MAE 620-01

- jan 23 – monday –
- jan 25 – wednesday
  - panopto recorded lectures –
  - pdf of power point presentations –

2. Integral Forms of the Conservation Equations for Inviscid Flows

[Eulerian perspective

integral perspective

2.1. Philosophy

In aerodynamics, lift, drag, and moments on a vehicle are caused by collisions of atoms and molecules with that vehicle. We don't count all those collisions, we instead relate those at the macroscopic level to things we can measure, pressure, temperature, density, and velocity. We want to know these properties in the flowfield so that we know what they are at the surface. From that knowledge, we can calculated forces and heating rates.

Lagrangian perspective –

differential perspective

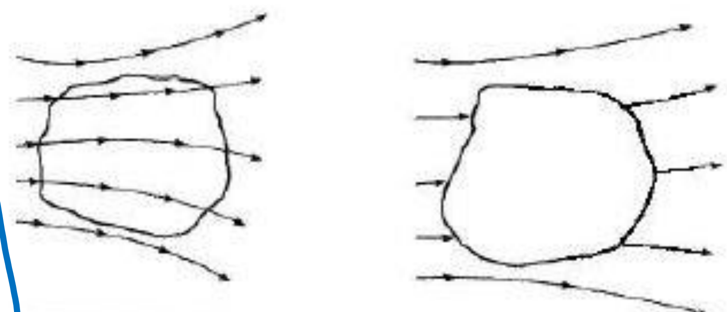
2.2. Approach

Everything in fluid mechanics is rooted in thermodynamics, which is rooted in fundamental conservation laws. If you understand these very well, you can understand quite a few other disciplines. Think of thermodynamics as front row tickets to 100's of sporting events.

The control volume approach will not yield detailed results, but are based on one dimensional approximations applied at boundaries. It is great for 'ballparking' performance of a system or subsystem.

Systems versus Control Volumes

A *system* is whatever we are going to study, usually defined by a fixed mass with well-defined *boundaries*, outside of which is called the *surroundings*. The laws of mechanics govern what happens when there is an interaction with the system at its surroundings. In Anderson's book, we use a fixed control volume as shown below.



Finite control volume fixed in space with the fluid moving through it

Finite control volume moving with the fluid such that the same fluid particles are always in the same control volume

FIGURE 2.1
Finite control volume approach

So virtually everything dealing with fluid flow is modeled using conservation laws (mass or continuity, momentum, and energy). We are going to derive these here in integral form.

Reynold's transport theorem (not in your 420 book, but from 310)

So we are going to take a fixed control volume approach. Let Q (B is your 310 book) be any extensive property of the fluid and $\beta = dQ/dm$ be the intensive value (Q per unit mass). The total Q in the control volume is

$$Q_{CV} = \int_{CV} \beta\, dm = \int_{CV} \beta \rho\, d\mathcal{V} \tag{1}$$

There are 3 sources of change for Q,

$$\begin{aligned} &\text{A change within the control volume} && \frac{d}{dt}\left(\int_{CV} \beta \rho\, d\mathcal{V}\right) \\ &\text{Outflow of } \beta \text{ from the control volume} && \int_{CS} \beta \rho V \cos\theta\, dA_{out} \\ &\text{Inflow of } \beta \text{ to the control volume} && \int_{CS} \beta \rho V \cos\theta\, dA_{in} \end{aligned} \tag{2}$$

ρ = density

$\mathcal{V}$ = total volume

$d\mathcal{V}$ = differential volume

velocity component in direction of interest

In the limit has dt → 0, the time rate of change of the property Q is

apply to a mass balance

$\beta = 1$

part of inflow/out flow terms

$$\underbrace{\rho V \cos\theta\, dA}_{d(\text{mass})} \rightarrow \text{which is crossing system or control volume boundary}$$

$$\frac{d}{dt}(Q_{syst})=\frac{d}{dt}\left(\int_{CV}\beta\rho\, d\mathcal{V}\right)+\int_{CS}\beta\rho V\cos\theta\, dA_{out}-\int_{CS}\beta\rho V\cos\theta\, dA_{in} \tag{3}$$

Positive - items leaving C.V.

negative - items entering C.V.

aka Reynold's transport theorem. This can also be written in more compact form $\vec{V}\cdot\vec{n}$

$$\frac{d}{dt}(Q_{syst})=\frac{d}{dt}\left(\int_{CV}\beta\rho\, d\mathcal{V}\right)+\int_{CS}\beta\rho(\mathbf{V}\cdot\mathbf{n})\, dA \tag{4}$$

If we fix the control volume d$\mathcal{V}$ does not change with time so the time derivative term on the RHS can be written as

balance/amounts β crossing system boundary

$$\frac{d}{dt}\left(\int_{CV}\beta\rho\, d\mathcal{V}\right)=\int_{CV}\frac{\partial}{\partial t}(\beta\rho)\, d\mathcal{V} \tag{5}$$

We are going to follow this approach for the conservation of mass, momentum, and energy. Here's the process:

1. Make a statement about the fundamental principle behind the conservation law. .
2. use the fixed control volume to determine the rate of change of some property due to both volumetric and surface effects.
3. summarize the final formula.

- time derivative -
- unsteady part -
- related change of β within the C.V. -

$\vec{V}\cdot\vec{n}$

$\vec{n}$ = unit vector outward & normal to surface

vector dot product

component of $\vec{V}$ normal to the surface

2.3. Continuity Equation / conservation of mass

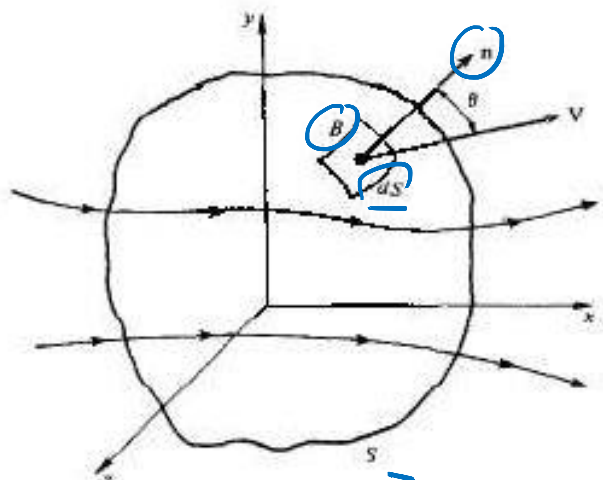
1. Mass can never be created or destroyed. Discuss.
2. $\mathcal{V}$ is the volume of the fixed control volume

$\rho = 1$

S is the surface area
B is a point on the control surface
dS is the elemental area around B
n is the unit vector normal to the surface B
d**S** = **n** dS.
V, ρ are the local velocity, density at B
ρV_n is the mass flux

$\dot{m}$ = mass flow rate (kg/sec.)

$V_n = V\cos\theta$ = component normal to surface



$\vec{S}$ same direction as $\vec{n}$ vector

The mass flow rate is

$$d\dot{m} = \rho(V\cos\theta)dS = \rho V_n dS = \rho \mathbf{V}\cdot d\mathbf{S} \tag{6}$$

Net mass across the control volume surface is

$$\oiint_S \rho \mathbf{V}\cdot d\mathbf{S} \tag{7}$$

The total mass is

$$\oiiint_{\mathcal{V}} \rho \, d\mathcal{V} \tag{8}$$

The time rate of change of this mass is

$$\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho \, d\mathcal{V} \tag{9}$$

3. Therefore, the sum of the net rate increase in mass plus the net mass flux across the surface must be zero to satisfy conservation of mass:

mass balance equations

$$\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho \, d\mathcal{V} + \oiint_{S} \rho \mathbf{V} \cdot d\mathbf{S} = 0$$

or

$$-\oiint_{S} \rho \underbrace{\mathbf{V} \cdot d\mathbf{S}}_{} = \frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho \, d\mathcal{V} \tag{10}$$

$$\frac{dm_{cv}}{dt} = \frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

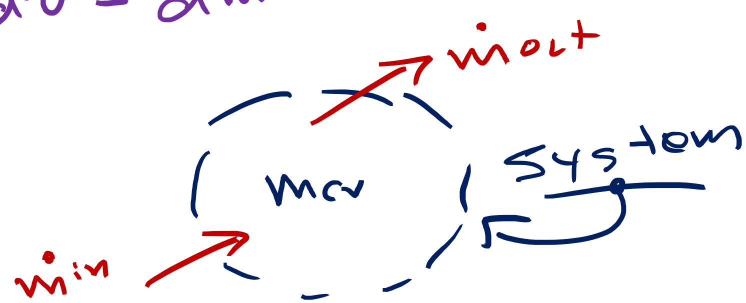
$$\dot{m}_{in} = \underbrace{\frac{dm_{cv}}{dt} + \dot{m}_{out}}$$

positive for $\dot{m}_{out}$

$$\rho \vec{V} \cdot d\vec{S} = d\dot{m}$$

$$\rho \, d\mathcal{V} = dm$$

$$m = m_{cv}$$



2.4. Momentum Equation

1. The time rate of change of momentum on a body is equal to the net force exerted on it
2. in vector form,

$$\frac{d}{dt}(m\mathbf{V})=\mathbf{F} \tag{11}$$

side point: for constant mass this is F=ma

$\vec{F}$ = force vector

There are two types of forces, body and surface. Body forces are due to fields, like gravity, electromagnetic, and the force. surface forces act at the surface of a control volume, and are due to pressure and shear stress.

$$\text{Total body force} = \iiint_{\mathcal{V}} \rho \mathbf{f} \, d\mathcal{V} \tag{12}$$

$B = \vec{V}$

$$\text{Total surface force due to pressure} = -\oiint_{S} p \, d\mathbf{S} \tag{13}$$

The minus sign is because the pressure force acts inward, opposite in direction from direction of S.

Now we need to account for the change in momentum. There are two sources. First, momentum can move across the surface. Second, momentum can change inside the volume. So the time rate of change of momentum is the sum of these, the net flow of momentum across the surface plus the rate of change of momentum inside the volume. The time rate of change of momentum is thus

$$\frac{d}{dt}(m\mathbf{V}) = \oiint_{S} (\rho \mathbf{V} \cdot d\mathbf{S})\mathbf{V} + \iiint_{\mathcal{V}} \frac{\partial(\rho \mathbf{V})}{\partial t} d\mathcal{V} \tag{14}$$

3. Putting it all together,

$$\oiint_{S} (\rho \mathbf{V} \cdot d\mathbf{S})\mathbf{V} + \iiint_{\mathcal{V}} \frac{\partial(\rho \mathbf{V})}{\partial t} d\mathcal{V} = \iiint_{\mathcal{V}} \rho \mathbf{f} \, d\mathcal{V} - \oiint_{S} p \, d\mathbf{S} + \mathbf{F}_{viscous} \tag{15}$$

We are going to neglect the viscous term, but put it in here for completeness.

creation of momentum

transport of momentum across system boundary

time derivative of momentum within system

net forces acting on c.v.

body force

pressure force

viscous force

2.5. A Comment

See comment in the TEXTBOOK.

energy transport across system boundary

$\dot{Q}$ = heat transfer rate → positive into system

$\dot{W}$ = work rate → positive into system

2.6. Energy Equation

1. Energy can neither be created nor destroyed. It can only change form.
2. There are several ways to change energy of the system: heat transfer, work, energy flux into the system, source terms (like radiation). You can also convert one form of energy into another, like dissipation or acceleration.

time derivative of energy within system

Let's just jump right into it

First law is

e tot transport across system boundary

$$\boxed{\dot{Q}+\dot{W}=\dot{E}=\frac{\partial}{\partial t}\left(\iiint_{V}\rho e_{tot}\,dV\right)+\oiint_{S}\rho e_{tot}\,\mathbf{V}\cdot d\mathbf{S}} \tag{16}$$

where e_{tot} is the total specific internal energy (see a few sections from now). In words, the processes for changing energy are equal to the time rate of change of energy in the system. Positive Q/W denote heat added to/work done on the system. Now we are going to break this down into individual terms, and put it all back together.

energy transport
with in
across system
boundary

2.6.1. Heat transfer

The heat transfer per unit mass is $\dot{q}$. Thus the heat transfer rate to the control volume is

$$\dot{Q}=\iiint_{V}\dot{q}\,\rho\,dV \tag{17}$$

e total = e internal + e potential + e kinetic + e other

2.6.2. Work

Work is the dot product of force with the distance. The rate of change of work, or power, is then the force acting on the body times the velocity,

$$\dot{W} = F\,V \tag{18}$$

$\dot{w}$ = work rate = power

Work is split into

$$\begin{aligned}\dot{W} &= \dot{W}_{shaft} + \dot{W}_{pressure} + \dot{W}_{body} + \dot{W}_{viscous\ stresses} \\ &= \dot{W}_s + \dot{W}_p + \dot{W}_b + \dot{W}_v\end{aligned} \tag{19}$$

Shaft work will be specified or computed, but no details are involved. Pressure work is

$$\dot{W}_p = -\oiint_S (p\,d\mathbf{S}) \cdot \mathbf{V} \tag{20}$$

p = pressure

and acts on the surface of the control volume. I call this the flow work term, as is done in Moran and Shapiro. You add this to internal energy, and you get the enthalpy.

$\vec{f}$ = body force vector

$\vec{\tau}$ = shear stress vector

For gravitation and other body forces, we have

$$\dot{W}_b = \oiiint_{\mathcal{V}} (\rho\,\mathbf{f}\,d\mathcal{V}) \cdot \mathbf{V} \tag{21}$$

Shear work is similar, and given by

$$\dot{W}_v = -\int_{CS} \tau \cdot \mathbf{V}\,dA \tag{22}$$

This term is often small and neglected. At solid surfaces, V = 0 so this term vanishes at the wall.

2.6.3. Energy

On a per unit mass basis, the system energy:

intensive energy total → kJ/kg

$$e_{tot} = e_{\text{internal}} + e_{\text{potential}} + e_{\text{kinetic}} + e_{\text{other}} \tag{23}$$

or

e = internal energy

$$e_{tot} = e + \frac{1}{2}V^2 + gz \tag{24}$$

We will neglect potential energy and other sources of stored energy, and drop the subscript 'internal' so that 'e' implies internal energy. Just like conservation of mass and momentum, there are two ways the energy can change. The energy can change inside the volume, plus there can be energy flux across the surface. The elemental flux of energy across dS is $(\rho \mathbf{V} \cdot d\mathbf{S})\left(e + \frac{V^2}{2}\right)$ Thus we have → neglect shear work term

$$\left[\begin{array}{c}\text{net rate of flow}\\ \text{of energy across}\\ \text{the control surface}\end{array}\right] + \left[\begin{array}{c}\text{time rate of change}\\ \text{of energy inside}\\ \text{volume}\end{array}\right] = \oiint_S (\rho \mathbf{V} \cdot d\mathbf{S})\left(e + \frac{V^2}{2}\right) + \iiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V} \tag{25}$$

→ neglect potential energy

3. Putting it all together

$$\underbrace{\iiint_{\mathcal{V}} \dot{q}\rho\, d\mathcal{V}}_{\text{heat transfer rate term}} - \underbrace{\oiint_S (p\, d\mathbf{S}) \cdot \mathbf{V}}_{\text{pressure work term}} + \iiint_{\mathcal{V}} \underbrace{\rho \mathbf{f} \cdot \mathbf{V}\, d\mathcal{V}}_{\text{body force work term}} = \overbrace{\oiint_S \rho \left(e + \frac{V^2}{2}\right) \mathbf{V} \cdot d\mathbf{S}}^{\text{energy transport crossing system boundary}} + \overbrace{\iiint_{\mathcal{V}} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] d\mathcal{V}}^{\text{time rate change of energy within system}} \tag{26}$$

You can also add in shaft work or viscous stress terms, but we leave them out.

e total = e internal + e kinetic