Chapter 4

Rotating Coordinate Systems

4.1 General

At any instant we have $\{\mathbf{v}\}_2 = T_{2,1}\{\mathbf{v}\}_1$ and $\{\mathbf{v}\}_1 = T_{2,1}^T\{\mathbf{v}\}_2$. If F_2 is rotating with respect to F_1 with angular rate ω_2^1 then the relative orientation of the axes of the two reference frames must be changing, and $T_{2,1}$ is time-varying. To find the rate of change of $T_{2,1}$ we will consider particular vectors and see how the relative rotation of the reference frames affects their representation. By then considering arbitrary vectors we may infer $\dot{T}_{2,1}$.

We use the dot notation and associated subscript to indicate the time-derivative of a quantity as seen from a particular reference frame. Thus the notation $\dot{\mathbf{v}}_2$ indicates the rate of change of the vector $\mathbf{v}$ relative to F_2. The entity is itself another vector and may be represented in any coordinate system, so that $\{\dot{\mathbf{v}}_2\}_1$ means the vector defined as the rate of change of $\mathbf{v}$ relative to F_2 but represented by its components in F_1. The simplest case is that of a vector $\mathbf{v}$ whose components are given in a particular reference frame, whose derivative is taken with respect to that reference frame, and then represented in that reference frame. In this case the result is found by taking the derivative of each of the components:

$$\begin{aligned}
\{\mathbf{v}\}_2 &= v_{x_2}\mathbf{i}_2 + v_{y_2}\mathbf{j}_2 + v_{z_2}\mathbf{k}_2 \\
\{\dot{\mathbf{v}}_2\}_2 &= \dot{v}_{x_2}\mathbf{i}_2 + \dot{v}_{y_2}\mathbf{j}_2 + \dot{v}_{z_2}\mathbf{k}_2
\end{aligned}$$

Having found this vector and its representation in F_2 we could then calculate (at the instant the derivative is valid)

$$\{\dot{\mathbf{v}}_2\}_1 = T_{1,2}\{\dot{\mathbf{v}}_2\}_2$$

Note that in general this is *not* the rate of change of $\mathbf{v}$ relative to F_1.

With the notation established, consider $\{\mathbf{v}\}_2 = T_{2,1}\{\mathbf{v}\}_1$. At first we might just write down $\dot{\mathbf{v}}_2 = \dot{T}_{2,1}\mathbf{v}_1 + T_{2,1}\dot{\mathbf{v}}_1$ (?), but it is by no means clear who is taking the derivatives of the vectors and where they are represented. What we want is $\{\dot{\mathbf{v}}_2\}_2$, so we will use the expression $\{\mathbf{v}\}_2 = v_{x_2}\mathbf{i}_2 + v_{y_2}\mathbf{j}_2 + v_{z_2}\mathbf{k}_2$ and see what the derivative of each term is. We have

$$\begin{Bmatrix} v_{x_2} \\ v_{y_2} \\ v_{z_2} \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}_{2,1} \begin{Bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \end{Bmatrix}$$

The components are each similar to $v_{x_2} = t_{11}v_{x_1} + t_{12}v_{y_1} + t_{13}v_{z_1}$. The derivative of this component is

$$\dot{v}_{x_2} = \dot{t}_{11}v_{x_1} + t_{11}\dot{v}_{x_1} + \dot{t}_{12}v_{y_1} + t_{12}\dot{v}_{y_1} + \dot{t}_{13}v_{z_1} + t_{13}\dot{v}_{z_1}$$

The terms $\dot{t}_{ij}$ are clearly the elements of $\dot{T}_{2,1}$, and the terms $\dot{v}_{x_1}, \dot{v}_{y_1}, \dot{v}_{z_1}$ are those from $\{\dot{\mathbf{v}}_1\}_1 = \dot{v}_{x_1}\mathbf{i}_1 + \dot{v}_{y_1}\mathbf{j}_1 + \dot{v}_{z_1}\mathbf{k}_1$. Reassembling we have

$$\{\dot{\mathbf{v}}_2\}_2 = \dot{T}_{2,1}\{\mathbf{v}\}_1 + T_{2,1}\{\dot{\mathbf{v}}_1\}_1$$

In words, then, the vector defined as the rate of change of $\mathbf{v}$ relative to F_2 and represented by its components in F_2 equals the rate of change of the transformation times $\mathbf{v}$ represented in F_1, plus the transformation of the vector that is the rate of change of $\mathbf{v}$ relative to F_1 and represented by its components in F_1.

Working from $\{\mathbf{v}\}_1 = T_{2,1}^T\{\mathbf{v}\}_2$, it is easy to show that

$$\{\dot{\mathbf{v}}_1\}_1 = \dot{T}_{2,1}^T\{\mathbf{v}\}_2 + T_{2,1}^T\{\dot{\mathbf{v}}_2\}_2 \tag{4.1}$$

Now recalling that the direction cosine matrix depends only on the relative orientation of two reference frames, and not on any particular vector in

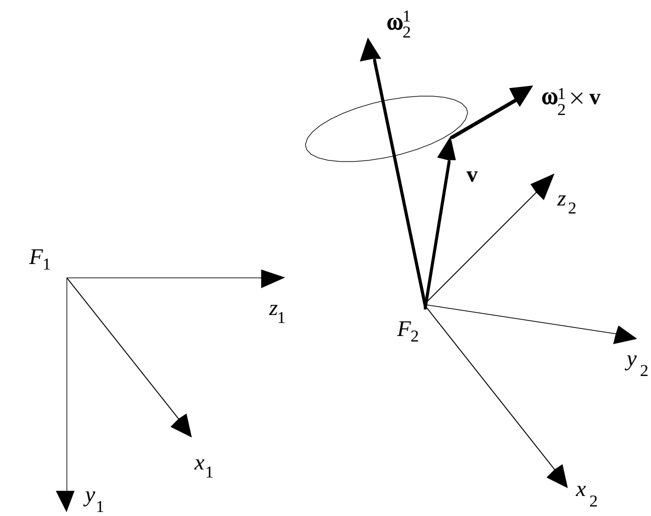

Figure 4.1: Vector $\mathbf{v}$ Fixed in F_2

either one, it follows that the above equations involving $T_{2,1}$ and $\dot{T}_{2,1}$ must hold for any choice of $\mathbf{v}$. So we may pick $\mathbf{v}$ fixed with respect to F_1 so that $\{\dot{\mathbf{v}}_1\}_1 = 0$, or fixed in F_2 so that $\{\dot{\mathbf{v}}_2\}_2 = 0$. Let $\{\dot{\mathbf{v}}_2\}_2 = 0$, and consider what $\{\dot{\mathbf{v}}_1\}_1$ looks like:

$$\{\dot{\mathbf{v}}_1\}_1 = \dot{T}_{2,1}^T \{\mathbf{v}\}_2 + T_{2,1}^T \{\dot{\mathbf{v}}_2\}_2 = \dot{T}_{2,1}^T \{\mathbf{v}\}_2$$

Figure 4.1 gives us another way to express $\{\dot{\mathbf{v}}_1\}_1$. From the point of view of an observer in F_1 $\mathbf{v}$ is rotating about the instantaneous rotation vector $\boldsymbol{\omega}_2^1$, so that it appears to be changing according to the cross product $\boldsymbol{\omega}_2^1 \times \mathbf{v}$. From this we may immediately write, for a vector $\mathbf{v}$ fixed in F_2, $\dot{\mathbf{v}}_1 = \boldsymbol{\omega}_2^1 \times \mathbf{v}$. We need to represent this vector in F_1 so the parts in the cross product must be in F_1, or $\{\dot{\mathbf{v}}_1\}_1 = \{\boldsymbol{\omega}_2^1\}_1 \times \{\mathbf{v}\}_1$. Combining these results we have

$$\dot{T}_{2,1}^T \{\mathbf{v}\}_2 = \left\{\boldsymbol{\omega}_2^1\right\}_1 \times \{\mathbf{v}\}_1 \tag{4.2}$$

To more easily manipulate this equation we will replace the cross product operation with the product of a matrix and a vector. Consider $\mathbf{u} \times \mathbf{v}$, with

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}, \qquad \mathbf{v} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$$

Define the matrix U as

$$U = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

It is then easy to verify that $\mathbf{u} \times \mathbf{v} = U\mathbf{v}$. This matrix is *skew symmetric*, meaning that $U^T = -U$. We may therefore rewrite $\dot{T}_{2,1}^T \{\mathbf{v}\}_2 = \{\boldsymbol{\omega}_2^1\}_1 \times \{\mathbf{v}\}_1$ as

$$\dot{T}_{2,1}^T \{\mathbf{v}\}_2 = \left\{\Omega_2^1\right\}_1 \{\mathbf{v}\}_1$$

We then replace $\{\mathbf{v}\}_1$ with $\{\mathbf{v}\}_1 = T_{1,2} \{\mathbf{v}\}_2 = T_{2,1}^T \{\mathbf{v}\}_2$,

$$\dot{T}_{2,1}^T \{\mathbf{v}\}_2 = \left\{\Omega_2^1\right\}_1 T_{2,1}^T \{\mathbf{v}\}_2$$

This must hold for all $\{\mathbf{v}\}_2$, so

$$\dot{T}_{2,1}^T = \left\{\Omega_2^1\right\}_1 T_{2,1}^T$$

Transposing each side,

$$\dot{T}_{2,1} = T_{2,1} \left\{\Omega_2^1\right\}_1^T = -T_{2,1} \left\{\Omega_2^1\right\}_1$$

Now we go the other direction. If F_2 is rotating with respect to F_1 with angular rate ω_2^1 then F_1 is rotating with respect to F_2 with angular rate $\omega_1^2 = -\omega_2^1$. We start with $\{\dot{\mathbf{v}}_2\}_2 = \dot{T}_{2,1} \{\mathbf{v}\}_1 + T_{2,1} \{\dot{\mathbf{v}}_1\}_1$. We pick $\mathbf{v}$ fixed in F_1 so that $\{\dot{\mathbf{v}}_1\}_1 = 0$. Then $\{\dot{\mathbf{v}}_2\}_2 = \dot{T}_{2,1} \{\mathbf{v}\}_1 = \{\boldsymbol{\omega}_1^2\}_2 \times \{\mathbf{v}\}_2 = \{\Omega_1^2\}_2 \{\mathbf{v}\}_2$, and finally

$$\dot{T}_{2,1} = \left\{\Omega_1^2\right\}_2 T_{2,1} = -\left\{\Omega_2^1\right\}_2 T_{2,1}$$

This gives us four ways of evaluating the derivative of the direction cosine matrix in terms of the relative rotation of two reference frames:

$$\begin{aligned}\dot{T}_{2,1} &= T_{2,1}\,\{\Omega_1^2\}_1 \\ &= -T_{2,1}\,\{\Omega_2^1\}_1 \\ &= \{\Omega_1^2\}_2\,T_{2,1} \\ &= -\,\{\Omega_2^1\}_2\,T_{2,1}\end{aligned} \tag{4.3}$$

Which expression is to be used depends on how $\boldsymbol{\omega}$ is most naturally expressed. For reasons that will become clear later, we will most often assume we know $\{\boldsymbol{\omega}_2^1\}_2$ or equivalently $\{\Omega_2^1\}_2$. In the following application of the equations developed we will therefore use:

$$\dot{T}_{2,1} = -\left\{\Omega_2^1\right\}_2 T_{2,1}$$

With this result we can easily write an expression for the transformation of the derivative of a vector. Using $\{\dot{\mathbf{v}}_2\}_2 = \dot{T}_{2,1}\,\{\mathbf{v}\}_1 + T_{2,1}\,\{\dot{\mathbf{v}}_1\}_1$ we have

$$\begin{aligned}\{\dot{\mathbf{v}}_2\}_2 &= T_{2,1}\,\{\dot{\mathbf{v}}_1\}_1 - \{\Omega_2^1\}_2\,T_{2,1}\,\{\mathbf{v}\}_1 \\ &= T_{2,1}\,\{\dot{\mathbf{v}}_1\}_1 - \{\Omega_2^1\}_2\,\{\mathbf{v}\}_2\end{aligned} \tag{4.4}$$

To avoid overworking the subscripts we will define:

$$\left\{\boldsymbol{\omega}_2^1\right\}_2 = \begin{Bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{Bmatrix}$$

$$\left\{\Omega_2^1\right\}_2 = \begin{bmatrix}0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0\end{bmatrix}$$

4.2 Direction Cosines

Here we have

$$T_{2,1} = \begin{bmatrix} \cos\theta_{x_2x_1} & \cos\theta_{x_2y_1} & \cos\theta_{x_2z_1} \\ \cos\theta_{y_2x_1} & \cos\theta_{y_2y_1} & \cos\theta_{y_2z_1} \\ \cos\theta_{z_2x_1} & \cos\theta_{z_2y_1} & \cos\theta_{z_2z_1} \end{bmatrix}$$

Term-by-term differentiation yields $\dot{T}_{2,1} = \dot{t}_{ij}, i, j = 1\ldots3$ where, for example, $\dot{t}_{11} = -\dot{\theta}_{x_2x_1}\sin\theta_{x_2x_1}$

The right hand side of the equation is

$$\dot{T}_{2,1} = -\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_{x_2x_1} & \cos\theta_{x_2y_1} & \cos\theta_{x_2z_1} \\ \cos\theta_{y_2x_1} & \cos\theta_{y_2y_1} & \cos\theta_{y_2z_1} \\ \cos\theta_{z_2x_1} & \cos\theta_{z_2y_1} & \cos\theta_{z_2z_1} \end{bmatrix}$$

We could multiply these matrices to evaluate each $\dot{t}_{ij}$, for example

$$\dot{t}_{11} = -\dot{\theta}_{x_2x_1}\sin\theta_{x_2x_1} = \omega_z\cos\theta_{y_2x_1} - \omega_y\cos\theta_{z_2x_1}$$

This would result in nine nonlinear ordinary differential equations. The six nonlinear constraining equations based on the orthogonality of the rows (or columns) of the matrix could probably be used to simplify the relationships, but not without some trouble. The use of direction cosines in evaluating $\dot{T}_{2,1}$ is rare and has nothing to offer over the use of Euler angles or Euler parameters, and will not be used in this course.

4.3 Euler angles

The Euler angle representation of $T_{2,1}$ (for a 321 rotation from F_1 to F_2) is

$$T_{2,1} =$$
$$\begin{bmatrix} (\cos\theta_y \cos\theta_z) & (\cos\theta_y \sin\theta_z) & (-\sin\theta_y) \\ \begin{pmatrix} \sin\theta_x \sin\theta_y \cos\theta_z \\ -\cos\theta_x \sin\theta_z \end{pmatrix} & \begin{pmatrix} \sin\theta_x \sin\theta_y \sin\theta_z \\ +\cos\theta_x \cos\theta_z \end{pmatrix} & (\sin\theta_x \cos\theta_y) \\ \begin{pmatrix} \cos\theta_x \sin\theta_y \cos\theta_z \\ +\sin\theta_x \sin\theta_z \end{pmatrix} & \begin{pmatrix} \cos\theta_x \sin\theta_y \sin\theta_z \\ -\sin\theta_x \cos\theta_z \end{pmatrix} & (\cos\theta_x \cos\theta_y) \end{bmatrix}$$

Clearly we will have some trouble performing the element-by-element differentiation of this matrix for the left hand side of $\dot{T}_{2,1} = -\{\Omega_2^1\}_2 T_{2,1}$. The right hand side is not much better, and equating the various elements to find expressions for $\dot{\theta}_x$, $\dot{\theta}_y$, and $\dot{\theta}_z$ would be tedious. It is easier to proceed in a different way. First we note that the axes z_1, y', and x'' are not orthogonal, but can be used to form a basis for the space in which the rotations occur. That is, we may represent $\boldsymbol{\omega}_2^1$ in terms of the unit vectors in the directions z_1, y', and x''. The vector $\boldsymbol{\omega}_2^1$ is made up of a rate $\dot{\theta}_z$ which is about z_1, plus a rate $\dot{\theta}_y$ which is about y', and $\dot{\theta}_x$ which is about x'', or

$$\omega_2^1 = \dot{\theta}_z \mathbf{k}_1 + \dot{\theta}_y \mathbf{j}' + \dot{\theta}_x \mathbf{i}''$$

There is some savings in noting that $\mathbf{k}_1 = \mathbf{k}'$, $\mathbf{j}' = \mathbf{j}''$, and $\mathbf{i}'' = \mathbf{i}_2$, and so

$$\omega_2^1 = \dot{\theta}_z \mathbf{k}' + \dot{\theta}_y \mathbf{j}'' + \dot{\theta}_x \mathbf{i}_2$$

We require $\{\omega_2^1\}_2$, so we transform the unit vectors $\mathbf{k}'$ and $\mathbf{j}''$ ($\mathbf{i}_2$ is good the way it is) into F_2 using the intermediate rotations previously developed (equations 3.3 and 3.4). That is,

$$\begin{aligned} \{\mathbf{k}'\}_2 &= T_{2,F'} \{\mathbf{k}'\}_{F'} \\ \{\mathbf{j}''\}_2 &= T_{2,F''} \{\mathbf{j}''\}_{F''} \end{aligned}$$

This is straightforward, since ,

$$\{\mathbf{k}'\}_{F'} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \{\mathbf{j}''\}_{F''} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Also we have

$$T_{2,F''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}$$

$$\begin{aligned} T_{2,F'} &= T_{2,F''}T_{F'',F'} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ \sin\theta_x\sin\theta_y & \cos\theta_x & \sin\theta_x\cos\theta_y \\ \cos\theta_x\sin\theta_y & -\sin\theta_x & \cos\theta_x\cos\theta_y \end{bmatrix} \end{aligned}$$

From this we evaluate

$$\begin{aligned} \left\{\boldsymbol{\omega}_2^1\right\}_2 &= \dot\theta_z \{\mathbf{k}'\}_2 + \dot\theta_y \{\mathbf{j}''\}_2 + \dot\theta_x \mathbf{i}_2 \\ &= \dot\theta_z T_{2,F'} \{\mathbf{k}'\}_{F'} + \dot\theta_y T_{2,F''} \{\mathbf{j}''\}_{F''} + \dot\theta_x \mathbf{i}_2 \\ &= \dot\theta_z \begin{Bmatrix} -\sin\theta_y \\ \sin\theta_x\cos\theta_y \\ \cos\theta_x\cos\theta_y \end{Bmatrix} + \dot\theta_y \begin{Bmatrix} 0 \\ \cos\theta_x \\ -\sin\theta_x \end{Bmatrix} + \dot\theta_x \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \end{aligned}$$

With $\{\boldsymbol{\omega}_2^1\}_2^T = \{\omega_x, \omega_y, \omega_z\}$ we evaluate each term as

$$\left\{\boldsymbol{\omega}_2^1\right\}_2 = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} -\sin\theta_y\dot\theta_z + \dot\theta_x \\ \sin\theta_x\cos\theta_y\dot\theta_z + \cos\theta_x\dot\theta_y \\ \cos\theta_x\cos\theta_y\dot\theta_z - \sin\theta_x\dot\theta_y \end{Bmatrix}$$

We may rewrite this as a matrix times a vector as

$$\{\boldsymbol{\omega}_2^1\}_2 = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta_y \\ 0 & \cos\theta_x & \sin\theta_x\cos\theta_y \\ 0 & -\sin\theta_x & \cos\theta_x\cos\theta_y \end{bmatrix} \begin{Bmatrix} \dot\theta_x \\ \dot\theta_y \\ \dot\theta_z \end{Bmatrix}$$

We wish to solve this equation for $\dot\theta_x$, $\dot\theta_y$, and $\dot\theta_z$ in terms of the Euler angles and rotational components. The determinant of the matrix on the right hand side is easily verified to be $\cos\theta_y$, which means the inverse does not exist if $\theta_y = \pm 90\,\mathrm{deg}$. This condition results in $\dot\theta_x$ and $\dot\theta_z$ being undefined. If $\theta_y \neq \pm 90\,\mathrm{deg}$, we have

$$\begin{Bmatrix} \dot\theta_x \\ \dot\theta_y \\ \dot\theta_z \end{Bmatrix} = \begin{bmatrix} 1 & \sin\theta_x\tan\theta_y & \cos\theta_x\tan\theta_y \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x\sec\theta_y & \cos\theta_x\sec\theta_y \end{bmatrix} \{\boldsymbol{\omega}_2^1\}_2 \tag{4.5}$$

This completely general result applies to any two coordinate systems with a defined 321 transformation. All that is needed to apply it to a specific case are (1) the right names for the Euler angles (e.g., θ, ϕ, and ψ for local horizontal to body) and (2) the right relationships for the relative angular rotation rates $\{\boldsymbol{\omega}_2^1\}_2$ (e.g., p, q, and r for the inertial to body transformation).

The singularity involving θ_y is the fatal flaw of which we spoke earlier. It is also true that no matter what sequence is taken for the Euler angle rotations, the angle of the second rotation displays a similar singularity at either zero or $\pm 90\,\mathrm{deg}$. To avoid this, one could use the nine direction cosines themselves, or use Euler parameters.

4.4 Euler parameters

In evaluating the left and right hand sides of $\dot T_{2,1} = -\{\Omega_2^1\}_2 T_{2,1}$, we have

$$T_{2,1} =$$

$$\begin{bmatrix} (q_0^2+q_1^2-q_2^2-q_3^2) & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & (q_0^2-q_1^2+q_2^2-q_3^2) & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & (q_0^2-q_1^2-q_2^2+q_3^2) \end{bmatrix}$$

Clearly it should not be necessary to take the derivative of each of the elements in this matrix. We will use the three diagonal terms, combined with $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ to eliminate q_0. For the (1,1) entry we have

$$\begin{aligned} t_{11} &= q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ &= \left(1 - q_1^2 - q_2^2 - q_3^2\right) + q_1^2 - q_2^2 - q_3^2 \\ &= 1 - 2q_2^2 - 2q_3^2 \end{aligned}$$

Similarly for the (2,2) and (3,3) entries,

$$\begin{aligned} t_{22} &= q_0^2 - q_1^2 + q_2^2 - q_3^2 = 1 - 2q_1^2 - 2q_3^2 \\ t_{33} &= q_0^2 - q_1^2 - q_2^2 + q_3^2 = 1 - 2q_1^2 - 2q_2^2 \end{aligned}$$

The derivatives of these three terms are

$$\begin{aligned} dt_{11}/dt &= -4\left(q_2\dot{q}_2 + q_3\dot{q}_3\right) \\ dt_{22}/dt &= -4\left(q_1\dot{q}_1 + q_3\dot{q}_3\right) \\ dt_{33}/dt &= -4\left(q_1\dot{q}_1 + q_2\dot{q}_2\right) \end{aligned}$$

Now the right hand side of $\dot{T}_{2,1} = -\left\{\Omega_2^1\right\}_2 T_{2,1}$ is

$$\left\{\Omega_2^1\right\}_2 T_{2,1} =$$

$$\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} \begin{pmatrix} q_0^2 + q_1^2 \\ -q_2^2 - q_3^2 \end{pmatrix} & 2\left(q_1 q_2 + q_0 q_3\right) & 2\left(q_1 q_3 - q_0 q_2\right) \\ 2\left(q_1 q_2 - q_0 q_3\right) & \begin{pmatrix} q_0^2 - q_1^2 \\ +q_2^2 - q_3^2 \end{pmatrix} & 2\left(q_2 q_3 + q_0 q_1\right) \\ 2\left(q_1 q_3 + q_0 q_2\right) & 2\left(q_2 q_3 - q_0 q_1\right) & \begin{pmatrix} q_0^2 - q_1^2 \\ -q_2^2 + q_3^2 \end{pmatrix} \end{bmatrix}$$

Evaluating just the diagonal terms,

$$
\left\{\Omega_2^1\right\}_2 T_{2,1} =
$$

$$
\begin{bmatrix}
\begin{pmatrix} -2\omega_z\left(q_1q_2 - q_0q_3\right) \\ +2\omega_y\left(q_1q_3 + q_0q_2\right) \end{pmatrix} & ? & ? \\
? & \begin{pmatrix} 2\omega_z\left(q_1q_2 + q_0q_3\right) \\ -2\omega_x\left(q_2q_3 - q_0q_1\right) \end{pmatrix} & ? \\
? & ? & \begin{pmatrix} -2\omega_y\left(q_1q_3 - q_0q_2\right) \\ +2\omega_x\left(q_2q_3 + q_0q_1\right) \end{pmatrix}
\end{bmatrix}
$$

Equating the diagonal terms on the left and right hand sides,

$$
\begin{aligned}
4\left(q_2\dot{q}_2 + q_3\dot{q}_3\right) &= -2\omega_z\left(q_1q_2 - q_0q_3\right) + 2\omega_y\left(q_1q_3 + q_0q_2\right) \\
4\left(q_1\dot{q}_1 + q_3\dot{q}_3\right) &= 2\omega_z\left(q_1q_2 + q_0q_3\right) - 2\omega_x\left(q_2q_3 - q_0q_1\right) \\
4\left(q_1\dot{q}_1 + q_2\dot{q}_2\right) &= -2\omega_y\left(q_1q_3 - q_0q_2\right) + 2\omega_x\left(q_2q_3 + q_0q_1\right)
\end{aligned}
$$

Now placing these expressions in matrix form, we have

$$
\begin{bmatrix} 0 & q_2 & q_3 \\ q_1 & 0 & q_3 \\ q_1 & q_2 & 0 \end{bmatrix}
\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} =
$$

$$
\frac{1}{2}\begin{bmatrix} 0 & \left(q_1q_3 + q_0q_2\right) & \left(q_0q_3 - q_1q_2\right) \\ \left(q_0q_1 - q_2q_3\right) & 0 & \left(q_1q_2 + q_0q_3\right) \\ \left(q_2q_3 + q_0q_1\right) & \left(q_0q_2 - q_1q_3\right) & 0 \end{bmatrix}
\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}
$$

Solving this set of equations is not as messy as it first appears, and it simplifies very nicely. The result is

$$
\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \frac{1}{2}
\begin{bmatrix} q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}
\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}
$$

We may recover $\dot{q}_0$ by using $q_0^2 = 1 - q_1^2 - q_2^2 - q_3^2$ whence

$$q_0\dot{q}_0 = -q_1\dot{q}_1 - \dot{q}_2 q_2 - \dot{q}_3 q_3$$

Substitutions and a little manipulation yield

$$\dot{q}_0 = \frac{1}{2}\left(-q_1\omega_x - q_2\omega_y - q_3\omega_z\right)$$

$$\begin{Bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

This expression can also be written as

$$\begin{Bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} -q_0 & -q_1 & -q_2 & -q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix}\begin{Bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{4.6}$$

It is easy to verify that there are no singularities in this expression.

4.5 Customs and Conventions

4.5.1 Angular Velocity Components

Inertial angle rates in body and wind axes occur frequently, and the components associated with them are given special symbols. These are as follows:

$$\{\boldsymbol{\omega}_B\}_B = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

$$\{\boldsymbol{\omega}_W\}_W = \left\{ \begin{array}{c} p_W \\ q_W \\ r_W \end{array} \right\}$$