CM12006: Problem Sheet

1. $$\begin{cases} 3X - Y = 8 \\ -2X + Y + Z = 9 \\ 2X - Y + 4Z = -5 \end{cases} \qquad A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

$$A_{\text{x}} = \begin{pmatrix} 8 & -1 & 0 \\ 9 & 1 & 1 \\ -5 & -1 & 4 \end{pmatrix} \qquad A_{\text{y}} = \begin{pmatrix} 3 & 8 & 0 \\ -2 & 9 & 1 \\ 2 & -5 & 4 \end{pmatrix} \qquad A_{\text{z}} = \begin{pmatrix} 3 & -1 & 8 \\ -2 & 1 & 9 \\ 2 & -1 & -5 \end{pmatrix}$$

According to Cramer's rule, if $detA \neq 0$, then:

$$X = \frac{detA_{\text{x}}}{detA} \qquad Y = \frac{detA_{\text{y}}}{detA} \qquad Z = \frac{detA_{\text{z}}}{detA}$$

$$detA = 3 * 1 * 4 + 0 + 2 * (-1) * 1 - 0 - 3 * (-1) * 1 - 4 * (-2) * (-1) = 5$$

$$detA_{\text{x}} = 8 * 1 * 4 + 0 + 1 * (-1) * (-5) - 0 - 9 * (-1) * 4 - 8 * (-1) * 1 = 81$$

$$detA_{\text{y}} = 3 * 9 * 4 + 0 + 2 * 8 * 1 - 0 - 8 * (-2) * 4 - 1 * 3 * (-5) = 203$$

$$detA_{\text{z}} = 3 * 1 * (-5) + 8 * (-2) * (-1) + 2 * 9 * (-1) - 2 * 1 * 8 - (-2) * (-1) * (-5) - 9 * (-1) * (3) = 4$$

$$X = \frac{81}{5} \qquad Y = \frac{203}{5} \qquad Z = \frac{4}{5}$$

2. $A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}$

$$detA = 3 * 1 * 4 + 0 + 2 * (-1) * 1 - 0 - 3 * (-1) * 1 - 4 * (-2) * (-1) = 5$$

Let $C^T = \text{adj}(A)$ be calculated as follows:

$$C^T = \text{adj}(A) = \begin{pmatrix} (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} & (-1)^{1+2} \cdot \det \begin{pmatrix} -1 & 0 \\ -1 & 4 \end{pmatrix} & (-1)^{1+3} \cdot \det \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \\ (-1)^{2+1} \cdot \det \begin{pmatrix} -2 & 1 \\ 2 & 4 \end{pmatrix} & (-1)^{2+2} \cdot \det \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix} & (-1)^{2+3} \cdot \det \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} \\ (-1)^{3+1} \cdot \det \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} & (-1)^{3+2} \cdot \det \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} & (-1)^{3+3} \cdot \det \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \end{pmatrix}$$

Simplifying, we have:

$$C^T = \text{adj}(A) = \begin{pmatrix} +\det \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} & -\det \begin{pmatrix} -1 & 0 \\ -1 & 4 \end{pmatrix} & +\det \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \\ -\det \begin{pmatrix} -2 & 1 \\ 2 & 4 \end{pmatrix} & +\det \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix} & -\det \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} \\ +\det \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} & -\det \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} & +\det \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \end{pmatrix}$$

$$A^{-1} = \frac{1}{detA} * C^T = \frac{1}{5} * \begin{pmatrix} 5 & 4 & \text{-1} \\ 10 & 12 & \text{-3} \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{5} & -\frac{1}{5} \\ 2 & \frac{12}{5} & -\frac{3}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \frac{4}{5} & -\frac{1}{5} \\ 2 & \frac{12}{5} & -\frac{3}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 8 \\ 9 \\ \text{-5} \end{pmatrix} = \begin{pmatrix} 1*8 + 9 * \frac{4}{5} \text{ - } \frac{1}{5}*(\text{-5}) \\ 2*8 + 9 * \frac{12}{5} \text{ - } \frac{3}{5}*(\text{-5}) \\ 0 + 9 * \frac{1}{5} + \frac{1}{5} * (\text{-5}) \end{pmatrix} = \begin{pmatrix} \frac{81}{5} \\ \frac{203}{5} \\ \frac{4}{5} \end{pmatrix}$$

3. $A = \left(\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 3 & 1 & -1 & 1 \end{array} \right)$

i. $1 \cdot X + 3 = 0, \quad X = -3$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 0 & 4*(-3)+1 & 1*(-3)+(-4) & 1*(-3)+1 \end{array} \right)$$

ii. $X = 1$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 6 & 3 & 1 \\ 0 & -11 & -4 & -2 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 6 & 3 & 1 \\ -11 & -4 & -\frac{1}{6} \end{array} \right)$$

iii. $6 \cdot X = 11, \quad X = \dfrac{11}{6}$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 6 & 3 & 1 \\ 0 & -11 & -4 & -2 \end{array} \right)$$

iv. A in echelon form $= \left(\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 6 & 3 & 1 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{6} \end{array} \right)$

$$\begin{cases} X_1 + 4X_2 + X_3 = 1, \\ 6X_2 + 3X_3 = 1, \\ \frac{3}{2}X_3 = -\frac{1}{6} \end{cases} \qquad \begin{cases} X_1 + 4X_2 + X_3 = 1, \\ 6X_2 - \frac{1}{3} = 1, \\ X_3 = -\frac{1}{9} \end{cases} \qquad \begin{cases} X_3 = 1 + \frac{1}{9} - \frac{8}{9} = \frac{2}{9}, \\ X_2 = \frac{2}{9}, \\ X_3 = -\frac{1}{9} \end{cases}$$

4. Identify Non-Zero Rows: In the REF or RREF, the rows that are not entirely zero indicate the presence of a leading 1 or any non-zero number in the case of REF and represent linearly independent vectors.

$$A = \begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ -2 & 4 & 2 & 4 \end{pmatrix}$$

Transformation steps to reach RREF:

1. **Initial operation**: $-1 \cdot X - 2 = 0$, giving $X = -2$.

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ 0 & 2 \cdot (-2) + 4 & 1 \cdot (-2) + 2 & 2 \cdot (-2) + 4 \end{pmatrix}$$

2. **Next step**: $-1 \cdot X + 1 = 0$, solving for X gives $X = 1$.

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. **Operation**: $2 \cdot \text{Row1} + \text{Row2}$.

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4. **Operation**: $\text{Row3} - \text{Row2}$.

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. **Operation**: $-\frac{1}{5} \cdot \text{Row2}$.

$$\begin{pmatrix} -1 & 2 & 1 & 2 \\ 0 & -1 & -\frac{2}{5} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

6. **Operation**: $\text{Row1} \cdot (-1)$.

$$\begin{pmatrix} 1 & -2 & -1 & -2 \\ 0 & -1 & -\frac{2}{5} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

7. **Operation**: $\text{Row1} - 2 \cdot \text{Row2}$.

$$\begin{pmatrix} 1 & 0 & -\frac{1}{5} & 0 \\ 0 & -1 & -\frac{2}{5} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

8. **Final operation**: $(-1) \cdot \text{Row2}$.

$$\begin{pmatrix} 1 & 0 & -\frac{1}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank is 2.

5.

$$\begin{cases} 3X_1 - 2X_2 + 3X_3 - X_4 = 1, \\ X_2 + X_4 = 3, \\ X_1 + X_2 - 2X_3 + 4X_4 = 1. \end{cases}$$

The augmented matrix A representing this system is:

$$A = \begin{pmatrix} 3 & -2 & 3 & -1 & | & 1 \\ 0 & 1 & 0 & 1 & | & 3 \\ 1 & 1 & -2 & 4 & | & 1 \end{pmatrix}$$

Performing Gaussian elimination to solve:

(a) **Operation:** Swap Row1 and Row2 to get the leading one in the upper left corner.

$$\begin{pmatrix} 1 & 1 & -2 & 4 & | & 1 \\ 0 & 1 & 0 & 1 & | & 3 \\ 3 & -2 & 3 & -1 & | & 1 \end{pmatrix}$$

(b) **Operation:** Row3 = Row3 - 3Row1 to eliminate X_1 from Row3.

$$\begin{pmatrix} 1 & 1 & -2 & 4 & | & 1 \\ 0 & 1 & 0 & 1 & | & 3 \\ 0 & -5 & 9 & -13 & | & -2 \end{pmatrix}$$

(c) **Operation:** Row1 = Row1 - Row2 to make X_2 coefficient zero in Row1.

$$\begin{pmatrix} 1 & 0 & -2 & 3 & | & -2 \\ 0 & 1 & 0 & 1 & | & 3 \\ 0 & -5 & 9 & -13 & | & -2 \end{pmatrix}$$

(d) **Operation:** Row3 = Row3 + 5Row2 to eliminate X_2 from Row3.

$$\begin{pmatrix} 1 & 0 & -2 & 3 & | & -2 \\ 0 & 1 & 0 & 1 & | & 3 \\ 0 & 0 & 9 & -8 & | & 13 \end{pmatrix}$$

(e) **Operation:** Row3 $\times \frac{1}{9}$ to normalize the coefficient of X_3 in Row3.

$$\begin{pmatrix} 1 & 0 & -2 & 3 & | & -2 \\ 0 & 1 & 0 & 1 & | & 3 \\ 0 & 0 & 1 & -\frac{8}{9} & | & \frac{13}{9} \end{pmatrix}$$

(f) **Operation:** Row1 = Row1 + 2Row3 to make X_3 coefficient zero in Row1.

$$\begin{pmatrix} 1 & 0 & 0 & \frac{11}{9} & | & \frac{8}{9} \\ 0 & 1 & 0 & 1 & | & 3 \\ 0 & 0 & 1 & -\frac{8}{9} & | & \frac{13}{9} \end{pmatrix}$$

Given one solution for $X_4 = 0$, the solutions are:

$$\begin{cases} X_1 = \frac{8}{9} - \frac{11}{9}X_4, \\ X_2 = 3 - X_4, \\ X_3 = \frac{13}{9} + \frac{8}{9}X_4. \end{cases}$$

Thus, the vector form of the solution is:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\frac{11}{9} \\ -1 \\ \frac{8}{9} \end{pmatrix} X_4 + \begin{pmatrix} \frac{8}{9} \\ 3 \\ \frac{13}{9} \end{pmatrix}.$$