Abstraction of Continuous-time Systems Based on Feedback Controllers and Mixed Monotonicity

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***Abstract*—In this paper, we consider the problem of computation of efficient symbolic abstractions for continuous-time control systems. The new abstraction algorithm builds symbolic models with the same number of states but fewer transitions in comparison to the one produced by the standard algorithm. At the same time, the new abstract system is at least as controllable as the standard one. The proposed algorithm is based on the solution of a region-to-region control synthesis problem. This solution is formally obtained using the theory of viscosity solutions of the dynamic programming equation and the theory of differential equations with discontinuous righthand side. In the new abstraction algorithm, the symbolic controls are essentially the feedback controllers that solve this control synthesis problem. The improvement in the number of transitions is achieved by reducing the number of successors for each symbolic control. For a certain class of control systems, with a suitable set of discretization parameters, the new algorithm may even produce deterministic abstract systems or systems with a singleton input alphabet. The approach is illustrated by examples that compare the two abstraction algorithms.**

I. INTRODUCTION

Synthesis of feedback controllers for nonlinear dynamical systems is one of the key problems in control theory. Formal methods approach suggests splitting this problem into several subproblems with the first one being the construction of a symbolic abstract system (or abstraction), which is usually a system with finite number of states and transitions (see [1], [2]). These abstractions capture the behavior of the original system in such a way that a controller built to solve the control problem for an abstract system can be refined to a respective controller for the original system. The notions of an alternating simulation relation, an approximate alternating simulation relation and a feedback refinement relation are used to formalize such properties.

There are several known methods of abstraction. Some of those methods require the control system to satisfy certain sets of conditions to be applicable. One of the more general methods is based on partitioning of the state space and on discretizing the control space [3]. Due to its generality and popularity we will refer to this method as "standard" throughout the paper. This abstraction method utilizes the notion of alternating simulation relation and can be applied to a very wide class of systems but is especially efficient when the reachable sets originated from partition elements can be efficiently computed or approximated (see [4], [5], [6], [7], [8]). One of such types of control systems is mixed monotone systems [9]. Mixed monotonicity is a very mild assumption on the system. Essentially, every practically meaningful system is mixed monotone [10]. The trick is in that to utilize mixed monotonicity, one has to compute the corresponding decomposition function, which is a whole separate problem. In particular, evaluating a so called tight decomposition function is equivalent to solving a finite-dimensional optimization problem. Over the course of this paper we assume that the decomposition function is known. In Section V we calculate the decomposition function analytically for a practical example.

The method we present here also utilizes the partitioning of the state space. Unlike in the standard algorithm, each symbolic control in this method corresponds to a certain feedback controller for the original system as opposed to an open-loop control function (see e.g. [11]). Intuitively, we use a feedback controller such that the interval approximation of the reachable set (of the closed-loop system) from a partition element is the smallest in size or, more precisely, that it is minimal with respect to inclusion in a certain class $\mathscr{A}$ of interval sets for which we are able to construct the respective controllers. That way we expect to have fewer transitions correponding to a single symbolic control. The considered class $\mathscr{A}$ of interval sets has a description in terms of viscosity solutions of the related Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation (see [12], [13]). These intervals can be also described by certain differential equations with discontinuous right-hand side (see [14]). We utilize both frameworks to establish the existence and uniqueness of the minimal element as well as the method of its practical construction. Similarly to the standard algorithm, the new algorithm requires computing a solution of a system of ODEs for each symbolic state and control. The system of ODEs has the dimension $3n_x$ as opposed to $2n_x$ in the standard algorithm where n_x is the dimension of the control system. For a certain subclass of systems the new algorithm may produce deterministic abstractions or abstractions with a singleton input alphabet. These properties may be especially beneficial when considering synthesis problems for complex specifications such as those that are given by LTL formulas or non-deterministic Büchi automata [2].

The problem of polytope-to-polytope control for nonlinear control systems in relation with symbolic control has been considered extensively in the literature (see [15], [16], [17], [18], [19]). It has been shown (see e.g. [20]) that for controllability reasons it is sometimes important to consider "flat" partition elements. Moreover, depending on the system and the partition element, a minimal reachable set may be also

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flat. In the deterministic case, each interval even shrinks to a single point at the corresponding sampling time τ. These considerations pose the main technical difficulty in the proof of correctness of our construction.

The paper is structured as follows. In Section II we define the problem of calculating the minimal (in a certain class $\mathscr{A}$) target set to which we can control the system from a given initial set (Problem 1). After that we give a short introduction into mixed monotone systems. The main result of Section III.A suggests that every target set in the considered class $\mathscr{A}$ corresponds to a viscosity supersolution (upper solution) of the related backward HJBI equation. Once we have a supersolution, the feedback controller can be constructed (or verified) using the idea of extremal aiming (see, e.g. [21]). In Section III.B we first obtain the description of $\mathscr{A}$ in terms of differential equations with discontinuous righthand side. Then we prove the existence and uniqueness of the minimal element of class $\mathscr{A}$. Finally, in Section III.C we define the controller and prove that it solves Problem 1.

In Section IV we utilize the controllers obtained in Section III to define the new abstraction. Each symbolic control input v is associated with a particular controller $u(t,x)$ (instead of an open-loop or a constant control as in the standard algorithm). The transitions from a state q with a control v in the abstract system are enabled for every partition element that intersects the respective reachable set over-approximation. Then in Theorems 5 and 6 we consider two special subclasses of systems: for the first one there are deterministic abstractions, for the other one there are abstractions without inputs. In Section V.A we comment on the numerical method utilized to solve the aforementioned ODE system. In Sections V.B and V.C we compare the standard and the new abstraction algorithms on two examples: a temperature regulation problem and an autonomous boat docking problem. For the sake of clarity of presentation, the proofs are put into Appendix section.

The present paper is an extension of work [22]. The results of Theorems 1-4 were established there for a class of monotone systems. The numerical simulations for the temperature regulation example also first appeared in that paper. In the present paper, we extend the results to a much wider class of control systems, introduce new results on deterministic abstractions and abstractions with a singleton input set, address numerical issues that arise in interval computations, update the simulations for the temperature regulation example, and investigate the new autonomous boat example.

Notations: Let $\operatorname{card}(A)$ be the cardinality of A. For a vector function f: $A \to \mathbb{R}^n$, in expressions $\max_{a\in A} f(a)$ and $\min_{a\in A} f(a)$ the maximum and minimum operators are applied to each component independently. For a scalar function f, expressions $\operatorname{Arg\,min}_{a\in A} f(a)$ and $\operatorname{Arg\,max}_{a\in A} f(a)$ denote the sets of all minimizers and maximizers, respectively, of function f over the set A. For a function $f : X \to Y$ and a set $X' \subseteq X$, $f(X')$ denotes the image $\{y \in Y \mid \exists x \in X' : \ y = f(x)\}$. For $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_i |x_i|$ is the infinity norm. For vectors $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, let $z = [x; y] \in \mathbb{R}^{n_1+n_2}$ denote their concatenation. Let $d(x, X)$ denote the distance $\inf_{z\in X} \|x-z\|_\infty$ between $x \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. Given vectors $x,\, x' \in \mathbb{R}^n$, $x \preceq x'$ stands for $x_i \le x'_i$ for all $i = 1, \ldots, n$. Using this partial order, we define multi-dimensional interval sets as follows: for $\underline{x}, \overline{x} \in \mathbb{R}^n$, $[\underline{x}, \overline{x}] = \{x \mid x \succeq \underline{x},\ x \preceq \overline{x}\}$. For a compact set $W \subset \mathbb{R}^{n_w}$, the space of all Lebesgue measurable functions $w(\cdot)$ on $[0, T]$ such that $w(t) \in W$ a.e. is denoted by $L^\infty([0, T], W)$.

II. Interval-to-interval controller synthesis problem

A. Problem statement

Consider a nonlinear system of the following type:

$$\dot{x} = f(t, x, u, w), \quad t \in [0, T]. \tag{1}$$

Here $x \in \mathbb{R}^{n_x}$ is the state, $u \in U = [\underline{u}, \overline{u}] \subset \mathbb{R}^{n_u}$ is the control and $w \in W = [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$ is the disturbance.

The set of admissible open-loop controls is $\mathcal{U}(t, \tau) = L^\infty([t, \tau], U)$. The set of admissible realizations of the disturbance is $\mathcal{W}(t, \tau) = L^\infty([t, \tau], W)$. Let $\mathbf{x}(t; \tau, x, u(\cdot), w(\cdot))$ denote a trajectory of the system satisfying the initial condition $x(\tau) = x$ and corresponding to the control $u(\cdot)$ and disturbance $w(\cdot)$. Finally, let $X^{u(\cdot)}(t; t_0, X^0)$ denote the reachable set

$$\{x \in \mathbb{R}^{n_x} \mid \exists x^0 \in X^0,\ \exists w(\cdot) \in \mathcal{W}(t_0, t) : \\ \mathbf{x}(t; t_0, x^0, u(\cdot), w(\cdot)) = x\}.$$

Assumption 1: The conditions on the considered class of systems are summarized in the following.

1) Function f is continuous in (t, x, u, w), globally Lipschitz in (x, u) uniformly in (t, w) with a constant L:
$$\|f(t, x^1, u^1, w) - f(t, x^2, u^2, w)\|_\infty \le \\ L \left[\|x^1 - x^2\|_\infty + \|u^1 - u^2\|_\infty\right].$$
2) Isaacs minimax condition is satisfied: for all $p \in \mathbb{R}^{n_x}$
$$\min_{u\in U} \max_{w\in W} \langle p, f(t, x, u, w)\rangle = \max_{w\in W} \min_{u\in U} \langle p, f(t, x, u, w)\rangle; \tag{2}$$
3) For all $(t, x, w) \in [0, T] \times \mathbb{R}^{n_x} \times W$, $f(t, x, U, w)$ is an interval set:
$$f(t, x, U, w) = [\min_{u\in U} f(t, x, u, w),\ \ \max_{u\in U} f(t, x, u, w)].$$

The first assumption is quite standard. It holds for almost any practically meaningful system. The second assumption means that there is no information advantage in the respective differential game: at each position (t, x) it does not matter whether the control or the disturbance is chosen first. It will be instrumental below in the proofs of Lemmas 4 and 5 and the subsequent results. The last assumption is more specific. However, when it does not hold it is always possible to underapproximate sets $f(t, x, U, w)$ with intervals (see autonomous boat example in Section V). Here we assume that it has been already done for system (1).

Let us consider a class $\mathscr{A}$ of target interval sets X^1 that we will define below. For a controller u: $[0, T] \times \mathbb{R}^{n_x} \to U$ and a disturbance realization $w(\cdot)$, we will consider the closed-loop system:

$$\dot{x} = f(t, x, u(t, x), w(t)), \quad t \in [0, T]. \tag{3}$$

Problem 1: Given a system (1) satisfying Assumption 1, an initial interval set $X^0 = [\underline{x}^0, \overline{x}^0] \subset \mathbb{R}^{n_x}$ and a time horizon $T > 0$, find a minimal by inclusion set X^1 in a class $\mathscr{A}$ and a controller $u(t,x)$ such that

- the closed-loop system has a solution for all initial data and all admissible disturbances and every solution exists on the whole interval $[0,T]$;
- all trajectories of the closed-loop system originated from X^0 at $t = 0$ reach X^1 at $t = T$.

Since the inclusion relation $\subseteq$ induces only a partial order on subsets of $\mathbb{R}^{n_x}$, a minimal by inclusion set X^1 may be not unique in general. However, we will prove that it is in the class $\mathscr{A}$ considered below.

Following [13], a map γ: $\mathcal{W}(t,T) \longrightarrow \mathcal{U}(t,T)$ is called *a progressive strategy* if for any two disturbance realizations $w(\cdot), \tilde{w}(\cdot) \in \mathcal{W}(t,T)$ with $w(s) = \tilde{w}(s)$ for $s \in [t,\tau]$ it follows that $\gamma(w)(s) = \gamma(\tilde{w})(s)$ for $s \in [t,\tau]$.

Let us now introduce the type of classes $\mathscr{A}$ of target sets under consideration. Fix a trajectory $\hat{x}(\cdot) = \mathbf{x}(\cdot; 0, x^0, \hat{u}(\cdot), \hat{w}(\cdot))$ of system (1) such that $x^0 \in X^0$. Consider a class $\mathscr{A}^{\hat{x}(\cdot)}$ consisting of all interval sets X^1 for which there exists a Lipschitz continuous interval-valued map $X(t)$ satisfying the following properties:

(a) $X(0) = X^0$, $X(T) = X^1$;
(b) for all $t \in [0,T]$, $x \in X(t)$ there exists a progressive strategy γ such that for all $w(\cdot) \in \mathcal{W}(t,T)$, we have $\mathbf{x}(\tau; t, x, \gamma(w)(\cdot), w(\cdot)) \in X(\tau)$ for all $\tau \in [t,T]$;
(c) $\hat{x}(t) \in X(t)$ for all $t \in [0,T]$.

Remark 1: Property (b) is sometimes called *weak invariance* (also known as *robust control invariance*) of $X(t)$ with respect to differential inclusion $\dot{x}_i \in f_i(t,x,U,w(t))$. This property is what allows us to synthesize a controller for the target set X^1. Property (c) implies the following: consider $X(t) = [\underline{x}(t), \overline{x}(t)]$ and let $\overline{x}_j(\tau) = \hat{x}_j(\tau)$ for some j and $\tau \in [0,T)$. If $\overline{x}_j(\cdot)$ and $\hat{x}_j(\cdot)$ are differentiable at τ then $\dot{\hat{x}}_j(\tau) \le \dot{\overline{x}}_j(\tau)$. Similarly, one may prove that if $\underline{x}_j(\tau) = \hat{x}_j(\tau)$ then $\dot{\underline{x}}_j(\tau) \le \dot{\hat{x}}_j(\tau)$ if both derivatives exist at τ.

B. Mixed monotone decomposition

The notion of mixed monotonicity ([9], [8], [10]), provides a clear and consize way of formulating the main results of the present paper. For this reason, let us introduce the notion of a decomposition function.

Definition 1: Function g: $[0,T] \times \mathbb{R}^{2n_x} \times U^2 \times W^2 \to \mathbb{R}^{n_x}$ is called a *decomposition function* for f if

1) $g(t,[x;x],[u;u],[w;w]) = f(t,x,u,w)$;
2) $g_i(t,[x;y],\theta,\omega)$ is nondecreasing in x_j, nonincreasing in y_j when $i \neq j$;
3) $g_i(t,z,[u^1;u^2],[w^1;w^2])$ is nondecreasing in u^1 and w^1, nonincreasing in u^2 and w^2.

We call f a mixed monotone function and system (1) a mixed monotone system if there exists such decomposition function g.

Definition 2: Let f be a mixed monotone function and g be a decomposition of f. Decomposition function g is called *tight* if for all $i \in \{1,\dots,n_x\}$, for all $t \in [0,T]$, for all $\underline{x}, \overline{x} \in \mathbb{R}^{n_x}$, for all $u^1, u^2 \in U$, and all $w^1, w^2 \in W$ such that $\underline{x} \preceq \overline{x}$, $u^1 \preceq u^2$, $w^1 \preceq w^2$, it follows that

$$\begin{gathered} g_i(t,[\overline{x};\underline{x}],[u^1;u^2],[w^2;w^1]) = \\ \max_{x\in[\underline{x},\overline{x}]} \min_{u\in[u^1,u^2]} \max_{w\in[w^1,w^2]} f_i(t,x,u,w), \\ g_i(t,[\underline{x};\overline{x}],[u^2;u^1],[w^1;w^2]) = \\ \min_{x\in[\underline{x},\overline{x}]} \max_{u\in[u^1,u^2]} \min_{w\in[w^1,w^2]} f_i(t,x,u,w). \end{gathered}$$

In particular, when g is a tight decomposition function, $[g(t,[\underline{x};\overline{x}],[u;u],[w^1;w^2]), g(t,[\overline{x};\underline{x}],[u;u],[w^2;w^1])]$ is the minimal (by set inclusion) interval that contains $f(t,[\underline{x},\overline{x}],u,W)$. One situation when a tight decomposition function is trivially available is the case of *monotone systems* [23]. System (1) is called monotone if for all i, f_i is nondecreasing in x_j, $j \neq i$ and in (u,w). Then $g(t,[x;y],[u^1;u^2];[w^1;w^2]) = f(t,x,u^1,w^1)$ is a decomposition function.

Let us now recall the explicit construction of a tight decomposition function from [10]. For $x, y \in \mathbb{R}$ and a function h: $\mathbb{R} \to \mathbb{R}$, define

$$\mathrm{opt}_\alpha^{(x,y)} h(\alpha) = \begin{cases} \min_{\alpha\in[x,y]} h(\alpha), & x \le y, \\ \max_{\alpha\in[y,x]} h(\alpha), & x > y. \end{cases}$$

Then function g defined by

$$\begin{gathered} g_i(t,[x^1;x^2],[u^1;u^2],[w^1;w^2]) = \\ \mathrm{opt}_{x_1}^{(x_1^1,x_1^2)} \dots \mathrm{opt}_{x_{n_x}}^{(x_{n_x}^1,x_{n_x}^2)} \mathrm{opt}_{u_1}^{(u_1^1,u_1^2)} \dots \mathrm{opt}_{u_{n_u}}^{(u_{n_u}^1,u_{n_u}^2)} \\ \mathrm{opt}_{w_1}^{(w_1^1,w_1^2)} \dots \mathrm{opt}_{w_{n_w}}^{(w_{n_w}^1,w_{n_w}^2)} f_i(t,x,u,w), \quad i = 1,\dots,n_x \end{gathered} \tag{4}$$

is a tight decomposition function of f. Thus, for any function f satisfying condition 1) of Assumption 1, there exists a tight decomposition function g. Moreover, g also satisfies this condition.

In present paper, we are only interested in decomposition functions (not necessarily tight) that satisfy this assumption.

Assumption 2: Let g be a decomposition function of f. Function g is continuous in (t,z,θ,ω), globally Lipschitz in (z,θ) uniformly in (t,ω) with a constant L:

$$\begin{gathered} \|g(t,z^1,\theta^1,\omega) - g(t,z^2,\theta^2,\omega)\|_\infty \le \\ L\left[\|z^1 - z^2\|_\infty + \|\theta^1 - \theta^2\|_\infty\right]. \end{gathered}$$

The notion of mixed monotonicity allows for simple calculation of over-approximations of forward reachable sets. Indeed, let $X^0 = [\underline{x}^0, \overline{x}^0]$ and let us introduce functions ζ_i that map $\mathbb{R}^{2n_x}$ into itself:

$$(\zeta_i(x,y)))_j = \begin{cases} x_j, & j \le n_x, \\ y_j, & j > n_x, \ j \neq n_x + i, \\ x_i, & j = n_x + i. \end{cases}$$

Let us also define the following convenient notations that we will use throughout the paper:

$$\overline{\theta} = [\overline{u};\underline{u}], \quad \underline{\theta} = [\underline{u};\overline{u}], \quad \overline{\omega} = [\overline{w};\underline{w}], \quad \underline{\omega} = [\underline{w};\overline{w}].$$

Now consider a system of equations $(i = 1,\dots,n_x)$

$$\begin{aligned} \dot{\overline{x}}_i &= g_i(t,\zeta_i(\overline{x},\underline{x}),[\hat{u};\hat{u}],\overline{\omega}), \quad \overline{x}_i(0) = \overline{x}_i^0 \\ \dot{\underline{x}}_i &= g_i(t,\zeta_i(\underline{x},\overline{x}),[\hat{u};\hat{u}],\underline{\omega}), \quad \underline{x}_i(0) = \underline{x}_i^0. \end{aligned} \tag{5}$$

Denoting the components of its solution as $\overline{\mathbf{x}}(t;[\underline{x}^0,\overline{x}^0],\hat{u})$ and $\underline{\mathbf{x}}(t;[\underline{x}^0,\overline{x}^0],\hat{u})$, one may observe that (5) is monotone with respect to state $(\overline{x},-\underline{x})$ and input $(\overline{w},-\underline{w})$. Therefore, by applying Theorem 1 of [23], the interval

$$X^+(t)=[\underline{\mathbf{x}}(t;[\underline{x}^0,\overline{x}^0],\hat{u}),\overline{\mathbf{x}}(t;[\underline{x}^0,\overline{x}^0],\hat{u})] \tag{6}$$

is an over-approximation of the forward reachable set $X^{\hat{u}}(t;0,X^0)$.

The interval $X^+(t)$ introduced in (6) gives an example of an interval-valued map that satisfies properties (a), (b), (c) of the class $\mathscr{A}^{\hat{x}(\cdot)}$. It corresponds to a constant strategy $\gamma(w)(\cdot)=\hat{u}$.

It is known that the problem of controller synthesis for a reachability specification can be solved by considering the corresponding problem of dynamic optimization (see [21], [24]). Namely, given a supersolution of the backward Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, a reachability controller can be obtained, for example, by utilizing the idea of extremal aiming. With this in mind, let us formally translate our description of the problem into the Hamilton-Jacobi setting.

III. Solution of the synthesis problem

In this section we provide the solution to Problem 1. From this point onward we consider Assumptions 1 and 2 being satisfied.

A. Preliminaries on the Hamiltonian formalism

Consider an arbitrary initial set X^0 and let us represent it as a sublevel set of some function $\sigma(\cdot)$:

$$X^0=\{x\in\mathbb{R}^{n_x}\mid\sigma(x)\le 0\}.$$

Similarly, given an arbitrary target set X^1, let us represent it as a sublevel set of some other function $\psi(\cdot)$:

$$X^1=\{x\in\mathbb{R}^{n_x}\mid\psi(x)\le 0\}.$$

Consider now the HJBI equation

$$V_t+H(t,x,V_x)=0 \tag{7}$$

where $H(t,x,p)$ is given by the expression

$$\min_{u\in U}\max_{w\in W}\big\langle p,f(t,x,u,w)\big\rangle.$$

Let us now remind precisely the definitions of viscosity solutions in the considered cases (see [12], [13]). For equation (7) considered in forward time we have

- A function V is a forward viscosity subsolution of (7) if and only if for all $(t,x)\in(0,T]\times X$
$$q+H(t,x,p)\le 0\quad\forall(q,p)\in D^+V(t,x); \tag{8}$$
- A function V is a forward viscosity supersolution of (7) if and only if for all $(t,x)\in(0,T]\times X$
$$q+H(t,x,p)\ge 0\quad\forall(q,p)\in D^-V(t,x); \tag{9}$$
- V is a forward viscosity solution if it is both a sub- and a supersolution.

Here $D^+V(t,x)$ and $D^-V(t,x)$ are the Dini superdifferential and the Dini subdifferential of V at (t,x).

For equation (7) considered in backward time we have

- A function V is a backward viscosity subsolution of (7) if and only if for all $(t,x)\in[0,T)\times X$
$$q+H(t,x,p)\ge 0\quad\forall(q,p)\in D^+V(t,x); \tag{10}$$
- A function V is a backward viscosity supersolution of (7) if and only if for all $(t,x)\in[0,T)\times X$
$$q+H(t,x,p)\le 0\quad\forall(q,p)\in D^-V(t,x); \tag{11}$$
- V is a backward viscosity solution if it is both a sub- and a supersolution.

As mentioned above, we may obtain a controller, which steers system (1) to X^1 at $t=T$, by computing a supersolution (or the actual solution) of equation (7) with the terminal condition

$$V(T,x)=\psi(x) \tag{12}$$

backwards in time. To guarantee that every point of X^0 is controllable, condition $V(0,x)\le\sigma(x)$ for all $x\in\mathbb{R}^{n_x}$ must be satisfied.

However, since X^1 is an unknown part of the solution of Problem 1, we have to employ another approach. Intuitively, one may try to consider equation (7) forward in time with the initial condition

$$V(0,x)=\sigma(x) \tag{13}$$

and put

$$X^1=\{x\in\mathbb{R}^{n_x}\mid V(T,x)\equiv\psi(x)\le 0\}.$$

In general, the forward solution V of (7), (13) is not even a backward supersolution of (7), (12). However, for system (1) the forward subsolutions, which we construct below, turn out to be backward supersolutions indeed.

The next lemma and the following proposition show the connection between Problem 1 and the HJBI equation (7).

Lemma 1: Consider a continuous set-valued map $X(t)$, $t\in[0,T]$ with closed values. $X(t)$ satisfies property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ if and only if the function

$$V(t,x)=e^{-Lt}d(x,X(t))$$

is a backward supersolution of equation (7).

Proposition 1: Let the assumptions of Lemma 1 hold. If $X(t)$ satisfies property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ and $X(\cdot)$ is Lipschitz continuous and convex-valued then the function $V(t,x)=e^{-Lt}d(x,X(t))$ is a forward subsolution of equation (7).

In the next subsection we utilize this proposition to obtain a description of $\mathscr{A}^{\hat{x}(\cdot)}$ in terms of equations with discontinuous right-hand side (Proposition 2).

B. Minimal reachable sets

In this subsection we find equations that define the minimal target set X^1 in Problem 1.

Given an arbitrary interval $X^0=[\underline{x}^0,\overline{x}^0]$, let us consider a Lipschitz continuous interval-valued map $X(t)=[\underline{x}(t),\overline{x}(t)]$ such that $X(0)=X^0$. We introduce the function $\sigma(\cdot)$:

$$\sigma(x)\equiv d(x,X^0)=\max_i\max\{x_i-\overline{x}_i^0,\ \underline{x}_i^0-x_i,0\}. \tag{14}$$

Now let us define the function

$$V(t,x) = e^{-Lt} \max_i \max\{x_i - \overline{x}_i(t),\ \underline{x}_i(t) - x_i, 0\}. \quad (15)$$

As mentioned in the previous subsection, to obtain a controller that solves the reachability problem for a target set $X^1 = [\underline{x}(T), \overline{x}(T)]$, we need a backward supersolution of (7), (12). Under the assumptions of Proposition 1, a backward supersolution of the form (15) is also a forward subsolution of (7), (13). Therefore, let us now give a criterion for (15) to be a forward subsolution.

Lemma 2: Function V is a viscosity subsolution of (7), (13) in forward time if

$$\begin{array}{rcl} \dot{\overline{x}}_i(t) & \geq & g_i(t, \zeta_i(\overline{x}(t), \underline{x}(t)), \underline{\theta}, \overline{\omega}), \\ \dot{\underline{x}}_i(t) & \leq & g_i(t, \zeta_i(\underline{x}(t), \overline{x}(t)), \overline{\theta}, \underline{\omega}) \end{array} \quad (16)$$

a.e. on $[0,T]$. In addition, under assumption that g is a tight decomposition function, conditions (16) are equivalent to V being a forward viscosity subsolution of (7), (13).

Thus, for every interval-valued map $X(t)$ in the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ inequalities (16) must hold for the tight decomposition function g. This observation leads to the following.

Proposition 2: If $X^1 \in \mathscr{A}^{\hat{x}(\cdot)}$ then there exist $X(t) = [\underline{x}(t), \overline{x}(t)]$ and $\xi(\cdot) = (\overline{\xi}(\cdot), \underline{\xi}(\cdot)) \in L^\infty([0,T], \mathbb{R}^{2n_x})$ with $\overline{\xi}(t) \succeq 0$, $\underline{\xi}(t) \preceq 0$ satisfying equations

$$\begin{array}{l} \dot{\overline{x}}_i = \left\{ \begin{array}{rl} g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}) + \overline{\xi}_i(t), & \hat{x}_i(t) < \overline{x}_i, \\ \max\{g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\}, & \hat{x}_i(t) \geq \overline{x}_i, \end{array} \right. \\ \dot{\underline{x}}_i = \left\{ \begin{array}{rl} g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}) + \underline{\xi}_i(t), & \underline{x}_i < \hat{x}_i(t), \\ \min\{g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\}, & \underline{x}_i \geq \hat{x}_i(t) \end{array} \right. \end{array} \quad (17)$$

for a tight decomposition function g a.e. on $[0,T]$, initial conditions

$$\underline{x}(0) = \underline{x}^0, \quad \overline{x}(0) = \overline{x}^0 \quad (18)$$

and such that $X(T) = X^1$.

This result gives a useful description of the considered class $\mathscr{A}^{\hat{x}(\cdot)}$. Intuitively, the interval-valued map $X(t)$ that satisfies differential equations (17) with $\xi(t) \equiv 0$ should produce the minimal element of the respective class $\mathscr{A}^{\hat{x}(\cdot)}$. To formally establish it, we need to prove that (17) is monotone in state $(\overline{x}, -\underline{x})$ and input $(\overline{\xi}, -\underline{\xi})$ and has a solution for $\xi(t) \equiv 0$. First, we provide the following two lemmas.

Lemma 3:

1) System of equations (17) has a unique solution on $[0,T]$ in the sense of Filippov (see [14], §4, definition a)). Moreover, the solution is Lipschitz continuous.
2) For any solution of (17), (18), the following relation holds:
$$\underline{x}(t) \preceq \hat{x}(t) \preceq \overline{x}(t).$$

Lemma 4: The function V defined by (15), (17), (18) is a viscosity supersolution of (7), (12) in backward time.

Remark 2: The statements of these two lemmas hold for an arbitrary decomposition function g satisfying Assumption 2.

Thus, for every solution of (17), (18) with an arbitrary decomposition function g, the corresponding set $X(T) \in \mathscr{A}^{\hat{x}(\cdot)}$. However, when a tight decomposition function is available, we are able to solve Problem 1. For $\xi(t) \equiv 0$ equations (17) take the form:

$$\begin{array}{l} \dot{\overline{x}}_i = \left\{ \begin{array}{rl} g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}), & \hat{x}_i(t) < \overline{x}_i, \\ \max\{g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}), \dot{\hat{x}}_i(t)\}, & \hat{x}_i(t) \geq \overline{x}_i, \end{array} \right. \\ \dot{\underline{x}}_i = \left\{ \begin{array}{rl} g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}), & \underline{x}_i < \hat{x}_i(t), \\ \min\{g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}), \dot{\hat{x}}_i(t)\}, & \underline{x}_i \geq \hat{x}_i(t). \end{array} \right. \end{array} \quad (19)$$

Theorem 1: Let g be a tight decomposition function. Consider the solution $(\underline{x}(\cdot), \overline{x}(\cdot))$ of (19), (18). The set $X^1 = [\underline{x}(T), \overline{x}(T)]$ is the unique minimal element of class $\mathscr{A}^{\hat{x}(\cdot)}$.

Corollary 1: Let g be an arbitrary decomposition function. For $X(t) = [\underline{x}(t), \overline{x}(t)]$ defined by (19), (18) and $X^+(t)$ defined by (6), the inclusion $X(t) \subseteq X^+(t)$ holds for all $t \in [0,T]$.

C. Controller constructions

Let us consider the interval-valued map $X(t) = [\underline{x}(t), \overline{x}(t)]$ defined by (19), (18). Now we define a controller $u(t,x)$ that solves Problem 1 for the target set $[\underline{x}(T), \overline{x}(T)]$. Let $u(t,x)$ be a function satisfying the following condition:

$$\begin{array}{l} p_i(t,x) = \left\{ \begin{array}{ll} 1, & x_i > \overline{x}_i(t), \\ -1, & x_i < \underline{x}_i(t), \\ 0, & \underline{x}_i(t) \leq x_i \leq \overline{x}_i(t), \end{array} \right. \\ u(t,x) \in \operatorname{Arg} \min\limits_{u \in U} \max\limits_{w \in W} \langle p(t,x), f(t,x,u,w) \rangle. \end{array} \quad (20)$$

Lemma 5: If the interior of $[\underline{x}(t), \overline{x}(t)]$ is not empty for all $t \in [0,T]$ then there exists a controller $u(t,x)$ satisfying (20) such that it is Lipschitz in x uniformly in t.

Now consider the following special case. Let $\cup_{1 \leq i \leq n_x} \mathcal{J}_i = \{1, \ldots, n_u\}$ and $\mathcal{J}_{i_1} \cap \mathcal{J}_{i_2} = \emptyset$ when $i_1 \neq i_2$. Suppose that each function f_i only depends on components u_j, $j \in \mathcal{J}_i$ of the control vector. Then there is an alternative, more explicit controller construction:

$$\begin{array}{c} x_i^c(t) = (\underline{x}_i(t) + \overline{x}_i(t))/2, \quad x_i^r(t) = (\overline{x}_i(t) - \underline{x}_i(t))/2, \\ u_j^c = (\underline{u}_j + \overline{u}_j)/2, \quad u_j^r = (\overline{u}_j - \underline{u}_j)/2, \\ u_j(t,x) = \left\{ \begin{array}{rl} \underline{u}_j, & x_i > \overline{x}_i(t), \\ u_j^c + u_j^r \frac{x_i - x_i^c(t)}{x_i^r(t)}, & \underline{x}_i(t) \leq x \leq \overline{x}_i(t), \\ \overline{u}_j, & x_i < \underline{x}_i(t) \end{array} \right. \end{array} \quad (21)$$

for all $j \in \mathcal{J}_i$, for all $i \in \{1, \ldots, n_x\}$. If $\underline{x}_i(t) = \overline{x}_i(t)$ we formally put $u_j(t, \hat{x}(t)) = \hat{u}_j(t)$.

Theorem 2: Consider the closed-loop system (3) with a controller defined by (20) or by (21). Then the following propositions hold.

1) Closed-loop system (3) has a unique solution on $[0,T]$ (in the sense of Filippov) for all admissible disturbances $w(\cdot)$. Every solution $x(\cdot)$ emanating from $X^0 = [\underline{x}^0, \overline{x}^0]$ satisfies the inclusions $x(t) \in [\underline{x}(t), \overline{x}(t)]$ for all $t \in [0,T]$;
2) If the interior of $[\underline{x}(t), \overline{x}(t)]$ is not empty for all $t \in [0,T]$ then the closed-loop system (3) has a solution (in the sense of Carathéodory) for all admissible disturbances $w(\cdot)$. Every solution $x(\cdot)$ emanating from $X^0 = [\underline{x}^0, \overline{x}^0]$ satisfies the inclusions $x(t) \in [\underline{x}(t), \overline{x}(t)]$ for all $t \in [0,T]$.

As we mentioned earlier, the first item of Assumption 1 is very general and non-restrictive. Let us now give some examples showing that conditions 2) and 3) of Assumption 1 are essential for the construction.

Example 1. Consider the system $\dot{x}_1 = \dot{x}_2 = u$, $u \in [-1, 1]$ and let $X^0 = [-1, 1]^2$. Conditions 1) and 2) do hold while condition 3) fails. Calculating the solution of (19), (18), we obtain $\underline{x}(1) = \overline{x}(1) = [0; 0]$. However, state $x = [1; -1]$ is not controllable to $[0; 0]$ by any controller.

Example 2. Consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 - 1 + \begin{cases} \max\{w, u_3\}, & w \le 1 \\ \min\{w, u_4 + 1 + \lambda\}, & w > 1, \end{cases} \\ \dot{x}_2 &= u_2 - 1 + \begin{cases} \max\{w, u_4 - \lambda\}, & w \le 1 \\ \min\{w, u_3\}, & w > 1 \end{cases} \end{aligned}$$

where $u_1, u_2 \in [-\lambda, \lambda]$, $u_3, u_4 \in [-1, 1]$, and $w \in [0, 2]$. Here $\lambda > 0$ is a constant parameter. Observe that conditions 1) and 3) are satisfied, and the system is actually monotone in (u, w). A direct calculation shows that

$$\begin{aligned} \lambda = \max_{u \in U} \min_{w \in W} \big\langle p, f(t, x, u, w) \big\rangle < \\ \min_{w \in W} \max_{u \in U} \big\langle p, f(t, x, u, w) \big\rangle = 2\lambda. \end{aligned} \tag{22}$$

for $p = [1; -1]$. Now let $\underline{x}^0 = \overline{x}^0 = [0; 0]$. Then one may check that $\overline{x}(t) = \underline{x}(t) = \hat{x}(t) = [\lambda; -\lambda]t$ is the solution of (19), (18). However, from (22) it follows that state $[0; 0]$ at $t = 0$ cannot be controlled to $[\lambda; -\lambda]$ at $t = 1$ unless the controller has access to $w(t)$ at each time instant t.

IV. Abstraction algorithm

A. General case

In this section we consider the time-invariant version of system (1):

$$\dot{x} = f(x, u, w), \quad i = 1, \dots, n_x. \tag{23}$$

Here $u \in U = [\underline{u}, \overline{u}]$, $w \in W = [\underline{w}, \overline{w}]$ as before.

Given a controller u: $[0, T] \times \mathbb{R}^{n_x} \to U$, let $\mathbf{x}(t; x, u, w(\cdot))$ denote the set of all solution endpoints (in the sense of Filippov) of the closed-loop system satisfying the initial condition $x(0) = x$ and corresponding to the disturbance $w(\cdot) \in \mathcal{W}(0, T)$.

Let us denote $\mathscr{U}_T^0(x)$ the set of all controllers such that for $x(0) = x$ and for every $w(\cdot) \in \mathcal{W}(0, T)$ there is at least one Filippov solution of the closed-loop system and every such solution exists on $[0, T]$.

Let us consider a set $X \subseteq \mathbb{R}^{n_x}$, which we call the state space, and restrict the dynamics of system (23) to this set. Let the state space X be covered by a finite set of intervals $(X_q)_{q \in Q}$: $X = \cup_{q \in Q} X_q$, $X_q = [\underline{x}^q, \overline{x}^q]$.

Definition 3: A *transition system* is a tuple (X, U, Y, Δ, H), where

- X is a set of states;
- U is a set of inputs;
- Y is a set of outputs;
- $\Delta : X \times U \rightrightarrows X$ is a set-valued transition map;
- H: $X \longrightarrow Y$ is an output map.

An input $u \in U$ is called *enabled* at $x \in X$ if $\Delta(x, u) \neq \emptyset$. Let $\mathrm{enab}_\Delta(x) \subseteq U$ denote the set of all inputs enabled at x. If $\mathrm{enab}_\Delta(x) = \emptyset$ the state x is called *blocking*. A transition system is called *deterministic* if for all $(x, u) \in X \times U$, $\mathrm{card}(\Delta(x, u)) \le 1$.

Given the cover $(X_q)_{q \in Q}$, system (23) may be written as a transition system as follows:

$$S = (X, \mathscr{U}, Q, \delta, H)$$

where

$$\begin{gathered} \mathscr{U} = \{(T, u),\ T \in [0, +\infty),\ u :\ [0, T] \times \mathbb{R}^{n_x} \to U\}, \\ q = H(x) \quad \Leftrightarrow \quad x \in X_q \end{gathered}$$

and transition relation δ is defined as follows:

$$x' \in \delta(x, T, u), \quad (T, u) \in \mathrm{enab}_\delta(x)$$

if and only if there exists $w \in \mathcal{W}(0, T)$ such that $x' \in \mathbf{x}(T; 0, x, u, w(\cdot))$. Here the set of enabled inputs is defined as follows

$$\begin{gathered} \mathrm{enab}_\delta(x) = \{(T, u) \in \mathscr{U} \mid u \in \mathscr{U}_T^0(x) \quad \text{and} \\ \forall w \in \mathcal{W}(0, T), \forall t \in [0, T],\ \mathbf{x}(t; 0, x, u, w(\cdot)) \subseteq X\}. \end{gathered}$$

We now define an abstract transition system S_a using the cover $(X_q)_{q \in Q}$, a sampling parameter $\tau > 0$ and a finite set of control inputs $\mathscr{V}$:

$$S_a = (Q, \mathscr{V}, Q, \Delta, \mathrm{Id}).$$

Here Id is the identity map on Q. In a state $q \in Q$ a symbolic control $v \in \mathscr{V}$ corresponds to a pair $(\tau, u^{(q,v)}) \in \mathscr{U}$ such that $u^{(q,v)}$ is defined by (20) or (21) and the corresponding interval $X^{(q,v)}(t) = [\underline{x}^{(q,v)}(t), \overline{x}^{(q,v)}(t)]$ is defined by (19) with the initial conditions

$$\underline{x}^{(q,v)}(0) = \underline{x}^q \preceq \overline{x}^q = \overline{x}^{(q,v)}(0).$$

The corresponding reference trajectories $\hat{x}(\cdot)$ and reference controls $\hat{u}(\cdot)$ in (19) depend on the pair (q, v). Below we provide a particular choice of those that guarantees the comparison result in Theorem 4.

Observe that $u^{(q,v)} \in \mathscr{U}_\tau^0(x)$. Transition relation Δ is defined as follows: $q' \in \Delta(q, v)$ for $v \in \mathscr{V}$ if and only if

$$X_{q'} \cap [\underline{x}^{(q,v)}(\tau), \overline{x}^{(q,v)}(\tau)] \neq \emptyset$$

and

$$[\underline{x}^{(q,v)}(t), \overline{x}^{(q,v)}(t)] \subseteq X$$

for all $t \in [0, \tau]$.

Definition 4: Let $S_a = (X_a, U_a, Y_a, \Delta_a, H_a)$ and $S_b = (X_b, U_b, Y_b, \Delta_b, H_b)$ be two transition systems with $Y_a = Y_b$. A relation $R \subseteq X_a \times X_b$ is an alternating simulation relation from S_a to S_b if the following conditions are satisfied:

1) for every $(x_a, x_b) \in R$ we have $H_a(x_a) = H_b(x_b)$;
2) for every $(x_a, x_b) \in R$ and for every $u_a \in \mathrm{enab}_{\Delta_a}(x_a)$ there exists $u_b \in \mathrm{enab}_{\Delta_b}(x_b)$ such that for every $x_b' \in \Delta_b(x_b, u_b)$ there exists $x_a' \in \Delta_a(x_a, u_a)$ satisfying $(x_a', x_b') \in R$.

It is said that S_b alternatingly simulates S_a, denoted by $S_a \preceq_{AS} S_b$, if there exists an alternating simulation relation $R \neq \emptyset$ from S_a to S_b.

Theorem 3: Transition system S alternatingly simulates abstract system S_a: $S_a \preceq_{AS} S$.

Let us now introduce the standard abstract system S_{std}. Consider a finite approximation $\hat{U}$ of the control space: $\hat{U} \subset U$. We define the abstraction

$$S_{std} = (Q, \hat{U}, Q, \hat{\Delta}, \mathrm{Id})$$

where transition relation $\hat{\Delta}$ is defined as follows: $q' \in \hat{\Delta}(q, \hat{u})$ for $\hat{u} \in \hat{U}$ if and only if

$$X_{q'} \cap [\underline{\mathbf{x}}(\tau; [\underline{x}^q, \overline{x}^q], \hat{u}), \overline{\mathbf{x}}(\tau; [\underline{x}^q, \overline{x}^q], \hat{u})] \neq \emptyset$$

and

$$[\underline{\mathbf{x}}(t; [\underline{x}^q, \overline{x}^q], \hat{u}), \overline{\mathbf{x}}(t; [\underline{x}^q, \overline{x}^q], \hat{u})] \subseteq X$$

for all $t \in [0, \tau]$.

To provide a comparison result between S_a and S_{std}, let us specify the set $\mathscr{V}$ and the corresponding controls $u^{(q,v)}$. Let $\mathscr{V} = \hat{U}$ and $u^{(q,v)}$ corresponds to the reference trajectory $\hat{x}(\cdot)$ satisfying the following conditions:

$$\dot{\hat{x}} = f(\hat{x}, \hat{u}, \hat{w}), \quad \hat{x}(0) \in [\underline{x}^q, \overline{x}^q], \quad \hat{w} \in [\underline{w}, \overline{w}]. \tag{24}$$

Theorem 4: Transition system S_a alternatingly simulates S_{std}: $S_{std} \preceq_{AS} S_a$.
Theorems 3 and 4 give us the relation

$$S_{std} \preceq_{AS} S_a \preceq_{AS} S. \tag{25}$$

Analyzing the proofs of those theorems, we conclude that the relations above are also *feedback refinement relations* [3].

Given an arbitrary control specification, every symbolic state $q \in Q$, which is controllable for S_{std}, is also controllable for S_a. We emphasize that by construction the number of transitions in the new abstraction does not exceed the number of transitions in the standard abstraction. The schematic visualization of both abstraction algorithms is given on Figures 1 and 2 respectively.

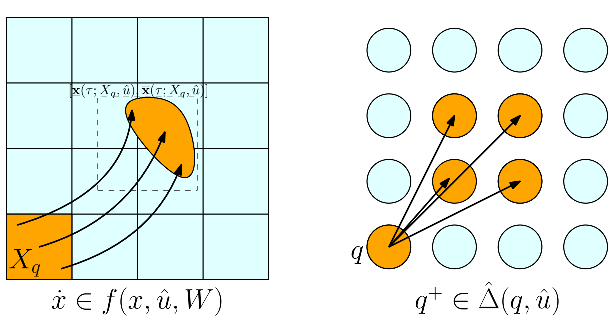

Fig. 1. Visualization of the standard abstraction algorithm.

B. Special cases

In this subsection we focus on two special cases when the constructed abstractions possess special properties that, in addition to improvement in the transition number, enable more efficient synthesis algorithms.

To obtain the results, let us expand the class of the considered reference trajectories. We will call $\hat{x}(\cdot)\colon [0, \tau] \to \mathbb{R}^{n_x}$ a *generalized reference trajectory* if there exists a partition

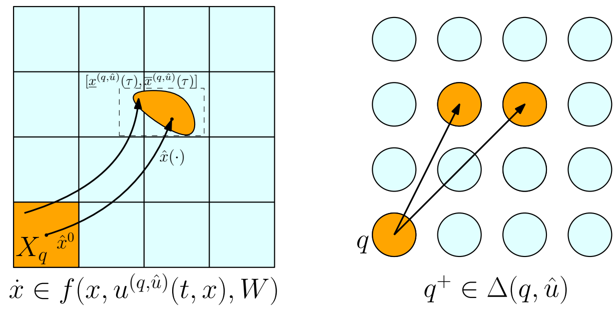

Fig. 2. Visualization of the new abstraction algorithm.

$0 = \tau_0 < \tau_1 < \cdots < \tau_m = \tau$ such that the differential equation in (24) is satisfied on each interval $[\tau_{j-1}, \tau_j)$, $1 \leq j \leq m$. Thus, functions $\hat{x}(\cdot)$ may have discontinuities on the set $\{\tau_j,\ 1 \leq j \leq m\}$. These trajectories can be used to obtain feedback controllers (20) and (21) by solving equations (19), (18) as we did with ordinary reference trajectories. While an ordinary reference trajectory is defined by a triple $(\hat{x}^0, \hat{u}, \hat{w})$, a generalized reference trajectory also has jumps $\hat{x}(\tau_j)$ as free parameters. All those parameters may be chosen in a particular way so that the abstraction has a certain special structure provided that system (1) satisfies some additional assumptions.

Let us now specify a subclass of systems, for which there exists a deterministic abstraction.

Theorem 5: (Sufficient condition for determinism) Let $d_x > 0$ be such that for all i, $\overline{x}_i^q - \underline{x}_i^q < d_x$. Assume that there exists $r > 0$ such that for all $x, y \in X$, $y \preceq x$, $x_i - y_i \leq d_x$, the following condition holds:

$$g_i(\zeta_i(x,y), \underline{\theta}, \overline{\omega}) - g_i(\zeta_i(y,x), \overline{\theta}, \underline{\omega}) \leq -r.$$

Then for any time sampling parameter $\tau \geq d_x/r$, there exists a deterministic transition system S_a such that (25) holds.

The theorem provides an existence result. A practical algorithm for constructing such abstractions is given in the next section. Now let us specify a subclass of systems and a set of abstraction parameters, for which there exists an abstraction with a singleton set of symbolic controls.

Theorem 6: (Sufficient condition for singleton input alphabet) Let $d_x > 0$ be such that for all i, $\overline{x}_i^q - \underline{x}_i^q > d_x$. Assume that there exists $r \geq 0$ such that for all $x, y \in X$, $y \preceq x$ the following condition holds:

$$g_i(\zeta_i(x,y), \underline{\theta}, \overline{\omega}) - g_i(\zeta_i(y,x), \overline{\theta}, \underline{\omega}) \geq -r.$$

Then for any time sampling parameter $0 < \tau \leq d_x/r$ (or for any $\tau > 0$ in the case of $r = 0$) there exists a transition system S_a with a singleton set of inputs such that (25) holds.

The proof of this theorem is constructive: equations (32) determine the single interval $[\underline{x}^q(\tau), \overline{x}^q(\tau)]$ that describes the set of successor states for each symbolic state q. From the theorem it follows, in particular, that

$$[\underline{x}^q(\tau), \overline{x}^q(\tau)] \subseteq \cap_{\hat{u} \in \hat{U}} [\underline{\mathbf{x}}(\tau; [\underline{x}^q, \overline{x}^q], \hat{u}), \overline{\mathbf{x}}(\tau; [\underline{x}^q, \overline{x}^q], \hat{u})].$$

The abstractions constructed in Theorems 5 and 6 not only better than the standard abstraction S_{std}, but also possesses properties useful for formal controller synthesis. For example, for LTL specifications, control synthesis for such abstractions is reduced to a verification problem [2]. Suppose τ_d is a time sampling such that conditions of Theorem 5 hold, and τ_u is a time sampling such that conditions of Theorem 6 hold. Then we necessarily have $\tau_u \leq \tau_d$. The type of evolution of the controllable intervals in both special cases is depicted on Figures 3 and 4.

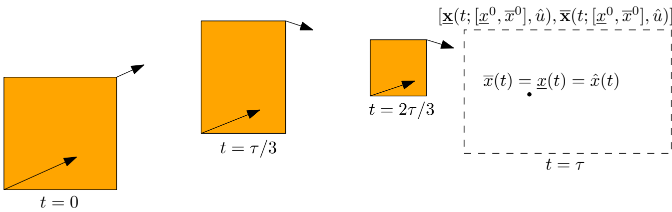

Fig. 3. Evolution of $[\underline{x}(t), \overline{x}(t)]$ in the deterministic case. Comparison with the standard over-approximation corresponding to a discretized control $\hat{u}$.

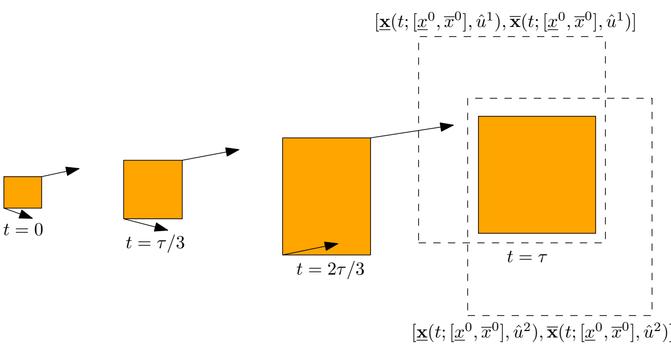

Fig. 4. Evolution of $[\underline{x}(t), \overline{x}(t)]$ in the singleton case. Comparison with the standard over-approximations corresponding to different discretized controls.

V. Numerical simulations

We begin this section by giving some insights on the numerical computation of intervals $[\underline{x}(t), \overline{x}(t)]$ and the construction of abstractions. Then we proceed with the examples.

A. Numerical method

Let us now discuss the numerical procedure for solving equation (19). Let $(\overline{x}(\cdot), \underline{x}(\cdot))$ denote the actual solution of (19), and let $(\overline{z}(\cdot), \underline{z}(\cdot))$ denote its numerical approximation. Furthermore, let $\hat{z}(\cdot)$ be a numerical approximation of $\hat{x}(\cdot)$.

Consider the time step $h > 0$. To numerically solve (19), (18) with the desired convergence rate, the time instants of switches must be determined with the corresponding accuracy. Let us use a base numerical method for integrating ODEs with polynomial convergence rate $O(h^p)$. Let function $\mathrm{Solve}(t_0, t_1)$ denote the numerical solution of (19), (18) on $[t_0, t_1]$ for a given initial condition $(\overline{z}(t_0), \underline{z}(t_0), \hat{z}(t_0))$ and a time step parameter $h > 0$ using this base numerical method. According to Lemma 3, functions $\overline{x}(\cdot)$, $\underline{x}(\cdot)$, and $\hat{x}(\cdot)$ are Lipschitz continuous. Therefore, there exist a constant $L_z > 0$, which depends on the base numerical method, such that $\|\overline{z}(t+h_*)-\overline{z}(t)\| \leq L_z h_*$, $\|\underline{z}(t+h_*)-\underline{z}(t)\| \leq L_z h_*$, and $\hat{z}(t+h_*) - \hat{z}(t)\| \leq L_z h_*$. Finally, for a fixed approximation parameter $\varepsilon > 0$, let P denote the following predicate:

$$\exists i,\ \overline{z}_i(t) - \underline{z}_i(t) \geq \varepsilon \wedge [\overline{z}_i(t) \leq \hat{z}_i(t) + h^p \vee \underline{z}_i(t) \geq \hat{z}_i(t) - h^p].$$

Our numerical scheme for computing $(\overline{z}(\cdot), \underline{z}(\cdot))$ is presented below (Algorithm 1).

Algorithm 1: Numerical method for solving (19), (18)

Input: Decomposition function g, Lipschitz constant $L_z > 0$, time horizon τ, initial value $(\overline{x}^0, \underline{x}^0)$, time step $h > 0$, approximation parameter $\varepsilon > 0$
Output: Numerical solution $(\overline{z}(\cdot), \underline{z}(\cdot))$ of the initial value problem (19), (18)
begin
  $t = 0$, $\overline{z}(t) := \overline{x}^0$, $\underline{z}(t) := \underline{x}^0$;
  while $t < \tau$ **do**
    $h := \min\{h, t - \tau\}$;
    if P **then**
      $\hat{z}(t) := \frac{1}{2}(\overline{z}(t) + \underline{z}(t))$;
    for $1 \leq i \leq n_x$ **do**
      if $\overline{z}_i(t) \leq \hat{z}_i(t) + h^p$ **then**
        $\overline{z}_i(t) := \hat{z}_i(t)$;
      if $\underline{z}_i(t) \geq \hat{z}_i(t) - h^p$ **then**
        $\underline{z}_i(t) := \hat{z}_i(t)$;
    $h_* := \min_i \min\{\overline{z}_i(t) - \hat{z}_i(t), \hat{z}_i(t) - \underline{z}_i(t)\}/(2L_z)$;
    $h_* := \min\{h_*, h\}$;
    $(\overline{z}(t+h_*), \underline{z}(t+h_*), \hat{z}(t+h_*)) := \mathrm{Solve}(t, t+h_*)$;
    $t := t + h_*$;
  return $(\overline{z}(\cdot), \underline{z}(\cdot))$;

Under conditions of Theorem 5, Algorithm 1 has the same convergence rate as the base method.

Theorem 7: Let $d_x > 0$ be such that for all i, $\overline{x}_i^0 - \underline{x}_i^0 < d_x$. Assume that there exists $r \geq 0$ such that for all $x, y \in \mathbb{R}^{n_x}$, $y \preceq x$ the following condition holds:

$$g_i(\zeta_i(x,y), \underline{\theta}, \overline{\omega}) - g_i(\zeta_i(y,x), \overline{\theta}, \underline{\omega}) \leq -r.$$

Suppose that the jumps of $\hat{x}(\cdot)$ coincide with the resets of $\hat{z}(\cdot)$. Then $\|\overline{z}(\tau) - \overline{x}(\tau)\| = O(h^p)$ and $\|\underline{z}(\tau) - \underline{x}(\tau)\| = O(h^p)$. In addition, if $\tau \geq d_x/r$ then $\max_i[\overline{z}_i(\tau) - \underline{z}_i(\tau)] \leq \varepsilon + L_z h$.

Now let us show how to find the limit point $\overline{x}(\tau) = \underline{x}(\tau)$ in the deterministic case with the same convergence rate $O(h^p)$. Define the following parameters:

$$\tau_0 = \frac{2\varepsilon}{r} \cdot \frac{1-h^p}{1-h},\ \tau_1 = \tau_0 - \frac{2\varepsilon}{r},\ \ldots,\ \tau_{p-1} = \frac{2\varepsilon}{r} h^{p-1},$$
$$\varepsilon_0 = \varepsilon,\ \varepsilon_1 = \varepsilon h,\ \ldots,\ \varepsilon_p = \varepsilon h^p,$$
$$h_0 = h,\ h_1 = h^2, \ldots, h_p = h^{p+1}.$$

Corollary 2: Under the assumptions of Theorem 7, suppose that
$$\tau \geq \frac{d_x}{r} + \frac{2\varepsilon}{r} \cdot \frac{1-h^p}{1-h}, \quad \varepsilon \geq L_z h.$$
Consider the application of Algorithm 1 consecutively on intervals $[0, \tau - \tau_0]$, $[\tau_0, \tau_1]$, ..., $[\tau_{p-1}, \tau]$ with parameters ε_0, ε_1, ..., ε_p and h_0, h_1, ..., h_p to obtain the numerical solution of (19), (18) on $[0, \tau]$. Then $\max_i[\overline{z}_i(\tau) - \underline{z}_i(\tau)] = O(h^p)$.

On the other end of the spectrum, under the conditions of Theorem 6, system (19) has a continuous righthand side in the neighborhood of the solution. Therefore, one may utilize any suitable numerical integration method. Moreover, the computation of the reference trajectories is not needed in that case.

Our final comment is that to improve numerical stability, one may also utilize *overlapping covers* of the state space. Suppose $(X_q)_{q\in Q}$ is an arbitrary cover of X. Then for $\varepsilon > 0$, $(X'_q)_{q\in Q}$ with $X'_q = \{x \in X \mid d(x, X_q) \leq \varepsilon\}$ is an overlapping cover. In particular, if $\max_i[\overline{x}_i - \underline{x}_i] < 2\varepsilon$ and $[\underline{x}, \overline{x}] \subseteq X$ then there exists $q \in Q$ such that $[\underline{x}, \overline{x}] \subseteq X'_q$, which guarantees accurate computation of deterministic abstractions.

B. Temperature regulation model

Let us consider a temperature regulation model of a circular n_x room building, which was adapted from [25]. The system is given by equations:
$$\dot{\mathbf{T}}_i(t) = \alpha(\mathbf{T}_{i+1}(t) + \mathbf{T}_{i-1}(t) - 2\mathbf{T}_i(t))$$
$$+\beta(\mathbf{T}_e(t) - \mathbf{T}_i(t)) + \gamma(\mathbf{T}_h - \mathbf{T}_i(t))u_i(t).$$
Here $\mathbf{T}_i$ is the temperature in room i, $\mathbf{T}_e(t) \in [\mathbf{T}_e^{\min}, \mathbf{T}_e^{\max}]$ is the outside temperature, which is considered as disturbance, α, β and γ are the corresponding conduction factors. The heater powers $u_i(t) \in [0, 1]$ are the control parameters whereas the maximal heater temperature is $\mathbf{T}_h$. We utilize the following values for conduction factors: $\alpha = 0.05$, $\beta = 0.005$, $\gamma = 0.01$. The system is monotone in state and inputs.

The state space $X = [\mathbf{T}_1^{\min}, \mathbf{T}_1^{\max}] \times \cdots \times [\mathbf{T}_{n_x}^{\min}, \mathbf{T}_{n_x}^{\max}]$. Here we consider a simple safety problem of keeping trajectories of the system in X at all times.

For both abstraction algorithms we use a uniform partition with 10 discretization intervals per state space dimension and the sampling paramter τ. We compare the two algorithms for $\tau = 1, 5, 40$. We utilize $\hat{U} = \{0, \frac{1}{2}, 1\}^{n_x}$ as a finite approximation of U in the standard abstraction algorithm. In the new algorithm we use $|\hat{U}| = 3^{n_x}$ reference trajectories chosen according to (24), except for the singleton case where the reference trajectories are not needed, and there is only one symbolic control.

For the simulations below we choose the following parameters: $n_x = 3$, $\mathbf{T}_i^{\min} = 19°\text{C}$, $\mathbf{T}_i^{\max} = 23°\text{C}$, $\mathbf{T}_e^{\min} = -1°\text{C}$, $\mathbf{T}_e^{\max} = 10°\text{C}$, $\mathbf{T}_h = 50°\text{C}$, $N_i = 10$.

Table 1 gives the total count of transitions and controllable states for the standard and the new abstraction algorithms. Both abstract systems utilize the same number of symbolic controls but the overall number of transitions is greatly reduced for the new abstraction. The higher reduction is achieved for bigger values of sampling parameter τ. Coincidentally, for big enough values of τ the standard abstract system in this example becomes completely uncontrollable while the new abstract system is still controllable.

τ	Algorithm	# of transitions	# of cont. states	Property
1	Standard	235167	1000 / 1000	-
1	New	(< 1.5%) 3128	1000 / 1000	singleton
5	Standard	403094	1000 / 1000	-
5	New	(< 7%) 26590	1000 / 1000	deterministic
40	Standard	549192	0 / 1000	-
40	New	(< 2%) 9515	1000 / 1000	deterministic

TABLE I
TEMPERATURE CONTROL EXAMPLE.

τ	Algorithm	# of transitions	# of cont. states	Property
3	Standard	10775060	79313 / 125000	-
3	New	(< 11%) 1104316	83233 / 125000	-
5	Standard	19993932	80052 / 125000	-
5	New	(< 4%) 797158	84089 / 125000	deterministic
7	Standard	23141064	87775 / 125000	-
7	New	(< 3.2%) 732828	97619 / 125000	deterministic

TABLE II
AUTONOMOUS BOAT DOCKING EXAMPLE.

One may observe that the conditions of Theorem 5 are satisfied here. We estimate that for a time sampling $\tau \geq 2.29$ we should have a deterministic abstraction. Our simulations show that for $\tau = 5$ and $\tau = 40$ the constructed abstract system is indeed deterministic.

On the other hand, conditions of Theorem 6 hold for small τ. We estimate that for a time sampling $\tau \leq 1.55$ we should have an abstraction with a singleton input alphabet.

C. Autonomous boat docking

Now let us consider the boat control problem adapted from [26]. Kinematic boat model is given by
$$\dot{x} = R(x_3)\tilde{u} + w. \tag{26}$$
where the state $x = [x_1; x_2; x_3]$ are the South-North and West-East positions and heading of the boat ($x_3 = 0$ points North and $x_3 = \pi/2$), the control $\tilde{u} = [\tilde{u}_1; \tilde{u}_2; \tilde{u}_3]$ are the surge and sway velocities, and yaw rate of the ship. $R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a rotation matrix. The disturbance w corresponds to current velocity.

Let us denote $\tilde{u} = \mathrm{diag}(\lambda)(u + u_c)$ where $u = [u_1; u_2; u_3]$ is the new centered and normalized control, $\lambda = [\lambda_1; \lambda_2; \lambda_3] = [0.09; 0.05; 0.1]$ and $u_c = [1; 0; 0]$ are the constant vectors.

The state space $X = [0, 10] \times [0, 6.5] \times [-\pi, \pi]$. The control problem here is to reach the target set $X_f = [7, 10] \times [0, 6.5] \times [\pi/3, 2\pi/3]$ while avoiding the obstacles $X_{o1} = [2, 2.5] \times [0, 3] \times [-\pi, \pi]$ and $X_{o2} = [5, 5.5] \times [3.5, 6.5] \times [-\pi, \pi]$. The control and disturbance spaces are $U = [-1; 1]^3$ and $W = [-0.01, 0.01]^3$.

To construct the abstraction, we conservatively under-approximate $R(x_3)\mathrm{diag}(\lambda)(U + u_c)$ with an interval set. This results in the following system:

$$\dot{x} = \begin{bmatrix} \lambda_2 f(x_3) u_1 \\ \lambda_2 f(x_3) u_2 \\ \lambda_3 u_3 \end{bmatrix} + \lambda_1 \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} + w \tag{27}$$

where $f(\phi) = \frac{1}{|\cos\phi| + |\sin\phi|}$. Now to construct a mixed-monotone decomposition function for this system, we compute tight decompositions of each vector in the righthand side. In particular, the tight decomposition of $f(x)u$ is given by the following:

$$g(x, y, u) = \begin{cases} \tilde{g}(x, y)u, & u \geq 0, \\ \tilde{g}(y, x)u, & u < 0, \end{cases}$$

$$\tilde{g}(x, y) = \begin{cases} \min\{f(x), f(y)\}, & x \leq y,\ [x, y] \cap Z_{\min} = \emptyset, \\ \frac{\sqrt{2}}{2}, & x \leq y,\ [x, y] \cap Z_{\min} \neq \emptyset, \\ \max\{f(x), f(y)\}, & x > y,\ [y, x] \cap Z_{\max} = \emptyset, \\ 1, & x > y,\ [y, x] \cap Z_{\max} \neq \emptyset, \end{cases}$$

$$Z_{\min} = \{\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{4}\}, \quad Z_{\max} = \{-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi\}.$$

The overall decomposition is not tight but it ensures controllability for a large portion of state symbols. Note that for system (27) a tight mixed monotone decomposition can be also constructed analytically.

For the discrete abstraction, we use uniform partition with 50 discretization intervals per state space dimension and 2 extremal values for each control. This results in a transition system with 125000 symbolic states and 8 symbolic controls. We compute the abstractions for three values of the time sampling parameter $\tau = 1, 3, 5$. Here the sufficient conditions for determinism are fulfilled again. We obtained the following conservative estimate for the time sampling: if $\tau \geq 6.12$ then the abstract system is deterministic. This is supported by our simulations: the abstractions for $\tau = 5$ and $\tau = 7$ are deterministic. Conditions of Theorem 6 are satisfied only for a small time sampling: for $\tau \leq 1.28$ we have an abstraction with a singleton input alphabet. Total count of transitions and controllable states for both algorithms is summarized on Table 2.

VI. Conclusion

We introduced a novel abstraction algorithm for a class of continuous-time control systems. This algorithm produces more efficient symbolic systems with fewer transitions than the standard partition based algorithm. The improvement is achieved by considering interval-to-interval feedback controllers instead of open-loop (or constant) controls. In the extreme cases, the new abstractions may even be either deterministic or without inputs, which opens up the posibility to use more efficient synthesis algorithms in the case of complex specifications such as LTL formulas or non-deterministic Büchi automata.

The feedback controllers are constructed in such a way to keep the trajectories of the closed-loop system from leaving the corresponding interval-valued tube. This tube is described by a new system of ODEs that generalizes the equations of reachable set interval over-approximations for mixed monotone systems. By construction, the new abstraction is at least as controllable as the standard one regardless of the control specification. A special numerical method for approximately solving this type of ODEs was discussed and utilized on the examples.

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VII. Appendix

Proof of Lemma 1. *Necessity*. In property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ let us take $\tau = t+h$, $h>0$. We obtain

$$V(t+h, \mathbf{x}(t+h;t,x,\gamma(w)(\cdot),w(\cdot))) - V(t,x) \leq 0$$

for all $x \in X(t)$. Dividing by h and passing to the limit $h \to +0$ gives us inequality (11) for all $x \in X(t)$ (see, e.g., [13], Lemma XI.6.2 for details). The Lipschitz condition on f then ensures that (11) holds on $[0,T] \times \mathbb{R}^{n_x}$:

$$\begin{gathered} q + H(t,x,p) = \\ -LV(t,x) + \tilde{q} + \min_{u\in U}\max_{w\in W} \langle \tilde{p}, f(t,x,u,w)\rangle \leq \\ \tilde{q} + \min_{u\in U}\max_{w\in W} \langle \tilde{p}, f(t,\tilde{x},u,w)\rangle + \\ L\big[e^{-Lt}\|x-\tilde{x}\|_\infty - V(t,x)\big] \leq 0. \end{gathered}$$

Here $\tilde{x}$ is a projection of x onto $X(t)$ and $(\tilde{q},\tilde{p}) \in D^-V(t,\tilde{x})$. We utilized that $q = \tilde{q} - LV(t,x)$, $p = \tilde{p}$ and the expression in brackets is equal to zero.

Sufficiency. If V is a backward viscosity supersolution then it is known that its level set is weakly invariant (see [21]). Therefore, property (b) holds. □

Proof of Proposition 1. Since $X(\cdot)$ is convex-valued, V is convex in x. Using the representation

$$D^+V(t,x) = \left\{ (q,p) \mid \limsup_{(s,y)\to(t,x)} \frac{V(s,y) - V(t,x) - q(s-t) - \langle p, x-y\rangle}{|s-t| + \|x-y\|} \leq 0 \right\}$$

we may infer that for all $(q,p) \in D^+V(t,x)$, $V(t,y) \leq V(t,x) + \langle p, x-y\rangle$ for all y in some neighborhood of x. Therefore, if (q,p), $(q',p') \in D^+V(t,x)$ then $p = p'$ due to convexity of V in x. Thus, from the definition of V it follows that either $D^+V(t,x) = \emptyset$ or $V(t,\cdot)$ is continuously differentiable at x and V_x is continuous in the neighborhood of (t,x).

Let $D^+V(t,x) \neq \emptyset$. Then $D^+V(t,x) \subseteq \partial_C V(t,x)$ where $\partial_C V(t,x)$ denotes the Clarke generalized gradient [27]. Since the left-hand side of (8) is linear in q, its maximum over $\partial_C V(t,x)$ is achieved at a corner point. Using Lipschitz continuity of $X(\cdot)$, we obtain that V is Lipschitz in (t,x). For such function V, for all (t,x), for any corner point $(q_c,p_c) \in \partial_C V(t,x)$, there exists a sequence of points $\{(t^k,x^k)\}$ converging to (t,x) such that V is differentiable at (t^k,x^k) and $(V_t(t^k,x^k), V_x(t^k,x^k))$ converges to (q_c,p_c). Note that $p_c = V_x(t,x)$ since $V(t,\cdot)$ is continuously differentiable at x and $V_x(s,y)$ is continuous in the neighborhood of (t,x). Therefore, for condition (8) to be satisfied it is necessary and sufficient that

$$V_t + H(t,x,V_x) \leq 0$$

a.e. on $[0,T]\times\mathbb{R}^{n_x}$. Here we also utilized continuity of H in its variables. The inequality $V_t + H(t,x,V_x) \leq 0$ is true since V is a backward supersolution according to Lemma 1. □

Proof of Lemma 2. *Sufficiency*. Let us consider the case when $V(t,x)e^{Lt} = x_i - \overline{x}_i(t)$:

$$-LV(t,x)e^{Lt} - \dot{\overline{x}}_i(t) + \min_{u\in U}\max_{w\in W} f_i(t,x,u,w) \leq 0$$

Using the notion of a decomposition function, we have

$$\dot{\overline{x}}_i(t) \geq g_i(t,[x;x],\underline{\theta},\overline{\omega}) - LV(t,x)e^{Lt}.$$

Let for $j = 1,\dots,n$,

$$\overline{\chi}_j(t,x) = \begin{cases} x_j, & \underline{x}_j(t) \leq x_j \leq \overline{x}_j(t), \\ \overline{x}_j(t), & x_j > \overline{x}_j(t), \\ \underline{x}_j(t), & x_j < \underline{x}_j(t). \end{cases}$$

Then we estimate

$$\begin{gathered} g_i(t,[x;x],\underline{\theta},\overline{\omega}) - LV(t,x)e^{Lt} \leq \\ g_i(t,[\overline{\chi}(t,x);\overline{\chi}(t,x)],\underline{\theta},\overline{\omega}) + \\ \big[L\|x-\overline{\chi}(t,x)\|_\infty - LV(t,x)e^{Lt}\big] \leq \\ g_i(t,\overline{\zeta}_i(\overline{x}(t),\underline{x}(t)),\underline{\theta},\overline{\omega}) + \big[L\|x-\overline{\chi}(t,x)\|_\infty - LV(t,x)e^{Lt}\big]. \end{gathered}$$

For the first inequality, we utilized Lipschitz continuity of g_i. For the second inequality, we used mixed monotonicity and the equality $\overline{\chi}_i(t,x) = \overline{x}_i(t)$, which holds in the considered case. One may observe that the expression in the square brackets is equal to zero. Thus, we obtain the relation

$$\dot{\overline{x}}_i(t) \geq g_i(t,\zeta_i(\overline{x}(t),\underline{x}(t)),\underline{\theta},\overline{\omega}).$$

Similar reasoning in the case when $V(t,x)e^{Lt} = \underline{x}_i(t) - x_i$ gives us the differential inequality

$$\dot{\underline{x}}_i(t) \leq g_i(t,\zeta_i(\underline{x}(t),\overline{x}(t)),\overline{\theta},\underline{\omega}).$$

For V to be a forward viscosity subsolution, it is sufficient that these two relations hold.

Necessity. Let V be a viscosity subsolution of (7), (13) in forward time. For all $t \in [0,T]$, all indices i, and all points $\tilde{x} \in [\underline{x}(t),\overline{x}(t)]$, $\tilde{x}_i = \overline{x}_i(t)$, there exists a sequence of points $\{(t^k,x^k)\}_{k=0}^\infty$ converging to $(t,\tilde{x})$ such that V is differentiable at (t^k,x^k) and $V(t^k,x^k) = (x_i^k - \overline{x}_i(t^k))e^{-Lt^k}$. Then plugging (t^k,x^k) into inequality (8), we obtain

$$\dot{\overline{x}}_i(t^k) \geq g_i(t^k,[x^k;x^k],\underline{\theta},\overline{\omega}) - LV(t,x^k)e^{Lt^k}.$$

Now by passing to the limit and taking the maximum over all such points $\tilde{x}$, we establish that the first condition in (16) follows directly from the definition of a tight decomposition function g. The second condition holds by the same argument. □

Proof of Proposition 2. The statement directly follows from Remark 1, Lemma 1, Proposition 1, and Lemma 2. □

Proof of Lemma 3. According to the definition from [14], a pair of absolutely continuous functions $(\overline{x}(\cdot),\underline{x}(\cdot))$ is a solution of (17) if and only if it satisifies the initial condition

$X(0) = X^0$ and is a solution of the corresponding differential inclusion, which is defined by (17), when $\overline{x}_i \neq \hat{x}_i(t)$, $\underline{x}_i \neq \hat{x}_i(t)$ and

$$\dot{\overline{x}}_i \in \big[g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}) + \overline{\xi}_i(t), \\ \max\{g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\}\big]$$

for $\overline{x}_i = \hat{x}_i(t)$,

$$\dot{\underline{x}}_i \in \big[\min\{g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\}, \\ g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}) + \underline{\xi}_i(t)\big]$$

for $\underline{x}_i = \hat{x}_i(t)$. The right hand side $F(t, \overline{x}, \underline{x})$ of this differential inclusion is nonempty, compact, convex and for some α, $\beta > 0$ satisfies the bound

$$\|F(t, \overline{x}, \underline{x})\| \leq \alpha(\|\overline{x}\| + \|\underline{x}\|) + \beta \tag{28}$$

for all $(t, \overline{x}, \underline{x})$. The set-valued map F is measurable in t and upper semicontinuous in $(t, \overline{x}, \underline{x})$. Thus, applying Theorem 3.3 of [28] we obtain global existence of a solution of (17). From relation (28) it follows that all solutions of (17) are bounded: $\|\overline{x}(t)\| \leq M$, $\|\underline{x}(t)\| \leq M$, $t \in [0, T]$. Therefore, the solution is Lipschitz continuous (see [14], §7, Lemma 2 and Theorem 2).

Next, one may verify that for all $(\overline{x}, \underline{x}), (\overline{y}, \underline{y}) \in \mathbb{R}^{2n_x}$ such that $\underline{y} \preceq \overline{y}$ and $\underline{x} \preceq \overline{x}$, the following estimate holds:

$$\frac{d}{dt}\big[\|\overline{x} - \overline{y}\|_2^2 + \|\underline{x} - \underline{y}\|_2^2\big] \leq L'\big[\|\overline{x} - \overline{y}\|_\infty^2 + \|\underline{x} - \underline{y}\|_\infty^2\big] \tag{29}$$

a.e. on $[0, T]$ for $L' = 2Ln_x$. Therefore, the uniqueness follows from [14], §10, Theorem 1.

To prove the second statement, let us assume that there exists a solution of (17), a number i and a time instant $t_2 \in (0, T]$ such that

$$\hat{x}_i(t_2) > \overline{x}_i(t_2).$$

Then there exists $t_1 \in [0, T)$, $t_1 < t_2$ such that $\hat{x}_i(t_1) = \overline{x}_i(t_1)$ and

$$\hat{x}_i(t) > \overline{x}_i(t) \quad \forall t \in (t_1, t_2]$$

On the other hand, from (17) we have

$$\dot{\hat{x}}_i(t) \leq \dot{\overline{x}}_i(t) \quad \text{a. e. on } [t_1, t_2].$$

Integrating this on $[t_1, t_2]$, we arrrive at

$$\hat{x}_i(t_2) - \hat{x}_i(t_1) \leq \overline{x}_i(t_2) - \overline{x}_i(t_1),$$

which contradicts the assumption above. Similarly, one may obtain $\underline{x}_i(t) \leq \hat{x}_i(t)$ on $[0, T]$. Thus, we obtained

$$\underline{x}_i(t) \leq \hat{x}_i(t) \leq \overline{x}_i(t)$$

for all $t \in [0, T]$. □

Proof of Lemma 4. Let us first assume that $\underline{x}^0 \prec \overline{x}^0$ and let $\varepsilon > 0$ be such that $\underline{x}^0 \preceq \overline{x}^0 - \varepsilon$. Consider now the following modification of system (17):

$$\dot{\overline{x}}_i = \begin{cases} g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}) + \overline{\xi}_i(t) \\ \qquad \text{if} \quad \hat{x}_i(t) < \overline{x}_i - \varepsilon, \\ \max\{f_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}) + \overline{\xi}_i(t),\ \dot{\hat{x}}_i(t)\} \\ \qquad \text{if} \quad \hat{x}_i(t) \geq \overline{x}_i - \varepsilon, \end{cases}$$

$$\dot{\underline{x}}_i = \begin{cases} g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}) + \underline{\xi}_i(t) \\ \qquad \text{if} \quad \underline{x}_i < \hat{x}_i(t) - \varepsilon, \\ \min\{g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}) + \underline{\xi}_i(t),\ \dot{\hat{x}}_i(t)\} \\ \qquad \text{if} \quad \underline{x}_i \geq \hat{x}_i(t) - \varepsilon \end{cases}$$

Let us denote a solution of this system by $(\overline{x}^\varepsilon(\cdot), \underline{x}^\varepsilon(\cdot))$. Repeating the argument of Lemma 3, we obtain the global existence of the solution as well as the following inequalities:

$$\dot{\overline{x}}^\varepsilon(t) \geq g_i(t, \overline{\zeta}_i(\overline{x}^\varepsilon(t), \underline{x}^\varepsilon(t)), \underline{\theta}, \overline{\omega}), \\ \dot{\underline{x}}^\varepsilon(t) \leq g_i(t, \underline{\zeta}_i(\overline{x}^\varepsilon(t), \underline{x}^\varepsilon(t)), \overline{\theta}, \underline{\omega}), \\ \underline{x}^\varepsilon(t) \prec \overline{x}^\varepsilon(t).$$

Let us first prove the statement of this lemma for the approximation V^ε of V, which is defined as follows:

$$V^\varepsilon(t, x) = e^{-Lt} \max_i \max\{x_i - \overline{x}_i^\varepsilon(t),\ \underline{x}_i^\varepsilon(t) - x_i, 0\}.$$

Let us consider an arbitrary point (t, x). Without loss of generality let us assume that

$$x_j - \overline{x}_j^\varepsilon(t) \geq \underline{x}_j^\varepsilon(t) - x_j, \quad 1 \leq j \leq j^*, \\ x_j - \overline{x}_j^\varepsilon(t) \leq \underline{x}_j^\varepsilon(t) - x_j, \quad j^* < j \leq n_x$$

for some j^*. Let us then approximate the subdifferential:

$$D^- V^\varepsilon(t, x) \subseteq \{(q, p) \mid p_j = \lambda_j e^{-Lt} \operatorname{sgn}(j - j^* + \tfrac{1}{2}), \\ q = -\sum_{1 \leq j \leq j^*} \lambda_j q_j^\varepsilon e^{-Lt} + \sum_{j^* < j \leq n_x} \lambda_j q_j^\varepsilon e^{-Lt} - \\ LV^\varepsilon(t, x),\ \textstyle\sum_j \lambda_j \leq 1,\ \lambda_j \geq 0, \\ q_j^\varepsilon \in \partial_C \overline{x}_j^\varepsilon(t) \text{ for } j \leq j^* \text{ and } q_j^\varepsilon \in \partial_C \underline{x}_j^\varepsilon(t) \text{ for } j > j^*\}. \tag{30}$$

Here we utilized the strict relation $\underline{x}^\varepsilon(t) \prec \overline{x}^\varepsilon(t)$. By plugging this into (11), we obtain

$$-LV^\varepsilon(t, x)e^{Lt} - \sum_{1 \leq j \leq j^*} \lambda_j q_j^\varepsilon + \sum_{j^* < j \leq n_x} \lambda_j q_j^\varepsilon + \\ \min_{u \in U} \max_{w \in W} \Big\{ \sum_{1 \leq j \leq j^*} \lambda_j f_j(t, x, u, w) + \\ \sum_{j^* < j \leq n_x} (-\lambda_j) f_j(t, x, u, w) \Big\} \leq 0.$$

Since the left-hand side is decreasing in q_j^ε for $j \leq j^*$, increasing in q_j^ε for $j > j^*$, function f is continuous and function V^ε is Lipschitz, it is sufficient to consider this inequality only a. e. on $[0, T]$. Therefore, after doing some rearrangements, we have

$$\max_{w \in W} \Big\{ \sum_{1 \leq j \leq j^*} \lambda_j \big[-\dot{\overline{x}}_j^\varepsilon(t) + g_j(t, [x; x], \underline{\theta}, [w; w])\big] + \\ \sum_{j^* < j \leq n_x} \lambda_j \big[\dot{\underline{x}}_j^\varepsilon(t) - g_j(t, [x; x], \overline{\theta}, [w; w])\big] \Big\} \\ -LV^\varepsilon(t, x)e^{Lt} \leq 0.$$

Here we utilized items 2) and 3) of Assumption 1. For this relation to hold, it is sufficient that

$$\sum_{1\le j\le j^*} \lambda_j \big[-\dot{\overline{x}}^\varepsilon_j(t) + g_j(t,[x;x],\underline{\theta},\overline{\omega})\big] + \sum_{j^*<j\le n_x} \lambda_j \big[\dot{\underline{x}}^\varepsilon_j(t) - g_j(t,[x;x],\overline{\theta},\underline{\omega})\big] - LV^\varepsilon(t,x)e^{Lt} \le 0.$$

Now we take the maximum over all (q,p) in the righthand side of (30), which is the same as maximizing over λ from

$$\Lambda = \{\lambda \in \mathbb{R}^{n_x} \mid \sum_j \lambda_j \le 1,\ \lambda_j \ge 0\}.$$

Since the expression that is being maximized depends linearly on λ, the maximum is achieved at a corner point. For instance, let $i \le j^*$ be such that

$$\lambda_i = 1,\ \lambda_j = 0 \quad \text{for} \quad i \ne j$$

is a maximizer. Then

$$-\dot{\overline{x}}^\varepsilon_i(t) + g_i(t,x,\underline{\theta},\overline{\omega}) - LV^\varepsilon(t,x)e^{Lt} \le 0.$$

By a similar reasoning as in Lemma 2, for this to hold it is sufficient that

$$\dot{\overline{x}}^\varepsilon_i(t) \ge g_i(t,\overline{\zeta}_i(\overline{x}^\varepsilon(t),\underline{x}^\varepsilon(t)),\underline{\theta},\overline{\omega})$$

a. e. on $[0,T]$. For the case $i > j^*$ we obtain the sufficient condition

$$\dot{\underline{x}}^\varepsilon_i(t) \le g_i(t,\underline{\zeta}_i(\overline{x}^\varepsilon(t),\underline{x}^\varepsilon(t)),\overline{\theta},\underline{\omega})$$

a. e. on $[0,T]$. Thus, V^ε is a viscosity supersolution of (7), (12) in backward time.

Let us now consider a sequence $(\varepsilon_k)_{k=0}^\infty$ such that $\varepsilon_k > 0$, $\varepsilon_k \longrightarrow 0$. Let $(\overline{x}^{\varepsilon_k}(\cdot),\underline{x}^{\varepsilon_k}(\cdot))$ be a sequence of the corresponding solutions. Note that for any $\delta > 0$ there exists $K \in \mathbb{N}$ such that for all $k \ge K$ the pair $(\overline{x}^{\varepsilon_k}(\cdot),\underline{x}^{\varepsilon_k}(\cdot))$ is also a *δ-solution* of (17) (see [14], §7). Just as in Lemma 3 it follows that every solution of ε-equation exists on $[0,T]$. Therefore, the set of δ-solutions of (17) is compact in $C([0,T],\mathbb{R}^{2n_x})$. Hence, there is a converging subsequence of solutions whose limit is a solution of the limiting system (17). The corresponding subsequence of functions $V^{\varepsilon_{k_m}}$ then converges uniformly to function V. The statement of the lemma then follows from the stability property of the HJBI equation (see [13], Section II.6, Lemma 6.2). □

Proof of Theorem 1. According to Lemma 3, for every $\xi(\cdot)$ there is a unique solution of (17) on $[0,T]$. Therefore, to prove the statement of the theorem it is then sufficient to establish monotonicity of system (17) with respect to state $(\overline{x},-\underline{x})$ and input $(\overline{\xi},-\underline{\xi})$.

For any $\varepsilon > 0$ one may construct a continuous monotone approximation of the right-hand side of (17) such that

$$\dot{\overline{x}}_i = \begin{cases} g_i(t,\zeta_i(\overline{x},\underline{x}),\underline{\theta},\overline{\omega}) + \overline{\xi}_i(t) \\ \qquad \text{if} \quad \hat{x}_i(t) + \varepsilon_k < \overline{x}_i, \\ \max\{g_i(t,\zeta_i(\overline{x},\underline{x}),\underline{\theta},\overline{\omega}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\} \\ \qquad \text{if} \quad \hat{x}_i(t) \ge \overline{x}_i, \end{cases}$$

$$\dot{\underline{x}}_i = \begin{cases} g_i(t,\zeta_i(\underline{x},\overline{x}),\overline{\theta},\underline{\omega}) + \underline{\xi}_i(t) \\ \qquad \text{if} \quad \underline{x}_i < \hat{x}_i(t) - \varepsilon_k, \\ \min\{g_i(t,\zeta_i(\underline{x},\overline{x}),\overline{\theta},\underline{\omega}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\} \\ \qquad \text{if} \quad \underline{x}_i \ge \hat{x}_i(t). \end{cases}$$

Consider a pair of solutions $(\overline{x}^{j,\varepsilon_k}(\cdot),\underline{x}^{j,\varepsilon_k}(\cdot))$, $j = 1,2$ of the approximation system corresponding to inputs $\overline{\xi}^1(t) \preceq \overline{\xi}^2(t)$, $\underline{\xi}^2(t) \preceq \underline{\xi}^1(t)$. Then $\underline{x}^{2,\varepsilon_k}(t) \preceq \underline{x}^{1,\varepsilon_k}(t) \preceq \overline{x}^{1,\varepsilon_k}(t) \preceq \overline{x}^{2,\varepsilon_k}(t)$ due to monotonicity. For a sequence $\varepsilon_k \to +0$ there is a subsequence of pairs of solutions that converges to some pair of solutions of (17). Since the solution of (17) is unique for every $\xi(\cdot)$, we obtain the monotonicity property: $\underline{x}^2(t) \preceq \underline{x}^1(t) \preceq \overline{x}^1(t) \preceq \overline{x}^2(t)$. □

Proof of Corollary 1. The proof of Theorem 1 may be adapted to prove this corollary. Indeed, we may prove that system (19), (18) is monotone with respect to state $(\overline{x},-\underline{x})$ and input $(\overline{u},-\underline{u})$. To finish the proof, we note that (19) reduces to (5) if we put $\overline{u} = \underline{u} = \hat{u}$. □

Proof of Lemma 5. First, since the Isaacs minimax condition holds, there exists a map $w(t,x)$ such that

$$\operatorname{Arg}\min_{u\in U}\max_{w\in W} \big\langle p(t,x), f(t,x,u,w)\big\rangle \equiv \operatorname{Arg}\min_{u\in U} \big\langle p(t,x), f(t,x,u,w(t,x))\big\rangle$$

for all $t \in [0,T]$, $x \in \mathbb{R}^{n_x}$. Define

$$\Omega_i(t,x) = \operatorname{Arg}\min_{u\in U} p_i(t,x) f_i(t,x,u,w(t,x)).$$

Then, from Assumption 1, we observe that

$$\operatorname{Arg}\min_{u\in U}\max_{w\in W} \big\langle p(t,x), f(t,x,u,w)\big\rangle = \cap_i \Omega_i(t,x).$$

Note that $\Omega_i(t,x)$ is constant in regions $\{x \mid x_i > \overline{x}_i(t)\}$ and $\{x \mid x_i < \underline{x}_i(t)\}$. Moreover, $\Omega_i(t,x) = U$ in $\{x \mid \underline{x}_i(t) \le x_i \le \overline{x}_i(t)\}$. By the assumption of the lemma, there is some minimal distance (independent of t) between the two former regions. Therefore, there exists a selector $u(t,x)$ such that it is Lipschitz in x uniformly in t. □

Proof of Theorem 2. Consider a partition of time interval $[0,T]$:

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

We define a piecewise-constant approximate controller as follows ($k = 1,\ldots,N$)

$$u^N(t,x) = u(t_{k-1},x(t_{k-1})), \quad t \in [t_{k-1},t_k).$$

The number $\delta = \max_k |t_k - t_{k-1}|$ is called the diameter of the partition. The corresponding closed-loop system

$$\dot{x}_i = f_i(t,x,u^N(t,x),w(t)), \quad t \in [0,T] \tag{31}$$

has a solution (in the sense of Carathéodory) for all admissible disturbances $w(\cdot)$.

As in Lemma 3, one may observe that the right-hand side of the differential inclusion corresponding to (3) is nonempty, compact, convex and satisfy linear growth bound in x. The set-valued map is also measurable in t and upper semicontinuous in x. Therefore, at least one solution of (3) exists and every solution can be extended on the whole interval $[0,T]$ and is bounded on it. Consider now a converging sequence of solutions of (31) with the diameter $\delta \to 0$. Then from [14], §7, Lemma 3, it follows that the limiting function is a solution of

(3). The uniqueness follows from [14], §10, Theorem 1 since for some $L'' > 0$ we have

$$(x_i - y_i)(f_i(t, x, u(t, x), w(t)) - f_i(t, y, u(t, y), w(t))) \le L'' \|x - y\|_2^2 \quad t \in [0, T],\ x, y \in \mathbb{R}^{n_x}.$$

As a level set of a backward viscosity supersolution (Lemma 4), $X(t) = [\underline{x}(t), \overline{x}(t)]$ is weakly invariant. One may check that $u(t, x)$ is an extremal aiming controller for $X(t)$. Therefore, by Theorem 13.3 of [21], we obtain the first statement.

To prove the second statement we note that in this case $u(t, x)$ is Lipschitz in x uniformly in t. Thus, the Carathéodory solution of the closed-loop system exists. The result then follows from the previous statement. □

Proof of Theorem 3. Consider a relation $R \subset X \times Q$ defined by

$$(x, q) \in R \quad \Leftrightarrow \quad x \in X_q.$$

Let us prove that it is an alternating simulation relation. Condition 1) of the definition does obviously hold. Condition 2) reads: for every $q \in Q$, $x \in X_q$ and every $v \in \mathrm{enab}_\Delta(q)$ there exists $(T, u) \in \mathrm{enab}_\delta(x)$ such that for every $x' \in \delta(x, T, u)$ there exists $q' \in \Delta(q, v)$, $x' \in X_{q'}$. By Theorem 2, this condition holds for $T = \tau$ and $u = u^{(q,v)}$. □

Proof of Theorem 4. For the statement to hold, it is sufficient that for every $q \in Q$ and every $\hat{u} \in \hat{U}$ there exists $v \in \mathscr{V}$ such that for every $q' \in \Delta(q, v)$ the inclusion $q' \in \hat{\Delta}(q, \hat{u})$ holds. Let us take $v = \hat{u}$. From Corollary 1 it follows that

$$[\underline{x}^{(q,\hat{u})}(t), \overline{x}^{(q,\hat{u})}(t)] \subseteq [\underline{\mathbf{x}}(t; [\underline{x}^q, \overline{x}^q], \hat{u}), \overline{\mathbf{x}}(t; [\underline{x}^q, \overline{x}^q], \hat{u})]$$

for all $t \in [0, \tau]$. Therefore, $\emptyset \ne \Delta(q, \hat{u}) \subseteq \hat{\Delta}(q, \hat{u})$ for all $q \in Q$, $\hat{u} \in \mathrm{enab}_{\hat{\Delta}}(q) \subseteq \hat{U}$. □

Proof of Theorem 5. Consider an arbitrary generalized reference trajectory $\hat{x}(\cdot)$ such that the state $\hat{x}(\tau_j)$ after the jump belongs to an interval $[\underline{x}(\tau_j), \overline{x}(\tau_j)]$ defined by equations (19), (18). Observe that Lemma 3 is still true. Applying Lemma 4 to each interval $[\tau_{j-1}, \tau_j]$ separately, we obtain that $[\underline{x}(\tau), \overline{x}(\tau)] \in \mathscr{A}^{\hat{x}(\cdot)}$. Therefore, Theorem 2 holds in this case as well. Then it follows that S_a is properly defined, and Theorem 3 and Theorem 4 still hold.

As we mentioned before, for a given control $\hat{u} \in \hat{U}$, there are many generalized reference trajectories. Fix $\tau \ge d_x/r$ and $\varepsilon > 0$. Assume that $\max_i[\overline{x}_i(t) - \underline{x}_i(t)] \ge \varepsilon$ on $[0, \tau]$ for any choice of the reference trajectory. Let us choose $\hat{x}(\cdot)$ so that $\underline{x}_i(t) < \hat{x}_i(t) < \overline{x}_i(t)$ for all $i \in \mathrm{Arg}\max_i[\overline{x}_i(t) - \underline{x}_i(t)]$, for all $t \in [0, \tau]$. Since $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are Lipschitz uniformly in $\hat{x}(\cdot)$, this can be done. Assuming that there exists $0 < \tau' < \tau$ such that $\overline{x}_i(\tau') - \underline{x}_i(\tau') \ge d_x$ for some i leads to a contradiction with the assumption of the theorem. Therefore, we estimate

$$\frac{d}{dt} \max_i [\overline{x}_i(t) - \underline{x}_i(t)] \le -r.$$

Integrating from 0 to τ, we obtain $\max_i[\overline{x}_i(\tau) - \underline{x}_i(\tau)] \le -\tau r + d_x$. It follows that $\tau \le (d_x - \varepsilon)/r$. We arrive at a contradiction with the assumption that $\max_i[\overline{x}_i(t) - \underline{x}(t)] \ge \varepsilon$ on $[0, \tau]$ for any choice of $\hat{x}(\cdot)$.

Therefore, $\min_{t \in [0,\tau]} \max_i[\overline{x}_i(t) - \underline{x}_i(t)] \le \varepsilon$ for some choice of $\hat{x}(\cdot)$. It is straightforward to show that τ can be chosen as the minimizer. Then, we may consider a sequence $\varepsilon_k \to 0$ and the corresponding sequence of reference trajectory. The elements of this sequence are uniformly bounded and Lipschitz continuous. Let $(\overline{x}(\cdot), \underline{x}(\cdot))$ be the limit point of the corresponding sequence of solutions of (19). It follows that $\overline{x}(\tau) = \underline{x}(\tau)$. Clearly, property (b) of the definition of $\mathscr{A}^{\hat{x}(\cdot)}$ still holds as it does not depend on $\hat{x}(\cdot)$. Then Theorem 2 and Theorem 3 do hold as well, and S_a is properly defined. Lastly, inspecting the proof of Theorem 4, one may observe that since it holds for any choice of $\varepsilon > 0$, the result still stands for the limit case. □

Proof of Theorem 6. Consider the abstraction S_a constructed as above. Let us estimate the difference $\dot{\overline{x}}_i - \dot{\underline{x}}_i$:

$$\dot{\overline{x}}_i - \dot{\underline{x}}_i \ge g_i(\zeta_i(x, y), \underline{\theta}, \overline{\omega}) - g_i(\zeta_i(y, x), \overline{\theta}, \underline{\omega}) \ge -r.$$

Integrating from 0 to τ, we obtain that for $\tau \le d_x/r$, the interval $[\underline{x}(t), \overline{x}(t)]$ has non empty interior on $[0, \tau]$. Clearly, for such an interval map, there exists a generalized reference trajectory such that $\hat{x}(t)$ lies in its interior for all t. For such a reference trajectory, system (19) turns into

$$\begin{aligned} \dot{\overline{x}}_i &= g_i(t, \zeta_i(\overline{x}, \underline{x}), \underline{\theta}, \overline{\omega}), \\ \dot{\underline{x}}_i &= g_i(t, \zeta_i(\underline{x}, \overline{x}), \overline{\theta}, \underline{\omega}). \end{aligned} \tag{32}$$

By monotonicity argument, such a choice of reference trajectory results in the unique minimal by inclusion interval $[\underline{x}(\tau), \overline{x}(\tau)]$ among all generalized reference trajectories corresponding to all possible values $\hat{u} \in \hat{U}$. Thus, in abstraction S_a for every symbolic state q and all symbolic controls $\hat{u}^1, \hat{u}^2 \in \hat{U}$, we have $\Delta(q, \hat{u}^1) = \Delta(q, \hat{u}^2)$. Such a transition system is equivalent to the one with a singleton set of symbolic controls. □

Proof of Theorem 7. First, observe that h_* is chosen in such a way that the righthand side of (19) is continuous on $[t, t + h_*]$ in the neighborhood of the trajectory that starts at $(\overline{z}(t), \underline{z}(t), \hat{z}(t))$. This follows from the definition of L_z and the assumption of the theorem.

The corrections in $\overline{z}(\cdot)$ and $\underline{z}(\cdot)$ happen only on the value that is bounded by h^p, and there is only a finite number of those provided that the number of resets of $\hat{z}(\cdot)$ is bounded uniformly in h. The latter follows from the definitions of P and L_z.

To finish the proof of the first statement, we need to show that the number of time steps $h_* < h$ scales as $O(1/h)$ so that the total number of time steps is $O(1/h)$ as well. Suppose that $h_* = (\overline{z}_i(t) - \hat{z}_i(t))/(2L_z) < h$. Then

$$\overline{z}_i(t + h_*) - \hat{z}_i(t + h_*) \le (1 - \frac{r}{2L_z})(\overline{z}_i(t) - \hat{z}_i(t)).$$

Therefore, the number of reduced time steps k needed to find the switch for component i could be obtained from the condition:

$$(1 - \frac{r}{2L_z})^k \le \frac{1}{2L_z} h^{p-1}.$$

We observe that k actually scales as $O(\log \frac{1}{h})$.

To prove the second part, observe that

$$\max_i [\overline{z}_i(t + h_*) - \underline{z}_i(t + h_*)] \ge \max_i [\overline{z}_i(t) - \underline{z}_i(t)]$$

only if the expression on the right is less than ε. $\square$

Proof of Corollary 2. Observe that $\varepsilon_j \geq L_z h_j$ for all $0 \leq j \leq p$. Therefore, $\max_i[\overline{z}_i(\tau) - \underline{z}_i(\tau)] \leq 2\varepsilon_p = 2\varepsilon h^p$. To complete the proof, we note that on each time segment the number of time steps is $O(1/h)$. $\square$