Convergence of a PI Coordination Protocol in Networks with Switching Topology and Quantized Measurements

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***Abstract*—This paper analyzes the convergence properties of a proportional-integral protocol for coordination of a network of agents with dynamic information flow and quantized measurements. In the setup adopted, each agent is only required to exchange its coordination state with its neighboring agents, and the desired reference rate is only available to a group of leaders. We show that the integral term of the protocol allows the agents to learn the reference rate, rather than have it available a priori, and also provides disturbance rejection capabilities. The paper addresses the convergence of the collective dynamics when the graph that captures the network topology is not connected during some interval of time or even fails to be connected at all times.**

I. Introduction

Worldwide, there has been growing interest in the use of autonomous vehicles to execute complex missions without constant supervision of human operators. A key enabling element for the execution of such missions is the availability of advanced systems for motion control of autonomous vehicles.

In [1], for example, the authors address the development of robust strategies for cooperative missions in which a fleet of fixed-wing UAVs is required to follow collision-free paths and arrive at their respective final destinations at the same time, or at different times so as to meet a desired inter-vehicle schedule. The decentralized protocol used for group coordination, which was first introduced in [2], has a proportional-integral (PI) structure in which each agent is only required to exchange its coordination state with its neighbors, and the constant reference rate is only available to a single leader. The integral term in the consensus algorithm allows the follower UAVs to learn the reference rate from the leader, and also provides disturbance rejection capabilities due to, for instance, winds.

A generalization of this PI protocol was proposed in [3], where the authors developed an adaptive algorithm to reconstruct a time-varying reference velocity that is available only to a single leader. The paper used a passivity framework to show that a network of nonlinear agents with fixed connected topology asymptotically achieves coordination. The work in [4] also used a (discrete-time) PI consensus protocol to synchronize networks of clocks with fixed connected information flow. In this application, the integral part of the controller was critical to eliminate the different initial clock offsets.

This paper modifies the PI protocol in [1], [2] to include *multiple leaders*, and analyzes the convergence properties of the protocol for coordination of a network of agents with *dynamic information flow* and *quantized measurements*, a topic that has received increased attention in recent years [5]–[9]. On one hand, the use of multiple leaders in the protocol improves the robustness of the network to a single-point failure. On the other hand, the use of finite-rate communication links and/or coarse sensors motivates the interest in quantized consensus problems. The main contribution of this paper is twofold. First, we present lower bounds on the convergence rate of the collective dynamics as a function of the number of leaders and the *quality of service* (QoS) of the network, which in the context of this work represents a measure of the level of connectivity of the dynamic graph that captures the underlying network topology. And second, we analyze the existence of equilibria as well as the convergence properties of the collective dynamics under quantized feedback.

The paper is organized as follows. Section II describes the problem formulation. Section III presents the PI protocol adopted in this paper and analyzes its convergence properties. In Section IV, we study the collective dynamics under quantization. Simulation results are presented in Section V, while Section VI summarizes concluding remarks.

II. Problem Formulation

Consider a network of n *integrator-agents*

$$\dot{x}_i(t) = u_i(t) + d_i, \; x_i(0) = x_{i0}, \quad i \in \mathcal{I}_n := \{1, \ldots, n\}, \quad (1)$$

with *dynamic information flow* $\mathcal{G}_0(t) := (\mathcal{V}_0, \mathcal{E}_0(t))$. In the above formulation, $x_i(t) \in \mathbb{R}$ is the *coordination state* of the ith agent, $u_i(t) \in \mathbb{R}$ is its control input, and $d_i \in \mathbb{R}$ is an unknown constant disturbance.

The control objective is to design a *distributed protocol* that asymptotically solves the following *coordination problem*:

$$x_i(t) - x_j(t) \overset{t\to\infty}{\longrightarrow} 0\,, \qquad \forall\, i,j \in \mathcal{I}_n\,, \quad (2a)$$

$$\dot{x}_i(t) \overset{t\to\infty}{\longrightarrow} \rho\,, \qquad \forall\, i \in \mathcal{I}_n\,, \quad (2b)$$

where ρ is the desired (constant) reference rate.

The network and the communications between agents satisfy the following assumptions:

Assumption 1: The ith agent can only exchange information with a set of neighboring agents, denoted by $\mathcal{N}_i^0(t)$.

Assumption 2: Communications between two agents are bidirectional ($\mathcal{G}_0(t)$ is undirected) and the information is transmitted continuously with no delays.

Assumption 3: The connectivity of the graph $\mathcal{G}_0(t)$ at time t satisfies the persistency of excitation (PE)-like condition

$$\frac{1}{n}\frac{1}{T}\int_t^{t+T} \boldsymbol{Q_n L_0}(\tau)\boldsymbol{Q_n^\top} d\tau \geq \mu \mathbb{I}_{\boldsymbol{n-1}}\,, \qquad \forall\, t \geq 0\,, \quad (3)$$

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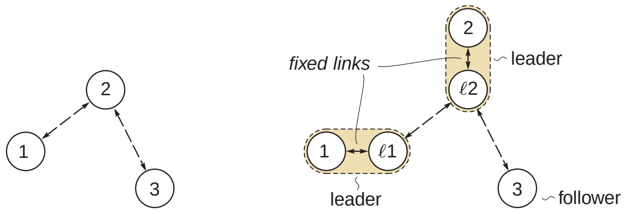


(a) Original network. (b) Network with two virtual agents.

Fig. 1. Addition of $n_\ell = 2$ virtual agents in a network of $n = 3$ agents.

where $\boldsymbol{L_0}(t) \in \mathbb{R}^{n\times n}$ is the piecewise-constant Laplacian of the graph $\mathcal{G}_0(t)$, and $\boldsymbol{Q_n}$ is any $(n-1)\times n$ matrix satisfying $\boldsymbol{Q_n}\boldsymbol{1_n} = \boldsymbol{0}$ and $\boldsymbol{Q_n}\boldsymbol{Q_n}^\top = \mathbb{I}_{\boldsymbol{n-1}}$, with $\boldsymbol{1_n}$ being the vector in $\mathbb{R}^n$ whose components are all 1. The parameters $T > 0$ and $\mu \in [0, 1]$ characterize the QoS of the communications network, which in the context of this paper represents a measure of the level of connectivity of the dynamic graph $\mathcal{G}_0(t)$.

Remark 1: The PE-like condition (3) requires only the graph $\mathcal{G}_0(t)$ to be connected in an integral sense, not pointwise in time. In fact, the graph may be disconnected during some interval of time or may even fail to be connected at all times. Similar type of conditions can be found in [10] and [11].

III. Distributed Consensus Protocol

A. Addition of Virtual Leaders

The consensus protocol adopted in this paper introduces n_ℓ *virtual agents* ($1 \le n_\ell \le n$) in the network, associated with n_ℓ agents. These virtual agents are implemented in n_ℓ distinct agents and have the following dynamics:

$$\dot{x}_{\ell i}(t) = u_{\ell i}(t),\ \ x_{\ell i}(0) = x_{\ell i0}, \qquad i \in \mathcal{I}_\ell := \{1, \dots, n_\ell\},$$

where the virtual control laws $u_{\ell i}(t)$, $i \in \mathcal{I}_\ell$, are yet to be defined. Without loss of generality, we assume that these virtual agents are implemented in agents 1 to n_ℓ, that is, the ith virtual agent is implemented in the ith agent. In the context of this paper, these n_ℓ agents are referred to as *leaders*, while the remaining agents are *followers*.

To limit the amount of information transmitted over the network, each leader is only allowed to exchange the state of its virtual agent with its neighbors; in other words, the ith leader can only transmit the state $x_{\ell i}(t)$, rather than transmitting both $x_{\ell i}(t)$ and $x_i(t)$. Finally, we note that the agent and the virtual agent of a leader can exchange information uninterruptedly, as these two agents do not communicate over the network. Figure 1 presents an example illustrating the addition of two virtual agents in a network of three agents.

The inclusion of these n_ℓ virtual agents results in a new *extended network* of $N := n + n_\ell$ agents with a new dynamic topology $\mathcal{G}(t)$. According to the description above, this new topology is characterized by the following neighboring sets:

$$\begin{aligned}
\mathcal{N}_i &:= \{\ell i\}, && i \in \mathcal{I}_\ell, \\
\mathcal{N}_i(t) &:= (\mathcal{N}_i^0(t) \setminus \mathcal{I}_\ell) \cup \mathcal{L}_i(t), && i \notin \mathcal{I}_\ell, \\
\mathcal{N}_{\ell i}(t) &:= (\mathcal{N}_i^0(t) \setminus \mathcal{I}_\ell) \cup \mathcal{L}_i(t) \cup \{i\}, && i \in \mathcal{I}_\ell,
\end{aligned}$$

where the vertex set $\mathcal{L}_i(t)$ is defined as

$$\mathcal{L}_i(t) := \{\ell j \ : \ j \in (\mathcal{N}_i^0(t) \cap \mathcal{I}_\ell)\}.$$

The Laplacian $\boldsymbol{L}(t)$ of the new extended graph with vertex set $\mathcal{V} := \{\ell 1, \dots, \ell n_\ell, 1, \dots, n\}$ is given by

$$\boldsymbol{L}(t) = \boldsymbol{P_\ell}^\top \left[\begin{smallmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L_0}(t) \end{smallmatrix}\right] \boldsymbol{P_\ell} + \boldsymbol{L_v} \quad \in \mathbb{R}^{N\times N},$$

where $\boldsymbol{P_\ell}$ is the $(0,1)$-*permutation matrix*

$$\boldsymbol{P_\ell} := \begin{bmatrix} \boldsymbol{0} & \mathbb{I}_{\boldsymbol{n_\ell}} & \boldsymbol{0} \\ \mathbb{I}_{\boldsymbol{n_\ell}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \mathbb{I}_{\boldsymbol{n-n_\ell}} \end{bmatrix} \quad \in \mathbb{R}^{N\times N};$$

while $\boldsymbol{L_v}$ is defined as

$$\boldsymbol{L_v} := \begin{bmatrix} \mathbb{I}_{\boldsymbol{n_\ell}} & -\mathbb{I}_{\boldsymbol{n_\ell}} & \boldsymbol{0} \\ -\mathbb{I}_{\boldsymbol{n_\ell}} & \mathbb{I}_{\boldsymbol{n_\ell}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \quad \in \mathbb{R}^{N\times N}.$$

The lemma below shows that the connectivity of the graph $\mathcal{G}(t)$ satisfies a PE-like condition similar to (3).

Lemma 1: Consider a network with n agents and n_ℓ virtual agents, added according to the description above. If the connectivity of the original network satisfies Assumption 3, then the connectivity of the extended network verifies

$$\frac{1}{N}\frac{1}{T}\int_t^{t+T} \boldsymbol{Q_N}\boldsymbol{L}(\tau)\boldsymbol{Q_N}^\top d\tau \ge \mu_{n_\ell}\mathbb{I}_{\boldsymbol{N-1}}, \qquad \forall\ t \ge 0, \quad (4)$$

where $\boldsymbol{Q_N}$ is any $(N-1)\times N$ matrix such that $\boldsymbol{Q_N}\boldsymbol{1_N} = \boldsymbol{0}$ and $\boldsymbol{Q_N}\boldsymbol{Q_N}^\top = \mathbb{I}_{\boldsymbol{N-1}}$; while the constant μ_{n_ℓ} characterizes the QoS of the extended network. Indeed, this paramater μ_{n_ℓ} can be determined recursively from the relation

$$\mu_i = F(n + i - 1, \mu_{i-1}), \qquad i = 1, \dots, n_\ell,$$

together with the initial condition $\mu_0 = \mu$, and

$$F(k, x_\mu) := \tfrac{(k x_\mu + 2) - \sqrt{(k x_\mu + 2)^2 - 4 x_\mu (k+1)}}{2(k+1)}.$$

Proof. The proof is given in Appendix A. □

B. Proportional-Integral Protocol

To solve the consensus problem (2), we adopt the protocol

$$u_{\ell i}(t) = k_P \sum_{j\in\mathcal{N}_{\ell i}} (x_j(t) - x_{\ell i}(t)) + \rho, \qquad i \in \mathcal{I}_\ell, \quad \text{(5a)}$$

$$u_i(t) = k_P \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)) + \chi_i(t), \qquad i \in \mathcal{I}_n, \quad \text{(5b)}$$

$$\dot{\chi}_i(t) = k_I \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)),\ \ \chi_i(0) = \chi_{i0}, \quad i \in \mathcal{I}_n, \quad \text{(5c)}$$

where $k_P > 0$ and $k_I > 0$ are positive coordination gains. This protocol has a PI structure in which each agent is only required to exchange its coordination state $x_\bullet(t)$ with its neighbors, and the reference rate ρ is only available to the n_ℓ leaders. We also note that the virtual agents adjust their dynamics according to information exchanged with their neighboring agents.

The protocol (5) can be rewritten in compact form as

$$\begin{aligned}
\boldsymbol{u}(t) &= -k_P \boldsymbol{L}(t)\boldsymbol{x}(t) + \begin{bmatrix} \rho \boldsymbol{1_{n_\ell}} \\ \boldsymbol{\chi}(t) \end{bmatrix}, \\
\dot{\boldsymbol{\chi}}(t) &= -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{x}(t), \qquad \boldsymbol{\chi}(0) = \boldsymbol{\chi_0},
\end{aligned} \quad (6)$$

where $\boldsymbol{u}(t)$, $\boldsymbol{x}(t)$, and $\boldsymbol{\chi}(t)$ are defined as

$$\begin{aligned}
\boldsymbol{u}(t) &:= [u_{\ell 1}(t), \dots, u_{\ell n}(t), u_1(t), \dots, u_n(t)]^\top \in \mathbb{R}^N, \\
\boldsymbol{x}(t) &:= [x_{\ell 1}(t), \dots, x_{\ell n}(t), x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^N, \\
\boldsymbol{\chi}(t) &:= [\chi_1(t), \dots, \chi_n(t)]^\top \in \mathbb{R}^n;
\end{aligned}$$

and $\boldsymbol{C}^\top := [\ \boldsymbol{0} \quad \mathbb{I}_{\boldsymbol{n}}\ ] \in \mathbb{R}^{n\times N}$.

C. Collective Dynamics and Convergence Analysis

Protocol (6) leads to the *closed-loop collective dynamics*

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= -k_P \boldsymbol{L}(t)\boldsymbol{x}(t) + \begin{bmatrix}\rho \boldsymbol{1}_{\boldsymbol{n_\ell}}\\ \boldsymbol{\chi}(t)+\boldsymbol{d}\end{bmatrix}, && \boldsymbol{x}(0) = \boldsymbol{x_0}\,,\\ \dot{\boldsymbol{\chi}}(t) &= -k_I \boldsymbol{C}^\top \boldsymbol{L}(t)\boldsymbol{x}(t)\,, && \boldsymbol{\chi}(0) = \boldsymbol{\chi_0}\,,\end{aligned}$$

where $\boldsymbol{d} := [d_1, \ldots, d_n]^\top \in \mathbb{R}^n$ is the disturbance vector. Note that the solutions (in the sense of Carathéodory [12]) of the collective dynamics above exist and are unique, since the Laplacian $\boldsymbol{L}(t)$ is piecewise constant in t.

To analyze the convergence properties of the algorithm (6), we reformulate the consensus problem (2) into a stabilization problem. To this end, we define the *projection matrix* $\boldsymbol{\Pi_N}$ as

$$\boldsymbol{\Pi_N} := \mathbb{I}_{\boldsymbol{N}} - \tfrac{\boldsymbol{1_N}\boldsymbol{1_N}^\top}{N} \quad \in \mathbb{R}^{N\times N}\,,$$

and note that the following equalities hold:

$$\boldsymbol{\Pi_N} = \boldsymbol{\Pi_N}^\top = \boldsymbol{\Pi_N}^2\,, \qquad \boldsymbol{Q_N}^\top \boldsymbol{Q_N} = \boldsymbol{\Pi_N}\,,$$
$$\boldsymbol{L}(t)\boldsymbol{\Pi_N} = \boldsymbol{\Pi_N}\boldsymbol{L}(t) = \boldsymbol{L}(t)\,.$$

Moreover, we have that the spectrum of the matrix

$$\bar{\boldsymbol{L}}(t) := \boldsymbol{Q_N}\boldsymbol{L}(t)\boldsymbol{Q_N}^\top \quad \in \mathbb{R}^{(N-1)\times(N-1)}$$

is equal to the spectrum of the extended Laplacian $\boldsymbol{L}(t)$ without the eigenvalue $\lambda_1 = 0$ corresponding to the eigenvector $\boldsymbol{1_N}$. Finally, we define the *consensus error state* $\boldsymbol{\zeta}(t) := [\boldsymbol{\zeta_1}^\top(t), \boldsymbol{\zeta_2}^\top(t)]^\top$ as

$$\begin{aligned}\boldsymbol{\zeta_1}(t) &:= \boldsymbol{Q_N}\boldsymbol{x}(t) && \in \mathbb{R}^{N-1}\,,\\ \boldsymbol{\zeta_2}(t) &:= \boldsymbol{\chi}(t) - \rho\boldsymbol{1_n} + \boldsymbol{d} && \in \mathbb{R}^n\,.\end{aligned}$$

Note that, by definition, $\boldsymbol{\zeta_1}(t) = \boldsymbol{\zeta_2}(t) = \boldsymbol{0}$ is equivalent to $\boldsymbol{x}(t) \in \operatorname{span}\{\boldsymbol{1_N}\}$ and $\dot{\boldsymbol{x}}(t) = \rho\boldsymbol{1_N}$.

With the above notation, the closed-loop collective dynamics can be reformulated as (see Appendix B)

$$\dot{\boldsymbol{\zeta}}(t) = \boldsymbol{A_\zeta}(t)\boldsymbol{\zeta}(t)\,, \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta_0}\,, \tag{7}$$

where $\boldsymbol{A_\zeta}(t) \in \mathbb{R}^{(N+n-1)\times(N+n-1)}$ is given by

$$\boldsymbol{A_\zeta}(t) := \begin{bmatrix} -k_P\bar{\boldsymbol{L}}(t) & \boldsymbol{Q_N}\boldsymbol{C}\\ -k_I\boldsymbol{C}^\top\boldsymbol{Q_N}^\top\bar{\boldsymbol{L}}(t) & \boldsymbol{0}\end{bmatrix}.$$

Next we show that, if the connectivity of the graph $\mathcal{G}_0(t)$ verifies the PE-like condition (3), then protocol (6) solves the consensus problem (2). The next theorem proves this result.

Theorem 1: Consider the closed-loop collective dynamics (7) and suppose that the topology $\mathcal{G}_0(t)$ verifies the PE-like condition (3) for some parameters μ and T. Then, there exist coordination gains k_P and k_I such that the inequality

$$\|\boldsymbol{\zeta}(t)\| \le \alpha_\zeta \|\boldsymbol{\zeta}(0)\|\, \mathrm{e}^{-\lambda_c t} \tag{8}$$

holds for some positive constant $0 < \alpha_\zeta < \infty$, and with

$$\lambda_c \ge \bar{\lambda}_c := \tfrac{k_P N \mu_{n_\ell}}{(1+k_P N T)^2}\left(1 + 2\tfrac{N}{n_\ell}\sqrt{\tfrac{N}{n_\ell}}\right)^{-1}.$$

Also, the coordination states and their rates of variation satisfy

$$\begin{aligned}\lim_{t\to\infty} |x_i(t) - x_j(t)| &= 0\,, && i, j \in \mathcal{I}_n\,,\\ \lim_{t\to\infty} \dot{x}_i(t) &= \rho\,, && i \in \mathcal{I}_n\,.\end{aligned} \tag{9}$$

Proof. The proof is given in Appendix C. □

Remark 2: Theorem 1 above indicates that the QoS of the network (characterized by T and μ) limits the achievable (guaranteed) rate of convergence of the closed-loop collective dynamics. According to the theorem, for a given QoS of the network, the maximum (guaranteed) rate of convergence $\bar{\lambda}_c^*$ is achieved by setting $k_P = \frac{1}{TN}$, which results in

$$\bar{\lambda}_c^* := \tfrac{\mu_{n_\ell}}{4T}\left(1 + 2\tfrac{N}{n_\ell}\sqrt{\tfrac{N}{n_\ell}}\right)^{-1}.$$

Also, it is important to mention that, as T goes to zero (graph connected pointwise in time), the convergence rate can be set arbitrarily fast by increasing the coordination gains k_P and k_I.

Finally, we notice that $\bar{\gamma}_c := \frac{k_P N \mu_{n_\ell}}{(1+k_P N T)^2}$ represents the (guaranteed) convergence rate for the collective dynamics with a proportional protocol (see proof of Theorem 1). One can verify that, for a given k_P, we have that $\bar{\lambda}_c < \bar{\gamma}_c$, which implies that a proportional protocol can provide higher rates of convergence than the PI protocol adopted. However, the integral term in the consensus algorithm is important as it allows the followers to learn the reference rate ρ from the leaders and also provides disturbance rejection capabilities.

IV. Convergence under Quantization

In this section we analyze the stability and performance characteristics of the distributed PI protocol presented in the previous section when the agents exchange quantized measurements. For the sake of simplicity, in this paper we consider only *uniform quantizers* with step size Δ.

A. Protocol and Collective Dynamics

When only quantized information from the other agents is available, the PI protocol introduced in (6) becomes

$$\begin{aligned}\boldsymbol{u}(t) &= -k_P\left(\tilde{\boldsymbol{D}}(t)\boldsymbol{x}(t) - \tilde{\boldsymbol{A}}(t)\,\mathrm{q}(\boldsymbol{x}(t))\right) + \begin{bmatrix}\rho\boldsymbol{1}_{\boldsymbol{n_\ell}}\\ \boldsymbol{\chi}(t)\end{bmatrix},\\ \dot{\boldsymbol{\chi}}(t) &= -k_I\boldsymbol{C}^\top\Big(\tilde{\boldsymbol{D}}(t)\boldsymbol{x}(t) - \tilde{\boldsymbol{A}}(t)\,\mathrm{q}(\boldsymbol{x}(t))\Big),\ \boldsymbol{\chi}(0) = \boldsymbol{\chi_0},\end{aligned} \tag{10}$$

where $\mathrm{q}(\boldsymbol{x}(t)) \in \mathbb{Z}^N\Delta$ is the quantized coordination state

$$\mathrm{q}(\boldsymbol{x}(t)) := [\,\mathrm{q}_\Delta(x_{\ell 1}(t)), \ldots, \mathrm{q}_\Delta(x_n(t))]^\top\,,$$

with $\mathrm{q}_\Delta(\cdot) : \mathbb{R} \to \mathbb{Z}\Delta$ being defined as

$$\mathrm{q}_\Delta(\xi) := \operatorname{sgn}(\xi)\Delta\left\lfloor \tfrac{|\xi|}{\Delta} + \tfrac{1}{2}\right\rfloor, \qquad \xi \in \mathbb{R}\,.$$

The time-varying matrices $\tilde{\boldsymbol{D}}(t)$ and $\tilde{\boldsymbol{A}}(t)$ are defined as

$$\tilde{\boldsymbol{D}}(t) := \boldsymbol{D}(t) + \boldsymbol{D_\ell}\,, \qquad \tilde{\boldsymbol{A}}(t) := \boldsymbol{A}(t) + \boldsymbol{D_\ell}\,,$$

where $\boldsymbol{D}(t)$ and $\boldsymbol{A}(t)$ are respectively the *degree* and *adjacency matrices* of $\boldsymbol{L}(t)$, while $\boldsymbol{D}_\ell$ is given by

$$\boldsymbol{D}_\ell := \begin{bmatrix}\boldsymbol{0} & -\mathbb{I}_{\boldsymbol{n_\ell}} & \boldsymbol{0}\\ -\mathbb{I}_{\boldsymbol{n_\ell}} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \quad \in \mathbb{R}^{N\times N}.$$

Note that only the information exchanged over the network is subject to quantization; in fact, each agent has access to its own unquantized state, and leaders also have access to the unquantized state of its virtual agent (and viceversa).

Then, noting that $\boldsymbol{L}(t) = \tilde{\boldsymbol{D}}(t) - \tilde{\boldsymbol{A}}(t)$, the collective dynamics can be written as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= -k_P\boldsymbol{L}(t)\boldsymbol{x}(t) + \begin{bmatrix}\rho\boldsymbol{1}_{\boldsymbol{n_\ell}}\\ \boldsymbol{\chi}(t)+\boldsymbol{d}\end{bmatrix}\\ &\quad + k_P\tilde{\boldsymbol{A}}(t)\boldsymbol{e_x}(t)\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x_0}\,,\\ \dot{\boldsymbol{\chi}}(t) &= -k_I\boldsymbol{C}^\top\boldsymbol{L}(t)\boldsymbol{x}(t) + k_I\boldsymbol{C}^\top\tilde{\boldsymbol{A}}(t)\boldsymbol{e_x}(t)\,, \quad \boldsymbol{\chi}(0) = \boldsymbol{\chi_0}\,,\end{aligned}$$

where $\boldsymbol{e_x}(t) := \mathrm{q}(\boldsymbol{x}(t)) - \boldsymbol{x}(t)$ is the *quantization error vector*. In terms of the consensus error state $\boldsymbol{\zeta}(t)$, the collective dynamics can be expressed as:

$$\dot{\boldsymbol{\zeta}}(t) = \boldsymbol{A}_{\boldsymbol{\zeta}}(t)\boldsymbol{\zeta}(t) + \boldsymbol{B}_{\boldsymbol{\zeta}}(t)\boldsymbol{e}_{\boldsymbol{x}}(t)\,, \qquad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta_0}\,, \tag{11}$$

where $\boldsymbol{A}_{\boldsymbol{\zeta}}(t)$ was introduced in (7) and $\boldsymbol{B}_{\boldsymbol{\zeta}}(t)$ is given by

$$\boldsymbol{B}_{\boldsymbol{\zeta}}(t) := \begin{bmatrix} k_P \boldsymbol{Q_N} \tilde{\boldsymbol{A}}(t) \\ k_I \boldsymbol{C}^\top \tilde{\boldsymbol{A}}(t) \end{bmatrix}.$$

Note that, in this case, the right-hand side of the collective dynamics is discontinuous not only due to the time-varying topology, but also due to the presence of quantized states. Moreover, as proven in [8], Carathéodory solutions might not exist for quantized consensus problems, implying that a weaker concept of solution has to be considered. Similar to [8], we will consider solutions *in the sense of Krasowskii* [12].

To show that Krasowskii solutions to (11) exist (at least) locally, we note that, during continuous evolution of the system between "quantization jumps", the network dynamics (11) are linear time varying, with the quantized state $\mathrm{q}(\boldsymbol{x}(t))$ acting as a bounded exogenous input. This implies that the solutions $\boldsymbol{x}(t)$ are locally bounded (no *finite escape time* occurs). Then, local existence of a Krasowskii solution is guaranteed by the fact that the right-hand side of (11) is measurable and locally bounded [12]. At this point, we cannot claim that Krasowskii solutions to (11) are complete; for this, we will need to prove that solutions are bounded (Theorem 2).

B. (Krasowskii) Equilibria

Before investigating the convergence properties of the quantized collective dynamics (11), in this section we analyze the existence of equilibria for these dynamics. To simplify the analysis, we assume (only in this section) that the topology of the network is static and connected. Under this assumption, one can easily show that the unquantized collective dynamics (7) have one isolated equilibrium point at $\boldsymbol{\zeta_{eq}} = \boldsymbol{0}$. However, when quantized information is exchanged over the network, $\boldsymbol{\zeta_{eq}} = \boldsymbol{0}$ is not an equilibrium point of the collective dynamics anymore and other (undesirable) equilibria might exist, depending on the step size of the quantizers.

To show this, we first notice that $\dot{\boldsymbol{\zeta}}(t) \equiv \boldsymbol{0}$ is equivalent to $\dot{\boldsymbol{x}}(t) \in \mathrm{span}\{\boldsymbol{1_N}\}$ and $\dot{\boldsymbol{\chi}}(t) \equiv \boldsymbol{0}$ holding simultaneously. This implies that $\boldsymbol{\zeta_{eq}} := [\boldsymbol{\zeta}_{\boldsymbol{1eq}}^\top, \boldsymbol{\zeta}_{\boldsymbol{2eq}}^\top]^\top$ is an equilibrium of (7) if

$$\gamma(t)\boldsymbol{1_N} \in \mathcal{K}\left(-k_P\left(\tilde{\boldsymbol{D}}\boldsymbol{x_{eq}}(t) - \tilde{\boldsymbol{A}}\,\mathrm{q}(\boldsymbol{x_{eq}}(t))\right) + \begin{bmatrix} \rho \boldsymbol{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{\chi_{eq}} + \boldsymbol{d} \end{bmatrix}\right),$$

$$\boldsymbol{0} \in \mathcal{K}\left(-k_I \boldsymbol{C}^\top\left(\tilde{\boldsymbol{D}}\boldsymbol{x_{eq}}(t) - \tilde{\boldsymbol{A}}\,\mathrm{q}(\boldsymbol{x_{eq}}(t))\right)\right),$$

where $\gamma(t) \in \mathbb{R}$ is an arbitrary signal; $\boldsymbol{x_{eq}}(t)$ is a continuous coordination-state trajectory satisfying $\boldsymbol{\zeta_{1eq}} = \boldsymbol{Q_N}\boldsymbol{x_{eq}}(t)$; while $\boldsymbol{\chi_{eq}} := \boldsymbol{\zeta_{2eq}} - \rho\boldsymbol{1_n} + \boldsymbol{d}$. The second inclusion above and continuity of $\boldsymbol{x_{eq}}(t)$, along with the fact that the network is assumed to be static and connected, preclude the existence of equilibria involving time-varying coordination-state trajectories, i.e. $\gamma(t) \equiv 0$ (or equivalently $\dot{\boldsymbol{x}}(t) \equiv 0$). Then, the set of (Krasowskii) equilibria of (7) can be defined as:

$$E := \Bigg\{ (\boldsymbol{x_{eq}}, \boldsymbol{\chi_{eq}}) \in \mathbb{R}^N \times \mathbb{R}^n \ : \ \boldsymbol{0} \in \mathcal{K}\left(\begin{bmatrix} -k_P\left(\tilde{\boldsymbol{D}}\boldsymbol{x_{eq}} - \tilde{\boldsymbol{A}}\,\mathrm{q}(\boldsymbol{x_{eq}})\right) + \begin{bmatrix} \rho\boldsymbol{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{\chi_{eq}} + \boldsymbol{d}\end{bmatrix} \\ -k_I\boldsymbol{C}^\top\left(\tilde{\boldsymbol{D}}\boldsymbol{x_{eq}} - \tilde{\boldsymbol{A}}\,\mathrm{q}(\boldsymbol{x_{eq}})\right) \end{bmatrix}\right) \Bigg\}. \tag{12}$$

Next, we show that, under sufficiently fine quantization, the set E is empty.

Lemma 2: Consider the quantized collective dynamics (11), and assume the network topology is static and connected. If the step size of the quantizers satisfies

$$\Delta < \frac{2n_\ell}{n(n-1)}\,\frac{|\rho|}{k_p}\,, \tag{13}$$

then the set of equilibria E is empty.

Proof. The proof is given in Appendix D. □

C. Convergence Analysis

Next we show that, if the connectivity of the graph $\mathcal{G}_0(t)$ verifies the PE-like condition (3), then the protocol (10) solves the consensus problem (2) *in a practical sense*. Moreover, the consensus error state degrades gracefully with the value of the quantizer step size. The next theorem summarizes this result.

Theorem 2: Consider the closed-loop collective dynamics (11) and suppose that the topology $\mathcal{G}_0(t)$ verifies the PE-like condition (3) for some parameters μ and T. Then, there exist coordination gains k_P and k_I ensuring that there is a finite time $T_b \geq 0$ such that the bounds

$$|x_i(t) - x_j(t)| \leq \alpha_\eta \Delta\,, \tag{14}$$

$$|\dot{x}_i(t) - \rho| \leq \alpha_\rho \Delta\,, \tag{15}$$

hold for all $t \geq T_b$ and some constants $\alpha_\eta, \alpha_\rho \in (0, \infty)$.

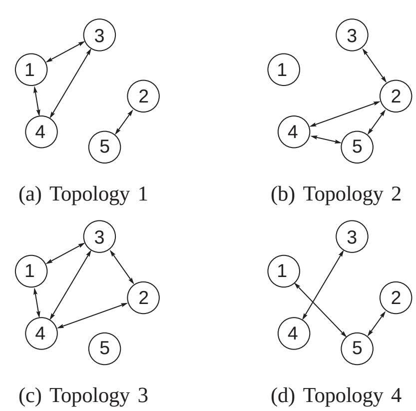
Proof. The proof is given in Appendix E. □

V. Simulation Results

In this section, we present simulation results that illustrate the theoretical findings of the paper. To this end, we consider a network of 5 agents with dynamics (1). At a given time t, the information flow of the network is characterized by one of the graphs in Figure 2; note that all four graphs are *not* connected. The control objective is to design a PI distributed protocol that solves the consensus problem (2) with $\rho = 1$ (in a practical sense). In all of the simulations, the initial coordination-state vector $\boldsymbol{x_0}$ and the disturbance vector $\boldsymbol{d}$ are given by

$$\boldsymbol{x_0} = \left[-1,\ 2,\ 4,\ -4,\ 3\right]^\top, \qquad \boldsymbol{d} = \left[0,\ 5,\ -3,\ 4,\ 1\right]^\top.$$

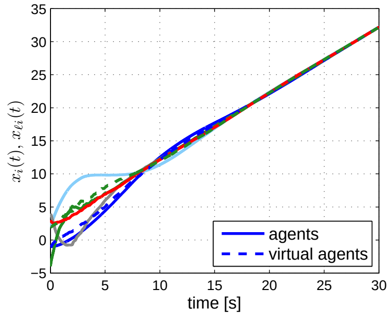
To solve the consensus problem, we add 2 virtual agents to the network, and implement the (quantized) protocol (10) with PI gains $k_P = 0.50$ and $k_I = 0.28$, and initial integrator state $\boldsymbol{\chi_0} = \boldsymbol{0}$. Figure 3 presents the computed evolution of the closed-loop collective dynamics with quantizer step



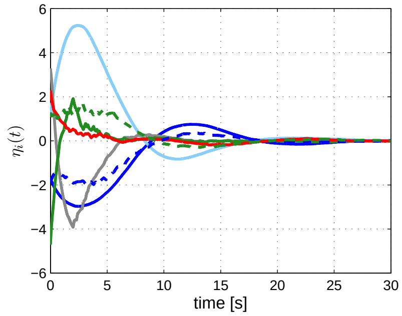
(a) Topology 1 (b) Topology 2

(c) Topology 3 (d) Topology 4

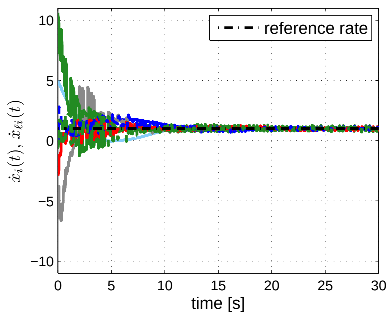
Fig. 2. Network topologies.



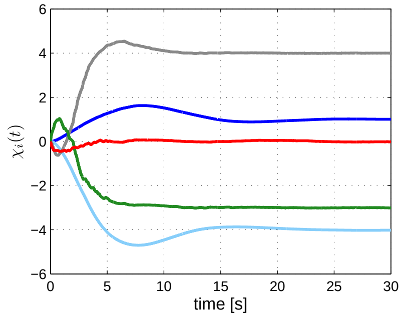
(a) Coordination states, $\boldsymbol{x}(t)$



(b) Disagreement vector, $\boldsymbol{\eta}(t)$

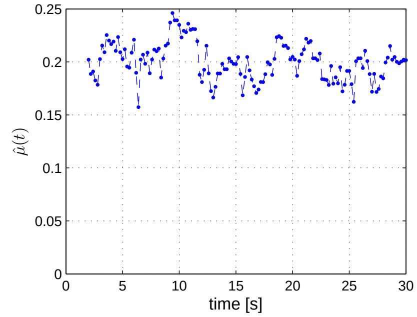


(c) Rate of the coordination states, $\dot{\boldsymbol{x}}(t)$

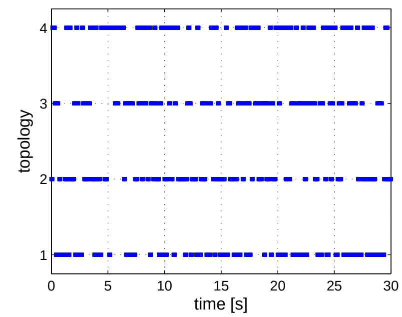


(d) Integrator states, $\boldsymbol{\chi}(t)$

Fig. 3. Quantized closed-loop collective dynamics with fine quantization ($\Delta = 0.35$).



(a) Estimate of the QoS, $\hat{\mu}(t)$



(b) Topology index

Fig. 4. QoS and information flow.

size $\Delta = 0.35$ (note that this step size verifies inequality (13)). The figure shows the time evolution of the coordination states, their time-derivative, the integrator states, and the disagreement vector $\boldsymbol{\eta}(t) := \boldsymbol{\Pi_N x}(t)$. Additionally, Figure 4 shows an estimate of the QoS of the network, computed as

$$\hat{\mu}(t) := \lambda_{\min}\left(\frac{1}{n}\,\frac{1}{T}\int_{t-T}^{t} \boldsymbol{Q_n L_0}(\tau)\boldsymbol{Q_n}^\top d\tau\right), \qquad t \geq T\,,$$

with $T = 2$ sec. The results demonstrate that the PI distributed protocol allows the followers to learn the reference rate command ρ and reach agreement with the leaders, while effectively compensating for the disturbances present in the network.

Finally, we use the same simulation scenario to illustrate the existence of undesirable attractors in the presence of coarse quantization. For this purpose, we change the quantizer step size to $\Delta = 3$. The computed response of the closed-loop collective dynamics is shown in Figure 5. In this case, the agents do not reach the desired agreement and, in fact, converge to a neighborhood of the equilibrium point (34) mentioned in the proof of Lemma 2 in Appendix D.

VI. Conclusions

In this paper we analyzed the convergence properties of a PI distributed protocol to coordinate a network of agents subject to constant disturbances. We addressed explicitly the situation where each agent transmits only its coordination state to only a subset of the other agents, as determined by the network topology. Furthermore, we considered the case where the graph that captures the information flow is not connected during some interval of time or even fails to be connected at all times. Finally, we analyzed the convergence properties of the protocol when the agents exchange quantized measurements. Simulation results verified the main theoretical findings.

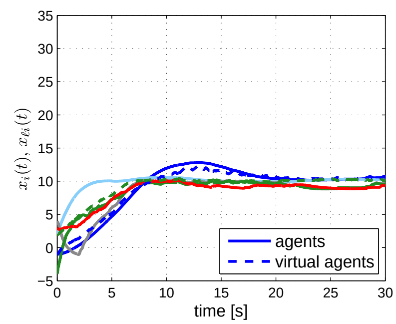


Fig. 5. Time-response with coarse quantization ($\Delta = 3$).

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Appendix A
Proof of Lemma 1

To prove the result in Lemma 1, we first characterize the quality of service of the original network augmented with a *single* virtual agent. For this purpose, we consider a network of n agents and assume that a virtual agent, denoted here by ℓ, is implemented in the jth agent. Communications between this virtual agent and the other agents are assumed to conform to the topology described in Section III-A.

Proposition 1: Consider a network with n agents and a single virtual agent, implemented in the jth agent, $j \in \mathcal{I}_n$. If the connectivity of the original network satisfies Assumption 3, then the connectivity of the network extended with a single virtual agent verifies

$$\frac{1}{n+1}\,\frac{1}{T}\int_t^{t+T} \boldsymbol{Q_{n+1}} \boldsymbol{L_1}(\tau) \boldsymbol{Q}_{\boldsymbol{n+1}}^\top d\tau \geq \mu_1 \mathbb{I}_{\boldsymbol{n}}, \qquad \forall\, t \geq 0, \quad (16)$$

where $\boldsymbol{L_1}(t)$ is the Laplacian of the extended graph; $\boldsymbol{Q_{n+1}}$ is any $n \times (n+1)$ matrix satisfying $\boldsymbol{Q_{n+1}} \boldsymbol{1_{n+1}} = \boldsymbol{0}$ and $\boldsymbol{Q_{n+1}} \boldsymbol{Q}_{\boldsymbol{n+1}}^\top = \mathbb{I}_{\boldsymbol{n}}$; and μ_1 characterizes the quality of service of the extended network and is given by

$$\mu_1 := \tfrac{(n\mu+2)-\sqrt{(n\mu+2)^2-4\mu(n+1)}}{2(n+1)}\,. \quad (17)$$

Proof. We start by noting that the Laplacian $\boldsymbol{L_1}(t)$ of the extended graph with vertex set $\mathcal{V}_1 := \{\ell, 1, \ldots, n\}$ is given by

$$\boldsymbol{L_1}(t) = \boldsymbol{P}_{\boldsymbol{1}}^\top \left[\begin{smallmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L_0}(t) \end{smallmatrix}\right] \boldsymbol{P_1} + \boldsymbol{L_{v1}} \quad \in \mathbb{R}^{(n+1)\times(n+1)},$$

where $\boldsymbol{P_1}$ is the $(0,1)$-*permutation matrix*

$$\boldsymbol{P_1} := \begin{bmatrix} 0 & \boldsymbol{0} & 1 & \boldsymbol{0} \\ \boldsymbol{0} & \mathbb{I}_{\boldsymbol{j-1}} & \boldsymbol{0} & \boldsymbol{0} \\ 1 & \boldsymbol{0} & 0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathbb{I}_{\boldsymbol{n-j}} \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)};$$

while $\boldsymbol{L_{v1}} \in \mathbb{R}^{(n+1)\times(n+1)}$ is defined as

$$\boldsymbol{L_{v1}} := \begin{array}{c} \begin{array}{cccc} \scriptstyle \ell & \scriptstyle 1\ldots(j-1) & \scriptstyle j & \scriptstyle (j+1)\ldots n \end{array} \\ \begin{bmatrix} 1 & \boldsymbol{0} & -1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -1 & \boldsymbol{0} & 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \end{array} \begin{array}{l} \scriptstyle \ell \\ \scriptstyle 1\ldots(j-1) \\ \scriptstyle j \\ \scriptstyle (j+1)\ldots n \end{array}.$$

We also note that a matrix $\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \in \mathbb{R}^{(k-1)\times k}$ satisfying $\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \boldsymbol{1_k} = \boldsymbol{0}$ and $\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \tilde{\boldsymbol{Q}}_{\boldsymbol{k}}^\top = \mathbb{I}_{\boldsymbol{k-1}}$ can be obtained recursively from the relation

$$\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} = \begin{bmatrix} \sqrt{\frac{k-1}{k}} & -\frac{1}{\sqrt{k(k-1)}} \boldsymbol{1}_{\boldsymbol{k-1}}^\top \\ \boldsymbol{0} & \tilde{\boldsymbol{Q}}_{k-1} \end{bmatrix}, \quad k = 3, 4, 5 \ldots,$$

together with the initial condition $\tilde{\boldsymbol{Q}}_{\boldsymbol{2}} = \left[\, \frac{1}{\sqrt{2}} \; -\frac{1}{\sqrt{2}} \,\right]$. Moreover, one can easily verify that, for any $k \times k$ $(0,1)$-permutation matrix $\boldsymbol{P_\pi}$, the matrix $(\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \boldsymbol{P_\pi})$ also satisfies the equalities $(\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \boldsymbol{P_\pi}) \boldsymbol{1_k} = \boldsymbol{0}$ and $(\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \boldsymbol{P_\pi})(\tilde{\boldsymbol{Q}}_{\boldsymbol{k}} \boldsymbol{P_\pi})^\top = \mathbb{I}_{\boldsymbol{k-1}}$, as $\boldsymbol{P_\pi} \boldsymbol{1_k} = \boldsymbol{1_k}$ and $\boldsymbol{P_\pi} \boldsymbol{P}_{\boldsymbol{\pi}}^\top = \mathbb{I}_{\boldsymbol{k}}$.

Then, letting $\hat{\boldsymbol{Q}} := \tilde{\boldsymbol{Q}}_{\boldsymbol{n+1}} \boldsymbol{P_1}$ and noting that $\boldsymbol{P_1} \boldsymbol{L_{v1}} \boldsymbol{P}_{\boldsymbol{1}}^\top = \boldsymbol{L_{v1}}$, we find that

$$\hat{\boldsymbol{Q}} \boldsymbol{L_1}(t) \hat{\boldsymbol{Q}}^\top = \left[\begin{smallmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{Q}}_{\boldsymbol{n}} \boldsymbol{L_0}(t) \tilde{\boldsymbol{Q}}_{\boldsymbol{n}}^\top \end{smallmatrix}\right] + \begin{bmatrix} \frac{n+1}{n} & -\sqrt{\frac{n+1}{n}}\,\tilde{\boldsymbol{q}}_j^\top \\ -\sqrt{\frac{n+1}{n}}\,\tilde{\boldsymbol{q}}_j & \tilde{\boldsymbol{q}}_j \tilde{\boldsymbol{q}}_j^\top \end{bmatrix},$$

where $\tilde{\boldsymbol{q}}_j$ is used to denote the jth column of $\tilde{\boldsymbol{Q}}_{\boldsymbol{n}}$. This implies that the following bound holds for all $t \geq 0$:

$$\begin{aligned} \frac{1}{n+1}\,\frac{1}{T}\int_t^{t+T} \hat{\boldsymbol{Q}} \boldsymbol{L_1}(t) \hat{\boldsymbol{Q}}^\top d\tau \quad &\geq \\ \frac{n}{n+1} \left[\begin{smallmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \mu\mathbb{I}_{\boldsymbol{n-1}} \end{smallmatrix}\right] &+ \begin{bmatrix} \frac{1}{n} & -\sqrt{\frac{1}{n(n+1)}}\,\tilde{\boldsymbol{q}}_j^\top \\ -\sqrt{\frac{1}{n(n+1)}}\,\tilde{\boldsymbol{q}}_j & \frac{1}{n+1}\tilde{\boldsymbol{q}}_j \tilde{\boldsymbol{q}}_j^\top \end{bmatrix}. \end{aligned}$$

Noting that $\lambda_{\max}(\tilde{\boldsymbol{q}}_j \tilde{\boldsymbol{q}}_j^\top) = \frac{n-1}{n}$, one can use Schur complements to show that the following inequality holds:

$$\frac{n}{n+1} \left[\begin{smallmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \mu\mathbb{I}_{\boldsymbol{n-1}} \end{smallmatrix}\right] + \begin{bmatrix} \frac{1}{n} & -\sqrt{\frac{1}{n(n+1)}}\,\tilde{\boldsymbol{q}}_j^\top \\ -\sqrt{\frac{1}{n(n+1)}}\,\tilde{\boldsymbol{q}}_j & \frac{1}{n+1}\tilde{\boldsymbol{q}}_j \tilde{\boldsymbol{q}}_j^\top \end{bmatrix} \;\geq\; \mu_1 \mathbb{I}_{\boldsymbol{n}},$$

where μ_1 was defined in (17). Therefore, we obtain

$$\frac{1}{n+1}\,\frac{1}{T}\int_t^{t+T} \hat{\boldsymbol{Q}} \boldsymbol{L_1}(t) \hat{\boldsymbol{Q}}^\top d\tau \;\geq\; \mu_1 \mathbb{I}_{\boldsymbol{n}}\,. \quad (18)$$

The inequality above can be used to show that PE-like condition (16) holds for arbitrary $n \times (n+1)$ matrix $\boldsymbol{Q_{n+1}}$ satisfying $\boldsymbol{Q_{n+1}} \boldsymbol{1_{n+1}} = \boldsymbol{0}$ and $\boldsymbol{Q_{n+1}} \boldsymbol{Q}_{\boldsymbol{n+1}}^\top = \mathbb{I}_{\boldsymbol{n}}$. In fact, recalling the properties of the projection matrix $\boldsymbol{\Pi_{n+1}}$, we can write

$$\begin{aligned} \frac{1}{n+1}\,\frac{1}{T}\int_t^{t+T} \boldsymbol{Q_{n+1}} \boldsymbol{L_1}(t) \boldsymbol{Q}_{\boldsymbol{n+1}}^\top d\tau &= \\ \boldsymbol{Q_{n+1}} \hat{\boldsymbol{Q}}^\top \left(\frac{1}{n+1}\,\frac{1}{T}\int_t^{t+T} \hat{\boldsymbol{Q}} \boldsymbol{L_1}(t) \hat{\boldsymbol{Q}}^\top d\tau \right) &\hat{\boldsymbol{Q}} \boldsymbol{Q}_{\boldsymbol{n+1}}^\top \end{aligned}$$

and, since $\hat{\boldsymbol{Q}} \boldsymbol{L_1}(t) \hat{\boldsymbol{Q}}^\top$ is a symmetric matrix, inequality (18) yields

$$\begin{aligned} \frac{1}{n+1}\,\frac{1}{T}\int_t^{t+T} &\boldsymbol{Q_{n+1}} \boldsymbol{L_1}(t) \boldsymbol{Q}_{\boldsymbol{n+1}}^\top d\tau \\ &\geq \mu_1 \boldsymbol{Q_{n+1}} \hat{\boldsymbol{Q}}^\top \hat{\boldsymbol{Q}} \boldsymbol{Q}_{\boldsymbol{n+1}}^\top \\ &= \mu_1 \boldsymbol{Q_{n+1}} \boldsymbol{\Pi_{n+1}} \boldsymbol{Q}_{\boldsymbol{n+1}}^\top \\ &= \mu_1 \boldsymbol{Q_{n+1}} \left(\mathbb{I}_{\boldsymbol{n+1}} - \tfrac{\boldsymbol{1_{n+1}} \boldsymbol{1}_{\boldsymbol{n+1}}^\top}{n+1} \right) \boldsymbol{Q}_{\boldsymbol{n+1}}^\top \\ &= \mu_1 \mathbb{I}_{\boldsymbol{n+1}}\,. \end{aligned}$$

This concludes the proof of the proposition. □

Proof of Lemma 1. The result in Lemma 1 follows by consecutively applying Proposition 1 n_ℓ times. □

Appendix B
Closed-Loop Collective Dynamics

From the definition of $\boldsymbol{\zeta_1}(t)$ and $\boldsymbol{\zeta_2}(t)$ and the coordination-state dynamics, it follows that

$$\begin{aligned} \dot{\boldsymbol{\zeta}}_{\boldsymbol{1}}(t) &= -k_P \boldsymbol{Q_N} \boldsymbol{L}(t) \boldsymbol{x}(t) + \rho \boldsymbol{Q_N} \boldsymbol{1_N} + \boldsymbol{Q_N} \left[\begin{smallmatrix} \boldsymbol{0} \\ \boldsymbol{\zeta_2}(t) \end{smallmatrix}\right] \\ &= -k_P \boldsymbol{Q_N} \boldsymbol{L}(t) \boldsymbol{x}(t) + \boldsymbol{Q_N} \boldsymbol{C} \boldsymbol{\zeta_2}(t)\,. \end{aligned}$$

The properties of the projection matrix $\boldsymbol{\Pi_N}$, along with the fact that $\boldsymbol{Q_N} \boldsymbol{Q}_{\boldsymbol{N}}^\top = \mathbb{I}_{\boldsymbol{N-1}}$, imply that

$$\begin{aligned} \dot{\boldsymbol{\zeta}}_{\boldsymbol{1}}(t) &= -k_P \boldsymbol{Q_N} \boldsymbol{\Pi_N} \boldsymbol{L}(t) \boldsymbol{\Pi_N} \boldsymbol{x}(t) + \boldsymbol{Q_N} \boldsymbol{C} \boldsymbol{\zeta_2}(t) \\ &= -k_P \boldsymbol{Q_N} \boldsymbol{Q}_{\boldsymbol{N}}^\top \boldsymbol{Q_N} \boldsymbol{L}(t) \boldsymbol{Q}_{\boldsymbol{N}}^\top \boldsymbol{Q_N} \boldsymbol{x}(t) + \boldsymbol{Q_N} \boldsymbol{C} \boldsymbol{\zeta_2}(t) \\ &= -k_P \boldsymbol{Q_N} \boldsymbol{L}(t) \boldsymbol{Q}_{\boldsymbol{N}}^\top \boldsymbol{\zeta_1}(t) + \boldsymbol{Q_N} \boldsymbol{C} \boldsymbol{\zeta_2}(t) \\ &= -k_P \bar{\boldsymbol{L}}(t) \boldsymbol{\zeta_1}(t) + \boldsymbol{Q_N} \boldsymbol{C} \boldsymbol{\zeta_2}(t)\,. \end{aligned} \quad (19)$$

Similarly, it follows that

$$\dot{\boldsymbol{\zeta}}_{\boldsymbol{2}}(t) = -k_I \boldsymbol{C}^\top \boldsymbol{L}(t) \boldsymbol{x}(t) = -k_I \boldsymbol{C}^\top \boldsymbol{Q}_{\boldsymbol{N}}^\top \bar{\boldsymbol{L}}(t) \boldsymbol{\zeta_1}(t)\,. \quad (20)$$

Equations (19) and (20) lead to (7).

APPENDIX C
PROOF OF THEOREM 1

From Lemma 1, it follows that the connectivity of the extended network $\mathcal{G}(t)$ verifies the PE-like condition (16).

To prove that the origin of the closed-loop collective dynamics (7) is globally uniformly exponentially stable (GUES) under this connectivity condition, we first consider the system

$$\dot{\boldsymbol{\phi}}(t) = -k_P \bar{\boldsymbol{L}}(t)\boldsymbol{\phi}(t)\,, \qquad \boldsymbol{\phi}(t) \in \mathbb{R}^{N-1}\,, \tag{21}$$

where k_P is the proportional coordination gain. Letting $\boldsymbol{\Lambda}(t)$ be the time-varying *incidence matrix*, $\boldsymbol{L}(t) = \boldsymbol{\Lambda}(t)\,\boldsymbol{\Lambda}^\top(t)$, we can rewrite the above system as

$$\dot{\boldsymbol{\phi}}(t) = -k_P(\boldsymbol{Q_N}\boldsymbol{\Lambda}(t))(\boldsymbol{Q_N}\boldsymbol{\Lambda}(t))^\top\boldsymbol{\phi}(t)\,.$$

Then, since $\boldsymbol{Q_N}\boldsymbol{\Lambda}(t)$ is piecewise constant in time and

$$\|\boldsymbol{Q_N}\boldsymbol{\Lambda}(t)\|^2 \le N\,,$$

one can prove that, if the PE-like connectivity condition (16) is verified, the system in (21) is GUES, and the following bound holds:

$$\|\boldsymbol{\phi}(t)\| \le \alpha_\phi\|\boldsymbol{\phi}(0)\|\,\mathrm{e}^{-\gamma_c t}$$

with $\alpha_\phi = 1$ and $\gamma_c \ge \bar{\gamma}_c := \frac{k_P N \mu_{n_\ell}}{(1+k_P N T)^2}$. This result can be proven along the same lines as Lemma 5 in [13] or Lemma 3 in [14]. Since $\bar{\boldsymbol{L}}(t)$ is continuous for almost all $t \ge 0$ and uniformly bounded, and the system in (21) is GUES, Lemma 1 in [14] and a similar argument as in [15, Theorem 4.12] imply that, for any constants $\bar{c}_3$ and $\bar{c}_4$ satisfying $0 < \bar{c}_3 \le \bar{c}_4$, there exists a continuous, piecewise-differentiable matrix $\boldsymbol{P_0}(t) = \boldsymbol{P_0}^\top(t)$ such that

$$\bar{c}_1\mathbb{I}_{\boldsymbol{N-1}} := \tfrac{\bar{c}_3}{2k_P N}\mathbb{I}_{\boldsymbol{N-1}} \;\le\; \boldsymbol{P_0}(t) \;\le\; \tfrac{\bar{c}_4}{2\gamma_c}\mathbb{I}_{\boldsymbol{N-1}} =: \bar{c}_2\mathbb{I}_{\boldsymbol{N-1}}\,, \tag{22}$$

$$\dot{\boldsymbol{P}}_{\boldsymbol{0}}(t) - k_P\bar{\boldsymbol{L}}(t)\boldsymbol{P_0}(t) - k_P\boldsymbol{P_0}(t)\bar{\boldsymbol{L}}(t) \le -\bar{c}_3\mathbb{I}_{\boldsymbol{N-1}}\,. \tag{23}$$

Next, we apply the similarity transformation

$$\boldsymbol{z}(t) = \boldsymbol{S_\zeta}\boldsymbol{\zeta}(t) = \begin{bmatrix} \mathbb{I}_{\boldsymbol{N-1}} & 0 \\ -\frac{k_I}{k_P}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top & \mathbb{I}_{\boldsymbol{n}} \end{bmatrix}\boldsymbol{\zeta}(t) \tag{24}$$

to the original collective dynamics (7), which leads to

$$\begin{aligned}\dot{\boldsymbol{z}}(t) &= \boldsymbol{S_\zeta}\boldsymbol{A_\zeta}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}^{-1}\boldsymbol{z}(t) \\ &= \begin{bmatrix} -k_P\bar{\boldsymbol{L}}(t) + \frac{k_I}{k_P}\boldsymbol{Q_N}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top & \boldsymbol{Q_N}\boldsymbol{C} \\ -\frac{k_I^2}{k_P^2}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q_N}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top & -\frac{k_I}{k_P}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q_N}\boldsymbol{C} \end{bmatrix}\boldsymbol{z}(t)\,.\end{aligned} \tag{25}$$

Consider now the Lyapunov function candidate

$$V(t,\boldsymbol{z}) := \boldsymbol{z}^\top\boldsymbol{P}(t)\,\boldsymbol{z}\,, \tag{26}$$

where $\boldsymbol{P}(t)$ is defined as

$$\boldsymbol{P}(t) := \begin{bmatrix} \boldsymbol{P_0}(t) & \boldsymbol{0} \\ \boldsymbol{0} & \frac{k_P^3}{k_I^3}N\,\mathbb{I}_{\boldsymbol{n}} \end{bmatrix} \quad \in \mathbb{R}^{(N+n-1)\times(N+n-1)}\,.$$

The time derivative of V along the trajectories of system (25) is given in (27), at the top of next page. Inequality (23) implies that $\dot{V}(t)$ can be bounded as in (28), also at the top of next page. Now, letting

$$\begin{gathered} k_P > 0\,, \qquad \lambda_c = \frac{\gamma_c}{1+2\frac{N}{n_\ell}\sqrt{\frac{N}{n_\ell}}}\,, \qquad k_I = 2k_P\lambda_c\frac{N}{n_\ell}\,, \\ \bar{c}_3 = \bar{c}_4 = \frac{\gamma_c}{\lambda_c}\left(\frac{\gamma_c}{\lambda_c} - 1\right)\frac{n_\ell^3}{2N^2}\,, \end{gathered} \tag{29}$$

and noting that $\|\boldsymbol{Q_N}\boldsymbol{C}\| = 1$ and $\lambda_{\min}(\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q_N}\boldsymbol{C}) = \frac{n_\ell}{N}$, one can use Schur complements to prove that the inequality in (30) holds for all $t \ge 0$; see top of next page. Then, for the choice of parameters in (29), inequality (30) implies that

$$\dot{V}(t) \le -2\lambda_c\,\boldsymbol{z}^\top(t)\begin{bmatrix} \bar{c}_2\mathbb{I}_{\boldsymbol{N-1}} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{k_P^3}{k_I^3}\frac{N^2}{n_\ell}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q_N}\boldsymbol{C} \end{bmatrix}\boldsymbol{z}(t)\,,$$

which, along with the fact that $\boldsymbol{P_0}(t) \le \bar{c}_2\mathbb{I}_{\boldsymbol{N-1}}$ and $\lambda_{\min}(\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q_N}\boldsymbol{C}) = \frac{n_\ell}{N}$, leads to

$$\dot{V}(t) \le -2\lambda_c\,\boldsymbol{z}^\top(t)\begin{bmatrix} \boldsymbol{P_0}(t) & \boldsymbol{0} \\ \boldsymbol{0} & \frac{k_P^3}{k_I^3}N\mathbb{I}_{\boldsymbol{n}} \end{bmatrix}\boldsymbol{z}(t) = -2\lambda_c\,V(t)\,.$$

Application of the comparison lemma [15, Lemma 3.4] leads to

$$V(t) \le V(0)\mathrm{e}^{-2\lambda_c t}\,,$$

and since

$$\min\{\bar{c}_1, \tfrac{k_P^3}{k_I^3}N\}\|\boldsymbol{z}(t)\|^2 \le V(t) \le \max\{\bar{c}_2, \tfrac{k_P^3}{k_I^3}N\}\|\boldsymbol{z}(t)\|^2\,,$$

we find that

$$\|\boldsymbol{z}(t)\| \le \left(\frac{\max\{\bar{c}_2, \frac{k_P^3}{k_I^3}N\}}{\min\{\bar{c}_1, \frac{k_P^3}{k_I^3}N\}}\right)^{\frac{1}{2}}\|\boldsymbol{z}(0)\|\mathrm{e}^{-\lambda_c t}\,.$$

The similarity transformation in (24) implies that

$$\|\boldsymbol{\zeta}(t)\| \le \|\boldsymbol{S}_{\boldsymbol{\zeta}}^{-1}\|\left(\frac{\max\{\bar{c}_2, \frac{k_P^3}{k_I^3}N\}}{\min\{\bar{c}_1, \frac{k_P^3}{k_I^3}N\}}\right)^{\frac{1}{2}}\|\boldsymbol{S_\zeta}\|\|\boldsymbol{\zeta}(0)\|\mathrm{e}^{-\lambda_c t}\,, \tag{31}$$

and thus system (7) is GUES. Noting that $\lambda_{\max}(\boldsymbol{Q_N}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top) = 1$, one can use Fact 9.14.12 in [16] to show that

$$\max\{\|\boldsymbol{S_\zeta}\|, \|\boldsymbol{S}_{\boldsymbol{\zeta}}^{-1}\|\} \le \left(1 + \frac{k_I^2}{2k_P^2}\right) + \frac{k_I}{2k_P}\sqrt{\frac{k_I^2}{k_P^2} + 4}\,.$$

Moreover, since $\gamma_c \ge \bar{\gamma}_c$, we have that

$$\lambda_c = \frac{\gamma_c}{1+2\frac{N}{n_\ell}\sqrt{\frac{N}{n_\ell}}} \ge \frac{\bar{\gamma}_c}{1+2\frac{N}{n_\ell}\sqrt{\frac{N}{n_\ell}}} = \bar{\lambda}_c\,.$$

This proves the bound in (8). The limits in (9) follow from the definition of $\boldsymbol{\zeta_1}(t)$ and $\boldsymbol{\zeta_2}(t)$ and the bound in (31). □

APPENDIX D
PROOF OF LEMMA 2

Let $\hat{\boldsymbol{x}} \in \mathbb{R}^N$ be a vector such that $\mathrm{q}(\hat{\boldsymbol{x}}) = \boldsymbol{k}\Delta$, $\boldsymbol{k} \in \mathbb{Z}^N$, and let $\boldsymbol{v} \in \mathcal{K}(\mathrm{q}(\hat{\boldsymbol{x}}))$. Then, we have

$$v_i = \begin{cases} k_i\Delta\,, & \hat{x}_i \ne k_i\Delta - \frac{\Delta}{2} \\ [(k_i-1)\Delta, k_i\Delta]\,, & \hat{x}_i = k_i\Delta - \frac{\Delta}{2} \end{cases}\,,$$

where $v_i \in \mathbb{R}$, $\hat{x}_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ are the ith components of $\boldsymbol{v}$, $\boldsymbol{x}$, and $\boldsymbol{k}$, respectively. Note that $|v_i - \hat{x}_i| \le \frac{\Delta}{2}$. Also, let $\hat{\boldsymbol{x}}_{\boldsymbol{\ell}}$ and $\boldsymbol{v_\ell}$ be the vectors with the first n_ℓ components of $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ as their elements, respectively. Similarly, let $\hat{\boldsymbol{x}}_{\boldsymbol{f}}$ and $\boldsymbol{v_f}$ be the vectors with the last $(n - n_\ell)$ components of $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ as their elements.

To prove the result of the lemma, it is enough to show that, if the bound (13) holds, then there exists no pair $(\boldsymbol{v}, \boldsymbol{\chi})$, $\boldsymbol{v} \in \mathcal{K}(\mathrm{q}(\hat{\boldsymbol{x}}))$ and $\boldsymbol{\chi} \in \mathbb{R}^n$, such that the following equality holds:

$$\boldsymbol{0} = \begin{bmatrix} -k_P\left(\tilde{\boldsymbol{D}}\hat{\boldsymbol{x}} - \tilde{\boldsymbol{A}}\boldsymbol{v}\right) + \begin{bmatrix} \rho\boldsymbol{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{\chi}+\boldsymbol{d} \end{bmatrix} \\ -k_I\boldsymbol{C}^\top\left(\tilde{\boldsymbol{D}}\hat{\boldsymbol{x}} - \tilde{\boldsymbol{A}}\boldsymbol{v}\right) \end{bmatrix}\,. \tag{32}$$

To this end, we first consider the first N rows of the equality (32) and multiply them on the left by $C^\top$ to obtain

$$-k_P\boldsymbol{C}^\top\left(\tilde{\boldsymbol{D}}\boldsymbol{x} - \tilde{\boldsymbol{A}}\boldsymbol{v}\right) + \boldsymbol{\chi} + \boldsymbol{d} = \boldsymbol{0}\,.$$

Then, noting that the last n rows of (32) imply that

$$\boldsymbol{C}^\top\left(\tilde{\boldsymbol{D}}\boldsymbol{x} - \tilde{\boldsymbol{A}}\boldsymbol{v}\right) = \boldsymbol{0}\,,$$

it follows that equality (32) can be satisfied only if $\boldsymbol{\chi} = -\boldsymbol{d}$. With this result, one can reformulate the problem in terms of the vectors $\boldsymbol{v_\ell} \in \mathbb{R}^{n_\ell}$ and $\boldsymbol{v_f} \in \mathbb{R}^{n-n_\ell}$ defined above. In fact, it can be shown that existence of a pair $(\boldsymbol{v}, \boldsymbol{\chi})$ satisfying (32) is equivalent to existence of $\boldsymbol{v_\ell}$ and $\boldsymbol{v_f}$ such that the following equality is satisfied:

$$\boldsymbol{L_0}\begin{bmatrix} \boldsymbol{v_\ell} \\ \boldsymbol{v_f} \end{bmatrix} - \begin{bmatrix} \frac{\rho}{k_P}\boldsymbol{1}_{\boldsymbol{n_\ell}} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{D_0}\begin{bmatrix} \boldsymbol{v_\ell} - \hat{\boldsymbol{x}}_{\boldsymbol{\ell}} \\ \boldsymbol{v_f} - \hat{\boldsymbol{x}}_{\boldsymbol{f}} \end{bmatrix}\,, \tag{33}$$

where $\boldsymbol{L_0} \in \mathbb{R}^{n\times n}$ is the Laplacian of the "original" network topology $\mathcal{G}_0$ without the additional virtual agents, and $\boldsymbol{D_0} \in \mathbb{R}^{n\times n}$ is its degree matrix. Recall that, in this lemma, we assume that the topology is static, and thus both $\boldsymbol{L_0}$ and $\boldsymbol{D_0}$ are constant matrices.

$$\dot{V}(t) = \boldsymbol{z}^\top(t) \begin{bmatrix} \dot{\boldsymbol{P}}_{\boldsymbol{0}}(t) - k_P\big(\bar{\boldsymbol{L}}(t)\boldsymbol{P}_{\boldsymbol{0}}(t) + \boldsymbol{P}_{\boldsymbol{0}}(t)\bar{\boldsymbol{L}}(t)\big) + \frac{k_I}{k_P}\big(\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{P}_{\boldsymbol{0}}(t) + \boldsymbol{P}_{\boldsymbol{0}}(t)\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\big) & \boldsymbol{P}_{\boldsymbol{0}}(t)\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} - \frac{k_P}{k_I}N\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \\ \boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{P}_{\boldsymbol{0}}(t) - \frac{k_P}{k_I}N\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top & -2\frac{k_P^2}{k_I^2}N\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \end{bmatrix} \boldsymbol{z}(t). \tag{27}$$

$$\dot{V}(t) \leq \boldsymbol{z}^\top(t) \begin{bmatrix} -\bar{c}_3\mathbb{I}_{\boldsymbol{N-1}} + \frac{k_I}{k_P}\big(\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{P}_{\boldsymbol{0}}(t) + \boldsymbol{P}_{\boldsymbol{0}}(t)\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\big) & \boldsymbol{P}_{\boldsymbol{0}}(t)\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} - \frac{k_P}{k_I}N\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \\ \boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{P}_{\boldsymbol{0}}(t) - \frac{k_P}{k_I}N\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top & -2\frac{k_P^2}{k_I^2}N\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \end{bmatrix} \boldsymbol{z}(t)\,. \tag{28}$$

$$\begin{bmatrix} -\bar{c}_3\mathbb{I}_{\boldsymbol{N-1}} + \frac{k_I}{k_P}\big(\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{P}_{\boldsymbol{0}}(t) + \boldsymbol{P}_{\boldsymbol{0}}(t)\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\big) & \boldsymbol{P}_{\boldsymbol{0}}(t)\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} - \frac{k_P}{k_I}N\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \\ \boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{P}_{\boldsymbol{0}}(t) - \frac{k_P}{k_I}N\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top & -2\frac{k_P^2}{k_I^2}N\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \end{bmatrix} \leq -2\lambda_c \begin{bmatrix} \bar{c}_2\mathbb{I}_{\boldsymbol{N-1}} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{k_P^3}{k_I^3}\frac{N^2}{n_\ell}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{Q}_{\boldsymbol{N}}\boldsymbol{C} \end{bmatrix} \tag{30}$$

The existence of vectors $\boldsymbol{v_\ell}$ and $\boldsymbol{v_f}$ such that equality (33) holds, depends on the quantizer precision. For instance, if $\|\frac{\rho}{k_P}\boldsymbol{D_0}^{-1}\left[\begin{smallmatrix}\boldsymbol{1_{n_\ell}}\\ \boldsymbol{0}\end{smallmatrix}\right]\|_\infty < \frac{\Delta}{2}$, then the vectors

$$\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{\ell}}\\ \hat{\boldsymbol{x}}_{\boldsymbol{f}}\end{bmatrix} = k\Delta\boldsymbol{1_n} + \boldsymbol{D_0}^{-1}\begin{bmatrix}\frac{\rho}{k_P}\boldsymbol{1_{n_\ell}}\\ \boldsymbol{0}\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{v_\ell}\\ \boldsymbol{v_f}\end{bmatrix} = k\Delta\boldsymbol{1_n}, \tag{34}$$

verify equality (33) for any $k \in \mathbb{Z}$. However, if the bound (13) holds, then there exist no vectors $\boldsymbol{v_\ell}$ and $\boldsymbol{v_f}$ such that equality (33) holds. To see this, consider the scalar equality

$$\frac{\rho}{k_P}n_\ell = \boldsymbol{1}_{\boldsymbol{n}}^\top\boldsymbol{D_0}\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{\ell}} - \boldsymbol{v_\ell}\\ \hat{\boldsymbol{x}}_{\boldsymbol{f}} - \boldsymbol{v_f}\end{bmatrix}, \tag{35}$$

which has been obtained from (33) by multiplying on the left by $\boldsymbol{1}_{\boldsymbol{n}}^\top$. The right-hand side of this equality can be bounded as

$$\left|\boldsymbol{1}_{\boldsymbol{n}}^\top\boldsymbol{D_0}\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{\ell}} - \boldsymbol{v_\ell}\\ \hat{\boldsymbol{x}}_{\boldsymbol{f}} - \boldsymbol{v_f}\end{bmatrix}\right| \leq n(n-1)\frac{\Delta}{2}\,.$$

If the step size of the quantizers is bounded as in (13), then we have

$$\left|\boldsymbol{1}_{\boldsymbol{n}}^\top\boldsymbol{D_0}\begin{bmatrix}\boldsymbol{x_\ell} - \boldsymbol{v_\ell}\\ \boldsymbol{x_f} - \boldsymbol{v_f}\end{bmatrix}\right| < \frac{|\rho|}{k_p}n_\ell\,,$$

which implies that no vectors $\boldsymbol{v_\ell}$ and $\boldsymbol{v_f}$ satisfy (35), and thus (33). In turn, this implies that there is no pair $(\boldsymbol{v}, \boldsymbol{\chi})$ such that equality (32) holds, and therefore the set E defined in (12) is empty. □

Appendix E
Proof of Theorem 2

Let the function $\boldsymbol{\zeta}(t) = \begin{bmatrix}\boldsymbol{Q_N}\boldsymbol{x}(t)\\ \boldsymbol{\chi}(t) - \rho\boldsymbol{1_n} + \boldsymbol{d}\end{bmatrix}$, $t \in I_t \subset \mathbb{R}$ be a Krasowskii solution to (11) on I_t, that is, $\boldsymbol{\zeta}(t)$ is absolutely continuous and satisfies the differential inclusion [12]

$$\dot{\boldsymbol{\zeta}}(t) - \boldsymbol{A}_{\boldsymbol{\zeta}}(t)\boldsymbol{\zeta}(t) + \boldsymbol{B}_{\boldsymbol{\zeta}}(t)\boldsymbol{x}(t) \in \boldsymbol{B}_{\boldsymbol{\zeta}}(t)\mathcal{K}\left(\mathrm{q}(\boldsymbol{x}(t))\right)\,,$$

for almost every $t \in I_t$. Then, letting $\boldsymbol{z}(t) := \boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{\zeta}(t)$, where $\boldsymbol{S}_{\boldsymbol{\zeta}}$ was defined in (24), we have

$$\dot{\boldsymbol{z}}(t) - \boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{A}_{\boldsymbol{\zeta}}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}^{-1}\boldsymbol{z}(t) + \boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\boldsymbol{x}(t) \in \boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\mathcal{K}\left(\mathrm{q}(\boldsymbol{x}(t))\right)\,.$$

Consider now the same Lyapunov function candidate (26) as in the proof of Theorem 1 with $k_P > 0$ and $k_I = 2k_P\lambda_c\frac{N}{n_\ell}$. Then, letting $\boldsymbol{v}(t) \in \mathcal{K}(\mathrm{q}(\boldsymbol{x(t)}))$ and following the same steps as in the proof of Theorem 1, we have that

$$\begin{aligned}\dot{V}(t, \boldsymbol{z}(t)) &\leq -2\lambda_c V(t, \boldsymbol{z}(t)) \\ &\quad + 2\|\boldsymbol{P}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\|\|\boldsymbol{z}(t)\|\|\boldsymbol{v}(t) - \boldsymbol{x}(t)\|\,.\end{aligned}$$

Noting that $\|\boldsymbol{v}(t) - \boldsymbol{x}(t)\| \leq \sqrt{N}\frac{\Delta}{2}$, it follows that

$$\dot{V}(t, \boldsymbol{z}(t)) \leq -2\lambda_c V(t, \boldsymbol{z}(t)) + \sqrt{N}\Delta\|\boldsymbol{P}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\|\|\boldsymbol{z}(t)\|\,.$$

We can now rewrite the above inequality as

$$\begin{aligned}\dot{V}(t, \boldsymbol{z}(t)) &\leq -2(\lambda_c - \theta)V(t, \boldsymbol{z}(t)) \\ &\quad - 2\theta V(t, \boldsymbol{z}(t)) + \sqrt{N}\Delta\|\boldsymbol{P}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\|\|\boldsymbol{z}(t)\|\,,\end{aligned}$$

where $0 < \theta < \lambda_c$. Then, for all $\boldsymbol{z}$ satisfying

$$-2\theta V(t, \boldsymbol{z}) + \sqrt{N}\Delta\|\boldsymbol{P}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\|\|\boldsymbol{z}\| \leq 0\,, \tag{36}$$

we have $\dot{V}(t, \boldsymbol{z}(t)) \leq -2(\lambda_c - \theta)V(t, \boldsymbol{z}(t))$. Inequality (36) holds outside the bounded set D_Δ defined by:

$$D_\Delta := \left\{\boldsymbol{z} \in \mathbb{R}^{N+n-1} \;:\; \|\boldsymbol{z}\| \leq \frac{\sqrt{N}\Delta\|\boldsymbol{P}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\|}{2\theta\min\{\bar{c}_1, \frac{k_P^3}{k_I^3}N\}}\right\}.$$

Then, noting that

$$\|\boldsymbol{P}(t)\boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{B}_{\boldsymbol{\zeta}}(t)\| \leq \sqrt{2}(n-1)k_P\max\left\{\bar{c}_2, \frac{k_P^3}{k_I^3}N, \frac{k_P^2}{k_I^2}N\right\} =: \sigma_B\,,$$

it can be shown that the set D_Δ is in the interior of the compact set Ω_Δ given by:

$$\Omega_\Delta := \{\boldsymbol{z} \in \mathbb{R}^{N+n-1} \;:\; V(z, t) \leq L_V\Delta^2\}$$

where L_V is defined as

$$L_V := \frac{N\sigma_B^2}{4\theta^2}\,\frac{\max\left\{\bar{c}_2, \frac{k_P^3}{k_I^3}N\right\}}{\left(\min\{\bar{c}_1, \frac{k_P^3}{k_I^3}N\}\right)^2}\,.$$

With this results and using a proof similar to that of Theorem 4.18 in [15], it can be shown that there is a time $T_b \geq 0$ such that

$$\begin{aligned}V(t, \boldsymbol{z}(t)) &\leq V(0, \boldsymbol{z}(0))\mathrm{e}^{-2(\lambda_c - \theta)t}\,, && 0 \leq t < T_b\,, \\ V(t, \boldsymbol{z}(t)) &\leq L_V\Delta^2\,, && t \geq T_b\,.\end{aligned}$$

In particular, noting that

$$\bar{c}_1\|\boldsymbol{\zeta_1}\|^2 + \frac{k_P^3}{k_I^3}N\|\boldsymbol{\zeta_2} - \frac{k_I}{k_P}\boldsymbol{C}^\top\boldsymbol{Q}_{\boldsymbol{N}}^\top\boldsymbol{\zeta_1}\|^2 \leq V(t, \boldsymbol{S}_{\boldsymbol{\zeta}}\boldsymbol{\zeta})\,,$$

we have that

$$\|\boldsymbol{\zeta_1}(t)\| \leq \left(\frac{L_V}{\bar{c}_1}\right)^{\frac{1}{2}}\Delta\,, \tag{37}$$

$$\|\boldsymbol{\zeta_2}(t)\| \leq \frac{k_I}{k_P}\left(1 + \sqrt{\frac{k_I}{k_P}\frac{\bar{c}_1}{N}}\right)\left(\frac{L_V}{\bar{c}_1}\right)^{\frac{1}{2}}\Delta\,, \tag{38}$$

for all $t \geq T_b$. Moreover, since the solution $\boldsymbol{\zeta}(t)$ to the quantized consensus problem (11) is bounded, then it is also complete.

To prove inequalities (14) and (15), we introduce the *disagreement vector* $\boldsymbol{\eta}(t) := \boldsymbol{\Pi_N}\boldsymbol{x}(t)$ and use the facts that

$$x_i(t) - x_j(t) = \eta_i(t) - \eta_j(t)\,, \qquad \forall\, i, j \in \mathcal{I}_n\,, \tag{39}$$

$$\|\boldsymbol{\eta}(t)\| = \|\boldsymbol{\zeta_1}(t)\|\,. \tag{40}$$

From (39) and (40), it follows that

$$|x_i(t) - x_j(t)| \leq 2\|\boldsymbol{\eta}(t)\| = 2\|\boldsymbol{\zeta_1}(t)\|\,, \qquad \forall\, i, j \in \mathcal{I}_n\,,$$

and thus equation (37) leads to (14) with $\alpha_\eta = 2\left(\frac{L_V}{\bar{c}_1}\right)^{\frac{1}{2}}$. On the other hand, from the quantized collective dynamics and the definition of $\boldsymbol{\zeta_2}(t)$, one obtains

$$|\dot{x}_i(t) - \rho| \leq 2k_P(n-1)\|\boldsymbol{\zeta_1}(t)\| + \|\boldsymbol{\zeta_2}(t)\| + k_P(n-1)\frac{\Delta}{2}, \quad \forall i \in \mathcal{I}_n,$$

which leads to the bound in (15) with

$$\alpha_\rho = k_P(n-1)\left(\frac{1}{2} + 2\left(\frac{L_V}{\bar{c}_1}\right)^{\frac{1}{2}}\right) + \frac{k_I}{k_P}\left(1 + \sqrt{\frac{k_I}{k_P}\frac{\bar{c}_1}{N}}\right)\left(\frac{L_V}{\bar{c}_1}\right)^{\frac{1}{2}}.$$

□