Chapter 3: Airplane Basic Equations of Motion and Open-Loop Dynamics[1]

The goal of this Chapter is to present fundamental background information related to the derivation of the basic equations of motion of a traditional airplane, explain how the airplane's position and orientation are determined with respect to a reference frame (Earth-fixed inertia reference frame), derive the aerodynamic forces that act on the airplane, define the corresponding control angles, and conclude with derivation of the open-loop dynamics. The material included in this Chapter is a very concise version of what may be found in any related textbook, and follows the same notation and derivation approach described in the references.

3.1 Introduction

The overall objective of this Chapter is to discuss the fundamental behavior of a traditional airplane in flight. It describes the kinematics properties and basic equations of motion of a generic airplane, where the term generic is used to emphasize that the airplane's structural components and flight control systems may be found in every 'traditional' airplane design.

Equations of motion are derived by implementing Newton's second law that deals with vector summations of all forces and moments as applied to the airplane relative to an inertial reference frame. However, for practical reasons, analysis may be significantly simplified if motion is described relative to a body-fixed reference frame attached to the airplane. When this is the case, the equations of motion are derived relative to this non-inertial frame. Further, Euler angles are used to define the airplane orientation relative to a general Earth-fixed inertial frame.

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The equations of motion are nonlinear. Aerodynamic forces and moments are also nonlinear functions of motion characteristics and airplane controls. Linearization of the nonlinear equations is based on considering a specific configuration of airplane non-accelerating motion that is subject to small perturbations of linear and angular velocities from the reference non-accelerating steady flight. Under such constraints, the resulting perturbed aerodynamic forces and moments may be considered as linear functions of the perturbed linear and angular velocities, the airplane control angles, and their associated derivatives. This is a common practical approximation of real flight behavior, despite the fact that it is not based on a rigorous mathematical background.

This linearization results in obtaining a set of linear differential equations (for the perturbed model). Using Laplace transform one may obtain a set of algebraic equations for controller design purposes. This controller may be used for disturbance rejection. Subsequently, closed-loop controllers may be designed that meet set performance criteria and stability of flight. However, this is beyond the scope of this Chapter.

3.2 Equations of Motion

The equations of motion include derivation of the respective equations with respect to the body-fixed reference frame that is attached to the airplane, as well as position and orientation of the airplane relative to an Earth-fixed inertial frame.

The first step towards dynamic modeling of an airplane is to consider it as a rigid body with six degrees of freedom (DOF), followed by application of Newton's laws to the rigid body (airplane). As previously mentioned, an Earth-fixed inertial frame makes analysis impractical since moments and products of inertia vary with time. This is not the case when a body-fixed reference frame is considered, where moments and products of inertia are constant.

Figure 3.1 depicts the body-fixed reference frame (moving frame) that is attached to the airplane. The center C of the body-fixed reference frame C_{xyz} coincides with the center of gravity (CG) of the airplane. The C_{xz} plane coincides with the plane of symmetry of the airplane with the C_x and C_z axes pointing forward and downward, respectively. The C_y axis is perpendicular to the plane of symmetry in the direction of the right wing. The C_{xyz} body-fixed reference frame is a right-handed Cartesian coordinate system.

The linear velocity components of the CG along the C_x, C_y and C_z axes are defined as *U*, *V* and *W*, respectively. The angular velocity components about the axes of the body-fixed reference frame are defined as *P*, *Q* and *R*, respectively.

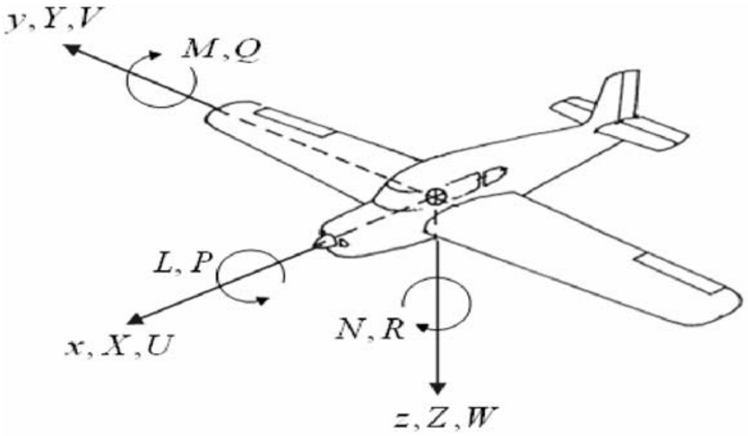

Fig. 3.1. Body-fixed coordinate system.

However, it is important to clarify that the linear and angular velocity vectors of the CG of the airplane are vectors relative to the Earth-fixed inertial frame, that is, vectors viewed by a stationary observer in the Earth-fixed inertial frame. The values of *U*, *V* and *W* are the instantaneous components of that vector relative to the body-fixed reference frame. The same holds for the angular velocities as well.

External aerodynamic forces components along the axes are denoted by *X*, *Y* and *Z*. The components about the axes of the external aerodynamic moments are denoted by *L*, *M* and *N* as shown in Figure 3.1.

Positive direction of the angular velocity components and of the moment components refers to the clockwise direction about the respective axis.

Basic concepts of kinematics analysis for rotating frames are used to derive the equations of motion. A more detailed presentation may be found in [8]. The first step is to define an Earth-fixed reference frame. It is a right-handed Cartesian system denoted by $O_{x'y'z'}$. The underlying assumption is that the Earth is fixed in space, so $O_{x'y'z'}$ is an inertia frame.

As illustrated in Figure 3.2, $\vec{R}_0$ is the position vector of the origin *C* relative to the Earth-fixed reference frame. The set of the unit vectors for the body-fixed reference frame is denoted by $\{\hat{I}, \hat{J}, \hat{K}\}$. Point P is the position in space of a mass element dm of the airplane. Point P is rigidly attached to the body-fixed reference frame. The position vector of point P relative to the body-fixed reference frame is denoted by $\vec{r}$. If the coordinates of P relative to the body-fixed reference frame are (x, y, z) then:

$$\vec{r} = x\hat{I} + y\hat{J} + z\hat{K} \tag{3.1}$$

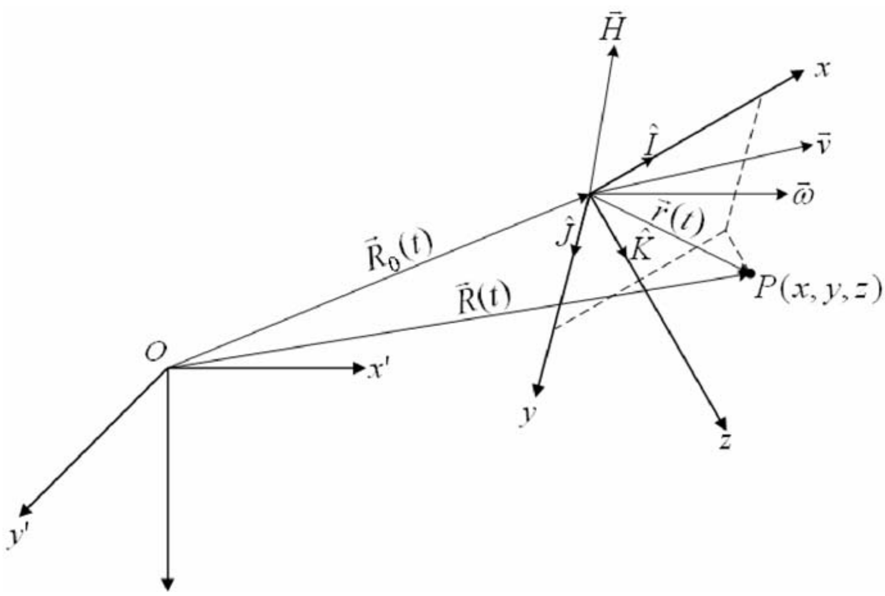

Fig. 3.2. Motion of the airplane relative to the Earth-fixed reference frame.

If $\vec{R}(t)$ represents the position vector of the mass element dm relative to the Earth-fixed reference frame, then:

$$\vec{R} = \vec{R}_0 + \vec{r} \tag{3.2}$$

The velocity of the mass element at point P relative to the Earth-fixed reference system is given by:

$$\vec{v}_P = \left.\frac{d\vec{R}}{dt}\right|_E = \left.\frac{d\vec{R}_0}{dt}\right|_E + \left.\frac{d\vec{r}}{dt}\right|_E \tag{3.3}$$

where $\left.\frac{d(\circ)}{dt}\right|_E$ denotes the time derivative of a vector in space relative to the Earth-fixed reference frame, as viewed by an observer in the Earth-fixed reference frame. The derivative of the position vector $\vec{R}_0$, relative to the Earth-fixed reference frame equals the velocity of the CG. The linear velocity of the airplane's CG is measured with respect to the Earth-fixed frame. Since the components of the linear velocity along the axes of the body-fixed reference frame are *U*, *V* and *W*, it follows that:

$$\vec{v} = \left.\frac{d\vec{R}_0}{dt}\right|_E = U\hat{I} + V\hat{J} + W\hat{K} \tag{3.4}$$

and $\hat{v}$ denotes the instantaneous velocity of the CG of the airplane relative to the Earth-fixed reference frame. The vector $\vec{r}$ is a position vector of the rotating body-fixed reference frame. According to [8], the time derivative of $\vec{r}$ with respect to the Earth-fixed reference frame is:

$$\left.\frac{d\vec{r}}{dt}\right|_E = \left.\frac{d\vec{r}}{dt}\right|_B + \vec{\omega} \times \vec{r} \tag{3.5}$$

where $\vec{\omega} = P\hat{I} + Q\hat{J} + R\hat{K}$ denotes the angular velocity of the body-fixed frame with respect to the Earth-fixed reference frame. The operator $(\times)$ is the vector cross product. The term $\left.\frac{d\vec{r}}{dt}\right|_B$ denotes the time derivative of the position vector $\vec{r}(t)$ with respect to the body-fixed reference frame. In general, $\left.\frac{d(\circ)}{dt}\right|_B$ denotes the derivative of a vector from the viewpoint of an observer in the body-fixed reference frame. Since point P is rigidly attached to the body-fixed reference frame, it follows that $\left.\frac{d\vec{r}}{dt}\right|_B = \vec{0}$. Hence, the velocity of the airplane's arbitrary element mass placed at the point P is given by:

$$\vec{v}_P = \left.\frac{d\vec{R}_O(t)}{dt}\right|_E + \vec{\omega} \times \vec{r}(t) \tag{3.6}$$

If u_{Px}, u_{Py} and u_{Pz} are the velocity components of the element mass dm along the axes of the body-fixed frame, then by equating both sides of (3.6) one obtains:

$$\begin{aligned} u_{Px} &= U + Qz - Ry \\ u_{Py} &= V + Rx - Pz \\ u_{Pz} &= W + Py - Qx \end{aligned} \tag{3.7}$$

The acceleration vector $\vec{a}$ of the airplane's CG is:

$$\vec{a} = \left.\frac{d\vec{v}(t)}{dt}\right|_E \tag{3.8}$$

Since $\vec{v}(t)$ is expressed in terms of the body-fixed frame unit vectors, and the body-fixed frame is rotating, following analysis presented in [3] and [8], the acceleration vector of the CG is given by the following equation:

$$\vec{a} = \left.\frac{d\vec{v}(t)}{dt}\right|_E = \left.\frac{d\vec{v}(t)}{dt}\right|_B + \omega \times \vec{v}(t) \tag{3.9}$$

But $\vec{v} = U\hat{I} + V\hat{J} + W\hat{K}$, therefore, $\left.\frac{d\vec{v}}{dt}\right|_B = \dot{U}\hat{I} + \dot{V}\hat{J} + \dot{W}\hat{K}$ since $\left.\frac{d\vec{v}}{dt}\right|_B$ is the time derivative of the velocity with respect to the body-fixed frame. It is clarified that the vector $\vec{a}$ is the instantaneous acceleration of the airplane's CG with respect to the Earth-fixed inertia frame. If a_x, a_y and a_z denote the instantaneous components of the vector $\vec{a}$ along the axis of the body-fixed reference frame, then from (3.9) the following algebraic equations are derived:

$$\begin{aligned} a_x &= \dot{U} - RV + QW \\ a_y &= \dot{V} - PW + RU \\ a_z &= \dot{W} - QU + PV \end{aligned} \tag{3.10}$$

If the vectors of all forces acting on the airplane are expressed in terms of their components $\sum X, \sum Y$ and $\sum Z$ along the respective axes of the body-fixed reference frame, then:

$$\begin{aligned}\sum X &= m(\dot{U} - RV + QW) \\ \sum Y &= m(\dot{V} - PW + RU) \\ \sum Z &= m(\dot{W} - QU + PV)\end{aligned} \tag{3.11}$$

To conclude derivation of the equations of motion, Newton's second law is applied to all moments that act on the CG. Let $\vec{H} = h_x\hat{I} + h_y\hat{J} + h_z\hat{K}$ be the vector of the airplane's angular momentum expressed in the body-fixed frame unit vectors. From [8], the angular momentum components of the body-fixed reference frame are expressed as a function of moments of inertia and products of inertia as:

$$\begin{aligned}h_X &= I_{XX}P - I_{XY}Q - I_{XZ}R \\ h_Y &= -I_{YX}P + I_{YY}Q - I_{YZ}R \\ h_Z &= -I_{ZX}P - I_{ZY}Q + I_{ZZ}R\end{aligned} \tag{3.12}$$

where $I_{XX} = \sum dm(y^2 + z^2)$, $I_{YY} = \sum dm(x^2 + z^2)$, $I_{ZZ} = \sum dm(x^2 + y^2)$ and the products of inertia are $I_{XY} = \sum dmxy = I_{YX}$, $I_{XZ} = \sum dmxz = I_{ZX}$, $I_{YZ} = \sum dmyz = I_{ZY}$.

The above sums apply to all elementary masses of the airplane, and x, y and z are the distances of each elementary mass from the origin (the CG). Moreover, since C_{xz} is a plane of symmetry for the airplane, it follows that $I_{XY} = I_{YX} = 0$ and $I_{YZ} = I_{ZY} = 0$.

The external moments equal the time rate of change of the angular momentum with respect to the Earth-fixed reference frame. Since the angular momentum is described by the unit vectors of the body-fixed frame, the following is true:

$$\left.\frac{d\vec{H}}{dt}\right|_E = \left.\frac{d\vec{H}}{dt}\right|_B + \vec{\omega} \times \vec{H} \tag{3.13}$$

The term $\left.\frac{d\vec{H}}{dt}\right|_E$ is the time rate of change of the angular momentum with respect to the Earth-fixed reference frame. Regarding $\left.\frac{d\vec{H}}{dt}\right|_B$, the time derivative of the angular momentum with respect to the body-fixed reference frame is derived as:

$$\begin{aligned} \dot{h}_X &= I_{XX}\dot{P} - I_{XZ}\dot{R} \\ \dot{h}_Y &= I_{YY}\dot{Q} \\ \dot{h}_Z &= -I_{ZX}\dot{P} + I_{ZZ}\dot{R} \end{aligned} \tag{3.14}$$

Let $\sum L, \sum M$ and $\sum N$ denote the moments of all forces about the axes of the body-fixed reference frame. Then:

$$\begin{aligned} \sum L &= \frac{dH_X}{dt} = I_{XX}\dot{P} + QR(I_{ZZ} - I_{YY}) - I_{XZ}(\dot{R} + PQ) \\ \sum M &= \frac{dH_Y}{dt} = I_{YY}\dot{Q} + PR(I_{XX} - I_{ZZ}) + I_{XZ}(P^2 - Q^2) \\ \sum N &= \frac{dH_Z}{dt} = I_{ZZ}\dot{R} + PQ(I_{YY} - I_{XX}) + I_{XZ}(QR - \dot{P}) \end{aligned} \tag{3.15}$$

Therefore, the final form of the equations of motion with respect to the Earth-fixed frame but expressed in the body-fixed frame unit vectors is given by (3.11) for the forces and (3.15) for the moments.

3.3 Position and Orientation of the Airplane

The main disadvantage of using a body-fixed reference frame C_{xyz} attached to the airplane relates to the inability to express the airplane's position and orientation with respect to this body-fixed frame. Position and orientation of rigid bodies is defined with respect to fixed, inertial reference frames. Therefore, the airplane position and orientation equations should and will be derived relative to a generic Earth-fixed inertial reference frame. Derivation follows [4] but with additional details for clarification purposes.

A right-handed Cartesian system $O_{x'y'z'}$ is first defined as the Earth-fixed reference frame. Airplane directions at specific time instances are described by the orientation of body-fixed frames relative to the Earth-fixed reference frame. The origin of those frames is the CG of the airplane. At time instant $t = 0$ the CG of the airplane coincides with the origin of the frame $O_{x'y'z'}$.The initial position of the airplane is described by the frame $C_{x_1y_1z_1}$ that is aligned with $O_{x'y'z'}$. The final orientation of the airplane at time t is described by the body-fixed frame C_{xyz}. Figure 3.3 shows the schematics of deriving the orientation of the airplane.

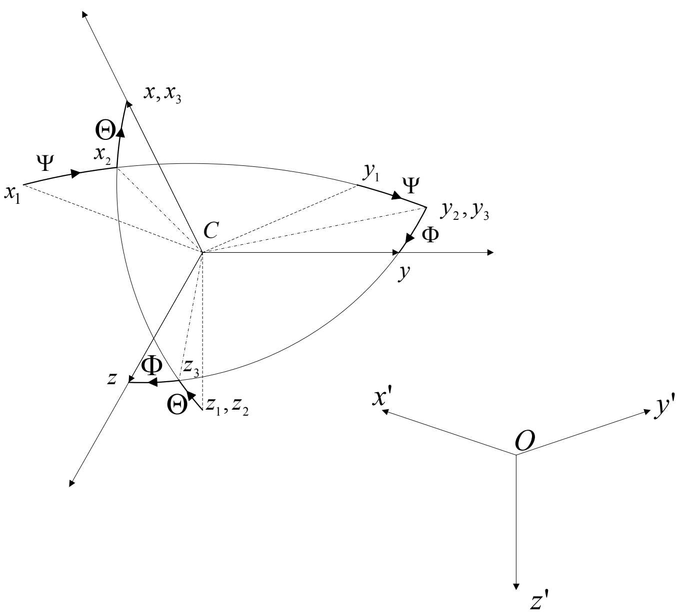

Fig. 3.3. Airplane orientation.

The airplane orientation at any time instant may be obtained by performing three consecutive rotations relative to the Earth-fixed frame; rotations are performed at a specific order, they cannot be considered as vectors and they are not commutative [8]. Therefore, the rotation order is important for consistency, as follows:

- A counterclockwise rotation of an angle Ψ about axis C_{z_1}. This rotation moves the airplane to the position described by $C_{x_2y_2z_2}$, bringing C_{x_2} parallel to the plane C_{xz_2}.

- A counter clockwise rotation of an angle Θ about axis C_{y_2} .This rotation moves the airplane to the position described by $C_{x_3y_3z_3}$, aligning C_{x_3} with the C_x axis.
- A counterclockwise rotation of an angle Φ about axis C_{x_3} bringing the axes to their final direction C_{xyz} .

3.3.1 Airplane Position

The position of the airplane may be calculated by integrating velocity components at any time instant. Let $\hat{I},\hat{J},\hat{K}$ denote unitary vectors of C_{xyz}, $\hat{I}',\hat{J}',\hat{K}'$ unitary vectors of $O_{x'y'z'}$, and $\hat{I}_i,\hat{J}_i,\hat{K}_i$ unitary vectors of frames $Cx_iy_iz_i$, where $i=1,2,3$ .The linear velocity of the airplane relative to the C_{xyz} frame and relative to the Earth-fixed frame is, respectively:

$$\hat{v}=U\hat{I}+V\hat{J}+W\hat{K} \tag{3.16a}$$

$$\hat{v}=\frac{dx'}{dt}\hat{I}'+\frac{dy'}{dt}\hat{J}'+\frac{dz'}{dt}\hat{K}' \tag{3.16b}$$

The unit vectors of the body-fixed reference frame C_{xyz} are written relative to the frame $C_{x_3y_3z_3}$ as:

$$\begin{aligned}
\hat{I} &= \hat{I}_3 \\
\hat{J} &= \cos\Phi\hat{J}_3+\sin\Phi\hat{K}_3 \\
\hat{K} &= -\sin\Phi\hat{J}_3+\cos\Phi\hat{K}_3
\end{aligned} \tag{3.17}$$

The unit vectors of the frame $C_{x_3y_3z_3}$ are expressed relative to the frame $C_{x_2y_2z_2}$ as:

$$\begin{aligned}
\hat{I}_3 &= \cos\Theta\hat{I}_3-\sin\Theta\hat{K}_2 \\
\hat{J}_3 &= \hat{J}_2 \\
\hat{K}_3 &= \sin\Theta\hat{I}_2+\cos\Theta\hat{K}_2
\end{aligned} \tag{3.18}$$

Finally, the unit vectors of the frame $C_{x_2y_2z_2}$ relative to $C_{x_1y_1z_1}$ are expressed as:

$$\begin{aligned} \hat{I}_2 &= \cos\Psi\hat{I}_1 + \sin\Psi\hat{J}_1 \\ \hat{J}_2 &= -\sin\Psi\hat{I}_1 + \cos\Psi\hat{J}_1 \\ \hat{K}_2 &= \hat{K}_1 \end{aligned} \tag{3.19}$$

Substituting (3.17) to (3.19) to (3.16a) and equating with (3.16b) the following algebraic equations are obtained for the velocity components of the airplane relative to the Earth-fixed frame:

$$\begin{aligned} \frac{dx'}{dt} &= U\cos\Theta\cos\Psi + V(\sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi) + W(\cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi) \\ \frac{dy'}{dt} &= U\cos\Theta\sin\Psi + v(\sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi) + W(\cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi) \\ \frac{dz'}{dt} &= -U\sin\Theta + V\sin\Phi\cos\Theta + W\cos\Phi\cos\Theta \end{aligned} \tag{3.20}$$

Therefore, the position coordinates of the CG of the airplane are obtained by integrating (3.20). Due to integration complexity, (3.20) is usually linearized for simplification purposes.

3.3.2 Airplane Orientation

The orientation of the airplane is expressed with respect to the angular velocity components (P, Q, R) of the body-fixed reference frame.

Consider that during an infinitesimal time interval dt the airplane is subjected to three infinitesimal rotations $d\Psi, d\Theta$ and $d\Phi$ resulting in a position defined by angles $\Psi + d\Psi$, $\Theta + d\Theta$ and $\Phi + d\Phi$. Although finite rotations cannot be treated as vectors, infinitesimal rotations may be treated as such, thus, according to [4] the vector that represents the above rotation is:

$$\hat{n} = d\Phi\hat{I} + d\Theta\hat{J}_3 + d\Psi\hat{K}_2 \tag{3.21}$$

The angular velocity is:

$$\vec{\omega} = \frac{d\vec{n}}{dt} = \dot{\Phi}\hat{I} + \dot{\Theta}\hat{J}_3 + \dot{\Psi}\hat{K}_2 \tag{3.22}$$

The unit vectors of the frame $C_{x_3 y_3 z_3}$ are expressed with respect to the unit vectors of the reference frame C_{xyz} as:

$$\begin{aligned} \hat{I}_3 &= \hat{I} \\ \hat{J}_3 &= \cos\Phi \hat{J} - \sin\Phi \hat{K} \\ \hat{K}_3 &= \sin\Phi \hat{J} + \cos\Phi \hat{K} \end{aligned} \tag{3.23}$$

The unit vectors of the frame $C_{x_2 y_2 z_2}$ are expressed with respect to the unit vectors of the reference frame $C_{x_3 y_3 z_3}$ as:

$$\begin{aligned} \hat{I}_2 &= \cos\Theta \hat{I}_3 + \sin\Theta \hat{K}_3 \\ \hat{J}_3 &= \hat{J}_2 \\ \hat{K}_2 &= -\sin\Theta \hat{I}_3 + \cos\Theta \hat{K}_3 \end{aligned} \tag{3.24}$$

After substituting equations (3.23) and (3.24) to (3.22), equating with $\vec{\omega} = P\hat{I} + Q\hat{J} + R\hat{K}$ and then solving for $\dot{\Psi}, \dot{\Theta}, \dot{\Phi}$, one obtains the differential equations that relate the orientation angles of the airplane relative to the Earth-fixed frame with the components of the angular velocity expressed in the body-fixed frame unit vectors:

$$\begin{aligned} \dot{\Psi} &= Q\sin\Phi\sec\Theta + R\cos\Phi\sec\Theta \\ \dot{\Theta} &= Q\cos\Phi - R\sin\Phi \\ \dot{\Phi} &= P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta \end{aligned} \tag{3.25}$$

Integration leads to the final orientation equations:

$$\begin{aligned} \Psi &= \int_0^t (Q\sin\Phi\sec\Theta + R\cos\Phi\sec\Theta)ds \\ \Theta &= \int_0^t (Q\cos\Phi - R\sin\Phi)ds \\ \Phi &= \int_0^t (P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta)ds \end{aligned} \tag{3.26}$$

3.4 Aerodynamic Forces Acting on the Airplane

An important and very common term in flight terminology is *trimmed flight* that refers to the airplane being at equilibrium or in a non-accelerating motion. An airplane is at trim when the vector sum of all aerodynamic and gravitational forces acting on the CG equals zero. At trim, the sum of all aerodynamic moments acting on the CG is zero, too. Subsequent analysis follows the method found in [5].

Figure 3.4 depicts an airplane in longitudinal trim flight. V is the total velocity of the airplane CG that lies in its plane of symmetry. The angle between the body-fixed frame axis that points forward and the Earth-fixed frame plane $O'_{x'y'}$ is defined as Θ. In general, aerodynamic forces are not applied to the CG of the airplane [5]. However, since the airplane is performing a trimmed flight, all acting forces are in equilibrium. Since weight is acting on the CG, it may be considered, for that specific case, that aerodynamic forces act on the CG as well.

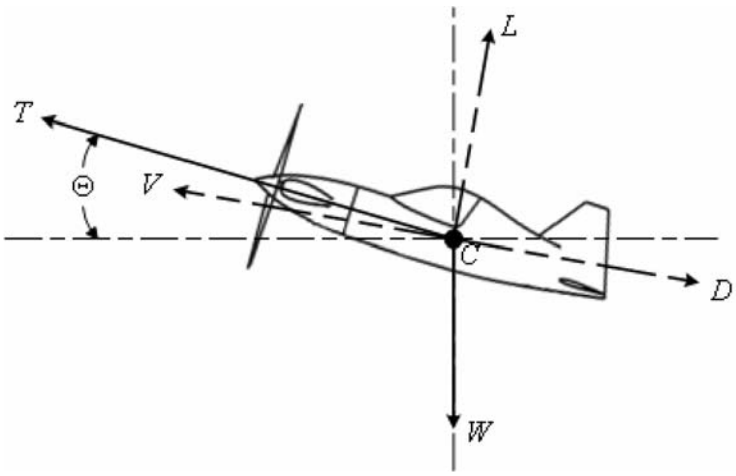

Fig. 3.4. Acting forces on the airplane during a trimmed longitudinal flight.

The first aerodynamic force is the thrust T acting on the body-fixed frame axis that points to forward direction. Thrust is produced by a propeller, a turbojet, or a rocket engine.

The second force is the lift L, normal to the velocity vector. The main component of the lift is the vertical force acting on the wings, facing upwards. The actual lift is the sum of all lifts produced in the wings, tail, even in the propellers.

The third force is the drug D, collinear with the velocity vector of the airplane. Its direction opposes the motion of the airplane. Drug is mainly composed of the parasite and the induced drug. The induced drug is a side effect of the lift generation. Parasite drug is created by airplane surfaces exposed to air.

The fourth force is the airplane weight W pointing downwards.

3.5 Angle of Attack and Sideslip Angle

The *angle of attack* and the *sideslip angle* shown in Figure 3.5 affect the aerodynamic forces of the airplane. Both angles are useful for describing the airplane dynamics and they are characteristics of its dynamic behavior.

The accurate definition of the angle of attack is given in [3] as the angle between the velocity vector of the relative wind and the wing cord. However, since this Chapter does not focus on specifics and details of aerodynamic forces, for simplicity purposes, the angle of attach will be defined in terms of the linear velocity components of the body-fixed reference frame as the angle between the velocity components of the C_x and the C_z axes:

$$\alpha = \tan^{-1}\frac{W}{U} \tag{3.27}$$

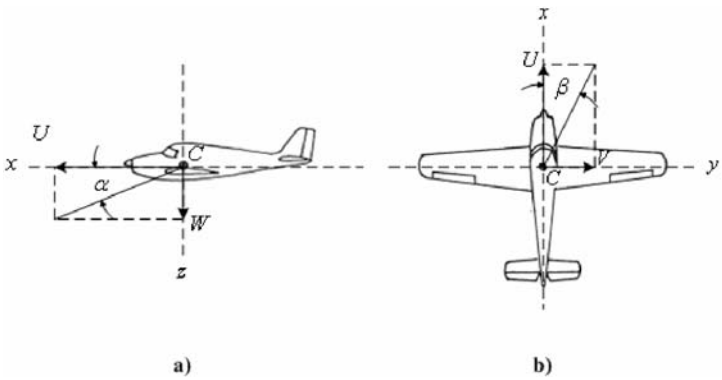

Fig. 3.5. a) Angle of attack; b) Sideslip angle.

The sideslip angle is defined for the lateral motion as:

$$\beta = \tan^{-1}\frac{V}{U} \tag{3.28}$$

The above definition is simplified; the strict definition may be found in [3] and [6]. The sideslip angle is not the same as the yaw angle. The definition of this angle is different depending on the type of motion the airplane performs. The first definition corresponds to longitudinal steady motion that experiences a small disturbance and moves the airplane about the C_z axis. The second definition is a result of asymmetric flight.

Due to the influence of the angle of attack on the lift force, it is an important parameter to control. On the contrary, the sideslip angle is considered a negative effect for airplane stability.

3.6 Weight Components in the Body-Fixed Frame

Weight along the body-fixed reference frame is calculated by the following equations:

$$\vec{W} = mg\hat{K}_1 = -mg\sin\Theta\hat{I} + mg\cos\Theta\sin\Phi\hat{J} + mg\cos\Theta\cos\Phi\hat{K}$$

$$\begin{aligned} X_g &= -mg\sin\Theta \\ Y_g &= mg\cos\Theta\sin\Phi \\ Z_g &= mg\cos\Theta\cos\Phi \end{aligned} \tag{3.29}$$

3.7 Airplane Control Angles

Airplane behavior is controlled by manipulating the angles of several movable parts on its surface. The airplane structural components associated with control actions are depicted in Figure 3.6. Control actions manipulate mainly the three moments about the CG; a secondary action of this type of control changes the external aerodynamic forces.

The horizontal tail is composed of: the forward part, which is fixed, called the *horizontal stabilizer*, and the rare part of the tail wing, which is a movable flap, called the *elevator*. The elevator angle, denoted as δ_e, affects longitudinal motion. Changes of elevator angle affect the equilibrium lift force and, more importantly, the pitching moment about the CG. When the elevator is deflected downwards, the elevator angle has a positive sign; when the deflection is upwards, the elevator angle has a minus sign.

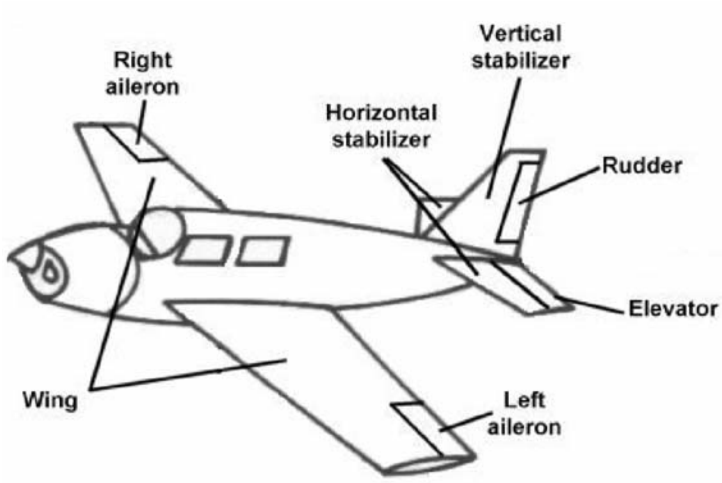

Fig. 3.6. Airplane configuration associated with control actions.

The movable aft portion of the vertical stabilizer is called the *rudder*. By controlling the rudder angle, denoted by δ_r, one manipulates the yawing moment about the C_z moving axis of the body-fixed frame. The effect of the rudder is useful for lateral directional motion and maneuvering. The rudder angle has a positive sign when the rudder is deflected at the left side of the vertical stabilizer's symmetry axis, and a negative sign when it is deflected on the right side.

The *ailerons* are movable parts hinged in the rear side of the wing. By controlling the ailerons angle one can change the airplane roll. The angles of the two ailerons have opposite directions in the two wings and they are not necessarily the same. The total aileron angle is considered to be the sum of the two, left and right aileron angle. The aileron is denoted as δ_a and $\delta_a = \delta_L + \delta_R$ where δ_R and δ_L are the absolute values of the left and right aileron angle, respectively. The aileron angle is considered positive when the right aileron is deflected downwards.

3.8 Open-Loop Dynamics

Equations describing the airplane motion are nonlinear differential equations. Linearizing these equations, under specific assumptions, is a common practice that simplifies greatly calculations and at the same time provides an accurate description of the actual behavior of the airplane. Derivations follow work described in [4] and [5].

Model linearization is based on small disturbance theory. According to the theory, analysis is done under small perturbations of motion characteristics (related to forces, momentums, velocities, angular velocities, etc.) from a steady non-accelerating reference flight. The rationale behind this approach is the fact that external aerodynamic forces and moments acting on the CG depend mainly on control angles and flight characteristics such as linear and angular velocities, angle of attack and sideslip angle. When this is the case, the perturbed aerodynamic forces and moments may be considered as linear functions of the disturbances [4].

Notation wise, parameter values with a zero subscript refer to the trimmed value of the variable, while lower case letters denote the perturbed value of that variable. The airplane is assumed to perform a reference trimmed flight when the disturbances occur. For simplicity purposes it is assumed that the body-fixed frame axes are identical to the airplane *stability axes.* When the airplane performs a longitudinal motion and the C_x axis is collinear and in the same direction with the total velocity of the airplane, with no sideslips, then those axes are termed as stability axes. It is obvious that for the stability axes the trim values of V and W are zero. Moreover, the trim values of the angular velocities and the moments about the CG will be zero as well. Summarizing the above leads to:

$$v_0 = w_0 = p_0 = q_0 = r_0 = \phi_0 = L_0 = M_0 = N_0 = 0$$

The motion of the airplane before the effect of the disturbances can be seen in Figure 3.7. The perturbation quantities and their derivatives will have very small values; therefore, their products are negligible. Without loss of generality, it is assumed that the trigonometric quantities of the perturbed variables, for example θ, will be $\cos\theta = 1$ and $\sin\theta = \theta$. Therefore:

$$\begin{aligned}
\sin(\theta_0 + \theta) &= \sin\theta_0 \cos\theta + \cos\theta_0 \sin\theta = \sin\theta_0 + \theta\cos\theta_0 \\
\cos(\theta_0 + \theta) &= \cos\theta_0 \cos\theta - \sin\theta_0 \sin\theta = \cos\theta_0 - \theta\sin\vartheta_0
\end{aligned}$$

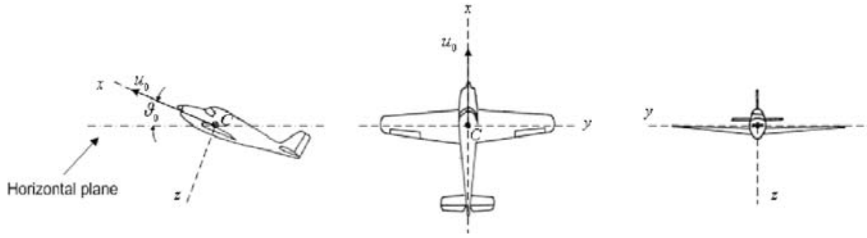

Fig. 3.7. Airplane motion before the effect of the disturbances.

Based on the above assumptions, substitutions into (3.11) and (3.15) results in the following perturbed equations:

$$
\begin{aligned}
&X_0 + \Delta X - mg(\sin\theta_0 + \theta\cos\theta_0) = m\dot{u} \\
&Y_0 + \Delta Y + mg\phi\cos\theta_0 = m(\dot{v} + u_0 r) \\
&Z_0 + \Delta Z + mg(\cos\theta_0 - \theta\sin\theta_0) = m(\dot{w} - u_0 q)
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
&L_0 + \Delta L = Ixx\dot{p} - Ixz\dot{r} \\
&M_0 + \Delta M = I_{YY}\dot{q} \\
&N_0 + \Delta N = I_{ZZ}\dot{r} - I_{XZ}\dot{p}
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
&\dot{\vartheta} = q \\
&\dot{\phi} = p + r\tan\theta_0 \\
&\dot{\psi} = r\sec\theta_0 \\
&p = \dot{\phi} - \dot{\psi}\sin\theta_0
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
&\frac{dx'}{dt} = (u_0 + u)\cos\theta_0 - u_0\theta\sin\theta_0 + w\sin\theta_0 \\
&\frac{dy'}{dt} = u_0\psi\cos\theta_0 + r \\
&\frac{dz'}{dt} = -(u_0 + u)\sin\theta_0 - u_0\theta\cos\theta_0 + w\cos\theta_0
\end{aligned}
\tag{3.33}
$$

In the above equations $\Delta X, \Delta Y, \Delta Z$ denote the perturbed values of the external aerodynamic forces and $\Delta L, \Delta M, \Delta N$ denote the perturbed values of the moments about the CG. When the airplane is at trim, the vector sum of all forces applied to the airplane is zero. Moreover the momentums about CG will be zero as well. Hence at trim:

$$\begin{aligned} &X_0 - mg\sin\theta_0 = 0 \\ &Y_0 = 0 \\ &Z_0 + mg\cos\theta_0 = 0 \\ &L_0 = M_0 = N_0 \end{aligned} \tag{3.34}$$

Substituting (3.34) in the perturbed model, the following two sets of equations are obtained:

$$\begin{aligned} &\Delta X - mg\theta\cos\theta_0 = m\dot{u} \\ &\Delta Z - mg\theta\sin\theta_0 = m(\dot{w} - u_0 q) \quad \text{(a)} \\ &\Delta M = I_{YY}\dot{q} \\ \\ &\Delta Y + mg\phi\cos\theta_0 = m(\dot{v} + u_0 r) \\ &\Delta L = Ixx\dot{p} - Ixz\dot{r} \qquad\qquad\quad \text{(b)} \\ &\Delta N = I_{ZZ}\dot{r} - I_{XZ}\dot{p} \end{aligned} \tag{3.35}$$

The first set, (3.35a) includes terms and moments which lie in the C_{xy} plane and they describe the longitudinal motion of the airplane. The second set (3.35b) includes terms which tend to move the plane of symmetry itself. Those equations are the equations of lateral-directional motion, and they may be solved independently [5].

3.8.1 Stability Derivatives

Definition of the stability derivatives is based on [4] and [5]. Analysis of the perturbed aerodynamic forces and moments follows the assumption that the latter are functions of the angle of attack, the sideslip angle, the linear perturbed and the angular perturbed velocities, the control angles, and their respective derivatives. An additional assumption is that the disturbed forces and moments can be described in a linear manner. An example of the aerodynamic force in the C_x axis is:

$$\Delta X = (\frac{\partial X}{\partial u})_0 u + (\frac{\partial X}{\partial v})_0 v + (\frac{\partial X}{\partial \alpha})_0 \alpha + (\frac{\partial X}{\partial \dot{\alpha}})_0 \dot{\alpha} + (\frac{\partial X}{\partial p})_0 p + (\frac{\partial X}{\partial q})_0 q + (\frac{\partial X}{\partial r})_0 r + (\frac{\partial X}{\partial \delta})_0 \delta + (\frac{\partial X}{\partial \phi})_0 \phi + (\frac{\partial \vartheta}{\partial \vartheta})_0 \vartheta + (\frac{\partial X}{\partial \psi})_0 \psi$$

where $(\frac{\partial X}{\partial u})_0$ denotes the partial derivative of the external aerodynamic force in the C_x axis, with respect to u and calculated under the trim flight condition. Stability derivatives are obtained by dividing the above partial derivatives by the airplane mass when the partial derivative corresponds to an external force. When the partial derivative corresponds to a moment, then the stability derivative is obtained by dividing by the appropriate mass inertia. An example is:

$$X_u = \frac{1}{m}(\frac{\partial X}{\partial u})_0$$
$$M_\alpha = \frac{1}{I_{YY}}(\frac{\partial M}{\partial \alpha})_0$$

The calculation of the stability derivatives is beyond the scope of this Chapter; however, details may be found in [2], [4] and [5], [6], [7].

In general not all stability derivatives are necessary for linearization of the forces or moments. The significance of each stability derivative in approximating forces and moments is summarized in Table 3.1**,** duplicated from [5]. The (+) entry indicates that the value of the specific stability derivative can be significant for calculation of force or moment. A (-) entry indicates that the value of the corresponding stability derivative is zero or insignificant for calculation of force or moment.

Independent variable	Dependent variable					
	X	Y	Z	L	M	N
u	-	-	+	-	-	-
v	-	+	-	+	-	+
α	+	-	+	-	+	-
$\dot{\alpha}$	-	-	+	-	+	-
p	-	-	-	+	-	+
q	-	-	+	-	+	-
r	-	+	-	+	-	+
δ_e	-	-	+	-	+	-
δ_r	-	+	-	+	-	+
δ_a	-	-	-	+	-	-
ϕ	-	+	-	-	-	-
θ	+	-	-	-	-	-
ψ	-	-	-	-	-	-

Table 3.1. Stability derivatives significance (duplicated from [5]).

3.8.2 Longitudinal Motion

Due to perturbation in the velocity of the C_z axis a small angle of attack is generated, which may be approximated by:

$$\alpha = \frac{w}{u_0 + u} \approx \frac{w}{u_0} \tag{3.36}$$

since $u << u_O$. Under this assumption, and observing Table 3.1, the equations describing the longitudinal motion are written as:

$$\begin{aligned} &X_u u + X_\alpha \alpha - g\theta\cos\theta_0 = \dot{u} \\ &Z_u u + Z_\alpha \alpha + Z_{\dot{\alpha}}\dot{\alpha} + Z_q \dot{\theta} - g\theta\sin\theta_0 + Z_{\delta_e}\delta_e = u_0(\dot{\alpha} - \dot{\theta}) \\ &M_\alpha \alpha + M_{\dot{\alpha}}\dot{\alpha} + M_q \dot{\theta} + M_{\delta_e}\delta_e = \ddot{\vartheta} \end{aligned} \tag{3.37}$$

The value of the angular velocity q has been replaced by $\dot{\vartheta}$. The Laplace transform of the above equations is:

$$\begin{bmatrix} g\cos\theta_0 & s - X_u & -X_a \\ -s(Z_q + u_0) + g\sin\theta_0 & -Z_u & s(u_0 - Z_{\dot{\alpha}}) - Z_\alpha \\ s^2 - sM_q & 0 & -(sM_{\dot{\alpha}} + M_\alpha) \end{bmatrix} \begin{bmatrix} \theta(s) \\ u(s) \\ \alpha(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix} \delta_e(s) \tag{3.38}$$

3.8.3 Lateral-Directional Motion

Due to perturbation in the velocity of the C_y axis, a small sideslip angle is generated, which may be approximated by:

$$\beta = \frac{v}{u_0 + u} \approx \frac{v}{u_0} \tag{3.39}$$

since $u << u_O$. Under this assumption, and from Table 3.1, the equations describing the longitudinal motion are written as:

$$\begin{aligned} &Y_\beta \beta + Y_r r + Y_p p + Y_{\delta_r} \delta_r + g\phi\cos\theta_0 = u_0(\dot{\beta} + r) \\ &L_\beta \beta + L_r r + L_p p + L_{\delta_r} \delta_r + L_{\delta_a} \delta_a = \dot{p} - \frac{Ixz}{I_{XX}} \dot{r} \\ &N_\beta \beta + N_r r + N_p p + N_{\delta_r} \delta_r = \dot{r} - \frac{I_{XZ}}{I_{ZZ}} \dot{p} \end{aligned} \tag{3.40}$$

by substituting $r = \dot{\psi}\cos\theta_0$ and $p = \dot{\phi} - \dot{\psi}\sin\theta_0$ (3.40) takes the form:

$$\begin{aligned} &Y_\beta \beta + \left(Y_r \cos\theta_0 - Y_p \sin\theta_0 - u_0 \cos\theta_0\right)\dot{\psi} + Y_p\dot{\phi} + Y_{\delta_r}\delta_r + g\phi\cos\theta_0 = u_0\dot{\beta} \\ &L_\beta \beta + \left(L_r \cos\theta_0 - L_p \sin\theta_0\right)\dot{\psi} + L_p\dot{\phi} + L_{\delta_r}\delta_r + L_{\delta_a}\delta_\alpha = \ddot{\phi} - \left(\sin\theta_0 + \frac{Ixz}{I_{XX}}\cos\theta_0\right)\ddot{\psi} \\ &N_\beta \beta + \left(N_r \cos\theta_0 - N_p \sin\theta_0\right)\dot{\psi} + N_p\dot{\phi} + N_{\delta_r}\delta_r = \left(\cos\theta_0 + \frac{I_{XZ}}{I_{ZZ}}\sin\theta_0\right)\ddot{\psi} - \frac{I_{XZ}}{I_{ZZ}}\ddot{\phi} \end{aligned} \tag{3.41}$$

The Laplace transform of the above equations will be:

$$\begin{bmatrix} su_0 - Y_\beta & \left(u_0 \cos\theta_0 + Y_p \sin\theta_0 - Y_r \cos\theta_0\right)s & -(Y_p s + g\cos\theta_0) \\ -L_\beta & -\left\{\left(\sin\theta_0 + \dfrac{I_{XZ}}{I_{XX}}\cos\theta_0\right)s^2 + \left(L_r \cos\theta_0 - L_p \sin\theta_0\right)s\right\} & s^2 - L_p s \\ -N_\beta & \left(\cos\theta_0 + \dfrac{I_{XZ}}{I_{ZZ}}\sin\theta_0\right)s^2 + (N_p \sin\theta_0 - N_r \cos\theta_0)s & -\dfrac{I_{XZ}}{I_{ZZ}}s^2 - N_p s \end{bmatrix} \begin{bmatrix} \beta(s) \\ \psi(s) \\ \phi(s) \end{bmatrix} =$$

$$\begin{bmatrix} Y_{\delta_r} \\ L_{\delta_r} \\ N_{\delta_r} \end{bmatrix} \delta_r(s) + \begin{bmatrix} 0 \\ L_{\delta_a} \\ 0 \end{bmatrix} \delta_a(s) \tag{3.42}$$

Equations (3.38) and (3.42) are the two sets of equations that are used for controller design.

3.9 Conclusions

This Chapter presented background information related to how an airplane behaves in flight. Control of the airplane is accomplished by manipulating the angles of the airplane moving surfaces in the rare of the wings, the horizontal and vertical stabilizers.

The first challenge was to express equations with respect to the body-fixed reference frame attached to the airplane. By doing so, the moments and products of inertia have constant values over time. The difficulty in this derivation is that Newton's second law applies to inertial frames. Therefore, the idea is to express appropriately the inertia frame instantaneous linear and angular velocities and their derivatives relative to the body-fixed reference frame; then, Newton's second law is applied.

The derivation of the equations of motion relative to the body-fixed reference frame is not sufficient to describe the position and orientation of the airplane in space. Therefore, still using an Earth-fixed reference frame, the orientation of the airplane is given in terms of three Euler angles. More specifically, the airplane's orientation at any instant is determined by three sequential finite rotations, starting with a frame initially aligned with the Earth-fixed frame. The final position and orientation equations relative to the Earth-fixed frame are nonlinear and coupled. Since integration of those equations is highly complicated, numerical integration may be followed, or analysis should be restricted to special motion case studies.

The next step was the analysis of the aerodynamic forces and moments acting on the airplane under the assumption of a trimmed longitudinal flight subjected to small perturbations. Perturbations are small changes in orientation, linear and angular velocities and their derivatives. The perturbed aerodynamic forces may be expressed as linear combinations of the perturbed position, linear and angular velocities their derivatives and control angles. The resulting equations are linearized around conditions of trimmed longitudinal flight. Apart from the linearity advantage, it appears that the equations of motion may be split in two uncoupled subsets. The first set describes the longitudinal motion while the second set describes the lateral-directional motion. The two sets may be treated separately.

Since those equations are linear differential equations, Laplace transform is used to obtain the equations for designing closed-loop controllers.

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