

LECTURE  
2023-01-09

# Compressible Flow

hello  
everyone!

## 1. Introduction



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green

## 1. INTRODUCTION

### *Nomenclature and Units*

Dimension	English	SI
force (F)	pound (lb) or (lbf)	newton (N)
mass (m)	slug	kilogram (kg)
time (t)	second (s)	second (s)
length (l)	foot (ft)	meter (m)
pressure (p)	lb/ft <sup>2</sup>	pascal = Pa = N/m <sup>2</sup>
velocity (u,v,w)	ft/s	m/s
acceleration (a)	ft/s <sup>2</sup>	m/s <sup>2</sup>
viscosity ( $\mu$ )	lb-s/ft <sup>2</sup>	Pa-s = N-s/m <sup>2</sup>
absolute temperature (T)	Rankine (°R)	Kelvin (K)
density ( $\rho$ )	slug/ft <sup>3</sup>	kg/m <sup>3</sup>

## 1.1. Notation

We are dealing mostly with Euler's equations, which involve a system of partial derivatives (PDE's). We'll use the notation

$$\phi_t = \frac{\partial \phi}{\partial t}, \phi_x = \frac{\partial \phi}{\partial x}, \phi_y = \frac{\partial \phi}{\partial y}, \phi_z = \frac{\partial \phi}{\partial z}, \quad (1)$$

The dot product of two vectors  $\mathbf{A} = (a_1, a_2, a_3)$  and  $\mathbf{B} = (b_1, b_2, b_3)$  gives the scalar quantity

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (2)$$

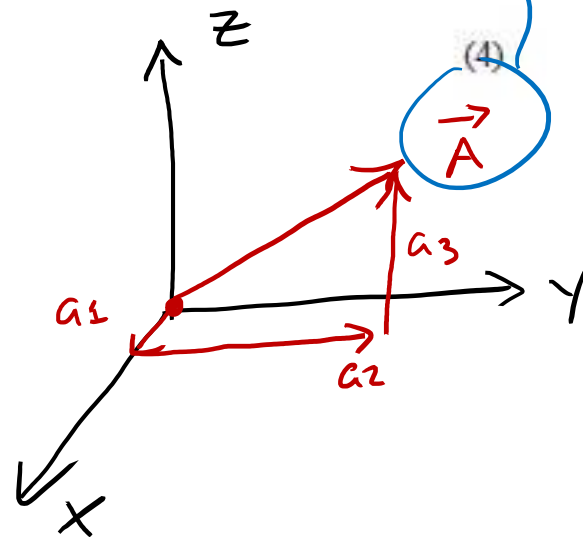
A scalar quantity  $\phi$  based on coordinates  $x$ ,  $y$ , and  $z$  will have a gradient

$$\text{grad} \phi = \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \quad (3)$$

The divergence operator applies to vectors,

$$\text{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \quad (4)$$

$\nabla = \text{"del"}$



## 1.2. Definition of compressible flow

This means that the density in a flow can change significantly. It implies that

$$\nabla \cdot \mathbf{V} \neq 0$$

and since density changes, Bernoulli's equation

$$p + \frac{1}{2} \rho V^2 = \text{const}$$

is no longer valid.

The compressibility of a fluid  $\tau$  is

$$\tau = -\frac{1}{v} \frac{dv}{dp}$$

Specifically, this depends on the process. Isothermal compressibility is

$$\tau_T = -\frac{1}{v} \left( \frac{\partial v}{\partial p} \right)_T$$

If the process is adiabatic and reversible (viscosity and diffusion are unimportant)

$$\tau_s = -\frac{1}{v} \left( \frac{\partial v}{\partial p} \right)_s$$

sonic velocity

The compressibility of a gas is about 10,000 times more than that of a liquid. So what causes a large pressure change sufficient to compress a gas? For aerospace applications, this is usually high velocities. To be compressible, the velocity needs to be at least 0.3 times the speed of sound. So virtually all cars and all aircraft before WWII encountered incompressible flow. Most jet aircraft and all rockets are fast enough that the flow dynamics are compressible.

Mach number?

$\rho$  = density (5)

$v = 1/\rho$  (6)

$v$  = specific volume

$P$  = static pressure (7)

$T$  = static temperature (8)

$s$  = entropy (9)



### 1.3. Flow regimes

The motivation for development of compressible flow theory comes primarily from high speed flow, but it also applies in astrophysics, weapons physics, and inertial fusion. Primarily we'll discuss flow regimes from the point of view of aircraft.

The freestream flow is upstream of the aircraft. All freestream properties are denoted with subscript  $\infty$ . The flow is characterized by the ratio of the freestream velocity to sound speed, Mach number  $M_\infty$ .

#### 1.3.1. Subsonic Flow

$M_\infty \leq 0.8$  The flow is subsonic everywhere. Stream lines are perturbed far upstream of the disturbance. All streamlines are continuous.

#### 1.3.2. Transonic flow

There are two separate regimes in transonic flow, and both flows have mixed supersonic and subsonic regions.

1.  $0.8 < M_\infty < 1.0$  flow speed locally supersonic over part of airfoil or aircraft, producing a shock wave. This region moves backwards as the mach number approaches 1.0
2.  $1.0 \leq M_\infty < 1.2$  Bow shock forms off of leading edge and trailing edge shock forms

#### 1.3.3. Supersonic flow

A flowfield where  $M > 1$  everywhere is supersonic. In a flow over a wedge, the stream lines are straight upstream of the

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oblique shock and are turned discontinuously by the shock.

#### 1.3.4. Hypersonic flow

$M > 5$ . The shock is very close to the body. The shocked flow is very hot and is typically chemically reacting.

"characteristics"  
mathematical  
characterization

hyperbolic  
partial differential  
eqns.  
Mach number =  $\frac{v}{c}$

$$= \frac{v}{c}$$

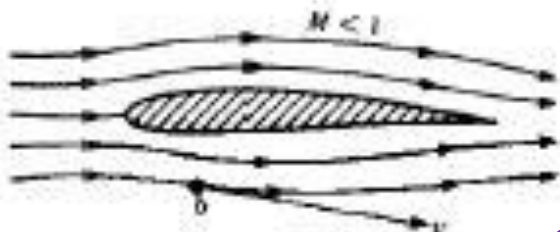
$c$  = sonic  
velocity

$$v = \text{flow velocity}$$

why this limit/hypersonic flow?

$M > 5$   
viscous shear &  
viscous dissipation  
- friction  
- inc. in temp.  
- plasma - gas dissociates -

$V_{\infty}$   
 $M_{\infty} < 0.8$



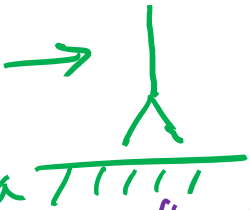
subsonic flow

$0.8 < M_{\infty} < 1.0$



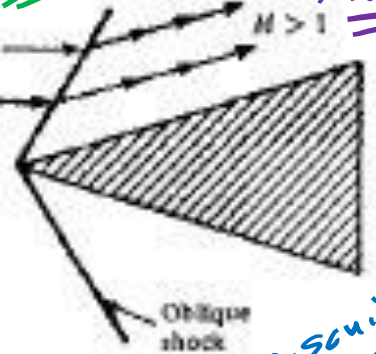
supersonic flow /  $M > 1$   
transonic flow /  $M < 1$

⊛ approximately normal to surface with a lambda foot  
freestream streamwise direction



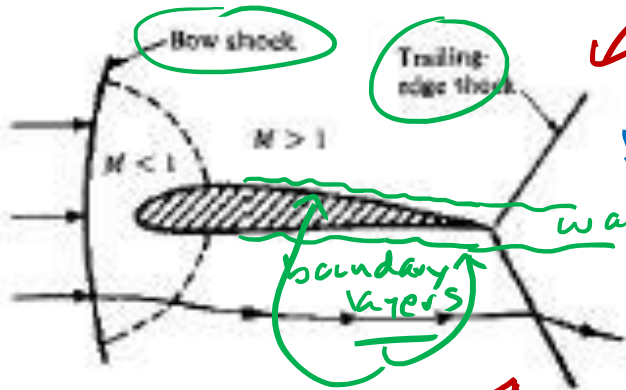
flow follows contour/body

$M_{\infty} > 1.2$



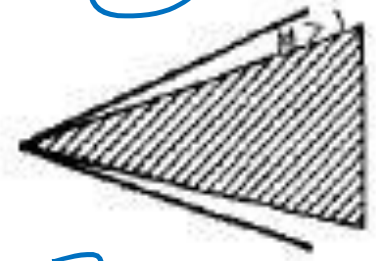
supersonic flow

$1.0 \leq M_{\infty} < 1.2$



transonic flow

$M_{\infty} > 5$



hypersonic flow

dramatic & important changes across shock waves

oblique shock waves

-  $P_0$ ,  $P_s$ ,  $T_s$ ,  $T_0$ , streamlines, flow direction,  $S$

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We will assume that the flow is inviscid and is a continuum. Viscous flows are important, especially in the boundary layer where velocity gradients are large. Continuum flow means that the mean free path between molecular collisions is small compared to the scale length of interest. This is applicable for most aircraft. Rarefied flow occurs at high altitudes (~100 kft) where the mean free path becomes large (~1 m or more). Free molecular flow is encountered at much higher altitudes (~200 kft) where the gas basically consists of remotely spaced particles.

Continuum  
flow

no slip  
boundary  
condition

L. Prandtl

led to  
concept  
of  
boundary  
layer

rarefaction  
flow

upper  
atmosphere

regimes

→ slip flow

→ free molecular  
flow

$$Kn = \frac{\lambda}{L}$$

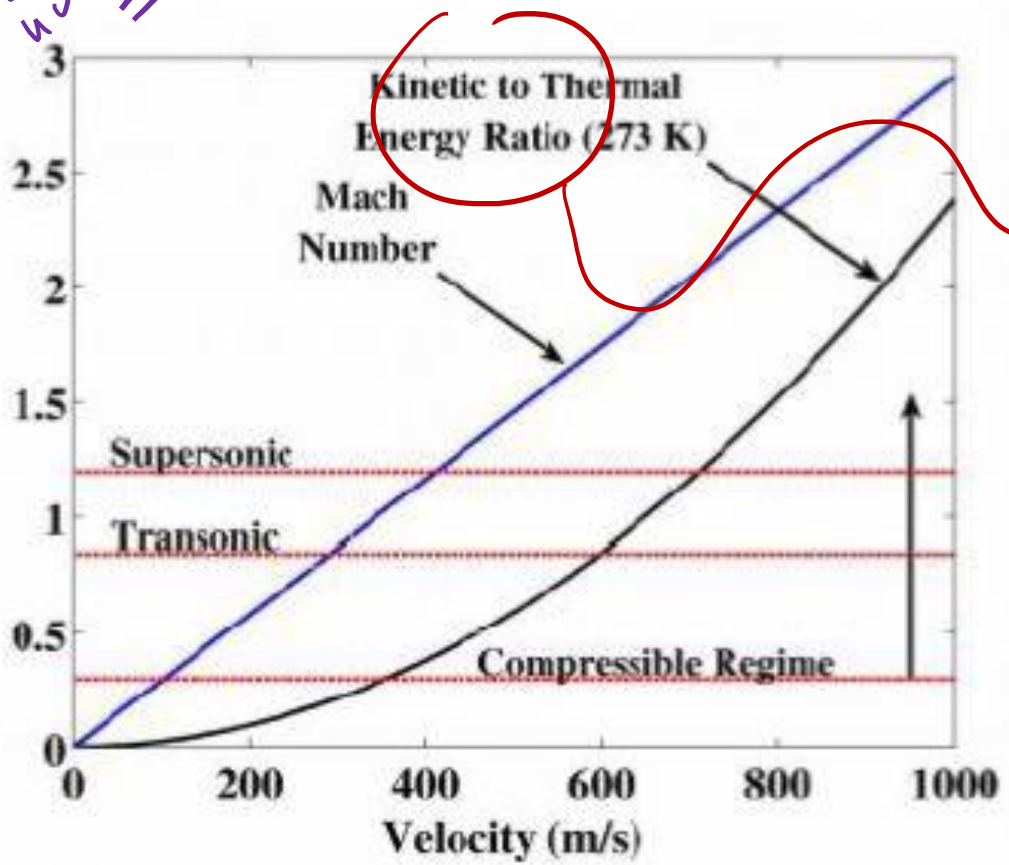
$Kn =$  ~~Knudsen~~  
Kundsen  
number



# 1.4. A brief review of thermodynamics

Mach number

→ conversion mechanical energy to thermal energy  
→ viscous dissipation



Here is a comparison of kinetic to thermal energy of a gas vs. velocity. For low speed flow (below compressible regime), the kinetic energy per unit mass is less than 2% of the thermal energy per unit. Above this, the kinetic energy, as it slows or speeds up, can drastically affect the other properties. Thus, compressible flow requires thermodynamics so that we can determine the relationships among all these properties.

→ important as flow approaches object

Figure 1.

kinetic energy  
↓  
isentropic  
constant entropy  
reversible

→ PART  
viscous  
dissipation  
irreversible



## 1.4.1. Perfect gas

### Nomenclature

symbol	name	units	
		(SI)	(English)
$C$	concentration	$\text{kmol/m}^3$	$\text{slug mol/ft}^3$
$c_p$	constant pressure specific heat	$\text{J/kg K}$	$\text{ft lb/slug R}$
$c_v$	constant volume specific heat	$\text{J/kg K}$	$\text{ft lb/slug R}$
$e$	specific internal energy	$\text{J/kg}$	$\text{ft lb/slug}$
$F$	force	$\text{N}$	$\text{lb}$
$h$	specific enthalpy	$\text{J/kg}$	$\text{ft lb/slug}$
$k$	Boltzmann's constant	$\text{J/K}$	$\text{ft lb/R}$
$M$	molecular weight	$\text{kg/kmol}$	$\text{slug/slug mol}$
$N$	number of moles	$\text{kmol}$	$\text{slug mol}$
$n$	number density	$\text{\#/m}^3$	$\text{\#/ft}^3$
$p$	pressure	$\text{N/m}^2$	$\text{lb/ft}^2$
$R$	specific gas constant	$\text{J/kg K}$	$\text{ft-lb/slug R}$
$\mathcal{R}$	universal gas constant	$\text{J/kmol-K}$	$\text{ft lb/slug mol-R}$
$T$	temperature	$\text{K}$	$\text{R}$
$V$	volume	$\text{m}^3$	$\text{ft}^3$
$v$	specific volume	$\text{m}^3/\text{kg}$	$\text{ft}^3/\text{slug}$
$\eta$	mole mass ratio	$\text{kmol/kg}$	$\text{kmol/slug}$
$\rho$	density	$\text{kg/m}^3$	$\text{slug/ft}^3$

A perfect gas is one in which 1. intermolecular forces are negligible and 2. particle volume is negligible. Fundamentally,

$$\underline{pV = MRT} \quad m = \text{mass} \quad (10)$$

R is the specific gas constant (J/kg-K or ft lb / slug °R). The forms that we will be using mostly are

$$pv = RT \quad (11)$$

and

$$p = \rho RT \quad (12)$$

For chemically reacting systems, it is better to put this in terms of the number of moles of a system.

$$pV = NRT \quad (13)$$

A particularly useless form is

$$pv = \eta RT \quad (14)$$

For chemically reacting flows, typically one uses

$$p = CRT \quad (15)$$

In plasma physics, one almost always uses

$$p = nkT \quad (16)$$

These equations all mean the same thing, and utilize the same assumptions. Get to know your gas constants.

1. Universal ( $R$ ) (use whenever ideal gas involves properties on a molar basis)

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(17)

2. When the equations deal with mass (density and specific volume), you use the specific gas constant,  $R = R/M$  which depends on the gas you are studying. For air at standard conditions,

$$\begin{aligned} R &= 287 \text{ J/kg} \cdot \text{K} \\ R &= 1716 \text{ ft} \cdot \text{lb/slug} \cdot {}^\circ R \end{aligned} \quad (18)$$

3. When the equation deals with particles, use the Boltzmann constant  $k$

$$\begin{aligned} k &= 1.38 \times 10^{-23} \text{ J/K} \\ k &= 0.565 \times 10^{-23} \text{ ft} \cdot \text{lb/} {}^\circ R \end{aligned} \quad (19)$$

Real gas effects are rarely important unless you deal with very cold temperatures or very high pressures (1 Mbar). Sometimes in high temperature flows, chemistry is referred to as a real gas effect. Strictly speaking, it is not a real gas effect (intermolecular forces). Just beware that sometimes the phrase 'real gas' is used for something other than accounting for molecular volume or intermolecular forces.

The internal energy is a measure of all the energy captured in a molecule. A molecule can store translational, rotational, vibrational, and electronic energy. An atom has translational and electronic. An electron or fully stripped ion only has translational and spin.

$$\begin{aligned} e &= e(T, v) \\ h &= h(T, p) \end{aligned}$$

$e = \text{interval energy}$  (20)  
 $h = \text{enthalpy}$

$$\begin{aligned} e &= e(T) \\ h &= h(T) \\ de &= c_v dT \\ dh &= c_p dT \end{aligned}$$

(21)

$$\begin{aligned} e &= c_v T \\ h &= c_p T \end{aligned}$$

Thermodynamic  
definition  
of  
enthalpy

$T =$  absolute temperature



## Other useful relations

$$c_p - c_v = R$$

(23)

$$c_p = \left( \frac{\partial h}{\partial T} \right)_p = \text{specific heat at constant pressure}$$

$$c_v = \left( \frac{\partial e}{\partial T} \right)_v = \text{specific heat at constant volume}$$

(24)

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$$\gamma = \frac{c_p}{c_v}$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$c_v = \frac{R}{\gamma - 1}$$

(25)

### 1.4.3. First law of thermodynamics

The first law is

$$\delta q + \delta w = de \quad (26)$$

This says an incremental amount of heat added to a system plus incremental amount of work done on the same system is equal to the differential change in internal energy. Note that we use delta for q, w and 'd' for e. e is a property, and q, w are not (they are processes). It's like combing my hair is a process, and the awesomeness of the hairdo is a property. Combing changes the amount of awesomeness. Formerly, e is a state variable, and changes are independent of the path. work and heat transfer are not, and depend on the process. Speaking of processes

1. adiabatic-no heat transfer
2. reversible-no dissipative effects occur (viscosity, thermal conductivity, and mass diffusion are absent)
3. isentropic- when processes 1 and 2 occur together

### 1.4.4. Entropy and 2<sup>nd</sup> law of thermodynamics

2<sup>nd</sup> law tells us the direction of a process. i.e. water flows downhill, energy goes from hot to cold sources, etc. Entropy is a way of tracking possible directions for a process to take.

$$ds = \frac{\delta q}{T} + ds_{irrev} \quad (27)$$

No matter what

$$ds_{irrev} \geq 0 \quad (28)$$

Thus

$$ds \geq \frac{\delta q}{T} \quad (29)$$

For an adiabatic process,

$$ds \geq 0 \quad (30)$$

### 1.4.5. Calculation of entropy

$S = \text{entropy}$

You have seen the derivations in 341/342 so I'm skipping this and just giving you the equations.

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \quad (31)$$

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1} \quad (32)$$

ideal gas -  
perfect gas -  
constant  
 $c_p \neq c_v$

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### 1.4.6. Isentropic relations

constant entropy / isentropic flow

flow across a shock wave is not isentropic.

For isentropic processes,  $ds=0$  (no heat transfer and no irreversibilities). See section 1.4.6 for the derivation, which again you have seen in your thermo classes.

$$\frac{p_2}{p_1} = \left( \frac{\rho_2}{\rho_1} \right)^\gamma = \left( \frac{T_2}{T_1} \right)^{\gamma/(\gamma-1)}$$

$$\gamma = \frac{c_p}{c_v} = \text{ratio of specific heats} \quad (33)$$

Isentropic processes are common in aerodynamics. The inviscid flow outside the boundary layer over a wing is typically isentropic. Expansion in a rocket nozzle can be accurately treated as isentropic as well.

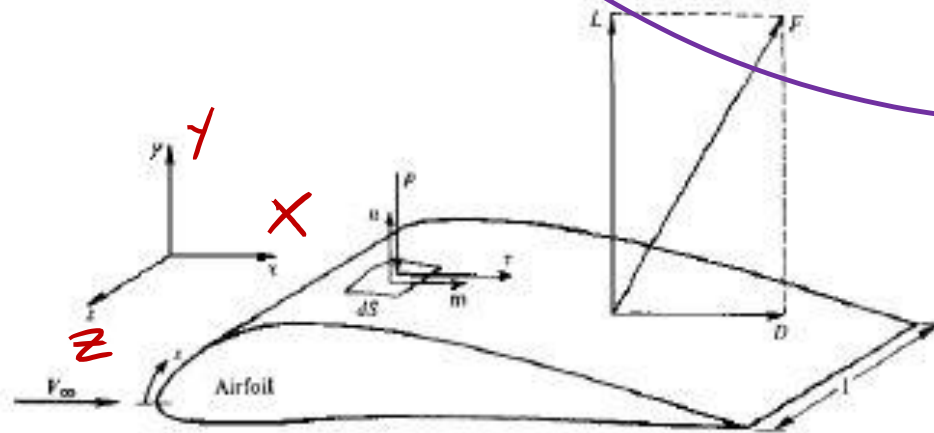
## 1.5. Aerodynamic forces on a body

There are only two aerodynamic forces which act on a surface, aerodynamic and shear forces. To develop the theory, let's use a wing of unit width. Measure the length along the airfoil from the leading edge with the coordinate 's'. Both the pressure  $p$  and shear stress  $\tau$  are a function of  $s$ . The force acting on the differential surface  $dS$  is (derived from Newton's 2<sup>nd</sup> law)

$$dF = -pndS + \tau mdS = -pdS + \tau mdS \quad (34)$$

To determine total force, we just sum up all the differential forces. As  $dS$  gets very small, we can replace the sum with an integral, obtaining

$$F = \oint dF = -\oint p dS + \oint \tau mdS \quad (35)$$



drag / lift  
 = [parallel to surface]  
 → shear / viscous forces } viscous drag  
 → aerodynamic forces / pressure } [normal to surface] form drag  
 n } unit vectors  
 m }

Lift is mostly created by the surface pressure forces since it is normal to the surface. The shear stress contributes some but it is small.

$$\begin{aligned} \text{Lift} &= y \text{ component of } [-\oint p dS] \\ \text{Drag} &= x \text{ component of } [-\oint p dS] + x \text{ component of } [\oint \tau mdS] \end{aligned} \quad (36)$$

Viscous effects are important for drag, so our drag calculations will be appropriate for calculating inviscid component but it is not accurate for the total drag. Just keep that in mind.