Project Euler Problem #6

The problem #6 asks us to find a solution for the expression

$$A(x) = (\sum_{x=0}^{N} x)^2 - \sum_{x=0}^{N} x^2 \tag{1}$$

and evaluate this at x = 100. The first term is relatively straightforward, as it's well known that

$$\sum_{x=0}^{N} x = \frac{N(N+1)}{2} \tag{2}$$

The hardest part of this problem lies in deriving a similar expression for the second term. Let F(x) be a function that satisfies

$$F(x) - F(x-1) = x^2 \tag{3}$$

From this definitition it's trivial to show that

$$F(x) = \sum_{n=1}^{x} n^2 \tag{4}$$

To find F, start with the ansatz

$$F(x) = ax^3 + bx^2 + cx \tag{5}$$

Inserting (5) into (3), we find a system of equations for a,b,c:

$$\begin{gathered}(b - 3a - b - 1)x^2 = 0 \\ (2b - 3a)x = 0 \\ a - b + c = 0\end{gathered}$$

With solutions $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$. F(x) can now be evaluated

$$\sum_{n=1}^{x} n^2 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x \tag{6}$$

Combining (6) with (2), and inserting back into (1), we arrive at the expression

$$A(x) = \frac{x^2(x+1)^2}{4} - \frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{6}x \tag{7}$$

Evaluating this for x=100, we find

$$A(100) = 25164150 \tag{8}$$