Assignment 1 - Solving a linear system in python

Neural Networks

Due date: 14-10-2024

10 points

In this homework, you will familiarize yourself with key linear algebra concepts and Python programming by solving a system of linear equations. You will explore multiple methods for solving such systems, including Cramer's rule and matrix inversion. By the end of this assignment, you will have a good understanding of how to represent and manipulate matrices and vectors in Python.

We begin with the following system of 3 linear equations with 3 unknowns:

$$2x + 3y - z = 5$$

$$x - y + 4z = 6$$

$$3x + y + 2z = 7$$

This system can be vectorized in the following form:

$$A \cdot X = B$$

where:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 4 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Considerations

- do not use any linear algebra framework such as *numpy*
- use python lists as data structures for matrices and vectors
- experiment with other values for the coefficients and free terms

1 Parsing the System of Equations (1 point)

The first task is to implement a Python script that reads a system of linear equations from a text file and parses it into a matrix A and a vector B. You will use the input format described below to extract the coefficients for A and B.

Input File Format:

```
2x + 3y - z = 5
x - y + 4z = 6
3x + y + 2z = 7
```

Note that the coefficients are always in the order x, y and z and the terms are always space separated

2 Matrix and Vector Operations (5 points)

Once you have successfully parsed the matrix and vector, complete the following exercises to manipulate and understand basic matrix and vector operations. Write Python functions for each of these tasks:

1. **Determinant**: Write a function to compute the determinant of matrix A. Recall one of the formulae for the determinant of a $3x3$ matrix:

   $$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

2. **Trace**: Compute the sum of the elements along the main diagonal of matrix A. For a matrix A, this is:

   $$\text{Trace}(A) = a_{11} + a_{22} + a_{33}$$

3. **Vector norm**: Compute the Euclidean norm of vector B, which is:

   $$||B|| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

4. **Transpose of matrix**: Write a function to compute the transpose of matrix A. The transpose of a matrix A is obtained by swapping its rows and columns.

5. **Matrix-vector multiplication**: Write a function that multiplies matrix A with vector B.

3 Solving using Cramer's Rule (1 points)

Now that you have explored basic matrix operations, solve the system of linear equations using Cramer's rule.

Cramer's Rule: Cramer's rule allows you to solve for each unknown x, y, and z using determinants. For example:

$$x = \frac{\det(A_x)}{\det(A)}, \quad y = \frac{\det(A_y)}{\det(A)}, \quad z = \frac{\det(A_z)}{\det(A)}$$

where A_x, A_y, and A_z are matrices formed by replacing the respective column of matrix A with vector B.

4 Solving using Inversion (3 points)

Finally, solve the system by computing the inverse of matrix A and multiplying it by vector B.

$$A \cdot X = B \rightarrow X = A^{-1} \cdot B$$

Adjugate Method for Matrix Inversion: To find the inverse of matrix A, you can use the adjugate method:

$$A^{-1} = \frac{1}{\det(A)} \times \operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the adjugate (or adjoint) matrix, which is the transpose of the cofactor matrix of A.

Cofactor Matrix: The cofactor matrix is a matrix where each element is replaced by its cofactor. The cofactor of an element a_{ij} is given by:

$$(-1)^{i+j} \times \det(M_{ij})$$

where M_{ij} is the minor of element a_{ij}, which is the matrix obtained by removing the i-th row and j-th column from matrix A.

Bonus: Can you see any similarities between the cofactor and the provided formula for the determinant of the matrix A?