Playing pool with π (the number π from a billiard point of view)

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PLAYING POOL WITH π (THE NUMBER π FROM A BILLIARD POINT OF VIEW)

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Counting collisions in a simple dynamical system with two billiard balls can be used to estimate π to any accuracy.

π

It's irrational but well-rounded.
– found on a printed T-shirt

1. Introduction

The remarkable book of V. I. Arnold on differential equations [1] starts with the following sentence: "*The notion of the configuration space alone let us solve a very difficult mathematical problem.*" Then the problem is formulated and solved. The result of this article confirms Arnold's idea to use configuration space for another problem, the problem of calculating the number π with any precision.

There are many ways to calculate π with a good precision; some of them are known from ancient times, some are pretty recent. The methods use various elegant ideas [2]: *geometric* (inscribing and circumscribing regular polygons around a circle gives, in particular, the ancient values $3\frac{1}{7}$ and $3\frac{10}{71}$ for π); *number theory* (continued fractions allow us to find the regular fraction 355/113 as the simplest approximation for π accurate to the one millionth place); *analytical* (that use series, integrals, and infinite products); and many others (e.g., the *Monte Carlo Method*) which require modern electronic devices – powerful calculators and computers.

There is also an interesting experimental method for finding π discovered by a French mathematician Georges Louis Leclerc Comte de Buffon (1707–1788) in his article "Sur le jeu de franc-carreau" [3] published in 1777. Buffon suggested dropping a needle of length $L = D/2$ at random on a grid of parallel lines of spacing D. One drops the needle N times and counts the number of intersections, R, with the grid lines (note that since the needle is shorter than the distance between two consecutive grid lines, it intersects each time either exactly one line or none of the lines). The frequency of intersection with a line is R/N; on the other hand, one can show that the probability for the needle to intersect a grid line is $1/\pi$ (for an arbitrary needle of length, L, the probability equals $2L/\pi D$). Equating the

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frequency and the probability, we obtain that π approximately equals N/R, the ratio of the number of drops to the number of intersections with the grid. Surely, greater accuracy is obtained by increasing the number of drops to infinity.

The reader can play "Buffon's game" and find good approximations of π on the web site [4]. He/she can easily observe that it takes a lot of drops to get a more or less good precision for π; in addition to that, one needs to be sure that the dropping is *equi-distributed*. The disadvantage of Buffon's method is in its probabilistic nature, and no one can guarantee any specific precision in calculation of π with the use of this method.

In this article, we present an absolutely new idea for calculating π, a *billiard* one. As with Buffon's method, our method is also experimental and does not require use of any modern device at all. However, in contrast to the Buffon's method, this method is entirely *deterministic*: the only thing you have to do is to "launch" the dynamical system consisting of just two billiard balls and an absolutely elastic obstacle (a wall), count the number of collisions in that system, and then write down this number on a (perhaps long!) sheet of paper. The integer that is going to be written on the paper will be

$$31415926535897932384626433832795028841971693993751 0\ldots \quad ;$$

it consists of the first N digits of $\pi = 3.14159265\ldots$, where N represents the number of decimal digits of π you want to know.

On the one hand, our method is purely mathematical and, most likely, will never be used as a practical way for finding approximations of π. On the other hand, this method is the *simplest* one among all the known methods (beginning from the ancient Greeks!). Moreover, it gives a possibility to find π with an *arbitrary precision*, i.e., allows us to find every single digit of π.

To obtain an accuracy of N decimal digits, one needs just to take the balls with the appropriate masses: *the ratio of the masses should be chosen to be the Nth power of* 100.[1]

2. Procedure

Consider two point-like balls with masses m and M, $M \geqslant m$. The balls will move along the positive x-axis and collide with each other at every encounter, and the small ball, m, will reflect off a vertical wall located at point $x = 0$.

Each collision in the system is supposed to be *absolutely elastic*. This means that a collision between the balls satisfies two mechanical laws: the law of conservation of *momentum*, and the law of conservation of *kinetic energy*. In addition, the small ball reflects from the wall by changing its velocity vector to the opposite vector. In other words, the wall can be thought of as a non-moving billiard ball of infinite mass.

Let us follow the following

PROCEDURE:

1. Let N be a fixed positive whole number. Take two billiard balls with the ratio of their masses $M/m = 100^N$.

2. Put the small ball, m, between the wall at the origin and the big ball, M.

3. Push the big ball towards the small ball very fast[2].

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[1] As will become evident to the reader, the number $100 = 10^2$ is taken only because we find the *decimal* representation of π. For π represented in the counting system base b, one needs to take $M/m = b^{2N}$.

[2] The latter condition is not necessary: if we change the time scaling, the big ball may move slowly. The changing of the time scaling alters the energy of the whole system; so it is equivalent to substituting one level of energy by another. As you will see later in section 8, the change of the time scaling preserves the parallelism of the unfolding of the configuration trajectory in the configuration space.

4. Calculate the total number of hits in the system: the number of collisions between the balls plus the number of reflections of the small ball from the wall.

5. Write down the number $\mathbf{\Pi}$ of hits obtained from item #4 on a sheet of paper.

(Note that we do not know a'priori if the number of hits is finite or infinite; we will prove it is finite.)

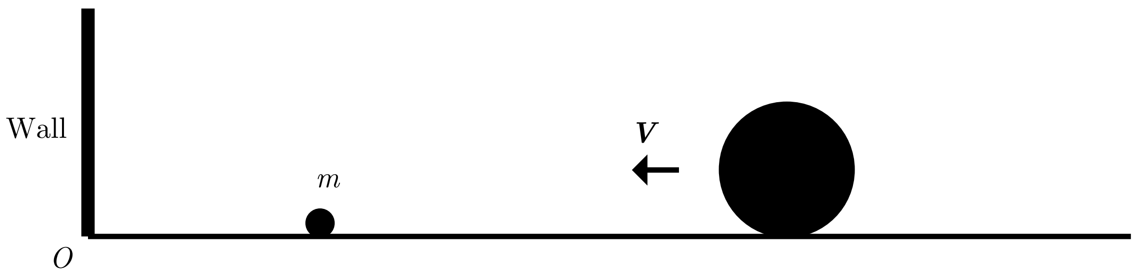

Fig. 1. Pushing the ball M towards the ball m

3. Investigation of a particular case

For different values of N the preceding Procedure gives us different values for the number $\mathbf{\Pi}$ (some of which could perhaps be infinity). Thus $\mathbf{\Pi} = \mathbf{\Pi}(N)$ is a function of the exponent N of the number 100^N.

Let us investigate the simplest case $N = 0$, which corresponds to the equality of the masses: $M = m$. The laws of conservation yield the following description of the system's behavior: if one ball is in the static position and the other one collides with it, then after the collision, the stationary ball starts moving with exactly the same velocity in the same direction as the second ball moved previously, while the second ball stops. It looks like the moving ball penetrates through the still ball without changing its velocity or affecting the still ball[3].

Then the moving ball hits the wall and reflects from it. The ball's velocity changes to the opposite one and, after that, it passes through the "transparent" ball and goes to infinity.

As you can see, the total number of hits in the system with $M = m$ is 3: two collisions and one reflection. Thus, $\mathbf{\Pi}(0) = 3$.

Note that 3 is the first digit of π. In what follows, the number of hits, $\mathbf{\Pi}$, is 31 (two first digits of π) and 314 (three digits of π) for, respectively, $M = 100m$ and $M = 100^2 m$, i.e., $\mathbf{\Pi}(1) = 31$ and $\mathbf{\Pi}(2) = 314$.

4. The main result

Theorem. *The number of hits, $\mathbf{\Pi} = \mathbf{\Pi}(N)$, in the system described in the Procedure is always finite and equal to a number with $N + 1$ digits,*

$$\mathbf{\Pi}(N) = 314159265358979323846264338327950288419716939937510\ldots,$$

whose first N digits coincide with the first N decimal digits of the number π (starting with 3).

The rest of the article is devoted to the proof of the theorem.

[3]Actually, the same behavior occurs when both identical balls move: after the collision, they just exchange their velocities; or, equivalently, they penetrate through each other without any effect upon each other. This reasoning solves easily the following fairly hard (at the first glance) problem: n identical balls move with the same speed along a line from left to right, and m other balls of the same kind move with the same speed along that line from right to left; how many collisions could occur in the system? The answer is now evident: mn, because each left ball penetrates each right ball.

5. The configuration space of the system. The behavior of the configuration point

Let the balls m and M be located at points x_0 and y_0 on the horizontal line ℓ at the initial moment of time $t = 0$. While moving, the balls' coordinates x and y change in time t, so $x = x(t)$ and $y = y(t)$, $t \geqslant 0$.

Fig. 2. The coordinates of the balls

In particular, $x(0) = x_0$, $y(0) = y_0$. Note that at each moment t, the small ball is situated between the wall and the big ball M. Therefore

$$0 \leqslant x(t) \leqslant y(t), \qquad \forall t \geqslant 0.$$

At a moment of reflection of the small ball from the wall, this ball m is always situated at the origin: $x\left(t_{\text{refl}}\right) = 0$.

Consider the ordered pair $(x(t), y(t))$ of the balls' positions. It can be thought of as a point on the xy-plane $\mathbf{R}^2$. Such a point is called *the configuration point* and the set of all possible configuration points is said to be *the configuration space*.[4] (See book [5].)

The configuration space for the dynamical system of the two balls in question is the 45° angle with its interior formed by the positive y-axis and the angle bisector of the first quadrant of the xy-plane, i.e. the ray $\{(x, y) \mid x = y,\ x \geqslant 0\}$.

Thus, along with the initial ("physical") system of two balls on the semi-line, we will consider also another ("mathematical") system — the mathematical "shadow" or "picture" of the initial system. Both of the systems are called *dynamical systems*; they are different formal descriptions of the same phenomena. We will jump freely from one description to the other in our future reasoning.[5]

Let us watch the behavior of the configuration point $P(t) = (x(t), y(t))$.

Step 0 (*before the first collision*). At the initial moment $t = 0$, the configuration point P is located on the plane at the geometric point $P_0 = (x_0, y_0)$. The time changes and point $P = P(t)$ starts moving. The small ball is fixed before the first collision, therefore the x-coordinate of the moving point P does not change. On the other hand, the big ball M moves towards the small ball m, and hence its coordinate y decreases, remaining, however, bigger than x_0 during the entire time period before the first collision with m. Consequently, the configuration point P moves directly down toward the x-axis (parallel to the y-axis) until the first collision.

Step 1 (*the first collision*). Thereupon the big ball collides with the small one; at this moment, t_1,

$$y(t_1) = x_0.$$

Then the balls bounce off each other instantaneously, and the next step begins.

[4] A configuration space may have dimension less or greater than 3; we keep the standard term "space" even for the case of a two-dimensional plane.

[5] The same idea is used often in mathematical investigations. The given model is substituted by its mathematical code that formally describes the model in new terms. Such a description often helps to investigate the problem entirely. Another bright example is a game of chess and its symbolic description in letters.

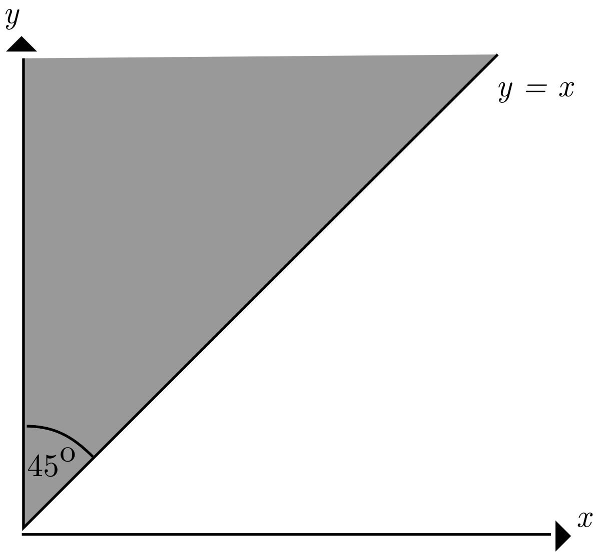

Fig. 3. The configuration space of the system

Step 2 (*between the first collision and the first reflection*). At the moment t_1, both balls begin to move along the horizontal line ℓ. The small ball moves with some velocity u and the big ball with velocity v, so that the laws of conservation of momentum and energy hold:

$$\begin{cases} mu + Mv = MV, \\ mu^2/2 + Mv^2/2 = MV^2/2, \end{cases}$$

where V is the initial (huge!) velocity of the ball M. We will not solve this system of equations quantitatively with respect to the variables u and v. Our aim is just to describe the behavior of the configuration point *after* the first collision.

Coming back to the initial dynamical system and using the system of equations (1), we can conclude that, after the first collision, the ball m will move very fast towards the wall (since the big ball gives it a big momentum) and the ball M also continues to move, a little bit slower than before, towards the wall. Both coordinates $x(t)$ and $y(t)$ are decreasing on the time interval after the first collision but before the reflection of the ball m from the wall. Therefore the configuration point $P(t)$ moves along a straight line segment inside the angle AOB, where O is the origin, OA is the positive y-axis, and OB is the ray $y = x$ outgoing from the origin in the first quadrant. Point $P(t)$ travels from the side OB to the side OA, approaching the origin O.

Step 3 (*reflection from the wall*). It is not hard to check that the small ball m moves faster than the big ball after the first collision, i.e. $u > v$.

Indeed, first note that $v < V$. This happens because the big ball gives some momentum to the small ball and accelerates it, so its new velocity v becomes smaller.

Multiplying the first equation of the system (1) by V and subtracting the double of the second equation yield

$$\begin{array}{rcccl} & Mu \cdot V & + & Mv \cdot V & = MV^2 \\ - & mu^2 & + & Mv^2 & = MV^2 \\ \hline & mu(V-u)^2 & + & Mv(V-v) & = 0 \\ & & \Downarrow & & \\ & & Mv(V-v) = mu(u-V). & & \end{array}$$

As $V - v > 0$, $Mv > 0$, and $mu > 0$, we conclude that $u - V > 0$, so $u > V$. Thus $u > V > v$.[6]

[6]For simplicity, we consider the velocity of a ball moving from right to left to be positive, and from left to right to be negative. One can consider just speeds (the absolute values of velocities) instead of the velocities.

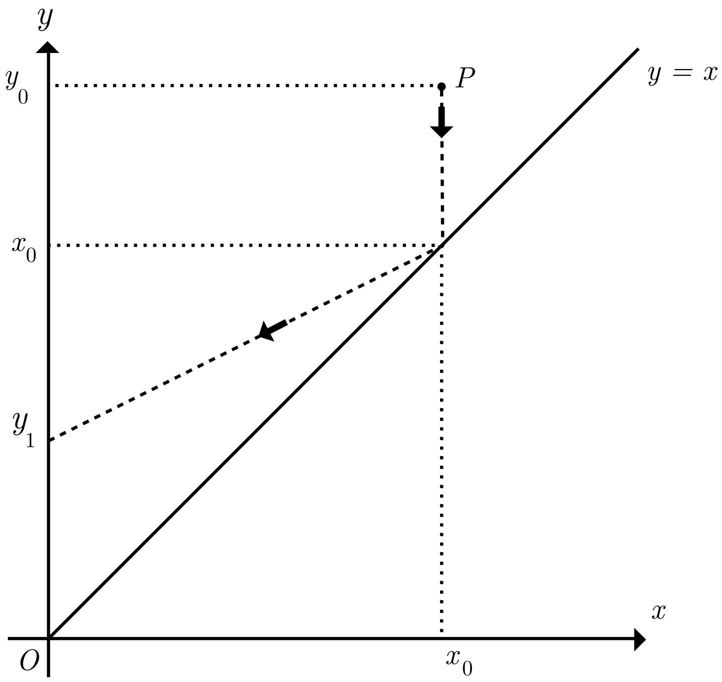

Fig. 4. Before and after the first collision

Therefore, the ball m reaches the wall at some moment t_2 at which the ball M is still moving toward the wall. At t_2, the ball reflects off the wall, its velocity instantaneously jumps from u to $(-u)$; the momentum of the small ball becomes $(-mu)$ whereas its energy remains the same: $m(-u)^2/2 = mu^2/2$. The angle of incidence and the angle of reflection formed by the segments of the trajectory with the x-axis is equal.

Step 4 (*after the first reflection off the wall*). Point $P(t)$ moves after the first reflection from the y-axis along a straight line segment towards the line $y = x$ approaching the origin O: its x-coordinate increases and the y-coordinate decreases. This corresponds to the balls approaching each other when the ball m bounces off the wall.

It is important to note that, since the wall is not considered part of the ball system, the momentum of the system after the wall reflection has been changed from $mu + Mv = MV$ to $-mu + Mv$ whereas the energy remains the same, $MV^2/2$. The momentum does not change between two successive reflections off the wall.

Step 5 (*the second collision of the balls*). At the moment when P reaches the side $y = x$ of the configuration angle AOB, the second collision of the balls occurs. Let it happen at point $x_1 > 0$ on line ℓ. The balls change their velocities after the collision. If the new velocities are u_1 and v_1, then

$$\begin{cases} mu_1 + Mv_1 = -mu + Mv \\ mu_1^2/2 + Mv_1^2/2 = MV^2/2 \end{cases} \tag{5.1}$$

By an argument similar to the given above, we have $|u_1| > |v| > |v_1|$.

Steps 6, 7, … . Starting from Step 5, the initial situation repeats:

1. The small ball, m, moves with velocity u_1 towards the wall, while the big ball, M, reduces its velocity from v to v_1 but continues to move, as before, towards the wall;
2. The small ball reflects from the wall, changes its velocity, u_1, to the opposite $(-u_1)$; and
3. The small ball meets the big ball and collides with it.

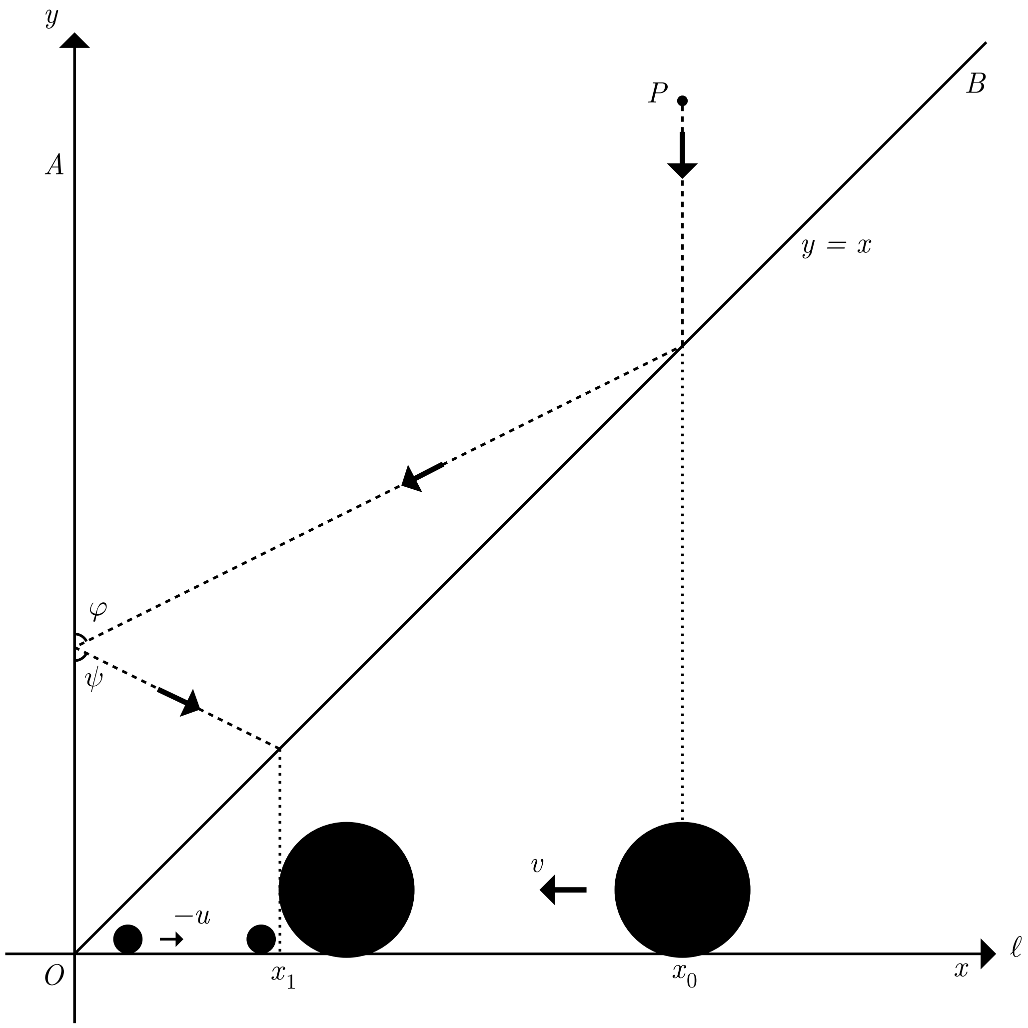

Fig. 5. $\varphi = \psi$, and the second collision

After that, the process repeats once again. This behavior of the system of the balls is reflected in the motion of the configuration point P (Figure 6).

It is absolutely unclear what kind behavior P exhibits:

(i) either P approaches the vertex O of the configuration angle AOB forever;

(ii) or P approaches O for a finite period of time, then moves away from O and reflects off the sides of the angle AOB infinitely many times; [7]

(iii) or P makes only finitely many reflections off the angle's sides, and, from some moment T_0 moves freely and rectilinearly.

Cases (i) and (ii) correspond to infinitely many hits in the system, i.e., collisions between the balls and reflections from the wall, and case (iii) to finitely many collisions and reflections. The theorem states that only the third case can occur, where the total number of hits in the system is $\mathbf{\Pi} = 314159265\ldots$.

[7] By the way, although P approaches vertex O during a finite period of time, P could make infinitely many reflections before it will start to move away from vertex O.

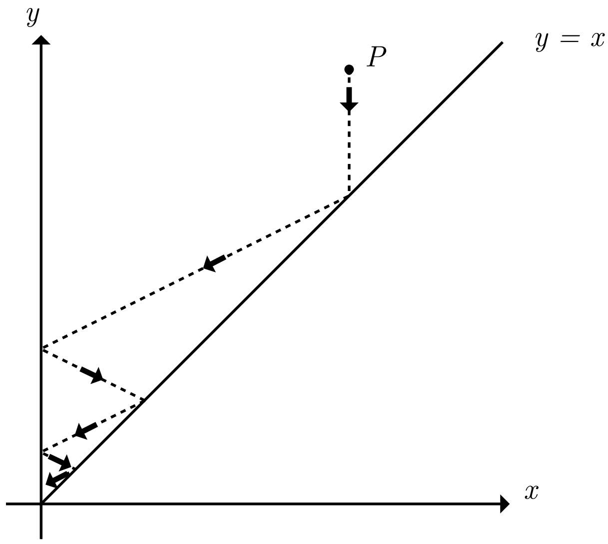

Fig. 6. The configuration path

6. Reduction to the billiard problem in an angle

To prove the theorem, we reduce the problem on the motion of the configuration point P in the 45° angle AOB to a billiard problem in some other angle α.

A *billiard system* is a dynamical system that consists of a domain (closed, like a circle or a square, or open, like an angle) and a moving point inside the domain. The point moves along a straight line with constant speed and reflects off the domain's boundary by the billiard (= optics) law: *the angle of incidence equals the angle of reflection.* In other words, the point in a billiard system behaves as a ray of light in a room (domain) with mirror walls (boundary).

In order to carry out the reduction to a billiard problem, let us make a special linear transformation, T, of the xy-plane. (This transformation was considered first by Ya. G. Sinai in book [6].) The matrix of the transformation T in the xy-coordinates is

$$T = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{M} \end{pmatrix}.$$

Transformation T expands the x-coordinate of any vector by a factor of $\sqrt{m}$ and the y-coordinate by a factor of $\sqrt{M}$. Consider the new variables,

$$\begin{cases} X = \sqrt{m} \cdot x \ , \\ Y = \sqrt{M} \cdot y \ . \end{cases} \tag{6.1}$$

The linear transformation T maps the 45° angle AOB into the angle $\alpha = \mathcal{AOB}$ satisfying

$$\tan\alpha = \frac{X}{Y} = \frac{\sqrt{m} \cdot x}{\sqrt{M} \cdot y} = \sqrt{\frac{m}{M}},$$

since $y = x$ for the points on the oblique side of the angle AOB (see Figure 7).

The broken line corresponding to the trajectory of the configuration point $P(x, y)$ inside the 45° angle AOB will be mapped into a broken line corresponding to the trajectory of the point $\mathcal{P} = \mathcal{P}(X, Y) = P\left(\sqrt{m}\ x, \sqrt{M}\ y\right)$ inside the angle $\alpha = \mathcal{AOB}$.

Lemma 1. *The behavior of the new configuration point $\mathcal{P}$ inside angle α obeys the billiard law.*

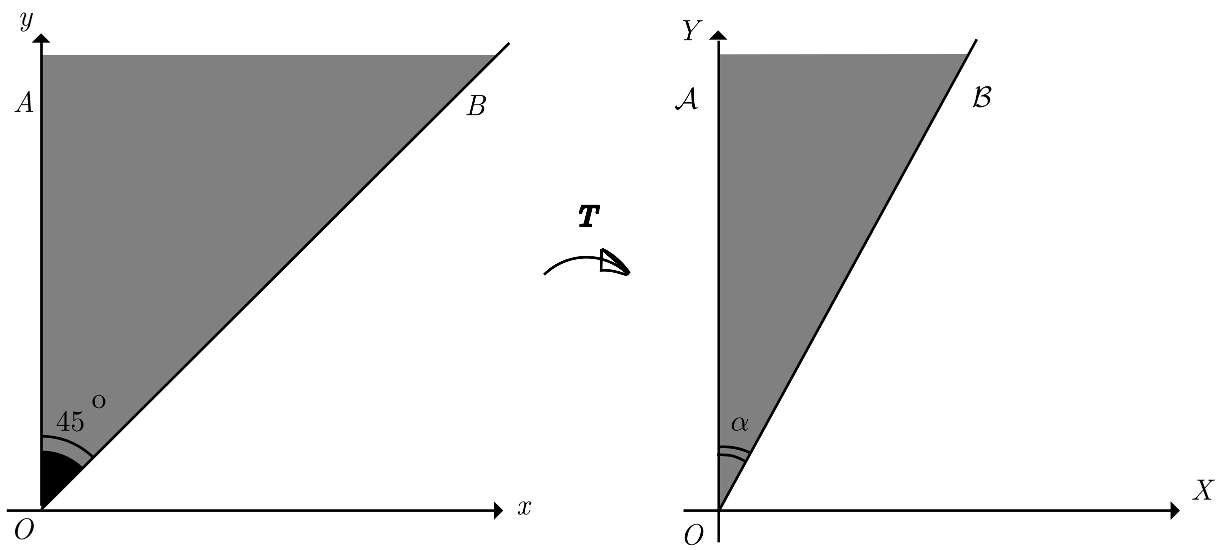

Fig. 7. Linear transformation T

Proof. Note that if point $P(x, y)$ has the velocity vector $\overrightarrow{\mathbf{w}} = (u, v) = (\dot{x}(t), \dot{y}(t))$ at moment t (different from a reflection), then point $\mathcal{P}(t)$ has the velocity vector

$$\overrightarrow{\mathbf{v}} = \left(\sqrt{m}\ \dot{x}(t), \sqrt{M}\ \dot{y}(t)\right) = \left(\sqrt{m}\ u, \sqrt{M}\ v\right)$$

at that moment. Thus the linear transformation T of the configuration space $\{(x, y)\}$ induces the same linear transformation in the velocity space $\{(\dot{x}, \dot{y})\}$.

Consider the following two cases: Case 1, in which point $\mathcal{P}(X, Y)$ reflects from the vertical side $X = 0$ (the Y-axis) of the angle α; and Case 2, in which $\mathcal{P}(X, Y)$ reflects off the side $Y = \sqrt{M/m}\ X$.

Case 1: Reflection from the Y-axis.

When the small ball reflects from the wall, its velocity u changes to $(-u)$. Then vector $\overrightarrow{\mathbf{v}}$ converts into vector

$$\overrightarrow{\mathbf{v}'} = \left(\sqrt{m}(-u), \sqrt{M}v\right).$$

which means $\varphi = \psi$ — the billiard reflection law (Figure 8).

Case 2: Reflection from the side $Y = \sqrt{M/m}\ X$.

This reflection corresponds to the ball collision. We will consider an interval of time in which only this collision occurs (i.e., the interval between two successive reflections of the ball m from the wall).

The system of moving balls has unchanging momentum during this interval of time, and the collision of the balls does not change it. The energy also doesn't change (it is always constant during the whole process).

For the sake of convenience, denote the momentum, p, by const_1 and twice the energy, $2E$, by const_2. Suppose the small ball has velocity u and the big ball has velocity v. The system (1) can be written as follows:

$$(1') \qquad \begin{cases} mu + Mv = \text{const}_1\ , \\ mu^2 + Mv^2 = \text{const}_2\ . \end{cases}$$

Some interesting geometry is hidden in the system $(1')$. Namely, consider the constant vector

$$\overrightarrow{\mathbf{m}} = (\sqrt{m}, \sqrt{M})$$

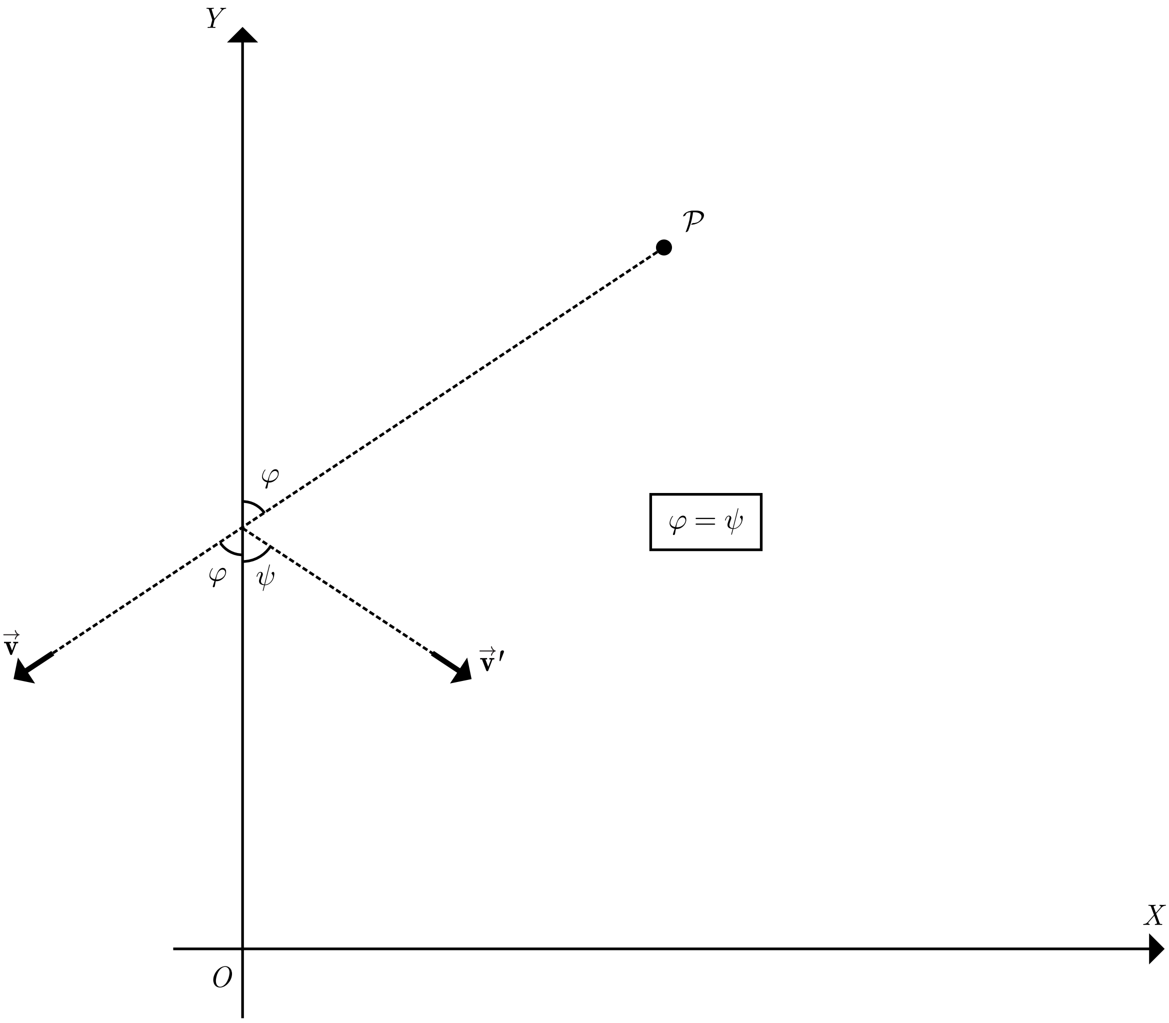

Fig. 8. Reflection from the Y-axis

in the xy-plane (it goes along the line $Y = \sqrt{\dfrac{M}{m}}\, X$), and the time-variant vector

$$\overrightarrow{\mathbf{v}} = \left(\sqrt{m}\, u, \sqrt{M}\, v\right).$$

Then $(1')$ can be rewritten as follows:

$$(1'') \qquad \begin{cases} \overrightarrow{\mathbf{m}} \bullet \overrightarrow{\mathbf{v}} = \mathrm{const}_1\ , \\ |\ \overrightarrow{\mathbf{v}}\ | = \mathrm{const}_2\ , \end{cases}$$

where "$\bullet$" is the dot product in the xy-plane, and "$|\quad.|$" is the Euclidean metric on this plane. The formulas $(1'')$ reflect the fact that the Euclidean metric is substituted by another Riemannian metric called "the kinetic energy metric"(in which the square of the length of vector (u, v) is the kinetic energy $mu^2 + Mv^2$).

Since

$$\overrightarrow{\mathbf{m}} \bullet \overrightarrow{\mathbf{v}} = |\ \overrightarrow{\mathbf{m}}\ |\ |\ \overrightarrow{\mathbf{v}}\ |\ \cos\varphi = \mathrm{const}_1,$$

$$|\ \overrightarrow{\mathbf{m}}\ | = \sqrt{m + M}\ ,$$

$$|\ \overrightarrow{\mathbf{v}}\ | = \mathrm{const}_2,$$

we obtain

$$\cos\varphi = \frac{\mathrm{const}_1}{\mathrm{const}_2}\cdot(m+M)^{-1/2} = \mathrm{const}_3.$$

After reflection, point $\mathcal{P}$ moves with a new velocity, $\overrightarrow{\mathbf{v}'}$, satisfying the same system ($1''$). Therefore, the same reasoning for the angle ψ of $\mathcal{P}$'s reflection from the α's side $Y = \sqrt{M/m}\,X$ show that

$$\cos\psi = \mathrm{const}_3 \qquad \text{(see Figure 9.)}$$

Consequently,

$$\psi = \varphi,$$

and the billiard law is proven for that reflection. The reduction to the billiard system in the angle α is finished. We shall call the angle $\mathcal{AOB}$ "the billiard configuration space" for the initial system.

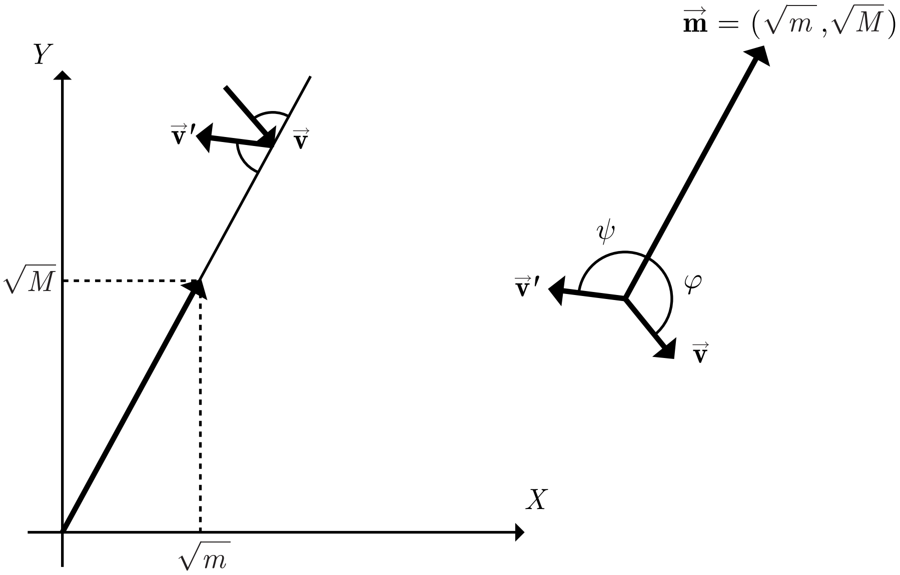

Fig. 9. *Reflection from the side* $Y = \sqrt{M/m}\,X$

7. The number of billiard reflections inside angle α

Lemma 2. (a) *The maximal number of reflections of a billiard point inside an angle* α*, over all possible billiard trajectories, is finite.*

(b) *This number equals* π/α *if* π/α *is an integer, and equals* $[\pi/\alpha]+1$ *if* π/α *is not an integer (where the "[ ]" is the greatest integer function).*

(c) *If the initial billiard ray is parallel to one side of the angle* α*, then the total number of reflections for this particular trajectory is one fewer than the maximum (i.e., equals* $\pi/\alpha - 1$ *if* $\pi/\alpha \in \mathbf{Z}$ *and* $[\pi/\alpha]$ *if* $\pi/\alpha \notin \mathbf{Z}$*) .*

Proof. Let us unfold the angle α together with the billiard trajectory γ in it. We just reflect angle α in its sides which the particle hits and consider the image of the trajectory γ under those reflections. The trajectory's image is a straight line, $\mathbf{k}$, that lies in the corridor of reflected angles (see Figure 10).

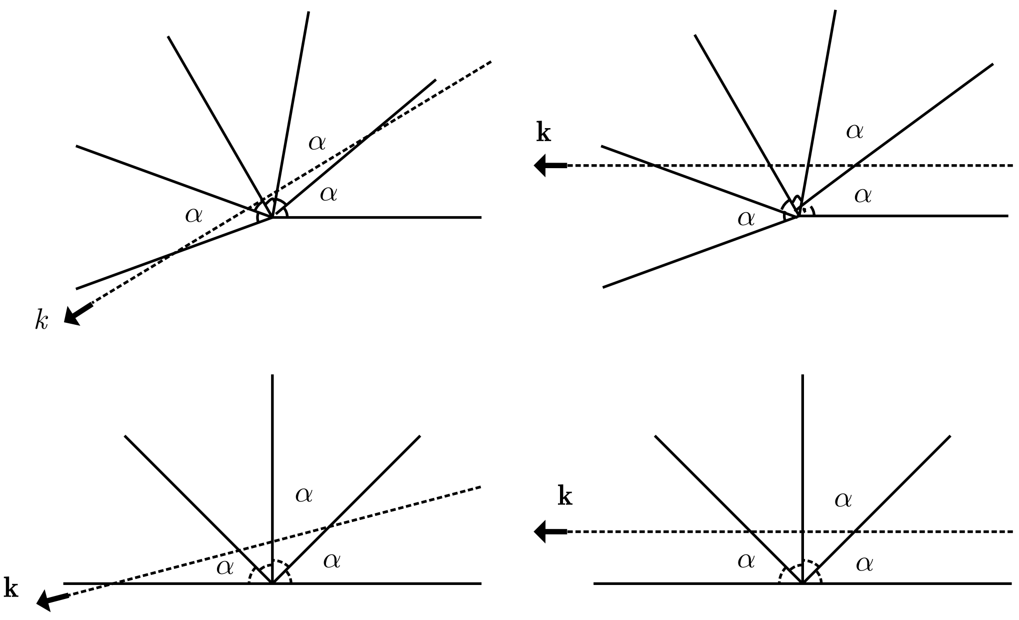

Fig. 10. *The unfolding of a billiard trajectory*

Line **k** intersects only finitely many copies of the angle α in the corridor; the number of intersections with the sides of the reflected angles equals the number of reflections of the trajectory γ inside the angle α. Hence the number of billiard reflections is finite. If n is the maximal possible number of reflections, then either $n\alpha = \pi$ or $n\alpha > \pi > (n-1)\alpha$ (see Figure 10). In the first case, $n = \pi/\alpha$, whereas in the second case $n = [\pi/\alpha] + 1$. If line **k** is parallel to the angle's side, then the final possible intersection is missed. The lemma is proven. ►

8. Proof of the theorem: $\Pi = 314159265\ldots$

It is useful to direct the reader's attention to the following exceptionality of the system "wall-ball-ball". The motion in this system corresponds to a special ray **k** in the unfolding of the billiard configuration space $\alpha = \mathcal{AOB}$: this ray must be parallel to the Y-axis.

Indeed, the first segment of the billiard trajectory is directed down to the X-axis, hence parallel to the Y-axis, because the small ball was at rest at the very beginning of the process. Therefore the ray **k** — the unfolding of the billiard trajectory γ and hence the extension of the first, vertical, segment of γ — is parallel to the Y-axis. Note that the parallelism of **k** to the Y-axis does not depend on the value of the initial velocity V of the ball M (see the footnote to the item **3** of the Procedure).

Therefore, according to Lemma 2 **(c)**, the number of reflections in the sides of the angle α is $[\pi/\alpha]$, unless π/α is an integer. In our case,

$$\alpha = \arctan\sqrt{m/M} = \arctan\sqrt{m/(m\cdot 100^N)} = \arctan(10^{-N})\ .$$

For $N=0$, $\alpha = \arctan 1 = \pi/4$ and $\pi/\alpha = 4 \in \mathbf{Z}$; so the first part of Lemma 2 **(c)** is applicable: the number of reflections equals $[4]-1=3$. But if $N \geqslant 1$, then the angle $\alpha = \arctan(10^{-N})$ is not of the form π/k with an integer $k \geqslant 1$; actually, this angle, as well as an arbitrary angle of the form $\arctan(p/q)$, where p and q are distinct positive integers, cannot measure a *rational* number of degrees, in particularly, $(180/k)^\circ$ (see the proof of this fact in [7]). Therefore the second part of Lemma 2 **(c)**

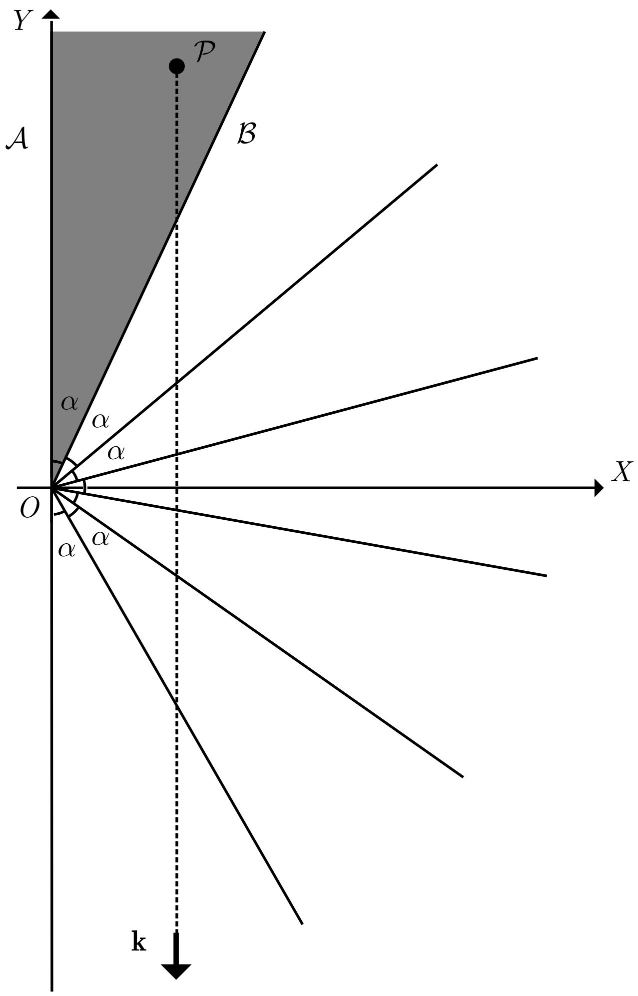

Fig. 11. *Ray* **k** *is parallel to the* Y*-axis*

is applicable in this case: the number of reflections equals

$$\mathbf{\Pi}(N) = \left[\frac{\pi}{\arctan(10^{-N})}\right]. \tag{8.1}$$

This is the precise formula; we will now use an approximation for $\arctan(10^{-N})$ to simplify (4).

The idea of the rest part of this section is as follows. If we replace the denominator of the fraction $\pi/\arctan(10^{-N})$ with the slightly larger number 10^{-N}, we get the slightly smaller fraction $\pi/10^{-N} = \pi \cdot 10^{N}$; if the integer part of the initial fraction and the perturbed fraction are the same, we can substitute $[\pi/\arctan(10^{-N})]$ by $[\pi \cdot 10^{N}]$. This will give us N correct digits of π.

However, the situation is much more delicate than it seems at first glance: *the integer parts of the two fractions could be different*! Fortunately, if N is sufficiently large, then they will differ by at most 1. Let us explain what occurs in more detail.

Denote $x = 10^{-N}$ for a moment. Recall that

$$\begin{aligned}\arctan x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n dt = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ ,\end{aligned}$$

so $\arctan x \approx x$ as $x \to 0$. We will substitute x for $\arctan x$ in the formula (3).

Lemma 3. (a). $\lim\limits_{x\to 0}\left(\dfrac{1}{\arctan x}-\dfrac{1}{x}\right)=0.$

(b). $\left(\dfrac{1}{\arctan x}-\dfrac{1}{x}\right)<x \quad \text{for} \quad x>0.$

Proof. ◀ **(a).** Substituting $\arctan x$ by its Taylor's series yields

$$\frac{1}{\arctan x}-\frac{1}{x}=\frac{x-\arctan x}{x\cdot\arctan x}=\frac{x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right)}{x\cdot\left(x-\frac{x^3}{3}+\frac{x^5}{5}\ldots\right)}=x\cdot\frac{\frac{1}{3}-\frac{x^2}{5}+\ldots}{1-\frac{x^2}{3}}\longrightarrow 0$$

as $x\to 0$, since the limit of the fraction factor is $1/3$.

(b). The inequality can be easily extracted from the proof of **(a)**. Here is another, independent, proof. Consider function $f(x)=(1+x^2)\arctan x-x$. Since $f'(x)=2x\arctan x>0$ for $x>0$ and $f(0)=0$, we have $f(x)>0$ for all $x>0$. ▶

From Lemma 3, $\dfrac{\pi}{\arctan(10^{-N})}-\dfrac{\pi}{10^{-N}}\to 0 \quad \text{as} \quad N\to\infty.$

Thus $\dfrac{\pi}{\arctan(10^{-N})}\approx\pi\cdot 10^N$ if N is big enough, so either

$$\mathbf{\Pi}(N)=\left[\frac{\pi}{\arctan(10^{-N})}\right]=[\pi\cdot 10^N]= \tag{8.2}$$

$$=[(3.1415\ \ldots\ a_{N-1}a_N\ldots)\ \cdot 10^N]=31415\ \ldots\ a_{N-1}a_N\ ,$$

or

$$\begin{aligned}\mathbf{\Pi}(N)\ &=\left[\frac{\pi}{\arctan(10^{-N})}\right]=[\pi\cdot 10^N]+1=\\ &=31415\ldots a_{N-1}a_N+1.\end{aligned} \tag{8.3}$$

This possibility causes some additional trouble. Indeed, formulas (5) and (6) guarantee that the $(N+1)$ digit number $\mathbf{\Pi}(N)$ has either the first N or $N+1$ exact digits of π (starting from the digit 3),

unless the k last digits in $31415\ \ldots\ a_N$ are nines.

We conjecture below (see the end of Section 10) that the latter case cannot occur (at least for $N\leqslant 50$ million) or occurs with probability zero. Anyway,

$$\mathbf{\Pi}(N)\ -\ [\pi\cdot 10^N]\ =\ \begin{cases}0\\1\end{cases}. \tag{8.4}$$

Thus the Theorem, modulo Conjecture, is proven. ■

9. The arithmetical nature of π

If

$$\pi=3.1415\ldots a_k\ \underbrace{99\ldots 99}_{(N-k)\text{ nines}}\ a_{N+1}\ldots\ ,$$

and

$$a_k<9,\ a_{N+1}<9\ ,$$

then formula (6) yields

$$\begin{aligned}\mathbf{\Pi}(N) &= [\pi\cdot 10^N]+1 = 31415\ldots a_k\ \underbrace{999\ldots 9}_{N-k\ \text{nines}}\ +1\\ &= 31415\ldots(a_k+1)\ \underbrace{000\ldots 0}_{N-k\ \text{zeros}}\ .\end{aligned}$$

So, $\mathbf{\Pi}(N)$ gives only $k<N$ correct digits of π, while the last $N-k$ digits in $\mathbf{\Pi}(N)$ are incorrect, compared to those in π.

Fortunately, this happens (if it happens!) very rarely and does not affect the general result concerning *the whole sequence of integers* $\mathbf{\Pi}(N)$ as $N\to\infty$: the sequence of first digits in $\mathbf{\Pi}(0)$, $\mathbf{\Pi}(1)$, $\mathbf{\Pi}(2)$, …, $\mathbf{\Pi}(N)$, … will stabilize from some number, $\mathbf{\Pi}(N)$, and eventually we will be able to know **all the digits** of number π.

But most likely formula (6) is false for all N. Then (5) would always work and $\mathbf{\Pi}(N)$ would always give the precise N digits of π. We discuss this in the next section.

10. Some arithmetical questions and a conjecture

It would be very nice if only formula (5), which gives the correct value for $\mathbf{\Pi}(N)$, always applied and formula (6) was always false.

Let us pose four related questions in this connection.

Question 1. Is the formula (5),

$$\left[\frac{\pi}{\arctan(10^{-N})}\right]=\left[\frac{\pi}{10^{-N}}\right],$$

true for any natural N big enough?

Question 2. Is it true that for any positive *irrational number* a and for any positive x small enough,

$$\left[\frac{a}{\arctan x}\right]=\left[\frac{a}{x}\right]\quad ?$$

Question 3. Is the equality

$$\left[\frac{\pi}{\arctan(1/N)}\right]=\left[\frac{\pi}{1/N}\right]$$

true for any natural N big enough?

Question 4. Is the equality

$$\left[\frac{\sqrt{2}}{\arctan(10^{-N})}\right]=\left[\frac{\sqrt{2}}{10^{-N}}\right]$$

true for any natural N big enough?

Question 2 comes as a broad generalization of Question 1, while Questions 3 and 4 arise from Question 1 if you substitute 10^{-N} for $1/N$ and, respectively, π for $\sqrt{2}$. I am confident that the answer to the Question 1 is "YES" but cannot prove this. I know, with rigorous proofs, the exact answers to the other three questions (see the answers together with their proofs in the next section).

As for Question 1, the modern mathematics is powerless to answer it; in any event, several leading specialists in number theory and related topics explained to the author of this article that the modern mathematics is far from a solution to this problem. Here is a quotation from an e-mail of a famous Australian mathematician, Alf van der Poorten, to the author: "[...] One state of knowledge is so weak that I would today need some $7N$ nines to follow the first N digits of π before I could be certain that the phenomenon is absurd. Indeed, I'm sorry to say, I fear that the situation is that fewer than $N+\varepsilon$ such nines will forever (given anything remotely accessible today) be feasible to allege. It's back to conjecture for you, I regret to have to tell you. The simplest reference I know is my old paper [8] "A proof that Euler missed", in Math. Intelligencer, 1979."

On the other hand, there are some evidences that confirm my assurance for the affirmative answer to the Question 1.

First of all, one can show that *the equality (6) holds if and only if the string of* $2N$ *first decimal digits of* π *contains* $N-1$ *nines in its right half.* Indeed, for $x>0$, the inequality (see Lemma 3,**(b)** above)

$$\frac{1}{\arctan x} - \frac{1}{x} < x$$

implies, by setting $x=\pi\cdot 10^{-N}$,

$$-10^N\pi + \frac{\pi}{\arctan(10^{-N})} < \pi\cdot 10^{-N} .$$

For $[10^N\pi]$ and $\left[\dfrac{\pi}{\arctan(10^{-N})}\right]$ to differ (and then only by 1), there must be $(N-1)$ nines following the first N digits in the decimal expansion of π. If this is possible for *infinitely many values* of N, then the (upper) density of 9's in the decimal expansion of π is at least $1/2$. This is very unlikely!

The reader can easily check that "*The string* 999999999 *did not occur in the first* 100,000,000 *digits of* π *after position* 0": see the site [9]

$$\text{http:}//cgi.aros.net/\ angio/pi_stuff/bigpi.cgi?UsrQuery = 999999999\ .$$

So, not only 100 million digits of π are 9's (as it should be if the equality (6) held), but the maximal string of successive 9's has merely the length 8! This observation leads us to the idea that, most likely, π is indeed a *normal* number: the string of a given sequence of 8 digits (in our case this is the string of 8 nines) appears with the probability $(1/10)^8$ in the decimal representation of π; hence, the longest string of all nines in the segment of length 10^8 must be eight!

Thus our method for finding π is surely accurate up to 100 million digits.

It's also not very difficult to calculate the probability of the event that an integer is contained in the union of all the intervals with the ends $\pi/10^{-n}$ and $\pi/\arctan(10^{-n})$, where $n>10^8$. The length of one interval is about $\pi\cdot 10^{-n}/3$; let us assume that this is the probability for the interval to contain an integer. Summing up all these values for $n>10^8$, we get an estimate for the total probability

$$p \approx \pi\cdot 10^{-10^8}/27 \ .$$

It is almost zero!

The whole discussion allows us to restate Question 1 as the following conjecture.

Conjecture:

(A). *For any natural number* N,

$$\left[\frac{\pi}{\arctan(10^{-N})}\right] = \left[\frac{\pi}{10^{-N}}\right] .$$

(B). *For any natural number* N, *the* $2N$ *first consecutive digits of* π *do not contain a string of* $N-1$ *consecutive nines in its right half, i.e.* $\pi \neq 3.\underbrace{1415\ldots a_N}_{N\text{ digits}}\ \underbrace{99\ldots99}_{N-1\text{ nines}}\ \ldots$

We have proven above the Conjecture for $N\leqslant 10^8$ and with the probability $1-\pi\cdot 10^{-10^8}/27$ for $N>10^8$.

11. Solutions to Questions 2, 3, and 4

Solution to Question 2. The answer to Question 2 is negative.

Statement 1. The equality $\left[\dfrac{a}{\arctan x}\right] = \left[\dfrac{a}{x}\right]$ cannot hold for all values of x.

Proof.

Let a be a positive number (no matter, rational or irrational). Let us set $x = \tan(a/n)$. Then the left hand side of the equality equals n, while the right hand side is strictly less than n (since $a/n < \tan(a/n)$). (A small shift of the value $x = \tan a$ does not change the inequality $\left[\dfrac{a}{\arctan x}\right] > \left[\dfrac{a}{x}\right]$.) ■

Solution to Question 3. Let us denote $L_N = \left[\dfrac{\pi}{\arctan(1/N)}\right]$.

Statement 2. For each $N \geqslant 23$, either $L_N = [\pi N]$ or $L_{N+1} = [\pi(N+1)]$.

Proof.

Assume the contrary: $L_N \neq [\pi N]$ and $L_{N+1} \neq [\pi(N+1)]$. Then $L_N > \pi N$ and $L_{N+1} > \pi(N+1)$. By Lemma 3, **(b)**, we have

$$\frac{\pi}{\arctan x} < \pi x + \frac{\pi}{x} \quad ,$$

so, plugging $x = 1/N$ and $x = 1/(N+1)$ we get two inequalities:

$$\pi N < L_N < \pi N + \frac{\pi}{N} \quad ,$$

and

$$\pi(N+1) < L_{N+1} < \pi(N+1) + \frac{\pi}{N+1} \quad .$$

Subtracting of the inequalities yields

$$\pi - \frac{\pi}{N} < L_{N+1} - L_N < \pi + \frac{\pi}{N+1} \quad .$$

If $N \geqslant 23$, then $\pi - \dfrac{\pi}{N} > 3$ and $\pi + \dfrac{\pi}{N+1} < 4$, so we obtain

$$3 < L_{N+1} - L_N < 4 \quad ,$$

which is a contradiction because $L_{N+1} - L_N$ is an integer. Statement 2 is proven. ■

Solution to Question 4. The answer to Question 4 is "YES":

Statement 3. For natural N big enough, $\left[\dfrac{\sqrt{2}}{\arctan(10^{-N})}\right] = \left[\dfrac{\sqrt{2}}{10^{-N}}\right]$.

Proof.

We write $g(x) = \overline{o}\,(f(x))$ as $x \to 0$ if $\lim\limits_{x\to 0} \dfrac{g(x)}{f(x)} = 0$. As $x \to 0$,

$$\frac{\sqrt{2}}{\arctan x} = \frac{\sqrt{2}}{x - \frac{x^3}{3} + \overline{o}(x^4)} = \frac{\frac{\sqrt{2}}{x}}{1 - \frac{x^2}{3} + \overline{o}(x^3)} = \frac{\sqrt{2}}{x}\left(1 + \frac{x^2}{3} + \overline{o}(x^3)\right).$$

Let $M_N = \left[\dfrac{\sqrt{2}}{\arctan(x_N)}\right]$, where $x_N = 10^{-N}$. Then $\dfrac{\sqrt{2}}{\arctan(x_N)} \geqslant M_N$, so

$$\frac{\sqrt{2}}{x_N}\left(1+\frac{x_N^2}{3}+\overline{o}(x_N^3)\right) \geqslant M_N \quad \text{and} \quad \frac{\sqrt{2}}{x_N} \geqslant \frac{M_N}{1+\dfrac{x_N^2}{3}+\overline{o}(x_N^3)}.$$

Suppose the contrary, that $\left[\dfrac{\sqrt{2}}{x_N}\right] < M_N$ for some N. (Note that in this case, $M_N - \left[\dfrac{\sqrt{2}}{x_N}\right] = 1$.) Then $\dfrac{\sqrt{2}}{x_N} < M_N$ as well (the integer M_N is situated between non integers $\sqrt{2}/x_N$ and $\sqrt{2}/\arctan x_N$), and hence

$$M_N > \frac{\sqrt{2}}{x_N} \geqslant \frac{M_N}{1+\dfrac{x_N^2}{3}+\overline{o}(x_N^3)} \quad .$$

This follows

$$\frac{\sqrt{2}}{x_N}\left(1+\frac{x_N^2}{3}+\overline{o}(x_N^3)\right) \geqslant M_N > \frac{\sqrt{2}}{x_N}.$$

Let us plug in 10^{-N} for x_N:

$$\begin{aligned}\sqrt{2}\cdot 10^N < M_N &\leqslant \sqrt{2}\cdot 10^N\left(1+\frac{10^{-2N}}{3}+\overline{o}(10^{-3N})\right)\\ &= \sqrt{2}\cdot\left((10^N+\frac{10^{-N}}{3}+\overline{o}(10^{-2N})\right)\end{aligned}$$

Taking the square of all the expressions yields

$$\begin{aligned}2\cdot 10^{2N} < M_N^2 &\leqslant 2\left(10^{2N}+\frac{2}{3}+\frac{10^{-2N}}{9}+\overline{o}(10^{-N})\right)\\ &= 2\cdot 10^{2N}+\frac{4}{3}+\overline{o}(10^{-N}).\end{aligned}$$

There is only one integer in the interval

$$\left(2\cdot 10^{2N}, \quad 2\cdot 10^{2N}+\frac{4}{3}\right),$$

namely, $2\cdot 10^{2N}+1$. Hence

$$M_N^2 = 2\cdot 10^{2N}+1.$$

But $2\cdot 10^{2N}+1 \equiv 2\cdot 1+1 \equiv 0 \pmod 3$ and $2\cdot 10^{2N}+1 \equiv 2\cdot 1+1 = 3 \not\equiv 0 \pmod 9$. This contradicts the fact that if a perfect square is divisible by 3, it must be divisible by 9. Therefore $M_N = \left[\dfrac{\sqrt{2}}{10^{-N}}\right]$, and the statement is proven. ∎

12. A related dynamical system that also counts π

The system "wall—small ball—big ball" can be substituted by another system that also counts the same number $\mathbf{\Pi}(N)$. This new system consists of only balls and does not contain an obstacle.

Let us think of the obstacle as a plane mirror. Then the two balls have their mirror images on the other side of the mirror. When the ball m reflects off the mirror, its image also reflects off the mirror from the other side; when the two balls collide with each other, their images also collide.

Let us simply remove the mirror and substitute the images of the two balls by real balls with the same masses, m and M, as their preimages. When the two small balls move towards each other with the same velocities and then collide, they just exchange their velocities, or, in other words, penetrate each other without any effect. The pairs of balls $(m - -M)$ and $(M - -m)$ collide simultaneously and we fix this event as one, not two, collisions. Thus the dynamical system

$$M - - m \;- - m \;- - M$$

counts the same number, $\mathbf{\Pi}$, of hits as the previous system " **wall** $- -\; m \;- - M$ " does.

The difference between the two systems consist of the following:

1. The wall in the first system plays the role of a ball of infinite mass, and there is no obstacle or infinite mass in the second system;
2. The configuration space of the first system is 2-dimensional whereas that of the second system is 4-dimensional: it is the direct sum of two identical copies of the space for the first system and has the natural symmetries;
3. In the first system, one should distinguish the hits off the wall and the collisions of the balls. After a collision with the wall, the momentum of the system (of balls alone) changes, which does not occur when the balls collide with each other. In the second system one counts only the collisions of balls and uses only formula (1) or (1′).

The commonality between the systems is that they carry out the same function:

count the same number $\mathbf{\Pi}$ of hits!

13. Closing remarks

The author created the billiard method for finding π when he was preparing a mathematical colloquium talk at Eastern Illinois University about balls' collisions (the so-called "Sinai's Problem"). When the procedure was presented to the audience, no one believed it at first, but then the author gave a proof, the ease of which convinced everyone.

Later, the author talked about his discovery in several other American universities, with the same reaction of the audience: first complete distrust and then complete acceptance, due to the obviousness of the proof.

From the experimental (physical) point of view, the central theorem of the presented article (see sections "Procedure" and "The main result") is completely proven: the ratio of two real masses, M/m, that is used in the procedure of calculating π, cannot exceed the number of atoms in our Universe, which is much smaller than $10^{100,000,000}$ (actually, it is even less than 10^{200}); but we know (from the Internet) that our method gives correct first 100,000,000 digits of π. However, from mathematical point of view, the Conjecture is a real challenge.

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