Chapter 7

Equations of Motion

7.1 General

The rigid body equations of motion are the differential equations that describe the evolution of the twelve basic states of an aircraft: the scalar components of $\mathbf{v}_B$, $\boldsymbol{\omega}_B$, and $\mathbf{r}_C$, plus the three Euler angles that define $T_{B,I}$. Most of the differential equations needed were derived either in Chapter 4, "Rotating Coordinate Systems" or as problems following Chapter 5, "Inertial Accelerations".

The usual coordinate systems in which we represent the equations of motion are the wind and body axis systems. Mixed systems using both coordinate systems are common. Any such set of equations of motion must be *complete*. This means that in every expression $d\,(State)\,/dt = \ldots$, everything on the right-hand side must be given by either an algebraic equation (including $variable = constant$), by another differential equation, or by some external input to the system (e.g., the pilot's application of controls).

7.2 Body Axis Equations

7.2.1 Body Axis Force Equations

The body axis force equations were previously derived, and are

$$\{\mathbf{F}\}_B = m\,\{\dot{\mathbf{v}}_B\}_B + m\,\{\Omega_B\}_B\,\{\mathbf{v}\}_B$$

In terms of the rates-of-change of the inertial components of velocity, as seen in the body-axes:

$$\{\dot{\mathbf{v}}_B\}_B = \frac{1}{m}\{\mathbf{F}\}_B - \{\Omega_B\}_B\{\mathbf{v}\}_B$$

All of the terms in this equation have been defined, so all we have to do is make the necessary substitutions and expand the equations to solve for $\dot{u}$, $\dot{v}$, and $\dot{w}$. On the right-hand side, the net applied force is comprised of aerodynamic forces, the aircraft weight, and the force due to thrust:

$$\begin{aligned}\{\mathbf{F}\}_B &= \{\mathbf{F}_A\}_B + \{\mathbf{W}\}_B + \{\mathbf{T}\}_B \\ &= \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} + \begin{Bmatrix} -mg\sin\theta \\ mg\sin\phi\cos\theta \\ mg\cos\phi\cos\theta \end{Bmatrix} + \begin{Bmatrix} T\cos\epsilon \\ 0 \\ T\sin\epsilon \end{Bmatrix}\end{aligned} \tag{7.1}$$

For the motion variables we substitute

$$\{\mathbf{v}\}_B = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}, \ \{\dot{\mathbf{v}}_B\}_B = \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix}$$

$$\{\Omega_B\}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

The cross-product term is easily evaluated, and is

$$\{\Omega_B\}_B\{\mathbf{v}\}_B = \begin{Bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{Bmatrix}$$

As a result we may write the body axis force equations,

$$\dot{u} = \frac{1}{m}(X + T\cos\epsilon_T) - g\sin\theta + rv - qw \tag{7.2a}$$

$$\dot{v} = \frac{1}{m}(Y) + g\sin\phi\cos\theta + pw - ru \tag{7.2b}$$

$$\dot{w} = \frac{1}{m}(Z + T\sin\epsilon_T) + g\cos\phi\cos\theta + qu - pv \tag{7.2c}$$

7.2.2 Body Axis Moment Equations

The moment equations are given by

$$\{\mathbf{M}\}_B = I_B\{\dot{\boldsymbol{\omega}}_B\}_B + \{\Omega_B\}_B I_B\{\boldsymbol{\omega}_B\}_B$$

In terms of the rates of change of the inertial components of angular rotation, as seen in the body axes,

$$\{\dot{\boldsymbol{\omega}}_B\}_B = I_B^{-1}\left[\{\mathbf{M}\}_B - \{\Omega_B\}_B I_B\{\boldsymbol{\omega}_B\}_B\right]$$

The externally applied moments are those due to aerodynamics and thrust. We will ignore rolling and yawing moments due to thrust as these are special cases (note, however, that yawing moment due to thrust is an important consideration in multi-engine aircraft, and may have to be added in for analysis of the consequences of engine failure). As a result the moments are

$$\{\mathbf{M}\}_B = \begin{Bmatrix} L \\ M + M_T \\ N \end{Bmatrix}$$

We assume a plane of symmetry so in the inertia matrix the cross-products involving y become zero,

$$I_B = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

The inverse is then given by

$$I_B^{-1} = \frac{1}{I_D} \begin{bmatrix} I_{zz} & 0 & I_{xz} \\ 0 & I_D/I_{yy} & 0 \\ I_{xz} & 0 & I_{xx} \end{bmatrix}$$

Here, $I_D = I_{xx}I_{zz} - I_{xz}^2$. The rest is just substitution and expansion:

$$\dot{p} = \frac{I_{zz}}{I_D} \left[L + I_{xz}pq - \left(I_{zz} - I_{yy}\right) qr\right] + \frac{I_{xz}}{I_D} \left[N - I_{xz}qr - \left(I_{yy} - I_{xx}\right) pq\right] \tag{7.3a}$$

$$\dot{q} = \frac{1}{I_{yy}} \left[M + M_T - \left(I_{xx} - I_{zz}\right) pr - I_{xz} \left(p^2 - r^2\right)\right] \tag{7.3b}$$

$$\dot{r} = \frac{I_{xz}}{I_D} \left[L + I_{xz}pq - \left(I_{zz} - I_{yy}\right) qr\right] + \frac{I_{xx}}{I_D} \left[N - I_{xz}qr - \left(I_{yy} - I_{xx}\right) pq\right] \tag{7.3c}$$

If we are lucky enough to be working the problem in principal axes, then

$$\begin{aligned} \dot{p} &= \left[L - \left(I_{zp} - I_{yp}\right) qr\right] / I_{xp} \\ \dot{q} &= \left[M + M_T - \left(I_{xp} - I_{zp}\right) pr\right] / I_{yp} \\ \dot{r} &= \left[N - \left(I_{yp} - I_{xp}\right) pq\right] / I_{zp} \end{aligned}$$

7.2.3 Body Axis Orientation Equations (Kinematic Equations)

The required equations were derived in Chapter 4, "Rotating Coordinate Systems". We will use the Euler angle relationships, with the Euler angles (flat Earth assumption) from the transformation from the local horizontal to the body axes. The resulting matrix equation is

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

As three scalar equations we have

$$\dot{\phi} = p + (q \sin\phi + r \cos\phi) \tan\theta \tag{7.4a}$$
$$\dot{\theta} = q \cos\phi - r \sin\phi \tag{7.4b}$$
$$\dot{\psi} = (q \sin\phi + r \cos\phi) \sec\theta \tag{7.4c}$$

7.2.4 Body Axis Navigation Equations

The position of the aircraft relative to the Earth is found by integrating the aircraft velocity along its path, or by representing the velocity in Earth-fixed coordinates and integrating each component. The latter is easier, and is given by

$$\begin{Bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{Bmatrix} = T_{H,B} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

The needed transformation is $T_{B,H}^T$. Expanding the equations,

$$\dot{x}_E = u\,(\cos\theta\cos\psi) + v\,(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\ + w\,(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \tag{7.5a}$$

$$\dot{y}_E = u\,(\cos\theta\sin\psi) + v\,(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) \\ + w\,(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \tag{7.5b}$$

$$\dot{h} = -\dot{z}_E = u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \tag{7.5c}$$

7.3 Wind Axis Equations

7.3.1 Wind Axis Force Equations

Development of the wind axis force equations begins similarly to the body axis equations, with

$$\{\dot{\mathbf{v}}_W\}_W = \frac{1}{m}\{\mathbf{F}\}_W - \{\Omega_W\}_W\{\mathbf{v}\}_W$$

The external forces are

$$\{\mathbf{F}_A\}_W = \begin{Bmatrix} -D \\ -C \\ -L \end{Bmatrix}$$

$$\{\mathbf{T}\}_W = T_{W,B}\begin{Bmatrix} T\cos\epsilon_T \\ 0 \\ T\sin\epsilon_T \end{Bmatrix} = \begin{Bmatrix} T\cos\beta\cos(\epsilon_T-\alpha) \\ -T\sin\beta\cos(\epsilon_T-\alpha) \\ T\sin(\epsilon_T-\alpha) \end{Bmatrix}$$

$$\{\mathbf{W}_W\} = \begin{Bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{Bmatrix}$$

Again inserting the linear and angular velocity components,

$$\{\mathbf{v}\}_W = \begin{Bmatrix} V \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\boldsymbol{\omega}_W\}_W = \begin{Bmatrix} p_W \\ q_W \\ r_W \end{Bmatrix}$$

$$\{\Omega_W\}_W\{\mathbf{v}\}_W = \begin{Bmatrix} 0 \\ V r_W \\ -V q_W \end{Bmatrix}$$

The resulting scalar equations are

$$\dot{V} = \frac{1}{m}\left[-D - mg\sin\gamma + T\cos\beta\cos\left(\epsilon_T - \alpha\right)\right] \tag{7.6a}$$
$$r_W = \frac{1}{mV}\left[-C - mg\sin\mu\cos\gamma - T\sin\beta\cos\left(\epsilon_T - \alpha\right)\right] \tag{7.6b}$$
$$q_W = \frac{1}{mV}\left[L - mg\cos\mu\cos\gamma - T\sin\left(\epsilon_T - \alpha\right)\right] \tag{7.6c}$$

This is a little different from the body axis results, in that we have one differential equation and two algebraic equations. In body axes we had differential equations for each of the three velocity components so that angle-of-attack and sideslip could be calculated at any instant for use in determining forces and moments. What is needed for the wind axis force equations are separate relationships for α and β.

We know that the angle rates $\dot{\alpha}$ and $\dot{\beta}$ will be related to the relative rotation of the wind and body axes through the previously derived result

$$\left\{\boldsymbol{\omega}_2^1\right\}_2 = \begin{bmatrix} 1 & 0 & -\sin\theta_y \\ 0 & \cos\theta_x & \sin\theta_x\cos\theta_y \\ 0 & -\sin\theta_x & \cos\theta_x\cos\theta_y \end{bmatrix} \begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix}$$

We substitute $(\theta_x, \theta_y, \theta_z) = (0, \alpha, -\beta)$ to arrive at

$$\left\{\boldsymbol{\omega}_B^W\right\}_B = \begin{bmatrix} 1 & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\alpha} \\ -\dot{\beta} \end{Bmatrix} = \begin{Bmatrix} -\dot{\beta}\sin\alpha \\ \dot{\alpha} \\ \dot{\beta}\cos\alpha \end{Bmatrix}$$

This is the rate at which the body axes rotate relative to the wind axes. From the body axis moment equations we can determine $\{\boldsymbol{\omega}_B\}_B$. We already have algebraic expressions for two components of $\{\boldsymbol{\omega}_W\}_W$ (q_W and r_W, equation 7.6 above) so with a little luck we should be able to assemble these results to get equations for $\dot{\alpha}$ and $\dot{\beta}$. We begin with

$$\begin{aligned} \{\boldsymbol{\omega}_W\}_W &= \{\boldsymbol{\omega}_B\}_W - \left\{\boldsymbol{\omega}_B^W\right\}_W \\ &= T_{W,B}\left[\{\boldsymbol{\omega}_B\}_B - \left\{\boldsymbol{\omega}_B^W\right\}_B\right] \end{aligned}$$

In terms of their components these equations are

$$\begin{Bmatrix} p_W \\ q_W \\ r_W \end{Bmatrix} = T_{W,B} \left[\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} - \begin{Bmatrix} -\dot{\beta}\sin\alpha \\ \dot{\alpha} \\ \dot{\beta}\cos\alpha \end{Bmatrix} \right]$$
$$= \begin{Bmatrix} p\cos\alpha\cos\beta + \sin\beta\,(q - \dot{\alpha}) + r\sin\alpha\cos\beta \\ -p\cos\alpha\sin\beta + \cos\beta\,(q - \dot{\alpha}) - r\sin\alpha\sin\beta \\ -p\sin\alpha + r\cos\alpha + \dot{\beta} \end{Bmatrix}$$

These equations are easily solved for p_W and two differential equations for $\dot{\alpha}$ and $\dot{\beta}$:

$$p_W = p\cos\alpha\cos\beta + \sin\beta\,(q - \dot{\alpha}) + r\sin\alpha\cos\beta \qquad (7.7a)$$
$$\dot{\alpha} = q - \sec\beta\,(q_W + p\cos\alpha\sin\beta + r\sin\alpha\sin\beta) \qquad (7.7b)$$
$$\dot{\beta} = r_W + p\sin\alpha - r\cos\alpha \qquad (7.7c)$$

The three differential equations for use in wind axes are therefore

$$\dot{V} = \frac{1}{m}\left[-D - mg\sin\gamma + T\cos\beta\cos(\epsilon_T - \alpha)\right] \qquad (7.8a)$$
$$\dot{\alpha} = q - \sec\beta\,(q_W + p\cos\alpha\sin\beta + r\sin\alpha\sin\beta) \qquad (7.8b)$$
$$\dot{\beta} = r_W + p\sin\alpha - r\cos\alpha \qquad (7.8c)$$

The wind axis angular rates to be used in these equations are given by the two algebraic equations from equation 7.6,

$$q_W = \frac{1}{mV}\left[L - mg\cos\mu\cos\gamma - T\sin(\epsilon_T - \alpha)\right]$$
$$r_W = \frac{1}{mV}\left[-C + mg\sin\mu\cos\gamma - T\sin\beta\cos(\epsilon_T - \alpha)\right]$$

As a result *we may mix the wind axis force equations with the body axis moment equations, and never have to worry about the wind axis moment equations.*

It should be noted that the differential equation for $\dot{\alpha}$ depends on q_W, which depends on C_L through the lift L. If C_L is dependent on $\dot{\alpha}$ then this dependency must be considered in solving for $\dot{\alpha}$. If C_L has some simple dependency on $\dot{\alpha}$, such as a linear relationship, then it may be easy to factor out $\dot{\alpha}$ and combine it and its factor with the explicit $\dot{\alpha}$ on the left-hand side of the equation. Moreover, if additionally C_m is dependent on $\dot{\alpha}$, then the body-axis pitching moment equation will contain not only $\dot{q}$ but $\dot{\alpha}$ as well, in which case the wind-axis $\dot{\alpha}$ equation must be solved simultaneous with the body-axis pitching moment equation to determine two equations for $\dot{\alpha}$ and $\dot{q}$.

If neither C_L nor C_m is functionally dependent on $\dot{\alpha}$ then a somewhat simpler approach is possible. Rather than use the mixed formulations in 7.8 we may relate $\dot{V}$, $\dot{\alpha}$, and $\dot{\beta}$ directly to the body-axis force equations. With a few derivatives we proceed as follows:

$$\alpha \equiv \tan^{-1}\left(\frac{w}{u}\right) \Rightarrow \dot{\alpha} = \frac{u\dot{w} - w\dot{u}}{u^2 + w^2} \tag{7.9a}$$

$$V \equiv \sqrt{u^2 + v^2 + w^2} \Rightarrow \dot{V} = \frac{u\dot{u} + v\dot{v} + w\dot{w}}{\sqrt{u^2 + v^2 + w^2}} = \frac{u\dot{u} + v\dot{v} + w\dot{w}}{V} \tag{7.9b}$$

The $\dot{\beta}$ expression is best formulated in terms of V and $\dot{V}$

$$\beta \equiv \sin^{-1}\left(\frac{v}{V}\right) \Rightarrow \dot{\beta} = \frac{V\dot{v} - v\dot{V}}{V\sqrt{u^2 + w^2}} \tag{7.9c}$$

Some care must be taken when mixing the two systems (u, v, w) and (V, α, β). It should be understood that one first evaluates $\dot{u}$, $\dot{v}$, and $\dot{w}$ from equation 7.2, then applies those results to equations 7.9a and 7.9b, and then V and $\dot{V}$ from equation 7.9b are applied to equation 7.9c.

7.3.2 Wind-Axis Orientation Equations (Kinematic Equations)

With the appropriate substitutions,

$$\begin{Bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\chi} \end{Bmatrix} = \begin{bmatrix} 1 & \sin\mu\tan\gamma & \cos\mu\tan\gamma \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sin\mu\sec\gamma & \cos\mu\sec\gamma \end{bmatrix} \begin{Bmatrix} p_W \\ q_W \\ r_W \end{Bmatrix}$$

The previously derived results for p_W, q_W, and r_W (equations 7.6 and 7.7) may be used. In scalar form,

$$\dot{\mu} = p_W + (q_W \sin\mu + r_W \cos\mu)\tan\gamma \tag{7.10a}$$

$$\dot{\gamma} = q_W \cos\mu - r_W \sin\mu \tag{7.10b}$$

$$\dot{\chi} = (q_W \sin\mu + r_W \cos\mu)\sec\gamma \tag{7.10c}$$

7.3.3 Wind Axis Navigation Equations

This proceeds exactly like the body axis equations, only with simpler results (Flat Earth):

$$\begin{Bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{Bmatrix} = T_{H,W} \begin{Bmatrix} V \\ 0 \\ 0 \end{Bmatrix}$$

$$\dot{x}_E = V\cos\gamma\cos\chi \tag{7.11a}$$

$$\dot{y}_E = V\cos\gamma\sin\chi \tag{7.11b}$$

$$\dot{z}_E = -V\sin\gamma \tag{7.11c}$$

7.4 Steady-State Solutions

7.4.1 General

The equations of motion developed above are all nonlinear first order ordinary differential equations. In addition they are highly coupled, i.e., each differential equation depends upon variables which are described by other differential equations. Analytical solutions to such equations are not known.

We may, however, gain some insight into the equations of motion by examining steady-state solutions, which then are algebraic equations. The term "steady-state" as applied to the equations of motion is not quite accurate. If taken literally it means that the time rate of change of each state is zero. Considering just the navigation equations it is clear that this makes for an uninteresting case, since it implies the aircraft is parked somewhere. As the term is normally used "steady-state" should be taken to mean the linear and angular motion variables are constant. That is,

$$\dot{u} = \dot{v} = \dot{w} = \dot{p} = \dot{q} = \dot{r} = 0 \tag{7.12}$$

and

$$\dot{V} = \dot{\alpha} = \dot{\beta} = \dot{p_W} = \dot{q_W} = \dot{r_W} = 0 \tag{7.13}$$

In addition we will require that all the controls be constant, or

$$\delta_\ell, \delta_m, \delta_n, \delta_T : Constant \tag{7.14}$$

These conditions alone can tell us a lot about the force and moment equations. The fact that all twelve equations are coupled has implications for the other equations as well, which we will examine.

Forces and Moments

Consider the aerodynamic and thrust forces and moments. With the exception of altitude (density) dependencies each is dependent only on independent variables that are constant in steady-state flight. If we restrict our analysis to flight at nearly constant altitude then we may conclude that all these forces and moments are constant as well:

$$\begin{aligned} &D, C, L : Constant \\ &X, Y, Z : Constant \\ &L, M, N : Constant \\ &T, M_T : Constant \end{aligned} \tag{7.15}$$

Euler Angles

In steady-state flight the body axis force equations become

$$X + T\cos\epsilon_T - mg\sin\theta + m\,(rv - qw) = 0 \tag{7.16a}$$
$$Y + mg\sin\phi\cos\theta + m\,(pw - ru) = 0 \tag{7.16b}$$
$$Z + T\sin\epsilon_T + mg\cos\phi\cos\theta + m\,(qu - pv) = 0 \tag{7.16c}$$

In these equations the only terms that have not been assumed or shown to be constant are those involving θ and ϕ. It is easy to show that they must be constant as well. Aside from the mathematics, the physical reason is that if θ or ϕ change then the orientation of the gravity vector with respect to the aircraft changes and the forces are no longer balanced. The Euler angle ψ does not appear in the equations, but with θ, ϕ, q, and r all constant, the kinematic equations (equation 7.4) tell us that $\dot{\psi}$ must be constant as well. We therefore add to our list of conditions for steady-state flight,

$$\dot{\phi} = 0 \tag{7.17a}$$
$$\dot{\theta} = 0 \tag{7.17b}$$
$$\dot{\psi} = Constant \tag{7.17c}$$

By similar reasoning with the wind axis force equations we arrive at

$$\dot{\mu} = 0 \tag{7.18a}$$
$$\dot{\gamma} = 0 \tag{7.18b}$$
$$\dot{\chi} = Constant \tag{7.18c}$$

With only one of the three Euler angles permitted to vary with time, the relationships between the body- and wind-axis rates is easily evaluated:

$$p = -\dot{\psi}\sin\theta = Constant \tag{7.19a}$$
$$q = \dot{\psi}\sin\phi\cos\theta = Constant \tag{7.19b}$$
$$r = \dot{\psi}\cos\phi\cos\theta = Constant \tag{7.19c}$$

$$p_W = -\dot{\chi}\sin\gamma = Constant \tag{7.20a}$$
$$q_W = \dot{\chi}\sin\mu\cos\gamma = Constant \tag{7.20b}$$
$$r_W = \dot{\chi}\cos\mu\cos\gamma = Constant \tag{7.20c}$$

7.4.2 Special Cases

Straight Flight

Straight flight means the aircraft is not turning relative to the Earth. This condition is satisfied only if $\dot{\psi} = 0$ and $\dot{\chi} = 0$. From our discussion above, this implies that $p = q = r = 0$ and $p_W = q_W = r_W = 0$.

The longitudinal equations (body-axis moment, wind-axis force) in steady, straight flight reduce to $M + M_T = 0$, $-D + T\cos\beta\cos(\epsilon_T - \alpha) = mg\sin\gamma$, and $L - T\sin(\epsilon_T - \alpha) = mg\cos\mu\cos\gamma$.

In the lateral-directional moment equations (body axes) [1] we have $I_{zz}L + I_{xz}N = 0$ and $I_{xz}L + I_{xx}N = 0$. From these relationships it is easy to show that $L = N = 0$. Also, the body-axis side force equation yields $Y = -mg\sin\phi\cos\theta$. Thus, the general requirements and conditions for straight flight are

$$C_m(M, \alpha, \delta_m) + C_{m_T}\left(\hat{V}, \delta_T\right) = 0 \tag{7.21a}$$
$$C_D(M, \alpha, \delta_m) - C_T\left(\hat{V}, \delta_T\right)\cos\beta\cos(\epsilon_T - \alpha) + mg\sin\gamma/\bar{q}S = 0 \tag{7.21b}$$
$$C_L(M, \alpha, \delta_m) - C_T\left(\hat{V}, \delta_T\right)\sin(\epsilon_T - \alpha) - mg\cos\mu\cos\gamma/\bar{q}S = 0 \tag{7.21c}$$

and

$$C_\ell(\beta, \delta_\ell, \delta_n) = 0 \tag{7.22a}$$
$$C_n(\beta, \delta_\ell, \delta_n) = 0 \tag{7.22b}$$
$$C_Y(\beta, \delta_n) + mg\sin\phi\cos\theta/\bar{q}S = 0 \tag{7.22c}$$

[1] Throughout the discussion in the text it is assumed there are no externally applied asymmetric moments. This would be the case, for example, if a multi-engine aircraft experienced an engine failure on one wing or the other. Such abnormal flight conditions require additional analysis.

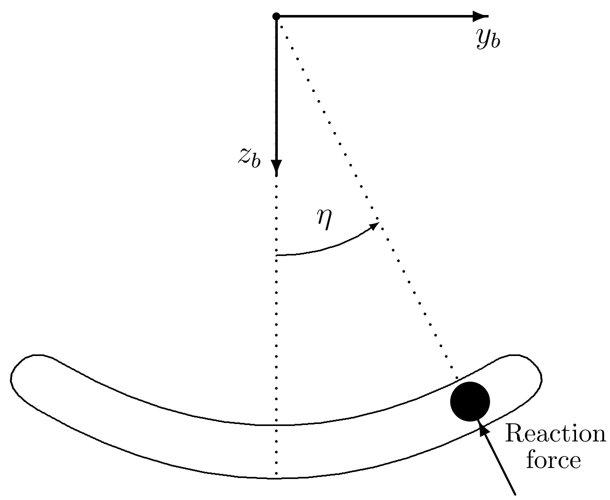

Figure 7.1: Aircraft Slip Indicator

Equations 7.21 and 7.22 are six equations that may be satisfied for various combinations of the ten independent variables V, α, β, ϕ, θ, δ_ℓ, δ_m, δ_n, δ_T, and h. Note that V and h determine M and $\bar{q}$, and that when α, β, ϕ, and θ are specified then μ and γ are determined through $T_{H,W} = T_{H,B}T_{B,W}$.

Symmetric Flight

Symmetric flight means the aircraft has no sideslip, or

$$\beta = v = 0 \tag{7.23}$$

Balanced Flight

Balanced flight is related to and often confused with symmetric flight. Almost every airplane built has a so-called "slip indicator"[2]. It consists of a ball in a curved tube mounted on the pilots intrument panel. The tube is filled with a fluid that dampens the motion of the ball, and prevents it from sticking so the ball is free to move within the tube, as depicted in figure 7.4.2.

[2] The slip indicator is normally combined as a single instrument with a gyroscopic turn indicator, all of which is referred to as the turn-and-slip indicator.

Balanced flight is normally taken to mean flight in which the ball in the slip indicator is centered. The confusion between symmetric and balanced flight comes from the fact that, despite its name, the slip indicator does not in general measure sideslip. Under normal conditions a centered ball means there is very little sideslip, however.

To analyze the conditions of balanced flight, assume the tube is a circular arc, and measure the displacement of the ball from the center of the arc by the angle η, positive in the sense shown. The instrument is mounted in the $y-z$ plane of the body-fixed coordinate system as shown; assume the ball is always at the center of mass of the airplane.

With small thrust angle ($\epsilon_T \approx 0$) the aircraft force equations in body axes (equations 7.2) are

$$\begin{aligned}
\dot{u} &= \frac{1}{m}(X+T) - g\sin\theta + (rv - qw) \\
\dot{v} &= \frac{Y}{m} + g\sin\phi\cos\theta + (pw - ru) \\
\dot{w} &= \frac{Z}{m} + g\cos\phi\cos\theta + (qu - pv)
\end{aligned}$$

The ball is a rigid body in steady flight, and it follows the same trajectory as the CG of the aircraft. Moreover, if a coordinate system for the ball is taken as parallel to the body-axis coordinate system, then the quantities $\dot{u}$, $\dot{v}$, $\dot{w}$, u, v, w, p, q, and r are the same for the ball as for the aircraft, since these accelerations and velocities are those of the coordinate system itself. Likewise, the angles ϕ and θ describe the orientation of the coordinate system, so they are the same for ball and aircraft.

With subscript "b" to denote the ball, the force equations of the ball in body-axes are:

$$\begin{aligned}
\dot{u}_b &= \frac{X_b}{m_b} - g\sin\theta_b + (r_b v_b - q_b w_b) \\
\dot{v}_b &= \frac{Y_b}{m_b} + g\sin\phi_b\cos\theta_b + (p_b w_b - r_b u_b) \\
\dot{w}_b &= \frac{Z_b}{m_b} + g\cos\phi_b\cos\theta_b + (q_b u_b - p_b v_b)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{X_b}{m_b} &= \dot{u}_b + g\sin\theta_b - (r_b v_b - q_b w_b) = \dot{u} + g\sin\theta - (rv - qw) = \frac{(X+T)}{m} \\
\frac{Y_b}{m_b} &= \dot{v}_b - g\sin\phi_b\cos\theta_b - (p_b w_b - r_b u_b) = \dot{v} - g\sin\phi\cos\theta - (pw - ru) = \frac{Y}{m} \\
\frac{Z_b}{m_b} &= \dot{w}_b - g\cos\phi_b\cos\theta_b - (q_b u_b - p_b v_b) = \dot{w} - g\cos\phi\cos\theta - (qu - pv) = \frac{Z}{m}
\end{aligned}
$$

$$
\begin{aligned}
\frac{(X+T)}{m} &= \frac{X_b}{m_b} \\
\frac{Y}{m} &= \frac{Y_b}{m_b} \\
\frac{Z}{m} &= \frac{Z_b}{m_b}
\end{aligned}
$$

Since motion of the ball is constrained to the y-z plane, the X force equations are always satisfied. From the Y and Z force equations we arrive at

$$\frac{Y_b}{m_b} = \frac{Y}{m}, \ \frac{Z_b}{m_b} = \frac{Z}{m} \Rightarrow \frac{Y_b}{Z_b} = \frac{Y}{Z}$$

Hence,

$$\tan\eta = \frac{Y_b}{Z_b} = \frac{Y}{Z}$$

From this we conclude that the slip indicator is actually a body-axis side-force indicator. The side force Y consists of aerodynamic and thrust forces, but does not include the component of weight. If there is no component of thrust in the y-body direction then the ball deflection is proportional to aerodynamic side force only. For our purposes,then,

$$Y = 0 \qquad \text{(Balanced flight)} \tag{7.28}$$

The aerodynamic side force Y is typically modeled as

$$Y = \bar{q}SC_y(\beta, \hat{p}, \hat{r}, \delta_r)$$

The effect of β is usually the much larger than that of the other independent variables, so the ball actually does respond more to sideslip than to other effects. Moreover, in straight flight $p = r = 0$, and often the effect of δ_r may be neglected. Under these conditions and assumptions the slip indicator is appropriately named.

Straight, Symmetric Flight

We combine the requirements for straight and symmetric flight (with symmetric thrust), for which equations 7.22 become

$$C_\ell(\delta_\ell, \delta_n) = 0 \tag{7.29a}$$
$$C_n(\delta_\ell, \delta_n) = 0 \tag{7.29b}$$
$$C_Y(\delta_n) + mg\sin\phi\cos\theta/\bar{q}S = 0 \tag{7.29c}$$

From these equations the implication is that given some δ_n that satisfies the side force requirement, there is some rolling moment controller δ_ℓ setting that will simultaneously satisfy both the rolling and yawing moment equations. Even if such a combination of controls existed (and that is unlikely) it is clear that we are talking about a very particular case. In general, then, we will require $\delta_\ell = 0$, $\delta_n = 0$, and $Y = 0$.

Since we now require $Y = 0$, we have $mg\sin\phi\cos\theta = 0$ which, for $\theta \neq \pm 90\,\text{deg}$, means $\phi = 0$. Applying $\beta = 0$ and $\phi = 0$ to $T_{H,W} = T_{H,B}T_{B,W}$, and equating the (3,2) entries on the left and right sides yields $\sin\mu\cos\gamma = 0$, which, for $\gamma \neq \pm 90\,\text{deg}$, means $\mu = 0$.

Summarizing the conditions and requirements for straight, symmetric flight,

$$C_\ell = C_n = C_Y = 0$$
$$\beta = v = \phi = \mu = \delta_\ell = \delta_n = 0$$

Turning, Balanced Flight

With $\dot{v} = 0$ and $Y = 0$, the body-axis side-force equation becomes

$$g \sin\phi \cos\theta = ru - pw$$

Using the kinematic requirements for steady flight, equations 7.19a and 7.19c,

$$g \sin\phi \cos\theta = \dot{\psi} \left(u \cos\phi \cos\theta + w \sin\theta \right)$$

To get manageable relationships, we note that in nearly level flight at low angles of attack and moderate angles of bank,

$$u \cos\phi \cos\theta \gg w \sin\theta$$

and

$$u \approx V$$

we have

$$\tan\phi \approx \frac{\dot{\psi} V}{g} \tag{7.30}$$

Continuing with the assumption of small α, z_B and z_W are nearly coincident so that $L = -Z$. The definition of *load factor* (n) is

$$n = \frac{L}{W} = \frac{L}{mg} \tag{7.31}$$

Then, using the body-axis Z-force equation, with $L = -Z$,

$$L = mg \cos\phi \cos\theta + m \left(qu - pv \right)$$

We now use equations 7.19a and 7.19b, and make similar assumptions regarding the angles, yielding

$$qu - pv \approx \dot{\psi} V \sin\phi$$

$$L \approx mg\cos\phi + m\dot{\psi} V \sin\phi$$

Finally, using equation 7.30 ($\dot{\psi} V \approx g\tan\phi$), we have

$$L \approx mg\cos\phi + mg\tan\phi\sin\phi = mg\left(\cos\phi + \frac{\sin^2\phi}{\cos\phi}\right) = mg\sec\phi$$

This yields the approximate relationship between load factor and bank angle,

$$n \approx \sec\phi \qquad \text{(Many assumptions)} \tag{7.32}$$

Level Flight

Level flight simply means the aircraft is neither climbing nor descending, or $\gamma = 0$. Level flight by itself has little effect on the equations of motion, save to eliminate one variable and to ensure that the effects of changing altitude may be safely neglected. If level flight is not assumed, then the effects of altitude changes on the forces and moments (change of density, or ground effect) may require consideration.

7.4.3 The Trim Problem

The trim problem is the problem of determining the values of those non-zero variables such that a specified steady-state flight condition results. For example, if we specify straight steady-state flight we require that

$$\begin{gathered} C_\ell\left(\beta, \delta_\ell, \delta_n\right) = 0 \\ C_n\left(\beta, \delta_\ell, \delta_n\right) = 0 \\ C_Y\left(\beta, \delta_n\right) + mg\sin\phi\cos\theta/\bar{q}S = 0 \end{gathered}$$

These three equations must be satisfied simultaneously by some combination of sideslip, rolling and yawing controls, bank angle, pitch angle, velocity, and altitude. Most flight dynamics problems begin with a specification of some velocity and altitude of interest, so we may expect Mach and dynamic pressure to be given. For the rolling and yawing moment equations we may therefore specify one of the variables sideslip, rolling and yawing controls and solve for the other two. The remaining question is to find a bank angle and pitch angle that satisfies the side force equation. One of these (normally pitch angle) may be specified. Because the functional dependencies of the force and moment coefficients are usually quite complex, the trim problem can not be solved analytically. With the linear assumption, however, closed form solutions are possible. Using the same example the rolling moment coefficient is approximated by

$$C_\ell\left(\beta, \hat{p}, \hat{r}, \delta_\ell, \delta_n\right) = C_{\ell_{Ref}} + C_{\ell_\beta}\beta + C_{\ell_p}\hat{p} + C_{\ell_r}\hat{r} + C_{\ell_{\delta_\ell}}\delta_\ell + C_{\ell_{\delta_n}}\delta_n$$

Now, if we take the reference condition to be that in which all independent variables are zero, and assume this approximation is good for a suitable range of the variables about that reference condition, then (with $p = 0$ and $r = 0$, due to straight flight):

$$C_\ell = C_{\ell_\beta}\beta + C_{\ell_{\delta_\ell}}\delta_\ell + C_{\ell_{\delta_n}}\delta_n$$

In a similar fashion we have

$$C_n = C_{n_\beta}\beta + C_{n_{\delta_\ell}}\delta_\ell + C_{n_{\delta_n}}\delta_n$$

$$C_Y = C_{Y_\beta}\beta + C_{Y_{\delta_n}}\delta_n$$

This part of the trim problem becomes

$$\begin{aligned}
C_{\ell_\beta}\beta + C_{\ell_{\delta_\ell}}\delta_\ell + C_{\ell_{\delta_n}}\delta_n &= 0 \\
C_{n_\beta}\beta + C_{n_{\delta_\ell}}\delta_\ell + C_{n_{\delta_n}}\delta_n &= 0 \\
C_{Y_\beta}\beta + C_{Y_{\delta_n}}\delta_n + mg\sin\phi\cos\theta/\bar{q}S &= 0
\end{aligned}$$

For example, if we now assume that ϕ, θ, and $\bar{q}$ are given, then we need to solve for β, δ_ℓ, and δ_n in the linear system

$$\begin{bmatrix} C_{\ell_\beta} & C_{\ell_{\delta_\ell}} & C_{\ell_{\delta_n}} \\ C_{n_\beta} & C_{n_{\delta_\ell}} & C_{n_{\delta_n}} \\ C_{Y_\beta} & 0 & C_{Y_{\delta_n}} \end{bmatrix} \begin{Bmatrix} \beta \\ \delta_\ell \\ \delta_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -mg\sin\phi\cos\theta/\bar{q}S \end{Bmatrix}$$