AUJOURD'HUI : <u>EXAMEN BLANC.</u>

DEMAIN : <u>16H – 18H</u> EXAMEN EN DISTANCIEL

(COMPRESHBLE : 14H - 16H)

EXERCICE 1

$$\forall t > 0, \quad y'''(t) - y''(t) + 2y'(t) - y(t) + 2 = 0$$

avec pour condition initiale :

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 2 \end{cases}$$

EDO d'ordre 3 donc 3 composantes

① $\vec{q} : \mathbb{R}^+ \longrightarrow \mathbb{R}^3$

$$t \longmapsto \vec{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}$$

et on identifie, $\forall t \in \mathbb{R}^+$,
$$q_1(t) = y(t)$$
$$q_2(t) = y'(t)$$
$$q_3(t) = y''(t)$$

$$\frac{d\vec{q}}{dt} = \begin{pmatrix} \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \frac{dq_3}{dt} \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{pmatrix} = \begin{pmatrix} q_2(t) \\ q_3(t) \\ y''(t) - 2y'(t) + y(t) - 2 \end{pmatrix}$$

$$\Rightarrow \quad \frac{d\vec{q}}{dt} = \boxed{\begin{pmatrix} q_2(t) \\ q_3(t) \\ q_3(t) - 2q_2(t) + q_1(t) - 2 \end{pmatrix}}$$

De la forme " $\dot{\vec{q}}(t) = f(t, \vec{q}(t))$ "

(2) rhs(t, q_1, q_2, q_3) = (q_2, q_3, q_3 - 2q_2 + q_1 - 2)

Problème de Cauchy :

$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases}$$

Exercice 2

(1) $\forall t > 0 \quad y'(t) = 2t - y(t) \quad (I)$

$y(0) = 1$

Ici, $f : (t, y) \longmapsto 2t - y$.

ÉTAPE 1 : Résolution du problème homogène :

$$y_H'(t) = -y_H(t)$$

$$\Leftrightarrow \quad y_H'(t) + y_H(t) = 0$$

$$\Leftrightarrow \quad \boxed{y_H(t) = A\exp(-t)}$$

ÉTAPE 2 : Recherche d'une solution particulière (méthode de variation de la constante)

$$y_p(t) = g(t)\exp(-t)$$

$$y_p'(t) = g'(t)\exp(-t) + g(t)\left(-\exp(-t)\right)$$
$$= \left[g'(t) - g(t)\right]\exp(-t)$$

On substitue dans l'EDO (I)

$$y'(t) = 2t - y(t) \Rightarrow \left[g'(t) - g(t)\right]\exp(-t)$$
$$= 2t - g(t)\exp(-t)$$

$$\Rightarrow \quad g'(t)\exp(-t) = 2t$$
$$\Rightarrow \quad g'(t) = 2t\exp(t) \qquad \times \exp(t)$$

Primitive de $x \longmapsto x\exp(x)$

$$\left[(x-1)\exp(x)\right]' = \cancel{\exp(x)} + (x-\cancel{1})\exp(x)$$
$$= x\exp(x)$$

$$\Rightarrow \quad g(t) = 2(t-1)\exp(t) + A$$

On en déduit :

$$y_p(t) = g(t)\exp(-t)$$
$$= \left[2(t-1)\exp(t) + A\right]\exp(-t)$$
$$= 2(t-1) + A\exp(-t)$$

ÉTAPE 3 : Application de la condition initiale pour déterminer A.

$$y_p(0) = 1$$
$$= 2(0-1) + A\exp(-0)$$
$$= -2 + A$$

$$\Rightarrow \quad -2 + A = 1$$

$$\Rightarrow \quad A = 3$$

$$\Rightarrow \quad y_p(t) = 2(t-1) + 3\exp(-t)$$

② $\dot{y}(t) = f[t, y(t)]$

E.E. : $y_{n+1} = y_n + \tau f[t_n, y_n]$

$$\frac{y_{n-1} - y_n}{\tau} = f[t, y_n]$$

$$0{,}3/\tau = \frac{0{,}3}{0{,}1} = 3$$

$$\begin{aligned} y_0 &= 1 \\ &\downarrow n=0 \\ y_1 &= \ldots \\ &\downarrow n=1 \\ y_2 &= \ldots \\ &\downarrow n=2 \\ y_3 &= \ldots \end{aligned}$$

$$f : (t, y) \longmapsto 2t - y$$

$n=0$

$$y_1 = \underbrace{y_0}_{=1} + \overbrace{\tau}^{=0{,}1} f(\overbrace{t_0}^{=0}, \underbrace{y_0}_{=1})$$

$$\Rightarrow y_1 = 1 + 0{,}1 \times (2 \times 0 - 1)$$

$$= 1 + 0{,}1 \times (-1)$$

$$= 0{,}9$$

$\Rightarrow \quad y_1 - y(t_1) = -0{,}012$

$n=1 \quad y_2 = y_1 + \underset{=0{,}1}{\tau} f(\underset{=0{,}1}{t_1}, \underset{=0{,}9}{y_1})$

$= 0{,}9 + 0{,}1\,(2\times 0{,}1 - 0{,}9)$

$= 0{,}9 + 0{,}1\,(-0{,}7)$

$= 0{,}83$

$y_2 - y(t_2) = -0{,}026$

$n=2 \quad y_3 = \underset{=0{,}83}{y_2} + \underset{=0{,}1}{\tau} f(\underset{=0{,}2}{t_2}, y_2)$

$= 0{,}83 + 0{,}1\,(2\times t_2 - y_2)$

$= 0{,}83 + 0{,}1\,(2\times 0{,}2 - 0{,}83)$

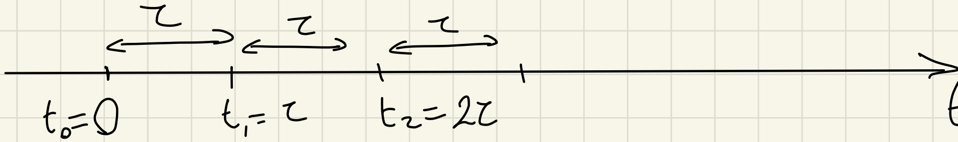
$= 0{,}83 + 0{,}1\,(0{,}4 - 0{,}83)$

$= 0{,}83 + 0{,}1\,(-0{,}43)$

$= 0{,}83 - 0{,}043$

$= 0{,}787$

$y_3 - y(t_3) = -0{,}035$



EXERCICE 3

Étude d'un nouveau schéma :

$$y_{n+1} = y_n + \tau \left[(1-\omega) f(t_n, y_n) + \omega \; f(t_{n+1}, y_{n+1}) \right]$$

avec $\omega \in [0,1]$

① $\omega = 0$ $\qquad y_{n+1} = y_n + \tau f(t_n, y_n)$ $\qquad$ EXPLICITE EULER

$\omega = 1$ $\qquad y_{n+1} = y_n + \tau f(t_{n+1}, y_{n+1})$ $\qquad$ IMPLICITE EULER

$\omega = 2$ $\qquad y_{n+1} = y_n + \tau \left[\dfrac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} \right]$

REGLE DES TRAPÈZES

② $f : (t, y) \longmapsto \lambda y$

$$\begin{cases} \dot{y}(t) = \lambda y(t) \\ y(0) = y_0 \end{cases} \Rightarrow \boxed{y(t) = \exp(\lambda t) y_0}$$

$$y(t_{n+1}) = \exp(\lambda t_{n+1}) y_0$$

$$= \exp\left[\lambda(1+n)\tau\right] y_0$$

$$= \exp(\lambda\tau)\exp(\lambda \underbrace{n\tau}_{=t_n}) y_0$$

$$= \exp(\lambda\tau) \underbrace{\exp(\lambda t_n) y_0}_{= y(t_n)}$$

$$\Rightarrow \quad y(t_{n+1}) = \exp(\lambda\tau)\, y(t_n)$$

$$= \sigma(\lambda\tau) = \sigma(z) \quad \text{en posant } z = \lambda\tau$$

③ $$y_{n+1} = y_n + \tau\left[(1-\omega)\lambda y_n + \omega\lambda y_{n+1}\right] \qquad (f(t,y) \leftarrow \lambda y)$$

$$y_{n+1} - \omega \underbrace{\lambda\tau}_{=z} y_{n+1} = y_n + (1-\omega)\underbrace{\lambda\tau}_{=z} y_n$$

$$[1 - \omega z]\, y_{n+1} = [1 + (1-\omega)z]\, y_n$$

$$\boxed{y_{n+1} = \underbrace{\frac{1+(1-\omega)z}{1-\omega z}}_{=\overline{\sigma}(z,\omega)} y_n}$$

$$\sigma(z) = \exp(z)$$
$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$
$$\left(\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \quad \text{avec } 0! = 1 \right)$$

$$\bar{\sigma}(z,\omega) = \frac{1+(1-\omega)z}{1-\omega z}$$
$$= [1+(1-\omega)z]\left[\frac{1}{1-x}\right]$$
où $x = \omega z$

Rappel : $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

donc $\frac{1}{1-x} = \frac{1}{1-\omega z} = 1 + \omega z + (\omega z)^2 + (\omega z)^3 + \dots$

D'où : $\bar{\sigma}(z,\omega) = [1+(1-\omega)z][1 + \omega z + (\omega z)^2 + (\omega z)^3 + \dots]$

$$= 1 + \omega z + (\omega z)^2 + (\omega z)^3$$
$$+ (1-\omega)z + \omega(1-\omega)z^2$$
$$+ \omega^2(1-\omega)z^3 + \dots$$

$$= 1 + z + \omega z^2 + \omega^2 z^3 + \dots$$

La condition à satisfaire pour obtenir un schéma d'ordre 2 est $\omega = 1/2$ (règle des trapèzes)

(5) $\bar{\sigma}(z,\omega) = \frac{1+(1-\omega)z}{1-\omega z}$ avec $z = x + iy$

$$|\bar{\sigma}(z,\omega)|^2 = \left| \frac{1+(1-\omega)(x+iy)}{1-\omega(x+iy)} \right|^2$$

$$= \left| \frac{1 + (1-\omega)x + (1-\omega)y\, i}{1 - \omega x - \omega y\, i} \right|^2$$

$$= \frac{[1 + (1-\omega)x]^2 + [(1-\omega)y]^2}{(1-\omega x)^2 + (\omega y)^2}$$

EXERCICE 4

Formule de Taylor-Young de y au voisinage de a

$$y(x) = \sum_{n=0}^{+\infty} \frac{(x-a)^n}{n!} y^{(n)}(a)$$

$y \leftarrow z'$

$$z'(x) = \sum_{n=0}^{+\infty} \frac{(x-a)^n}{n!} z^{(n+1)}(a)$$

ORDRE 4

$$y(x) = y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2} y''(a) + \frac{(x-a)^3}{6} y'''(a) + \frac{(x-a)^4}{24} y^{(iv)}(a) + \dots \quad (II)$$

ORDRE 3

$$y'(x) = y'(a) + (x-a)y''(a) + \frac{(x-a)^2}{2} y'''(a) + \frac{(x-a)^3}{6} y^{(iv)}(a) + \dots \quad (III)$$

$a = x_{1/2} = h/2$

$y'_0 \; : \quad x = x_0 = 0$

À partir de III, et avec $x - a = 0 - h/2 = -h/2$

$$y'_0 = y'_{1/2} - \frac{h}{2} y''_{1/2} + \frac{h^2}{8} y'''_{1/2} - \frac{h^3}{48} y^{(iv)}_{1/2}$$

$y_1 \; : \quad x = x_1 = h$

À partir de I et avec $x - a = h - h/2 = h/2$

$$y_1 = y_{1/2} + \frac{h}{2} y'_{1/2} + \frac{h^2}{8} y''_{1/2} + \frac{h^3}{48} y'''_{1/2} + \frac{h^4}{384} y^{(iv)}_{1/2}$$

$y_2 \; : \quad x = x_2 = 2h$

$$\Rightarrow x - a = 2h - h/2 = 3h/2$$

$$y_2 = y_{1/2} + \frac{3h}{2} y'_{1/2} + \frac{9h^2}{8} y''_{1/2} + \frac{9h^3}{16} y'''_{1/2} + \frac{27h^4}{128} y^{(iv)}_{1/2}$$

$$y''_{1/2} = \alpha y'_0 + \beta y_1 + \gamma y_2$$

$$\Rightarrow y''_{1/2} = \alpha \left(y'_{1/2} - \frac{h}{2} y''_{1/2} + \frac{h^2}{8} y'''_{1/2} - \frac{h^3}{48} y^{(iv)}_{1/2} \right)$$
$$+ \beta \left(y_{1/2} + \frac{h}{2} y'_{1/2} + \frac{h^2}{8} y''_{1/2} + \frac{h^3}{48} y'''_{1/2} + \frac{h^4}{384} y^{(iv)}_{1/2} \right)$$
$$+ \gamma \left(y_{1/2} + \frac{3h}{2} y'_{1/2} + \frac{9h^2}{8} y''_{1/2} + \frac{9h^3}{16} y'''_{1/2} + \frac{27h^4}{128} y^{(iv)}_{1/2} \right)$$

$$0 = (\beta + \gamma) y_{1/2} + \left(\alpha + \beta \frac{h}{2} + \frac{3\gamma h}{2} \right) y'_{1/2}$$
$$+ \left(-\alpha \frac{h}{2} + \beta \frac{h^2}{8} + \frac{9\gamma h^2}{8} - 1 \right) y''_{1/2}$$

$$+\left(\alpha \frac{h^2}{8} + \beta \frac{h^3}{48} + \frac{9\gamma h^3}{16}\right) y'''_{1/2} + \dots$$

$$\Rightarrow \begin{cases} \beta + \gamma = 0 & (A) \\ \alpha + \beta \frac{h}{2} + \frac{3\gamma h}{2} = 0 & (B) \\ -\alpha \frac{h}{2} + \beta \frac{h^2}{8} + \frac{9\gamma h^2}{8} = 1 & (C) \end{cases}$$

$$(A) \Rightarrow \gamma = -\beta \quad (A')$$

$$(A') \to (B) \Rightarrow \alpha + \beta \frac{h}{2} - \frac{3\beta h}{2} = 0$$

$$\Rightarrow \alpha - \beta h = 0$$

$$\Rightarrow \alpha = \beta h \quad (B')$$

$$\left.\begin{matrix} (B') \to (C) \\ (A') \to (C) \end{matrix}\right) \Rightarrow -\alpha \frac{h}{2} + \alpha \frac{h}{8} - \frac{9\alpha h}{8} = 1$$

$$\Rightarrow \alpha h\left(-\frac{1}{2} + \frac{1}{8} - \frac{9}{8}\right) = 1$$

$$\Rightarrow \alpha h\left(\frac{-4+1-9}{8}\right) = 1$$

$$\Rightarrow \alpha h\left(-\frac{12}{8}\right) = 1$$

$$\Rightarrow \alpha h\left(-\frac{3}{2}\right) = 1 \Rightarrow \alpha = -\frac{2}{3h}$$

$$(B') : \alpha = \beta h \Rightarrow \beta = \frac{\alpha}{h} \Rightarrow \beta = -\frac{2}{3h^2}$$

$(A') : \gamma = -\beta \Rightarrow \gamma = \frac{2}{3h^2}$

$$y''_{1/2} = -\frac{2}{3h} y'_0 + \frac{2}{3h^2}(y_2 - y_1)$$

ERREUR DE TRONCATURE = ?

$$\left(\alpha \frac{h^2}{8} + \beta \frac{h^3}{48} + \frac{9\gamma h^3}{16}\right) y'''_{1/2}$$

$$= \left(\alpha \frac{h^2}{8} + \frac{\alpha h^2}{48} - \frac{9\alpha h^2}{16}\right) y'''_{1/2}$$

$$= \alpha h^2 \left(\frac{1}{8} + \frac{1}{48} - \frac{1}{16}\right) y'''_{1/2}$$

$$= -\frac{2h}{3}\left(\frac{6 + 1 - 3}{48}\right) y'''_{1/2}$$

$$= -\frac{2h}{3}\left(\frac{4}{48}\right) y'''_{1/2}$$

$$= -\frac{h}{18} y'''_{1/2}$$

de la forme $A h^p$ avec $p = 1$

$\boxed{\text{SCHEMA D'ORDRE 1}}$