Synchronization of Complex Networks via Hybrid Adaptive Coupling and Evolving Topologies ⋆

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Abstract: We discuss a novel approch to synchronization of complex networks based on the use of a hybrid adaptive strategy where the coupling gains between neighbouring oscillators are time-varying and the links of the network become activated only when certain conditions are satisfied. We study the novel strategy on some representative example showing its viability and effectiveness in achieving synchronization. The emerging network topologies are studied and compared via appropriate numerical simulations.

Keywords: Synchronization, adaptive gains, networks, hybrid systems, chaotic behaviour.

1. INTRODUCTION

Networked systems abound in nature and in applied science. Networks of dynamical systems have been recently proposed as models in many diverse fields of applications (see for instance Boccaletti et al. (2006), Newman et al. (2006) and references therein).

Recently particular attention has been focused on the problem of making a network of dynamical systems synchronize onto a common evolution. Typically, the network consists of N identical nonlinear dynamical systems coupled through the edges of the network itself (Boccaletti et al. (2006), Newman et al. (2006)). Each uncoupled system is described by a nonlinear set of ordinary differential equations (ODEs) of the form $\dot{x} = f(x)$, where $x \in R^n$ is the state vector and $f : R^n \mapsto R^n$ is a sufficiently smooth nonlinear vector field describing the system dynamics. Because of the coupling with the neighboring nodes in the network, the dynamics of each oscillator is affected by a nonlinear input representing the interaction of all neighboring nodes with the oscillator itself. Hence, the equations of motion for the generic i-th system in the network become:

$$\frac{dx_i}{dt} = f(x_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} h(x_j), \qquad i = 1, 2, ...N, \tag{1}$$

where x_i represents the state vector of the i-th oscillator, σ the overall strength of the coupling, $h(x) : R^n \mapsto R^n$ the output function through which the systems in the network are coupled and $\mathcal{L}_{ij}$ the elements of the Laplacian matrix $\mathcal{L}$ describing the network topology. In particular, $\mathcal{L}$ is such that its entries, $\mathcal{L}_{ij}$, are zero if node i is not connected to node $j \neq i$, while are negative if node i is connected to node j, with $\mid \mathcal{L}_{ij} \mid$ giving a measure of the strength of the interaction.

In general, the coupling gain is chosen to be identical for all edges in the network and constant in time and the topology of the connection is given. Many real-world networks are characterized instead by evolving, adapting coupling gains and self-emergent topologies, varying in time according to different environmental conditions. For example we can mention wireless networks of sensors that gather and communicate data to a central base station (Moallemi and Roy (2004)). Adaptation is also necessary to control networks of robots when the conditions change unexpectedly (i.e. a robot loses a sensor) (Stanley et al. (2003)). Moreover, examples of adaptive networks could be found in biology. Social insect colonies, for instance, have many of the properties of adaptive networks (Fewell (2003)).

A completely decentralized strategy of gain adaptation, named *edge-based*, was presented in de Lellis et al. (2008c), and its effectiveness was proved in de Lellis et al. (2008b). The aim of this paper is to introduce a new hybrid adaptive strategy for gain adaptation, where while the gains continue to be estimated adaptively, the actual connection between two neighboring nodes is only activated if the gain estimate is above a certain threshold. In so doing, starting with a set of unconnected nodes, the network topology and coupling gains will both be observed to evolve in time. A network topology guaranteeing synchronization is therefore evolved together with the appropriate mutual strength between oscillators to guarantee the achievement of the synchronous orbit.

The rest of the paper is outlined as follows. In Sec. 2 we introduce the *edge-based* decentralized adaptive strategy firstly presented in (de Lellis et al. (2008c)). Then in Sec. 3, we present the hybrid adaptive approach to synchronization. In Sec. 4, extensive numerical analysis of the novel hybrid strategy is performed to show the effectiveness of the strategy. Then, a detailed analysis of the properties of

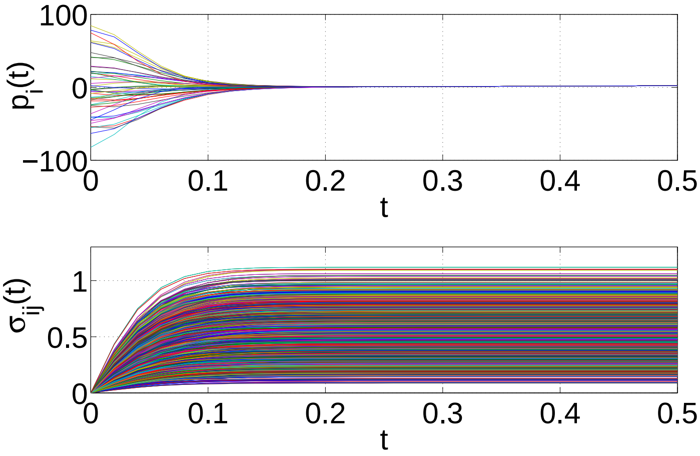

Fig. 1. All-to-all network of 50 Chua's circuits. Edge-based strategy. Evolution of x (top), σ (bottom).

the emerging topology is presented. Conclusions are finally drawn in Sec. 5.

2. ADAPTIVE SYNCHRONIZATION OF NETWORKS

In this section we recall the *edge-based* strategy firstly presented in de Lellis et al. (2008c). Let us denote with $\mathcal{E}$ the set of edges of the network, containing pairs of indices associated to nodes connected by an existing link. For example $(i,j) \in \mathcal{E}$ will indicate there exists an edge connecting node i to node j. Moreover, let us indicate with $\mathcal{E}_i \subseteq \mathcal{E}$ the subset of all edges connected to node i. We consider the following network model:

$$\frac{dx_i}{dt} = f(x_i) - \sum_{j \in \mathcal{E}_i} \sigma_{ij}(t)(h(x_j) - h(x_i)),\ i = 1, 2, \ldots, N, \tag{2}$$

The only difference with the network described by (1) is that now the coupling gains are the adaptive, time-varying functions $\sigma_{ij}(t)$ associated to each edge (i,j). In other words, each pair of connected nodes negotiates the strength of their interaction. The adaptive law is the following:

$$\dot{\sigma}_{ij} = \alpha ||e_{ij}||, \tag{3}$$

where $e_{ij} = x_j - x_i$ is the error between node i and node j. This strategy was proved to be effective in de Lellis et al. (2008b) and then extended in de Lellis et al. (2008a). In figure 1, a representative example is shown of a network of 50 Chua's circuits (see section 4 for further details) coupled through all-to-all network. We can see that synchronization is achieved and that the various σ_{ij} settle to constant values.

3. HYBRID ACTIVATION OF ADAPTIVE GAINS

The adaptation strategy presented in the previous section concerns the case of a topology given *a priori*. It would be interesting to introduce a new adaptive strategy, in which not only the gains but also the topology is adapted. In what follows we will introduce a hybrid adaptive mechanism that, starting from an uncoupled network, adds links on the basis of the estimated coupling gains. Thus, the evolution of the adaptive gains is described by a discontinuous law. In particular, we are going to consider the following mechanism, in which there is an activation threshold on the coupling gains:

$$\sigma_{ij}(t) := \begin{cases} \hat{\sigma}_{ij}, & \text{if } \Phi(\hat{\sigma}_{ij}, e_{ij}) > l, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Our choice of the *reference gains* $\hat{\sigma}_{ij}$ and of the *link activation function* Φ is the following:

$$\dot{\hat{\sigma}}_{ij} = \alpha ||e_{ij}|| \tag{5}$$

$$\Phi(e_{ij}, \hat{\sigma}_{ij}) = \hat{\sigma}_{ij} \tag{6}$$

According to these laws, new links are activated only when the edge-based estimates of the coupling gains exceed the threshold. Therefore at time zero all oscillators will be uncoupled with $\sigma_{ij}(0) = 0$. Then, according to the evolution of the estimated gains, new links will be activated to the network which will evolve until synchronization is reached. As will be shown later, the resulting network topology once synchronization is attained is not all-to-all but is characterized by some interesting topological features. To illustrate this point we now move to the validation of the strategy onto some representative example.

4. NUMERICAL RESULTS

In what follows, we consider networks of N Chua's circuits coupled on all the three state variables. Given that the state of the circuit is $x = (p, q, r)^T$ The dynamics of each isolated circuit are described by the following differential equation:

$$f(p, q, r) = (\beta(-q - \gamma(p)), p - q + r, \xi q)^T, \tag{7}$$

where $\gamma(p) = m_0 p + \frac{1}{2}(m_1 - m_0)(|p+1| - |p-1|)$. To ensure the chaotic behaviour of the circuit the parameters are chosen as in Bartissol and Chua (1988): $\beta = 0.59/0.12$, $\xi = 0.59/0.162$, $m_0 = -0.07$, $m_1 = 1.5$. Moreover, the initial conditions of the coupling gains between the networked circuits σ_{ij} are considered to be null. In all the simulations, we set $h(x_i) = x_i$ and we choose $\alpha = 0.1$. In what follows, we shall consider a network of $N = 50$ nodes. The network is initially disconnected as all gains are initially set to 0 and evolves adaptively reaching a final topological configuration. Thus, we are going to investigate not only the stability properties of the hybrid strategy, but also the properties of the possible emerging network topology; in so doing, we have to take into account that clearly these properties depend also on the initial conditions of the chaotic oscillators.

In our simulations, the initial conditions are chosen randomly from a normal distribution with zero mean and standard deviation equal to 40. First of all, we have focused our attention on the stability properties of the presented strategy. The first thing that we have noticed is that the stability is not affected by the threshold l for the activation of the links. Indeed, the threshold seems to affect only the speed of convergence, that is higher the lower is the threshold, as we can see from figures 2, 3 and 4.

4.1 Analysis of the emerging topologies

Given that, from the above simulations, stability seems to be guaranteed, the main issue to be addressed is the

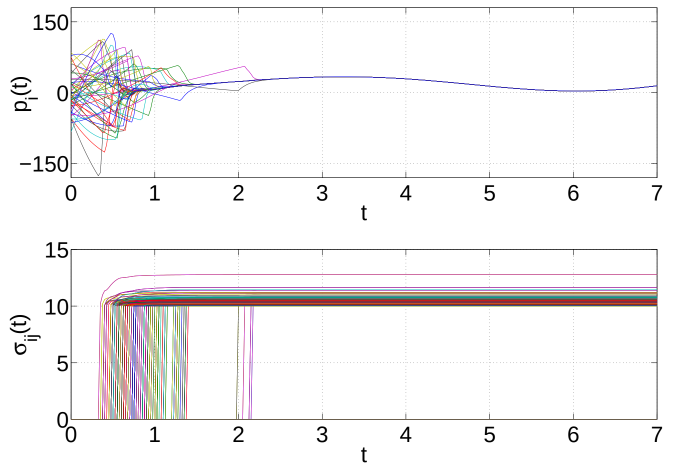

Fig. 2. 50 Chua's circuits. $l = 10$. Evolution of x (top), σ (bottom).

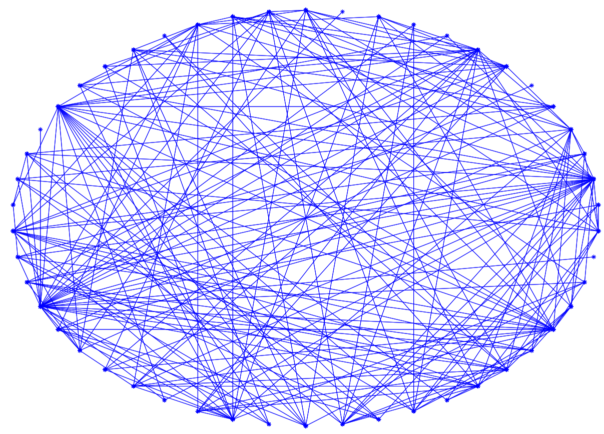

Fig. 5. 50 Chua's circuits. $l = 10$. Topology of the emerging network.

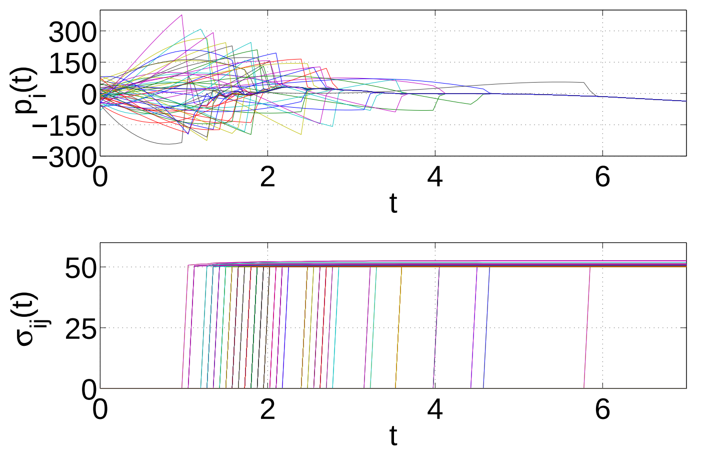

Fig. 3. 50 Chua's circuits. $l = 50$. Evolution of x (top), σ (bottom).

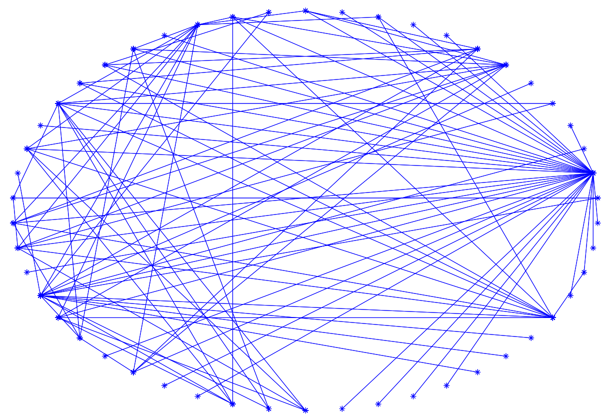

Fig. 6. 50 Chua's circuits. $l = 50$. Topology of the emerging network.

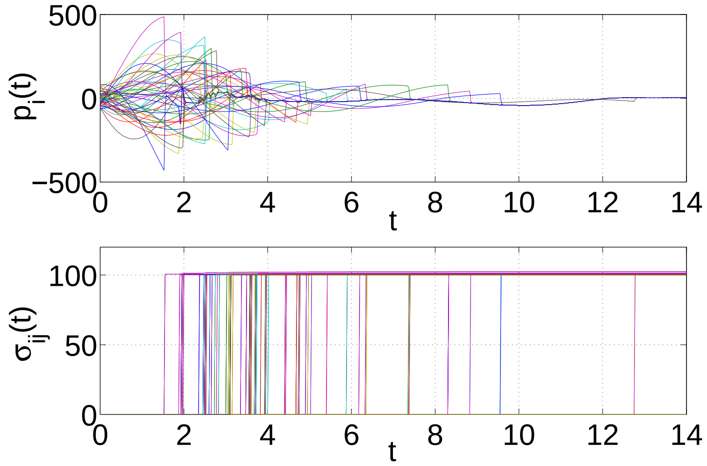

Fig. 4. 50 Chua's circuits. $l = 100$. Evolution of x (top), σ (bottom).

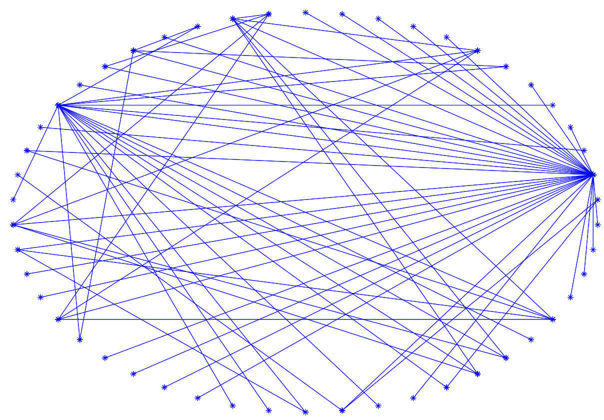

Fig. 7. 50 Chua's circuits. $l = 100$. Topology of the emerging network.

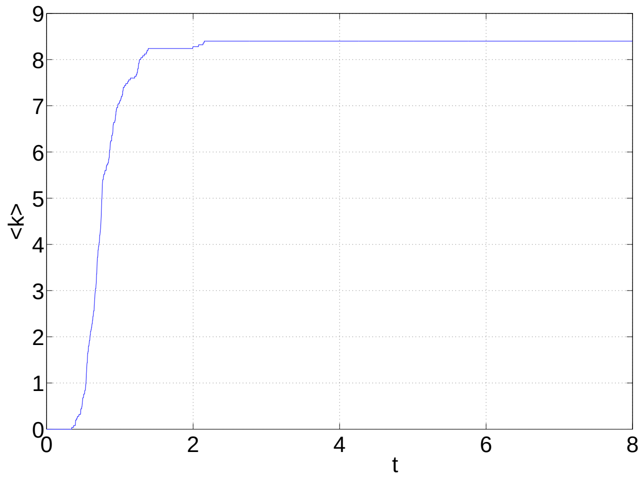

Fig. 8. 50 Chua's circuits. $l = 10$. Evolution of the average degree.

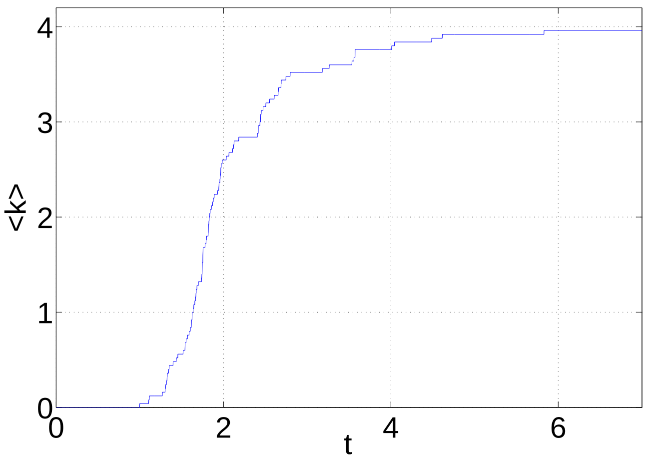

Fig. 9. 50 Chua's circuits. $l = 50$. Evolution of the average degree.

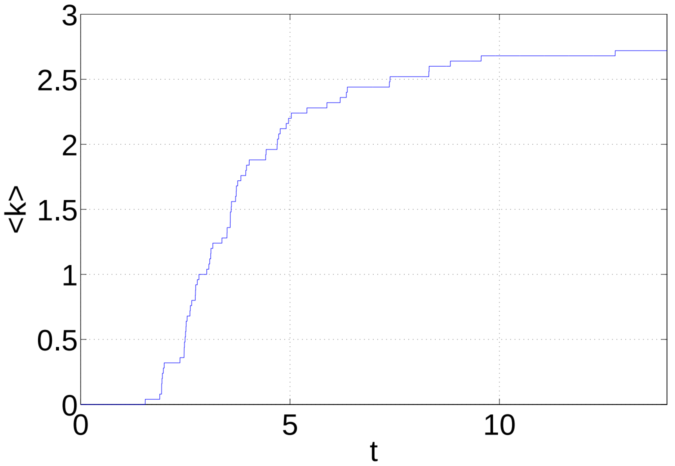

Fig. 10. 50 Chua's circuits. $l = 100$. Evolution of the average degree.

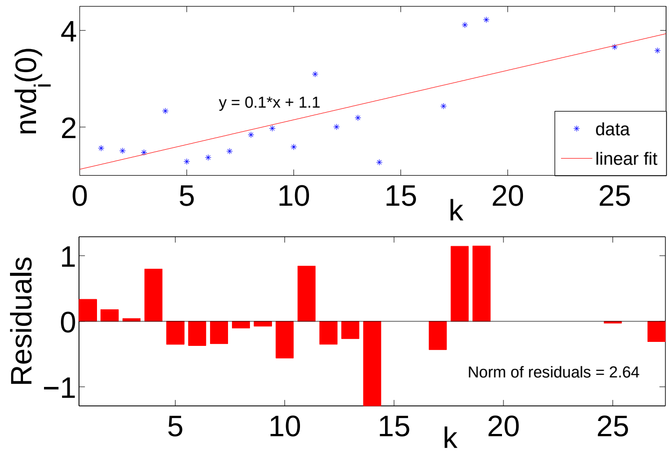

Fig. 11. $l = 10$. Relationship between the initial distance from synchronization $nvd_i(0)$ and the final degree k_i of a node (top). Residuals of the linear fitting (bottom).

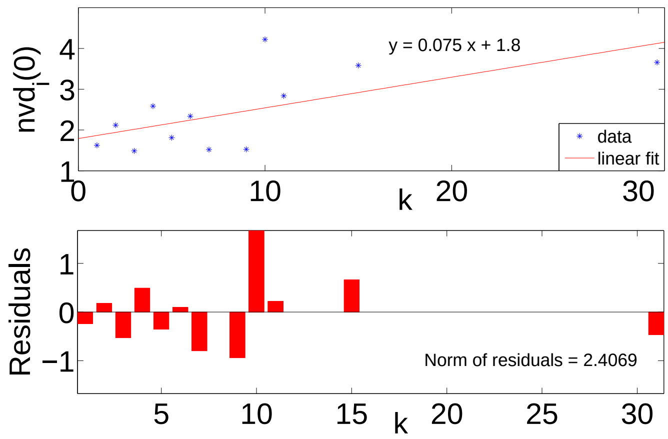

Fig. 12. $l = 50$. Relationship between the initial distance from synchronization $nvd_i(0)$ and the final degree k_i of a node (top). Residuals of the linear fitting (bottom).

analysis of the properties of the emerging topology of the network when synchronization is reached. As we can notice from the graphical representation of the networks obtained for different values of the threshold (see figures 5, 6, 7) at steady-state the network seems to possess an average degree dependent on the threshold value. This is confirmed by looking at the trend of the average degree $< k >$ (see figures 8, 9, 10) while adaptation of links and gains is taking place. We can observe that, the threshold is increased, the network becomes more sparse.

This is confirmed by several numerical experiments as reported in table 14, where the settling time needed to reach synchronization is reported together with the average degree of the resulting network topology and its clustering coefficient and geodesic. We notice that as the activation threshold is increased the average degree of the emerging network decreases with an increase of the transient time needed to reach synchronization. Hence, there is a trade-off between the speed of convergence and

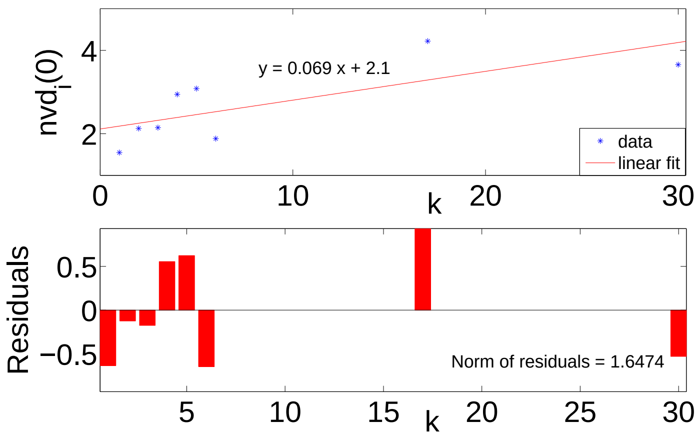

Fig. 13. $l = 100$. Relationship between the initial distance from synchronization $nvd_i(0)$ and the final degree k_i of a node (top). Residuals of the linear fitting (bottom).

Characteristics of the emerging topology				
l	t_s	<k>	Clus_coef	Geodesic
10	2,6	8,4	0,0815	2,0873
20	4,8	6,4	0,0281	2,4122
30	5,95	5,48	0,0360	2,3576
40	6,35	4,88	0,0404	2,3339
50	6,05	3,96	0	2,3641
60	6,15	3,52	0	2,4302
70	7,3	3,04	0	2,4743
80	8,85	2,88	0	2,4824
90	10,9	2,88	0	2,5102
100	12,8	2,72	0	2,4384

Fig. 14. Characteristic of the emergent topology in a network of 50 Chua's cicuits.

Table 2			
l	p1	p2	Res. norm
10	0,1024	1,1258	2,6395
20	0,1138	1,2844	2,7852
30	0,0990	1,5354	2,7000
40	0,0847	1,5749	2,0836
50	0,0752	1,7906	2,4069
60	0,0661	1,8810	1,5663
70	0,0447	2,4735	2,7933
80	0,0637	2,1154	2,2300
90	0,0649	2,1382	1,6407
100	0,0692	2,1122	1,6474

Fig. 15. Parameters p_1 and p_2 and norm of the residuals of the linear fitting for different values of the threshold.

the number of connection: if we increase the threshold, the speed of convergence increases, but the network becomes sparser (see table 14).

Another interesting observation is that the average degree seems to be an exponential function of the threshold l as it rapidly decreases to a value just below 3 as l is increased. This seems to indicate that for very large threshold values, the emergence of an asymptotic minimal topology of the network can be observed. This is the subject of ongoing investigation and will be reported elsewhere. From a preliminary analysis on our small size network of 50 nodes it seems to be evident that the topology, for high values of the threshold, has an average degree between 2 and 3 and is characterized by the absence of triples (the clustering coefficient is null for $l \geq 50$, as reported in table 14).

Finally, it is important to notice that the initial states of the nodes play a key role in determining the final topology of the coupling network. To investigate this relationship, we need to introduce a measure of the initial location of each node, based on the initial errors. Let us define the *vertex distance from synchronization* as the quantity $vd_i(t) = \sum_{j=1}^{N} ||e_{ij}(t)||$. We can then introduce the normalized parameter $nvd_i(t) = \frac{vd_i(t)}{vd_{min}(t)}$ where $vd_{min}(t) = \min_i vd_i(t)$. Thus we can think of using $nvd_i(0)$ as a parameter to measure the initial distance from synchronization of node i. In figures 11, 12 and 13, we plot this parameter $nvd_i(0)$ as a function of the degree $< k >$. When more nodes share the same degree we consider an averaged value. As we can observe, there is a fairly linear dependence between $nvd_i(0)$ and the average degree $< k >$. Namely, the nodes that achieve an higher final degree are typically the ones which are further away from the others. On average, the gain estimates for these nodes tend to increase more rapidly as the norm of their errors is higher and, therefore, more links departing from these node are activated. In table 15, the values are shown of the fitting parameters p_1 and p_2 used to obtain the best linear fit $y = p_1 x + p_2$ of the collected data.

5. CONCLUSIONS

We have discussed a new strategy to synchronize complex networks of mutually coupled nonlinear dynamical systems. The main idea is to use a combined hybrid adaptive strategy characterized by evolving gains and network topology. After presenting the novel approach to synchronization, we validated its performance on the representative case of a network of Chua's circuits. We showed that, starting with a fully disconnected network, the coupling gains evolve adaptively with new links being activated as a certain activation threshold on the gains is reached. This, in turn, was shown to cause the emergence of a network of interconnections between nodes (with associated gain values) that guarantees synchronization. A striking observation is that some features of the emerging topologies seem to converge towards certain values when the link activation threshold is varied. Ongoing work is aimed at proving analytically the stability of the novel hybrid adaptive strategy presented in this paper as well as validating the preliminary conclusions on the emerging topologies on a much larger set of oscillators.

REFERENCES

Bartissol, P. and Chua, L.O. (1988). The double hook. *IEEE Transactions on Circuits and Systems*, 35(12).

Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., and Hwang, D.U. (2006). Complex networks: structure and dynamics. *Physics Reports*, 424, 175–308.

de Lellis, P., di Bernardo, M., and Garofalo, F. (2008a). On a novel decentralized adaptive strategy for the synchronization of complex networks. Accepted for publication on Automatica.

de Lellis, P., di Bernardo, M., and Garofalo, F. (2008b). Synchronization of complex networks through local adaptive coupling. *Chaos*, 18, 037110.

de Lellis, P., di Bernardo, M., Sorrentino, F., and Tierno, A. (2008c). Adaptive synchronization of complex networks. *International Journal of Computer Mathematics*, 85(8), 1189–1218.

Fewell, J.H. (2003). Social insect networks. *Science*, 301, 1867–1870.

Moallemi, C.C. and Roy, B.V. (2004). Distributed optimization in adaptive networks. *Advances in Neural Information Processing Systems*, 16.

Newman, M.E.J., Barabàsi, A.L., and Watts, D.J. (2006). *The structure and dynamics of complex networks*. Princeton University Press.

Stanley, K.O., Bryant, B.D., and Mikkulainen, R. (2003). Evolving adaptive neural networks with and without adaptive synapses. *Proceedings of the 2003 IEEE Congress on Evolutionary Computation*.