

# Lagrangian and Hamiltonian Mechanics

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## Introduction

In this course we have been studying classical mechanics as formulated by Sir Isaac Newton; this is called *Newtonian mechanics*. Newtonian mechanics is mathematically fairly straightforward, and can be applied to a wide variety of problems. It is not a unique formulation of mechanics, however; other formulations are possible. Here we will look at two common alternative formulations of classical mechanics: *Lagrangian mechanics* and *Hamiltonian mechanics*.

It is important to understand that all of these formulations of mechanics are equivalent. In principle, any of them could be used to solve any problem in classical mechanics. The reason they're important is that in some problems one of the alternative formulations of mechanics may lead to equations that are much easier to solve than the equations that arise from Newtonian mechanics.

Unlike Newtonian mechanics, neither Lagrangian nor Hamiltonian mechanics requires the concept of *force*; instead, these systems are expressed in terms of energy.

Although we will be looking at the equations of mechanics in one dimension, all these formulations of mechanics may be generalized to two or three dimensions.

## Newtonian Mechanics

We begin by reviewing Newtonian mechanics in one dimension. In this formulation, we begin by writing Newton's second law, which gives the force  $F$  required to give an acceleration  $a$  to a mass  $m$ :

$$F = ma. \tag{1}$$

Generally the force is a function of  $x$ . Since the acceleration  $a = d^2x/dt^2$ , Eq. (1) may be written

$$F(x) = m \frac{d^2x}{dt^2}. \tag{2}$$

This is a second-order ordinary differential equation, which we solve for  $x(t)$  to find the position  $x$  at any time  $t$ . Solving a problem in Newtonian mechanics then consists of these steps:

1. Write down Newton's second law (Eq. 2);
2. Substitute for  $F(x)$  the specific force present in the problem;
3. Solve the resulting differential equation for  $x(t)$ .

## Partial Derivatives

The equations of Lagrangian and Hamiltonian mechanics are expressed in the language of partial differential equations. We will leave the methods for solving such equations to a more advanced course, but we can still write down the equations and explore some of their consequences. First, in order to understand these equations, we'll first need to understand the concept of *partial derivatives*.

You've already learned in a calculus course how to take the derivative of a function of one variable. For example, if

$$f(x) = 3x^2 + 7x^5 \quad (3)$$

then

$$\frac{df}{dx} = 6x + 35x^4. \quad (4)$$

But what if  $f$  is a function of more than one variable? For example, if

$$f(x, y) = 5x^3y^5 + 4y^2 - 7xy^6 \quad (5)$$

then how do we take the derivative of  $f$ ? In this case, there are *two* possible first derivatives you can take: one with respect to  $x$ , and one with respect to  $y$ . These are called *partial derivatives*, and are indicated using the “backward-6” symbol  $\partial$  in place of the symbol  $d$  used for ordinary derivatives.

To compute a partial derivative with respect to  $x$ , you simply treat all variables except  $x$  as constants. Similarly, for the partial derivative with respect to  $y$ , you treat all variables except  $y$  as constants. For example, if  $g(x, y) = 3x^4y^7$ , then the partial derivative of  $g$  with respect to  $x$  is  $\partial g / \partial x = 12x^3y^7$ , since both 3 and  $y^7$  are considered constants with respect to  $x$ .

As another example, the partial derivatives of Eq. (5) are

$$\frac{\partial f}{\partial x} = 15x^2y^5 - 7y^6 \quad (6)$$

$$\frac{\partial f}{\partial y} = 25x^3y^4 + 8y - 42xy^5 \quad (7)$$

Notice that in Eq. (6), the derivative of the term  $4y^2$  with respect to  $x$  is 0, since  $4y^2$  is treated as a constant.

## Lagrangian Mechanics

The first alternative to Newtonian mechanics we will look at is *Lagrangian mechanics*. Using Lagrangian mechanics instead of Newtonian mechanics is sometimes advantageous in certain problems, where the equations of Newtonian mechanics would be quite difficult to solve.

In Lagrangian mechanics, we begin by defining a quantity called the *Lagrangian* ( $L$ ), which is defined as the difference between the kinetic energy  $K$  and the potential energy  $U$ :

$$L \equiv K - U \quad (8)$$

Since the kinetic energy is a function of velocity  $v$  and potential energy will typically be a function of position  $x$ , the Lagrangian will (in one dimension) be a function of both  $x$  and  $v$ :  $L(x, v)$ .

The motion of a particle is then found by solving *Lagrange's equation*; in one dimension it is

$$\boxed{\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial x} = 0} \quad (9)$$

### Example: Simple Harmonic Oscillator

As an example of the use of Lagrange's equation, consider a one-dimensional simple harmonic oscillator. We wish to find the position  $x$  of the oscillator at any time  $t$ .

We begin by writing the usual expression for the kinetic energy  $K$ :

$$K = \frac{1}{2}mv^2 \quad (10)$$

The potential energy  $U$  of a simple harmonic oscillator is given by

$$U = \frac{1}{2}kx^2 \quad (11)$$

The Lagrangian in this case is then

$$L(x, v) = K - U \quad (12)$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \quad (13)$$

Lagrange's equation in one dimension is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial x} = 0 \quad (14)$$

Substituting for  $L$  from Eq. (13), we find

$$\frac{d}{dt} \left[ \frac{\partial}{\partial v} \left( \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \right) \right] - \frac{\partial}{\partial x} \left( \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \right) = 0 \quad (15)$$

Evaluating the partial derivatives, we get

$$\frac{d}{dt} (mv) + kx = 0 \quad (16)$$

or, since  $v = dx/dt$ ,

$$m \frac{d^2x}{dt^2} + kx = 0, \quad (17)$$

which is a second-order ordinary differential equation that one can solve for  $x(t)$ . Note that the first term on the left is  $ma = F$ , so this equation is equivalent to  $F = -kx$  (Hooke's Law). The solution to the differential equation (17) turns out to be

$$x(t) = A \cos(\omega t + \delta), \quad (18)$$

where  $A$  is the *amplitude* of the motion,  $\omega = \sqrt{k/m}$  is the angular frequency of the oscillator, and  $\delta$  is a phase constant that depends on where the oscillator is at  $t = 0$ .

### Example: Plane Pendulum

Part of the power of the Lagrangian formulation of mechanics is that one may define any coordinates that are convenient for solving the problem; those coordinates and their corresponding velocities are then used in place of  $x$  and  $v$  in Lagrange's equation.

For example, consider a simple plane pendulum of length  $\ell$  with a bob of mass  $m$ , where the pendulum makes an angle  $\theta$  with the vertical. The goal is to find the angle  $\theta$  at any time  $t$ . In this case we replace

$x$  with the angle  $\theta$ , and we replace  $v$  with the pendulum's angular velocity  $\omega$ . The kinetic energy  $K$  of the pendulum is the rotational kinetic energy

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}m\ell^2\omega^2, \quad (19)$$

where  $I$  is the moment of inertia of the pendulum,  $I = m\ell^2$ . The potential energy of the pendulum is the gravitational potential energy

$$U = mg\ell(1 - \cos \theta) \quad (20)$$

The Lagrangian in this case is then

$$L(\theta, \omega) = K - U \quad (21)$$

$$= \frac{1}{2}m\ell^2\omega^2 - mg\ell(1 - \cos \theta) \quad (22)$$

Lagrange's equation becomes

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (23)$$

Substituting for  $L$ ,

$$\frac{d}{dt} \left\{ \frac{\partial}{\partial \omega} \left[ \frac{1}{2}m\ell^2\omega^2 - mg\ell(1 - \cos \theta) \right] \right\} - \frac{\partial}{\partial \theta} \left[ \frac{1}{2}m\ell^2\omega^2 - mg\ell(1 - \cos \theta) \right] = 0 \quad (24)$$

Computing the partial derivatives, we find

$$\frac{d}{dt} (m\ell^2\omega) + mg\ell \sin \theta = 0. \quad (25)$$

Since  $\omega = d\theta/dt$ , this gives

$$m\ell^2 \frac{d^2\theta}{dt^2} + mg\ell \sin \theta = 0, \quad (26)$$

which is a second-order ordinary differential equation that one may solve for the motion  $\theta(t)$ . The first term on the left-hand side is the torque  $\tau$  on the pendulum, so this equation is equivalent to  $\tau = -mg\ell \sin \theta$ .

The solution to the differential equation (26) is quite complicated, but we can simplify it if the pendulum only makes *small* oscillations. In that case, we can approximate  $\sin \theta \approx \theta$ , and the differential equation (26) becomes a simple harmonic oscillator equation with solution

$$\theta(t) \approx \theta_0 \cos(\omega t + \delta), \quad (27)$$

where  $\theta_0$  is the (angular) amplitude of the pendulum,  $\omega = \sqrt{g/\ell}$  is the angular frequency, and  $\delta$  is a phase constant that depends on where the pendulum is at  $t = 0$ .

## Hamiltonian Mechanics

The second formulation we will look at is *Hamiltonian mechanics*. In this system, in place of the Lagrangian we define a quantity called the *Hamiltonian*, to which Hamilton's equations of motion are applied. While Lagrange's equation describes the motion of a particle as a single second-order differential equation, Hamilton's equations describe the motion as a coupled system of two first-order differential equations.

One of the advantages of Hamiltonian mechanics is that it is similar in form to *quantum mechanics*, the theory that describes the motion of particles at very tiny (subatomic) distance scales. An understanding of Hamiltonian mechanics provides a good introduction to the mathematics of quantum mechanics.

The *Hamiltonian*  $H$  is defined to be the *sum* of the kinetic and potential energies:

$$H \equiv K + U \quad (28)$$

Here the Hamiltonian should be expressed as a function of position  $x$  and momentum  $p$  (rather than  $x$  and  $v$ , as in the Lagrangian), so that  $H = H(x, p)$ . This means that the kinetic energy should be written as  $K = p^2/2m$ , rather than  $K = mv^2/2$ .

Hamilton's equations in one dimension have the elegant nearly-symmetrical form

$\frac{dx}{dt} = \frac{\partial H}{\partial p} \quad (29)$
$\frac{dp}{dt} = -\frac{\partial H}{\partial x} \quad (30)$

### Example: Simple Harmonic Oscillator

As an example, we may again solve the simple harmonic oscillator problem, this time using Hamiltonian mechanics. We first write down the kinetic energy  $K$ , expressed in terms of momentum  $p$ :

$$K = \frac{p^2}{2m} \quad (31)$$

As before, the potential energy of a simple harmonic oscillator is

$$U = \frac{1}{2}kx^2 \quad (32)$$

The Hamiltonian in this case is then

$$H(x, p) = K + U \quad (33)$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad (34)$$

Substituting this expression for  $H$  into the first of Hamilton's equations, we find

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \quad (35)$$

$$= \frac{\partial}{\partial p} \left( \frac{p^2}{2m} + \frac{1}{2}kx^2 \right) \quad (36)$$

$$= \frac{p}{m} \quad (37)$$

Substituting for  $H$  into the second of Hamilton's equations, we get

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} \quad (38)$$

$$= -\frac{\partial}{\partial x} \left( \frac{p^2}{2m} + \frac{1}{2}kx^2 \right) \quad (39)$$

$$= -kx \quad (40)$$

Equations (35) and (38) are two coupled first-order ordinary differential equations, which may be solved simultaneously to find  $x(t)$  and  $p(t)$ . Note that for this example, Eq. (35) is equivalent to  $p = mv$ , and Eq. (38) is just Hooke's Law,  $F = -kx$ .

### Example: Plane Pendulum

As with Lagrangian mechanics, more general coordinates (and their corresponding momenta) may be used in place of  $x$  and  $p$ . For example, in finding the motion of the simple plane pendulum, we may replace the position  $x$  with angle  $\theta$  from the vertical, and the linear momentum  $p$  with the angular momentum  $\mathcal{L}$ .

To solve the plane pendulum problem using Hamiltonian mechanics, we first write down the kinetic energy  $K$ , expressed in terms of angular momentum  $\mathcal{L}$ :

$$K = \frac{\mathcal{L}^2}{2I} = \frac{\mathcal{L}^2}{2m\ell^2}, \quad (41)$$

where  $I = m\ell^2$  is the moment of inertia of the pendulum. As before, the gravitational potential energy of a plane pendulum is

$$U = mg\ell(1 - \cos \theta). \quad (42)$$

The Hamiltonian in this case is then

$$H(\theta, \mathcal{L}) = K + U \quad (43)$$

$$= \frac{\mathcal{L}^2}{2m\ell^2} + mg\ell(1 - \cos \theta) \quad (44)$$

Substituting this expression for  $H$  into the first of Hamilton's equations, we find

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial \mathcal{L}} \quad (45)$$

$$= \frac{\partial}{\partial \mathcal{L}} \left[ \frac{\mathcal{L}^2}{2m\ell^2} + mg\ell(1 - \cos \theta) \right] \quad (46)$$

$$= \frac{\mathcal{L}}{m\ell^2} \quad (47)$$

Substituting for  $H$  into the second of Hamilton's equations, we get

$$\frac{d\mathcal{L}}{dt} = -\frac{\partial H}{\partial \theta} \quad (48)$$

$$= -\frac{\partial}{\partial \theta} \left[ \frac{\mathcal{L}^2}{2m\ell^2} + mg\ell(1 - \cos \theta) \right] \quad (49)$$

$$= -mg\ell \sin \theta \quad (50)$$

Equations (45) and (48) are two coupled first-order ordinary differential equations, which may be solved simultaneously to find  $\theta(t)$  and  $\mathcal{L}(t)$ . Note that for this example, Eq. (45) is equivalent to  $\mathcal{L} = I\omega$ , and Eq. (48) is the torque  $\tau = -mg\ell \sin \theta$ .