Chapter 5

Inertial Accelerations

5.1 General

We need expressions for the inertial accelerations of our aircraft in order to apply Newton's Second Law. Our point of view will be from some local coordinate system, usually attached to the aircraft. The problem is that this local coordinate system rotates with respect to Earth, which rotates with respect to inertial space. The first problem we will tackle is to relate inertial acceleration to our locally observed acceleration. This will re-introduce the familiar concepts of tangential, coriolis, and centripetal accelerations. We will then address the way in which inertial acceleration is related to the entire mass of the aircraft, its distribution, and our choice of coordinate systems.

5.2 Inertial Acceleration of a Point

5.2.1 Arbitrary Moving Reference Frame

We begin by hypothesizing an inertial reference F_I and some moving reference frame F_M. The origin of F_M may be accelerated ($\mathbf{a}_M$) and the reference frame itself may be rotating ($\boldsymbol{\omega}_M$) with respect to inertial space, both assumed known. It is also assumed that we know how some point P moves around with respect to F_M. We are seeking an expression for the inertial acceleration of the point P expressed in F_M ($\{\mathbf{a}_P\}_M$). Later we will use this expression twice: first with the Earth-centered frame as F_M and a point on the Earth's surface as P, then with an Earth-fixed frame as F_M and the aircraft CG as

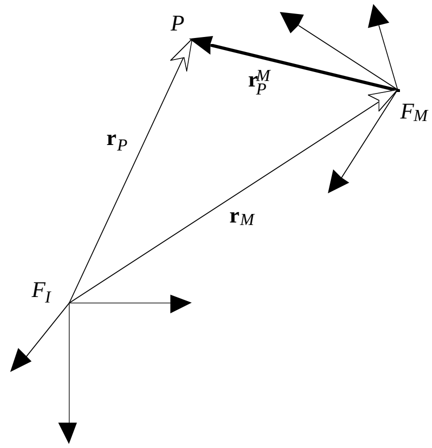

Figure 5.1: A Point in Inertial Space

P. These two results will enable us to write the inertial acceleration of the CG in a local coordinate system. The same results will be used to evaluate certain rotational accelerations and decide whether they are small enough to be neglected.

Figure 5.1 shows a generic picture of F_M and P in inertial space. It is clear from this figure that we may express the inertial position of P as $\mathbf{r}_P = \mathbf{r}_M + \mathbf{r}_P^M$. The inertial velocity of P is the first derivative of $\mathbf{r}_P$ with respect to time as seen from the inertial frame, and its inertial acceleration is the second derivative, $\mathbf{v}_P = \dot{\mathbf{r}}_P$ and $\mathbf{a}_P = \dot{\mathbf{v}}_P$. What we are after is $\{\mathbf{a}_P\}_M$, but first we formulate $\{\mathbf{v}_P\}_M$.

$$\begin{aligned} \mathbf{v}_P &= \tfrac{d}{dt}\mathbf{r}_P \\ &= \tfrac{d}{dt}\left(\mathbf{r}_M + \mathbf{r}_P^M\right) \\ &= \tfrac{d}{dt}\mathbf{r}_M + \tfrac{d}{dt}\mathbf{r}_P^M \end{aligned}$$

The first term on the right is the inertial velocity of the moving reference frame, or $\mathbf{v}_M$, and the second is the derivative of the vector $\mathbf{r}_P^M$ that is in a rotating reference frame. With the dot notation for time derivatives we have $\mathbf{v}_P = \mathbf{v}_M + \dot{\mathbf{r}}_P^M$. Assume these vectors are expressed in F_I ($\{\mathbf{v}_P\}_I$, $\{\mathbf{v}_M\}_I$, and $\left\{\dot{\mathbf{r}}_P^M\right\}_I$ with the I subscript just to be clear) then we may transform them to F_M using $T_{M,I}$:

$$\begin{aligned}\{\mathbf{v}_P\}_M &= T_{M,I}\{\mathbf{v}_P\}_I \\ &= T_{M,I}\{\mathbf{v}_M\}_I + T_{M,I}\left\{\tfrac{d\mathbf{r}_P^M}{dt}{}_I\right\}_I \\ &= \{\mathbf{v}_M\}_M + T_{M,I}\left\{\dot{\mathbf{r}}_P^M\right\}_I\end{aligned}$$

The expression $T_{M,I}\{\mathbf{v}_M\}_I$ is a straightforward transformation, $T_{M,I}\{\mathbf{v}_M\}_I = \{\mathbf{v}_M\}_M$. The other term on the right is the transformation of the inertial derivative of a vector. We can express the derivative in the moving reference frame using the previously derived result, $\{\dot{\mathbf{v}}_2\}_2 = T_{2,1}\{\dot{\mathbf{v}}_1\}_1 - \{\Omega_2^1\}_2\{\mathbf{v}\}_2$, or $T_{2,1}\{\dot{\mathbf{v}}_1\}_1 = \{\dot{\mathbf{v}}_2\}_2 + \{\Omega_2^1\}_2\{\mathbf{v}\}_2$. Here we let $F_1 = F_I$, $F_2 = F_M$, $\mathbf{v}_1 = \left\{\mathbf{r}_P^M\right\}_I$, and $\mathbf{v}_2 = \left\{\mathbf{r}_P^M\right\}_M$. This results in

$$T_{M,I}\left\{\dot{\mathbf{r}}_P^M\right\}_I = \left\{\frac{d\mathbf{r}_P^M}{dt}{}_M\right\}_M + \{\Omega_M\}_M\left\{\mathbf{r}_P^M\right\}_M$$

The first term on the right seems to have overworked the superscripts and subscripts a bit. In words, it means the derivative of the vector $\mathbf{r}_P^M$ as seen from F_M, with the answer expressed in F_M coordinates. This is just the velocity of P relative to F_M expressed in F_M coordinates, $\left\{\mathbf{v}_P^M\right\}_M$. We therefore have

$$\{\mathbf{v}_P\}_M = \{\mathbf{v}_M\}_M + \left\{\mathbf{v}_P^M\right\}_M + \{\Omega_M\}_M\left\{\mathbf{r}_P^M\right\}_M \tag{5.1}$$

This is probably more familiar with the cross product,

$$\{\Omega_M\}_M\left\{\mathbf{r}_P^M\right\}_M = \{\boldsymbol{\omega}_M\}_M \times \left\{\mathbf{r}_P^M\right\}_M$$

To express the inertial acceleration of P in F_M we need the transformation of the inertial derivative of $\mathbf{v}_P$. Using our previous result for the needed transformation we have

$$\begin{aligned}\{\mathbf{a}_P\}_M &= T_{M,I}\tfrac{d\mathbf{v}_P}{dt} \\ &= \left\{\tfrac{d\mathbf{v}_P}{dt}{}_M\right\}_M + \{\Omega_M\}_M\{\mathbf{v}_P\}_M\end{aligned}$$

Again the overworked superscripts and subscripts. This one means the derivative of the vector $\mathbf{v}_P$ as seen from F_M, with the answer expressed in

F_M coordinates. We already have $\mathbf{v}_P$ in F_M coordinates, $\{\mathbf{v}_P\}_M = \{\mathbf{v}_M\}_M + \{\mathbf{v}_P^M\}_M + \{\Omega_M\}_M \{\mathbf{r}_P^M\}_M$, so:

$$\begin{aligned}
\left\{\tfrac{d\mathbf{v}_P}{dt}_M\right\}_M &= \left\{\tfrac{d\mathbf{v}_M}{dt}_M\right\}_M + \{\mathbf{a}_P^M\}_M + \left[\frac{d\left(\{\Omega_M\}_M\{\mathbf{r}_P^M\}_M\right)}{dt}_M\right]_M \\
&= \left\{\begin{array}{c} \left[T_{M,I}\frac{d\mathbf{v}_M}{dt} - \{\Omega_M\}_M\{\mathbf{v}_M\}_M\right] + \{\mathbf{a}_P^M\}_M \\ + \left[\left\{\dot{\Omega}_M\right\}_M \{\mathbf{r}_P^M\}_M + \{\Omega_M\}_M\{\mathbf{v}_P^M\}_M\right] \end{array}\right\} \\
&= \left\{\begin{array}{c} \left[\{\mathbf{a}_M\}_M - \{\Omega_M\}_M\{\mathbf{v}_M\}_M\right] + \{\mathbf{a}_P^M\}_M \\ + \left[\left\{\dot{\Omega}_M\right\}_M \{\mathbf{r}_P^M\}_M + \{\Omega_M\}_M\{\mathbf{v}_P^M\}_M\right] \end{array}\right\}
\end{aligned}$$

The other term is $\{\Omega_M\}_M \{\mathbf{v}_P\}_M$, whence

$$\begin{aligned}
\{\Omega_M\}_M\{\mathbf{v}_P\}_M &= \{\Omega_M\}_M\left[\{\mathbf{v}_M\}_M + \{\mathbf{v}_P^M\}_M + \{\Omega_M\}_M\{\mathbf{r}_P^M\}_M\right] \\
&= \{\Omega_M\}_M\{\mathbf{v}_M\}_M + \{\Omega_M\}_M\{\mathbf{v}_P^M\}_M + \{\Omega_M\}_M^2\{\mathbf{r}_P^M\}_M
\end{aligned}$$

Assembling and canceling the two $\{\Omega_M\}_M \{\mathbf{v}_M\}_M$ terms that appear,

$$\begin{aligned}
\{\mathbf{a}_P\}_M = \{\mathbf{a}_M\}_M &+ \{\mathbf{a}_P^M\}_M \\
&+ \left\{\dot{\Omega}_M\right\}_M \{\mathbf{r}_P^M\}_M \\
&+ 2\{\Omega_M\}_M\{\mathbf{v}_P^M\}_M \\
&+ \{\Omega_M\}_M^2\{\mathbf{r}_P^M\}_M
\end{aligned} \tag{5.2}$$

In words this says that the inertial acceleration of a point P is the inertial acceleration of the (origin of the) moving reference frame, plus the acceleration of the point relative to the moving reference frame, plus the tangential, coriolis, and centripetal accelerations at the point. All quantities are expressed in the coordinates of the moving reference frame.

5.2.2 Earth-Centered Moving Reference Frame

As promised we now will use the expression for $\{\mathbf{a}_P\}_M$ twice: first with the Earth-centered frame as F_M and a point on the Earth's surface as P, then with an Earth-fixed frame as F_M and the aircraft CG as P. In the first case we have $F_M = F_{EC}$ and $P = O_E$, or, to save subscripts, $P = E$, understood to be the origin of some F_E . Substituting we have

$$\begin{aligned}\{\mathbf{a}_E\}_{EC} = \{\mathbf{a}_{EC}\}_{EC} &+ \left\{\mathbf{a}_E^{EC}\right\}_{EC} \\ &+ \left\{\dot{\Omega}_{EC}\right\}_{EC} \left\{\mathbf{r}_E^{EC}\right\}_{EC} \\ &+ 2\left\{\Omega_{EC}\right\}_{EC} \left\{\mathbf{v}_E^{EC}\right\}_{EC} \\ &+ \{\Omega_{EC}\}_{EC}^2 \left\{\mathbf{r}_E^{EC}\right\}_{EC}\end{aligned}$$

On the right hand side, $\mathbf{a}_{EC}$ is the inertial accleration of the Earth's center. If we neglect the annual rotation of the Earth about the sun, and any inertial acceleration the sun may have, this term is zero. The term $\mathbf{r}_E^{EC}$ is the position vector of the Earth-fixed origin from the center of the Earth and (barring earthquakes) is constant. Therefore $\mathbf{v}_E^{EC}$ and $\mathbf{a}_E^{EC}$ are zero. If the Earth's rotation is constant (it changes only on a geological scale) then $\dot{\Omega}_{EC}$ is zero as well. This leaves just the centripetal acceleration,

$$\{\mathbf{a}_E\}_{EC} = \{\Omega_{EC}\}_{EC}^2 \left\{\mathbf{r}_E^{EC}\right\}_{EC}$$

The rotation of the Earth is about 360 deg per 24 hours, or $7.27 \times 10^{-5} rad/s$. If we take the diameter of the Earth to be 6,875.5 nautical miles then $\mathbf{r}_E^{EC}$ has magnitude of roughly $2.09 \times 10^7 ft$. For F_E on the equator the magnitude of the centripetal acceleration is about $0.11 ft/s^2$, and it is zero at the poles.

5.2.3 Earth-Fixed Moving Reference Frame

Now with an Earth-fixed frame as F_M and the aircraft CG as P, we substitute $F_M = F_E$ and $P = C$ and write

$$\begin{aligned}\{\mathbf{a}_C\}_E = \{\mathbf{a}_E\}_E &+ \left\{\mathbf{a}_C^E\right\}_E \\ &+ \left\{\dot{\Omega}_E\right\}_E \left\{\mathbf{r}_C^E\right\}_E \\ &+ 2\left\{\Omega_E\right\}_E \left\{\mathbf{v}_C^E\right\}_E \\ &+ \{\Omega_E\}_E^2 \left\{\mathbf{r}_C^E\right\}_E\end{aligned}$$

The coordinate system F_E is fixed relative to F_{EC} and thus rotates with it, so $\dot{\Omega}_E$ is zero. This leaves

$$
\begin{aligned}
\{\mathbf{a}_C\}_E = \{\mathbf{a}_E\}_E + &\left\{\mathbf{a}_C^E\right\}_E \\
&+2\{\Omega_E\}_E\left\{\mathbf{v}_C^E\right\}_E \\
&+\{\Omega_E\}_E^2\left\{\mathbf{r}_C^E\right\}_E
\end{aligned}
$$

The question is, do we really need to keep track of all of the terms? We need the inertial acceleration of our aircraft's CG for Newton's Second Law, and it would be convenient to ignore the rotational accelerations and just use the local acceleration $\mathbf{a}_C^E$ for our aircraft's inertial acceleration. For instance, the centripetal acceleration $\mathbf{a}_E$ was just derived in F_{EC}, so $\{\mathbf{a}_E\}_E = T_{E,EC}\{\mathbf{a}_E\}_{EC}$, and it has maximum magnitude of $0.11ft/s^2$. Whether it and other terms should be retained depends on what kind of problem is being addressed.

The terms $\mathbf{r}_C^E$, $\mathbf{v}_C^E$, and $\mathbf{a}_C^E$ are respectively the position, velocity, and acceleration of our aircraft's CG relative to a point on the Earth, and are what would be measured by a ground tracking station at F_E's origin. The position $\mathbf{r}_C^E$ may be on the order of hundreds of nautical miles; if it is very large then the problem should probably be referred to an Earth-centered reference frame. With $7.27 \times 10^{-5} rad/s$ for the magnitude of Ω_E and $1.5 \times 10^6 ft$ for the magnitude of $\mathbf{r}_C^E$, the maximum centripetal component of acceleration is on the order of $8 \times 10^{-3} ft/s^2$. The velocity could easily be greater than $1,000kts$, and may be much greater if, for example, a reentry vehicle is being tracked. For rough estimates, the maximum coriolis acceleration at a speed of $2,000ft/s$ is around $0.3ft/s^2$.

In problems that span a large time or distance or both, and in which great accuracy is required, each term may be significant. In calculations for artillery projectiles they are all retained. If the objective is to understand the short-term dynamics of an aircraft in response to atmospheric disturbances or control inputs, it is usually assumed that $\{\mathbf{a}_C\}_E = \left\{\mathbf{a}_C^E\right\}_E$. Because this expression ignores all terms resulting from the rotation of the Earth and does not depend on the Earth's curvature, it is commonly called the flat-Earth approximation.

$$
\{\mathbf{a}_C\}_E = \left\{\mathbf{a}_C^E\right\}_E \qquad \text{(Flat, non-rotating Earth)} \tag{5.3}
$$

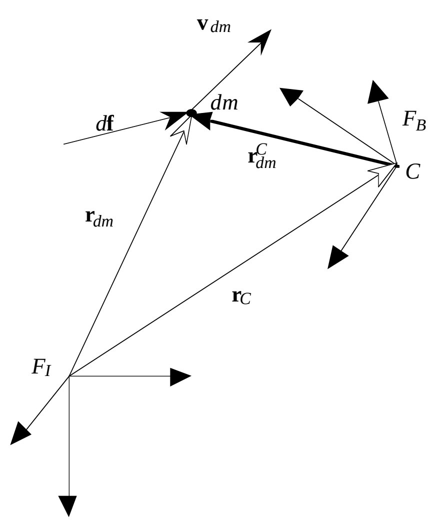

Figure 5.2: A Particle in Inertial Space

5.3 Inertial Acceleration of a Mass

The previous equations dealt only with a point moving with respect to a rotating reference frame in inertial space. Newton's Second Law applies to an infinitesimal mass dm at such a point. Any physical body, such as an aircraft, has finite mass that is the summation of an infinite number of such infinitesimal masses. We begin by examining how Newton's Second Law applies to one such infinitesimal mass, and sum these relationships over the whole mass. Figure 5.2 is the same as figure 5.1, except now the point P is replaced by dm located at that point. The mass dm is acted on by infinitesimal force $d\mathbf{f}$ and has inertial velocity $\mathbf{v}_{dm}$. We replace F_M by a coordinate system with its origin at the CG of the aircraft for reasons that will become clear shortly. Here we have used F_B, but any suitably defined coordinate system would do as well.

5.3.1 Linear Acceleration

Newton tells us that the infinitesimal force $d\mathbf{f}$ is proportional to the (inertial) time rate of change of the momentum of dm, or

$$d\mathbf{f} = \dot{\mathbf{v}}_{dm} dm$$

We wish to integrate both sides of this equation over the entire mass of the aircraft, and relate the result to the acceleration of the CG. First note that the CG is defined such that

$$\int_m \mathbf{r}^C_{dm} dm = 0$$

Now, $\dot{\mathbf{v}}_{dm} = \ddot{\mathbf{r}}_{dm}$ so we relate $\mathbf{r}_{dm}$ to $\mathbf{r}^C_{dm}$ and $\mathbf{r}_C$ to obtain $\mathbf{r}_{dm} = \mathbf{r}^C_{dm} + \mathbf{r}_C$. We will form the integrals and then differentiate with respect to time as we go. First,

$$\int_m \mathbf{r}_{dm} dm = \int_m \mathbf{r}^C_{dm} dm + \int_m \mathbf{r}_C dm$$

On the right hand side, the first integral is zero because we have chosen our coordinate system's origin to be at the CG. In the second integral, we observe that, for a given mass distribution, the inertial position of the CG is fixed and so $\mathbf{r}_C$ can be taken out of the integral. The result is

$$\int_m \mathbf{r}_{dm} dm = \mathbf{r}_C \int_m dm = m\mathbf{r}_C$$

Differentiating both sides with respect to time, and noting that the integrals do not depend on time,

$$m\dot{\mathbf{r}}_C = m\mathbf{v}_C = \int_m \dot{\mathbf{r}}_{dm} dm = \int_m \mathbf{v}_{dm} dm$$

Differentiating again,

$$m\dot{\mathbf{v}}_C = m\mathbf{a}_C = \int_m \dot{\mathbf{v}}_{dm} dm$$

But we have $d\mathbf{f} = \dot{\mathbf{v}}_{dm} dm$, so

$$m\mathbf{a}_C = \int_m d\mathbf{f}$$

The integral on the right accounts for all the forces acting on particles of mass in the body of the aircraft. The internal forces are taken to cancel one another out, so the net result is the vector sum of all the externally applied forces, $\mathbf{F}$. In other words,

$$\mathbf{F} = m\mathbf{a}_C \tag{5.4}$$

The freshman form of Newton's Second Law is valid for a mass only if applied to the mass center of gravity. We drop the subscript C and write the inertial equation as

$$\mathbf{F} = m\mathbf{a} = m\dot{\mathbf{v}}$$

In the body axes we have $\{\mathbf{F}\}_B = T_{B,I}\mathbf{F}$, or

$$\begin{aligned} \{\mathbf{F}\}_B &= mT_{B,I}\dot{\mathbf{v}} \\ &= m\{\dot{\mathbf{v}}_B\}_B + m\{\Omega_B\}_B\{\mathbf{v}\}_B \end{aligned} \tag{5.5}$$

(Referenced to CG)

5.3.2 Rotational Acceleration

We also will need the rotational equivalent of Newton's Second Law. The moment of $d\mathbf{f}$ about O_I is $d\mathbf{M}$, and

$$d\mathbf{M} = \mathbf{r}_{dm} \times d\mathbf{f} = \mathbf{r}_{dm} \times \dot{\mathbf{v}}_{dm} dm$$

The inertial angular momentum of dm is denoted $d\mathbf{h}$ and is

$$d\mathbf{h} = \mathbf{r}_{dm} \times \mathbf{v}_{dm} dm$$

Relating the two through the Second Law,

$$d\mathbf{M} = \mathbf{r}_{dm} \times d\mathbf{f} = \frac{d}{dt} d\mathbf{h} = \frac{d}{dt}\left(\mathbf{r}_{dm} \times \mathbf{v}_{dm}\right) dm$$

To get the moment and time rate of change of angular momentum with respect to the CG, we again integrate and use $\mathbf{r}_{dm} = \mathbf{r}^C_{dm} + \mathbf{r}_C$. We have

$$\int_m \mathbf{r}_{dm} \times d\mathbf{f} = \frac{d}{dt}\int_m (\mathbf{r}_{dm} \times \mathbf{v}_{dm})\, dm$$

Working first on the left hand side,

$$\begin{aligned}
\textstyle\int_m \mathbf{r}_{dm} \times d\mathbf{f} &= \textstyle\int_m \left(\mathbf{r}^C_{dm} + \mathbf{r}_C\right) \times d\mathbf{f} \\
&= \textstyle\int_m \left(\mathbf{r}^C_{dm} \times d\mathbf{f}\right) + \int_m (\mathbf{r}_C \times d\mathbf{f}) \\
&= \textstyle\int_m \left(\mathbf{r}^C_{dm} \times d\mathbf{f}\right) + \mathbf{r}_C \times \int_m d\mathbf{f} \\
&= \mathbf{M}_C + (\mathbf{r}_C \times \mathbf{F})
\end{aligned}$$

Here we have defined M_C to be the moment about the CG.

On the right hand side,

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\int_m (\mathbf{r}_{dm} \times \mathbf{v}_{dm})\, dm &= \tfrac{d}{dt}\textstyle\int_m \left(\mathbf{r}^C_{dm} \times \mathbf{v}_{dm}\right) dm + \tfrac{d}{dt}\int_m (\mathbf{r}_C \times \mathbf{v}_{dm})\, dm \\
&= \tfrac{d}{dt}\mathbf{h}_C + \tfrac{d}{dt}\left[\mathbf{r}_C \times \textstyle\int_m \mathbf{v}_{dm} dm\right]
\end{aligned}$$

We have defined $\mathbf{h}_C$ as the angular momentum with respect to the CG. For the remaining term we have

$$\frac{d}{dt}\left[\mathbf{r}_C \times \int_m \mathbf{v}_{dm} dm\right] = \mathbf{v}_C \times \int_m \mathbf{v}_{dm} dm + \mathbf{r}_C \times \int_m \dot{\mathbf{v}}_{dm} dm$$

We previously showed that

$$m\mathbf{v}_C = \int_m \mathbf{v}_{dm} dm$$

and

$$m\mathbf{a}_C = \int_m \dot{\mathbf{v}}_{dm} dm$$

so

$$\begin{aligned}\tfrac{d}{dt}\left[\mathbf{r}_C \times \textstyle\int_m \mathbf{v}_{dm}\right] &= (\mathbf{v}_C \times m\mathbf{v}_C) + (\mathbf{r}_C \times m\mathbf{a}_C)\\ &= \mathbf{r}_C \times m\mathbf{a}_C\\ &= \mathbf{r}_C \times \mathbf{F}\end{aligned}$$

On the left,

$$\int_m (\mathbf{r}_{dm} \times d\mathbf{f}) = \mathbf{M}_C + (\mathbf{r}_C \times \mathbf{F})$$

And on the right,

$$\frac{d}{dt}\int_m (\mathbf{r}_{dm} \times \mathbf{v}_{dm})\, dm = \frac{d}{dt}\mathbf{h}_C + (\mathbf{r}_C \times \mathbf{F})$$

Upon equating the left and right sides,

$$\mathbf{M}_C = \frac{d}{dt}\mathbf{h}_C \tag{5.6}$$

That is, the externally applied moments about the CG are equal to the inertial time-rate of change of angular momentum about the CG. We drop the subscript C and use the dot notation so that $\mathbf{M} = \dot{\mathbf{h}}$. In the body-fixed coordinate system this is:

$$\begin{aligned}\{\mathbf{M}\}_B &= T_{B,I}\mathbf{M} = T_{B,I}\dot{\mathbf{h}}\\ &= \left\{\dot{\mathbf{h}}_B\right\}_B + \{\Omega_B\}_B\,\mathbf{h}_B\end{aligned}$$

To evaluate the angular momentum we have

$$\mathbf{h} = \int_m \left(\mathbf{r}^C_{dm} \times \mathbf{v}_{dm}\right) dm = \int_m (\mathbf{R}_{dm}\mathbf{v}_{dm})\, dm$$

In this expression $\mathbf{R}_{dm}$ is the matrix equivalent of the operation $\left[\mathbf{r}^C_{dm}\times\right]$ (the superscript C will be dropped in all subsequent discussion, as everything will be referenced to the CG). The vector $\mathbf{v}_{dm}$ is the inertial velocity of dm, or

$\mathbf{v}_{dm} = \dot{\mathbf{r}}dm$. To relate the angular momentum to the moving reference frame we use $\{\dot{\mathbf{r}}_{dm_B}\}_B = T_{B,I}\{\dot{\mathbf{r}}_{dm}\} - \{\Omega_B\}_B\{\mathbf{r}_{dm}\}_B$. The angular momentum in inertial coordinates is then

$$\begin{aligned}
\mathbf{h} &= \textstyle\int_m (R_{dm}\mathbf{v}_{dm})\, dm \\
&= \textstyle\int_m (R_{dm}\dot{\mathbf{r}}_{dm})\, dm \\
&= \textstyle\int_m \{R_{dm}T_{I,B}\left[\{\dot{\mathbf{r}}_{dm_B}\}_B + \{\Omega_B\}_B\{\mathbf{r}_{dm}\}_B\right]\}\, dm \\
&= \textstyle\int_m R_{dm}T_{I,B}\{\dot{\mathbf{r}}_{dm_B}\}_B\, dm + \int_m R_{dm}T_{I,B}\{\Omega_B\}_B\{\mathbf{r}_{dm}\}_B\, dm
\end{aligned}$$

The same angular momentum in body coordinates is just $\{\mathbf{h}\}_B = T_{B,I}\mathbf{h}$:

$$\begin{aligned}
\{\mathbf{h}\}_B &= T_{B,I}\mathbf{h} \\
&= \textstyle\int_m T_{B,I}R_{dm}T_{I,B}\{\dot{\mathbf{r}}_{dm_B}\}_B\, dm + \int_m T_{B,I}R_{dm}T_{I,B}\{\Omega_B\}_B\{\mathbf{r}_{dm}\}_B\, dm \\
&= \textstyle\int_m \{R_{dm}\}_B\{\dot{\mathbf{r}}_{dm_B}\}_B\, dm + \int_m \{R_{dm}\}_B\{\Omega_B\}_B\{\mathbf{r}_{dm}\}_B\, dm
\end{aligned}$$

We note that

$$\begin{aligned}
\{\Omega_B\}_B\{\mathbf{r}_{dmB}\}_B &= \{\boldsymbol{\omega}_B\}_B \times \{\mathbf{r}_{dm}\}_B \\
&= -\{\mathbf{r}_{dm}\}_B \times \{\boldsymbol{\omega}_B\}_B \\
&= -\{R_{dm}\}_B\{\boldsymbol{\omega}_B\}_B
\end{aligned}$$

and

$$\{R_{dm}\}_B\{R_{dm}\}_B = \{R_{dm}\}_B^2$$

So,

$$\{\mathbf{h}\}_B = \int_m \{\mathbf{r}_{dm}\}_B \times \{\dot{\mathbf{r}}_{dm_B}\}_B\, dm - \int_m \{R_{dm}\}_B^2\{\boldsymbol{\omega}_B\}_B\, dm$$

Now, since $\{\boldsymbol{\omega}_B\}_B$ does not affect the integration, we move it outside the integral,

$$\{\mathbf{h}\}_B = \int_m \{\mathbf{r}_{dm}\}_B \times \{\dot{\mathbf{r}}_{dmB}\}_B\, dm + \left[-\int_m \{R_{dm}\}_B^2\, dm\right]\{\boldsymbol{\omega}_B\}_B$$

The first integral depends on $\{\dot{\mathbf{r}}_{dmB}\}_B$ which is the rate of change of $\mathbf{r}_{dm}$ as seen from F_B with components in F_B. If we denote

$$\{\mathbf{r}_{dm}\}_B \equiv \left\{ \begin{array}{c} x_{dm} \\ y_{dm} \\ z_{dm} \end{array} \right\}$$

Then

$$\{\dot{\mathbf{r}}_{dm_B}\}_B = \left\{ \begin{array}{c} \dot{x}_{dm} \\ \dot{y}_{dm} \\ \dot{z}_{dm} \end{array} \right\}$$

The vector $\{\dot{\mathbf{r}}_{dm}\}_B$ represents the relative motion of the particles of mass that make up the aircraft: fuel sloshing, spinning rotors, fluttering control surfaces, etc. If the aircraft is treated as a perfectly rigid body then the term and the integral vanish. In any event we will call the integral the *deformation component* of the angular momentum and denote it $\mathbf{h}^*_B$:

$$\mathbf{h}^*_B \equiv \int_m \{\mathbf{r}_{dm}\}_B \times \{\dot{\mathbf{r}}_{dmB}\}_B \, dm$$

The other integral may be evaluated term-by-term. First,

$$\begin{aligned} -\{R_{dm}\}^2_B &= -\left[\begin{array}{ccc} 0 & -z_{dm} & y_{dm} \\ z_{dm} & 0 & -x_{dm} \\ -y_{dm} & x_{dm} & 0 \end{array} \right]^2 \\ &= \left[\begin{array}{ccc} (y^2_{dm} + z^2_{dm}) & -x_{dm}y_{dm} & -x_{dm}z_{dm} \\ -x_{dm}y_{dm} & (x^2_{dm} + z^2_{dm}) & -y_{dm}z_{dm} \\ -x_{dm}z_{dm} & -y_{dm}z_{dm} & (x^2_{dm} + y^2_{dm}) \end{array} \right] \end{aligned}$$

The integral of this matrix is the matrix of the integrals of its terms, so

$$-\int_m \{R_{dm}\}^2_B \, dm = \left[\begin{array}{ccc} \int_m (y^2_{dm} + z^2_{dm}) \, dm & -\int_m x_{dm}y_{dm}dm & -\int_m x_{dm}z_{dm}dm \\ -\int_m x_{dm}y_{dm}dm & \int_m (x^2_{dm} + z^2_{dm}) \, dm & -\int_m y_{dm}z_{dm}dm \\ -\int_m x_{dm}z_{dm}dm & -\int_m y_{dm}z_{dm}dm & \int_m (x^2_{dm} + y^2_{dm}) \, dm \end{array} \right]$$

We recognize the integrals on the right as the moments of inertia in the particular body axis system we have chosen,

$$-\int_m \{R_{dm}\}_B^2 \, dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \equiv I_B$$

Finally, for the angular momentum about the CG in body axes,

$$\mathbf{h}_B = \mathbf{h}_B^* + I_B \{\boldsymbol{\omega}_B\}_B$$

This is the expression we needed for $\{\mathbf{M}\}_B = \left\{\dot{\mathbf{h}}_B\right\}_B + \{\Omega_B\}_B \{\mathbf{h}\}_B$. Everything is in body coordinates, so we have

$$\{\mathbf{M}\}_B = \left[\dot{\mathbf{h}}_B^* + \dot{I}_B \{\boldsymbol{\omega}_B\}_B + I_B \{\dot{\boldsymbol{\omega}}_B\}_B\right] + \left[\{\Omega_B\}_B \mathbf{h}_B^* + \{\Omega_B\}_B I_B \{\boldsymbol{\omega}_B\}_B\right]$$

If we have a rigid body then $\mathbf{h}_B^*$, $\dot{\mathbf{h}}_B^*$, and $\dot{I}_B$ vanish leaving

$$\{\mathbf{M}\}_B = I_B \{\dot{\boldsymbol{\omega}}_B\}_B + \{\Omega_B\}_B I_B \{\boldsymbol{\omega}_B\}_B \tag{5.7}$$

5.4 States

The position and velocity variables for which we have derived differential equations are collectively referred to as the *states* of the aircraft. The states are a minimum set of variables that completely describe the aircraft position, orientation, and velocity. The basic states derived so far are the inertial linear velocity, the inertial angular velocity, and a set of three transformation variables that uniquely describe the orientation of the aircraft with respect to inertial space. For the latter we usually take Euler angles, since they already constitute the minimum set of orientation variables. The only thing missing is the inertial position of the aircraft which is simply the integral of the inertial velocity. The twelve basic states of an aircraft may therefore be written as the scalar components of $\mathbf{v}_C$, $\boldsymbol{\omega}_B$, and $\mathbf{r}_C$, plus the three Euler angles that define $T_{B,I}$.

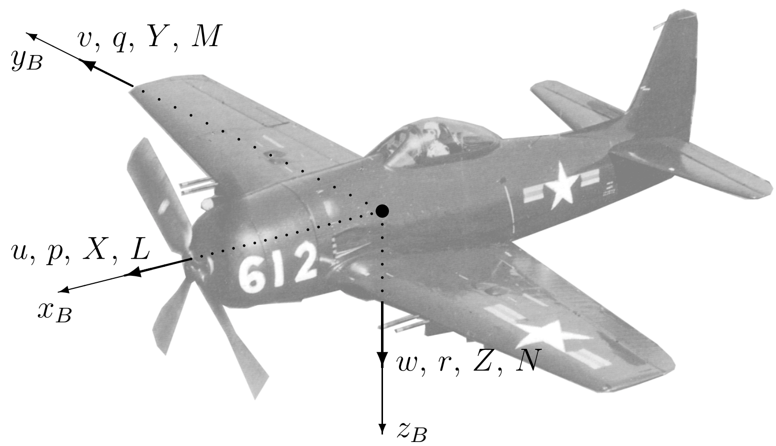

Figure 5.3: Linear velocity (u, v, w), angular velocity (p, q, r), aerodynamic force (X, Y, Z), and aerodynamic moment (L, M, N) components in body axes.

5.5 Customs and Conventions

5.5.1 Linear Velocity Components

Inertial velocities in body and wind axes occur frequently, and the components associated with them are given special symbols. These are shown in figures 5.3 and 5.4, and are defined as follows:

$$\{\mathbf{v}_C\}_B = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}, \ \{\mathbf{v}_C\}_W = \begin{Bmatrix} V_c \\ 0 \\ 0 \end{Bmatrix}$$

The position and velocity of the aircraft CG with respect to some Earth-fixed coordinate system F_E is obviously of importance in navigation. The position $\left\{\mathbf{r}_C^E\right\}_E$ is denoted by x_E, y_E, and z_E, and the velocity

$$\left\{\mathbf{v}_C^E\right\}_E = \left\{ \left.\frac{d\mathbf{r}_C^E}{dt}\right._E \right\}_E = \begin{Bmatrix} \dot{\mathbf{x}}_E \\ \dot{\mathbf{y}}_E \\ \dot{\mathbf{z}}_E \end{Bmatrix}$$

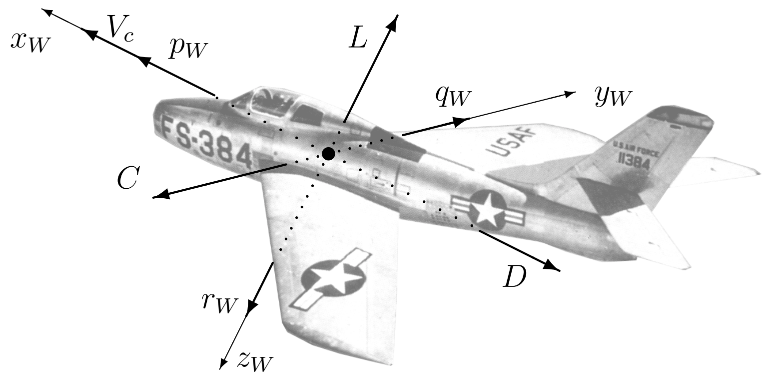

Figure 5.4: Linear velocity (V_c), angular velocity (p_W, q_W, r_W), and aerodynamic force (D, C, L) components in wind axes.

With the flat-Earth assumption the aircraft's inertial velocity is the same as its velocity with respect to F_E, and since F_E is always parallel to F_H under this assumption,

$$\begin{Bmatrix} \dot{\mathbf{x}}_E \\ \dot{\mathbf{y}}_E \\ \dot{\mathbf{z}}_E \end{Bmatrix} = T_{B,H}^T \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = T_{W,H}^T \begin{Bmatrix} V_c \\ 0 \\ 0 \end{Bmatrix} \qquad \text{(Flat Earth)}$$

5.5.2 Angular Velocity Components

Angular velocity components were previously defined and are:

$$\{\boldsymbol{\omega}_B\}_B = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}, \ \{\boldsymbol{\omega}_B\}_W = \begin{Bmatrix} p_W \\ q_W \\ r_W \end{Bmatrix}$$

5.5.3 Forces

The external forces that accelerate an aircraft are made up of its weight, thrust of the propulsive system, and aerodynamic forces.

The *weight* is naturally represented in the local horizontal reference frame, and is converted by the appropriate transformation to the desired system:

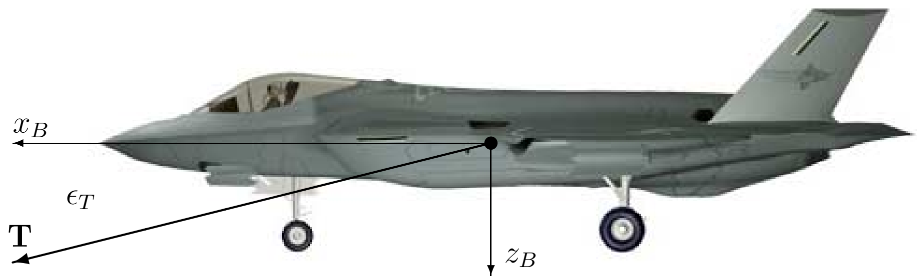

Figure 5.5: Thrust Vector Orientation (exaggerated).

$$\{\mathbf{W}\}_H = \begin{Bmatrix} 0 \\ 0 \\ mg \end{Bmatrix}$$

$$\{\mathbf{W}\}_B = \begin{Bmatrix} -mg\sin\theta \\ mg\sin\phi\cos\theta \\ mg\cos\phi\cos\theta \end{Bmatrix},\ \{\mathbf{W}\}_W = \begin{Bmatrix} -mg\sin\gamma \\ mg\sin\mu\cos\gamma \\ mg\cos\mu\cos\gamma \end{Bmatrix}$$

The *thrust* is usually naturally represented in some body axis system. The direction of thrust may vary with respect to that system (as in the case of vectored thrust, such as the Harrier). If the thrust vector $\mathbf{T}$ varies in its relationship to the aircraft then a separate coordinate system analogous to the wind axes should be defined. For the analysis of a particular flight condition the thrust vector $\mathbf{T}$ is normally considered fixed with respect to the body. The direction of thrust usually lies in the plane of symmetry but may not align with the x-axis. The symbol ϵ_T will be used to denote the difference between $\mathbf{T}$ and x_B, positive as shown in figure 5.5.

In this special case we have

$$\{\mathbf{T}\}_B = \begin{Bmatrix} T\cos\epsilon_T \\ 0 \\ T\sin\epsilon_T \end{Bmatrix}$$

$$\{\mathbf{T}\}_W = T_{W,B} \left\{ \begin{array}{c} T\cos\epsilon_T \\ 0 \\ T\sin\epsilon_T \end{array} \right\} = \left\{ \begin{array}{c} T\cos\beta\cos(\epsilon_T - \alpha) \\ -T\sin\beta\cos(\epsilon_T - \alpha) \\ T\sin(\epsilon_T - \alpha) \end{array} \right\}$$

We will represent the *Aerodynamic Forces* by the vector $\mathbf{F}_A$. From the point of view of the aerodynamicist the aerodynamic forces are naturally represented in the wind axes as the familiar components lift, drag, and side force. The customary definitions of lift (L) and drag (D) have positive lift in the negative z_W direction and positive drag in the negative x_W direction. We will adopt the symbol C for the aerodynamic side force and for consistency define positive side force in the negative y_W direction, as shown in figure 5.4. Thus,

$$\{\mathbf{F}_A\}_W = \left\{ \begin{array}{c} -D \\ -C \\ -L \end{array} \right\}$$

The relationship (angle-of-attack and sideslip) of the wind axes to some body axis system greatly influences the magnitude of the aerodynamic forces, so the wind axes alone are insufficient to characterize these forces. Representation of the aerodynamic forces in the body axes is at least as "natural" as their representation in wind axes, and these forces have their own symbols in body axes, X, Y, and Z. The relationships are shown in figure 5.3 and defined as follows:

$$\{\mathbf{F}_A\}_B = \left\{ \begin{array}{c} X \\ Y \\ Z \end{array} \right\}$$

The two representations are obviously related by $\{\mathbf{F}_A\}_B = T_{B,W}\{\mathbf{F}_A\}_W$, so that data given in either system may represented in the other. In fact, aerodynamic data are often presented in a hybrid system, in which lift and drag are the familiar wind axis quantities, but the side force is in the body axis system. The reason for using this non-orthogonal system is related to the way the forces are actually measured in wind tunnel experiments. Clearly this representation will suffer at very high angles of sideslip β.

5.5.4 Moments

Since the gravity force acts through the CG it does not generate any moments about it. Aerodynamic moments about the CG are normally represented in a body-axis system for two reasons: first, this is how they are normally measured in wind-tunnel experiments, and second, the moments of inertia of the aircraft are reasonably constant in body axes and accelerations are easier to formulate than in wind axes. The components of aerodynamic moments $\mathbf{M}_A$ about the CG in body axes are given the names L, M, and N. The relationships are shown in figure 5.3 and defined as follows:

$$\{\mathbf{M}_A\}_B = \begin{Bmatrix} L \\ M \\ N \end{Bmatrix}$$

Thrust-generated moments about the CG will arise if the net thrust vector $\mathbf{T}$ does not pass directly through the CG. In aircraft with variable thrust vectoring this is intentional, and serves as an additional control. In multi-engine aircraft the loss of one or more engines can create quite large moments. Even if the net thrust vector is fixed with respect to the body, the moment it generates will be proportional to the magnitude of the thrust, which can vary greatly. Thrust generated moments will be denoted $\mathbf{M}_T$, and will normally be available in the body-axes, $\{\mathbf{M}_T\}_B$,

$$\{\mathbf{M}_T\}_B = \begin{Bmatrix} L_T \\ M_T \\ N_T \end{Bmatrix}$$

5.5.5 Groupings

Longitudinal

The assumption that the aircraft has a plane of symmetry leads to a grouping of variables that are associated with motion in that plane. These variables are the X and Z forces, the pitching moment M, the velocities u and w, and the pitch rate q. Collectively these variables are called *longitudinal* variables.

Lateral-Directional

Variables associated with motion about a body-fixed x-axis, loosely those thought of as rolling the aircraft, are called *lateral* variables. These are the rolling moment L and the roll rate p. The remaining variables are the side force Y, the yawing moment N, and the yaw rate r, collectively called *directional* variables. For reasons we will see later there are strong relationships between lateral and directional variables, and they are often grouped together as *lateral/directional* variables, or in the jargon, *lat-dir* variables.

Symmetric Flight

If an aircraft experiences no sideslip β it is said to be in *symmetric* flight. The longitudinal motion of an aircraft is often analyzed under the assumption of no sideslip and, for purposes of such analysis, the lift L and drag D are spoken of as longitudinal variables as well.