KINEMATICS AND DYNAMICS OF FIXED WING UAVS

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Abstract

The chapter provides a review of fundamental knowledge required for accurate mathematical modelling of flight of a fixed wing UAV. The key building blocks required for accurate modelling include the kinematics and dynamics of motion, and the transformation of forces and moments acting on the airplane. The detailed discussion of the "Kinematics-Dynamics-Actions" triad in application to a generic fixed wing UAV is the main objective of this chapter. Therefore, the presentation starts with an introduction to the coordinate frames, their transformations and differential rotations. Kinematics of the coordinate frames is what connects states of a fixed wing UAV and transforms forces and moments acting in different coordinate frames. Understanding of reference frames and their dynamics is essential for the guidance, navigation and control systems design. Next, the chapter provides a detailed derivation of the the equations of motion using Newtonian approach. Assuming that a fixed wing UAV can be represented as a rigid body moving in an inertial space, allows ~~deriving the~~ linear for the derivation of the and angular momentum equations. Starting in an inertial frame, it is shown how the final form of translational and rotational equations of motion become written in a body fixed coordinate frame. The development of both~~,~~ the kinematic and dynamic equations~~,~~ is carried out first in a general vector form; ~~and~~ then, using simplifying assumptions applicable to a generic fixed wing symmetric UAV, the vector equations are expanded into a scalar form to better represent the details of remaining terms. Finally, the chapter presents the principles of defining the forces and moments acting on a generic fixed wing airplane. Since the forces and moments ~~acting~~ on an airplane act in a number of coordinate frames including inertial, body fixed and wind frames, the chapter utilizes the concepts and tools built in the kinematics section to transform the forces and moments into the body fixed frame. Such transformations complete the presentation of the "Kinematics-Dynamics-Actions" triad.

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Introduction

a The chapter objective is to provide an overview of the necessary theoretical material to enable reliable mathematical modelling of the free and controlled motion of a generic fixed wing UAV. Besides the equations of motion describing the kinematics and dynamics of a rigid body motion, the tools and methods developed in this chapter contribute significantly into the UAV flight dynamics, system identification, control, guidance and navigation. Although the subject is not new and is well presented in the existing literature, the rapid advancements of last decade in research and development of fixed wing UAV technologies open new applications that require revision of the a existing assumptions. New materials, novel structural designs, new aerodynamic configurations, advanced onboard instrumentation including miniature sensors, envelopment?

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actuators, and tremendous onboard processing power enable much wider operational envelop of fixed wing UAVs and significantly higher utility of their payloads. Depending on the UAV configuration and it's intended operational use the standard 12 equations of motion might not suffice the task at hand and require deeper consideration of the UAV components' interaction.

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The chapter starts with some preliminaries required to describe kinematics of a rigid body motion in three dimensional (3D) space. Thus, the kinematics of 3D rotation is introduced first. The most commonly used coordinate frames that are utilized in the description of UAV states are presented next. Applying the kinematics of rotating frames to a set of specific coordinate frames builds the basis for a convenient description of the forces and moments acting on a fixed wing airplane. Understanding of reference frames and their dynamics is essential for eventual development of the guidance, navigation and control systems architectures. Next, the chapter provides a detailed derivation of the equations of motion using classical Newtonian approach. Assuming that a fixed wing UAV can be represented as a rigid body moving in an inertial space allows deriving the linear and angular momentum equations. Starting in an inertial frame, it is shown how the final form of translational and rotational equations of motion become written in a body fixed coordinate frame. The development of both the kinematic and dynamic equations is carried out first in a general vector form, and then, using simplifying assumptions applicable to a generic fixed wing symmetric UAV, the vector equations are expanded into a scalar form to better represent the details of remaining terms. A conceptual review of the approaches used to model the aerodynamic, propulsion, gravity and turbulent atmosphere forces and moments completes the formal definition of the equations of motion. Since the forces and moments acting on an airplane act in a number of coordinate frames including inertial, body fixed and wind frames, the chapter utilizes the concepts and tools built in the kinematics description to transform the forces and moments into the body fixed frame. Thus the complete derivation of linear and angular momentum equations along with the accurate definition of the forces and moments acting on a rigid body results in the generalized set of 6 Degree of Freedom (6DOF) equations of motion.

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Reference Frames and Coordinate Transformations

In order to accurately describe a body motion it is required to define (i) the forces and moments acting on the body and thus resulting in the body motion, and (ii) the coordinate system that can be used as a reference for the motion states definition. It is important to note that there two types of forces acting on a body in free motion. First, the inertial forces and moments that depend on the velocities and accelerations relative to an inertial reference frame; inertial is the frame, where the classical dynamics Newtonian equations hold. Second group consists of the aerodynamic forces and moments resulting from interaction of the body with the surrounding airflow and therefore relative to the air. Since the airflow might not be stationary and in turn can be arbitrarily moving with respect to the body, it is therefore convenient to describe the resulting aerodynamics in the coordinate frames connected to the body and to the surrounding air. The resulting motion can be conveniently described in terms of the position, velocity, acceleration and attitude coordinates which comprise the states of

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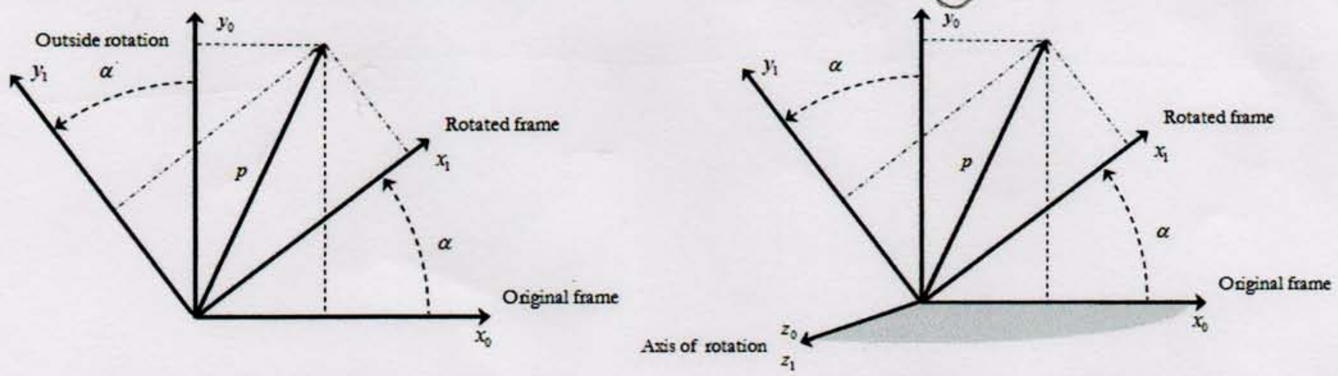
the moving body. Some of these states, in turn, need to be defined with respect to a in which the reference frame ~~which~~ choice is defined by the specifics of the UAV application. Thus, the information carried by various reference frames is what facilitates the complete and convenient definition of the free body motion.

Therefore, the section starts with a generalized definition of a coordinate frame and the description of the coordinate frame rotation. The reference frames required to represent the aerodynamic forces and moments and facilitating the solution of the ? follow navigation states ~~are introduced next~~. Communication of the states' information occurring during the coordinate frame transformation is presented for the major coordinate frames. The section ends with a set of kinematic equations required to represent the transition of linear and angular accelerations.

Kinematics of moving frames

The objective of this subsection is to define a coordinate frame transformation and the associated mathematical formalism. Namely, the direct cosine matrix is introduced and its key properties are presented. The rotational matrix formalism is then followed by a differential rotation that defines the rate of change of the rotation matrix. A the fundamental property of simple summation of angular rates is introduced next. The section ends with a detailed presentation of the coordinate frames used to describe the 6DOF motion of a rigid body. The formal results of this initial development are heavily throughout utilized ~~along~~ the entire chapter.

An arbitrary motion of a rigid body can be described by a transformation that consists of (Goldstein 1980) translational and rotational components. First, address the pure rotation of a rigid body. Consider a vector **P** defined in two orthogonal coordinate frames rotated with respect to their mutual origin by angle α, as shown in Figure 1.a.



a) two dimensional coordinates b)three dimensional coordinates

Figure 1. The same plane rotation considered with respect to two and three axes

From this geometrical setup it can be demonstrated that vector $\mathbf{P}=[x_0, y_0]$ can be uniquely defined in both frames as follows:

$$x_1 = x_0 \cos\alpha + y_0 \sin\alpha$$
$$y_1 = -x_0 \sin\alpha + y_0 \cos\alpha$$

Introducing the matrix notation for the linear transformation above results in a simple form that relates the vector $\mathbf{p}$ components in (x_0, y_0) frame to the corresponding components in (x_1, y_1) frame:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = R_0^1 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad R_0^1 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

The resulting rotation matrix is called a directional cosine matrix (DCM). The DCM matrix R consists of the cosine and sine functions which are the direction cosines between corresponding axes of the new and old coordinate systems denoted in the superscript and the subscript correspondingly. Following the same approach, it can be easily demonstrated that for the case of three orthogonal axes (see Figure 1.b), the same right hand rotation results in transformation[1]

$${}_zR_0^1 = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where, for clarity, the subscript z denotes the axes of rotation. Proceeding similarly, right handed rotations of the coordinate frame about the y_0 and x_0 axis give

$${}_yR_0^1 = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}, \quad {}_xR_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}.$$

It is worth noting that the DCM transformation has the following easy-to-remember properties that simplify its application, see more details in (Rogers 2003):

1. The transformed vector components along the axis of rotation remain unchanged with the rotation about that axis; elements of DCM are either 0 or1.
2. The remaining elements of DCM are either $\sin$ or $\cos$ functions of the angle of rotation.
3. The $\cos$ elements are on the main diagonal with $\sin$ elements on off-diagonal.
4. The negative $\sin$ component corresponds to the component rotated "outside" of the quadrant formed by the original frames.
5. Columns (rows) of a DCM matrix form an orthonormal set.

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It is straightforward to verify that a DCM matrix have the following properties:

$$\det(R) = 1; \quad R^T = R^{-}; \quad R^T R = I; \; R = [c_1, c_2, c_3] \Rightarrow c_i \cdot c_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

and it therefore belongs to a general class of orthonormal transformation matrices. For a sequence of rotations performed with respect to each ~~of the~~ orthogonal axis, the resulting transformation can be obtained by a matrix composed of three sequential ?

[1] A coordinate system in which the axes satisfy the right-hand rule is called a right-handed coordinate system. The right-hand rule defines the orientation of the resulting vector in the vector cross product multiplication.

rotations, called Euler angles, starting from the original frame of reference, see Figure 2.

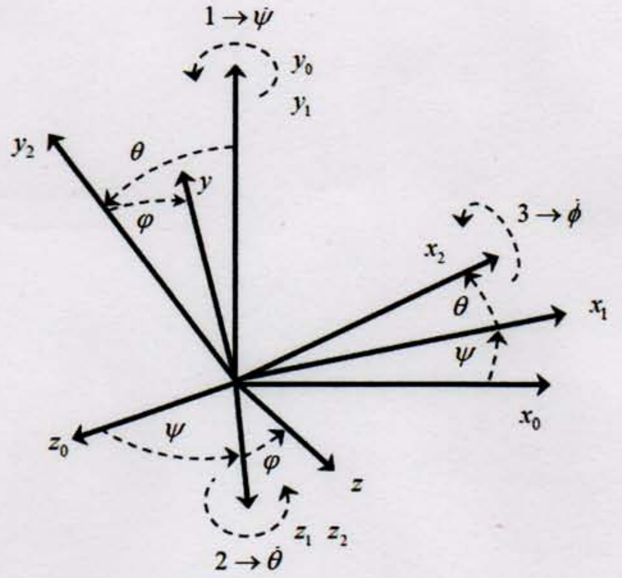

Figure 2. Three consecutive rotations.

Formally, this transformation is accomplished by rotating through the ordered sequence of Euler angles $[\psi, \theta, \phi]$, where the numerical indexes define the ordered sequence of rotations.

$$p^y = R^y_{x_0} p^x = R^y_{x_2} R^{x_2}_{x_1} R^{x_1}_{x_0} p^x$$

It t is worth noting here, that the corresponding Euler angles are also widely used in notations, so that in the following notation ~~is possible~~ $R_\phi = R^y_{x_2}, R_\theta = R^{x_2}_{x_1}, R_\psi = R^{x_1}_{x_0}$ is possible.

Therefore, a vector $\mathbf{p} = [x_0, y_0, z_0]$ described in one coordinate frame can be described in another coordinate frame of arbitrary orientation with respect to the original frame by a transformation matrix $R^y_{x_0}$ composed of three sequential rotations as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R^y_{x_0}} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$R^y_{x_0} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}.$$

This matrix, which ~~that~~ represents a transformation resulting from three sequential Euler angles rotations, will be used throughout the chapter.

Overall, any DCM matrix has a number of properties. They are summarized here for completeness; an interested reader is referred to reference (Goldstein 1980) for more details:

- Rotation matrices are orthogonal.

- The determinant of a rotation matrix is unity.
- Successive rotations can be represented by the ordered product of the individual rotation matrices.
- Rotation matrices are not commutative, hence, in general case $R_b^c R_a^b \neq R_a^b R_b^c$.
- A nontrivial[2] rotation matrix has only one eigenvalue equal to unity with other two, being a complex conjugate pair with unity magnitude.

The time rate of change of the DCM matrix that defines the dynamics of the attitude states is important in the derivation of the kinematic equations of motion. As it will be shown shortly, it enables relating the sensor measurements obtained in a body fixed frame to the time derivatives of the Euler angles describing the attitude of a body in an inertial frame.

Deriving the time derivative of a DCM matrix can be obtained by utilizing a linearization technique. Consider the variation of the DCM matrix R_x^y resulting from three infinitely small $\delta\psi$, $\delta\theta$, $\delta\phi$ consecutive rotations performed over an interval of time δt. Utilizing the small angle ε approximation of $\sin(\varepsilon)=\varepsilon$ and $\cos(\varepsilon)=1$ and neglecting the higher order terms, it follows that:

$$R_x^y = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}_{\delta\phi,\delta\theta,\delta\psi} \approx$$

$$= \begin{bmatrix} 1 & \delta\psi & -\delta\theta \\ -\delta\psi & 1 & \delta\phi \\ s\theta & -\delta\phi & 1 \end{bmatrix} = I - \begin{bmatrix} 0 & -\delta\psi & \delta\theta \\ \delta\psi & 0 & -\delta\phi \\ -s\theta & \delta\phi & 0 \end{bmatrix} = I - S(\delta\psi,\delta\theta,\delta\phi)$$

Utilizing at any given time t, the values of $\mathbf{p}^y$ at the time $t+\delta t$ can be expressed by

$$\mathbf{p}^y(t+\delta t) = (I - S(\cdot))R_x^y\mathbf{p}^x = R_x^y(t+\delta t)\mathbf{p}^x,$$

where $S(\cdot) = S(\delta\psi,\delta\theta,\delta\phi)$. The time rate of change of the DCM matrix is then defined as

$$\dot{R}_x^y = \lim_{\delta t\to 0}\frac{R_x^y(t+\delta t) - R_x^y(t)}{\delta t} = \lim_{\delta t\to 0}\frac{R_x^y(t) - S(\cdot)R_x^y(t) - R_x^y(t)}{\delta t} = \lim_{\delta t\to 0} - \frac{S(\cdot)R_x^y(t)}{\delta t} = -S(\dot{\mathbf{u}})R_x^y(t)$$

$$\dot{R}_x^y = -S(\dot{\mathbf{u}})R_x^y(t) \qquad (1)$$

where

$$S(\dot{\mathbf{u}}) = \begin{bmatrix} 0 & -\dot\psi & \dot\theta \\ \dot\psi & 0 & -\dot\phi \\ -\dot\theta & \dot\phi & 0 \end{bmatrix}$$

and $\dot{\mathbf{u}} = [\dot\phi, \dot\theta, \dot\psi]$ is the vector of angular velocities. The rotation matrix $S(\dot{\mathbf{u}})$ can be viewed as a rotation of the frame $\{y\}$ with respect to frame $\{x\}$ measured in the $\{y\}$

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[2] Trivial rotation is the one described by an identity matrix, thus no rotation takes place

frame. It can be observed that matrix $S(\grave{\mathbf{u}})$ is a skew symmetric matrix $(S^T(\grave{\mathbf{u}}) = -S(\grave{\mathbf{u}}))$ and therefore the transposed equivalent of the rate of rotation is

$$R^{\&}_y = -(S(\grave{\mathbf{u}})R^y_x(t))^T = R^x_y(t)S(\grave{\mathbf{u}}) .$$

Another useful general property of angular velocities is called the angular velocities addition theorem (Rogers 2003). The theorem states that for angular velocity vectors coordinated in a common frame, the resulting angular velocity of the cumulative rotation is a plain sum of the contributing rotations. ~~An~~ A application to $\grave{\mathbf{u}}$ in the theorem results in $\grave{\mathbf{u}} = \phi\&\mathbf{i} + \theta\&\mathbf{j} + \psi\&\mathbf{k}$, where $[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ are the directional unity vectors defining the intermediate coordinate frames. Now, if a rotating frame $\{y\}$ is given by a set of time varying Euler angles $[\psi, \theta, \phi]$ defined with respect to a stationary frame $\{x\}$, then it is straightforward to determine the components of the angular velocity vector $[p, q, r]$ as though ~~if it was~~ measured in the rotating frame $\{y\}$. Starting from an initial stationary frame $\{x\}$ (see Figure 2) and using two intermediate frames whose relative angular velocities are defined by the Euler angle rates, and utilizing the angular velocities addition theorem, we obtain

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = R_{\ddot{o}} R_{\grave{e}} \begin{bmatrix} 0 \\ 0 \\ \& \end{bmatrix} + R_{\ddot{o}} \begin{bmatrix} 0 \\ \& \\ 0 \end{bmatrix} + \begin{bmatrix} \& \\ 0 \\ 0 \end{bmatrix} .$$

Substituting the corresponding DCM matrices from results in

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \\ 0 & \cos & \cos\ \sin \\ 0 & -\sin & \cos\ \cos \end{bmatrix} \begin{bmatrix} \& \\ \& \\ \& \end{bmatrix} .$$

Inverting the last equation results in

$$\begin{bmatrix} \& \\ \grave{e}\& \\ \& \end{bmatrix} = \begin{bmatrix} 1 & \sin\ddot{o}\dfrac{\sin\grave{e}}{\cos\grave{e}} & \cos\ddot{o}\dfrac{\sin\grave{e}}{\cos\grave{e}} \\ 0 & \cos\ddot{o} & -\sin\ddot{o} \\ 0 & \sin\ddot{o}\dfrac{1}{\cos\grave{e}} & \cos\ddot{o}\dfrac{1}{\cos\grave{e}} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} ,$$

Plural. the angles singular? or themselves

which defines the derivatives of the Euler angles in terms of the angle~~s~~ itself and the rates $[p, q, r]$ as they were measured in the frame $\{y\}$. These equations define the rotational kinematics of a rigid body; they contribute to the final set of 6DOF equations of motion.

space

angles

Analysis of the equation shows that four elements of the inverted matrix become singular when the second rotation angle θ approaches $\pi/2$. This problem is usually called a kinematic singularity, or a gimbal lock in navigation, and is one of the issues associated with the use of Euler angles for the attitude determination. For differently ordered Euler rotation sequences, the kinematic singularity will occur at a different point. Therefore, one way to avoid the singularity is to switch or change the Euler angle the sequences when approaching the singularity. Next, depending on the available computing power, the integration of kinematic equation space can be computationally expensive because it involves calculation of trigonometric functions. Furthermore, it can be observed that the Euler angles based DCM matrix is redundant; it requires only 3 out of 9 elements of the DCM matrix to uniquely define the Euler angles. These shortcomings usually result in applying different parameters describing the attitude and its dynamic transformation.

In applications to the fixed wing UAV attitude determination, the Rodriguez-Hamilton parameters, or quaternion, is one of the most widely used alternatives (Goldstein 1980). Utilizing the quaternion approach is very powerful because it gives a singularity free attitude determination at any orientation of a rigid body. Next, since it can be shown that the equations of motion of a rigid body are linear differential equations in the components of quaternion, then it (linearity) is a desirable property, especially when developing estimation and control algorithms. Furthermore, the quaternion is a relatively computationally efficient approach since it does not involve trigonometric functions to compute the attitude matrix, and has only one redundant parameter, as opposed to the six redundant elements of the attitude matrix. However, it is also worth noting that the quaternion and Euler angles techniques are well connected with simple analytical representations of the DCM matrix and Euler angles through the parameters of quaternion. An interested reader is referred to an extensive historical survey of attitude representations (Shuster 1993) and references (Rogers 2003, Goldstein 1980) for more details in the alternative methods of attitude determination.

Coordinate frames

Deriving equations of motion of a fixed wing UAV requires a definition of coordinate frames where forces and moments acting on the airplane can be conveniently defined, and where the motion states including the position, velocity, acceleration and attitude can be suitably described. Considering the desired nomenclature of coordinate frames, it is also important to account for a maximum duration of UAV mission and the corresponding operational range. With the latest advances in power technologies, a mission not long duration ~~mission~~ becomes a reality. As an example, ~~the~~ solar power technology is one of the alternatives that can make this 24/7 flight of a fixed wing solar powered autonomous soaring gliders feasible. Thus, long duration and great operational distances require considering the UAV flight operations with respect to the rotating Earth. Therefore, in this subsection we define the following coordinate frames:

1. Earth-Centered-Earth-Fixed Frame {e}
2. Geodetic Coordinate System {λ, φ, h}
3. Tangent Plane Coordinate System {u}
4. Body-Carried Frame {n}
5. Body-Fixed Frame {b}
6. Wind frame {w}

the

Depending on the duration of flight and operational range, both dictated by a specific UAV application, first three frames can be considered ~~as~~ inertial frames, with the remaining three frames being body fixed. The inertial and body frames are related by a plain translation, while the body frames relate to each other by pure rotations. Details of the frames definition and their relations are the subject of this section.

Earth-Centered Earth Fixed and Geodetic Coordinate Frames

The Earth Centered Earth Fixed (ECEF) orthogonal coordinate system is fixed to the Earth and therefore it rotates at the Earth's sidereal rate. The frame is usually marked with $\{e\}$ in subscript. It has its origin at the center of the Earth with x_e and y_e axes placed in the equatorial plane and z_e axis aligned with the direction of the Earth's rotation vector, see Figure 3. The x_e axis is usually attached to the intersection of the Greenwich meridian and the equator, and the y_e axis completes the right hand system. It is worth noting, that the ECEF axes definition may vary; however, the definition always states (or: lists) the attachment of two vectors to the direction of the earth rotation and the Greenwich meridian as the inherent Earth properties. The sidereal rate Ω_e is the rate of Earth's rotation with respect to the distant stars (the true inertial frame). If necessary, for the purpose of UAV flight description, this rate can be accurately approximated by one full rotation in 23h56'4.099", thus resulting in 15.04106718 deg/h.

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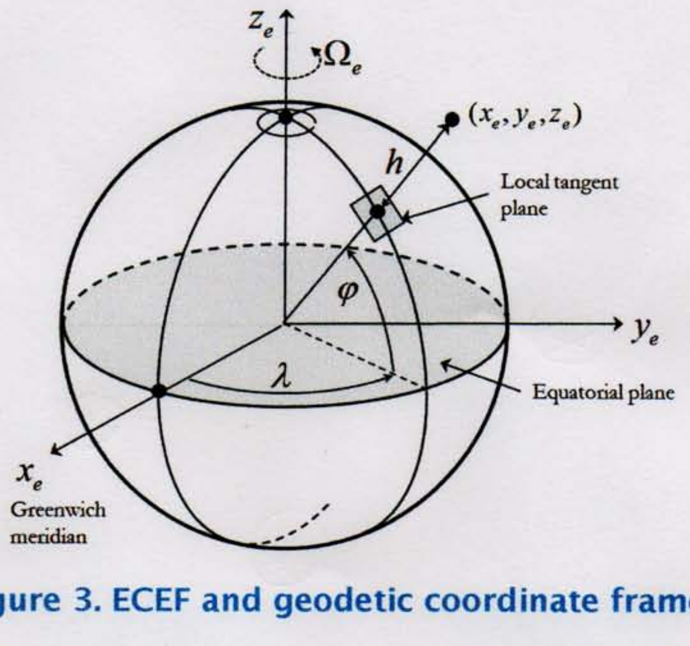

should be the Earth's unless there's a technical reason I don't know

Figure 3. ECEF and geodetic coordinate frames.

The local Local Geodetic $\{\lambda, \varphi, h\}$ frame is usually associated with the ECEF frame, see Figure 3. It has the same origin at the center of the Earth. The frame defines the orientation of the line normal to ~~the~~ Earth's surface and passing through the point of interest. The orientation of the line is defined by two angles, λ - geographic latitude and φ - geographic longitude, with the height h above the Earth's surface; these three parameters, along with the components of velocity vector, are the major navigation states. For ~~the~~ most UAV applications it is sufficiently accurate to model ~~the~~ Earth's ? surface as an oblate spheroid with given r_e -equatorial and r_p -polar radiuses, or one of the radiuses and the e-ellipticity. Last revisited in 2004 the datum of World Geodetic System (WGS-84) provides the following parameters for the oblate spheroid modeling: $r_e = 6378137.00\ m$, $r_p = 6356752.314\ m$. The resulting transformation from the geodetic $\{\lambda, \varphi, h\}$ to the ECEF frame is as follows:

$$x^e = (r_{\bar{\varepsilon}} + h)\cos \quad \cos$$
$$y^e = (r_{\bar{\varepsilon}} + h)\cos \quad \sin$$
$$z^e = ((1 - \ ^2)\ _{\bar{\varepsilon}} + \ )\sin$$

where ε – the eccentricity of oblate ellipsoid is defined as

$$r_{\bar{\varepsilon}} = \frac{r_e}{\sqrt{1 - \ddot{a}^2 \sin^2 \ddot{o}}}; \quad = \sqrt{1 - \frac{r_p^2}{r_e^2}}$$

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Local Tangent Plane Coordinate System

The origin of the Local Tangent Plane (LTP) is fixed to the surface of the Earth with two of its axes attached to the plane tangent to the surface, see Figure 4. The frame is usually marked with the subscript $\{u\}$ ~~and serves the~~ purpose of an inertial frame in most ~~of~~ UAV applications. The frame's x_u and y_u axes are in the tangent plane and most often aligned with the North and East directions correspondingly; the z_u axis completes the right hand coordinate system, thus pointing down. Quite often the order and alignment of the LTP frame principal axes change. In such cases the LTP coordinate system explicitly specifies its type; in the nominal case presented above, it can be also defined as an NED frame.

Connected ??

When the origin of the LTP frame is defined in terms of its geographic latitude, longitude and altitude above ~~the~~ ground surface, then the equations can be applied to define the kinematics of navigation states.

Space

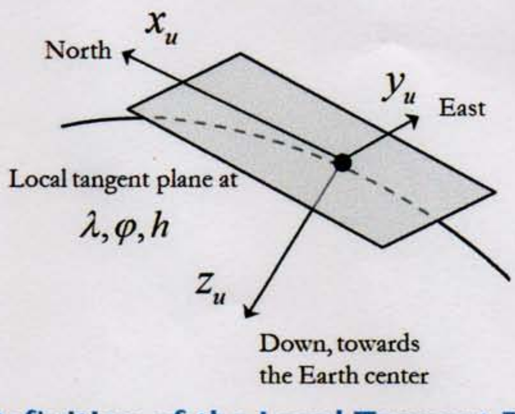

Figure 4. Definition of the Local Tangent Plane; NED.

Body- Carried and Fixed Frames

In flight dynamics, the body-attached reference frames usually have their origin at the center of gravity (CG) of an airplane; therefore, these frames are moving. The body-carried frame $\{n\}$ is an orthogonal frame originated at the CG of the UAV. All its axes are permanently stabilized and aligned with the LTP frame axes as it was connected to the CG, see Figure 5. This frame is connected to the LTP frame by means of a plain translation $\mathbf{r} = [r_n, r_e, r_d]$.

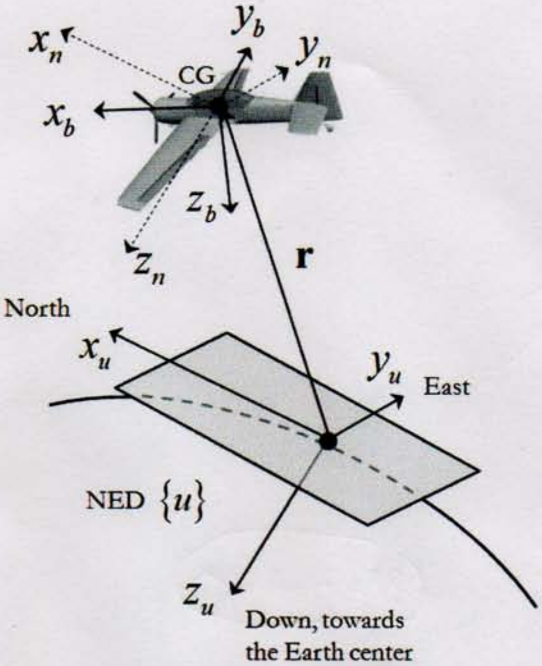

Figure 5. Definition of the body fixed frame with respect to LTP frame.

The body-fixed frame is an orthogonal frame defined with respect to the body-carried frame. Its origin is at the CG of UAV and its axes are rigidly connected to the body, therefore the frame rotates with the body. The frame is usually marked with the subscript $\{b\}$. It can be proven (Goldstein 1980) that for every rigid body there is always an orthogonal coordinate system, usually called principal, in which the cross-products of inertia terms are zero. This feature is typical to bodies with planes of symmetry. Assuming that a typical UAV has at least one plane of symmetry (geometric and mass symmetry), ~~results~~ in two of the body-fixed axes lying in the plane of symmetry. When the axes are aligned along the principal axes of inertia of the body, as it will be shown in the following chapter, the dynamic equations of motion become significantly more simple. ~~simpler~~. In a majority of fixed wing UAV configurations, the axes of $\{b\}$ frame match the principal axes of inertia. The typical orientation of the body fixed axes is as follows (see Figure 5): if the UAV has a vertical plane of symmetry, then x_b and z_b lie in that plane of symmetry; x_b ~~is~~ points towards the direction of flight and z_b points downward and y_b points right, thus completing the right hand system.

the results would show/depict ? ?

As the body moves, its attitude is defined with reference to the body-carried frame $\{n\}$ by three consecutive Euler rotations by Ψ -yaw, θ -pitch and Φ -roll angles. See their graphical illustration in Figure 2 where frames {0} and {1} relate to the frames $\{n\}$ and $\{b\}$ correspondingly. The formal definition of the Euler angles in the application to an airplane attitude specification is presented here for completeness:

- ψ – yaw is the angle between x_n and the projection of x_b on the local horizontal plane.
- θ –pitch is the angle between the local horizon and the x_b axis measured in the vertical plane of symmetry of the UAV.
- ϕ –roll is the angle between the body fixed y_b axis and the local horizon measured in the plane $y_b z_b$.

As it follows from the attitude representation section, the DCM matrix transforming the body-carried $\{n\}$ to the body-fixed $\{b\}$ frame can be constructed as follows:

$$R_u^b = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\theta\sin\psi+\sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi+\cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

Here subscripts (u ® b) denote the rotation from the LTP $\{u\}$ to the body fixed frame; $\{u\}$ and $\{n\}$ frames are always aligned by the definition of body-carried frame.
The application of the rotation matrix space immediately follows from the need to describe the UAV translational motion in an inertial frame of reference by utilizing the inertial velocity measurements taken in the body fixed frame – $\mathbf{V}_b = [u, \upsilon, \omega]$. To this end, consider Figure 5 where vector $\mathbf{r} = [r_n, r_e, r_d]$ denotes the position of an airplane CG with respect to the LTP (NED) frame attached to the Earth. Relating the translational velocity and position, and accounting for the fact that body-carried frame $\{n\}$ is stabilized with respect to the non-rotating $\{u\}$ frame results in

$$\frac{d\mathbf{r}}{dt} = R_b^u V_b$$

$$\frac{d}{dt}\begin{bmatrix} r_n \\ r_e \\ r_d \end{bmatrix} = R_b^u \begin{bmatrix} u \\ \upsilon \\ \omega \end{bmatrix}$$

The relation between the Euler angles defining the relation between the stabilized $\{n\}$ frame and the body fixed frame $\{b\}$ were already derived in –. They define the dynamics of Euler angles defined in an inertial frame with respect to the rates measured in the body fixed frame. Thus, the kinematic equations ~~and~~ represent the dynamics of translational and rotational coordinates, and therefore are the part of the final set of equations of motion.

Wind Frame

Aerodynamic forces and moments resulting from the body-air interaction as the airframe moves through the air depend on the body orientation with respect to the surrounding air. In other words, they depend on the vector representing the wind. The velocity vector calculated with respect to the possibly moving surrounding air (wind) is
possible movement of ?

the radiuses and the e-ellipticity. Last revisited in 2004 the datum of World Geodetic System (WGS-84) provides the following parameters for the oblate spheroid modeling: $r_e = 6378137.00\ m$, $r_p = 6356752.314\ m$. The resulting transformation from the geodetic $\{\lambda, \varphi, h\}$ to the ECEF frame is as follows:

$$x^e = (r_{\bar{\varepsilon}} + h)\cos\ \ \cos$$
$$y^e = (r_{\bar{\varepsilon}} + h)\cos\ \ \sin$$
$$z^e = ((1 - \ ^2)\ _{\bar{\varepsilon}} + \ )\sin$$

where ε – the eccentricity of oblate ellipsoid is defined as

$$r_{\bar{\varepsilon}} = \frac{r_e}{\sqrt{1 - \ddot{a}^2 \sin^2 \ddot{o}}}; \quad = \sqrt{1 - \frac{r_p^2}{r_e^2}}$$

I'm not completely sure if you need these periods. But I'd suggest using them.

Local Tangent Plane Coordinate System

The origin of the Local Tangent Plane (LTP) is fixed to the surface of the Earth with two of its axes attached to the plane tangent to the surface, see Figure 4. The frame is usually marked with the subscript $\{u\}$ connected ?? ~~and serves the~~ purpose of an inertial frame in most ~~of~~ UAV applications. The frame's x_u and y_u axes are in the tangent plane and most often aligned with the North and East directions correspondingly; the z_u axis completes the right hand coordinate system, thus pointing down. Quite often the order and alignment of the LTP frame principal axes change. In such cases the LTP coordinate system explicitly specifies its type; in the nominal case presented above, it can be also defined as an NED frame.

When the origin of the LTP frame is defined in terms of its geographic latitude, longitude and altitude above ~~the~~ ground surface, then the equations can be applied to define the kinematics of navigation states. Space

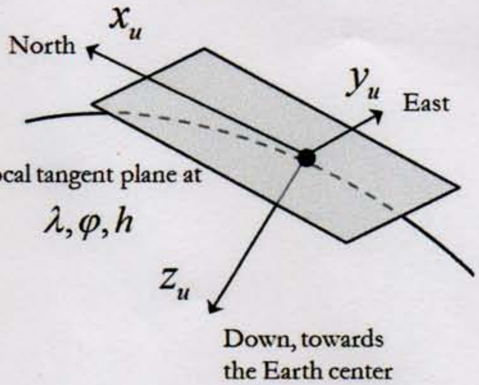

Figure 4. Definition of the Local Tangent Plane; NED.

Body- Carried and Fixed Frames

In flight dynamics, the body-attached reference frames usually have their origin at the center of gravity (CG) of an airplane; therefore, these frames are moving. The body-carried frame $\{n\}$ is an orthogonal frame originated at the CG of the UAV. All its axes are permanently stabilized and aligned with the LTP frame axes as it was connected to the CG, see Figure 5. This frame is connected to the LTP frame by means of a plain translation $\mathbf{r} = [r_n, r_e, r_d]$.

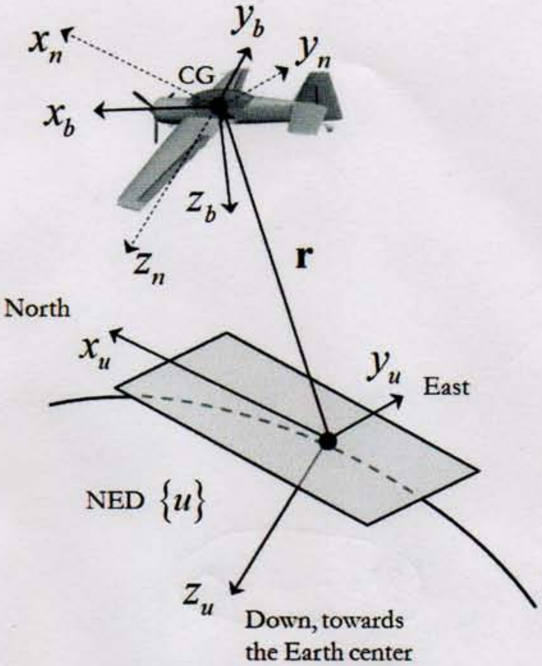

Figure 5. Definition of the body fixed frame with respect to LTP frame.

the results would show/depict ? ?

The body-fixed frame is an orthogonal frame defined with respect to the body-carried frame. Its origin is at the CG of UAV and its axes are rigidly connected to the body, therefore the frame rotates with the body. The frame is usually marked with the subscript $\{b\}$. It can be proven (Goldstein 1980) that for every rigid body there is always an orthogonal coordinate system, usually called principal, in which the cross-products of inertia terms are zero. This feature is typical to bodies with planes of symmetry. Assuming that a typical UAV has at least one plane of symmetry (geometric and mass ~~symmetry),~~ ~~results~~ in two of the body-fixed axes lying in the plane of symmetry. When the axes are aligned along the principal axes of inertia of the body, as it will be shown in the following chapter, the dynamic equations of motion become significantly more simple. ~~simpler~~. In a majority of fixed wing UAV configurations, the axes of $\{b\}$ frame match the principal axes of inertia. The typical orientation of the body fixed axes is as follows (see Figure 5): if the UAV has a vertical plane of symmetry, then x_b and z_b lie in that plane of symmetry; x_b ~~is~~ points towards the direction of flight and z_b points downward and y_b points right, thus completing the right hand system.

As the body moves, its attitude is defined with reference to the body-carried frame $\{n\}$ by three consecutive Euler rotations by Ψ -yaw, θ -pitch and Φ -roll angles. See their graphical illustration in Figure 2 where frames {0} and {1} relate to the frames $\{n\}$ and $\{b\}$ correspondingly. The formal definition of the Euler angles in the application to an airplane attitude specification is presented here for completeness:

- ψ – yaw is the angle between x_n and the projection of x_b on the local horizontal plane.
- θ –pitch is the angle between the local horizon and the x_b axis measured in the vertical plane of symmetry of the UAV.
- ϕ –roll is the angle between the body fixed y_b axis and the local horizon measured in the plane $y_b z_b$.

As it follows from the attitude representation section, the DCM matrix transforming the body-carried $\{n\}$ to the body-fixed $\{b\}$ frame can be constructed as follows:

$$R_u^b = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix} .$$

Here subscripts (u ® b) denote the rotation from the LTP $\{u\}$ to the body fixed frame; $\{u\}$ and $\{n\}$ frames are always aligned by the definition of body-carried frame.
The application of the rotation matrix space immediately follows from the need to describe the UAV translational motion in an inertial frame of reference by utilizing the inertial velocity measurements taken in the body fixed frame – $\mathbf{V}_b = [u, \upsilon, \omega]$. To this end, consider Figure 5 where vector $\mathbf{r} = [r_n, r_e, r_d]$ denotes the position of an airplane CG with respect to the LTP (NED) frame attached to the Earth. Relating the translational velocity and position, and accounting for the fact that body-carried frame $\{n\}$ is stabilized with respect to the non-rotating $\{u\}$ frame results in

$$\frac{d\mathbf{r}}{dt} = R_b^u V_b$$

$$\frac{d}{dt}\begin{bmatrix} r_n \\ r_e \\ r_d \end{bmatrix} = R_b^u \begin{bmatrix} u \\ \upsilon \\ \omega \end{bmatrix} .$$

The relation between the Euler angles defining the relation between the stabilized $\{n\}$ frame and the body fixed frame $\{b\}$ were already derived in –. They define the dynamics of Euler angles defined in an inertial frame with respect to the rates measured in the body fixed frame. Thus, the kinematic equations ~~and~~ represent the dynamics of translational and rotational coordinates, and therefore are the part of the final set of equations of motion.

Wind Frame

Aerodynamic forces and moments resulting from the body-air interaction as the airframe moves through the air depend on the body orientation with respect to the surrounding air. In other words, they depend on the vector representing the wind. The velocity vector calculated with respect to the possibly moving surrounding air (wind) is

possible movement of ?

① comma only

denoted $\mathbf{V}_a$, see Figure 6. The magnitude of $\mathbf{V}_a$ is called an airspeed, as opposed to the velocity vector defined in LTP with respect to the ground – ground speed vector $\mathbf{V}_g$. The orientation of the wind frame $\{w\}$ defined by $\mathbf{V}_a$ with respect to the body fixed $\{b\}$ is defined by two angles.

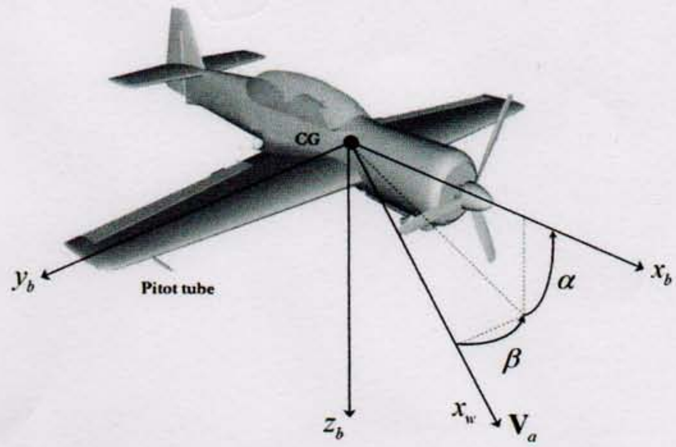

Figure 6. Wind frame and Body fixed frames. Definition of the angle of attack and the side slip.

To generate the lift force in flight, the wing of the UAV must be oriented at a positive angle α with respect to the $\mathbf{V}_a$ vector. This angle is called the angle of attack. The angle of attack α is also one of the key parameters that define the longitudinal stability of an airplane. Therefore, quite often, the coordinate frame that results from a single rotation from the body-fixed $\{b\}$ frame on angle α is called a stability frame (Beard and McLain 2012, Etkin and Reid 1995). As illustrated in Figure 6, the angle of attack is defined by the projection of $\mathbf{V}_a$ into a vertical plane of symmetry of the UAV (spanned by axes x_b, z_b in frame $\{b\}$) and the longitudinal axis x_b of the UAV. It is positive when a leading edge of the wing rotates upward with respect to the $\mathbf{V}_a$. In turn, the angle between the velocity vector $\mathbf{V}_a$ projected into the "wing level" plane (spanned by axes x_b, y_b in frame $\{b\}$) and the longitudinal axis x_b of UAV is called the side-slip angle. It is denoted by β. Applying the DCM matrix approach to represent the complete transformation from the body fixed frame $\{b\}$ to the wind frame $\{w\}$ results in the following:

$$R_b^w = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}.$$

The inverse transformation from the wind frame $\{w\}$ to the body fixed frame $\{b\}$ is the transposition ? transpose of: $R_w^b = \left(R_b^w\right)^T$.

The importance of the wind frame in application to the UAVs flying in wind conditions that might contribute up to 100% of the nominal airplane speed cannot be overestimated. As an example, imagine an autonomous glider that is designed to utilize the wind energy to sustain the long duration of flight. Therefore, it is necessary to clearly understand the difference between airspeed, represented by the velocity vector $\mathbf{V}_a$ and defined with respect to the air and the ground speed $\mathbf{V}_g$, which is represented with respect to the LTP frame. Consider the graphical representation of the relation between these vectors in Figure 7. In the presence of constant wind, these velocities are related by the equation that is often called the "wind triangle":

$$\mathbf{V}_a = \mathbf{V}_g - \mathbf{V}_w$$

where $\mathbf{V}_w$ is the wind velocity defined in the LTP frame.

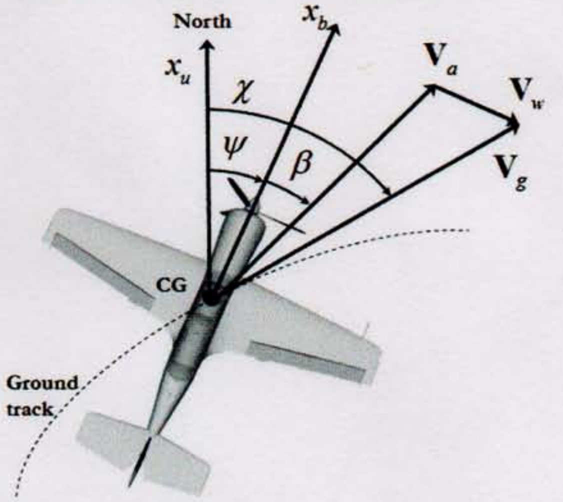

Figure 7. Wind triangle in 2D plane. Definition of the yaw, side slip and course over the angles.

The objective of the following development is to define the relations among these velocities defined in three different frames, while being measured or estimated by the algorithms and sensors installed in the body fixed and in the LTP frames. First, define the components of ~~all~~ three vectors in the body fixed frame $\{b\}$. Let the UAV velocity in the LTP frame expressed in the body frame be $\mathbf{V}_g^b = [u, \upsilon, \omega]^T$, and let the wind velocity in the LTP frame expressed in the body frame be $\mathbf{V}_w^b = [u_\omega, \upsilon_\omega, \omega_\omega]^T$. Observe, that $\mathbf{V}_a$ defined in $\{w\}$ frame can be expressed as $\mathbf{V}_a^w = [V_a \ 0 \ 0]^T$ and let $\mathbf{V}_a^b = [u_w \ \upsilon_w \ \omega_w]^T$ be its components expressed in the body frame. Utilizing the definition of the angles of attack and sideslip relating the wind frame to the body-fixed frame and the "wind triangle" equation expressed in the body frame results in the following;

$$\mathbf{V}_a^b = \mathbf{V}_g^b - \mathbf{V}_w^b = \begin{bmatrix} u \\ \upsilon \\ \omega \end{bmatrix} - \begin{bmatrix} u_\omega \\ \upsilon_\omega \\ \omega_\omega \end{bmatrix}$$

$$\mathbf{V}_a^b = \begin{bmatrix} u_w \\ \upsilon_w \\ \omega_w \end{bmatrix} = R_w^b \begin{bmatrix} V_a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} V_a \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_w \\ \upsilon_w \\ \omega_w \end{bmatrix} = V_a \begin{bmatrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{bmatrix}$$

This last equation relates the airspeed components of $\mathbf{V}_a$ resolved in the body frame with the airspeed and the angles of attack and sideslip. In turn, if the wind components resolved in the body frame are known, then inverting the last equation allows for calculation of the airspeed V_a and the α, β angles.

$$V_a = \sqrt{u_w^2 + \upsilon_w^2 + \omega_w^2}$$

$$\alpha = a\tan\left(\frac{\omega_w}{u_w}\right)$$

$$\beta = a\sin\left(\frac{\upsilon_w}{\sqrt{u_w^2 + \upsilon_w^2 + \omega_w^2}}\right)$$

Generalized Motion.

In the development of dynamic equations of motion it will be necessary to calculate the absolute time derivative of a vector defined in coordinate frames that rotate and move with respect to each other. In an application to the UAV kinematics this can be justified by a necessity to relate the absolute time derivative of a position vector in inertial space (inertial velocity) that is defined based on the measurements taken in a body frame. Similarly, the second time derivative defines the body inertial acceleration.

Consider two coordinate frames $\{F_i\}$ and $\{F_r\}$, where i – stands for an inertial not rotating frame, and r –stands for the rotating frame. The first objective is to calculate the derivative of a unity vector $\mathbf{r}_r$ defined in $\{F_r\}$ attached to a rigid body rotating with respect to the $\{F_i\}$ with angular speed **ù** , see Figure 8. Denote the DCM transformation from $\{F_r\}$ to $\{F_i\}$ as R.

$$\mathbf{r}_i = R\mathbf{r}_r$$

Taking the derivative results in

? looks messy → $\dot{\mathbf{r}}_i = \dot{R}\mathbf{r}_r + R\dot{\mathbf{r}}_r = \dot{R}\mathbf{r}_r = S(\ )_r = \ \times\ _r$

where the time derivative of $\mathbf{r}_r$ is zero due to the rigid body assumption.