$\underline{\text{Recall}}$: 1) $d_\beta(x|y) = \underbrace{\check{d}_\beta(x|y)}_{\text{convex part}} + \underbrace{\hat{d}_\beta(x|y)}_{\text{convex term}} + \underbrace{\bar{d}_\beta}_{\text{constant term}}$

for $x, y > 0$

2) Auxiliary function for $D_\beta(v|wH) = \sum_n d_\beta(v_n | [wH]_n)$.

at $\tilde{w}$: $$G(w|\tilde{w}) = \sum_n \left[\sum_k \frac{\tilde{w}_k h_{kn}}{\tilde{v}_n} \check{d}_\beta\left(v_n \Big| \tilde{v}_n \frac{w_k}{\tilde{w}_k}\right)\right] + \bar{d}(v_n)$$
$$+ \left[\hat{d}'(v_n|\tilde{v}_n) \sum_k (w_k - \tilde{w}_k) h_{kn} + \hat{d}(v_n|\tilde{v}_n)\right]$$

where $\tilde{v} = \tilde{w}H$ and > 0, $\tilde{w}_k > 0$ for all k

3) for $\beta \in [1,2]$: $\times$ $\check{d}_\beta(x|y) = d_\beta(x|y) = \begin{cases} \frac{1}{\beta(\beta-1)}(x^\beta + (\beta-1)y^\beta - \beta x y^{\beta-1}) & \beta \in \mathbb{R}\setminus\{0,1\} \\ x \log \frac{x}{y} - x + y & \text{for } \beta = 1 \end{cases}$

$\times$ $\hat{d}_\beta(x|y) = 0$

Gradient of $G(w|\tilde{w}) + \frac{1}{2}\mu_w \|w\|_F^2$ w.r.t w_k (one entry)

$\underline{\text{Cases } \beta \in [1,2]}$: $\nabla_{w_k} G(w|\tilde{w}) = \sum_n \frac{\tilde{w}_k h_{kn}}{\tilde{v}_n} \nabla_{w_k} d_\beta\left(v_n \Big| \tilde{v}_n \frac{w_k}{\tilde{w}_k}\right)$

Since $\nabla_y d_\beta(x|y) = y^{\beta-1} - x y^{\beta-2}$, we obtain

$$\nabla_{w_k} G(w|\tilde{w}) = \sum_n h_{kn}\left[\left(\tilde{v}_n \frac{w_k}{\tilde{w}_k}\right)^{\beta-1} - v_n \left(\tilde{v}_n \frac{w_k}{\tilde{w}_k}\right)^{\beta-2}\right]$$

$$\Rightarrow \nabla_{w_k}\left(G(w|\tilde{w}) + \frac{1}{2}\mu_w\|w\|_F^2\right)$$
$$= a\, w_k + b\, w_k^{\beta-1} - c\, w_k^{\beta-2} = \underbrace{p(w_k)}_{\text{to set to zero}}$$

where $a = \mu_w > 0$

$b = \sum_n h_{kn}\left(\frac{\tilde{v}_n}{\tilde{w}_k}\right)^{\beta-1} \geq 0$ (very likely positive)

$c = \sum_n h_{kn} v_n \left(\frac{\tilde{v}_n}{\tilde{w}_k}\right)^{\beta-2} \geq 0$ //

Assume b and c positive for the following. Let us use the Descartes rules of sign :

$\times$ $\beta = 1$: $p(w_k) = a w_k + b - c/w_k = 0$

$\iff a w_k^2 + b w_k - c = 0$ for $w_k > 0$

Therefore : $v = 1$ and $m_p = 1$ $\Rightarrow$ one positive real root

$\times$ $\beta = 2$: $p(w_k) = (a+b) w_k - c = 0$ $\rightarrow$ one positive real root

$\times$ $\beta \in (1,2)$: $\beta - 1 > 0$ and $\beta - 2 < 0$, we will therefore have to compute w_k such that : $a w_k + b w_k^{\beta-1} - c/w_k^{2-\beta} = 0$

for $w_k > 0$, $\iff a w_k^{1+(2-\beta)} + b w_k - c = 0$

$\boxed{\text{Question}}$: is the Descartes rules still ok for rationale exponent? If yes : $v = 1$ and $m_p = 1$ $\underline{\text{again}}$

$\rightarrow$ should be check numerically