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Use of P_τ-Nets for the Approximation of the Edgeworth–Pareto Set in Multicriteria Optimization

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Abstract. Engineering optimization problems, design problems among them, are multicriteria in their essence. When designing machines, mechanisms, and structures, one has to deal with numerous contradictory criteria.

Experience gained in solving engineering optimization and optimal design problems shows that the designer cannot formulate them correctly. Unfortunately, the known optimization methods offer little assistance. To assure a correct formulation and solution of engineering optimization problems, the parameter space investigation (PSI) method and methods of approximation have been developed and widely integrated into different fields of industry, science, and technology. The PSI method has become one of the basic working tools for choosing optimal parameters in many fields of the national economy in Russia.

Key Words. Parameter space investigation method, feasible solutions set, Edgeworth–Pareto optimal set, design variables, performance criteria, approximation, topology.

1. Introduction

In this paper, the reader will find a rather complete presentation of an original approach to multicriteria optimization that has been used and implemented into engineering practice for many years in Russia. These works were started in the 1970's; see Refs. 1–2.

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The parameter space investigation (PSI) method and methods of approximation presented here not only give a new point of view for general problems of engineering optimization, but also allow their successful solution. In the Western literature, one can find the PSI method expounded in Ref. 3 and others. The applied aspects of the PSI method are discussed in Refs. 4–5.

Let us enumerate some basic features of the problems to be considered:

(i) The problems are essentially multicriteria.

(ii) The determination of the feasible solutions set is one of the fundamental issues of the analysis of engineering problems. The construction of this set is an important step in the formulation and solution of such problems.

(iii) The problem formulation and solution comprise a single process. The customary approach is that first the designer formulates a problem and then a computer is employed to solve it. However, in the case under consideration, this approach is untenable, because only in rare cases can one formulate a problem completely and correctly before its solution. Thus, problems should be formulated and solved interactively.

(iv) Note that the goal functions may be nondifferentiable. For this reason, the methods considered here are zeroth-order methods.

The PSI method and methods of approximation have been developed to formulate and solve engineering optimization problems. In the Russian literature (Refs. 6–8), one can find numerous examples of application of these methods. The limited size of this paper does not allow us to present these examples here.

2. Formulation of Multicriteria Optimization Problems

Let us consider a system whose operation is described by a set of equations (differential, algebraic, etc.) or whose performance criteria may be calculated directly. We assume that the system depends on r design variables $a_1, \dots, a_r$ representing a point $a = (a_1, \dots, a_r)$ of an r-dimensional space. Usually, a appears in the above-mentioned equations.

Design variable constraints have the form

$$a_j^* \leq a_j \leq a_j^{**}, \qquad j = 1, \dots, r. \tag{1}$$

In the case of mechanical systems, the variables a_j might represent stiffness coefficients, moments of inertia, masses, damping factors, geometric dimensions, etc.

Functional constraints may be written in the form

$$C_\ell^* \le f_\ell(a) \le C_\ell^{**}, \qquad \ell = 1, \dots, t. \tag{2}$$

In the operation of the system, there exist particular performance criteria such as productivity, material consumption, efficiency, etc. All other things being equal, it is desired that these criteria, denoted by $F_\nu(a)$, $\nu = 1, \dots, k$, attain extreme values. For simplicity, we suppose that the functions $F_\nu(a)$ are to be minimized.

Obviously, the constraints (1) determine parallelepiped Π in the r-dimensional space of the design variables.

To avoid the situation where the values of some criteria are unacceptable, we introduce criteria constraints of the form

$$F_\nu(a) \le F_\nu^{**}, \qquad \nu = 1, \dots, k, \tag{3}$$

where F_ν^{**} is the worst value of the criterion $F_\nu(a)$ acceptable to the designer. Criteria constraints differ from functional constraints in that the former are adjusted while solving a problem and as a rule are repeatedly revised.

The constraints (1)–(3) define the feasible solutions set D, i.e., the set of design solutions that satisfy the constraints. If the functions $f_\ell(a)$ and $F_\nu(a)$ are continuous in Π, then the set D is closed.

Let us formulate one of the basic problems of multicriteria optimization. Let $F = (F_1, F_2, \dots, F_k)$ be the vector of criteria.

Definition 2.1. See Ref. 9. A vector $a \in D$ is Edgeworth–Pareto optimal iff $b \in D$ and

$$F(b) \le F(a) \rightarrow F(a) = F(b), \tag{4}$$

for every P-comparable $b \in D$. The set P of Edgeworth–Pareto optimal vectors consists of all such a.

In solving this problem, one still has to determine the vector design variables $a^0 \in P$ which is the most preferred among the vectors belonging to the set P.

The EP-optimal set plays an important role in vector optimization problems.

3. Uniformly Distributed Sequences in Multidimensional Domains

The features of the problems under consideration make it necessary to represent the vectors a by points of uniformly distributed sequences in the space of design variables; see Ref. 6. We briefly consider this issue below.

For many applied problems, the following situation is typical. There exists a multidimensional domain in which a function or a system of functions is considered whose values are calculated at certain points. Suppose that we wish to get some information on the behavior of the function in the entire domain or in a subdomain. Then, in the absence of additional information about the function, it is natural to wish that the points where the function is calculated be uniformly distributed in some sense within the domain. Suppose that we consider a sequence of points $Q_1, Q_2, \ldots, Q_\ell, \ldots$ belonging to a unit r-dimensional cube K^r. We denote by G an arbitrary domain in K^r; and we denote by $S_N(G)$ the number of points Q_ℓ belonging to G, $1 \leq i \leq N$. A sequence Q_ℓ is called uniformly distributed in K^r, if

$$\lim_{N \to \infty} (S_N(G)/N) = V(G), \tag{5}$$

where $V(G)$ is the volume of the r-dimensional domain G. If a parallelepiped Π is considered, instead of the unit cube, then the right-hand side of (5) changes to $V(G)/V(\Pi)$.

The use of more uniform sequences/nets has the following practical advantages. If we wish to solve a problem (for instance, to obtain the EP-optimal set and feasible solutions set) with a prescribed accuracy, then the use of a more uniform sequence assures a better convergence rate. However, if the time allowed for solving the problem is very short (hence, N is small), then the problem cannot be studied in this way. Nevertheless, using more uniform sequences, one may distribute the points in such a way that they would represent satisfactorily the whole domain G. As a result, the designer would have sufficiently reliable information about the problem under consideration.

In the following discussion, we consider two different classes of uniform sequences whose uniformity characteristics are among the best presently known. These are the so-called LP_τ-sequences and the novel P_τ-nets. The P_τ-nets were first described in Ref. 8. They have the minimum value of the index τ and the best uniform distribution of initial segments used for the optimization.

Remark 3.1. In addition to the LP_τ-sequences and the P_τ-nets, there exist other sequences and nets which can be employed successfully.

4. Parameter Space Investigation Method

Now, we proceed by describing the parameter space investigation (PSI) method allowing the correct determination of F_v^{**}, and hence the feasible solutions; see Refs. 6 and 10.

Stage 1. Compilation of Test Tables via Computer. First, one chooses N trial points $a^1, \ldots, a^N$ that satisfy (2). Then, all the particular criteria $F_v(a^i)$ are calculated at each of the points a^i; for each of the criteria, a test table[3] is compiled so that the values of $F_v(a^1), \ldots, F_v(a^N)$ are arranged in increasing order, i.e.,

$$F_v(a^{i_1}) \leq F_v(a^{i_2}) \leq \cdots \leq F_v(a^{i_N}), \qquad v=1, \ldots, k, \tag{6}$$

where $i_1, i_2, \ldots, i_N$ are the numbers of trials (a separate set for each v). Taken together, the k tables form a complete test table. In the following discussion, the latter is called the test table.

Stage 2. Preliminary Selection of Criteria Constraints. This stage includes interaction with the designer. By analyzing Table (6), the designer specifies the criteria constraints F_v^{**}.

The F_v^{**} are the maximum values of the criteria $F_v(a)$ which guarantee an acceptable level of the system operation. If the selected values of F_v^{**} are not maxima, then this may result in the loss of desirable solutions, since some of the criteria are contradictory. As a rule, the designer may put F_v^{**} equal to a criterion value $F_v(\bar{a})$ whose feasibility is beyond doubt. However, if he starts by determining the maximum possible value of F_v^{**}, then he has to pass to Stage 3.

Stage 3. Solvability of Problem (4) via Computer. Let us fix a criterion, say $F_{v_1}(a)$, and consider the corresponding Table (6). Let S_1 be the number of values in the table satisfying the selected criteria constraints,

$$F_{v_1}(a^{i_1}) \leq \cdots \leq F_{v_1}(a^{i_{S_1}}) \leq F_{v_1}^{**}. \tag{7}$$

One should choose the criterion F_{v_1} for which S_1 is the minimum among the analogous numbers calculated for each of the criteria F_v.

Then, the criterion F_{v_2} is selected similarly, and the values $F_{v_2}(a^{i_1}), \ldots, F_{v_2}(a^{i_{S_1}})$, of F_{v_2} in the test table are considered. Let the table contain $S_2 \leq S_1$ values such that

$$F_{v_2}(a^{i_j}) \leq F_{v_2}^{**}, \qquad 1 \leq j \leq S_2.$$

The procedure continues until all of the criteria have been considered. Then, if one point can be found for which all of the inequalities (3) are satisfied simultaneously, the set D defined by (1)–(3) is nonempty and problem (4) is solvable. Otherwise, one should return to Stage 2 and ask the designer to

[3]This is called an ordered test table.

make certain concessions in the specification of F_ν^{**}. However, if the concessions are highly undesirable, one may return to Stage 1 and increase the number of points in order to repeat Stage 2 using extended test tables.

The procedure is to be continued until D proves to be nonempty. Then, the EP-optimal set is constructed in accordance with the definition presented in Section 2. This is done by removing those feasible points which can be improved with respect to all of the criteria simultaneously.

Let us consider the case where it is difficult to decide whether the value of a criterion F_ν^{**} is a maximum. Usually, one is not sure whether the values of $F_\nu(a)$ from the interval

$$F_\nu(\bar{a}) \leq F_\nu(a) \leq \tilde{F}_\nu^{**}$$

are feasible. Here, $\tilde{F}_\nu^{**}$ is the value of the νth criterion for which the values $F_\nu(a) > \tilde{F}_\nu^{**}$ are known to be unacceptable. In such a case, one has to go to Stage 3 and construct the feasible solutions set D, under the constraints $F_\nu^{**} = F_\nu(\bar{a})$, and the corresponding EP-optimal set P. Furthermore, the set $\tilde{D}$ is constructed subject to the constraints $\tilde{F}_{\nu_j}^{**}$, $\nu = 1, \ldots, k$, as well as the corresponding EP-optimal set $\tilde{P}$. Let us compare $F(P)$ and $F(\tilde{P})$. If the vectors belonging to $F(\tilde{P})$ do not improve the criterion value compared to vectors from $F(P)$ substantially, then one may put $F_\nu^{**} = F_\nu(\bar{a})$. Otherwise, if the improvement is significant, then the values of the criteria constraints may be put equal to $\tilde{F}_\nu^{**}$. In this case, one has to make sure that the optimal solution thus obtained is feasible.[4] If the designer is unable to do this, then the criteria constraints are put equal to their previous values, $F_\nu^{**} = F_\nu(\bar{a})$. This scheme can be used for all possible values of $F_\nu(\bar{a})$ and $\tilde{F}_\nu^{**}$.

5. Estimation of the Convergence Rate

The term "feasible solution" for multicriteria optimization problems was introduced in Refs. 6 and 10. An algorithm for the simple and efficient selection of feasible points from the design set was discussed in Refs. 6 and 10. We shall discuss ways in which the algorithm may be used to construct a feasible solution set D with a given accuracy. Indeed, it is known that, for problems involving continuous design variables, the set D also varies continuously, and we may construct D by isolating subsets of D which approach each criterion value in the region $F(D)$ to within a specified accuracy.

[4]To do this, the designer will possibly have to analyze the mathematical model anew or, if necessary, conduct additional experimental studies.

Let ϵ_v be an admissible designer-specified error in the criterion F_v. Let ϵ denote the error set $\{\epsilon_v: v=1, \ldots, k\}$. We shall say that the region $F(D)$ is approximated by a finite set $F(D_\epsilon)$ with accuracy ϵ iff, for each vector $a \in D$, there exists a vector $b \in D_\epsilon$ such that

$$|F_v(a)-F_v(b)| \leq \epsilon_v, \qquad v=1, \ldots, k.$$

For specified errors ϵ_v, $F(D)$ or D may be properly constructed only if the number of points of D_ϵ is sufficiently large. The construction then may require a large amount of computer time. However, an incomplete construction of the feasible solution set is equally undesirable, since this affects the solution quality.

We shall thus constrain our discussion to functions which are continuous and satisfy the following Lipschitz condition:

(L) For every pair of vectors a and b belonging to the domain of definition of F_v, there exists a number L_v such that

$$|F_v(a)-F_v(b)| \leq L_v \max_j |a_j-b_j|,$$

or equivalently, there exists a number L'_v such that

$$|F_v(a)-F_v(b)| \leq L'_v \sum_{j=1}^{r} |a_j-b_j|.$$

This is one of the less stringent conditions encountered in optimization theory. In practice, only more or less pathological functions violate the condition; few such functions arise in engineering applications.

It is convenient to reduce the class of functions considered still more with the following special Lipschitz condition:

(SL) For every pair of vectors a and b in the domain of F_v, there exist numbers $L^j_v, j=1, \ldots, r$, such that

$$|F_v(a)-F_v(b)| \leq \sum_{j=1}^{r} L^j_v |a_j-b_j|,$$

where at least two of the L^j_v are different.

This latter class is of interest for the following reasons:

(i) The class L incorporates all of the functions belonging to the class SL. In the majority of practical cases, these classes coincide, since the functions encountered in engineering problems have different sensitivities with respect to the design variables with different constants L^j_v as a consequence.

(ii) The convergence rate of the approximation process is greater for the class SL than it is for the class L; see Theorem 5.1.

(iii) The P_τ-nets (see Refs. 6 and 8), used for the calculation of the criteria values, are optimal for the class of SL functions; see Ref. 11.

The following theorem provides an estimate for the number of points of an r-dimensional P_τ-net required for the approximation of $F(D)$ with specified accuracy for the criteria $F_\nu(a) \in \mathrm{L}$ or SL. The notation $[a]$ will be used to denote a rational dyadic with property related to a. Recall that a rational dyadic is a number of the form $p/2^n$, where p and n are natural numbers.

Theorem 5.1. See Ref. 8. Suppose that the criteria $F_\nu(a)$ are continuous and belong to either the class L or the class SL. Then, $F(D)$ may be approximated with accuracy ϵ provided the number of points in the P_τ-net is at least

$$\max_\nu 2^\tau([L_\nu]/[\epsilon_\nu])^r \quad \text{or} \quad \max_\nu 2^\tau\left(\left[\sum_{j=1}^{r} L_\nu^j\right] \Big/ [\epsilon_\nu]\right)^r.$$

Proof. Let $[L_\nu]$ be a rational dyadic sufficiently close to but exceeding L_ν with equivalent requirement for $[\sum_{j=1}^r L_\nu^j]$; and let $[\epsilon_\nu]$ be the maximum rational dyadic less than or equal to ϵ_ν whose numerator is the same as that of $[L_\nu]$ or $[\sum_{j=1}^r L_\nu^j]$. Let $\mathfrak{L}_a^\nu$ be the r-dimensional cube with edge length $[\epsilon_\nu]/[L_\nu]$; and let $a=(a_1, \ldots, a_r)$ be any $a \in \mathfrak{L}_a^\nu$. The volume of the cube is $([\epsilon_\nu]/[L_\nu])^r$. Since this number is a rational dyadic and its numerator is unity, it may be represented in the form

$$([\epsilon_\nu]/[L_\nu])^r = 2^\tau/2^{\gamma_\nu},$$

where $\gamma_\nu > \tau$ is unknown and τ is the subscript of the P_τ-net corresponding to an r-dimensional cube K^r. From the latter equality, we get

$$2^{\gamma_\nu} = 2^\tau [L_\nu]^r/[\epsilon_\nu]^r. \tag{8}$$

According to the definition of P_τ-nets, any binary paralleliped of the cube K^r of volume $2^\tau/2^{\gamma_\nu}$ contains 2^τ points of the 2^{γ_ν} points of the P_τ-net (Ref. 12). Hence, if γ_ν satisfies Eq. (8), then the cube $\mathfrak{L}_a^\nu$ contains 2^τ points. In view of the Lipschitz condition and the definition of $\mathfrak{L}_a^\nu$, the inequality

$$|F_\nu(a) - F_\nu(b)| \leq \epsilon_\nu$$

is satisfied for any point $b \in \mathfrak{L}_a^r$. Thus, an arbitrary value of $F_\nu(a)$ may be approximated to within an accuracy ϵ_ν by 2^{γ_ν} points of the P_τ-net. The required value of τ may be calculated based on formulas in Ref. 12.

Suppose that, for some v_i and v_j, we have

$$[\epsilon_{v_i}]/[L_{v_i}]<[\epsilon_{v_j}]/[L_{v_j}].$$

Then, $\mathfrak{L}_a^{v_i}\subset\mathfrak{L}_a^{v_j}$; by choosing a value of n satisfying

$$2^n=\max_v 2^{\gamma_v}, \qquad v=1,\ldots,k,$$

we obtain a finite ϵ-approximation $F(D_\epsilon)$ of the set $F(D)$. In that case, the inequality

$$|F_v(a)-F_v(b)|\leq\epsilon_v, \qquad v=1,\ldots,k,$$

is satisfied, where $a\in K^r$ and b is one of the 2^n points. □

Remark 5.1. Theorem 5.1 provides a conservative estimate for the convergence of the approximation process. Once the admissible errors ϵ_v for the criteria F_v have been specified and the constants L_v or $\sum_{j=1}^r L_v^j$ are known, the whole feasible domain may be approximated with a given accuracy for any function corresponding to these constants. To do so, one has to calculate the criteria at the points of the P_τ-net whose number is specified by the theorem. However, this provides an estimate from above, since it takes into account even the worst functions of the class. For a concrete problem, the number of trials required for the approximation is less than that prescribed by the estimate.

The previous estimate of the convergence rate is generally applicable for the determination of the number of trials. Unfortunately, it is not of great help in solving engineering problems, since the number of points needed for the calculation of the performance criteria may be so large that even the speed of today's computers may prove to be inadequate. One way to overcome this difficulty may be through the development of fast algorithms which take into account special features of the functions in a particular problem, rather than dealing only with a whole function class.

6. Approximation of a Feasible Solution Set

Assume that the Lipschitz constants L_v, $v=1,\ldots,k$, have been specified, and let N_1 be the subset of points from D which are either EP-optimal or lie within an ϵ-neighborhood of an EP-optimal point with respect to at least one criterion. In other words,

$$F_v(a^0)\leq F_v(a)\leq F_v(a^0)+\epsilon_v,$$

where $a^0\in P$ and P is the set of EP-optima. Furthermore, let $N_2=D\backslash N_1$ and, for a given accuracy ϵ_v, let $\bar{\epsilon}_v>\epsilon_v$.

Definition 6.1. A set $F(D_\epsilon)$ is a normal approximation of the set $F(D)$ iff N_1 is approximated to an accuracy ϵ and any point of the set N_2 is approximated to an accuracy $\bar{\epsilon}$.

Theorem 6.1. See Ref. 13. There exists a normal approximation $F(D_\epsilon)$ of a feasible solution set $F(D)$.

Proof. Let the criteria values be calculated at N points of the P_τ-net; denote those which are feasible by D_N and those which are EP-optimal by P_N. Let N'_1 be a subset of N specified points whose images are either EP-optimal or belong to the ϵ-neighborhood of an EP-optimal point $F(b)$ with respect to at least one criterion. Furthermore, let $N'_2 = D_N \backslash N'_1$; let those of the N points which satisfy the functional and criteria constraints to an accuracy of ϵ be added to D.

Step 1. Consider an arbitrary point $\bar{a} \in N'_2$. Let $b \in P_N$

$$K_\nu^b = |F_\nu(\bar{a}) - F_\nu(b) - \epsilon_\nu|.$$

If $\bar{a} \in D$, then $F_\nu(b)$ is replaced by F_ν^{**}. Now, choose $\bar{a}$ as the center of a cube $K_{\bar{a}}$ whose edge length is $2K_\nu^b/L_\nu$. Then, for any $a \in K_{\bar{a}}$,

$$|F_\nu(\bar{a}) - F_\nu(a)| \leq L_\nu \max_j |\bar{a}_j - a_j| \leq K_\nu^b.$$

If the edge length of the cube is $\min_\nu 2K_\nu^b/L_\nu$, then the latter inequality holds for all ν. Perform this operation for all points of P_N, and choose $K_{\bar{a}}$ with edge length $\min_{b \in P_N} \min_\nu 2K_\nu^b/L_\nu$. We thus arrive at a cube with center at $\bar{a}$ such that

$$K_{\bar{a}} \cap N'_1 = \phi.$$

Construct the cube for all $\bar{a} \in N'_2$, and form the union

$$K_1 = \bigcup_{\bar{a} \in N'_2} K_{\bar{a}}.$$

Next, choose a point $\hat{a} \in N'_1$ and construct a cube $K_{\hat{a}}$ with center $\hat{a}$ and edge length $\min_\nu 2\epsilon_\nu/L_\nu$. Then, the inequality

$$|F_\nu(\hat{a}) - F_\nu(a)| \leq L_\nu \max_j |\hat{a}_j - a_j| \leq \epsilon_\nu, \qquad \nu = 1, \ldots, k,$$

is satisfied for all $a \in K_{\hat{a}}$. Let

$$K_2 = \bigcup_{a \in N'_1} K_{\hat{a}} \quad \text{and} \quad K^1 = K_1 \cup K_2.$$

Step 2. Consider the complement $K^r \backslash K^1$, where K^r is the initial parallelepiped. Since K^1 is a union of cubes, $K^r \backslash K^1$ may be represented as a finite number of nonintersecting parallelepipeds. Define K_i^1 and $K^2 = \bigcup_i K_i^1$ in a similar fashion for all of the parallelepipeds Π_i above, and the result is the region $K^1 \cup K^2$, again the union of a finite number of cubes. The most promising points of the region, belonging to N_1, are approximated to ϵ-accuracy. The remaining points are approximated to the worse accuracy $\bar{\epsilon}$ and are of no interest in constructing the EP-optimal set. It should be noted that the EP-optimal set as computed in Step 2 must be formed from the union with the EP-optimal set as obtained in Step 1 and the set of feasible points added in Step 2.

Step m. The mth step is performed in a similar way. After n steps and the determination of K^i, $i = 1, \ldots, n$, we obtain

$$K^r \setminus \bigcup_{i=1}^{n} K^i = \varnothing.$$

That is, we have covered all of the region K^r with a union of cubes whose points are approximated to the desired accuracy. This completes the proof. □

We emphasize that much of the previous proof is patterned after similar proofs given by other authors in connection with single-criterion problems. However, we were able to obtain a faster algorithm by using a set of functions satisfying a special Lipschitz condition with

$$\sum_{j=1}^{r} L_v^j \leq L_v' \quad \text{or} \quad \sum_{j=1}^{r} L_v^j \leq L_v^r.$$

Hence, the cubes covering the region K^r will be larger than for the case of functions subjected only to a standard Lipschitz condition, and the whole cube will be covered more economically. Finally, the use of highly uniform P_τ-nets also improves the rate of approximation of the feasible region.

7. Approximation of the EP-Optimal Set

The EP-optimal set is computationally unstable, so that even slight errors in the criteria $F_v(a)$ may lead to a drastic change in the set. Thus, the approximation of the feasible set with a given accuracy is not sufficient to guarantee a corresponding approximation of the EP-optimal set.

This instability of the EP-optimal set is one of the major difficulties encountered in attempting to obtain an approximation thereto. Although treatments of the problem appeared as early as 1950, a complete solution acceptable for the majority of practical problems is still outstanding. Nevertheless, promising methods have been proposed for some classes of functions (Refs. 3, 4, 14).

For problems which are neither linear nor concave, the majority of the approximation methods for the EP-optimal set may be divided into two classes. The first class incorporates methods based on the minimization of various functions (Refs. 5, 15–17). In Ref. 18, the so-called ill-posed problem for the approximation of the EP-optimal set is analyzed. The proposed solution there is based on a Hausdorff metric. Further interesting results relating to the applications of these methods are obtained in Refs. 19–22.

The second class consists of methods based on the covering of the feasible solution set with subsets of a special shape (Refs. 23 and 24): cubes, spheres, and the like. Our approach falls within this second category.

The approach is not based on the use of a Hausdorff metric, nor is it based on the imposition of constraints on the designer preferences. We shall use uniformly distributed sequences of points which allow us to conjecture that the resulting algorithms for the approximation of the EP-optimal set are among the fastest ones. We shall require only the continuity of the criteria and the satisfaction of Lipschitz conditions (Ref. 8).

We pose the following general problem. Let P be the EP-optimal set in the design space, let $F(P)$ be its image, and let ϵ be a set of admissible errors. It is desired to construct a finite EP-optimal set $F(P_\epsilon)$ approximating $F(P)$ with ϵ-accuracy.

Let $F(D_\epsilon)$ be the ϵ-approximation of $F(D)$, and let P_ϵ be the EP-optimal subset of D_ϵ. Then, $F(P_\epsilon)$ generally does not approximate $F(P)$ with the same accuracy. This is due to the previously mentioned computational instability of approximations of the EP-optimal set. Such problems are said to be ill-posed in the sense of Tikhonov (Ref. 25). We recall this notion here, since our approach is similar.

Let F be a functional on X, $F: X \to Y$. We suppose that there exists

$$y^* = \inf F(x),$$

with $N_\epsilon(y^*)$ as a neighborhood of the required solution y^*. Now, select an element x^* or a set of elements from the space X and its δ-neighborhood $N_\delta(x^*)$, and call x_δ^ϵ a solution of the problem of finding the extremum of F when the solution satisfies both $x_\delta^\epsilon \in N_\delta(x^*)$ and $F(x_\delta^\epsilon) \in N_\epsilon(y^*)$. If at least one of these conditions is not satisfied for arbitrary values of ϵ and δ, then the problem is called ill-posed in the sense of Tikhonov.

An analogous definition may be formulated for the case where P is an operator mapping X into Y. Define

$$X = \{F(D_\epsilon), F(D)\}, \qquad Y = \{F(P_\epsilon), F(P)\},$$

with ϵ eventually arbitrarily small, and let $P: X \to Y$ be an operator relating any element of X to its EP-optimal subset. Then, the problem of constructing sets $F(D_\epsilon)$ and $F(P_\epsilon)$ belonging simultaneously to the ϵ-neighborhoods of $F(D)$ and $F(P)$ is ill posed. Of course, the metric of topology introduced on the spaces X and Y should be compatible with the preferences on $F(D)$.

Define the N_ϵ-neighborhood of a point $F(a^0) \in F(\Pi)$ by

$$N_\epsilon = \{F(a) \in F(\Pi): |F_v(a^0) - F_v(a)| \leqslant \epsilon_v,\ v = 1, \ldots, k\}.$$

Next, we construct an EP-optimal set $F(P_\epsilon)$ such that, for any point $F(a^0) \in F(P)$ and any of its ϵ-neighborhoods N_ϵ, there exists a point $F(b) \in F(P_\epsilon)$ belonging to N_ϵ. Conversely, the ϵ-neighborhood of any point $F(b) \in F(P_\epsilon)$ must contain a point $F(a^0) \in F(P)$. Then, the set $F(P_\epsilon)$ is called an approximation possessing the M-property. An approximation $F(P_\epsilon)$ possess the M_1-property iff, for any point $F(a^0) \in F(P)$ and any of its neighborhoods N_ϵ, there exists a point $F(b) \in F(P_\epsilon)$ belonging to N_ϵ.

Suppose that an approximation $F(D_\epsilon)$ of $F(D)$ has been constructed.

Lemma 7.1. See Ref. 8. If the criteria $F_v(a)$ are continuous and satisfy a Lipschitz condition, then there exists an approximation $F(P'_\epsilon)$ possessing the M_1-property.

Proof. The proof will be based on the analysis of likely points of $F(D_\epsilon)$, that is, points whose neighborhoods are the most likely to contain the actual EP-optimal vectors. If a neighborhood of a likely point yields new EP-optimal vectors, these may be added to $F(P_\epsilon)$. Taken together with $F(P_\epsilon)$, they form the ϵ-approximation of the EP-optimal set.

We begin with the determination of a set of likely points. Consider $F(a) \in F(P_\epsilon)$, and let

$$M_a^+ = \{F(b) \in F(D_\epsilon): \forall v F_v(b) \geqslant F_v(a)\},$$

$$M_a^\epsilon = \{F(b) \in M_a^+ : \exists v F_v(b) - F_v(a) \leqslant \epsilon_v/2\}.$$

Here, we have arbitrarily chosen $\epsilon_v/2$. Any number less than ϵ_v is equally acceptable. Let

$$M_a^- = D(D_\epsilon) \backslash (M_a^+ \cup F(P_\epsilon)),$$

$$M_a = M_a^\epsilon \cup M_a^-, \qquad M = \bigcup_a M_a.$$

Now, consider a point $F(a^0) \in M$. If $F(P_\epsilon)$ contains no point $F(b)$ such that

$$F_v(b) \leq F_v(a^0) - \epsilon_v,$$

for any ν, then $F(a^0)$ is termed a likely point. The set of such likely points will be denoted by LM. Clearly, points of $F(P)$ which are not approximated by the set $F(P_\epsilon)$ with ϵ-accuracy must lie in ϵ-neighborhoods of vectors from LM. Thus, if we construct a cube $\mathfrak{L}_{a^1}$ with center at a^1 and edge length $\min_\nu 2\epsilon_\nu/L_\nu$, $\nu=1,\ldots,k$, for any point $a^1 \in D_\epsilon$ such that $F(a^1) \in LM$, then the cube may contain EP-optimal points from $F(P)$, approximated with ϵ-accuracy, none of which are points of $F(P_\epsilon)$.

Let ϵ'_ν denote negligible errors. Use P_τ-net points to approximate $F(\mathfrak{L}_{a^1}) \cap F(D)$ to ϵ'-accuracy, as before. Since the volume $\mathfrak{L}_{a^1}$ is quite small compared to K^r, the number of points needed to attain the approximation is rather small. At least one of the points of the P_τ-net in $\mathfrak{L}_{a^1}$ belongs to the neighborhood $N \subset \mathfrak{L}_{a^1}$ of an EP-optimal point $F(a^0)$, provided such a point exists; denote it by $F(g)$. Thus, if $F(a^0)$ is an EP-optimal point, then $F(g)$ definitely improves the value of at least one criterion for an arbitrary point $F(a) \in F(P_\epsilon)$. If such a point exists, it is added to $F(P_\epsilon)$. Conversely, if $\mathfrak{L}_{a^1}$ does not contain a point $F(a^0) \in F(P)$ within the approximation ϵ', then the operation is repeated for all vectors belonging to LM.

Let

$$\bigcup_i F(g^i) \cup F(P_\epsilon) = F(P'_\epsilon),$$

and let

$$\bigcup_i g^i \cup D_\epsilon = D'_\epsilon,$$

where g^i is a point obtained by the preceding procedure. Then, $F(P'_\epsilon)$ may contain points which are not EP-optimal and are thus to be discarded. Consequently, we arrive at the set $F(P'_\epsilon)$, an EP-optimal subset of $F(\bigcup_i g^i \cup D_\epsilon)$, and an approximation of $F(P)$. This completes the proof. □

The approximation $F(P'_\epsilon)$ obtained in this fashion possesses the M_1-property. However, the inverse statement is generally not true, since $F(P'_\epsilon)$ may contain additional points whose analysis would be fruitless.

We now refine the previous result. The ϵ-approximation of the EP-optimal set $F(P'_\epsilon)$, as constructed in Lemma 7.1, has the M_2-property if and only if there exists a point $F(b) \in F(P)$ within an ϵ-neighborhood of any point $F(a) \in F(P_\epsilon)$.

Lemma 7.2. There exists a subset $F(P''_\epsilon)$ of the set $F(P'_\epsilon)$ which possesses the M_2-property.

Proof. Let $F(a) \in F(P'_\epsilon)$, let B be an arbitrary subset of $\{1, \ldots, k\}$, and let

$$V_{F(a)} = \{F(b) \in F(P'_\epsilon): \forall v \in B, F_v(a) \leqslant F_v(b) \leqslant F_v(a) + \epsilon_v,$$
$$\text{and } \forall v \in \{1, \ldots, k\} \setminus B, F_v(b) \leqslant F_v(a) - \epsilon_v\}.$$

As before, we begin with the investigation of the neighborhoods of points a and b for which $F(b) \in V_{F(a)}$.

Suppose that the condition

$$\forall \tilde{a} \in \mathfrak{L}_a \cap D, \exists g \in \mathfrak{L}_b \cap D \text{ such that } F_v(g) \leqslant F_v(\tilde{a}) + \epsilon'_v, \qquad \forall v \in B,$$

for sufficiently small ϵ'_v, is satisfied for all P_τ-net points belonging to the cubes $\mathfrak{L}_a$, $\mathfrak{L}_b$ and satisfying the previous requirement for the approximation of the feasible solution set. Then, the point $F(a)$ may be excluded from $F(P'_\epsilon)$, since its ϵ-neighborhood will not contain any points from $F(P)$. By carrying out a similar procedure for all $F(a) \in F(P'_\epsilon)$, we arrive at the set $F(P''_\epsilon) \subset F(P'_\epsilon)$ possessing the required M_2-property. □

The preceding two lemmas, Lemmas 7.1 and 7.2, may be combined to yield the following theorem.

Theorem 7.1. See Ref. 8. $F(P''_\epsilon)$ is an approximation of the EP-optimal set $F(P)$ possessing the M-property.

Thus, we have constructed the desired approximation of the EP-optimal set. As previously mentioned, the problem of approximating the EP-optimal set is ill posed in the sense of Tikhonov, and one must thus select the metric/topology which is suited to the solution space as well as the system of preferences on $F(D)$.

Let us pursue this idea in more detail. Clearly, for each concrete case, one has to consider a metric which suits the mathematical model of the physical problem as well as the preferences. There is no single metric which can adequately represent every preference system. Even if such a metric were to exist, its construction would generally be more complicated than the construction of the approximation of the EP-optimal set. However, one may introduce a topology which is well suited to the preference system on $F(D)$.

Define a topology on X (Ref. 26) by specifying the neighborhood

$$W_\epsilon(x) = \{F(D_\epsilon): \forall F(a) \in X, \exists F(b) \in F(D_\epsilon): |F_v(a) - F_v(b)| \leqslant \epsilon_v,$$
$$\text{and } \forall F(g) \in F(D_\epsilon), \exists F(h) \in X: |F_v(g) - F_v(h)| \leqslant \epsilon_v, \ v = 1, \ldots, k\},$$

for every $x \in X$. A neighborhood and hence a topology are now established for Y, $F(D) \subset Y$, in a similar fashion. The topology that we have introduced is a Hausdorff topology satisfying the second countability axiom. Convergence may thus be described in terms of sequences.

It is well known that the solution of an ill-posed problem reduces to the construction of a regularizing sequence. In the present case, this is a sequence of sets $F(P_{\epsilon^j}), j=1, \ldots, \infty$, such that, for the corresponding sequence $F(D_{\epsilon^j})$ and any ϵ^j-neighborhoods of the sets $F(P)$ and $F(D)$, the sets $F(P_{\epsilon^j})$ and $F(D_{\epsilon^j})$, beginning with some j_0, belong to the respective neighborhoods.

Suppose now that sequences $F(P''_{\epsilon^j})$ and $F(D''_{\epsilon^j})$, $P''_{\epsilon^j} \subset D''_{\epsilon^j}$, as introduced in Lemmas 7.1 and 7.2, are constructed for the sequences of sets $\epsilon^j, j=1, \ldots, \infty$.

Theorem 7.2. The sequence $F(P''_{\epsilon^j})$ is regularizing.

Proof. Based on the M-property formulated above, valid for any term of the sequence $F(P''_{\epsilon^j})$, and based on the definition of the neighborhoods $W_\epsilon(F(P))$ and $W_\epsilon(F(D))$, it follows directly that the regularizability conditions are such for this sequence. □

Remark 7.1. In other references (Ref. 18, for example), the problem of regularizing the approximation of the EP-optimal set is solved by using a Hausdorff metric defined by the distance

$$d(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \rho(a, b) \text{ or } \sup_{b \in B} \inf_{a \in A} \rho(a, b)\right\},$$

$$\text{where } \rho(a, b) = \max_{v} |a_v - b_v|,$$

and where a_v, b_v are the coordinates of vectors a and b, with $A, B \subset X$.

The class of problems amenable to the use of this metric is rather limited, since it may distort the designer preference system, because of the different significance of the performance criteria (for example) and because changes in ρ may affect convergence. This poses the questions: Why use a Hausdorff metric generated by some distance $\rho(a, b)$? Why not introduce a topology as was done above? This topology is a generalization of the Hausdorff metric. Roughly speaking, it operates in the same fashion as the Hausdorff metric, without however distorting the designer preference system.

The method that we have proposed may be used for the simultaneous approximation of feasible set and EP-set both in criteria space and in parameter space. One only needs to introduce additional criteria $F_{k+j} = a_j$, where

a_j is the jth parameter (design variable) of the system, $j=\overline{1, r}$. This approach is treated in detail in Ref. 8.

8. Conclusions

The PSI method and methods of approximation provide solutions of multicriteria optimization problems that are completely acceptable for many applications.

References

1. STATNIKOV, R. B., *Solution of Multicriteria Machine Design Problems on the Basis of the Parameter Space Investigation*, Multicriteria Decision-Making Problems, Edited by J. M. Gvishiani and S. V. Yemelyanov, Mashinostroyeniye, Moscow, Russia, pp. 148–155, 1978 (in Russian).
2. SOBOL', I. M., and STATNIKOV, R. B., *Statement of Some Problems of Computer-Aided Optimal Design*, Keldysh Institute of Applied Mathematics, Moscow, Russia, 1977 (in Russian).
3. OZERNOY, V. M., *Multiple-Criteria Decision Making in the USSR: A Survey*, Naval Research Logistics, Vol 35, pp. 543–566, 1988.
4. STADLER, W., and DAUER, J. P., *Multicriteria Optimization in Engineering: A Tutorial and Survey*, Structural Optimization: Status and Promise, Edited by M. P. Kamat, Progress in Aeronautics and Astronautics, AIAA, Washington, DC, Vol. 150, pp. 209–249, 1992.
5. DYER, J. S., FISHBURN, P. C., STEUER, R. E., WALLENIUS, J., and ZIONTS, S., *Multiple-Criteria Decision Making, Multiattribute Utility Theory: The Next Ten Years*, Management Science, Vol. 38, pp. 645–654, 1992.
6. SOBOL, I. M., and STATNIKOV, R. B., *The Choice of Optimal Parameters in Multicriteria Problems*, Nauka, Moscow, Russia, 1981 (in Russian).
7. SOBOL, I. M., and STATNIKOV, R. B., *The Best Solutions: Where They May Be Found*, Znaniye, Moscow, Russia, 1982 (in Russian).
8. STATNIKOV, R. B., and MATUSOV, I. B., *Multicriteria Machine Design*, Znaniye, Moscow, Russia, 1989 (in Russian).
9. STADLER, W., *Fundamentals of Multicriteria Optimization*, Multicriteria Optimization in Engineering and in the Sciences, Edited by W. Stadler, Plenum Press, New York, New York, pp. 1–25, 1988.
10. STATNIKOV, R. B., and MATUSOV, I. B., *General-Purpose Finite-Element Programs in Search for Optimal Solutions*, Doklady Rossiyskoy Akademii Nauk, Vol. 336, pp. 441–443, 1994 (in Russian).
11. SOBOL, I. M., *On Functions Satisfying the Lipschitz Condition in Multidimensional Problems of Computational Mathematics*, Doklady AN SSSR, Vol. 293, pp. 1314–1319, 1987 (in Russian).

12. Sobol, I. M., *Multidimensional Quadrature Formulas and Haar Functions*, Nauka, Moscow, Russia, 1969 (in Russian).
13. Matusov, I. B., and Statnikov, R. B., *Approximation and Vector Optimization of Large Systems*, Doklady AN SSSR, Vol. 296, pp. 532–536, 1987 (in Russian).
14. White, D. J., *A Bibliography on the Applications of Mathematical Programming: Multiple-Objective Methods*, Journal of the Operations Research Society, Vol. 8, pp. 669–691, 1990.
15. Steuer, R. E., and Choo, E. U., *An Interactive Weighted Tchebycheff Procedure for Multiple-Objective Programming*, Mathematical Programming, Vol. 26, pp. 326–344, 1983.
16. Benson, H. P., *A Finite, Nonadjacent Extreme-Point Search Algorithm for Optimization over the Efficient Set*, Journal of Optimization Theory and Applications, Vol. 73, pp. 47–64, 1992.
17. Merkuriev, V. V., and Moldavskii, M. A., *A Family of Convolutions of a Vector-Valued Criterion for Finding Points in the Pareto Set*, Avtomatika i Telemekhanika, No. 1, pp. 110–121, 1979 (in Russian).
18. Molodtsov, D. A., and Fedorov, V. V., *Stability of Optimality Principles*, Modern State of Operations Research Theory, Edited by N. N. Moiseyev, Nauka, Moscow, Russia, pp. 236–262, 1979 (in Russian).
19. Eschenauer, H. A., *Multicriteria Optimization Techniques for Highly Accurate Focusing Systems*, Multicriteria Optimization in Engineering and in the Sciences, Edited by W. Stadler, Plenum Press, New York, New York, pp. 309–354, 1988.
20. Koski, J., *Multicriteria Truss Optimization*, Multicriteria Optimization in Engineering and in the Sciences, Edited by W. Stadler, Plenum Press, New York, New York, pp. 263–308,1988.
21. Ester, J., *Some Applications of MCDM to Engineering Problems*, Operations Research Spectrum, Vol. 9, pp. 59–80, 1987 (in German).
22. Stadler, W., and Krishman, V., *Natural Structural Shapes for Shells of Revolution in the Membrane Theory of Shells*, Structural Optimization, No. 1, pp. 19–27, 1989.
23. Sukharev, A. G., *Optimal Search for an Extremum*, Zhurnal Vychislitelnoy Matematiki i Matematicheskoy Fiziki, Vol. 11, pp. 265–269, 1971 (in Russian).
24. Yevtushenko, Yu. G., and Mazurik, V. P., *Software for Optimization Systems*, Znaniye, Moscow, Russia, 1989 (in Russian).
25. Vasil'ev, F. P., *Methods of Solving Extremum Problems*, Nauka, Moscow, Russia, 1981 (in Russian).
26. Kelley, J. L., *General Topology*, Van Nostrand Reinhold, New York, New York, 1957.