Chapter 3

Coordinate System Transformations

3.1 Problem Statement

We have met the various coordinate systems that will be of primary interest in the study of flight dynamics. Now we address the subject of how these coordinate systems are related to one another. For instance, when we begin to sum the external forces acting on the aircraft, we will have to relate all these forces to a common reference frame. If we take some body-axis system to be the common frame, then we need to be able to take the gravity vector (weight) from the local horizontal reference frame, the thrust from some other body-axis frame, and the aerodynamic forces from the wind-axis frame, and represent all these forces in the body-axis frame.

Some of these relationships are easy because they are fixed: Two body-axis systems, once defined, are always related by a single rotation around their common y axis. Others are much more complicated because they vary with time: The orientation of a given body-axis system with respect to the wind axes determines certain aerodynamic forces and moments which change that orientation.

First we must characterize the relationship between two coordinate systems at some frozen instant in time. The instantaneous relationship between two coordinate systems will be addressed by determining a *transformation* that will take the representation of an arbitrary vector in one system and convert it to its representation in the other.

Three approaches to finding the needed transformations will be presented. The first is called *Direction Cosines,* a sort of brute-force approach to the problem. Next we will discuss *Euler angles,* by far the most common approach but one with a potentially serious problem. Finally we will examine *Euler parameters,* an elegant solution to the problem found with Euler angles.

3.2 Transformations

3.2.1 Definitions

Consider two reference frames, F_1 and F_2, and a vector $\mathbf{v}$ whose components are known in F_1, represented as $\{\mathbf{v}\}_1$:

$$\{\mathbf{v}\}_1 = \left\{ \begin{array}{c} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{array} \right\}$$

We wish to determine the representation of the same vector in F_2, or $\{\mathbf{v}\}_2$:

$$\{\mathbf{v}\}_2 = \left\{ \begin{array}{c} v_{x_2} \\ v_{y_2} \\ v_{z_2} \end{array} \right\}$$

These are linear spaces, so the transformation of the vector from F_1 to F_2 is simply a matrix multiplication which we will denote $T_{2,1}$, such that $\{\mathbf{v}\}_2 = T_{2,1}\,\{\mathbf{v}\}_1$. Transformations such as $T_{2,1}$ are called similarity transformations. The transformations involved in simple rotations of orthogonal reference frames have many special properties that will be shown.

The order of subscripts of $T_{2,1}$ is such that the left subscript goes with the system of the vector on the left side of the equation and the right subscript with the vector on the right. For the matrix multiplication to be conformal $T_{2,1}$ must be a $3x3$ matrix:

$$T_{2,1} = \left[\begin{array}{ccc} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{array} \right]$$

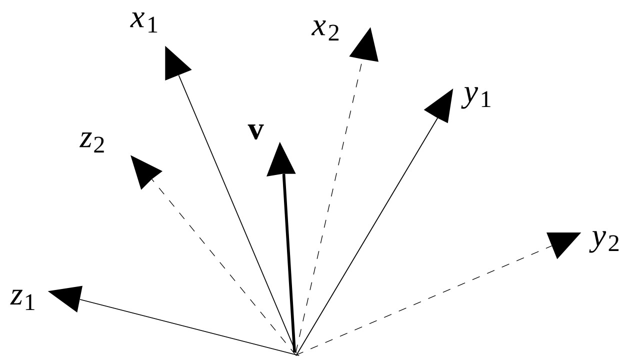

Figure 3.1: Two Coordinate Systems

(The t_{ij} probably need more subscripting to distinguish them from those in other transformation matrices, but this gets cumbersome.)

The whole point of the three approaches to be presented is to figure out how to evaluate the numbers t_{ij}. It is important to note that for a fixed orientation between two coordinate systems, the numbers t_{ij} are the same quantities no matter how they are evaluated.

3.2.2 Direction Cosines

Derivation

Consider our two reference frames F_1 and F_2, and the vector $\mathbf{v}$, shown in figure 3.1. We claim to know the representation of $\mathbf{v}$ in F_1. The vector $\mathbf{v}$ is the vector sum of the three components $v_{x_1}\mathbf{i}_1$, $v_{y_1}\mathbf{j}_1$, and $v_{z_1}\mathbf{k}_1$ so we may replace the vector by those three components, shown in figure 3.2

Now, the projection of $\mathbf{v}$ onto x_2 is the same as the vector sum of the projections of each of its components $v_{x_1}\mathbf{i}_1$, $v_{y_1}\mathbf{j}_1$, and $v_{z_1}\mathbf{k}_1$ onto x_2, and similarly for y_2 and z_2. Define the angle generated in going from x_2 to x_1 as $\theta_{x_2x_1}$. The projection of $v_{x_1}\mathbf{i}_1$ onto x_2 therefore has magnitude $v_{x_1}\cos\theta_{x_2x_1}$ and direction $\mathbf{i}_2$, as shown if figure 3.3.

To the vector $v_{x_1}\cos\theta_{x_2x_1}\mathbf{i}_2$ must be added the other two projections, $v_{y_1}\cos\theta_{x_2y_1}\mathbf{i}_2$ and $v_{z_1}\cos\theta_{x_2z_1}\mathbf{i}_2$. Similar projections onto y_2 and z_2 result in:

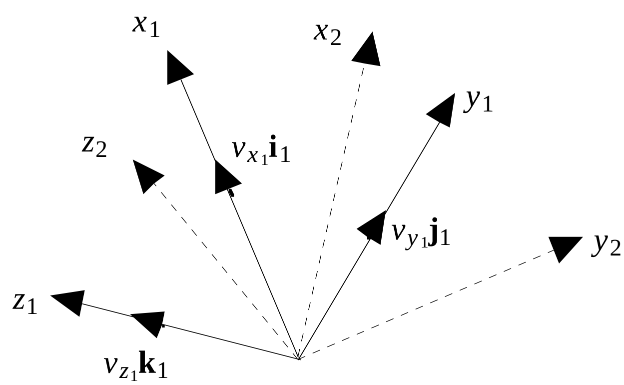

Figure 3.2: Vector in Component Form

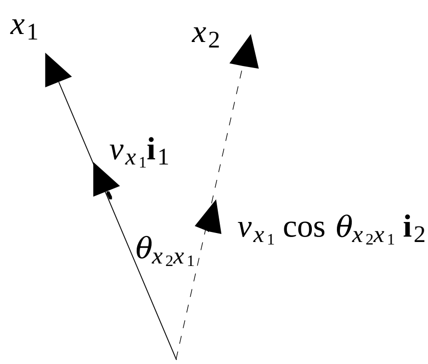

Figure 3.3: One Component of the Vector

$$
\begin{aligned}
v_{x_2} &= v_{x_1}\cos\theta_{x_2x_1} + v_{y_1}\cos\theta_{x_2y_1} + v_{z_1}\cos\theta_{x_2z_1}\\
v_{y_2} &= v_{x_1}\cos\theta_{y_2x_1} + v_{y_1}\cos\theta_{y_2y_1} + v_{z_1}\cos\theta_{y_2z_1}\\
v_{z_2} &= v_{x_1}\cos\theta_{z_2x_1} + v_{y_1}\cos\theta_{z_2y_1} + v_{z_1}\cos\theta_{z_2z_1}
\end{aligned}
$$

In vector-matrix notation, this may be written

$$
\left\{ \begin{array}{c} v_{x_2} \\ v_{y_2} \\ v_{z_2} \end{array} \right\} = \begin{bmatrix} \cos\theta_{x_2x_1} & \cos\theta_{x_2y_1} & \cos\theta_{x_2z_1} \\ \cos\theta_{y_2x_1} & \cos\theta_{y_2y_1} & \cos\theta_{y_2z_1} \\ \cos\theta_{z_2x_1} & \cos\theta_{z_2y_1} & \cos\theta_{z_2z_1} \end{bmatrix} \left\{ \begin{array}{c} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{array} \right\}
$$

Clearly this is $\{\mathbf{v}\}_2 = T_{2,1}\{\mathbf{v}\}_1$, so we must have $t_{ij} = \cos\theta_{(axis)_2(axis)_1}$ in which i or j is 1 if the corresponding $(axis)$ is x, 2 if $(axis)$ is y, and 3 if $(axis)$ is z.

$$
\boxed{T_{2,1} = \begin{bmatrix} \cos\theta_{x_2x_1} & \cos\theta_{x_2y_1} & \cos\theta_{x_2z_1} \\ \cos\theta_{y_2x_1} & \cos\theta_{y_2y_1} & \cos\theta_{y_2z_1} \\ \cos\theta_{z_2x_1} & \cos\theta_{z_2y_1} & \cos\theta_{z_2z_1} \end{bmatrix}} \tag{3.1}
$$

Properties of the Direction Cosine Matrix

The transformation matrix (often called the direction cosine matrix, regardless of how it is derived or represented) does not depend on the vector $\mathbf{v}$ for its existence. It is therefore the same no matter what we choose for $\mathbf{v}$. If we take

$$
\{\mathbf{v}\}_1 = \left\{ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\}
$$

Then $\{\mathbf{v}\}_2$ is just the first column of the direction cosine matrix:

$$
\{\mathbf{v}\}_2 = \left\{ \begin{array}{c} \cos\theta_{x_2x_1} \\ \cos\theta_{y_2x_1} \\ \cos\theta_{z_2x_1} \end{array} \right\}
$$

Since the length of $\mathbf{v}$ is 1, we have shown the well-known property of direction cosines, that

$$\cos^2\theta_{x_2x_1} + \cos^2\theta_{y_2x_1} + \cos^2\theta_{z_2x_1} = 1$$

The same obviously holds true for any column of the direction cosine matrix.

If we need to go the other way, $\{\mathbf{v}\}_1 = T_{1,2}\{\mathbf{v}\}_2$, it is clear that $T_{1,2} = T_{2,1}^{-1}$, and we might be tempted to invert the direction cosine matrix. On the other hand, had we begun by assuming $\{\mathbf{v}\}_2$ was known, and figuring out how to get $\{\mathbf{v}\}_1 = T_{1,2}\{\mathbf{v}\}_2$ using similar arguments to the above, we would have arrived at

$$\left\{\begin{array}{c} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{array}\right\} = \left[\begin{array}{ccc} \cos\theta_{x_1x_2} & \cos\theta_{x_1y_2} & \cos\theta_{x_1z_2} \\ \cos\theta_{y_1x_2} & \cos\theta_{y_1y_2} & \cos\theta_{y_1z_2} \\ \cos\theta_{z_1x_2} & \cos\theta_{z_1y_2} & \cos\theta_{z_1z_2} \end{array}\right] \left\{\begin{array}{c} v_{x_2} \\ v_{y_2} \\ v_{z_2} \end{array}\right\}$$

Here we observe that $\cos\theta_{x_1x_2}$ is the same as $\cos\theta_{x_2x_1}$, $\cos\theta_{x_1y_2}$ is the same as $\cos\theta_{y_2x_1}$, etc., since (even if we had defined positive rotations of these angles) the cosine is an even function. So the same transformation is

$$\left\{\begin{array}{c} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{array}\right\} = \left[\begin{array}{ccc} \cos\theta_{x_2x_1} & \cos\theta_{y_2x_1} & \cos\theta_{z_2x_1} \\ \cos\theta_{x_2y_1} & \cos\theta_{y_2y_1} & \cos\theta_{z_2y_1} \\ \cos\theta_{x_2z_1} & \cos\theta_{y_2z_1} & \cos\theta_{z_2z_1} \end{array}\right] \left\{\begin{array}{c} v_{x_2} \\ v_{y_2} \\ v_{z_2} \end{array}\right\}$$

Obviously the columns of $T_{1,2}$ are the rows of $T_{2,1}$ (and have unit length as well). This leads us to another nice property of these transformation matrices, that the inverse of the direction cosine matrix is equal to its transpose,

$$T_{1,2} = T_{2,1}^{-1} = T_{2,1}^{T}$$

Since $T_{2,1}T_{2,1}^{-1} = T_{2,1}T_{2,1}^{T} = I_3$, the 3×3 identity matrix, it must be true that the scalar (dot) product of any row of $T_{2,1}$ with any other row must be zero. It is easy to show that $T_{1,2}T_{1,2}^{-1} = T_{1,2}T_{1,2}^{T} = I_3$, so the scalar (dot) product of any column of $T_{2,1}$ with any other column must be zero since the columns of $T_{2,1}$ are the rows of $T_{1,2}$. When the rows (or columns) of the direction cosine matrix are viewed as vectors, this means the rows (or the columns) form orthogonal bases for 3-dimensional space.

Also, with $T_{2,1}T_{2,1}^T = I_3$, if we take the determinant of each side and note that $\left|T_{2,1}^T\right| = |T_{2,1}|$, we have

$$\begin{aligned}\left|T_{2,1}T_{2,1}^T\right| &= |T_{2,1}|\left|T_{2,1}^T\right| \\ &= |T_{2,1}|\,|T_{2,1}| \\ &= |T_{2,1}|^2 = 1\end{aligned}$$

The only conclusion we can reach at this point is that $|T_{2,1}| = \pm 1$. Because the identity matrix is a transformation matrix with determinant $+1$, and since all other transformations may be reached through continuous rotations from the identity transformation, it seems unreasonable to think that the sign of the determinant would be different for some rotations and not others. We will show in our discussion of Euler angles that the right answer is indeed $|T_{2,1}| = +1$.

While we have 9 variables in $T_{2,1} = \{t_{ij}\}, i, j = 1\ldots 3$, there are 6 nonlinear constraining equations based on the orthogonality of the rows (or columns) of the matrix. For the rows these are:

$$\begin{aligned}t_{11}^2 + t_{12}^2 + t_{13}^2 &= 1 \\ t_{21}^2 + t_{22}^2 + t_{23}^2 &= 1 \\ t_{31}^2 + t_{32}^2 + t_{33}^2 &= 1 \\ t_{11}t_{21} + t_{12}t_{22} + t_{13}t_{23} &= 0 \\ t_{11}t_{31} + t_{12}t_{32} + t_{13}t_{33} &= 0 \\ t_{21}t_{31} + t_{22}t_{32} + t_{23}t_{33} &= 0\end{aligned}$$

The result of these constraints is that there are only 3 independent variables. In principle all nine may be derived given any three that do not violate a constraint.

3.2.3 Euler angles

If there are only three independent variables in the direction cosine matrix, then we should be able to express each of the t_{ij} in terms of some set of three

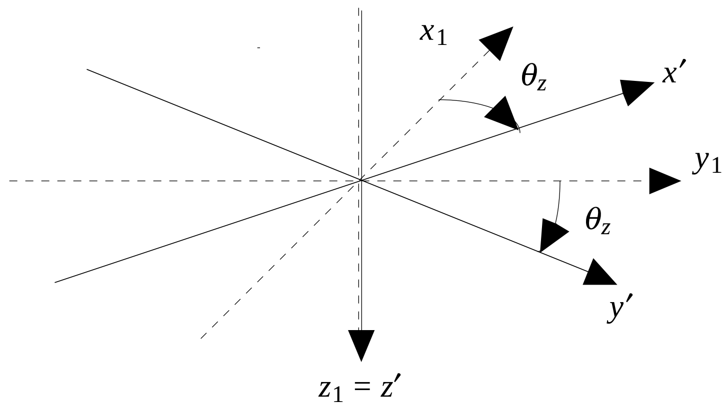

Figure 3.4: Rotation Through θ_z

(not necesarily unique) independent variables. One means of determining these variables is by use of a famous theorem due to the Swiss mathematician Leonhard Euler (1707-83). Briefly, this theorem holds that any arbitrarily oriented reference frame may be placed in alignment with (made to have axes parallel to) any other reference frame by three successive rotations about the axes of the reference frame. The order of selection of axes in these rotations is arbitrary, but the same axis may not be used twice in succesion. The rotation sequences are usually denoted by three numbers, 1 for x, 2 for y, and 3 for z. The twelve valid sequences are 123, 121, 131, 132, 213, 212, 231, 232, 312, 313, 321, and 323. The angles through which these rotations are performed (defined as positive according to the right hand rule for right handed coordinate systems) are called generically *Euler angles*.

The rotation sequence most often used in flight dynamics is the 321, or $z - y - x$. Considering a rotation from F_1 to F_2 the first rotation (figure 3.4) is about z_1 through an angle θ_z which is positive according to the right hand rule about the z_1 axis. With two rotations to go, the resulting alignment in general is oriented with neither F_1 or F_2, but some intermediate reference frame (the first of two) denoted F'. Since the rotation was about z_1, z' is parallel to it but neither of the other two primed axes is.

The next rotation (figure 3.5) is through an angle θ_y about the axis y' of the first intermediate reference frame to the second intermediate reference frame, F''. Note that $y'' = y'$, and neither y'' or z'' are necessarily axes of either F_1 or F_2.

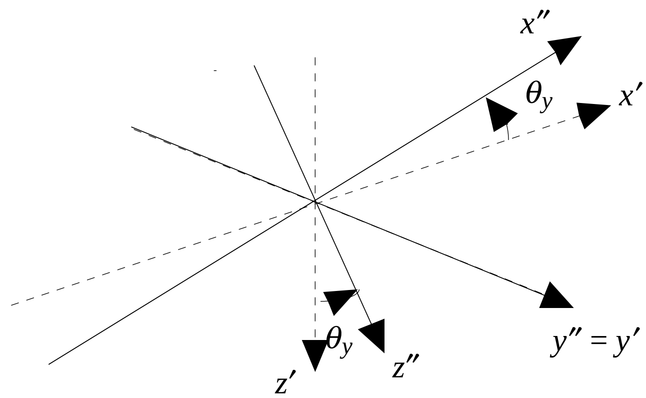

Figure 3.5: Rotation Through θ_y

The final rotation (figure 3.6) is about x'' through angle θ_x and the final alignment is parallel to the axes of F_2.

Now assuming we know the angles θ_x, θ_y, and θ_z we need to relate them to the elements of the direction cosine matrix $T_{2,1}$. We will do this by seeing how the arbitrary vector $\mathbf{v}$ is represented in each of the intermediate and final reference frames in terms of its representation in the prior reference frame.

Consider first the rotation about z_1 (Figure 3.7). In terms of the direction cosines previously defined, the angles between the axes are as follows: between z_1 and z' it is zero; between either z and any x or y it is 90 deg; between x_1 and x' or y_1 and y' it is θ_z; between x_1 and y' it is 90 deg $+\theta_z$; and between y_1 and x' it is 90 deg $-\theta_z$. We therefore may write

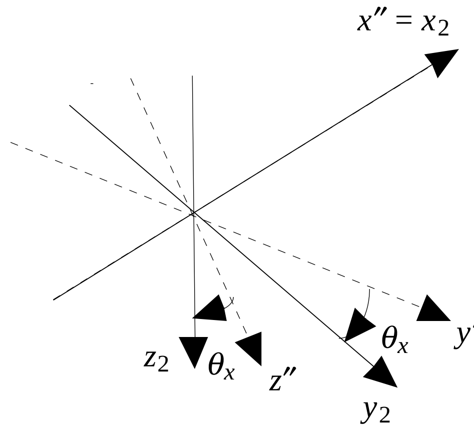

Figure 3.6: Rotation Through θ_x

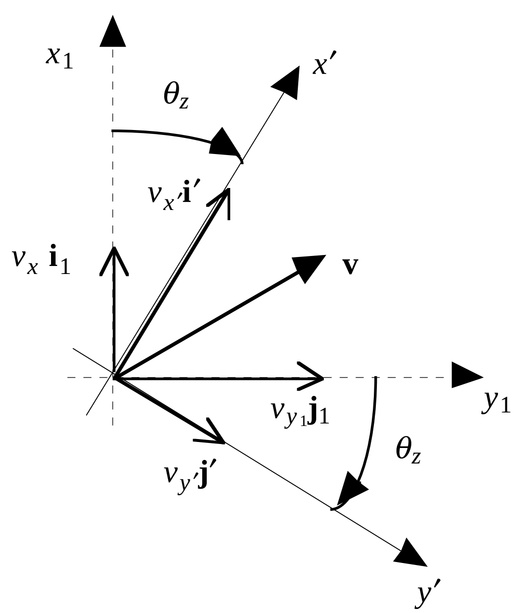

Figure 3.7: Rotation from F_1 to F'

$$\begin{Bmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{Bmatrix} = \begin{bmatrix} \cos\theta_{x'x_1} & \cos\theta_{x'y_1} & \cos\theta_{x'z_1} \\ \cos\theta_{y'x_1} & \cos\theta_{y'y_1} & \cos\theta_{y'z_1} \\ \cos\theta_{z'x_1} & \cos\theta_{z'y_1} & \cos\theta_{z'z_1} \end{bmatrix} \begin{Bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{Bmatrix}$$
$$= \begin{bmatrix} \cos\theta_z & \cos(90\deg - \theta_z) & \cos 90\deg \\ \cos(90\deg + \theta_z) & \cos\theta_z & \cos 90\deg \\ \cos 90\deg & \cos 90\deg & \cos 0 \end{bmatrix} \begin{Bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{Bmatrix}$$
$$= \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{Bmatrix}$$

In short, $\{\mathbf{v}\}' = T_{F',1}\{\mathbf{v}\}_1$ in which

$$T_{F',1} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

From the rotation about y' we get $\{\mathbf{v}\}'' = T_{F'',F'}\{\mathbf{v}\}'$ in which

$$T_{F'',F'} = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \tag{3.3}$$

From the rotation about x'', finally, $\{\mathbf{v}\}_2 = T_{2,F''}\{\mathbf{v}\}''$ with

$$T_{2,F''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \tag{3.4}$$

We now cascade the relationships: $\{\mathbf{v}\}_2 = T_{2,F''}\{\mathbf{v}\}''$, $\{\mathbf{v}\}'' = T_{F'',F'}\{\mathbf{v}\}'$ to arrive first at $\{\mathbf{v}\}_2 = T_{2,F''}T_{F'',F'}\{\mathbf{v}\}'$ and then at $\{\mathbf{v}\}_2 = T_{2,F''}T_{F'',F'}T_{F',1}\{\mathbf{v}\}_1$, or

$$\{\mathbf{v}\}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \{\mathbf{v}\}_1$$

The transformation matrix we seek is the product of the three sequential transformations (in the correct order!), or $T_{2,1} = T_{2,F''}T_{F'',F'}T_{F',1}$. The details are slightly tedious, but the result is

$$\boxed{T_{2,1} = \begin{bmatrix} \cos\theta_y\cos\theta_z & \cos\theta_y\sin\theta_z & -\sin\theta_y \\ \begin{pmatrix} \sin\theta_x\sin\theta_y\cos\theta_z \\ -\cos\theta_x\sin\theta_z \end{pmatrix} & \begin{pmatrix} \sin\theta_x\sin\theta_y\sin\theta_z \\ +\cos\theta_x\cos\theta_z \end{pmatrix} & \sin\theta_x\cos\theta_y \\ \begin{pmatrix} \cos\theta_x\sin\theta_y\cos\theta_z \\ +\sin\theta_x\sin\theta_z \end{pmatrix} & \begin{pmatrix} \cos\theta_x\sin\theta_y\sin\theta_z \\ -\sin\theta_x\cos\theta_z \end{pmatrix} & \cos\theta_x\cos\theta_y \end{bmatrix}}$$

Since this is just a different way of representing the direction cosine matrix, it must be true that $\cos\theta_y\cos\theta_z = \cos\theta_{x_2x_1}$, $\cos\theta_y\sin\theta_z = \cos\theta_{x_2y_1}$, etc. At this point we may observe that since $T_{2,1} = T_{2,F''}T_{F'',F'}T_{F',1}$, we must have

$$|T_{2,1}| = |T_{2,F''}T_{F'',F'}T_{F',1}| = |T_{2,F''}|\,|T_{F'',F'}|\,|T_{F',1}|$$

The determinant of each of the three intermediate transformations is easily verified to be $+1$, so that we have the expected result,

$$|T_{2,1}| = +1$$

It is very important to note that the Euler angles θ_x, θ_y, and θ_z have been defined for a rotation from F_1 to F_2 using a 321 rotation sequence. Thus, a 321 rotation from F_2 to F_1 ($T_{1,2}$) with suitably defined angles (say, ϕ_x, ϕ_y, and ϕ_z) would have the same form as given for $T_{2,1}$, *but the angles involved would be physically different from those in* $T_{2,1}$. So, for the rotation from F_1 to F_2 using a 321 rotation sequence:

$$T_{1,2} = \begin{bmatrix} \cos\phi_y \cos\phi_z & \cos\phi_y \sin\phi_z & -\sin\phi_y \\ \begin{pmatrix} \sin\phi_x \sin\phi_y \cos\phi_z \\ -\cos\phi_x \sin\phi_z \end{pmatrix} & \begin{pmatrix} \sin\phi_x \sin\phi_y \sin\phi_z \\ +\cos\phi_x \cos\phi_z \end{pmatrix} & \sin\phi_x \cos\phi_y \\ \begin{pmatrix} \cos\phi_x \sin\phi_y \cos\phi_z \\ +\sin\phi_x \sin\phi_z \end{pmatrix} & \begin{pmatrix} \cos\phi_x \sin\phi_y \sin\phi_z \\ -\sin\phi_x \cos\phi_z \end{pmatrix} & \cos\phi_x \cos\phi_y \end{bmatrix}$$

Alternatively we may note that $T_{1,2} = T_{2,1}^{-1} = T_{2,1}^T$ and may write the matrix

$$T_{1,2} = \begin{bmatrix} \cos\theta_y \cos\theta_z & \begin{pmatrix} \sin\theta_x \sin\theta_y \cos\theta_z \\ -\cos\theta_x \sin\theta_z \end{pmatrix} & \begin{pmatrix} \cos\theta_x \sin\theta_y \cos\theta_z \\ +\sin\theta_x \sin\theta_z \end{pmatrix} \\ \cos\theta_y \sin\theta_z & \begin{pmatrix} \sin\theta_x \sin\theta_y \sin\theta_z \\ +\cos\theta_x \cos\theta_z \end{pmatrix} & \begin{pmatrix} \cos\theta_x \sin\theta_y \sin\theta_z \\ -\sin\theta_x \cos\theta_z \end{pmatrix} \\ -\sin\theta_y & \sin\theta_x \cos\theta_y & \cos\theta_x \cos\theta_y \end{bmatrix}$$

The two matrices are the same, only the definitions of the angles are different. Clearly the relationships among the two sets of angles are non-trivial.

In short, the definitions of Euler angles are unique to the rotation sequence used and the decision as to which frame one goes from and which one goes to in that sequence. We will normally define our Euler angles going in only one direction using a 321 rotation sequence, and rely on relationships like $T_{1,2} = T_{2,1}^{-1} = T_{2,1}^T$ to obtain the other.

3.2.4 Euler parameters

The derivation of Euler parameters is at appendix A. They are based on the observation that any two coordinate systems are instantaneously related by a single rotation about some axis that has the same representation in each system. The axis, called the eigenaxis, has direction cosines ξ, ζ, and χ; the angle of rotation is η. Then, with the definition of the Euler parameters

$$\begin{aligned} q_0 &\doteq \cos(\eta/2) \\ q_1 &\doteq \xi \sin(\eta/2) \\ q_2 &\doteq \zeta \sin(\eta/2) \\ q_3 &\doteq \chi \sin(\eta/2) \end{aligned} \tag{3.5}$$

the transformation matrix becomes:

$$T_{2,1} = \begin{bmatrix} (q_0^2 + q_1^2 - q_2^2 - q_3^2) & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & (q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix} \tag{3.6}$$

Euler parameters have one great disadvantage relative to Euler angles: Euler angles may in most cases be easily visualized. If one is given values of θ_x, θ_y, and θ_z it is not hard to visualize the relative orientation of two coordinate systems. A given set of Euler parameters, however, conveys almost no information about how the systems are related. Thus even in applications in which Euler parameters are preferred (such as in flight simulation), the results are very often converted to Euler angles for ease of interpretation and visualization.

The direct approach to convert a set of Euler parameters to the corresponding set of Euler angles is to equate corresponding elements of the two representations of the transformation matrix. We may combine the (2,3) and (3,3) entries to obtain

$$\frac{\sin\theta_x \cos\theta_y}{\cos\theta_x \cos\theta_y} = \tan\theta_x = \frac{2(q_2q_3 + q_0q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}$$

$$\theta_x = \tan^{-1}\left(\frac{t_{23}}{t_{33}}\right) = \tan^{-1}\left[\frac{2(q_2q_3 + q_0q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}\right], -\pi \le \theta_x < \pi$$

From the (1,3) entry we have

$$-\sin\theta_y = 2(q_1q_3 - q_0q_2)$$

$$\theta_y = -\sin^{-1}(t_{13}) = -\sin^{-1}(2q_1q_3 - 2q_0q_2), -\pi/2 \le \theta_y \le \pi/2$$

Finally we use the (1,1) and (1,2) entries to yield

$$\frac{\cos\theta_y \sin\theta_z}{\cos\theta_y \cos\theta_z} = \tan\theta_z = \frac{2(q_1q_2 + q_0q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}$$

$$\theta_z = \tan^{-1}\left(\frac{t_{12}}{t_{11}}\right) = \tan^{-1}\left[\frac{2(q_1q_2 + q_0q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}\right], 0 \le \theta_z \le 2\pi$$

With the arctangent functions one has to be careful to not divide by zero when evaluating the argument. Most software libraries have the two-argument arctangent function which avoids this problem and helps keep track of the quadrant. In summary,

$$\begin{aligned} \theta_x &= \tan^{-1}\left[\tfrac{2(q_2q_3+q_0q_1)}{q_0^2-q_1^2-q_2^2+q_3^2}\right], -\pi \le \theta_x < \pi \\ \theta_y &= -\sin^{-1}(2q_1q_3 - 2q_0q_2), -\pi/2 \le \theta_y \le \pi/2 \\ \theta_z &= \tan^{-1}\left[\tfrac{2(q_1q_2+q_0q_3)}{q_0^2+q_1^2-q_2^2-q_3^2}\right], 0 \le \theta_z \le 2\pi \end{aligned} \tag{3.7}$$

Going the other way, from Euler angles to Euler parameters, has been studied extensively. It may be done in a somewhat similar manner to equating elements of the transformation matrix. Another more elegant method due to Junkins and Turner [1] yields a very nice result, here presented without proof:

$$\begin{aligned} q_0 &= \cos(\theta_z/2)\cos(\theta_y/2)\cos(\theta_x/2) + \sin(\theta_z/2)\sin(\theta_y/2)\sin(\theta_x/2) \\ q_1 &= \cos(\theta_z/2)\cos(\theta_y/2)\sin(\theta_x/2) - \sin(\theta_z/2)\sin(\theta_y/2)\cos(\theta_x/2) \\ q_2 &= \cos(\theta_z/2)\sin(\theta_y/2)\cos(\theta_x/2) + \sin(\theta_z/2)\cos(\theta_y/2)\sin(\theta_x/2) \\ q_3 &= \sin(\theta_z/2)\cos(\theta_y/2)\cos(\theta_x/2) - \cos(\theta_z/2)\sin(\theta_y/2)\sin(\theta_x/2) \end{aligned} \tag{3.8}$$

[1] Junkins, J.L. and Turner, J.D., "Optional Continuous Torque Attitude Maneuvers," AIAA-AAS Astrodynamics Conference, Palo Alto, California, August 1978.

Just one final note on Euler parameters. Frequently in the literature Euler parameters are referred to as quaternions. However, quaternions are actually defined as half-scalar, half-vector entities in some coordinate system as $\tilde{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. The algebra of quaternions is useful for proving theorems regarding Euler parameters, but will not be used in this course.

3.3 Transformations of Systems of Equations

Suppose we have a linear system of equations in some reference frame F_1:

$$\{\mathbf{y}\}_1 = A_1 \{\mathbf{x}\}_1$$

We know the transformation to F_2 ($T_{2,1}$) and wish to represent the same equations in that reference frame. Each side of $\{\mathbf{y}\}_1 = A_1 \{\mathbf{x}\}_1$ is a vector in F_1 , so we transform both sides as $T_{2,1} \{\mathbf{y}\}_1 = \{\mathbf{y}\}_2 = T_{2,1} A_1 \{\mathbf{x}\}_1$. Then we may insert the identity matrix in the form of $T_{2,1}^T T_{2,1}$ in the right hand side to yield $\{\mathbf{y}\}_2 = T_{2,1} A_1 T_{2,1}^T T_{2,1} \{\mathbf{x}\}_1$. The reason for doing this is to transform the vector $\{\mathbf{x}\}_1$ to $\{\mathbf{x}\}_2 = T_{2,1} \{\mathbf{x}\}_1$. Grouping terms we have the result,

$$\{\mathbf{y}\}_2 = T_{2,1} A_1 T_{2,1}^T \{\mathbf{x}\}_2 = A_2 \{\mathbf{x}\}_2$$

With the conclusion that the transformation of a matrix A_1 from F_1 to F_2 (or the equivalent operation performed by A_1 in F_2) is given by

$$\boxed{A_2 = T_{2,1} A_1 T_{2,1}^T} \tag{3.9}$$

This is sometimes spoken of as the transformation of a matrix. Such transformations may be very useful. For example, if we could find F_2 and $T_{2,1}$ such that A_2 is diagonal, then the system of equations $\{\mathbf{y}\}_2 = A_2 \{\mathbf{x}\}_2$ is very easy to solve. The original variables can then be recovered by $\{\mathbf{y}\}_1 = T_{2,1}^T \{\mathbf{y}\}_2 , \{\mathbf{x}\}_1 = T_{2,1}^T \{\mathbf{x}\}_2$.

3.4 Customs and Conventions

3.4.1 Names of Euler angles

Some transformation matrices occur frequently, and the 321 Euler angles associated with them are given special symbols. These are summarized as follows:

T_{F_2,F_1}	θ_x	θ_y	θ_z
$T_{B,H}$	ϕ	θ	ψ
$T_{W,H}$	μ	γ	χ
$T_{B,W}$	0	α	$-\beta$

(3.10)

3.4.2 Principal Values of Euler angles

The principal range of values of the Euler angles is fixed largely by convention and may vary according to the application. In this course the convention is

$$\boxed{\begin{aligned} -\pi \leq &\theta_x < \pi \\ -\pi/2 \leq &\theta_y \leq \pi/2 \\ 0 \leq &\theta_z < 2\pi \end{aligned}} \tag{3.11}$$

These ranges are most frequently used in flight dynamics, although occasionaly one sees $-\pi < \theta_z \leq \pi$ and $0 < \theta_x \leq 2\pi$.