Stufenkalkül” (Hilbert & Ackermann, 1928, pp.114–15).

[12] “Keine von diesen Behauptungen stellt eine logicische Identität dar”. (Hilbert & Ackermann, 1928, p.115).

discussion of the theory of types because they believed that Russell had abandoned the ramified theory of types for the simple theory with extensionality, this discussion may have led others to believe that is what happened. It is interesting to note that Hilbert and Ackermann, already in their 1928 book, express the view that if definitions of functions are interpreted as picking out pre-existing classes, then the axiom of reducibility should be true already, and not be needed as an additional axiom. This agrees with the criticisms of the vicious circle principle on just this ground in Ramsey (1926) and Gödel (1944). The interpretation of extensionality as asserting that co-extensive functions simply refer to the same set, or extension seems common to many in the 1920s.

## 7.4 Other "German authorities"

Ramsey's visit to Vienna from May through October of 1924, during which he visited Wittgenstein and worked on "Foundations of mathematics" was also the opportunity to learn about current investigations of logic in Germany. From the names on the list he sent to Russell, including Hilbert, Weyl, Behmann, Bernays and "Schönwinkel", it is clear that Ramsey somehow learned about the study of logic and foundations in Göttingen. Hans Hahn led a seminar in Vienna on *Principia Mathematica* during the 1924–1925 academic year, but most likely Ramsey had returned to England before it began.[13] Still, Ramsey was correct in reporting that the "German authorities" had abandoned the logicist program of *PM*, even though it might be melodramatic for him to describe them as having "deserted altogether its line of approach" (Ramsey, 1926).

Abraham (then "Adolf") Fraenkel's *Einleitung in die Mengenlehre* (1928) devotes a chapter to responses to the crisis caused by the antinomies of set theory, with Brouwer's intutionism and Russell's logicism as the main subjects.[14] Fraenkel presents the now common view of axiomatic formulation of set theory as the basis for foundational theory in response to the paradoxes. He views the axiomatic method as a cautious approach to the paradoxes by which mathematicians proceed carefully, adding individual axioms as needed for the construction of mathematics. Fraenkel's rejection of Russell's "logicist" approach is subtle. First Fraenkel presents Russell as trying to steer between the Scylla of the antinomies and the Charybdis of the sacrifice of analysis that is required by the restrictions of ramified type theory. Fraenkel reports the proof in Appendix B as showing that the theory of the natural numbers ("complete induction") can be saved within the

[13] See Stadler (2001, p.648).

[14] The chapter is titled "Erschütterungen der Grundlagen und ihre Folgen" (The shaking of the foundations and its consequences) and contains what may be the first use of the term "Logicism" for the school. (Fraenkel, 1928, p.263)

theory of types. Adopting the axiom of reducibility to obtain further results is then presented as a solution which abandons the logicist strategy, not because it is not a principle of logic, but simply because it is an "axiom". Fraenkel compares it with Hilbert's "completeness axioms", in that it postulates that the predicative functions already generate all the classes which can be defined by higher level, impredicative definitions (Fraenkel, 1928, p.260).[15] This seems to follow Hilbert's own views about the axiom of reducibility and logicism.[16] The axiom of reducibility is thus to be judged by comparison with other axioms of set theory, and as nothing particularly justifies its truth beyond the need for such an axiom to recover analysis, alternative axioms are preferable. Fraenkel adds that the simple theory of types would then suffice for mathematical purposes, citing Ramsey as showing that the antinomies which require the ramification will have solutions in a theory of symbolism.

Fraenkel's views thus seem to express a consensus of views in 1928, and probably represent the views that Ramsey reports from four years earlier as representing the "German authorities". That these views about logicism, in particular about the theory of types, soon became common in Germany, is revealed by careful attention to Gödel's famous paper (1931) which cites *Principia Mathematica* in its title, and is generally taken to mark the rejection of the early foundational projects of logicism and Hilbert's program. The system of arithmetic in which the theorems of "On formally undecidable propositions of *Principia Mathematica* and related systems I" are proved is formulated in an extensional simple theory of types. All of the struggles that Russell went through to adapt the theory of types to extensionality are cut through with two axioms in a simple theory of types. After axioms of groups I (Peano's axioms), II (four axiom schemas for propositional logic), and III (two schemas for the logic of the universal quantifier $\Pi$) we have:

IV. Every formula that results from the schema

1. $(Eu)(v\Pi(u(v) \equiv a))$
when for $v$ we substitute any variable of type $n$, for $u$ one of type $n+1$, and for $a$ any formula that does not contain $u$ free. This axiom plays the role of the axiom of reducibility (the comprehension axiom of set theory).

V. Every formula that results from

1. $x_1\Pi(x_2(x_1) \equiv y_2(x_1)) \supset x_2 = y_2$
by type elevation (as well as this formula itself). This axiom states that a class is completely determined by its elements.

[15] He does not rely on, but does cite Wittgenstein's objections that the axiom is not a logical truth from the *Tractatus*, a work Fraenkel describes as "profound but obscure" ("tiefschüfende aber dunkle") (Fraenkel, 1928, p.262).

[16] See Sieg (1999) for Hilbert's views on the axiom of reducibility.

The final abandonment of type theory as the basis for logic, in favor of first order logic, was only completed much later, by 1948 or so, as described above.

The effect of the appearance of the second edition of *Principia Mathematica*, drawing attention to the issues of ramified versus simple type theory in its discussion of both the axiom of reducibility and extensionality, seems to have solidified this growing consensus about logic among the "German authorities". The reception was in some sense a "desertion" of the line of investigation represented by *PM*, but not a wholesale rejection of the approach. The positive results of *Principia Mathematica* were accepted, and then work on both foundations of mathematics and logic advanced, rather than changing direction or backing up to start over again.

## 7.5 Reviews of the second edition

The reviews of the second edition were directed primarily towards evaluating the general logicist project of *Principia Mathematica* and hardly notice the innovations in the new introduction and appendices. In addition to numerous brief notices, there were short reviews by Ramsey (twice), B. A. Bernstein, Rudolf Carnap, Alonzo Church, C. I. Lewis, C. H. Langford and others.[17] Harry T. Costello (1928), for example, in the *Journal of Philosophy*, writes in defense of the project of *PM*, beginning, however, with a defensive remark that may be the source of talk of symbolic logic as devoted to "chicken scratches":

> When this great treatise first came out, with its pages that look something like hen-tracks on the barnyard snow of a winter morning, there were those who accused the authors of trying to conceal the secrets of the universe under an incomprehensible hieroglyphic cipher, and who said that the universe itself was more comprehensible than was this book about it.
> *(Costello, 1928, p.438)*

Costello does notice the changes made in the second edition. It is said to "... investigate an alternative to the 'axiom of reducibility' ... " There is also a discussion of the Sheffer stroke and Nicod's axiom, as well as the abandonment of real variables ("In fact variables are whatever does not vary") but the review is devoted primarily to a discussion of the place of symbolic logic in philosophy. The discussion is sophisticated, relating the role of atomic sentences to logical atomism and the dispute with C. I. Lewis over strict implication and similar points. Still, no great change in direction in the second edition is noted.

[17] In a list in the Bertrand Russell Archives, Harry Ruja reports these additional short notices of the second edition of *PM*: R. B. Braithwaite, *Nat. & Athenaeum* **37**, 1925, 299–300; W. Dubislav, *Arch. Rectsphil.*, **26**, 1933, 252; H. Hahn, *Mh. Math. Physik*, 1932, *Lit.-Berichte*, 41–2; H. Neubert, *Grundwissenschaft* **12**, 1933, 237; R. Poirier, *Rev. phil. France, Étrang.* **103**, 1927, 300–2; R. Potinie, *Scientia* **53**, 1933, 417–18; G. Rabeau, *Rev. Sci. phil. théol.* **16**, 1927, 375–7; A. Schmidt, *Zentbl. Math. Grenzgeb.* **4**, 1932, 1; H. Scholz, *Dt. Lit. Zeit.* **53**, 1932, cols. 2488–90; and H. Sheffer, *Unterrichtsbl. Math. Naturwiss.* **40**, 1934, 302.

Frank Ramsey's reviews in *Nature* (1925a) and *Mind* (1925b) are brief, but point at the key issues for interpreting the second edition. In *Nature*, Ramsey concludes that despite the developments due to Sheffer and Nicod: "... the main interest of this new edition of *Principia Mathematica* lies in its treatment of the axiom of reducibility". (Ramsey, 1925a, p.128) Ramsey reads the lesson of Appendix B, which he reports to be a successful "new treatment" of induction, as showing that a new axiom must be found to replace the axiom of reducibility, lest one must follow Weyl and "reject the theory of real numbers as groundless". The discussion of types in the Introduction is noted as a "much simplified exposition of the theory of types", while the new theory to replace the axiom of reducibility is to be replaced by a "new assumption" based on ideas of Wittgenstein. In *Mind* he is more explicit. He says that the assumption is: "... that any function of functions $F(\phi!\hat{z})$ can be constructed from the values of $\phi!z$ by truth-operations and generalisation". (Ramsey, 1925b, p.506). Of course Ramsey's disagreements with Russell over identity would have made him unaware of the problems with interpreting the new theory in conjunction with the old theory of $*11$. Ramsey does notice the defense of extensionality in Appendix C, but offers an objection, arguing that it cannot be consistently maintained that *all* propositions are truth-functions of atomic propositions, for what are we to make of the very analysis in terms of "resemblance" that is given?

For instance, what account could they give of the proposition "the fact that Socrates is Greek and that fact that Socrates is wise have particular-resemblance"? How is this to be analysed as a truth-function of atomic propositions?

*(Ramsey, 1925b, p.507)*

Carnap also focusses on the abandonment of the axiom of reducibility in his reviews, repeating the formula about functions only appearing through their values, and then pointing out the consequent extensionality.[18] He describes extensionality as the view that co-extensive functions are identical, reminding us that for Russell that is identified as sharing all properties. Carnap also does not notice any problem for the introduction of extensionality into the ramified theory of types, and mentions the proof of Appendix B as successful in proving induction without the axiom of reducibility.

C. I. Lewis, understandably, given his views in the *Survey of Symbolic Logic* (1918), criticises Russell's use of "implication" and "incompatibility" to read '$\supset$' and the Sheffer stroke, but also pays attention to the abandonment of the axiom of reducibility. He also quotes Russell's formula, but partially: "a function can only

[18] "... alle Funktionen von Sätzen Wahrheitsfunktionen sind, daß eine Funktion in einem Satz nur durch ihre *Werte* auftreten kann, daß alle Funktionen von Funktionen extensional sind." (Carnap, 1932, p.73).

enter into a proposition though its values". (Lewis, 1928, p.204), but ventures his own assessment of the issues.

> The real purpose both of the axiom of reducibility and of the proposed substitute is to be able to treat all the propositional functions which figure in mathematics in a manner which would be valid without further assumption if it were established that all such functions can be analyzed into purely logical relations of their ultimate constituents.
>
> *(Lewis, 1928, p.204)*

Lewis, however, sees this as a mistake based on ignoring the "form" of mathematical propositions through seeing them as simply truth-functional combinations of those constituents.

C. H. Langford (1928) goes into more detail about the presentation of the theory of types in the Introduction to the second edition, and Russell's discussion of matrices, the apparent ban on higher order matrices, and of the derivation of second order functions $\phi_2$.

> Now it appears that the chief novelty of the new edition is the following. It is held that every logical proposition can be derived from some matrix by generalization, that matrix being such that it does not involve any constituents which take values themselves involving generalized constituents.
>
> *(Langford, 1928, p.515)*

He interprets Russell's argument as showing that from a proposition such as $\phi x \,.\vee .\sim \phi x$ where $\phi$ is predicative, one can derive each instance for higher order functions. The completely general version $(f)\ f .\vee .\sim f$ is then concluded by interpreting the universal quantifier $(f)$ as an infinite conjunction of these cases. Langford thus sees the discussion of quantifiers as infinite conjunctions as Russell's way of presenting what we would now give as a meta-logical proof by induction on formulas. One instance of this higher order law of the excluded middle is presented as a proof beginning with a generalized form of initial form

$$(\phi)(x)\ [\phi x \,.\vee .\sim \phi x]$$

and then leading via a series of implications and equivalences:

$$\begin{aligned}
&(\phi)(y)[(\exists x)\phi x \,.\vee .\sim \phi y]\\
&(\phi)[(\exists x)\phi x \,.\vee .(y) \sim \phi y]\\
&(\phi)(\exists\psi)[(\exists x)\phi x \,.\vee .(y) \sim \psi y]\\
&(\phi)(\exists x)\phi x \,.\vee .(\exists\psi)(y) \sim \psi y\\
&(\phi)(\exists x)\phi x \,.\vee .\ \sim (\psi)(\exists y)\psi y
\end{aligned}$$

to the higher order instance:

$$(\phi)(\exists x)\phi x \;.\vee.\; \sim (\phi)(\exists x)\phi x.$$ [19]

This reveals careful attention to those passages in the Introduction to the second edition which is unique among the discussions. Langford's interpretation is not completely accurate, however, as it leads him to accuse Russell of inferring $(\exists y):(x).f(x, y)$ from $(x).f(x, x)$ in some of these cases, though the inference is transparently invalid. (Langford, 1928, p.517).[20]

The other reviewers pay more attention to different issues. In one of his first publications, Alonzo Church reviewed the second edition, commenting on the project of *PM* in both editions. He proposes simply adopting a postulate of "abstraction" to provide a set as object where functions are co-extensive, rather than going by way of the "no-classes" theory as Whitehead and Russell did. He does remark on the proposals in the new edition, however, repeating Russell's terminology, reporting as well that in the new edition a function can only appear in a proposition "through its values" and that "functions of propositions are always truth functions", and simply asserts that this implies that "equivalent propositional functions are identical" (Church, 1928, p.239). Church may here reveal a source of some of his later thinking about syntactic interpretations of intensional contexts by criticising the view in Appendix C.

The statement that $A$ asserts $p$ is correctly analyzed not as meaning that $A$ utters certain sounds (the analysis proposed by Whitehead and Russell) but that $A$ utters sounds which have a certain content of meaning, and it is this content of meaning which constitutes the proposition $p$, in the usual sense of the word proposition. The proposal under discussion can mean only that the word proposition is to be used in some quite different sense, in which case the *Principia Mathematica* would fail to give any account of propositions in the usual sense of the word.

*(Church, 1928, p.239)*

B. A. Bernstein (1926) mentions the additions of the new edition but then retreats to a more general criticism of the logicist approach to logic as a fundamental discipline. Bernstein opposes this to a position that he attributes to Peano, Pieri, Hilbert, Veblen, and Huntington by which propositional logic is rather an algebra in need of a meta-linguistic interpretation to be seen to be about assertions, truth, and logical consequence.[21] He quotes Whitehead and Russell (*PM*, p.91) as denying

[19] Langford presents the whole argument in proper *PM* notation rather than this modernized, summary, version.
[20] This would be, presumably in the case, to be added into the "infinite conjunction" of $(\exists y):(x).f(x, y) \;.\vee.\; \sim (\exists y):(x).f(x, y)$.
[21] Huntington (1933) presents the propositional logic of *PM* in just such as way, as a Boolean algebra. Huntington presents B. A. Bernstein's contributions to this project, also mentioning Sheffer as contributing. The proof of $*1{\cdot}05$ from the others in Bernays (1926) is also cited and was clearly well known by 1933.

that independence or even consistency results are possible for foundational work.[22] These views, especially the criticism of *PM* for not distinguishing the language of the system from the language which talks about it thus have a continuous history from the "algebraic" approach to logic that preceeded the work of Frege and Russell, to the later criticisms by Quine and the earlier Hilbert school, for insufficient attention to meta-mathematical issues, and even simple use–mention distinctions.

The response to the second edition in these reviews was thus generally just to report the new proposals. It was observed that the Sheffer stroke and Nicod axiom were not really fundamental changes, but that instead the real interest was in replacing the axiom of reducibility with the new idea of extensionality derived from Wittgenstein. Ramsey presented his own alternative view, going his own way with Wittgenstein's ideas in his "The foundations of mathematics". No one caught the error in Appendix B, although the proof of induction was noted, and seen as showing the need for some replacement for the axiom of reducibility, rather than as a preliminary attempt to develop a "constructive" mathematics following Weyl. The argument of Appendix C was criticized by Lewis, Langford, and Church who maintained the need for an intensional logic. Carnap, however, seems to have approved of the extensional turn. It seems that the second edition of *Principia Mathematica* was read, and appreciated, although the technical problems were not found until Gödel (1944). The developments in the second edition of *Principia Mathematica* were not ignored, as Monk suggests they were by saying that the new edition made "no significant technical advances", even if it did have only a small influence on the prevalent views about logicism that had been formed in reaction to the first edition.

The subsequent reception of the second edition of *PM* has been described above. After these initial reviews there was no substantial discussion of the new material in the second edition until Gödel's (1944) and then the papers by Myhill, Landini, and Hazen, years later. The time is now right for a reassessment of the second edition. New archival work has shown that *Principia Mathematica* was important in discussions in Hilbert's school in the 1920s, so we can no longer dismiss the second edition as out of touch with the field, or as obsolete from its publication.[23] The material printed here, including the "Amended list of propositions" and "Hierarchy of propositions and functions" manuscripts show that the "improvements" that Russell intended were deliberate, although still unclear even in the manuscript material. It seems certain that a major revision of the theory of types was being

[22] This is the footnote to *PoM* §17 at (*PM*, p.91) discussed above, in Section 3.2.2 in connection with Bernay's proof of the independence of the axioms for propositional logic in *PM*.

[23] Such as Sieg (1999) and Mancosu (2003) relating *PM* to the Hilbert school.

presented in the Introduction to the second edition. The manuscript material transcribed below, and the letters cited above, together present all of the material in the Bertrand Russell Archives which is directly related to the second edition of *Principia Mathematica*. The timely reassessment of that work now has materials with which to work.

# 8

# The list of definitions for Carnap

## 8.1 Editorial note

This manuscript is item 111-01-01 A in the Carnap archive at the University of Pittsburgh, and is published here by their permission. The manuscript consists of 35 leaves foliated by Russell from *1* to *35*. "Russell" is written by Carnap in the upper left corner of the first leaf. Russell used underlining for emphasizing what was set in print as capitalized section heads and italics. These underlined words are set in italics below. Some numbers are not in numerical order, but rather follow the order in which they are discussed in the "summary" of a given chapter. Russell adds comments and picks out theorems in a manner that reveals that he was also reading through the symbolic portion of each chapter as he composed the list. The selection of definitions is partial, but self contained, with no evidence of Russell slipping, and using symbols which were not defined earlier in the list.

Whitehead and Russell generally define a notion as a relation, and then present a more familiar form as a theorem, e.g. D‘$R$. He says "It is easier, though less correct, to give the latter forms as Dfs. This plan is adopted forthwith." (p.*5*)[1] Notice that ∗56, the last number of the paperback reprint of *PM* (1962), is completed on page *7*. Carnap's *Abriss* uses this notation and follows definitions into Volume II.

## 8.2 The list

*1*

*Principia Mathematica*

∗1. Primitive ideas: $\sim p$ (not-$p$), $p \vee q$ ($p$ or $q$)
[Sheffer (*Trans. Am. Math. Soc.* **XIV** pp. 484–488) showed that these could

[1] This is a striking example of the view that the distinction between definitions and axioms is arbitrary which Quine later discusses in his criticisms of Carnap in "Two dogmas of empiricism" (1951b).

be replaced by $p|q$ (not-$p$ or not-$q$), with the Dfs $\sim p = p|p$ Df $p \vee q = (p|p)|(q|q)$ Df ]

Df $p \supset q . = \sim p \vee q$ Df

Primitive Props (formal):

$\vdash : p \vee p . \supset . p$ $\quad \vdash : q . \supset . p \vee q$

$\vdash : p \vee q . \supset . q \vee p$ $\quad \vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$

$\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r$

[Nicod (*Proc. Camb. Phil. Soc.* **XIX**, 1, Jan 1917) has reduced these to one, namely: putting $P = p|(q|r)$, $\pi = t|(t|t)$, $Q = (s|q)|\sim(p|s)$, the one formal Pp is $P|(\pi|Q)$.]

$*3{\cdot}01$ $p . q . = . \sim(\sim p \vee \sim q)$ Df

$*4{\cdot}01$ $p \equiv q . = . p \supset q . q \supset p$ Df

*2*

$*10$. Primitive Idea: $(x). \phi x . = .$ All values of $\phi x$ are true.

$\cdot 01$ $(\exists x). \phi x . = \sim\{(x). \sim \phi x\}$ Df In these forms, $x$ is

$\cdot 02$ $\phi x \supset_x \psi x . = . (x). \phi x \supset \psi x$ Df called an "apparent variable."

$\cdot 1$ $\vdash : (x). \phi x . \supset . \phi y$ Pp

$\cdot 11$ If $\phi y$ is true whatever argument $y$ may be, $(x). \phi x$ is true. Pp

$\cdot 12$ $\vdash : (x). p \vee \phi x . \supset : p \vee (x). \phi x$

$*11{\cdot}01$ $(x, y). \phi(x, y) . = : (x) : (y). \phi(x, y)$ Df etc.

$*12$. Primitive idea: $f!x$ stands for "a predicative function of $x$."

Primitive Props: $\vdash : (\exists f) : \phi x . \equiv_x . f!x$ Pp

$\vdash : (\exists f) : \phi(x, y). \equiv_{x,y} . f!(x, y)$ Pp

[A predicative function can be an apparent variable; a general function cannot]

$*13{\cdot}01$ $x = y . = : (\phi) : \phi!x . \supset . \phi!y$ Df

$*14{\cdot}01$ $[(\iota x)(\phi x)]. \psi(\iota x)(\phi x) . = : \exists b : \phi x . \equiv_x . x = b : \psi b$ Df

[This defines: the $x$ such that $\phi x$ is true]

$*14{\cdot}02$ $E!(\iota x)(\phi x). = : (\exists b) : \phi x. \equiv_x . x = b$ Df [This defines *existence*, i.e. "the $x$ such that $\phi x$ is true exists. This Df shows in what sense existence is not a predicate. Cf Kant.]

*Convention*. The symbol "$[(\iota x)(\phi x)]$" is omitted when the $\phi x$ to which it would apply is the smallest prop. enclosed in dots or brackets which occurs.

*3*

$*20{\cdot}01$ $f\{\hat{z}(\psi z)\}. = : (\exists \phi) : \phi!x . \equiv_x . \psi x : f(\phi!\hat{z})$ Df

$\cdot 02$ $x \in (\phi!\hat{z}) . = . \phi!x$ Df

$\cdot 03$ $\mathrm{Cls} = \hat{\alpha}\{(\exists \phi). \alpha = \hat{z}(\phi z)\}$ Df

These Dfs found the theory of classes. Classes are denoted by Greek letters for short; but a rigid symbolism would require that a class should always be denoted by $\hat{z}(\phi z)$ or $\hat{z}(\phi ! z)$. When a class is to be an apparent variable, i.e. to occur in the form $(\alpha).f\alpha$ or $(\exists\alpha).f\alpha$, it must always be replaceable by $\hat{z}(\phi ! z)$, not $\hat{z}(\phi z)$.

$\cdot 06 \quad x \sim\in \alpha . = . \sim(x \in \alpha) \quad \mathrm{Df}$

$*21 \cdot 01 \quad f\{\hat{x}\hat{y}\ \psi(x,y)\} . = : (\exists\phi) : \phi!(x,y) . \equiv_{x,y} . \psi(x,y) : f(\phi!(\hat{u},\hat{v}))\} \quad \mathrm{Df}$

$\cdot 02 \quad a\{\phi!(\hat{x}\hat{y})\}b . = . \phi!(a,b) \quad \mathrm{Df}$

$\cdot 03 \quad \mathrm{Rel} = \hat{R}\{(\exists\phi) . R = \hat{x}\hat{y}\ \phi!(x,y)\} \quad \mathrm{Df}$

These Dfs found the theory of relations. The same remarks apply as to the Dfs of $*20$. Relations are denoted by capitals.

*4*

$*22 \cdot 01 \quad \alpha \subset \beta . = : x \in \alpha . \supset_x . x \in \beta \quad \mathrm{Df}$

$\cdot 02 \quad \alpha \cap \beta = \hat{x}\,(x \in \alpha . x \in \beta) \quad \mathrm{Df}$

$\cdot 03 \quad \alpha \cup \beta = \hat{x}\,(x \in \alpha .\vee. x \in \beta) \quad \mathrm{Df}$

$\cdot 04 \quad -\alpha = \hat{x}\,(x \sim\in \alpha) \quad \mathrm{Df}$

$\cdot 05 \quad \alpha - \beta = \alpha \cap -\beta \quad \mathrm{Df}$

$*23 \cdot 01 \quad R \mathrel{\dot\subset} S . = : xRy . \supset_{x,y} . xSy \quad \mathrm{Df}$

$R \mathbin{\dot\cap} S$, $R \mathbin{\dot\cup} S$, $\dot{-}R$, $R \dot{-} S$, as above.

$*24 \cdot 01 \quad \mathrm{V} = \hat{x}(x = x) \quad \mathrm{Df} \qquad \cdot 02 \quad \Lambda = -\mathrm{V} \quad \mathrm{Df}$

$\cdot 03 \quad \exists!\alpha . = . (\exists x) . x \in \alpha \quad \mathrm{Df}$

$*25 \cdot 01 \quad \dot{\mathrm{V}} = \hat{x}\hat{y}(x = x . y = y) \ \mathrm{Df} \qquad \dot{\Lambda} = \dot{-}\dot{\mathrm{V}} \ \mathrm{Df}$

$\cdot 03 \quad \dot{\exists}!R . = . (\exists x, y) . xRy \quad \mathrm{Df}$

$*30 \cdot 01 \quad R`y = (\imath x)(xRy) \quad \mathrm{Df}$

[$R`y$ is read as "the $R$ of $y$"; e.g. "the father of $y$", "the square of $y$".]

$\cdot 02 \quad R`S`y = R`(S`y) \quad \mathrm{Df}$

$*31 \cdot 01 \quad \mathrm{Cnv} = \hat{Q}\hat{P}\{xQy . \equiv_{x,y} . yPx\} \quad \mathrm{Df}$

$\cdot 02 \quad \breve{P} = \hat{x}\hat{y}(yPx) \quad \mathrm{Df}$

Hence $\breve{P} = \mathrm{Cnv}`P$

*5*

$*32 \cdot 01 \quad \overrightarrow{R} = \hat{\alpha}\hat{y}\{\alpha = \hat{x}(xRy)\} \quad \mathrm{Df}$

$*32 \cdot 02 \quad \overleftarrow{R} = \hat{\beta}\hat{x}\{\beta = \hat{y}(xRy)\} \quad \mathrm{Df}$

Hence $\overrightarrow{R}`y = \hat{x}(xRy)$, $\overleftarrow{R}`x = \hat{y}(xRy)$

It is easier, though less correct, to give the latter forms as Dfs.
This plan is adopted henceforth.

$*33 \cdot 01 \quad \mathrm{D}`R = \hat{x}\{(\exists y) . xRy\} \quad \mathrm{Df}$[2]

$\cdot 02 \quad \mathrm{Ɑ}`R = \hat{y}\{(\exists x) . xRy\} \quad \mathrm{Df}$[3]

$\cdot 03 \quad C`R = \mathrm{D}`R \cup \mathrm{Ɑ}`R \quad \mathrm{Df}$[4]

$\cdot 04 \quad F = \hat{x}\hat{R}\{x \in C`R\} \quad \mathrm{Df}$[5]

[2] This is theorem $*33 \cdot 11$ in *PM*. [3] Theorem $*33 \cdot 111$ in *PM*. [4] Theorem $*33 \cdot 16$ in *PM*.
[5] Carnap has written a familiar looking capital 'F' above Russell's handwritten letter which looks like '₮'.

$*34{\cdot}01\ \ R|S \ = \ \hat{x}\hat{z}\{(\exists y).\ xRy.\ ySz\}\quad \mathrm{Df}$

$\quad{\cdot}02\ \ R^2 \ = \ R|R\ \ \mathrm{Df} \qquad {\cdot}03\ \ R^3 = R^2|R\ \ \mathrm{Df}$

$*35{\cdot}01\ \ \alpha \upharpoonleft R \ = \ \hat{x}\hat{y}(x \epsilon \alpha\ .\ xRy)\ \ \mathrm{Df}$

$\quad{\cdot}02\ \ R \upharpoonright \beta \ = \ \hat{x}\hat{y}(xRy\ .\ y \epsilon \beta)\ \ \mathrm{Df}$

$\quad{\cdot}03\ \ \alpha \upharpoonleft R \upharpoonright \beta \ = \ \hat{x}\hat{y}(x \epsilon \alpha\ .\ xRy\ .\ y \epsilon \beta)\ \ \mathrm{Df}$

$\quad{\cdot}04\ \ \alpha \uparrow \beta \ = \ \hat{x}\hat{y}(x \epsilon \alpha\ .\ y \epsilon \beta)\ \ \mathrm{Df}$

$\quad{\cdot}05\ \ R\text{‘}x \uparrow \beta \ = \ (R\text{‘}x) \uparrow \beta\ \ \mathrm{Df}$

$*36{\cdot}01\ \ P \upharpoonright\!\!\downharpoonleft \alpha \ = \ \alpha \upharpoonleft P \upharpoonright \alpha\ \ \mathrm{Df}$

6

$*37{\cdot}01\ \ R\text{“}\beta \ = \ \hat{x}\{(\exists y).\ y \epsilon \beta\ .\ xRy)\ \ \mathrm{Df}$

$\quad{\cdot}02\ \ R_\epsilon\text{‘}\beta \ = \ R\text{“}\beta\ \ \mathrm{Df}$[6]

$\quad{\cdot}03\ \ \breve{R}_\epsilon \ = \ \mathrm{Cnv}\text{‘}R_\epsilon\ \ \mathrm{Df}$

$\quad{\cdot}04\ \ R\text{“‘}\kappa \ = \ R_\epsilon\text{“}\kappa\ \ \mathrm{Df}$

$\quad{\cdot}05\ \ E!!R\text{“}\beta\ .\ =:y \epsilon \beta.\supset_y .\ E!\ R\text{‘}y\ \ \mathrm{Df}$

$*38{\cdot}01\ \ x\,♀ = \hat{u}\hat{y}(u = x\,♀\,y)\ \mathrm{Df}$

$\quad{\cdot}02\ \ ♀\,y \ = \ \hat{u}\hat{x}(u = x\,♀\,y)\ \ \mathrm{Df}$

$\quad{\cdot}03\ \alpha \underset{,,}{♀} y \ = \ ♀\,y\text{“}\alpha \quad \mathrm{Df}$

Here “♀” stands for any functional sign which can be put between two letters, e.g. $\alpha \cup \beta$, $R|S$, $\mu + \nu$, etc. Thus “$\alpha \underset{,,}{+} 2$” e.g. will be the class of numbers resulting from adding 2 to each member of the class $\alpha$.

$*40{\cdot}01\ \ p\text{‘}\kappa = \hat{x}(\alpha \epsilon \kappa\ .\supset_\alpha .\ x \epsilon \alpha)\ \mathrm{Df}$

$\quad{\cdot}02\ \ s\text{‘}\kappa = \hat{x}\{(\exists \alpha).\ \alpha \epsilon \kappa\ .\ x \epsilon \alpha)\}\ \ \mathrm{Df}$

$*41{\cdot}01\ \ \dot{p}\text{‘}\lambda = \hat{x}\hat{y}(R \epsilon \lambda\ .\supset_R .\ xRy)\ \mathrm{Df}$

$\quad{\cdot}02\ \ \dot{s}\text{‘}\lambda = \hat{x}\hat{y}\{(\exists R).\ R \epsilon \lambda\ .\ xRy)\}\ \ \mathrm{Df}$

$*43{\cdot}01\ \ R \,||\, S \ = \ (R|)\,|\,(|S)\ \ \mathrm{Df}$

The meanings of $R|$ and $|S$ result from $*38{\cdot}01{\cdot}02$

7

*Part II. Prolegomena to Cardinal Arithmetic*

$*50{\cdot}01\ \ I = \hat{x}\hat{y}(x = y)\ \ \mathrm{Df} \qquad\qquad J = \dot{-}I\ \ \mathrm{Df}$

$*51{\cdot}01\ \ \iota \ = \ \vec{I} \quad \mathrm{Df} \qquad\qquad \text{Hence } \iota\text{‘}x = \hat{y}(y = x)$

$*52{\cdot}02\ \ 1 = \mathrm{D}\text{‘}\iota\ \ \mathrm{Df}$[7] [This defines the cardinal number 1.]

$*54{\cdot}01\ \ 0 = \iota\text{‘}\Lambda\ \ \mathrm{Df} \qquad {\cdot}02\ \ 2 = \hat{\alpha}\{(\exists x, y).\ x \neq y.\ \alpha = \iota\text{‘}x \cup \iota\text{‘}y\}\ \ \mathrm{Df}$

$*55{\cdot}01\ \ x \downarrow y = \iota\text{‘}x \uparrow \iota\text{‘}y\ \ \mathrm{Df}$

*obs.* $\iota\text{‘}x \cup \iota\text{‘}y$ is a cardinal couple, $x \downarrow y$ is an ordinal couple.

$\quad{\cdot}02\ \ R\text{‘}x \downarrow y = R\text{‘}(x \downarrow y)\quad \mathrm{Df}$

[6] Theorem $*37{\cdot}11$ in *PM*. [7] Theorem $*52{\cdot}13$ in *PM*.

$*56{\cdot}01\ \ \dot{2} = \hat{R}\{(\exists x, y).\ R = x \downarrow y\}$ Df

$\cdot 02\ \ 2_r = \hat{R}\{(\exists x, y).\ x \neq y\,.\ R = x \downarrow y\}$ Df

$\cdot 03\ \ 0_r = \iota`\dot{\Lambda}$ Df

*Note*. $0_r$ and $2_r$ are ordinal numbers 0 and 2. $\dot{2}$ is not a number of any sort. There is no ordinal number 1. [cf. $*153$, below.]

$*60{\cdot}01\ \ \mathrm{Cl}`\alpha = \hat{\beta}(\beta \subset \alpha)$ Df $\quad \mathrm{Cl\ ex}`\alpha = \hat{\beta}(\beta \subset \alpha\ .\ \exists!\beta)$ Df[8]

$\cdot 03\ \ \mathrm{Cl}^2 = \mathrm{Cl}`\mathrm{Cls}$ Df $\quad \mathrm{Cls}^3 = \mathrm{Cl}`\mathrm{Cls}^2$ Df[9]

$*61{\cdot}01\ \ \mathrm{Rl}`P = \hat{R}(R \mathrel{\dot\subset} P)$ Df $\quad \mathrm{Rl\ ex}`P = \hat{R}(R \mathrel{\dot\subset} P.\ \dot{\exists}!R)$ Df[10]

$\cdot 02\ \ \mathrm{Rel}^2 = \mathrm{Rl}`(\mathrm{Rel} \uparrow \mathrm{Rel})$ Df $\quad \mathrm{Rel}^3 = \mathrm{Rl}`(\mathrm{Rel}^2 \uparrow \mathrm{Rel}^2)$ Df[11]

$*62{\cdot}01\ \ \epsilon\ =\ \hat{x}\hat{\alpha}(x \in \alpha)$ Df

$*63{\cdot}01\ \ t`x = \iota`x \cup -\iota`x$ Df $\quad t_0`\alpha = \alpha \cup -\alpha$ Df $\quad t_1`\kappa = t`s`\kappa$ etc.[12]

$*64{\cdot}01\ \ t_{00}`\alpha = t`(t_0`\alpha \uparrow t_0`\alpha)$ Df $\quad t^{11}x = t`(t`x \uparrow t`x)$ Df

$t_{11}`\alpha = t`(t_1`\alpha \uparrow t_1\alpha)$ Df etc.[13]

$*65{\cdot}01\ \ \alpha_x = \alpha \cap t`x$ Df $\quad \alpha(x) = \alpha \cap t`t`x$ Df $\quad R_x = (t`x) \upharpoonleft R$

$R_{xy} = (t`x \upharpoonleft R \upharpoonright t`y)$ etc.[14]

*8*

$*70{\cdot}01\ \ \alpha \to \beta\ =\ \hat{R}(\overrightarrow{R}``\mathrm{C\!\!\!I}`R \subset \alpha\ .\ \overleftarrow{R}``\mathrm{D}`R \subset \beta)$ Df

used in the forms $1 \to \mathrm{Cls}$, $\mathrm{Cls} \to 1$, $1 \to 1$. We have

$\vdash :.\ R \in 1 \to \mathrm{Cls}\ .\equiv:\ xRy.\ x'Ry\ .\supset_{x,x',y}.\ x = x'$ [15]

$\vdash :.\ R \in \mathrm{Cls} \to 1\ .\equiv:\ xRy.\ x'Ry\ .\supset_{x,x',y}.\ x = x'$ [16]

$\vdash :.\ 1 \to 1 = (1 \to \mathrm{Cls}) \cap (\mathrm{Cls} \to 1)$[17]

Also $\vdash :.\ R \in 1 \to \mathrm{Cls}\ .\equiv: y \in \mathrm{C\!\!\!I}`R\ .\supset_y.\ E!\ R`y$ [18]

$*73{\cdot}01\ \ \alpha\ \overline{\mathrm{sm}}\ \beta\ =\ (1 \to 1) \cap\ \overleftarrow{\mathrm{D}}`\alpha\ \cap\ \overleftarrow{\mathrm{C\!\!\!I}}`\beta$ Df

i.e. a member of $\alpha\ \overline{\mathrm{sm}}\ \beta$ is a one-one relation whose domain is $\alpha$ and whose converse domain is $\beta$.

$\cdot 02\ \ \mathrm{sm} = \hat{\alpha}\hat{\beta}(\exists!\ \alpha\ \overline{\mathrm{sm}}\ \beta)$ Df

*Selections*

$*80{\cdot}01\ \ P_\Delta`\kappa = (1 \to \mathrm{Cls}) \cap \mathrm{Rl}`P \cap \overleftarrow{\mathrm{C\!\!\!I}}`\kappa$ Df[19]

In practice, $P$ is almost always either $\epsilon$ or $F$, i.e. we have to deal with $\epsilon_\Delta`\kappa$ or $F_\Delta`\kappa$, of which a member of the former picks out a representative from each of a set of classes, and the latter from the fields of each of a set of relations.

[8] These are theorems $*60{\cdot}12$ and $*60{\cdot}13$ in *PM*.
[9] Df. $*60{\cdot}04$ in *PM*.
[10] Theorem $*61{\cdot}12$ and $*61{\cdot}13$ in *PM*.
[11] Dfs $*61{\cdot}03$ and $*61{\cdot}04$ in *PM*.
[12] These are numbered separately in *PM* as Dfs $*63{\cdot}01$, $\cdot 02$, $\cdot 03$.
[13] Dfs $*63{\cdot}01$, $\cdot 02$, $\cdot 03$ in *PM*.
[14] Dfs $*64{\cdot}01$, 02, 03 $\cdot 1$ in *PM*.
[15] Theorem $*71{\cdot}17$, with different variables, in *PM*.
[16] Theorem $*71{\cdot}171$, with different variables, in *PM*.
[17] Theorem $*71{\cdot}04$ in *PM*.
[18] Theorem $*71{\cdot}163$, with $\supset_y$ replaced by $\equiv_y$, in *PM*.
[19] Theorem $*80{\cdot}11$ in *PM*.

$*84{\cdot}01$ $\text{Cls}^2\text{excl} = \hat{\kappa}(\alpha, \beta \epsilon \kappa \,.\, \alpha \neq \beta \,.\, \supset_{\alpha,\beta} .\, \alpha \cap \beta = \Lambda)$ Df

$\cdot 02$ $\text{Cl excl}`y = \text{Cls}^2\text{excl} \cap \text{Cl}`y$ Df

$\cdot 03$ $\text{Cl ex}^2\text{excl}`y = \text{Cls}^2\text{excl} - \overleftarrow{\epsilon}`\Lambda$ Df

$*85{\cdot}5$ $P \downarrow y = \downarrow y``\overrightarrow{P}`y$ Df

$*88{\cdot}01$ $\text{Rel Mult} = \hat{P}(\exists \,!\, P_\Delta`\text{Cl}`P)$ Df

$\cdot 02$ $\text{Cls}^2\text{Mult} = \hat{\kappa}(\exists \,!\, \epsilon_\Delta`\kappa)$ Df

$\cdot 03$ $\text{Mult ax} \,.=:.\, \kappa \epsilon \text{Cls}^2\text{excl} \,.\supset_\kappa : (\exists\mu) : \alpha \epsilon \kappa \,.\supset_\alpha .\, \alpha \cap \mu \epsilon 1$ Df

This is the "multiplicative axiom," equivalent to Zermelo's axiom, i.e.

$(\alpha).\, \exists \,!\, \epsilon_\Delta`\text{Cl ex}`\alpha$

*9*

In $*85$ we prove the associative law for cardinal multiplication, viz.

$*85{\cdot}44$ $\vdash : \kappa \epsilon \text{Cls}^2\text{excl}. \supset .\, \epsilon_\Delta`s`\kappa \text{ sm } \epsilon_\Delta`\epsilon_\Delta``\kappa$

This is a particular case of

$*85{\cdot}43$ $\vdash : \kappa \epsilon \text{Cls}^2\text{excl}. \supset .\, P_\Delta`s`\kappa \text{ sm } \epsilon_\Delta`P_\Delta``\kappa$ which is derived from

$*85{\cdot}27$ $\vdash : \kappa \epsilon \text{Cls}^2\text{excl}. \supset .\, P_\Delta`s`\kappa = \dot{s}``D``\epsilon_\Delta`P_\Delta``\kappa$ and

$*85{\cdot}42$ $\vdash : \kappa \epsilon \text{Cls}^2\text{excl} \,.\, M, N \epsilon\ \epsilon_\Delta`P_\Delta``\kappa \,.\, \dot{s}`D`M = \dot{s}`D`N \,.\supset .\, M = N$

In $*88$ we prove that the following are all equivalent to Multax:

$(P, \kappa) :\ \kappa \subset \text{Cl}`P. \supset .\, \exists \,!\, P_\Delta`\kappa$

$(P, \kappa) :\ P \epsilon \text{Cls} \to 1 \,.\, \kappa \subset \text{Cl}`P \,.\supset .\, \exists !\, P_\Delta`\kappa$

$(P).\, \exists \,!\, P_\Delta`\text{Cl}`P$

$(\kappa) :\ \Lambda \sim\epsilon \kappa \,.\supset .\, \exists \,!\, \epsilon_\Delta`\kappa$

$(\kappa) : \kappa \epsilon \text{Cls ex}^2\text{excl} \,.\supset .\, \exists \,!\, \epsilon_\Delta`\kappa$

$(\alpha).\, \exists \,!\, \epsilon_\Delta`\text{Cl ex }`\alpha$ [Zermelo's axiom]

$(\kappa) :.\, \kappa \epsilon \text{Cls ex}^2\text{excl} \,.\supset : (\exists\mu) : \alpha \epsilon \kappa \,.\supset_\alpha .\, \mu \cap \alpha \epsilon 1$

In virtue of Zermelo's Theorem, these are all equivalent to the proposition that every class can be well-ordered. It is not known whether these various propositions are true or false.

*10*

*Inductive Relations.*

$*90{\cdot}01$ $R_* = \hat{x}\hat{y}\{x \epsilon \text{C}`R : \breve{R}``\mu \subset \mu \,.\, x \epsilon \mu \,.\supset_\mu .\, y \epsilon \mu\}$ Df

This is the "ancestral relation." If $R$ = parent, then $R_*$ = ancestor or self.

[Cf. Frege, *Begriffschrift* and *Grundgesetze der Arithmetik*.]

$*91{\cdot}01$ $R_{\text{st}} = (R|)_*$ [cf. $*38$ for Df of $R|$] Df

$\cdot 02$ $R_{\text{ts}} = (|R)_*$ $\ldots\ldots\ldots |R$ Df

$\cdot 03$ $\text{Pot}`R = \overrightarrow{R}_{\text{ts}}`R$ Df

$\cdot 04$ $\text{Potid}`R = \overrightarrow{R}_{\text{ts}}`(I \upharpoonright C`R)$ Df

$\cdot 05$ $R_{\text{po}} = \dot{s}`\text{Pot}`R$ Df

Pot‘$R$ is the set of relations $R$, $R^2$, $R^3$, $R^4$, … [Observe that finite cardinals have not yet been defined.] Potid‘$R$ is the same set together with $I \restriction C$‘$R$, which counts as the 0th power of $R$, $R^0$. We have $R_* = I \restriction C$‘$R \mathbin{\dot\cup} R_{\text{po}}$. If $R$ = parent, $R_{\text{po}}$ = ancestor.

$*93{\cdot}01 \;\; B \;=\; \hat{x}\hat{P}(x \in \mathrm{D}‘P - \mathrm{Ɑ}‘P) \qquad \text{Df}$

Here $B$ stands for “beginning”. If $P$ is a series having a first term, the first term is $B$‘$P$.

$\quad {\cdot}02 \;\; \min_P \;=\; \min(P) \;=\; \hat{x}\hat{\alpha}(x \in \alpha \cap C‘P - \breve{P}“\alpha) \qquad \text{Df}$

If $\alpha$ is a set of terms in a series, $\min_P$‘$\alpha$ is the 1st term (if any).

$\quad {\cdot}021 \;\; \max_P \;=\; \max(P) \;=\; \min(\breve{P}) \qquad \text{Df}$

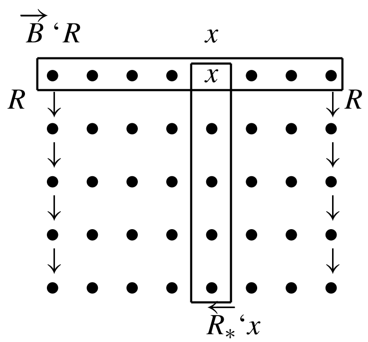
$\quad {\cdot}03 \;\; \mathrm{gen}‘P \;=\; \overrightarrow{\min_P}“\mathrm{Ɑ}“\mathrm{Potid}‘P \qquad \text{Df}$

This defines “generations”. E.g. if $P$ = parent, one member of gen‘$P$ is Adam and Eve. The next member their children, the next their grandchildren, and so on.

*11*

Note arrangement of rows and columns: suppose each dot in a rectangle of dots has the relation R to the one immediately below it:

The top row is $\overrightarrow{B}$‘$R$,
each row is a
member of *gen*‘$R$,
The column of which
$x$ is the top member
is $\overleftarrow{R_*}$‘$x$. Thus
the rows are *gen*‘$R$ and
the columns are $\overleftarrow{R_*}“\overrightarrow{B}$‘$R$.




$*96{\cdot}01 \;\; I_R‘x \;=\; \overleftarrow{R_*}‘x \cap \hat{z}(z_{po}z) \qquad \text{Df}$

$\quad {\cdot}02 \;\; J_R‘x \;=\; \overleftarrow{R_*}‘x - I_R‘x \qquad \text{Df}$

$*97{\cdot}01 \;\; \overleftrightarrow{R} \;=\; \overrightarrow{R_*}‘x \cup (\iota‘x \cap \mathrm{C}‘R) \cup \overleftarrow{R_*}‘x \qquad \text{Df}$

Note that $\iota$‘$x \cap$ C‘$R$ is the class whose only member is $x$ if $x$ is a member of the field of $R$ and is null otherwise. $\overleftrightarrow{R}\, x$ is the family of $x$, ancestry and posterity together.

*12*

*Vol. II. Part III. Cardinal Arithmetic*

$*100{\cdot}01 \;\; \mathrm{Nc} \;=\; \overrightarrow{\mathrm{sm}} \quad \text{Df} \quad {\cdot}02 \;\; \mathrm{NC} \;=\; \mathrm{D}‘\mathrm{Nc} \quad \text{Df}$ [cf. Frege, *Grundgesetze*]
In virtue of these Dfs, Nc‘$\alpha$ is the number of terms in a class $\alpha$, and NC is the class

of cardinal numbers. The Df of cardinals thus may be summed up in the following Dfs:

$I = \hat{x}\hat{y}(x = y),\ \iota = \overrightarrow{I},\ 1 = \mathrm{D}`\iota,$

$\alpha \to \beta = \hat{R}(\overrightarrow{R}``\mathrm{Cl}`R \subset \beta\ .\ \overleftarrow{R}``D`R \subset \beta)$

$\mathrm{sm} = \mathrm{D} \upharpoonright (1 \to 1) | \breve{\mathrm{Cl}}$ [This is equivalent to previous Df]

$\mathrm{Nc} = \overrightarrow{\mathrm{sm}},\ \mathrm{NC} = \mathrm{D}`\mathrm{Nc}.$

*Addition, Multiplication, and Exponentiation*

$*110{\cdot}01\ \alpha + \beta = \downarrow (\Lambda \cap \beta)``\iota``\alpha \cup (\Lambda \cap \alpha) \downarrow ``\iota``\beta$ Df

Here $\alpha, \beta$ are classes, not numbers. The Df is an artificial one, designed to produce a class of the number of whose members shall be the sum of the number of members of $\alpha$ and the number of members of $\beta$, even if $\alpha$ and $\beta$ are of different types, or overlap, or are identical.

$\cdot 02\ \mu +_c \nu = \hat{\xi}\{(\exists\alpha, \beta).\ \mu = N_0c`\alpha\ .\ \nu = N_0c`\beta\ .\ \xi\ \mathrm{sm}(\alpha + \beta)\}$ Df

We have, without hypothesis, $\vdash .\ \mu +_c \nu \in \mathrm{NC}$, $\vdash .\ \mu +_c \nu = \nu +_c \mu$, $\vdash .\ (\mu +_c \nu) + \omega = \mu +_c (\nu +_c \omega)$, which are proved in $*110$.

*13*

$*111$. *Double Similarity* (i.e. similar classes of similar classes).

This number is mainly concerned with avoiding Mult ax.

$\cdot 01\ \kappa\ \overline{\mathrm{sm}}\ \overline{\mathrm{sm}}\ \lambda = (1 \to 1) \cap \overleftarrow{\mathrm{Cl}}`s`\lambda \cap \hat{T}(\kappa = T_\epsilon``\lambda)$ Df

[cf.$*37$ for Df of $T_\epsilon$]

$\cdot 02\ \mathrm{Crp}(S)`\beta = (s`\beta)\ \overline{\mathrm{sm}}\ \beta$ Df

$\cdot 03\ \mathrm{sm\ sm} = \hat{\kappa}\hat{\lambda}\ (\exists!\ \kappa\ \overline{\mathrm{sm}}\ \overline{\mathrm{sm}}\ \lambda)$ Df

We have $\vdash : \kappa\ \mathrm{sm\ sm}\ \lambda\ .\supset .\ \kappa\ \mathrm{sm}\ \lambda\ .\ s`\kappa\ \mathrm{sm}\ s`\lambda$

$*111{\cdot}51\ \vdash :$ Mult ax $.\ \kappa, \lambda \in \mathrm{Cls}^2\mathrm{excl}\ .\ \exists!\ \kappa\ \overline{\mathrm{sm}}\ \lambda \cap Rl`\mathrm{sm}\ .\supset .\ s`\kappa\ \mathrm{sm}\ s`\lambda$

$*111{\cdot}53\ \vdash :$ Mult ax $.\ \mu, \nu \in \mathrm{NC}\ .\ \kappa, \lambda \in \mu \cap \mathrm{Cl\ excl}`\nu\ .\supset .\ \kappa\ \mathrm{sm\ sm}\ \lambda$

Mult ax is essential to these props. E.g. we cannot prove that if two men have the same number of pairs of socks, they have the same number of socks, unless we know that they have a finite number of pairs of socks.

$*112$. *Arithmetical sum of a class of classes.*

$\cdot 01\ \Sigma`\kappa = s`\epsilon \downarrow``\kappa$ Df (cf $*85{\cdot}5$ for Df of $\epsilon\downarrow$)

$\cdot 02\ \Sigma\ \mathrm{Nc}`\kappa = \mathrm{Nc}`\Sigma`\kappa$ Df

[Here we are constructing non-overlapping classes out of the set $\kappa$, which may overlap.] We prove the associative law:

$\cdot 41 \quad \vdash . s`\Sigma``\lambda = \Sigma`s`\lambda$

$\cdot 43 \quad \vdash : \lambda \in \mathrm{Cls}^2\mathrm{excl} . \supset . \mathrm{Nc}`\Sigma`\Sigma``\lambda = \mathrm{Nc}`\Sigma`s`\lambda$

*14*

$*$113. *Arithmetical product of two classes or cardinals.*

$\cdot 02 \quad \beta \times \alpha = s`\alpha \downarrow ``\beta \qquad$ Df (cf. $*38$)

$\cdot 03 \quad \mu \times_c \nu = \hat{\xi}\{(\exists \alpha, \beta). \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \xi \ \mathrm{sm}(\alpha \times \beta)\}$

$*$114 *Arithmetical product of a class of classes.*

$\cdot 01 \quad \Pi\, \mathrm{Nc}`\kappa = \mathrm{Nc}`\epsilon_\Delta`\kappa \qquad$ Df

[the Df in $*113$ can only be extended to a finite number of factors; the Df in $*114$ allows an infinite number of factors.]

$\cdot 26 \quad \vdash :. \mathrm{Mult\ ax} . \equiv : \Pi\, \mathrm{Nc}`\kappa = 0 . \equiv_\kappa . \Lambda \in \kappa$

i.e. Mult ax is equivalent to the hypothesis that a product only vanishes when one of its factors vanishes.

$\cdot 31 \quad \vdash : \kappa \cap \lambda = \Lambda . \supset . \mathrm{Nc}`\kappa \times_c \mathrm{Nc}`\lambda = \mathrm{Nc}``(\kappa \cup \lambda)$

$*$115 *Multiplicative classes.*

$\cdot 01 \quad \mathrm{Prod}`\kappa = \mathrm{D}``\epsilon_\Delta`\kappa \qquad$ Df

$\cdot 02 \quad \mathrm{Cls}^3\mathrm{arithm} = \hat{\kappa}(\kappa, s`\kappa \in \mathrm{Cls}^2\mathrm{excl}) \qquad$ Df

We prove

$\vdash : \kappa \in \mathrm{Cls}^3\mathrm{arithm} . \supset . \mathrm{Nc}`\mathrm{Prod}`\mathrm{Prod}``\kappa = \mathrm{Nc}`\mathrm{Prod}`s`\kappa = \Pi\, \mathrm{Nc}`s`\kappa$

$\mathrm{Prod}`\mathrm{Prod}``\kappa = \mathrm{D}```\mathrm{Prod}`\epsilon_\Delta`\kappa = \mathrm{D}```\mathrm{D}``\epsilon_\Delta`\epsilon_\Delta```\kappa$

$\mathrm{Prod}`s`\kappa = s``\mathrm{Prod}`\mathrm{Prod}``\kappa$

$\vdash : \kappa \ \mathrm{sm\,sm}\ \lambda . \supset . \mathrm{Prod}`\kappa \ \mathrm{sm\,sm}\ \mathrm{Prod}`\lambda$

*15*

$*$116. *Exponentiation.*

$\cdot 01 \quad \alpha \exp \beta = \mathrm{Prod}`\alpha \downarrow ``\beta \qquad$ Df

$\cdot 01 \quad \mu^\nu . = . \hat{\gamma}\{(\exists \alpha, \beta). \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta . \gamma \ \mathrm{sm}(\alpha \exp \beta)\}$

We have $*16\cdot 15 \ \vdash : (\alpha \exp \beta)\, \mathrm{sm}\, (\alpha + \beta)_\Delta`\beta$ which connects our definition with Cantor's, as $(\alpha + \beta)_\Delta`\beta$ is the class of his "Belegungen".

We prove the usual laws:

$\cdot 52 \quad \vdash : \mu^\nu \times_c \mu^\varpi = \nu^{\nu +_c \varpi}$

$\cdot 55 \quad \vdash : \mu^\varpi \times_c \nu^\varpi = (\mu \times_c \nu)^\varpi$

$\cdot 63 \quad \vdash : \mu^{\nu \times_c \varpi} = (\mu^\nu)^\varpi$

Also $\cdot 72 \ \vdash . \mathrm{Nc}`\mathrm{Cl}`\alpha = 2^{\mathrm{Nc}`\alpha}$

$*$117. *Greater and Less.*

$\cdot 01 \quad \mu > \nu . = . (\exists\, \alpha, \beta). \mu = \mathrm{N_0c}`\alpha . \nu = \mathrm{N_0c}`\beta .$

$\exists\, !\, \mathrm{Cl}`\alpha \cap \mathrm{Nc}`\beta . \sim\exists\, !\, \mathrm{Cl}`\beta \cap \mathrm{Nc}`\alpha \quad$ Df

We prove (following Cantor)

$\cdot 661 \quad \vdash : \mu \in \mathrm{N_0C} \,.\, \supset .\, 2^{\mu} > \mu$ (using $*116{\cdot}72$)[20]

$*119$. *Subtraction.*

$\cdot 01 \quad \gamma -_c \nu \;=\; \hat{\xi}\,\{\mathrm{Nc}`\xi +_c \nu = \gamma \,.\, \exists\,!\,\mathrm{Nc}`\xi +_c \nu\}$ Df

*16*

*Finite and Infinite.*

$*120{\cdot}01 \quad \mathrm{NC\ induct} \;=\; \hat{\alpha}\,\{\alpha(+_\mathrm{c}1)_*0\}$ Df

[The meaning of $+_\mathrm{c}1$ results from $*38$.]

$\cdot 02 \quad \mathrm{Cls\ induct} = s`\mathrm{NC\ induct}$ Df

$\cdot 03 \quad \mathrm{Infin\ ax} \,.=:\, \alpha \in \mathrm{NC\ induct} \,.\, \supset_\alpha .\, \exists\,!\,\alpha$ Df

$\cdot 04 \quad \mathrm{Infin\ ax}(x) \,.=:\, \alpha \in \mathrm{NC\ induct} \,.\, \supset_\alpha .\, \exists\,!\,\alpha(x)$ Df

(For Df of $\alpha(x)$, cf $*65$)

From the Df we have immediately

$\cdot 101 \quad \vdash :: \alpha \in \mathrm{NC\ induct} \,.\equiv :.\, \xi \in \mu \,.\supset_\xi .\, \xi +_\mathrm{c} 1 \in \mu : 0 \in \mu : \supset_\mu . \alpha \in \mu$

which is the usual formula for mathematical induction.

$*121{\cdot}01 \quad P(x - y) = \overleftarrow{P}_{\mathrm{po}}`x \cap \overrightarrow{P}_{\mathrm{po}}`y$ Df $\quad \cdot 011 \; P(\dashv y) = \overleftarrow{P}_{\mathrm{po}}`x \cap \overrightarrow{P}_{*}`y$ Df

$\cdot 012 \; P(x \vdash y) = \overleftarrow{P}_{*}`x \cap \overrightarrow{P}_{\mathrm{po}}`y$ Df $\quad \cdot 013 \; P(x \vdash\!\dashv y) = \overleftarrow{P}_{*}`x \cap \overrightarrow{P}_{*}`y$ Df

These Dfs define intervals without or with their end terms.

$\cdot 02 \quad P_\nu \;=\; \hat{x}\hat{y}\,\{\mathrm{N_0c}`P(x \vdash\!\dashv y) = \nu +_\mathrm{c} 1\}$ Df

$\cdot 03 \quad \mathrm{finid}`P \;=\; \hat{R}\,\{(\exists\nu).\,\nu \in \mathrm{NC\ induct} - \iota`\Lambda \,.\, R = P_\nu\}$ Df

$\cdot 031 \; \mathrm{fin}`P \;=\; \hat{R}\,\{(\exists\nu).\,\nu \in \mathrm{NC\ induct} - \iota`\Lambda - \iota`0 \,.\, R = P_\nu\}$ Df

$\cdot 04 \quad \nu_P \;=\; \breve{P}_{\nu -_\mathrm{c} 1}`B`P$ Df

$\nu_P$ is the $\nu^{\mathrm{th}}$ term of the series $P$, if it exists.

We have

$\cdot 47 \quad \vdash : R \in (\mathrm{Cls} \to 1) \cup (1 \to \mathrm{Cls}) \,.\, \supset .\, R(x \vdash\!\dashv z) \in \mathrm{Cls\ induct}$

which is a very useful proposition.[21]

*17*

$*122$. *Progressions.*

$\cdot 01 \; \mathrm{Prog} \;=\; (1 \to 1) \cap \hat{R}\,(\mathrm{D}`R = \overleftarrow{R}_{*}`B`R)$ Df

We prove

$\cdot 34 \quad \vdash :.\, R \in \mathrm{Prog} \,.\, \supset : \nu \in \mathrm{NC\ induct} - \iota`0 \,.\equiv .\, \mathrm{E}\,!\,\nu_\mathrm{R}$

$\cdot 341 \; \vdash : R \in \mathrm{Prog} \,.\, \supset .\, \mathrm{D}`R = \hat{x}\,\{(\exists\nu)\,.\,\nu \in \mathrm{NC\ induct} - \iota`0 \,.\, x = \nu_\mathrm{R}\}$

[20] There is a discussion of how “Cantor’s Theorem” depends on the axiom of reducibility in the Introduction to the second edition (pp.*xlii–xliii*). See 2.6.3 above.

[21] The topic of intervals is taken up again in Appendix B. See $*89{\cdot}21$ and $*89{\cdot}25$ for related results which are claimed to be provable without the use of the axiom of reducibility.

Thus the field of a progression R consists of the terms

$$1_R, 2_R, 3_R, \ldots \nu_R, \ldots$$

where every inductive cardinal (and no others) occurs.

$*123{\cdot}01 \quad \aleph_0 \;=\; \mathrm{D}``\mathrm{Prog} \qquad \mathrm{Df}$

The usual properties of $\aleph_0$ are here proved.

$*124$. *Reflexive classes and cardinals.*

$\cdot 01\ \mathrm{Cls\,refl} \;=\; \hat{\rho}\,\{(\exists R).\, R \in (1 \to 1).\, \text{Ɑ}`R \subset \mathrm{D}`R\,.\,\exists\,!\,\overrightarrow{B}`R\,.\,\rho = \mathrm{D}`R\}\ \mathrm{Df}$

$\cdot 02\ \mathrm{Nc\,refl} \;=\; \mathrm{N_0c}``\mathrm{Cls\,refl} \qquad \mathrm{Df}$

$\cdot 03\ \mathrm{Nc\,mult} \;=\; \mathrm{NC} \cap \hat{\alpha}\,\{\kappa \in \alpha \cap \mathrm{Cls\,ex^2excl}\,.\,\supset_\kappa.\,\exists\,!\,\epsilon_\Delta`\kappa\} \qquad \mathrm{Df}$

It is commonly assumed that all classes and cardinals are either inductive or reflexive, but there is no reason to believe this. It follows from Multax, or from the smaller hypothesis $\aleph_0 \in \mathrm{Nc\,mult}$, but we have no justification for assuming either.[22] We prove

$\cdot 56 \vdash : \aleph_0 \in \mathrm{Nc\,mult}\,.\supset.\, - \mathrm{Cls\,induct} = \mathrm{Cls\,refl}\,.\,\mathrm{N_0C} - \mathrm{Cls\,induct} = \mathrm{Nc\,refl}$

$\cdot 6 \vdash : \rho \sim\in \mathrm{Cls\,induct}\,.\equiv.\, \mathrm{Cl}`\mathrm{Cl}`\rho \in \mathrm{Cls\,refl}$

But neither of these yields what is commonly assumed.

$*125$ gives various forms of Infin ax (See $*120$).

$*126{\cdot}01 \quad \mathrm{NC\,ind} \;=\; \mathrm{NC\,induct} - \iota`\Lambda \qquad \mathrm{Df}$

*18*

*Part IV. Relation-Arithmetic.*

[This includes ordinal arithmetic as a special case.]

$*150{\cdot}01 \quad S\,\dot{;}\,Q \;=\; S|Q|\breve{S} \quad \mathrm{Df} \qquad \cdot 02 \quad S\dagger Q \;=\; S|Q|\breve{S} \quad \mathrm{Df}$

[Alternative notation.]

$$\begin{tikzcd} \bullet \arrow[r, "S\dot{;}Q"] \arrow[d, "S"'] & \bullet \\ \bullet \arrow[r, "Q"'] & \bullet \arrow[u, "\breve{S}"'] \end{tikzcd}$$

$\cdot 03 \quad Q\,\underset{,\,,}{♀}\,y \;=\; ♀\,y\,\dot{;}\,Q\ \mathrm{Df}$ [analogous to Dfs in $*38$]

$\cdot 04 \quad R`S\,\dot{;}\,Q \;=\; R`(S\,\dot{;}\,Q)\ \mathrm{Df} \qquad \cdot 05 \quad R\,\dot{;}\,S\,\dot{;}\,Q \;=\; R\,\dot{;}\,(S\,\dot{;}\,Q)\ \mathrm{Df}$

[22] Boolos (1994) has a detailed explanation of this result, which follows J. R. Littlewood in describing as "Perfectly good mathematics" that is proved in *PM*. A "reflexive" class can be mapped into a proper subset of itself, while an "inductive" set can be mapped into the integers. The "good mathematics" lies in showing how the two sorts of infinity are related. Russell is drawing attention to this result. The notion is that we have "no justification for assuming either". These are the multiplicative axiom (choice) or the weaker hypothesis that $\aleph_0$ is a multiplicative cardinal. "We will call a cardinal $\nu$ a "multiplicative cardinal" if a product of $\nu$ factors none of which is zero is never zero." (*PM* II, p.270). The expression "no justification" suggests more than just that they should be explicitly stated as hypotheses when used. Perhaps he means that there is no purely logical justification for these assumptions.

Important props:

$\cdot 22 \ \vdash : C`Q \subset \text{Ɑ}`S \,.\, \supset .\, C`S \dot{;} Q \ = \ S``C`Q$

$\cdot 4 \ \ \vdash :.\ S \in 1 \to \text{Cls} \,.\, \supset :\ x(S\dot{;}Q)y \,.\, \equiv .\, (\exists z, w).\ x = S`z \,.\, y = S`w \,.\, zQw$

$\cdot 41 \ \vdash :.\ S \in \text{Cls} \to 1 \,.\, \supset :\ x(S\dot{;}Q)y \,.\, \equiv .\, (\breve{S}`x)Q(\breve{S}`y)$

$\cdot 71 \ \vdash :.\ S \in 1 \to \text{Cls} \,.\, z, w \in \text{Ɑ}`S \,.\, \supset :\ S\dot{;}(z \downarrow w) = (S`z) \downarrow (S`w)$

$\cdot 16 \ \vdash .\ \dot{s}`R\dagger``\lambda = R\dagger(\dot{s}`\lambda) = R\dot{;}\dot{s}`\lambda$

cf $*40\cdot 38$, $\vdash .\, s`R```\kappa = R``s`\kappa$. [$R\dagger$ is the analogue of $R_\epsilon$ ($*37$)]

$*151$. *Ordinal Similarity*. "smor" stands for "similar ordinally".

$\cdot 01 \ P\, \overline{\text{smor}}\, Q = \hat{S}\{S \in 1 \to 1 \,.\, C`Q \ = \text{Ɑ}`S \,.\, p = S\dot{;}Q\}$ Df

$\cdot 02 \quad \text{smor} \ = \ \hat{P}\hat{Q}\{\exists !\ P\, \overline{\text{smor}}\, Q\}$ Df

Relation arithmetic is based on "smor" as cardinal is on "sm".

$*152\cdot 01 \ \ \text{Nr} \ = \ \overrightarrow{\text{smor}}$ Df $\qquad \cdot 02 \ \ \text{NR} \ = \ \text{D}`\text{Nr}$ Df

$*153\cdot 01 \ \ 1_s \ = \ \hat{R}\{(\exists x) \,.\, R = x \downarrow x\}$ Df

This is a relation number, but not an ordinal number.

Subsequent numbers analogous to $*102$ ff.

*19*

$*160$. *The sum of two relations*.

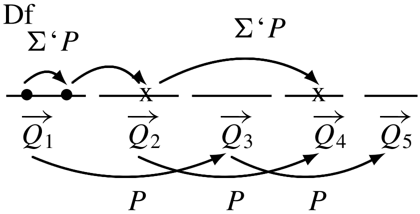
$\cdot 01 \quad P \dotplus Q \ = \ P \cup Q \cup C`P \uparrow C`Q$ Df

$*161\cdot 01 \ \ P \dotplus\!\!\to x \ = \ P \cup (C`P) \uparrow (\iota`x)$ Df

$\cdot 02 \ \ x \leftarrow\!\!\dotplus P \ = \ (\iota`x) \uparrow (C`P) \cup P$ Df

$*162\cdot 01 \ \ \Sigma`P \ = \ \dot{s}`C`P \cup F\dot{;}P$ Df

[ For Df of $F$, cf. $*33$]




$*163\cdot 01 \ \text{Rel}^2\text{excl} \ = \ \hat{P}\{Q, R \in C`P \,.\, Q \neq R \,.\, \supset_{Q,R} .\, C`Q \cap C`R = \Lambda\}$ Df

$\cdot 12 \ \ \vdash : P \in \text{Rel}^2\text{excl} \,.\, \equiv .\, F \upharpoonright C`P \in \text{Cls} \to 1$

$*164\cdot 01 \ P\, \overline{\text{smor}}\, \overline{\text{smor}}\, Q = (1 \to 1) \cap \overleftarrow{\text{Ɑ}}`C`\Sigma`Q \cap \hat{S}(P = S\dagger\dot{;}Q)$ Df

$\cdot 02 \ \ \text{smor smor} \ = \ \hat{P}\,\hat{Q}\,(\exists !\ P\, \overline{\text{smor}}\, \overline{\text{smor}}\, Q)$ Df

[Analogous to $*111$]

$*166\cdot 01 \ \ Q \times P \ = \ \Sigma`P \underset{,,}{\downarrow}\dot{;}\, Q$ Df $\qquad$ [Analogous to $*113$]

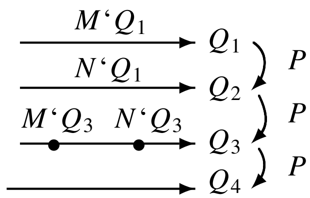
*Principle of* 1[st] *differences, multiplication and exponentiation of relations.*

$*170{\cdot}01 \quad P_{\mathrm{cl}} \;=\; \hat{\alpha}\,\hat{\beta}\,\{\alpha, \beta \,\epsilon\, \mathrm{Cl}`C`P \,.\, \exists\,!\,\alpha - \beta - \breve{P}``(\beta - \alpha)\} \qquad \mathrm{Df}$

$\quad {\cdot}02 \quad P_{\mathrm{lc}} \;=\; \mathrm{Cnv}`(\breve{P})_{\mathrm{cl}} \qquad \mathrm{Df}$

$*171\cdot 01 \; P_{\mathrm{df}} = \hat{\alpha}\,\hat{\beta}\,\{\alpha, \beta \,\epsilon\, \mathrm{Cl}`C`P : (\exists z).\, z \,\epsilon\, \alpha - \beta .\, \overrightarrow{P}`z \cap \alpha - \iota`z = \overrightarrow{P}`z \cap \beta\} \;\; \mathrm{Df}$

$\quad {\cdot}02 \; P_{\mathrm{fd}} = \mathrm{Cnv}`(\breve{P})_{\mathrm{cl}} \quad \mathrm{Df}$




$*172{\cdot}01 \quad \Pi`P \;=\; \hat{M}\hat{N}\{M, N \,\epsilon\, F_\Delta`C`P :.$
$\qquad (\exists Q) : (M`Q) Q (N`Q) : R P Q \,.\, R \neq Q \,.\, \supset_R .\, M`R = N`R\} \quad \mathrm{Df}$

This is the Df required for ordinal multiplication when the number of factors is not known to be finite.

*20*

$*173{\cdot}01\ \mathrm{Prod}`P \;=\; \mathrm{D}{}^{\raisebox{0pt}{;}}\Pi`P \quad \mathrm{Df} \quad$ (Analogous to $*115$)

$*174$. *Associative Law.*

$\quad {\cdot}01 \;\; \mathrm{Rel^3arithm} \;=\; \hat{P}\,(P, \Sigma`P \,\epsilon\, \mathrm{Rel^2excl}) \quad \mathrm{Df}$

$\quad {\cdot}25 \;\; \vdash : P \,\epsilon\, \mathrm{Rel^2excl} \,.\, P \subset J \,.\, \supset .$
$\qquad \Pi`\Sigma`P \ \mathrm{smor}\ \Pi`\Pi;P \,.\, \Pi`\Sigma`P \ \mathrm{smor}\ \mathrm{Prod}`\Pi;P$

This is the associative law for multiplication.

$*176$. *Exponentiation.*

$\quad {\cdot}01 \;\; P \exp Q \;=\; \mathrm{Prod}`P \underset{\cdot\cdot}{\downarrow};Q \qquad \mathrm{Df} \qquad$ (Cf $*116$)[23]

$\quad {\cdot}02 \;\; P^Q \;=\; \dot{s};(P \exp Q) \quad \mathrm{Df}$

$\quad {\cdot}181 \;\vdash .\; P^Q \ \mathrm{smor}\,(P \exp Q)$

$\quad {\cdot}22 \;\; \vdash : \; P \ \mathrm{smor}\ R \,.\, Q \ \mathrm{smor}\ S \,.\supset .\, (P \exp Q)\ \mathrm{smor}\,(R \exp S)$[24]

$\quad {\cdot}42 \;\; \vdash : \; \dot{\exists}!\,Q \,.\, \dot{\exists}!\,R \,.\, C`R \cap C`R = \Lambda \,.\supset .\; P^Q \times P^R \ \mathrm{smor}\ P^{Q \dagger R}$

$\quad {\cdot}44 \;\; \vdash : S \,\epsilon\, \mathrm{Rel^2excl} \,.\, S \subset J \,.\supset .\, \{\mathrm{Prod}`(P \exp);S\}\ \mathrm{smor}\,\{P \exp(\Sigma`S)\}$

$\quad {\cdot}57 \;\; \vdash : R \subset J \,.\supset .\, (P \exp Q) \exp R \ \mathrm{smor}\,\{P \exp(R \times Q)\}.$
$\qquad (P^Q)^R \ \mathrm{smor}\ p^{R \times Q}$

[23] $Q \underset{\cdot\cdot}{\female} y$ was defined by: $*150{\cdot}03$. $Q \underset{\cdot\cdot}{\female} y \;=\; \female y;Q$ Df.

[24] Russell miscopied the proposition, leaving off the last clause. It should be:
$*176{\cdot}22 \vdash : \; P \ \mathrm{smor}\ R \,.\, Q \ \mathrm{smor}\ S \,.\supset .\, (P \exp Q)\ \mathrm{smor}\,(R \exp S) \,.\, P^Q \ \mathrm{smor}\ R^S$.

These propositions yield the usual formal laws for exponentiation.

$*177{\cdot}13 \ \vdash : x \neq y \,.\supset .\ P_{\mathrm{df}}\,\mathrm{smor}\,\{(x \downarrow y)^P\}$
Hence, e.g., $2^{\aleph_0} = C``\,2^{\omega}$, where $\omega$ has its usual meaning.
(See $*186{\cdot}4$, below.)

*21*

*Arithmetic of Relation-Numbers*

$*180{\cdot}01 \quad P + Q \ = \ \{\downarrow (\Lambda \cap C`Q) \,;\!\iota\,;\, P\} \mathrel{\dot{\uparrow}} \{(\Lambda \cap C`P) \downarrow \,;\!\iota\,;\, Q\} \quad \mathrm{Df} \quad (\mathrm{cf}*110)$
$\cdot 02 \quad \mu \dot{+} \nu \ = \ \hat{R}\,\{(\exists P, Q).\, \nu = \mathrm{N_0r}`P.\, \nu = \mathrm{N_0r}`Q.\, R\,\mathrm{smor}\,(P+Q)\} \quad \mathrm{Df}$

$*181$. *Addition of unity to a relation-number.* Here we define $\mu \dot{+} \dot{1}$ etc., without defining $\dot{1}$ separately. cf $*161$.
$\cdot 01 \quad P \dot{\nrightarrow} x \ = \downarrow \Lambda_x \,;\!\iota\,;\, P \nrightarrow (\Lambda \cap C`P) \downarrow \iota`x \quad \mathrm{Df} \qquad \cdot 011\ x \dot{\nleftarrow} P$ similar
$\cdot 02 \quad \mu \dot{+} \dot{1} = \hat{R}\,\{(\exists P, x)\,.\,\mathrm{N_0r}`P = \mu\,.\, R\,\mathrm{smor}\,(P \dot{\nrightarrow} x)\}\mathrm{Df}$
$\cdot 021 \ \dot{1} \dot{+} \dot{\mu}$, similar.[25]
$\cdot 04 \quad \dot{1} \dot{+} \dot{1} = 2_r \qquad \mathrm{Df}$

$*182$. *Separated Relations* (cf $*112$)

$\cdot 01 \quad \overset{\frown}{\varphi} = \ \hat{y}\,\hat{x}\,(y = x\, \varphi\, x) \qquad \mathrm{Df}$

Thus e.g. $\overset{\frown}{|}\,`R = R^2,\ \overset{\frown}{\downarrow}\,`x = x \downarrow x,\ \overset{\frown}{\dot{\downarrow}}\,`Q \ = \ Q \dot{\downarrow} Q \ = \downarrow Q\,\dot{;}\,Q$

The notation is introduced for the sake of $\overset{\frown}{\dot{\downarrow}}\,;P$. We have

$\cdot 1 \quad \vdash .\ \ \overset{\frown}{\dot{\downarrow}}\,;P \ = \ \hat{X}\,\hat{Y}\,\{(\exists Q, R)\,.\, X = \ Q \dot{\downarrow} Q\ .\ Y = \ R \dot{\downarrow} R\ .\ QPR\}$

$*183{\cdot}01 \quad \Sigma\mathrm{Nr}`P \ = \ \mathrm{Nr}`\Sigma`\,\overset{\frown}{\dot{\downarrow}}\,;P \qquad \mathrm{Df}$

$*184{\cdot}01\ \mu \dot{\times} \nu = \hat{R}\,\{(\exists\, P, Q).\, \mu = \mathrm{N_0r}`P.\ \nu = \mathrm{N_0r}`Q.\ R\,\mathrm{smor}\,(P \times Q)\} \ \mathrm{Df}$

$*185{\cdot}01\ \Pi\,\mathrm{Nr}`P \ = \ \mathrm{Nr}`\Pi`P \qquad \mathrm{Df}$

$*186 \cdot 01\, \mu \exp_r \nu \ = \ \hat{R}\{(\exists\, P, Q).\, \mu = \mathrm{N_0r}`P.\, \nu = \mathrm{N_0r}`Q.\, R\,\mathrm{smor}\,(P \exp Q)\}\ \mathrm{Df}$
We have[26]
$\cdot 21 \ \vdash .\ \mu \exp_r 2_r = \mu \dot{\times} \mu, \quad \cdot 22 \ \vdash .\ \alpha \exp_r (\beta \dot{+} \dot{1}) = (\alpha \exp_r \beta) \dot{\times} \alpha$
$\cdot 14 \ \vdash :\ \nu \neq 0_r \,.\, \varpi \neq 0_r \,.\supset .\ \mu \exp_r(\nu \dot{+} \varpi\,\nu) = (\mu \exp_r \nu) \dot{\times} (\mu \exp_r \varpi)$
$\cdot 15 \ \vdash :\ \varpi \subset \mathrm{Rl}`J \,.\supset .\ \mu \exp_r(\varpi \dot{\times} \nu) \ = \ (\mu \exp_r \nu) \exp_r \varpi$

[25] The last dot is an error. This defines: $\dot{1} \dot{+} \mu$.
[26] These are in the order in which they are listed in the *Summary of* $*186$.

·4 $\vdash : \mathrm{Nr}`P_{\mathrm{df}} = 2_r \exp_r (\mathrm{Nr}`P)$
·5 $\vdash : \mu, \nu \in \mathrm{N}_0 R \,.\, \nu \neq 0_r \,.\, \supset .\, C``(\mu \exp_r \nu) = (C``\mu)^{C``\nu}$

*22*

*Part V. Series.*

*200. *Relations contained in diversity.* $(\mathrm{Rl}`J)$
N.B. $P$ is "asymmetrical" if $P^2 \mathrel{\dot{\subset}} J$.

*201. *Transitive Relations.* ·01 $\mathrm{Trans} = \hat{P}\,(P^2 \mathrel{\dot{\subset}} P)$ Df

*202. *Connected Relations.*
·01 $\mathrm{connex} = \hat{P}\,\{x \in C`P \,.\, \supset_x .\, \overleftrightarrow{P}`x = C`P\}$ Df [cf *97 for Df of $\overleftrightarrow{P}`x$]

*204·01 $\mathrm{Ser} = \mathrm{Rl}`J \cap \mathrm{trans} \cap \mathrm{connex}$ Df [*Definition of Series*]

*205. *Maximum and Minimum Points* [Dfs in *93]
·31 $\vdash : P \in \mathrm{connex} . \supset .\, \min_P, \max_P \in 1 \to \mathrm{Cls}$

*206. *Sequent Points.*
·01 $\mathrm{seq}_P = \hat{x}\,\hat{\alpha}\,\{x \min_P p`\overleftarrow{P}``(\alpha \cap C`P)\}$ Df
·02 $\mathrm{prec}_P = \hat{x}\,\hat{\alpha}\,\{x \max_P p`\overrightarrow{P}``(\alpha \cap C`P)\}$ Df

*207. *Limits.*
·01 $\mathrm{lt}_P = \mathrm{lt}(P) = \mathrm{seq}_P \upharpoonright (-\mathrm{CI}`\max_P)$ Df
·02 $\mathrm{tl}_P = \mathrm{lt}(\breve{P})$ Df
·03 $\mathrm{limax}_P = \max_P \mathrel{\dot{\cup}} \mathrm{lt}_P$ Df
·04 $\mathrm{limin}_P = \min_P \mathrel{\dot{\cup}} \mathrm{tl}_P$ Df

*208. *Correlation of Series.* ·01 $\mathrm{cror}`P = s`(\overline{\mathrm{smor}}P)``\mathrm{Rl}`P$ Df

*Segments.*
*210·12 $\vdash :: \alpha, \beta \in \kappa \,.\, \supset_{\alpha,\beta} : \alpha \subset \beta \,.\vee.\, \beta \subset \alpha :.$
$Q = \hat{\alpha}\,\hat{\beta}\,(\alpha, \beta \in \kappa \,.\, \alpha \subset \beta \,.\, \alpha \neq \beta) :. \supset .\, Q \in \mathrm{Ser}$
·232·233 $\vdash :$ Same hp $. \kappa \sim\in 1 . \supset : \lambda \subset \kappa \,.\, p`\lambda \in \kappa \,.\, \supset .\, p`\lambda = \mathrm{limin}_P`\lambda :$
$\lambda \subset \kappa \,.\, s`\lambda \in \kappa \,.\, \supset .\, s`\lambda = \mathrm{limax}_P`\lambda$

These props lead to the Dedekindian character of series of segments.

*23*

*211·01 $\mathrm{sect}`P = \hat{\alpha}\,(\alpha \subset C`P \,.\, P``\alpha \subset \alpha)$ Df
This defines *sections*; *segments* are $\mathrm{D}`P_\epsilon$. [cf. *37 for Df of $P_\epsilon$]

*212·01 $\varsigma`P = P_{\mathrm{lc}} \upharpoonright \mathrm{D}`P_\epsilon$ Df
·02 $\mathrm{sgm}`P = P_{\mathrm{lc}} \upharpoonright \mathrm{D}`(P_\epsilon \mathrel{\dot{\cap}} I)$ Df
For Df of $P_{\mathrm{lc}}$ see *170. $\varsigma`P$ is the series of segments, $\mathrm{sgm}`P$ the series of those segments that have no maximum.

$*213{\cdot}01 \quad P_\varsigma \ = \ P \upharpoonright \dot{\text{'}}(\varsigma\text{‘}P_*) \upharpoonright (-\iota\text{‘}\Lambda) \qquad \text{Df}$

This defines the series of segmental *series*, as opposed to the former Df of the series of segmental *classes*.

$*214$. *Dedekindian Relations*.

$\cdot 01 \ \ \text{Ded} \ = \ \hat{P}\,\{(\alpha)\,.\,\alpha \in \text{ꓷ‘max}_P \cup \text{ꓷ‘seq}_P\} \qquad \text{Df}$

$\cdot 02 \ \ \text{semi Ded} \ = \ \hat{P}\{\text{sect‘}P \ - \ \iota\text{‘}C\text{‘}P \subset \text{ꓷ‘max}_P \cup \text{ꓷ‘seq}_P\} \qquad \text{Df}$

$\cdot 32 \ \ \vdash : \ P \in \text{connex} \,.\, \dot{\exists}\,!\,P \,.\, \supset .\, \varsigma\text{‘}P_* \in \text{Ser} \cap \text{Ded}$

$*215$. *Stretches*. $\quad \cdot 01 \ \ \text{str‘}P \ = \ \hat{\alpha}\,(\alpha \subset C\text{‘}P \,.\, P\text{“}\alpha \cap \breve{P}\text{“}\alpha \subset \alpha) \qquad \text{Df}$

$*216$. *Derivatives*. $\cdot 01 \ \ \delta_P\text{‘}\alpha \ = \ \text{lt}_P\text{“Cl ex‘}(\alpha \cap C\text{‘}P) \qquad \text{Df}$

$\cdot 02 \ \ \text{dense‘}P \ = \ \hat{\alpha}(\alpha - \overrightarrow{\text{min}}_P\text{‘}\alpha \subset \delta_P\text{‘}\alpha) \qquad \text{Df}$

$\cdot 03 \ \ \text{closed‘}P \ = \ \hat{\alpha}\{\text{Cl ex‘}(\alpha \cap C\text{‘}P) \subset \text{ꓷ‘limax}_P \,.\, \delta_P\text{‘}\alpha \subset \alpha\} \quad \text{Df}$

$\cdot 04 \ \ \text{perf‘}P \ = \ \text{dense‘}P \cap \text{closed‘}P \qquad \text{Df}$

$\cdot 05 \ \ \nabla\text{‘}P \ = \ P \upharpoonright \text{D‘lt}_P \qquad \text{Df}$

$*217$. *Segments of Sums and Converses*.[27]

*24*

*Convergence, and Limits of Functions.*

Here limits of functions and continuity of functions are defined without using numbers.

$*230{\cdot}01 \quad R\,\overline{Q}_{\text{cn}}\,\alpha \ = \ C\text{‘}Q \cap \text{ꓷ‘}R \cap \hat{y}\,(R\text{“}\overleftarrow{Q}_*\text{‘}y \ \subset \alpha) \qquad \text{Df}$

$\cdot 02 \ \ Q_{\text{cn}} \ = \ \hat{R}\,\hat{\alpha}\,(\exists\,!\,R\,\overline{Q}_{\text{cn}}\,\alpha) \qquad \text{Df}$

Here $R$ is the function, $Q$ the series to which its arguments belong, $P$ (introduced later) the series to which its values belong. A value of the function is $R\text{‘}y$, where $y \in C\text{‘}Q$.

$*231{\cdot}01 \ \ P\overline{R}_{\text{sc}}\,Q \ = \ p\text{‘}P_*\text{“‘}\overleftarrow{Q_{\text{cn}}}\text{‘}R \cap C\text{‘}P \qquad \text{Df}$

$\cdot 02 \ \ P\overline{R}_{\text{os}}\,Q \ = \ P\,\overline{R}_{\text{sc}}\,Q \cap \breve{P}\,R_{\text{sc}}\,Q \qquad \text{Df}$[28]

$P\,\overline{R}_{\text{os}}\,Q$ is (roughly) the ultimate oscillation of the function as the argument approaches “infinity”.

$*232{\cdot}01 \ \ (P\overline{R}\,Q)_{\text{sc}}\text{‘}\alpha \ = \ P\,\overline{R}_{\text{sc}}(Q_* \upharpoonright \alpha) \qquad \text{Df}$

$\cdot 02 \ \ (P\overline{R}\,Q)_{\text{os}}\text{‘}\alpha \ = \ P\,\overline{R}_{\text{os}}(Q_* \upharpoonright \alpha) \qquad \text{Df}$

This defines the ultimate oscillation as the argument approaches the limit of $\alpha$.

$*233{\cdot}01 \ (P\overline{R}\,Q)_{\text{lmx}} \ = \ \text{limax}_P|(P\overline{R}\,Q)_{\text{sc}} \qquad \text{Df}$

$\cdot 02 \quad R(PQ) \ = \ (P\overline{R}\,Q)_{\text{lmx}} \,|\, \overrightarrow{Q} \qquad \text{Df}$

[27] $*217$ is devoted to the proof of a theorem, $*217{\cdot}43$, “. . . which is required in the theory of real numbers . . . ” and so contains no definitions.

[28] An error. $\breve{P}R_{\text{sc}}$ should read $\breve{P}\overline{R}_{\text{sc}}$.

The four limits of the function for an argument $y$ are $R(PQ)`y$, $R(\breve{P}Q)`y$, $R(P\breve{Q})`y$, $R(\breve{P}\breve{Q})`y$.

$*234{\cdot}01$ $\mathrm{sc}(P, Q)`R \ = \ C`P \cap \hat{x}\,(\overrightarrow{P}_{\mathrm{po}}``\overleftarrow{P}_{\mathrm{po}}`x \subset \overleftarrow{Q}_{\mathrm{cn}}`R)$ Df

$\cdot 02$ $\mathrm{os}(P, Q)`R \ = \ \mathrm{sc}(P, Q)`R \cap \mathrm{sc}(\breve{P}, Q)`R$ Df

$\cdot 03$ $\mathrm{ct}(P, Q)`R \ = \ \hat{\alpha}\,\{R`\alpha \in \mathrm{os}(P, Q_* \upharpoonright \overrightarrow{Q}_{\mathrm{po}}`\alpha)`R - C`P_1\}$ Df

[cf $*121$ for Df of $P_1$].

$\cdot 04$ $\mathrm{contin}(P, Q)`R \ = \ \mathrm{ct}(P, Q)`R \cap \mathrm{ct}(P, \breve{Q})`R$ Df

$\cdot 05$ $P\,\overline{\mathrm{contin}}\,Q \ = \ \hat{R}\,\{\exists\,!\,C`Q \cap \text{Ɑ}`R\,.\,C`Q \cap \text{Ɑ}`R \subset \mathrm{contin}(P, Q)`R\}$ Df

$P\,\overline{\mathrm{contin}}\,Q$ is the class of continuous functions. They are those for which, if $\alpha \in C`Q \cap \text{Ɑ}`R$, $R(PQ)`\alpha = R(\breve{P}Q)`\alpha = R(\breve{P}\breve{Q})`\alpha = R`\alpha$.

*25*

*Vol. III*

*Well-ordered Series.*

$*250{\cdot}01$ $\mathrm{Bord} \ = \ \hat{P}\,(\mathrm{Cl\,ex}`C`P \subset \text{Ɑ}`\mathrm{min}_P)$ Df

$\cdot 02$ $\Omega \ = \ \mathrm{Ser} \cap \mathrm{Bord}$ Df

$\Omega$ is the class of well-ordered series.

$*251{\cdot}01$ $\mathrm{NO} \ = \ \mathrm{Nr}`\Omega$ Df This defines ordinal numbers.

$*252$. Segments of well-ordered series. $*253$ Ditto.

$*254$. *Greater and Less.*

$\cdot 01$ $\mathrm{less} \ = \ \hat{P}\hat{Q}\,\{P, Q \in \Omega\,.\,\exists\,!\,\mathrm{Rl}`Q \cap \mathrm{Nr}`P.\ \sim(P\ \mathrm{smor}\ Q)\}$ Df

$\cdot 02$ $P_{\mathrm{sm}} \ = \ (\mathrm{D}`P_{\varsigma}) \upharpoonright \mathrm{smor}$ Df

[This number contains much of Cantor, *Math. Annalen* **46**][29]

$*255 \cdot 01$ $\lessdot \ = \ \mathrm{N_0r}\,\dot{;}\,\mathrm{less}$ Df $\cdot 02$ $\gtrdot \ = \ \mathrm{Cnv}`\lessdot$ $\cdot 03$ $\mathrm{N_0O} = \mathrm{NO} \cap \mathrm{N_0R}$ Df

$\cdot 04$ $\leqslant \ = \ \lessdot \ \dot{\cup}\ \mathrm{smor}_{\epsilon} \upharpoonright \mathrm{N_0O}$ Df (Less than or equal to)

$*256$. *The series of ordinals*. Including solution of Burali-Forti, viz.[30]

$\cdot 56$ $\vdash .\ (N \upharpoonright \lambda)\,\mathrm{less}\,\{N \upharpoonright t`t_{00}`\lambda)\}$ where

$N \ = \ \lessdot \dot{\cup}\, 0_r \downarrow 1_s \dot{\cup} (\iota`1_s) \uparrow \text{Ɑ}`\lessdot$ Dft[$*256$][31]

[29] This is a reference to Cantor (1895), translated as “Contributions to the founding of the theory of transfinite numbers”.

[30] The Summary of $*256$ in *PM* refers to Burali-Forti (1897), the source of the “Burali-Forti Paradox”. Burali-Forti presents his argument as a disproof of the trichotomy principle used in the derivation. At (*PM* III, p.74) we find “The conclusion drawn by Burali-Forti is that the above proposition A [the trichotomy principle] is false. This, however, cannot be maintained in view of Cantor’s proof, reproduced above ($*255{\cdot}112$, depending on $*254{\cdot}4$). The solution of the paradox must therefore be sought elsewhere”. Neither appear in these notes for Carnap.

$*255{\cdot}112. \vdash :.\ \mu, \nu \in \mathrm{N_0O}. \supset : \mu \lessdot \nu\,.\vee.\,\mu = \mathrm{smor}``\nu\,.\vee.\,\nu \lessdot \mu.$

Whitehead and Russell find their solution in the theory of types. “Thus in higher types there are greater ordinals than any to be found in lower types. This fact is what gave rise to the paradox, as the corresponding fact in cardinals gave rise to the paradox of the greatest ordinal”.(*PM* III, p.75) For a discussion of the emergence of the Burali-Forti “paradox” and Zermelo’s reaction to it, see Moore (1978).

[31] $N$ is here given a “temporary definition” for the purposes of this number: Dft[$*256$]

$*257$. Transfinite Ancestral Relation. (Leading to Zermelo's theorem)

$\cdot 01 \ \ (R * Q)`x \ = \ C`Q \cap \hat{y}\{x \in \sigma \,.\, \breve{R}``\sigma \cup \delta_Q`\sigma \subset \sigma \,.\, \supset_\sigma .\, y \in \sigma\}$ Df

$\cdot 02 \ \ Q_{Rx} \ = \ Q(R, x) \ = \ Q \upharpoonright (R * Q)`x$ Df

$*258$. Zermelo's Theorem, viz.[32]

$\cdot 36 \ \ \vdash : \ \mu \in C``\Omega \cup 1 \,.\equiv .\, \exists\,!\, \epsilon_\Delta`\mathrm{Cl\ ex}`\mu$

$\cdot 37 \ \ \vdash : \ \mathrm{Mult\ ax} \,.\equiv .\, C``\Omega \cup 1 = \mathrm{Cls}$

$*259$. Inductively defined correlations. $A \ = \ \hat{S}\hat{T}(S \mathrel{\subset\!\!\!\cdot} T \,.\, S \neq T)$ Dft[$*259$]

$\cdot 02 \ \ A_W \ = \ \hat{S}\hat{T}(T = S \mathrel{\dot\cup} W`S)$ Dft $\quad \cdot 03 \ \ W_A \ = \ \dot{s}`(A_W * A)`\dot\Lambda$ Df

In the important case, $W = \hat{X}\hat{T}\{X = \mathrm{seq}_P`\mathrm{D}`T \downarrow \mathrm{seq}_Q`\mathrm{ᗡ}`T\}$ where $P$, $Q \in$ Ser.

*26*

*Finite and Infinite Series and Ordinals.*

$*260\cdot 01 \ P_{\mathrm{fn}} \ = \ \dot{s}`\mathrm{fin}`P$ Df [Cf $*121$ for Df of fin$`P$]

$\cdot 27 \ \ \vdash : P_{\mathrm{po}} \in \mathrm{Ser} \,.\supset .\, P_{\mathrm{fn}} = (P_1)_{\mathrm{po}}$ *N.B.* $\vdash : P \in \mathrm{Ser} \,.\supset .\, P_1 = P \mathbin{\dot-} P^2$

$*261\cdot 01$ Ser infin $=$ Ser $\cap\, \breve{C}``$Cls refl Df $\quad \cdot 03$ Ser fin $=$ Ser $-$ Ser infin Df

$\cdot 02 \ \ \Omega$ infin $= \Omega \cap \breve{C}``$Cls refl Df $\quad \cdot 04\ \Omega$ fin $= \Omega - \Omega$ infin Df

$\cdot 05 \ \ \Omega$ induct $= \Omega \cap \breve{C}``$Cls induct Df

$\cdot 42 \ \ \vdash .\, \Omega$ fin $= \Omega$ induct

$\cdot 33 \ \ \vdash : \ P, Q \in \Omega \,.\, Q \mathrel{\subset\!\!\!\cdot} \breve{P} \,.\supset .\, Q \in \Omega$ induct

$*262\cdot 01$ NO fin $=$ $\mathrm{N_0r}``\Omega$ fin Df $\quad \cdot 02$ NO infin $=$ $\mathrm{N_0r}``\Omega$ infin Df

$\cdot 03 \ \ \mu_r \ = \ \Omega \cap \breve{C}``\mu$ Df

If $\mu$ is a finite cardinal, $\mu_r$ is the corresponding ordinal ($*262\cdot 24$)

$*263$. *Progressions.* $\omega \ = \ \hat{P}\{(\exists R) \,.\, R \in \mathrm{Prog} \,.\, P = R_{\mathrm{po}}\}$ Df

$\omega$ is Cantor's ordinal number $\omega$.

$*264$. *Derivatives of* $\Omega$*s.* $\quad \cdot 01 \ P_{\mathrm{pr}} \ = \ P \upharpoonright \overleftarrow{(P_1)_*} ; \nabla`P$ Dft($*264$)

$*265$. *The series of Alephs.*

$\cdot 01 \ \ \omega_1 \ = \ \hat{P}\{\overrightarrow{\mathrm{less}}`P \ = \ (\aleph_0)_r \cup \Omega\,\mathrm{fin}\}$ Df (cf $*262\cdot 03$)

$\cdot 02 \ \ \aleph_1 \ = \ C``\omega_1$ Df

$\cdot 03 \ \ \omega_2 \ = \ \hat{P}\{\overrightarrow{\mathrm{less}}`P \ = \ (\aleph_1)_r \cup (\aleph_0)_r \cup \Omega\,\mathrm{fin}\}$ Df

$\cdot 04 \ \ \aleph_2 \ = \ C``\omega_2$ Df *etc.*

[32] This is presented with "Hence the assumption that a selection can be made from all the existent sub-classes of $\mu$ is equivalent to the assumption that $\mu$ can be well-ordered or is a unit class, i.e. . . . " (*PM* III, p.97). Zermelo's theorem is described as the assertion that assuming the multiplicative axiom as a hypothesis, every set can be well ordered and Zermelo (1908a) is cited in the Summary of $*257$.

Much of the usually accepted theory of transfinite ordinals is fallacious.[33] We cannot prove that $\omega_1$ is not the limit of a progression, without Mult ax, since the proof depends on $*113{\cdot}32$, *viz.*

$\vdash :. \text{Mult ax} . \supset : \mu, \nu \in \text{NC} . \kappa \in \nu \cap \text{Cl excl}`\mu . \supset . s`\kappa \in \mu \times_c \nu.$

Whence $\vdash :. \text{Mult ax} . \supset : \kappa \in \aleph_0 \cap \text{Cl excl}`\aleph_0 . \supset . s`\kappa \in \aleph_0$ . Mult ax is essential here.

*27*

*Compact Series.*

$*270{\cdot}01 \ \ \text{comp} = \hat{P}\,(P \mathrel{\subset\!\!\!\cdot} P^2)$ Df

Thus Ser $\cap$ comp is the class of compact series.

$*271$. Median Classes. $\cdot 01 \ \ \text{med} = \hat{\alpha}\,\hat{P}\,(\alpha \subset C`P \,.\, P \mathrel{\subset\!\!\!\cdot} P \restriction \alpha | P)$ Df

$*272$. Similarity of Position.

$\cdot 01 \ \ T_{\text{PQ}} = \hat{x}\,\hat{y}\,\{x \in C`P . y \in C`Q . \text{D}`T \cap \overrightarrow{P}`x \subset T``\overrightarrow{Q}`y .$
$\text{D}`T \cap \overleftarrow{P}`x \subset T``\overleftarrow{Q}`y . \text{D}`T \cap \iota`x \subset \overrightarrow{T}`y\}$ Df

$*273$. Rational Series.

$\cdot 01 \ \ \eta = \text{Ser} \cap \text{comp} \cap \breve{C}``\aleph_0 \cap \breve{P}(\text{D}`P = \text{Ꭰ}`P)$ Df [Cantor's $\eta$]

$\cdot 02 \ \ R_{SPQ}`T = T \mathbin{\dot\cup} \text{seq}_R`\text{D}`T \downarrow \text{min}_S`\overleftarrow{T}_{PQ}`\text{seq}_R`\text{D}`T$ (Dft, $*273$)

$\cdot 03 \ \ (RS)_{PQ} = \overrightarrow{(R_{SPQ})_*}`\dot{\Lambda}$ (Dft, $*273$)

$\cdot 04 \ \ T_{RSPQ} = \dot{s}`(RS)_{PQ}$ (Dft, $*273$)

These Dfs are needed for proving $P, Q \in \eta . \supset . P$ smor $Q$ $(*273{\cdot}4)$

$*274$. Series of finite sub-classes of a series.

$\cdot 01 \ \ P_\eta = P_{\text{cl}} \mathbin{\restriction\!\!\!\cdot} (\text{Cls induct} - \iota`\Lambda)$ Df

$\cdot 02 \ \ P_m`\kappa = \text{max}_P`\text{min}_P``\kappa$ Dft($*274$)

$\cdot 03 \ \ \breve{T}_P`\kappa = (-\iota`P_m`\kappa)``(\kappa \cap \overleftarrow{\text{min}}_P`P_m`\kappa) - \iota`\Lambda$ Dft($*274$)

$\cdot 04 \ \ M_P`\kappa = P_m``\overleftarrow{(T_P)_*}`\kappa$ Dft($*274$)

We prove

$\cdot 25 \ \ \vdash : \breve{P} \in \omega . \supset . \breve{P}_\eta \in \omega$

$\cdot 25 \ \ \vdash : P \in \omega . \supset . P_\eta \in \eta$

$*275$. *Continuous Series* $\cdot 01 \ \theta = \text{Ser} \cap \text{Ded} \cap \breve{\text{med}}``\aleph_0$ Df (cf $*271{\cdot}01$)

$\cdot 21 \ \vdash : P \in \eta . \supset . \varsigma`P \in \theta$ $\quad$ $\cdot 32 \ \vdash : P \in \theta . \supset . \theta = \text{Nr}`P$

$*276{\cdot}01 \ P_\theta = P_{\text{cl}} \mathbin{\restriction\!\!\!\cdot} (-\text{Cls induct} \cup \iota`\Lambda)$ Df

Here we prove $\vdash : P \in \Omega\,\text{infin} . \supset . P_\theta \in \text{Ded}$

[33] Notice that the theory of transfinite ordinals is fallacious because it relies on an additional assumption, Mult ax, not because it cannot be derived from logic alone. See Boolos (1994) for the issue of "logicism" in *PM*.

$\vdash : P \in \omega . \supset . P_\theta \in \theta \qquad \vdash . C``\theta = 2^{\aleph_0}$
$\vdash . \theta = (\omega \exp_r \omega) \dot{+} \dot{1}$[34]

*28*

*Part VI. Quantity.*

This part is concerned with ratios, and, as a preliminary to measurement, with families of vectors.

$*300$. *Positive and Negative Integers.*

$\cdot 01 \;\; U = (+_c 1)_{\mathrm{po}} \restriction (\mathrm{NC\ induct} - \iota`\Lambda)$ Df
$\cdot 02 \;\; \mathrm{Rel\,num} = (1 \to 1) \cap \hat{R}\,(\mathrm{Pot}`R \subset \mathrm{Rl}`J)$ Df
$\cdot 03 \;\; \mathrm{Rel\,num\,id} = (1 \to 1) \cap \hat{R}\,(\mathrm{Potid}`R - \iota`R_0 \subset \mathrm{Rl}`J)$ Df

*N.B.* If Infin ax ($*125$) is false, $\Lambda \in$ NC induct. If the number of individuals in the world is $\nu$, $\nu +_c 1 = \Lambda$ in the type of individuals, $2^\nu +_c 1 = \Lambda$ in the type of classes of individuals, etc. The truth or falsehood of Infin ax is obviously an empirical question, and therefore its truth ought not to be assumed in pure mathematics.[35]

$*301$. *Numerically defined powers of relations.*

$\cdot 01 \;\; R_P = (|R) \parallel (\breve{U}_1 \restriction t^3`R)$ Dft($*301$) [cf $* 43\cdot 01$]
$\cdot 02 \;\; \mathrm{num}(R) = \overrightarrow{(R_P)_*}`\{I \restriction C`R) \downarrow (0 \cap t^2`R)\}$ Dft($*301$)
$\cdot 03 \;\; R^\sigma = \{\dot{s}`\mathrm{num}(R)\}`\dot{\sigma}$ Df

This is the same meaning as before for $R^2$ and $R^3$.

*29*

$*302$. *Relative Primes.*

$\cdot 01\ \mathrm{Prm} = \hat{\rho}\,\hat{\sigma}\{\rho, \sigma \in \mathrm{NC\,induct} : \rho = \xi \times_c \tau . \sigma = \eta \times_c \tau . \supset_{\xi,\eta,\tau} . \tau = 1\}$ Df
$\cdot 02 \;\; (\rho, \sigma)\,\mathrm{Prm}_\tau\,(\mu, \nu) . =$
$. \rho\,\mathrm{Prm}\,\sigma . \tau \in \mathrm{NC\ induct} - \iota`0 . \mu = \rho \times_c \tau . \nu = \sigma \times_c \tau$ Df
$\cdot 03 \;\; (\rho, \sigma)\,\mathrm{Prm}\,(\mu, \nu) . = . (\exists T) . (\rho, \sigma)\,\mathrm{Prm}_\tau\,(\mu, \nu)$ Df
$\cdot 04 \;\; \mathrm{hcf}(\mu, \nu) . = . (\imath\tau)\{(\exists\,\rho, \sigma) . (\rho, \sigma)\,\mathrm{Prm}_\tau\,(\mu, \nu)\}$ Df
$\cdot 05 \;\; \mathrm{lcm}(\mu, \nu) . = . (\imath\xi)\{(\exists\,\rho, \sigma, \tau) . (\rho, \sigma)\,\mathrm{Prm}_\tau\,(\mu, \nu) . \xi = \rho \times_c \sigma \times_c \tau\}$ Df

$*303$. *Ratios.*

$\cdot 01 \;\; \mu/\nu = \hat{R}\,\hat{S}\{(\exists \rho, \tau) . (\rho, \sigma)\,\mathrm{Prm}\,(\mu, \nu) . \dot{\exists}\,!\,R^\sigma \dot{\cap} S^\rho$ Df

In important cases, $R$, $S$ will be vectors. E.g. we might have
$R = (+_c 2) \restriction \mathrm{NC\ induct}$, $S = (+_c 3) \restriction \mathrm{NC\ induct}$. Then $\dot{\exists}\,!\,R^3 \dot{\cap} S^2$

[34] The first three are $*276\cdot 4\cdot 41$, $\cdot 42$, $\cdot 43$, and the last is stated and proved, without a star number, at the end of $*276$ (*PM* III, pp.228–30).

[35] However, it may still be stated and used, as an hypothesis, even though it is not a logical truth, contrary to Wittgenstein (*TLP* 5.535).

$\cdot 02 \quad 0_q \;=\; \dot{s}`0/``\mathrm{NC\ induct}$ Df

$\cdot 03 \quad \infty_q \;=\; \dot{s}`/0``\mathrm{NC\ induct}$ Df

$\cdot 04 \quad \mathrm{Rat} \;=\; \hat{X}\,\{(\exists\mu,\nu)\,.\,\mu,\nu\in \mathrm{NC\ induct}\,.\,\nu\neq 0\,.\,X=\mu/\nu\}$ Df

$\cdot 05 \quad \mathrm{Rat\,def} \;=\; \hat{X}\,\{(\exists\mu,\nu)\,.\,\mu,\nu\in \mathrm{D}`U\cap \mathrm{G}`U\,.\,X=(\mu/\nu)\restriction t_{11}`\mu\}\,.$ Df

$*304$. *The Series of Ratios*.

$\cdot 01 \quad X <_r Y\,.=.\,(\exists\mu,\nu,\rho,\sigma).\,\mu,\nu,\rho,\sigma\in\mathrm{NC\ induct}-\iota`0\,.\,\sigma\neq 0\,.$
$\mu\times_{\mathrm{c}}\sigma<\mu\times_{\mathrm{c}}\rho\,,\,X=\mu/\nu\,.\,Y=\rho/\sigma\}$ Df

$\cdot 02 \quad H \;=\; \hat{X}\,\hat{Y}\,\{X,Y\in\mathrm{Rat\,def}\,.\,X<_r Y\}$ Df

$\cdot 03 \quad H' \;=\; \hat{X}\,\hat{Y}\,\{X,Y\in\mathrm{Rat\,def}\cup\iota`0_q\,.\,X<_r Y$ Df

$\cdot 33 \quad \vdash :\mathrm{Infin\,ax}\,.\supset.\,H\in\eta$

*30*

$*305\cdot 01 \quad X\times_s Y \;=\; \hat{R}\hat{S}\{(\exists\mu,\nu,\rho,\sigma).\,\mu,\nu,\rho,\sigma\in\mathrm{NC\ induct}\,.\,\nu\neq 0.\,\sigma\neq 0\,.$
$X=\mu/\nu\,.\,Y=\rho/\nu\,.\,R(\mu\times_{\mathrm{c}}\rho/\nu\times_{\mathrm{c}}\sigma)S\}$ Df

$*306\cdot 01 \quad X+_s Y \;=\; \hat{R}\hat{S}\{(\exists\mu,\nu,\rho).\,\mu,\nu,\rho\in\mathrm{NC\ induct}\,.\,\nu\neq 0\,.$
$X=\mu/\nu\,.\,Y=\rho/\nu\,.\,R(\mu+_{\mathrm{c}}\rho/\nu)S\}$ Df

$*307$. *Generalized Ratios*. (i.e. including negative ratios)

$\cdot 01 \quad \mathrm{Rat}_n \;=\; |\,\mathrm{Cnv}``\mathrm{Rat}$ (Df of negative ratios) Df

$\cdot 011 \quad \mathrm{Rat}_g \;=\; \mathrm{Rat}\cup\mathrm{Rat}_n$ Df

$\cdot 02 \quad <_n \;=\; |\,\mathrm{Cnv}\dot{;}<_r$ Df

$\cdot 021 \quad >_n \;=\; \mathrm{Cnv}`<_n$ Df

$\cdot 03 \quad <_g \;=\; (>_n)\,\dot{\cup}\,(<_r)\,\dot{\cup}\,(\mathrm{Rat}_n-\iota`0_q)\uparrow\mathrm{Rat}$ Df

$\cdot 031 \quad >_g \;=\; \mathrm{Cnv}`<_g$ Df

$\cdot 04 \quad H_n \;=\; |\,\mathrm{Cnv}\dot{;}H$ Df

$\cdot 05 \quad H_g \;=\; \breve{H}_n \,\dagger\, H'$ Df

$*308$. *Addition of Generalized Ratios*.

$\cdot 01 \quad X-_s Y \;=\; \hat{R}\,\hat{S}\,\{(\exists Z):\,X,Y,Z\in\mathrm{Rat}:\,Z+_s Y=X\,.\,RZS\,.$
$\vee.\,Z+_s X=Y\,.\,RZ\breve{S}\}$ Df

$\cdot 02 \quad X+_g Y \;=\; (X+_s Y)\,\dot{\cup}\,(X-_s Y|\mathrm{Cnv})\,\dot{\cup}\,(Y-_s X|\mathrm{Cnv})\,\dot{\cup}$
$(X|\mathrm{Cnv}+_s Y|\mathrm{Cnv})$ Df

$*309\cdot 01 \quad X\times_g Y \;=\; (X\times_s Y)\,\dot{\cup}\,(X|\mathrm{Cnv}\times_s Y|\mathrm{Cnv})\,\dot{\cup}$
$(X\times_s Y|\mathrm{Cnv})|\mathrm{Cnv}\,\dot{\cup}\,(X|\mathrm{Cnv}\times_s Y)|\mathrm{Cnv}))$ Df

*31*

$*310$. *Real Numbers*.

$\cdot 01 \quad \Theta \;=\; (\varsigma`H)\restriction(-\iota`\Lambda-\iota`\mathrm{D}`H)$ Df

$\cdot 011 \quad \Theta' \;=\; \iota`0_q \dotplus \Theta$ Df

$\cdot02 \quad \Theta_n \;=\; (\varsigma\text{‘}H_n) \upharpoonright (-\iota\text{‘}\Lambda - \iota\text{‘}\mathrm{D}\text{‘}H_n) \qquad \mathrm{Df}$
$\cdot021 \;\; \Theta'_n \;=\; \iota\text{‘}0_q \nleftarrow \Theta_n \qquad \mathrm{Df}$
$\cdot03 \quad \Theta_g \;=\; \breve{\Theta}_n \;\,\uparrow\!\!\!\!+\;\, \Theta' \qquad \mathrm{Df}$
$\cdot15 \quad \vdash:\ \text{Infin ax}\,.\supset.\,\Theta' \nrightarrow C\text{‘}H,\ \Theta'_n \nrightarrow C\text{‘}H'_n,\ C\text{‘}H_n \nleftarrow \Theta_g \nrightarrow C\text{‘}H \in \theta$
$\cdot151 \;\; \vdash:\ \text{Infin ax}\,.\supset.\,\Theta',\ \Theta'_n \in \text{Ser} \cap \text{comp} \cap \text{semi Ded}$

$*311$. *Addition of concordant real numbers.*

$\cdot01 \quad \text{concord}(\mu, \nu, \ldots).\ =:\mu, \nu, \ldots \in C\text{‘}\Theta'.\vee.\,\mu, \nu, \ldots \in C\text{‘}\Theta'_n \quad \mathrm{Df}$
$\cdot02 \quad \mu +_p \nu \;=\; \hat{X}\{\text{concord}(\mu, \nu)\,.\,X \in s\text{‘}\mu \underset{,,}{+}_g\text{“}\nu\} \quad \mathrm{Df}$

$*312$. *Algebraic addition of real numbers.*

$\cdot01 \;\; \mu -_p \nu \;=\; \hat{X}\{(\exists\lambda):\lambda, \mu, \nu \in C\text{‘}\Theta_g : \nu +_p \lambda = \mu\,.\,X \in \lambda\,.\vee.$
$\mu +_p \lambda \;=\; \nu\,.\,X|\text{Cnv“}\lambda\} \quad \mathrm{Df}$
$\cdot02 \;\; \mu +_a \nu \;=\; (\mu +_p \nu) \cup (\mu -_p |\text{Cnv“}\nu) \quad \mathrm{Df}$

$*313\cdot01 \;\; \mu \times_a \nu \;=\; \hat{X}\{\mu, \nu \in C\text{‘}\Theta_g\,.\,X \in s\text{‘}\mu \underset{,,}{\times}_g\text{“}\nu\} \qquad \mathrm{Df}$

These Dfs give addition and multiplication of real numbers as required by ordinary analysis. A possible alternative Df of a real number is as $\dot{s}\text{‘}\mu$, where $\mu$ is a real number as formerly defined. We put

$*314\cdot01 \; X +_r Y = \hat{R}\,\hat{S}\{(\exists\,\mu, \nu)\,.\,X = \dot{s}\text{‘}\mu\,.\,Y = \dot{s}\text{‘}\nu\,.\,R(\dot{s}\text{‘}\overline{\mu +_a \nu})S\} \quad \mathrm{Df}$
$\cdot02 \;\; X \times_r Y \;=\; \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \times_a \quad \ldots \quad \mathrm{Df}$

These Dfs are used in $*356$.

*32*

*Vector Families.*

$*330\cdot01 \;\; \text{cr‘}\alpha \;=\; (1 \to 1) \cap \overleftarrow{\mathrm{Cl}}\text{‘}\alpha \cap \breve{\mathrm{D}}\text{“Cl‘}\alpha \qquad \mathrm{Df}$
$\cdot02 \;\; \text{Abel} \;=\; \hat{\kappa}\,\{R, S \in \kappa\,.\supset_{R,S}.\,R|S = S|R) \qquad \mathrm{Df}$
$\cdot03 \;\; \text{fm‘}\alpha \;=\; \text{Abel} \cap \text{Cl ex‘cr‘}\alpha \qquad \mathrm{Df}$
$\cdot04 \;\; FM \;=\; s\text{‘D‘fm} \qquad \mathrm{Df}$
$\cdot05 \;\; \kappa_\iota \;=\; s\text{‘}(\text{Cnv“}\kappa)\underset{,,}{|}\text{“}\kappa \qquad \mathrm{Df}$

$*331$. *Connected Families.*

$\cdot01 \;\; \text{conx‘}\kappa \;=\; s\text{‘}\mathrm{Cl}\text{“}\kappa \cap \hat{\alpha}\,(\overrightarrow{\dot{s}\text{‘}\kappa}\text{‘}\alpha \cup \overleftarrow{\dot{s}\text{‘}\kappa}\text{‘}\alpha \;=\; s\text{‘}\mathrm{Cl}\text{“}\kappa\,) \qquad \mathrm{Df}$
$\cdot02 \;\; \text{FM conx} \;=\; \text{FM} \cap \hat{\kappa}\,\{\exists\,!\,\text{conx‘}\kappa\,) \qquad \mathrm{Df}$

$*332$. *The Representative of a relation in a family.*

$\cdot01 \;\; \text{rep}_\kappa\text{‘}P \;=\; \dot{s}\text{‘}(\kappa_\iota \cap \overleftarrow{\in}\text{‘}P\,) \qquad \mathrm{Df}\ \ [\text{with suitable hp, rep}_\kappa\text{‘}P \in \kappa_\iota]$

$*333$. *Open Families.*

$\cdot01 \;\; \kappa_\partial \;=\; \kappa - \text{Rl‘}I \quad \mathrm{Df} \qquad \cdot011 \;\; \kappa_{\iota\partial} \;=\; (\kappa_\iota)_\partial \quad \mathrm{Df}$
$\cdot02 \;\; FM\ \text{ap} \;=\; FM \cap \hat{\kappa}\,\{s\text{‘Pot“}\kappa_{\iota\partial} \subset \text{Rl‘}J\,\} \qquad \mathrm{Df}$
$\cdot03 \;\; FM\text{ap conx} \;=\; FM\ \text{ap} \cap FM\text{conx} \;\; \mathrm{Df}\ [\text{Similiar Dfs assumed later}]$

$*334$. *Serial Families.*

$\cdot 01$ $\text{trs}`\kappa \;=\; s`\text{Ɑ}``\kappa \cap \hat{\alpha}\,\{(\dot{s}`\kappa_\partial)``\overrightarrow{\dot{s}`\kappa_\partial}`\alpha \subset \overrightarrow{\dot{s}`\kappa_\partial}`\alpha\}$ Df

$\cdot 02$ $FM\,\text{trs} \;=\; FM \cap \hat{\kappa}\,(\exists\,!\,\text{trs}`\kappa)$ Df

$\cdot 03$ $FM\,\text{connex} \;=\; FM \cap \hat{\kappa}\,(\exists\,!\,\text{conx}`\kappa \cap p`C``\kappa_\iota)$ Df

$\cdot 04$ $FM\,\text{sr} \;=\; FM\,\text{trs} \cap FM\,\text{connex}$ Df

$\cdot 05$ $FM\,\text{asym} \;=\; FM \cap \hat{\kappa}(\kappa \cap \text{Cnv}``\kappa \subset \text{Rl}`I)$ Df

$\cdot 3$ $\vdash :\ \kappa \in FM\,\text{sr}\,.\supset.\,\dot{s}`\kappa_\partial \in \text{Ser}$

*33*

$*335$. *Initial Families.*

$\cdot 01$ $\text{init}`\kappa \;=\; \breve{\iota}\,`(\text{conx}`\kappa - s`\text{D}``\kappa_\partial)$ Df

$\cdot 02$ $FM\,\text{init} \;=\; FM \cap \text{Ɑ}\,`\text{init}$ Df

$*336$. *The Series of Vectors.*

$\cdot 01$ $V_\kappa \;=\; \hat{M}\,\hat{N}\{M, N \in \kappa_\iota : (\exists x).(M`x)(\dot{s}`\kappa_\partial)(N`x)\}$ Df

$\cdot 011$ $U_\kappa \;=\; V_\kappa \upharpoonright \kappa$ Df

[These Dfs define greater and less among vectors.]

$\cdot 02$ $A_a \;=\; \hat{x}\,\hat{R}(xRa)$ Df

$*337$. *Multiples and Submultiples of vectors.*

$\cdot 13$ $\vdash :.\ \kappa \in FM\,\text{sr}\,.\,\breve{P} = \dot{s}`\kappa_\partial\,.\,P \in \text{Semi Ded}\,.\,R \in \kappa_\partial\,.\,a \in C`P\,.\supset:$
$x \in C`P\,.\supset.\,(\exists\nu)\,.\,\nu \in \text{NC induct} - \iota`0\,.\,xP(R^\nu`a)$

[Axiom of Archimedes]

$\cdot 27$ $\vdash :.\ \kappa \in FM\,\text{sr}\,.\,\text{Cnv}`\dot{s}`\kappa_\partial \in \text{comp} \cap \text{Semi Ded}\,.\supset:$
$S \in \kappa\,.\,\nu \in \text{NC ind} - \iota`0\,.\supset.\,(\exists L)\,.\,L \in \kappa\,.\,S = L^\nu$

[Axiom of divisibility]

*Measurement*

If $X$ is a ratio as previously defined, and $\kappa$ a vector-family, $X \upharpoonright \kappa$ is the ratio $X$ as applied to the family $\kappa$.

$*350\cdot 43$ $\vdash :.\ \kappa \in FM\,\text{ap conx}\,.\,\mu, \nu \in \text{NC ind}\,.\,\sim(\mu = \nu = 0)\,.$
$R, T \in \kappa\,.\supset:\ R(\mu/\nu)T\,.\equiv.\,R^\nu = T^\mu$

$*351$ *Submultiple Families.*

$\cdot 01$ $FM\,\text{subm} \;=$
$FM \cap \hat{\kappa}\,\{R \in \kappa\,.\,\nu \in \text{NC ind} - \iota`0\,.\supset_{R,\nu}.(\exists S)\,.\,S \in \kappa\,.\,R = S^\nu\}$ Df

We prove

$\cdot 31$ $\vdash :\ \kappa \in FM\,\text{ap subm conx}\,.\,X, Y \in C`H'\,.\supset.$
$(X \upharpoonright \kappa_\iota)|(Y \upharpoonright \kappa_\iota) = (X \times_s Y) \upharpoonright \kappa_\iota$

$\cdot 43 \ \vdash :\ \text{Same hp} \,.\, s`\text{Pot}``\kappa \subset \kappa \,.\, R \in \kappa \ \supset .$

$$(X \lfloor \kappa`R)|(Y \lfloor \kappa`R) = (X \times_s Y) \lfloor \kappa`R$$

*34*

$*352$. *Rational multiples of a given vector.*

$\cdot 01 \ \ T_\kappa \ = \ \kappa \rceil A_T \dot{;} H' \qquad \text{Df}$

$\cdot 02 \ \ T_{\kappa\iota} \ = \ \kappa_\iota \rceil A_T \dot{;} H_g \qquad \text{Df}$

These Dfs enable us to construct "rational nets" as in assigning coordinates in projective geometry. With $T$ as unit, the vectors whose measure is $X$ (where $X \in C`H'$) is $\kappa \rceil A_T`X$. Thus $\kappa \rceil A_T \dot{;} H'$ is the series of those vectors which have a rational ratio to the unit $T$.

$*353$. *Rational Families.*

$\cdot 01 \ \ FM\,\text{rt} \ = \ FM \cap \hat{\kappa}\{(\exists T).\, T \in \kappa_\partial \,.\, \kappa \subset A_T``C`H'\} \qquad \text{Df}$

$\cdot 02 \ \ FM\,\text{cx} \ = \ FM \cap \hat{\lambda}\{(\exists \kappa).\, \kappa \in FM\text{conx}.\, \lambda \subset \kappa\} \qquad \text{Df}$

$*354$. *Rational Nets.*

$\cdot 01 \ \ \kappa_g \ = \ \kappa \cup \text{Cnv}``(\kappa \cap \overleftarrow{D}`s`\text{Ɑ}``\kappa) \qquad \text{Df}$

$\cdot 02 \ \ \text{cx}_a`\lambda \ = \lfloor (A_a``\lambda_\iota)``\lambda \qquad \text{Df}$

$\cdot 03 \ \ FM\text{grp} \ = \ FM \cap \hat{\kappa}(s`\kappa \underset{,,}{|}``\kappa \subset \kappa) \qquad \text{Df}$

[$s`\kappa \underset{,,}{|}``\kappa \subset \kappa$ is the usual Df of a group]

$*356$. *Measurement by Real Numbers.*

$\cdot 01 \ \ X_\kappa \ = \ \text{prec}``(U_\kappa) \,|\, \overrightarrow{X} \upharpoonright \kappa \qquad \text{Df}$

With suitable hps, $X$, $Y$ being relational real numbers,

$$X_\kappa | Y_\kappa = (X \times_r Y)_\kappa \ , \ (X_\kappa`R) \,|\, (Y_\kappa`R) = (X +_r Y)_\kappa`R$$

$*359$. *Existence-Theorems for Vector-Families.* (All follow from Infin ax.)

*35*

*Cyclic Families.*

[E.g. Angles in a plane round a point. The preceding theories do not apply to these without adaptation.]

$*370\cdot 01 \ FM\text{cycl} = (FM\,\text{conx} - 2) \cap \hat{\kappa}\{\kappa_\partial \rceil U_\kappa \in \text{trans} : (\exists K).\, K \in \kappa_\partial .\, K = \breve{K}\}\ \text{Df}$

$\cdot 02 \ \ K_\kappa \ = \ (\imath K)(K \in \kappa_\partial \,.\, K = \breve{K}) \quad \text{Df}$ [ In the case of angles, $K$ is $\pi$]

$\cdot 03 \ \ I_\kappa \ = \ I \upharpoonright s`\text{Ɑ}``\kappa \qquad \text{Df}$

$*371$. *The Series of Vectors.*

$\cdot 01 \ \ W_\kappa \ = \ K_\kappa |\dot{;} U_\kappa \lfloor \kappa_\partial \ \dot{+} \ U_\kappa \lfloor \kappa_\partial \quad \text{Df}$

$*372$. *Integral Sections of the Series of Vectors.*

[In angles, submultiples of $2\pi$]

$\cdot 01 \quad \nu_\kappa \;=\; (\kappa \cup \text{Cnv}``\kappa) \cap \hat{R}(\sigma < \nu \,.\, \sigma \neq 0 \,.\supset_\sigma .\, R^{\sigma+1} W_\kappa R^\sigma) \qquad \text{Df}$

[E.g. $\nu_\kappa$ is angles not greater than $2\pi/\nu$]

$*373$. *Submultiples of identity*. [In angles, submultiples of $2\pi$]

$\cdot 01 \quad M_{\nu\kappa} \;=\; \hat{Q}\,\hat{P}(Q \in \kappa_\partial \,.\, Q^\nu = P) \qquad \text{Dft } (*373-5)$

$\cdot 02 \quad \text{Prime} \;=\; \text{NC induct} \cap \hat{\mu}(\mu = \sigma \times_c \tau \,.\supset_{\sigma,\tau}: \sigma = 1 \,.\vee.\, \sigma = \mu) \quad \text{Df}$

$\cdot 03 \quad (S, \nu) \;=\; \hat{P}\,(P^\nu = S : \sigma < \nu \,.\, \sigma \neq 0 \,.\supset_\sigma .\, P^\sigma \neq S) \;\; \text{Dft } (*373-5)$

We prove, with suitable hp, $\nu_\kappa \cap \hat{S}(S^\nu = I_\kappa) \in 1$.

$*374$. *Principal Submultiples.*

Here we prove

$*374\cdot 2 \;\; \vdash : \kappa \in FM\text{cycl subm} \,.\, R \in \kappa \cup \text{Cnv}``\kappa \,.\supset .\, \nu_\kappa \cap \hat{P}(P^\nu = R) \in 1$

$*375$. *Principal Ratios.*

$\cdot 01 \quad (\mu/\nu)_\kappa \;=\; \hat{R}\hat{S}\{(\exists T) \,.\, T \in \mu_\kappa \cap \nu_\kappa \,.\, R = T^\mu \,.\, S = T^\nu\} \qquad \text{Df}$

This Df gives us a method of measurement for cyclic families.

[ $\mu_\kappa \cap \nu_\kappa$ is $\mu_\kappa$ or $\nu_\kappa$, whichever is smaller.]

# Introduction to the second edition

### Editorial note

The manuscripts of the Introduction and Appendices A, B, and C are transcribed below with almost all the changes due to the publisher's "house style" imposed, as in the use of capitals in the title. More substantial changes which Russell made as corrections to the printer's proof sheets are indicated in footnotes as " '. . . ' becomes '. . . ' ". Russell struck out some material with a single line (reproduced here in the text as ~~deletions~~) and sometimes added some material above the line, and that is indicated as well. These insertions, usually above the line with a caret to indicate their position, are inserted in the line with '⌊' and '⌋' around the inserted material, as follows: ⌊insertions⌋. A sequence below such as "Given $p$ and $p|(q|r)$, we can infer ⌊$r$⌋ ~~$q$~~" indicates that the letter $q$ was deleted and changed to $r$ probably immediately after it was first written. Footnotes in the manuscript are indicated by a line across the page but are presented here as they appear in print. The printed Introduction begins on page *xiii*, indicated below in parentheses. Subsequent page numbers are inserted in the text where the page begins in the printed version. The number in the manuscripts ("foliation") is reproduced in italics in the upper right hand corner of the material from each page. When Russell reused a leaf, giving it a new number, this is indicated in a footnote. Thus page *38* of the manuscript was previously *49a*. That it was probably from the HPF manuscript can be determined from the Editorial note to HPF below. Corrections or other changes between the manuscript and the final published version are indicated in the footnotes using the expressions "becomes" or "corrected to" to distinguish changes in the manuscript from those made later in the publication process.

This manuscript is catalogued as 230.031350-F1, F2 in the Bertrand Russell Archives, and is stored in the same box, 3.84, as the other manuscript material to follow.

## The manuscript

(xiii)

### *INTRODUCTION TO THE SECOND EDITION*∗[1]

In preparing this new edition of *Principia Mathematica*, the authors have thought it best to leave the text unchanged, except as regards misprints and minor errors †, even where they were aware of possible improvements. The chief reason for this decision is that any alteration of the propositions would have entailed alterations of the references, which would have meant a very great labour. It seemed preferable, therefore, to state in an introduction the main improvements which appear desirable. Some of these are scarcely open to question; others are, as yet, a matter of opinion.

The most definite improvement resulting from work in mathematical logic during the past fourteen years is the substitution, in Part I, Section A, of the one indefinable "$p$ and $q$ are incompatible" (or, alternatively, "$p$ and $q$ are both false") for the two indefinables "not-$p$" and "$p$ or $q$". This is due to Dr. H. M. Sheffer.‡ Consequentially, M. Jean Nicod§ showed that one primitive proposition could replace the five primitive propositions $*1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6$.

From this there follows a great simplification in the building up of molecular propositions and matrices; ∗9 is replaced by a new chapter, ∗8, given in the Appendix to this Volume.[2]

* In this Introduction, as well as in the Appendices, the authors are under great obligations to Mr. F. P. Ramsey of King's College, Cambridge, who has read the whole in MS. and contributed valuable criticisms and suggestions.
† In regard to these we are indebted to many readers, but especially to Drs. Behmann and Boscovitch, of Göttingen.
‡ *Trans. Amer. Math. Soc.* Vol. XIV, pp. 481-488.
§ A reduction in the number of the primitive propositions of logic. *Proc. Camb. Phil. Soc.* Vol. XIX.

*2*

Another point about which there can be no doubt is that there is no need of the distinction between real and apparent variables, nor of the primitive idea "assertion of a propositional function". On all occasions where, in *Principia Mathematica*, we have an asserted proposition of the form "$\vdash . fx$" or "$\vdash . fp$", this is to be taken as meaning "$\vdash . (x). fx$" or "$\vdash . (p). fp$". Consequently the primitive proposition $*1\cdot 11$ is no longer required.[3] All that is necessary, in order to adapt the propositions as printed to this change, is the convention that, when the scope of an apparent variable is the whole of the asserted proposition in which it occurs, this fact will not be explicitly indicated unless "some" is involved instead of "all". That is to

[1] 'Symonds', underlined, is written in pencil by a different hand at an angle in the upper left, indicating the typesetter.
[2] This becomes 'Appendix A to this Volume', in print.
[3] $*1\cdot 11$. When $\phi x$ can be asserted, where $x$ is a real variable, and $\phi x \supset \psi x$ can be asserted, where $x$ is a real variable, then $\psi x$ can be asserted, where $x$ is a real variable. Pp.

say, "$\vdash . \phi x$" is to mean "$\vdash . (x). \phi x$"; but in "$\vdash . (\exists x). \phi x$" it is still necessary to indicate explicitly the fact that "some" $x$ (not "all" $x$'s) is involved.

It is possible to indicate more clearly than was done formerly what are the novelties introduced in Part I Section B as compared with Section A. (xiv) They are three in number, two being essential logical novelties, and the third merely notational.

(1). For the "$p$" of Section A, we substitute "$\phi x$", so that in place of "$\vdash .(p). fp$" we have "$\vdash .(\phi, x). f(\phi x)$". Also, if we have "$\vdash . f(p, q, r, \ldots)$", we may substitute $\phi x, \phi y, \phi z, \ldots$ for $p, q, r, \ldots$ or $\phi x, \phi y$ for $p, q$, and $\psi z, \ldots$ for $r, \ldots$, and so on. We thus obtain a number of new general propositions ⌊different⌋ from those of Section A.

*3*

(2). We introduce in Section B the new primitive idea "$(\exists x). \phi x$", i.e. existence-propositions, which do not occur in Section A. In virtue of the abolition of the real variable, general propositions of the form "$(p). fp$" do occur in section A, but "$(\exists p). fp$" does not occur.

(3). By means of definitions, we introduce in Section B general propositions which are molecular constituents of other propositions; thus "$(x). \phi x . \vee . p$" is to mean "$(x). \phi x \vee p$".

It is these three novelties which distinguish Section B from Section A.

One point in regard to which improvement is obviously desirable is the axiom of reducibility ($*12{\cdot}1 \cdot 11$). This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content. On this subject, however, it cannot be said that a satisfactory solution is as yet obtainable. Dr. Leon Chwistek* took the heroic course of dispensing with the axiom without adopting any substitute; from his work, it is clear that this course compels us to sacrifice a great deal of ordinary mathematics. There is another course, recommended by Wittgenstein† for philosophical reasons. This is to assume that functions of propositions are always truth-functions, and that a function can only occur in a

*In his "Theory of Constructive Types". See references at the end of this Introduction.

†*Tractatus Logico-Philosophicus*, $*5{\cdot}54$ ff.

*4*

proposition through its values. There are difficulties in the way of this view, but perhaps they are not insurmountable.‡ It involves the consequence that all functions of functions are extensional. It requires us to maintain that "A believes $p$" is not a function of $p$. How this is possible is shown in *Tractatus Logico-Philosophicus* (*loc. cit.* and pp. 19–21). We are not prepared to assert that this theory is certainly right, but it has seemed worth while to work out its consequences in the following pages. It appears that everything in Vol. I remains true (though often new proofs are required); the theory of inductive cardinals and ordinals survives; but ⌊it seems that⌋ the theory of infinite Dedekindian and well-ordered series largely collapses, so that irrationals ⌊and real numbers generally,⌋ can no longer be adequately dealt with ~~and it is even doubtful whether we can prove that $\sqrt{2}$ is irrational.~~ Also Cantor's proof that $2^n > n$ breaks down unless $n$ is finite. Perhaps some further axiom, less objectionable than the axiom of reducibility, might give these results, but we have not succeeded in finding such an axiom.

(xv) It should be stated that a new and very powerful method in mathematical logic has been invented by Dr. H. M. Sheffer. This method, however, would demand a complete re-writing of *Principia Mathematica*. We recommend this task to Dr. Sheffer, since what has so far been published by him is scarcely sufficient to enable others to undertake the necessary reconstruction.

We now proceed to the detailed development of the above general sketch.

‡ See Appendix C.

*5*[4]

## *I. ATOMIC AND MOLECULAR PROPOSITIONS*

Our system begins with "atomic propositions". We accept these as a datum, because the problems which arise concerning them belong to the philosophical part of logic, and are not amenable (at any rate at present) to mathematical treatment.

Atomic propositions may be defined negatively as propositions containing no parts that are propositions, and not containing the notions "all", or "some". Thus "this is red", "this is earlier than that", are atomic propositions.

Atomic propositions may also be defined positively – and this is the better course – as propositions of the following sorts:

$R_1(x)$, meaning "$x$ has the predicate $R_1$";

$R_2(x, y)$ [or $x R_2 y$], meaning "$x$ has the relation $R_2$ (in intension) to $y$";

$R_3(x, y, z)$, meaning "$x, y, z$ have the triadic relation $R_3$ (in intension)";

$R_4(x, y, z, w)$, meaning "$x, y, z, w$ have the tetradic relation $R_4$ (in intension)";

and so on *ad infinitum*, or at any rate as long as possible. Logic does not know whether there are in fact $n$-adic relations (in intension); this is an empirical question. We know as an empirical fact that there are at least dyadic relations (in intension),

[4] 'Dawson', underlined, is written, in the same hand and position as the 'Symonds' on the first page.

because without them series would be impossible. But logic is not interested in this fact; it is concerned solely with the *hypothesis* of there being propositions of such-and-such a form. In certain cases, this hypothesis is itself of the form in question, or contains a part which is of the form in question; in these cases, the fact that the hypothesis can be framed proves that it is true. But even when a hypothesis occurs in logic, the fact that it can be framed does not itself belong to logic.

*6*[5]

Given all ⌊true⌋ atomic propositions, together with the fact that they are all, every other true proposition can theoretically be deduced by logical methods. That is to say, the apparatus of crude fact required in proofs can all be condensed into the true atomic propositions together with the fact that every true atomic proposition is one of the following: (here the list should follow). In practice, this method would presumably involve an infinite enumeration, since it seems natural to suppose that the number of ⌊true⌋ atomic propositions is infinite, though this should not be regarded as certain.[6] In practice, generality is not obtained by the method of complete enumeration, because this method requires more knowledge than we possess.

(xvi) We must now advance to molecular propositions. Let $p$, $q$, $r$, $s$, $t$ denote, to begin with, atomic propositions. We introduce the primitive idea

$$p|q$$

which may be read "$p$ is incompatible with $q$"*, and is to be true whenever either or both are false. Thus it may also be read "$p$ is false or $q$ is false"; or again, "$p$ implies not-$q$". But as we are going to define disjunction, implication, and negation in terms of $p|q$, these ways of reading $p|q$ are better avoided to begin with. The symbol "$p|q$" is pronounced: "$p$ stroke $q$". We now put

$$\sim p \;.=.\; p|p \quad \text{Df}$$
$$p \supset q \;.=.\; p|\sim q \quad \text{Df}$$
$$p \vee q \;.=.\; \sim p|\sim q \quad \text{Df}$$
$$p \,.\, q \;.=.\; \sim(p|q) \quad \text{Df}$$

Thus all the usual truth-functions can be constructed by means of the stroke. Note that by the above, $p \supset q \;.=.\; p|(q|q)$ Df We ~~shall~~ find that

$$p \,.\supset.\, q \,.\, r :\equiv.\, p|(q|r)$$

Thus $p \supset q$ is a degenerate case of a function of *three* propositions.

[5] Previously *5a*. [6] 'In practice ... ' becomes 'If used ... ' .

*For what follows, see Nicod, "A reduction in the number of primitive propositions of logic", *Proc. Camb. Phil. Soc.* Vol. XIX, pp. 32–41.[7]

*7*[8]

We can construct new propositions indefinitely by means of the stroke; for example, $(p|q)|r$, $p|(q|r)$, $(p|q)|(r|s)$, and so on. Note that the stroke obeys the permutation law $(p|q) = (q|p)$ but not the associative law $(p|q)|r = p|(q|r)$.[9] (These of course are results to be proved later.) Note also that, when we construct a new proposition by means of the stroke, we cannot know its truth or falsehood unless either (a) we know the truth or falsehood of some of its constituents, or (b) at least one of its constituents occurs ~~at least twice~~ ⌊several times in a suitable manner⌋. The case (a) interests logic as giving rise to the *rule of inference*, viz.,

Given $p$ and $p|(q|r)$, we can infer $\lfloor r \rfloor$ $\cancel{q}$.

This or some variant must be taken as a primitive proposition. For the moment, we are applying it only when $p$, $q$, $r$ are atomic propositions, but we shall extend it later. We shall consider (b) in a moment.

In constructing new propositions by means of the stroke, we assume that the stroke can have on either side of it any proposition so constructed, and need not have an atomic proposition on either side. Thus given three atomic propositions $p$, $q$, $r$, we can form, first, $p|q$ and $q|r$ and thence $(p|q)|r$ and $p|(q|r)$. Given four, $p, q, r, s$, we can form

$$\{(p|q)|r\}|s, \ \ (p|q)|(r|s), \ \ p|\{q|(r|s)\}$$

and of course others by permuting $p, q, r, s$. The above three are substantially (xvii) different propositions. We have in fact

$$\{(p|q)|r\}|s \ . \equiv \ :. \sim p \vee \sim q \ . \ r \ : \vee : \sim s$$
$$(p|q)|(r|s) \ . \equiv \ : p \ . \ q \ .\vee. \ r \ . \ s$$
$$p|\{q|(r|s)\}. \equiv \ :. \sim p \ : \vee : \ q \ . \sim r \vee \sim s$$

*8*[10]

All the propositions obtained by this method follow from one rule: in "$p|q$", substitute, for $p$ or $q$ or both, propositions already constructed by means of the stroke. This rule generates a definite assemblage of new propositions out of the original assemblage of atomic propositions. All the propositions so generated (excluding the

[7] This becomes '... of the primitive ...' in print. [8] Previously *5b*.
[9] In print these formulas become '$(p|q) \equiv (q|p)$' and '$(p|q)|r \equiv p|(q|r)$.' [10] Previously *5c*.

original atomic propositions) will be called “molecular propositions”, thus molecular propositions are all of the form $p|q$, but the $p$ and $q$ may now themselves be molecular propositions. If $p$ is $p_1|p_2$, $p_1$ and $p_2$ may be molecular; suppose $p_1 = p_{11}|p_{12}$. $p_{11}$ may be of the form $p_{111}|p_{112}$, and so on; but after a finite number of steps of this kind, we ~~shall~~ ⌊are to⌋ arrive at atomic constituents. In a proposition $p|q$, the stroke between $p$ and $q$ is called the “principal” stroke; if $p = p_1|p_2$, the stroke between $p_1$ and $p_2$ is a secondary stroke; so is the stroke between $q_1$ and $q_2$ if $q = q_1|q_2$. If $p_1 = p_{11}|p_{12}$, the stroke between $p_{11}$ and $p_{12}$ is a tertiary stroke, and so on.

Atomic and molecular propositions together are “elementary propositions”. Thus elementary propositions are atomic propositions together with all that can be generated from them by means of the stroke applied any finite number of times. This is a definite assemblage of propositions. We shall now, until further notice, use the letters $p, q, r, s, t$ to denote elementary propositions, not necessarily atomic propositions. The rule of inference stated above is to hold still; i.e.:

If $p, q, r$ are elementary propositions, given $p$ and $p|(q|r)$,
we can infer $\cancel{q}$ $r$.

This is a primitive proposition.

*9*[11]

We can now take up the point (b) mentioned above. When a molecular proposition contains repetitions of a constituent proposition in a suitable manner, it can be known to be true without our having to know the truth or falsehood of any constituent. The simplest instance is

$$p|(p|p)$$

which is always true. It means “$p$ is incompatible with the incompatibility of $p$ with itself”, which is obvious. Again, take “ $p\ .\ q\ .\supset .\ p$”. This is

$$\{(p|q)|(p|q)\}|(p|p)$$

Again, take “$\sim p\ .\supset .\sim p \vee \sim q$.” This is

$$(p|p)|\{(p|q)|(p|q)\}$$

Again, “$p\ .\supset .\ p \vee q$” is

$$p \mid [\{(p|p)|(q|q)\} \mid \{p|p)|(q|q)\}]$$[12]

[11] ‘Symonds’, underlined, is written in pencil in the same hand and position as the ‘Dawson’ four pages before. Previously *5d*.

[12] The missing left parenthesis is provided in print.

All these are true however $p$ and $q$ may be chosen. It is the fact that we can build up invariable truths of this sort that makes molecular propositions important to logic. Logic is helpless with atomic propositions, because their (xviii) truth or falsehood can only be known empirically. But the truth of molecular propositions of suitable form can be known universally without empirical evidence.

*10*[13]

The laws of logic, so far as elementary propositions are concerned, are all assertions to the effect that, whatever elementary propositions $p, q, r, \ldots$ may be, a certain function

$$F(p, q, r, \ldots)$$

~~which is itself a~~ ⌊whose values are⌋ molecular propositions, built up by means of the stroke, is always true.[14] The proposition "$F(p)$ is true, whatever elementary proposition $p$ may be" is denoted by

$$(p).\ F(p)$$

~~We can extend this step by step to any finite number of variables. Thus~~

$$\cancel{(p):(q).\ F(p, q)}$$

~~is to be understood as follows: first, keeping $p$ constant,~~

Similarly the proposition "$F(p, q, r, \ldots)$ is true, whatever elementary propositions $p, q, r, \ldots$ may be" is denoted by

$$(p, q, r, \ldots).\ F(p, q, r, \ldots)$$

When such a proposition is *asserted*, we shall omit the "$(p, q, r, \ldots)$" at the beginning. Thus

$$\text{“} \vdash .\ F(p, q, r, \ldots) \text{”}$$

denotes the assertion (as opposed to the hypothesis) that $F(p, q, r, \ldots)$ is true whatever elementary propositions $p, q, r, \ldots$ may be.

(The distinction between real and apparent variables, which occurs in Frege and in *Principia Mathematica*, is unnecessary. Whatever appears as a real variable in *Principia Mathematica* is to be taken as an apparent variable whose scope is the whole of the asserted proposition in which it occurs.)

[13] Previously *6*. [14] This page was previously *6*.

*11*[15]

The rule of inference, in the form given above, is never required within logic, but only when logic is applied. Within logic, the rule required is different. In the logic of propositions, which is what concerns us at present, the rule used is:

Given, whatever elementary propositions $p, q, r$ may be, both "$\vdash . F(p, q, r, \ldots)$" and "$\vdash . F(p, q, r, \ldots)|\{G(p, q, r, \ldots)|H(p, q, r, \ldots)\}$" we can infer "$\vdash . \cancel{G} \lfloor H \rfloor (p, q, r, \ldots)$".

Other forms of the rule of inference will meet us later. For the present, the above is the form we shall use.

Nicod has shown that the logic of propositions ($*1 - *5$) can be deduced, by the help of the rule of inference, from two primitive propositions:

$$\vdash . p|(p|p)$$

and

$$\vdash : p \supset q . \supset . s|q \supset p|s$$

The first of these may be interpreted as "$p$ is incompatible with not-$p$", or as "$p$ or not-$p$", or as "not ($p$ and not-$p$)", or as "$p$ implies $p$". The second may be interpreted as

$$p \supset q . \supset : q \supset \sim s . \supset . p \supset \sim s$$

(xix) which is a form of the principle of the syllogism. Written wholly in terms of the stroke, the principle becomes

$$\{p|(q|q)\} \mid [\{(s|q)|((p|s)|(p|s))\} \mid \{(s|q)|((p|s)|(p|s))\}]$$

*12*[16]

Nicod has shown further that these two principles may be replaced by one. Written wholly in terms of the stroke, this one principle is

$$\{p|(q|r)\} \mid [\{r|(r|r)\} \mid \{(s|q)|((p|s)|(p|s))\}]$$

It will be seen that, written in this form, the principle is less complex than the second of the above principles written wholly in terms of the stroke. When interpreted into the language of implication, Nicod's one principle becomes

$$\cancel{p . \supset . q . r \ :\supset : r \supset r : q \supset \sim s . \supset . p \supset \sim s}$$

~~or~~ $$p . \supset . q . r : \supset . r \supset r . s|q \supset p|s$$

[15] Previously *7*. [16] Previously *8*.

In this form, it looks more complex than

$$p \supset q \,.\supset.\, s|q \supset p|s$$

but in itself it is less complex.

From the above primitive proposition, together with the rule of inference, everything that logic can ascertain about elementary propositions can be proved, provided we add one other primitive proposition, viz. that, given a proposition $(p, q, r, \ldots)$. $F(p, q, r, \ldots)$, we may substitute for $p, q, r, \ldots$ functions of the form

$$f_1(p, q, r, \ldots)\ ,\ \ f_2(p, q, r, \ldots)\ ,\ \ f_3(p, q, r, \ldots)$$

and assert

$$(p, q, r, \ldots)\,.\,F\{f_1(p, q, r, \ldots),\ \ f_2(p, q, r, \ldots),\ \ f_3(p, q, r, \ldots),\ \ldots\}$$

where $f_1$, $f_2$, $f_3, \ldots$ are functions constructed by means of the stroke. Since the former assertion applied to all elementary propositions, while the latter applies only to some, it is obvious that the former implies the latter.

A more general from of this principle will concern us later.

*13*[17]

## *II. ELEMENTARY FUNCTIONS OF INDIVIDUALS*

### (1). *Definition of "individual."*

We saw that atomic propositions are of one of the series of forms:

$$\cancel{\alpha(x)}\ \ R_1(x),\ \ R_2(x, y),\ \ R_3(x, y, z),\ \ R_4(x, y, z, w), \ldots$$

Here $R_1$, $R_2$, $R_3$, $R_4 \ldots$ are each characteristic of the special form in which they are found; that is to say, $R_n$ cannot occur in an atomic proposition $R_m(x_1, x_2, \ldots x_m)$ unless $n = m$, and then can only occur as $R_m$ occurs, not as $x_1, x_2, \ldots x_m$ occur. On the other hand, any term which can occur as the $x$'s occur in $R_n(x_1, x_2, \ldots x_n)$ can also occur as one of the $x$'s in $R_m(x_1, x_2, \ldots x_m)$, ~~whatever *m* and *n* may be.~~ ⌊even if $m$ is not equal to $n$.⌋ Terms which can occur in any form of atomic proposition are called "individuals" or "particulars"; terms which occur as the $R$'s occur are called "universals".

We might state our definition compendiously as follows: An "individual" is anything that can be the subject of an atomic proposition.

(xx) Given an atomic proposition $R_n(x_1, x_2, \ldots x_n)$, we shall call any of the $x$'s a "constituent" of the proposition, and $R_n$ a "component" of the proposition.* We shall say the same ~~of~~ ⌊as regards⌋ any molecular proposition in which

17 'Dawson', underlined, is written, in the same hand and position as the 'Symonds' on the first page. Previously p. *9*.

$R_n(x_1, x_2, \ldots x_n)$ occurs. Given an elementary proposition $p|q$, where $p$ and $q$ may be atomic or molecular, we shall call $p$ and $q$ "parts" of $p|q$; and any parts of $p$ or $q$ will in turn be called parts of $p|q$, and so on until we reach the atomic ~~constituents~~ ⌊parts⌋ of $p|q$. Thus to say that a proposition $r$ "occurs in" $p|q$ and to say that $r$ is a "part" of $p|q$ will be synonymous.

*This terminology is taken from Wittgenstein.

*14*[18]

(2). *Definition of an elementary function of an individual.*

Given any elementary proposition which contains a part of which an individual $a$ is a constituent, other propositions can be obtained by replacing $a$ by other individuals in succession. We thus obtain a certain assemblage of elementary propositions. We may call the original proposition $\phi a$, and then the proposition⌊al function⌋ obtained by putting ⌊a variable⌋ $x$ in the place of $a$ will be called $\phi x$.[19] Thus $\phi x$ is a function of which the argument is $x$ and the values are elementary propositions. The essential use of "$\phi x$" is that it collects together a certain set of propositions, namely all those that are its values ~~for~~ ⌊with⌋ different arguments.

We have already had ⌊various special⌋ functions of propositions. If $p$ is a part of some molecular proposition, we may consider the set of propositions resulting from the substitution of other propositions for $p$. If we call the original molecular proposition $fp$, the result of substituting $q$ is called $fq$.

When an individual or a proposition occurs ~~more than once~~ ⌊twice⌋ in a proposition, ~~several~~ ⌊three⌋ functions can be obtained, by varying only one, or only another, or both, of the occurrences. For example, $p|p$ is a value of any one of the three functions $p|q$, $q|p$, $q|q$, where $q$ is the argument. Similar considerations apply when an argument occurs more than twice. Thus $p|(p|p)$ is a value of $q|(r|s)$, or $q|(r|q)$, or $q|(q|r)$, or $q|(r|r)$, or $q|(q|q)$. When we assert a proposition "$\vdash .(p).\ Fp$", the $p$ is to be varied whenever it occurs. We may similarly assert a proposition of the form "$(x).\ \phi x$", meaning "all propositions of the assemblage indicated by $\phi x$ are true"; here also, every occurrence of $x$ is to be varied.

*15*[20]

(3). *"Always true" and "sometimes true."*

Given any function, it may happen that all its values are true; again, it may happen that at least one of its values is true. The proposition that all the values of a function $\phi(x, y, z, \ldots)$ are true is expressed by the symbol

$$\text{“}(x, y, z, \ldots).\ \phi(x, y, z, \ldots)\text{”}$$

[18] Previously *10*.

[19] This sentence originally read "... then the proposition obtained by putting $x$ in the place of $a$ will be called $\phi x$".

[20] Previously *11*.

unless we wish to assert it, in which case the assertion is written

$$\text{“} \vdash . \phi(x, y, z, \ldots)\text{”}$$

(xxi) We have already had assertions of this kind where the variables were elementary propositions. We want now to consider the case where the variables are individuals and the function is elementary, i.e. all its values are elementary propositions. We no longer wish to confine ourselves to the case in which it is *asserted* that all the values of $\phi(x, y, z, \ldots)$ are true; we desire to be able to make the proposition

$$(x, y, z, \ldots). \phi(x, y, z, \ldots)$$

a part of a stroke function. For the present, however, we will ignore this desideratum, which will occupy us in section III of this Introduction.

*16*[21]

In addition to ~~assert~~ the proposition that a function ~~of $x$~~ $\lfloor \phi x \rfloor$ is "always true" (i.e. $(x). \phi x$), we need also the proposition that $\phi x$ is "sometimes true", i.e. is true for at least one value of $x$. This we denote by

$$\text{“} (\exists x). \phi(x)\text{”}$$

Similarly the ~~hypothesis~~ proposition that $\phi(x, y, z, \ldots)$ is "sometimes true" is denoted by

$$\text{“} (\exists x, y, z, \ldots). \phi(x, y, z, \ldots)\text{”}.$$

We need, in addition to $(x, y, z, \ldots). \phi(x, y, z, \ldots)$ and $(\exists x, y, z, \ldots). \phi(x, y, z, \ldots)$, various other propositions of an analogous kind. Consider first a function of two variables. We can form

$$(\exists x) : (y). \phi(x, y), \ \ (x) : (\exists y). \phi(x, y), \ \ (\exists y) : (x). \phi(x, y), \ \ (y) : (\exists x). \phi(x, y).$$

These are substantially different propositions, of which no two are always equivalent. It would seem natural, in forming these propositions, to regard the function $\phi(x, y)$ as framed in two stages. Given $\phi(a, b)$, where $a$ and $b$ are constants, we can first form a function $\phi(a, y)$, containing the one variable $y$; we can then form

$$(y). \phi(a, y) \text{ and } (\exists y). \phi(a, y).$$

We can now vary $a$, obtaining again a function of one variable, and leading to the four propositions

$$(x) : (y). \phi(x, y), \ \ (\exists x) : (y). \phi(x, y), \ \ (x) : (\exists y). \phi(x, y), \ \ (\exists x) : (\exists y). \phi(x, y)$$

[21] Previously *12*.

On the other hand, we might have gone from $\phi(a,b)$ to $\phi(x,b)$, thence to $(x).\,\phi(x,b)$ and $(\exists x).\,\phi(x,b)$, and thence to

$$(y):(x).\,\phi(x,y),\ \ (\exists y):(x).\,\phi(x,y),\ \ (y):(\exists x).\,\phi(x,y),\ \ (\exists y):(\exists x).\,\phi(x,y).$$

*17*[22]

All of these will be called "general propositions"; thus 8 general propositions can be derived from the function $\phi(x,y)$. We have

$$(x):(y).\,\phi(x,y):\equiv:(y):(x).\,\phi(x,y)$$

$$(\exists x):(\exists y).\,\phi(x,y):\equiv:(\exists y):(\exists x).\,\phi(x,y)$$

But there are no other equivalences that always hold. For example, the distinction between "$(x):(\exists y).\,\phi(x,y)$" and "$(\exists y):(x).\,\phi(x,y)$" is the same as the distinction in analysis between "For every $\epsilon$, however small, there is a $\delta$ such that . . . " and "there is a $\delta$ such that, for every $\epsilon$, however small, . . . "

(xxii) Although it might seem easier, in view of the above considerations, to regard every function of several variables as obtained by successive steps, each involving only ⌊a⌋ function~~s~~ of one variable, yet there are powerful considerations on the other side. There are two grounds in favour of the step-by-step method: first, that only functions of *one* variable need be taken as a primitive idea; secondly, that such distinctions as the above seem to require *either* that we should first vary $x$, keeping $y$ constant, *or* that we should first vary $y$, keeping $x$ constant. The former seems to be involved when "$(y)$" or "$(\exists y)$" appears to the left of "$(x)$" or "$(\exists x)$", the latter in the converse case. The ground against the step-by-step method is that it interferes with the method of matrices, which brings order into the successive generation of types of propositions and functions demanded by the theory of types, and that it requires us, from the start, to deal with such propositions as $(y).\,\phi(a,y)$, which are not elementary.[23] Take for example, the ~~primitive~~ proposition "$\vdash : q\,.\supset.\,p \vee q$." This

*18*[24]

will be

$$\vdash :.\,(p):.\,(q):q\,.\supset.\,p \vee q$$

or

$$\vdash :.\,(q):.\,(p):q\,.\supset.\,p \vee q$$

[22] Previously *12a*. 'Symonds' underlined in the upper left.
[23] 'The ground . . . is' becomes 'The grounds . . . are' .
[24] Previously *12b*.

and will thus involve all values of either

$$(q) : q \,.\supset.\, p \vee q \text{ considered as a function of } p\text{, or}$$
$$(p) : q \,.\supset.\, p \vee q \text{ considered as a function of } q.$$

This makes it impossible to start our logic with elementary propositions, as we wish to do. It is useless to enlarge the definition of elementary propositions, since that only increases the values of $q$ or $p$ in the above functions. Hence it seems necessary to start with an elementary function

$$\phi(x_1, x_2, x_3, \ldots x_n)$$

before which we ~~may~~ write, for each $x_r$, either "$(x_r)$" or "$(\exists x_r)$", the variables in this ~~prefix~~ ⌊process⌋ being taken in any order we like. Here $\phi(x_1, x_2, x_3, \ldots x_n)$ is called the "matrix", and what comes before it is called the "prefix". Thus in

$$(\exists x) : (y).\ \phi(x, y)$$

"$\phi(x, y)$" is the matrix and "$(\exists x) : (y)$." is the prefix. It thus appears that a matrix containing $n$ variables gives rise to $n!2^n$ propositions by taking its variables in all possible orders and distinguishing "$(x_r)$" and "$\exists x_r$" in each case. (Some of these, however, are equivalent.) The process of obtaining such propositions from a matrix will be called "generalization", whether we take "all values" or "some value", and the propositions which result will be called "general propositions".

*19*[25]

~~(4). *General propositions and matrices.* When a function~~

$$\cancel{\phi(x_1, x_2, \ldots x_n)}$$

~~is used in order to give rise to general propositions, it is called a "matrix."~~
We shall later have occasion to consider matrices containing variables that are not individuals; we may therefore say:

A "matrix" is a function of any number of variables (which may or may not be individuals), which has elementary propositions as its values, and is used for the purpose of generalization.

(xxiii) A "general proposition" is one derived from a matrix by generalization. We shall add one further definition at this stage:

A "first order proposition" is one derived by generalization from a matrix in which all the variables are individuals.

[25] Previously *13*.

(~~5~~ ⌊4⌋). *Methods of proving general propositions.*

There are two fundamental methods of proving general propositions, one for universal propositions, the other for such as assert existence. The method of proving universal propositions is as follows. Given a proposition

$$\text{“} \vdash . \, F(p, q, r, \ldots)\text{”}$$

where $F$ is built up by the stroke, and $p, q, r, \ldots$ are elementary, we may replace them by elementary functions of individuals in any way we like, putting

$$p = f_1(x_1, x_2, \ldots x_n)$$
$$q = f_2(x_1, x_2, \ldots x_n)$$

and so on, and then assert the result for all values of $x_1, x_2, \ldots x_n$. What we thus assert is less than the original assertion, since $p, q, r \ldots$ could originally take all values that are elementary propositions, whereas now they can only take such as are values of $f_1$, $f_2$, $f_3$, ... (Any two or more of $f_1$, $f_2$, $f_3$, ... may be identical.)

*20*[26]

For proving existence-theorems we have two primitive propositions, namely

$$*8\cdot 1 \quad \vdash . \, (\exists x, y). \, \phi a|(\phi x|\phi y) \quad \text{and}$$
$$*8\cdot 11 \quad \vdash . \, (\exists x). \, \phi x|(\phi a|\phi b)$$

Applying the definitions to be given shortly, these assert respectively

$$\phi a \, . \supset . \, (\exists x). \, \phi x \qquad \text{and}$$
$$(x). \, \phi x \, . \supset . \, \phi a \, . \, \phi b$$

These two primitive propositions are to be assumed, not only for one variable, but for any number. Thus we assume

$$\phi(a_1, a_2, \ldots a_n) \, . \supset . \, (\exists x_1, x_2, \ldots x_n). \, \phi(x_1, x_2, \ldots x_n)$$
$$(x_1, x_2, \ldots x_n). \, \phi(x_1, x_2, \ldots x_n) \, . \supset . \, \phi(a_1, a_2, \ldots a_n) \, . \, \phi(b_1, b_2, \ldots b_n)$$

The proposition $(x). \, \phi x \, . \supset . \, \phi a. \, \phi b$, in this form, does not look suitable for proving existence theorems. But it may be written

$$** \, (\exists x). \sim\!\phi x \, . \vee . \, \phi a \, . \, \phi b$$

or $$\sim\!\phi a \vee \sim \phi b \, . \supset . \, (\exists x). \sim \phi x,$$

[26] Previously *14*.

in which form it is identical with $*9\cdot 11$, writing $\phi$ for $\sim\phi$. Thus our two primitive propositions are the same as $*9\cdot 1$ and $*9\cdot 11$.

For purposes of inference, we still assume that from $(x).\ \phi x$ and $(x).\ \phi x \supset \psi x$ we can infer $(x).\ \psi x$, and from $p$ and $p \supset q$ we can infer $q$, even when the functions or propositions involved are not elementary.

*21*[27]

(xxiv) Existence-theorems are very often obtained from the above primitive propositions in the following manner. Suppose we know a proposition

$$\vdash .\ f(x, x)$$

Since $\phi x\ .\supset .\ (\exists y).\ \phi y$, we can infer

$$\vdash .\ (\exists y).\ f(x, y)$$

i.e. $$\vdash :(x):(\exists y).\ f(x, y)$$

Similarly $$\vdash :(y):(\exists x).\ f(x, y)$$

Again, since $\phi(x, y)\ .\supset .\ (\exists z, w).\ \phi(z, w)$, we can infer

$$\vdash .\ (\exists x, y).\ f(x, y)$$

and $$\vdash .\ (\exists y, x).\ f(x, y)$$

We may illustrate the proofs both of universal and of existence propositions by a simple example. We have

$$\vdash .\ (p).\ p \supset p$$

Hence, substituting $\phi x$ for $p$,

$$\vdash .\ (x).\ \phi x \supset \phi x$$

Hence, as in the case of $f(x, x)$ above,

$$\vdash :(x):(\exists y).\ \phi x \supset \phi y$$

$$\vdash :(y):(\exists x).\ \phi x \supset \phi y$$

$$\vdash :\ (\exists x, y).\ \phi x \supset \phi y$$

Apart from special axioms asserting existence-theorems (such as the axiom of reducibility, the multiplicative axiom, and the axiom of infinity), the above two primitive propositions give the sole method of proving existence-theorems in logic. They are, in fact, always derived from general propositions of the form $(x).\ f(x, x)$

[27] Previously *14a*. ‘Dawson’ underlined in the upper left.

or $(x).\ f(x, x, x)$ or etc., by substituting other variables for some of the occurrences of $x$.

*22*[28]

### *III. GENERAL PROPOSITIONS OF LIMITED SCOPE*

In virtue of a primitive proposition, given $(x).\ \phi x$ and $(x).\ \phi x \supset \psi x$, we can infer $(x).\ \psi x$. So far, however, we have introduced no notation which would enable us to state the corresponding *implication* (as opposed to *inference*). Again, $(\exists x).\ \phi x$ and $(x, y).\ \phi x \supset \psi y$ enable us to infer $(y).\ \psi y$; here again, we wish to be able to state the corresponding ~~inference~~ implication. So far, we have only defined occurrences of general propositions as complete asserted propositions. Theoretically, this is their only use and there is no need to define any other. But practically, it is highly convenient to be able to treat them as parts of stroke-functions. This is entirely a matter of definition. By introducing suitable definitions, first-order propositions can be shown to satisfy all the propositions of $*1 - *5$. Hence in using the propositions of $*1 - *5$, it will no longer be necessary to assume that $p, q, r, \ldots$ are elementary.

The fundamental definitions are given ~~on the next page~~ ⌊below⌋.

(xxv) When a general proposition occurs as part of another, it is said to have limited scope. If it contains an apparent variable $x$, the scope of $x$ is said to be limited to the general proposition in question. Thus in $p|\{(x).\ \phi x\}$, the scope of $x$ is limited to $(x).\ \phi x$, whereas in $(x).\ p|\phi x$ the scope of $x$ extends to the whole proposition. Scope is indicated by dots.

The ~~following~~ ⌊new⌋ chapter $*8$ ⌊(given in the Appendix)⌋[29] should replace $*9$ in *Principia Mathematica*. Its general procedure will, however, be explained now.

*23*[30]

The occurrence of a general proposition as part of a stroke-function is defined by means of the following definitions:

$$\{(x).\ \phi x\}|q\ .=.\ (\exists x).\ \phi x|q \qquad \text{Df}$$

$$\{(\exists x).\ \phi x\}|q\ .=.\ (x).\ \phi x|q \qquad \text{Df}$$

$$p|\{(y).\ \psi y\}\ .=.\ (\exists y).\ p|\psi y \qquad \text{Df}$$

$$p|\{(\exists y).\ \psi y\}\ .=.\ (y).\ p|\psi y \qquad \text{Df}$$

These define, in the first place, only what is meant by the stroke when it occurs between two propositions of which one is elementary while the other is of the first

[28] Previously *15*.

[29] ‘given in Appendix A’ in print. This addition indicates that before the division into an introduction and three appendices was completed, Russell had planned to reorganize HPF into an introduction and one appendix.

[30] Previously *16*.

order. When the ~~proposition~~ ⌊stroke⌋ occurs between two propositions which are both of the first order, we shall adopt the convention that the one on the left is to be eliminated first, treating the one on the right as if if were elementary; then the one on the right is to be eliminated – in each case, in accordance with the above definitions. Thus

$$\begin{aligned}\{(x).\ \phi x\}|\{(y).\ \psi y\} \ .=:\ & (\exists x):\phi x|\{(y).\ \psi y\}: \\ =:\ & (\exists x):(\exists y).\ \phi x|\psi y \\ \{(x).\ \phi x\}|\{(\exists y).\ \psi y\} \ .=:\ & (\exists x):\phi x|\{(\exists y).\ \psi y\}: \\ =:\ & (\exists x):(y).\ \phi x|\psi y \\ \{(\exists x).\ \phi x\}|\{(y).\ \psi y\} \ .=:\ & (x):(\exists y).\ \phi x|\psi y\end{aligned}$$

*24*[31]

The rule about the order of elimination is only required for the sake of definiteness, since the two orders give equivalent results. For example, in the last of the above instances, if we had eliminated $y$ first we should have obtained

$$(\exists y):(x).\ \phi x|\psi y$$

which requires either $(x).\sim\phi x$ or $(\exists y).\sim\psi y$, and is then true. And

$$(x):(\exists y).\ \phi x|\psi y$$

is true in the same circumstances. This possibility of changing the order of the variables in the prefix is only due to the way in which they occur, i.e. to the fact that $x$ only occurs on one side of the stroke and $y$ only on the other. The order of variables in the prefix is indifferent whenever the occurrences of one variable are all on one side of a ⌊certain⌋ stroke, while those of the other are all on the other side of it. We do not have in general

$$(\exists x):(y).\ \chi(x,y) :\equiv: (y):(\exists x).\ \chi(x,y)\ ;$$

(xxvi) here the right-hand side is more often true than the left-hand side. But we do have

$$(\exists x):(y).\ \phi x|\psi y :\equiv: (y):(\exists x).\ \phi x|\psi y\ .$$

The possibility of altering the order of the variables in the prefix when they are separated by a stroke is a primitive proposition. In general it is convenient to put on the left the variables of which “all” are involved, and on the right those of which

[31] Previously *17*.

"some" are involved, after the elimination has been finished – always assuming that the variables occur in a way to which our primitive proposition is applicable.

*25*[32]

It is not necessary for the above primitive proposition that the stroke separating $x$ and $y$ should be the principal stroke. E.g.

$$\begin{aligned} p \mid [\{(\exists x).\ \phi x\}|\{(y).\ \psi y\}] \ .=\ &.\ p \mid [(x):(\exists y).\ \phi x|\psi y]. \\ =&:(\exists x):(y).\ p\ |(\phi x|\psi y): \\ =&:(y):(\exists x).\ p\ |(\phi x|\psi y) \end{aligned}$$

All that is necessary is that there should be *some* stroke which separates $x$ from $y$. When this is not the case, the order cannot in general be changed. Take e.g. the matrix

$$\phi x \vee \psi y\ .\ {\sim}\phi x \vee {\sim}\psi y$$

This may be written $(\phi x \supset \psi y) \mid (\psi y \supset \phi x)$ or

$$\{\phi x \mid (\psi y|\psi y)\} \mid \{\psi y \mid (\phi x|\phi x)\}$$

Here there is no stroke which separates all the occurrences of $x$ from all those of $y$, and in fact the two propositions

$$(y):(\exists x).\ \phi x \vee \psi y\ .\ {\sim}\phi x \vee {\sim}\psi y$$

and

$$(\exists x):(y).\ \phi x \vee \psi y\ .\ {\sim}\phi x \vee {\sim}\psi y$$

are not equivalent except for special values of $\phi$ and $\psi$.

*26*[33]

By means of the above definitions, we are able to derive all propositions, of whatever order, from a matrix of elementary propositions combined by means of the stroke. Given any such matrix, containing a part $p$, we may replace $p$ by $\phi x$ or $\phi(x, y)$ or etc. and proceed to add the prefix $(x)$ or $(\exists x)$ or $(x, y)$ or $(x):(\exists y)$ or $(y):(\exists x)$ or etc. If $p$ and $q$ both occur, we may replace $p$ by $\phi x$ and $q$ by $\psi y$, or we may replace both by $\phi x$, or one by $\phi x$ and another by some stroke-function of $\phi x$.

In the case of a proposition such as

$$p \mid \{(x):(\exists y).\ \psi(x, y)\}$$

[32] Previously *18*. 'Symonds' underlined in the upper left.

[33] Previously *18a*, before that *19*.

we must treat it as a case of $p|\{(x).\ \phi x\}$, and first eliminate $x$. Thus

$$p \mid \{(x):(\exists y).\ \psi(x, y)\} \ .=:\ (\exists x):(y).\ p \mid \psi(x, y)$$

That is to say, the definitions of $\{(x).\ \phi x\}|q$ etc. are to be applicable unchanged when $\phi x$ is not an elementary function.

*27*[34]

(xxvii) The definitions of $\sim p$, $p \vee q$, $p.q$ , $p \supset q$ are to be taken over unchanged. Thus

$$\begin{aligned}
\sim\{(x).\ \phi x\} \ .=:&\ \{(x).\ \phi x\} \mid \{(x).\ \phi x\} \\
=:&\ (\exists x):\ \phi x \mid \{(x).\ \phi x\} \\
=:&\ (\exists x):(\exists y).\ (\phi x|\phi y) \\
\sim\{(\exists x).\ \phi x\} \ .=:&\ (x):(y).\ (\phi x|\phi y) \\
p \ .\supset.\ (x).\ \phi x :=:&\ p \mid [\{(x).\ \phi x\}|\{(x).\ \phi x\}] \\
=:&\ p \mid \{(\exists x):(\exists y).\ (\phi x|\phi y)\} \\
=:&\ (x):(y).\ p \mid (\phi x|\phi y) \\
(x).\ \phi x \ .\supset.\ p :=:&\ \{(x).\ \phi x\}|\ (p|p) : \\
=:&\ (\exists x).\ \phi x|\ (p|p) :=:\ (\exists x).\ \phi x \supset p \\
(x).\ \phi x \ .\vee.\ p :=:&\ [\sim\{(x).\ \phi x\}] \mid\ \sim p \\
=:&\ \{(\exists x):(\exists y).\ (\phi x|\phi y)\} \mid (p|p) \\
=:&\ (x).\ \{(\exists y).\ (\phi x|\phi y)\} \mid (p|p) \\
=:&\ (x):(y).\ (\phi x|\phi y) \mid (p|p) \\
p \ .\vee.\ (x).\ \phi x :=:&\ (x):(y).\ (p|p) \mid (\phi x|\phi y)
\end{aligned}$$

*28*[35]

It will be seen that in the above two variables appear where only one might have been expected. We shall find, before long, that the two variables can be reduced to one; i.e. we shall have

$$(\exists x):(\exists y).\ \phi x|\phi y : \equiv .\ (\exists x).\ \phi x|\phi x$$
$$(x):(y).\ \phi x|\phi y : \equiv .\ (x).\ \phi x|\phi x$$

These lead to

$$\sim\{(x).\ \phi x\} \ .\equiv.\ (\exists x).\ \sim\phi x$$
$$\sim\{(\exists x).\ \phi x\} \ .\equiv.\ (x).\ \sim\phi x$$

[34] Previously *18b*, before that *3*. [35] Previously *19*.

But we cannot prove these propositions at our present stage; nor, if we could, would they be of much use to us, since we do not yet know that, when two general propositions are equivalent, either may be substituted for the other as part of a stroke-proposition without changing the truth-value.

For the present, therefore, suppose we have a stroke-function in which $p$ occurs several times, say $p|(p|p)$, and we wish to replace $p$ ~~for~~ by $(x).\ \phi x$, we shall have to write the second occurrence of $p$ "$(y).\ \phi y$", and the third "$(z).\ \phi z$." Thus the resulting proposition will contain as many separate variables as there are occurrences of $p$.

*29*[36]

The primitive propositions required, which have been already mentioned, are four in number. They are as follows:

(1) $\vdash .\ (\exists x, y).\ \phi a \mid (\phi x | \phi y)$, i.e. $\vdash : \phi a\ .\supset .\ (\exists x).\ \phi x$

(2) $\vdash .\ (\exists x).\ \phi x \mid (\phi a | \phi b)$, i.e. $\vdash : (x).\ \phi x\ .\supset . \phi a\ .\ \phi b$

(3) The extended rule of inference, i.e. from $(x).\ \phi x$ and $(x).\ \phi x \supset \psi x$ we can infer $(x).\ \psi x$, even when $\phi$ and $\psi$ are not elementary.

(4) If all the occurrences of $x$ are separated from all the occurrences of $y$ by a certain stroke, the order of $x$ and $y$ can be changed in the prefix; i.e.

(xxviii) For $(\exists x):(y).\ \phi x | \psi y$ we can substitute $(y):(\exists x).\ \phi x | \psi y$, and vice versa, even when this is only a part of the whole asserted proposition.

The above primitive propositions are to be assumed, not only for one variable, but for any number.

*30*[37]

By means of the above primitive propositions it can be proved that all the propositions of $*1 - *5$ apply equally when one or more of the propositions $p, q, r, \ldots$ involved are not elementary. For this purpose, we make use of the work of Nicod, who proved that the primitive propositions of $*1$ can all be deduced from

$\vdash .\ p \supset p$ and
$\vdash : p \supset q\ .\supset .\ s|q \supset p|s$

together with the rule of inference: "Given $p$ and $p|(q|r)$, we can infer $r$."

[36] Where the previous numbers are on earlier pages, a *20* overwritten to become a *29*. 'Dawson' underlined in the upper left.
[37] Previously *21*.

Thus all we have to do is to show that the above propositions remain true when $p, q, s$, or some of them, are not elementary. This is done in $*8$ in the Appendix.

*31*[38]

## *IV. FUNCTIONS AS VARIABLES*

The essential use of a variable is to pick out a certain assemblage of elementary propositions, and enable us to assert that all members of this assemblage are true, or that at least one member is true. We have already used functions of individuals, by substituting $\phi x$ for $p$ in the propositions of $*1 - *5$, and by the primitive propositions of $*8$. But hitherto we have always supposed that the function is kept constant while the individual is varied, and we have not considered cases where we have "$\exists\phi$", or where the scope of "$\phi$" is less than the whole asserted proposition. It is necessary now to consider such cases.

Suppose $a$ is a constant. Then "$\phi a$" will denote, for the various values of $\phi$, all the various elementary propositions of which $a$ is a constituent. This is a different assemblage of elementary propositions from any that can be obtained by variation of individuals; consequently it gives rise to new general propositions. The values of the function are still elementary propositions, just as when the argument is an individual; but they are a new ~~set~~ ⌊assemblage⌋ of elementary propositions, different from previous assemblages.

As we shall have occasion later to consider functions whose values are not elementary propositions, we will distinguish those that have elementary propositions for their values by a note of exclamation between the letter denoting the function and the letter denoting the argument. Thus "$\phi!x$" is a function of two variables, $x$ and $\phi!\hat{z}$. It is a matrix, since it contains no apparent variable and has elementary propositions for its values. We shall henceforth write "$\phi!x$" where we have hitherto written $\phi x$.

*32*[39]

If we replace $x$ by a constant $a$, we can form such propositions as

$$(\phi).\ \phi!a, \quad (\exists\phi).\ \phi!a.$$

(xxix) These are not elementary propositions, and are therefore not of the form $\phi!a$. The assertion of such propositions is derived from the matrices by the method of $*8$. The primitive propositions of $*8$ are to apply when the variables, or some of them, are elementary functions as well as when they are are all individuals.

[38] Previously *43* and that appears to be written over *41*.
[39] Previously *44*. In the upper left, at an angle, is written "c" and under it "p. xxix", presumably by the typesetter.

*A function can only appear in a matrix through its values.** To obtain a matrix, proceed, as before, by writing $\phi!x$, $\psi!y$, $\chi!z, \ldots$ in place of $p, q, r, \ldots$ in some molecular proposition built up by means of the stroke. We can then apply the rules of $*8$ to $\phi$, $\psi$, $\chi, \ldots$ as well as to $x$, $y$, $z, \ldots$ . The difference between a function of an individual and a function of an elementary function of individuals is that, in the former, the passage from one value to another is effected by making the same statement about a different individual, while in the latter it is effected by making a different statement about the same individual. Thus the passage from "Socrates is mortal" to "Plato is mortal" is a passage from $f!x$ to $f!y$, but the passage from "Socrates is mortal" to "Socrates is wise" is a passage from $\phi!a$ to $\psi!a$.[40] Functional variation is involved in such a proposition as: ~~"All that Xenophon says about Socrates is true."~~ "Napoleon had all the characteristics of a great general".

* This assumption is fundamental in the following theory. It has its difficulties, but for the moment we ignore them. It takes the place (not quite adequately) of the axiom of reducibility. It is discussed in Appendix C.

*33*[41]

Taking the collection of elementary props, every matrix has values all of which belong to this collection. Every ⌊general⌋ prop ~~required~~ results from some matrix by generalization.† Every matrix intrinsically determines a certain classification of elementary props, which in turn determines the scope of the generalization of that matrix. Thus "$x$ loves Socrates" picks out a certain collection of props, generalized in "$(x)$. $x$ loves Soc" and "$(\exists x)$. $x$ loves Soc". But "$\phi!$ Soc" picks out those, among elementary props, which mention Soc. The generalizations "$(\phi)$. $\phi!$ Soc" and "$(\exists\phi)$. $\phi!$ Soc" involve a class of elementary props which cannot be obtained from an individual-variable. But any value of "$\phi!$ Soc" is an ordinary elementary prop; the novelty introduced by the variable $\phi$ is a novelty of classification, not of material classified. On the other hand, $(x)$. $x$ loves Soc, $(\phi)$. $\phi!$ Soc, etc. are new props, not contained among elementary props. It is the business of $*8$ ~~and *9~~ to show that these props obey the same rules as elementary props. The method of proof makes it irrelevant what the variables are, so long as all the functions concerned have values which are elementary props. The variables may themselves be elementary props, as they are in $*1 - *5$.

† In a proposition of logic, all the variables in the matrix must be generalized. In other general propositions, such as "all men are mortal", some of the variables in the matrix are replaced by constants.

[40] '$f!x$' and '$f!y$' are originally '$\phi!x$' and '$\phi!y$', respectively.

[41] Previously *45*. In the upper left, at an angle, is once again 'Symonds', but in a different hand than that which previously been writing 'Symonds' and 'Dawson'. Across the top Russell has written '[ To printer: put "proposition" wherever I have written "prop", "Socrates" . . . "Soc" ].' These changes were made in the printed version.

A variable function which has values that are not elementary props starts a new set. But variables of this sort seem unnecessary. Every elementary prop is a value of $\hat{\phi}!\hat{x}$; therefore

$$(p).\ fp\ .\equiv .\ (\phi, x).\ f(\phi!x) : (\exists p).\ fp\ .\equiv .\ (\exists\phi, x).\ f(\phi!x)$$

(xxx) Hence all second-order props⌊in which the variable is an elementary proposition⌋ can be derived from elementary matrices. The question of other second-order props will be dealt with in the next section.

*34*[42]

A function of 2 variables, say $\phi(x, y)$, picks out a certain class of classes of props. We shall have the class $\phi(a, y)$, for given $a$ and variable $y$; then the class of all classes $\phi(a, y)$ as $a$ varies. Whether we are to regard our function as giving classes $\phi(a, y)$ or $\phi(x, b)$ depends upon the order of generalization adopted. Thus "$(\exists x):(y)$" involves $\phi(a, y)$, but "$(y):(\exists x)$" involves $\phi(x, b)$.

Consider now the matrix $\phi!x$, as a function of two variables. If we first vary $x$, keeping $\phi$ fixed (which seems the more natural order), we form a class of propositions $\phi!x, \phi!y, \phi!z, \ldots$ which differ solely by the substitution of one individual for another. Having made one such class, we make another, and so on, until we have done so in all possible ways. But now suppose we vary $\phi$ first, keeping $x$ fixed and equal to $a$. We then first form the class of all props of the form $\phi!a$, i.e. all elementary props of which $a$ is a constitutent; we then form the class $\phi!b$, and so on. The set of props which are values of $\phi!a$ is a set not obtainable by variation of individuals, i.e. not of the form $fx$ [for constant $f$ and variable $x$]. This is what makes $\phi$ a new sort of variable, different from $x$. This also is why generalization of the form $(\phi).\ F!(\phi!\hat{z}, x)$ gives a function not of the form $f!x$ [for constant $f$]. Observe also that whereas $a$ is a constituent of $f!a$, $f$ is not; thus the matrix $\phi!x$ has the peculiarity that, when a value is assigned to $x$, this value is a constituent of the result, but when a value is assigned to $\phi$, this value is absorbed in the resulting prop, and completely disappears. We may define a function ⌊$\phi!\hat{x}$⌋ as that kind of similarity between props which exists when one results from the other by the substitution of one individual for another.

*35*[43]

We have seen that there are matrices containing, as variables, functions of individuals. We may denote any such matrix by

$$f!(\phi!\hat{z}, \psi!\hat{z}, \chi!\hat{z}, \ldots x, y, z, \ldots).$$

[42] Previously *46*. '2' in the first line is written out as 'two'. [43] Previously *47*.

Since a function can only occur through its values, $\phi!\hat{z}$ (e.g.) can only occur in the above matrix through the occurrence of $\phi!x$, $\phi!y$, $\phi!z$, ... or of $\phi!a$, $\phi!b$, $\phi!c$, ... where $a$, $b$, $c$ are constants. Constants do not occur in logic, that is to say, the $a$, $b$, $c$ which we have been supposing constant are to be regarded as obtained by an extra-logical assignment of values to variables. They may therefore be absorbed into the $x$, $y$, $z$, .... Now $x$, $y$, $z$ themselves will only occur in logic as arguments to variable functions. Hence any matrix which contains the variables $\phi!\hat{z}$, $\psi!\hat{z}$, $\chi!\hat{z}$, $x$, $y$, $z$ and no others, if it is of the sort that can occur ⌊explicitly⌋ in logic, will result from substituting

$$\phi!x,\ \phi!y,\ \phi!z,\ \psi!x,\ \psi!y,\ \psi!z,\ \chi!x,\ \chi!y,\ \chi!z, \text{ or some of them,}$$

for elementary propositions in some stroke-function.

(xxxi) It is necessary here to explain what is meant when we speak of a "matrix that can occur ⌊explicitly⌋ in logic", or, as we may call it, a "logical matrix". A logical matrix is one that contains no constants. Thus $p|q$ is a logical matrix; so is $\phi!x$, where $\phi$ and $x$ are both variable. Taking any elementary proposition, we shall obtain a logical matrix if we replace all its components and constituents by variables. ⌊Other matrices result from logical matrices by assigning values to some of their variables.⌋ There are, however, various ways of analysing a proposition, and therefore various logical matrices can be derived from a given proposition. Thus a proposition which is a value of $p|q$ will also be a value of $(\phi!x)|(\psi!y)$ and of $\chi!(x, y)$. Different forms are required for different purposes; but all the forms of matrices required ⌊explicitly⌋ in logic are logical matrices as above defined. This is merely an illustration of the fact that logic aims always at complete generality. The test of a logical matrix is that it can be expressed without introducing any symbols other than those of logic, e.g. we must not require the symbol "Socrates".

*36*[44]

Consider the expression

$$f!(\phi!\hat{z}, \psi!\hat{z}, \chi!\hat{z}, \ldots x, y, z, \ldots).$$

When a value is assigned to $f$, this represents a matrix containing the variables $\phi, \psi, \chi, \ldots x, y, z, \ldots$ But while $f$ remains unassigned, it is a matrix of a new sort, containing the new variable $f$. We call $f$ a "second-order function", because it takes functions among its arguments. When a value is assigned, not only to $f$, but also to $\phi, \psi, \chi, \ldots x, y, z, \ldots$, we obtain an elementary proposition; but when a value is assigned to $f$ alone, we obtain a matrix containing ⌊as variables⌋ only first-order functions and individuals. This is analogous to what happens when we

[44] Previously *48*.

consider the matrix $\phi!x$. If we give values to both $\phi$ and $x$, we obtain an elementary proposition; but if we give a value to $\phi$ alone, we obtain a matrix containing only an individual as variable.

There is no logical matrix of the form $f!(\phi!\hat{z})$. The only matrices in which $\phi!\hat{z}$ is the only argument are those containing $\phi!a$, $\phi!b$, $\phi!c, \ldots$ where $a, b, c, \ldots$ are constants; but these are not logical matrices, being derived from the logical matrix $\phi!x$. Since $\phi$ can only appear through its values, it must appear, in a logical matrix, with one or more variable arguments. The simplest logical functions of $\phi$ alone are $(x).\ \phi!x$ and $(\exists x).\ \phi!x$, but these are not matrices. A ~~matrix~~ logical matrix

$$f!(\phi!\hat{z}, x_1, x_2, \ldots x_n)$$

is always derived from a stroke-function

$$F(p_1, p_2, p_3, \ldots p_n)$$

by substituting $\phi!x_1, \phi!x_2, \ldots \phi!x_n$ for $p_1, p_2, p_3, \ldots p_n$. This is the sole method of constructing such matrices. (We may however have $x_r = x_s$ for some values of $r$ and $s$.)

*37*[45]

Second-order functions have two connected properties which first-order functions do not have ~~; t~~ ⌊.T⌋he first of these is that, when a value is assigned to (xxxii) $f$, the result may be a logical matrix; the second is that certain constant values of $f$ can be assigned without going outside logic.

To take the first point first: $f!(\phi!\hat{z}, x)$, for example, is a matrix containing three variables, $f$, $\phi$, and $x$. The following logical matrices (among an infinite number) result from the above by assigning a value to $f$: $\phi!x$ , $(\phi!x)|(\phi!x)$, $\phi!x \supset \phi!x$, etc. Similarly $\phi!x \supset \phi!y$, which is a logical matrix, ~~is a value of~~ ⌊results from assigning a value to $f$ in⌋ $f!(\phi!\hat{z}, x, y)$. In all these cases, the constant value assigned to $f$ is one which can be expressed in logical symbols alone ⌊(which was the second property of $f$)⌋. This in not the case with $\phi!x$: in order to assign a value to $\phi$, we must introduce what we may call "empirical constants", such as "Socrates" and "mortality" and "being Greek". The functions of $x$ that can be formed without going outside logic must involve a function as a generalized variable; they are (in the simplest case) such as $(\phi).\ \phi!x$ and $(\exists\phi).\ \phi!x$.

To some extent, however, the above peculiarity of functions of the second and higher orders is arbitrary. We might have adopted in logic the symbols

$$R_1(x),\ \ R_2(x, y),\ \ R_3(x, y, z), \ldots$$

[45] Previously *49*. 'Dawson' in the upper left, at an angle.

where $R_1$ represents a variable predicate, $R_2$ a variable dyadic relation (in intension), and so on. Each of the symbols $R_1(x)$, $R_2(x, y)$, $R_3(x, y, z)$, ... is a logical matrix, so that, if we used them, we should

*38*[46]

have logical matrices ~~of~~ not containing variable functions. It is perhaps worth while to remind ourselves of the meaning of "$\phi!a$", where $a$ is a constant. The meaning is as follows. Take any finite number of propositions of the various forms $R_1(x)$, $R_2(x, y)$, ... and combine them by means of the stroke in any way desired, allowing any one of them to be repeated any finite number of times. If at least one of them has $a$ as a constituent, i.e. is of the form

$$R_n(a, b_1, b_2, \ldots b_{n-1}),$$

then the molecular proposition we have constructed is of the from $\phi!a$, i.e. is a value of "$\phi!a$" with a suitable $\phi$. This of course also holds of the proposition $R_n(a, b_1, b_2, \ldots b_{n-1})$ itself. It is clear that the logic of propositions, and still more of general propositions concerning a given argument, would be intolerably complicated if we abstained from the use of variable functions; but it can hardly be said that it would be impossible. As for the question of matrices, we could form a matrix $f!(R_1, x)$, of which $R_1(x)$ would be a value. That is to say, the properties of second-order matrices which we have been discussing would also belong to matrices containing variable universals. They cannot belong to matrices containing only variable individuals.

By assigning $\phi!\hat{z}$ and $x$ in $f!(\phi!\hat{z}, x)$, while leaving $f$ variable, we obtain an assemblage of elementary propositions not to be obtained by means of variables representing individuals and first-order functions. This is why the new variable $f$ is useful.

(xxxiii) We can proceed in like manner to matrices

$$F!\{f!(\hat{\phi}!\hat{z}, \hat{x}), g!(\hat{\phi}!\hat{z}, \hat{x}), \ldots \psi!\hat{z}, \chi!\hat{z}, \ldots x, y, \ldots\}$$

and so on indefinitely. These merely represent new ways of grouping elementary propositions, leading to new kinds of generality.

*39*[47]

## *V. FUNCTIONS OTHER THAN MATRICES*

When a matrix contains several variables, functions of some of them can be obtained by turning the others into apparent variables. Functions obtained in this way are not

[46] Previously *49a*. [47] Previously *50*.

matrices, and their values are not elementary propositions. The simplest examples are

$$(y).\ \phi!(x, y) \text{ and } (\exists y).\ \phi!(x, y)$$

When we have a general proposition $(\phi).\ F\{\phi!\hat{z}, x, y, \ldots\}$, the only values $\phi$ can take are matrices, so that functions containing apparent variables are not included. We can, if we like, introduce a new variable, to denote not only functions such as $\phi!\hat{x}$, but also such as

$$(y).\ \phi!(\hat{x}, y), (y, z).\ \phi!(\hat{x}, y, z), \ldots (\exists y).\ \phi!(\hat{x}, y), \ldots;$$

in a word, all such functions of one variable as can be derived ⌊by generalization⌋ from matrices containing only individual variables. Let us denote any such function by $\phi_1 x$, or $\psi_1 x$, or $\chi_1 x$, or etc. Here the suffix 1 is intended to indicate that the values of the functions may be first-order propositions resulting from generalization in respect of individuals. In virtue of $*8$, no harm can come from including such functions along with matrices as values of simple variables.

*40*[48]

Theoretically, it is unnecessary to introduce such variables as $\phi_1$, because they can be replaced by an infinite conjunction or disjunction. Thus, e.g.

$$(\phi_1).\ \phi_1 x\ .\equiv: (\phi).\ \phi!x : (\phi, y).\ \phi!(x, y) : (\phi) : (\exists y).\ \phi!(x, y) : \text{etc.}$$

$$(\exists\phi_1).\phi_1 x\ .\equiv: (\exists\phi).\phi!x : \vee : (\exists\phi) : (y).\phi!(x, y) : \vee : (\exists\phi, y).\phi!(x, y) : \vee : \text{etc.}$$

and generally, given any matrix $f!(\phi!\hat{z}, x)$, we shall have the following process for interpreting $(\phi_1).\ f!(\phi_1\hat{z}, x)$ and $(\exists\phi_1).\ f!(\phi_1\hat{z}, x)$. Put

$$(\phi^1).\ f!(\phi^1\hat{z}, x)\ . =: (\phi).\ f!\{(y).\ \phi!(\hat{z}, y), x\} : (\phi).\ f!\{(\exists y).\ \phi!(\hat{z}, y), x\}$$

where $f!\{(y).\ \phi!(\hat{z}, y), x\}$ is constructed as follows: wherever, in $f!\{\phi!\hat{z}, x\}$, a value of $\phi$, say $\phi!a$, occurs, substitute $(y).\ \phi!(a, y)$, and develop by the definitions at the beginning of $*8$. $f!\{(\exists y).\ \phi!(\hat{z}, y), x\}$ is similarly constructed. Similarly put

$$\begin{aligned}(\phi^2).\ f!(\phi^2!\hat{z}, x)\ . =: (\phi).\ f!\{(y, w).\ \phi!(\hat{z}, y, w), x\} :&\\ (\phi).\ f!\{(y) : (\exists w).\ \phi!(\hat{z}, y, w), x\} : \text{etc.}&\end{aligned}$$

where "etc" covers the prefixes $(\exists y) : (w).$, $(\exists y, w).$, $(w) : (\exists y)$. We define $\phi^3, \phi^4, \ldots$ similarly. Then

$$(\phi_1).\ f!(\phi_1\hat{z}, x)\ . =: (\phi^1).\ f!(\phi^1\hat{z}, x) : (\phi^2).\ f!(\phi^2\hat{z}, x) : \text{etc.}$$

[48] Previously *51* and before that *52*. See the history of the HPF manuscript.

This process depends upon the fact that $f!(\phi!\hat{z}, x)$, for each value of $\phi$ and $x$, is a proposition constructed out of elementary propositions by the stroke, and (xxxiv) that $*8$ enables us to replace any of these by a proposition which is not elementary. $(\exists\phi_1).\ f!(\phi_1\hat{z}, x)$ is defined by an exactly analogous *disjunction*.

*41*[49]

It is obvious that, in practice, an infinite conjunction or disjunction such as the above cannot be manipulated without assumptions *ad hoc*. We can work out results for any segment of the infinite conjunction or disjunction, and we can "see" that these results hold throughout. But we cannot prove this, because mathematical induction is not applicable. We therefore adopt certain primitive propositions, which assert only that what we can prove in each case holds generally. By means of these it becomes possible to manipulate such variables as $\phi_1$. ~~A similar process applies to $\phi_2, \phi_3, \ldots$~~

In like manner we can introduce $f_1(\phi_1\hat{z}, x)$, where any number of individuals and functions $\psi_1, \chi_1, \ldots$ may appear as apparent variables.

No essential difficulty arises in this process so long as the apparent variables involved in a function are not of higher order than the argument to the function. For example, $x \in \text{Ɑ}‘R$, which is $(\exists y).\ xRy$, may be treated without danger as if it were of the form $\phi!x$. In virtue of $*8$, $\phi_1 x$ may be substituted for $\phi!x$ without interfering with the truth of any logical proposition in which $\phi!x$ is a part.[50] Similarly whatever logical proposition holds concerning $f!(\phi_1\hat{z}, x)$ will hold concerning $f_1(\phi_1\hat{z}, x)$.

*42*[51]

But when the apparent variable is of higher order than the argument, a new situation arises. The simplest cases are

$$(\phi).\ f!(\phi!\hat{z}, x)\ ,\ \ (\exists\phi).\ f!(\phi!\hat{z}, x).$$

These are functions of $x$, but are obviously not included among the values ~~of~~ ⌊for⌋ $\phi!x$ (where $\phi$ is the argument ). If we adopt a new ~~function~~ ⌊variable⌋ $\phi_2$ which is to include functions in which $\phi!\hat{z}$ can be an apparent variable, we shall obtain other new functions

$$(\phi_2).\ f!(\phi_2\hat{z}, x)\ ,\ \ (\exists\phi_2).\ f!(\phi_2\hat{z}, x)$$

which are again not ~~of~~ among values ~~of~~ ⌊for⌋ $\phi_2 x$ (where $\phi_2$ is the argument), because the totality of values of $\phi_2\hat{z}$, which is now involved, is different from the totality of values of $\phi!\hat{z}$, which was formerly involved. However much we may

[49] Previously *52* and before that *53*. ‘Symonds’ in the upper left.
[50] Strangely, ‘proposition in which $\phi!x$ is a part’ becomes ‘proposition which $\phi!x$ is a part’ in print, an error.
[51] Previously *53*.

enlarge the meaning of $\phi$, a function of $x$ in which $\phi$ occurs as apparent variable has a correspondingly enlarged meaning, so that, however $\phi$ may be defined

$$(\phi).\ f!(\phi\hat{z}, x) \text{ and } (\exists\phi).\ f!(\phi\hat{z}, x)$$

can never be values for $\phi x$. To attempt to make them so is like attempting to catch one's own shadow. It is impossible to obtain one variable which embraces among its values all possible functions of individuals.

We denote by $\phi_2 x$ a function of $x$ in which $\phi_1$ is an apparent variable, but there is no variable of higher order. Similarly $\phi_3 x$ will contain $\phi_2$ as apparent variable, and so on.

*43*[52]

(xxxv) The essence of the matter is that a variable may travel through any well-defined totality of values, provided these values are all such that any one can replace any other significantly in any context. In constructing $\phi_1 x$, the only totality involved is that of individuals, which is already presupposed. But when we allow $\phi$ to be an apparent variable in a function of $x$, we enlarge the totality of functions of $x$, however $\phi$ may have been defined. It is therefore always necessary to specify what sort of $\phi$ is involved, whenever $\phi$ appears as an apparent variable.

The other condition, that of significance, is fully provided for by the definitions of $*8$, together with the principle that a function can only occur through its values. In virtue of the principle, a function of a function is a stroke-function of values of the function. And in virtue of the definitions in $*8$, a value of any function can significantly replace any proposition in a stroke-function, because propositions containing any number of apparent variables can always be substituted for elementary propositions and for each other in any stroke-function. What is necessary for significance is that every complete asserted proposition should be derived from a matrix by generalization, and that, in the matrix, the substitution of constant values for the variables should always result, ultimately, in a stroke-function of atomic propositions. We say "ultimately", because, when such variables as $\phi_2\hat{z}$ are admitted, the substitution of a value for $\phi_2$ may yield a proposition still containing apparent variables, and in this proposition the apparent variables must be replaced by constants before we arrive at a stroke-function of atomic propositions. We may introduce variables requiring several such stages, but the end must always be the same: a stroke-function of atomic propositions.

*44*[53]

It seems, however, though it might be difficult to prove formally, that the functions $\phi_1$, $f_1$ introduce no propositions that cannot be expressed without them.

[52] Previously *53a*. [53] Previously *53b*.

Let us take first a very simple illustration. Consider the proposition

$$(\exists\phi_1).\ \phi_1 x\ .\ \phi_1 a, \text{ which we will call } f(x, a).$$

Since $\phi_1$ includes all possible values of $\phi!$ and also a great many other values in its range, $f(x, a)$ might seem to make a smaller assertion than would be made by

$$(\exists\phi).\ \phi!x\ .\ \phi!a, \text{ which we will call } f_0(x, a).$$

But in fact $f(x, a)\ .\supset .\ f_0(x, a)$. This may be seen as follows : $\phi_1 x$ has one of the ~~two~~ ⌊various⌋ sets of forms:

$$(y).\ \phi!(x, y), (y, z).\ \phi!(x, y, z), \ldots$$

$$(\exists y).\ \phi!(x, y), (\exists y, z).\ \phi!(x, y, z), \ldots$$

$$(y) : (\exists z).\ \phi!(x, y, z), (\exists y) : (z).\ \phi!(x, y, z), \ldots$$

Suppose first that $\phi_1 x\ . = .\ (y).\ \phi!(x, y)$. Then

$$\begin{aligned} \phi_1 x\ .\ \phi_1 a\ . \equiv : (y).\ \phi!(x, y) : (y).\ \phi!(a, y) : \\ \supset : \phi!(x, b)\ .\ \phi!(a, b) : \\ \supset : (\exists\phi).\ \phi!x\ .\ \phi!a \end{aligned}$$

(xxxvi) Next suppose $\phi_1 x\ . = .\ (\exists y).\ \phi!(x, y)$. Then

$$\begin{aligned} \phi_1 x\ .\ \phi_1 a\ . \equiv : (\exists y).\ \phi!(x, y) : (\exists z).\ \phi!(a, z) : \\ \supset : (\exists y, z) : \phi!(x, y) \vee \phi!(x, z)\ .\ \phi!(a, y) \vee \phi!(a, z) : \\ \supset : (\exists\phi).\ \phi!x\ .\ \phi!a, \end{aligned}$$

because $\phi!(x, y) \vee \phi!(x, z)$ is of the form $\phi!x$, when $y$ and $z$ are fixed. It is obvious that this method of proof applies to the other cases mentioned above. Hence

$$(\exists\phi_1).\ \phi_1 x\ .\ \phi_1 a\ . \equiv .\ (\exists\phi).\ \phi!x\ .\ \phi!a$$

*45*[54]

We can satisfy ourselves that the same result holds in general form

$$(\exists\phi_1).\ f!(\phi_1\hat{z}, x)\ . \equiv .\ (\exists\phi).\ f!(\phi!\hat{z}, x)$$

by a similar argument. We know that $f!(\phi!\hat{z}, x)$ is derived from some stroke-function

$$F(p, q, r, \ldots)$$

[54] Previously *54*.

by substituting $\phi!x, \phi!a, \phi!b, \ldots$ (where $a, b, \ldots$ are constants) for some of the propositions $p, q, r, \ldots$ and $g_1!x, g_2!x, g_3!x, \ldots$ (where $g_1, g_2, g_3 \ldots$ are constants) for others of $p, q, r, \ldots$, while replacing any remaining propositions $p, q, r, \ldots$ by constant propositions. Take a typical case; suppose[55]

$$f!(\phi!\hat{z}, x) \,.=.\, (\phi!a) \mid \{(\phi!x)|(\phi!b)\}$$

We then have to prove

$$\phi_1 a \mid (\phi_1 x|\phi_1 b) \,.\supset.\, (\exists\phi).\ \phi!a \mid (\phi!x|\phi!b)$$

where $\phi_1 x$ may have any of the forms enumerated above.

Suppose first that $\phi_1 x \,.=.\, (y).\ \phi!(x, y)$. Then

$$\begin{aligned} \phi_1 a \mid (\phi_1 x|\phi_1 b) \,.=:\, & (\exists y):(z, w).\ \phi!(a, y) \mid \{\phi!(x, z) \mid \phi!(b, w)\} : \\ \supset :\, & (\exists y).\ \phi!(a, y) \mid \{\phi!(x, y) \mid \phi!(b, y)\} : \\ \supset :\, & (\exists\phi).\ \phi!a \mid (\phi!x|\phi!b) \end{aligned}$$

because, for a given $y$, $\phi!(x, y)$ is of the form $\phi!x$.

Suppose next that $\phi_1 x \,.=.\, (\exists y)\ \phi!(x, y)$. Then

$$\begin{aligned} \phi_1 a \mid (\phi_1 x|\phi_1 b) \,.=:\, & (y):(\exists z, w).\ \phi!(a, y) \mid \{\phi!(x, z) \mid \phi!(b, w)\} : \\ & \cancel{\supset : (\exists z, w).\ \phi!(a, a) \mid \{\phi!(x, z) \mid \phi!(b, w)\}} \\ \supset :\, & (\exists\psi).\ \psi!a \mid (\psi!x|\psi!b) \\ & \text{putting } \psi!x \,.=.\, \phi!(x, z) \vee \phi!(x, w) \end{aligned}$$

Similarly the other cases can be dealt with. Hence the result follows.

*46*[56]

Consider next the correlative proposition

$$(\phi_1).\ f!(\phi_1\hat{z}, x) \,.\equiv.\, (\phi).\ f!(\phi!\hat{z}, x)$$

Here it is the converse implication that needs proving, i.e.

$$(\phi).\ f!(\phi!\hat{z}, x) \,.\supset.\, (\phi_1).\ f!(\phi_1\hat{z}, x)$$

$$\cancel{\text{i.e. } (\phi_1) :.\ (\exists\phi):\ f!(\phi!\hat{z}, x) \,.\supset.\, f!(\phi_1\hat{z}, x)}$$

[55] In the manuscript '... $(\phi!x)|\phi!b)\}$' , with a missing left parenthesis. This is a rare slip in symbolism.
[56] Previously *55*.

⌊This follows from the previous case by transposition. It can also be seen independently as follows.⌋ Suppose, as before, that

$$f!(\phi_1\hat{z}, x) \,.=.\, (\phi_1 a)|(\phi_1 x|\phi_1 b)$$

and put first $\phi_1 x \,.=.\, (y).\, \phi!(x, y)$. Then

$$(\phi_1 a) \mid (\phi_1 x|\phi_1 b) \,.=:\, (\exists y):(z, w).\, \phi!(a, y) \mid \{\phi!(x, z)|\phi!(b, w)\}$$

(xxxvii) Thus we require that, ~~for a suitable value of $\psi$, we should have~~ given

$$(\psi).\, (\psi!a) \mid (\psi!x \mid \psi!b)$$

we should have $(\exists y):(z, w).\, \phi!(a, y) \mid \{\phi!(x, z)|\phi!(b, w)\}$

Now[57]

$$\begin{aligned}
&(\psi).\, (\psi!a) \mid (\psi!x \mid \psi!b).\\
&\quad \supset :.\, \phi!(a, z) \,.\supset.\, \phi!(x, z) \,.\, \phi!(b, z) : \phi!(a, w) \,.\supset.\, \phi!(x, w) \,.\, \phi!(b, w) :.\\
&\qquad\qquad \supset :\ \phi!(a, z) \,.\, \phi!(a, w) \,.\supset.\, \phi!(x, z) \,.\, \phi!(b, w)\\
&\qquad\qquad \supset :.\, \phi!(a, w) \,.\supset : \phi!(a, z) \,.\supset.\, \phi!(x, z) \,.\, \phi!(b, w) \quad (1)\\
&\text{Also} \qquad \sim\phi!(a, w) \,.\supset : \phi!(a, w) \,.\supset.\, \phi!(x, z) \,.\, \phi!(b, w) \quad (2)
\end{aligned}$$

$$(1).(2) \,.\supset :.\, (\psi).\, \psi!a \mid (\psi!x|\psi!b) : \supset :.\, (\exists y) : \phi!(a, y) \,.\supset.\, \phi!(x, z) \,.\, \phi!(b, w)$$

which was to be proved.

*47*[58]

Put next $\qquad \phi_1 x \,.=.\, (\exists y).\, \phi!(x, y)$.

Then $(\phi_1 a) \mid (\phi_1 x|\phi_1 b) \,.=:\, (y):(\exists z, w).\, \phi!(a, y) \mid \{\phi!(x, z)|\phi!(b, w)\}$

In this case we merely put $z = w = y$ and the result follows.

The method will be the same in any other case. Hence generally:

$$(\phi_1).\ f!(\phi_1\hat{z}, x). \equiv .(\phi).\ f!(\phi!\hat{z}, x).$$

Although the above arguments do not amount to formal proofs, they suffice to make it clear that, in fact, any general propositions about $\phi!\hat{z}$ are also true about $\phi_1\hat{z}$. This gives us, so far as such functions are concerned, all that could have been got from the axiom of reducibility.

[57] The third line below is corrected in print to begin with '$\supset :.$ ' and end with '$:.$'
[58] Previously *56*.

Since the proof can only be conducted in each separate case, it is necessary to introduce a primitive proposition stating that the result holds always. This primitive proposition is

$$\vdash : (\phi).\ f!(\phi!\hat{z}, x) \,.\supset.\, f!(\phi_1\hat{z}, x) \qquad \text{Pp}$$

As an illustration: suppose we have proved some property of all classes defined as functions of the form $\phi!\hat{z}$, the above primitive proposition enables us to substitute the class D‘$R$, where $R$ is the relation defined by $\phi!(\hat{x}, \hat{y})$, or by $(\exists z).\ \phi!(\hat{x}, \hat{y}, z)$, or etc. Wherever a class or relation is defined by a function containing no apparent variables except individuals, the above primitive proposition enables us to treat it as if it were defined by a matrix.

*48*[59]

We have now to consider functions of the form $\phi_2 x$, where

$$\phi_2 x \,.=.\, (\phi).\ f!(\phi!\hat{z}, x) \text{ or } \phi_2 x \,.=.\, (\exists\phi).\ f!(\phi!\hat{z}, x).$$

We want to discover whether, or under what circumstances, we have

$$(\phi).\ g!(\phi!\hat{z}, x) \,.\supset.\, g!(\phi_2\hat{z}, x). \qquad \text{(A)}$$

Let us begin with an important particular case. Put

$$g!(\phi!\hat{z}, x) \,.=.\, \phi!a \supset \phi!x.$$

Then $(\phi).\ g!(\phi!\hat{z}, x) \,.=.\, x = a$, according to $*13{\cdot}1$.

(xxxviii) We want to prove

$$(\phi).\ \phi!a \supset \phi!x \,.\supset.\, \phi_2 a \supset \phi_2 x$$

i.e.[60] $(\phi).\ \phi!a \supset \phi!x \,.\supset: (\phi).f!(\phi!\hat{z}, a) \,.\supset.\, (\phi).\ f!(\phi!\hat{z}, a) :$

$$(\exists\phi).f!(\phi!\hat{z}, a) \,.\supset.\, (\exists\phi).\ f!(\phi!\hat{z}, a)$$

Now $f!(\phi!\hat{z}, x)$ must be derived from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values $\phi!x, \phi!b, \phi!c, \ldots$ where $b, c, \ldots$ are constants. As soon as $\phi$ is assigned, this is of the form $\psi!x$. Hence

$$\begin{aligned}(\phi).\ \phi!a \supset \phi!x \,.\supset: (\phi) : f!(\phi!\hat{z}, a) \,.\supset.\, f!(\phi!\hat{z}, x) :\\ \supset: (\phi).\ f!(\phi!\hat{z}, a) \,.\supset.\, (\phi).\ f!(\phi!\hat{z}, x) :\\ (\exists\phi).\ f!(\phi!\hat{z}, a) \,.\supset.\, (\exists\phi).\ f!(\phi!\hat{z}, x)\end{aligned}$$

[59] Previously *57*. [60] The next two lines conclude with ‘$x$’ replacing the last ‘$a$’.

Thus generally $(\phi).\ \phi!a \supset \phi!x\ .\supset.\ (\phi_2).\ \phi_2 a \supset \phi_2 x$ without the need of any axiom of reducibility.

*49*[61]

It must not, however, be assumed that (A) is always true. The procedure is as follows: $f!(\phi!\hat{z}, x)$ results from some stroke-function

$$F(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values $\phi!x, \phi!a, \phi!b, \ldots$ ( $a, b, \ldots$ being constants). We assume that, e.g.,

$$\phi_2 x\ .=.\ (\phi).\ f!(\phi!\hat{z}, x)$$

Thus $$\phi_2 x\ .=.\ (\phi).\ F(\phi!x, \phi!a, \phi!b, \ldots) \qquad \text{(B)}$$

What we want to discover is whether

$$(\phi).\ g!(\phi!\hat{z}, x)\ .\supset.\ g!(\phi_2\hat{z}, x)$$

Now $g!(\phi!\hat{z}, x)$ will be derived from a stroke-function

$$G(p, q, r, \ldots)$$

by substituting $\phi!x, \phi!a', \phi!b', \ldots$ for some of $p, q, r, \ldots$ To obtain $g!(\phi_2\hat{z}, x)$, we have to put $\phi_2 x$ , $\phi_2 a'$, $\phi_2 b'$, ... in $G(p, q, r, \ldots)$, instead of $\phi!x$, $\phi!a'$, $\phi!b'$, ... We shall thus obtain a new matrix.

If $(\phi).\ g!(\phi!\hat{z}, x)$ is known to be true because $G(p, q, r, \ldots)$ is always true, then $g!(\phi_2\hat{z}, x)$ is true in virtue of $*8$, because it is obtained from $G(p, q, r, \ldots)$ by substituting for some of $p, q, r, \ldots$ the propositions $\phi_2 x$, $\phi_2 a'$, $\phi_2 b'$ which contain apparent variables. Thus in this case the inference is warranted.

*50*[62]

We have thus the following important proposition:

Whenever $(\phi).\ g!(\phi!\hat{z}, x)$ is known to be true because $g!(\phi_2\hat{z}, x)$ is always a value of a stroke-function

$$G(p, q, r, \ldots)$$

which is true for all values of $p, q, r, \ldots$, then $g!(\phi_2\hat{z}, x)$ is also true, and so ~~is~~ (of course) is $(\phi_2).\ g!(\phi_2\hat{z}, x)$.

(xxxix) This, however, does not cover the case where $(\phi).\ g!(\phi!\hat{z}, x)$ is not a truth of logic, but a hypothesis, which may be true for some values of $x$ and false for

[61] Previously *58*. [62] Previously *59*.

others. When this is the case, the inference to $g!(\phi_2\hat{z}, x)$ is sometimes legitimate and sometimes not; the various cases must be investigated separately. We shall have an important illustration of the failure of the inference in connection with mathematical induction.

*51*

## *VI. CLASSES*

The theory of classes is at once simplified in one direction and complicated in another by the assumption that functions only occur through their values and ~~that~~ by the abandonment of the axiom of reducibility.

According to our present theory, all functions of functions are extensional, i.e.

$$\phi x \equiv_x \psi x \,.\supset.\, f(\phi\hat{z}) \equiv f(\psi\hat{z})$$

This is obvious, since $\phi$ can only occur in $f(\phi\hat{z})$ by the substitution of values of $\phi$ for $p, q, r, \ldots$ in a stroke-function, and, if $\phi x \equiv \psi x$, the substitution of $\phi x$ for $p$ in a stroke-function gives the same truth-value to the truth-function as the substitution of $\psi x$. Consequently there is no longer any reason to distinguish between functions and classes, for we have, in virtue of the above,

$$\phi x \equiv_x \psi x \,.\supset.\, \phi\hat{x} = \psi\hat{x}$$

We shall continue to use the notation $\hat{x}(\phi x)$, which is often more convenient than $\phi\hat{x}$; but there will no longer be any difference between the meanings of the two symbols. Thus classes, as distinct from functions, lose even that shadowy being which they retain in $*20$. The same, of course, applies to relations in extension. This, so far, is a simplification.

*52*

On the other hand, we now have to distinguish classes of different orders composed of members of the same order. Taking classes of individuals as the simplest case, $\hat{x}(\phi!x)$ must be distinguished from $\hat{x}(\phi_2 x)$ and so on. In virtue of the proposition at the end of the last section, the general logical properties of classes will be the same for classes of all orders. Thus e.g.

$$\alpha \subset \beta \,.\, \beta \subset \gamma \,.\supset.\, \alpha \subset \gamma$$

will hold whatever may be the orders of $\alpha$, $\beta$, $\gamma$ respectively. In other kinds of cases, however, trouble arises. Take, as a first instance, $p`\kappa$ and $s`\kappa$. We have

$$x \in p`\kappa \,.\equiv:\, \alpha \in \kappa \,.\supset_\alpha.\, x \in \alpha$$

thus $p`\kappa$ is a class of a higher order than any of the members of $\kappa$. Hence the hypothesis $(\alpha).\ f\alpha$ may not imply $f(p`\kappa)$, if $\alpha$ is of the order of the members of $\kappa$. There is a kind of proof invented by Zermelo, of which the simplest example is his second proof of the Schröder–Bernstein theorem (given in $*73$). This kind of proof consists in defining a certain class of classes $\kappa$, and then showing that $p`\kappa \in \kappa$. On the face of it, "$p`\kappa \in \kappa$" is impossible, since $p`\kappa$ is (xl) not of the same order as members of $\kappa$. This, however, is not all that is to be said. A class of classes $\kappa$ is always defined by some function of the form

$$(x_1, x_2, \ldots) : (\exists y_1, y_2, \ldots).\ F(x_1 \in \alpha,\ x_2 \in \alpha, \ldots y_1 \in \alpha,\ y_2 \in \alpha, \ldots)$$

where $F$ is a stroke-function, and "$\alpha \in \kappa$" means that the above function is true. It may well happen that the above function is true when $p`\kappa$ is substituted for $\alpha$, and the result is interpreted by $*8$. Does this justify us in asserting $p`\kappa \in \kappa$?

*53*[63]

Let us take an illustration which is important in connection with mathematical induction. Put

$$\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha \,.\, a \in \alpha)$$

Then $\breve{R}``p`\kappa \subset p`\kappa \,.\, a \in p`\kappa$ (see $*40{\cdot}81$) ~~But if we have ( as in the definition of~~ $\cancel{aR_*x)}$

$$\cancel{\breve{R}``\alpha \subset \alpha \,.\, a \in \alpha \,.\, \supset_\alpha .\, x \in \alpha, \text{i.e. } \alpha \in \kappa \,.\, \supset_\alpha .\, x \in \alpha,}$$

~~we are not justified in inferring~~ $\cancel{x \in p`\kappa}$~~, although~~ $\cancel{p`\kappa}$~~, in a sense, satisfies the defining function of~~ $\cancel{\kappa}$~~.~~ so that, in a sense $p`\kappa \in \kappa$. That is to say, if we substitute $p`\kappa$ for $\alpha$ in the defining function of $\kappa$, and apply $*8$, we obtain a true proposition. By the definitions of $*90$,

$$\overleftarrow{R_*}`a = p`\kappa.$$

Thus $\overleftarrow{R_*}`a$ is a second-order class. Consequently, if we have a ⌊hypothesis⌋ $(\alpha).\ f\alpha$, where $\alpha$ is a first-order class, we cannot assume

$$(\alpha).\ f\alpha \,.\supset.\, f(\overleftarrow{R_*}`a). \qquad \text{(A)}$$

By the proposition at the end of the previous section, if $(\alpha).\ f(\alpha)$ is deduced by logic from a universally-true stroke-function of elementary propositions, $f(\overleftarrow{R_*}`a)$ will also be true. Thus we may substitute $\overleftarrow{R_*}`a$ for $\alpha$ in any ⌊asserted⌋ proposition "$\vdash .\ f(\alpha)$" which occurs in *Principia Mathematica*. But when $(\alpha).\ f(\alpha)$ is a

[63] 'Dawson' in the upper left.

hypothesis, not a universal truth, the implication (A) is not, *prima facie*, necessarily true.

*54*

For example, if $\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha \,.\, a \in \alpha)$, we have

$$\alpha \in \kappa \,.\, \supset : \alpha \cap \beta \in \kappa \,.\, \equiv \,.\, \breve{R}``(\alpha \cap \beta) \subset \beta \,.\, \alpha \in \beta$$

Hence $$\alpha \in \kappa \,.\, \breve{R}``(\alpha \cap \beta) \subset \beta \,.\, \alpha \in \beta \,.\, \supset \,.\, p`\kappa \subset \beta \qquad (1)$$

In many of the propositions of $*90$, as hitherto proved, we substitute $p`\kappa$ for $\alpha$, whence we obtain

$$\breve{R}``(\beta \cap p`\kappa) \subset \beta \,.\, \alpha \in \beta \,.\, \supset \,.\, p`\kappa \subset \beta \qquad (2)$$

i.e. $$z \in \beta \,.\, aR_*z \,.\, \supset_{z,w} \,.\, w \in \beta : a \in \beta \,.\, aR_*x : \supset \,.\, x \in \beta$$

or $$aR_*x . \supset :. \, z \in \beta \,.\, aR_*z \,.\, \supset_{z,w} \,.\, w \in \beta : a \in \beta : \supset \,.\, x \in \beta$$

This is a more powerful form of induction than that used in the definition of $aR_*x$. But the proof is not valid, because we have no right to substitute $p`\kappa$ for $\alpha$ in passing from (1) to (2). Therefore the proofs which use this form of induction have to be reconstructed.

~~Thus speaking broadly we may say that, if $\kappa = \hat{\alpha}(f\alpha)$ and we have $f(p`\kappa)$, we are justified in *inferring* $g(p`\kappa)$ from an asserted logical proposition "$\vdash : \alpha \in \kappa \,.\, \supset_\alpha \,.\, g(\alpha)$", but we are not justified in the implication~~

$$\cancel{\alpha \in \kappa \,.\, \supset_\alpha \,.\, g(\alpha) : \supset \,.\, g(p`\kappa)}$$

~~if "$\alpha \in \kappa \,.\, \supset_\alpha \,.\, g(\alpha)$" is a mere hypothesis, not a logical truth.~~[64]

*55*

(xli) It will be found that the form to which we can reduce most of the fallacious inferences that seem plausible is the following.

Given "$\vdash .(x).\ f(x,x)$" we can infer "$\vdash : (x) : (\exists y).\ f(x,y)$". Thus given "$\vdash .(\alpha).\ f(\alpha,\alpha)$" we can infer "$\vdash : (\alpha) : (\exists\beta).\ f(\alpha,\beta)$". But this depends upon the possibility of $\alpha = \beta$. If, now, $\alpha$ is of one order and $\beta$ of another, we do not know that $\alpha = \beta$ is possible. Thus suppose we have

$$\alpha \in \kappa \,.\, \supset_\alpha \,.\, g\alpha$$

and we wish to infer $g\beta$, where $\beta$ is a class of higher order satisfying $\beta \in \kappa$. The proposition

$$(\beta) :. \, \alpha \in \kappa \,.\, \supset_\alpha \,.\, g\alpha : \supset : \beta \in \kappa \,.\, \supset \,.\, g\beta$$

[64] Changes of numbering on the leaves suggest that Russell deleted this section after having written one more folio, *55*, now *56*. He then added a new folio *55* and continued the composition with *57*.

becomes, when developed by $*8$,

$$(\beta) :: (\exists\alpha) :. \alpha \in \kappa . \supset . g\alpha : \supset : \beta \in \kappa . \supset . g\beta$$

This is only valid if $\alpha = \beta$ is possible. Hence the inference is fallacious if $\beta$ is of higher order than $\alpha$.

*56*[65]

Let us apply these considerations to Zermelo's proof of the Schröder–Bernstein theorem, given in $*73{\cdot}8$ ff. We have a class of classes

$$\kappa = \hat{\alpha}(\alpha \subset \mathrm{D}`R . \beta - \mathrm{C\!I}`R \subset \alpha . \breve{R}``\alpha \subset \alpha)$$

and we prove $p`\kappa \in \kappa$ ($*73{\cdot}81$), which is admissible in the limited sense explained above. We then add the hypothesis

$$x \sim\in (\beta - \mathrm{C\!I}`R) \cup \breve{R}``p`\kappa$$

and proceed to prove $p`\kappa - \iota`x \in \kappa$ (in the fourth line of the proof of $*73{\cdot}82$). This also is admissible in the limited sense. But in the next line of the same proof we make a use of it which is not admissible, arguing from $p`\kappa - \iota`x \in \kappa$ to $p`\kappa \subset p`\kappa - \iota`x$, because

$$\alpha \in \kappa . \supset_\alpha . p`\kappa \subset \alpha$$

The inference from

$$\alpha \in \kappa . \supset_\alpha . p`\kappa \subset \alpha \quad \text{to} \quad p`\kappa - \iota`x \in \kappa . \supset . p`\kappa \subset p`\kappa - \iota`x$$

is only valid if $p`\kappa - \iota`x$ is a class of the same order as the members of $\kappa$. For, when $\alpha \in \kappa . \supset_\alpha . p`\kappa \subset \alpha$ is written out, it becomes

$$(\alpha) ::: (\exists\beta) ::. (x) :: \alpha \in \kappa . \supset :. \beta \in \kappa . \supset . x \in \beta : \supset . x \in \alpha$$

This is deduced from $\alpha \in \kappa . \supset :. \alpha \in \kappa . \supset . x \in \alpha : \supset . x \in \alpha$ by the principle that $f(\alpha, \alpha)$ implies $(\exists\beta) . f(\alpha, \beta)$. But here the $\beta$ must be of the same order as the $\alpha$, while in our case $\alpha$ and $\beta$ are not of the same order, if $\alpha = p`\kappa - \iota`x$ and $\beta$ is an ordinary member of $\kappa$. At this point, therefore, where we infer $p`\kappa \subset p`\kappa - \iota`x$, the proof breaks down.

*57*[66]

It is easy, however, to remedy this defect in the proof. All we need is

$$x \sim\in (\beta - \mathrm{C\!I}`R) \cup \breve{R}``p`\kappa . \supset . x \sim\in p`\kappa$$

[65] Previously *55*. [66] 'Dawson' in the upper left.

or, conversely,

$$x \in p`\kappa \,.\supset.\, x \in (\beta - \text{Ɑ}`R) \cup \breve{R}``p`\kappa$$

(xlii) Now $\quad x \in p`\kappa \,.\supset:.\, \alpha \in \kappa \,.\supset_\alpha: \alpha - \iota`x \sim\in \kappa :$

$$\supset_\alpha : \sim(\beta - \text{Ɑ}`R \subset \alpha - \iota`x) \,.\vee.\, \sim\{\breve{R}``(\alpha - \iota`x) \subset \alpha - \iota`x\} :$$

$$\supset_\alpha :\; x \in \beta - \text{Ɑ}`R \,.\vee.\, x \in \breve{R}``(\alpha - \iota`x)$$

$$\supset:.\, x \in \beta - \text{Ɑ}`R : \vee : \alpha \in \kappa \,.\supset_\alpha.\, x \in \breve{R}``\alpha$$

Hence, by $*72\cdot341$,

$$x \in p`\kappa \,.\supset.\, x \in (\beta - \text{Ɑ}`R) \cup \breve{R}``p`\kappa$$

which gives the required result.

We assume that $\alpha - \iota`x$ is of no higher order than $\alpha$; this can be secured by taking $\alpha$ to be of at least the second order, since $\iota`x$, and therefore $-\iota`x$, is of the second order. We may always assume our classes raised to a given order, but not raised indefinitely.

Thus the Schröder–Bernstein theorem survives.

*58*

Another difficulty arises in regard to sub-classes. We put

$$\text{Cl}`\alpha = \hat{\beta}(\beta \subset \alpha) \quad \text{Df}$$

Now "$\beta \subset \alpha$" is significant when $\beta$ is of higher order than $\alpha$, provided its members are of the same type as those of $\alpha$. But when we have

$$\beta \subset \alpha \,.\supset_\beta.\, f\beta,$$

the $\beta$ must be of some definite type. As a rule, we shall be able to show that a proposition of this sort holds whatever the type of $\beta$, if we can show that it holds when $\beta$ is of the same type as $\alpha$. Consequently no difficulty arises until we come to Cantor's proposition $2^n > n$, which results from the proposition

$$\sim\{(\text{Cl}`\alpha) \text{ sm } \alpha\}$$

which is proved in $*102$. The proof is as follows:

$$R \in 1\to 1 \,.\, \text{D}`R = \alpha \,.\, \text{Ɑ}`R \subset \text{Cl}`\alpha \,.\, \xi = \hat{x}\{x \in \alpha - \breve{R}`x\} \,.\supset:$$

$$y \in \alpha \,.\, y \in \breve{R}`y \,.\supset_y.\, y \sim\in \xi : y \in \alpha \,.\, y \sim\in \breve{R}`y \,.\supset_y.\, y \in \xi :$$

$$\supset: y \in \alpha \,.\supset_y.\, \xi \neq \breve{R}`y :$$

$$\supset: \xi \sim\in \text{Ɑ}`R$$

As this proposition is crucial, we shall enter into it somewhat minutely.[67]

*59*

Let $\alpha = \hat{x}(\mathrm{A}!x)$, and let

$$xR\{\hat{z}(\phi!z)\} \,.=.\, f!(\phi!\hat{z}, x)$$

Then by our data

$$\mathrm{A}!x \,.\supset.\, (\exists\phi).\, f!(\phi!\hat{z}, x)$$

$$f!(\phi!\hat{z}, x) \,.\supset.\, \mathrm{A}!x \,.\, \phi!y \supset_y \mathrm{A}!y$$

$$f!(\phi!\hat{z}, x) \,.\, f!(\phi!\hat{z}, y) \,.\supset.\, x = y$$

$$f!(\phi!\hat{z}, x) \,.\, f!(\psi!\hat{z}, x) \,.\supset.\, \phi!y \equiv_y \psi!y$$

With these data,

$$x \in \alpha - \breve{R}\text{‘}x \,.\equiv:\, \mathrm{A}!x : f!(\phi!\hat{z}, x) \,.\supset_\phi.\, \sim\phi!x$$

Thus $$\xi = \hat{x}\{(\phi) : \mathrm{A}!x : f!(\phi!\hat{z}, x) \,.\supset.\, \sim\phi!x\}$$

(xliii) Thus $\xi$ is defined by a function in which $\phi$ appears as apparent variable. If we enlarge the initial range of $\phi$, we shall enlarge the range of values involved in the definition of $\xi$. There is therefore no way of escaping from the result that $\xi$ is of higher order than the sub-classes of $\alpha$ contemplated in the definition of Cl‘$\alpha$. Consequently the proof of $2^n > n$ collapses when the axiom of reducibility is not assumed. We shall find, however, that the proposition remains true when $n$ is finite.

*60*

With regard to relations, exactly similar questions arise as with regard to classes. A relation is no longer to be distinguished from a function of two variables, and we have

$$\phi(\hat{x}, \hat{y}) = \psi(\hat{x}, \hat{y}) \,.\equiv:\, \phi(x, y) \,.\equiv_{x,y}.\, \psi(x, y).$$

The difficulties as regards $\dot{p}$‘$\lambda$ and Rl‘P are less important than those concerning $p$‘$\kappa$ and Cl‘$\alpha$, because $\dot{p}$‘$\lambda$ and Rl‘P are less used. But a very serious difficulty arises as regards similarity. We have

$$\alpha \ \mathrm{sm}\ \beta. \equiv.\, (\exists R).\, R \in 1 \to 1 \,.\, \alpha = \mathrm{D}\text{‘}R \,.\, \beta = \text{ᗡ}\text{‘}R$$

[67] In the first edition the comment after Cantor's theorem, $*117{\cdot}661 \quad \vdash : \mu \in \mathrm{N_0C} \,.\supset.\, 2^\mu > \mu$ is "The above proposition is important". In the second edition this is added "(See, however, the Introduction to the second edition.)", a reference to this discussion.

Here $R$ must be confined within some type; but whatever type we choose, there may be a correlation of higher type ~~for~~ ⌊by⌋ which $\alpha$ and $\beta$ can be correlated. Thus we can never prove $\sim(\alpha \text{ sm } \beta)$, except in such special cases as when either $\alpha$ or $\beta$ is finite. This difficulty was illustrated by Cantor's theorem $2^n > n$, which we have just examined. Almost all our propositions are concerned in proving that two classes *are* similar, and these can all be interpreted so as to remain valid. But the few propositions which are concerned with proving that two classes are *not* similar collapse, except where one at least of the two is finite.

*61*[68]

### VII. MATHEMATICAL INDUCTION.

All the propositions on mathematical induction in Part II Section E and Part III Section C remain valid, when suitably interpreted.[69] But the proofs of many of them become fallacious when the axiom of reducibility is not assumed, and in some cases new proofs can only be obtained with considerable labour. The difficulty becomes at once apparent on observing the definition of "$x R_* y$" in $*90$. Omitting the factor "$x \in C‘R$", which is irrelevant for our purposes, the definition of "$x R_* y$" may be written

$$zRw \,.\supset_{z,w}.\, \phi!z \supset \phi!w :\supset_\phi .\, \phi!x \supset \phi!y \qquad \text{(A)}$$

i.e. "$y$ has every elementary hereditary property possessed by $x$." We may instead of elementary properties, take any other order properties; as we shall see later, it is advantageous to take 3rd-order properties ⌊when $R$ is one-many or many-one, and fifth-order properties in other cases.⌋[70] But for preliminary purposes it makes no difference what order of properties we take, and therefore for the sake of definiteness we take elementary properties to begin with. The difficulty is that, if $\phi_2$ is any second-order property, we cannot deduce from (A)

$$zRw \,.\supset_{z,w}.\, \phi_2 z \supset \phi_2 w :\supset_\phi .\, \phi_2 x \supset \phi_2 y \qquad \text{(B)}$$

(xliv) Suppose, for example, that $\phi_2 z \,.=.(\phi).\, f!(\phi!\hat{z}, z)$; then from (A) we can deduce

$$zRw \,.\supset_{z,w}.\, f!(\phi!\hat{z}, z) \supset_\phi f!(\phi!\hat{z}, w) :\supset: f!(\phi!\hat{z}, x) .\supset_\phi . f!(\phi!\hat{z}, y) : \\ \supset : \phi_2 x \,.\supset.\, \phi_2 y \qquad \text{(C)}$$

68 'Symonds' in the upper left.
69 Russell means here that the proofs remain correct after both assuming the extensionality of the logic and confirming that they do not rely on the axiom of reducibility.
70 '3rd-order' becomes 'third-order'.

But ⌊in general⌋ our hypothesis is not implied by the hypothesis of (B) ~~in general~~. If we put $\phi_2 z \,.=.(\exists\phi).\, f!(\phi!\hat{z}, z)$, we get exactly analogous results.

*62*

Hence in order to apply mathematical induction to a second-order property, it is not sufficient that it should be itself hereditary, but it must be composed of hereditary elementary properties. That is to say, if the property in question is $\phi_2 z$, where $\phi_2 z$ is either

$$(\phi).\, f!(\phi!\hat{z}, z) \text{ or } (\exists\phi).\, f!(\phi!\hat{z}, z),$$

it is not enough to have

$$zRw \,.\supset_{z,w}.\, \phi_2 z \supset \phi_2 w$$

but we must have, for each elementary $\phi$,

$$zRw \,.\supset_{z,w}.\, f!(\phi!\hat{z}, z) \supset f!(\phi!\hat{z}, w).$$

One inconvenient consequence is that, *prima facie*, an inductive property must not be of the form

$$xR_*z \,.\, \phi!z$$

or $$S \in \text{Potid`}R \,.\, \phi!S$$

or $$\alpha \in \text{NCinduct} \,.\, \phi!\alpha \,.$$

This is inconvenient, because often such properties are hereditary when $\phi$ alone is not, i.e. we may have

$$xR_*z \,.\, \phi!z \,.\, zRw \,.\supset_{z,w}.\, xR_*w \,.\, \phi!w$$

when we do not have

$$\phi!z \,.\, zRw \,.\supset_{z,w}.\, \phi!w$$

and similarly in the other cases.

These considerations make it necessary to re-examine all inductive proofs. In some cases they are still valid, in others they are easily rectified; in still others, the rectification is laborious, but it is always possible. The method of rectification is explained in the Appendix to this volume.[71]

*63*

There is, however, so far as we can discover, no way by which our present primitive propositions can be made adequate to Dedekindian and well-ordered relations. The

[71] 'in the Appendix' becomes 'in Appendix B'.

practical uses of Dedekindian relations depend upon $*211{\cdot}63 - {\cdot}692$, which lead to $*214{\cdot}3 - {\cdot}34$, showing that the series of segments of a series is Dedekindian. It is upon this that the theory of real numbers rests, real numbers being defined as segments of the series of rationals. This subject is dealt with in $*310$. If we were to regard as doubtful the proposition that the series of real numbers is Dedekindian, analysis would collapse.

The proofs of this proposition in *Principia Mathematica* depend upon the axiom of reducibility, since they depend upon $*211{\cdot}64$, which asserts

$$\lambda \subset \mathrm{D}`P_\epsilon \,.\supset.\, s`\lambda \in \mathrm{D}`P_\epsilon .$$

(xlv) For reasons explained above, if $\alpha$ is of the order of members of $\lambda$, $(\alpha).f\alpha$ may not imply $f(s`\lambda)$, because $s`\lambda$ is a class of higher order than the members of $\lambda$. Thus although we have

$$\mathrm{D}`P_\epsilon = \hat{\alpha}\{(\exists\beta).\, \alpha = P``\beta\}$$
$$s`\lambda = P``s`\breve{P}_\epsilon``\lambda$$

yet we cannot infer $s`\lambda \in \mathrm{D}`P_\epsilon$ except when $s`\lambda$ or $s`\breve{P}_\epsilon``\lambda$ is, for some special reason, of the same order as the members of $\lambda$. This will be the case when ~~every member of~~ $\lambda$ is finite, but not necessarily otherwise. Hence the theory of irrationals will require reconstruction.

*64*

Exactly similar difficulties arise in regard to well-ordered series. The theory of well-ordered series rests on the definition of $*250{\cdot}01$:

$$\mathrm{Bord} = \hat{P}(\mathrm{Cl\,ex}`\mathrm{C}`P \subset \text{Ɑ}`\mathrm{min}_P) \qquad \mathrm{Df}$$

whence $\quad P \in \mathrm{Bord} \,.\equiv:\, \alpha \subset \mathrm{C}`P \,.\, \exists!\alpha \,.\supset_\alpha.\, \exists!\alpha - \breve{P}``\alpha$

In making deductions, we constantly substitute for $\alpha$ some constructed class of higher order than $\mathrm{C}`P$. For instance, in $*250{\cdot}122$ we substitute for $\alpha$ the class $\mathrm{C}`P \cap p`\overleftarrow{P}``(\alpha \cap \mathrm{C}`P)$, which is in general of higher order than $\alpha$. If this substitution is illegitimate, we cannot prove that a class contained in $\mathrm{C}`P$ and having successors must have an immediate successor, without which the theory of well-ordered series becomes impossible. This particular difficulty might be overcome, but it is obvious that many important propositions must collapse.

It might be possible to sacrifice infinite well-ordered series to logical rigour, but the theory of real numbers is an integral part of ordinary mathematics, and can hardly be the object of a reasonable doubt. We are therefore justified in supposing that some logical axiom which is true will justify it. The axiom required may be more restricted than the axiom of reducibility, but, if so, it remains to be discovered.

*65*

The following are among the more important contributions to mathematical logic since the publication of the first edition of *Principia Mathematica*.[72]

D. HILBERT. Axiomatisches Denken, *Mathematische Annalen*, Vol. 78.
Die logischen Grundlagen der Mathematik, *ib.* Vol. 88.
Neue Begründung der Mathematik, *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, 1922.

P. BERNAYS. Ueber Hilbert's Gedanken zur Grundlegung der Arithmetik, *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. 31.

H. BEHMANN. Beiträge zur Algebra der Logik. *Mathematische Annalen*, Vol. 86.

L. CHWISTEK. Ueber die Antinomien der Prinzipien der Mathematik, *Mathematische Zeitschrift*, Vol. 14.
⌊The Theory of Constructive Types. *Annales de la Societe Mathematique de Pologne*, 1923. (Dr. Chwistek has kindly allowed us to read in MS. a longer work with the same title.)[73]⌋

H. WEYL. *Das Kontinuum*. Veit, 1918.
Ueber die neue Grundlagenkrise der Mathematik, *Mathematische Zeitschrift*, Vol. 10.[74]

(xlvi)

L. E. J. BROUWER. Begründung der Mengenlehre unabhängig vom Gesetz des ausgeschlossenen Dritten. *Verhandlingen d. K. Akademie der Wetenschappen*, Amsterdam, 1918.[75]

[72] 'among the more important contributions' becomes 'among the contributions', following Ramsey's suggestion.
[73] The title of the journal is not correct. See the discussion above in Chapter 3.
[74] Ramsey suggests this in his list of three "Additions" which were added to the printed version: "H. Weyl. Randbemerkungen zu Hauptproblemen der Mathematik. *Mathematische Zeitschrift* vol 20".
[75] In print this becomes: L. E. J. Brouwer. Begründung der Mengenlehre unabhängig vom logischen Satz des ausgeschlossenen Dritten. *Verhandlingen d. K. Akademie v, Wetenschappen*, 1918, 1919. Intuitionistische

*66*

A. TAJTELBAUM-TARSKI. Sur le terme primitif de la logistique, *Fundamenta Mathematicae*, Tome IV[76]
Sur les "truth-functions" au sens de MM. Russell et Whitehead, *ib.* Tome V.
Sur quelques théorèmes qui équivalent à l'axiome du choix, *ib.*

F. BERNSTEIN. Die Mengenlehre Georg Cantor's und der Finitismus, *Jahresberichte der deutschen Mathematiker-Vereinigung*, Vol. 28.

J. KÖNIG. *Neue Grundlagen der Logik, Arithmetik und Mengenlehre*. Veit, 1914.

C. I. LEWIS. *Types of Logical Theory*, University of California, 1921.[77]

H. M. SHEFFER. Total determinations of deductive systems with special reference to the Algebra of Logic. *Bulletin of the American Mathematical Society*, 1910.[78]
*Trans. Amer. Math. Soc.* Vol. XIV, pp. 481–488.
*The General Theory of Notational Relativity*. Cambridge, Mass. 1921.

J. G. P. NICOD. A reduction in the number of the primitive propositions of logic. *Proc. Camb. Phil. Soc.* Vol. XIX.[79]

Mengenlehre, *Jahresbericht der deutschen Mat-ematiker-Vereinigung*, Vol. 28. The corrections to the first are due to Ramsey, who also adds the second in his proofreading corrections.

76 'Tom.' replaces 'Tome' in *PM*.

77 There is a question mark, '?' to the left of this entry. It is corrected to *A Survey of Symbolic Logic*, University of California, 1918'.

78 '?' to the left of this entry as well. '*Mathematical Society*, 1910' corrected to '*Mathematical Society* Vol. XVI'.

79 "L. Wittgenstein. *Tractatus Logico-Philosophicus*, Kegan Paul, 1922." and "M. Schönwinkel. Ueber die Bausteine der mathematische Logick. *Math. Annalen*, Vol. 92" are added, apparently at Ramsey's suggestion. This latter contains a mistake, for the name is "Schönfinkel".

# Appendix A

## Editorial note

This manuscript is RA 230.031360. The beginning of each new page of print is indicated with a short divider line by the typesetter in the left margin. There are 30 folio leaves, numbered 1–10, 10a, 11–29. (16, formerly 30, has a verso, also numbered 30, and just $*8 \cdot 33 \vdash :. \{(x).\phi x\}$.) The pages with new numbers are from the HPF manuscript. The transcription follows the practices indicated in the note to the "Introduction" above.

## Appendix A: manuscript

*1*[1]

(635)

*APPENDIX A*

$*8$. *THE THEORY OF DEDUCTION FOR PROPOSITIONS CONTAINING APPARENT VARIABLES**

ALL propositions, of whatever order, are derived from a matrix composed of elementary propositions combined by means of the stroke. Given such a matrix, any constituent may be left constant or turned into an apparent variable; the latter may be done in two ways, by taking "all values" or "some values". Thus, if $p$ and $q$ are elementary propositions, giving rise to $p|q$, we may replace $p$ by $\phi x$ or $q$ by $\psi y$ or both, where $\phi x$, $\psi y$ are propositional functions whose values are elementary propositions. We thus arrive, to begin with, at four new propositions:

$$(x).\,(\phi x|q),\ \ (\exists x).\,(\phi x|q),\ \ (y).\,(p|\psi y),\ \ (\exists y).\,(p|\psi y)$$

[1] 'Pica' is written by a different hand next to the title and the 'S.P.' on the left margin, presumably by the typesetters. This first page was previously *16*.

By means of definitions, we can separate out the constant and the variable part in these expressions; we put

·01 $\{(x).\ \phi x\}|q\ .=.\ (\exists x).\ (\phi x|q)$ Df

·011 $\{(\exists x).\ \phi x\}|q\ .=.\ (x).\ (\phi x|q)$ Df

·012 $p|\{(y).\ \psi y\}\ .=.\ (\exists y).\ (p|\psi y)$ Df

·013 $p|\{(\exists y).\ \psi y\}\ .=.\ (y).\ (p|\psi y)$ Df[2]

These definitions define the meaning of the stroke when it occurs between two propositions of which one is elementary while the other is of the first order.

*This chapter is to replace ∗9 of the text.

*2*[3]

When the stroke occurs between two propositions which are both of the first order, we shall adopt the rule that the above definitions are to be applied first to the one on the left, treating the one on the right as if it were elementary, and are then to be applied to the one on the right. Thus

$$\{(x).\ \phi x\}|\{(y).\ (\psi y)\}.=:(\exists x).\ \phi x|\{(y).\ \psi y\}$$
$$=:(\exists x):(\exists y).\ (\phi x|\psi y)$$

The same rule can be applied to *n* propositions; they are to be eliminated from left to right. If a proposition occurs ~~twice~~ ⌊more than once⌋, ~~the first of~~ its occurrences ~~is the one to be counted~~ ⌊must be eliminated successively as if they were different propositions⌋. These rules are only required for the sake of definiteness, as different orders of elimination give equivalent results. This is only true because we are dealing with various functions each containing one variable, ⌊and no variable occurs on both sides of the stroke;⌋ it would not be true if we were dealing with functions of several variables. We have e.g.

$$(\exists x):(y).\ (\phi x|\psi y):\equiv:(y):(\exists x).\ (\phi x|\psi y)$$

(636) But we do not have in general

$$(\exists x):(y).\ \chi(x,y):\equiv:(y):(\exists x).\ \chi(x,y)\ ;$$

here the right-hand side is more likely to be true than the left-hand side. For the present, however, we are not concerned with variable functions of two variables.

*3*[4]

It should be observed that this possibility of changing the order of the variables is a merit of the stroke. We have

$$(\exists x):(y).\ \phi x|\psi y:\equiv:(y):(\exists x).\ \phi x|\psi y:\equiv:(\exists x).\sim\phi x\ .\vee.\ (y).\sim\psi y$$

[2] These are numbered ∗8·01, . . . , ∗8·013 in print, and similarly through the rest of this appendix.
[3] Previously *17.* [4] Previously *17a.*

That is, ~~both~~ ⌊these equivalent propositions ⌋ are true when, and only when, either $\phi$ is sometimes false or $\psi$ is always false. But if we take e.g.

$\phi x \vee \psi y \,.\, \sim \phi x \vee \sim \psi y$

we shall not get the same results. For

$(\exists x):(y).\ \phi x \vee \psi y \,.\, \sim \phi x \vee \sim \psi y : \supset : (y).\ \psi y \,.\vee.\, \cancel{\sim (y).\ \psi y}\ (y).\ \sim \psi y,$

whereas $(y):(\exists x).\ \phi x \vee \psi y \,.\, \sim \phi x \vee \sim \psi y$ does not imply this.

Written in the stroke notation, after some reduction, the above matrix is

$\{\phi x|(\psi y|\psi y)\} \mid \{\psi y|(\phi x|\phi x)\}.$

Here ⌊both⌋ $x$ and $y$ occur on both sides of the principal matrix. Thus in order to be able to change the order of "$(\exists x)$" and "$(y)$", it is sufficient (though not *always* necessary) that the matrix should contain some ~~consitu~~ part of the form $\phi x|\psi y$, and that $x$ and $y$ should not occur in any other part of the matrix. (This part may of course be the whole matrix.) We assume the legitimacy of this interchange by a primitive proposition, and in practice arrange to have all the $\exists$-prefixes as far to the right as possible, because this facilitates proofs.

*4*

Our primitive propositions are the following:

$*8\cdot1 \quad \vdash .\ (\exists x, y).\ \phi a|(\phi x|\phi y)$ Pp

On applying the definitions, this is seen to be

$\vdash : \phi a \,.\supset.\, (\exists x).\ \phi x.$

$\cdot11 \quad \vdash .\ (\exists x).\ \phi x|(\phi a|\phi b)$ Pp

On applying the definitions, this becomes

$\vdash : (x).\ \phi x \,.\supset.\, \phi a \,.\, \phi b.$

We have $\phi a|(\phi a|\phi b) \,.\vee.\, \phi b|(\phi a|\phi b)$

and by $*8\cdot1$ $\vdash : \phi a|(\phi a|\phi b) \,.\supset.\, (\exists x).\ \phi x|(\phi a|\phi b) :$

$\phi b|(\phi a|\phi b) \,.\supset.\, (\exists x).\ \phi x|(\phi a|\phi b)$

but we cannot deduce $(\exists x).\ \phi x|(\phi a|\phi b)$ without $*8\cdot11$ or an equivalent.

$\cdot12$ From "$(x).\ \phi x$" and "$(x).\ \phi x \supset \psi x$" we can infer "$(x).\ \psi x$", even when $\phi$ and $\psi$ are not elementary. Pp

$\cdot13$ If all occurrences of $x$ are separated from all occurrences of $y$ by a certain stroke, we can change the order of $x$ and $y$ in the prefix, i.e. we can replace "$(y):(\exists x).\ \phi x|\phi y$" by "$(\exists x):(y).\ \phi x|\phi y$" and vice versa. Pp.

(637) The above primitive propositions are to be assumed, not only for one or two variables, but for any number. Thus e.g. $*8\cdot1$ allows us to assert

$\vdash : \phi(a_1, a_2, \ldots a_n) . \supset . (\exists x_1, x_2, \ldots x_n). \phi(x_1, x_2, \ldots x_n).$

$\cdot 2 \quad \vdash : (x). \phi x . \supset . \phi a \quad [*8 \cdot 11 \frac{a}{b}]$

*5*

In what follows, the method of proof is invariably the same. We first apply the definitions until the whole asserted proposition is brought into the form of a matrix with a prefix. If necessary, we apply $*8 \cdot 13$ to change the order of the variables in the prefix. When the proposition to be proved has been brought into this form, we deduce it by means of $*8 \cdot 1 \cdot 11$, using $*8 \cdot 12$ in the deduction if necessary. ⌊It will be observed that $*8 \cdot 1$ is $\vdash : \phi a . \supset . (\exists x). \phi x$. Hence, by $*8 \cdot 12$, whenever we know $\phi a$, we can assert $(\exists x). \phi x$ ; $*8 \cdot 1$ is often used in this way.⌋

$*8 \cdot 21 \quad \vdash :. (x). \phi x \supset \psi x . \supset : (\exists x). \phi x . \supset . (\exists x). \psi x$

*Dem.*

Applying the definitions, and using $*8 \cdot 13$, the proposition to be proved becomes[5]

$(y, y') : (\exists x, z, w, z', w'). \{\phi x | (\psi x | \psi x)\}[\{\phi y | (\psi z | \psi w)\} | \{\phi y' | (\psi z' | \psi w')\}]$

Putting $z = w = z' = w' = x$, the above becomes

$(y, y') : (\exists x). \{\phi x | (\psi x | \psi x)\} | [\{\phi y | (\psi x | \psi x)\} | \{\phi y' | (\psi x | \psi x)\}]$

By $*8 \cdot 1$, the proposition to be proved is true if this is true. But this is true by $*8 \cdot 11$, putting $y$, $y'$ for $a$,$b$ and $\phi y | (\psi x | \psi x)$ for $\phi a$. Hence the proposition is true.

$\cdot 22 \quad \vdash : \phi a \vee \phi b . \supset . (\exists x). \phi x$

*Dem.*

$\vdash . *8 \cdot 11. \supset \vdash . (\exists z). (\sim \phi z) | (\sim \phi a | \sim \phi b)$ (1)

$\text{Transp.} \supset \vdash : (\sim \phi z) | (\sim \phi a | \sim \phi b) . \supset . (\phi a \vee \phi b) | (\phi z | \phi z)$ (2)

$\vdash .(1).(2). *8 \cdot 21. \supset \vdash . (\exists z). (\phi a \vee \phi b) | (\phi z | \phi z)$ (3)

$\vdash .(3). *8 \cdot 1 \cdot 21. \quad \supset \vdash . (\exists z, w). (\phi a \vee \phi b) | (\phi z | \phi w)$

$[(*8 \cdot 012 \cdot 013)] \quad \supset \vdash : \phi a \vee \phi b . \supset . (\exists x). \phi x : \supset \vdash . \text{Prop}$

These propositions, as well as all the others in $*8$, apply to any number of variables, since the primitive propositions do so.

*6*

$*8 \cdot 23 \quad \vdash : \ (\exists x). \phi x \vee \phi c . \supset . (\exists x). \phi x$

*Dem.*

Applying the definitions, this proposition is

$(x) : (\exists y, z). (\phi x \vee \phi c) | (\phi y | \phi z)$

i.e. $(x) : \phi x \vee \phi c . \supset . (\exists x). \phi x$

[5] The missing main connective of the matrix, a | right before the [, has been added in the printed version.

which follows from $*8 \cdot 22$.

(638)

The following propositions are concerned with forms of the syllogism.

$\cdot 24 \quad \vdash :: p \supset q \,.\supset :.\, q \,.\supset.\, (\exists x).\, \phi x : \supset : p \,.\supset.\, (\exists x).\, \phi x$

*Dem.*

Applying the definitions, we obtain a matrix

$(p \supset q) \mid [\{(q|(\phi x|\phi y)) \mid (p|(\phi z|\phi w)|p|(\phi u|\phi v))\}|$

{the same with accented letters}]

with a prefix

$(x, y, x', y') : (\exists z, w, u, v, z', w', u', v')$.

By $*8 \cdot 1$, this will be true if it is true for chosen values of $z, w, u, v, z', w', u', v'$. Put $z = u = x \,.\, w = v = y \,.\, z' = u' = x' \,.\, w' = v' = y'$. Then what has to be proved becomes

$p \supset q \,.\supset :.\, q \,.\supset.\, \phi x . \phi y : \supset : p \,.\supset.\, \phi x . \phi y :.\, q \,.\supset.\, \phi x' . \phi y' : \supset : p \,.\supset$
$.\, \phi x' . \phi y'$

which is true by Syll. Hence the proposition follows.

$\cdot 241 \quad \vdash :: (x).\, \phi x \,.\supset.\, p : \supset :.\, p \supset q \,.\supset :\ (x).\, \phi x \,.\supset.\, q$

Putting $f(y, z) \,.=.\, \{p|(q|q)\} \mid [\{\phi y|(q|q)\} \mid \{\phi z|(q|q)\}]$,

the matrix of the proposition to be proved is

$\{\phi x|(p|p)\} \mid \{f(y, z)|f(y', z')\}$

and the prefix is $(x) : (\exists y, z, y', z')$. Putting $y = z = y' = z' = x'$, the matrix reduces to $\phi x \supset p \,.\supset :\ p \supset q \,.\supset.\, \phi x \supset q$, which is true by Syll.[6] Hence the proposition is true by $*8 \cdot 1$.

7

$*8 \cdot 25 \ \vdash :: p \,.\supset.\, (\exists x).\, \phi x : \supset :.\, (\exists x).\, \phi x \,.\supset.\, (\exists x).\, \psi x : \supset : p \,.\supset.\, (\exists x).\, \psi x$

*Dem.*

Put $f(x, y, z, u, v, m, n) \,.=.\, \{\phi x|(\psi y|\psi z)\} \mid [\{p|(\psi u|\psi v)\} \mid p|(\psi m|\psi n)\}]$.[7]
Then the proposition to be proved, on applying the definitions, is found to have a matrix

$\{p|\phi a|\phi b)\}|\{f(x, y, z, u, v, m, n)|f(x', y', z', u', v', m', n')\}$

with the prefix

$(a, b, y, z, y', z') : (\exists x, u, v, m, n, x', u', v', m', n')$.

Put $x = a \,.\, x' = b \,.\, u = v = y \,.\, m = n = z \,.\, u' = v' = y' \,.\, m' = n' = z'$.

Then the matrix reduces to

$p \,.\supset.\, \phi a . \phi b : \supset :.\, \phi a \,.\supset.\, \psi y . \psi z : \supset : p \,.\supset.\, \psi y . \psi z \ :.$
$\phi b \,.\supset.\, \psi y' . \psi z' : \supset : p \,.\supset.\, \psi y' . \psi z'$

6 '$y = z = y' = z' = x'$' is changed to '$y = z = y' = z' = x$' .
7 '$\ldots \mid p|(\psi m|\psi n)\}]$' corrected to '$\ldots \mid \{p|(\psi m|\psi n)\}]$'.

which is true by Syll. Hence our proposition results by repeated applications of $*8\cdot1\cdot13$.

Analogous proofs apply to other forms of the syllogism.

*8*

$*8\cdot26$ $\vdash : \phi a \vee \phi b \vee \phi c \,.\supset .\, (\exists x).\, \phi x \vee \phi c$

*Dem.*

$\vdash : \phi a \vee \phi b \vee \phi c \,.\supset .(\phi a \vee \phi c) \vee (\phi b \vee \phi c)$ (1)

$\vdash .\, * \,8\cdot22 \,.\supset \vdash : (\phi a \vee \phi c) \vee (\phi b \vee \phi c) \,.\supset .\, (\exists x).\, \phi x \vee \phi c$ (2)

$\vdash .(1).(2).\, * \,8\cdot24 \,.\supset \vdash$ . Prop

(639)

$\cdot261$ $\vdash : \phi a \vee \phi b \vee \phi c \,.\supset .\, (\exists x).\, \phi x$

$[*8\cdot25\cdot26\cdot23]$

It is obvious that we can prove in like manner

$\phi a \vee \phi b \vee \phi c \;\vee \phi d \,.\supset .\, (\exists x).\, \phi x$

and so on.

*9*

$*8\cdot27$ $\vdash :: q \,.\supset .\, (\exists x).\, \phi x : \supset :.\, p \supset q \,.\supset : p \,.\supset .\, (\exists x).\, \phi x$

*Dem.*

Put $f(x, y, u, v) \,.=.\, \{p \mid (\phi x|\phi y)\} \mid \{p \mid (\phi u|\phi v)\}$

Then the matrix is

$\{q \mid (\phi a|\phi b)\} \mid [\{(p \supset q)|f(x, y, u, v)\} \mid \{(p \supset q)|f(x', y', u', v')\}]$

and the prefix is

$(a, b) : (\exists x, y, u, v, x', y', u', v')$.

Putting $x = u = x' = u' = a \,.\, y = v = y' = v' = b$, the matrix becomes

$q \,.\supset .\, \phi a.\, \phi b : \supset :.\, p \supset q \,.\supset : p \,.\supset .\, \phi a.\, \phi b$

which is true. Hence the proposition.

$\cdot271$ $\vdash :: q \,.\supset .\, (\exists x, y).\, \phi(x, y) : \supset :.\, p \supset q \,.\supset : p \,.\supset .\, (\exists x, y).\, \phi(x, y)$

[ Proof as in $*8\cdot27$ ]

It is obvious that we can prove similarly the analogous proposition with $\phi(x_1, x_2, \ldots x_n)$ in place of $\phi(x, y)$.

$\cdot272$ $\vdash ::.\, p \,.\supset : q \,.\supset .\, (\exists x).\, \phi x :.\supset :: r \supset p \,.\supset :.\, r. \supset : q \,.\supset .\, (\exists x).\, \phi x$

*Dem.*

$q \,.\supset .\, (\exists x).\, \phi x$ is $(\exists x, y).\, q|(\phi x|\phi y)$. Hence the proposition results from $*8{\cdot}271$ by the substitution of $p$ for $q$, $r$ for $p$, and $q \mid (\phi x|\phi y)$ for $\phi(x, y)$.

*10*

$*8{\cdot}28 \ \vdash :: p \,.\supset.\, (\exists x).\, \phi x : \supset :.\, q \,.\supset.\, (\exists x).\, \phi x : \supset : p \vee q \,.\supset.\, (\exists x).\, \phi x$

*Dem.*

Put $f(x, y, z, w) \,.=.\, \{(p \vee q) \mid (\phi x|\phi y)\} \mid \{(p \vee q) \mid (\phi z|\phi w)\}$.

Then the matrix is

$\{p \mid (\phi a|\phi b)\} \mid [\{(q|(\phi c|\phi d))|f(x, y, z, w)\} \mid \{(q|(\phi c'|\phi d'))|f(x', y', z', w')\}]$

and the prefix is

$(a, b, c, d, c', d') : (\exists x, y, z, w, x', y', z', w')$.

The matrix is

$$p \,.\supset.\, \phi a.\, \phi b : \supset :.\, q \,.\supset.\, \phi c.\, \phi d : \supset .\, f(x, y, z, w) :. \\ q \,.\supset.\, \phi c'.\, \phi d' : \supset .\, f(x', y', z', w')$$

while

$$f(x, y, z, w) \,.\equiv : p \vee q \,.\supset.\, \phi x.\, \phi y.\, \phi z.\, \phi w \,.$$

Call the matrix

$$F(x, y, z, w, x', y', z', w').$$

Then

$$\vdash : \quad p \,.\supset.\, F(a, b, a, b, a, b, a, b)$$
$$\vdash : \sim p \,.\supset.\, F(c, d, c, d, c', d', c', d')$$

Hence $\vdash : F(a, b, a, b, a, b, a, b) \,.\vee.\, F(c, d, c, d, c', d', c', d')$.

(640) Hence, by the extension of $*8{\cdot}261$ to eight variables,

$\vdash .\, (\exists\, x, y, z, w, x', y', z', w') \,.\, F(x, y, z, w, x', y', z', w')$

which was to be proved.

~~$\cdot 29 \ \vdash .\, (x).\, \phi x \,.\supset.\, \phi a$~~

~~*Dem.*~~

~~$\vdash .\, *8\cdot 11 \frac{a}{b} .\supset \vdash .\, (\exists x).\, \phi x|(\phi a|\phi a)$ (1)~~

~~$\vdash .(1).(*8{\cdot}01). \supset \vdash$ . Prop~~

*10a*[8]

$*8{\cdot}29 \ \vdash :.\, (x).\, \phi x \supset \psi x \,.\supset : (x).\, \phi x \,.\supset.\, (x).\, \psi x$

*Dem.*

Applying the definitions, our proposition is found to have a matrix

$$(\phi x \supset \psi x) \mid [\{\phi y|(\psi u|\psi v)\} \mid \{\phi y'|(\psi u'|\psi v')\}]$$

with a prefix (after using $*8{\cdot}13$)

$$(u, v, u', v') : (\exists\, x, y, y').$$

The matrix is equivalent to

$$\phi x \supset \psi x \,.\supset : \phi y \,.\supset.\, \psi u.\, \psi v : \phi y' .\supset.\, \psi u'.\, \psi v'$$

[8] Previously *5a*.

Calling this $M(x, y, y')$, we have to prove

$$(\exists\, x, y, y').\ M(x, y, y').$$

If $\psi u.\ \psi v.\ \psi u'.\ \psi v',\ M(x, y, y')$ is always true. (1)

If $\sim\psi u$, put $x = y = y' = u$. Then if $\phi u$ is true, $\phi u \supset \psi u$ is false and $M(u, u, u)$ is true. But if $\phi u$ is false, $\phi u\ .\supset .\ \psi u.\ \psi v$ and $\phi u\ .\supset .\ \psi u'.\psi v'$ are true, so that $M(u, u, u)$ is true. Hence

$$\sim\psi u\ .\supset .\ M(u, u, u)\ .\supset .\ (\exists\, x, y, y').\ M(x, y, y') \quad (2)$$

Similarly if $\quad \sim\psi v \vee \sim\psi u' \vee \sim\psi v'$ (3)

(1), (2), and (3) exhaust possible cases. Hence result by $*8{\cdot}28$.

*11*[9]

We are now in a position to prove that all the propositions of $*1 - *5$ remain true when one or more of the propositions $p, q, r, \ldots$ are first-order propositions instead of being elementary propositions. For this purpose, we take, not the one primitive proposition which Nicod has shown to be sufficient, but the two which he has shown to be equivalent to it, namely:

$$p \supset p \text{ and } p \supset q\ .\supset .\ s|q \supset p|s$$

We show that these are true when one, or two, or three, of the propositions $p, q, r$ are first-order propositions. From this, the rest follows. The first of these primitive propositions, $p \supset p$, gives rise to two cases, according as we substitute $(x).\ \phi x$ or $(\exists x).\ \phi x$ for $p$; the second primitive proposition gives rise to 26 cases. These have to be considered one by one.

$*8{\cdot}3\ \vdash : (x).\ \phi x\ .\supset .\ (x).\ \phi x$

Applying the definitions, this is $(\exists x) : (y, z).\ \phi x|(\phi y|\phi z)$, which follows from $*8{\cdot}11$ by $*8{\cdot}13$.

$\cdot 31\ \vdash : (\exists x).\ \phi x\ .\supset .\ (\exists x).\ \phi x$

Applying the definitions, this is $(x) : (\exists\, y, z).\ \phi x|(\phi y|\phi z)$

This ~~follows from $\phi x|(\phi x|\phi x)$ by~~ is $*8{\cdot}1$.

This completes the proof of $p \supset p$.

*12*[10]

(641)

$*8{\cdot}32\ \vdash :.\ (x).\ \phi x\ .\supset .\ q : \supset : s|q\ .\supset .\ \{(x).\ \phi x\}|s$

Putting $p\ .=.\ (x).\ \phi x$, the proposition to be proved is

$$(p|\sim q)\ |\ \sim\{(s|q)|\sim(p|s)\}.$$

[9] Previously *25*.

[10] ‘Dawson’ underlined in the upper left . ‘R. & W. I-41 p.641’ is next to the name on the manuscript, likely the beginning of the the new signature.

By the definitions,

$$p|\sim q \;.=.\; (\exists a).\ \phi a|(q|q) \qquad (1)$$

$$p|s \;.=.\; (\exists x).\ \phi x|s$$

$$\sim(p|s) \;.=.\; (x,y).\ (\phi x|s) \mid (\phi y|s)$$

$$(s|q)|\sim(p|s) \;.=.\; (\exists\, x,y).\ (s|q)|\{(\phi x|s) \mid (\phi y|s)\}$$

Put $\quad f(x,y) \;.=.\; (s|q)|\{(\phi x|s) \mid (\phi y|s)\}.$

Then $\quad \sim\{(s|q)|\sim(p|s)\} \;.=.\; (x,y,x',y').\ f(x,y)|f(x',y') \qquad (2)$

By (1) and (2), the proposition to be proved is

$(a):(\exists\, x,y,x',y').\ \{\phi a|(q|q)\}|\{f(x,y)|f(x',y')\}.$

Putting $x = y = x' = y' = a$, the matrix of this proposition reduces to

$$\phi a \supset q \;.\supset.\; s|q \supset \phi a|s$$

which is our primitive proposition with $\phi a$ substituted for $p$, and is therefore true. Hence the proposition follows by $*8\cdot1$.

In what follows, the reduction of the proposition to be proved to a matrix and prefix, by means of the definitions, proceeds always by the same method, and the steps will usually be omitted.

*13*

$*8\cdot321 \;\vdash :.\ (\exists x).\ \phi x \;.\supset.\; q :\supset: s|q \;.\supset.\; \{(\exists x).\ \phi x\}|s$

We obtain the same matrix as in $*8\cdot32$, but the opposite prefix, i.e. the prefix is

$$(x,y,x',y'):(\exists a).$$

The matrix is equivalent to

$$\phi a \supset q \;.\supset: q \supset \sim s \;.\supset.\; \phi x \supset \sim s \;.\; \phi y \supset \sim s \;.\; \phi x' \supset \sim s \;.\; \phi y' \supset \sim s$$

Calling this $fa$, we have to prove $(\exists a).\ fa$, for any $x, y, x', y'$. We have

$$\phi a \;.\sim q \;.\supset.\; fa$$

Also

$$\phi a.\, q. \supset :.\ fa \;.\equiv: \sim s \;.\supset.\; \phi x \supset \sim s \;.\; \phi y \supset \sim s \;.\; \phi x' \supset \sim s \;.\; \phi y' \supset \sim s :.$$
$$\supset :.\ fa$$

Hence $\qquad \phi a \;.\supset.\; fa$

Hence by $*8\cdot1\cdot24 \qquad \phi x \;.\supset.\; (\exists a).\ fa$

and similarly for $\phi y, \phi x', \phi y'$. Hence by $*8\cdot261$

$$\phi x \vee \phi y \vee \phi x' \vee \phi y' \;.\supset.\; (\exists a).\ fa$$

Also $\qquad \sim\phi x \;.\sim\phi y \;.\sim\phi x' \;.\sim\phi y' \;.\supset.\; fa.$

$[*8\cdot1\cdot24] \qquad\qquad \supset.\ (\exists a).\ fa$

Hence by $*8\cdot28$

$$\phi x \vee \phi y \vee \phi x' \vee \phi y' \vee \sim\phi x \;.\sim\phi y \;.\sim\phi x' \;.\sim\phi y' :\supset.\ (\exists a).\ fa$$

Hence, by $*8\cdot12$, $(\exists a).\ fa$, which was to be proved.

*14*

(642)
$*8\cdot322\ \vdash :.\ p\ .\supset .\ (x).\ \psi x : \supset : s|\{(x).\ \psi x\}\ .\supset .\ p|s$

*Dem.*

Put $fy\ . = .\ (s|\psi y)\ |\ \{(p|s)|(p|s)\}$

Then the proposition to be proved is

$$\lfloor(y, y'):\rfloor(\exists b, c)\ \cancel{:(y, y')}\ \{p|(\psi b|\psi c)\}|(fy|fy')$$

The matrix here is equivalent to

$$p\ .\supset .\ \psi b\ .\ \psi c : \supset : s|\psi y\ .\supset .\ p|s : s|\psi y'\ .\supset .\ p|s$$

Putting $b = y\ .\ c = y'$, this follows at once from the primitive proposition, which gives

$$p \supset \psi y\ .\supset : s|\psi y\ .\supset .\ p|s$$
$$p \supset \psi y'.\supset : s|\psi y'.\supset .\ p|s$$

Hence the proposition.

$\cdot323\ \vdash :.\ p\ .\supset .\ (\exists x).\ \psi x : \supset : s|\{(\exists x).\ \psi x\}\ .\supset .\ p|s$

We have the same matrix as in $*8\cdot322$, but the opposite prefix, i.e.

$$(b, c):(\exists y, y').$$

Putting $y = b\ .\ y' = c$, the matrix is satisfied, as in $*8\cdot322$.

*15*

$*8\cdot324\ \vdash :.\ p \supset q\ .\supset : \{(x).\ \chi x\}|q\ .\supset .\ p|\{(x).\chi x\}$

*Dem.*

Put $f(x, y, z). = .\ (\chi x|q)\ |\ \{(p|\chi y)|(p|\chi z)\}$. Then the matrix is

$$\{p|(q|q)\}|\{f(x, y, z)|f(x', y', z')\}$$

and the prefix is $(x, x'):(\exists y, z, y', z')$. $\cancel{(\exists x, x')}$ Putting

$$y = z = x\ .\ y' = z' = x',$$

the matrix is equivalent to

$$p \supset q.\supset : \chi x|q\ .\supset .\ p|\chi x : \chi x'|q\ .\supset .\ p|\chi x'$$

which follows from our primitive proposition by Comp.

$\cdot325\ \vdash :.\ p \supset q\ .\supset : \{(\exists x).\ \chi x\}|q\ .\supset .\ p|\{(\exists x).\ \chi x\}$

*Dem.*

The matrix is the same as in $*8\cdot324$, but the prefix is the opposite, i.e.

$$(y, z, y', z'):(\exists x, x')$$

Calling the matrix $M(x, x')$, we have, if $\theta w\ . \equiv_w .\ {\sim}\chi w$,

$M(x, x'). \equiv$

$:: p \supset q . \supset :. q \supset \theta x . \supset : p . \supset . \theta y . \theta z :. q \supset \theta x' . \supset : p . \supset . \theta y' . \theta z'$

Hence $\theta y . \theta z . \theta y' . \theta z' . \supset . M(x, x') . \supset . (\exists\, x, x') . M(x, x')$ (1)

But $\sim\theta x . \sim\theta x' . \supset . M(x, x')$. Hence

$$\sim\theta x . \supset . M(x, x) . \supset . (\exists\, x, x') . M(x, x') \qquad (2)$$

Similarly with $\theta y, \theta x', \theta y'$. Hence the result follows as in $*8{\cdot}321$.

This ends the cases in which only one of $p, q, r$ in

$$p \supset q . \supset : s|q . \supset . p|s$$

is of the first order instead of being elementary. We have now to deal with the cases in which two, but not three, are of the first order.

*16*[11]

(643)

$*8{\cdot}33 \ \vdash :. (x). \phi x . \supset . (x). \psi x : \supset : s|\{(x). \psi x\} . \supset . \{(x). \phi x\}|s$

Putting $f(x, y, z) . = . (s|\psi x) \mid \{\phi y|s)|(\phi z|s)\}$, the matrix is

$$\{\phi a|(\psi b|\psi c)\}|\{f(x, y, z)|f(x', y', z')\}$$

and the prefix is $(a, x, x') : (\exists\, b, c, y, z, y', z')$. The matrix is satisfied by

$$\cancel{x} \lfloor b \rfloor = \cancel{b} \lfloor x \rfloor . \cancel{x'} \lfloor c \rfloor = \cancel{c} \lfloor x' \rfloor . \ y = z = y' = z' = a$$

in which case it is equivalent to[12]

$\phi a . \supset . \psi x . \psi x' : \supset : \psi x \supset \sim s . \supset . \phi a \supset \sim s : \psi x' \supset \sim s . \supset . \phi a \supset \sim s$

Hence prop.

We have the same matrix in the three following propositions, only with different prefixes.

$\cdot 331 \ \vdash :. (x). \phi x . \supset . (\exists x). \psi x : \supset : s|\{(\exists x). \psi x\} . \supset . \{(x). \phi x\}|s$

Here the prefix to the matrix is $(a, b, c) : (\exists x, y, z, x', y', z')$. The matrix is satisfied by $x = b . x' = c . y = z = y' = z' = a$. Hence Prop.

$\cdot 332 \ \vdash :. (\exists x). \phi x . \supset . (x). \psi x : \supset : s|\{(x). \psi x) . \supset . \{(\exists x). \phi x\}|s$[13]

The prefix here is $\cancel{(\exists a, b, c)}$ $(x, y, z, x', y', z') : \lfloor (\exists a, b, c). \rfloor$ Writing $r$ for $\sim s$, matrix becomes

$\phi a . \supset . \psi b . \psi c : \supset :. \psi x \supset r . \supset . \phi y \vee \phi z \supset r : \psi x' \supset r . \supset . \phi y' \vee \phi z' \supset r.$

(Here only $a, b, c$ can be chosen arbitrarily.) This is true if $\phi y, \phi z, \phi y', \phi z'$ are all false. Suppose $\phi y$ is true. Put $a = y$. Then if $\psi b$ or $\psi c$ is false $\phi a . \supset . \psi b . \psi c$ is false, and the matrix is true. Therefore if $\psi x$ is false, put $b = c = x$; if $\psi x'$ is

[11] Previously *30*. 'Symonds' in the upper left. The verso is also foliated *30* with, in the upper left, : '$*8 \cdot 33 \ \vdash :. \{(x).\phi x\}$', a false start.

[12] '$: \supset :$' in the next line is corrected to '$: \supset :.$' .

[13] Pairing { with ) is the only such slip in all the manuscripts. It is corrected in print.

false, put $b = c = x'$. If $\psi x$ and $\psi x'$ are both true, putting $a = y \,.\, b = c = x$, the matrix becomes equivalent to

$$r \,.\supset.\, \phi y \vee \phi z \supset r : r \,.\supset.\, \phi y' \vee \phi z' \supset r$$

which is true. Hence if $\phi y$ is true, the matrix can be made true. Similarly for $z$, $y'$, $z'$. This exhausts possible cases. Hence prop, by $*8{\cdot}28$.

*17*

$*8{\cdot}333 \vdash :. (\exists x).\, \phi x \,.\supset.\, (\exists x).\, \psi x :\supset: s|\{(\exists x).\, \psi x\} \,.\supset.\, \{(\exists x).\, \phi x\}|s$

*Dem.*

The matrix is as before, and the prefix (after using $*8{\cdot}13$) is

$$(b, c, y, z, y', z') : (\exists a, x, x').$$

Call the matrix $M(a, x, x')$. Then

$\vdash : \psi b \,.\supset.\, M(a, b, b) \,.\supset.\, (\exists a, x, x').\, M(a, x, x')$ (1)

$\vdash : \psi c \,.\supset.\, M(a, c, c) \,.\supset.\, (\exists a, x, x').\, M(a, x, x')$ (2)

$\vdash : \sim\psi b.\, \sim\psi c.\, \phi y \,.\supset.\, M(y, b, c) \,.\supset.\, (\exists a, x, x').\, M(a, x, x')$ (3)

(1).(2).(3). $\supset \vdash : \phi y \,.\supset.\, (\exists a, x, x').\, M(a, x, x')$ [using $*\,8{\cdot}28$] (4)

Similarly for $\phi y'$, $\phi z$, $\phi z'$. Hence by $*8{\cdot}28$

$\vdash : \phi y \vee \phi y' \vee \phi z \vee \phi z' \,.\supset.\, (\exists a, x, x').\, M(a, x, x')$ (5)

But $\vdash :. \sim\phi y \,.\, \sim\phi y' \,.\, \sim\phi z \,.\, \sim\phi z' \,.\supset: \phi y \vee \phi z \supset r \,.\, \phi y' \vee \phi z' \supset r :$

$\supset : M(a, x, x')$

[$*8{\cdot}1$] $\supset : (\exists a, x, x').\, M(a, x, x')$ (6)

$\vdash$ .(5).(6). $*\,8{\cdot}28 \,.\supset \vdash .(\exists a, x, x').\, M(a, x, x').\ \supset \vdash$ . Prop

(644)

This ends the cases in which $p$ and $q$ but not $s$ contain apparent variables. We take next the four cases in which $p$ and $s$, but not $q$, contain apparent variables.

*18*[14]

$*8{\cdot}34 \vdash :. (x).\phi x \,.\supset.\, q :\supset: \{(x).\chi x\}|q \,.\supset.\, \{(x).\phi x\}|\{(x).\chi x\}$

Putting $f(x, y, z, u, v) \,.=.\, (\chi x|q) \mid \{(\phi y|\chi z)|(\phi u|\chi v)\}$, the matrix is

$$(\phi a| \sim q) \mid \{f(x, y, z, u, v)|f(x', y', z', u', v')\}.$$

(This is also the matrix of the three following propositions.)

The prefix is $(a, x, x') : (\exists y, z, u, v, y', z', u', v')$.

The matrix is equivalent to

$$\phi a \supset q \,.\supset.\, f(x, y, z, u, v) \,.\, f(x', y', z', u', v')$$

and $\quad f(x, y, z, u, v). \equiv : \chi x|q \,.\supset.\, \phi y|\chi z \,.\, \phi u|\chi v :$

$\equiv : q \supset \sim\chi x \,.\supset.\, \phi y \supset \sim\chi z \,.\, \phi u \supset \sim\chi v$

[14] Previously *32*.

Putting $y = u = y' = u' = a \,.\, z = v = x \,.\, z' = v' = x'$, the matrix is satisfied. Hence Prop.

*19*[15]

$*8{\cdot}341 \;\vdash :.\, (x).\, \phi x \,.\supset .\, q :\supset : \{(\exists x).\, \chi x\}|q \,.\supset .\, \{(x).\, \phi x\}|\{(\exists x).\, \chi x\}$

Matrix as in $*8{\cdot}341$.[16] Prefix $(a, z, v, z', v') : (\exists x, y, u, x', y', u')$.

Matrix is equivalent to

$$\phi a \supset q \,.\supset :.\, q \supset \sim\chi x \,.\supset .\, \phi y \supset \sim\chi z \,.\, \phi u \supset \sim\chi v :$$
$$q \supset \sim\chi x' \,.\supset .\, \phi y' \supset \sim\chi z' .\, \phi u' \supset \sim\chi v'$$

If $\phi a$ is false, this becomes true by putting $y = u = y' = u' = a$. If $\phi a$ is true, the matrix is ~~equivalent to~~ true if $q$ is false. ~~$q \,.\supset :\, q \supset \sim\chi x \,.\supset .$ etc. This is true if $q$ is false.~~ Suppose $q$ true. Then the matrix is equivalent to

$$\sim\chi x \,.\supset .\, \phi y \supset \sim\chi z \,.\, \phi u \supset \sim\chi v : \sim\chi x' \,.\supset .\, \phi y' \supset \sim\chi z' .\, \phi u' \supset \sim\chi v'$$

This is true if $\chi x, \chi v, \chi z', \chi v'$ are false. If one of them, say $\chi z$, is true, put $x = x' = z$, and the matrix is true. This exhausts possible cases. Hence Prop, by $*8{\cdot}28$.

~~$\cdot 342 \;\vdash :.\, (\exists x).\, \phi x \,.\supset .\, q :\supset : \{(x).\, \chi x\}|q \,.\supset .\, \{(\exists x).\, \phi x\}|\{(x).\, \chi x\}$~~

~~Here we have $(\exists a, z, v, z', v') : (x, y, u, x', y', u')$, with same matrix. If $\sim\chi x$, put $z = v = z' = v' = x$ and matrix holds, If $\sim\chi x'$, put $z = v = z' = v' = x'$ and matrix holds. Assume next $\chi x \,.\, \chi x'$. Then if $q$ is true, $q \supset \sim\chi x$ and $q \supset \sim\chi x'$ are false , and whole is true. Therefore assume $\sim q$. Then if $\phi a$ is true, $\phi a \supset q$ is false and whole is true. Hence if one of $\phi y, \phi u, \phi y', \phi u'$ is true, say $\phi y$, put $a = y$ and whole is true. But if all four are false, $\phi y \supset \sim\chi z$, etc. are true, and whole is true. This exhausts possible cases, and therefore proves prop, by $*8{\cdot}28$.~~

*20*[17]

$*8{\cdot}342 \;\vdash :.\, (\exists x).\, \phi x \,.\supset .\, q :\supset : \{(x).\, \chi x\}|q \,.\supset .\, \{(\exists x).\, \phi x\}|\{(x).\, \chi x\}$

Matrix as before. Prefix (after assuming $8{\cdot}13$)

$$(x, y, u, x', y', u') : (\exists a, z, v, z', v').$$

Call the matrix $M(a, z, v, z', v')$. Then

$$\vdash : \sim\chi x \,. \qquad \supset .\, M(a, x, x, x, x) \qquad (1)$$
$$\vdash : \sim\chi x' . \qquad \supset .\, M(a, x', x', x', x') \qquad (2)$$
$$\vdash :\, q \,.\, \chi x \,.\, \chi x' \,.\supset .\, \sim(q \supset \sim\chi x) \,.\, \sim(q \supset \sim\chi x').$$
$$\supset .\, M(a, z, v, z', v') \qquad (3)$$
$$\vdash : \sim q \,.\, \phi y \,.\supset .\, \sim(\phi y \supset q).$$
$$\supset .\, M(y, z, v, z', v') \qquad (4)$$

Similarly if $\sim q \,.\, \phi u$ or $\sim q \,.\, \phi y'$ or $\sim q \,.\, \phi u'$. Hence by $*8{\cdot}1{\cdot}28$

$$\vdash : \sim q \,.\, \phi y \vee \phi u \vee \phi y' \vee \phi u' \,.\supset .\, (\exists\, a, z, v, z', v').\, M(a, z, v, z', v') \qquad (5)$$

[15] Previously *33*. [16] Corrected to '$*8{\cdot}34$' . [17] Previously *33*. 'Dawson' in the upper left.

$\vdash : \sim\phi y \,.\, \sim\phi u \,.\, \sim\phi y' .\, \sim\phi u' .\supset .$

$$\phi y \supset \sim\chi z \,.\, \phi a \supset \sim\chi v \,.\, \phi y' \supset \sim\chi z' \,.\, \phi u' \supset \sim\chi v' .$$
$$\supset .\, M(a, z, v, z', v') \qquad (6)$$

(5).(6). $\supset \vdash : \sim q \,.$ $\quad \supset .\, (\exists\, a, z, v, z', v').\, M(a, z, v, z', v') \qquad (7)$

$\vdash$ .(1).(2).(3).(7). $\supset \vdash$ . Prop

*21*[18]

(645)

$*8{\cdot}343 \;\vdash :.\, (\exists x).\, \phi x \,.\supset .\, q : \supset : \{(\exists x).\, \chi x\}|q \,.\supset .\, \{(\exists x).\, \phi x\}|\{(\exists x).\, \chi x\}$

Prefix to matrix is $\cancel{(\exists\, a, x, x')}$ $(y, z, u, v, y', z', u', v') : (\exists\, a, x, x')$.

Call this matrix $f(a, x, x')$.

It is true if $\quad \sim\chi z .\, \sim\chi v .\, \sim\chi z' .\, \sim\chi v' .$ (1)

Also $\quad \chi z .\, q \,.\supset .\, f(a, z, z) \,.\supset .\, (\exists\, a, x, x').\, f(a, x, x')$ (2)

Similarly if we have $\quad \chi v .\, q$ or $\chi z' .\, q$ or $\chi v' .\, q$ (3)

From (1).(2).(3), by $*8{\cdot}28$, $q \,.\supset .\, (\exists\, a, x, x').\, f(a, x, x')$ (4)

Now $\phi a \,.\, \sim q \,.\supset .\, f(a, x, x')$. Hence

$$\phi y \,.\, \sim q \,.\supset .\, f(y, x, x') \,.\supset .\, (\exists\, a, x, x').\, f(a, x, x')$$

Similarly for $\phi z \,.\, \sim q$, $\phi y' .\, \sim q$, $\phi z \,.\, \sim q$. Hence

$$\phi y \vee \phi z \vee \phi y' \vee \phi z' \,.\, \sim q \,.\supset .\, (\exists a, x, x').\, f(a, x, x') \qquad (5)$$

But $\quad \sim\phi y \,.\, \sim\phi z \,.\, \sim\phi y' \,.\, \sim\phi z' \,.\supset .\, f(a, x, x')$ (6)

By (5) and (6), $\quad \sim q \,.\supset .\, (\exists a, x, x').\, f(a, x, x')$ (7)

$\vdash$ .(4).(7). $*\, 8{\cdot}28$. $\supset \vdash$ . Prop

In the next four propositions $q$ and $r$ are replaced by propositions containing apparent variables, while $p$ remains elementary.

*22*[19]

$*8{\cdot}35 \;\vdash :.\, p \,.\supset .\, (x).\, \psi x : \supset : \{(x).\, \chi x\}|\{(x).\, \psi x\} \,.\supset .\, p|\{(x).\, \chi x\}$

Putting $q \,.=.\, (x).\, \psi x$, $s \,.=.\, (x).\, \psi x$, the proposition is

$$(p \mid \sim q) \mid \; \sim\{(s|q)| \sim(p|s)\}$$

We have by the definitions

$$\sim q \,.=.\, (\exists\, b, c).\, \psi b|\psi c$$
$$p \mid\sim q \,.=.\, (b, c).\, p|(\psi b|\psi c)$$
$$s \mid q \,.=.\, (\exists x, y).\, \chi y|\psi x$$
$$p \mid s \,.=.\, (\exists z).\, p \mid\chi z$$
$$\sim(p|s) \,.=.\, (z, w).\, (p \mid\chi z)|(p \mid\chi w)$$
$$(s|q)|\sim(p|s) \,.=:\, (x, y) : (\exists\, z, w).\, (\chi y \mid\psi x) \mid \{(p \mid\chi z) \mid (p \mid\chi w)\}$$

Put $\quad f(x, y, z, w) \,.=.\, (\chi y \mid\psi x) \mid \{(p \mid\chi z) \mid (p \mid\chi w)\}$.

[18] Previously *34*. [19] Previously *35*.

Then $\sim\{(s|q)|\sim(p|s)\} . =: (\exists x, y, x', y')$

$:(z, w, z', w'). f(x, y, z, w)|f(x', y', z', w'),$

$(p \mid\sim q)|\sim\{(s|q)|\sim(p|s)\}. =: (x, y, x', y'):(\exists\, b, c, z, w, z', w').$

$\{p|(\psi b|\psi c)\}|\{f(x, y, z, w)|f(x', y', z', w')\}.$

Writing $\theta\hat{x}$ for $\sim\chi\hat{x}$, the matrix is equivalent to

$p. \supset . \psi b.\psi c :\supset :. \psi x \supset \theta y. \supset : p . \supset . \theta z.\theta w :. \psi x' \supset \theta y'. \supset : p . \supset . \theta z'.\theta w'$

This is satisfied by putting $b = x . c = x' . z = w = y . z' = w' = y'$. Hence Prop. The same matrix appears in the next three propositions; only the prefix changes.

*23*[20]

$*8{\cdot}351 \vdash :. p . \supset . (x). \psi x : \supset : \{(\exists x). \chi x\}|\{(x). \psi x\} . \supset . p|\{(\exists x). \chi x\}$

Same matrix as in $*8{\cdot}35$, but prefix $\cancel{(\exists\, b, c, y, y')}$ $(x, z, w, x', z', w'):(\exists\, b, c, y, y')$.

Matrix is true if $\theta z . \theta w . \theta z' . \theta w'$.

Assume $\sim\theta z$, and put $y = y' = z . b = x . c = x'$.

(646)

We now have $\psi x \supset \theta y. \equiv . \sim\psi x$ and $p . \supset . \theta z . \theta w : \equiv . \sim p$. Hence matrix is equivalent to

$p . \supset . \psi x . \psi x' : \supset :. \sim\psi x . \supset . \sim p :. \sim\psi x'. \supset : p . \supset . \theta z' . \theta w'$

which is true. Similarly if $\sim\theta w \vee \sim\theta z' \vee \sim\theta w'$. Hence prop, by $*8{\cdot}1{\cdot}28$.

$\cdot 352 \vdash :. p . \supset . (\exists x). \psi x : \supset : \{(x). \chi x\}|\{(\exists x). \psi x\} . \supset . p|\{(x). \chi x\}$

Same matrix, but prefix $(b, c, y, y'):(\exists\, x, z, w, x', z', w')$.

Satisfied by $x = b . x' = c . z = w = y . z' = w' = y'$. Hence Prop.

$\cdot 353 \vdash :. p . \supset . (\exists x). \psi x : \supset : \{(\exists x). \chi x\}|\{(\exists x). \psi x\} . \supset . p|\{(\exists x). \chi x\}$

Same matrix, with prefix $\cancel{(\exists\, x, y, x', y')}$ $(b, c, z, w, z', w'):(\exists\, x, y, x', y')$.

If $\psi b$ is true and $\theta z$ false, matrix is satisfied by $x = x' = b . y = y' = z$, because these values make $\psi x \supset \theta y$ and $\psi x' \supset \theta y'$ false. Similarly if $\psi b$ is true and $\theta w$ or $\theta z'$ or $\theta w'$ is false, and if $\psi c$ is true and $\theta z, \theta w, \theta z'$ or $\theta w'$ is false.

It remains to consider $\sim\psi b . \sim\psi c : \vee : \theta z . \theta w . \theta z' . \theta w'$

The second alternative makes the matrix true, because it gives

$$p . \supset . \theta z . \theta w : p . \supset . \theta z' . \theta w'$$

The first alternative gives

$$p . \supset . \psi b . \psi c : \supset : \sim p :$$
$$\supset : p . \supset . \theta z . \theta w : p . \supset . \theta z' . \theta w'$$

so that again the matrix is true. Hence Prop.

[20] Previously *36*.

This finishes the cases in which one or two of the three constituents of $p \supset q \,.\supset.\, s|q \supset p|s$ remain elementary. It remains to consider the eight cases in which none remains elementary. These all have the same matrix.

*24*[21]

$*8{\cdot}36 \vdash :.\ (x).\phi x \,.\supset.\, (x).\psi x : \supset : \{(x).\chi x\}|\{(x).\psi x\} \,.\supset.\, \{(x).\phi x\}|\{(x).\chi x\}$

Putting $p \,.=.\, (x).\ \phi x,\ \ q \,.=.\, (x).\ \psi x,\ \ s \,.=.\, (x).\ \chi x$, we have

$$\sim q \,.=.\, (\exists\, b, c).\ \psi b|\psi c$$
$$p\,|\sim q \,.=:\, (\exists a):(b, c).\ \phi a|(\psi b|\psi c)$$
$$s\,|\,q \,.=.\, (\exists x, y).\ \chi y|\psi x$$
$$p\,|\,s \,.=.\, (\exists z, w).\ \phi z|\chi w$$
$$\sim(p|s) \,.=.\, (z, w, u, v).\ (\phi z|\chi w)|(\phi u|\chi v)$$
$$(s|q)|\sim(p|s) \,.=:\, (x, y):(\exists\, z, w, u, v).\ (\chi y\,|\psi x)\,|\,\{(\phi z|\chi w)|(\phi u|\chi v)\}$$

Put $\quad f(x, y, z, w, u, v) \,.=.\, (\chi y\,|\psi x)\,|\,\{(\phi z\,|\chi w)|(\phi u|\chi v)\}$. Then

$$\sim\{(s|q)|\sim(p|s)\} \,.=:\, (\exists x, y, x', y'):(z, w, u, v, z', w', u', v').$$
$$f(x, y, z, w, u, v)|f(x', y', z', w', u', v')$$
$$(p\,|\sim q)|\sim\{(s|q)|\sim(p|s)\}.=:(a, x, y, x', y'):(\exists\, b, c, z, w, u, v, z', w', u', v').$$
$$\{\phi a|(\psi b|\psi c)\}|\{f(x, y, z, w, u, v)|f(x', y', z', w', u', v')\}$$

Writing $\theta\hat{x}$ for $\sim\chi\hat{x}$, the matrix is equivalent to

$$\phi a \,.\supset.\psi b.\psi c : \supset :.\ \psi x \supset \theta y. \supset .\phi z \,.\supset.\, \theta w.\ \phi u \supset \theta v :$$
$$\psi x' \supset \theta y'. \supset .\ \phi z' \supset \theta w' \,.\, \phi u' \supset \theta v'$$

This is satisfied by $b = x.\ c = x'.\ z = u = z' = u' = a.\ w = v = y.\ w' = v' = y'$.

Hence Prop.

*25*[22]

(647)

$*8{\cdot}361 \vdash :.\ (x).\phi x \,.\supset.\, (x).\psi x : \supset :$

$$\{(\exists x).\chi x\}|\{(x).\psi x\} \,.\supset.\, \{(x).\phi x\}|\{(\exists x).\chi x\}$$

Same matrix, but "all" and "some" are interchanged in arguments to $\chi$, i.e. in $y, w, v, y', w', v'$. The $\exists$ - variables are therefore $b, c, y, y', z, z', u, u'$.

If $\sim\phi a$, put $z = u = z' = u' = a$, and matrix is satisfied.

If $\phi a$ is true, matrix is true if $\sim\psi b \vee \sim\psi c$,⌊i.e. if $\sim\psi x \vee \sim\psi x'$, since $b, c$ are arbitrary.⌋ Assume $\psi x \,.\, \psi x'$. Then matrix reduces to

$$\theta y \,.\supset.\, \phi z \supset \theta w \,.\, \phi u \supset \theta v : \theta y' \,.\supset.\, \phi z' \supset \theta w' \,.\, \phi u' \supset \theta v'$$

If $\theta w, \theta v, \theta w', \theta v'$ are all true, this is true.

If $\sim\theta w$, put $y = y' = w$, and matrix is satisfied.

[21] Previously *37*. [22] Previously *38*.

Similarly if $\sim\theta v$, $\sim\theta w'$ or $\sim\theta v'$. Hence Prop.

$\cdot 362 \vdash :. (x).\phi x . \supset . (\exists x).\psi x :\supset: \{(x).\chi x\}|\{(\exists x).\psi x\} . \supset . \{(x).\phi x\}|\{(x).\chi x\}$

Matrix as in $8\cdot 36$. Prefix results from $8\cdot 36$ by interchanging "all" and "some" among $\psi$ - arguments, i.e. $b, c, x, x'$. Hence Prop results from same substitutions as in $8\cdot 36$.

$\cdot 363 \vdash :. (x).\phi x . \supset . (\exists x).\psi x :\supset:$

$$\{(\exists x).\chi x\}|\{(\exists x).\psi x\} . \supset . \{(x).\phi x\}|\{(\exists x).\chi x\}$$

Results from interchanging "all" and "some", in $*8\cdot 361$, in the $\psi$- arguments, viz. $b, c, x, x'$. The $\exists$ - variables are therefore $x, x', y, y', z, z', u, u'$, and the proof proceeds exactly as in $*8\cdot 361$, interchanging $x, x'$ and $b, c$.

*26* [23]

$*8\cdot 364 \vdash :. (\exists x).\phi x . \supset . (x).\psi x :\supset: \{(x).\chi x\}|\{(x).\psi x\} . \supset . \{(\exists x).\phi x\}|\{(x).\chi x\}$

The proposition is what results from $*8\cdot 36$ by interchanging "all" and "some" in the $\phi$ - arguments, viz. $a, z, u, z', u'$. Hence the $\exists$ - arguments are $a, b, c, w, v, w', v'$. If $\theta y$ is true, put $w = v = w' = v' = y$, and the matrix is satisfied. If $\theta y'$ is true, put $w = v = w' = v' = y'$ and the matrix is satisifed. Assume $\sim\theta y . \sim\theta y'$. The matrix is true if $\psi x \supset \theta y$ and $\psi x' \supset \theta y'$ are false, i.e. since $\theta y$, $\theta y'$ are false, if $\psi x$ and $\psi x'$ are true. If $\psi x$ is false, put $b = c = x$ and $a = y$; then $\phi a . \supset . \psi b . \psi c$ is false and the matrix is true. If $\psi x'$ is false, similarly. Hence Prop.

$\cdot 365 \vdash :. (\exists x).\phi x . \supset . (x).\psi x :\supset:$

$$\{(\exists x).\chi x\}|\{(x).\psi x\} . \supset . \{(\exists x).\phi x\}|\{(\exists x).\chi x\}$$

Prop is what results from $*8\cdot 364$ by interchanging "all" and "some" in the $\chi$ - arguments, viz. $y, w, v, y', w', v'$. Hence the $\exists$ - arguments are $a, b, c, y, y'$. Matrix is true if $\cancel{\theta y . \theta w . \theta y' . \theta w'}$ $\theta w . \theta v . \theta w' . \theta v'$. Assume $\sim\theta w$ and put $y = y' = w$. Matrix is true if $\psi x \supset \theta y$ and $\psi x' \supset \theta y'$ are false, i.e. in the present case, if $\psi x$ and $\psi x'$ are true. Suppose one of them false, and put $b = x . c = x'$. Then $\psi b . \psi c$ is false. Therefore $\phi a . \supset . \psi b . \psi c$ is false if $\phi a$ is true; therefore the matrix is true if $\phi a$ is true. Therefore if $\phi z$ is true, the matrix is true for $a = z$. Similarly if $\phi u$, $\phi z'$ or $\phi u'$ is true. But if all are false, matrix is also true. Hence matrix is true when we have $\sim\theta w$ and $\sim\psi x \vee \sim\psi x'$. Similarly for $\sim\theta v$, $\sim\theta w'$ or $\sim\theta v'$ with $\sim\psi x \vee \sim\psi x'$. We saw that matrix ~~is true~~ ⌊can be satisfied⌋ (648) for $\sim\theta w$, $\sim\theta v$, $\sim\theta w'$, for $\sim\theta v'$ with $\psi x . \psi x'$. Hence it ~~is true~~⌊can be satisfied⌋

[23] Previously *39*.

for $\sim\theta w \vee \sim\theta v \vee \sim\theta w \vee \sim\theta v'$ And we saw that it is true for $\theta w \,.\, \theta v \,.\, \theta w' \,.\, \theta v'$. This completes the cases. Hence Prop.

*27*[24]

$*8\cdot366 \vdash :. (\exists x).\phi x \,.\supset.\, (\exists x).\psi x :\supset:$
$\{(x).\chi x\}|\{(\exists x).\psi x\} \,.\supset.\, \{(\exists x).\phi x\}|\{(x).\chi x\}$

Prop is what results from $*8\cdot364$ by interchanging "all" and "some" among $\psi$ -arguments, viz. $b, c, x, x'$. Hence $\exists$-arguments are $a, x, x', w, v, w', v'$. The proof proceeds as in $*8\cdot364$, interchanging $b, c$ and $x, x'$.

$*8\cdot367 \vdash :.(\exists x).\phi x \,.\supset.(\exists x).\psi x :\supset:$
$\{(\exists x).\chi x\}|\{(\exists x).\psi x\} \,.\supset.\{(\exists x).\phi x\}|\{(\exists x).\chi x\}$

Prop is what results from $*8\cdot365$ by interchanging "all" and "some" among $\psi$-arguments, viz. $b, c, x, x'$. Hence the $\exists$-arguments are $a, x, x', y, y'$. The proof proceeds as in $*8\cdot365$, interchanging $b, c$ and $x, x'$.

This completes the 26 cases of $p \supset q \,.\supset.\, s|q \supset p|s$. Hence in all the propositions of $*1 - *5$ we can substitute propositions containing one variable. The proofs for propositions containing 2 or 3 or 4 or ... variables are step-by-step the same. Hence the propositions of $*1 - *5$ hold of all first-order propositions.

The extension to second-order propositions, and thence to third-order propositions, and so on, is made by exactly analogous steps. Hence all stroke-functions which can be demonstrated for elementary propositions can be demonstrated by propositions of any order.

*28*[25]

It remains to prove $\sim \{(x).\ \phi x\} \,.\equiv.\, (\exists x).\ \sim\phi x$ and similar propositions.

$*8\cdot4 \ \vdash : \sim \{(x).\ \phi x\} \,.\equiv.\, (\exists x).\ \sim\phi x$

*Dem.*

$\vdash .\quad *8\cdot1. \supset \vdash : \phi x|\phi x \,.\supset.\, (\exists y).\ \phi x|\phi y$ (1)
$\vdash .(1). *8\cdot21. \supset \vdash : (\exists x).\ \phi x|\phi x \,.\supset.\, (\exists x, y).\ \phi x|\phi y$
$[(*8\cdot01\cdot012)] \supset \vdash : (\exists x).\ \sim \phi x \,.\supset.\, \sim \{(x).\ \phi x\}$ (2)
We have $\vdash : p|q \,.\equiv.\, p|p \vee q|q$ (3)
$\vdash .(3).\ \supset\ \vdash\ :\ \phi x|\phi y \,.\equiv.\, \phi x|\phi x \vee \phi y|\phi y$ (4)
$\vdash .(4). *8\cdot22\cdot24 \,.\supset \vdash : \phi x|\phi y \,.\supset.\, (\exists x).\ \phi x|\phi x$ (5)
$[(*8\cdot011)]\ \vdash :. (\exists x, y).\ f(x, y) \,.\supset.\, p :\equiv: (x, y).\ f(x, y) \supset p$ (6)
$\vdash .(5).(6). \supset \vdash : (\exists x, y).\ \phi x|\phi y \,.\supset.\, (\exists x).\ \phi x|\phi x$
$[(*8\cdot01\cdot012)] \supset \vdash : \sim\{(x).\ \phi x\} \,.\supset.\, (\exists x).\ \sim\phi x$ (7)
$\vdash .(2).(7). \supset \vdash$ . Prop

[24] Previously *40.* [25] Previously *41.*

$\cdot 41 \ \vdash : \sim \{(\exists x).\ \phi x\} \ . \equiv . \ (x).\ \sim\phi x$

[ Similar proof ]

*29*[26]

$*8\cdot 42 \ \vdash :.\ p\ .\supset .\ (\exists x).\ \phi x : \equiv : (\exists x).\ p \supset \phi x$

*Dem.*

$\vdash :.\ p\ .\supset .\ (\exists x).\ \phi x : \equiv : p|\{\sim(\exists x).\ \phi x\} :$

$[*8\cdot 41] \qquad \equiv : p|\{(x).\ \sim\phi x\} :$

$[(*8\cdot 011)] \qquad \equiv : (\exists x).\ p|\sim\phi x$

$[*8\cdot 21] \qquad \equiv : (\exists x).\ p \supset \phi x \ :.\ \supset \vdash .\ \text{Prop}$

(649)

$\cdot 43 \ \vdash :.\ p\ .\supset .\ (x).\ \phi x : \ \equiv : (x).\ p \supset \phi x$

[ Similar proof ]

Other propositions of this type may be taken for granted.

$\cdot 44 \ \vdash :.\ (x).\ \phi x\ .\supset : (x).\ \psi x\ .\supset .\ (x).\ \phi x\ .\ \psi x$

*Dem.*

$\vdash :.\ \phi z\ .\supset : \psi z\ .\supset .\ \phi z.\ \psi z \quad (1)$

$\vdash .(1).*8\cdot 1.\supset \vdash ::.(\exists x) ::.(\exists y) :: (z) :.\ \phi x\ .\supset : \psi y\ .\supset .\ \phi z.\ \psi z \quad (2)$

$\vdash .(2).*8\cdot 42\cdot 43\ .\supset \vdash .\ \text{Prop.}$

$\cdot 5$ If $F(p, q, r, \ldots)$ is a stroke-function of elementary propositions, and $p, q, r, \ldots$ are replaced by first-order propositions $p_1, q_1, r_1, \ldots,$ we shall have

$p \equiv p_1\ .\ q \equiv q_1\ .\ r \equiv r_1 \ldots . \supset : F(p, q, r, \ldots) : \equiv .\ F(p_1, q_1, r_1, \ldots)$

This follows from

$$p_1. = .(x).\ \phi x : \supset : p \equiv p_1\ .\supset .\ p_1|q \equiv p|q\ .\ q|p_1 \equiv q|p$$
$$p_1. = .(\exists x).\ \phi x : \supset : p \equiv p_1\ .\supset .\ p_1|q \equiv p|q\ .\ q|p_1 \equiv q|p$$

both of which are very easily proved.

[26] Previously *42*.

# Appendix B

### Editorial note

This material is RA 230.03170. The theorems in this section were all numbered $*98$, then that was struck out, then $*89$ was inserted. These changes were systematic, and are not indicated in my transcription of the manuscript. Some pages have numbers starting with $*120$, as indicated. $*98$ would be at the very end of volume I. "Part II, Section E, Inductive relations" begins with "$*90$ On the ancestral relation", and ends with "$*97$ Analysis of the field of a relation into families". "$*120$ Inductive cardinals" is in "Part III Cardinal arithmetic", which runs from $*100$ to $*126$. Russell's parenthetical remarks, such as "[Proof as in $*90 \cdot 311$]" are written with square brackets in the manuscript and that is carried over into the printed version. Page 9 contains the error that Gödel identified in line (3). This page was renumbered twice, starting as 71, then changed to 8, before becoming page 9 in the final manuscript. Appendix B starts at page 650 of Volume I.

### Appendix B: manuscript

*1*

(650)

*APPENDIX B*

$*89$. *MATHEMATICAL INDUCTION.*

The difficulties which arise in connection with mathematical induction when the axiom of reducibility is rejected have been explained in the Introduction to the present edition. Retaining the definition of $R_*$ ($*90 \cdot 01$), we have

$$\vdash :. \, x R_* y \, . \equiv : x \in C`R : \breve{R}``\mu \subset \mu \, . \lfloor x \in \mu \, . \rfloor \supset_\mu . \, y \in \mu$$

The "$\mu$" which occurs here as apparent variable must be of some definite order. If $\kappa$ is a class of classes, and the members of $\kappa$ are of the order contemplated in the definition of $R_*$, we cannot infer

$$xR_*y\ .\supset:\breve{R}“p‘\kappa\subset p‘\kappa\ .\ x\in p‘\kappa\ .\supset.\ y\in p‘\kappa$$

nor yet $\qquad xR_*y\ .\supset:\breve{R}“s‘\kappa\subset s‘\kappa\ .\ x\in s‘\kappa\ .\supset.\ y\in s‘\kappa$

It is necessary, *prima facie*, to have

$$\alpha\in\kappa\ .\supset_\alpha.\ \breve{R}“\alpha\subset\alpha$$

in order to be able to argue from $x\in p‘\kappa$ to $y\in p‘\kappa$ or from $x\in s‘\kappa$ to $y\in s‘\kappa$. In the following pages, we shall show how to avoid the resulting complications.

*2*

Let us denote by "$\mu_m$" a variable class of the m[th] order, and put

$$*89\cdot 01\quad xR_{*m}y.=:x\in C‘R:\breve{R}“\mu_m\subset\mu_m\ .\ x\in\mu_m\ .\supset_{\mu_m}.\ y\in\mu_m\quad \text{Df}$$[1]

Since every class of a lower order is equal to some class of any given higher order, $R_{*m}\mathrel{\subset\!\!\!\cdot}R_{*n}$ if $m>n$. We shall show that

$$m>5\ .\supset.\ R_{*m}=R_{*5}$$

Hence we take $R_{*5}$ as $R_*$, and the complications disappear.

In $*90$, substituting $R_{*m}$ for $R_*$ and $\mu_m$ for $\mu$ and $\phi_m z$ for $\phi z$, the first proposition involving an invalid induction is $*90\cdot 17$, where we use the fact that $\overleftarrow{R}_*‘x$ is hereditary.[2] It is obvious that $\overleftarrow{R}_{*m}‘x$ is a class of order $m+1$, and therefore, although

$$\breve{R}“\overleftarrow{R}_{*m}‘x\ \subset\ \overleftarrow{R}_{*m}‘x,\ \cancel{x\in\overleftarrow{R}_{*m}‘x,}$$

we cannot infer

$$y\in\overleftarrow{R}_{*m}‘x\ .\ yR_{*m}z\ .\supset.\ z\in\overleftarrow{R}_{*m}‘x$$

In this case, however, as in many others, there is no difficulty in substituting a valid induction. Put

$$\kappa=\hat{\mu}_m\{\breve{R}“\mu_m\subset\mu_m\ .\ x\in\mu_m\}$$

then $\overleftarrow{R}_{*m}‘x=p‘\kappa$. Now we have not merely $\breve{R}“p‘\kappa\ \subset\ p‘\kappa$ but also

$$\mu_m\in\kappa\ .\supset.\ \breve{R}“\mu_m\subset\mu_m.$$

[1] This replaces $*90\cdot 131$ which is without the index '$m$'.

[2] $*90\cdot 17.\ \vdash.\ R_*^2=R_*$.

Hence the induction is valid.

(651) The proofs of $R_*^2 \mathrel{\subset\!\!\!\cdot} R_*$ and analogous propositions are easily re-written so as to be valid.

*3*

The next difficulty – and this one is more serious – arises in connection with $*90{\cdot}31$.[3] The present proof uses the fact that

$$x(I \restriction C`R \cup R_*|R)z$$

is a hereditary property of $z$. But it is a property of a higher order than those by which $R_*$ is defined; i.e. if $R_*$ is $R_{*m}$, then $x(I \restriction C`R \cup R_{*m}|R)z$ is of order $m+1$. Let us prove first

$$R_0 \cup R_*|R \mathrel{\subset\!\!\!\cdot} R_*$$

where ~~$R_0 = I \restriction C`R \quad \mathrm{Df}$~~

$*89{\cdot}02.\quad R_0 = I \restriction C`R \quad \mathrm{Df}$

The proof is as follows:

$*89{\cdot}1.\quad \vdash . R_0 \cup R_*|R \mathrel{\subset\!\!\!\cdot} R_*$

*Dem.*

$\vdash :: x \in \mu . \breve{R}``\mu \subset \mu . \supset :. x = z . \vee . u \in \mu . uRz : \supset . z \in \mu \qquad (1)$

$\vdash .(1).\mathrm{Comm.}$

$\supset \vdash :. x = z . \vee . u \in \mu . uRz : \supset : x \in \mu . \breve{R}``\mu \subset \mu . \supset . z \in \mu :. \quad \cancel{(2)}$[4]

$\supset \vdash :: x = z :. \vee :. x \in \mu . \breve{R}``\mu \subset \mu . \supset . u \in \mu : uRz :$

$\supset : x \in \mu . \breve{R}``\mu \subset \mu . \supset . z \in \mu ::$

$\supset \vdash :: x = z :. \vee :. x \in \mu . \breve{R}``\mu \subset \mu . \supset_\mu . u \in \mu : uRz :.$

$\supset : x \in \mu . \breve{R}``\mu \subset \mu . \supset_\mu . z \in \mu ::$

$\supset \vdash :: xR_0z . \vee . xR_*u . uRz : \supset . xR_*z :. \supset \vdash . \mathrm{Prop}$

$\cdot 101 \quad \vdash . R_0 \cup R|R_* \mathrel{\subset\!\!\!\cdot} R_* \quad$ [Proof as in $*90{\cdot}311$]

*4*

$*89{\cdot}102 \quad \vdash : R \in \mathrm{Cls} \to 1 . \supset . R* = R_0 \cup R|R_*$

*Dem.*

$\vdash :. \mathrm{Hp} . \breve{R}`x \in \beta . \breve{R}``\beta \subset \beta . \supset : x \in \iota`x \cup \beta . \breve{R}``(\iota`x \cup \beta) \subset \beta :$

$\supset : xR_*y . \supset . y \in \iota`x \cup \beta \qquad (1)$

$\vdash .(1).\mathrm{Comm.} \supset \vdash :. \mathrm{Hp}. y \neq x . xR_*y \supset . \breve{R}`x \in \beta . \breve{R}``\beta \subset \beta . \supset . y \in \beta \ (2)$

[3] $*90{\cdot}31. \vdash . R_* = I \restriction C`R \cup R_*|R.$ [4] $uRz \ :\supset :$ in the next line is corrected to $uRz \ :. \supset :.$

$\vdash .(2). \supset \vdash :\ \text{Hp}.\ xR_*y\ .\ x \neq y.\ \supset .\ x(R|R_*y)$ (3)[5]
$\vdash .(3). *89{\cdot}101\ . \supset \vdash\ .\ \text{Prop}$
$\cdot 103\ \ \vdash :\ R \epsilon 1 \to \text{Cls}\ . \supset .\ R* = R_0 \cup R_*|R \quad [*89{\cdot}102\tfrac{\breve{R}}{R}]$
$\cdot 104\ \vdash :.\ \cancel{R \epsilon \text{Cls} \to 1 \supset}$
$\kappa = \hat{\alpha}(x \epsilon \alpha\ .\ \breve{R}``\alpha \subset \alpha)\ . \supset : x(R|R_*)z\ . \supset .\ z \epsilon p`\breve{R}```\kappa$

*Dem.*

$\vdash : \text{Hp}\ .\ xRy\ . \supset .\ y \epsilon p`\breve{R}```\kappa$ (1)
$\vdash : \text{Hp}\ .\ \alpha \epsilon \kappa.\ y \epsilon \breve{R}``\alpha\ .\ yR_*z\ . \supset .\ z \epsilon \breve{R}``\alpha$ (2)
$\vdash .(2).\text{Comm}. \supset \vdash :\ \text{Hp}.\ y \epsilon p`\breve{R}```\kappa\ .\ yR_*z.\ \supset .\ z \epsilon p`\breve{R}```\kappa$ (3)
$\vdash .\ (1).(3). \supset \vdash .\ \text{Prop}$

(652)[6]

$\cdot 105\ \vdash :.\ \text{Hp} *89{\cdot}104\ .\ R \epsilon \text{Cls} \to 1\ . \supset : x(R|R_*)z\ . \equiv .\ z \epsilon p`\breve{R}```\kappa$

*Dem.*

$\vdash :\ \text{Hp}.\ \breve{R}`x \epsilon \mu.\ \breve{R}``\mu \subset \mu\ .\ \beta = \iota`\breve{R}`x \cup \breve{R}``\mu\ .$
$\supset :\ y \epsilon R``\beta \cup -\text{D}`R\ .\ E!\breve{R}`y\ . \supset .\ \breve{R}`y \epsilon R``\beta \cup -\text{D}`R :$
$\supset :\ \breve{R}``(R``\beta \cup -\text{D}`R) \subset R``\beta \cup -\text{D}`R$ (1)
$\vdash :\ \text{Hp}(1). \supset .\ x \epsilon R``\beta$ (2)
$\vdash .(1).(2). \supset \vdash :.\ \text{Hp}(1).\ \ z \epsilon p`\breve{R}```\kappa\ . \supset .\ z \epsilon \breve{R}``(R``\beta \cup -\text{D}`R)$
$\supset .\ z \epsilon \beta$ (3)
$\vdash :\ \text{Hp}(1)\ . \supset .\ \beta \subset \mu$ (4)
$\vdash .(3).(4).\ \supset \vdash :.\ \text{Hp}\ . \supset : \breve{R}`x \epsilon \mu\ .\ \breve{R}``\mu \subset \mu\ . \supset .\ z \epsilon \mu :$
$. \supset :\ x(R|R_*)z$ (5)[7]
$\vdash .(5). *89{\cdot}104\ .\ \supset \vdash .\ \text{Prop}$

*5*

$*89{\cdot}106\ \ \vdash :\ R \epsilon \text{Cls} \to 1\ . \supset .\ R_*|R \ \subseteq\ R|R_*$[8]

*Dem*

$\vdash :\ x(R_*|R)z\ . \equiv .\ z \epsilon \breve{R}``p`\kappa$ (1)
$\vdash\ .\ (1).\ *89{\cdot}105\ .*40{\cdot}37.\ \supset \vdash .\ \text{Prop}$

It is now necessary to take up the subject of intervals (cf.$*121$). Our further progress depends upon the fact that in suitable circumstances the $R$-interval between $x$ and $y$, i.e. $\overleftarrow{R}_*`x \cap \overrightarrow{R}_*`y$, is an inductive class.

$\cdot 11\ \vdash :\ R \epsilon \text{Cls} \to 1\ .\ xRz\ .\ zR_*y\ . \supset .\ R(x \vdash\!\dashv y) = \iota`x \cup R(z \vdash\!\dashv y)$[9]

5 '$x(R|R_*y)$' corrected to '$x(R|R_*)y$' .
6 A tick here indicates the end of the printed page as the manuscript was checked against the proofs.
7 'Hp . $\supset : \breve{R}`x \epsilon \mu$' corrected to 'Hp . $z \epsilon p`\breve{R}```\kappa\ . \ \supset : \breve{R}`x \epsilon \mu$'.
8 $*89{\cdot}106$ is a weakened version of $*90\cdot 32.\vdash .\ R|R_* = R \cup R|R_* = R_*|R$, which, however, still holds.
9 This is $*121{\cdot}4$, with only a change of variables.

*Dem*

$\vdash . *89{\cdot}102 \supset \vdash :: \text{Hp} . \supset :. xR_*u . \equiv : x = u .\vee. zR_*u$ (1)

$\vdash : \text{Hp} . x = u . \supset . u \in R(x \mapsto y)$ (2)

$\vdash :. \text{Hp} . zR_*u . \supset : uR_*y . \supset . u \in R(x \mapsto y)$ (3)

$\vdash . (2).(3). \supset \vdash : \text{Hp} . \supset . \iota\text{‘}x \cup R(z \mapsto y) \subset R(x \mapsto y)$ (4)

$\vdash . (1). \quad \supset \vdash : \text{Hp} . \supset . R(x \mapsto y) \subset \iota\text{‘}x \cup R(z \mapsto y)$ (5)

$\vdash . (4).(5). \supset \vdash . \text{Prop}$

$\cdot 111 \vdash : \sim (zR_*y). \supset . R(z \mapsto y) = \Lambda$

$\cdot 112 \vdash : R \in \text{Cls} \to 1 . xRz . xR_*y . \sim(zR_*y) . \supset .$

$x = y . R(x \mapsto y) = R(x \mapsto x)$

[$*98{\cdot}102$][10]

$\cdot 113 \vdash : R \in \text{Cls} \to 1 . \sim(xR|R_*x) . \supset . R(x \mapsto x) = \iota\text{‘}x$[11]

*Dem*

$\vdash :. \text{Hp}. \supset : yR_*x . \supset . \sim(xR|R_*y)$

$\supset : xR_*y . yR_*x . \supset . xR_*y . \sim(xR|R_*y).$

[$*89{\cdot}102$] $\supset . x = y :. \supset \vdash . \text{Prop}$

$\cdot 114 \vdash : R \in \text{Cls} \to 1. \breve{R}\text{“}\alpha \subset \alpha . x \in \alpha - \breve{R}\text{“}\alpha. \supset . \sim(xR|R_*x)$[$*89{\cdot}105$]

$\cdot 115 \vdash :. R \in \text{Cls} \to 1 . \breve{R}\text{“}\alpha \subset \alpha . x \in \alpha - \breve{R}\text{“}\alpha . \supset . R(x \mapsto x) = \iota\text{‘}x$

[$*89{\cdot}113 \cdot 114$]

~~$\cdot 116 \vdash :. R \in \text{Cls} \to 1 . xR_*y . \sim(yR|R_*y) . \supset . \sim(xR|R_*y)$ [$*98 \cdot 113$]~~

*6*[12]

(653) We now take as the definition of an inductive class the property proved in $*121{\cdot}24$, i.e. we put

Cls induct $= \hat{\rho}\{ \eta \in \mu . \supset_{\eta,y} . \eta \cup \iota\text{‘}y \in \mu : \Lambda \in \mu : \supset_\mu . \rho \in \mu\}$ Df.

That is to say, if $M = \hat{\eta}\hat{\zeta}\{(\exists y) . \zeta = \eta \cup \iota\text{‘}y\}$,

we put Cls induct $= \overleftarrow{M_*}\text{‘}\Lambda$ Df

There will be different orders of inductive classes according to the order of $\mu$. $\mu$ must be at least of the second order, since $\iota\text{‘}y$ is of the second order; at least, not much could be proved if we took $\mu$ to be of the first order. We put

Cls induct$_m$ $= \overleftarrow{M_{*m}}\text{‘}\Lambda$ Df

We have $(\exists\mu_2). \Lambda = \mu_2 : (\exists\mu_2). \eta = \mu_2 . \supset . (\exists\mu_2). \eta \cup \iota\text{‘}y = \mu_2$

Now $(\exists\mu_2). \eta = \mu_2$ is a third-order property. Hence

$*89{\cdot}12 \quad \vdash : \rho \in \text{Cls induct}_3. \supset . (\exists\mu_2). \rho = \mu_2$

[10] Corrected to ‘[$*89 \cdot 102$]’ .

[11] ‘$R \in \text{Cls} \to 1 . \sim(xR|R_*x)$’ corrected to ‘$R \in \text{Cls} \to 1 . x \in C'R . \sim(xR|R_*x)$’ .

[12] ‘Dawson’ underlined in the upper left.

This proposition is fundamental.

*7*

$*89{\cdot}13 \vdash :. R \in \mathrm{Cls} \to 1 : \eta \in \mu . \supset_{\eta,y} . \eta \cup \iota`y \in \mu : \Lambda \in \mu :$
$\sim(xR|R_*x) . xRz : \supset : R(z \longmapsto y) \in \mu . \supset . R(x \longmapsto y) \in \mu$

[$*89{\cdot}11 \cdot 111 {\cdot} 113$][13]

Put

$\cdot 131.\ R_m(x \longmapsto y) = \overleftarrow{R}_{*m}`x \cap \overrightarrow{R}_{*m}`y$ Df

Then[14]

$\kappa = \hat{\alpha}_m(\breve{R}``\alpha_m \subset \alpha_m . x \in \alpha_m) . \lambda = \hat{\beta}(R``\beta_m \subset \beta_m . y \in \beta_m) . \supset .$
$R_m(x \longmapsto y) = p`\kappa \cup p`\lambda.$

Thus $R_m(x \longmapsto y)$ is a class of order $m+1$. Moreover we have

$\cdot 132 \quad \vdash :. R \in \mathrm{Cls} \to 1 . xRy . \supset : \sim(yR|R_*y) . \supset . \sim(xR|R_*x)$

*Dem*

$\vdash : \sim(yR|R_*y) . xRy . \supset . (\exists\alpha). \breve{R}``\alpha \subset \alpha . y \in \alpha - \breve{R}``\alpha . xRy$ (1)

$\vdash :$ Hp $. \breve{R}``\alpha \subset \alpha . y \in \alpha - \breve{R}``\alpha . xRy . \gamma = \iota`x \cup \iota`y \cup \breve{R}``\alpha .$
$\supset . \breve{R}``\gamma = \iota`y \cup \overleftarrow{R}`y \cup \breve{R}``\breve{R}``\alpha.$ (2)
$\supset . \breve{R}``\gamma \subset \gamma$ (3)

$\vdash :. \sim(yR|R_*y) . \supset : \sim(yRy) :$
$\supset : xRy . \supset . x \neq y$ (4)

$\vdash : y \in \alpha - \breve{R}``\alpha . xRy. \supset . x \sim\in \alpha$ (5)

$\vdash :. \sim(yR|R_*y) . \supset : \sim(yR^2y) :$
$\supset : xRy . \supset . x \sim\in \overleftarrow{R}`y$ (6)

$\vdash .(2).(4).(5).(6). \supset \vdash :$ Hp(2) $. \supset . x \in y - \breve{R}``y$ (7)[15]

$\vdash .(3).(7). \supset \vdash :$ Hp(2) $. \supset . \sim(xR|R_*x)$ (8)

$\vdash .(1).(8). \supset \vdash .$ Prop

*7a*

(654)

$*89{\cdot}133 \vdash :. R \in \mathrm{Cls} \to 1 : \eta \in \mu . \supset_{\eta,y} . \eta \cup \iota`y \in \mu : \Lambda \in \mu : xRz : \supset :$
$\sim(zR|R_*x) . R(z \longmapsto y) \in \mu . \supset . \sim(xR|R_*x) . R(x \longmapsto y) \in \mu$

[$*89{\cdot}13 \cdot 132$]

$\cdot 14 \vdash :. R \in \mathrm{Cls} \to 1 . \sim(yR|R_{*m}y) . \supset :$
$xR_{*(m+1)}y . \supset . R_m(x \longmapsto y) \in \mathrm{Cls\ induct}_{m+1}$

[13] This becomes '[$*89{\cdot}11 \cdot 111 {\cdot} 112 {\cdot} 113$]' .
[14] '$\lambda = \hat{\beta}(R``\beta_m$' in the next line becomes '$\lambda = \hat{\beta_m}(R``\beta_m$' .
[15] 'Hp(2) $. \supset . x \in y$' becomes 'Hp(2) $. \sim(yR|R_*y) \supset . x \in y$' .

*Dem*

By $*89{\cdot}133$, $\sim(zR|R_{*m}z) \,.\, R_m(z \longmapsto y) \,\epsilon\, \mu_{m+1}$ is a hereditary property of $z$ ~~moreover it is of order $m+1$~~ if

$$\eta \,\epsilon\, \mu_{m+1} \,.\, \supset_{\eta,y} \,.\, \eta \cup \iota`y \,\epsilon\, \mu_{m+1} : \Lambda \,\epsilon\, \mu_{m+1}.$$

Moreover this property is of order $m+1$. And by $*89{\cdot}113$, $y$ has this property if $\sim(yR|R_{*m}y)$. Hence $x$ has this property if $xR_{*(m+1)}y$. Hence with this hypothesis we have

$$\eta \,\epsilon\, \mu_{m+1} \,.\, \supset_{\eta,y} \,.\, \eta \cup \iota`y \,\epsilon\, \mu_{m+1} : \Lambda \,\epsilon\, \mu_{m+1} \;:\; \supset_{\mu_{m+1}} \;.\; R_m(x \longmapsto y) \,\epsilon\, \mu_{m+1},$$

i.e. $\qquad R_m(x \longmapsto y) \,\epsilon$ Cls induct$_{m+1}$

which was to be proved.

*8*

$*89{\cdot}15 \vdash :\; R \,\epsilon\, \mathrm{Cls} \to 1 \,.\, \breve{R}``\alpha_m \subset \alpha_m \,.\, y \,\epsilon\, \alpha_m - \breve{R}``\alpha_m \,.\, \supset :$

$\qquad xR_{*(m+1)}y \,.\, \supset \,.\, R_m(x \longmapsto y) \,\epsilon$ Cls induct$_{m+1}$ $\quad[*98 \cdot 114 \cdot 14]$[16]

We have $\qquad R_{m+1}(x \longmapsto y) \subset R(x \longmapsto y)$

$\qquad$ Cls induct$_{m+1} \subset$ Cls induct$_m$.

The next point is to prove

$$\rho \,\epsilon\, \text{Cls induct}_m \,.\, \gamma \subset \rho \,.\, \supset \,.\, \gamma \,\epsilon\, \text{Cls induct}_m.$$

This can be proved for Cls induct$_3$, and extended to any other order of inductive classes. The proof is as follows.

*9*[17]

$*89{\cdot}16 \;\vdash :\; \alpha \sim\epsilon$ Cls induct$_3$ $.\; \gamma \,\epsilon$ Cls induct$_3$ $.\, \supset .\, \exists!\, \alpha - \gamma$

*Dem*

$\mathrm{Hp} \,.\, \supset : (\exists\mu_3) : \Lambda \,\epsilon\, \mu_3 : \beta \,\epsilon\, \mu_3 .\, \supset_{\beta,y} .\, \beta \cup \iota`y \,\epsilon\, \mu_3 \;: \gamma \,\epsilon\, \mu_3 \,.\, \alpha \sim\epsilon\, \mu_3 \qquad (1)$

$\Lambda \,\epsilon\, \mu_3 : \beta \,\epsilon\, \mu_3 .\, \supset_{\beta,y} .\, \beta \cup \iota`y \,\epsilon\, \mu_3 : \gamma \,\epsilon\, \mu_3 .\, \alpha \sim\epsilon\, \mu_3 : \supset : \alpha \neq \Lambda \,.\, \Lambda \,\epsilon\, \mu_3$

$\qquad \supset : \exists\,!\, \alpha - \Lambda \,.\, \Lambda \,\epsilon\, \mu_3 \qquad (2)$

$\exists\,!\, \alpha - \beta \,.\, \alpha \subset \beta \cup \iota`y \,.\; \supset .\, \alpha = \beta \cup \iota`y \qquad (3)$

$(3).\, \supset :.\lfloor \mathrm{Hp}(2) \,.\, \supset :\rfloor \beta \,\epsilon\, \mu_3 \,.\, \alpha \sim\epsilon\, \mu_3 \,.\, \exists\,!\, \alpha - \beta \,.\, \supset .$

$\qquad \beta \cup \iota`y \,\epsilon\, \mu_3 \,.\, \alpha \neq \beta \cup \iota`y \,.\, \exists\,!\, \alpha - (\beta \cup \iota`y) \qquad (4)$

$(4).\, \supset :.\, \mathrm{Hp}(2) \,.\, \supset : \beta \,\epsilon\, \mu_3 \,.\, \exists\,!\, \alpha - \beta \,.\, \supset .\, \beta \cup \iota`y \,\epsilon\, \mu_3 \,.\, \exists\,!\alpha - (\beta \cup \iota`y) \qquad (5)$

$(2).(5).\, \supset \vdash :.\, \mathrm{Hp}(2) \,.\, \supset : \beta \,\epsilon$ Cls induct$_3$ $.\, \supset .\, \beta \,\epsilon\, \mu_3 \,.\, \exists\,!\, \alpha - \beta \qquad (6)$

$(1).(6).\, \supset$ Prop[18]

[16] Corrected to '$*89$'.

[17] 'Symonds' underlined in the upper left. The annotation of the page and first theorem number tell that this page began as *71* of the HPF manuscript, with theorem $*120{\cdot}823$, then was renumbered as *8*, then finally *9*, while the theorem went through renumbering as $*98{\cdot}16$, then finally $*89{\cdot}16$. Line (3) is the error found by Gödel.

[18] This is changed to '(1).(6). $\supset \vdash$ . Prop'.

$\cdot 17 \quad \vdash : \ \gamma \in \text{Cls induct}_3 \ . \ \alpha \subset \gamma . \supset . \ \alpha \in \text{Cls induct}_3 \qquad [*89 \cdot 16.\text{Transp}]$[19]

$\cancel{\cdot 825 \quad \vdash : \ \exists \, ! \, \text{NC induct}_3 - \text{NC induct}_4 \ . \supset . \ \text{NC induct}_4 \in \text{Cls induct}_3}$
$\cancel{[*120 \cdot 822 \cdot 824]}$
$\cancel{\cdot 826 \quad \vdash : \ \exists \, ! \, \text{NC induct}_3 - \text{NC induct}_4 \ . \supset . \ (\exists \mu_2). \ \text{NC induct}_4 = \mu_2}$
$\cancel{[*120 \cdot 81 \cdot 825]}$
$\cancel{\cdot 827 \quad \vdash : \ \exists \, ! \, \text{NC induct}_3 - \text{NC induct}_4 \ . \supset . \ (\exists \mu_2). - \text{NC induct}_4 = \mu_2}$
$\cancel{[*120 \cdot 826]}$
$\cancel{\cdot 828 \ \vdash : \ \mu_2 \subset \text{NC induct}_3 \ . \ \exists \, ! \, \mu_2 \cap \text{NC induct}_3 \ . \supset .}$
$\cancel{\exists \, ! \, \mu_2 \lfloor \cap \text{NC induct}_3 \rfloor - (+_c 1)``\mu_2}$
$\cancel{\textit{Dem}. \ \mu_2 \lfloor \cap \text{NC induct}_3 \rfloor \subset (+_c 1)``\mu_2 \ . \supset . \ (+_c 1)`` - \mu_2 \ \subset - \mu_2 \ . \ 0 \sim\in \mu_2 .}$
$\cancel{\supset . \ \text{NC induct}_3 \subset - \mu_2 \ \ (1)}$
$\cancel{(1).\text{Transp} \ . \supset . \ \text{Prop}}$

*10*

It follows that, with the hypothesis of $*89 \cdot 15$, $R_m(x \longmapsto y)$, $R_{m+1}(x \longmapsto y)$, etc. are all of the inductive classes of the $(m+1)^{\text{th}}$ or any lower order.

(655)
$*89 \cdot 18 \ \vdash :. \ R \in \text{Cls} \to 1 \ . \ y, z \in \overleftarrow{R}_{*3}`x . \ \sim(yR|R_{*2}y) \ . \supset : yR_{*3}z \ .\vee. \ zR_{*3}y$
*Dem.*
Put $\xi = R_3(x \longmapsto y) \cap R_3(x \longmapsto z)$.
$\vdash . \ * \, 89 \cdot 14 \cdot 17 . \supset \vdash : \text{Hp}. \supset . \ \xi \in \text{Cls}_2$, i.e. $\xi$ is a class of second order. (1)
$\vdash :. \ \text{Hp}. \sim(yR_{*3}z). \sim(zR_{*3}y). \supset : u \in \xi . \supset . \ uR_{*3}y . \ uR_{*3}z . \ u \neq y . \ u \neq z .$
$[\ 89 \cdot 102] \qquad \supset . \ \breve{R}`uR_{*3}y \ . \ \breve{R}`uR_{*3}z .$
$[\text{Hp}] \qquad \supset . \ \breve{R}`u \in \xi \qquad (2)$
$\vdash .(1).(2). \ \supset \vdash : \text{Hp}(2). \supset . \ y \in \xi \qquad (3)$
$\vdash : \text{Hp}(2). \supset . \ y \ \sim\in \xi \qquad (4)$
$\vdash .(3).(4). \ \supset \vdash :. \ \text{Hp}. \supset : \ yR_{*3}z \ .\vee. \ zR_{*3}y \ :. \ \supset \vdash . \ \text{Prop}$

$\cdot 19 \ \vdash : \ R \in \text{Cls} \to 1 \ . \ \breve{R}``\mu_2 \subset \mu_2 \ . \ \lambda = \ \overleftarrow{R}_{*3}`x \cap \mu_2 - \breve{R}``\mu_2 . \supset . \ \lambda \in 0 \cup 1$
*Dem.*
$\vdash : \text{Hp}. \ y, z \in \lambda . \ y \neq z \ . \ \xi = R_2(x \longmapsto y) \cap R_2(x \longmapsto z). \supset . \breve{R}``\xi \ \subset \ \xi \ . \ x \in \xi$
[as above] $\cancel{(1)}$
$[\ 89 \cdot 15 \cdot 17] \qquad \supset . \ y, z \in \xi \qquad (1)$[20]

[19] This underwent the same order of alterations as first $*120 \cdot 824$, then $*98 \cdot 17$, finally $*89 \cdot 17$.
[20] Corrected to '[ 89· 12· 15·17]' .

$\vdash :\ \mathrm{Hp}(1) . \supset . y, z \ \sim\in \xi$ (2)
$\vdash .(1).(2). \supset \vdash . \mathrm{Prop}$

*11*

$*89\cdot 2 \ \vdash : R \in \mathrm{Cls} \to 1 . xR_{*3}y . R_2(y \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3 .$
$\supset . R_2(x \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3$

*Dem*

As in $*89\cdot 11\cdot 111\cdot 112$,
$\vdash :. \cancel{\mathrm{Hp}} .\lfloor R \in \mathrm{Cls} \to 1.\rfloor xRz . \supset :$
$R(x \vdash\!\dashv y) = \iota\text{‘}x \cup R(z \vdash\!\dashv y) .\vee. R(x \vdash\!\dashv y) = \iota\text{‘}x .\vee. R(x \vdash\!\dashv y) = \Lambda$ (1)
$\vdash .(1). \supset \vdash :. \mathrm{Hp}(1) : \Lambda \in \mu : \alpha \in \mu . \supset_{\alpha,\mu} . \alpha \cup \iota\text{‘}u \in \mu : \supset :$
$R(z \vdash\!\dashv y) \in \mu . \supset . R(x \vdash\!\dashv y) \in \mu$ (2)
$\vdash .(2). \supset \vdash : R \in \mathrm{Cls} \to 1 .\lfloor xR_{*3}y.\rfloor R_2(y \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3 . \supset .$
$R_2(x \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3 : \supset \vdash . \mathrm{Prop}$

*12*[21]

To deal $\lfloor$further$\rfloor$ with the case in which $y(R|R_{*2})y$, proceed as follows:
Having proved
$R \in \mathrm{Cls} \to 1 . xR_{*3}y . R_2(y \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3 . \supset .$
$R_2(x \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3$,
we have ~~then~~ to prove $R_2(y \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3$; ~~put~~ for this purpose, put

$$S = (-\iota\text{‘}y) \upharpoonleft R$$

Then $S \in \mathrm{Cls} \to 1 . S \Subset R.$

Observe that $yRy . \supset . R(y \vdash\!\dashv y) = \iota\text{‘}y$

$$yR^2y . \supset . R(y \vdash\!\dashv y) = \iota\text{‘}y \cup \iota\text{‘}\breve{R}\text{‘}y$$

Assume, therefore, $\sim(yRy). \sim(yR^2y).$

We have $\breve{S}\text{“}\mu = \breve{R}\text{“}(\mu - \iota\text{‘}y) . S\text{“}\mu = R\text{“}\mu - \iota\text{‘}y$. Hence

$$\breve{S}\text{“}\mu \subset \mu . \breve{R}\text{‘}y \in \mu . \equiv . \breve{R}\text{“}\mu \subset \mu . \breve{R}\text{‘}y \in \mu$$
$$S\text{“}\mu \subset \mu . y \in \mu . \equiv . R\text{“}\mu \subset \mu . y \in \mu$$

(656)
Hence $\overleftarrow{S_*}\text{‘}\breve{R}\text{‘}y = \overleftarrow{R}_*\text{‘}\breve{R}\text{‘}y . \overrightarrow{S_*}\text{‘}y = \overrightarrow{R_*}\text{‘}y$
Hence $S_2(\breve{R}\text{‘}y \vdash\!\dashv y) = R_2(\breve{R}\text{‘}y \vdash\!\dashv y) = R_2(y \vdash\!\dashv y)$
because $y(R|R_{*2})y$.

Moreover we have $\sim (yS|S_*y)$ because $y \sim\in \mathrm{D}\text{‘}S$.

Hence by $*89\cdot 14$, $R_2(y \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3$. Hence generally:

$*89\cdot 201 \ \vdash : R \in \mathrm{Cls} \to 1 .\lfloor xR_{*3}y .\rfloor \supset . R_2(x \vdash\!\dashv y) \in \mathrm{Cls\ induct}_3$
~~because~~ $\cancel{\sim (xR_{*3}y) . \supset . R_3(x \vdash\!\dashv y) = \Lambda}$

[21] Previously *11*.

~~Hence~~

~~$\cdot 21 \vdash :. R \epsilon \text{Cls} \to 1 . y, z \epsilon \overleftarrow{R}_{*3}`x . \supset : yR_{*3}z .\vee. zR_{*3}y$~~

~~[Proof as in $*98 \cdot 18$, using $*98 \cdot 2$ instead of $*98 \cdot 14$]~~

~~Henceforth, when no numerical suffix is given, the suffix 3 is understood.~~

~~$\cdot 22 \vdash :. S, T \epsilon \text{Potid}`R . \supset : SR_{ts}T .\vee. TR_{ts}R$~~

~~$[*98 \cdot 21 \frac{\text{Cnv}`|R}{R}]$~~

*13*

We have $R_3(x \longmapsto y) \subset R_2(x \longmapsto y)$

Hence by $*89 \cdot 17$,

$$R_2(x \longmapsto y) \epsilon \text{ Cls induct}_3 . \supset . R_3(x \longmapsto y) \epsilon \text{ Cls induct}_3.$$ Hence

$*89 \cdot 21 \vdash : R \epsilon \text{ Cls} \to 1 . \supset . R_3(x \longmapsto y) \epsilon \text{ Cls induct}_3$

because $\sim (xR_{*3}y) . \supset . R_3(x \longmapsto y) = \Lambda$.

~~Henceforth, when no numerical suffix is given, the suffix 3 is to be understood.~~

$\cdot 22 \vdash :. R \epsilon \text{ Cls} \to 1 . y, z \epsilon \overleftarrow{R}_{*3}`x . \supset : yR_{*3}z . \vee . zR_{*3}y$

[Proof as in $*89 \cdot 18$, using $*89 \cdot 21$ instead of $*89 \cdot 14$]

$\cdot 221 \text{ Potid}_m`R = \overrightarrow{(R_{ts})_m}`R_0 \quad \text{Df}$

$\cdot 23 \vdash :. S, T \epsilon \text{ Potid}_3`R . \supset : SR_{ts3}T . \vee . TR_{ts3}S \quad [*89 \cdot 22 \frac{\text{Cnv}`|R}{R}]$

$\cdot 24 \vdash : R \epsilon \text{ Cls} \to 1 . \breve{R}``\lambda \subset \lambda . x \epsilon \lambda . \supset . \overleftarrow{R}_{*3}`x \subset \lambda$

Here $\lambda$ is assumed to be of more than the 3rd order.

*Dem.*

$\vdash :. \text{Hp} . y \epsilon \overleftarrow{R}_{*3}`x - \lambda . \supset : z \epsilon \lambda \cap R_3(x \longmapsto y) . \supset . z \neq y.$
$\supset . \breve{R}`z \epsilon \lambda \cap R_3(x \longmapsto y) \quad (1)$

$\vdash . \cancel{(1)} *89 \cdot 21 \cdot 17 \cdot 12 . \supset \vdash : \text{Hp} . \supset . (\exists \mu_2) . \lambda \cap R_3(x \longmapsto y) = \mu_2 \quad (2)$

$\vdash .(1).(2). \supset \vdash : \text{Hp}(1). \supset .(\exists \mu_2) . \lambda \cap R_3(x \longmapsto y) = \mu_2 . \breve{R}``\mu_2 \subset \mu_2 . x \epsilon \mu_2.$
$\supset . \overleftarrow{R}_{*3}`x \subset \lambda \cap R_3(x \longmapsto y) \quad (3)$

$\vdash .(3). \supset \vdash :. \text{Hp} . \supset : y \epsilon \overleftarrow{R}_{*3}`x - \lambda . \supset . y \epsilon \lambda :$
$\supset : \overleftarrow{R}_{*3}`x \subset \lambda :. \supset \vdash . \text{Prop}$

Hence if $\lambda$ is an inductive class, it can be used in an induction no matter what its order may be, if $R \epsilon \text{ Cls} \to 1$. ~~It remains only to extend this result to the case when $R \sim \epsilon \text{ Cls} \to 1$.~~

~~For this purpose, we need use $R_\epsilon$, which is a $1 \to \text{Cls}$. We have~~

*14*

$*89 \cdot 25 \vdash : R \epsilon 1 \to \text{Cls} . \supset . R_3(x \longmapsto y) \epsilon \text{ Cls induct}_3 \quad [*89 \cdot 21 \frac{\breve{R}}{R}]$

$\cdot 26 \vdash :. R \epsilon 1 \to \text{Cls} . y, z \epsilon \overrightarrow{R}_{*3}`x . \supset : yR_{*3}z .\vee. zR_{*3}y \quad [*89 \cdot 22 \frac{\breve{R}}{R}]$

$\cdot 27$[22] $\vdash : \ R \epsilon 1 \to \mathrm{Cls} \,.\, R``\lambda \subset \lambda \,.\, x \epsilon \lambda \,.\, \supset .\overrightarrow{R}_{*3}`x \subset \lambda$ $\quad [*89 \cdot 24 \frac{\breve{R}}{R}]$
(657)

$\cdot 28 \ \vdash : \ R \epsilon (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) \,.\supset .\, R_{*3} = \dot{s}`\mathrm{Potid}_3`R$

*Dem.*

$\vdash : \ T \epsilon \mathrm{Potid}_3`R \,.\, xTy \,.\, yRz \,.\supset .\, T|R \epsilon \mathrm{Potid}_3`R \,.\, x(T|R)z$

~~Hence~~ $\cancel{\vdash .\ \breve{R}``\dot{s}`\mathrm{Potid}`R}$

Hence $\vdash : \dot{s}`\mathrm{Potid}_3`R = S \,.\supset .\, \breve{R}``\overleftarrow{S}`x \subset \overleftarrow{S}`x$ (1)

$\vdash .(1). * 89 \cdot 24. \supset \vdash : R \epsilon \mathrm{Cls} \to 1 \,.\, \mathrm{Hp}(1). \supset .\overleftarrow{R}_{*3}`x \subset \overrightarrow{S}`x$ (2)

$\vdash :. \lfloor \mathrm{Hp}(1) \,.\rfloor \breve{R}``\mu \subset \mu \,.\supset : \overleftarrow{T}`x \subset \mu \,.\supset .\, \breve{R}``\overleftarrow{T}`x \subset \mu :$

$\cancel{\supset : \overleftarrow{S}`x \subset \mu}$

$\supset : \ x \epsilon \mu \,.\supset .\, \overleftarrow{S}`x \subset \mu$ (3)

$\vdash .\,(3).\ \supset \vdash : \ \mathrm{Hp}(1) \,.\supset .\, \overleftarrow{S}`x \subset \overleftarrow{R}_{*3}`x$ (4)

$\vdash .(2).(4).\ \supset \vdash : \ \mathrm{Hp}(2) \,.\supset .\, \overleftarrow{S}`x = \overleftarrow{R}_{*3}`x$ (5)

$\vdash .(5).\ \supset \vdash : \ R \epsilon \mathrm{Cls} \to 1 \,.\supset .\, R_{*3} = \dot{s}`\mathrm{Potid}`R$ (6)

Similarly $\vdash : \ R \epsilon 1 \to \mathrm{Cls} \,.\supset .\, R_{*3} = \dot{s}`\mathrm{Potid}`R$ (7)

$\vdash .(6).(7). \supset \vdash .$ Prop

$\cdot 29 \ \vdash : \ R \epsilon (1 \to \mathrm{Cls}) \cup (\mathrm{Cls} \to 1) \,.\supset .\, R_{*(3+m)} = R_{*3} \quad [*89 \cdot 24 \cdot 27]$

We have now to obtain an analogous result when $R$ is not one-many or many-one. For this purpose, we use $R_\epsilon$, which is one-many.

*15*

We ~~now~~ prove $\overrightarrow{R_{*(m+2)}}`x = s`\overrightarrow{(R_\epsilon)_{*m}}`\iota`x$
whence, since $(R_\epsilon)_{*(3+m)} = (R_\epsilon)_{*3}$
it follows that $R_{*(5+m)} = R_{*5}$

So that for a relation which is not one-many or many-one we obtain the advantages of unlimited induction by proceeding to $R_{*5}$. The proof is as follows.

$*89 \cdot 3 \ \vdash : \ R_\epsilon = S \,.\supset .\, s`\overrightarrow{S_{*m}}`\iota`\kappa \subset \overrightarrow{R_{*m}}`x$

*Dem.*

$\vdash ::.\, \mathrm{Hp} \,.\supset :: \ \alpha S_* \iota`x \,.\equiv :.\, \iota`x \epsilon \mu \ : \ \xi \epsilon \mu \,.\supset_\xi .\, R``\xi \epsilon \mu : \supset_\mu .\, \alpha \epsilon \mu \ :.$

$\supset :.\, \iota`x \epsilon \mathrm{Cl}`\gamma : \xi \epsilon \mathrm{Cl}`\gamma \,.\supset_\xi .\, R``\xi \epsilon \mathrm{Cl}`\gamma : \supset_\gamma .\, \alpha \epsilon \mathrm{Cl}`\gamma :.$

$\supset :.\, x \epsilon \gamma.\, R``\gamma \subset \gamma \,.\supset_\gamma .\, \alpha \subset \gamma \ :.$

$\supset :.\, \alpha \subset \overrightarrow{R}_*`x \ ::.\ \supset \vdash$ . Prop

*16*

$*89 \cdot 31 \ \vdash : \ R_\epsilon = S \,.\supset .\, \overrightarrow{R_{*(m+2)}}`x \subset s`\overrightarrow{S_{*m}}`\iota`x$

[22] "R &W I. 42 f.657." is in the left margin and appears at the bottom of p.657.

*Dem.*

$$\vdash . *89\cdot101 . \supset \vdash . S|S_* \in S_*$$
$$\supset \vdash . S``\overrightarrow{S_*}`\iota`x \subset \overrightarrow{S_*}`\iota`x$$
$$\supset \vdash . s`S``\overrightarrow{S_*}`\iota`x \subset s`\overrightarrow{S_*}`\iota`x \quad (1)$$
$$\vdash .(1). *40\cdot38 . \supset \vdash : \text{Hp.} \supset . R``s`\overrightarrow{S_*}`\iota`x \subset s`\overrightarrow{S_*}`\iota`x \quad (2)$$
$$\vdash : \lambda = \hat{\mu}(\iota`x \in \mu . S``\mu \subset \mu) . \supset . s`\overrightarrow{S_*}`\iota`x = s`p`\lambda \quad (3)$$
$$\vdash .(2). \supset \vdash : s`\overrightarrow{S_*}`\iota`x \in \mathrm{Cls}_n . \supset . \overrightarrow{R_{*n}}`x \subset s`\overrightarrow{S}_*`\iota`x \quad (4)$$
$$\vdash .(3). \supset \vdash : s`\overrightarrow{S_{*m}}`\iota`x \in \mathrm{Cls}_{m+2} \quad (5)$$
$$\vdash .(4).(5). \supset \vdash . \text{Prop}$$

$$\cdot32 \ \vdash . \ \overrightarrow{R_{*5}}`x = s`(\overrightarrow{R_\epsilon})_{*3}`x$$

*Dem.*

$$\vdash . *89\cdot3\cdot29 . \supset \vdash . s`(\overrightarrow{R_\epsilon})_{*3}`x \subset \overrightarrow{R_{*5}}`x \quad (1)$$[23]
$$\vdash .(1). *89\cdot31 . \supset \vdash . \text{Prop}$$

$$\cdot33 \ \vdash . R_{*(5+m)} = R_{*5}$$

*Dem*

$$\text{As in } *89\cdot32,\ \vdash . \ \overrightarrow{R_{*(5+m)}}`x = s`(\overrightarrow{R_\epsilon})_{*(3+n)}`x$$
$$[*89\cdot29] \qquad = s`(\overrightarrow{R_\epsilon})_{*3}`x$$
$$[*89\cdot32] \qquad = \overrightarrow{R_{*5}}`x . \supset \vdash . \text{Prop}$$

$$\cdot34 \ \vdash : yR_{*5}x . x \in \lambda . R``\lambda \subset \lambda . \supset . y \in \lambda \ [*89\cdot33]$$ [24]

Here λ is supposed to be of any order, however high. Hence, so far as mathematical induction is concerned, all proofs remain valid without the axiom of reducibility provided “$R_*$” is understood to mean “$R_{*5}$”.

23 ‘$s`(\overrightarrow{R_\epsilon})_{*3}`x$’ corrected to ‘$s`(\overrightarrow{R'_\epsilon})_{*3}`x$’.

24 The first conjunct of the antecedent should be $xR_{*5}y$. This error is preserved in the printed version.

# Appendix C

### Editorial note

This manuscript is RA 230.031380. The text is in blue ink, with compositor's marks in pencil. All other manuscripts are in a (now faded) black ink. None of this material originated in the "Hierarchy" manuscript. There are no preliminary drafts or notes for this material in the Archives other than the papers "What is meant by '*A* believes *p*?' " and "Truth-functions and meaning-functions", published in *Papers 9*. Appendix C begins at page 659 of Volume 1.

### Appendix C: manuscript

(659)

*APPENDIX C*[1]

*TRUTH FUNCTIONS AND OTHERS*

In the Introduction to the present edition we have assumed that a function can only enter into a proposition through its values. We have in fact assumed that a matrix $f!(\phi!\hat{z})$ always arises through some stroke-function

$$F(p, q, r, \ldots)$$

by substituting $\phi!a$, $\phi!b$, $\phi!c$, ... for some or all of $p, q, r, \ldots$, and that all other functions of functions are derivable from such matrices by generalization – i.e. by replacing some or all of $a$, $b$, $c$, ... by variables, and taking "all values" or "some value".

The uses which we have made of this assumption can be validated by definition, even if the assumption is not universally true. That is to say, we can decide that mathematics is to confine itself to functions of functions which obey the above

[1] 'Dawson' is underlined, in the upper left.

assumption. This amounts to saying that mathematics is essentially extensional rather than intensional. We might, on this ground, abstain from the inquiry whether our assumption is universally true or not.[2] The inquiry, however, is important on its own account, and we shall, in what follows, suggest certain considerations without arriving at a dogmatic conclusion.

*2*[3]

There is a prior question which is simpler, and that is the question whether all functions of propositions are truth functions. Or, more precisely, can all propositions which do not contain apparent variables be built up from atomic propositions by means of the stroke? If this were the case, we should have, if $f\hat{p}$ is any function of propositions,

$$p \equiv q \,.\supset .\, fp \equiv fq$$

Consequently, according to the definition $*13{\cdot}01$,

$$p \equiv q \,.\supset .\, p = q$$

There will thus be only two propositions, one true and one false. This was Frege's point of view, but it is one which cannot easily be accepted. Frege maintained that every proposition is a proper name, either for the true or for the false. On grounds not connected with our present question, we cannot regard propositions as names; but that does not decide the question whether equivalent propositions are identical. It is this latter question that concerns us. That is to say, we have to consider whether, or in what sense, there are functions of $p$ which are true for some true values of $p$ and false for other true values of $p$.

*3*

Two obvious *prima facie* instances are "$A$ believes $p$" and "$p$ is about $A$". We may take these instances as crucial. If $A$ believes $p$, and $p$ is true, it does (660) not follow that $A$ believes every other true proposition $q$; nor, if $A$ believes $p$ and $p$ is false, does it follow that $A$ believes every other false proposition $q$. Again, the proposition " $A$ is mortal" is about $A$; but the proposition "$B$ is mortal", which is equally true, is not about ~~$B$~~ $A$. Thus the function "$p$ is about $A$" is not a truth-function of $p$. This instance is important, because the notation "$\phi x$" is used to denote a proposition about $x$, and thus the conception involved *seems* to be ~~involved~~⌊presupposed⌋in the whole procedure of propositional functions.

We must, to begin with, distinguish between a proposition as a fact and a proposition as a vehicle of truth or falsehood. The following series of black marks: "Socrates is mortal", is a fact of geography. The noise which I should make if I were

[2] A short vertical line here, the height of the line of the text, is accompanied by an 'X' in the right margin.
[3] 'Symonds' in the same hand, underlined, in the upper left.

to say "Socrates is mortal" would be a fact of acoustics. The mental occurrence when I entertain the belief "Socrates is mortal" is a fact of psychology. None of these introduces the notion of truth or falsehood, which is, for logic, the essential characteristic of propositions. We shall return in a moment to the consideration of propositions as facts.

*4*

When we say that truth or falsehood is, for logic, the essential characteristic of propositions, we must not be misunderstood. It does not matter, for mathematical logic, what consititutes truth or falsehood; all that matters is that they divide propositions into two classes according to certain rules. Let us take a set of marks

$$x_1, x_2, \ldots x_{2n-1}, x_{2n}.$$

Let us put, as unexplained assertions,

$$T(x_{2m+1}) \quad m < n$$
$$F(x_{2m}) \qquad m \leq n$$

Let us further introduce the symbol $x_r|x_s$, and assume

$$T(x_r|x_s) \text{ if } F(x_r) \text{ or } F(x_s);$$
$$F(x_r|x_s) \text{ if } T(x_r) \text{ and } T(x_s).$$

Assume further that, if $p, q, s$ are any one of the $x$s or any combination of them by means of the stroke, the above rules are to apply to $p|q$ etc., and further we are to have:

$$T\{p|(p|p)\}$$
$$\cancel{T[\{p|(q|q)\}]}$$
$$T\{p \supset q \;.\supset .\; \cancel{q|s} \; \lfloor s|q \rfloor \supset \cancel{s|p} \; \lfloor p|s \rfloor\}$$

where "$p \supset q$" means "$p|(q|q)$". Further: given $T\{p|(q|r)\}$ and $T(p)$, we are to have $T(r)$.

*5*

Taking the above as mere conventional rules, all the topic of molecular propositions follows, replacing "$\vdash . \; p$" by "$T(p)$".

Thus from the formal point of view it is irrelevant what constitutes truth or falsehood: all that matters is that propositions are divided into two classes according to certain rules. It does not matter what propositions are, so long as we are content to regard our primitive propositions as defining hypotheses, (661) not as truths. (From a philosophical point of view, this formal procedure may be shown to presuppose the non-formal interpretation of our primitive propositions; but that does not matter for our present purpose.)

Throughout the logic of molecular propositions, we do not want to know anything about propositions except whether they are true or false. Further, we are concerned only with those combinations of propositions which are true in virtue of the rules, whether their constituent propositions are true or false. That is – to take the simplest illustration – we assert $p|(p|p)$, but we never assert any proposition $p$ that has not some suitable molecular structure, although we believe that half of such propositions are true. Our assertions depend always upon structure, never upon the mere fact that some proposition is true.

*6*

A new situation arises, however, when we replace $p$ by $\phi!x$. For example, we have

$$\vdash . \ p|(p|p)$$

and we infer

$$\cancel{\vdash . \ (\phi x)|\{(\phi x)|\phi x}$$

$$\vdash . \ \phi!x|(\phi!x|\phi!x)$$

We cannot *explain* the notation $\phi!x$ without introducing characteristics of propositions other than their truth or falsehood. Take for example the primitive proposition ($*8{\cdot}11$)

$$\vdash . \ (\exists x). \ \phi!x|(\phi!a|\phi!b)$$

The truth of this proposition depends upon the *form* of the constituent propositions $\phi!x$, $\phi!a$, $\phi!b$, not simply upon their truth or falsehood. It cannot be replaced by

$$\text{“} \vdash . \ (\exists p). \ p|(q|r)\text{”}$$

which is true but does not have the desired consequences. We are therefore compelled to consider what is meant by saying that a proposition is of the form $\phi!a$ (where $a$ is some constant). This brings us back to ~~the instance of~~ “$A$ occurs in $p$”, which we gave above as an example of a function which is not a truth-function.[4] And this, we shall find, brings us back to the proposition as fact, in opposition to the proposition as true or false.

*7*

Let us revert to our two instances: “$A$ believes $p$” and “$p$ is about $A$”. We shall avoid ~~irrelevant~~ ⌊certain⌋ psychological difficulties if we take, to begin with, “$A$ asserts $p$” instead of “$A$ believes $p$”. Suppose “$p$” is “Socrates is Greek”. A word is a class of similar noises. Thus a person who asserts “Socrates is Greek” is a person who makes, in rapid succession, three noises, of which the first is a member of the class “Socrates”, the second a member of the class “is”, and the third a member of the class “Greek”. This series of events is part of the series of events which constitutes the person. If $A$ is the series of events constituting the person, $\alpha$ is the class of noises “Socrates”, $\beta$ is the class “is”, and $\gamma$ is the class “Greek”, then “$A$ asserts that Socrates is Greek” is (omitting the rapidity of the succession)

$$(\exists x, y, z). \ x \epsilon \alpha \ . \ y \epsilon \beta \ . \ z \epsilon \gamma \ . \ x \downarrow y \ \cup \ x \downarrow z \cup y \downarrow z \sqsubset A$$

It is obvious that this is not a function of $p$ as $p$ occurs in a truth-function.

4 Russell uses a different example here, “$A$ occurs in $p$”, rather than “$p$ is about $A$”.

(662) If we now take up "$A$ believes $p$", we find the matter rather more complicated, owing to doubt as to what constitutes belief. Some people maintain that a proposition must be expressed in words before we can believe it; if that were so, there would not, from our point of view, be any vital difference between believing and asserting. But if we adopt a less unorthodox standpoint, we shall say that when a man believes "Socrates is Greek" he has simultaneously two thoughts, one of which "means" Socrates while the other "means" Greek, and these two thoughts are related in the way we call "predication". It is not necessary for our

*8*

purposes to define "meaning", beyond noticing that two different thoughts may "have the same meaning". The relation "having the same meaning" is ~~reflex.~~ symmetrical and transitive; moreover, if two thoughts "have the same meaning", either can replace the other in any belief without altering its truth-value. Thus we have one class of thoughts, called "Socrates", which all "have the same meaning"; call this class $\alpha$. We have another class of thoughts, ⌊called "Greek";⌋ which all "have the same meaning"; call this class $\beta$. Call the relation of predication between two thoughts $P$. (This is the relation which holds between our thought of the subject and our thought of the predicate when we believe that the subject has the predicate. It is wholly different from the relation which holds between the subject and the predicate when our belief is true.) Then "$A$ believes that Socrates is Greek" is

$$(\exists x, y) . x \epsilon \alpha . y \epsilon \beta . xPy . x, y \epsilon C`A$$

Here, again, the proposition as it occurs in truth-functions has dissappeared.

It is not necessary to lay any stress upon the above analysis of belief, which may be completely mistaken. All that is intended is to show that "$A$ believes $p$" may very well not be a function of $p$, in the sense in which $p$ occurs in truth-functions.

*9*

We have now to consider "$p$ is about $A$", e.g. " 'Socrates is Greek' is about Socrates". Here we have to distinguish (1) the fact, (2) the belief, (3) the verbal proposition. The fact and the belief, however, do not raise separate problems, since it is fairly clear that Socrates is a constituent of the fact in the same sense in which the thought of Socrates is a constitutent of the belief. And the verbal proposition raises no difficulty, since each instance of the verbal proposition is a series containing a part which is an instance of "Socrates". That is to say "Socrates" (the word) is a class of series of noises, say $\lambda$; and "Socrates is Greek" is another class of series, say $\mu$; and the fact that "Socrates" occurs in "Socrates is Greek" is

$$\cancel{P \epsilon \lambda . \supset . \exists Q}$$

$$P \epsilon \mu . \supset . (\exists Q). Q \epsilon \lambda . Q \Subset P.$$

Thus we are left with the question: what do we mean by saying that Socrates is a constituent of the fact that Socrates is Greek? This raises the whole problem of analysis. But we do not need an ultimate answer; we only need (663) an answer sufficient to throw light on the question whether there are functions of propositions which are not truth-functions.

*10*

There are those who deny the legitimacy of analysis. Without admitting that they are in the right, we can frame a theory which they need not reject. Let us assume that facts are capable of various kinds of resemblances and differences. Two facts may have particular-resemblance; then we shall say that they are about the same particular. Again they may have predicate-resemblance, or dyadic-relation-resemblance, or etc. We shall say that a fact is about only one ~~individual~~⌊particular⌋if any two facts which have ~~individual~~ ⌊particular⌋-resemblance to the given fact have ~~individual~~ ⌊particular⌋-resemblance to each other. Given such a fact, we may define its one ~~individual~~ ⌊particular⌋ as the class of all facts having ~~individual~~ ⌊particular⌋-resemblance to the given fact. In that case, to say that Socrates is a constituent of the fact that Socrates is Greek ⌊(assuming conventionally that Socrates is a particular)⌋is to say that the fact is a member of the class of facts which is Socrates. In the case of a belief about Socrates, which is itself a fact composed of thoughts, we shall say that a belief is about Socrates if it is one of the class of facts constituting a certain ~~thought~~ ⌊idea⌋which "means" Socrates in ~~the~~ whatever sense we may give to "meaning". Here an "idea" is taken to be a class of psychical facts, say all the beliefs which "refer to" Socrates.

*11*

We can define predicates by a similar procedure. Take a fact which is only capable of two kinds of resemblance such as we are considering, namely particular-resemblance and predicate-resemblance; such a fact will be a subject-predicate fact. The predicate involved in it is the class of facts to which it has predicate-resemblance.

We shall assume also various kinds of difference: particular-difference, predicate-difference, etc. These are not necessarily incompatible with the corresponding kind of resemblance; e.g. $R(x, x)$ and $R(x, y)$ have both particular-resemblance in respect of $x$ and particular-difference in respect of $y$. This enables us to define what is meant by saying that a particular occurs twice in a fact, as $x$ occurs twice in $R(x, x)$. First: $R(x, x)$ is a dyadic-relation-fact because it is capable of dyadic-relation-resemblance to other facts; second: any two facts having individual-resemblance to $R(x, x)$ have individual resemblance to each other.[5] This is what we mean by saying that $R(x, x)$ is a dyadic-relation-fact in which $x$ occurs twice, not a subject-predicate fact. Take next a triadic-relation-fact $R(x, x, z)$. This is, by definition, a triadic-relation-fact because it is capable of triadic-relation-resemblance. The facts having particular-resemblance to $R(x, x, z)$ can be divided into two groups (not three) such that any two members of one group have particular-resemblance to each other. This shows that there is repetition, but not whether it is $x$ or $z$ that is repeated. The facts of one group are $R(x, x, c)$ for varying $c$; the facts of the other are $R(a, b, z)$ for varying $a$ and $b$. Each fact of the

*12*

group $R(x, x, c)$ belongs to only two groups constituted by particular-resemblance, whereas (664) the facts of the group $R(a, b, z)$, except when it happens that $a = b$, belong to three groups constituted by particular-resemblance. This defines what is meant by saying that $x$ occurs twice and $z$ once in the fact $R(x, x, z)$. It is obvious that we can deal with tetradic etc. relations in the same way.

According to the above, when we say that Socrates is a constituent of the fact that Socrates is Greek, we mean that this fact is a member of the class of facts which is Socrates.

When we use the notation "$\phi!x$" to denote a proposition in which "$x$" occurs, it is a fact that "$x$" occurs in "$\phi!x$", but we do not need to assert the fact; the fact does its work without having to be asserted. It is also a fact that, if "$x$" occurs in a proposition $p$, and $p$ asserts a fact, then $x$ is a constituent of that fact. This is not a law of logic, but a law of language. It might be false in some languages. For instance, in former days, when a crime was committed in India, the indictment stated that it was committed "in the manor of East Greenwich". These words did not denote any constituents of the fact. But a logical language avoids fictions of this kind.

*12a*

The notation for functions is an illustration of Wittgenstein's principle, that a logical symbol must, in certain ⌊formal⌋ respects, resemble what it symbolizes. All the facts of which $x$ is a constituent, according to the above, constitute a certain class defined by particular-resemblance. The various symbols $\phi x$, $\psi x$, $\chi x$, ... also all resemble each other in a certain respect, namely that their right-hand halves are very similar (not ~~identical~~⌊*exactly* similar⌋, because no two $x$s are exactly alike). The symbols $R(x, x)$, $R(x, x, z)$, etc., are appropriate to their meanings for similar reasons. The

[5] Both occurrences of 'individual-resemblance' in this sentence changed to 'particular-resemblance'.

symbols are *used* before their suitability can be explained. To explain *why* "$\phi x$" is a suitable symbol for a proposition about $x$ is, as we have seen, a complicated matter. But to use the symbol is not a complicated matter. Our symbolism, as a set of facts, resembles, in certain logical respects, the facts which it is to symbolize. This makes it a good symbolism. But in using it we do not presuppose the explanation of why it is good, which belongs to a later stage. And so the notation "$\phi x$" can be used without first explaining what we mean by "a proposition about $x$".

*13*

We are now in a position to deal with the difference between propositions considered factually and propositions as vehicles of truth and falsehood. When we say " 'Socrates' occurs in the proposition 'Socrates is Greek' ", we are taking the proposition factually. Taken in this way, it is a class of series, and 'Socrates' is another class of series. Our statement is only true when we take the proposition and the name as classes. The particular 'Socrates' that occurs at the beginning of our sentence does not occur in the proposition 'Socrates is Greek'; what is true is that another particular closely resembling it occurs in the proposition. It is therefore absolutely essential to all such statements to take words and propositions as classes of similar occurrences, not as single occurrences. But when we assert a proposition, the single occurrence is all (665) that is relevant. When I assert "Socrates is Greek", the particular occurrences of the words have meaning, and the assertion is made by the particular occurrence of that sentence. And to say of that sentence " 'Socrates' occurs in it" is simply false, if I mean the 'Socrates' that I have just written down, since it was a different 'Socrates' that occurred in it. Thus we conclude:

A proposition as the vehicle of truth or falsehood is a particular occurrence, while a proposition considered factually is a class of similar occurrences. It is the proposition considered factually that occurs in such statements as "A believes $p$" and "$p$ is about A".

*14*

Of course it is possible to make statements about the particular fact "Socrates is Greek". We may say how many centimetres long it is; we may say it is black; and so on. But these are not the statements that a philosopher or logician is tempted to make.

When an assertion occurs, it is made by means of a particular fact, which is an instance of the proposition asserted. But this particular fact is, so to speak, "transparent"; nothing is said about it, but by means of it something is said about something else. It is this "transparent" quality which belongs to propositions as they occur in truth-functions. This belongs to $p$ when $p$ is asserted, but not when we say "$p$ is true". Thus suppose we say: "All that Xenophon said about Socrates is true". Put

$$\mathrm{X}(p) \,.=.\, \text{Xenophon asserted } p.$$
$$\mathrm{S}(p) \,.=.\, p \text{ is about Socrates.}$$

Then our statement is

$$\mathrm{X}(p) \,.\, \mathrm{S}(p) \,.\supset_p.\, p \text{ is true.}$$

Here the occurrence of $p$ is not "transparent". But if we say

$$x \epsilon \alpha \,.\supset_x.\, \phi!x$$

we are asserting $\phi!x$ for a whole class of values of $x$, and yet "$\phi!x$" still has a "transparent" occurrence. The essential difference is that in the former case we speak *about* the symbol or belief, whereas in the latter we merely use it to speak about something else. This is the point which distinguishes the occurrences of propositions in mathematical logic from their occurrences in non-truth-functions.

*15*

Let us endeavour to give greater definiteness to this point. Take the statement "Socrates had all the predicates that Xenophon said he had". Let the series of events which was Xenophon be called X. Then if Xenophon attributed the predicate $\alpha$ to Socrates, we ⌊might appear to⌋ have (writing $x \downarrow y \downarrow z \downarrow w$ for the series $x, y, z, w$)

$$\text{Socrates}{\downarrow}\text{had}{\downarrow}\text{predicate}{\downarrow}\alpha \Subset \mathrm{X}.$$

Thus our assertion would be

$$\text{Socrates}{\downarrow}\text{had}{\downarrow}\text{predicate}{\downarrow}\alpha \Subset \mathrm{X} \,.\supset_\alpha.\, \text{Socrates had predicate } \alpha.$$

Here, however, there is an ambiguity. On the left, "Socrates", "had", "predicate", and "$\alpha$" occur as noises; on the right they occur as symbols. This (666) ambiguity amounts to a fallacy. For, in fact, what I write on paper is not the noise that Xenophon made, but a symbol for that noise. Thus I am using one symbol "Socrates" in two senses: (a) to mean the noise that Xenophon made on a certain occasion, (b) to mean a certain man. We must say:

If Xenophon made a series of noises which mean what is meant by "Socrates had the predicate $\alpha$", then what this means is true.

For example: if Xenophon said "Socrates was wise", then what is meant by "Socrates is wise" is true.

But this does not assert that Socrates was wise. When I actually assert that Socrates was wise, I say something which cannot be said by talking about the words I use in saying it; and ~~conversely~~ when I assert that Socrates was wise, although an instance of the proposition occurs, yet I do not say anything whatever about the proposition, in particular I do not say that it is true. This is an inference, not logical, but linguistic.

*16*

If the above considerations in any way approximate to the truth, we see that there is an absolute gulf between the assertion of a proposition and an assertion about the proposition. The $p$ that occurs when we assert $p$ and the $p$ that occurs in "$A$ asserts $p$" are by no means identical. The occurrence of propositions as asserted is simpler than their occurrence as something spoken about. In the assertion of a proposition, and in the assertion of any molecular function of a proposition, the proposition does not occur, if we mean by the proposition ~~what~~ ⌊the $p$ that⌋ occurs in such propositions as "$A$ asserts $p$" or "$p$ is about $A$". When these latter are analysed, they are found not to conflict with the view that propositions, in the sense in which they occur when they are asserted, only occur in truth-functions.

When $p$ is asserted, $p$ does not really occur, but the constituents of $p$ occur, or an instance of $p$ occurs. The same is true when a molecular proposition containing $p$ is asserted. Thus we cannot infer $p = q$, because here $p$ and $q$ occur in a sense in which they do not occur when molecular propositions containing them are asserted.

Similar considerations apply to propositional functions. Suppose there are two predicates $\alpha$ and $\beta$ which are always found together; we may still say that they are two, on the ground that $\alpha(x)$ and $\beta(x)$ are facts which do not have predicate-resemblance. But the propositional function $\alpha(\hat{x})$ is solely to be used in building up matrices by means of the stroke. The predicate $\alpha$ is a class of facts, whereas the propositional function $\alpha(\hat{x})$ is ~~a~~ merely ⌊a⌋ symbolic convenience in speaking about certain propositions. Thus we may have $\alpha(\hat{x}) = \beta(\hat{x})$ without having $\alpha = \beta$. In this way we escape the *prima facie* paradoxes of the theory that propositions only occur in truth-functions and propositional functions only occur through their values. The paradoxes rest on the confusion between factual and assertive propositions.

# Hierarchy of propositions and functions

## Reconstructing the original manuscript

The "Hierarchy of propositions and functions" (HPF) consists of 53 leaves, all but one with writing on only one side. The item is in file RA 230.031400. The leaves are all numbered (foliated) with only one exception, from *1* to *78*. There are some duplicate numberings and some numbers out of order at the end of the manuscript. Almost all of the 25 missing folios reappear renumbered in the manuscripts for the Introduction and Appendix A. The numbers are written in the upper right of each page, underlined following Russell's practice, and they are reproduced here as italic. There are very few deletions, except for the last few pages. Some entire pages were deleted with a single line, or large 'X' through the whole page, and that is indicated where it happens.

At one point this HPF manuscript was the entire draft of the new material for the second edition. Russell then decided to split the new material into an introduction and an appendix.[1] Long sections of the manuscript were moved to become parts of the Introduction to the second edition, and the core of Appendix A. Several folios were added at the beginning of each of those two. The rest of the material in the HPF manuscript was revised to differing extents and turns up in the published version there and in the Introduction, Appendix A, and Appendix B. Appendix C appears to be new material, added after the HPF manuscript was divided.

A note at the bottom of each page states what became of the material. That record is collected below. Given the small number of pages that are declared "missing" it is likely that this file contains all the remnants of the original manuscript that were not reused in the final version. Pages that remain in the "Hierarchy" file are numbered in italics, those moved are in roman font, with an identification of how they were renumbered for use in the Introduction to the second edition,

[1] Russell uses the expression "the appendix" in the manuscript of the Introduction to the second edition, so the division into A, B, and C may have been quite late.

Appendix A and Appendix B manuscripts. Four leaves were numbered "*52*" at some point, then renumbered and replaced. The final numberings are identified by the numeral which strikes the eye as most prominent. Material that is indicated as "revised" was rewritten for the new introduction and appendices, and so remained in the "Hierarchy" folder. Many of these revised pages are replaced with reformulated proofs or other symbolic material. Very little of the prose is revised.

The fate of the "Hierarchy of propositions and functions" (HPF) manuscript, as reconstructed from surviving folios:

*1* : revised as *5* of the Introduction manuscript.
2–4 : missing
5 : missing, revised as *5a–5d*, renumbered *6–9*, Introduction.
6–10 : renumbered *10–14*, Introduction.
*11* : revised as *11*, renumbered *15*, Introduction.
*12* : revised as *12*, *12a*, *12b*, renumbered *16–18*, Introduction.
13 : renumbered *19*, Introduction.
*14* : revised as *14* and *14a*, renumbered *20*, *21*, Introduction.
[no foliation] : with two revisions in print, all are theorems of *Principia Mathematica*.
15–17 : renumbered *22–24*, Introduction.
18 : missing, revised as *18*, *18a*, *18b*, renumbered *25–27*, Introduction.
19 : missing, revised as *19–21*, renumbered as *28–30*, Introduction.
*20* : revised as *4*, *5*, Appendix A manuscript.
*21* : revised as *6*, Appendix A.
*22* : revised as *7*, *8*, Appendix A.
*23* : revised as *9*, Appendix A.
*24* : revised as *10*, *5a* (then *10a*), Appendix A.
25 : renumbered *11*, Appendix A.
*26–29*, *29a* : revised as *12–15*, Appendix A.
30 : renumbered *16*, Appendix A.
*31* : revised as following *31*
*31* : deleted with a single stroke, revised as *17*, Appendix A.
32, 33 : renumbered *18–19*, Appendix A.
*33a*, *33b* : revised as *20*, Appendix A.
34–42 : renumbered *21–29*, concluding Appendix A.
43–48 : renumbered *31–36*, Introduction.
*49* : revised as *49*, *49a*, renumbered *37*, *38*, Introduction.
50 : renumbered *39*, Introduction.
*52* : originally *51*, then switched with next leaf and renumbered.

51 : first *52*, then *51* after preceding was moved, finally *40*, Introduction.
*52* : the second folio *52* of HPF.
*53* : renumbered *52* after preceding was removed, finally *41*, Introduction.
53 : renumbered *42*, Introduction.
53a, 53b : renumbered *43*, *44*, Introduction.
54 : renumbered *45*, Introduction.
55 : renumbered *46*, Introduction.
*56–59* : revised as *47–50* of Introduction.
*60–61* : revised as *61–62* of Introduction.
*61a* : revised as *3–4* of Appendix B manuscript.
*61a–65* : not used because unnecessary for Appendix B.
*66* : revised as very early material on induction for Introduction.
*67* : early material on orders, for Introduction.
*68* : revised as *13*, Appendix B.
*69–70* : revised as *6*, Appendix B.
*70–78* : material on $*120{\cdot}81$–$*120{\cdot}832$, for Appendix B, not used.
*74–76* : material on $*121{\cdot}47$–$*121{\cdot}84$, for Appendix B, not used.

## The remaining HPF leaves

*1*[2]

The hierarchy of propositions and functions

I.

We begin with "atomic propositions". These may be defined negatively as propositions containing no parts that are propositions, and not containing the notions "all" or "some". They may also be defined positively – and this is the better course – as propositions of the following sorts:

$\alpha(x)$, ⌊or $R_1(x)$, ⌋ meaning: $x$ has the predicate $\alpha$ (or $R_1$);

$x R_2 y$ or $R_2(x, y)$, meaning: $x$ has the relation $R_2$ (in intension) to $y$;

$R_3(x, y, z)$, meaning: $x, y, z$ have the triadic relation $R_3$ (in intension);

$R_4(x, y, z, w)$, meaning: $x, y, z, w$ have the tetradic relation $R_4$ (in intension);

[2] HPF *1* is revised as Introduction *5*. HPF *2–5* were revised as *5a–5d*, then foliated as *6–9* in the Introduction. The next folios of HPF, *6–10*, are used in Introduction, with numbers changed to 10–14.

and so on *ad infinitum*. Logic does not know whether there are in fact $n$-adic relations (in intension); this is an empirical question. We know as an empirical fact that there are dyadic relations (in intension), because without them series would be impossible. But logic is not interested in this fact; it is concerned solely with the *hypothesis* of there being propositions of such-and-such a form. In certain cases, this hypothesis is itself of the form in question, or contains a part ⌊which is⌋ of the form in question; in these cases, the fact that the hypothesis can be framed proves that it is true. But even when a hypothesis occurs in logic, the fact that it can be framed does not itself belong to logic.

*11*[3]

(3). "Always true" and "sometimes true". An elementary function of one or more individuals has a set of values which are some among elementary propositions. It may happen that all these values are true; again, it may happen that at least one is true. ~~If~~ ⌊The proposition that⌋ all the values of a function $\phi(x, y, z, \ldots)$ are true ~~, we ex~~ is expressed by the symbol

$$(x, y, z \ldots).\ \phi(x, y, z \ldots)$$

The proposition that at least one value of the function is true is expressed by the symbol

$$(\exists x, y, z, \ldots).\ \phi(x, y, z, \ldots)$$

When a function contains several variables, it may be regarded as obtained step by step. Thus $\phi(x, y)$ may be regarded as derived from a proposition $\phi(a, b)$, where $a$ and $b$ are constants, by first varying $b$ and obtaining a function $\phi(a, y)$, containing one variable $y$. We may then proceed to the proposition $(y).\phi(a, y)$, and then, by varying $a$, obtain a function of one variable, namely $(y).\phi(x, y)$. We can then proceed to

$$(x):(y).\phi(x, y)$$

which will be equivalent to $(x, y).\phi(x, y)$. This method has the advantage of only requiring functions of one variable as a primitive idea, but it has disadvantages in that it requires us to deal with such propositions as $(y).\phi(a, y)$, which are not elementary. ~~Connected with this is the disadvantage that it does not permit us to use the method of "matrices", to be explained shortly.~~ We ~~shall~~ ⌊are⌋ therefore ⌊tempted to⌋ admit as a primitive idea the notion of an elementary function of any number

[3] This was revised as a distinct *11*, which in turn was renumbered as *15* for the Introduction manuscript.

of variables, as well as the propositions asserting that it is "always true", i.e. $(x, y, z, \ldots).\phi(x, y, z, \ldots)$, or "sometimes true", i.e. $(\exists x, y, z, \ldots).\phi(x, y, z, \ldots)$.

*12*[4]

These, however, are not the only notions required. Take an elementary function of two variables, $\phi(x, y)$. We may wish to assert that for every value of $x$ this is true for some value of $y$, which we symbolize by

$$(x):(\exists y).\phi(x, y)$$

Again we may wish to assert that, for some value of $y$, $\phi(x, y)$ is true for every value of $x$, which we symbolize by

$$(\exists y):(x).\phi(x, y)$$

(Both of these are forms which occur constantly in analysis. The first says: "For every $\epsilon$, however small, there is a $\delta$ such that ... ", while the second says: "There is a $\delta$ such that, for every $\epsilon$, however small, ... ".) It may be held that these notions demand the step-by-step variation of one argument at a time; certainly they are easier with the help of this notion. However that may be, what we require in practice is that, given an elementary function

$$\phi(x_1, x_2, x_3, \ldots x_n)$$

we may write before it "$(x_m)$" or "$(\exists x_m)$" for all values of $m$ from 1 to $n$, taken in any order we please, and obtain in this way a definite proposition, containing a definite assertion, to the effect that all, or at that at least one, of some definite collection of propositions, are true. It will be seen that this gives rise to $n!.2^n$ propositions derived from a given function of $n$ variables. Some of these, however, are equivalent. The process of deriving such propositions from a function will be called "generalization", whether we take "all values" or "some value". The propositions which result will be called "general propositions".

*14*[5]

The method of proving existence-propositions is: Given, by the previous method, a proposition[6]

$$\text{“} \vdash .\ (x).\ f(x, x) \text{”}$$

[4] This page was revised as three pages; a new *12*, *12a*, and *12b*, which were then renumbered as *16*, *17*, and *18* in the Introduction. The next folio of HPF,*13*, appears renumbered as *19* in the Introduction.

[5] Revised as *14* and *14a*, then renumbered as *20* and *21* of Introduction. The next three pages, *15*, *16*, and *17* are renumbered as *22*, *23*, and *24* of Introduction. *18* was revised as *18*, *18a*, and *18b*, which were renumbered as *25*, *26*, and *27* of Introduction. *19* appears renumbered as *28* of Introduction.

[6] The deletion below is in pencil, with the following material as an insertion in a darker ink over an original pencil version.

we can infer "$\vdash : (x) : (\exists y) .\ f(x, y)$"
and "$\vdash : (\exists x, y) .\ f(x, y)$"
and "$\vdash : \cancel{(\exists y) : (x) .}\ \lfloor (y) : (\exists x) . \rfloor f(x, y)$"

These two methods cover our practice, by the help of certain primitive propositions and definitions.

Both methods may be illustrated by a simple example. We have

$$\vdash . (p) .\ p \supset p$$

Hence, by the first principle,

$$\vdash . (x) .\ \phi x \supset \phi x$$

Hence, by the second principle,

$$\vdash : (x) : (\exists y) .\ \phi x \supset \phi y$$

$$\vdash : (\exists x, y) .\ \phi x \supset \phi y$$

and $\vdash : (y) : (\exists x) .\ \phi x \supset \phi y$

Apart from special axioms asserting existence-theorems (such as the axiom of reducibility, the multiplicative axiom, and the axiom of infinity), the above is the sole method of proving existence-theorems, except for the primitive proposition $*8 \cdot 11$ (below).

[no foliation][7]

$*8$. List of Props.

$\cdot 01$ $\{(x) .\ \phi x\} | q \ . = .\ (\exists x) .\ \phi x | q$ Df
$\cdot 011$ $\{(\exists x) .\ \phi x\} | q \ . = .\ (x) .\ \phi x | q$ Df etc.
$\cdot 1$ $\vdash : \phi a \ . \supset .\ (\exists x) .\ \phi x$ Pp i.e. $\vdash . (\exists x, y) .\ \phi a \mid (\phi x | \phi y)$
$\cdot 11$ $\vdash . (\exists x) .\ \phi x \mid (\phi a | \phi b)$ Pp
$\cdot 12$ $p$ and $p \supset q$ allow inference to $q$ Pp[8]
$\cdot 13$ $(\exists x) : (y) .\ \phi x | \psi y$ ~~allows inference to~~ $\lfloor$may be substituted for$\rfloor$ $(y) : (\exists x) .\ \phi x | \psi y$, and vice versa. Pp[9]
$\cdot 2\cancel{9}$ $\vdash : (x) .\ \phi x \ . \supset .\ \phi a$
$\cdot 21$ $\vdash :. (x) .\ \phi x \supset \psi x \ . \supset : (\exists x) .\ \phi x \ . \supset .\ (\exists x) .\ \psi x$
$\cdot 22$ $\vdash : \phi a \vee \phi b \ . \supset .\ (\exists x) .\ \phi x$

[7] There is no comparable list of propositions in the printed version. With two exceptions, namely $*8 \cdot 12$ and $*8 \cdot 13$, these "props." are the same as those in Appendix A.

[8] In print: $*8 \cdot 12$ From "$(x) .\ \phi x$" and "$(x) .\ \phi x \supset \psi x$" we can infer "$(x) .\ \psi x$", even when $\phi$ and $\psi$ are not elementary. Pp

[9] In *PM:* If all occurrences of $x$ are separated from all occurrences of $y$ by a certain stroke, we can change the order of $x$ and $y$ in the prefix, i.e. we can replace "$(y) : (\exists x) .\ \phi x | \phi y$" by "$(\exists x) : (y) .\ \phi x | \phi y$" and vice versa. Pp.

$\cdot 23 \quad \vdash : (\exists x).\ \phi x \vee \phi c\ .\supset .\ (\exists x).\ \phi x$
$\cdot 24 \quad \vdash :: p \supset q\ .\supset :.\ q\ .\supset .\ (\exists x).\ \phi x : \supset : p\ .\supset .\ (\exists x).\ \phi x$
$\lfloor \cdot 241 \vdash :: (x).\ \phi x.\supset .\ p\ : \supset :.\ p \supset q\ .\supset : (x).\ \phi x\ .\supset .\ q \rfloor$
$\cdot 25 \quad \vdash :: p\ .\supset .\ (\exists x).\ \phi x : \supset :.\ (\exists x).\ \phi x.\supset .(\exists x).\ \psi x : \supset : p\ .\supset .\ (\exists x).\ \psi x$
$\cdot 26 \quad \vdash : \phi a \vee \phi b \vee \phi c\ .\supset .\ (\exists x).\ \phi x \vee \phi c$
$\cdot 261 \vdash : \phi a \vee \phi b \vee \phi c\ .\supset .\ (\exists x).\ \phi x$
$\cdot 27 \quad \vdash :: q\ .\supset .\ (\exists x).\ \phi x : \supset :.\ p \supset q\ .\supset : p.\ \supset .\ (\exists x).\ \phi x$
$\cdot 271 \vdash :: q\ .\supset .\ (\exists x, y).\ \phi(x, y) : \supset :.\ p \supset q\ .\supset : p\ .\supset .\ (\exists x, y).\ \phi(x, y)$
$\cdot 272 \vdash ::.\ p\ .\supset : q : \supset .\ (\exists x).\ \phi x : \supset :: r \supset p\ .\supset :.\ r\ .\supset : q\ .\supset .\ (\exists x).\ \phi x$
$\cdot 28 \quad \vdash :: p\ .\supset .\ (\exists x).\ \phi x : \supset :.\ q\ .\supset .\ (\exists x).\ \phi x : \supset : p \vee q\ .\supset .\ (\exists x).\ \phi x$
$\cdot 29 \quad \vdash :.\ (x).\ \phi x \supset \psi x\ .\supset : (x).\ \phi x\ .\supset .\ (x).\ \psi x$

*20*[10]

The primitive propositions required are:

$*8\cdot 1 \quad \vdash : \phi a\ .\supset .\ (\exists x).\ \phi x$ i.e. $\vdash .\ (\exists x, y).\ \phi a \mid (\phi x | \phi y)$
$\cdot 11 \quad \vdash : (\exists x).\ \phi x \mid (\phi a | \phi b)$ i.e. $\vdash : (x).\ \phi x\ .\supset .\ \phi a\ .\ \phi b$
$\cdot 12$ From $p$ and $p \supset q$ we can infer $q$, even when $p$, $q$ are not elementary.[11]
$\cdot 13$ For $(\exists x) : (y).\ (\phi x | \psi y)$ we can ~~infer~~ ⌊substitute⌋ $(y) : (\exists x).\ \phi x | \psi y$, and vice versa, in any stroke proposition.[12]
$*8\cdot 1\cdot 11$ have to be assumed, not only for one variable, but for any number. I.e. we assume

$\vdash : \phi(a_1 a_2 \ldots a_n)\ .\supset .\ \exists(x_1 x_2 \ldots x_n).\ \phi(x_1 x_2 \ldots x_n)$ and
$\vdash : \exists(x_1 x_2 \ldots x_n).\ \phi(x_1 x_2 \ldots x_n) | \{\phi(a_1 a_2 \ldots a_n) | \phi(b_1 b_2 \ldots b_n)\}$

~~$*8\cdot 2 \quad \vdash : (x).\ \phi x.\supset .\ \phi a \quad [*8\cdot 11 \frac{a}{b}]$~~

$\cdot 21 \ \vdash :.\ (x).\ \phi x \supset \psi x.\supset : (\exists x).\ \phi x\ .\supset .\ (\exists x).\ \psi x$

Applying Dfs, prop is (using $*8\cdot 13$)
$(y, y') : (\exists x, z, w, z', w').\ (\phi x \supset \psi x) | [\{\phi y | (\psi z | \psi w)\} | \{\phi y' | (\psi z' | \psi w')\}]$
Putting $z, w, z', w'$ all $= x$, above (by $*8\cdot 1$) is true if
$(y, y') : (\exists x).\ (\phi x \supset \psi x) | \{(\phi y \supset \psi x) | (\phi y' \supset \psi x)\}$
which results from $*8\cdot 11$ by putting $y, y'$ for $a, b$
and $\phi y \supset \psi x$ for $\phi y$. Hence prop.

[10] Revised to become *4* and *5* of Appendix A.
[11] In print: $*8\cdot 12$ From "$(x).\ \phi x$" and "$(x).\ \phi x \supset \psi x$" we can infer "$(x).\ \psi x$", even when $\phi$ and $\psi$ are not elementary. Pp
[12] In print: If all occurrences of $x$ are separated from all occurrences of $y$ by a certain stroke, we can change the order of $x$ and $y$ in the prefix, i.e. we can replace "$(y) : (\exists x).\ \phi x | \phi y$" by "$(\exists x) : (y).\ \phi x | \phi y$" and vice versa. Pp.

$\cdot 22 \quad \vdash : \phi a \vee \phi b \,.\supset.\, (\exists x).\, \phi x$

By $*8\cdot 11$, $\vdash .(\exists z).\, (\sim \phi z)|(\sim \phi a| \sim \phi b)$ (1)
Also $\sim \phi z|(\sim \phi a| \sim \phi b) \,.\supset.\, (\phi a \vee \phi b)|(\phi z|\phi z)$ (2) [by Transp]
Hence by $*8\cdot 21$, (1), (2), give $\vdash .(\exists z).\, (\phi a \vee \phi b)|(\phi z|\phi z)$ (4)
$\vdash .(4). * 8\cdot 1 \supset \vdash .$ Prop.

These propositions, as well as all the others in $*8$, apply to any number of variables, since the primitive propositions do so.

*21*[13]

$*8\cdot 23 \quad \vdash : (\exists x).\, \phi x \vee \phi c \,.\supset.\, (\exists x).\, \phi x$

Applying Dfs, this prop is
$$(\exists x).\, \phi x \vee \phi c. \mid (y, z).\, \phi y|\phi z$$
i.e. $(x) : (\exists y, z).\, (\phi x \vee \phi c) \mid (\phi y|\phi z)$ i.e.
$(x) : \phi x \vee \phi c \,.\supset.\, (\exists x).\, \phi x$
Now $\vdash . * 8\cdot 22 \,.\supset \vdash : (x) : \phi x \vee \phi c \,.\supset.\, (\exists x).\, \phi x$. Hence prop.

$\cdot 24 \quad \vdash :: p \supset q \,.\supset :.\, q \,.\supset.\, (\exists x).\, \phi x : \supset : p \,.\supset.\, (\exists x).\, \phi x$

Applying the Dfs, this becomes
$(x, y, x', y') : (\exists z, w, u, v, z', w', u', v').$
$$(p \supset q) \mid [\{(q|(\phi x|\phi y)) \mid (p|(\phi z|\phi w))|(p|(\phi u|\phi v))\} \mid \{q|(\phi x'|\phi y') \mid (p|\phi z'|\phi w')(p|(\phi u'|\phi v)\}]$$
By $*8\cdot 1$ this will be true if it is true for chosen values of $z, w, u, v, z', w', u', v'$.
Put $z, u = x \,.\, w, v = y \,.\, z', u' = x' \,.\, w', v' = y'$.
Then what has to be proved is
$(p \supset q)|[\{(q|(\phi x|\phi y) \,.\supset.\, p|(\phi x|\phi y)\} \mid \{q|(\phi x'|\phi y') \,.\supset.\, p|(\phi x'|\phi y')\}]$
i.e. $p \supset q \,.\supset :.\, q \,.\supset.\, \phi x \,.\, \phi y : \supset : p \,.\supset.\, \phi x \,.\, \phi y$
$:.\, q \,.\supset.\, \phi x' \,.\, \phi y' : \supset : p \,.\supset.\, \phi x' \,.\, \phi y'$
which is true by Syll. Hence prop.

$\cdot 241 \quad \vdash :: (x).\, \phi x \,.\supset.\, p : \supset :.\, p \supset q \,.\supset : (x).\, \phi x \,.\supset.\, q$[14]

Applying Dfs, and putting
$f(y, z) \,.=.\, (p \supset q)|\{(\phi y \supset q)|(\phi z \supset q)\}$,
matrix is $\{\phi x|(p|p)\} \mid \{f(y, z)|f(y', z')\}$ and prefix is $(x) : (\exists y, z, y', z')$.
Truth is secured by $y = z = y' = z' = x$. Hence prop.

*22*[15]

$*8\cdot 25 \quad \vdash :: p. \supset .(\exists x).\phi x : \supset :.\, (\exists x).\phi x. \supset .(\exists x).\psi x : \supset : p. \supset .(\exists x).\psi x$

[13] Revised to become *6* of Appendix A.
[14] Here to the bottom of the page is in a lighter, blue, ink, indicating that it is a later addition to the rest, which is in a black ink.
[15] Revised to become *7* and *8* of Appendix A.

*Dem.* Put
$f(x, y, z, u, v, m, n). = .\{\phi x|(\psi y|\psi z)\}|[\{p|(\psi u|\psi v)\}|\{p|(\psi m|\psi n)\}]$
Then applying Dfs, the prop. to be proved is
$(a, b, y, z, y', z') : (\exists x, u, v, m, n, x', u', v', m', n').$
$\{p|(\phi a|\phi b)\}|\{f(x, y, z, u, v, m, n)|f(x', y', z', u', v', m', n')\}$
Put
$x = a.\ x' = b.\ u = v = y.\ m = n = z.\ u' = v' = y'.\ m' = n' = z'.$
Then the matrix reduces to
$p\ .\supset .\ \phi a\ .\ \phi b : \supset :.\ \phi a\ .\supset .\ \psi y\ .\ \psi z : \supset : p\ .\supset .\ \psi y\ .\ \psi z :.$
$\phi b\ .\supset .\ \psi y'\ .\ \psi z' : \supset : p\ .\supset .\ \psi y'\ .\ \psi z'$
which is true by Syll. Hence prop by $*8\cdot 1\cdot 13$ repeatedly.
Similar proofs apply to other forms of the syllogism, which will be assumed.[16]

$\cdot 26\quad \vdash : \phi a \vee \phi b \vee \phi c\ .\supset .\ (\exists x).\ \phi x \vee \phi c$

*Dem.*
$\vdash : \phi a \vee \phi b \vee \phi c\ .\supset .\ (\phi a \vee \phi c) \vee (\phi b \vee \phi c)$ (1)
$\vdash .\ *8\cdot 22\ .\supset \vdash : (\phi a \vee \phi c) \vee (\phi b \vee \phi c)\ .\supset .\ (\exists x).\ \phi x \vee \phi c$ (2)
$\vdash .(1).(2).\ *8\cdot 24\ .\supset \vdash .$ Prop

$\cdot 261\quad \vdash : \phi a \vee \phi b \vee \phi c\ .\supset .\ (\exists x).\ \phi x$

[$*8\cdot 23\cdot 26\cdot 25$]
It is obvious that we can prove in like manner
$\phi a \vee \phi b \vee \phi c \vee \phi d\ .\supset .\ (\exists x).\ \phi x$ and so on.

*23*[17]

$*8\cdot 27\quad \vdash :: q.\supset .\ (\exists x).\ \phi x : \supset :.\ p \supset q\ .\supset : p\ .\supset .\ (\exists x).\ \phi x$

Put $\{p|(\phi x|\phi y)\}|\{p|(\phi u|\phi v)\} = f(x, y, u, v)$. Then applying Dfs, prop is
$(a, b) : (\exists\ x, y, u, v, x', y', u, v').$
$\{q|(\phi a|\phi b)\}\ |\ [\{(p \supset q)|f(x, y, u, v)\}|\{(p \supset q)|f(x', y', u', v')\}]$
Putting $x = u = x' = u' = a,\ \ y = v = y' = v' = b$, this becomes
$(a, b) : q\ .\supset .\ \phi a\ .\ \phi b : \supset :.\ p \supset q\ .\supset : p\ .\supset .\ \phi a\ .\ \phi b$
which is true. Hence prop.

$\cdot 271\quad \vdash :: q\ .\supset .\ (\exists\ x, y).\ \phi(x, y) : \supset :.\ p \supset q\ .\supset : p\ .\supset .\ (\exists\ x, y).\ \phi(x, y)$

[Proof as in $*8\cdot 271$]
This proposition can be proved similarly for
$(\exists\ x_1 x_2 \ldots x_n).\ \phi(x_1 x_2 \ldots x_n)$
We shall quote it as $*8 \cdot 271$ for any number of variables $x_1 x_2 \ldots x_n$.

$\cdot 272\quad \vdash ::.\ p\ .\supset : q\ .\supset .(\exists x).\ \phi x :.\supset :: r \supset p\ .\supset :.\ r\ .\supset : q\ .\supset .\ (\exists x).\ \phi x$

[16] This line is a late addition in blue ink. [17] Revised to become *9* of Appendix A.

[$*8\cdot271$]
This proposition also holds for $(\exists\, x_1x_2\ldots x_n).\ \phi(x_1x_2\ldots x_n)$

*24*[18]

$*8\cdot28 \quad \vdash :: p\ .\supset .\ (\exists x).\ \phi x\ :\supset :.\ q\ .\supset .\ (\exists x).\ \phi x\ :\supset :\ p \vee q\ .\supset .\ (\exists x).\ \phi x$

*Dem.*
Applying Dfs, this prop becomes, putting

$f(x,y,z,w)\ .=.\ \{(p \vee q)|(\phi x|\phi y)\}|\{(p \vee q)|(\phi z|\phi w)\},$

$(a,b,c,d,c',d'):(\exists x,y,z,w,x',y',z',w').$
$\{p|(\phi a|\phi b)\}|[\{q|(\phi c|\phi d)\}|f(x,y,z,w)]\ |$
$[\{q|(\phi c'|\phi d')\}|f(x',y',z',w')\}]$

The matrix here is
$p\ .\supset .\ \phi a\ .\ \phi b :\supset :.$
$q\ .\supset .\ \phi c\ .\ \phi d :\supset .\ f(x,y,z,w) :.$
$q\ .\supset .\ \phi c'\ .\ \phi d' :\supset .\ f(x',y',z',w')$
where $f(x,y,z,w)\ .\equiv :\ p \vee q\ .\supset .\ \phi x\ .\ \phi y\ .\ \phi z\ .\ \phi w.$
Call the matrix $F(x,y,z,w,x',y',z',w')$. We have
$\vdash : p\ .\supset .\ F(a,b,a,b,a,b,a,b)$
$\vdash : \sim p\ .\ F(c,d,c,d,c',d',c',d')$
Hence $\vdash :\ F(a,b,a,b,a,b,a,b)\ .\vee .\ F(c,d,c,d,c',d',c',d')$
Hence, by the extension of $*8\cdot22$ to eight variables:

$$\vdash .\ (\exists x,y,z,w,x',y',z',w')\ .\ F(x,y,z,w,x',y',z',w')$$

which was to be proved.

$\cdot29 \quad \vdash : (x).\ \phi x\ .\supset .\ \phi a \qquad \cancel{[*8\cdot11\tfrac{a}{b}\cdot *8\cdot241]}$[19]

*Dem.*
$\vdash .\ *\,8\cdot11\tfrac{a}{b}\ .\supset \vdash .\ (\exists x).\ \phi x|(\phi a|\phi a)$
$[(*8\cdot01)] \supset \vdash : (x).\ \phi x\ .\supset .\ \phi a$

*26*[20]

$*8\cdot32 \quad \vdash :.\ (x).\ \phi x\ .\supset .\ q :\supset : s|q\ .\supset .\ \{(x).\ \phi x\}|s$

*Dem*
The proposition to be proved is
$\{p|(q|q)\}\ |\ [\{(s|q)|((p|s)|(p|s))\}\ |\ \{(s|q)|((p|s)|(p|s))\}]$
where $p\ .=.\ (x).\ \phi x$. It may also be written
$(p|\sim q)\ |\ \sim\{(s|q)\ |\ \sim(p|s)\}$

[18] Revised to become *10* and *10a* (previously *5a*) of Appendix A. The next folio, *25*, is renumbered and becomes *11* of Appendix A.
[19] A late addition, in blue ink. [20] Revised to become *12* of Appendix A.

Applying the definitions,

$$\begin{aligned}
&p|\sim q \;.=.\; (\exists a).\ \phi a \mid \sim q\\
&p|s \;.=.\; (\exists x).\ \phi x|s\\
&\sim(p|s) \;.=.\; (x, y).\ (\phi x|s) \mid (\phi y|s)\\
&(s|q)|\sim(p|s) \;.=.\; (\exists x, y).\ (s|q) \mid \{(\phi x|s)|(\phi y|s)\}\\
&\sim\{(s|q)|\sim(p|s)\} \;.=.\\
&\quad (x, y, x', y').\ \{(s|q) \mid ((\phi x|s)|(\phi y|s))\} \mid \{(s|q) \mid ((\phi x'|s)|(\phi y'|s))\}
\end{aligned}$$

$$\begin{aligned}
&(p|\sim q)|\sim\{(s|q)|\sim(p|s)\} \;.=:\\
&\qquad (a):(\exists\, x, y, x', y').\ (\phi a|\sim q)\\
&\qquad\qquad \mid [\{(s|q)|((\phi x|s)|(\phi y|s))\} \mid \{(s|q)|((\phi x'|s)|(\phi y'|s))\}]
\end{aligned}$$

The matrix here is equivalent to

$\phi a \supset q \;.\supset:.\; s|q \;.\supset.\; \phi x|s \;.\; \phi y|s : s|q \;.\supset.\; \phi x'|s \;.\; \phi y'|s$

which is equivalent to

$\phi a \supset q \;.\supset$

$\quad :.\; q \supset \sim s \;.\supset.\; \phi x \supset \sim s \;.\; \phi y \supset \sim s \;.\; \phi x' \supset \sim s \;.\; \phi y' \supset \sim s$

This is true when $x = x' = y = y' = a$. Hence Prop, by $*8{\cdot}1$.

*27*[21]

$*8{\cdot}321 \quad \vdash :.\ (\exists x).\ \phi x \;.\supset.\; q : \supset : s|q \;.\supset.\; \{(\exists x).\ \phi x\} \mid s$

Putting $p \;.=.\; (\exists x).\ \phi x$, we proceed as in $*8{\cdot}32$. We have

$$\begin{aligned}
&p \mid \sim q \;.=.\; (a).\ \phi a \mid \sim q\\
&p|s \;.=.\; (x).\ \phi x|s\\
&\sim(p|s) \;.=.\; (\exists\, x, y).\ (\phi x|s) \mid (\phi y|s)\\
&(s|q)|\sim(p|s) \;.=.\; (x, y).\ (s|q) \mid \{(\phi x|s)|(\phi y|s)\}\\
&\sim\{(s|q)|\sim(p|s)\} \;.=.\\
&\quad (\exists\, x, y, x', y').\ [(s|q)|\{(\phi x|s)|(\phi y|s)\}] \mid [(s|q)|\{(\phi x'|s)|(\phi y'|s)\}]
\end{aligned}$$

$$\begin{aligned}
&(p|\sim q)|\sim\{(s|q)|\sim(p|s)\} \;.=:\\
&\qquad (\exists a):(x, y, x', y').\ (\text{Same matrix as in } *8{\cdot}32)
\end{aligned}$$

Thus we have to prove, for any $x, y, x', y'$,

$(\exists a) :.\ \phi a \supset q \;.\supset:$

$\qquad q \supset \sim s \;.\supset.\; \phi x \supset \sim s \;.\; \phi y \supset \sim s \;.\; \phi x' \supset \sim s \;.\; \phi y' \supset \sim s$

Call this proposition $(\exists a).\ fa$.

We have $\phi a \;.\sim q \;.\supset. fa$ $\cancel{:\ \phi a \;.\; q \;.\supset:}$ Also

$\phi a \;.\; q \;.\supset$

$:.\ fa \;.\equiv: \sim s \;.\supset.\; \phi x \supset \sim s \;.\; \phi y \supset \sim s \;.\; \phi x' \supset \sim s \;.\; \phi y' \supset \sim s :.$

$\supset :.\ fa$

Hence $\phi a \;.\supset.\; fa$. Hence $\phi x \;.\supset.\; fx \;.\supset. (\exists a).\ fa$

(using $*8{\cdot}1{\cdot}24$ )

Hence by $*8{\cdot}28 \quad \phi x \vee \phi y \vee \phi z \vee \phi w \;.\supset.\; (\exists a).\ fa$

[21] Revised to become *13* of Appendix A.

*Also* $\sim\phi x\ .\sim\phi y\ .\sim\phi z\ .\sim\phi w\ .\supset .\ fa\ .\ \cancel{\text{etc}} \supset .\ (\exists a).\ fa$
(using $*8\cdot1\cdot24$ ) Hence by $*8\cdot28$
$\vdash :.\ \phi x \vee \phi y \vee \phi z \vee \phi w\ .\vee .\sim\phi x\ .\sim\phi y\ .\sim\phi z\ .\sim\phi w : \supset$
$.\ (\exists a).\ fa$
Hence by $*8\cdot12$, $\vdash .\ (\exists a).\ fa$, which was to be proved.

*28*[22]

$*8\cdot322$ $\vdash :.\ p\ .\supset .\ (x).\ \psi x\ : \supset : s \mid \{(x).\psi x\}\ .\supset .\ p|s$

Proceed as before, with $q\ .=.\ (x).\psi x$

$\sim q\ .=.\ (\exists\, b, c).\ \psi b|\psi c$
$p \mid \sim q\ .=.\ (b, c).\ p \mid (\psi b|\psi c)$
$s|q\ .=.\ (\exists y).\ s \mid \psi y$
$(s|q)|\sim(p|s)\ .=.\ (y).\ (s|\psi y) \mid \sim(p|s)$

$\sim\{(s|q)|\sim(p|s)\}.=.\ (\exists y, y').\{(s|\psi y)|\sim(p|s)\}|\{(s|\psi y')|\sim(p|s)\}$

$(p \mid\sim q) \mid \sim\{(s|q)|\sim(p|s)\}.$
$=: (\exists b, c) : (y, y').\ \{p \mid (\psi b|\psi c)\}|$
$[\{(s|\psi y)|\sim(p|s)\} \mid \{(s|\psi y')|\sim(p|s)\}]$
The matrix here is equivalent to

$p\ .\supset .\ \psi b\ .\ \psi c : \supset : s|\psi y\ .\supset .\ p|s : s|\psi y'\ .\supset .\ p|s$

i.e. to
$p.\supset .\ \psi b.\ \psi c : \supset :\ \psi y \supset \sim s.\supset .\ p \supset \sim s : \psi y' \supset \sim s.\supset .\ p \supset \sim s$
This is verified by putting $b = y\ .\ c = y'$. Hence prop.

$\cdot323$ $\vdash :.\ p\ .\supset .\ (\exists x).\ \psi x : \supset : s \mid \{(\exists x).\ \psi x\}\ .\supset .\ p|s$

We arrive at the same matrix as in $*8\cdot322$, but with
"$(b, c) : (\exists y, y')$" before it. Putting $y = b,\ \ y' = c$, the matrix is satisfied, as in $*8\cdot322$.

*29*[23]

$*8\cdot324$ $\vdash :.\ p \supset q\ .\supset : \{(x).\ \chi x\} \mid q\ .\supset .\ p \mid \{(x).\ \chi x\}$

As before, with $s\ .=.\ (x).\ \chi x$, we have
$s|q\ .=.\ (\exists x).\ \chi x|q$
$p|s\ .=.\ (\exists y).\ p|\chi y$
$\sim(p|s).=.(y, z).\ (p|\chi y) \mid (p|\chi z)$
$(s|q) \mid \sim(p|s)\ .=: (x) : (\exists\ y, z).(\chi x|q) \mid \{(p|\chi y)|(p|\chi z)\}$
$\sim\{(s|q) \mid \sim(p|s)\}\ .=: (\exists x, x') : (y, z, y', z').$
$[(\chi x|q) \mid \{(p|\chi y)|(p|\chi z)\}] \mid [(\chi x'|q) \mid \{(p|\chi y')|(p|\chi z')\}]$

[22] Revised to become *14* of Appendix A. [23] Revised as *15* of Appendix A.

The required matrix is obtained by prefixing $p \mid \sim q$ to the matrix in the last line, and the prefix is "$(x, x'):(\exists\, y, z, y', z')$".
The matrix is equivalent to
$p \supset q \,.\supset :.\, q \supset \sim\chi x \,.\supset :\, p \,.\supset .\sim\chi y \,.\sim\chi z$
$:.\, q \supset \sim\chi x' . \supset :\, p \,.\supset . \sim\chi y' . \sim\chi z'$
This is satisfied by putting $y = z = x \,.\, y' = z' = x'$. Hence prop.

$\cdot 325 \quad \vdash :.\, p \supset q \,.\supset : \{(\exists x).\, \chi x\}|q \,.\supset .\, p|\{(\exists x).\, \chi x\}$

Same matrix as in $*8\cdot 324$, but with prefix "$(\exists\, x, x'):(y, z, y', z')$" Call the matrix $M(x, x')$. We have
$\sim\chi y. \sim\chi z. \sim\chi y'. \sim\chi z' \,.\supset .\, M(x, x') \,.\supset .\, (\exists\, x, x').\, M(x, x')$ (1)
But $\chi x \,.\, \chi x' \,.\supset .\, M(x, x')$. Hence
$\chi y \,.\supset .\, M(y, y) \,.\supset .\, (\exists\, x, x').\, M(x, x')$ (2)
Similarly $\chi z$, $\chi y'$, $\chi z'$ all imply $(\exists\, x, x').\, M(x, x')$ (3)
$\vdash$ .(1).(2).(3). $*\, 8\cdot 28 \,.\supset \vdash$ . Prop
This ends the cases in which only one of $p, q, r$ is of the first order instead of being elementary.

*29a*[24]

~~Put~~ $\cancel{\chi x|q = \phi x \,.\, p|\chi x = \psi x}$
~~Then matrix is~~
$\cancel{p \supset q . \supset : \phi x \supset \psi x\ \psi z \,.\, \phi x' \supset \psi y'\ \psi z'}$

$*8\cdot 325$ Matrix is

$$p \supset q \,.\supset :\ q \supset \sim\chi x \,.\supset .\, p \supset \sim\chi y\ \sim\chi z :\ q \supset \sim\chi x' \,.\supset .\, p \supset \sim\chi y'\ \sim\chi z'$$

and prefix is $(\exists x, x')::(y, z, y', z') :.$
Matrix is true if $\sim(p \supset q)$ and if $\sim p$, i.e. if $\sim p \vee \sim q$.
If $p \,.\, q$, matrix $\equiv\ \sim\chi x \supset \sim\chi y \sim\chi z \,.\sim\chi x' \supset \sim\chi y' \sim\chi z'$
This, with required prefix, follows from $*8\cdot 11\cdot 13$. Hence Prop by $*8\cdot 28$

*31*[25]

$*8\cdot 333 \quad \vdash :.\, (\exists x).\, \phi x \,.\supset .\, (\exists x).\, \psi x : \supset : s|\{(\exists x).\, \psi x\} \,.\supset .\, \{(\exists x).\, \phi x\}|s$

Put $f(x, y, z) \,.=.\, (s|\psi x) \mid \{(\phi y|s)|(\phi z|s)\}$. Then matrix is
$\{\phi a|(\psi b|\psi c)\} \mid \{f(x, y, z)|f(x', y', z')\}$
and prefix is $(\exists\, a, x, x'):(b, c, y, z, y', z')$
By $*8\cdot 13$, this prefix may be replaced by
$(\exists x, x')::(y, z, y', z') :.\, (\exists a):(b, c)$. Matrix is, putting $\tau = \sim s$,
$\phi a \,.\supset .\, \psi b \,.\, \psi c : \supset :.$

[24] This is revised as the remainder of *15* from Appendix A. The next folio, *30*, is renumbered as *16* of Appendix A.
[25] This folio is replaced by the following, also numbered *31*.

$\psi x \supset \tau \,.\supset.\, \phi y \supset \tau \,.\, \phi z \supset \tau : \psi x' \supset \tau \,.\supset.\, \phi y' \supset \tau \,.\, \phi z' \supset \tau$

We have to prove

$(\exists x) :: (y, z) :. (\exists a) : (b, c).$

$$\{\phi a \,.\supset.\, \psi b \,.\, \psi c : \supset : \psi x \supset \tau. \supset. \phi y \supset \tau. \phi z \supset \tau\}$$

If $\tau$, this is true. If $\sim\tau$, matrix $\equiv$

$\phi a \,.\supset.\, \psi b \,.\, \psi c : \supset : \sim\psi x \,.\supset.\, \sim\phi y \,.\, \sim\phi z$

i.e. $\phi a \,.\supset.\, \psi b \,.\, \psi c : \supset : \phi y \vee \phi z \,.\supset.\, \psi x$

If $\psi b$, this is true. If $\psi c$, it is true. If not, it is true if $\phi a$.

Since $a$ is arbitrary, it follows it is true if $\phi y \vee \phi z$. If not, it is again true. Hence prop.

*31*[26]

$*8{\cdot}333 \quad \vdash :. (\exists x).\, \phi x \,.\supset.\, (\exists x).\, \psi x : \supset : s|\{(\exists x).\, \psi x\} \,.\supset.\, \{(\exists x).\, \phi x\}|s$

Prefix to matrix is $(\exists\, a, x, x') : (b, c, y, z, y', z')$. Call the matrix $f(a, x, x')$. Then

$\psi b \,.\supset.\, f(a, b, b) : \psi c \,.\supset.\, f(a, c, c) \qquad (1)$

$\sim\psi b \,.\, \sim\psi c \,.\, \phi y \,.\supset.\, f(y, b, c)$

Similarly for $\phi z, \phi y', \phi z'$. Hence by $*8{\cdot}28$

$\sim\psi b \,.\, \sim\psi c \,.\, \phi y \vee \phi y' \vee \phi z \vee \phi z' \,.\supset.\, (\exists a, x, x') \,.\, f(a, x, x') \quad (2)$

Also $\sim\phi y \,.\, \sim\phi y' . \sim\phi z \,.\, \sim\phi z' \,.\supset.\, f(a, x, x').$

$\supset. (\exists a, x, x').\, f(a, x, x') \quad (3)$

$\vdash .(1).(2).(3). *8{\cdot}28 \,.\supset \vdash .$ Prop.

This ends the cases in which $p$ and $q$ but not $s$ contain apparent variables.

*33a*[27]

$*8{\cdot}342 \quad \vdash :. (\exists x).\, \phi x \,.\supset.\, q : \supset : \{(x).\, \chi x\}|q \,.\supset.\, \{(\exists x).\, \phi x\}|\{(x).\, \chi x\}$

Matrix is

$\phi a \supset q \,.\supset:.\, q \supset \sim\chi x \,.\supset.\, \phi y \supset \sim\chi z \,.\, \phi u \supset\sim \chi v :$

$q \supset \sim\chi x'. \supset. \phi y' \supset \sim\chi z' \,.\, \phi u' \supset \sim\chi v'$

and prefix is $(\exists a, z, v, z', v') : (x, y, u, x', y', u')$

or (by $*8{\cdot}13$) $\quad (\exists\, z, v, z', v') :. (x, y, u, x', y', u') : (\exists a).$

Put $\theta = \sim\chi$. We have to prove

$$\phi a \supset q \,.\supset:\, q \supset \theta x \,.\supset.\, \phi y \supset \theta z \,.\, \phi u \supset \theta u$$

with prefix $(\exists\, z, v) :. (x, y, u) : (\exists a)$

This holds if $\phi a \,.\, \sim q$. Since $a$ is arbitrary, it holds if

$\phi y \vee \phi u \,.\, \sim q$

It also holds if $\sim\phi y \,.\, \sim\phi u$. Therefore it holds if $\sim q$. Assume $q$.

Then it $\equiv$ $\cancel{\phi a \,.\supset.}$ $\theta x \,.\supset.\, \phi y \supset \theta z \,.\, \phi u \supset \theta v$

~~If $\theta c$, this is true, taking $z = v = x$. Hence prop.~~

~~If $\sim (\exists c).\, \theta c$, $\theta x$ is false and prop is true.~~

[26] Though deleted with a single line, this folio, with its proof of $*8{\cdot}333$, is closer to *17* of Appendix A than the preceding folio. *32* and *33* are renumbered as *18* and *19*, respectively.

[27] This proof of $*8{\cdot}342$ and the next are completely rewritten as *20* of Appendix A.

This requires $\theta x \,.\supset.\, \phi y \supset \theta z$, with prefix $(\exists z):(x, y)$.
If $(\exists c).\, \theta c, \quad z = c$ will do. If $(c).\sim\theta c$, any value of $z$ will do.
Hence prop follows from
$\vdash :: (x).\, \phi x \,.\supset.\, p :\supset:.\, (\exists x).\sim\phi x \,.\supset.\, p :\supset.\, p$
which is easily proved. [See p. 33b]

*33b*[28]

*To prove*

$(x).\, \phi x \,.\supset.\, p :\supset:.\, (\exists x).\sim\phi x \,.\supset.\, p :\supset.\, p$
i.e. $(\exists x).\, \phi x \supset p \,.\supset: (x).\sim\phi x \supset p \,.\supset.\, p$
i.e. $(x):.\, \phi x \supset p \,.\supset: (\exists y):\sim\phi y \supset p \,.\supset.\, p$
i.e. $(x)::(\exists y):.\, \phi x \supset p \,.\supset:\sim\phi y \supset p \,.\supset.\, p$
which follows putting $y = x$.

*To prove*

$(x).\, \phi x \,.\supset.\, (a).\, \psi a :\supset:.\, (\exists x).\sim\phi x \,.\supset.\, (a).\, \psi a :\supset.\, (a).\, \psi a$
This is $(\exists x):(a).\, \phi x \supset \psi a :\supset:.\, (x, b):\sim\phi x \supset \psi b \,.\supset.\, (c).\, \psi c$
i.e. $(\exists x):(a).\, \phi x \supset \psi a :\supset:.(c):.(\exists\, y, b):\sim\phi y \supset \psi b \,.\supset.\, \psi c$
i.e. $(x, c)::(\exists\, a, y, b):.\, \phi x \supset \psi a \,.\supset:\sim\phi y \supset \psi b \,.\supset.\, \psi c$
which is obvious $(a = b = c\,,\ \ y = x)$

~~But suppose we had~~
$\cancel{(\exists\, a, y, b)::(x, c):.\, \phi x \supset \psi a \,.\supset:\sim\phi y \supset \psi b \,.\supset.\, \psi c}$

*49*[29]

Second-order functions $f$ have a property which first-order functions do not have, namely that, when a value is assigned to $f$, the result may be a logical matrix. Take e.g. $f!(\phi!\hat{z}, x)$. A value of this is $\phi!x$, which is a logical matrix. Another value is $(\phi!x)|(\phi!x)$, which is again a logical matrix, and so on. Similarly $\phi!x \supset \phi!y$, which is a logical matrix, is a value of $f!(\phi!\hat{z}, x, y)$. This is a new property of second order functions. A matrix containing only first-order functions and individuals does not remain a logical matrix when values are assigned to the functions. This, however, is partly arbitrary. If we were to use in logic, as we might do, the symbols

$$R_1(x),\ R_2(x, y),\ R_3(x, y, z), \text{etc.}$$

[28] The concluding nine folios, *34* to *42*, are renumbered as *21* to *29* respectively, and constitute the remainder of the Appendix A manuscript. *43* to *48* are renumbered as *31* to *36* of the Introduction.

[29] Revised as *49* and *49a*, renumbered as *37* and *38* of the Introduction. The revision introduces the examples of "Socrates", "mortality", and "being Greek" which figure in Appendix C. The following folio, *50*, which begins section "V. Functions other than matrices", is renumbered as *39* in the Introduction, where it is still used to begin the new section.

where $R_1$ represents a variable predicate, $R_2$ a variable dyadic relation, $R_3$ a variable triadic relation, and so on, we should have logical matrices not containing variable functions. This, however, involves the introduction of variable predicates and relations, which is rendered unnecessary by the use of variable functions.

By assigning $\phi!\hat{z}$ and $x$ in $f!(\phi!\hat{z}, x)$, while leaving $f$ variable, we obtain an assemblage of ⌊elementary⌋propositions not to be obtained by means of variables representing individuals and first-order functions. This is why the new variable $f$ is useful.

We can proceed in like manner to matrices

$$F!\{f!(\hat{\phi}!\hat{z}, \hat{x}), g!(\hat{\phi}!\hat{z}, \hat{x}), \ldots \psi!\hat{z}, \chi!\hat{z}, \ldots x, y, \ldots\}$$

and so on indefinitely. These merely represent new ways of grouping elementary propositions, leading to new kinds of generality.

*52*[30]

But when functions occur as apparent variables, while the argument remains an individual, a new situation arises. The cases concerned are such as

$$(\phi).\ f!(\phi!\hat{z}, x)\ ,\ (\exists\phi).\ f!(\phi!\hat{z}, x).$$

Suppose we introduce, instead, the functions $\phi_1$, so as to obtain

$$(\phi_1).\ f!(\phi_1\hat{z}, x)\ ,\ (\exists\phi_1).\ f!(\phi_1\hat{z}, x)$$

The totality of $\phi$s referred to now is different from what it was before, and the new functions are not the same as those in which matrices were the apparent variables. However much we may enlarge the meaning of $\phi$, a function of $x$ in which $\phi$ occurs as apparent variable will be correspondingly enlarged, so that, however $\phi$ may be defined,

$$(\phi).\ f!(\phi\hat{z}, x)\ \text{ and }\ (\exists\phi).\ f!(\phi\hat{z}, x)$$

can never be values for $\phi x$. To attempt to make them so is like attempting to catch one's own shadow. Let us illustrate by the simplest possible example. Put

$$fx\ .=.\ (\exists\phi).\phi!x$$

[30] The second digit was overwritten on the original '*51*'. The order of composition and numbering of the next pages is not completely certain. It is likely that that the next to be written was *40* of the Introduction, originally numbered *52*, then changed to *51* when this leaf was first switched with it and so renumbered as *52*, finally to be dropped from the manuscript. The next folio composed was the second *52* of HPF, which follows here.

Then $fx$ is not of the form $\phi!x$. But suppose that we adopt a new kind of variable $\phi_2\hat{x}$, which is to include

$$(\phi_1).f!(\phi_1\hat{z},\hat{x}), (\phi_1,\psi_1).f!(\phi_1\hat{z},\psi_1\hat{z},\hat{x}), \ldots (\exists\phi_1).f!(\phi_1\hat{z},\hat{x}), \ldots$$

and put

$$f'x \,.=.\, (\exists\phi_2).\,\phi_2 x$$

Then $f'x$ is ~~still~~⌊again⌋ not of the form $\phi_2 x$. If we went on similarly to $\phi_3$, and put

$$f''x \,.=.\, (\exists\phi_3).\,\phi_3 x$$

$f''x$ would not be of the form $\phi_3 x$, and so on. The attempt to obtain one variable which embraces all possible functions of individuals among its values is therefore hopeless, and must be abandoned.

*52*[31]

It is not necessary to introduce new variables in order to be able to express what is expressed above by means of $\phi_1$, $\phi_2$, $\phi_3$, etc. Everything that they can express can also be expressed in terms of matrices. There is need, however, ⌊*prima facie*,⌋ of an infinite disjunction or conjunction, unless we introduce such functions as $\phi_1$. For example,

$$(\exists\phi_1).\phi_1 x.\equiv:(\exists\phi).\phi!x:\vee:(\exists\phi,y).\phi!(x,y):\vee:(\exists\phi):(y).\phi!(x,y):\vee\text{ etc.}$$

$$(\phi_1).\phi_1 x.\equiv:(\phi).\phi!x:(\underline{\phi_1},y).\phi!(x,y):(\phi):(\exists y).\phi!(x,y):\quad\text{etc.}$$

To avoid this, ~~it is worth while to~~⌊we will, for the moment,⌋ introduce the functions $\phi_1\hat{x}$. Similarly ~~it is worth while to~~⌊we will⌋ introduce $f_1(\phi_1\hat{z},x)$, where any number of individuals and of functions $\psi_1, \chi_1, \ldots$ may appear as ~~independent~~⌊apparent⌋variables. The essence of the matter is that a variable may travel through any well-defined totality of values, provided these values are all such that any one can replace any other significantly in any context. In virtue of the definitions at the beginning of $*8$, propositions containing any number of apparent variables can replace each other and elementary propositions in stroke-functions. What is necessary for significance is that every complete asserted proposition should be derived from a matrix by generalization, and that, in the matrix, the

[31] This page was followed by by the second *52* and *53*. The former was removed from the manuscript and so the latter was renumbered *52*, then finally was renumbered as *41* in the Introduction. *53* was written next, ending as *42* in the Introduction. *53a* and *53b* were then added, becoming *43* and *44* in the Introduction. The next two folios, *54* and *55*, became *45* and *46* in the Introduction.

substitution of constant values for the variables should always result, ultimately, in a stroke-function of atomic propositions. We say "ultimately" because, when such variables as $\phi_1\hat{x}$ are admitted, the substitution of a value for $\phi_1$ may yield a proposition still containing apparent variables, and in this proposition the apparent variables must be replaced by constants before we arrive at a stroke-function of atomic propositions. We may introduce variables requiring several such stages, but the end must always be the same: a stroke-function of atomic propositions.

*56*[32]

Put next $\phi_1 x \,.=.\, (\exists y).\, \phi!(x, y)$.

Then $(\phi_1 a) \mid (\phi_1 x | \phi_1 b) \,.=:\, (y) : (\exists z, w).\, \phi!(a, y) \mid \{\phi!(x, z) | \phi!(b, w)\}$

In this case we merely have to put $z = w = y$ and the result follows. The method will be the same in any other case. Hence generally

$$(\phi_1).\ f!(\phi_1\hat{z}, x) \,.\equiv.\, (\phi).\ f!(\phi!\hat{z}, x).$$

Although the above arguments do not amount to formal proofs, they suffice to make it clear that, in fact, such variables as $\phi_1$, $f_1$, do not introduce any substantially new propositions. We may therefore dispense with them in such general logical propositions as we have been considering.

Nevertheless, they have still a certain importance in connection with classes. The above argument does not prove

$$(\phi) :. (\exists\psi) : \psi!x \,.\equiv_x.\, (y).\, \phi!(x, y)$$

or

$$(\phi) :. (\exists\psi) : \psi!x \,.\equiv_x.\, (\exists y).\, \phi!(x, y)$$

or, more generally,

$$(\phi) :. (\exists\psi) : \psi!x \,.\equiv_x.\, \phi_1 x$$

Thus when we come to deal with classes it is necessary to take account of such functions as $\phi_1\hat{x}$, unless we have an axiom (such as the axiom of reducibility) to

[32] These next pages; *56*, *57*, *58r*, and *58v*, through *59*, were thoroughly revised as *47–50* of the Introduction. The first version presents the argument that any theorem true of predicated functions will be true of higher order functions, because all functions ultimately "appear through their values" as truth-functions of atomic propositions. It appears that at the time of this revision Russell identified the problem of the occurrences of higher order functions in hypotheses, as in instances of induction, or in the use of identity. The next section of the Introduction, Section VI, "Classes" begins with folio *51* and runs to *60* with newly written material that is not renumbered. The remainder of the HPF manuscript, from *60* on, is used in the Introduction manuscript in a separate section VII, "Mathematical induction" at *61*, Introduction and in Appendix B.

enable us to avoid them. Suppose, for example, that we have a matrix $\phi!(x, y)$ containing two variables. Let $R$ be the relation of $x$ and $y$ defined by $\phi!(x, y)$. Then

$$x \in \text{D}`R \ . \equiv . \ (\exists y).\ \phi!(x, y).$$

We do not know, apart from an axiom, that D‘$R$ can be defined by means of a function $\psi!\hat{x}$. Nevertheless, we do know, in virtue of $*8$,

*57*

that $(\exists y).\phi!(x, y)$ may be substituted for $p$ in any ~~truth~~ function of $p$ generated by means of the stroke; and since stroke-functions are truth-functions, $x \in \text{D}`R$ may be similarly substituted. It follows, from what has just been proved, that, give any general proposition

$$(\phi).\ f!(\phi!\hat{z}, x) \quad \cancel{\text{or} \quad (\exists\phi).\ f!(\phi!\hat{z}, x)}$$

we may infer the proposition

$$f!(\hat{z} \in \text{D}`R, x).$$

This inference is all that is ⌊usually⌋ needed in practice. Therefore, so far at least as concerns the functions $\phi_1$, the axiom of reducibility is not needed.

Let us next consider functions of the form $\phi_2 x$, where

$\phi_2 x \ . = . \ (\phi).f!(\phi!\hat{z}, x)$ or $\phi_2 x \ . = . \ (\exists\phi).f!(\phi!\hat{z}, x)$

We want to prove

$$(\phi).\ f!(\phi!\hat{z}, x) \ . \supset . \ f!(\phi_2\hat{z}, x).$$

Let us suppose, for the sake of illustration, that

$$f!(\phi!\hat{z}, x) \ . = . \ \phi!a|(\phi!x|\phi!x)$$

so that $\qquad (\phi).f!(\phi!\hat{z}, x) \ . = . \ a = x$ according to $*13\cdot1$.

We want to prove that

$$(\phi).(\phi!a)|(\phi!x|\phi!x) \ . \supset . \ (\phi_2 a)|(\phi_2 x|\phi_2 x)$$

i.e. $\qquad a = x \ . \supset . \ \phi_2 a \supset \phi_2 x,$

where $\qquad \phi_2 x \ . = . \ (\phi).\ F!(\phi!\hat{z}, x)$

or $\qquad \phi_2 x \ . = . \ (\exists\phi).\ F!(\phi!\hat{z}, x)$

*58*

We have to prove, taking the first value of $\phi_2$,

$$(\phi).\ F!(\phi!\hat{z}, a) \ . \supset . \ (\phi).\ F!(\phi!\hat{z}, x) \qquad (1)$$

given

$$(\phi) : \phi!a \,.\supset.\, \phi!x \qquad (2)$$

Now $F!(\phi!\hat{z}, x)$ must be derived from a stroke-function

$$G(p, q, r, \ldots)$$

by substituting for some of $p, q, r, \ldots$ the values

$$\phi!x, \phi!b, \phi!c, \ldots$$

where $b, c, \ldots$ are constants. As soon as $\phi$ is assigned, this is of the form $\psi!x$. Hence by (2), $F!(\phi!\hat{z}, a) \,.\supset_\phi.\, F!(\phi!\hat{z}, x)$. Hence

$$(\phi).\ F!(\phi!\hat{z}, a) \,.\supset.\, (\phi).\ F!(\phi!\hat{z}, x)$$

which was to be proved. Similarly, taking the second value of $\phi_2$,

$$(\exists\phi).\ F!(\phi!\hat{z}, a) \,.\supset.\, (\exists\phi).\ F!(\phi!\hat{z}, x)$$

Thus generally

$$(\phi).\ \phi!a \supset \phi!x \,.\supset.\, (\phi_2).\ \phi_2 a \supset \phi_2 x$$

without the need of any axiom of reducibility.

*58*[33]

~~We find~~ $\cancel{f!(\phi_2\hat{z}, x) \,.\equiv:\, \phi_2 a \supset \phi_2 x :}$
$\cancel{\equiv : (\exists\phi) : (\psi, \chi).(\phi!a \supset \phi!a) | \{(\psi!a \supset \psi!x) | (\chi!a \supset \chi!x)\}}$
$\cancel{\equiv : (\psi, \chi) : \psi!a \supset \psi!x \,.\, \chi!0 \supset \chi!x}$
~~which follows from~~ $\cancel{(\phi).f!(\phi!\hat{z}, x)}$.
~~Hence without assuming the axiom of reducibility,~~
$\cancel{a = x \,.\supset.\, \phi_2 a \supset \phi_2 x}$
~~in this particular case.~~

*59*[34]

The above dealt with a special case. In general, given

$$(\phi).\ f!(\phi!\hat{z}, x)$$

we want to prove

$$f!(\phi_2\hat{z}, x)$$

[33] The verso of the preceding leaf.
[34] The topic of induction is mentioned at page *50* of the Introduction manuscript. *60* of the Introduction ends with a mention of Cantor's theorem, thus *59* of HPF was replaced by the whole section "VI. Classes" of *51–60* of the Introduction.

where $\phi_2 x \ . = . \ (\phi). \ F!(\phi!\hat{z}, x)$ or $\phi_2 x \ . = . \ (\exists\phi). \ F!(\phi!\hat{z}, x)$.

As before, $f!(\phi!\hat{z}, x)$ will be derived from a stroke-function

$$g(p, q, r, \ldots)$$

by substituting $\phi!x, \phi!a, \phi!b, \ldots$ for some of $p, q, r, \ldots$ ($a$, $b$, … being constants). Similarly $f!(\phi_2\hat{z}, x)$ is obtained by substituting $\phi_2 x, \phi_2 a, \phi_2 b, \ldots$ in the same places. Take the case where $\phi_2 x \ . = .(\phi).F!(\phi!\hat{z}, x)$. We have to apply the definitions at the beginning of $*8$. We shall obtain a matrix which is the value of $g(p, q, r, \ldots)$ obtained by substituting $\phi!x$, $\psi!a$, $\chi!b, \ldots$ where before we substituted $\phi!x$, $\phi!a$, $\phi!b, \ldots$ and the prefix will be of the form "$(\phi, \psi, \ldots) : (\exists\chi, \theta, \ldots)$". The method of proof, in each particular case, will be analogous to the methods employed in $*8$. It is not, however, always possible, as we shall see.[35]

Difficulties arise in $*73$ as to the definition of similarity, which contains a relation as apparent variable. So far as appears, two classes may have a correlative of the second order without having one of the first, or of the third but not the second order, and so on. Thus the notion of similarity becomes blurred. Nevertheless the propositions of $*73$ remain valid, even those proving the Schröder–Bernstein theorem, though such statements as "$p$‘$\kappa \in \kappa$" require explanation. ⌊Cantor's proof of $2^\nu > \nu$ fails, because the sub-class omitted from the correlation is of a higher type than those included. (See proof of $*102{\cdot}71$.)⌋

In other cases, ~~however~~⌊also⌋, the proofs of propositions as given become actually fallacious. The most important of these is mathematical induction. This may be made evident as follows.

*60*[36]

~~It cannot be assumed, however, that a proof is always possible. Take, as one of the most important illustrations, the case of mathematical induction~~. The hypothesis "$x R_* y$", upon which mathematical induction turns, becomes (omitting "$x \in C$‘$R$", which is unnecessary for our purposes)

$$R(z, w). \supset_{z,w} . \ \phi!z \supset \phi!w : \supset_\phi . \ \phi!x \supset \phi!y \qquad \text{(A)}$$

i.e., "$y$ has every⌊elementary⌋hereditary property possessed by $x$". If we now attempt to deduce

$$R(z, w). \supset_{z,w} . \ \phi_2 z \supset \phi_2 w : \supset . \ \phi_2 x \supset \phi_2 y,$$

where $\phi_2 z \ . = .(\phi).f!(\phi!\hat{z}, z)$ or $\phi_2 z \ . = .(\exists\phi).f!(\phi!\hat{z}, z)$, we shall find that the deduction is impossible. If, however, we take "$\phi!z \supset \phi!w$" and "$\phi!x \supset \phi!y$" as our unit propositions, the result is different. From the hypothesis ($A$) we can

[35] This sentence to the end of the page is a late addition in blue ink.
[36] This is revised as *61* of the Introduction.

deduce

$$R(z, w) . \supset_{z,w} : (\phi) : f!(\phi!\hat{z}, z) . \supset . f!(\phi!\hat{z}, w) :. \supset : (\phi) : f!(\phi!\hat{z}, x) . \supset . f!(\phi!\hat{z}, y). \tag{B}$$

And, of course, ~~if $\phi_2 z . =$~~ $(\phi) : f!(\phi!\hat{z}, x) . \supset . f!(\phi!\hat{z}, y) : \supset . \phi_2 x \supset \phi_2 y$,

whichever of the two meanings is adopted for $\phi_2 x$. The point is that, in (B), we have

$$(\phi) : f!(\phi!\hat{z}, z) . \supset . f(\phi!\hat{z}, w),$$

instead of

$$(\phi) : f!(\phi!\hat{z}, z) . \supset . (\phi) . f(\phi!\hat{z}, w)$$

which ~~appears~~ ⌊does⌋ not ~~to~~ yield the required result.

*61*[37]

The effect of this is that, when we wish to use induction in the form

$$x R_* y . \supset . \phi_2 x \supset \phi_2 y \tag{C}$$

$\phi_2$ must not merely be itself hereditary, but must be composed of hereditary functions. For example, if

$$\phi_2 z . = . (\phi) . f!(\phi!\hat{z}, z).$$

we shall have to have:

$$(\phi) :. z R w . \supset : f!(\phi!\hat{z}, z) . \supset . f!(\phi!\hat{z}, w).$$

If this fails, the implication (C) is not necessarily true. It follows that some of the proofs in Part II Section E and Part III Section C are invalid. Fortunately, all the proofs, in Part II Section E, of propositions subsequently used, can be amended so as to become valid.

In $*90$, the first invalid induction is in $*90\cdot31$.[38] To remedy this, $*90 \cdot 31 - \cdot341$ must be put after $*90\cdot36$, and the proof of $*90\cdot351$ must be slightly altered, because at present it uses $*90\cdot32$. It will be found that $*90\cdot15\cdot16$ can replace $*90\cdot32$ in this proof. Hence $*90\cdot351$ and $*90\cdot36$.

It should be observed that, *prima facie*, an inductive property must not have the form

$$x R_* y . \phi y$$

[37] This is revised as *62* of the Introduction.

[38] $*90\cdot31 . \vdash . R_* = I \restriction C`R \cup R_* | R$. In Appendix B, the first correction is to $*90\cdot17$.

or $\qquad S \epsilon \text{ Potid}`R \,.\, \phi S$

or $\qquad \alpha \epsilon \text{ NC induct} \,.\, \phi\alpha.$

This is inconvenient, because often such properties are inductive when $\phi$ alone is not, i.e. we may have $xR_*y \,.\, \phi y \,.\, yRz \,.\supset .\, xR_*z \,.\, \phi z$ but not $\phi y \,.\, yRz \,.\supset .\, \phi z$, and similarly in other cases.

*61a*[39]

Returning now to $*90{\cdot}31$, the proof of

$$I \upharpoonright \text{C}`R \;\dot{\cup}\; R_*|R \;\dot{\subset}\; R_*$$

remains as at present. The converse requires (by $*90{\cdot}36$)

$$xR_*z \,.\supset :.\, x = z : \vee : \breve{R}``\mu \subset \mu \,.\, \overleftarrow{R}`\kappa \epsilon \mu \,.\supset_\mu .\, z \epsilon \mu$$

Put $\;\phi(x, z, \mu) \,.=:.\, x = z : \vee : \breve{R}``\mu \subset \mu \,.\, \overleftarrow{R}`x \subset \mu \,.\supset .\, z \epsilon \mu$

Then obviously $\phi(x, x, \mu)$.
Also $\;x = z \,.\supset .\, zRw. \supset .\phi(x, w, \mu)$ because $\overleftarrow{R}`x \subset \mu$.

What is wanted further for induction is

$$\breve{R}``\mu \subset \mu \,.\, \overleftarrow{R}`x \subset \mu \,.\supset .\, z \epsilon \mu : zRw : \supset : \breve{R}``\mu \subset \mu \,.\, \overleftarrow{R}`x \subset \mu \,.\supset .\, w \epsilon \mu$$

This reduces to

$$\breve{R}``\mu \subset \mu.\, \overleftarrow{R}`x \subset \mu \,.\, z \epsilon \mu \,.\, zRw. \supset .\, w\epsilon\mu$$

which is obvious. Hence, collecting cases,

$$\phi(x, x, \mu) : \phi(x, z, \mu).\, zRw. \supset .\, \phi(x, w, \mu)$$

Hence $\quad xR_*z \,.\supset_\mu .\, \phi(x, z, \mu)$
Hence $\quad xR_*z \,.\supset .\, (\mu).\, \phi(x, z, \mu)$
which was to be proved. This finishes the proof of $*90{\cdot}31$.[40]

*61b*[41]

The next difficulty occurs in proving $(R_*)_* = R_*$ ($*90{\cdot}4$).[42] Hitherto we have assumed that $R$ was elementary, but $R_*$ cannot be elementary. Therefore in forming

[39] This is revised as *3–4* of the Appendix B manuscript.
[40] In Appendix B only the weaker $89{\cdot}103.\ \vdash : R \epsilon 1 \to \text{Cls} \,.\supset .\, R* = R_0 \cup R_*|R$ is proved.
[41] The material from this and the next few pages in HPF, i.e., *61b, 61c, 62, 63, 64, 65*, and *66*, does not appear in the second edition, presumably because Russell found it unnecessary for his final proof. Preliminary material on these topics is in ALP, and is indicated leaf by leaf below.
[42] This is not proved in Appendix B, even in a weakened form.

$(R_*)_*$ we have to apply induction to a relation which is not elementary. However, $*90\cdot14\cdot22$ cause the result to follow without any fresh use of induction.

Thus all the propositions of $*90$ are valid.

In $*91$, the first induction requiring correction is in $*91\cdot241$,[43] which states

$$TR_{ts}P\ .\supset.\ (Q|T)R_{ts}(Q|P)$$

By the definitions,

$$TR_{ts}\ .\ P\ \equiv:.\ f!S\ .\supset_S.\ f!(S|R):\supset_f:f!P\ .\supset.\ f!T$$

The induction used in the proof substitutes for $f!S$ the value

$$(Q|S)R_{ts}(Q|P)$$

which is invalid, because $RT_{ts}$ contains $f$ as apparent variable. Put

$$(f).\ F!(f,S)\ .=.\ (Q|S)R_{ts}(Q|P)$$

Then we have to prove

$$(f):\ F!(f,S)\ .\supset.\ F!(f,S|R)$$

not merely

$$(f).\ F!(f,S)\ .\supset.\ (f)F!(f,S|R).$$

Substituting for $F$, we have to prove

$$\begin{aligned}(f)::f!T\ .\supset_T.\ f!(T|R):\supset:f!(Q|P)\ .\supset.\ f!(Q|S):.\\ \supset:.\ f!T\ .\supset_T.f!(T|R):\supset:f!(Q|P)\ .\supset.\ f!(Q|S|R)\end{aligned}$$

which is true. Hence $*91\cdot241$ is true.

*61c*[44]

$*91\cdot242\cdot33\cdot341\cdot342\cdot37$ are proved by inductions which need exactly similar corrections. None of them offer any difficulty.

The next difficulty occurs with $*91\cdot44$, but this difficulty is more serious.[45] In order to overcome it, several lemmas are necessary, since the existing proof cannot be modified so as to become valid.

43 $*91\cdot02\ R_{ts}=(|R)_*$ Df. $|R$ is the relation of $S$ to $S|R$ and $R_{ts}$ is its ancestral, i.e. the relation of $R$, $R^2$, $R^3$, etc. $R_{ts}$ is mentioned in ALP, at *28*, *30r*, *31*, *40v*, *47v*, and *62r*.

44 This is not used in *PM*.

45 $*91\cdot44.\ \vdash:.\ P,Q\in\text{Potid}`R\ .\supset:QR_{ts}\ .\vee.\ PR_{ts}Q.$

A property $f$ is hereditary among powers of $R$ if

$$P \in \text{Potid}`R \,.\, f!P \,.\, \supset_P \,.\, f!(P|R)$$

Sometimes it is convenient to write "$H!f$" for this hypothesis.

It will be seen that hereditary properties belonging to $R$ belong to all powers of $R$, but in general there is a first power of $R$ to which a given hereditary property belongs. That is to say, there will be some power of $R$, say $P$, such that $\sim f!P \,.\, f!(P|R)$. We prove first that, for each $f$, there is at most one such power. We then prove that, if $f$ and $g$ are two hereditary properties, then, throughout the powers of $R$, either $f$ always implies $g$, or $g$ always implies $f$. From this $*91{\cdot}44$ follows easily. The stages in the proof are given below.

*62*[46]

$*91{\cdot}432$

Proof of $\lfloor P, Q \in \text{Potid}`R.\rfloor H!f. \sim f!P.f!(P|R). \sim f!Q.f(Q|R). \supset . \, P = Q$

$$\text{Put } \lambda = \text{Potid}`R \cap \hat{P}\{\sim f!P \,.\, f!(P|R)\}.$$

$$P \in \lambda \,.\, T R_{ts}(P|R) \,.\, \supset \,.\, f!T$$

$[*91{\cdot}212]$ $\quad P \in \lambda \,.\, T R_{ts} P \,.\, \supset : f!T \,.\vee .\, T = P$

$$P \in \lambda \,. \sim f!T \,.\supset : T = P . \vee . \sim (T R_{ts} P)$$

$\therefore \qquad P, Q \in \lambda . \supset : P = Q \,.\vee .\, \sim (Q R_{ts} P). \sim (P R_{ts} Q) \qquad$ (I)

$$P \in \lambda . \supset . \,\cancel{(I \restriction C`R)} P R_{ts}(I \restriction C`R)$$

$\therefore \qquad (\exists M) : P \in \lambda \,.\supset_P .\, P R_{ts} M : M \in \text{Potid}`R \qquad$ (II)

$\text{I} \,.\supset : \lambda \sim\in 0 \cup 1 . \supset (\exists M, P). \, P \in \lambda \,. \sim (P R_{ts} M) \,.\, M \in \text{Potid}`R$

Hence
$P \in \lambda \,.\supset_P .\, P R_{ts} M$ is not a hereditary property of $M$ if $\lambda \sim\in 0 \cup 1$.

Hence
$\lfloor \lambda \in 0 \cup 1. \supset : \rfloor (\exists M) : P\in\lambda . \supset_P P R_{ts} M : (\exists P).P\in\lambda. \sim\{P R_{ts}(M|R)\}$ (III)

By I, $\qquad \lambda \sim\in 0 \cup 1 : P\in\lambda. \supset_P . \, P R_{ts} M : \supset M \sim\in\lambda \qquad$ (IV)

[46] This material was not used. In *PM* $*91{\cdot}44$ is immediately preceded by $*91{\cdot}431 \vdash :. \, P \in \text{Potid}`R \,.\, Q R_{ts} P \,.\supset .\, P \in \text{Potid}`R$. There is no $*91{\cdot}432$.

Hence $\lfloor \lambda \sim\epsilon\, 0 \cup 1 :\rfloor P \epsilon \lambda \,.\supset_P .\, P R_{ts} M : (\exists P).\, P \epsilon \lambda \,.$
$$\sim\{P R_{ts}(M|R)\} : \supset : P\epsilon\lambda .\supset_P . M \neq P \quad \text{(V)}$$

By $*91\cdot212$, $\quad P R_{ts} M \,.\, M \neq P \,.\supset .\, P R_{ts}(M|R)$ (VI)

By V and VI, $\lambda \sim\epsilon\, 0 \cup 1 : P \epsilon \lambda \,.\supset_P .\, P R_{ts} M :$
$$\cancel{(\exists P).} \supset : P \epsilon \lambda \,.\supset_P .\, P R_{ts}(M|R) \quad \text{(VII)}$$

III and VII contradict each other. Hence, using II, $\lambda \epsilon\, 0 \cup 1$. Q.E.D.

*63*[47]

If $f!P$ and $g!P$ are hereditary properties, so is $f!P \vee g!P$.
Hence by above
$$\hat{P}\{\sim f!P .\sim g!P : f!(P|R) \vee g!(P|R)\} \epsilon\ 0 \cup 1 \quad (1)$$
Call this class $\mu$. Then
$$\sim f!P .\sim g!P \,.\, f!(P|R) .\sim g!(P|R) \,.\supset .\, P \epsilon \mu \quad (2)$$
$$\sim f!Q .\sim g!Q .\sim f!(Q|R) \,.\, g!(Q|R) \,.\supset .\, Q \epsilon \mu \quad (3)$$
Also these hyps $\supset\ P \neq Q$. Hence, since $\mu \epsilon\, 0 \cup 1$,

$$(\exists P).\sim f!P \,.\, \sim g!P .\, f!(P|R) .\sim g!(P|R) \,.\supset .$$
$$\sim(\exists Q).\sim f!Q .\sim g!Q .\sim f!(Q|R) \,.\, g!(Q|R) \quad (4)$$

Now assume $(\exists P).\, P \epsilon \text{ Potid}`R .\sim f!P .\sim g!P : (\exists P).\, P \epsilon \text{ Potid}`R .f!P \vee g!P$

It follows that $(\exists P).\, P \epsilon \text{ Potid}`R .\sim f!P .\sim g!P : f!(P|R) .\vee .\, g!(P|R)$ (5)

Assume $P\epsilon \text{ Potid}`R .\sim f!P .\sim g!P : f!(P|R) .\vee .\, g!(P|R) : \supset .\, g!(P|R)$
This gives $\quad \sim f!P .\sim g!P \,.\, f!(P|R) \,.\supset .\, g!(P|R)$
But $\quad \sim f!P \,.\, g!P \,.\, f!(P|R) \,.\supset .\, g!(P|R)$ because $g$ is hereditary.
Hence $\quad \sim f!P \,.\, f!(P|R) \,.\supset .\, g!(P|R)$.
Also $\quad g!P \,.\, f!(P|R) .\supset .\, g!(P|R)$ because $g$ is hereditary
Hence $\quad f!P \supset g!P \,.\supset : f!(P|R) .\supset .\, g!(P|R)$
if $P \epsilon \text{ Potid}`R .\sim f!P .\sim g!P : f!(P|R) .\vee .\, g!(P|R) : \supset_P .\, g!(P|R)$.
Hence on this hypothesis, $\lfloor$by induction,$\rfloor$ $P \epsilon \text{ Potid}`R \,.\supset_P . f!P \supset g!P$
because $\sim f!(I \restriction C`R) .\sim g!(I \restriction C`R)$, so that $f!(I \restriction C`R) \supset g!(I \restriction C`R)$
i.e. $*91\cdot433 \ \vdash :: \ \lfloor H!f.H!g :.\rfloor$

[47] This material was not used.

$$P \,\epsilon\, \text{Potid}`R \,.\, \sim f!P \,.\, \sim g!P : f!(P|R) \,.\vee.\, g!(P|R) : \supset_P .\, g!(P|R)$$
$$:. \supset : P \,\epsilon\, \text{Potid}`R \,.\, \supset_P .\, f!P \supset g!P$$

*64*[48]

If the above hypothesis is false, we have

$(\exists P).\ P \,\epsilon\, \text{Potid}`R \,.\, \sim f!P \,.\, \sim g!P \,.\, f!(P|R) \,.\, \sim g!(P|R)$

Hence by (4), $\sim(\exists Q).\ Q \,\epsilon\, \text{Potid}`R \,.\, \sim f!Q \,.\, \sim g!Q \,.\, \sim f!(Q|R) \,.\, g!(Q|R)$

i.e. $Q \,\epsilon\, \text{Potid}`R \,.\, \sim f!Q \,.\, \sim g!Q \,.\, g!(Q|R) \,.\supset_Q.\, f!(Q|R)$

Hence by $*91{\cdot}433$, interchanging $f$ and $g$, on above hp,

$$Q \,\epsilon\, \text{Potid}`R \,.\supset_Q.\, g!Q \supset f!Q \qquad (6)$$

From (6) and $*91{\cdot}433$, together with (5),

$$(\exists P).\ P \,\epsilon\, \text{Potid}`R \,.\, \sim f!P \,.\, \sim g!P : (\exists P).\ P \,\epsilon\, \text{Potid}`R \,.\, f!P \vee g!P : \supset :$$
$$P \,\epsilon\, \text{Potid}`R \,.\supset_P.\, f!P \,.\supset g!P : \vee : P \,\epsilon\, \text{Potid}`R \,.\supset_P.\, g!P \,.\supset f!P \quad (7)$$

If the hyp of (7) fails, either

$P \,\epsilon\, \text{Potid}`R \,.\supset_P.\, \sim f!P \,.\, \sim g!P \,.\supset_P.\, f!P \supset g!P$

or $P \,\epsilon\, \text{Potid}`R \,.\, \sim f!P \,.\supset_P.\, \sim g!P$

In the latter case, if $f!(I \upharpoonright \text{C}`R)$ is false, $g!P$ is always true, so that $f!P \supset g!P$, which if $f!(I \upharpoonright \text{C}`R)$ is true, $f!P$ is always true, so that $g!P \supset f!P$. Thus finally, collecting cases,

$$*91{\cdot}434 \ \vdash :: P \,\epsilon\, \text{Potid}`R \,.\supset_P.\, f!P \supset f!(P|R) \,.\, g!P \,.\supset g!(P|R) \ :. \supset :.$$
$$P \,\epsilon\, \text{Potid}`R \,.\, f!P \supset_P .\, g!P : \vee : \lfloor P \,\epsilon\, \text{Potid}`R . \rfloor g!P \,.\supset_P.\, f!P$$

This proposition is fundamental.[49]

*65*[50]

$*91{\cdot}44 \ \vdash :.\ P, Q \,\epsilon\, \text{Potid}`R \,.\supset : QR_{ts}P \,.\vee.\, PR_{ts}Q$

*Dem.*

$$Hp. \supset :. \sim (QR_{ts}P). \equiv : (\exists f).\ H!f \,.\, f!P \,.\sim f!Q \qquad (1)$$
$$*91{\cdot}434. \supset :. \lfloor H!f.\ H!g. \rfloor \sim f!Q \,.\, g!Q \,.\supset.\, f!P \supset g!P \ :.$$
$$\supset :. \lfloor H!f.\ H!g. \rfloor f!P \,.\ \sim f!Q \,.\supset.\, g!Q \supset g!P \ :.$$
$$\supset :. (\exists f).\ H!f \,.\, f!P \,.\sim f!Q \,.\supset :\ H!g \,.\supset_g.\, g!Q \supset g!P$$
$$\supset : PR_{ts}Q \qquad (2)$$

(1).(2). $\supset \ \vdash$ . Prop

[48] This material was not used. [49] $*91{\cdot}434$ does not appear in *PM*.

[50] This material was not used. $*91{\cdot}44$ occurs in *PM*, but with a different proof.

After this, the remaining propositions of $*91$ offer no difficulty.
~~From $*91{\cdot}434$ we easily deduce~~

$$\cancel{\vdash :. \breve{R}``\mu \subset \mu . \breve{R}``\nu \subset \nu . \supset : \overleftarrow{R_*}`x \cap \mu \subset \nu . \vee . \overleftarrow{R_*}`x \cap \mu \subset \mu}$$

~~by means of $R_* = \dot{s}`\mathrm{Potid}`R$~~

In $*92$, only $*92{\cdot}33$ uses a fallacious form of induction, and here the proof can be set right easily. No trouble arises in $*93$; $*94$,$*95$ are not used. No further difficulties arise in this section.

$*96{\cdot}13{\cdot}131$ are invalid as they stand, but are not used again in $*96$ or $*97$. The remaining propositions can easily be proved by amended proofs when necessary.

*66*[51]

~~It is not clear whether, in fact~~ By a somewhat complicated method, we can prove induction,⌊not quite⌋ in the form:

$$xR_*y . \supset :: (\phi). f!(\phi!\hat{z}, z) : zRw : \supset_{z,w} : (\phi). f!(\phi!\hat{z}, w) :. \supset : (\phi). f!(\phi!\hat{z}, x) . \supset . (\phi). f!(\phi!\hat{z}, y)$$

i.e. $xR_*y . \supset :. \phi_2 z . zRw. \supset_{z,w} . \phi_2 w : \supset : \phi_2 x. \supset . \phi_2 y$,
but in a form which is just as advantageous.
The following considerations may ~~lead to a proof~~ serve as an introduction.
Assume, as the only interesting case, $xR_*y. \phi_2 x. \sim \phi_2 y$. Put

$$\lambda = \mathrm{Potid}`R \cap \hat{T}(xTz. \supset_z . \phi_2 z) . \mu = \mathrm{Potid}`R \cap \hat{T}\{(\exists z). xTz. \sim \phi_2 z\}$$

Then $I \upharpoonright C`R \in \lambda . R \in \lambda . R^2 \in \lambda$ . etc.
Also ⌊by hypothesis⌋ $(\exists T). T \in \mu . xTy$. Hence $\exists ! \lambda . \exists ! \mu$.
We easily prove $|R``\lambda \subset \lambda . \mu \subset |R``\mu$.
Thus $\lambda$ has no maximum and $\mu$ no minimum.
Therefore $\lambda$ and $\mu$ represent an irrational Dedekind cut in the series of powers of $R$ generated by $R_{ts} \dot{\cap} J$. It ~~ought to be~~ ⌊might be thought⌋ possible to prove that this is impossible, but the obvious methods of proof appear to be invalid. It is possible, however, to obtain a result which is just as serviceable.

*67*[52]

We give the name "first-order function" to one containing only (at most) individuals as apparent variables. We will denote such a function, now, by "$\phi!\hat{x}$". A function of $x$ which involves one or more first-order functions (but no functions of

[51] This is a very early, optimistic, introduction to the issue of induction. The idea is to focus on the point at which induction on a given level, in this case 2, would break down when considering the first level descendants of the relation $R$.
[52] Yet another attempt to introduce the theory of types. See the various pages *52* above. These pages are given successive numbers although each starts with a new topic. The topics are connected, however, and so may be a series of notes in the order in which the ideas were sketched.

higher order) as apparent variables will be ⌊called a "second-order function" and⌋ denoted by $\phi_2 x$. Such a function of ⌊one of⌋ the forms:

$$(\phi).f!(\phi!\hat{z}, x)\ ,\ (\exists\phi).f!(\phi!\hat{z}, x)$$

or analogous forms with several function-variables. If we denote ~~by $\alpha$ a class determined by a first order function~~ by $\kappa$ a class of first-order functions such that the defining function of $\kappa$ contains no classes or functions as apparent variables, the classes corresponding to second-order functions are of the forms $s‘\kappa$ and $p‘\kappa$. A function involving a second-order function as apparent variable (but no function of higher order) will be called a "third-order function" and denoted by "$\phi_3 x$"; and so on. A third-order function can always be represented by taking a matrix

$$F!\{f!(\phi!\hat{z}, \psi!\hat{z}, \ldots x, y, \ldots)\}$$

and turning the $f$ (as well as some of $\phi, \psi, \ldots x, y, \ldots$) into an apparent variable, or by doing the same thing to a matrix containing several $f$s.

*68*[53]

We thus require *prima facie* a series of definitions of different forms of Potid‘$R$. There is the definition already given, which we will now write

Potid$_1$‘$R = \hat{T}\{\phi!(I \upharpoonright \mathrm{C}‘R) : \phi!S \,.\supset_S .\, \phi!(S|R) : \supset_\phi .\, \phi!T\}$ Df

For $(I \upharpoonright \mathrm{C}‘R)$ we will write "$R_0$". Then the other definitions are

Potid$_2$‘$R = \hat{T}\{\phi_2 R_0 : \phi_2 S \,.\supset_S .\, \phi_2(S|R) : \supset_{\phi_2} .\, \phi_2 T\}$ Df

Potid$_3$‘$R = \hat{T}\{\phi_3 R_0 : \phi_3 S \,.\supset_S .\, \phi_3(S|R) : \supset_{\phi_3} .\, \phi_3 T\}$ Df

and so on. It is obvious that the properties we require in Potid‘$R$ if it is to serve the purposes ~~required~~ of the theory of finite and infinite, progressions, etc., will only be demonstrable if we can define a class which belongs to Potid$_n$‘$R$ for any finite $n$. The usual finite powers $R$, $R^2$, $R^3$, ... belong to Potid$_n$‘$R$ for any finite $n$. But the doctrine of types forbids us to define the product of all classes Potid$_n$‘$R$. We shall show, however, that Potid$_{3+n}$‘$R$ = Potid$_3$‘$R$, so that we can take Potid$_3$‘$R$ as Potid‘$R$ and apply mathematical induction for classes of any order.

It is obvious that Potid$_{m+n}$‘$R \subset$ Potid$_m$‘$R$, because every $\phi_m$ is formally equivalent to some $\phi_{m+n}$. For the converse, in case $m \geq 3$, we need to consider the theory of inductive classes.

*69*[54]

$*120{\cdot}24$ states:

$$\rho \in \text{Cls induct.} \equiv :.\ \eta \in \mu \,.\supset_{\eta,y} .\, \eta \cup \iota‘y \in \mu :\ \Lambda \in \mu\ :\supset_\mu .\, \rho \in \mu$$

We may take this as the definition of an inductive class. We shall now have, *prima facie*, different orders of inductive classes according to the type of the apparent

[53] Replaced by *13* of *Appendix B*. [54] Replaced by *6* of Appendix B.

variable $\mu$. We will denote by "$\mu_1$" a class defined by a function $\phi!\hat{x}$; by "$\mu_2$" a class defined by a function $\phi_2\hat{x}$; and so on. We shall do the same for other letters. Thus

$$\text{Cls induct}_1 = \hat{\rho}\{\eta \in \mu_1 \,.\supset_{\eta,y}.\, \eta \cup \iota\text{‘}y \in \mu_1 : \Lambda \in \mu_1 \,:\supset_{\mu_1}.\, \rho \in \mu_1\} \quad \text{Df}$$

$$\text{Cls induct}_2 = \hat{\rho}\{\eta \in \mu_2 \,.\supset_{\eta,y}.\, \eta \cup \iota\text{‘}y \in \mu_2 : \Lambda \in \mu_2 \,:\supset_{\mu_2}.\, \rho \in \mu_2\} \quad \text{Df}$$

and so on. There will be a similar set of definitions of inductive numbers. Thus

$$\text{NC induct}_m = \hat{\alpha}\{(\exists T) \,.\, T \in \text{Potid}_m\text{‘}(+_c 1) \,.\, \alpha T 0\} \quad \text{Df}$$

We shall have

$$\text{Cls induct}_{m+n} \subset \text{Cls induct}_m \,.\, \text{NC induct}_{m+n} \subset \text{NC induct}_m.$$

We simplify the whole matter, however, by proving

$$\text{Cls induct}_{3+m} = \text{Cls induct}_3$$

This results from the proposition

$$\rho \in \text{Cls induct}_3 \,.\supset.\, (\exists \mu_2).\, \rho = \mu_2$$

which enables us to prove the equivalent of the axiom of reducibility where inductive classes are concerned.

*70*[55]

The proof of $\rho \in \text{Cls induct}_3 \,.\supset.\, (\exists \mu_2).\, \rho = \mu_2$ is very simple. We have $\Lambda = \hat{x}\{(\exists \phi).\, \phi!x.\, \sim\phi!x\}$, which is a second-order function. We have $\iota\text{‘}y = \hat{x}(\phi!y \supset_\phi \phi!x)$, which is again a second-order function. Moreover the sum of two second-order functions is a second-order function. Hence if $\eta$ is defined by a second-order function, so is $\eta \cup \iota\text{‘}y$. Thus

$$(\exists \mu_2).\, \rho = \mu_2$$

is an inductive property. It is of the third order. Hence it belongs to all inductive classes of the third order, i.e.

$*120{\cdot}81 \quad \vdash : \rho \in \text{Cls induct}_3 \,.\supset.\, (\exists \mu_2).\, \rho = \mu_2$

Since Cls $\text{induct}_{3+m} \subset$ Cls $\text{induct}_3$, this shows that inductive classes of the third and higher orders can all be defined by second-order functions. This proposition enables us, after some trouble, to secure results which otherwise would require the axiom of reducibility.

[55] Replaced by *6* of Appendix B. *71* survives renumbering and deletion of about half of its contents to become *9* of the Appendix B manuscript, with the fateful lemma $*89{\cdot}16$, which contains the mistake that Gödel spotted.

~~$\cdot 82 \vdash : \alpha \in \text{NC induct}_4 \,.\, \beta \in \text{NC induct}_3 - \text{NC induct}_4 \,.\, \supset \,.\, \beta(+_c 1)_{po3}\alpha$~~

~~*Dem.*~~

~~$\text{Hp.} \supset \,.\, \beta(+_c 1)_{po3} 0$~~ ~~(1)~~

~~$\text{Hp.} \supset \,.\, \alpha +_c 1 \in \text{NC induct}_4 \,.\, \supset \,.\, \alpha +_c 1 \neq \beta$~~ ~~(2)~~

~~$(2). \supset : \beta(+_c 1)_{po3}\alpha \,.\, \supset \,.\, \beta(+_c 1)_{po3}(\alpha +_c 1)$~~ ~~(3)~~

~~$(1).(3).\text{Induct.} \supset .\text{Prop.}$~~

~~$\cdot 821 \vdash : \beta \in \text{NC induct}_3. \supset \,.\, \overleftarrow{(+_c 1)}_{po3}\text{‘}\beta \in \text{Cls induct}_3$~~

~~Follows by induction from~~

~~$\overleftarrow{(+_c 1)}_{po3}\text{‘}(\beta +_c 1) = \overleftarrow{(+_c 1)}_{po3}\text{‘}\beta \cup \iota\text{‘}\beta$~~

~~$\cdot 822 \vdash : \exists !\, \text{NC induct}_3 - \text{NC induct}_4. \supset .(\exists y).y \in \text{Cls induct}_3.\text{NC induct}_4 \subset y$~~

~~[$*120 \cdot 82 \cdot 821$]~~

*70a*[56]

Further progress depends upon the use of intervals as defined in $*121$. Thus $(+_c 1)_m(\alpha \longmapsto\!\dashv 0)$ means all inductive members of the $m^{\text{th}}$ order from 0 to $\alpha$. If $\alpha$ is not an inductive number of the $m^{\text{th}}$ order, this is $\Lambda$. We now prove

$*120\cdot 811 \vdash : \alpha \in \text{NC induct}_4 \,.\, \supset \,.\, (+_c 1)_3(\alpha \longmapsto\!\dashv 0) \subset \text{NC induct}_4$

By $*121\cdot 371$, if $\alpha \in (+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv 0)$
$(+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv 0) = (+_c 1)_3\ (\alpha \longmapsto\!\dashv 0) \cup (+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv \alpha)$
But $\alpha \sim\in (+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv 0). \supset \,.\, (+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv 0) = \Lambda$
and $\alpha \in (+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv 0). \supset .$
$(+_c 1)_3\ (\alpha +_c 1 \longmapsto\!\dashv \alpha) = \iota\text{‘}\alpha \cup \iota\text{‘}(\alpha + 1).\ \alpha \in (+_c 1)_3(\alpha \longmapsto\!\dashv 0)$
Also $\alpha \in \text{NC induct}_4 \,.\, \supset \,.\, \alpha +_c 1 \in \text{NC induct}_4$
Hence $(+_c 1)_3(\alpha \longmapsto\!\dashv 0) \subset\ \text{NC induct}_4 \,.$
$\supset \,.\, (+_c 1)_3(\alpha +_c 1 \longmapsto\!\dashv 0) \subset \text{NC induct}_4$ (1)
Now $\text{NC induct}_4 = p\text{‘}\kappa$,
where $\alpha_4 \in \kappa. \equiv : 0 \in \alpha_4 : \mu \in \alpha_4. \supset_\mu \,.\, \mu +_c 1 \in \alpha_4$
(1) allows induction for each $\alpha_4$, hence prop.

$\cdot 82 \vdash : \alpha \in \text{NC induct}_4 \,.\, \beta \in \text{NC induct}_3 - \text{NC induct}_4 \,.\, \supset \,.\, \beta(+_c 1)_{po3}\alpha$

*Dem*

$\vdash . * 120 \cdot 811.\ \text{Transp} \,.\, \supset \vdash : \text{Hp.} \supset \,.\, \sim \{\alpha(+_c 1)_{po3}\beta\}$ (1)

$\vdash .(1). * 91\cdot 44 \,.\, \supset \vdash .\ \text{Prop}$

~~$\cdot 821 \vdash : \beta \in \text{NC induct}_3. \supset \,.\, \overleftarrow{(+_c 1)}_{po3}\text{‘}\beta \in \text{Cls induct}_3$~~

[56] Not used. There is no discussion of $+_c 1$, cardinal addition of 1, in Appendix B. The ancestral of this defines the natural numbers, and so it is the basis for mathematical induction, but the discussion in Appendix B is carried out with respect to the ancestral $R_*$ of an arbitrary relation $R$. Thus the notion of inductive class Cls induct is the focus of attention, rather than inductive cardinals, NC induct. This is, however, an early version of the important result, described on *70* as showing that one can “secure results which otherwise would require the axiom of reducibility”.

~~*Dem*~~

$\cancel{\vdash :\ \beta \in \text{NC induct}_3 . \supset . \overleftarrow{(+_c 1)}_{po3} `(\beta +_c 1) = \overleftarrow{(+_c 1)}_{po3} `\beta \cup \iota `\beta\ (1)}$

$\cancel{\vdash :\ \beta \sim\in \text{NC induct}_3 . \supset . \overleftarrow{(+_c 1)}_{po3} `(\beta +_c 1) = \Lambda}$

*70b*[57]

$*120{\cdot}821\ \vdash :\ \beta \in \text{NC induct}_3 . \supset . (+_c 1)_{po3}(\beta \longmapsto 0) \in \text{Cls induct}_3$

*Dem*

As in $*120{\cdot}811$,

$$\vdash : \beta \in \text{NC induct}_3 . \supset . (+_c 1)_{po3}(\beta +_c 1 \longmapsto 0) = (+_c 1)_{po3}(\beta \longmapsto 0) \cup \iota `(\beta +_c 1) \quad (1)$$

$$\vdash : \beta \in \text{NC induct}_3 . \supset . \beta +_c 1 \sim\in \text{NC induct}_3 . \supset . (+_c 1)_{po3}(\beta +_c 1 \longmapsto 0) = \Lambda \quad (2)$$

$$\vdash .(1).(2). \supset \vdash :\ (+_c 1)_{po3}(\beta \longmapsto 0) \in \text{Cls induct}_3 . \supset . (+_c 1)_{po3}(\beta +_c 1 \longmapsto 0) \in \text{Cls induct}_3 \quad (3)$$

$\vdash .(3). \text{Induct.} \supset \vdash .\text{Prop}$

$\cdot 822 \vdash : \exists ! \text{NC induct}_3 - \text{NC induct}_4 . \supset .(\exists y).y \in \text{Cls induct}_3 . \text{NC induct}_4 \subset y$
$[*120{\cdot}82{\cdot}821]$

*73*[58]

$*120{\cdot}83\ \vdash . \text{NC induct}_4 = \text{NC induct}_3 . \text{NC induct}_{3+n} = \text{NC induct}_3$

*Dem*

$$\vdash :. 0 \in \mu_2 :\ \nu \in \mu_2 . \supset_\nu . \nu +_c 1 \in \mu_2 : \supset . \text{NC induct}_3 \subset \mu_2 \quad (1)$$

$$(1). *120{\cdot}826. \supset \vdash : \exists ! \text{NC induct}_3 - \text{NC induct}_4 . \supset . \text{NC induct}_3 \subset \text{NC induct}_4 : \supset \vdash . \text{NC induct}_3 \subset \text{NC induct}_4\ \cancel{\supset \vdash .\text{Prop}}$$

$\cancel{\cdot 831\ \vdash :\ T \in \text{Potid}_n `R . \supset .}$

$\cancel{(\exists \alpha_n). |R``\alpha_n \subset \alpha_n . R_0 \in \alpha_n . \overleftarrow{R}_{ts} `T = \text{Potid}_n `R \cap \alpha_n - |T``|R``\alpha_n}$

Similarly $\text{NC induct}_{3+n} = \text{NC induct}_3$ . Hence prop.

~~*Dem*~~

[The remainder of the page is deleted with a single vertical line. Some lines were deleted before that, and are so indicated below:]

$\cdot 831\ \vdash :\ T \in \text{Potid}_n `R . \supset . \overleftarrow{R}_{ts} `T \in \text{Cls induct}_{n+1}$

*Dem*

$$\vdash . \overleftarrow{R}_{ts} `R_0 = \iota `R_0 \quad (1)$$

$$\cancel{\vdash . \overleftarrow{R}_{ts} `(}\vdash : T \in \text{Potid}_n `R . \supset . \overleftarrow{R}_{ts} `(T|R) = \overleftarrow{R}_{ts} `T \cup \iota `(T|R) \quad (2)$$

[57] Not used. A continuation of *70a*.

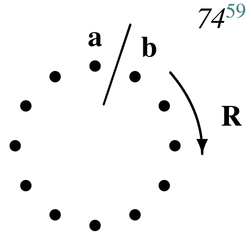
[58] Continuous with the last leaf. $*120{\cdot}83$ becomes $*89{\cdot}29$ in Appendix B, again generalized to the ancestral of an arbitrary relation rather than just to inductive cardinals.

~~$\vdash .(1).(2). \supset \vdash : \overleftarrow{R}_{ts}‘R_0 \epsilon \text{ Cls induct} : \overleftarrow{R}_{ts}‘T \epsilon \text{ Cls induct} . \supset .$~~
~~$\overleftarrow{R}‘(T|R) \epsilon \text{ Cls induct} \quad (3)$~~

$\vdash : T \epsilon \text{ Potid}_n‘R . \supset .$
$$\overleftarrow{R}_{ts}‘T = \hat{S}(\,|R“\alpha_n \subset \alpha_n \,.\, S \epsilon \alpha_n \,.\, \supset_{\alpha_n} .\, T \epsilon \alpha_n) \quad (3)$$
$\vdash .(1).(2).(3). \supset \vdash :. \Lambda \epsilon \alpha_{n+1} : \beta \epsilon \alpha_{n+1} . \supset_{\beta,y} . \beta \cup \iota‘y \epsilon \alpha_{n+1} :$
$$\supset : T \epsilon \text{ Potid}_n‘R . \supset . \overleftarrow{R}_{ts}‘T \epsilon \alpha_{n+1} \quad (4)$$
$\vdash .(4).\text{Comm}. \supset \vdash : T \epsilon \text{ Potid}_n‘R . \supset . \overleftarrow{R}_{ts}‘T \epsilon \text{ Cls induct}_{n+1} :$
$$\supset \vdash . \text{Prop}$$

$\cdot 832 \;\; \vdash :. n > 1 . T \epsilon \text{ Potid}_n‘R . \supset : \overleftarrow{R}_{ts}‘T \epsilon \text{ Cls induct}_3 : (\exists \mu_2) . \overleftarrow{R}_{ts}‘T = \mu_2$
$[*120\cdot 831\cdot 83\cdot 81]$

*74*[59]




For our further progress we require $*121\cdot 47$, i.e.

$R \epsilon (\text{Cls} \to 1) \cup (1 \to \text{Cls}) . \supset . R(x \longmapsto\!\dashv y) \epsilon \text{ Cls induct}.$

This becomes useful by substituting $|R$ for $R$ and $R_0$ for $x$.

The proof of $*121 \cdot 47$ given in the text requires modification. The essential step, which needs emendation, is the proof of

$R \epsilon \text{ Cls} \to 1 \,.\, aR_{po}a \,.\, \supset . R(a \longmapsto\!\dashv a) \epsilon \text{ Cls induct}.$

In this case $R(a \longmapsto\!\dashv a) = \overleftarrow{R}_*‘a$, because $\overleftarrow{R}_*‘a \subset \overrightarrow{R}_*‘a$.

If $aRa$, $\overleftarrow{R}_*‘a = \iota‘a$. Assume therefore that $a \neq \breve{R}‘a$. Put $b = \breve{R}‘a$.

Put $S = R \upharpoonright (-\iota‘b)$

$S$ and $R$ are both $1 \to 1$ when confined to $\overleftarrow{R}_*‘a$, but $R$ is cyclic, whereas $S$ begins with $b$ and ends with $a$. We have, as in $*121$,

$aR_*x \,.\, \supset . \, xR_*x \,.\, \overleftarrow{R}_*‘x = \overleftarrow{R}_*‘a$

We have first to prove

$S(b \longmapsto\!\dashv a) = R(b \longmapsto\!\dashv a) = R(a \longmapsto\!\dashv a)$

which follows at once from

$\overleftarrow{R}_*‘b = \overrightarrow{R}_*‘a \cap \overleftarrow{R}_*‘b = \overrightarrow{S}_*‘a \cap \overleftarrow{S}_*‘b$

The first of these equations is obvious. As for the second, obviously

$\overrightarrow{S}_*‘a \subset \overrightarrow{R}_*‘a \,.\, \overleftarrow{S}_*‘b \subset \overleftarrow{R}_*‘b$

It remains to prove

$\overleftarrow{R}_*‘b \subset \overleftarrow{S}_*‘b$ and $\overleftarrow{R}_*‘b \subset \overrightarrow{S}_*‘a$

*75*[60]

To prove $\overleftarrow{R}_*‘b \subset \overleftarrow{S}_*‘b$ we only need

[59] Not used. The final results about the level of intervals in Appendix B are $*89\cdot 21$ and $*89\cdot 25$ leading to the main result about many-one and one-many relations, $*89\cdot 29$.

[60] Not used. This continues the treatment of $*121\cdot 47$ and intervals.

$\breve{R}``\mu - \iota`b \subset \mu \,.\, b \in \mu \,.\, \supset \,.\, \breve{R}``\mu \subset \mu \,.\, b \in \mu$

which is obvious. To prove $\overleftarrow{R}_*`b \subset \overrightarrow{S}_*`a$ we prove first $bS_*a$, i.e.

$R``(\alpha - \iota`b) \subset \alpha \,.\, a \in \alpha. \supset_\alpha \,.\, b \in \alpha$

Take any one $\alpha$. If we had $R``(\alpha - \iota`b) \subset \alpha - \iota`b$, we should have (since $a \in \alpha - \iota`b$) $b \in \alpha - \iota`b$, because $bR_*a$. But $b \in \alpha - \iota`b$ is impossible. Therefore $R``(\alpha - \iota`b) \subset \alpha - \iota`b$ is impossible. Since $R``(\alpha - \iota`b) \subset \alpha$, by hypothesis, it follows that $b \in R``\alpha$. Since $aRb \,.\, a \neq b$ and $bR_{po}a$, it follows that $\sim(bRb)$. Hence $\cancel{b \in R``\alpha - \iota`b}$ $b \in R``(\alpha - \iota`b)$. Hence $b \in \alpha$, which was to be proved.

[The remainder of this page is deleted. The alterations before the whole was deleted are indicated.]

Again, $x \in \cancel{\overleftarrow{R}_*`b \,.\, x \neq a.}$ $\mathrm{D}`R \,.\, \breve{R}`x \neq b \,.\, \supset \,.\, \breve{R}`x = \breve{S}`x$

Also, since $S_{po} = S|S_*$, $xS_{po}a \,.\, \equiv \,.\, \breve{S}`xS_*a$

Moreover $S_* \dot{\cap} J \mathrel{\dot{\subset}} S_{po}$ . Hence

$xS_*a \,.\, x \in \overleftarrow{R}_*`b \,.\, \supset : \cancel{x = a}\ \breve{R}`x = b \,.\, \vee \,.\, \overleftarrow{R}_*`xS_*a$

$\qquad\qquad \supset : \breve{R}`xS_*a$

*76*[61]

To prove further $\overleftarrow{R}_*`b \subset \overrightarrow{S}_*`a$, we use

$S(b \longmapsto\!\dashv x) \in \text{Cls induct}$

which follows by $*121$, because $\sim (bS_{po}b)$. This allows us to prove by induction

$T \in \text{Potid}`S \,.\, \alpha = \mu \cup S(b \longmapsto\!\dashv \breve{T}`b) \,.\, \supset \,.\, \breve{R}``\alpha = \breve{R}``\mu \cup S(\breve{R}`b \longmapsto\!\dashv \breve{R}`\breve{T}`b)$

Whence, if $\breve{R}``\mu \subset \mu \cup \iota`b \,.\, \breve{R}`\breve{T}`b \in \mu$, it follows that $\breve{R}``\alpha \subset \alpha$. Hence, putting $x = \breve{R}`\breve{T}`b$, we find

$bS_*x \,.\, \alpha = \mu \cup S(b \vdash x) \,.\, \breve{R}``\mu \subset \mu \cup \iota`b \,.\, x \in \mu. \supset \,.\, \breve{R}``\alpha \subset \alpha \,.\, x \in \alpha$

But $bS_*x. \supset \,.\, xR_*a$. Hence $\breve{R}``\alpha \subset \alpha \,.\, x \in \alpha \supset \,.\, a \in \alpha$. Hence

$bS_*x \,.\, \breve{R}``\mu \subset \mu \cup \iota`b \,.\, x \in \mu \,.\, \supset \,.\, a \in \mu \cup S(b \vdash x)$

But $a \in S(b \vdash x)$ is impossible, because $a \sim\in \mathrm{D}`S$. Hence $a \in \mu$.

Hence $\cancel{xS_*a}$ $bS_*x \,.\, \supset \,.\, xS_*a$.

We have already proved $bR_*x \,.\, \supset \,.\, bS_*x$. Hence $bR_*x \,.\, \supset \,.\, xS_*a$

This completes the proof of

$\overleftarrow{R}_*`b = R(b \longmapsto\!\dashv a) = R(a \longmapsto\!\dashv a) = S(b \longmapsto\!\dashv a)$

and therefore of $R(a \longmapsto\!\dashv a) \in \text{Cls induct}$.

[61] More on $*121{\cdot}47$

Hence $*121{\cdot}47$ is valid if $R_*$ means $R_{*3}$. This is needed because we use induction to prove $S(b \vdash\!\dashv x) \mathbin{\epsilon} \mathrm{Cls\ induct}_3$.

*77*[62]

$*121{\cdot}8 \vdash : m > 2 \,.\ \supset .\ \mathrm{Potid}_m{}^{\backprime}R \cap \overleftarrow{R}_{tsm}{}^{\backprime}P \mathbin{\epsilon} \mathrm{Cls\ induct}_3 \quad [*121{\cdot}47\ \frac{|R, R_0, P}{R, y, x}]$

$\cdot 801\ \vdash : P \mathbin{\epsilon} \mathrm{Potid}_{3+n}{}^{\backprime}R \,.\supset .\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \ \subset\ \mathrm{Potid}_{3+n}{}^{\backprime}R$
Distinguish cases, as in $*96$ and $*121$. Put $\Sigma = (|R)_{po}$. We want
$\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \subset \mathrm{Potid}_{3+n}{}^{\backprime}R$
$\qquad . \supset .\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}(P|R) \subset \mathrm{Potid}_{3+n}{}^{\backprime}R$
If $P \Sigma P$, $\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \ =\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}(P|R)$,
If $\sim\{(P|R)\Sigma(P|R)\}$,
$\qquad \mathrm{Potid}_3{}^{\backprime}R \cup \overleftarrow{R}_{ts3}{}^{\backprime}(P|R) \ =\ (\mathrm{Potid}_3{}^{\backprime}R \cup \overleftarrow{R}_{ts3}{}^{\backprime}P) \cup \iota{}^{\backprime}(P|R)$
If $(P|R)\Sigma(P|R)$. $\sim(P \Sigma P)$, which can only happen for one value of $P$,
$\overrightarrow{R}_{ts3}{}^{\backprime}(P|R) \mathbin{\epsilon} \mathrm{Cls\ induct}_3$.
$\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}(P|R) \ =\ (\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P) \cup \overrightarrow{R}_{ts3}{}^{\backprime}(P|R)$
Now, $P|R \mathbin{\epsilon} \mathrm{Potid}_{3+n}{}^{\backprime}R \,.\, \overrightarrow{R}_{ts3}{}^{\backprime}(P|R) \mathbin{\epsilon} \mathrm{Cls\ induct}_3$ .
$\qquad \supset .\ \overrightarrow{R}_{ts3}{}^{\backprime}(P|R) \subset \mathrm{Potid}_{3+n}{}^{\backprime}R$
because $(\exists \mu_2).\overrightarrow{R}_{ts3}{}^{\backprime}(P|R) = \mu_2$. Hence collecting cases,
$\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \subset \mathrm{Potid}_{3+n}{}^{\backprime}R$.
$\qquad \supset .\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}(P|R) \subset \mathrm{Potid}_{3+n}{}^{\backprime}R$
Hence, splitting $\mathrm{Potid}_{3+n}{}^{\backprime}R$ into constituent hereditary classes
and applying induction to each,
$\vdash : P \mathbin{\epsilon} \mathrm{Potid}_{3+n}{}^{\backprime}R \,.\supset .\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \ \subset\ \mathrm{Potid}_{3+n}{}^{\backprime}R$

*77a*[63]

[ The first two theorems ($*121{\cdot}8$ and $*121{\cdot}801$) are deleted, apparently replaced by the proof on *77* ]
$*121{\cdot}8 \vdash : m > 2.\ \supset .\ \mathrm{Potid}_m{}^{\backprime}R \cap \overleftarrow{R}_{tsm}{}^{\backprime}P \mathbin{\epsilon} \mathrm{Cls\ induct}_3 \quad [*121{\cdot}47\ \frac{|R, P, R_0}{R, y, x}]$
We have now to prove
$*121{\cdot}801\ \vdash : P \mathbin{\epsilon} \mathrm{Potid}_{3+n}{}^{\backprime}R \,.\supset .\ \mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \ \subset\ \mathrm{Potid}_{3+n}{}^{\backprime}R$
We have $\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}R_0 \subset \mathrm{Potid}_{3+n}{}^{\backprime}R \qquad (1)$
Also by $*96$,
$\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}(P|R) \ =\ (\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P) \cup \iota{}^{\backprime}(P|R) \qquad (2)$
Hence
$\mathrm{Potid}_3{}^{\backprime}R \cap \overleftarrow{R}_{ts3}{}^{\backprime}P \subset \mathrm{Potid}_{3+n}{}^{\backprime}R$.
$\qquad \supset .\ \mathrm{Potid}_3{}^{\backprime}R \cap R_{ts3}{}^{\backprime}(P|R) \subset \mathrm{Potid}_{3+n}{}^{\backprime}R \qquad (3)$

[62] Not used. Corrections of theorems of $*121$ occupy this page, *77a*, and *78* following.
[63] This continues *77*.

Now $\text{Potid}_{3+n}\text{‘}R \;=\; p\text{‘}\hat{\mu}_{3+n}(|R\text{“}\mu_{3+n} \subset \mu_{3+n} \;.\; R_0 \in \mu_{3+n})$.
Hence, by (1) and (3),
$P \in \text{Potid}_{3+n}\text{‘}R$ .
$\supset : |R\text{“}\mu_{3+n} \subset \mu_{3+n} .\; R_0 \in \mu_{3+n} \;.\supset .\; \text{Potid}_3\text{‘}R \cap \overleftarrow{R}_{ts3}\text{‘}P \subset \mu_{3+n}$
$: \supset :$ $\cancel{\text{Potid}_3\text{‘}R \cap \overleftarrow{R}_{ts3}\text{‘}P \subset \text{Potid}_{3+n}\text{‘}R}$ ~~whence prop. Hence~~
[The deletion ends here.]

$*121\cancel{802}\ 81 \;\vdash :\; \exists!\, \text{Potid}_3\text{‘}R - \text{Potid}_{3+n}\text{‘}R \;.\supset .\; \text{Potid}_{3+n}\text{‘}R \in \text{Cls induct.}$
*Dem*
$\vdash .\; * 121{\cdot}801 \;.\supset$
$\vdash : P \in \text{Potid}_{3+n}\text{‘}R \;.\; S \in \text{Potid}_3\text{‘}R \;.\; P R_{ts3} S .\supset .\; S \in \text{Potid}_{3+n}\text{‘}R$
[Transp] $\supset \vdash :.\; S \in \text{Potid}_3\text{‘}R - \text{Potid}_{3+n}\text{‘}R .\supset :$
$P \in \text{Potid}_{3+n}\text{‘}R .\supset .\; \sim(P R_{ts3} S)$.
[$*91{\cdot}44$] $\supset .\; S R_{ts3} P$
$\supset : \text{Potid}_{3+n}\text{‘}R = \text{Potid}_{3+n}\text{‘}R \cap \overleftarrow{R}_{ts3}\text{‘}S :$
[$*121{\cdot}8 \cdot *120{\cdot}824$] $\supset : \text{Potid}_{3+n}\text{‘}R \in \text{Cls induct} \;:.\supset \vdash$ . Prop.
$\cancel{*121{\cdot}81 \;\vdash :\; \exists!}$

*78*[64]

It remains to prove ~~(C), namely~~
$*121{\cdot}82 \;\vdash .\; \text{Potid}_{3+n}\text{‘}R \in \text{Cls induct} \;.\supset .\; \text{Potid}_{3+n}\text{‘}R = \text{Potid}_3\text{‘}R$
Hp. $\supset .\; (\exists\mu_2) \;.\; \text{Potid}_{3+n}\text{‘}R = \mu_2$.
Also $R_0 \in \text{Potid}_{3+n}\text{‘}R : S \in \text{Potid}_{3+n}\text{‘}R \;.\supset_S .\; S|R \in \text{Potid}_{3+n}\text{‘}R$
Hence $R_0 \in \mu_2 : S \in \mu_2 \;.\supset_S .\; B|R \in \mu_2$
Hence $\text{Potid}_3\text{‘}R \subset \mu_2$ i.e. $\text{Potid}_3\text{‘}R \subset \text{Potid}_{3+n}\text{‘}R$.
Whence prop. Hence
$\cdot 83 \;\vdash .\; \text{Potid}_{3+n}\text{‘}R = \text{Potid}_3\text{‘}R$ [$*121{\cdot}8{\cdot}81$]

It follows that inductions of any order are valid whenever inductions of the third order are valid. Hence, by taking “Potid” as “$\text{Potid}_3$”, “Cls induct” as “$\text{Cls induct}_3$”, and “NC induct” as “$\text{NC induct}_3$”, all the propositions of Part II Section E and Part III Section C become valid without assuming the axiom of reducibility.

*74*[65]

$\cancel{*120\cdot 832 \;\vdash :\; \dot{\text{A}} \sim \in \text{Potid}_n\text{‘}R \;.\; R_0 \in \alpha_{n+1} \;.\; |R\text{“}\alpha_{n+1} \subset \alpha_{n+1} \;.}$
$\cancel{\exists!\, \text{Potid}\text{‘}R - \alpha_{n+1} \;.\supset .\; (\exists T) .\; T \in \text{Potid}_n\text{‘}R \;.\; \alpha_{n+1} \subset \overleftarrow{R}_{ts}\text{‘}T}$

[64] The end of this passage looks like a draft of the main theorems of Appendix B. It summarizes which theorems still hold without the use of the axiom of reducibility.

[65] The first number ($*120 \cdot 832$) is deleted with a line through the text. The remainder of the page is deleted with a single diagonal line. Some material deleted before that is so indicated. The material here and to the end, i.e. on *74a*, *75*, and *76* is a continuous series of theorems, none used in Appendix B.

$*120\cdot 833 \ \vdash : \ S \in \mathrm{Potid}_{3+n}`R \ . \ m > 2 \ . \supset . \overleftarrow{R}_{tsm}`S \subset \mathrm{Potid}_{3+n}`R$

*Dem*

$\vdash . * 120\cdot 832. \supset \vdash :. \mathrm{Hp}. \supset :$

$$\cancel{\overleftarrow{R}_{tsm}`S \in \mathrm{Cls\ induct}_3} : (\exists \mu_2). \overleftarrow{R}_{tsm}`S = \mu_2 \qquad (1)$$

$\vdash . \overleftarrow{R}_{tsm}`R_0 \subset \mathrm{Potid}_{3+n}`R$ (2)

$\vdash : T \in \mathrm{Potid}_1`R. \supset . \overleftarrow{R}_{tsm}`(T|R) \ = \ \overleftarrow{R}_{tsm}`T \cup \iota`(T|R)$ (3)

$\vdash .(3). \supset \vdash : T \in \mathrm{Potid}_1`R. \overleftarrow{R}_{tsm}`\cancel{(T|R)}\lfloor T \rfloor \subset \mathrm{Potid}_{3+n}`R \ .$

$$\supset . \overleftarrow{R}_{tsm}`(T|R) \subset \mathrm{Potid}_{3+n}`R \qquad (4)$$

$\vdash .(1).(2).(4). \supset \vdash . \mathrm{Prop}$

$\cdot 834 \ \vdash : \ S \in \mathrm{Potid}_{3+n}`R \ . \ T \in \mathrm{Potid}_3`R - \mathrm{Potid}_{3+n}`R \ . \supset . \ T R_{ts} S$

$\vdash . * 120\cdot 833.\mathrm{Transp}. \supset \vdash : \mathrm{Hp} \ . \supset . \sim(S R_{ts} T)$ (1)

$\vdash .(1). * 91\cdot 44. \ \supset \vdash . \mathrm{Prop}$

$\cdot 835 \ \vdash : \exists! \ \mathrm{Potid}_3`R - \mathrm{Potid}_{3+n}`R \ . \supset . \ \mathrm{Potid}_{3+n}`R \in \mathrm{Cls\ induct}_3$

*Dem*

$\vdash . * 120\cdot 834. \supset \vdash :$

$$\mathrm{Hp}. \supset .(\exists T). \ T \in \mathrm{Potid}_3`R \ . \ \mathrm{Potid}_{3+n}`R \subset \ \overleftarrow{R}_{ts}`T \quad (1)$$

$\vdash .(1). * 120\cdot 832\cdot 824. \ \supset \vdash . \mathrm{Prop}$

*74a*[66]

$*120\cdot 836 \ \vdash : (\mathrm{Potid}`R)\upharpoonleft(|R) \in J \ . \supset . \ \mathrm{Potid}_{3+n}`R \sim\in \mathrm{Cls\ induct}_3$

*Dem*

$\vdash . * 120\cdot 81. \supset \vdash :$

$$\mathrm{Potid}_{3+n}`R \in \mathrm{Cls\ induct}_3 \ . \supset . \ (\exists \mu_2). \ \mathrm{Potid}_{3+n}`R = \mu_2 \qquad (1)$$

$\vdash . \ \Lambda \ \subset \ |R``\Lambda$ (2)

$\vdash : \ \alpha \ \subset \ |R``\alpha. \supset : \ \alpha \cup \iota`T \subset |R``\alpha \cup \iota`T \ :$

$$\supset : T \in |R``\alpha \ . \vee . \ T \sim\in |R``(\alpha \cup \iota`T) :$$

$$\supset : \alpha \cup \iota`T \subset |R``(\alpha \cup \iota`T) \ . \vee . \ \exists!(\alpha \cup \iota`T) - |R``(\alpha \cup \iota`T) \quad (3)$$

$\vdash : \ M \in |R``\alpha - \alpha \ . \ M \neq T. \supset . \ M \in |R``(\alpha \cup \iota`T) - (\alpha \cup \iota`T)$ (4)

$\vdash : \ \iota`T \ = |R``\alpha - \alpha \ . \ \mathrm{Hp} \ . \supset . \ T|R \in |R``(\alpha \cup \iota`T) - (\alpha \cup \iota`T)$ (5)

$\vdash .(3).(4).(5). \supset \vdash :: \mathrm{Hp} \ . \supset :. \ \alpha \subset |R``\alpha \ .\vee. \ \exists! \ |R``\alpha - \alpha) : \supset :$

$$\alpha \cup \iota`T \subset |R``(\alpha \cup \iota`T) \ .\vee. \ \exists! \ |R``(\alpha \cup \iota`T) - (\alpha \cup \iota`T) \qquad (6)$$

$\vdash .(2).(6). \supset \vdash ::$

$$\mathrm{Hp} \ . \ \alpha \in \mathrm{Cls\ induct}_3 \ . \supset : \ \alpha \subset |R``\alpha \ .\vee. \ \exists! \ |R``\alpha - \alpha \qquad (7)$$

$\vdash .(1).(7). \supset \vdash :. \ \mathrm{Hp} \ . \ \mathrm{Potid}_{3+n}`R \in \mathrm{Cls\ induct}_3 \ . \supset :$

[66] This page is deleted with a single line through the text.

$\text{Potid}_{3+n}{}^{\backprime}R \subset |R``\text{Potid}_{3+n}{}^{\backprime}R \,.\vee.\, \exists!\, |R``\text{Potid}_{3+n}{}^{\backprime}R - \text{Potid}_{3+n}{}^{\backprime}R$ (8)

$\vdash .(8).\text{Transp.} \supset \vdash . \text{Prop}$

*75*[67]

$*120\cdot836 \;\vdash : (\text{Potid}{}^{\backprime}R) \upharpoonleft (|R) \mathrel{\sqsubset} J \,.\supset.\, \text{Potid}_{3+n}{}^{\backprime}R \sim\in \text{Cls induct}_3$

*Dem*

$|R``\text{Potid}_{3+n}{}^{\backprime}R \subset \text{Potid}_{3+n}{}^{\backprime}R$ (1)

$\vdash : \text{Hp} \,.\supset.\, R_0 \in \text{Potid}_{3+n}{}^{\backprime}R - |R``\text{Potid}_{3+n}{}^{\backprime}R$ (2)

$\vdash : \text{Hp} \,.\supset.\, (\text{Potid}_{3+n}{}^{\backprime}R) \upharpoonleft (|R) \in 1 \to 1$ (3)

$\vdash .(1).(2).(3). \supset \vdash . \text{Prop}$

$\cdot837 \;\vdash : \text{Potid}{}^{\backprime}R \upharpoonleft (|R) \mathrel{\sqsubset} J \,.\supset.\, \text{Potid}_{3+n}{}^{\backprime}R = \text{Potid}_3{}^{\backprime}R$ [$*120\cdot835\cdot836$]

$\cdot838 \;\vdash : \sim\{(\text{Potid}{}^{\backprime}R) \upharpoonleft (|R) \mathrel{\sqsubset} J\} \,.\supset.\, \text{Potid}_3{}^{\backprime}R \in \text{Cls induct}_3$

*Dem*

$\vdash : \text{Hp} \,.\supset.\, (\exists T).\, T \in \text{Potid}_3{}^{\backprime}R \,.\, T|R = T.$

$\supset .(\exists T).\, T \in \text{Potid}_3{}^{\backprime}R \,.\, \overrightarrow{R}_{ts}{}^{\backprime}T = \iota{}^{\backprime}T.$

$\supset .(\exists T).\, T \in \text{Potid}_3{}^{\backprime}R \,.\, \text{Potid}_3{}^{\backprime}R \subset \overleftarrow{R}_{ts}{}^{\backprime}T$ (1)

$\vdash .(1).\, *120\cdot832. \supset \vdash . \text{Prop}$

*76*[68]

$*120\cdot 839 \;\vdash : \text{Potid}_3{}^{\backprime}R \in \text{Cls induct}_3 \,.\supset.\, \text{Potid}_{3+m}{}^{\backprime}R = \text{Potid}_3{}^{\backprime}R$

*Dem*

$\vdash : \text{Potid}_{3+m}{}^{\backprime}R \subset \text{Potid}_3{}^{\backprime}R$ (1)

$\vdash .(1).\, * 120 \cdot 81 \cdot 824 \,.\supset \vdash : \text{Hp.} \supset .(\exists\mu_2).\, \text{Potid}_{3+m}{}^{\backprime}R = \mu_2$ (2)

$\vdash : R_0 \in \text{Potid}_{3+m}{}^{\backprime}R : T \in \text{Potid}_{3+m}{}^{\backprime}R \,.\supset_T.\, T|R \in \text{Potid}_{3+m}{}^{\backprime}R$ (3)

$\vdash .(2).(3). \supset \vdash :.\, \text{Hp.} \supset : T \in \text{Potid}_3{}^{\backprime}R. \supset .\, T \in \text{Potid}_{3+m}{}^{\backprime}R$ (4)

$\vdash .(1).(4). \supset \vdash . \text{Prop}$ $\cdot84 \;\vdash \;. \text{Potid}_{3+m}{}^{\backprime}R = \text{Potid}_3{}^{\backprime}R$
[$*120\cdot 837\cdot838\cdot839$]

Thus both in regard to Potid‘$R$ and in regard to Cls induct, it is unnecessary to go beyond functions of the third order. We may therefore retain all the propositions of *Principia Mathematica* on these subjects without assuming the axiom of reducibility, by merely supposing that Potid‘$R$ and Cls induct stand for $\text{Potid}_3{}^{\backprime}R$ and Cls $\text{induct}_3$; and the same of course applies to NC induct.

The analogous difficulties, however, which arise in regard to Dedekindian relations and well-ordered relations cannot, so far as we know, be removed. The accepted theory on these subjects must, therefore, be regarded as largely doubtful, except by those who accept some form of the axiom of reducibility.

67 This page is deleted with a large ‘X’ through $*120\cdot836$ and its proof, and a single line through the remainder.
68 This page is deleted with a single diagonal line.

# Amended list of propositions: notes

## Editorial note

The material given the title “Amended list of propositions” (RA 230.03190), is in a file labeled “working papers”. It consists of 62 leaves, 36 of them with writing on both sides, the other 26 on only one, for a total of 98 pages of notes. The paper is 24.9 cm by 20.2 cm, all in ink, with the exception of two pages, 51 and 54, which are on thinner paper, which is 24.4 cm by 20.1 cm; the rest of the notes and manuscripts are on the larger paper. With one exception, ‘*16*’, there is no foliation. Each page begins one or possibly two new sketches of a proof, and none runs onto a second page. The numbering provided here, at the top of each page in square brackets: [1r], etc., follows the order in which the leaves were found in the Archives so that the “recto” and “verso” of a leaf are determined by their placement in the folder as I found them in 2003. There are very few deletions. Sometimes, however, the remainder of a page is deleted, either with one diagonal line through the whole, or with single lines through the middle of each row of symbols. Most of the corrections seem to have been made immediately after a line of symbols was written down, or in some cases after a series of lines, then a new page follows correcting the perceived errors. Notes on any material are presented as footnotes to the page number which begins it.

## Russell’s notes

[1r]

*Amended List of Propositions*
We can omit $*120{\cdot}811{\cdot}82{\cdot}821{\cdot}822$. They are not needed.[1]

[1] $*120{\cdot}811{\cdot}\,82{\cdot}821\cdot 822$ are likely the numbers on 2r following. Those theorems that follow below do occur on 2r, but with different numbers. These are the crucial (though problematic) theorems of $*89$. Theorem $*120{\cdot}81$ here is $*89{\cdot}12$, $*120{\cdot}811$ is $*89{\cdot}16$, and $*120{\cdot}82$ is $*89{\cdot}17$ in Appendix B.

Then put

$*120{\cdot}81 \quad \vdash : \ \rho \in \text{Cls induct}_3 \ . \supset . \ (\exists \mu_2). \ \rho = \mu_2$

$\cdot 811 \quad \vdash : \alpha \sim\in \text{Cls induct}_3 \ . \ \gamma \in \text{Cls induct}_3 \ . \supset . \ \exists ! \ \alpha - \gamma$

$\cdot 82 \quad \vdash : \gamma \in \text{Cls induct}_3 \ . \ \alpha \subset \gamma \ . \supset . \ \alpha \in \text{Cls induct}_3$

$*121{\cdot}8 \quad \vdash . \ \text{Potid}_m \text{‘} R \cap \overleftarrow{R_{tsm}} \text{‘} P \in \text{Cls induct}_m$[2]

$\cdot 801 \quad \vdash : \ m > 2 \ . \ P \in \text{Potid}_m \text{‘} R \ . \supset . \ \text{Potid}_3 \text{‘} R \cap \overleftarrow{R_{t\,s_3}} \text{‘} P \ \subset \ \text{Potid}_m \text{‘} R$

$\cdot 81 \quad \vdash : \ \exists ! \ \text{Potid}_3 \text{‘} R - \text{Potid}_{3+n} \text{‘} R \ . \supset . \ \text{Potid}_{3+n} \text{‘} R \in \text{Cls induct}_3$

$\cdot 82 \quad \vdash : \text{Potid}_{3+n} \text{‘} R \in \text{Cls induct}_3 \ . \supset . \ \text{Potid}_3 \text{‘} R = \text{Potid}_{3+n} \text{‘} R$

$\cdot 83 \quad \vdash \ . \ \text{Potid}_3 \text{‘} R = \text{Potid}_{3+n} \text{‘} R$

$\cdot 84 \quad \vdash \ . \ R_{*_3} = \ R_{*n} \ . \ R_{po\,3} = \ R_{po\,n}$

$\cdot 85 \quad \vdash \ . \ \text{Cls induct}_3 = \text{Cls induct}_n \ . \ \text{NC induct}_3 = \text{NC induct}_n$

We use $*120{\cdot}81$ in $*121{\cdot}801$(not before). It is used again in $*121{\cdot}82$.

[1v]

We have to prove $bR_* x \ . \supset . \ xS_* a$[3] $\qquad p \supset r \ . \supset . \ q \supset r$

Take it by powers . Assume $\qquad p \ . \sim r \ .\vee. \ \sim q \vee r$

$T \in \text{Potid‘} R \ . \ bTx$. To prove $xS_* a \qquad q \supset p \vee r$

i.e. $\quad T \in \text{Potid‘} R : bTx. \ \breve{R}\text{“}\mu - \iota\text{‘}b \subset \mu \ . \ x \in \mu \ . \supset_x . \ a \in \mu$

and deduce this for $T|R$. This requires

$bTx \ . \ \breve{R}\text{“}\mu - \iota\text{‘}b \ \subset \ \mu \ . \ x \in \mu \ . \supset_x . \ a \in \mu \ :$

$\supset : \ bTx \ . \ \breve{R}\text{“}\mu - \iota\text{‘}b \subset \mu \ . \ \breve{R}\text{‘}x \in \mu \ . \supset_x . \ a \in \mu$

Would follow if

$bTx \ . \ \breve{R}\text{“}\mu - \iota\text{‘}b \ \subset \ \mu \ . \supset :. \ x \in \mu \ . \supset . \ a \in \mu \ : \supset : \ \breve{R}\text{‘}x \in \mu \ . \supset . \ a \in \mu$

i.e. $\supset :. \ \breve{R}\text{‘}x \in \mu \ . \supset : \ x \in \mu \ .\vee. \ a \in \mu$

Here $bTx$ is irrelevant. The rest is

$\breve{R}\text{“}\mu - \iota\text{‘}b \ \subset \ \mu \ . \ \breve{R}\text{‘}x \in \mu \ . \ x \sim\in \mu \ . \supset . \ a \in \mu$

---

We have to prove

$\breve{R}\text{“}\mu - \iota\text{‘}b \ \subset \ \mu \ . \ x \in \mu \ . \supset . \ a \in \mu \ : \supset : \ \breve{R}\text{“}\mu - \iota\text{‘}b \ \subset \ \mu \ . \ \breve{R}\text{‘}x \in \mu \ . \supset . \ a \in \mu$

i.e. $\breve{R}\text{“}\mu - \iota\text{‘}b \ \subset \ \mu \ . \ \supset :. \ \breve{R}\text{‘}x \in \mu \ . \supset : \ x \in \mu \ .\vee. \ a \in \mu$

---

What is necessary is to construct an $R$-hereditary class for $\breve{R}\text{“}\mu \ \subset \ \mu \cup \iota\text{‘}b$

If $b \in \mu$, $\mu$ or $\mu \cup \iota\text{‘}b$ will do.

[2] $*121{\cdot}8$ to $*121{\cdot}83$ occur, with different numbers, on 2r but $*121{\cdot}84$ and $*121{\cdot}85$ do not. None of these seven $*121$ theorems appear in Appendix B, although they do show up in the last pages of HPF.

[3] $S$ is defined at *PM*, p.655.

If $\breve{R}‘b \in \mu$, $\mu \cup \iota‘b$ will do.
We want one which will do if $b \sim \in \mu \,.\, \breve{R}‘b \sim \in \mu$.
Consider $\mu - \overleftarrow{R}‘b$. $\breve{R}“(\mu - \overleftarrow{R}‘b) = \hat{y}\{(\exists x).\, x \in \mu \,.\sim (xRb) \,.\, xRy\} = \breve{R}“\mu - \iota‘b$
This is contained in $\mu$. Is it contained in $-\overleftarrow{R}‘b$? ~~yes, if $b \sim \in \mu$.~~
~~Thus if $b \in \mu$, $\mu \cup \iota‘b$ is hereditary.~~
~~If $b \sim \in \mu$, $\mu - \overleftarrow{R}‘b$ is hereditary~~

[2r]

*List of Props*

$*120 \cdot 24 \;\; \vdash :.\, \rho \in \text{Cls induct.} \equiv :.\, \eta \in \mu \,.\supset_{\eta,y} .\, \eta \cup \iota‘y \in \mu :\supset_\mu : \Lambda \in \mu \,.\supset .\rho \in \mu$[4]

Cls $\text{induct}_1$ = above with $\mu_1$ (1st order) as a variable.
Cls $\text{induct}_2$ = $\mu_2$ (2nd order) etc.

$*120 \cdot 81 \;\; \vdash :\; \rho \in \text{Cls induct}_3 \,.\supset .\, (\exists \mu_2).\, \rho = \mu_2$[5]

$\text{Potid}_1‘R = \hat{T}\{\phi!(I \upharpoonright \iota‘R) : \phi!S \,.\supset_S .\, \phi!(S|R) : \supset_\phi .\, \phi!T\}$ Df
$R_0 = I \upharpoonright \iota‘R$ Df
$\text{Potid}_2‘R = \hat{T}\{\phi_2 \quad \phi_2 \quad \phi_2 \quad \phi_2 \}$ Df etc.
Similarly for $\text{Pot}_1$ and $\text{Pot}_2$ etc.
$R_{pom} = \dot{s}‘\text{Pot}_m‘R$ Df

$*120 \cdot 811 \;\; \vdash :\; \alpha \in \text{NCinduct}_4 \,.\supset .\, (+_c 1)_3(\alpha \longmapsto 0) \subset \text{NCinduct}_4$[6]

$*120 \cdot 82 \;\; \vdash :\; \alpha \in \text{NCinduct}_4 \,.\, \beta \in \text{NCinduct}_3 - \text{NCinduct}_4 \,.\supset .\, \beta(+1)_{po3}\alpha$[7]

$\cdot 821 \;\; \vdash :\; \beta \in \text{NCinduct}_3 \,.\supset .\, \lfloor \text{NCinduct}_3 \cap \rfloor (+_c 1)_{po3}‘\beta \in \text{Cls induct}_3$
$\cdot 822 \;\; \vdash : \exists!\, \text{NCinduct}_3 - \text{NCinduct}_4 \,.\supset .$
$(\exists \gamma).\, \gamma \in \text{Cls induct}_3 \,.\, \text{NCinduct}_4 \subset \gamma$
$\cdot 823 \;\; \vdash :\; \alpha \sim \in \text{Cls induct}_3 \,.\, \gamma \in \text{Cls induct}_3 \,.\supset .\, \exists!\, \alpha - \gamma$
$\cdot 824 \;\; \vdash :\; \gamma \in \text{Cls induct}_3 \,.\, \alpha \subset \gamma \,.\supset .\, \alpha \in \text{Cls induct}_3$
$\cdot 825 \;\; \vdash :\; \exists!\, \text{NCinduct}_3 - \text{NCinduct}_4 \,.\supset .\, \text{NCinduct}_4 \in \text{Cls induct}_3$
$\cdot 826 \;\; \vdash : \qquad \supset .(\exists \mu_2).\, \text{NCinduct}_4 = \mu_2$
$\cdot 83 \;\; \vdash \;.\, \text{NCinduct}_4 = \text{NCinduct}_3 \,.\, \text{NCinduct}_{3+n} = \text{NCinduct}_3$

[4] See HPF, p.69. [5] This is $*89{\cdot}12$. [6] This appears at HPF, p.70*a*.
[7] See deleted $*89{\cdot}828$ in the manuscript of Appendix B.

$*121{\cdot}8 \vdash . \mathrm{Potid}_m`R \cap \overleftarrow{R}_{t\,sm}`P \epsilon \mathrm{Cls\ induct}\ [*121{\cdot}47\ \frac{|R,P,R_0}{R,x,y}]$

$\cdot 801 \vdash :. m > 2 . \lfloor P \epsilon \mathrm{Potid}_m`R . \rfloor \supset : \mathrm{Potid}_3`R \cap \overleftarrow{R}_{t\,s3}`P\ \cancel{\epsilon\ \mathrm{Cls\ induct}_{\mathrm{T}}} \subset \mathrm{Potid}_m`R$

$\cdot 811 \vdash : \exists!\ \mathrm{Potid}_3`R - \mathrm{Potid}_{3+n}`R . \supset . \mathrm{Potid}_{3+n}`R \epsilon \mathrm{Cls\ induct}$

$\cancel{(\exists \mu_2) \ldots\ldots = \mu_2}$

$\cdot 82 \vdash : \mathrm{Potid}_{3+n}`R \epsilon \mathrm{Cls\ induct} . \supset . \mathrm{Potid}_3`R = \mathrm{Potid}_{3+n}`R$

$\cdot 83 \vdash . \mathrm{Potid}_{3+n}`R = \mathrm{Potid}_3`R$

[2v]

Consider $(\exists x):(y) . \phi x | \psi y : \equiv : (y):(\exists x) . \phi x | \psi y$

I think this is only true when $\phi x$ and $\psi y$ contain no common functional part, i.e. if no variable function occurs on *both* sides of the principal stroke.[8]

Consider $(y):(\exists x) . \phi x \vee \psi y . \sim \phi x \vee \sim \psi y$ (This is always true)

and $(\exists x):(y) . \phi x \vee \psi y . \sim \phi x \vee \sim \psi y$

i.e. $(\exists x): \phi x . \vee . (y) . \psi y : \sim \phi x . \vee . (y) . \sim \psi y$ ~~(This is always true too)~~

$(p \vee q . \sim p \vee r . \equiv : \sim p . q .\vee. p . r .\vee. q . r : \supset . q \vee r)$

which $\supset : (y). \psi y .\vee. (y). \sim \psi y$

---

$\{(\exists x).\phi x\}|\{(x).\psi x\}. \equiv : \sim (\exists x). \phi x .\vee. \sim (y). \psi y$

$\equiv : (x). \sim \phi x .\vee. (\exists y). \sim \psi y$

---

$p.q . \equiv . (p|q)|(p|q)$

Above $. \equiv .\{(\sim \phi x| \sim \psi y)|(\phi x \mid \psi y)\}|\{(\sim \phi x| \sim \psi y)|(\phi x \mid \psi y)\}$

Here both variables occur both sides of matrix.

[3r]

*Summary*

Can we prove directly

$\mathrm{Potid}_{3+n}`R \epsilon \mathrm{Cls\ induct}_3 . \supset . \mathrm{Potid}_{3+n}`R = \mathrm{Potid}_3`R$?[9]

In this case, $\mathrm{Potid}_{3+n}`R = \mu_2$.

Also $R_0 \epsilon \mu_2 : S \epsilon \mu . \supset_S . S|R \epsilon \mu_2$

Hence $\mathrm{Potid}_3`R \subset \mu_2$. Hence prop.

Hence $\mathrm{Potid}_3`R = \mathrm{Potid}_{3+n}`R$ . Q.E.D.

---

Thus method is as follows : By $*96$,

$R \epsilon \mathrm{Cls} \to 1 . \supset . (\exists S). S \epsilon 1 \to 1 . \overleftarrow{R}_*`x = \overleftarrow{S}_*`x$

[8] This appears at *PM*, p.*xxvi*. [9] Compare with last folio of HPF, numbered *76*.

$S \epsilon 1 \to 1 . y \epsilon \text{D}‘R . \supset . \overleftarrow{S_*}‘\breve{S}‘y = \overleftarrow{S_*}‘y \cup \iota‘\breve{S}‘y$
Hence by induction $x S_* y . \supset . (\exists \mu_2). \overleftarrow{S_*}‘y = \mu_2$
Also $\text{Potid}_3‘S \cap \overleftarrow{S_t}{}_{s3}“P \quad \subset \text{Potid}_{3+n}‘S$ if $P \epsilon \text{Potid}_{3+n}‘R$
where $S$ is $R$ arranged so that $|R \epsilon 1 \to 1$; whence prop for $|R$.
Hence $\exists ! \text{Potid}_3‘R - \text{Potid}_{3+n}‘R . \supset . \text{Potid}_{3+n}‘R \epsilon \text{Cls induct}_3$
Also $\text{Potid}_{3+n}‘R \epsilon \text{Cls induct}_3 . \supset . \text{Potid}_{3+n}‘R = \text{Potid}_3‘R$
Hence $\text{Potid}_{3+n}‘R = \text{Potid}_3‘R$ Q.E.D.

[3v][10]

Put $M = \hat{\alpha}\hat{\beta}\{(\exists y). \beta = \alpha \cup \iota‘y\}$[11]
Then Cls induct $= \overleftarrow{M_*}‘\Lambda$
If $N = \hat{\alpha}\hat{\beta}\{(\exists y). y \sim \epsilon \alpha . \beta = \alpha \cup \iota‘y\}$,
NC induct $= \hat{\mu}\{(\exists P). P \epsilon \text{Potid}‘N. \mu = \overleftarrow{P}‘\Lambda\}$

---

$$p \vee q = (p|q)|(q|q)$$

$(a, b) : (\exists x) . \phi x|(\phi a|\phi b) :$
$\equiv : (\exists x) : (a, b) . \phi x|(\phi a|\phi b)$
$(x).\phi x. \supset . (x).\phi x : \equiv : (x).\phi x .|. (a).\phi a|(b).\phi b$
$\equiv : (\exists x) : \phi x \mid (a)\phi a|(b)\phi b$
$\equiv : (\exists x) : (a, b). \phi x \mid (\phi a|\phi b)$

---

We have $\phi a \mid (\phi a|\phi b) . \vee . \phi b \mid (\phi a|\phi b)$

[4r]

Two cases in proving $p‘s“\lambda \subset s‘p‘\lambda$.[12] Assume $y \epsilon p‘s“\lambda ts . S = R_\epsilon$
First assume $(\exists \mu). \mu \epsilon \lambda . \beta \epsilon \mu - S“\mu . T \epsilon \text{Potid}‘S . y \epsilon T‘\beta$.
Then $\beta = \iota‘x$ whence $y \epsilon p‘s‘\lambda$
Second assume $\mu \epsilon \lambda . \beta \epsilon \mu . T \epsilon \text{Potid}‘S . y \epsilon T‘\beta . \supset . \beta \epsilon S“\mu$
or, what is equivalent, $\mu \epsilon \lambda . \alpha \epsilon \mu . y \epsilon \alpha . T \epsilon \text{Potid}‘S . \supset . \alpha \epsilon T“\mu$
In that case, putting $Q = \dot{s}‘\text{Potid}‘S$, put
$\gamma = \mu \cap \hat{\beta}\{y \epsilon s‘\overrightarrow{Q}‘\beta\}$. Then $y \sim \epsilon p‘s‘\lambda . \equiv . \iota‘x \sim \epsilon \gamma$
$\mu' = \mu - \gamma$
Then $\mu \epsilon \lambda . \supset . \mu' \epsilon \lambda$ if $y \sim \epsilon p‘s‘\lambda$. But $y \sim \epsilon s‘\mu'$. Hence $y \epsilon p‘s‘\lambda$
This proves our prop, if $\gamma$ could be more simply defined.

[10] Inverted with respect to 3r. [11] $M$ is so defined at *PM*, p.653.
[12] Russell repeatedly tries to prove $p‘s“\lambda \subset s‘p‘\lambda$, at ALP, p.[11r], [14r], [15r], and elsewhere. It is not in general true that the intersection of the union of the members of $\lambda$ is among the union of the product of the members of $\lambda$.

Within $\mu$, we have two cases: $\alpha \epsilon \mu \,.\, S`\alpha = \alpha$ and $\alpha \epsilon \mu \,.\, S`\alpha \neq \alpha$.
If $S`\iota`x = \iota`x$, $\quad \overrightarrow{S_*}`\iota`x = \iota`\iota`x \,.\, xRx \,.\, \overrightarrow{R}`x = \iota`x \,.\, \overrightarrow{R_*}`x = \iota`x$

Thus we may assume $S`\iota`x \neq \iota`x$

---

Can we define $\gamma$ more simply?
We have first $\beta \epsilon \mu \,.\, y \epsilon \beta$
second $\quad \beta \epsilon \mu \,.\, y \epsilon R``\beta$
third $\quad \beta \epsilon \mu \,.\, y \epsilon R``R``\beta$
generally $\beta \epsilon \mu \,.\, y \epsilon M``\beta$ where $M \epsilon \mathrm{Potid}`R$
i.e. if $\eta = \mu \cap \overleftarrow{\epsilon}`y$,
we shut out $\eta$, $\mu \cap \overleftarrow{S}`\eta$, $\mu \cap \breve{S}``\overleftarrow{S}`\eta$ etc. i.e. $\mu \cap \breve{S}_*``\eta$
Thus $\mu' = \mu - \breve{S}_*``\eta$

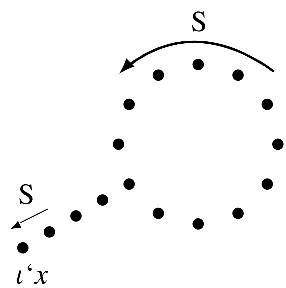
$$\begin{aligned}\breve{S}_*``\eta &= \hat{\alpha}\{(\exists\beta).\ \beta \epsilon \eta \,.\, \beta S_* \alpha\} \\ &= \hat{\alpha}\{(\exists\beta) :\ \beta \epsilon \mu \,.\, y \epsilon \beta : \beta \epsilon \nu \,.\, \breve{S}``\nu \subset \nu \,.\, \supset_\nu \,.\, \alpha \epsilon \nu\}\end{aligned}$$

[4v][13]

Apply $*97$

If $\xi = \mu \cap \hat{\alpha}\{\alpha(S_*|S)\alpha\}$,

Then if $\exists!\xi$, $\quad \overrightarrow{S_*}`\iota`x$ is finite.




We have $(\exists T) \,.\, T \epsilon \mathrm{Potid}`S \,.\, \overrightarrow{S_*}`\iota`x = S(T`\iota`x \ \longmapsto\!\!\!\dashv\ \iota`x)$

$$\begin{aligned}\gamma \epsilon \overrightarrow{S_*}`\iota`x \,.\equiv:\ \iota`x \epsilon \mu \,.\, S``\mu \subset \mu \,.\ &\supset_\mu .\gamma \epsilon \mu \\ &\supset_\mu \,.\, R``\gamma \epsilon \mu\end{aligned}$$

Hence $\gamma \epsilon \overrightarrow{S_*}`\iota`x \,.\supset.\, R``y \epsilon \overrightarrow{S_*}`\iota`x$.
This is using $S``\overrightarrow{S_*}`\iota`x \subset \overrightarrow{S_*}`\iota`x$ i.e. $S|S_* \in S_*$.
Hence $R``s`\overrightarrow{S_*}`\iota`x \subset s`\overrightarrow{S_*}`\iota`x$. Also $x \epsilon s`\overrightarrow{S_*}`\iota`x$.
Assume $yR_*x \,.\, y \sim\epsilon\, s`\overrightarrow{S_*}`\iota`x$

---

[5]

Consider $\iota`x \epsilon \mu \,.\, \beta \epsilon -\mu \,.\, S``\mu \subset \mu$
Then $\breve{S}`` - \mu \subset -\mu$. Hence $\overleftarrow{S_*}`\beta \subset -\mu$. Assume $\iota`x \sim\epsilon\, \overleftarrow{S_*}`\beta \,.\, y \epsilon \beta$
$\iota`x \epsilon \mu \,.\, \overleftarrow{\epsilon}`y \subset -\mu \,.\, S``\mu \subset \mu$

[13] A similar diagram appears in $*96$ at *PM*, p.607 on the posterity of a term.

Then $S_*``\overleftarrow{\epsilon}`y \subset -\mu$

$\breve{S}_*``\overleftarrow{\epsilon}`y = \hat{\alpha}\{(\exists\beta) : y \epsilon \beta : \beta \epsilon \mu . S``\mu \subset \mu . \supset_\mu . \alpha \epsilon \mu\}$

Thus $\breve{S}_*``\overleftarrow{\epsilon}`y \subset -\mu . \equiv :. y \epsilon \beta : \beta \epsilon \nu . S``\nu \subset \nu . \supset_\nu . \alpha \epsilon \nu : \supset_\alpha . \alpha \sim \epsilon \mu$

$\gamma \epsilon \overrightarrow{S_*}`\alpha . \equiv : \alpha \epsilon \mu . S``\mu \subset \mu . \supset_\mu . \gamma \epsilon \mu$

Any one $\mu$ consists of $\overrightarrow{S}_*`\alpha$ together with the $S_*$ s of other classes.

i.e. $\mu = S_*``(\mu - S``\mu)$, or, if $\mu = S``\mu$, $\mu = S_*``\mu$

$\mu = S``\mu . \equiv : \gamma \epsilon \mu . \equiv_\gamma . (\exists\delta). \delta \epsilon \mu . \gamma S \delta$

Put $\xi = \mu \cap \hat{\alpha}(\alpha = S`\alpha) . \eta = \mu \cap \hat{\alpha}(\alpha \neq S`\alpha)$

*No* . Put $\xi = \mu \cap \hat{\alpha}\{\alpha(S_*|S)\alpha\} . \eta = \mu \cap \hat{\alpha}\{\sim \alpha(S_*|S)\alpha\}$

$\alpha(S_*|S)\alpha . \supset : \alpha S_*(S`\alpha) : \supset : (S`\alpha)(S|S_*)(S`\alpha)$

Hence $S``\xi \subset \xi . \breve{S}``\eta \subset -\xi$

[6]

Given $\mu \epsilon \lambda . \alpha \epsilon \mu . y \epsilon \alpha . T \epsilon \text{Potid}`S . \supset . \alpha \epsilon T``\mu$

and putting $Q = \dot{s}`\text{Potid}`S$, put[14]

$\gamma = \hat{\beta}\{\beta \epsilon \mu : (\exists T). T \epsilon \text{Potid}`S . y \epsilon T`\beta\}$

$\mu' = \mu - \gamma$

We want to prove $\mu' \epsilon \lambda . y \sim \epsilon s`\mu'$

We assume $\iota`x \sim \epsilon \gamma$. Hence $\iota`x \epsilon \mu'$.

$\beta \epsilon \mu . \supset . s`\beta \epsilon \mu$

$\beta \epsilon \gamma . \beta = S`\alpha . \supset . T|S \epsilon \text{Potid}`S . y \epsilon (T|S)`\alpha$

$\beta \epsilon \gamma . \beta = S`\alpha . \alpha \epsilon \mu . \supset . \alpha \epsilon \gamma$

$\alpha \epsilon \mu - \gamma . \supset . S`\alpha \epsilon -\gamma$

Hence $\alpha \epsilon \mu - \gamma . \supset . S`\alpha \epsilon \mu - \gamma$

Hence $\iota`x \epsilon \mu - \gamma . \supset . \mu - \gamma \epsilon \lambda$ Also $y \sim \epsilon s`(\mu - \gamma)$. Hence prop, if $\gamma$ is a legitimate class.

Now $\gamma = \mu \cap \hat{\beta}\{(\exists T, \alpha). T \epsilon \text{Potid}`S . \alpha T \beta . y \epsilon \alpha\}$

$= \mu \cap \hat{\beta}\{y \epsilon s`\overrightarrow{Q}`\beta\} = \mu \cap \hat{\beta}\{y \epsilon s`\overrightarrow{S_*}`\beta\}$

Thus $\mu - \gamma = \mu \cap \hat{\beta}\{y \sim \epsilon s`\overrightarrow{S_*}`\beta\}$

$y \sim\epsilon s`\overrightarrow{S_*}`\beta . \equiv :. \beta \epsilon \mu . S``\mu \subset \mu . \supset_\mu . \alpha \epsilon \mu : \supset_\alpha . y \sim \epsilon \alpha$

[7]

Suppose now

$\mu \epsilon \lambda . \alpha \epsilon \mu . y \epsilon \alpha . \supset_{\alpha,\mu} : T \epsilon \text{Potid}`S . \supset_T . \alpha \epsilon T``\mu$

which is the only remaining case. What we have proved is

$\iota`x, \beta \epsilon \mu - S``\mu . T \epsilon \text{Potid}`S . y \epsilon T`\beta . \beta \neq \iota`x . u' = \mu - S(T`\beta \longmapsto \beta).$

$\supset . y \sim \epsilon S`\mu' . \mu' \epsilon \lambda$

14 $\dot{s}`\text{Potid}`S$ appears in $*89{\cdot}28$ (*PM*, p.657) and below in ALP, p.[7].

whence, since $y \epsilon p`s``\lambda$,

$\sim (\exists \beta, T). \beta \epsilon \mu - S``\mu - \iota`x . T \epsilon \text{Potid}`S . y \epsilon T`\beta$

we now assume

$\mu \epsilon \lambda . \beta \epsilon \mu . T \epsilon \text{Potid}`S . y \epsilon T`\beta . \supset . \beta \epsilon S``\mu$

$\beta = S`\alpha . \supset . T|S \epsilon \text{Potid}`S . y \epsilon (T|S)`\alpha$

Hence putting $\gamma = \hat{\beta}\{\beta \epsilon \mu : (\exists T) . T \epsilon \text{Potid}`S . y \epsilon T`\beta\}$,
we have $\gamma \subset S``\gamma$. But $\beta \epsilon \gamma . \supset . S`\beta \epsilon \mu$
Also $T \epsilon \text{Pot}`S . y \epsilon T`\beta . \supset . (\exists T) . T \epsilon \text{Potid}`S . y \epsilon T`S`\beta$
Hence $\beta \epsilon \gamma . y \sim \epsilon \beta . \supset . S`\beta \epsilon \gamma$ i.e. $S``(\gamma \cap - \overleftarrow{\epsilon}`y) \subset \gamma$

---

We have proved

$y \epsilon p`s``\lambda . \mu \epsilon \lambda . \beta \epsilon \mu - S``\mu . T \epsilon \text{Potid}`S . y \epsilon T`\beta . \supset . \beta = \iota`x$

The remaining case is where

$\mu \epsilon \lambda . \beta \epsilon \mu . T \epsilon \text{Potid}`S . y \epsilon T`\beta . \supset . \beta \epsilon S``\mu$

Then $\mu \epsilon \lambda . \alpha \epsilon \mu . y \epsilon \underline{\alpha} . T \epsilon \text{Potid}`S . \supset . \alpha \epsilon T``\mu$
We want to deduce $\mu - \overrightarrow{P}`\overleftarrow{\epsilon}`y \epsilon \lambda$ where $P = \dot{s}`\text{Potid}`S$

[8]

We have $\mu \epsilon \lambda . \supset_\mu . y \epsilon s`\mu$
Suppose $y \epsilon s`(T``\mu - R``T``\mu)$
i.e. $(\exists \alpha, \beta) . \beta \epsilon \mu - R``\mu . \alpha T \beta . y \epsilon \alpha$
If $\beta = \iota`x$, then $yPx$ where $P = \dot{c}`\text{Potid}`R$
If $\beta \neq \iota`x$,

take away $\beta - \iota`x$ from $\mu - R``\mu$, $R``(\beta - \iota`x)$ from $R``\mu - R``R``\mu$, etc.

The result should be a new member of $\lambda$, say $\mu`$, for which $y \sim \epsilon s`\mu`$.
If $\alpha T \beta$, take away $S\{(\beta - \iota`x) \longmapsto T`(\beta - \iota`x)\}$, which is a finite class.
We have to prove that what remains is a member of $\mu$
*To prove*

$\beta \epsilon \mu - R``\mu . \alpha T \beta . y \epsilon \alpha . \beta \neq \iota`x .$

$\mu' = \mu - S\{(\beta - \iota`x) \longmapsto T`(\beta - \iota`x)\} . \supset . \mu' \epsilon \lambda$

$\alpha \epsilon \mu . \supset . S`\alpha \epsilon \mu$

$\alpha \epsilon S(\gamma \longmapsto \delta) . \supset : S`\alpha \epsilon - S(\gamma \longmapsto \delta) . \vee . S`\alpha = \delta$

---

$\beta \epsilon \mu$, $R``\beta \epsilon \mu$, etc.
If $\beta \neq \iota`x$, take away $S(T`\beta \longmapsto \beta)$ from $\mu$. Put $\mu' = \mu - S(T`\beta \longmapsto \beta)$
$\alpha \epsilon \mu' . \supset . S`\alpha \epsilon \mu$
$\alpha \epsilon - S(T`\beta \longmapsto \beta) . \supset : S`\alpha \epsilon - S(T`\beta \longmapsto \beta) . \vee . S`\alpha = \beta$
But $\beta \epsilon \mu - S``\mu \therefore \alpha \epsilon \mu . \supset . S`\alpha \neq \beta \therefore \alpha \epsilon \mu' . \supset . S`\alpha \epsilon \mu'$.
It remains to prove $\iota`x \sim \epsilon S(T`\beta \longmapsto \beta)$.

We assume $\iota`x \in \mu - S``\mu$; hence prop follows since $\iota`x \neq \beta$.
This covers the case where $(\exists\beta, T) \,.\, \beta \in \mu - S``\mu \,.\, T \in \text{Potid}`S \,.\, y \in T`\beta$

[9]

Must prove $\mu \in \lambda \,.\, \alpha \in \mu - p`\lambda \,.\supset.\, \mu - \overrightarrow{S}`\alpha - \overleftarrow{S}`\alpha \in \lambda$
If $\mu \in \lambda$, $\mu$ consists of $\overrightarrow{S_*}`\iota`x$ together with $\overrightarrow{S_*}`\beta$, $\overrightarrow{S_*}`\gamma$ etc.
Thus we should first settle on $\mu - S``\mu$.
Suppose $(\exists\alpha) \,.\, \alpha \in \mu \,.\, \iota`x S_* \alpha$
Then $S(\iota`x \vdash\!\dashv \alpha)$ is a finite class. Suppose $\alpha \in \mu - S``\mu$
Then put $\mu' = \mu - S(\iota`x \vdash\!\dashv \alpha) \cup \iota`\iota`x$
~~Assume~~ Then $\iota`x \in \mu' \;:\; \beta \in \mu' \,.\supset.\, S`\beta \in \mu$
To prove $S`\beta \sim\in S(\iota`x \vdash\!\dashv \alpha)$, we want
$\sim\{\iota`x S_* \beta \,.\, \beta S_* \alpha\} \,.\supset.\, \sim\{\iota`x S_*(S`\beta) \,.\, (S`\beta) S_* \alpha\}$
If $\sim \beta S_* \alpha$, it is obvious unless $S`\beta = \alpha$
which is impossible because $\alpha \in \mu - S``\mu$.
If $\sim(\iota`x S_* \beta)$, we have $\sim(\iota`x = \beta) \,.\, \sim\{\iota`x(S|S_*)\beta\}$
$\sim\{\iota`x(S_*|S)\beta\}$
Hence proposition in either case.
Hence $\beta \in \mu' \,.\supset.\, S`\beta \in \mu'$. Hence $\mu' \in \lambda$.
Then $\mu' \cap \overleftarrow{S_*}`\iota`x = \iota`\iota`x \,.\, \iota`x \in \mu' - S``\mu'$
Divide $\mu$ into two parts $\xi, \eta$ such that $\xi = S``\xi \,.\, \eta \neq S``\eta$
The whole of $\mu$ can be divided into
$\mu - S``\mu,\; S``\mu - S``S``\mu,\; \ldots T``\mu - S``T``\mu,\; \ldots [T \in \text{Potid}`S]$
$\cancel{p`\text{D}``\text{Potid}`S}$
and a residue $\hat{\alpha}\{T \in \text{Potid}`S \,.\supset_T.\, \alpha \in T``\mu\}$

[10]

$y \sim\in s`p`\lambda \,.\supset:\; y \in \alpha \,.\supset_\alpha.\, (\exists\mu).\, x \in s`\mu \,.\, R``s`\mu \subset s`\mu \,.\, \alpha \sim\in \mu$
$y \sim\in p`s``\lambda \,.\supset:.\; y \in \lambda \,.\supset_\mu.\, (\exists\alpha).\, \alpha \in \mu \,.\, y \in \alpha \;:.$
$\supset:.\, y \in \alpha \,.\supset_\alpha.\, \alpha \sim\in \mu :\supset_\mu:\; \mu \sim\in \lambda$
$\cancel{\supset: (\exists\mu \underline{y}) \,.\, x \in s`\mu \,.\, R``s`\mu \subset s`\mu \,.\, \alpha \in \mu \,.\, y \in \alpha}$
$\cancel{\supset:\; \text{Cl}`s`\mu \in \lambda \,.\supset_\mu.\, (\exists\alpha). y \in \alpha \,.\, \alpha \subset s`\mu}$

---

"$y \sim\in s`p`\lambda$" implies: ~~If $s`\mu$ is an inductive class,~~
If $\alpha$ is any class of which $y$ is a member, there is a class $\mu$ such that $s`\mu$ is inductive but $\alpha \sim\in \mu$.
$y \sim\in s`p`\lambda \,.\equiv:.\, \alpha \in p`\lambda \,.\supset_\alpha.\, y \sim\in \alpha \;:.$
$\equiv:.\, y \in \alpha \,.\supset_\alpha: (\exists\mu).\, \mu \in \lambda \,.\, \alpha \sim\in \mu$

$y \in p‘s“\lambda \,.\, \equiv :.\, y \in \alpha \,.\, \supset_\alpha .\, \alpha \sim\in \mu \;:\supset_\mu .\, \mu \sim\in \lambda$
Now $\mathrm{Cl}‘s‘\mu \;\sim\in \lambda \,.\supset .\, \mu \sim\in \lambda$ Hence
$y \in p‘s“\lambda \,.\supset :.\, y \in \alpha \,.\supset_\alpha .\sim(\alpha \subset s‘\mu) : \supset_\alpha .\, \mu \sim\in \lambda$
$y \sim\in s‘p‘\lambda \,.\, \equiv :.\, y \in \alpha \,.\supset_\alpha .\, (\exists\mu).\, \mu \in \lambda \,.\, \alpha \sim\in \mu$
Consider $\hat\mu\{\mu \in \lambda \,.\, \alpha \sim\in \mu\}$ Call it $\lambda_\alpha$. Then $y \in \alpha \,.\supset_\alpha .\, \exists!\lambda_\alpha$
$\lambda - \lambda_\alpha = \hat\mu\{\mu \in \lambda \,.\, \alpha \in \mu\}$ $\qquad p‘(\lambda - \lambda_\alpha) = \hat\beta(\mu \in \lambda \,.\, \alpha \in \mu \,.\; \supset_\mu .\, \beta \in \mu)$
$\alpha \in p‘(\lambda - \lambda_\alpha) \,.\, y \in s‘p‘(\lambda - \lambda_\alpha)$
$p‘\lambda_\alpha = \hat\beta\{\mu \in \lambda \,.\, \alpha \sim\in \mu \,.\; \supset_\mu .\, \beta \in \mu\}$

[11r]

We have $\mu \in \lambda \,.\supset .\, \mathrm{Cl}‘s‘\mu \in \lambda$
Can we prove $s‘p‘\lambda = p‘s“\lambda$ by this means?
We want $p‘s“\lambda \subset s‘p‘\lambda$
$\mathrm{Cl}“s“\lambda \subset \lambda$
Assume $y \in p‘s“\lambda - s‘p‘\lambda$
Then $\mu \in \lambda \,.\supset .\, y \in s‘\mu \;:.\, \mu \in \lambda \,.\supset_\mu .\, \alpha \in \mu : \supset_\alpha .\, y \sim\in \alpha$
$\mathrm{Cl}‘s‘\mu \in \lambda \,.\supset_\mu .\, \alpha \in \mu : \supset :\; \mu \in \lambda \,.\supset_\mu .\, \alpha \in \mu$
$\mu \in \lambda \,.\supset_\alpha .\, \alpha \in \mu : \supset_\alpha .\, y \sim\in \alpha \;:.$
$\qquad\qquad \supset :.\, \mathrm{Cl}‘s‘\mu \in \lambda \,.\supset_\mu .\, \alpha \in \mu : \supset_\alpha .\, y \sim\in \alpha$
$\mathrm{Cl}‘s‘\mu \in \lambda \,.\, \equiv :.\, \iota‘x \subset s‘\mu : \alpha \subset s‘\mu \,.\supset_\alpha .\, R“\alpha \subset s‘\mu :.$
$\qquad \equiv :.\, x \in s‘\mu \,.\, R“s‘\mu \subset s‘\mu$
$\therefore\; y \sim\in s‘p‘\lambda \,.\supset :.\, x \in s‘\mu \,.\, R“s‘\mu \subset s‘\mu \;\supset_\mu .\, \alpha \in \mu : \supset_\alpha .\, y \sim\in \alpha$
Now $x \in s‘\mu \,.\, R“s‘\mu \subset s‘\mu \,.\supset_\mu .\, y \in s‘\mu$
And $y \sim\in s‘p‘\lambda \,.\supset :.\, y \in \alpha \,.\supset_\alpha .\, (\exists\mu).\, x \in s‘\mu \,.\, R“s‘\mu \subset s‘\mu \,.\, \alpha \sim\in \mu$
$\qquad\qquad \supset_\alpha .\, (\exists\mu).x, y \in s‘\mu \,.\, \alpha \sim\in \mu \,.\, R“s‘\mu \subset s‘\mu$
i.e. ~~if~~ $s‘\mu$ is an inductive class conatining $x \vee y$, ⌊but⌋no class of which $y$ is a member is a member of $\mu$
~~Now consider $\mu \cup \iota‘\alpha$~~
~~$R“\alpha \subset \alpha \,.\, y \in \alpha \,.\; \supset_\alpha \;.\, (\exists\mu) \,.\, x, y \in$~~

[11v]

We have $\mu \in \lambda \,.\supset .\, \exists\,!\, \mu \cap \overleftarrow{\in}‘y$.
Put $\gamma‘\mu = \mu \cap \overleftarrow{\in}‘y$ and consider $p‘\gamma“\lambda$
We have to prove $\mu \in \lambda \,.\supset .\, p‘\gamma“\lambda \in \mu : y \in p‘\gamma“\lambda$ The last is obvious.
$S“\overrightarrow{R}“\,\cancel{\iota“\xi}$
~~$z \in p‘\gamma“\lambda \,.\equiv:\; \mu \in \lambda \,.\, y \in \mu \,.\supset_\mu .\, z \in \mu$~~
$= \hat\gamma\{(\exists y) \,.\, y \in \xi \,.\, \gamma = S‘\overrightarrow{R}‘\,\cancel{\iota}‘y\}$
$\mu \in \lambda \,.\equiv .\, \iota‘x \in \mu \,.\, S“\mu \subset \mu$
$= \;\overrightarrow{R^2}“\xi = S“S“\iota“\xi$

$\xi \in \gamma‘\mu \,.\equiv.\, \xi \in \mu \,.\, y \in \xi$

$\overrightarrow{R}“\xi = S“\iota“\xi$

$\xi \in \gamma‘\lambda \,.\equiv.\, (\exists\mu) \,.\, \mu \in \lambda \,.\, \xi \in \mu \,.\, y \in \xi$

$R“\alpha = s‘\overrightarrow{R}“\alpha$

$z \in p‘\gamma“\lambda \,.\equiv:\, \mu \in \lambda \,.\, \xi \in \mu \,.\, y \in \xi \,.\supset_{\mu,\xi}.\, z \in \xi$

$R“‘\alpha = s“\overrightarrow{R}“‘\alpha$

$\equiv:\, \iota‘x \in \mu \,.\, S“\mu \subset \mu \,.\, \xi \in \mu \,.\, y \in \xi \,.\supset_{\mu,\xi}.\, z \in \xi$

~~$x \in \mu \,.\, R“\mu \subset \mu \,.\, \supset :.\, \iota‘x \in \iota“\mu : s‘\overrightarrow{R}“\iota“\mu \subset \mu$~~

~~$\alpha S_* \iota‘x. \equiv :.\, \iota‘x \in \xi : S“\xi \subset \xi : \supset_\xi .\, \alpha \in \xi \; :.$~~

~~$\supset :.\, \iota‘x \in \iota“\mu : S“\iota“\mu \subset \iota“\mu : \supset_\mu .\, \alpha \in \iota“\mu$~~

~~$\supset :.\, x \in \mu \,.\, \overrightarrow{R}“\mu \subset \iota“\mu \,.\supset_\mu.\, \alpha \in \iota“\mu$~~

~~$\equiv :.\, \iota‘x \in \xi : \gamma \in \xi \,.\supset_\gamma.\, s‘\overrightarrow{R}“\gamma \in \xi : \supset_\xi .\, \alpha \in \xi$~~

[12r]

We have $\mu \in \lambda \,.\supset_\mu.\, y \in s‘\mu$
Now suppose $y \sim\in s‘p‘\lambda$, i.e. $\alpha \in p‘\lambda \,.\supset.\, y \sim\in \alpha$
Then it seems clear we can construct a $\mu$ such that $y \sim\in s‘\mu$.

---

$x \in \alpha \,.\, R“\alpha \subset \alpha$. If $R“\alpha = \alpha$, then $\overrightarrow{S_*}‘\alpha = \iota‘\alpha$.
Hence $\overrightarrow{S_*}‘\iota‘x \subset \iota‘\alpha$ and $s‘\overrightarrow{S_*}‘\iota‘x \subset \alpha$
If $\exists!\alpha - R‘\alpha \,.\, R“\alpha = R“R“\alpha$, then $\overrightarrow{S_*}‘S‘\alpha = \iota‘S‘\alpha$.
Hence $s‘\overrightarrow{S_*}‘\alpha \subset \alpha \cup S‘\alpha \subset \alpha$ Hence $s‘\overrightarrow{S_*}‘\iota‘x \subset \alpha$
Divide $\alpha$ into $\alpha - R“\alpha,\ R“\alpha - R“R“\alpha, \ldots T“\alpha - R“T“\alpha, \ldots p‘\mathrm{D}“\mathrm{Potid}‘R$
Put $\beta = \alpha - R“\alpha$. ~~Then~~

$\overrightarrow{S_*}‘\iota‘x = \hat{y}\{\iota‘z \in \mu \,.\, S“\mu \subset \mu \,.\supset_\mu.\, y \in \mu\}$

$y \in \alpha \,.\supset:\, y \in (\alpha - R“\alpha) \cup R“(\alpha - R“\alpha) \cup \ldots$

$\cup\, \hat{z}\{T \in \mathrm{Potid}‘R \,.\supset_T.\, z \in T“\alpha\}$

Assume $\alpha - R“\alpha = \iota‘x$. Then the series is $\overrightarrow{S_*}‘\iota‘x$. The residue is
$\alpha \cap \hat{z}\{\sim(\exists T) \,.\, T \in \mathrm{Potid}‘R \,.\, z \in T“(\alpha - R“\alpha)\}$

---

$x \in \alpha \,.\, R“\alpha \subset \alpha \,.\supset:\, \iota‘x \in \mathrm{Cl}‘\alpha : \gamma \subset \alpha \,.\supset.\, R“\gamma \subset R“\alpha \subset \alpha$

$\supset:\, \iota‘x \in \mathrm{Cl}‘\alpha \,.\, S“\mathrm{Cl}‘\alpha \subset \mathrm{Cl}‘\alpha$

$\iota‘x \subset \beta \,.\supset.\, \overrightarrow{S_*}‘\iota‘x \subset \overrightarrow{S_*}‘\beta$ $\qquad$ $y \in \alpha \,.\supset.\, R“\iota‘y \subset \alpha = \overrightarrow{R}‘y$

$\iota“R“\iota‘y \subset \iota“\alpha$

~~$\iota x \in \mu \,.\, S“\mu \subset \mu$~~ $\qquad$ $x \in \alpha \,.\, R“\alpha \subset \alpha \,.\supset.\, \iota‘x \in \iota“\alpha \,.\, S“\iota“\alpha = \overrightarrow{R}“\alpha$

---

$P = \dot{s}‘\mathrm{Potid}‘R \,.\supset.\, R“\overrightarrow{P}‘x \subset \overrightarrow{P}‘x$ $\qquad$ $x \in \alpha \,.\, R“\alpha \subset \alpha \,.\supset.$

$\iota‘x \in \mathrm{Cl}‘\alpha \,.\, S“\mathrm{Cl}‘\alpha \subset \mathrm{Cl}‘\alpha$

Hence $\overrightarrow{R_{*4}}‘x \subset \overrightarrow{P_3}‘x$ $\qquad$ Hence $x \in \alpha \,.\, R“\alpha \subset \alpha \,.\supset.\, \mathrm{Cl}‘\alpha \in \lambda$
Perhaps $\mu \in \lambda \,.\supset.\, \mathrm{Cl}‘s‘\mu \in \lambda$? *yes*

$\mu \in \lambda . \supset : \alpha \in \mu . \supset . R``\alpha \in \mu :$
$\supset : \alpha \in \mu . \beta \subset \alpha . \supset . R``\beta \subset s`\mu : \supset : \beta \subset s`\mu . \supset . R``\beta \subset s`\mu$

[12v][15]

*General Hypothesis*
$R \in \mathrm{Cls} \to 1 . \mathrm{D}`R = V . \sim(xRx) . \kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha . x \in \alpha)$
$. \lambda = \hat{\beta}(R``\beta \subset \beta . z \in \beta)$

$*40{\cdot}37 \quad \vdash . R``p`\kappa \subset p`R```\kappa$
$*72{\cdot}341 \quad \vdash : R \in \mathrm{Cls} \to 1 . \exists ! \kappa . \supset . p`R```\kappa = R``p`\kappa$
We have $\breve{R}``p`\kappa \subset p`\breve{R}```\kappa \subset p`\kappa . p`R```\lambda = R``p`\lambda \subset p`\lambda$
$xR_*y . \equiv . y \in p`\kappa . \equiv . x \in p`\lambda$
$x(R_*|R)y . \equiv . y \in \breve{R}``p`\kappa$
$x(R|R_*)y . \equiv . y \in p`\breve{R}```\kappa$

*Dem.*
$x(R|R_*)y . \supset : \breve{R}`x \in \breve{R}``\alpha . \breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha . \supset_\alpha . y \in \breve{R}``\alpha :$
$\supset : x \in \alpha . \breve{R}``\alpha \subset \alpha . \supset_\alpha . y \in \breve{R}``\alpha :$
$\supset : y \in p`\breve{R}```\kappa$ (1)
$\breve{R}`x \in \breve{R}``\alpha . \gamma = \alpha \cup \iota`x . \supset . \breve{R}``\gamma = \breve{R}``\alpha . x \in \gamma$ (2)
$y \in p`\breve{R}```\kappa . \supset : \breve{R}``\alpha \subset \alpha . x \in \alpha . \supset_\alpha . y \in \breve{R}``\alpha$

[(2)] $\supset : \breve{R}``\breve{R}``\gamma \subset \breve{R}``\gamma . \breve{R}`x \in \breve{R}``\gamma . \supset_\gamma . y \in \breve{R}``\gamma :$
$\supset : \beta \subset \text{Ɑ}`R . \breve{R}``\beta \subset \beta . \breve{R}``x \in \beta . \supset_\beta . y \in \beta :$
$\supset : \breve{R}``(\delta \cap \text{Ɑ}`R) \subset \delta . \breve{R}`x \in \delta . \supset_\delta . y \in \delta :$
$\supset : \breve{R}``\delta \subset \delta . \breve{R}`x \in \delta . \supset_\delta . y \in \delta :$
$\supset : x(R|R_*)y$ (3)
(1).(3). $\supset \vdash$ . Prop
Hence $R_*|R \mathrel{\subset\!\!\!\cdot} R|R_*$

[13r]

*To prove* $\quad p`(s`\lambda \cap \overleftarrow{\epsilon}`y) \in p`\lambda$.[16]
$z \in p`(s`\lambda \cap \overleftarrow{\epsilon}`y) . \equiv : \alpha \in s`\lambda . y \in \alpha . \supset_\alpha . z \in \alpha$
$\supset : \alpha \in s`\lambda . y \in \alpha . y \neq z . \supset_\alpha . \alpha - \iota`z \sim \in s`\lambda :$
$\alpha \in s`\lambda . y \in \alpha - \beta . z \in \beta . \supset_{\alpha,\beta} . \alpha - \beta \sim \in s`\lambda$
$\mu \in \lambda . \equiv . \iota`x \in \mu . S``\mu \subset \mu$ Hence $s`\lambda = V$. Thus this won't do.

---

Try $\mu, \nu \in \lambda . \supset . p`(\mu \cap \overleftarrow{\epsilon}`y) = p`(\nu \cap \overleftarrow{\epsilon}`y)$

15 See the discussion of what must be "reproved" in *PM*, p.651.
16 $\overleftarrow{\epsilon}`x$ is defined as $\hat{\alpha}(x \in \alpha)$ at $*62{\cdot}21$. It doesn't appear in *PM* or in HPF.

$z \in p`(\mu \cap \overleftarrow{\in}`y) . \equiv : \alpha \in \mu . y \in \alpha . \supset_\alpha . z \in \alpha$

$\supset : \; y \in \alpha . z \in -\alpha . \supset_\alpha . \alpha \sim \in \mu$

~~Suppose~~ $\cancel{y \sim \in s`p`\lambda}$~~. Then~~ $\cancel{\beta \in p`\lambda . \supset . y \in -\beta}$~~. Hence~~ $\cancel{z \in \beta . \supset . -\beta \sim \in \mu}$

Hence $\iota`y \sim \in \mu$ unless $p`(\mu \cap \overleftarrow{\in}`y) = \iota`y$

Suppose $y \sim \in s`p`\lambda$. In that case, we can add $\overrightarrow{S_*}`(\iota`y \cup \iota`z)$ to $\mu$ and we still have a member of $\lambda$. We have

$$(\exists\alpha) . \alpha \in \mu . y \in \alpha \quad \text{but} \quad s`(\mu \cap \overleftarrow{\in}`y) \subset -p`\lambda$$

Hence $p`(\mu \cap \overleftarrow{\in}`y) \subset -p`\lambda$

Try $p`\gamma``\lambda$

$$\alpha \in p`\gamma``\lambda . \equiv : \mu \in \lambda . \supset_\mu . \alpha \in \mu \cap \overleftarrow{\in}`y$$

This assumes $y \in p`\lambda$

---

$\gamma`\mu = \mu \cap \overleftarrow{\in}`y \qquad\qquad y \in p`\gamma`\mu$

$y \in p`p``\gamma``\lambda$

Must try to prove $p`p``\gamma``\lambda \subset s`p`\lambda$

$p`p``\gamma``\lambda = p`s`\gamma``\lambda$ Hence $y \in p`s`\gamma``\lambda$

$s`\gamma``\lambda = \hat{\alpha}\{(\exists\mu) . \mu \in \lambda . \alpha \in \mu . y \in \alpha\}$

$p`s`\gamma``\lambda = \hat{z}\{\mu \in \lambda . \alpha \in \mu . y \in \alpha . \supset_{\mu,\alpha} . z \in \alpha\}$

[13v]

$\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha . x \in \alpha)$. $\lambda = \hat{\beta}(\breve{R}``\beta \subset \beta . x \in \beta)$. $R \in \text{Cls} \to 1 . \text{D}`R = V$

$\breve{R}``p`\kappa \subset p`\breve{R}```\kappa \subset p`\kappa \quad *72{\cdot}341 \vdash : R \in \text{Cls} \to 1 . \exists!\kappa . \supset . p`R```\kappa = R``p`\kappa$

$*40{\cdot}37 \;\; \breve{R}``p`\kappa \subset p`\breve{R}```\kappa$

$y \in \breve{R}``p`\kappa . \equiv . x(R_*|R)y$

$x(R|R_*)y . \supset . y \in p`\breve{R}```\kappa$

$y \in p`\breve{R}```\kappa . \supset : \breve{R}``\alpha \subset \alpha . x \in \alpha . \supset_\alpha . y \in \breve{R}``\alpha$

$\supset : \breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha . \breve{R}`x \in \breve{R}``\alpha . \supset_\alpha . y \in \breve{R}``\alpha$

if we can get $\breve{R}`x \in \breve{R}``\alpha . \supset . x \in \alpha$.

Suppose $x \sim \in \alpha$. Put $\gamma = \alpha \cup \iota`x$. Then $\breve{R}`\gamma \subset \gamma . x \in y . \breve{R}`x \in \breve{R}``\gamma$

Hence $\breve{R}``\alpha \subset \alpha . x \in \alpha . \supset_\alpha . y \in \breve{R}``\alpha : \supset : y \in \breve{R}``\gamma$. Hence

$y \in p`\breve{R}```\kappa . \supset : \breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha . \breve{R}`x \in \breve{R}``\alpha . \supset_\alpha . y \in \breve{R}``\alpha$

$\supset : \beta \subset \text{Ɑ}`R . \breve{R}``\beta \subset \beta . \breve{R}`x \in \beta . \supset_\beta . y \in \beta$

$\supset : \breve{R}``(\gamma \cap \text{Ɑ}`R) \subset \gamma . \breve{R}`x \in \gamma . \supset_\gamma . y \in \gamma \cap \text{Ɑ}`R$

$\supset : \breve{R}``\gamma \subset \gamma . \breve{R}`x \in \gamma . \supset_\gamma . y \in \gamma : \supset . x(R|R_*)y$

Thus $x(R|R_*)y . \equiv . y \in p`\breve{R}```\kappa$

~~Also~~ $\cancel{R|R_* \mathrel{\subset\!\!\cdot} R_*|R}$ Hence $R|R_* \mathrel{\subset\!\!\cdot} R|R_*$

We want $R|R_* \mathrel{\subset\!\!\cdot} R_*|R$[17]

[17] See ALP, p.[12v] above.

$x(R|R_*)y \,.\equiv.\, x \in R“p‘\lambda \,.\equiv.\, x \in p‘R“‘\lambda$

$x \in p‘R“‘\lambda \,.\equiv:\, R“\beta \subset \beta \,.\, y \in \beta \,.\supset_\beta.\, x \in R“\beta$

$\quad \supset : \beta \subset \breve{R}“\beta \,.\, y \in \breve{R}“\beta \,.\supset_\beta.\, x \in \beta$

$\quad \supset : \breve{R}“\beta \subset \beta \,.\, y \in \breve{R}“\beta \,.\supset_\beta.\, x \in \beta$

$\quad \supset : \breve{R}“\beta \subset \beta \,.\, z \in \beta \,.\, zRy \,.\supset_{z,\beta}.\, x \in \beta : \supset : zRy \,.\supset_z.\, xR_*z$

$\quad \supset : x(R_*|R)y$

Hence $R_*|R = R|R_* \,.\, p‘\breve{R}“‘\kappa = \breve{R}“p‘\kappa$

[14r][18]

$*98\cdot34 \quad \vdash :.\, yR_{*_4}x \,.\supset:\, R“\lambda \subset \lambda \,.\, x \in \lambda \,.\supset.\, y \in \lambda$ *16*

Here $\lambda$ is a class of any order.

*Dem.*

~~$\vdash .\, *98\cdot33 \,.\supset \vdash : \mathrm{Hp} \,.\, R_\epsilon = S \,.\supset.\, y \in s‘\overrightarrow{S_{*_3}}‘\iota‘x \quad (1)$~~

~~$\vdash .\, *98\cdot27 \,.\supset \vdash :: \mathrm{Hp}(1) .\supset :.\, S“\xi \subset \xi \,.\, \iota‘x \in \xi \,.\supset_\xi.\, \overrightarrow{S_{*_3}}‘\iota‘x \subset \xi :.$~~

~~$\supset :.\, S“\mathrm{Cl}‘\lambda \subset \mathrm{Cl}‘\lambda \,.\, \iota‘x \in \mathrm{Cl}‘\lambda \,.\supset_\lambda.\, \overrightarrow{S_{*_3}}‘\iota‘x \subset \mathrm{Cl}‘\lambda :.$~~

~~$\supset :.\, \xi \in \mathrm{Cl}‘\lambda \,.\supset_\xi.\, R“\xi \in \mathrm{Cl}‘\lambda : x \in \lambda : \supset_\lambda .\, \overrightarrow{S_{*_3}}‘\iota‘x \subset \mathrm{Cl}‘\lambda :.$~~

~~$\supset :.\, R“\lambda \subset \lambda \,.\, x \in \lambda \,.\supset_\lambda.\, \overrightarrow{S_{*_3}}‘\iota‘x \subset \lambda \quad (2)$~~

~~$\vdash .(1).(2). \supset \vdash .\, \mathrm{Prop}.$~~

$\vdash :.\, R“\lambda \subset \lambda \,.\, x \in \lambda \,.\equiv:\, \xi \subset \lambda \,.\supset_\xi.\, R“\xi \subset \lambda : x \in \lambda :$

$\quad \equiv : \xi \in \mathrm{Cl}‘\lambda \,.\supset_\xi.\, R“\xi \in \mathrm{Cl}‘\lambda : \iota‘x \in \mathrm{Cl}‘\lambda \qquad (1)$

$\vdash .\, (1). \supset \vdash :.\, R_\epsilon = S. \supset : R“\lambda \subset \lambda.\, x \in \lambda \,.\equiv.\, S“\mathrm{Cl}‘\lambda \subset \mathrm{Cl}‘\lambda.\, \iota‘x \in \mathrm{Cl}‘\lambda \quad (2)$

$\vdash .\, *98\cdot27. \supset \vdash :.\mathrm{Hp}(2). \supset : S“\mathrm{Cl}‘\lambda \subset \mathrm{Cl}‘\lambda.\, \iota‘x \in \mathrm{Cl}‘\lambda. \supset \overrightarrow{S_{*_3}}‘\iota‘x \subset \mathrm{Cl}‘\lambda.$

$\quad \supset .\, s‘\overrightarrow{S_{*_3}}‘\iota‘x \subset \lambda \qquad (3)$

$\vdash .\, *98\cdot33. \supset \vdash : \mathrm{Hp} \,.\, \mathrm{Hp}(2) \,.\supset.\, y \in s‘\overrightarrow{S_{*_3}}‘\iota‘x \qquad (4)$

$\vdash .(2).(3).(4). \supset \vdash .\, \mathrm{Prop}$

[14v]

*To prove* $p‘s“\lambda \subset s‘p‘\lambda$

We want to prove

$y \in p‘s“\lambda \,.\, \mu \in \lambda \,.\, \alpha \in \mu \,.\, y \in \alpha \,.\supset:\, \nu \in \lambda \,.\supset.\, \alpha \in \nu$

We have $\nu \in \lambda \,.\supset.\, (\exists\beta).\, \beta \in \nu \,.\, y \in \nu$

~~First: $\alpha, \beta \in \mu \,.\, y \in \alpha \cap \beta.$~~

$\alpha \in \mu \,.\supset.\, R“\alpha \in \mu \,.\, T“\alpha \in \mu$

$\alpha, \beta \in \mu \,.\, \alpha S_* \beta \,.\supset.\, \alpha = T“\beta$

Hence if $P \subset J,\ \alpha \cap \beta = \Lambda$

$\mu \in \lambda \,.\equiv.\, \iota‘x \in \mu \,.\, S“\mu \subset \mu \,.\, x \in s‘\mu \,.\, R“s‘\mu \subset s‘\mu$

[18] The number 16 on the page and the theorem number suggest that this was a late draft of the last page of the Appendix B manuscript. That page ends with the result listed here as 89·34, and with the subscript $*5$ rather than $*4$ in the antecedent.

Thus $\mu \in \lambda . \supset . R“s‘\mu \subset s‘\mu$

Put $\gamma ‘\mu_{\underline{y}} = \mu \cap \overleftarrow{\epsilon}‘y$. ~~Can we prove $p‘\gamma‘\mu \in \gamma‘\mu . \gamma‘\mu \in \lambda$?~~

~~We have $y \in p‘\gamma‘\mu$. We want~~

~~Take $p‘\gamma‘\mu$, and replace $\mu \cap \overleftarrow{\epsilon}‘y$ by $\overrightarrow{S_*}‘p‘\gamma‘\mu$. I.e. put~~

~~$\nu = \mu - \overleftarrow{\epsilon}‘y \cup \overrightarrow{S_*}‘p‘\gamma‘\mu$~~

~~We want to prove $\nu \in \lambda$~~

Consider $\hat{\alpha}(\alpha \in s‘\lambda . y \in \alpha)$ i.e. $s‘\lambda \cap \overleftarrow{\epsilon}‘y$

Perhaps $p‘(s‘\lambda \cap \overleftarrow{\epsilon}‘y) \in p‘\lambda$. This wants

$$\iota‘x \in \mu . S“\mu \subset \mu . \supset . p‘(s‘\lambda \cap \overleftarrow{\epsilon}‘y) \in \mu$$

$z \in p‘(s‘\lambda \cap \overleftarrow{\epsilon}‘y) . \equiv : \alpha \in s‘\lambda . y \in \alpha . \supset_\alpha . z \in \alpha$

[15r]

*To prove* $\quad yR_{*_4}x . R“\lambda \subset \lambda . x \in \lambda . \supset . y \in \lambda$[19]

We have $\quad y \in s‘\overrightarrow{S_{*_3}}‘\iota‘x$

Thus to prove $\quad y \in s‘\overrightarrow{S_{*_3}}‘\iota‘x . R“\lambda \subset \lambda . x \in \lambda . \supset . y \in \lambda$

i.e. $\quad R“\lambda \subset \lambda . x \in \lambda . \supset . \overrightarrow{S_{*_3}}‘\iota‘x \subset \lambda$

$$\begin{aligned} R“\lambda \subset \lambda . &\equiv : \xi \subset \lambda . \supset_\xi . R“\xi \subset \lambda \\ &\equiv : \xi \in \mathrm{Cl}‘\lambda . \supset_\xi . R“\xi \in \mathrm{Cl}‘\lambda \\ &\equiv : S“\mathrm{Cl}‘\lambda \subset \mathrm{Cl}‘\lambda \end{aligned}$$

---

$*98{\cdot}32$ is wrong. We prove $\overrightarrow{R_{*_2}}‘x \subset p‘s“\lambda$ when we want $s‘p‘\lambda$.[20]

Must try again to prove $p‘s“\lambda \subset s‘p‘\lambda$

Take $\overrightarrow{S_{*_3}}‘\iota‘x$. We have $\iota‘x \in \xi . S“\xi \subset \xi . \supset . \overrightarrow{S_{*_3}}‘\iota‘x \subset \xi$

Whatever the order of $\xi$.

$\lambda = \hat{\mu}(\iota‘x \in \mu . R“‘\mu \subset \mu)$

$y \in p‘s“\lambda . \equiv : \iota‘x \in \mu . R“‘\mu \subset \mu . \supset_\mu . y \in s‘\mu$

required $\quad \supset :. (\exists\alpha) : \iota‘x \in \mu . R“‘\mu \subset \mu . \supset_\mu . \alpha \in \mu : x \in \alpha$

~~Take $\overleftarrow{P}‘y \cap \overrightarrow{P}‘x$. This is a $\mu_2$?~~

~~$\overleftarrow{P}‘y \cap \overrightarrow{P}‘x = \hat{z}\{(\exists T) . T \in |R(R_0$~~

Take $\overleftarrow{P}‘y \cup \overrightarrow{P}‘x$, given $\sim(yPx)$.

$z \in \overleftarrow{P}‘y . \supset . \overleftarrow{R}‘z \subset \overleftarrow{P}‘y : z \in \overrightarrow{P}‘x . \supset . \overrightarrow{R}‘z \subset \overrightarrow{P}‘x$

[15v]

Given $\breve{R}“\overleftarrow{R}_*‘x = \breve{R}“p‘\kappa . \overleftarrow{R}_*‘\breve{R}‘x = p‘\breve{R}“‘\kappa = \breve{R}“p‘\kappa$

To prove $T \in \mathrm{Potid}‘R . \supset . \breve{T}“\overleftarrow{R}_*‘x = \breve{T}“p‘\kappa$ (obvious)

$$\overleftarrow{R}_*‘\breve{T}‘x = p‘\breve{T}“‘\kappa \qquad R|T = T|R$$

$$\breve{T}“p‘\kappa = p‘\breve{T}“‘\kappa \qquad \supset . R|(T|R) = (T|R)|R$$

[19] The same result appears on 14r.

[20] Perhaps the concern with $p‘s“\lambda$ and $s‘p‘\lambda$ has to do with the relation between early results such as the $*98{\cdot}32$ referred to here, and the final $*89{\cdot}32$, which is different.

$xT|R_* y \,.\equiv : \breve{T}‘x \;\epsilon\; \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha .\; y \;\epsilon\; \alpha$

$\quad\supset : \breve{T}‘x \;\epsilon\; \breve{T}“\alpha \,.\, \breve{R}“\breve{T}“\alpha \subset \breve{T}“\alpha \,.\supset_\alpha .\; y \;\epsilon\; \breve{T}“\alpha \qquad [R|T = T|R]$

$\quad\supset : \; x \;\epsilon\; \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha .\; y \;\epsilon\; \breve{T}“\alpha : \qquad \breve{R}|T = T|\breve{R}.$

$\qquad\qquad\qquad\qquad \supset .\; \breve{R}|T|R = T|\breve{R}|R = T$

$\quad\supset : \; y \;\epsilon\; p‘\breve{T}“‘\kappa \qquad\qquad = \breve{R}|R|T = \breve{R}|T|R$

$y \;\epsilon\; p‘\breve{T}“‘\kappa \,.\supset : \; x \;\epsilon\; \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha .\; y \;\epsilon\; \breve{T}“\alpha$

$\quad\supset : x \;\epsilon\; \underline{\breve{T}}\; T“\alpha \,.\, \breve{R}“T‘\alpha \subset T“\alpha \,.\supset_\alpha .\; y \;\epsilon\; \breve{T}“T“\alpha$

$\quad\supset : \breve{T}‘x \;\epsilon\; \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha .\; y \;\epsilon\; \alpha \; [\breve{R}|T = T|\breve{R} \,.\, \breve{T}“T“\alpha = \alpha]$

Hence $xT|R_* y \,.\equiv .\; y \;\epsilon\; p‘\breve{T}“‘\kappa$

We have $\cancel{p‘\breve{T}“‘\kappa}$ $\quad \breve{T}“p‘\kappa \subset p‘\breve{T}“‘\kappa$

We want $\qquad p‘\breve{T}“‘\kappa \subset \breve{T}“p‘\kappa$

Which is the same as $T|R_* \mathrel{\subset\!\!\!\cdot}\; R_*|T$

[16r]

~~To prove~~ $\cancel{x \;\epsilon\; \beta \,.\, R“\beta \subset \beta \,.\, \mu \;\epsilon\; \lambda \,.\supset .\, \cap\,\beta“\mu \;\epsilon\; \lambda}$

$\qquad \cancel{x \;\epsilon\; -\beta \,.\, \breve{R}“\beta \subset \beta \,.\, \mu \;\epsilon\; \lambda \,.\supset .\; -\beta“\mu \;\epsilon\; \lambda} \quad \}$

$\cancel{x \;\epsilon\; \beta \,.\supset .\, \iota‘x \cap \beta = \iota‘\beta \,.\supset .\, \iota‘x \;\epsilon\; \cap\,\beta“\mu}$

$\cancel{\alpha \;\epsilon\; \mu \,.\supset .\, R“\alpha \;\epsilon\; \mu \,.\, R“(\alpha \cap \beta) \subset R“\alpha \cap \beta}$

~~But we want~~ $\cancel{R“\alpha \cap \beta \subset R“(\alpha \cap \beta)}$

~~i.e.~~ $\cancel{y \;\epsilon\; \beta \,.\, yRz \,.\, z \;\epsilon\; \alpha \,.\supset .\, (\exists w) \,.\, w \;\epsilon\; \alpha \cap \beta \,.\, yRw}$

$x \;\epsilon\; \beta \,.\, R“\beta \subset \beta \,.\, x \;\epsilon\; \gamma \,.\, \breve{R}“\gamma \subset \gamma \,.\, \xi = \iota‘x \cup R“\beta \cup \breve{R}“\gamma \,.\supset .$

$\quad R“\xi \subset \beta \cup R“\breve{R}“\gamma$

---

Instead of $\breve{R}“\beta \subset \beta$ we may have $s‘\mu \cap \breve{R}“\beta \subset \beta$

Thus $\cancel{\breve{R}}$

$\breve{R}“\beta \subset \beta \,.\, s‘\mu \subset \beta \cup -\breve{R}“\beta \,.\, x \;\epsilon\; \beta \,.\supset : \mu \;\epsilon\; \lambda \,.\supset .\, \cap\,\beta“\mu \;\epsilon\; \lambda$

$\breve{R}“\beta \subset \beta \,.\, s‘\mu \subset -\beta \cup -\breve{R}“-\beta \,.\, x \;\epsilon\; -\beta \,.\supset : \mu \;\epsilon\; \lambda \,.\supset .\, -\beta“\mu \;\epsilon\; \lambda$

Now $\mu \;\epsilon\; \lambda \,.\supset .\, \overrightarrow{R_*}‘x \subset s‘\mu$. Hence

$R“\beta \subset \beta \,.\, x \;\epsilon\; \beta \,.\, \mu \;\epsilon\; \lambda \,.\, s‘\mu \subset \beta \cup -\breve{R}“\beta \,.\supset .\, \overrightarrow{R_*}‘x \subset \beta$

---

~~To prove~~ $\cancel{\overrightarrow{P}‘x \;\epsilon\; \lambda \,.\, \overrightarrow{P}‘x = p‘\lambda}$

Put $\mu = \hat{\alpha}\{(\exists T).\; T \;\epsilon\; \text{Potid}‘R \,.\, \alpha = \overrightarrow{T}‘x\}$

Then $\iota‘x \;\epsilon\; \mu : \alpha \;\epsilon\; \mu \,.\supset_\alpha .\; R“\alpha \;\epsilon\; \mu$ Hence $\mu \;\epsilon\; \lambda$. Also $\mu = p‘\lambda$

[16v]

To prove $R_* \mathrel{\subset\!\!\!\cdot}\; \dot{s}‘\text{Potid}‘R$[21]

$\qquad T \;\epsilon\; \text{Potid}‘R \,.\, xTy \,.\, yRz \,.\supset .\; T|R \;\epsilon\; \text{Potid}‘R \,.\, x(T|R)z$

Hence $\breve{R}“\,\overleftarrow{\dot{s}‘\text{Potid}‘R}‘x \subset \overleftarrow{\dot{s}‘\text{Potid}‘R}‘x$

[21] A restricted version of this result appears as $*89{\cdot}28$.

Assume $(\exists u).\ xR_*u\ .\ u \sim\epsilon\ \dot{s}\text{‘}\overleftarrow{\text{Potid‘}R}\ \text{‘}x$

Prove $\dot{s}\text{‘}\overleftarrow{\text{Potid‘}}R\ \text{‘}x \subset\ R(x \longmapsto u)$ (if $R \in \text{Cls} \to 1$)

Thence prop if $R \in \text{Cls} \to 1$.

For extension, take $R_\epsilon$. Put $R_\epsilon = S$. Then

$$S_* = \dot{s}\text{‘Potid‘}S$$

i.e. $\alpha S_*\beta\ .\equiv .\ (\exists T).\ T \in \text{Potid‘}S\ .\ \alpha = T\text{‘}\beta$

i.e. $\alpha S_*\iota\text{‘}x\ .\equiv .\ (\exists T).\ T \in \text{Potid‘}S\ .\ \alpha = T\text{‘}\iota\text{‘}x$

To prove $yR_*x\ .\supset .\ (\exists\alpha).\ \alpha S_*\iota\text{‘}x\ .\ y \in \alpha$

Assume $yR_*x : \alpha S_*\iota\text{‘}x\ .\supset_\alpha .\ y \sim\epsilon\ \alpha$

Then $\alpha S_*\iota\text{‘}x\ .\ z \in \alpha\ .\supset_{\alpha,z} .\ yR_*z$ i.e. $\alpha \subset\ \overleftarrow{R}_*\text{‘}y$

$z \in s\text{‘}\overrightarrow{S_*}\text{‘}\iota\text{‘}x\ .\supset .\ (\exists\mu_2)\ .\ R(z \longmapsto x) = \mu_2$

---

Assume $yR_*x\ .\ \sim y(\dot{s}\text{‘Potid‘}R)x$.

$yRz : \xi \in \mu\ .\supset_\xi .\ R\text{“}\xi \in \mu : \supset : z \in s\text{‘}\mu\ .\supset .\ y \in s\text{‘}\mu$

Hence $\iota\text{‘}x \in \mu : \xi \in \mu\ .\supset_\xi .\ R\text{“}\xi \in \mu : \supset : yR_*x\ .\supset .\ y \in s\text{‘}\mu$

Hence if $\lambda$ is the class of $\mu$'s, $y \in p\text{‘}s\text{“}\lambda$ But we want $y \in s\text{‘}p\text{‘}\lambda$

$y \in p\text{‘}s\text{“}\lambda\ .\equiv : \mu \in \lambda\ .\supset_\mu .\ y \in s\text{‘}\mu\ :\equiv : \mu \in \lambda\ .\supset_\mu .\ (\exists\xi).\ \xi \in \mu\ .\ y \in \xi$

$y \in s\text{‘}p\text{‘}\lambda\ .\equiv : (\exists\mu)\ \cancel{\mu \in \lambda\ .\ y \in \mu}\ .\lfloor \xi \in p\text{‘}\lambda\ .\ y \in \xi \rfloor.\ \equiv :$

$$(\exists\xi) : \mu \in \lambda\ .\supset_\mu .\ \xi \in \mu\ :\ y \in \xi$$

[17r]

$\dot{s}\text{‘Potid‘R} = \text{P}\ .\ \text{R}_\epsilon = \text{S}\ .\ \lambda = \hat{\mu}\{\iota\text{‘x} \in \mu\ .\ \text{S“}\mu \subset \mu\}$[22]

$p\text{‘}\lambda = \overrightarrow{S_*}\text{‘}\iota\text{‘}x = \hat{\alpha}\{(\exists T).\ T \in \text{Potid}R\ .\ \alpha = \overrightarrow{T}\text{‘}x\}$

$s\text{‘}p\text{‘}\lambda = \overrightarrow{P}\text{‘}x : \mu \in \lambda\ .\supset .\ x \in s\text{‘}\mu\ .\ R\text{“}s\text{‘}\mu \subset s\text{‘}\mu$

Suppose $\mu \in \lambda\ .\cap \beta\text{“}\mu \in \lambda$

Requires $\cancel{\iota\text{‘}x - \beta}$ $(\exists\alpha)\ .\ \alpha \in \mu\ .\ \iota\text{‘}x = \alpha \cap \beta$ which is satisfied if $x \epsilon \beta$

$$\alpha \in \mu\ .\supset .\ R\text{“}(\alpha \cap \beta) = R\text{“}\alpha \cap \beta$$

$R\text{“}(\alpha \cap \beta) \subset R\text{“}\alpha \cap R\text{“}\beta$

$R\text{“}\beta \subset \beta\ .\supset .\ R\text{“}(\alpha \cap \beta) \subset R\text{“}\alpha \cap \beta$

Required $R\text{“}\alpha \cap \beta \subset R\text{“}(\alpha \cap \beta)$

i.e. $y \in \beta\ .\ yRz\ .\ z \in \alpha\ .\supset .\ z \in \beta$ i.e. $\alpha \cap \breve{R}\text{“}\beta \subset \beta$

Thus $R\text{“}\beta \subset \beta\ .\ \breve{R}\text{“}\beta \subset \beta\ .\ \mu \in \lambda\ .\supset .\cap \beta\text{“}\mu \in \lambda$

Observe $R\text{“}\beta \supset \beta\ .\ \breve{R}\text{“}\beta \subset \beta\ .\equiv .\ R\text{“}-\beta \subset -\beta\ .\ \breve{R}\text{“}-\beta \subset -\beta$

[22] See ALP, p.7 above.

$\cancel{R``\beta \subset \beta \,.\, \breve{R}``\gamma \subset \gamma \,.\, \supset .}$

Hence $\cancel{\cap\beta``\mu \in \lambda \,.\, -\beta``\mu \in \lambda}$

$R``\beta \subset \beta.\ \breve{R}``\beta \subset \beta.\ \mu \in \lambda.\ \supset : x \in \beta.\ \supset .\ \cap\beta``\mu \in \lambda : x \sim\in \beta.\ \supset .\ -\beta``\mu \in \lambda$

Hence $x \sim\in \overrightarrow{P}`y \cup \overleftarrow{P}`y \,.\, \mu \in \lambda \,.\, \supset .\ -(\overrightarrow{P}`y \cup \overleftarrow{P}`y)``\mu \in \lambda$

$x \in \overrightarrow{P}`y \cup \overleftarrow{P}`y \,.\, \mu \in \lambda \,.\, \supset .\ (\overrightarrow{P}`y \cup \overleftarrow{P}`y)\cap``\mu \in \lambda$

$\mu \in \lambda \,.\, \supset .\ (\overrightarrow{P}`x \cup \overleftarrow{P}`x)\cap``\mu \in \lambda$

[17v]

To prove $\sim \breve{R}`\max_R`\gamma \in \alpha \,.\vee.\, y \in \alpha \cup \gamma$ by induction, i.e.[23]

$\sim p \vee q \,.\supset.\, \sim r \vee q \vee s$

$\equiv p \sim q \vee \sim r \vee q \vee s$

$\equiv p \vee \sim r \vee q \vee s$

$\sim \breve{R}`\max_R`\gamma \in \alpha \,.\vee.\, y \in \alpha \cup \gamma :$

$\supset : \sim \breve{R}`\breve{R}`\max_R`\gamma \in \alpha \,.\vee.\, y \in \alpha \cup \gamma \,.\vee.\, y = \breve{R}`\max_R`\gamma$

i.e. $\breve{R}`\max_R`\gamma \in \alpha \,.\vee.\, y \in \alpha \cup \gamma \,.\vee.\, \sim \breve{R}`\breve{R}`\max_R`\gamma \in \alpha \,.\vee.\, y = \breve{R}`\max_R`\gamma$

Try induction to $y$, i.e.

$\breve{R}`\max_R`\gamma \in \alpha \,.\supset.\, y \in \alpha \cup \gamma \ :\supset:\ \breve{R}`\max_R`\gamma \in \alpha \,.\supset.\, \breve{R}`y \in \alpha \cup \gamma$

$y \in \alpha \,.\supset.\, \breve{R}`y \in \alpha$ $\qquad p \supset q \,.\supset.\, p \supset r$

$\equiv .\ \sim p \vee \sim q \vee r$

$y \in \gamma \,.\supset.\, \breve{R}`y \in \gamma \cup \iota`\breve{R}`\max_R`\gamma \,.\supset.\, \breve{R}`y \in \alpha \cup \gamma$ $\qquad \equiv .\ pq \supset r$

Hence $\breve{R}`\max_R`\gamma \in \alpha \,.\supset.\, \overleftarrow{R}_*`x \subset \alpha \cup \gamma$

Hence $\overleftarrow{R}_*`x - \gamma \subset \overleftarrow{R}_*`\breve{R}`\max_R`\gamma$

and $\overleftarrow{R}_*`x - \overleftarrow{R}_*`\breve{R}`\max_R`\gamma \subset \gamma$

Can we now prove $\overleftarrow{R}_*`x \subset s`\overrightarrow{M}_*`\iota`x$?

We have $xR_*y \,.\supset:\, yR_*\max_R`\gamma \,.\vee.\, \max_R`\gamma R_* y$

because $\gamma \subset \overrightarrow{R}_*`\max_R`\gamma$

Thus $xR_*y \,.\, z \in s`\overrightarrow{M}_*`\iota`x \,.\supset:\ zR_*y \,.\vee.\, yR_*z$

Thus $xR_*y \,.\, y \sim\in s`\overrightarrow{M}_*`\iota`x \,.\supset: z \in s`\overrightarrow{M}_*`\iota`x \,.\supset_z.\, zR_*y$

$\gamma M_*\iota`x \,.\, xR_*y \,.\, y \sim\in \gamma \,.\supset.\, \gamma \subset \overrightarrow{R}_*`y$

[18r]

Must take out not only $y$ but $\breve{S}_*``\overleftarrow{\in}`y$ $\qquad z \in R``\alpha - \overrightarrow{P}`y.$

$\supset .\ (\exists u).\ u \in \alpha \,.\, zRu$

Consider $\nu = \hat{\beta}\{(\exists\alpha).\ \alpha \in \mu \,.\, \beta = \alpha - \breve{S}_*``\overleftarrow{\in}`y\}$ $\qquad u \in -\overrightarrow{P}`y$

$z \in R``\alpha - \overrightarrow{P}`y \,.\, zRu.$

[23] $\max_R$ is defined in $*93$, but does not appear in Appendix B. "Thus if $\alpha$ is the class of peers, and $P$ is the relation of father to son, $\overrightarrow{\min}_P`\alpha$ consists of those peers who are the first of their line, while $\overrightarrow{\max}_P`\alpha$ consists of those peers who are the last of their lines. (*PM*, p.578).

To prove $\nu \in \lambda$. $\qquad \supset . u \in -\overrightarrow{P}‘y \cup \overleftarrow{R}‘y$

$\beta \in \breve{S}_*“\overleftarrow{\epsilon}‘y . \equiv . (\exists\gamma). y \in \gamma . \gamma S_* \beta$ $\qquad xPy . uRz . \supset . zPy$

$u \in -\overrightarrow{P}‘y . zR\alpha . \supset . z \in -\overrightarrow{P}‘y$

$\equiv . (\exists T). T \in \text{Potid}‘R . y \in T‘\beta$ $\qquad R“(\alpha - P$

Put $\dot{s}‘\text{Potid}‘R = P$. To prove $P = R_*$, i.e. $R_* \mathrel{\dot{\subset}} P$
Put $R_\epsilon = S$. Then
$\quad p‘\lambda = \overrightarrow{S_*}‘\iota‘x = \hat{\alpha}\{(\exists T). T \in \text{Potid}‘R . \alpha = \overrightarrow{T}‘x\}$
$\quad s‘p‘\lambda = \overrightarrow{P}‘x$
$\mu \in \lambda . \equiv : \iota‘\kappa \in \mu . S“\mu \subset \mu : \equiv : \iota‘x \in \mu : \alpha \in \mu . \supset_\alpha . R“\alpha \in \mu$
$\mu \in \lambda . \supset : x \in s‘\mu . R“s‘\mu \subset \mu$

Assume $\mu \in \lambda . \alpha \in \mu . y \in \alpha . \sim(yPx)$
Consider $\alpha - \overleftarrow{P}‘y$ $\qquad \cancel{R“\alpha \subset \alpha} . R“ - \overleftarrow{P}‘y \subset -\overleftarrow{P}‘y$
Hence $R“(\alpha - \overleftarrow{P}‘y) \subset R“\alpha - \overleftarrow{P}‘y$ Also $\iota‘x = \iota‘x - \overleftarrow{P}‘y$
Put $\nu = \hat{\beta}\{(\exists\alpha) . \alpha \in \mu . \beta = \alpha - \overleftarrow{P}‘y\}$ Then
$\iota‘x \in \nu : \alpha \in \mu . \beta = \alpha - \overleftarrow{P}‘y . \supset . R“\beta = R“\alpha - \overleftarrow{P}‘y$?
[ We have $yPz . zRu . \supset . yPu$ Hence $u \in -\overleftarrow{P}‘y . zRu . \supset . z \in -\overleftarrow{P}‘y$
Hence $R“(\alpha - \overleftarrow{P}‘y) \subset R“\alpha - \overleftarrow{P}‘y$
Again $z \in -\overleftarrow{P}‘y . zRu . \supset . u \in -\overleftarrow{P}‘y \cup \iota‘y$
Thus we must put $R“\alpha - \overleftarrow{P}‘y \subset R“(\alpha - \overleftarrow{P}‘y \cup \iota‘y)$

[18v]

[Seven lines deleted, to first line across page.]
We have, putting $\mu_2 = -\overrightarrow{R_*}‘y \cup -\overrightarrow{R_*}‘z$
$R“\mu_2$, and $\mu_2 \subset \beta . R“\beta \subset \beta . \supset . \beta = V$ and $\mu_2 \subset \beta . \exists!\beta . \supset . \exists! R“\beta - \beta$
$\mu_2 = \hat{u}\{(\exists\beta) . R“\beta \subset \beta . y, z \in \beta . u \sim\in \beta\}$
$\quad = \hat{u}\{(\exists\beta) . \breve{R}“\beta \subset \beta . y, z \in -\beta . u \in \beta\}$
$\quad = s‘\hat{\beta}\{\breve{R}“\beta \subset \beta . y, z \in -\beta\} = s‘\kappa$ so,
We have $\breve{R}“\mu \in \kappa$. Hence $\breve{R}“\mu \subset \mu_2$
Hence $R“\breve{R}“\mu \subset \breve{R}“\mu_2 \subset \mu_2$ Hence $\mu \subset \mu_2$ and $y, z \in \mu_2$ which is absurd.

---

Prove first $\breve{R}“(\overrightarrow{R_*}‘y \cap \overrightarrow{R_*}‘z) \subset \overrightarrow{R_*}‘y \cap \overrightarrow{R_*}‘z$
Thence $R“(-\overrightarrow{R_*}‘y \cup -\overrightarrow{R_*}‘z) \subset -\overrightarrow{R_*}‘y \cup -\overrightarrow{R_*}‘z$
~~Hence $R“\breve{R}“\mu \subset -\alpha \cup -\beta$ if $y \in \alpha - \beta . z \in \beta - \alpha$~~
$u \in -\overrightarrow{R_*}‘y \cup -\overrightarrow{R_*}‘z . \equiv :. (\exists\beta) :. \breve{R}“\beta \subset \beta : y \in \beta . \vee . z \in \beta : u \sim\in \beta$
$\qquad \equiv :. (\exists\beta) : \breve{R}“\beta \subset \beta . \sim(y, z \in \beta) . u \in \beta$
i.e. $-\overrightarrow{R_*}‘y \cup -\overrightarrow{R_*}‘z = s‘\hat{\beta}(\breve{R}“\beta \subset \beta . \sim(y, z \in \beta)) = s‘\kappa$ so,

We have $R“s‘\kappa \subset s‘\kappa$ i.e. $s‘R“‘\kappa \subset s‘\kappa$
i.e. $\beta \in \kappa \,.\, u \in R“\beta \,.\supset.\, u \in s‘\kappa$
i.e. $\beta \in \kappa \,.\, v \in \beta \,.\, uRv \,.\supset.\, u \in s‘\kappa$
i.e. $\breve{R}“\beta \subset \beta \,.\, \sim(y, z \in \beta) \,.\, v \in \beta \,.\, uRv \,.$
$$\supset . (\exists\gamma).\ \breve{R}“\gamma \subset \gamma \,.\sim(y, z \in \gamma) \,.\, u \in \gamma$$

[19r]

$\mu \in \lambda \,.\, T \in \mathrm{Potid}‘R \,.\supset.\, \overrightarrow{T}‘x \in \mu$
$T \in \mathrm{Potid}‘R \,.\, yTx \,.\supset:\, \mu \in \lambda \,.\supset_\mu.\, y \in s‘\mu :\supset.\, y \in p‘s“\lambda$
Assume $y \sim\in s‘p‘\lambda$. Then $\alpha \in p‘\lambda \,.\supset_\alpha.\, y \sim\in \alpha$
But $\mu \in \lambda \,.\supset.\, y \in s‘\mu$
~~We want to prove that $(-\iota‘y)“\mu \in \lambda$ if $y \sim\in s‘p‘\lambda$~~
~~i.e. to prove $\iota‘x \in (-\iota‘y)“\mu \,.\, R“‘(-\iota‘y)“\mu \subset (-\iota‘y)“\mu$~~
~~First part obvious because $x \neq y \,.\, \iota‘x \in \mu$ Second requires~~
$$\cancel{\alpha \in \mu \,.\supset.\, (\exists\beta) \,.\, \beta \in \mu \,.\, R“(\alpha - \iota‘y) = \beta - \iota‘y}$$
~~This is true if $\alpha \in p‘\lambda$, because then $y \sim\in \alpha$~~
We have to prove
$\mu \in \lambda \,.\, y \in s‘\mu : \alpha \in p‘\lambda \,.\supset_\alpha.\, y \sim\in \alpha :\supset.\, (\exists\nu).\ \nu \in \lambda \,.\, y \sim\in s‘\nu$
Hp. $\supset :.\, \mu \in \lambda \,.\, \alpha \in \mu \,.\, y \in \alpha \,.\supset.\, \alpha \sim\in p‘\lambda \,.\supset.\, (\exists\nu).\ \nu \in \lambda \,.\, \alpha \sim\in \nu$
Perhaps $\mu \cap -\overleftarrow{\epsilon}‘y \in \lambda$?
$$\iota‘x \in \mu \cap -\overleftarrow{\epsilon}‘y : \alpha \in \mu \cap -\overleftarrow{\epsilon}‘y \,.\supset.\, R“\alpha \in \mu$$
Do we know that $y \sim\in R“\alpha$?

$\alpha S_{*3}\iota‘x \,.\supset:\, \iota‘x \in \mu \,.\; R“‘\mu \subset \mu \,.\supset.\, \alpha \in \mu$ whatever the order of $\mu$.[24]

$$\mu \in \lambda \,.\, \alpha \in \mu \,.\, y \in \alpha \,.\supset.\, (\exists\nu).\ \nu \in \lambda \,.\, \alpha \sim\in \nu$$
$$\alpha \sim\in \nu \,.\, \alpha = S‘\beta \,.\supset.\, \beta \sim\in \nu \text{ i.e. } \alpha \sim\in \nu \,.\supset.\, \overleftarrow{S}‘\alpha \subset -\nu$$

[19v]

To prove $\alpha \subset \mu_2 \,.\, \mu_2 \subset \beta \,.\, R“\mu_2 \subset \mu_2 \,.\supset.\, R_*“\alpha \subset \cancel{\mu_2}\, \beta$
$x \in R_*“\alpha \,.\equiv:\, (\exists z) : z \in \alpha : z \in \mu \,.\, R“\mu \subset \mu \,.\supset_\mu.\, x \in \mu$
Thus we have to prove
$$z \in \alpha : z \in \mu \,.\, R“\mu \subset \mu \,.\supset_\mu.\, x \in \mu :\supset.\, x \in \beta$$
or $x \in \mu \,.\, \breve{R}“\mu \subset \mu \,.\supset_\mu.\, z \in \mu :\supset:\, x \in \beta \,.\vee.\, z \sim\in \alpha$
$$\text{i.e.} \supset : x \in -\beta \,.\vee.\, z \in -\alpha$$
Now $-\beta \subset -\mu_2 \therefore \breve{R}“-\beta \subset \breve{R}“-\mu_2 \subset -\mu_2 \subset -\alpha$ Also $R“\alpha \subset \beta$

[24] This and p.[19v] are some of the few discussions of the order of inductive classes in all the notes.

Thus $x \epsilon -\beta . \supset . \breve{R}`x \epsilon -\alpha$
Consider $\alpha \cup \beta . R``\alpha \subset \beta$. Hence if $R``\beta \subset \beta$, $R``(\alpha \cup \beta) \subset \alpha \cup \beta$
Assume $R``\beta \subset \beta$. Then
$z \epsilon \alpha . xR_*z . \supset . x \epsilon \alpha \cup \beta . \supset . x \epsilon \beta$.
The essential condition is $R``\beta \subset \beta$.

[20r]

$\mu \epsilon \lambda . \equiv . \iota`x \epsilon \mu . S``\mu \subset \mu$ $\quad\quad$ $\xi \epsilon \gamma`\mu . \equiv . \xi \epsilon \mu . y \epsilon \cancel{\mu} \lfloor S \rfloor$
$z \epsilon p`\gamma``\lambda . \equiv . \iota`x \epsilon \mu . S``\mu \subset \mu . \xi \epsilon \mu . y \epsilon \xi . \supset_{\mu,\xi} . z \epsilon \xi$
We want to prove $\mu \epsilon \lambda . \supset . p`\gamma``\lambda \epsilon \mu$ i.e.
$\iota`x \epsilon \mu . S``\mu \subset \mu . \supset . p`\gamma``\lambda \epsilon \mu$ given $\mu \epsilon \lambda . \supset . (\exists \xi). \xi \epsilon \mu . y \epsilon \xi$
$\mu \epsilon \lambda . \xi \epsilon \mu . y \epsilon \xi . \supset . p`\gamma``\lambda \subset \xi$ $\quad\quad$ $\gamma`\mu \subset \mu$
Prove $\xi, \eta \epsilon \mu . y \epsilon \xi . y \sim \epsilon \eta . \supset . \xi \cap \eta = \Lambda$ $\quad\quad$ $\therefore\ p`\gamma``\lambda \subset p`\lambda$
$\xi \epsilon \mu . \supset . R``\xi \epsilon \mu$ $\quad\quad$ Required $p`s``\lambda = s`p`\lambda$[25]

---

$y \epsilon \xi . \xi \epsilon \mu . \supset . \overrightarrow{R}`y \subset s`\mu . \supset . \overrightarrow{R}_*`y \subset s`\mu$
The members of ~~λ~~ $\mu$ are $\overrightarrow{T}`x$ with additions.
Can we prove $\xi \epsilon \mu . y \epsilon \xi . \eta \epsilon \nu . y \epsilon \eta . \supset . \xi \cap \eta \epsilon \mu \cap \nu$?
We have $S`(\xi \cap \eta) \subset \xi \cap \eta$
$\mu, \nu \epsilon \lambda . \supset . \mu \cap \nu \epsilon \lambda$ $\quad\quad$ $\kappa \subset \lambda . \supset . p`\kappa \epsilon \lambda$ subject to types.

---

$\gamma`\mu = \mu \cap \overleftarrow{\epsilon}`y$
$\gamma``\lambda = \hat{\xi}\{(\exists \mu). \mu \epsilon \lambda . \xi = \mu \cap \overleftarrow{\epsilon}`y\}$
$\alpha \epsilon p`\gamma``\lambda . \equiv : (\exists \mu). \mu \epsilon \lambda . \xi = \mu \cap \overleftarrow{\epsilon}`y . \supset_\xi . \alpha \epsilon \xi$
$\equiv : \mu \epsilon \lambda . \supset_\mu . \alpha \epsilon \mu \cap \overleftarrow{\epsilon}`y : \equiv : \mu \epsilon \lambda . \supset_\mu . \alpha \epsilon \mu . y \epsilon \alpha$
$\equiv : y \epsilon \alpha . \alpha \epsilon p`\lambda$

[20v][26]

$R_1(x \longmapsto y)$ is a second-order class. Call it $\alpha_2(x, y)$.
Then $\alpha_2(x, \breve{R}`y) = \alpha_2(x, y) \cup \iota`\breve{R}`y$
$\beta \epsilon$ Cls induct $. \equiv :. \Lambda \epsilon \mu : \gamma \epsilon \mu . \supset_{\gamma,z} . \gamma \cup \iota`z \epsilon \mu : \supset_\mu . \beta \epsilon \mu$
$\gamma \epsilon \mu . \supset_{\gamma,z} . \gamma \cup \iota`z \epsilon \mu : \supset : \alpha_2(x, y) \epsilon \mu . \supset_{x,y} . \alpha_2(x, \breve{R}`y) \epsilon \mu$
provided $\mu$ has second-order classes as members.
Also $\alpha_2(x, x) \epsilon \mu$. Hence
$\alpha_2(x, x) \epsilon \mu : \alpha_2(x, y) \epsilon \mu . \supset_y . \alpha_2(x, \breve{R}`y) \epsilon \mu$
This is a third order property. Hence

[25] Compare [4r]. $\quad$ [26] These are results about the order of intervals.

$x R_{*_3} y \,.\supset.\, R_1(x \longmapsto y) \epsilon$ Cls induct$_3$

Also $R_m(x \longmapsto y) \subset R_1(x \longmapsto y)$. Hence

$x R_{*_3} y \,.\supset.\, R_m(x \longmapsto y) \epsilon$ Cls induct$_3$

Now we have

$x R_{*_3} y \,.\, y \sim\epsilon\, s‘\overrightarrow{M}_{*3}‘\iota‘x \,.\supset.\, s‘\overrightarrow{M}_{*3}‘\iota‘x \subset R_3(x \longmapsto y)$

$\supset .\, s‘\overrightarrow{M}_{*3}‘\iota‘x \,\epsilon$ Cls$_2$

Hence $s‘\overrightarrow{M}_{*3}‘\iota‘x \;=\; \overleftarrow{R}_{*3}‘x$ by induction.

[21r]

To prove

$x R_* y \,.\, \gamma M_* \iota‘x \,.\supset:.\, \breve{R}“\alpha \subset \alpha \,.\, \breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\supset.\, y \,\epsilon\, \alpha :\vee: y \,\epsilon\, \gamma$

May be proved by $M$-induction. It holds if $\gamma = \iota‘x$. Suppose it holds for $\gamma$. To prove

$\breve{R}“\alpha \subset \alpha \,.\, \breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\supset.\, y \,\epsilon\, \alpha :\vee: y \,\epsilon\, \gamma \;:.\supset$

$:.\, \breve{R}“\alpha \subset \alpha \,.\, \breve{R}‘\breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\supset.\, y \,\epsilon\, \alpha :\vee: y \,\epsilon\, \gamma \cup \iota‘\breve{R}‘\max_R‘\gamma$

$pq \supset r.\supset.\, pq' \supset r \vee s$

Form of proposition is $pq \supset r \,.\vee.\, s :\supset: pq' \supset r \,.\vee.\, s \vee t$

i.e. $\sim p \vee \sim q \vee r \vee s \,.\supset.\, \sim p \vee \sim q' \vee r \vee s \vee t$

$\sim q \vee r \,.\supset.\, \sim q' \vee r \vee t$ is

$\breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\vee.\, y \,\epsilon\, \alpha :\supset$

$: \breve{R}‘\breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\vee.\, y \,\epsilon\, \alpha \,.\vee.\, y = \breve{R}‘\max_R‘\gamma$

i.e. $\breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\, y \sim\epsilon\, \alpha \,.\vee$

$.\, \breve{R}‘\breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\vee.\, y \,\epsilon\, \alpha \,.\vee.\, y = \breve{R}‘\max_R‘\gamma$

~~We have~~ $\cancel{\breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\, y \sim\epsilon\, \alpha \,.\vee.\, \breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\vee.\, y \,\epsilon\, \alpha}$

$\cancel{\breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\, y = \breve{R}‘\max_R‘\gamma \,.\supset.\, y \sim\epsilon\, \alpha}$

This is because $\breve{R}‘\breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\supset.\, \breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha$

We have $y \,\epsilon\, \alpha \,.\vee.\, y = \breve{R}‘\max_R‘\gamma :\equiv: y \,\epsilon\, \alpha \,.\vee.\, y \sim\epsilon\, \alpha \,.\, y = \breve{R}‘\max_R‘\gamma$

$\equiv: y \,\epsilon\, \alpha \,.\vee.\, \breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha \,.\, y = \breve{R}‘\max_R‘\gamma$

$\supset: \; y \,\epsilon\, \alpha \,.\vee.\, \breve{R}‘\max_R‘\gamma \sim\epsilon\, \alpha$

Hence induction is valid?

To prove $\breve{R}“\alpha \subset \alpha \,.\, \breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\supset.\, y \,\epsilon\, \alpha :$

$\supset:.\, \breve{R}“\alpha \subset \alpha \,.\, \breve{R}‘\breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\supset.\, y \,\epsilon\, \alpha :\vee: y = \breve{R}‘\max_R‘\gamma$

i.e. $pq \supset r.\supset: pq' \supset r \,.\vee.\, s$

i.e. $\sim p \vee \sim q \vee r \,.\supset.\, \sim p \vee \sim q' \vee r \vee s$

i.e. $pq \sim r \vee \sim p \vee \sim q' \vee r \vee s$

This holds if $\sim q \supset \sim q' \vee s$ i.e. if $q' \supset q \vee s$

i.e. if $\breve{R}‘\breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\supset: \breve{R}‘\max_R‘\gamma \,\epsilon\, \alpha \,.\vee.\, y = \breve{R}‘\max_R‘\gamma$

or if $y \in \alpha \,.\vee .\, \breve{R}`\breve{R}`\max_R`\gamma \in \alpha$ :

$$\supset : y \in \alpha \,.\vee .\, \breve{R}`\max_R`\gamma \in \alpha \,.\vee .\, y = \breve{R}`\max_R`\gamma$$

Mustn't leave out the $\gamma$

[21v]

$yR_*x \,.\equiv : x \in \alpha \,.\, S`\alpha \subset \alpha \,.\supset_\alpha .\, y \in \alpha$

$S`\alpha \subset \alpha \,.\, \supset : \beta \subset \alpha \,.\, \gamma S\beta \,.\supset .\, \gamma \subset \alpha$ :

$\qquad \supset : \gamma S_*\alpha \,.\supset .\, \gamma \subset \alpha$

$S`\alpha \subset \alpha \,.\, \equiv \,.s`\overrightarrow{S_*}`\alpha \in \lfloor = \rfloor \alpha$ Hence

$yR_*x \,.\equiv : x \in \alpha \,.\, s`\overrightarrow{S_*}`\alpha \in \lfloor = \rfloor\alpha \,.\supset_\alpha .\, y \in \alpha$

$\qquad \equiv : x \in \alpha \,.\, s`\overrightarrow{S_*}`\alpha = \alpha \,.\supset_\alpha .\, y \in s`\overrightarrow{S_*}`\alpha$

$\beta S_*\iota`x \,.\equiv :.\, \iota`x \in \xi \;:\; \gamma \in \xi \,.\supset_\gamma .\, R``\gamma \in \xi \;:\supset_\xi .\, \beta \in \xi$

$\qquad \equiv :.\, \iota`x \in \xi \,.\, S``\xi \subset \xi . \supset_\xi .\, \beta \in \xi$

$R``\alpha \subset \alpha \,.\supset .\, R```\iota``\alpha = \hat{\gamma}\{(\exists y).\, y \in \alpha \,.\, \gamma = R``\iota`y\} = \overrightarrow{R}``\alpha$

$\qquad \supset .\, s`S``\iota``\alpha = R``\alpha \,.\supset .\, s`S``\iota``\alpha \subset \alpha$

---

$x \in \alpha \,.\, R``\alpha \subset \alpha \,.\supset .\, \iota`x \in \mathrm{Cl}`\alpha \,.\, R```\mathrm{Cl}`\alpha \subset \mathrm{Cl}`\alpha$

$\iota`x \in \xi \,.\, S``\xi \subset \xi \,.\supset .\, x \in s`\xi \,.\, R``s`\xi \subset s`\xi$

$x \in s`\xi \,.\, R``s`\xi \subset s`\xi \,.\supset .\, y \in s`\xi : \supset : \iota`x \in \xi \,.\, S``\xi \subset \xi \,.\supset .\, y \in s`\xi$

Hence $yR_{*_2}x \,.\supset .\, y \in p`s``\lambda$

Required $p`s`\lambda \subset s`p`\lambda$ i.e. required

$yR_*x : \mu \in \lambda \,.\supset_\mu .\, (\exists\xi).\, \xi \in \mu \,.\, y \in \xi : \supset : (\exists\xi) : \mu \in \lambda \,.\supset_\mu .\, \xi \in \mu : y \in \xi$

[22r]

$y(\dot{s}`\mathrm{Potid}`R)x \,.\equiv :.\, (\exists T) :.\, R_0 \in \mu \,.\, |R``\mu \subset \mu \,.\supset_\mu .\, T \in \mu : yTx$

Here $\mu$ is a class of relations.

---

Consider $\overrightarrow{M_*}`\iota`x$, where

$\alpha M_*\iota`x \,.\equiv :.\, \iota`x \in \mu : \gamma \in \mu \,.\supset_\gamma .\, \gamma \cup R``\gamma \in \mu : \supset_\mu .\, \alpha \in \mu$

$s`\overrightarrow{M_*}`\iota`x \subset \overrightarrow{R_*}`x$

Members of $\mu$ are $\iota`x,\ \iota`x \cup \overrightarrow{R}`x,\ \iota`x \cup \overrightarrow{R}`x \cup \overrightarrow{R^2}`x\ , \ldots \iota`x \cup \ldots \cup \overrightarrow{T}`x, \ldots$

Consider $\iota`x,\ \overrightarrow{R}`x\ , \overrightarrow{R^2}`x\ , \ldots\ \overrightarrow{T}`x \ldots$

This is $\overrightarrow{(R_\epsilon)_*}`\iota`x$ i.e. $s`\overrightarrow{S_*}`\iota`x$ if $S = R_\epsilon$

$yR_*x \,.\equiv : x \in \alpha \,.\, S`\alpha \subset \alpha \,.\supset_\alpha .\, y \in \alpha$

$S`\alpha \subset \alpha \,.\, \underline{\supset}$ <≡>: $\beta \subset \alpha \,.\supset_\beta .\, S`\beta \subset \alpha$

[The rest of this page is deleted with a single diagonal line.]

$S_*`\alpha \subset \alpha \,.\, \equiv :.\, \beta \subset \alpha \,.\supset_\beta .\, S`\beta \subset \alpha$

$S_*`\alpha \subset \alpha \,.\, \equiv \,.\, S_*`\alpha = \alpha$ Hence

$yR_*x \,.\, \equiv : x \in \alpha \,.\, s`S_*``\alpha = \alpha \,.\, \supset_\alpha \,.\, y \in \alpha$

$\equiv : x \in \alpha \,.\, s`S_*``\alpha = \alpha \,.\, \supset_\alpha \,.\, y \in s`S_*``\alpha$

$\supset : \iota`x \in \xi \,.\, s`S_*``s`\xi = s`\xi \,.\, \supset_\xi \,.\, y \in s`S_*``s`\xi$

$\supset : \iota`x \in \xi \,.\, s`s`S_*```\xi = s`\xi \,.\, \supset_\xi \,.\, y \in s`s`S_*```\xi$

[22v]

The thing to prove is

$xR_*y \,.\, \gamma M_*\iota`x \,.\, \supset : \sim \{(\breve{R}`\mathrm{max}_R`\gamma)R_*y\} \,.\, \supset \,.\, y \in \gamma$

i.e. $\supset :.\, \breve{R}``\alpha \subset \alpha \,.\, \breve{R}`\mathrm{max}_R`\gamma \in \alpha \,.\, \supset \,.\, y \in \alpha : \vee\ : y \in \gamma$

We have $\breve{R}``\gamma \subset \gamma \cup \iota`\breve{R}`\mathrm{max}_R`\gamma$

$\breve{R}`\mathrm{max}_R`\gamma \sim\in \gamma$. Hence $\breve{R}`\mathrm{max}_R`\gamma \in \alpha \,.\, \equiv \,.\, \breve{R}`\mathrm{max}_R`\gamma \in \alpha - \gamma$

~~Hence~~ $\cancel{\breve{R}``\alpha \subset \alpha \,.\, \breve{R}`\mathrm{max}_R`\gamma \in \alpha \,.\, \supset : \breve{R}``(\alpha \cup \gamma) \subset \alpha \cup \gamma \,.\, \breve{R}`\mathrm{max}_R`\gamma \in \alpha \cup \gamma}$

~~Hence~~ $\cancel{\{(\breve{R}`\mathrm{max}_R`\gamma)R_*y\}. \supset : R}$

Consider $\alpha - \gamma$. We have $\breve{R}``\alpha \subset \alpha \,.\, \breve{R}``\gamma \subset \gamma \cup \iota`\breve{R}`\mathrm{max}_R`\gamma \,.\, R``\gamma \subset \gamma$

$\cancel{z \in \breve{R}``(\alpha - \gamma) \,.\, \equiv \,.\, (\exists u).\, u \in \alpha - \gamma \,.\, z = \breve{R}`u}$

$\breve{R}`` - \gamma \subset -\gamma \,.\, \breve{R}``\alpha \subset \alpha$ Hence $\breve{R}``(\alpha - \gamma) \subset \alpha - \gamma$

Also $\breve{R}`\mathrm{max}_R`\gamma \in \alpha - \gamma$.

Hence $(\breve{R}\mathrm{max}_R`\gamma)R_*y \,.\, \supset \,.\, y \in -\gamma$

This is the converse of what we want.

~~Now consider~~ $\cancel{\gamma - \alpha}$~~. We have~~ $\cancel{R``\gamma \subset \gamma \,.\, R`` - \alpha \subset -\alpha \ \therefore\ R``(\gamma - \alpha) \subset \gamma - \alpha}$

~~Hence~~ $\cancel{y \in \gamma - \alpha \,.\, \{(\breve{R}\mathrm{max}_R`\gamma)R_*y\} \,.\, \supset \,.\, \breve{R}\mathrm{max}_R`\gamma \in \gamma - \alpha}$

~~Consider~~ $\cancel{\gamma \cap \alpha \,.\, \breve{R}``(\gamma \cap \alpha) \subset \gamma \cup \iota`\breve{R}`\mathrm{max}_R`\gamma \,.\, \breve{R}``(\gamma \cap \alpha) \subset \breve{R}``\alpha \subset \alpha}$

~~Hence~~ $\cancel{\breve{R}`\mathrm{max}_R`\gamma \sim\in \breve{R}``\alpha \,.\, \supset \,.\, \breve{R}``(\gamma \cap \alpha) \subset \gamma \cap \alpha}$

~~Hence~~ $\cancel{\breve{R}`\mathrm{max}_R`\gamma \in \alpha - \breve{R}``\alpha \,.\, \supset : (\breve{R}\mathrm{max}_R`\gamma)R_*y \,.\, \supset \,.\, y \in \gamma \cap \alpha}$

[23r]

To prove $R_* \mathrel{\subset\!\!\!\cdot}\ \dot{s}`\mathrm{Potid}`R$[27]

$z(\dot{s}`\mathrm{Potid}`R)x \,.\, yRz \,.\, \supset \,.\, y(\dot{s}`\mathrm{Potid}`R)x$

$z(\dot{s}`\mathrm{Potid}`R)x \,.\, \equiv \,.\, z \in s`p`\lambda$ where $\lambda = \hat{\mu}\{\iota`x \in \mu : \xi \in \mu \,.\, \supset_\xi \,.\, R``\xi \in \mu\}$

$R``s`p`\lambda \subset s`p`\lambda \,.\, x \in s`p`\lambda \,.\, s`p`\lambda \in \mathrm{Cls}_3$ Hence $R_{*_3} \mathrel{\subset\!\!\!\cdot}\ (\dot{s}`\mathrm{Potid}`R)_1$

Can we prove $(\dot{s}`\mathrm{Potid}`R)_1 = (\dot{s}`\mathrm{Potid}`R)_3$?

$p`\lambda = \overrightarrow{S_*}`\iota`x$ where $S = R_\epsilon$.

$\alpha S_*\iota`x. \equiv : \iota`x \in \mu \,.\, S``\mu \subset \mu \,.\, \supset_\mu \,.\, \alpha \in \mu$

$\equiv : \iota`x \in \mu \,.\, R``\mu \subset \mu \,.\, \supset_\mu \,.\, \alpha \in \mu$

Each $\mu$ has as members $\iota`x$, $\overrightarrow{R}`x$, $\overrightarrow{R^2}`x$, … $\overrightarrow{T}`x$, …

[27] See $*89{\cdot}28$.

~~Return to $M = (I \dot\cup R)_\epsilon$. Then~~

Given $R \in 1 \to \mathrm{Cls}$, we have $\overrightarrow{R_*}`x = s`\overrightarrow{S_*}`\iota`x$ without any fuss.

---

~~$(\dot s`\mathrm{Potid}`R)_\mathrm{T}$~~

$(\exists\alpha)\ :.$

$yR_*3x\ .\equiv ::\ \iota`x \in \mu : \xi \in \mu\ .\supset_\xi .\ R``\xi \in \mu : \supset_\mu .\ \alpha \in \mu :. \cancel{\supset_\alpha}\ .\ y \in \alpha$

~~$yR_*3x\ .\supset :.\ \iota`x \in \mu : \xi \in \mu\ .\supset_\xi .\ R``\xi \in \mu : \supset .\ y \in \mu$~~

I think this implies $y(\dot s`\mathrm{Potid}`R)_3 x$.

[23v]

$\overleftarrow{R}_*`x \subset s`\overrightarrow{M_*}`\iota`x$ is

$x \in \alpha\ .\ \breve{R}``\alpha \subset \alpha\ .\supset_\alpha .\ y \in \alpha :$

$\supset :.\ (\exists\gamma) :.\ \iota`x \in \xi : \beta \in \xi\ .\supset_\beta .\ \iota`x \cup \breve{R}``\beta \in \xi : \supset_\xi .\gamma \in \xi : y \in \gamma$

$\longrightarrow$ [Take as definition of $M_*$]

i.e. $\supset :.\ (\exists\gamma) :.\ x \in \mu : \beta \subset \mu\ .\supset_\beta .\ \iota`x \cup \breve{R}``\beta \subset \mu : \supset_\mu .\ \gamma \subset \mu : y \in \gamma$

i.e. $\supset :.\ (\exists\gamma) :.\ x \in \mu\ .\ \breve{R}``\mu \subset \mu\ .\ \supset_\mu .\ \gamma \subset \mu : y \in \gamma$

i.e. $\supset :.\ x \in \mu\ .\ \breve{R}``\mu \subset \mu\ .\ \supset_\mu .\ y \in \mu$

This would be all right but for the step which replaces $\beta \in \xi$ by $\beta \subset \mu$

$\beta \in \xi$ may have other forms, e.g. $\sim(\beta \subset \mu)$, $\alpha \in \beta$, $a, b \in \beta$,

$\beta \subset \mu\ .\vee.\ \beta \subset \nu$, etc., etc.

But $\iota`x \in \xi$ rules out some of these. Take $\sim(\beta \subset \mu)$.

Then $x \sim\in \mu : \sim(\beta \subset \mu)\ .\supset .\ \sim(\iota`x \cup \breve{R}``\beta \subset \mu)$

reduces to $x \sim\in \mu$ ; and $\gamma \in \xi$ is satisfied because $x \in \gamma - \mu$

Now take $a \in \beta$. This is impossible unless $a = x$.

Now take $a \sim\in \beta$. We shall have

$x \neq a : a \sim\in \beta\ .\supset_\beta .\lfloor a \sim\in \rfloor \iota`x \cup \breve{R}``\beta$

i.e. $x \neq a : a \in \breve{R}``\beta\ .\supset_\beta .\ a \in \beta$

This reduces to $a \sim\in \mathrm{G}`R$

Now take $\beta \subset \mu\ .\vee.\ \beta \subset \nu$ as $\beta \in \xi$. Then

$x \in \mu \vee \nu\ :.\ \beta \subset \mu\ .\vee.\ \beta \subset \nu : \supset_\beta : \iota`x \cup \breve{R}``\beta \subset \mu\ .\vee.\ \iota`x \cup \breve{R}``\beta \subset \nu\ :.$

$\supset_{\mu,\nu} : \gamma \subset \mu\ .\vee.\ \gamma \subset \nu$

[24r]

Put $\lambda = \hat\mu\{\iota`x \in \mu : \xi \in \mu\ .\supset_\xi .\ R``\xi \in \mu\}$

Then $\overrightarrow{S_*}`\iota`x = p`\lambda\ .\ s`\overrightarrow{S_*}`\iota`x = s`p`\lambda$

$\mu \in \lambda\ .\supset : \xi \in \mu\ .\ \alpha \subset \xi\ .\supset .\ R``\xi \in \mu\ .\ R``\alpha \subset R``\xi$

~~$\supset : \alpha = \iota`x\ .\supset .\ \overrightarrow{S_*}`\iota`x \subset \mu\ .\ \overrightarrow{S_*}\alpha \subset$~~

$\supset : (\exists\xi).\ \xi \in \mu\ .\ \alpha \subset \xi\ .\supset .(\exists\xi).\ \xi \in \mu\ .\ R``\alpha \subset \xi$

---

We have $y \epsilon p`s``\lambda$ and we want $y \epsilon s`p``\lambda$, i.e. we have
$yR_*x \,.\supset:\mu \epsilon \lambda \,.\supset_\mu .\,(\exists\xi).\,\xi \epsilon \mu \,.\, y \epsilon \xi$
and want $(\exists\xi): \mu \epsilon \lambda \,.\supset_\mu .\,\xi \epsilon \mu : y \epsilon \xi$
Consider a particular $\mu$. Take $\theta = \hat{\xi}(\xi \epsilon \mu . y \epsilon \xi)$
Consider $p`\theta$. We have $y \epsilon p`\theta \,.\, \theta \subset \mu \,.\, p`\mu \subset p`\theta$
We ought to have $p`\theta = \iota`y$.
This wants $z \neq y \,.\supset .(\exists\xi).\,\xi \epsilon \mu : y \epsilon \xi \,.\, z \sim\epsilon\, \xi$
If $z \sim\epsilon\, s`\mu$, This is obvious. Suppose $z \epsilon s`\mu$. Then

$(\exists\xi).\,\xi \epsilon \mu : y \epsilon \xi : (\exists\eta).\,\eta \epsilon \mu \,.\, z \epsilon \eta$
The members of $\mu$ are $\overrightarrow{T}`x$ for all $T$s in Potid$`R$, together with possible others.
Thus what we want is $(\exists T).\; yTx \,.\sim(zTx)$. This may be false.

[24v]

To prove $xR_*y \,.\, y \epsilon \mu - \breve{R}``\mu \,.\supset .\,(\exists\alpha).\,\breve{R}``\alpha \subset \alpha \,.\, x \epsilon \alpha - \breve{R}``\alpha$
$\cancel{x \epsilon \alpha . \breve{R}``\alpha \subset \alpha \,.\supset_\alpha .\, x \epsilon \breve{R}``\alpha : xR_*y :\supset: x \epsilon \alpha \,.\, y \epsilon \breve{R}``\alpha}$
$\cancel{xR_*y \,.\supset: y \epsilon \beta \,.\, R``\beta \subset \beta \,.\supset_\beta .\, x \epsilon \beta}$
$\overrightarrow{R_*}`x = p`\hat{\alpha}(R``\alpha \subset \alpha \,.\, x \epsilon \alpha)$
Hence if $\sim(xRx)$, $(\exists\alpha).\; R``\alpha \subset \alpha \,.\, x \epsilon \alpha \,.\, \breve{R}`x \sim\epsilon\, \alpha$
i.e. $(\exists\alpha).\; R``\alpha \subset \alpha \,.\, x \epsilon \alpha - \breve{R}``\alpha$
i.e. $(\exists\alpha).\; \breve{R}``-\alpha \subset -\alpha \,.\, x \epsilon \alpha \,.\, x \epsilon \breve{R}``-\alpha$
i.e. $(\exists\beta).\; \breve{R}``\beta \subset \beta \,.\, x \epsilon \breve{R}``\beta - \beta$
i.e. $(\exists\gamma).\; \breve{R}``\gamma \subset \gamma \,.\, x \epsilon \gamma - \breve{R}``\gamma \quad (\gamma = R``\beta)$
Now $y \epsilon \mu - \breve{R}``\mu \,.\supset .\; \sim(yRy)$.
Thus to prove $xRx . xR_*y \,.\supset .\, yRy$
$xRx \,.\, xRy \,.\supset .\, y = x \qquad\qquad xRx \,.\, xR_*y \,.\supset .\, y = x$
Hence prop. Thus $y \epsilon \mu - \breve{R}``\mu \,.\supset .\,(\exists\alpha).\, x \epsilon \alpha - \breve{R}``\alpha$
We want to prove $y \epsilon \alpha \,.\supset .\,(\exists S).\, S \epsilon \text{Potid}`R \,.\, y \epsilon \breve{S}``\alpha - \breve{R}``\breve{S}``\alpha$
Then we shall have $y \epsilon \breve{S}``\alpha - \breve{R}``\breve{S}``\alpha \,.\, z \epsilon \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha$
Also probably $y \sim\epsilon\, \breve{T}``\alpha \,.\, z \sim\epsilon\, \breve{S}``\alpha$
We ought to have $\mu - \breve{R}``\mu = \breve{S}``\alpha - \breve{R}``\breve{S}``\alpha$

[25]

Put $M`\alpha = \cancel{\alpha \cup} \breve{R}``\alpha$. Then $s`\overrightarrow{M_*}\,\iota`x \subset \overleftarrow{R}_*`x$.[28]
*To prove* $\overleftarrow{R}_*`x \subset s`\overrightarrow{M_*}\,\iota`x$ i.e.
$x \epsilon \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha .\, y \epsilon \alpha :\supset$
$\therefore (\exists\beta) \therefore \iota`x \epsilon \gamma : \xi \epsilon \gamma \,.\supset_\xi .\, \cancel{\xi \cup} \breve{R}``\xi \epsilon \gamma :\supset_\gamma .\, \beta \epsilon \gamma \therefore y \epsilon \beta$
$\breve{R}``\alpha \subset \alpha \,.\equiv: \xi \subset \alpha \,.\supset_\xi .\, \breve{R}``\xi \subset \alpha$ Hence

[28] This differs from the definition of $M$ in Appendix B.

$xR_*y\ .\supset :.\ \iota`x \subset \alpha : \xi \subset \alpha\ .\supset_\xi .\ \breve{R}``\xi \subset \alpha\ :\supset_\alpha .\ \iota`y \cancel{\subset_\beta}\ \alpha$

$\supset ::\ \lfloor(\exists\beta) ::\rfloor \iota`x \subset \alpha : \xi \subset \alpha\ .\supset_\xi .\ \breve{R}``\xi \subset \alpha : \supset_\alpha .\ \beta \subset \alpha :.\ \ni y \in \beta$

$R \in \mathrm{Cls} \to 1\ .\ \mathrm{D}`R = V\ .\supset .\ \overrightarrow{M_*}`\iota`x \subset 1$

*Dem.* $\iota`x \in 1 : \xi \in 1\ .\supset .\ \breve{R}``\xi \in 1.$

$\overrightarrow{R_*}`y \subset \beta.\supset :.\ xR_*y\ .\equiv : y \in \alpha\ .\ R``(\alpha \cap \beta) \subset \alpha\lfloor\cap\beta\rfloor.\supset_\alpha .\ x \in \alpha$

$y \in \alpha\ .\ R``(\alpha \cap \beta) \subset \alpha\ .\supset .\ y \in \alpha \cap \beta\ .\ R``(\alpha \cap \beta) \subset \alpha$

$[\ y \in \beta\ .\ R``\beta \subset \beta\ ] \qquad \supset .\ y \in \alpha \cap \beta\ .\ R``(\alpha \cap \beta) \subset \alpha \cap \beta$

Thus

$y \in \beta\ .\ R``\beta \subset \beta\ .\supset :.\ xR_*y\ .\ni\lfloor\equiv\rfloor : y \in \alpha\ .\ R``(\alpha \cap \beta) \subset \alpha\ .\supset_\alpha .\ x \in \alpha$

Hence $\beta M_*\iota`x\ .\equiv :.\ \iota`x \in y \cap 1 : \xi \in y \cap 1\ .\supset_\xi .\ \breve{R}``\xi \in \gamma : \supset_\gamma .\ \beta \in y \cap 1$

$\equiv :.\ \iota`x \in \iota`\alpha : \xi \in \iota`\alpha\ .\supset_\xi .\ \breve{R}``\xi \in \iota`\alpha : \supset_\alpha .\ \beta \in \iota`\alpha$

$(\exists\beta).\ \beta M_*\iota`x\ .\ y \in \beta\ .\equiv :.\ x \in \alpha : \xi \in \iota``\alpha\ .\supset_\xi .\ \breve{R}``\xi \in \iota`\alpha : \supset_\alpha .\ y \in \alpha$

$\xi \in \iota``\alpha\ .\supset_\xi .\ \breve{R}``\xi \in \iota`\alpha :$

$\equiv :.\ (\exists z).\ z \in \alpha\ .\ \xi = \iota`z\ .\ \supset_\xi .\ (\exists w).\ w \in \alpha\ .\ \breve{R}``\xi = \iota`w$

$\equiv :.\ z \in \alpha\ .\supset_z .\ (\exists w).\ w \in \alpha\ .\ \breve{R}``\iota`z = \iota`w$

$\equiv :.\ z \in \alpha\ .\supset_z .\ \breve{R}`z \in w\ :.$

$\equiv :.\ \breve{R}``\alpha \subset \alpha$

Hence $\overleftarrow{R}_*`x = s`\overrightarrow{M_*}`\iota`x$

[26][29]

$*98{\cdot}11\ \ \vdash : R \in \mathrm{Cls}\to 1\ .\ xRz\ .\ zR_*y\ .\supset .\ R(x \vdash\!\dashv y) = \iota`x \cup R(z \vdash\!\dashv y)$

$\cdot 111\ \ \vdash : \sim(zR_*y)\ .\supset .\ R(z \vdash\!\dashv y) = \Lambda$

$\cdot 112\ \ \vdash : R \in \mathrm{Cls}\to 1\ .\ xRz\ .\ zR_*y\ .\sim(zR_*y)\ .\supset$

$.\ x = y\ .\ R(x \vdash\!\dashv y) = R(x \vdash\!\dashv x)$

$\cdot 113\ \ \vdash : \qquad \sim(xR|R_*x)\ .\supset .\ R(x \vdash\!\dashv x) = \iota`x$

$\cdot 114\ \ \vdash : \qquad \breve{R}``\alpha \subset \alpha\ .\ x \in \alpha - \breve{R}``\alpha\ .\supset .\sim(xR|R_*x)$

$\cdot 115\ \ \vdash : \qquad \supset .\ R(x \vdash\!\dashv x) = \iota`x$

Hence $R \in \mathrm{Cls}\to 1\ .\ xRz\ .\sim(xR|R_*x)\ .\supset$

$: R(x \vdash\!\dashv y) = \Lambda\ .\vee.\ R(x \vdash\!\dashv y) = \iota`x \cup R(z \vdash\!\dashv y)$

We now want $xRz\ .\sim(zR|R_*z)\ .\supset .\ \sim(xR|R_*x)$

$xR|R_*x\ .\supset : \breve{R}`xR_*x : \supset : \breve{R}`x = x\ .\vee.\ \breve{R}`x(R|R_*)x$

$\supset : (\breve{R}`x)(R|R_*)(\breve{R}`x)$

Hence putting $\xi = \hat{x}\{\sim x(R|R_*)x\ .\ R(x \vdash\!\dashv y) \in \mu\}$

where $\Lambda \in \mu : \eta \in \mu\ .\supset_{\eta,y} .\ \eta \cup \iota`y \in \mu,$

we have $R``\xi \subset \xi\ .\ y \in \xi$ if $\sim y(R|R_*)y$

$\cdot 12\ \ \vdash :\ \rho \in \mathrm{Cls\ induct}_3\ .\supset .\ (\exists\mu_2).\ \rho = \mu_2$

[29] These are almost the same as the final version of $*89$, except for $*98{\cdot}113$ which differs significantly from $*89{\cdot}113$.

$\cdot 14 \quad \vdash : R \in \mathrm{Cls} \to 1 . \sim(yR|R_{*m}y) . \supset$

$: xR_{*(m+1)}y . \supset . R_m(x \longmapsto y) \in \mathrm{Cls\ induct}_{m+1}$

$\cdot 15 \quad \vdash : \qquad \breve{R}``\alpha_m \subset \alpha_m . y \in \alpha_m - \breve{R}``\alpha_m . \supset : \quad . \quad . \quad . \quad . \quad .$

$\cdot 16 \quad \vdash : \alpha \sim\in \mathrm{Cls\ induct}_3 . \gamma \in \mathrm{Cls\ induct}_3 . \supset . \exists ! \alpha - \gamma$

$\cdot 17 \quad \vdash : \gamma \in \mathrm{Cls\ induct}_3 . \alpha \subset \gamma . \supset . \alpha \in \mathrm{Cls\ induct}_3$

$\cdot 18 \quad \vdash : R \in \mathrm{Cls} \to 1 . y, z \in \overleftarrow{R_{*3}}`x . \sim(yR|R_*y) . \supset : yR_{*3}z . \vee . zR_{*3}y$

$\cdot 19 \quad \vdash : R \in \mathrm{Cls} \to 1 . \breve{R}``\mu_2 \subset \mu_2 . \lambda = \overleftarrow{R_{*3}}`x \cap \mu_2 - \breve{R}``\mu_2 . \supset . \lambda \in 0 \cup 1$

[26v]

We have $xR_*y . y \in \mu - \breve{R}``\mu . \breve{R}``\mu \subset \mu . \supset . (\exists\alpha) . x \in \alpha - \breve{R}``\alpha . \breve{R}``\alpha \subset \alpha$
Then $\overleftarrow{R_*}`x \cap \alpha - \breve{R}``\alpha \in 1$ because $\overleftarrow{R_*}`x - \iota`x \subset \breve{R}``\alpha$
$\iota`\breve{R}`x = \overleftarrow{R_*}`x \cap \breve{R}``\alpha - \breve{R}``\breve{R}``\alpha$
To prove $T \in \mathrm{Potid}`R . \supset . \iota`\breve{T}`x = \overleftarrow{R_*}`x \cap \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha$
$\breve{T}`x \in \breve{T}``\alpha . \breve{T}`x \in \overleftarrow{R_*}`x . \breve{T}`x \in \breve{T}`` - \breve{R}``\alpha$ i.e. $\in -\breve{R}``\breve{T}``\alpha$
Hence $\iota`\breve{T}`x \subset \overleftarrow{R_*}`x \cap \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha$
Also $\breve{T}``(\overleftarrow{R_*}`x \cap \alpha - \breve{R}``\alpha) \in 1$ and $\breve{T}``(\overleftarrow{R_*}`x \cap \alpha - \breve{R}``\alpha) \subset \overleftarrow{R_*}`x \cap \breve{T}``(\alpha - \breve{R}``\alpha)$
Thus $\iota`\breve{T}`x = \breve{T}``(\overleftarrow{R_*}`x \cap \alpha - \breve{R}``\alpha)$
Thus we can't prove $\overleftarrow{R_*}`x \cap \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha \subset \iota`\breve{T}`x$
because we might have $u \in \alpha - \overleftarrow{R_*}`x . \breve{T}`u \neq \breve{T}`x$
But perhaps we could get it from $\breve{T}`u \in -\breve{R}``\breve{T}``\alpha$
This requires $\sim(\breve{R}`\breve{T}`xR_*\breve{T}`u)$ while $\breve{T}`u \in \breve{T}``\alpha . \supset . \sim(\breve{R}`\breve{T}`uR_*\breve{T}`x)$
Hence $\breve{R}`u = \breve{R}`x$. Hence

$\overleftarrow{R_*}`x \cap \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha = \iota`\breve{T}`x$

We have now to prove
$xR_*y . \iota`x = \overleftarrow{R_*}`x \cap \alpha - \breve{R}``\alpha . \supset . (\exists T) . T \in \mathrm{Potid}`R . y \in \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha$
We have $y \in \alpha$. Hence $(\exists T) . T \in \mathrm{Potid}`R . y \in \breve{T}``\alpha \quad (T = R_0)$
Prove next $\qquad (\exists T) . T \in \mathrm{Potid}`R . y \in \breve{T}`` -\alpha$

[27r]

$*98\cdot 11 \quad \vdash : R \in \mathrm{Cls} \to 1 . xRz . zR_*y . \supset . R(x \longmapsto y) = \iota`x \cup R(z \longmapsto y).$

$\sim(zR_*y) . \supset . R(z \longmapsto y) = \Lambda$

$xRz . xR_*y . \sim(zR_*y) . \supset . x = y . R(x \longmapsto y) = \iota`x$

Hence $\vdash :. R \in \mathrm{Cls} \to 1 . xRz . \supset : R(x \longmapsto y) = \iota`x \cup R(z \longmapsto y) . \vee .$

$R(x \longmapsto y) = \Lambda . \vee . R(x \longmapsto y) = \iota`x$

Hence $\Lambda \in \mu : \alpha \in \mu . \supset_\alpha . \alpha \cup \iota`x \in \mu :$

$\supset : xRz . R(z \longmapsto y) \in \mu . \supset . R(x \longmapsto y) \in \mu$

$\supset : xR_{*(m+1)}y . R_m(y \longmapsto y) \in \mu . \supset . R_m(x \longmapsto y) \in \mu$

Hence $x\,R_{*(m+1)}y\,.\;R_m(y \mapsto y) \in \text{Cls induct}_{m+1}\;.\supset.\;R_m(x \mapsto y) \in \text{Cls induct}_{m+1}$

[27v]

*To prove* $\xi = \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z = \Lambda$
We have $R``\xi = \xi = \breve{R}``\xi\;.\;x \in \xi$
$y \in \alpha - \beta\;.\;z \in \beta - \alpha\;.\;\alpha \cup \beta \subset \mu\;.$
$\overrightarrow{R}`y \cup \overrightarrow{R}`z \subset -\mu \subset -\alpha - \beta\;.\;y \in -\breve{R}``\alpha\;.\;z \in -\breve{R}``\beta$
$\overrightarrow{R_*}`y \subset -\beta\;.\;\xi \subset R``\overrightarrow{R}`y \subset -R``\beta\;.\;\xi \subset -R``\alpha - R``\beta$
$\xi \subset T`` - \beta\;.\supset.\;\xi \subset R``T`` - \beta$
$R`` - \xi = -\xi = \breve{R}`` - \xi\;.\;y, z \in -\xi\;.\;\alpha \cup \beta \subset -\xi$
Try proving $\overrightarrow{R_*}`y \subset -\overrightarrow{R_*}`z$
Wants $y \in -\overrightarrow{R_*}`z : v \in -\overrightarrow{R_*}`z\;.\;uRv\;.\supset.\;u \in -\overrightarrow{R_*}`z$
Now $\overrightarrow{R_*}`z \subset -\alpha\;\;\therefore\;\;\alpha \subset -\overrightarrow{R_*}`z$
We want $xR_*y\;.\;xR_*z\;.\supset:\;yR_*z\;.\vee.\;zR_*y$
We have $x \in \alpha\;.\;\breve{R}``\alpha \subset \alpha\;.\supset.\;y, z \in \alpha$
$x \in \alpha\;.\;\breve{R}``\alpha \subset \alpha\;.\supset.\;y, z \in \alpha : y = x :\supset.\;z \in \alpha$
Assume $x \in \alpha - \breve{R}``\alpha$. It will do to prove $y(R_{\text{po}} \dot\cup I \dot\cup \breve{R}_{\text{po}})z$
$\alpha = (\alpha - \breve{R}``\alpha) \cup (\breve{R}``\alpha - \breve{R}``\breve{R}``\alpha) \cup \ldots \cup (\breve{T}``\alpha - \breve{R}``\breve{T}``\alpha) \cup \breve{R}``\breve{T}``\alpha$
i.e. $T \in \text{Potid}`R.\;\kappa = \hat{\hat{\xi}}\{(\exists S).\;S \in \text{Potid}`R\;.\;T\,R_{ts}\,S\;.\;\xi = \breve{S}``\alpha - \breve{R}``\breve{S}``\alpha\}\;.\supset$
$.\;\alpha = s`\kappa \cup \breve{R}``\breve{T}``\alpha$

[28]

Now prove $R_* \mathrel{\dot\subset} \dot{s}`\text{Potid}`R$[30]
This follows if $xR_*y\;.\;z \in R(x \mapsto y)\;.\supset.(\exists T).\;T \in \text{Potid}`R\;.\;z = \breve{T}`x$
We have $(\exists S, T).\;S, T \in \text{Potid}`R\;.\;z = \breve{S}`x = \breve{T}`y\;.\supset.\;z \in R(x \mapsto y)$
Hence this is an inductive class if $R \in \text{Cls} \to 1$.
$z = \breve{S}`x = \breve{T}`y\;.\supset.\;\breve{R}`z = \breve{R}`\breve{S}`x = \breve{R}`T`y$
Assume $\sim\{x\;(\dot{s}`\text{Potid}`R)\;y\}$. Then $\breve{R}`z \neq y$.
This property is inductive. Hence a contradiction.
This proof assumes $R \in \text{Cls} \to 1 \cup 1 \to \text{Cls}$. Take $R_\epsilon$
$\alpha(R_\epsilon)_*\iota`x\;.\supset.\;(\exists T).\;T \in \text{Potid}`R_\epsilon.\;\alpha = T`\iota`x$
$\alpha(R_\epsilon)_*\iota`x\;.\equiv:.\;\iota`x \in \mu : \xi \in \mu\;.\supset_\xi.\;R``\xi \in \mu :\supset_\mu.\;\alpha \in \mu\;:.$
$\supset:.\;\alpha \subset \overrightarrow{R_*}`x$
~~Perhaps we could prove~~ $\cancel{s`\overrightarrow{M_*}`\iota`x \subset \dot{s}`\text{Potid}`\overleftarrow{R}`x}$~~?~~
$\cancel{\alpha\;M_*\iota`x\;.\supset.\;(\exists T).\;T \in \text{Potid}`M\;.\;\alpha = T`\iota`x}$
As with $M$, prove $yR_*x\;.\supset.\;y \in s`(\overrightarrow{R_\epsilon})_*`\iota`x$

[30] Compare with $*89{\cdot}28$ where this result is proved for the case where $R$ is one to one.

Thus we have to prove

$T \in \text{Potid}`R_\epsilon \,.\supset.\, (\exists S).\, S \in \text{Potid}`R \,.\, T = S_\epsilon$

$T = S_\epsilon \,.\supset.\, T|R_\epsilon = (S|R)_\epsilon$

Assume $(\exists M): M \in \text{Potid}`R_\epsilon : \sim(\exists S).\, S \in \text{Potid}`R \,.\, M = S_\epsilon$

Prove $S \in \text{Potid}`R \,.\, S_\epsilon \in \text{Potid}`R_\epsilon \,.\supset.\, M R_{\text{ts}} S$

Hence $\hat{T}\{T \in \text{Potid}`R_\epsilon : (\exists S).\, S \in \text{Potid}`R \,.\, T = S_\epsilon\} \subset R_{\text{ts}}(M \mapsto R_0)$

Hence proposition by induction.

[29]

Prove first $x R_{*_3} y \,.\supset.\, R_m(x \mapsto y) \in \text{Cls induct}_3$[31]

Then $y, z \in \overleftarrow{R_{*_3}}`x \,.\supset: y \in R(x \mapsto z) \,.\vee.\, z \in R(x \mapsto y)$

*Dem.*

Put $R(x \mapsto y) = \mu_2 \,.\, R(x \mapsto z) = \nu_2$ Then

$u \in \mu_2 \,.\supset: \breve{R}`u \in \mu_2 \,.\vee.\, u = y$

$u \in \nu_2 \,.\supset: \breve{R}`u \in \nu_2 \,.\vee.\, u = z$

$u \in \mu_2 \cap \nu_2 \,.\supset: \breve{R}`u \in \mu_2 \cap \nu_2 \,.\vee.\, u = y. \vee.\, u = z \quad (1)$

$y \sim\in R(x \mapsto z) \,.\, z \sim\in R(x \mapsto y) \,.$

$\supset: y \sim\in \nu_2 \,.\, z \sim\in \mu_2 :$

$\supset: u \in \mu_2 \cap \nu_2 \,.\supset.\, u \neq y.\, u \neq z \quad (2)$

$(1).(2). \supset:. \text{Hp}(2) \,.\supset: u \in \mu_2 \cap \nu_2 \,.\supset.\, \breve{R}`u \in \mu_2 \cap \nu_2 :$

$\supset: \overleftarrow{R}`x \subset \mu_2 \cap \nu_2 :$

$\supset: y, z \in \mu_2 \cap \nu_2$

whence proposition by Transp.

Hence $y, z \in \overleftarrow{R_{*_3}}`x \,.\supset: y R_* z. \vee.\, z R_* y$

Where $R_*$ on the right may have any suffix. But still $R \in \text{Cls} \to 1$.

(Proposition not true otherwise.)

[30r]

Props changed[32]

$*90{\cdot}31$

$$\cdot 311 \begin{cases} \vdash .\, R_0 \mathbin{\dot\cup} R_*|R \mathbin{\dot\subset} R_* \\ \vdash : R \in \text{Cls} \to 1 \,.\supset.\, R_0 \mathbin{\dot\cup} R|R_* = R_* \\ \vdash : R \in 1 \to \text{Cls} \,.\supset.\, R_0 \mathbin{\dot\cup} R_*|R = R_* \end{cases}$$

$*90{\cdot}32 \quad \vdash : R \in \text{Cls} \to 1 \,.\supset.\, R_*|R \mathbin{\dot\subset} R|R_*$

$\quad \vdash : R \in 1 \to \text{Cls} \,.\supset.\, R|R_* \mathbin{\dot\subset} R_*|R$

[31] See HPF [74] and [75] at the end. This is proved for $m = 2$ and if $R \in Cls \to 1$ at $*89{\cdot}201$.

[32] See the discussion of the need to revise $*90$ at *PM*, p.651. $*90{\cdot}311$ is $\vdash .\, R_* = I \upharpoonright C`R \mathbin{\dot\cup} R|R_*$ and $*90{\cdot}32$ is $\vdash .\, R|R_* = R \mathbin{\dot\cup} R|R_*|R = R_*|R = R_*|R$.

$\cdot 351 \quad \vdash :. \ R \in \text{Cls} \to 1 : \breve{R}``\mu \subset \mu \ . \ \overleftarrow{R}`x \subset \mu \ . \supset_\mu . \ z \in \mu : \supset . \ xR|R_*z$
$\cdot 36 \quad \vdash :: \qquad\qquad . \supset :. \qquad\qquad \equiv$

Hence $|R``\mu \subset \mu \ . \ R \in \mu \ . \supset_\mu . \ P \in \mu : \equiv . \ R_0\{(|R)|R_{\text{ts}}\}P$
i.e. $P \in \text{Pot}`R. \equiv .R_0\{(|R)|R_{\text{ts}}\}P$
whence $|R``\text{Potid}`R \subset \text{Pot}`R$
$*91\cdot24$ is first doubtful proposition in $*90$
$\text{Potid}`R = \iota`R_0 \cup \text{Pot}`R$ Assume $R_0 \sim\in \text{Pot}`R$. Then
$\text{Pot}`R = \text{Potid}`R - \iota`R_0$
~~Now $|R``\text{Potid}`R \subset \text{Potid}`R$~~

$*91\cdot24$ becomes $\vdash .|R``\text{Potid}`R \subset \text{Pot}`R$ to begin with.[33]

$*91\cdot241$ is all right. But
$*91\cdot302 : |R``\text{Potid}`R = R|``\text{Potid}`R$
$\cdot264 \ \ \text{Pot}`R = \iota`R \cup R|``\text{Pot}`R$
whence $\text{Pot}`R \subset |R``\text{Potid}`R$ Hence $*91\cdot24$, after $*91\cdot302$.
Thence no trouble till $*91\cdot44$.

[30v]

$*90\cdot351$
$\sim x(R|R_*y) . \equiv : y \sim\in \mu \ . \ \breve{R}``\mu \subset \mu \ . \supset_\mu . \ \overleftarrow{R}`x \subset \mu$[34]
$\breve{R}``\mu \subset \mu \ . \ \overleftarrow{R}`x \subset \mu \ . \supset_\mu . \ y \in \mu : \sim x(R|R_*y) : \supset :$
$\breve{R}``\mu \subset \mu \ . \ y \sim\in \mu \ . \supset . \ y \in \mu : \supset : \breve{R}``\mu \subset \mu \ . \supset . \ y \in \mu : \supset : y \in \Lambda$
Hence prop.
Dem.
$\sim x(R|R_*)y \ . \equiv : zR_*y \ . \supset_z . \sim(xRz) :$
$\equiv : y \in \mu \ . \ \breve{R}``\mu \subset \mu \ . \supset_\mu . \ \mu \subset -\overleftarrow{R}`x :$
$\equiv : y \ \sim\in \mu \ . \ \breve{R}``\mu \subset \mu \ . \supset_\mu . \ \overleftarrow{R}`x \subset \mu \quad [\frac{-\mu}{\mu}]$
$\overrightarrow{R_*}`y \subset \alpha \ . \ \equiv :. \ y \in \mu \ . \ R``\mu \subset \mu \ . \supset_\mu . \ z \in \mu : \supset . \ z \in \alpha$
$\cancel{p`\kappa \subset \alpha \ . \equiv : \mu \in \kappa \ . \supset_\mu . \ \mu \subset \alpha}$

[31]

Proved $\quad \gamma M_*\iota`x \ . \supset . \overleftarrow{R}_*`x - \gamma \subset \overleftarrow{R}_*`\breve{R}`\text{max}_R`\gamma$[35]
Assuming $\quad \overleftarrow{R}_*`x - \overleftarrow{R}_*`\breve{R}`\text{max}_R`\gamma \subset \gamma$

[33] $*91\cdot24$ is originally $\text{Pot}`R = |R``\text{Potid}`R$.
[34] An arrow leads from the beginning of this formula to 'Dem.' below. The closing parenthesis after $y$ should precede it.
[35] See [22v].

throughout $\quad x R_* y \,.\supset:\, y R_* \max_R{}^{\backprime}\gamma \;.\vee.\; \max_R{}^{\backprime}\gamma R_* y$

$R \in \mathrm{Cls}\to 1 \quad z \in s{}^{\backprime}\overrightarrow{M_*}{}^{\backprime}\iota{}^{\backprime}x \,.\supset:\, y R_* z \;.\vee.\; z R_* y$

$y \sim\in s{}^{\backprime}\overrightarrow{M_*}{}^{\backprime}\iota{}^{\backprime}x \,.\supset:\, z \in s{}^{\backprime}\overrightarrow{M_*}{}^{\backprime}\iota{}^{\backprime}x \,.\supset_z.\, z R_* y$

$\breve{R}{}^{\backprime\backprime}s{}^{\backprime}\overrightarrow{M_*}{}^{\backprime}\iota{}^{\backprime}x \subset s{}^{\backprime}\overrightarrow{M_*}{}^{\backprime}\iota{}^{\backprime}x$

$x R_{*_3} y \,.\supset.\, R_1(x \longmapsto y) \in \mathrm{Cls\ induct}_3 \,.\supset.\, R_m(x \longmapsto y) \in \mathrm{Cls\ induct}_3$

because $\alpha \in \mathrm{Cls\ induct}_m \,.\, \beta \subset \alpha \,.\supset.\, \beta \in \mathrm{Cls\ induct}_m$

Hence by above

$\gamma M_{*_3}\iota{}^{\backprime}x \,.\, x R_{*_3} y \,.\, y \sim\in s{}^{\backprime}\overrightarrow{M_{*_3}}{}^{\backprime}\iota{}^{\backprime}x \,.\supset.\, s{}^{\backprime}\overrightarrow{M_{*_3}}{}^{\backprime}\iota{}^{\backprime}x \subset R_3(x \longmapsto y).$

$\supset.\, s{}^{\backprime}\overrightarrow{M_{*_3}}{}^{\backprime}\iota{}^{\backprime}x \in \mathrm{Cls\ induct}_3$

$\supset.(\exists\mu_2).\, s{}^{\backprime}\overrightarrow{M_{*_3}}{}^{\backprime}\iota{}^{\backprime}x = \mu_2$

Hence by induction $\quad s{}^{\backprime}\overrightarrow{M_{*_3}}{}^{\backprime}\iota{}^{\backprime}x = \overleftarrow{R_{*_3}}{}^{\backprime}x$

Hence $\quad x R_* y \,.\, x R_* z \,.\supset:\, y R_* z \;.\vee.\; z R_* y$

whence $\quad \breve{R}{}^{\backprime\backprime}\mu \subset \mu \,.\supset.\, \overleftarrow{R}_*{}^{\backprime}x \cap \mu - \breve{R}{}^{\backprime\backprime}\mu \in 0 \cup 1$

$P, Q \in \mathrm{Potid}{}^{\backprime}R \,.\supset:\, P R_{\mathrm{ts}} Q \;.\vee.\; Q R_{\mathrm{ts}} P$[36]

[32r]

Required $\quad x R_* y \,.\supset.(\exists\gamma).\, \gamma M_*\iota{}^{\backprime}x \,.\, y \in \gamma$

First prove $\gamma M_*\iota{}^{\backprime}x \,.\, x R_* y \,.\, y \in -\gamma \,.\supset.\, \gamma \subset \overrightarrow{R_*}{}^{\backprime}y$ assuming $R \in \mathrm{Cls}\to 1$

$\overrightarrow{R_*}{}^{\backprime}y = \hat{\beta}(y \in \beta \,.\, R{}^{\backprime\backprime}\beta \subset \beta)$

Thus to prove $y \in \beta \,.\, R{}^{\backprime\backprime}\beta \subset \beta \,.\supset.\, \gamma \subset \beta$

We have $x \in \beta$

We have to prove ~~$z \in \gamma \,.\, z \in \beta \,.\supset.$~~

$y \sim\in \gamma \,.\, \gamma \subset \beta \,.\supset.\, \gamma \cup \breve{R}{}^{\backprime\backprime}\gamma \subset \beta$

We have $E!\max_R{}^{\backprime}\gamma \,.\, \gamma \cup \breve{R}{}^{\backprime\backprime}\gamma = \gamma \cup \iota{}^{\backprime}\breve{R}{}^{\backprime}\max_R{}^{\backprime}\gamma$

$\breve{R}{}^{\backprime}\max_R{}^{\backprime}\gamma \in \beta \,.\supset.\, \gamma \subset \beta$

We have $\iota{}^{\backprime}x \subset \overrightarrow{R_*}{}^{\backprime}y$

$x R_* z \,.\, z \in y \,.\supset.\, z \neq y$

$\breve{R}{}^{\backprime}\max_R{}^{\backprime}\gamma = y \,.\supset.\, \breve{R}{}^{\backprime\backprime}\gamma \subset \beta$

$\breve{R}{}^{\backprime}\max_R{}^{\backprime}\gamma \neq y$

---

$\gamma M_*\iota{}^{\backprime}x \,.\supset.\, \gamma \cup \breve{R}{}^{\backprime\backprime}\gamma = \gamma \cup \iota{}^{\backprime}\breve{R}{}^{\backprime}\max_R{}^{\backprime}\gamma$

We want $x R_* y \,.\, y \sim\in \gamma \,.\supset.\, \gamma \subset \overrightarrow{R_*}{}^{\backprime}y$

i.e. $x R_* y \,.\supset:.\, y \in \gamma :\vee:\, y \in \beta \,.\, R{}^{\backprime\backprime}\beta \subset \beta \,.\supset.\, \gamma \subset \beta$

$y \sim\in \gamma. \supset:\, y \neq x :\supset:\, x R_* y \,.\supset.\, (\breve{R}{}^{\backprime}x) R_* y \,.\supset.\, M{}^{\backprime}\iota{}^{\backprime}x \subset \overrightarrow{R_*}{}^{\backprime}y$

[36] This is $*91{\cdot}44$.

$\gamma \subset \overrightarrow{R_*}`y \,.\, y \sim\epsilon\, \gamma \,.\supset.\, \gamma \subset \overrightarrow{R|R_*}`y \,.\supset.\, \breve{R}``\gamma \subset \overrightarrow{R_*}`y \,.\supset.\, M`\gamma \subset \overrightarrow{R_*}`y$
$\gamma \subset \beta \,.\, y \,\epsilon\, \beta - \gamma \,.\supset.\, \gamma \subset R``\beta \,.\supset.\, \breve{R}``\gamma \subset \beta$

[32v]

$R_m`\alpha - \alpha \subset R``\alpha$
$\cancel{R_m`R_m`\alpha - R_m`\alpha = R``R_m`\alpha = R``\alpha \cup R``R``\alpha}$
$R_m`\iota`x - \iota`x = \overrightarrow{R}`x$
$R_m^2`\iota`x - R_m`\iota`x = \overrightarrow{R^2}`x$
$x \,\epsilon\, \breve{R}^2``R_m^2`\iota`x$ i.e. $x \,\epsilon\, (\breve{R}_\epsilon|R_m)^2`\iota`x$
Consider powers of $\breve{R}_\epsilon|R_m$ where $\breve{R}_\epsilon = (\breve{R})_\epsilon$
To avoid confusion put
$R_m`\alpha = \alpha \cup \breve{R}``\alpha \qquad R_\epsilon`R_m`\alpha = R``(\alpha \cup \breve{R}``\alpha)$
Then $x \,\epsilon\, (R_\epsilon|R_m)`\iota`x$
$R_\epsilon`R_m`\iota`x = R``(\iota`x \cup \overleftarrow{R}`x)$
$\cancel{R_\epsilon`R_m`R_\epsilon`R_m`\iota`x = R``\{R``(\iota`x \cup \overleftarrow{R}`x) \cup \breve{R}``\overleftarrow{R}`x\}}$
$\cancel{R``R_m`\alpha = R_m`\alpha}$
$R_\epsilon`R_m`R_\epsilon`R_m`\iota`x = R``\{R``(\iota`x \cup \overleftarrow{R}`x) \cup \breve{R}``R``(\iota`x \cup \overleftarrow{R}`x)\}$

[33]

New Df. $\quad R_m`\alpha = \alpha \cup R``\alpha \quad$ Df
$x R_h y \,.=.\, x \,\epsilon\, s`\overrightarrow{(R_m)_*}\,\iota`y \quad$ Df
$\alpha(R_m)_*\iota`x \,.\equiv:.\, \iota`x \,\epsilon\, \xi : \beta \,\epsilon\, \xi \,.\supset_\beta.\, \beta \cup R``\beta \,\epsilon\, \xi : \supset_\xi .\, \alpha \,\epsilon\, \xi \;:.\; x \,\epsilon\, \mathrm{C}`R$
$\supset:.\, x \,\epsilon\, \mu \,.\, R``\mu \subset \mu \,.\supset_\mu.\, \alpha \subset \mu :.$
$\supset:.\, \alpha \subset \overrightarrow{R_*}`x$
Hence $R_h \mathrel{\subset\!\!\cdot} R_*$ without any hypothesis.
$\vdash : x \,\epsilon\, \mathrm{C}`R \,.\equiv.\, x R_h x \qquad \vdash .\, c`R_h = c`R$
Can we prove $S = \breve{R} \,.\supset.\, S_h = c`R_h$?
$\alpha(S_m)_*\iota`y \,.\equiv:.\, \iota`y \,\epsilon\, \eta : \gamma \,\epsilon\, \eta \,.\supset_\gamma.\, \gamma \cup \breve{R}``\gamma \,\epsilon\, \eta : \supset_\eta .\, \alpha \,\epsilon\, \eta \;:.\; y \,\epsilon\, \mathrm{C}`R$
Can't be proved yet.
Perhaps $x\epsilon$ C$`R$ is unnecessary in the Df?
Assume $x \sim\epsilon$ C$`R$. Assume $\beta \subset -$C$`R$. Then $\beta \cup R``\beta = \beta$
Put for $\xi$ Cl$`\mu$, where $\mu \subset -c`R$. Then $\alpha(R_m)_*\iota`x \,.\, x \,\epsilon\, \mu \,.\supset.\, \alpha \subset \mu$
Hence $\alpha \subset \iota`x$. In fact $\alpha = \iota`x$. Hence $x \,\epsilon\, c`R$ is necessary.

[34r]

Try $M`\alpha = \alpha \cup \breve{R}``\alpha$[37]
We want $\alpha - \breve{R}``\alpha,\ \breve{R}``\alpha - \alpha \,\epsilon\, 1$ if $\alpha M_*\iota`x$
$\alpha - \breve{R}``\alpha \,\epsilon\, 1 \,.\supset.\, (\alpha \cup \breve{R}``\alpha) - \breve{R}``(\alpha \cup \breve{R}``\alpha) = \alpha - \breve{R}``\alpha - \breve{R}``\breve{R}``\alpha = \alpha - \breve{R}``\alpha$

[37] $M$ is defined differently at *PM*, p.653.

In fact $\alpha - \breve{R}``\alpha = \iota`x$

$\breve{R}``\alpha - \alpha \in 1 . \supset . \breve{R}``(\alpha \cup \breve{R}``\alpha) - (\alpha \cup \breve{R}``\alpha) = \breve{R}``\breve{R}``\alpha - \breve{R}``\alpha - \alpha$

$= \breve{R}``(\breve{R}``\alpha - \alpha) - (\alpha - \breve{R}``\alpha)$

$= \breve{R}``(\breve{R}``\alpha - \alpha) - \iota`x = \breve{R}``(\breve{R}``\alpha - \alpha)$ wh. $\in 1$

Hence $\breve{R}``\alpha - \alpha \in 1$

$\breve{R}``\mu \subset \mu . \supset : \alpha \subset \mu . \supset . M`\alpha \subset \mu$

Hence $x \in \mu . \breve{R}``\mu \subset \mu . \supset . s`\overrightarrow{M_*}`\iota`x \subset \mu$ i.e. $s`\overrightarrow{M_*}`\iota`x \subset \overleftarrow{R}_*`x$

In fact, each member of $\overrightarrow{M_*}`\iota`x$ is wholly contained in every inductive class.

$y \in \overrightarrow{M_*}`\iota`x . \equiv :. \iota`x \in \xi : \alpha \in \xi . \supset_\alpha . \alpha \cup \breve{R}``\alpha \in \xi : \supset_\xi . \gamma \in \xi$

We want $x R_* y . \supset . (\exists\gamma). \gamma \in \overrightarrow{M_*}`\iota`x . y \in \gamma$

We have $\cancel{\gamma \in \overrightarrow{M_*}`\iota`x}. y \in \gamma . \supset . \breve{R}`y \in M`\gamma$

Hence $\breve{R}``s`\overrightarrow{M_*}`\iota`x \subset s`\overrightarrow{M_*}`\iota`x$

$\breve{R}``s`\kappa = s`\breve{R}```\kappa$. Hence

$s`\breve{R}```\overrightarrow{M_*}`\iota`x \subset s`\overrightarrow{M_*}`\iota`x$

Hence $\gamma \in \overrightarrow{M_*}`\iota`x . \supset . \breve{R}``\gamma \subset s`\overrightarrow{M_*}`\iota`x$

Try $\overrightarrow{M_*}`\iota`x \cap - \overleftarrow{\epsilon}`y$. Try proving that this has a maximum.

Then its successor will be what we want.

[34v]

~~Try~~ $\cancel{M`\alpha = \alpha \cup \breve{R}``(\alpha - R``\alpha)}$ Put $M`\alpha = \alpha \cup (\breve{R}``\alpha - \alpha)$[38]

Then $\alpha \subset M`\alpha : \cancel{\exists! \breve{R}``\alpha R``\alpha . \supset . \exists! M`\alpha - \alpha}$

Consider $\overrightarrow{M_*}`\iota`x$.

$\cancel{\iota`x = \iota`x - R``\iota`x}$

$\cancel{\alpha - R``\alpha \in 1 . \supset . \breve{R}``(\alpha - R``\alpha) \in 1}$

$\cancel{R``\{\alpha \cup \breve{R}``(\alpha - R``\alpha\} = R``\alpha \cup R``\breve{R}``(\alpha - R``\alpha)}$

$\cancel{\{\alpha \cup \breve{R}``(\alpha - R``\alpha)\} - R``\alpha - R``\breve{R}``(\alpha - R``\alpha)}$

$= \cancel{\breve{R}``(\alpha - R``\alpha)}$

~~Hence~~ $\cancel{\alpha M_* \iota`x . \supset . \alpha - R``\alpha \in 1}$

$\cancel{\alpha M_* \beta . \supset . \beta \subset \alpha}$

$\breve{R}``\iota`x - \iota`x \in 1$

To prove $\breve{R}``\alpha - \alpha \in 1 . \supset . \breve{R}``\{\alpha \cup (\breve{R}``\alpha - \alpha)\} - \alpha - (\breve{R}``\alpha - \alpha) \in 1$

i.e. $\{\breve{R}``\alpha \cup (\breve{R}``\breve{R}``\alpha - \breve{R}``\alpha)\} - \alpha - \breve{R}``\alpha \in 1$

i.e. $\breve{R}``\breve{R}``\alpha - \breve{R}``\alpha - \alpha \in 1$

This requires $\breve{R}``\breve{R}``\alpha - \breve{R}``\alpha \subset -\alpha$ i.e. $\breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha \cup -\alpha$

[35r]

Props proved. $\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha . x \in \alpha)$ General hp : $R \in \mathrm{Cls} \to 1 . \mathrm{D}`R = V$

$R|R_* = R_*|R . R_* = R_0 \dot{\cup} R|R_*$[39]

[38] Another proposal for $M$. See [32r] and *PM*, p.653.

[39] See 30r and $*89\cdot103$ in Appendix B.

$\breve{R}``\overleftarrow{R}_*`x = \breve{R}``p`\kappa = p`\breve{R}```\kappa = \overleftarrow{R}_*`\breve{R}`x$

$T \epsilon \text{ Potid}`R \,.\supset: x(T|R_*)y \,.\equiv.\, y \epsilon p`\breve{T}```x \,.\equiv.\, \breve{T}`xR_*y$

$x(R_*|T)y \,.\equiv.\, y \epsilon \breve{T}``p`\kappa : \breve{R}``p`\breve{T}```\kappa = p`\breve{R}```\breve{T}```\kappa$

$R_*|T \mathrel{\subset\!\!\cdot} T|R_* \,.\, \dot{s}`\text{Potid}`R \mathrel{\subset\!\!\cdot} R_*$

$xR_*y \,.\supset.\, (\exists T).\, T \epsilon \text{ Potid}`R \,.\, y \epsilon p`\breve{T}```\kappa$

$M`\alpha = \iota`x \cup \breve{R}``\alpha \,.\supset.\, \overrightarrow{M_*}`\iota`x \subset \text{Cls induct} \,.\, s`\overrightarrow{M_*}`\iota`x \subset \overleftarrow{R}_*`x$

[ The rest of this page is deleted with a single diagonal line.]

$\text{Int}(R, x) = \hat{\gamma}\{x \epsilon \mu : \beta \subset \mu \,.\supset_\beta.\, \iota`x \cup \breve{R}``\beta \subset \mu :\supset_\mu.\, \gamma \subset \mu\}$ Df

$\text{Int}(R, x) \subset$ Cls induct

$s`\text{Int}(R, x) = \overleftarrow{R}_*`x$

$\text{Int}(R, x) = \hat{\gamma}\{(\exists y).\, xR_*y \,.\, \gamma = R(x \longmapsto\!\!\dashv y)\}$

This proves that intervals are inductive classes.

Hence $R(x \longmapsto\!\!\dashv y) \cap R(x \longmapsto\!\!\dashv z) \epsilon$ Cls induct

Now in our original problem,

$\breve{R}``\{R(x \longmapsto\!\!\dashv y) \cap R(x \longmapsto\!\!\dashv z)\} \subset R(x \longmapsto\!\!\dashv y) \cap R(x \longmapsto\!\!\dashv z)$

Hence $\overleftarrow{R}_*`x \subset R(x \longmapsto\!\!\dashv y) \cap R(x \longmapsto\!\!\dashv z)$

which is impossible if $y \neq z$. Hence $\mu - \breve{R}``\mu \epsilon 0 \cup 1$. Q.E.D.

[35v]

Assuming $y, z \epsilon \overleftarrow{R}_*`x \cap \mu - \breve{R}``\mu$,

put $\alpha = \iota`y \cup \breve{R}``\mu \,.\, \gamma = R``{-\alpha} \cup (\mu - \iota`y) \cup \breve{R}``\mu \,.\, \breve{R}``\alpha \subset \alpha$

Then $\breve{R}``\gamma \subset -\alpha \cup \breve{R}``\mu \quad \cancel{\subset -\alpha - \breve{R}``\mu \cup \breve{R}``\mu}$

$-\alpha = -\iota`y - \breve{R}``\mu = \{(\mu - \iota`y) \cup -\mu\} - \breve{R}``\mu$

Now $\mu - \iota`y \subset \gamma \,.\, -\mu - \breve{R}``\mu = -\breve{R}``\mu$

Now $\alpha \subset \mu$ Hence $-\mu \subset -\alpha,\ -\breve{R}``\mu \subset -\breve{R}``\alpha \subset -\alpha \cup \alpha - \breve{R}``\alpha$

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$\alpha = \iota`y \cup \breve{R}``\mu \,.\supset$

$.\, \breve{R}``\alpha \subset \breve{R}``\mu \,.\, \alpha - \breve{R}``\alpha \subset \mu \cancel{- \breve{R}``\alpha} \,.\, \alpha - \breve{R}``\mu = \iota`y = \alpha - \breve{R}``\alpha$

$\beta = \iota`z \cup \breve{R}``\mu$

$R``{-\alpha}$ contains all $R``\mu$ except $\overrightarrow{R}`y$, which $\subset R``\alpha$ and $\subset -\beta$

We have $\overrightarrow{R_*}y \subset -\beta$. Consider $-\beta \cup \breve{R}``\mu$.

$\breve{R}``{-\beta} = \breve{R}``(\alpha - \beta) \cup \breve{R}``(-\alpha - \beta)$

$\beta \subset \mu \,.\, -\mu \subset -\beta \,.\, -\breve{R}``\mu \subset -\breve{R}``\beta \subset -\beta$

$\breve{R}``{-\beta} = \breve{R}``(R``\mu - \beta) \cup \breve{R}``(-R``\mu - \beta)$

$= \mu \cap \breve{R}`` - \beta \cup -\mu$

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$-\beta = (-\beta \cap \breve{R}`` - \beta) \cup (-\beta \cap \breve{R}``\beta)$

$\breve{R}`` - \beta \subset -\beta \cup \breve{R}``(R``\beta - \beta)$ i.e. $\subset -\beta \cup (\beta - \breve{R}``\beta) \quad \cancel{\subset -\beta \cup \iota`z \cup \breve{R}``\mu}$

$\beta - \breve{R}``\beta = \iota`z \cup (\breve{R}``\mu - \iota`\breve{R}`z)$
Hence $\breve{R}`` - \beta \subset -\beta \cup \iota`z \cup (\breve{R}``\mu - \iota`\breve{R}`z) \subset -\mu \cup (\mu - \iota`z) \cup (\breve{R}``\mu - \iota`\breve{R}`z)$

[36]

We have $\overrightarrow{M_*}`\iota`x \subset$ Cls induct.
Hence $\kappa \subset \overrightarrow{M_*}`\iota`x \,.\supset.\, p`\kappa \in$ Cls induct.
Hence induction can be applied to $p`\kappa$.
We must be able to prove
$\kappa \subset \overrightarrow{M_*}`\iota`x \,.\supset.\, p`\kappa \in \overrightarrow{M_*}`\iota`x$
i.e. $\gamma \in \kappa \,.\supset_\gamma :.\, x \in \mu : \beta \subset \mu \,.\supset_\beta .\, \iota`x \cup \breve{R}``\beta \subset \mu : \supset_\mu .\, \gamma \subset \mu :: \supset :.$
$x \in \mu : \beta \subset \mu \,.\supset_\beta .\, \iota`x \cup \breve{R}``\beta \subset \mu : \supset_\mu .\, p`\kappa \subset \mu$
This is obvious.
Now consider $p`\{\overrightarrow{M_*}`\iota`x \cap \overleftarrow{\in}`y\}$ This is $\in (\overrightarrow{M_*}`\iota`x \cap \overleftarrow{\in}`y)$
Also it is the smallest. Hence it $\sim\in M``(\overrightarrow{M_*}`\iota`x \cap \overleftarrow{\in}`y)$
Hence it $= R(x \longmapsto y)$.
Instead of $\overrightarrow{M_*}`\iota`x$, put
$\mathrm{Int}(R, x) = \hat{\gamma}\{x \in \mu : \beta \subset \mu \,.\supset_\beta .\, \iota`x \cup \breve{R}``\beta \subset \mu : \supset_\mu .\, \gamma \subset \mu\}$
This gives required results without doubtful steps.
From this it follows that $R(x \longmapsto y) \in$ Cls induct, whence all the rest.

[37r]

Put $M`\alpha = \iota`x \cup \breve{R}``\alpha$
We want to prove
$xR_*y \,.\supset.\, (\exists\beta).\, \beta M_*(\iota`x) \,.\, y \in \beta - R``\beta \,.\, \cancel{\iota`y}$
$M_0`\iota`x = \iota`x$
$M`\iota`x = \iota`x \cup \iota`\breve{R}`x$
$M^2`\iota`x = \iota`x \cup \iota`\breve{R}`x \cup \iota`\breve{R}`\breve{R}`x$ etc.
$\overrightarrow{M_*}`\iota`x = \hat{\gamma}\{\iota`x \in \xi : \beta \in \xi \,.\supset_\beta .\, \iota`x \cup \breve{R}``\beta \in \xi : \supset_\xi .\, \gamma \in \xi\}$
We ought to be able to prove $R(x \longmapsto y) \in \overrightarrow{M_*}`\iota`x$
$\overrightarrow{M_*}`\iota`x \subset$ Cls induct$_3$ $: \gamma \in \overrightarrow{M_*}`\iota`x \,.\supset.\, (\exists\mu_2).\, \alpha = \mu_2$
To prove $xR_*y \,.\supset.\, (\exists\beta).\, \beta M_*(\iota`x).\, y \in \beta$
$\gamma \in \overrightarrow{M_*}`\iota`x \,.\supset :\, x \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha .\, \gamma \subset \alpha : \supset .\, \gamma \subset \overleftarrow{R_*}`x$
Hence $s`\overrightarrow{M_*}`\iota`x \subset \overleftarrow{R_*}`x$
To prove $xR_*y \,.\supset.\, y \in s`\overrightarrow{M_*}`\iota`x$?
or to prove $\quad\supset.\, R(x \longmapsto y) \in \overrightarrow{M_*}`\iota`x$?
Prove first $\gamma \in \overrightarrow{M_*}`\iota`x \,.\supset.\, \gamma - R``\gamma \in 1$
$(\iota`x \cup \breve{R}``\beta) - R``(\iota`x \cup \breve{R}``\beta) = \breve{R}``\beta - \breve{R}``\breve{R}``\beta$
Assume $\beta - R``\beta = \iota`y$. Then $\breve{R}``\beta = \breve{R}``(\beta \cap R``\beta) \cup \iota`\breve{R}`y \subset R``\breve{R}``\beta \cup \iota`\breve{R}`y$

Hence $\breve{R}``\beta - R``\breve{R}``\beta = \iota`\breve{R}`y$ Hence prop by induction.
Also $M`\gamma - R``M`\gamma = \iota`\breve{R}`\breve{\iota}`(\gamma - R``\gamma) = \breve{R}``(\gamma - R``\gamma)$
Also $y = \breve{\iota}`(\gamma - R``\gamma) . \supset . \gamma \subset \overrightarrow{R_*}`y$ This must be easy.
Hence $\gamma \epsilon \overrightarrow{M_*}`\iota`x . \supset . (\exists y). xR_*y . \gamma = R(x \longmapsto y)$
We want the converse. This wants

$\iota`x \epsilon \xi : \beta \epsilon \xi . \supset_\beta . \iota`x \cup \breve{R}``\beta \epsilon \xi : xR_*y : \supset . y \epsilon \iota`\xi$

but this is not enough.

[37v]

Consider $R_0 \epsilon \mu . |R``\mu \subset \mu . yR_*x$. To pr. $y(\dot{s}`\mu)x$
We want $R_0 \epsilon \mu . |R``\mu \subset \mu . yR_*x . \supset . (\exists T). T \epsilon \mu . yTx$
We have $x \epsilon \alpha . R``\alpha \subset \alpha . \supset_\alpha . y \epsilon \alpha : y \epsilon \beta . \breve{R}``\beta \subset \beta . \supset_\beta . x \epsilon \beta$

$R_0 \epsilon \mu . \supset . \mathrm{D}`R_0 \epsilon D``\mu . \text{Ɑ}`R_0 \epsilon \text{Ɑ}``\mu$

$|R``\mu \subset \mu . \supset : T \epsilon \mu . \supset . T|R \epsilon \mu : \supset : z \epsilon s`\text{Ɑ}``\mu . zRw . \supset . w \epsilon s`\text{Ɑ}``\mu$

$\cancel{\supset : \iota`z \subset \text{Ɑ}``\mu}$

Thus $s`\text{Ɑ}``\mu$ is the natural class to use.
Suppose $T \epsilon \mu . \supset . \sim(yTx)$
Then $T \epsilon \mu . \supset . \sim(yT|Rx)$
$w \epsilon s`\text{Ɑ}``\mu . \equiv . (\exists R, z). R \epsilon \mu . zRw$
Must allow this if any relation of $R_{\mathrm{po}}$ and $R_*$ is to be demonstrable.
$T \epsilon \mu . \supset . T|R \epsilon \mu : \supset : (\exists T). yTz . T \epsilon \mu . \supset .(\exists T).$
$\cancel{R`zTy} \lfloor yT(\breve{R}`z)\rfloor . T \epsilon \mu$

$\supset : yR_*x . \supset . (\exists T). T \epsilon \mu . yTx$

Hence if $\lambda = \hat{\mu}(R_0 \epsilon \mu . |R``\mu \subset \mu)$, $R_* \mathrel{\dot\subset} \dot{p}`\dot{s}``\lambda . (R_{\mathrm{po}})_* = \dot{s}`p`\lambda$

[38]

Induction to minimum inductive classes
Assume $x \epsilon \alpha - \breve{R}``\alpha . \breve{R}``\alpha \subset \alpha$.
Put $M = \hat{\beta}\hat{\gamma}(\gamma = \iota`x \cup \breve{R}``\beta)$ $\qquad \breve{M}`\beta = (\iota`x \cup)`\breve{R}``\beta$
Then $\breve{M}`\alpha = \iota`x \cup \breve{R}``\alpha$ $\qquad$ For $T`\beta$ write $(\iota`x \cup)`T`\beta$
$\breve{M}`\breve{M}`\alpha = \iota`x \cup \iota`\breve{R}`x \cup \breve{R}``\breve{R}``\alpha$ etc. $\qquad$ ~~Where $T \epsilon$ Potid$`R_\epsilon$~~
$\breve{M}`\breve{T}`\alpha = \iota`x \cup \breve{R}``\breve{T}`\alpha$ $\qquad M = (\iota`x \cup)|\breve{R}_\epsilon$
The minimum inductive class containing $x$ is $p`\overleftarrow{M_*}`\alpha$ ; $\overleftarrow{R}_*`x = p`\overleftarrow{M_*}`\alpha$
$\breve{R}`\breve{M}`\breve{T}`\alpha = \iota`\breve{R}`x \cup \breve{R}``\breve{R}``\breve{T}`\alpha$
To prove this $\subset \breve{M}`\breve{T}`\alpha$ requires $\breve{R}`x \epsilon \breve{M}`\breve{T}`\alpha . \breve{R}``\breve{T}`\alpha \subset \breve{T}`\alpha$
We have $\breve{R}`x \epsilon \alpha : \underline{\breve{R}`} x \epsilon \breve{T}`\alpha . \supset . \breve{R}`x \epsilon \breve{M}`\breve{T}`\alpha$
Hence $\breve{R}``\breve{T}`\alpha \subset \breve{T}`\alpha$ if $T \epsilon$ Potid$`M$

Hence $x R_* y \,.\, T \in \text{Potid‘}M \,.\supset.\, y \in \breve{T}\text{‘}\alpha$

We now have to prove

$T \in \text{Potid‘}M \,.\supset_T.\, y \in \breve{T}\text{‘}\alpha :\supset .(\exists S).\ S \in \text{Potid‘}R \,.\, y = \breve{S}\text{‘}x$

Can't we prove $\overleftarrow{R}_*\text{‘}x \subset \iota\text{‘}x \cup \overleftarrow{R_{\text{po}}}\text{‘}x$?

The essential point is to prove there is no residue, i.e.

$\sim(\exists y) : T \in \text{Potid‘}R \,.\supset_T.\, y \in \breve{T}\text{“}\alpha$

or at least that, if such a $y$ exists, we cannot have

$y \in \mu - \breve{R}\text{“}\mu$

That is to say, we have to prove that, if $y$ belongs to the residue,

$y \in \mu \,.\, \breve{R}\text{“}\mu \subset \mu \,.\supset.\, y \in \breve{R}\text{“}\mu$

[39r]

Props proved $R|R* = R_*|R \,.\, R_* = R_0 \dot{\cup} R|R_*$[40]

$\breve{R}\text{“}\overleftarrow{R}_*\text{‘}x = \breve{R}\text{“}p\text{‘}\kappa = p\text{‘}\breve{R}\text{“‘}\kappa = \overleftarrow{R}_*\text{‘}\breve{R}x \quad [\kappa = \hat{\alpha}(\breve{R}\text{“}\alpha \subset \alpha \,.\, x \in \alpha)]$

$T \in \text{Potid‘}R \,.\supset: x(T|R_*)y \,.\equiv.\, y \in p\text{‘}\breve{T}\text{“‘}\kappa \,.\equiv.\, \breve{T}\text{‘}x R_* y$

$x(R_*|T)y \,.\equiv.\, y \in \breve{T}\text{“}p\text{‘}\kappa : \breve{R}\text{“}p\text{‘}\breve{T}\text{“‘}\kappa = p\text{‘}\breve{R}\text{“‘}\breve{T}\text{“‘}\kappa$

$R_*|T \mathrel{\dot\subset} T|R_* \,.\, \dot{s}\text{‘Potid‘}R \mathrel{\dot\subset} R_*$

$x R_* y \,.\supset.\, (\exists T).\ T \in \text{Potid‘}R \,.\, y \in p\text{‘}\breve{T}\text{“‘}\kappa$

Can we prove $(\exists T).\ T \in \text{Potid‘}R \,.\, y \in p\text{‘}\breve{T}\text{“‘}\kappa - p\text{‘}\breve{R}\text{“‘}\breve{T}\text{“‘}\kappa$?

In our original hp. we have

$x R_* y \,.\, x R_* z \,.\, \sim(y R_* z).\ \sim(z R_* y).$

$y, z \in \mu - \breve{R}\text{“}\mu.\ y \in \alpha - \breve{R}\text{“}\alpha - \beta.\ z \in \beta - \breve{R}\text{“}\beta - \alpha$

We now have

$(\exists S, T).\ S, T \in \text{Potid‘}R \,.\, y \in p\text{‘}\breve{S}\text{“‘}\kappa \,.\, z \in p\text{‘}\breve{T}\text{“‘}\kappa$

~~Now~~ $\cancel{y \in \alpha \,.\supset.\, x \in S\text{“}\alpha \,.\, y \in}$

~~Now~~ $\cancel{y \in \alpha \,.\, \breve{S}\text{‘}x = -u \ \supset.\, y \in \alpha \cup \iota\text{‘}x \,.\, x \in S\text{“}(\alpha \cup \iota\text{‘}x) \,.\, \breve{S}\text{“}(\alpha \cup \iota\text{‘}x) =}$

Also $x(S|R_*)y \,.\, x(T|R_*)z$

Assume $x \in \alpha \,.\, \breve{R}\text{“}\alpha \subset \alpha$. Then $y \in \breve{S}\text{“}\alpha$. Suppose $x \sim\in \alpha$.

Suppose $x \in \gamma \,.\, \breve{R}\text{“}\gamma \subset \gamma$. Then $y \in \breve{S}\text{“}(\alpha \cup \gamma) \,.\, x \in \alpha \cup \gamma$

We have $y \in -\breve{R}\text{“}\alpha$ ~~Now~~ $\cancel{\breve{R}\text{“}\alpha \subset \alpha \,.\, R\text{“}-\alpha \subset -\alpha}$ Hence $x \in -\breve{R}\text{“}\alpha$

If $\cancel{x \in \breve{S}}$ $x \in S\text{“}\alpha$, $y \in \breve{S}\text{“}S\text{“}\alpha$. Hence $x \in -\breve{R}\text{“}S\text{“}\alpha$

Thus $x \in \cancel{\breve{S}\text{“}\alpha} -\breve{R}\text{“}S\text{“}\alpha$ and $x \in \cancel{T\text{‘}\alpha} -\breve{R}\text{“}T\text{“}\alpha$

[39v]

To prove $p\text{‘}\breve{R}\text{“‘}\kappa \subset \breve{T}\text{“}p\text{‘}\kappa$

Follows if $T\text{“}p\text{‘}\breve{T}\text{“‘}\kappa \subset p\text{‘}\kappa$ i.e. $T|R_*|\breve{T} \mathrel{\dot\subset} R_*$

We have to prove

[40] See [35r].

$\breve{T}‘y \in p‘\breve{T}“‘\kappa \,.\, \breve{R}“\alpha \subset \alpha \,.\, x \in \alpha \,.\supset.\, y \in \alpha$

i.e. $\breve{R}“\beta \subset \beta \,.\, x \in \beta \,.\supset_\beta.\, \breve{T}‘y \in \breve{T}“\beta : \breve{R}“\alpha \subset \alpha \,.\, x \in \alpha \;:\supset.\, y \in \alpha$

~~Consider $\alpha \cup \beta$. We have $\breve{R}“(\alpha \cup \beta) \subset \alpha \cup \beta \,.\, x \in \alpha \cup \beta$~~

~~Hence $\breve{T}‘y \in \breve{T}“(\alpha \cup \beta)$~~

~~Consider $\beta \cup \iota‘y \,.\, \breve{R}“(\beta \cup \iota‘y) = \breve{R}“\beta \cup \iota‘\breve{R}‘y$~~

Assume $y \in \gamma \,.\, \breve{R}“\gamma \subset \gamma$. Then $\breve{R}“(\beta \cup \gamma) \subset \beta \cup \gamma \,.\, x \in \beta \cup \gamma$.

Hence $\breve{T}‘y \in \breve{T}“(\beta \cup \gamma)$

Must get absurdity from

$\breve{R}“\beta \subset \beta \,.\, x \in \beta \,.\supset_\beta.\, \breve{T}‘y \in \breve{T}“\beta : \breve{R}“\alpha \subset \alpha \,.\, x \in \alpha \,.\supset.\, y \in -\alpha$

Assume $\breve{R}“\beta \subset \beta \,.\, x \in \beta \,.\breve{T}‘y \in \breve{T}“\beta \,.\, y \in -\beta$

We have to deduce $(\exists S).\, S \in \text{Potid}‘R \,.\, \breve{S}‘y \in -\beta \,.\, \breve{R}‘\breve{S}‘y \in \beta$

[40r]

Given $\breve{R}“\overleftarrow{R}_*‘x = \breve{R}“p‘\kappa = p‘\breve{R}“‘\kappa = \overleftarrow{R}_*‘\breve{R}‘x$ $[\kappa = \hat{\alpha}(\breve{R}“\alpha \subset \alpha \,.\, x \in \alpha)]$

$xT|R_*y \,.\equiv.\, y \in p‘\breve{T}“‘\kappa : xR_*|Ty \,.\equiv.\, y \in \breve{T}“p‘\kappa$

To prove $T|R_* \mathrel{\subset\!\!\cdot} R_*|T$

i.e. $\breve{T}‘x \in \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha.\, y \in \alpha :\supset$

$:.\lfloor(\exists z) :\rfloor x \in \beta \,.\, \breve{R}“\beta \subset \beta \,.\supset_\beta.\, z \in \beta : \cancel{\supset_x} .zTy$

or $x \in \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha.\, y \in \breve{T}“\alpha :\supset$

$: (\exists z) : x \in \beta \,.\, \breve{R}“\beta \subset \beta \,.\supset_\beta.\, z \in \beta : zTy$

If not,

$x \in \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset_\alpha.\, y \in \breve{T}“\alpha : zTy \,.\supset_z.\, (\exists \beta).\, x \in \beta \,.\, \breve{R}“\beta \subset \beta \,.\, z \in -\beta$

Now $z \in -\beta \,.\supset.\sim(xR_*z) \,.\supset.\sim(\exists S).\, S \in \text{Potid}‘R \,.\, x = \cancel{\breve{S}}\, S‘z$

Hence $\sim(\exists S).\, S \in \text{Potid}‘R \,.\, x(\cancel{\breve{S}}S|T)y$. ~~Put $S = T$. Then $x \neq y$~~

~~Put $S = \breve{R}|T$. Then $\sim(xRy)$~~

~~Similarly $\sim(xR^2y)$~~

~~and generally $TR_{\text{ts}}S \,.\supset.\sim(xSy)$. Also $S \in \text{Potid}‘R \,.\supset.\sim(x\breve{S}y)$~~

Must prove $\cancel{T}|\, R_* = \dot{s}‘\overleftarrow{R_{\text{ts}}}‘T \cup T|R|R_*$

i.e. $xR_*y.\supset:(\exists S).\, TR_{\text{ts}}S \,.\, xSy \,.\vee.\, x(T|R|R_*)y$

Put $TR_{\text{ts}}S \,.=.\, p‘\breve{T}“‘\kappa \subset p‘\breve{S}“‘\kappa$ Dft. Then we want

$xR_*y.\supset:(\exists S).\, p‘\breve{T}“‘\kappa \subset p‘\breve{S}“‘\kappa \,.\, xSy \,.\vee.\, y \in p‘\breve{R}“‘\breve{T}“‘\kappa$

~~i.e. $\supset : y \sim\in p‘\breve{T}“‘\kappa \,.\supset.\, x \in p‘\breve{R}“‘\breve{T}“‘\kappa$~~

[40v]

We have now, if $\kappa = \hat{\alpha}(\breve{R}“\alpha \subset \alpha \,.\, x \in \alpha)$,

$x(R_*|R)y \,.\equiv.\, y \in \breve{R}“p‘\kappa$

$x(R|R_*)y \,.\equiv.\, y \in p‘\breve{R}“‘\kappa$

$R \in \text{Cls} \to 1 \,.\supset.\, \breve{R}“p‘\kappa = p‘\breve{R}“‘\kappa$ because of the special character of $\kappa$.

We ought to be able to prove, if $R \in \text{Cls} \to 1$,

$$T \in \text{Potid}‘R . \supset : x(R_*|T)y . \equiv . y \in \breve{T}“p‘\kappa \quad \text{(obvious)}$$
$$x(T|R_*)y . \equiv . y \in p‘\breve{T}“‘\kappa$$
$$\breve{T}“p‘\kappa = p‘\breve{T}“‘\kappa$$

We have $R_*|R = R|R_*$

If $\quad R_*|T = T|R_*,\ R_*|T|R = T|R_*|R = T|R|R_*$

Hence prop by induct. (Invalid)

[41]

To prove $R_* = R_0 \dot\cup R|R_*$ given $R_* = R_0 \dot\cup R_*|R$[41]

i.e. $xR_*y . \supset :. x = y : \vee : \breve{R}‘x \in \alpha . \breve{R}“\alpha \subset \alpha . \supset_\alpha . y \in \alpha$

i.e. $\supset :. x = y : \vee : \breve{R}‘x \in \alpha . \breve{R}“\alpha \subset \alpha . \supset . y \in \alpha$

i.e. $\quad x \neq y . x \in R“\alpha - \alpha . \breve{R}“\alpha \subset \alpha . \supset . y \in \alpha$

We have $y \in R“\alpha$. Thus what we have to prove is

$x, y \in R“\alpha - \alpha . \supset . x = y$

$x, y \in R“\alpha - \alpha . \supset . \cancel{\breve{R}‘x, \breve{R}‘y \in \alpha . \supset . \sim(\breve{R}‘xR_*y). \sim(\breve{R}‘yR_*x)}$

$\supset : xR_*z . yR_*w . \supset . z, w \in R“\alpha. \cancel{\supset . \sim(zR\breve{R}‘x). \sim(zR\breve{R}‘y)}$

$\cancel{\supset : \sim(xR_*|R_*\breve{R}‘y) . \sim(yR_*|R\breve{R}‘x)}$

$\cancel{\supset : \sim(xR_*y) . \sim(yR_*x)}$

$\supset : xR_*z . yR_*w . \supset . \breve{R}‘z, \breve{R}‘w \in \alpha . x, y \in -\alpha$

$\supset : \sim(xR_*|Ry) . \sim(y\cancel{(}R_*|Rx)$

$\supset : x = y$

Hence $R \in \text{Cls} \to 1 . \supset . R_* = R_0 \dot\cup R|R_*$

Hence $R|R_* \mathrel{\dot\subset} (R_* \dot\cap J) \dot\cup R_0 \mathrel{\dot\subset} R_*|R \dot\cup R_0$

Also $R_0 \dot\cap R|R_* = R_0 \dot\cap R_*|R$ Hence

$R \in \text{Cls} \to 1 . \supset . R|R_* = R_*|R . R_* = R_0 \dot\cup R|R_* . p‘\breve{R}“‘\kappa = \breve{R}“p‘\kappa.$

[42r]

Desirable to substitute $p‘\breve{R}“‘\kappa$ for $\breve{R}“p‘\kappa$ $\qquad \kappa = \hat{\alpha}(\breve{R}“\alpha \subset \alpha . x \in \alpha)$

$\qquad \lambda = \hat{\beta}(\breve{R}“\beta \subset \beta . y \in \beta)$

$y \in \breve{R}“p‘\kappa . \supset . y \in p‘\breve{R}“‘\kappa$ but not vice versa.

$y \in \breve{R}“p‘\kappa . \equiv . x(R_*|R)y$

$y \in p‘\breve{R}“‘\kappa . \equiv : x \in \alpha . \breve{R}“\alpha \subset \alpha . \supset_\alpha . y \in \breve{R}“\alpha :$

$\supset : x \in \alpha . \breve{R}“\alpha \subset \alpha . \supset_\alpha . y \in \alpha : \supset : xR_*y$

$x(R|R_*)y . \equiv : \breve{R}‘x \in \alpha . \breve{R}“\alpha \subset \alpha . \supset_\alpha . y \in \alpha$

$\equiv : x \in R“\alpha . \breve{R}“\alpha \subset \alpha . \supset_\alpha . y \in \alpha$

$\cancel{\supset : x \in R“\alpha . R“\breve{R}“\alpha \subset R“\alpha . \supset_\alpha . y \in \alpha}$

$\equiv : y \in -\alpha . R“-\alpha \subset -\alpha . \supset_\alpha . x \in R“-\alpha \qquad [\text{D}‘R = V]$

$\equiv : x \in p‘R“‘\lambda$

[41] Compare [39r].

$x(R|R_*)y \,.\supset: \breve{R}`x \,\epsilon\, \breve{R}``\alpha \,.\, \breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha$
$\supset: \breve{R}`x \,\epsilon\, \breve{R}``\alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha$
$\supset: x \,\epsilon\, \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha :\supset.\, y \,\epsilon\, p`\breve{R}```\kappa$

Hence $R|R_* \subseteq R_*|R$

Required $R_*|R \subseteq R|R_*$[42]

[42v]

$\breve{R}``R``\alpha \subset R``\alpha \,.\equiv.\, \alpha \subset R``\alpha$

$\alpha \subset R``\alpha \,.\supset.\, \breve{R}``\alpha \subset \breve{R}``R``\alpha \,.\supset.\, \breve{R}``\alpha \subset \alpha$

Thus $\breve{R}``R``\alpha \subset R``\alpha \,.\supset.\, \breve{R}``\alpha \subset \alpha$

Thus $\breve{R}``\alpha \subset \alpha.\supset$ etc $:\supset: \breve{R}``R``\alpha \subset R``\alpha \,.\supset.$ etc.

$y \,\epsilon\, p`\breve{R}```\kappa \,.\supset: x \,\epsilon\, \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha$
$\supset: x \,\epsilon\, \alpha \,.\, \breve{R}``R``\alpha \subset R``\alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha$

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$\breve{R}``\alpha \subset \alpha \,.\, x \,\epsilon\, \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha :$
$\supset: \breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha \,.\, x \,\epsilon\, \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha$ because
$\breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha \,.\equiv.\, \breve{R}``\alpha \subset \alpha$

Now $\breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha \,.\, \breve{R}`x \,\epsilon\, \breve{R}``\alpha \,.\, \gamma = \alpha \cup \iota`x.$
$\supset.\, \breve{R}``\gamma = \breve{R}``\alpha \,.\, x \,\epsilon\, \gamma.$

Hence $\breve{R}``\breve{R}``\alpha \subset \breve{R}``\alpha \,.\, x \,\epsilon\, \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{R}``\alpha :$
$\supset: \breve{R}``\breve{R}``\gamma \subset \breve{R}``\gamma \,.\, \breve{R}`x \,\epsilon\, \breve{R}``\gamma \,.\supset_\gamma.\, y \,\epsilon\, R``\gamma$

[43r]

Consider $T \,\epsilon$ Potid$`R \,.\supset_T.\, x \,\epsilon\, T`` - \beta$ and $xR_*z \,.\, z \,\epsilon\, \beta$

We do not have $\breve{R}`` - \beta \subset -\beta$ Hence $\exists!(\breve{R}`` - \beta) \cap \beta$

Suppose $u \,\epsilon\, \beta - \breve{R}``\beta$. Then $R``\overrightarrow{R_*}`u \subset -\beta \,.\, \overrightarrow{R}`u \subset -\beta$

We must have $T \,\epsilon$ Potid$`R \,.\supset.\, \sim(xTu)$, i.e. $\sim\{x(\dot{s}`\text{Potid}`R)u\}$

We have $\dot{s}`$Potid$`R \subseteq R_*$ Hence $\dot{-}R_* \subseteq \dot{-}\, \dot{s}`$Potid$`R$

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$xR_*y \,.\supset: (\exists T) : T \,\epsilon$ Potid$`R : x \,\epsilon\, \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \,\epsilon\, \breve{T}``\alpha$
$(\exists T)$ ... $y \,\epsilon\, \alpha \,.\, R``\alpha \subset \alpha \,.\supset_\alpha.\, x \,\epsilon\, T``\alpha$

Can we prove $(\exists T).\, y \,\epsilon\, \breve{T}``\alpha - \breve{R}``\breve{T}``\alpha$? Not necessarily.

Suppose we had $T \,\epsilon$ Potid$`R \,.\, x \,\epsilon\, \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_{T,\alpha}.\, y \,\epsilon\, \breve{T}``\alpha$?

This ought to be impossible.

Suppose $y \,\epsilon\, \beta - \breve{R}``\beta$ Then $\breve{R}``T``\beta \subset T``\beta \,.\, \overrightarrow{T}`y \subset \beta - \breve{R}``\beta$

Hence $y \sim\!\epsilon\, \breve{T}``\breve{R}``\beta$. Thus we must have $x \sim\!\epsilon\, T``R``\beta$ if hypothesis is to be satisfied. But this is impossible in our case.

[42] This is proved, on the assumption that $R \,\epsilon\, \text{Cls}\to 1$, at $*89\cdot106$.

Hence $(\exists T) : x \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \in \breve{T}``\alpha :$

$\sim\{x \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \in \breve{T}``\breve{R}``\alpha\}$

We want to prove that in this case $x \in T``\beta$ if $y \in \beta$

[43v]

$R_* \mathrel{\dot{\subset}} \dot{s}`\text{Potid}`R \,.\equiv ::$

$xR_*y : xSz \,.\, zRy \,.\supset_{z,S}.\, xSy :\supset_{x,y}.\, xR_0y$

$z \in \overrightarrow{R}`y - \iota`y \,.\supset.\, x(x\downarrow z)z \,.\, zRy \,.\, \sim x(x\downarrow z)y$

Hence hp. $\supset . \overrightarrow{R}`y \subset \iota`y \,.\supset.\, \overrightarrow{R_*}`y \subset \iota`y \,.\supset.\, xR_0y$ Q.E.D.

$x(\dot{s}`\text{Potid}`R)y \,.\equiv: (\exists S) : R_0 \in \mu \,.\, |R``\mu \subset \mu \,.\supset_\mu.\, S \in \mu : xSy$

$\sim\{x(\dot{s}`\text{Potid}`R)y\} \,.\equiv:.\, R_0 \in \mu \,.\, |R``\mu \subset \mu \,.\supset_\mu.\, S \in \mu :\supset_S.\, \sim(xSy)$

$\equiv:.\, xSy \,.\supset_S.\, (\exists\mu).\, R_0 \in \mu \,.\, |R``\mu \subset \mu \,.\, S \sim\in \mu$

We want to prove this implies

$(\exists\alpha).\, \breve{R}``\alpha \subset \alpha \,.\, x \in \alpha \,.\, y \in -\alpha$

$\breve{R}``\alpha \subset \alpha \,.\, x \in \alpha \cancel{-\breve{R}`\alpha} \,.\, \mu = x\downarrow``\alpha \,.\supset.\, |R``\mu = x\downarrow``\breve{R}``\alpha$ Hence $|R``\mu \subset \mu$

~~Suppose $\mu = R \in `(C`R \uparrow C`R)$.~~ Assume $D`R = V$

$D`(S|R) = D`S \,.\, \text{Ɑ}`(S|R) = \breve{R}``\text{Ɑ}`S$ i.e. $\text{Ɑ}`|R`S = \breve{R}``\text{Ɑ}`S$

Hence if $\mu = \hat{S}(\text{Ɑ}`S \subset \mu)$, $|R``\mu \subset \mu \,.\equiv.\, \breve{R}``\alpha \subset \alpha$

and $\text{Ɑ}``|R``\mu = \breve{R}```\text{Ɑ}``\mu$

Hence $|R``\mu \subset \mu \,.\supset.\, \breve{R}```\text{Ɑ}``\mu \subset \text{Ɑ}``\mu$

[44r]

$xR_*y \,.\equiv.\, x \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \in \alpha$

$x(R_*|R)y \,.\supset: x \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, y \in \breve{R}``\alpha$ Hence $R_*|R \mathrel{\dot{\subset}} R_*$

$xR_*y \,.\equiv.\, y \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset_\alpha.\, x \in \alpha$

$x(R|R_*)y \,.\supset: y \in \alpha \,.\, R``\alpha \subset \alpha \,.\supset_\alpha.\, x \in R``\alpha$ Hence $R|R_* \mathrel{\dot{\subset}} R_*$

Consider $x \in \alpha \,.\, \breve{T}``\alpha \subset \alpha \,.\supset_\alpha.\, y \in \breve{T}``\alpha$

$xR_*y \,.\supset.\, (\exists T).\, T \in \text{Potid}`R \,.\, y \in \breve{T}``\alpha$

This is obvious because $y \in \alpha$ and $\alpha = \breve{R}_0``\alpha$

The thing we want is different. We want

$xR_*y \,.\, y \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset.\, (\exists T).\, T \in \text{Potid}`R \,.\, x \in T``\alpha$

We have $y \in -\beta$. Hence $(\exists T).\, T \in \text{Potid}`R \,.\, x \in T``-\beta$

Suppose $T \in \text{Potid}`R \,.\supset_T.\, x \in T``-\beta$

Then $(\exists z).\, xTz \,.\, z \sim\in \beta \,.\supset.\, (\exists z).\, xTz \,.\, \breve{R}`z \sim\in \beta$

~~i.e. $\breve{R}``(\overleftarrow{T}`x \cap -\beta) \subset \overleftarrow{T}`x \cap -\beta$ Hence $\overleftarrow{R_*}`x \subset -\beta$ contra hp.~~

~~or $xTz \,.\, z \in -\beta \,.\supset.\, \breve{R}`z \in -\beta$~~

This is possible because it involves
$\breve{T}`x \in -\beta \,.\, \breve{T}`\breve{R}`x \in -\beta$
for every $T$ in Potid$`R$.

[44v]

$x \in S``R``\alpha \,.\supset_S .\, x \in S``\alpha : \supset :$
$S \in \text{Potid}`R. \supset_S : x \in S``R``\alpha \,.\supset .\, x \in S``\alpha : \supset : x \sim\in S``\alpha \,.\supset .\, x \sim\in R``S``\alpha$
$\supset : \breve{S}`x \sim\in \alpha \,.\supset .\, \breve{R}`\breve{S}`x \sim\in \alpha$

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Can we prove $R_* \subset \dot{s}`\text{Potid}`R$ ?[43]
i.e. $T \in \text{Potid}`R \,.\supset_T .\, \sim(xTy) : \supset .\, \sim(xR_*y)$
i.e. $\sim(xR_0y) : \sim(xSy) \,.\supset_S .\, \sim(xS|Ry) : \supset .\, \sim(xR_*y)$
~~i.e. $\sim(xR_0y) : y \sim\in \overleftarrow{S}`x \,.\supset_S .\, \breve{R}`y \sim\in \overleftarrow{S}`x : \supset .\, \sim(xR_*y)$~~
~~Consider $-\alpha \uparrow \breve{R}``\mu$~~
$\sim(xSy) \,.\supset_S .\, \sim(xS|Ry) : \equiv : xSz \,.\, zRy \,.\supset_{z,S} .\, xSy$
Take it by Transp: to prove
$xR_*y : xSz \,.\, zRy \,.\supset_{z,S} .\, xSy : \supset .\, xR_0y$
We have $xRy \,.\supset .\, xR_0y : xR^2y \,.\supset .\, xRy$ etc.
Assume $x \in \mu \,.\, \breve{R}``\mu \subset \mu$ Then $y \in \mu$. Assume $R \in \text{Cls}\to 1$.
~~For $S$ write $S|\breve{R}$ Then $xSy \,.zRy \,.\supset .\, xSz$~~
~~Thus hp. is $xR_*y \therefore zRy. \supset_z : xSy \,.\equiv_S .\, xSz$~~

[45]

We assume D$`R = V$. Hence $R`` - \mu = -R``\mu$ $\qquad$ $\alpha \subset R``\alpha \,.\, -R``\alpha \subset -\alpha$
~~$R``\breve{R}``\alpha \subset$~~
$(-\alpha \cup -\beta) \cap (R``\alpha \cup R``\beta) \subset -\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z$ $\qquad$ $\breve{R}``R``\alpha = \alpha \subset R``\alpha$

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We have $x \in \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z \,.\, \lfloor x \in T``\alpha \rfloor\, \breve{R}``T``\alpha \subset T``\alpha$ Hence $z \in T``\alpha$ which is impossible.
Thus to prove
$(\exists T).\, T \in \text{Potid}`R \,.\, x \in T``\alpha \,.\, z \sim\in T``\alpha$
~~$z \sim\in T``\alpha$ is obvious because $-\alpha \subset -T``\alpha$~~
~~Thus~~ $(\exists T).\, T \in \text{Potid}`R \,.\, x \in T``\alpha$ is the vital thing.
If not, $T \in \text{Potid}`R \,.\supset_T .\, x \sim\in T``\alpha$
which must imply $\sim(xR_*y)$
$T \in \text{Potid}`R \,.\supset .\, QT : \equiv : \phi R_0 : \phi S \,.\supset .\, \phi(S|R)$

[43] This is proved for $R$ a one to one relation and for level 3 at $*89{\cdot}23$.

Thus hp is $x \sim\epsilon\ \alpha : x \sim\epsilon\ S``\alpha\ .\supset_S .\ x \sim\epsilon\ S``R``\alpha$
i.e. $x \sim\epsilon\ \alpha : x\ \epsilon\ S``R``\alpha\ .\supset_S .\ x\ \epsilon\ S``\alpha$
~~i.e.~~ $\cancel{x \sim\epsilon\ \alpha : xSy\ .\ yRz\ .\ z\ \epsilon\ \alpha\ .\supset_{S,y,z} .\ xSz\ .\ z\ \epsilon\ \alpha}$
~~i.e.~~ $\cancel{x \sim\epsilon\ \alpha : (xSy\ .\ z\ \epsilon\ \alpha).}$

[46r]

Prove first
$uR_*y\ .\ y\ \epsilon\ u\ .\supset .\ (\exists T).\ T\ \epsilon$ Potid‘$R\ .\ u\ \epsilon\ T``\mu - R``T``\mu$
Then: $y\ \epsilon\ \alpha - \beta\ .\ z\ \epsilon\ \beta - \alpha\ .\supset .\overrightarrow{R_*}`y \subset -\beta \subset -\iota`z\ .\overrightarrow{R_*}`z \subset -\iota`y$
$R``\overrightarrow{R_*}`y \subset -R``\beta \subset -\overrightarrow{R}`z\ .\ R``\overrightarrow{R_*}`z \subset -R``\alpha \subset\ -\overrightarrow{R}`y$
Hence $\overrightarrow{R_*}`y \subset -\iota`z - \overrightarrow{R_*}`z$
$R``R``\overrightarrow{R_*}`y \subset -R``R``\beta \subset -R``\overrightarrow{R}`z$
Hence $\overrightarrow{R_*}`y \subset -\iota`z - \overrightarrow{R_*}`z - R``\overrightarrow{R}`z$
Hence in the end $T\ \epsilon$ Potid‘$R\ .\supset .\ \overrightarrow{R_*}`y \subset -\overrightarrow{T}`z$
We have $\overrightarrow{R_*}`y \subset -\beta\ .\ z\ \epsilon\ \beta$
$R``\overrightarrow{R_*}`y \subset -R``\beta\ .\overrightarrow{R}`z \subset R``\beta\ .\ \overrightarrow{R}`z \subset R`` -\alpha\ .\ \overrightarrow{R}`y \subset R``\alpha$
Suppose $\exists!\overrightarrow{R_*}`y \cap R``\beta$. Then $\iota`y \subset R``\beta$ i.e. $\breve{R}`y\ \epsilon\ \beta$. This is possible.
What we have is $\overrightarrow{R_*}`y \cap R``\beta \subset -\overrightarrow{R_*}`z$ and $\overrightarrow{R_*}`y - R``\beta \subset -\overrightarrow{R_*}`z$
We have $\overrightarrow{R}`y \subset R``\alpha\ .\ \overrightarrow{R}`z \subset R`` -\alpha \cup \iota`z$
$\overrightarrow{R}`y \subset R``\alpha \cap\ -\beta\ .\ \overrightarrow{R_*}`z \subset -R``\alpha \cup \beta$
Hence $\overrightarrow{R}`y \cap \overrightarrow{R_*}`z = \Lambda$
$R``\overrightarrow{R}`y \subset R``R``\alpha \cap\ -\beta \cap\ -R``\beta.\ \overrightarrow{R_*}`z \subset -R``R``\alpha \cup \beta \cup R``\beta$ etc.
and so on. Must try to get induction out of this.

[46v]

$xR_*y\ .\ xR_*z\ .\ \lambda = \overleftarrow{R}_*`x \cap \cancel{\overrightarrow{R_*}}`\mu - \breve{R}``\mu\ .\ y, z\ \epsilon\ \mu - \breve{R}``\mu$
$uR_*y\ .\ uR_*z\ .\supset$
$.\ u \neq y\ .\ u \neq z\ .\supset .\ \breve{R}`uR_*y\ .\ \breve{R}`\alpha R_*z\ .\supset .\ u\ \epsilon - \mu\ .\ \breve{R}`u\ \epsilon - \mu$
$uR_*y\ .\ u \neq y\ .\supset .\ u\ \epsilon - \mu$
$uR_*z\ .\ u \neq z\ .\supset .\ u\ \epsilon - \mu$
Hence $uR_*y\ .\ uR_*z\ .\supset .\ u\ \epsilon - \mu\ .\ \breve{R}`u\ \epsilon - \mu$
$\sim(uR_*y)\ .\supset .\ (\exists\alpha).\ R``\alpha \subset \alpha\ .\ y\ \epsilon\ \alpha\ .\ u\ \epsilon - \alpha$
$R``\alpha \subset \alpha\ .\ y\ \epsilon\ \alpha\ .\ u\ \epsilon - \alpha\ .\supset .\ \breve{R}`u\ \epsilon - \alpha$
$\sim(uR_*z)\ .\supset .\ (\exists\beta).\ R``\beta \subset \beta\ .\ z\ \epsilon\ \beta\ .\ u\ \epsilon - \beta$
$R``\beta \subset \beta\ .\ z\ \epsilon\ \beta\ .\ u\ \epsilon - \beta\ .\supset .\ \breve{R}`u\ \epsilon\ \ \beta$
Hence $u\ \epsilon - \mu\ .\ \breve{R}`u\ \epsilon - \mu\ : \vee$
$: (\exists\alpha) : R``\alpha \subset \alpha : y\ \epsilon\ \alpha\ .\vee .\ z\ \epsilon\ \alpha : u\ \epsilon - \alpha\ .\ \breve{R}`u\ \epsilon - \alpha$
Putting $u = y$, $(\exists\alpha).\ R``\alpha \subset \alpha\ .\ z\ \epsilon\ \alpha\ .\ y\ \epsilon - \alpha\ .\ \breve{R}`y\ \epsilon - \alpha$

$\sim(uR_*y) . \supset . (\exists\xi). \breve{R}``\xi \subset \xi . u \epsilon \xi . y \epsilon -\xi$

Hence $u \epsilon - \mu . \breve{R}`u \epsilon - \mu : \vee$

$: (\exists\xi) : \breve{R}``\xi \subset \xi : y \epsilon -\xi . \vee . z \epsilon -\xi : u \epsilon \xi . \breve{R}`u \epsilon \xi$

Putting $u = y$, $(\exists\xi). \breve{R}``\xi \subset \xi . z \epsilon -\xi . \breve{R}`y \epsilon \xi . y \epsilon \xi$

[47r]

$\cancel{xR_*y.}$
$u \epsilon \overleftarrow{R}_*`x \cap \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z . \supset . \breve{R}`u \epsilon$ ditto
Put ditto $= \xi$. Then $\breve{R}``\xi \subset \xi$.
Now consider $\xi \subset \alpha$. $\cancel{\breve{R}``\alpha \subset \alpha}$. Then $\breve{R}``\xi \subset \alpha$
We have $x \epsilon \xi . \breve{R}``\xi \subset \xi . y, z \epsilon -\xi$
Take $\overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z$ as $\xi$. Same holds
$\overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z = p`\hat{\mu}(\breve{R}``\mu \subset \mu : y \epsilon \mu . \vee . z \epsilon \mu)$
We have $\breve{R}``\mu \subset \mu : y \epsilon \mu . \vee . z \epsilon \mu : \supset . R``\mu \subset \mu . x \epsilon \mu$
The thing to prove is $\overrightarrow{R_*}`y \subset -\overrightarrow{R_*}`z$
We have $z \sim\epsilon \overrightarrow{R_*}`y . y \sim\epsilon \overrightarrow{R_*}`z$
$uRy . vRz . \supset : uR_*z . \supset . yR_*z : \supset : \sim uR_*z . \sim uR_*v . \sim vR_*u$
We have to prove $uR_*y . \supset . \sim(uR_*z)$
i.e. $R``\mu \subset \mu . y \epsilon \mu . \supset_\mu . u \epsilon \mu : \supset .(\exists\nu). R``\nu \subset \nu . z \epsilon \nu . u \sim\epsilon \nu$
We have $y \epsilon \alpha - \beta . z \epsilon \beta - \alpha . R`` - \alpha \subset -\alpha$
We have $\overrightarrow{R_*}`y \subset -\beta \subset -\iota`z$
$\overrightarrow{R_*}`y - \iota`y \subset -R``\beta \subset -\iota`z - \overrightarrow{R}`z$

[47v]

Put $\lambda =$ Potid$`R \cap \hat{P}(P \sim\epsilon \alpha . P|R \epsilon \alpha)$ where $|R``\alpha \subset \alpha$. To pr. $\lambda \epsilon 0 \cup 1$.
We consider the class Potid$`R \cap \hat{M}(\lambda \subset \overrightarrow{R_{\mathrm{ts}}}`M)$,
i.e. Potid$`R \cap p`\overleftarrow{R_{\mathrm{ts}}}``\lambda$
We prove that this class is hereditary but excludes $\lambda$, if $\lambda \sim\epsilon 0 \cup 1$.
$M \epsilon$ Potid$`R . \equiv : R_0 \epsilon \mu . |R``\mu \subset \mu . \supset_\mu . M \epsilon \mu$
$\lambda \subset \overrightarrow{R_{\mathrm{ts}}}`M . \equiv : P \epsilon$ Potid$`R . P \sim\epsilon \alpha . P|R \epsilon \alpha . \supset_P . PR_{\mathrm{ts}}M$
The two together give
$R_0 \epsilon \mu . |R``\mu \subset \mu . \supset_\mu . M \epsilon \mu :$
$M \epsilon \mu . |R``\mu \subset \mu . P \epsilon$ Potid$`R . P \sim\epsilon \alpha . P|R \epsilon \alpha . \supset_{P,\mu} . P \epsilon \mu$
We have $M \epsilon p`\overleftarrow{R_{\mathrm{ts}}}``\lambda . \supset . M \sim\epsilon \lambda . \supset . M|R \epsilon p`\overleftarrow{R_{\mathrm{ts}}}``\lambda$
We have $|R``\lambda \subset \alpha$; i.e. if $\beta = (\mathrm{Cnv}`|R)``\alpha$, $\lambda \subset \beta - |R``\beta$
Also $p`\overleftarrow{R_{\mathrm{ts}}}``\lambda \subset -\beta$. Consider
$P \epsilon \lambda . M\breve{R}_{\mathrm{ts}}P . M \sim\epsilon \beta$. Then $M|R\ \breve{R}_{\mathrm{ts}}\ P$ but perhaps not $M \sim\epsilon \beta$

Consider $P, Q \epsilon \lambda . M\breve{R}_{ts}P. M\breve{R}_{ts}Q$ Then $M \sim\epsilon \lambda$
Hence $(M|R)\breve{R}_{ts}P. (M|R)\breve{R}_{ts}Q$ this ought to involve $M|R \epsilon -\beta$

[48]

Props proved $p`\overrightarrow{R_*}``\lambda$ may be worth considering.
Also $R_*``\mu \cup \breve{R}_*``\mu$
General hp: $R \epsilon \mathrm{Cls}\to 1 . \text{Cl}`R = V. \lambda = \overleftarrow{R}_*`x \cap \mu - \breve{R}``\mu . y, z \epsilon \lambda . y \neq z.$
Then $\overleftarrow{R}_*`x \cap \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z = \xi . \supset . \breve{R}``\xi \subset \xi . R`` - \xi \subset -\xi$
$\xi \subset \alpha . \breve{R}``\alpha \subset \alpha . \supset . y, z \epsilon \alpha$
$\xi \subset -\mu$
$R(x \longmapsto\!\dashv y) = \mu_2 . R(x \longmapsto\!\dashv z) = \nu_2 . \supset . \xi = \mu_2 \cap \nu_2 . \supset . \xi = \overleftarrow{R}_*`x$
Whence Hp $. \supset . \sim(\exists\mu_2, \nu_2) . R(x \longmapsto\!\dashv y) = \mu_2 . R(x \longmapsto\!\dashv z) = \nu_2$
But $xR_*u . \supset . R(x \longmapsto\!\dashv \breve{R}`u) = R(x \longmapsto\!\dashv u) \cup \iota`\breve{R}`u.$
Again: Suppose $y \epsilon \alpha . z \epsilon -\alpha . \breve{R}``\alpha \subset \alpha . \alpha - \breve{R}``\alpha \subset \mu - \breve{R}``\mu.$
Then $\breve{R}``(-\alpha \cup \mu) \supset -\alpha \cup \mu$
Whence $\overleftarrow{R}_*`x \subset -\alpha \cup \mu$ ~~and~~ $\cancel{y \epsilon \alpha . \breve{R}``\alpha . \supset . \overrightarrow{R}_*`y \subset -\alpha}$
$y \epsilon \alpha - \beta . z \epsilon \beta - \alpha . \breve{R}``\alpha \subset \alpha. \breve{R}``\beta \subset \beta . \supset . \breve{R}``(\mu \cup -\alpha - \beta) \subset$
$\mu \cup -\alpha - \beta$
$\breve{R}``(\overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z) \subset \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z . R``(-\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z) \subset -\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z$
Put $\overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z = \mu_2$. Then $\breve{R}``\mu_2 \subset \mu_2 . R`` - \mu_2 \subset -\mu_2$
$\beta \subset \mu_2 . \exists!\beta . \supset . \exists!\breve{R}``\beta - \beta$
$\beta \subset \mu_2 . \breve{R}``\beta \subset \beta . \supset . \beta = \Lambda$ Also $\beta \subset \mu_2 . \supset . \breve{R}``\beta \subset \mu_2$
$\mu_2 \subset \beta . R``\beta \subset \beta . \supset . \beta = \mathrm{V}$
$\mu_2 \subset \beta . \exists! - \beta . \supset . \exists!R``\beta - \beta$

[49r]

Assume $x \epsilon \alpha - \breve{R}``\alpha . y \epsilon \beta - R``\beta . \breve{R}``\alpha \subset \alpha . R``\beta \subset \beta$
To pr. $R(x \longmapsto\!\dashv y) = \alpha \cap \beta$ or rather
$(\exists\alpha, \beta). \breve{R}``\alpha \subset \alpha . \cancel{\breve{R}}R``\beta \subset \beta . R(x \longmapsto\!\dashv y) = \alpha \cap \beta$
This can hardly be proved, but
$u \sim\epsilon R(x \longmapsto\!\dashv y) . \supset . (\exists\alpha, \beta) \ldots R(x \longmapsto\!\dashv y) . \subset . \alpha \cap \beta . u \sim\epsilon \alpha \cap \beta$

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Prop follows if $(\exists\mu_2) . R(x \longmapsto\!\dashv y) = \mu_2$
Put $R(x \longmapsto\!\dashv x) = \xi_2 = \iota`x$
Then $R(x \longmapsto\!\dashv \breve{R}'x) = \xi_2 \cup \breve{R}``\xi_2$
$R(x \longmapsto\!\dashv \breve{R}`y) = \xi_2 \cup \breve{R}``(x \longmapsto\!\dashv y)$
Thus we have a series
$\eta_1 = \xi_2, \eta_2 = \xi_2 \cup \breve{R}``\xi_2 . \eta_3 = \xi_2 \cup \breve{R}``\eta_2 . \eta_4 = \xi_2 \cup \breve{R}``\eta_3$ etc.
Relation is $S$ where
$S = \hat{\alpha}\hat{\beta}\{\beta = \xi_2 \cup \breve{R}``\alpha\}$

Hence the class of intervals is $\overleftarrow{S}_*\text{‘}\xi_2$ wh. $\subset$ Cls induct$_3$
Must prove $xR_*y\ .\supset .\ R(x\longmapsto\!\dashv y)\in\ \overleftarrow{S}_*\text{‘}\iota\text{‘}x$

[49v]

$\overleftarrow{R}_*\text{‘}x=\hat{y}(\breve{R}\text{“}\alpha\subset\alpha\ .\ x\in\alpha\ .\supset_\alpha .\ y\in\alpha)$
In our case, $\exists!\mu-\breve{R}\text{“}\mu\cap\ \overleftarrow{R}_*\text{‘}x$ Call this class $\lambda$. Also $R\in\text{Cls}\to 1$

$y\in\lambda\ .\supset .\ \sim(yRy)$
$xRx\ .\supset .\ \overleftarrow{R}_*\text{‘}x=\iota\text{‘}x\ .\supset .\ \sim\exists!\lambda$ Hence $\sim(xRx)$
$uR_*y\ .\ uRu\ .\supset .\ \overleftarrow{R}_*\text{‘}y=\overleftarrow{R}_*\text{‘}u=\iota\text{‘}u$. Hence $uR_*y\ .\supset .\ \sim(uRu)$

It follows that there are classes $\alpha$ for which $\breve{R}\text{“}\alpha\subset\alpha\ .\ \breve{R}\text{‘}x\in\alpha\ .\ x\sim\in\alpha$
Since $x\sim\in\alpha$, $\overrightarrow{R}_*\text{‘}x\subset-\alpha$.
Consider $\overleftarrow{R}_*\text{‘}x\cap\overrightarrow{R}_*\text{‘}y$
There must be a $\beta$ for which $R\text{“}\beta\subset\beta\ .\ y\in\beta\ .\ \breve{R}\text{‘}y\in-\beta\ .\ \overleftarrow{R}_*\text{‘}\breve{R}\text{‘}y\subset-\beta$
Then $\overrightarrow{R}_*\text{‘}y\subset\beta\ .\ x\in\beta\ .\ y\in\beta$
~~If $z\sim\in\beta$,~~
We have $xR_*y\ .\ xR_*z$

$u\in R(x\longmapsto\!\dashv y)\cap R(x\longmapsto\!\dashv z)\ .\supset .\ \breve{R}\text{‘}u\in R(x\longmapsto\!\dashv y)\cap R(x\longmapsto\!\dashv z)$

We ought to be able to find 2 classes $\alpha$, $\beta$ such that

$u\in\alpha-\beta\ .\supset .\ u\in R(x\longmapsto\!\dashv y)$

We have $y\in-\beta\ .\supset .\overrightarrow{R}_*\text{‘}y\subset-\beta : x\in\alpha\ .\supset .\overleftarrow{R}_*\text{‘}x\subset\alpha$. ~~Hence~~
~~$x\in\alpha\ .\ y\in-\beta\ .\ z\in-\gamma\ .\supset : u\in\alpha-\beta-\gamma\ .\supset .\ \breve{R}\text{‘}u\in R(x\longmapsto\!\dashv y)\cap R(x\longmapsto\!\dashv z)$~~

[50r]

$\mu_2=-\overrightarrow{R}_*\text{‘}y\cup-\overrightarrow{R}_*\text{‘}z\ .\ \mu_2\subset\beta\ .\ R\text{“}\beta\subset\beta\ .\supset .\ \beta=\text{V}$
Hence $-\beta\subset-\mu_2\ .\ \breve{R}\text{“}-\beta\subset-\beta\ .\supset .-\beta=\text{V}$
i.e. $\beta\subset-\mu_2\ .\ \breve{R}\text{“}\beta\subset\beta\ .\supset .\ \beta=\Lambda$
i.e. $\beta\subset\ \overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z.\ \breve{R}\text{“}\beta\subset\beta\ .\supset .\ \beta=\Lambda$
Now $\beta\subset\ \overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z\ .\supset .\ R\text{“}\beta\subset\ \overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z$
Can we deduce $\breve{R}_*\text{“}\beta\subset\ \overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z$?
We have $y\in\alpha\ .\ \cancel{\breve{R}}\ R\text{“}\alpha\subset\alpha\ .\supset .\ \beta\subset\alpha$
$\supset .\ \breve{R}\text{“}\beta\subset\breve{R}\text{“}\alpha$
Suppose $\exists!\ \breve{R}_*\text{“}\beta-(\overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z)$ say $-\overrightarrow{R}_*\text{‘}y$
Then $(\exists u,v)$. $u\in\beta\ .\ uR_*v\ .\sim(vR_*y)$
Here $u\neq v$. Hence $(\exists u,v)$. $u\in\beta\ .\ uR|R_*v\ .\sim v(R|R_*)y$

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$\beta\subset\ \overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z\ .\ \breve{R}\text{“}\beta\subset\beta\ .\supset .\ \beta=\Lambda$
Hence $x\in\beta\ .\ \breve{R}\text{“}\beta\subset\beta\ .\supset .\ \exists!\ \beta-(\overrightarrow{R}_*\text{‘}y\cap\overrightarrow{R}_*\text{‘}z)$

[50v]

$\overleftarrow{R}_*\text{‘}x=p\text{‘}\hat{\alpha}(\breve{R}\text{“}\alpha\subset\alpha\ .\ x\in\alpha)$
Thus taking any given term $u$ for which $\sim(xR_*u)$,

we have some class $\alpha$ for which $\breve{R}“\alpha \subset \alpha \,.\, x \epsilon \alpha \,.\, u \sim\epsilon \alpha$
Moreover $\overrightarrow{R_*}‘u \subset -\alpha \,.\, \alpha \subset -\overrightarrow{R_*}‘u \,.\, \overleftarrow{R}_*‘x \subset \alpha$
Similarly if $\sim(yR_*z)$, we shall have
$y \epsilon \alpha \,.\, z \epsilon -\alpha \,.\overrightarrow{R_*}‘z \subset -\alpha \,.\, \alpha \subset -\overrightarrow{R_*}‘z \,.\, \overleftarrow{R}_*‘y \subset \alpha$

$\mu_2 \subset \beta \,.\, R“\beta \subset \beta \,.\supset.\, \beta = \mathrm{V}$
Put $\beta = -\iota‘a$ where $a \sim\epsilon \mu_2$.
Then $\mu_2 \subset \beta$.
~~$R“\beta = \Lambda$ if $a \sim\epsilon \text{ᗡ}‘R$; hence $R“\beta \subset \beta$. Hence $\beta = \mathrm{V}$~~
~~This is impossible. Hence $a \sim\epsilon \mu_2 \,.\supset.\, a \epsilon \text{ᗡ}‘R$~~
$R“-\iota‘a \subset -\iota‘a \,.\equiv:\, \iota‘a \subset -R“-\iota‘a \,.\equiv:\, aRa \,.\vee.\, a \sim\epsilon \mathrm{D}‘R$
Assume $a \sim\epsilon \mathrm{D}‘R$. Then $-\iota‘a = \mathrm{V}$.
This is impossible. Hence $a \sim\epsilon \mu_2 \,.\supset.\, a \epsilon \mathrm{D}‘R$ i.e. $-\mu_2 \subset \mathrm{D}‘R$

[51r][44]

To prove $\alpha \subset \mu_2 \,.\, \mu_2 \subset \beta \,.\, R“\mu_2 \subset \mu_2 \,.\supset.\, R_*“\alpha \subset \mu_2$
i.e. to pr. $z \epsilon \alpha \,.\, xR_*z \,.\supset.\, x \epsilon \mu_2$
i.e. $xR_*z \,.\supset:\, x \epsilon \mu_2 \,.\vee.\, z \sim\epsilon \alpha$
We have $z \epsilon \mu_2 \,.\vee.\, z \sim\epsilon \alpha$ (1)
We want $y \epsilon \mu_2 \,.\vee.\, z \sim\epsilon \alpha : xRy :\supset: x\epsilon \mu_2 \,.\vee.\, z \sim\epsilon \alpha$ which is true.
But this uses a wrong induction.
Assume $\mu_2 \subset \beta$. Then to prove $R_*“\alpha \subset \beta$. Requires
$xR_*z \,.\supset:\, x \epsilon \beta \,.\vee.\, z \sim\epsilon \alpha$
i.e. $y \epsilon \beta \,.\vee.\, z \epsilon -\alpha : xRy :\supset: x \epsilon \beta \,.\vee.\, z \epsilon -\alpha$
or $z \epsilon \alpha \,.\supset:\, xR_*z \,.\supset.\, x \epsilon \beta$
i.e. $z \epsilon \alpha \,.\, x \epsilon -\beta \,.\supset.\, \sim(xR_*z)$

To prove $xR_*z \,.\, z \epsilon \alpha \,.\supset.\, x \epsilon \beta$
We have $R“\alpha \subset R“\mu_2 \subset \mu_2$
$R“R“\alpha \subset R“\mu_2 \subset \mu_2$ etc.
$\gamma \subset \alpha \,.\supset.\, R“\gamma \subset \mu_2$

[51v][45]

$\alpha \subset \beta \,.\, R“\beta \subset \beta \,.\supset.\, R_*“\alpha \subset \beta$
Hence $y \epsilon \beta \,.\, R“\beta \subset \beta \,.\supset.\, \overrightarrow{R_*}‘y \subset \beta$
$z \epsilon \beta \,.\, R“\beta \subset \beta \,.\supset.\, \overrightarrow{R_*}‘z \subset \beta$
Put $\mu_2 = -\overrightarrow{R_*}‘y \cup -\overrightarrow{R_*}‘z$. Then $R“\mu_2 \subset \mu_2$

44 This leaf differs in size from the rest of the notes, being about 5mm longer and 1mm narrower than all the others. Compare with [19v].
45 The initial material is inverted with respect to [51v]. The last two lines are thus oriented like [51r].

$y \epsilon \mu_2$. Hence $\mu_2 \subset \beta \;.\; R``\beta \subset \beta \;.\supset .\overrightarrow{R_*}`y \subset \beta \;.\overrightarrow{R_*}`z \subset \beta$

$\supset . -\mu_2 \subset \beta$

Hence $\mu_2 \subset \beta \;.\; R``\beta \subset \beta \;.\supset .\; \beta = \mathrm{V}$

i.e. $\mu_2 \subset \beta \;.\; \exists! -\beta \;.\supset .\; \exists!\cancel{\beta -} R``\beta - \beta$

Now $R``\beta \subset \beta \;.=.\; \breve{R}`` -\beta \subset -\beta \;.\equiv .\sim \exists!\cancel{-\beta}\; \breve{R}`` -\beta - \beta$ Hence

$\mu_2 \subset -\beta \;.\; \exists!\beta \;.\supset .\; \exists!\cancel{\beta -}\; \breve{R}``\beta - \beta$

i.e. $\beta \subset \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z \;.\; \exists!\beta \;.\supset .\; \exists!\cancel{\beta -}\breve{R}``\beta - \beta$

This does not apply only to inductive classes.

Assume $y \epsilon \xi - \eta \;.\; z \epsilon \eta - \xi$ Then

[The following two lines are inverted at the bottom of the page: ]

$R``(\overleftarrow{R_*}`x \cap \overrightarrow{R_*}`y - \overrightarrow{R_*}`z) \subset \overleftarrow{R_*}`x \cap \overrightarrow{R_*}`y - \overrightarrow{R_*}`z$

$R``(\overleftarrow{R_*}`x - \overrightarrow{R_*}`y) \subset (\overleftarrow{R_*}`x - \overrightarrow{R_*}`y) \cup (\overleftarrow{R_*}`x \cap \overrightarrow{R_*}`y - \overrightarrow{R_*}`z)$

[52r]

We have $R``(-\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z \cup \iota`y \cup \iota`z) \subset -\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z \cup \iota`y \cup \iota`z$

Hence

$\alpha \subset -\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z \cup \iota`y \cup \iota`z \;.\supset .\; \alpha \subset R_*``(-\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z \cup \iota`y \cup \iota`z)$

$-\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z \cup \iota`y \cup \iota`z \subset \beta \;.\supset .\; R``\alpha \subset \beta$

$v \epsilon \alpha \;.\; uRv \;.\supset .\; u \epsilon \beta$

To pr. $v \epsilon \alpha \;.\; uR_*v \;.\supset .\; u \epsilon \beta$ i.e. $uR_*v \;.\supset :\; v \epsilon \alpha \;.\supset .\; u \epsilon \beta$

$v \epsilon \alpha \;.\supset .\; u \epsilon \beta : u'Rv :\supset : v \epsilon \alpha \;.\supset .\; u' \epsilon \beta$

Hence $R_*``\alpha \subset \beta$. This must be proved impossible.

Put $\alpha = \iota`y$. Then $x \epsilon R_*``\alpha$. But $x \sim\epsilon \beta$? $\quad p \supset q \;.\supset .\; p \supset r$

We have only to prove $\quad \equiv : p \sim q \vee \sim p \vee r$

$x \sim\epsilon p`\hat{\alpha}(-\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z \subset \alpha)$ $\quad \equiv .\; pq \supset r$

or $x \sim\epsilon p`\underline{-}\,\hat{\alpha}(-\alpha \subset \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z)$

or $x \epsilon s`\hat{\alpha}(\alpha \subset \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z)$ which is true. Hence prop.

Essential point is to prove

$R``\mu_2 \subset \mu_2 \;.\; \alpha \subset \mu_2 \;.\; \mu_2 \subset \beta \;.\supset .\; R_*``\alpha \subset \beta$

i.e. $v \epsilon \alpha \;.\; uR_*v \;.\supset .\; u \epsilon \beta$ i.e. $uR_*v \;.\supset :\; v \epsilon \alpha \;.\supset .\; u \epsilon \beta$

~~i.e.~~ $\cancel{v \epsilon \alpha \;.\supset .\; u \epsilon \beta : wRv :\supset : w \epsilon \alpha \;.\supset .\; u \epsilon \beta}$

~~We have to pr.~~ $\cancel{wRu \;.\supset : v \epsilon \alpha \;.\; u \epsilon \beta \;.\supset .\; w \epsilon \beta}$

~~Now~~ i.e. $v \epsilon \alpha \;.\supset .\; u \epsilon \beta : wRv :\supset : w \epsilon \alpha \;.\supset .\; u \epsilon \beta$

which is true.

[52v]

$\beta \subset \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z \;.\; \exists!\,\beta \;.\supset .\; \exists!\breve{R}``\beta - \beta$

$\breve{R}``(\overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z) \subset \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z$

$\overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z = p\text{'}\hat{\alpha}(\breve{R}\text{``}\alpha \subset \alpha \,.\, y, z \,\epsilon\, \alpha)$
Hence $\breve{R}\text{``}\alpha \subset \alpha \,.\, y, z \,\epsilon\, \alpha \,.\supset .\, \exists!\, \breve{R}\text{``}\alpha - \alpha$ which is a contradiction.

$=$

Put $\mu_2 = -\overrightarrow{R}_*\text{'}y \cup -\overrightarrow{R}_*\text{'}z$ Prove $R\text{``}\mu_2 \subset \mu_2$.
$y, z \,\epsilon\, \mu_2$. Hence $\mu_2 \subset \beta \,.\, R\text{``}\beta \subset \beta \,.\supset .\, \overrightarrow{R}_*\text{'}y \subset \beta \,.\, \overrightarrow{R}_*\text{'}z \subset \beta \,.\supset .\, -\mu_2 \subset \beta$
Hence $\mu_2 \subset \beta \,.\, R\text{``}\beta \subset \beta \,.\supset .\, \beta = \mathrm{V}$
Hence $\mu_2 \subset -\beta \,.\, \breve{R}\text{``}\beta \subset \beta \,.\supset .\, -\beta = \mathrm{V}$
Hence $\mu_2 \subset -\beta \,.\, \exists!\, \beta \,.\supset .\, \exists!\, \breve{R}\text{``}\beta - \beta$
Hence $\beta \subset \overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z \,.\, \exists!\, \beta \,.\supset .\, \exists!\, \breve{R}\text{``}\beta - \beta$
Now $\overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z = p\text{'}\hat{\alpha}(\breve{R}\text{``}\alpha \subset \alpha \,.\, y, z \,\epsilon\, \alpha)$
Hence $\breve{R}\text{``}\alpha \subset \alpha \,.\, y, z \,\epsilon\, \alpha \,.\supset .\, \exists!\, \breve{R}\text{``}\alpha - \alpha$ which is a contradiction.

[53r]

$R \,\epsilon\, \mathrm{Cls} \to 1 \,.\, y, z \,\epsilon\, \mu - \breve{R}\text{``}\mu \,.\, y \neq z \,.\, \breve{R}\text{``}\mu \subset \mu.$
Hence $\sim(yR_*z) \,.\, \sim(zR_*y)$

$\mu_2 = -\overrightarrow{R}_*\text{'}y \cup -\overrightarrow{R}_*\text{'}z$ To pr. $R\text{'}\mu_2 \subset \mu_2$, i.e.
$\breve{R}\text{``}(\overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z) \subset \overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z$
We have
$\mu \,\epsilon\, \overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z \,.\supset .\, u \neq y \,.\, u \neq z \,.\supset .\, \breve{R}\text{'}u \,\epsilon\, \overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z$ whence prop.
We have $y, z \,\epsilon\, \mu_2$. Hence
$\mu_2 \subset \beta \,.\, R\text{``}\beta \subset \beta \,.\supset .\, \overrightarrow{R}_*\text{'}y \subset \beta \,.\, \overrightarrow{R}_*\text{'}z \subset \beta$
But by hp $-\overrightarrow{R}_*\text{'}y \subset \beta \,.\, -\overrightarrow{R}_*\text{'}z \subset \beta$
Hence $\mu_2 \subset \beta \,.\, R\text{``}\beta \subset \beta \,.\supset .\, \beta = \mathrm{V}$
Hence $\mu_2 \subset -\beta \,.\, \breve{R}\text{``}\beta \subset \beta \,.\supset .\, -\beta = \mathrm{V}$
Hence $\mu_2 \subset -\beta \,.\, \exists!\, \beta \,.\supset .\, \exists!\, \breve{R}\text{``}\beta - \beta$
Hence $\beta \subset \overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z \,.\, \exists!\, \beta \,.\supset .\, \exists!\, \breve{R}\text{``}\beta - \beta$
or $\beta \subset \overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z \,.\, \breve{R}\text{``}\beta \subset \beta \,.\supset .\, \beta = \Lambda$
~~Hence~~ $\cancel{y, z \,\epsilon\, \beta \,.\, R\text{``}\beta \subset \beta \,.\, \breve{R}\text{``}\beta \subset \beta \,.\supset .\, \beta = \Lambda}$
Put $\overrightarrow{R}_*\text{'}y \cap \overrightarrow{R}_*\text{'}z = p\text{'}\kappa$. Thus $\breve{R}\text{``}p\text{'}\kappa \subset p\text{'}\kappa : \beta \subset p\text{'}\kappa . \exists!\beta \,.\supset .\, \sim(\breve{R}\text{``}\beta \subset \beta)$
We have
$\mu_2 \subset \beta \,.\, R\text{``}\beta \subset \beta \,.\supset .\, \beta = \mathrm{V}$
$\mu_2 \subset \beta \,.\, \exists!\, -\beta \,.\supset .\, \exists!\, \breve{R}\text{``}\beta - \beta$

[53v]

$y \,\epsilon\, -\beta \,.\supset .\, \overrightarrow{R}_*\text{'}y \subset -\beta \,.\supset .\, \beta \subset -\overrightarrow{R}_*\text{'}y$
$z \,\epsilon\, -\alpha \,.\supset .\, \overrightarrow{R}_*\text{'}z \subset -\alpha \,.\supset .\, \alpha \subset -\overrightarrow{R}_*\text{'}z$
Hence $\alpha \cup \beta \subset -\overrightarrow{R}_*\text{'}y \cup -\overrightarrow{R}_*\text{'}z$

Hence $R\text{``}(\alpha \cup \beta) \subset -\overrightarrow{R}_*\text{'}y \cup -\overrightarrow{R}_*\text{'}z \cup \iota\text{'}y \cup \iota\text{'}z$

Assume $y \,\epsilon\, \alpha \,.\, z \,\epsilon\, \beta$
$\cancel{\breve{R}\text{``}-\alpha = \hat{y}\{(\exists x).\, x \sim\epsilon\, \alpha \,.\, xRy\}}$
$\cancel{= \mathrm{C\!\!\!I}\text{'}R - \breve{R}\text{``}\alpha}$
~~Hence~~ $\cancel{-\alpha \cap \mathrm{C\!\!\!I}\text{'}R \subset \breve{R}\text{``}-\alpha}$

Hence $\overrightarrow{R}_*‘y \subset -\alpha-\beta \,.\, \overrightarrow{R}_*‘z \subset -\alpha-\beta \,.\, y \in \alpha - \breve{R}“\alpha - \beta \,.\, z \in \beta - \breve{R}“\beta - \alpha.$

$y, z \in (\alpha\cup\beta) - \breve{R}“(\alpha\cup\beta)$

$\cancel{y \in \alpha \,.\, \breve{R}“\alpha \subset \alpha \,.\supset.\, R“-\alpha \subset -\alpha}$ Assuming D‘$R = \mathrm{V}$,

$R“-\alpha = -R“\alpha$

Thus we have the following prop: Hence $\alpha \subset R“\alpha$

$y, z \in \overleftarrow{R}_*‘x \cap \mu - \breve{R}“\mu \,.\supset:\, (\exists\alpha,\beta).\; y \in \alpha-\beta \,.\, z \in \beta-\alpha :$
$y \in \alpha-\beta \,.\, z \in \beta-\alpha \,.\supset.\, y \in \alpha - \breve{R}“\alpha \,.\, z \in \beta - \breve{R}“\beta \,.\, y, z \in (\alpha\cup\beta) - \breve{R}“(\alpha\cup\beta) :$
$\alpha \subset \mu \,.\, y \in \alpha \,.\supset.\, y \in \alpha - \breve{R}“\alpha$
$\overrightarrow{R}_*‘x \subset \overrightarrow{R}_*‘y \;\therefore\; -\overrightarrow{R}_*‘y \subset -\overrightarrow{R}_*‘x$
Hence $\alpha\cup\beta \subset -\overrightarrow{R}_*‘x$

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$\beta \subset -\overrightarrow{R}_*‘y \,.\, \overleftarrow{R}_*‘y \subset \alpha \,.\, \overrightarrow{R}_*‘y - \iota‘y \subset -\alpha$

[54][46]

$\kappa = \overleftarrow{R}_*‘x \cap \overrightarrow{R}_*‘y \cap \overrightarrow{R}_*‘z \,.\supset.\, \breve{R}“\kappa \subset \kappa \,.\, R“-\kappa \subset -\kappa$
$R“-\kappa = R“-\overleftarrow{R}_*‘x \cup R“(\overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y) \cup R“(\overleftarrow{R}_*‘x \cap \overrightarrow{R}_*‘y - \overrightarrow{R}_*‘z)$
$R“-\overleftarrow{R}_*‘x \subset -\overleftarrow{R}_*‘x \,.\, R“\overrightarrow{R}_*‘y \subset \overrightarrow{R}_*‘y$. Thus
$R“(\overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y) \cup R“(\overleftarrow{R}_*‘x \cap \overrightarrow{R}_*‘y - \overrightarrow{R}_*‘z)$
$\subset \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y \cup (\overleftarrow{R}_*‘x \cap \overrightarrow{R}_*‘y - \overrightarrow{R}_*‘z)$
$R“(\overleftarrow{R}_*‘x \cap \overrightarrow{R}_*‘y - \overrightarrow{R}_*‘z) \subset \overleftarrow{R}_*‘x \cap \overrightarrow{R}_*‘y - \overrightarrow{R}_*‘z$
$u \in R“(\overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y) \,.\equiv:\, \breve{R}‘u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y \,.\, u \in \overleftarrow{R}_*‘x$
$\equiv: u \in \overleftarrow{R}_*‘x : u = y \,.\vee.\, u \sim\in \overrightarrow{R}_*‘y$
$\supset: u \in (\overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y) \cup (\overleftarrow{R}_*‘x - \overrightarrow{R}_*‘z)$
Hence $u \in \overleftarrow{R}_*‘x \,.\supset:\, u \in -\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z \,.\equiv.\, \breve{R}‘u \in -\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z$
$\breve{R}‘u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y \,.\equiv.\, u \in \overleftarrow{R}_*‘x \,.\, u \in \iota‘y \cup -\overrightarrow{R}_*‘y$
$\breve{R}‘u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y - \breve{R}“\mu \,.\equiv.\, u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y - \cancel{R“}\,\mu$
$\cancel{\breve{R}‘u \in -\breve{R}“\mu \,.\supset.\, u \in -\mu}$
$\cancel{u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y - \mu \,.\supset.\, \breve{R}‘u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y}$
$u \in \overleftarrow{R}_*‘x \,.\, u \in \mu \cup -\overrightarrow{R}_*‘y \,.\supset.\, \breve{R}‘u \in \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y$
i.e. $u \in \overleftarrow{R}_*‘x .\supset: u \in \mu \cup -\overrightarrow{R}_*‘y \,.\supset.\, \breve{R}‘u \in \mu \cup -\overrightarrow{R}_*‘y$
$u \in \overleftarrow{R}_*‘x - \mu \,.\supset:\, u \in -\overrightarrow{R}_*‘y \,.\supset.\, \breve{R}‘u \in -\overrightarrow{R}_*‘y$

[55]

Put $\xi = -\overleftarrow{R}_*‘x \,.\, \eta = \overleftarrow{R}_*‘x - \overrightarrow{R}_*‘y \,.\, \zeta = \overleftarrow{R}_*‘x\, \cancel{\cap \overrightarrow{R}_*‘y} - \overrightarrow{R}_*‘z$
Then $R“(\xi\cup\eta\cup\zeta) \subset \xi\cup\eta\cup\zeta$
$R“\xi \subset \xi \;\cancel{\therefore\; R“(\eta\cup\zeta) \subset \eta\cup\zeta}$ $R_* = R_0 \,\dot{\cup}\, R|R_*$
$R“\eta \subset -\overrightarrow{R}_*‘y \cup \iota‘y$ $\dot{-}R_* = J \,\dot{\cap}\, \dot{-}(R|R_*)$

[46] This page of notes is on the larger size of paper: 24.4cm by 20.1cm.

$R“\eta \cap \overleftarrow{R}_*‘x \subset \eta \cup \iota‘y$
$R“\eta \cap \overleftarrow{R}_*‘x = \hat{u}\{xR_*u \,.\, \breve{R}‘u \mathbin{\dot{-}} R_*y\}$
$\qquad\qquad = \hat{u}\{xR_*u : u = y \,.\vee.\, u \in \eta\} = \eta \cup \iota‘y$
$R“\zeta \cap \overleftarrow{R}_*‘x = \zeta \cup \iota‘z$ Hence
$R“(\eta \cup \zeta) \cap \overleftarrow{R}_*‘x \subset \eta \cup \zeta \cup \iota‘y \cup \iota‘z$
$\quad R“\iota‘y \cap \overleftarrow{R}_*‘x \subset \zeta \,.\, R“\iota‘z \cap \overleftarrow{R}_*‘x \subset \eta$ Hence
$R“(\eta \cup \zeta \cup \iota‘y \cup \iota‘z) \cap \overleftarrow{R}_*‘x \subset \eta \cup \zeta \cup \iota‘y \cup \iota‘z$
Hence $\overleftarrow{R}_*‘x \subset \eta \cup \zeta \cup \iota‘y \cup \iota‘z \cup - R“(\eta \cup \zeta \cup \iota‘y \cup \iota‘z)$
and $R“\{-\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z \cup \iota‘y \cup \iota‘z\} \cap \overleftarrow{R}_*‘x \subset -\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z \cup \iota‘y \cup \iota‘z$
Consider $R“\{-\overrightarrow{R}_*‘y - \overleftarrow{R}_*‘x\}$ ; $y \sim\in -\overrightarrow{R}_*‘y - \overleftarrow{R}_*‘x$
$\breve{R}‘u \in -\overrightarrow{R}_*‘y - \overleftarrow{R}_*‘x \,.\supset:\, u \mathbin{\dot{-}} (R|R_*)y \,.\, u \neq y :\supset.\, \sim(uR_*y)$
Hence $R“\{-\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z \cup \iota‘y \cup \iota‘z\} \subset -\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z \cup \iota‘y \cup \iota‘z$
~~In fact~~ $\cancel{R“\{-\overrightarrow{R}_*‘y \cup -\overrightarrow{R}_*‘z}$
and $\breve{R}“\{\overrightarrow{R}_*‘y \cap \overrightarrow{R}_*‘z - \iota‘y - \iota‘z\} \subset \overrightarrow{R}_*‘y \cap \overrightarrow{R}_*‘z - \iota‘y - \iota‘z$

[56]

$*215{\cdot}01$ $\text{str}‘P = \hat{\alpha}(\alpha \subset \text{C}‘P \,.\, P“\alpha \cap \breve{P}“\alpha \subset \alpha)$ Df
$\cdot16 \vdash : P \in \text{trans.} \supset .\, \text{str}‘P = \hat{\gamma}\{(\exists\alpha,\beta).\, \alpha \in \text{sect}‘P.\, \beta \in \text{sect}‘\breve{P}.\, \gamma = \alpha \cap \beta\}$
$*211{\cdot}01$ $\text{sect}‘P = \hat{\alpha}(\alpha \subset \text{C}‘P \,.\, P“\alpha \subset \alpha)$ Df
$\cdot17 \vdash .\, \text{sect}‘P = \text{sect}‘P_{\text{po}} = \text{sect}‘P_*$[47]
Hence
$\gamma \subset \text{C}‘P \,.\, P_*“\gamma \cap \breve{P}_*“\gamma \subset \gamma \,.\equiv$
$\qquad\qquad .\,(\exists\alpha,\beta).\, \alpha \subset \text{C}‘P \,.\, \beta \subset \text{C}‘P \,.\, P“\alpha \subset \alpha \,.\, \breve{P}“\beta \subset \beta \,.\, \gamma = \alpha \cap \beta$
Here $\gamma \subset \text{C}‘P, \alpha \subset \text{C}‘P, \beta \subset \text{C}‘P$ are unnecessary. Thus
$\quad P_*“\gamma \cap \breve{P}_*“\gamma \subset \gamma \,.\equiv.\, (\exists\alpha,\beta).\, P“\alpha \subset \alpha \,.\, \breve{P}“\beta \subset \beta \,.\, \gamma = \alpha \cap \beta$
$\cancel{P_*“\gamma \cap \breve{P}_*“\gamma \subset \gamma \,.\supset_\gamma.\, \phi\gamma :\equiv:}$
$P_*“\gamma \cap \breve{P}_*“\gamma \subset \gamma \,.\, x, y \in \gamma \,.\supset_\gamma.\, u \in \gamma :\equiv$
$\qquad\qquad\qquad : P“\alpha \subset \alpha \,.\, \breve{P}“\beta \subset \beta \,.\, x, y \in \alpha \cap \beta \,.\supset_{\alpha,\beta}.\, u \in \alpha \cap \beta$
Hence $xR_*y \,.\supset.\, P(x \longmapsto\!\dashv y) = \hat{u}\{P_*“\gamma \cap \breve{P}_*“\gamma \subset \gamma.\, x, y \in \gamma \,.\supset_\gamma.\, u \in \gamma\}$

[57r]

To prove
$u \in R(x \longmapsto\!\dashv y). \equiv: \cancel{(\exists\alpha,\beta)}\ \breve{R}“\alpha \subset \alpha.\, \breve{R}“\beta \subset \beta.\, x \in \alpha \,.y \in -\beta \,.\supset_{\alpha,\beta} .u \in \alpha - \beta :$
$\qquad$ and $\equiv: R“\gamma \cap \breve{R}“\gamma \subset \gamma \,.\, x, y \in \gamma \,.\supset_\gamma.\, u \in \gamma$
First is obvious. As to second, it gives
$\quad R“\gamma \subset \gamma \,.\, x, y \in \gamma \,.\supset_\gamma.\, u \in \gamma : \breve{R}“\gamma \subset \gamma \,.\, x, y \in \gamma \,.\supset_\gamma.\, u \in \gamma$
Assume $xR_*y$. Then 2nd $\supset$ 1st. For converse,

[47] These four numbers are the same as in *PM*.

1st $\supset : \breve{R}``\alpha \subset \alpha \,.\, R``\beta \subset \beta \,.\, x \in \alpha \,.\, y \in \beta \,.\supset .\, u \in \alpha \cap \beta$

$\quad \supset : \breve{R}_*``\alpha \subset \alpha \,.\, R_*``\beta \subset \beta \,.\, x \in \alpha \,.\, y \in \beta \,.\supset .\ \ldots$

This doesn't work out. But we do have

$xR_*y \,.\supset :.\, R``\gamma \cap \breve{R}``\gamma \subset \gamma \,.\, x, y \in \gamma \,.\supset_\gamma .\, u \in \gamma : \supset .\, u \in R(x \longmapsto\!\dashv y)$

Thus we want to prove

$\quad R``\gamma \cap \breve{R}``\gamma \subset \gamma \,.\, x, y \in \gamma \,.\supset_\gamma .\, z \in \gamma$

Given $\quad xR_*y \,.\, xR_*z \,.\, y, z \in \mu - \breve{R}``\mu \,.\, \breve{R}``\mu \subset \mu$

This wants $\mu - \breve{R}``\mu \subset \gamma$ for any such $\gamma$.

[57v]

$\vdash : R \in \mathrm{Cls} \to 1 \,.\, \breve{R}``\mu \subset \mu \,.\, \lambda = \overleftarrow{R}_*`x \cap \mu - \breve{R}``\mu \,.\supset .\, \lambda \in 0 \cup 1$

*Dem.*

$\vdash :.\ \mathrm{Hp.}\ \kappa = \overleftarrow{R}_*`x \cap p`\overrightarrow{R_*}``\lambda \,.\, \lambda \sim\in 0 \cup 1 \,.\supset$

$\qquad\qquad .\, \breve{R}``\kappa \subset \kappa \,.\, x \in \kappa \,.\, \kappa \subset -\breve{R}``\mu - \mu \quad (1)$

$\cancel{\vdash :.\ \mathrm{Hp} : (\exists y).\, y \in \lambda \,.\, uR_*y : \supset .\, y \lfloor u \rfloor \in - \breve{R}``u \qquad (2)}$

$\vdash :.\mathrm{Hp} : y \in \lambda \,.\supset_y .\, \sim(uR_*y) : u \in \overleftarrow{R}_*`x : \supset : uR_*y. \supset_y .\, y \sim\in \lambda : u \in \overleftarrow{R}_*`x :$

$\qquad \supset : u \sim\in \lambda : u \in \overleftarrow{R}_*`x :$

$\qquad \supset : u \in \overleftarrow{R}_*`x : u \in -\mu \cup \breve{R}``\mu :$

$\qquad \supset : u \in \overleftarrow{R}_*`x : u \in -\breve{R}``\mu \,.\supset .\, \breve{R}`u \in -\breve{R}``\mu \ (2)$

$\vdash :.\mathrm{Hp} : (\exists y, z).\, y, z \in \lambda \,.\, y \neq z \,.\, uR_*y \,.\, uR_*z : \supset .\, u \sim\in \mu \,.\, \breve{R}`u \sim\in \breve{R}`\mu \quad (3)$

$\cancel{(\exists y) xR_*}\ (\exists y).\, y \in \lambda \,.\, \sim(uR_*y) :$

[58r]

Consider $\overleftarrow{R}_*`x \cap \overrightarrow{R_*}`y \cap \overrightarrow{R_*}`z$ Call this $\xi$.

Then $x \in \xi : w \in \xi \,.\supset_w .\, \breve{R}`w \in \xi : \iota`x = \xi - \breve{R}``\xi$

We have $y, z \in \mu - \breve{R}``\mu$. Hence $uRy \,.\supset .\, u \sim\in \mu \,.\supset .\, x \sim\in \mu$

Hence $\overrightarrow{R_*}`y - \iota`y \subset -\mu$. Hence $\xi \subset -\mu$.

We ought to be able to prove

$\quad xR_*y \,.\supset .\, (\exists \mu, \nu).\ \overleftarrow{R}_*`x \cap \overrightarrow{R_*}`y = \mu - \nu$

We have $y \in \mu - \breve{R}``\mu$. Hence $\overrightarrow{R_*}`y \subset -\mu$

Assume $\overleftarrow{R}_*`x \subset \nu$. Hence $R(x \longmapsto\!\dashv y) \subset \nu - \mu$

Thus $R(x \longmapsto\!\dashv y) \cap R(x \longmapsto\!\dashv z) \subset \nu - \mu$

If $y \in \alpha - \beta \,.\, z \in \beta - \alpha, \xi \subset \nu - \alpha - \beta$

$\quad u \in -\alpha - \beta \,.\supset : \breve{R}`u = y \,.\vee .\, \breve{R}`u = z \,.\vee .\, \breve{R}`u \in -\alpha - \beta$

$\qquad \supset : \breve{R}`u \in \mu \,.\vee .\, \breve{R}`u \in -\alpha - \beta$

Hence $u \in \mu \cup -\alpha - \beta \,.\supset .\, \breve{R}`u \in \mu \cup -\alpha \ -\beta$
Hence $\overleftarrow{R}_*`x \subset \mu \cup -\alpha - \beta$ Hence $\alpha \cup \beta \subset \mu$

[58v]

$xR_*y \,.\, xR_*z \,.\, y \neq x \,.\, z \neq x \,.\supset .\, \breve{R}`xR_*y \,.\, \breve{R}`xR_*z$
Hence $xR_*y \,.\, xR_*z \,.\, \sim(yR_*z) \,.\, \sim(zR_*y) \,.\supset .\, \breve{R}`xR_*y \,.\, \breve{R}`xR_*z$
$y, z \in \overleftarrow{R}_*`x \cap \mu - \breve{R}``\mu \,.\, y \neq z \,.\supset .\, x \sim\in \mu - \breve{R}``\mu$
$x \sim\in \mu - \breve{R}``\mu \,.\, \breve{R}`x \in \mu - \breve{R}``\mu$ is impossible.
Hence $\breve{R}`x \sim\in \mu - \breve{R}``\mu$
We want $\overrightarrow{R_*}`u \subset -\mu \,.\supset .\, u \sim\in \mu - \breve{R}``\mu$
But this won't do. We must have instead
$\mu \subset \alpha \,.\, u \in \alpha \,.\, \breve{R}``\alpha \subset \alpha \,.\supset .\, u \sim\in \mu - \breve{R}``\mu$

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$R(x \longmapsto\!\dashv \breve{R}`y) = R(x \longmapsto\!\dashv y) \cup \iota`\breve{R}`y$
$y \in \mu - \breve{R}``\mu. \supset : zR_*y. \supset_z . z \neq y : \supset . \cancel{R} \sim(yR|R_*y). \supset . R(y \longmapsto\!\dashv y) = \iota`y$
$\sim(yR|R_*y) \,.\supset .\, \sim(xR|R_*x) \,.\supset .\, R(x \longmapsto\!\dashv x) = \iota`x$
Hence $xR_*y \,.\, \breve{R}``\mu \subset \mu \,.\, y \in \mu - \breve{R}``\mu \,.\supset .\, (\exists\mu_2).\ R(x \longmapsto\!\dashv y) = \mu_2$
$(\exists\nu_2).\ R(x \longmapsto\!\dashv y) = \nu_2$
Hence $(\exists\mu_2).\ -R(x \longmapsto\!\dashv y) = \mu_2$
$-R(x \longmapsto\!\dashv y) = -\overleftarrow{R}_*`x \cup -\overrightarrow{R_*}`y \qquad -\mu_2 \cup -\nu_2 = -\overleftarrow{R}_*`x \cup -\overrightarrow{R_*}`y \cup -\overrightarrow{R_*}`z$
$R(x \longmapsto\!\dashv z) - R(x \longmapsto\!\dashv y) = \overleftarrow{R}_*`x \cap (\overrightarrow{R_*}`z \cap -\overrightarrow{R_*}`y) = \nu_2 - \mu_2$
$\mu_2 \cup -\nu_2 = \overleftarrow{R}_*`x \cap \overrightarrow{R_*}`y \cup -\overleftarrow{R}_*`x \cup -\overrightarrow{R_*}`z =$

[59]

Given $z, y \in \overleftarrow{R}_*`x \cap \mu - \breve{R}``\mu$, we have $(\exists\alpha, \beta).\ y \in \alpha - \beta \,.\, z \in \beta - \alpha$
Put $\lambda = \overleftarrow{R}_*`x \cap \mu - \breve{R}``\mu$. Consider $\kappa = \hat{\alpha}(\breve{R}``\alpha \subset \alpha \,.\, \exists!\ \lambda \cap \alpha \,.\, \exists!\ \lambda - \alpha)$
$y \in \lambda \,.\supset .\, (\exists\alpha).\ \alpha \in \kappa \,.\, y \in \alpha$ i.e. $\lambda \subset s`\kappa$

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Put $\xi = \hat{\alpha}\{xR_*u \,.\, uR_*y \,.\, uR_*z \,.\supset_u .\, u \in \alpha : \breve{R}``\alpha \subset \alpha\}$. Then $\mathrm{V} \in \xi.\ \therefore\ \exists!\ \xi.$
$R`` - \mu \subset -\mu$
Then $\alpha \in \xi \,.\supset : w \in \alpha \,.\supset .\, \breve{R}`w \in \alpha$ $\qquad \breve{R}``R`` - \mu = -\mu \cap \text{Cl}`R$
Do we have $\alpha \in \xi \,.\supset .\, x \in \alpha$ ? i.e. $\qquad \breve{R}`` - \mu \cup \breve{R}``\mu$
$xR_*u \,.\, uR_*y \,.\, uR_*z \,.\supset_u .\, u \in \alpha : \breve{R}``\alpha \subset \alpha : \supset .\, x \in \alpha$ yes.
Hence $\alpha \in \xi \,.\supset : \overleftarrow{R}_*`x \subset \alpha$
Hence $y, z \in p`\xi$ i.e.
$xR_*u \,.\, uR_*y \,.\, uR_*z \,.\supset_u .\, u \in \alpha : \breve{R}``\alpha \subset \alpha : \supset_\alpha .\, y, z \in \alpha$
Now $xR_*u \,.\, uR_*y \,.\, uR_*z \,.\supset_u .\, u \in -\mu \,.\, \breve{R}`u \in -\mu$
Thus $xR_*u \,.\, uR_*y \,.\, uR_*z \,.\supset_u .\, u \in R`` -\mu : \cancel{\breve{R}``R`` - \mu \subset R`` - \mu}$
~~Hence $y, z \in R`` - \mu$ which is possible.~~

~~Essential point is $y \in -\mu \cap \text{Ɑ}`R \,.\supset.\, y \in R“-\mu$~~
Consider $-\alpha \cup \mu \,.\, \breve{R}“(-\alpha \cup \mu) \subset -\alpha \cup \mu$
This follows if $\alpha - \breve{R}“\alpha \subset \mu - \breve{R}“\mu$
Thus we have to prove

$y \in \mu - \breve{R}“\mu \,.\, y \in \alpha \,.\, \exists!\, \mu - \alpha \,.\supset.\, \alpha - \breve{R}“\alpha \subset \mu - \breve{R}“\mu.$

[60r]

We have to prove

$y, z \in \overleftarrow{R}_*`x \,.\supset:\, y \in R(x \longmapsto z) \,.\vee.\, z \in R(x \longmapsto y)$

Put $R(x \longmapsto y) = \mu_2 \,.\, R(x \longmapsto z) = \nu_2$

$u \in \mu_2 \,.\supset:\, \breve{R}`u \in \mu_2 \,.\vee.\, u = y$

$u \in \nu_2 \,.\supset:\, \breve{R}`u \in \nu_2 \,.\vee.\, u = z$

$u \in \mu_2 \cup \nu_2 \,.\supset:\, \breve{R}`u \in \mu_2 \cup \nu_2 \,.\vee.\, u \in \iota`y \cup \iota`z$

$u \in \mu_2 \cap \nu_2 \,.\supset:\, \breve{R}`u \in \mu_2 \cap \nu_2 \,.\vee.\, u = y \cup u = z$

Hp $.\supset:\, y \sim\in \nu_2 \,.\, z \sim\in \mu_2$ Hence

$u \in \mu_2 \cap \nu_2 \,.\supset.\, \breve{R}`u \in \mu_2 \cup \nu_2$ which is impossible.[48]

Doubtful point is induction proving $R(x \longmapsto y) = \mu_2$.

[60v]

We have $\breve{R}`u \in \alpha \,.\, z \sim\in \alpha.\, u \neq z. \;\supset.\, \sim(uR_*z)$
Can we prove $xR_*y \,.\, y \in \alpha \,.\, z \sim\in \alpha . z \in \beta \,.\, y \sim\in \beta. \;\supset.\, \sim(xR_*z)$

$$\breve{R}“(\alpha - \beta) = \breve{R}“\alpha - \breve{R}“\beta \subset \alpha - \breve{R}“\beta$$

$xR_*y \,.\supset.\, x \in -\beta \,.\supset.\, x \neq z$

$xRy \,.\, y \in \alpha - \beta \,.\, z \in \beta - \alpha \,.\supset.\; \sim(\breve{R}`xR_*z) \,.\, x \neq z \,.\supset.\; \sim(xR_*z)$

[ $\breve{R}`u \in \alpha$ is a hereditary property belonging to $x$ but not to $z$ ]

$xRy \,.\supset.\, \breve{R}`x \sim\in \beta \,.\, \breve{R}`z \in \beta \,.\supset.\, \sim(zR_*x)$

$xRy \,.\supset.\, \breve{R}`x \in \alpha - \beta \,.\, \breve{R}`z \sim\in (\alpha - \beta) \,.\supset.\, \sim(xR_*z)$

~~Thus $\alpha - \beta$ and $\beta - \alpha$ are the hereditary classes to consider.~~

We have $y \in \alpha\lfloor -\beta \rfloor .\, xRy \,.\supset.\, \breve{R}`x \in \alpha - \beta$

$$\left.\begin{array}{l} T \in \text{Potid}`R \,.\, y \in \alpha - \beta \,.\, xTy \,.\supset.\, \breve{T}`x \in \alpha - \beta \\ S \in \text{Potid}`R \,.\, z \in \beta - \alpha \,.\, xSz \,.\supset.\, \breve{S}`x \in \beta - \alpha \end{array}\right\} \supset.\, \sim(xR_*y).\, \sim(xR_*z)$$

~~$y \in \alpha - \beta \,.\, yRw \,.\supset.\, w \in \alpha - \breve{R}“\beta$ Can we prove $w \sim\in \beta$?~~

We want to prove

$y, z \in \overleftarrow{R}_*`x \,.\supset.\, (\exists S, T).\, S, T \in \text{Potid}`R \,.\, y = \breve{T}`x \,.\, z = \breve{S}`x$

i.e. $R_* \Subset \dot{s}`\text{Potid}`R$.[49]

[61r]

Put $\kappa = \hat{\alpha}(\breve{R}“\alpha \subset \alpha \,.\, \cancel{\exists!\,\mu\cap\alpha\,.\,\exists!\,\mu - \alpha)}\; \exists!(\mu - \breve{R}“\mu)\cap\alpha \,.\, \exists!(\mu - \breve{R}“\mu) - \alpha)$
Then I think $s`\kappa = \mu$

[48] This attention to the order of intervals is rare. See *PM*, p.653. [49] Compare with $*89{\cdot}28$.

$y \epsilon \alpha \,.\, z \epsilon -\alpha \,.\supset.\, \sim(yR_*z)$
$z \epsilon \beta \,.\, y \epsilon -\beta \,.\supset.\, \sim(zR_*y)$
$\breve{R}`u \epsilon \alpha \,.\supset:\, u \epsilon \alpha \,.\vee.\, u \epsilon \alpha - \breve{R}``\alpha$

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$$\alpha - \breve{R}``\alpha \subset \mu - \breve{R}``\mu \,.\, \exists!\, \alpha - \breve{R}``\alpha$$

Put $\kappa = \hat{\alpha}\{\breve{R}``\alpha \subset \alpha \,.\, \cancel{\exists!\,(\mu - \breve{R}``\mu) \cap (\alpha - \breve{R}``\alpha)} \,.\, \exists!\,(\mu - \breve{R}``\mu) - \alpha\}$.
To prove $s`\kappa = \mu$.
$y \epsilon \alpha - \breve{R}``\alpha \,.\supset.\, y \epsilon \mu$ Hence $\overleftarrow{R}_*``(\alpha - \breve{R}``\alpha) \subset \mu$
Put $\kappa = \hat{\alpha}\{\breve{R}``\alpha \subset \alpha. \alpha \subset \mu. \exists!\overleftarrow{R}_*`x \cap (\mu - \breve{R}``\mu) \cap \alpha. \exists!\overleftarrow{R}_*`x \cap (\mu - \breve{R}``\mu) - \alpha\}$
Then $s`\kappa \subset \mu$. Also
$y \epsilon \mu - \breve{R}``\mu \,.\, z \epsilon \mu - \breve{R}``\mu \,.\, y \neq z \,.\supset.\, (\exists\alpha, \beta).\, \alpha, \beta \epsilon \kappa \,.\, y \epsilon \alpha - \beta \,.\, z \epsilon \beta - \alpha$
$\supset.\, y, z \epsilon s`\kappa$
Hence $\mu - \breve{R}``\mu \subset s`\kappa$ Hence $\breve{R}_*``(\mu - \breve{R}``\mu) \subset s`\kappa$

---

~~Put $\kappa = \hat{\alpha}\{\breve{R}``\alpha \subset \alpha. \alpha \subset \mu \cap \overleftarrow{R}_*`x \,.\, \exists!(\mu - \breve{R}``\mu) \cap \alpha. \exists!(\mu - \breve{R}``\mu) - \alpha\}$~~

[61v]

*To pr.* $y, z \epsilon \mu - \breve{R}``\mu \,.\, y \epsilon \alpha - \beta \,.\, z \epsilon \beta - \alpha \,.\supset.\, \alpha - \breve{R}``\alpha \subset \mu$
If $\alpha - \breve{R}``\alpha \not\subset \mu$, $\exists!\, \alpha - \breve{R}``\alpha - \mu$
$u \epsilon -\alpha \cup \mu \,.\supset:\, u \epsilon R``-\alpha \,.\vee.\, u \epsilon R``\alpha - \alpha \,.\vee.\, u \epsilon \mu$
$\supset:\, \breve{R}`u \epsilon -\alpha \,.\vee.\, \breve{R}`u \epsilon \alpha - \breve{R}``\alpha \,.\vee.\, \breve{R}`u \epsilon \mu$
Thus we have to prove $\alpha - \breve{R}``\alpha \subset \mu$ or $\alpha \subset \breve{R}``\alpha \cup \mu$
~~We have only to prove~~
~~$y, z \epsilon \mu - \breve{R}``\mu \,.\, y \neq z \,.\supset.\, (\exists\alpha).\, y \epsilon \alpha - \breve{R}``\alpha \,.\, z \sim\epsilon \alpha$~~
Most hopeful result is
$xR_*u \,.\, uR_*y \,.\, uR_*z \,.\supset_u.\, u \epsilon \alpha :\, \breve{R}``\alpha \subset \alpha :\supset_\alpha.\, y, z \epsilon \alpha$
Together with
$$\ldots\ldots \supset_u.\, u \epsilon -\mu \,.\, \breve{R}`u \epsilon -\mu$$

[62r][50]

Put $\lambda = \text{Potid}`R - \alpha \cap P\ (P|R \epsilon \alpha)$ To pr. $\lambda \epsilon 0 \cup 1$
$P \epsilon \lambda \,.\, TR_{\text{ts}}(P|R) \,.\supset.\, T \epsilon \underline{\mu}\alpha$ Hence
$P \epsilon \lambda \,.\, TR_{\text{ts}}P \,. \qquad \supset: T = P \,.\vee.\, T \epsilon \underline{\mu}\alpha \quad (*91{\cdot}212)$
$\therefore\ P \epsilon \lambda \,.\, T \sim\epsilon \underline{\mu}\alpha. \qquad \supset: T = P \,.\vee.\, \sim(TR_{\text{ts}}P)$
$\therefore \qquad P, Q \epsilon \lambda. \qquad \supset: P = Q \,.\vee.\, \sim(PR_{\text{ts}}Q). \sim(QR_{\text{ts}}P) \qquad (I)$
$P \epsilon \lambda \,.\supset.\, PR_{\text{ts}}R_0 \qquad (II)$
$\therefore\ (\exists M) : M \epsilon \text{Potid}`R : P \epsilon \lambda \,.\supset_P.\, PR_{\text{ts}}M \qquad (II)'$

50 Compare with HPF, p.61b, p.62 and p.65.

By $(I)$, $\lambda \sim\epsilon\ 0 \cup 1 . \supset : (\exists M) : M \epsilon \text{ Potid}`R : (\exists P) . P \epsilon \lambda . \sim(P R_{\text{ts}} M)$
Hence $\lambda \sim\epsilon\ 0 \cup 1 . \supset : (P \epsilon \lambda . \supset_P . P R_{\text{ts}} M)$ is not hereditary (resp. to $M$)
$\supset : (\exists M) : P \epsilon \lambda . \supset_P . P R_{\text{ts}} M : (\exists P) . P \epsilon \lambda . \sim\{P R_{\text{ts}}(M|R)\} (III)$
By $(I)$, $\lambda \sim\epsilon\ 0 \cup 1 : P \epsilon \lambda . \supset_P . P R_{\text{ts}} M : \supset : M \sim\epsilon \lambda :$ $(IV)$
$\supset : P \epsilon \lambda . \supset_P . P \neq M :$ $(V)$
[$*91 \cdot 212$] $\supset : P \epsilon \lambda . \supset_P . P R_{\text{ts}}(M|R)$ $(VI)$
$III$ and $VI$ contradict each other. Hence prop.
Is above valid?

[62v]

Must try to pr. $(|R)_*|(|R) \subset (|R)|(|R)_*$
$P\{(|R_*)|(|R)\}Q . \equiv . P(|R)_*(Q|R)$
$P\{(|R)|(|R)_*\}Q .$ $\cancel{\equiv .(\exists S).(S|R)(|R)_* Q}$
$\equiv .(\exists S) . P = S|R . S(|R)_* Q$
$P(|R)_*(Q|R) . \equiv : |R``\mu \subset \mu . Q|R \epsilon \mu . \supset_\mu . P \epsilon \mu$
$\supset .(\exists S) . P = S|R$
Hence $(\exists\mu) . \text{Pot}`P = |R``\mu$
Thus instead of $P$ we may put $S|R$, where $S \epsilon \mu$. To pr.
$(S|R)|R_*(Q|R) . S|R, Q \epsilon \text{ Pot}`P . \supset . (\exists S') . S'|R = S|R . S'(|R)_* Q$
or we may put $Q = R_0$. Then to pr.
$(S|R)|R_* R . \supset . (\exists S') . S'|R = S|R . S' \epsilon \text{ Potid}`R$
i.e. $S|R \epsilon \text{ Pot}`R . \supset . (\exists S') . S' \epsilon \text{ Potid}`R . S'|R = S|R$
Put $\text{Pot}`R = |R``\mu$. Then $R_0 \epsilon \mu$
Also $P \epsilon \mu . \supset . P|R \epsilon \text{ Pot}`R . \supset .$
~~Put $\kappa = \hat{\alpha}(\text{Pot}`R = |R``\mu)$ To pr. $p`\kappa = \text{Potid}`R$~~
$\cancel{\overrightarrow{R}_{\text{ts}}`R^2 \subset \text{Pot}`R}$

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The citations in the text use initials for important works of Russell, most notably *PM* for *Principia Mathematica*, following the scheme of *Russell: the Journal of Bertrand Russell Studies*, HPF for the “Hierarchy of propositions and functions” manuscript, and TLP for Wittgenstein’s *Tractatus Logico-Philosophicus*. Citations of *PM* in the text follow the *Principia Mathematica to ∗56* pagination, except for Appendix B, which is not reprinted in that edition, and sections after ∗56, where the pagination of the 1925 edition is used. Unpublished correspondence is cited below with missing information supplied by the editor, as in [1922?] for a letter from Carnap to Russell with no year, but likely written in 1922. The four letters from Carnap to Russell are from the Archives of Scientific Philosophy at the University of Pittsburgh, the others are in the Bertrand Russell Archives.

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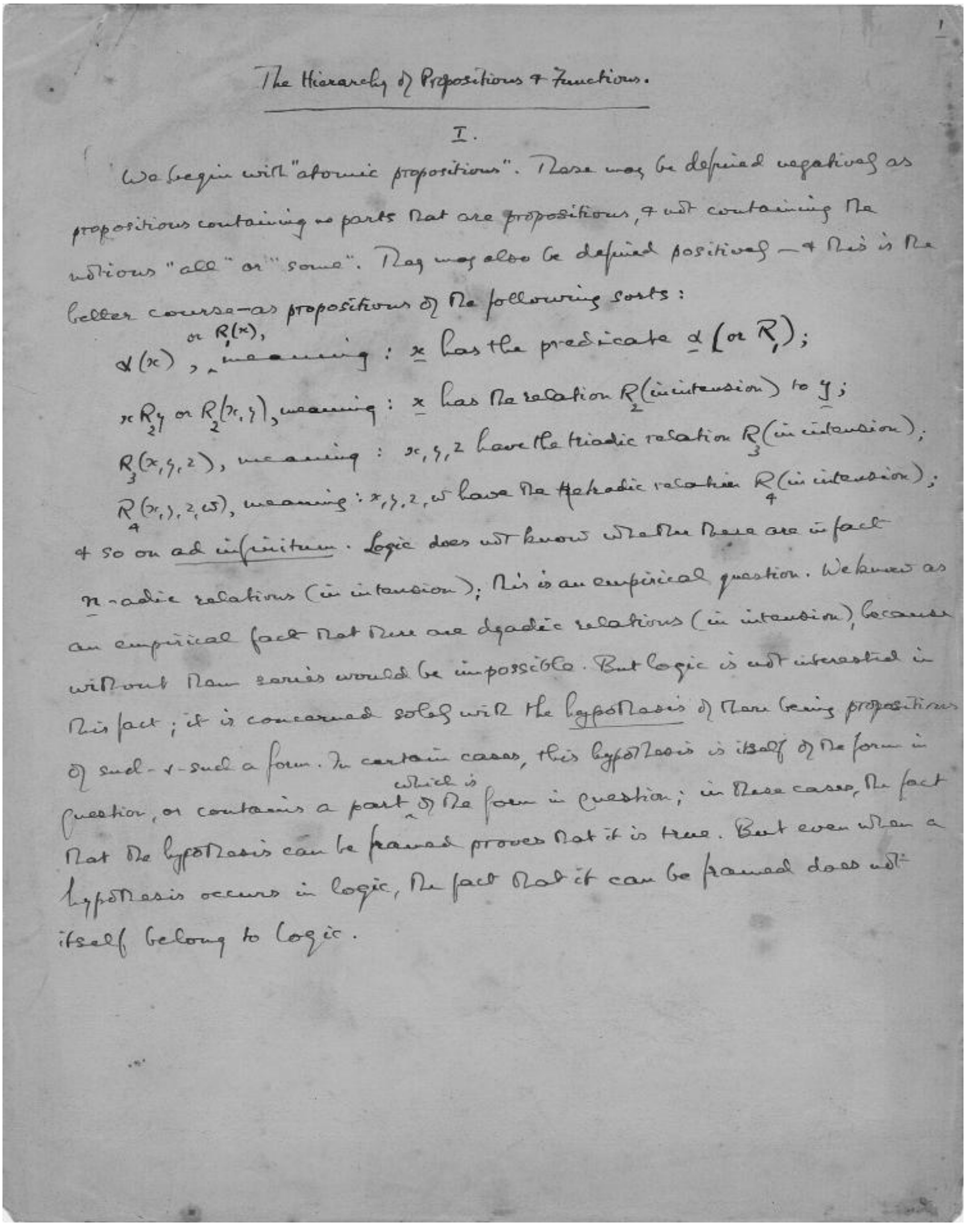
# Index



1

The Hierarchy of Propositions & Functions.

I.

We begin with "atomic propositions". These may be defined negatively as propositions containing no parts that are propositions, & not containing the notions "all" or "some". They may also be defined positively — & this is the better course — as propositions of the following sorts:

$\alpha(x)$, or $R_1(x)$, meaning: $x$ has the predicate $\alpha$ (or $R_1$);

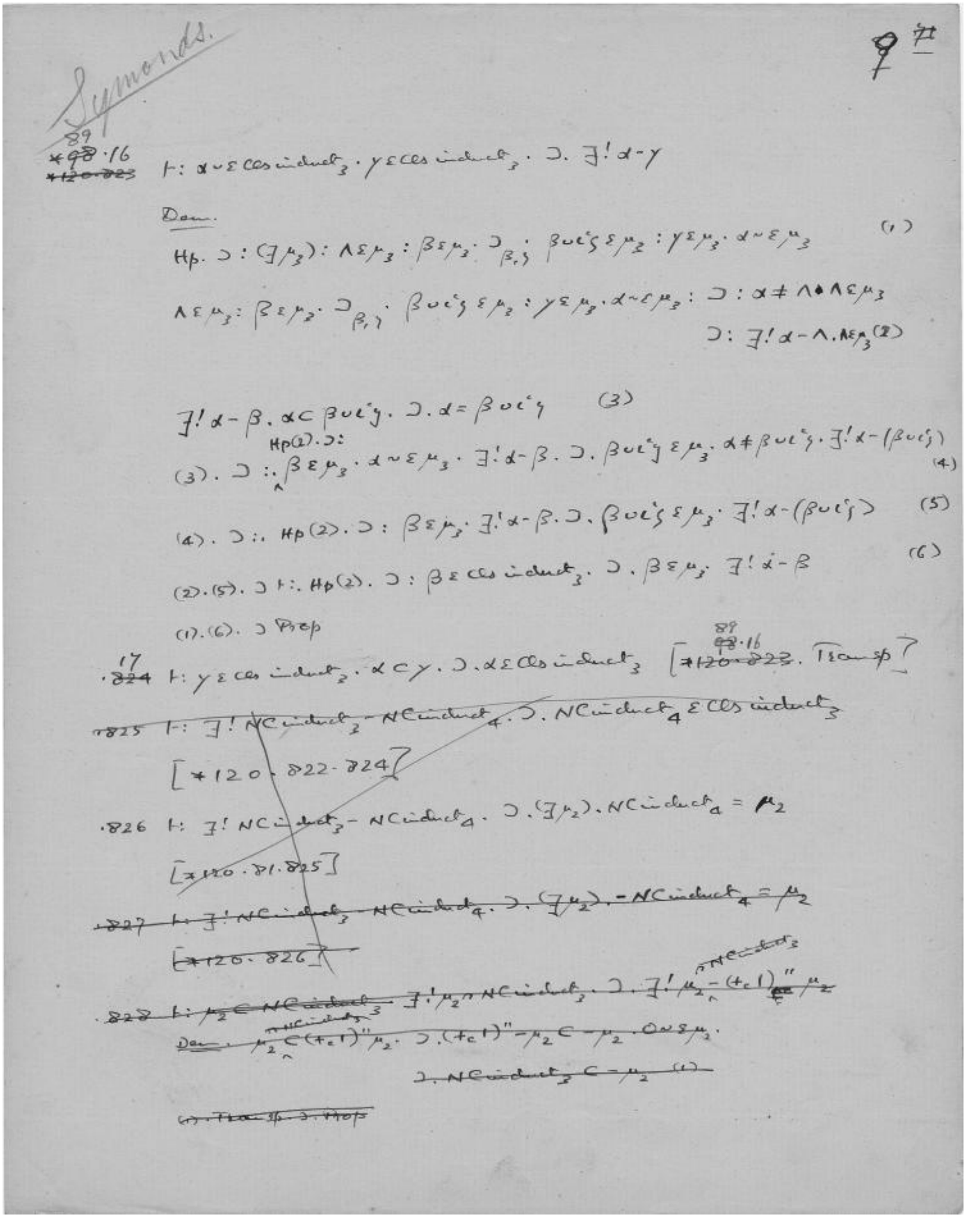
$xR_2y$ or $R_2(x,y)$, meaning: $x$ has the relation $R_2$ (in intension) to $y$;

$R_3(x,y,z)$, meaning: $x,y,z$ have the triadic relation $R_3$ (in intension);

$R_4(x,y,z,w)$, meaning: $x,y,z,w$ have the tetradic relation $R_4$ (in intension);

& so on *ad infinitum*. Logic does not know whether there are in fact $n$-adic relations (in intension); this is an empirical question. We know as an empirical fact that there are dyadic relations (in intension), because without them series would be impossible. But logic is not interested in this fact; it is concerned solely with the hypothesis of there being propositions of such-&-such a form. In certain cases, this hypothesis is itself of the form in question, or contains a part which is of the form in question; in these cases, the fact that the hypothesis can be framed proves that it is true. But even when a hypothesis occurs in logic, the fact that it can be framed does not itself belong to logic.

Plate 1. The HPF manuscript was revised as the Introduction to the Second Edition and Appendices A and B.


Symonds.

$\cancel{9}$ 7

89
$*\cancel{98}\cdot 16$ $\cancel{*120\cdot 823}$ $\vdash : \alpha \sim \varepsilon \, \mathrm{Cls\ induct}_3 \, . \, \gamma \, \varepsilon \, \mathrm{Cls\ induct}_3 \, . \supset . \, \exists ! \, \alpha - \gamma$

Dem.

$\mathrm{Hp} \, . \supset : (\exists \mu_3) : \Lambda \, \varepsilon \, \mu_3 : \beta \, \varepsilon \, \mu_3 \, . \supset_{\beta, y} . \, \beta \cup \iota ' y \, \varepsilon \, \mu_3 : \gamma \, \varepsilon \, \mu_3 \, . \, \alpha \sim \varepsilon \, \mu_3$ (1)

$\Lambda \, \varepsilon \, \mu_3 : \beta \, \varepsilon \, \mu_3 \, . \supset_{\beta, y} . \, \beta \cup \iota ' y \, \varepsilon \, \mu_3 : \gamma \, \varepsilon \, \mu_3 \, . \, \alpha \sim \varepsilon \, \mu_3 : \supset : \alpha \neq \Lambda \, . \, \Lambda \, \varepsilon \, \mu_3$

$\supset : \exists ! \, \alpha - \Lambda \, . \, \Lambda \varepsilon \mu_3$ (2)

$\exists ! \, \alpha - \beta \, . \, \alpha \subset \beta \cup \iota ' y \, . \supset . \, \alpha = \beta \cup \iota ' y$ (3)

$(3) \, . \supset :: \mathrm{Hp}(2) \, . \supset : \beta \, \varepsilon \, \mu_3 \, . \, \alpha \sim \varepsilon \, \mu_3 \, . \, \exists ! \, \alpha - \beta \, . \supset . \, \beta \cup \iota ' y \, \varepsilon \, \mu_3 \, . \, \alpha \neq \beta \cup \iota ' y \, . \, \exists ! \, \alpha - (\beta \cup \iota ' y)$ (4)

$(4) \, . \supset :. \, \mathrm{Hp}(2) \, . \supset : \beta \, \varepsilon \, \mu_3 \, . \, \exists ! \, \alpha - \beta \, . \supset . \, \beta \cup \iota ' y \, \varepsilon \, \mu_3 \, . \, \exists ! \, \alpha - (\beta \cup \iota ' y)$ (5)

$(2) . (5) \, . \supset \vdash :. \, \mathrm{Hp}(2) \, . \supset : \beta \, \varepsilon \, \mathrm{Cls\ induct}_3 \, . \supset . \, \beta \, \varepsilon \, \mu_3 \, . \, \exists ! \, \alpha - \beta$ (6)

$(1) . (6) \, . \supset$ Prop

17
$\cdot \cancel{824}$ $\vdash : \gamma \, \varepsilon \, \mathrm{Cls\ induct}_3 \, . \, \alpha \subset \gamma \, . \supset . \, \alpha \, \varepsilon \, \mathrm{Cls\ induct}_3$ [89
$*\cancel{98}\cdot 16$ $\cancel{*120\cdot 823}$. Transp]

~~$\cdot 825$ $\vdash : \exists ! \, N\mathrm{C\ induct}_3 - N\mathrm{C\ induct}_4 \, . \supset . \, N\mathrm{C\ induct}_4 \, \varepsilon \, \mathrm{Cls\ induct}_3$~~

~~[$*120 \cdot 822 \cdot 824$]~~

~~$\cdot 826$ $\vdash : \exists ! \, N\mathrm{C\ induct}_3 - N\mathrm{C\ induct}_4 \, . \supset . \, (\exists \mu_2) \, . \, N\mathrm{C\ induct}_4 = \mu_2$~~

~~[$*120 \cdot 81 \cdot 825$]~~

~~$\cdot 827$ $\vdash : \exists ! \, N\mathrm{C\ induct}_3 - N\mathrm{C\ induct}_4 \, . \supset . \, (\exists \mu_2) \, . - N\mathrm{C\ induct}_4 = \mu_2$~~

~~[$*120 \cdot 826$]~~

~~$\cdot 828$ $\vdash : \mu_2 \subset N\mathrm{C\ induct}_3 \, \exists ! \, \mu_2 \cap N\mathrm{C\ induct}_3 \, . \supset . \, \exists ! \, \mu_2 \cap N\mathrm{C\ induct}_3 - (+_c 1)'' \mu_2$~~

~~Dem. $\mu_2 \cap N\mathrm{C\ induct}_3 \subset (+_c 1)'' \mu_2 \, . \supset . \, (+_c 1)'' - \mu_2 \subset - \mu_2 \, . \, 0 \sim \varepsilon \, \mu_2 \, .$~~

~~$\supset . \, N\mathrm{C\ induct}_3 \subset - \mu_2$ (1)~~

~~(1) . Transp . $\supset$ . Prop~~

Plate 2.  *89.16 in Appendix B. Line (3) is the error identified by Gödel.