Chapter 6

Forces and Moments

6.1 General

In the previous section we derived $\mathbf{F} = m\dot{\mathbf{v}}$ and $\mathbf{M} = \dot{\mathbf{h}}$, valid if we are referring the problem to the CG of the aircraft. The aim is to solve these sets of equations for the derivative terms, and integrate them to yield velocity and position. On the left-hand side of these sets of equations we have $\mathbf{F}$ and $\mathbf{M}$ with major contributors from aerodynamic terms $\mathbf{F}_A$ and $\mathbf{M}_A$, and the thrust $\mathbf{T}$. These forces and moments in turn depend to a large extent on the very quantities we are trying to determine. For example, if everything else is held constant the aerodynamic forces and moments are functions of the velocity V. If we have any hope of solving the differential equations we have developed then we need some way to calculate the forces and moments.

The dependencies of forces and moments on other variables is usually very complex, and not amenable to setting forth analytical expressions that describe them. It is difficult enough to determine accurately the dependencies in steady cases, yet we need to know how forces and moments vary during dynamic changes in the flight condition that probably involve unsteady flow phenomena. Clearly some approximations will have to be made.

No attempt will be made here to present a detailed discussion of the sources of dependencies of the forces and moments. Instead sweeping generalizations and hand-waving will be offered to provide a high level rationale for the method we will adopt in characterizing those forces and moments. We will then examine how such data are likely to be made available and different ways in which the data can be used. Thereafter when necessary we will assume that such data are available.

6.1.1 Assumptions

Precise calculations of the external forces and moments an aircraft experiences will not be available. The magnitude of the errors we will introduce will vary from problem to problem, but in most cases will dwarf the error introduced by ignoring centripetal and coriolis accelerations, and the variation of gravity's magnitude with altitude. It is unnecessary and over-complicating to continue to relate the aircraft motion to the inertial reference frame. We therefore adopt the flat-Earth approximation for the remainder of our study.

Aerodynamic forces and moments depend on the interaction of the aircraft with the atmosphere, which may be in motion relative to the Earth. The large-scale motion of the atmosphere is of importance in navigation problems, but in flight dynamics and control it is the local motion of the atmosphere that is of concern. For the time being we will assume the atmosphere is stationary with respect to the Earth. Later we will introduce local variations to examine their influence on the aircraft's responses.

The upshot of these assumptions is that all the equations previously derived for the inertial motion of an aircraft may be used to describe its relationship to the atmosphere.

6.1.2 State Variables

Among the variables we will consider are the states of the aircraft. Generally these are all the velocity and position variables for which we have derived differential equations, namely V, α, β, p, q, r, θ, ϕ, ψ, x_E, y_E, and h. Instead of V, α, and β we could use u, v, and w, but V, α, and β are preferred. (Force and moment dependencies on linear velocities are almost always formulated in terms of the wind axis representation, V, α, and β. This is because it is easier in experiments to hold two of these quantities constant while varying the third, than it is with u, v, and w.)

From the list of variables we may remove from consideration the Euler angles θ, ϕ, and ψ as they are only used to keep track of the orientation of the gravity vector relative to the body. We may also remove x_E and y_E since (barring Bermuda triangle phenomena) the location of an aircraft with respect to the Earth's surface should not affect the forces or moments acting on it. The altitude h must be retained, since it influences the properties of the air mass, which will effect the aerodynamic forces and moments.

6.1.3 State Rates

Simple dependencies on states are often inadequate to describe the variation of forces and moments when angle-of-attack and sideslip are varying with time. Generally speaking this is because the flow field about the aircraft will not adjust immediately to such variations. This is a complicated phenomenon for which an approximation is required. The usual approximation is to add to the list of independent variables the time rate of change of angle-of-attack and sideslip, $\dot{\alpha}$ and $\dot{\beta}$. The former, $\dot{\alpha}$, appears quite frequently in aerodynamic force and moment data, while $\dot{\beta}$ is more rarely used.

6.1.4 Controls

In addition to the state variables, we will also consider as separate influences the aircraft *flight controls.* A flight control is any device or mechanism that may be used to directly change the external forces or moments. If we ignore any dynamics associated with the acceleration of the aircraft engine then the throttle is a flight control. The typical set of aerodynamic flight controls are the elevator, ailerons, and rudder. These will be referred to as the primary flight controls. Most aircraft also have secondary controls such as flaps and speed brakes that are used only in particular flight regimes.

It should be noted that modern aircraft may have more and different primary flight controls than the "classical" three controls. Possibilities include maneuvering trailing or leading edge flaps, canards, flaperons, thrust vectoring, and side-force generators. All of these are more or less redundant, meaning they may be combined to produce the same effects as elevator, ailerons, and rudder. We will therefore refer to the primary aerodynamic flight controls as the roll, pitch, and yaw controllers according to the moments they are intended to produce. The nomenclature to be used will be:

δ_T Thrust control
δ_ℓ Roll control
δ_m Pitch control
δ_n Yaw control

When discussing a particular aircraft we will replace the generic aerodynamic flight controls with more descriptive terms, such as δ_a (ailerons) for δ_ℓ, and δ_e (elevator) or δ_{HT} (horizontal tail) for δ_m.

6.1.5 Independent variables

We therefore assume the external forces and moments of the aircraft are functions of the independent variables V, α, β, p, q, r, h, $\dot{\alpha}$, $\dot{\beta}$, δ_T, δ_ℓ, δ_m, and δ_n. We will discuss the dependency on each variable as it affects the force or moment while holding all other variables constant. Care should be taken to not mix wind and body axis representations of velocity. In the wind axes velocity relative to the body is completely described by V, α, and β, and in the body axes by u, v, and w. So if we are speaking of the dependency of lift on α, we mean we are holding V and β constant. On the other hand if we speak of the dependency of lift on the velocity component w, then we are holding u and v constant.

6.2 Non-Dimensionalization

In basic aerodynamics we learned that forces are made non-dimensional by dividing them by the dynamic pressure $\bar{q}$ and a characteristic area S, and moments by $\bar{q}$, S, and a characteristic length. The length is the wing mean geometric chord $\bar{c}$ for the pitching moment and the span b for rolling and yawing moments. The dynamic pressure depends on the local density ρ of the atmosphere and the velocity V of the aircraft relative to the atmosphere,

$$\bar{q} = \frac{1}{2}\rho V^2$$

The non-dimensional quantity is denoted as a coefficient C plus a subscript identifying the force or moment in question. Thus,

$$\begin{array}{lll}
C_D = \frac{D}{\bar{q}S} & C_L = \frac{L}{\bar{q}S} & C_C = \frac{C}{\bar{q}S} \\
C_X = \frac{X}{\bar{q}S} & C_Y = \frac{Y}{\bar{q}S} & C_Z = \frac{Z}{\bar{q}S} \\
C_T = \frac{T}{\bar{q}S} & & C_W = \frac{W}{\bar{q}S} = \frac{mg}{\bar{q}S} \\
C_\ell = \frac{L}{\bar{q}Sb} & C_m = \frac{M}{\bar{q}S\bar{c}} & C_n = \frac{N}{\bar{q}Sb}
\end{array}$$

Theory then tells us that the non-dimensional coefficients are dependent themselves on other non-dimensional quantities such as Mach number or angle-of-attack. We will write the dependencies of forces and moments as the dependency of the associated non-dimensional coefficient on non-dimensional quantities, plus the dependence on the dynamic pressure. For example in $D = C_D \bar{q} S$, C_D will be a function of several non-dimensional quantities that are derived from the states, state-rates, and controls. All of the dimensional forces and moments will depend on h (density as a function of altitude) and V through the dynamic pressure $\bar{q}$.

The non-dimensional quantities that are derived from V, p, q, r, $\dot{\alpha}$, and $\dot{\beta}$ (all angular measurements in radians) are defined as follows:

$M = V/a$ (a is the local speed of sound in the atmosphere), and

$$\begin{array}{lll} \hat{p} = \frac{pb}{2V} & \hat{q} = \frac{q\bar{c}}{2V} & \hat{r} = \frac{rb}{2V} \\ \hat{\dot{\alpha}} = \frac{\dot{\alpha}\bar{c}}{2V} & \hat{\dot{\beta}} = \frac{\dot{\beta}b}{2V} & \end{array}$$

Because dependency on V appears explicitly in the dynamic pressure term, separate from the Mach dependency of the coefficients, we will later require one more nondimensional velocity term. This term is valid only when the current velocity V is compared to some reference velocity V_{Ref}, and is defined as

$$\hat{V} = \frac{V}{V_{Ref}}$$

6.3 Non-Dimensional Coefficient Dependencies

6.3.1 General

It is generally true that anything that changes the pressure distribution about an aircraft will cause changes in all the forces and moments. Thus a case could be made that each of the non-dimensional coefficients are dependent on all the listed states and controls. Clearly some of these effects are more important than others, and this is largely determined by common sense and experiment. The dependencies listed below are just those usually considered to have dominant effects for "conventional" aircraft. A particular aircraft in a particular flight condition may exhibit different dependencies, and other data may have to be acquired to adequately describe the effects.

The force coefficients described below are the wind axis representations of lift and drag (C_L and C_D) and the body axis representation of side force (C_Y) since these are the forces most frequently measured in experiments. The body axis force coefficients are related to the wind axis representations by transformation $T_{B,W}$ (since all dimensional force variables are non-dimensionalized by the same quantity). However, the mixed system of C_D, C_Y, and C_L does not so obviously transform to C_X, C_Y, and C_Z. However, it may be shown that

$$\begin{Bmatrix} C_X \\ C_Y \\ C_Z \end{Bmatrix} = \begin{bmatrix} \cos\alpha\sec\beta & -\cos\alpha\tan\beta & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha\sec\beta & -\sin\alpha\tan\beta & \cos\alpha \end{bmatrix} \begin{Bmatrix} -C_D \\ C_Y \\ -C_L \end{Bmatrix} \tag{6.1}$$

In the general case C_X and C_Z each must be a function of α and β irregardless of the dependencies of C_D, C_Y, and C_L. The dependency of C_X and C_Z on β introduces an unnecessary complication in later analysis. To continue to focus on the essentials, we neglect that dependency and take the relationships to be

$$\begin{Bmatrix} C_X \\ C_Y \\ C_Z \end{Bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{Bmatrix} -C_D \\ C_Y \\ -C_L \end{Bmatrix} \tag{6.2}$$

6.3.2 Altitude

The dimensional forces and moments are dependent on altitude through the dynamic pressure term, as previously discussed. There is another phenomenon known as *ground effect* that introduces an additional dependency of lift and pitching moment coefficients on altitude. Ground effect, loosely speaking, is a "cushioning" that occurs when an aircraft is at an altitude roughly equal to half its wing span. If needed we will nondimensionalize altitude by dividing by the chord length, $\hat{h} = h/\bar{c}$.

6.3.3 Velocity

Dependence of the nondimensional coefficients on velocity is due primarily to Mach effects. However, the case of the thrust coefficient is somewhat different. Depending on the powerplant being considered, we may require that the thrust coefficient be a function of velocity (or the nondimensional velocity, $\hat{V}$) directly. Consider for example a rocket powered aircraft: it is generally valid to consider the thrust T constant with velocity. Since $T = C_T \rho V^2 S/2$, C_T must decrease as V increases. For the case of the thrust coefficient C_T, we therefore assume a dependency on $\hat{V}$.

Mach dependency of the other coefficients is associated with compressibility phenomena which can create large variations in the pressure distribution over the aircraft. This will affect all of the aerodynamic coefficients. Most of our reference conditions will be in symmetric flight, meaning the airflow over the aircraft is the same on the left and right sides. For a change in Mach from this condition no additional sideforce, or rolling or yawing moments are generated. Even in asymmetric flight the effects of changes in Mach are relatively small and may be neglected. We therefore limit Mach dependency to longitudinal forces and moment, and do not consider it for the lateral/directional moments or force. Except for certain cases, at low subsonic speeds Mach dependency is usually ignored.

6.3.4 Angle-of-Attack

The angle-of-attack of an aircraft greatly influences the entire flow field and is likely to affect all force and moment coefficients. In symmetric flight, i.e. no sideslip, the effect of changes of angle-of-attack is generally the same on the left and right sides of the aircraft. In that case the rolling moment, or lateral coefficient C_ℓ, and the yawing moment and sideforce, or directional coefficients C_n and C_Y, will be unaffected. However, the coefficients C_L, C_D, and C_m have strong dependencies on angle-of-attack.

C_L Dependency on α:

The variation of lift coefficient of an airfoil with angle-of-attack should be familiar. Typically, the relationship is similar to that shown in figure 6.1.

All the surfaces of an aircraft — wing, horizontal tail, the fuselage itself — behave to some extent as airfoils, and each will have its own lift vs. angle-

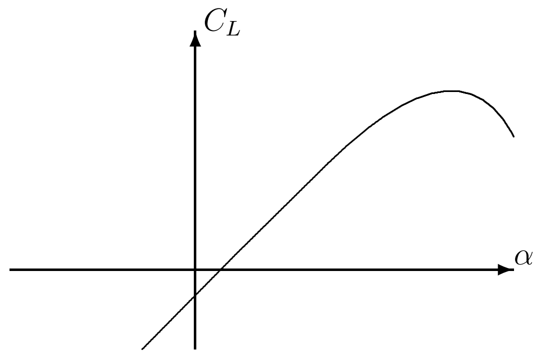

Figure 6.1: Typical Lift vs. Angle-of-Attack Curve

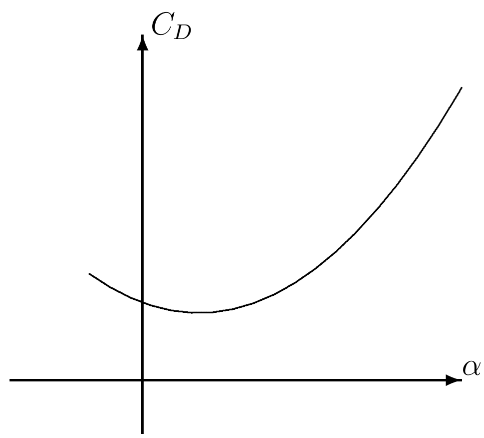

Figure 6.2: Typical Drag vs. Angle-of-Attack Curve

of-attack relationship. The net effect is not the sum of all the contributions because of mutual interference within the flow field, but overall will have a similar shape to that of a single airfoil.

C_D Dependency on α:

The drag coefficient is usually thought of as being a function of the lift coefficent according to $C_D = C_d + kC_L^2$. To the extent that C_L is linear in α this means C_D is a function of α^2, or roughly as shown in figure 6.2.

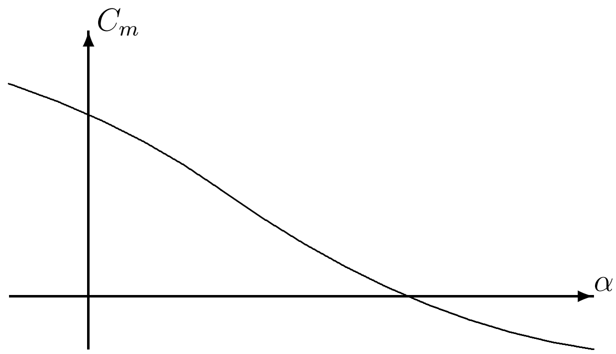

Figure 6.3: Typical Pitching Moment vs. Angle-of-Attack Curve

C_m **Dependency on** α**:**

The pitching moment coefficient dependence on angle-of-attack arises because changes in the net aerodynamic forces in the plane of symmetry due to changes in angle-of-attack do not generally act directly through the aircraft CG. In particular, a change in angle-of-attack on the horizontal tail creates a change in lift on the tail which is normally far removed from the CG, resulting in a change in pitching moment. In fact the horizontal tail is normally intended to provide moments that tend to counteract changes in angle-of-attack to provide static stability in pitch. This is seen in a typical C_m vs. α relationship in which the slope is negative, indicating that positive changes in α result in negative (nose down) changes in C_m, as shown in figure 6.3.

6.3.5 Sideslip

Sideslip affects an aircraft in a manner similar to angle-of-attack (but with a point-of-view from above the aircraft). The big differences are the absence of a large airfoil corresponding to the wing, the lack of symmetry about the plane of $x_B - y_B$, and the associated fact that the vertical tail is not symmetric about that plane. The fuselage acts roughly like a large, inefficient airfoil itself, but the primary influence of the fuselage is to partially alter the flow of air to surfaces on its lee (downwind) side. The vertical tail of course is an airfoil, and it generates aerodynamic forces in the presence of sideslip much like the horizontal tail.

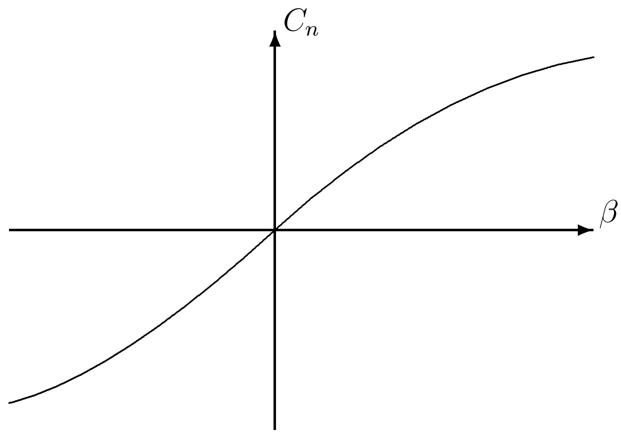

Figure 6.4: Typical Yawing Moment vs. Sideslip Curve

C_Y Dependency on β:

The fuselage and vertical tail combine to produce sideforce in the presence of sideslip, acting as airfoils as just described. The picture for C_Y vs. β is very much like that for C_L vs. α, except the slope is normally less and the stall occurs earlier due to the inefficiency of the fuselage as an airfoil.

C_n Dependency on β:

The generation of yawing moment in the presence of sideslip is analogous to the generation of pitching moment with angle-of-attack. For static directional stability, however, the slope of C_n vs. β should be positive. This is different from the negative slope required of C_m vs. α for static longitudinal stability because of the way positive sideslip is defined.

C_ℓ Dependency on β:

There are several consequences of sideslip that create rolling moments on aircraft. The most obvious is the fact that the vertical tail (fin) has more area above the x_B axis than below it. With positive sideslip the fin, as an airfoil, generates an aerodynamic force F_f in the negative y_B direction, hence a negative rolling moment. Thus the fin normally generates a negative contribution to C_ℓ with positive β as shown in figure 6.5.

A more subtle contributor to rolling moment due to sideslip is wing *dihedral*. Dihedral is the angle at which the left and right wings are inclined

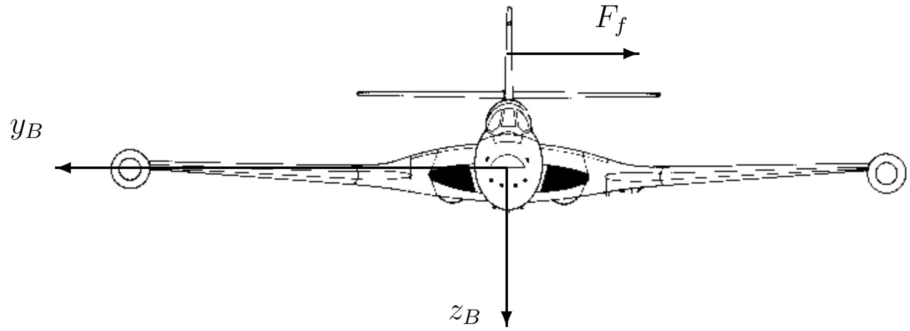

Figure 6.5: Vertical Fin Contribution to Rolling Moment

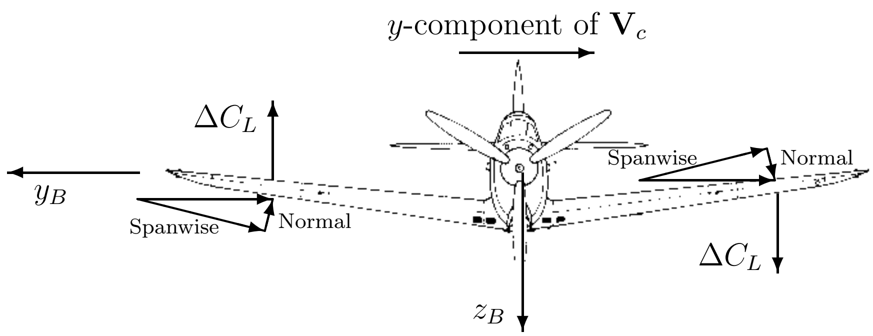

Figure 6.6: Effect of Dihedral

upward (positive) or downward (negative). Negative dihedral is sometimes called *anhedral* and (rarely) *cathedral.* In the presence of positive sideslip the plane of the right wing sees an addition vertical component of air flow that is upward, thus increasing the angle of attack (and lift) on that wing. The opposite is true on the left wing. Like the fin, dihedral normally generates a negative contribution to C_ℓ with positive β. This is easily seen by resolving the relative airflow due to sideslip into spanwise and normal components, as shown in figure 6.6

Other major contributions to rolling moment due to sideslip result from wing sweep and from wing position relative to the fuselage (high wing or low). Of these a low wing position is the only one which tends to create positive rolling moments with positive sideslip. The actual slope of the C_ℓ vs. β curve, however, is usually negative, as shown in figure 6.7.

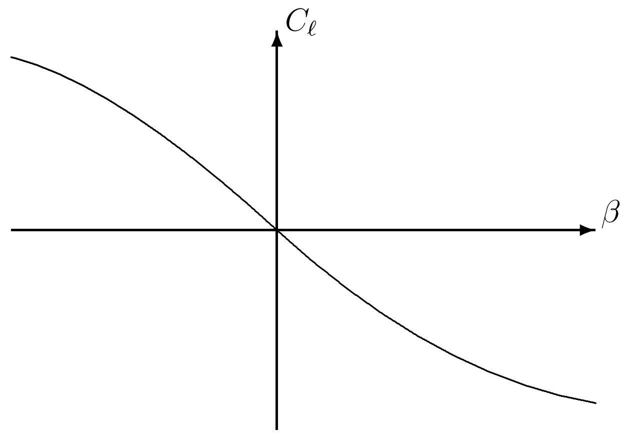

Figure 6.7: Typical Rolling Moment vs. Sideslip Curve

6.3.6 Angular Velocity Dependencies

The dependencies on $\dot{\alpha}$ and $\dot{\beta}$ were discussed in section 6.1.3. As mentioned there, $\dot{\beta}$ dependencies are rarely seen in the analysis of "conventional" aircraft. As for $\dot{\alpha}$ only lift and pitching moment are commonly assumed to have dependency (this in spite of the fact that drag is a function of lift). In the following we will therefore assume that C_L and C_m are functions of $\dot{\alpha}$, and that no coefficients are dependent on $\dot{\beta}$.

With respect to p, q, and r, all dependencies arise from similar phenomena: Superimposed on the linear flow of the air mass over the aircraft are components created by the rotation within the air mass.

C_m **Dependency on** $\hat{q}$**:**

This effect is illustrated by considering the longitudinal case, with pitch rate q and the horizontal tail some distance ℓ_t behind the CG. The vertical velocity at the tail due to q is $V_t = \ell_t q$, as shown in figure 6.8.

When the vertical velocity V_t is added to the free stream velocity, the effect is to increase the local angle-of-attack of the tail by some amount $\Delta\alpha_t$, shown in figure 6.9.

The increase in angle-of-attack $\Delta\alpha_t$ at the tail will generally increase the lift, which will create a nose-down (negative) pitching moment.

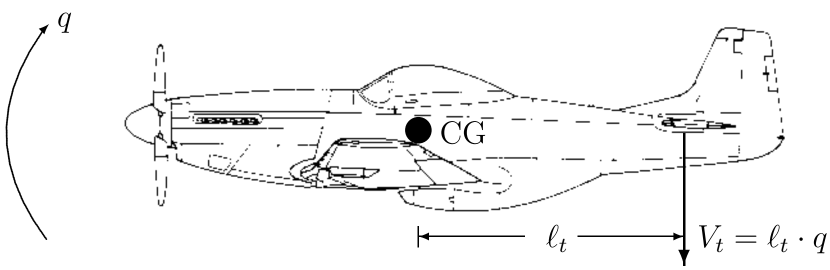

Figure 6.8: Vertical Velocity at Empenage Induced by Pitch Rate

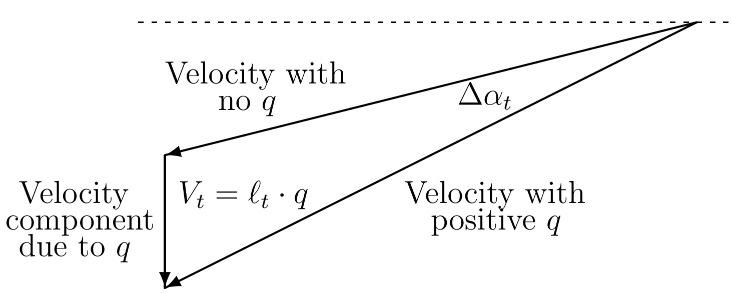

Figure 6.9: Change in Angle-of-Attack Due to Pitch Rate

C_ℓ and C_n Dependencies on $\hat{p}$ and $\hat{r}$:

Analogous to the effect of pitch rate upon the horizontal tail, a similar situation exists for the vertical tail in response to yaw rate r, and a negative yawing moment will be induced by positive yaw rate. In the case of the wings positive roll rate p increases the local angle-of-attack along the right (down-going) wing and decreases it along the left (up-going) wing. The increased lift on the right wing plus the decreased lift on the left wing generates a negative rolling moment.

Changes in lift on the wing are accompanied by changes in drag as well. In the case of an aircraft with a positive roll rate increased lift on the right wing increases its drag, while the decreased lift on the left wing decreases its drag. The differential in drag causes a yawing moment, in this case positive (nose right).

As in our discussion regarding rolling moments due to sideslip, the fact that the fin is not symmetrical about the $x_B - z_B$ plane means that any force generated by the fin will induce a rolling moment. Since a positive yaw rate generates a force in the positive y_B direction at some distance above the x_B axis, a positive rolling moment will result.

C_L and C_Y Dependencies on $\hat{p}$, $\hat{q}$, and $\hat{r}$:

Since each of the rotary motions creates a change in aerodynamic forces (that generated the moments described), we include dependency of C_L on $\hat{q}$ and of C_Y on $\hat{p}$ and $\hat{r}$.

6.3.7 Control Dependencies

C_T and C_m Dependencies on δ_T:

The throttle δ_T is assumed to directly control the thrust of the engine and therefore, the coefficient of thrust. We further assume the thrust line lies in the plane of symmetry $x_B - z_B$ so that the direction of thrust lies in the plane of symmetry as previously discussed, with ϵ_T defined as the angle between $\mathbf{T}$ and x_B. In terms of the thrust coefficient C_T and the throttle δ_T,

$$\{\mathbf{T}\}_B = \bar{q}S \begin{Bmatrix} T = C_T\left(\hat{V}, \delta_T\right)\cos\epsilon_T \\ 0 \\ T = C_T\left(\hat{V}, \delta_T\right)\sin\epsilon_T \end{Bmatrix}$$

It is usually the case that the thrust-line does not act exactly through the center of gravity of the aircraft. Thus there is a pitching moment due to thrust, and a change in C_m due to change in δ_T.

C_ℓ, C_m, and C_n Dependencies on δ_ℓ, δ_m, and δ_n:

Primary Effects. We have postulated generic aerodynamic controls that are intended as moment generators: δ_ℓ, δ_m, and δ_n. At a minimum therefore C_ℓ, C_m, and C_n will be functions of δ_ℓ, δ_m, and δ_n, respectively.

Coupled Effects. Most pitching moment control effectors (elevators, horizontal tails, canards, etc.) when operated symmetrically (left-right) produce pure pitching moment—no rolling or yawing moments. There is, however, a good amount of out-of-axis effects from typical rolling and yawing moment control effectors such as ailerons, spoilers, and rudders. It is therefore common to find that there is a dependency of C_ℓ on δ_n and of C_n on δ_ℓ.

C_D, C_Y, and C_L Dependencies on δ_ℓ, δ_m, and δ_n:

Roll Control All aerodynamic controls generate forces to produce the desired moments. In the case of roll control δ_ℓ, for implementations such as ailerons the forces generated are almost a pure couple, leaving no resultant effect on net lift. Other implementations, such as spoilers, change the lift on one wing but not the other, and there will be a net change in lift. In the conventional case of ailerons assumed subsequently, there will be no dependency of any force coefficients on δ_ℓ.

Pitch Control Every practical implementation of aircraft pitch control δ_m generates an unbalanced lift force. We therefore assume C_L is dependent upon δ_m. Since a change in lift creates a change in drag, we further assume C_D is dependent upon δ_m.

Yaw Control Similar to pitch control, most implementations of yaw control create unbalanced side forces on the aircraft. We therefore take C_Y to be dependent upon δ_n.

6.3.8 Summary of Dependencies

The force and moment coefficient dependencies assumed here and subsequently are representative of a "conventional" aircraft. Any given aircraft may require the assumption of other dependencies to adequately represent its aerodynamic and thrust forces and moments. For our purposes, however, we assume the following:

$$\begin{aligned}
C_D &= C_D\left(M, \alpha, \delta_m\right) \\
C_Y &= C_Y\left(\beta, \hat{p}, \hat{r}, \delta_n\right) \\
C_L &= C_L\left(M, \alpha, \hat{\dot{\alpha}}, \hat{q}, \hat{h}, \delta_m\right) \\
C_\ell &= C_\ell\left(\beta, \hat{p}, \hat{r}, \delta_\ell, \delta_n\right) \\
C_m &= C_m\left(M, \alpha, \hat{\dot{\alpha}}, \hat{q}, \hat{h}, \delta_m, \delta_T\right) \\
C_n &= C_n\left(\beta, \hat{p}, \hat{r}, \delta_\ell, \delta_n\right) \\
C_T &= C_T\left(\hat{V}, \delta_T\right)
\end{aligned} \tag{6.3}$$

6.4 The Linear Assumption

Even though the force and moment dependencies listed above can be quite nonlinear, we can restrict our analysis to a range of variables in which the nonlinearity is only slight, and consider the dependencies to be linear. This is completely equivalent to writing each coefficient as a Taylor series expansion about some reference condition, and discarding all but the constant and first order terms. Thus, for a coefficient C that is a function of several variables $v_1, v_2, \ldots v_n$, this results in the linear approximation given by:

$$\begin{aligned}
C\left(v_1, v_2, \ldots v_n\right) = {} & C\left(v_1, v_2, \ldots v_n\right)_{Ref} \\
& + \left(\frac{\partial C}{\partial v_1}\right)_{Ref} \Delta v_1 + \left(\frac{\partial C}{\partial v_2}\right)_{Ref} \Delta v_2 \cdots + \left(\frac{\partial C}{\partial v_n}\right)_{Ref} \Delta v_n
\end{aligned}$$

In this expression, $\Delta v_i = v_i - v_{i_{Ref}}$. If $v_{i_{Ref}} = 0$, then $\Delta v_i = v_i$. The customary notation in flight dynamics is to abbreviate the partial derivative of a nondimensional coefficient with respect to a nondimensional variable using successive subscripts, e.g.,

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_{Ref} \equiv C_{L_\alpha}$$

Using this notation, we may write:

$$C_D\left(M, \alpha, \delta_m\right) = C_D\left(M, \alpha, \delta_m\right)_{Ref} + C_{D_M}\Delta M + C_{D_\alpha}\Delta\alpha + C_{D_{\delta_m}}\Delta\delta_m$$

The other coefficients are represented in similar fashion.

With respect to the lateral-directional coefficients, the reference conditions are usually taken such that $C_{Y_{Ref}} = C_{\ell_{Ref}} = C_{n_{Ref}} = 0$. Due to the symmetry of the airplane, this condition generally requires $\beta_{Ref} = \hat{p}_{Ref} = \hat{r}_{Ref} = 0$. Suitably defined control deflection conventions also yield $\delta_{\ell_{Ref}} = \delta_{n_{Ref}} = 0$.

We have neglected the influence of Mach number on the lateral/directional coefficients; it is reasonable, however, to posit that the dependency of the coefficients on other states and controls will be influenced by changes in Mach number. This is accounted for by making some or all of the derivatives with respect to these states and controls functionally dependent on Mach. Thus one may see expressions such as

$$C_\ell = C_{\ell_\beta}(M)\,\Delta\beta + C_{\ell_p}(M)\,\Delta\hat{p} + \cdots$$

Here, $C_{\ell_\beta}(M)$ means C_{ℓ_β} is a function of Mach number, and similarly for the other derivatives.

6.5 Tabular Data

The force and moment coefficients are normally available in tabular form. Thus, for example, at a particular Mach number and control setting the drag coefficient may be measured at discrete values of angle-of-attack and the results tabulated. To obtain drag coefficients at angles-of-attack between the discrete values the data are interpolated. The interpolation is usually linear.

Within an interval of the table denote α_{Lower} the angle-of-attack at the lower end of the interval, α_{Upper} that at the upper end, and likewise for the coefficients $C_D(\alpha_{Lower})$ and $C_D(\alpha_{Upper})$. With linear interpolation for $C_D(\alpha)$ at arbitrary α in an interval we have:

$$\frac{C_D(\alpha) - C_D(\alpha_{Lower})}{C_D(\alpha_{Upper}) - C_D(\alpha_{Lower})} = \frac{\alpha - \alpha_{Lower}}{\alpha_{Upper} - \alpha_{Lower}}$$

This expression may be solved for the desired $C_D(\alpha)$,

$$C_D(\alpha) = C_D(\alpha_{Lower}) + \frac{\alpha - \alpha_{Lower}}{\alpha_{Upper} - \alpha_{Lower}} \left[C_D(\alpha_{Upper}) - C_D(\alpha_{Lower})\right]$$

This last expression is of the form

$$C_D = C_1(\alpha_{Lower}, \alpha_{Upper}) + C_2(\alpha_{Lower}, \alpha_{Upper})\,\alpha$$

The notation indicates that the values of C_1 and C_2 on the right hand side are valid only in the range of some particular α_{Lower} and α_{Upper}. Instead of recording the tabulated data, one may pre-calculate and record the values of C_1 and C_2 for each interval.

Because each coefficient is a function of several variables, the tabulation must be repeated for various discrete values of each of the independent variables. Continuing the drag coefficient example, the coefficient is (assumed to be) a function of Mach, angle-of-attack, and control setting, $C_D(M, \alpha, \delta_m)$. Therefore the tabulation will consist of C_D at several discrete values (called *breakpoints*) of α, repeated for several discrete settings of δ_m, all at some specific Mach number. The two-level tabulation must then be repeated at other discrete values of Mach number. For arbitrary combinations of M, α, and δ_m the tables are interpolated first with respect to one of the variables, then with respect to a second, and finally with respect to the third.

Interpolations of functions of more than three variables become computationally intensive and require fairly large amounts of data storage for adequate representation of an aircraft's forces and moments. It is customary to provide data for the more highly nonlinear dependencies, and to adjust the results as affine corrections for the remaining variables.

6.6 Customs and Conventions

The nondimensional derivatives introduced in 6.4 are used quite frequently in flight dynamics. Collectively those derivatives taken with respect to states (or state rates) are called *stability derivatives,* and those with respect to controls are called *control derivatives.* The sign of certain of the stability derivatives is an indication of the static stability of an aircraft with respect to disturbances in some variable, such as the requirement that C_{M_α} be negative for static stability in pitch, or that C_{n_β} should be positive for directional static stability. The contribution of the other stability derivatives is to the dynamic stability of the aircraft, and will be addressed later. Among the stability derivatives those that are with respect to rotary motion are called *rotary derivatives,* or *damping derivatives.*