CHAPTER

# THREE

## ONE-DIMENSIONAL FLOW

The Aeronautical engineer is pounding hard on the closed door leading into the field of supersonic motion.

*Theodore von Karman, 1941*

3. One-Dimensional Flow
   - 3.1.Introduction
   - 3.2.One-Dimensional Flow Equations
   - 3.3.Speed of Sound and Mach Number
   - 3.4.Some Conveniently Defined Flow Parameters
   - 3.5.Alternative Forms of the One-Dimensional Energy Equation

EXAMPLE PROBLEM. At a point in the flow over an F-15 high performance fighter airplane, the pressure, temperature, and Mach number are 1890 lbf/ft$^2$, 450$^o$R, and 1.5, respectively. At this point, calculate $T_o$, $P_o$, $T^*$, $P^*$, and the flow velocity.

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$P = 1890$ lbf./ft$^2$

$T = 450°R$

$M = 1.5$

determine

$T_o, P_o, T^*, P^*, V$

assumptions

- isentropic flow
- one-dimensional
- steady
- ideal gas/air - $\gamma = 1.4$

table A.1

$M = 1.5$

$P_o/p = 3.671$

$T_o/T = 1.45$

$P_o = 3.671\,P = 6938$ lbf./ft$^2$

$T_o = 1.45\,T = 652.5\,°R$

characteristic quantities $T^*$ $P^*$

table A.1 / $M = 1$

$P_o/p^* = 1.893$

$T_o/T^* = 1.2$

$$P^* = \frac{P^*}{P_o}\frac{P_o}{P}P = \left(\frac{1}{1.893}\right)(3.671)\left(1890\,\frac{\text{lbf}}{\text{ft}^2}\right) = 3665 \text{ lbf./ft}^2$$

$$T^* = \frac{T^*}{T_o}\frac{T_o}{T}T = \left(\frac{1}{1.2}\right)(1.45)(450\,°R) = 543.8\,°R$$

$$a = (\gamma R T)^{\frac{1}{2}} = \left[(1.4)\left(1716 \frac{ft^2}{sec^2 \, {}^\circ R}\right)(450^\circ R)\right]^{\frac{1}{2}}$$

$$= 1040 \ \frac{ft.}{sec.}$$

$$V = M a = 1.5\left(1040 \ \frac{ft.}{sec.}\right) = 1560 \ \frac{ft.}{sec.}$$

EXAMPLE PROBLEM. At a particular point in the flow over a high Performance aircraft wing, the local static pressure, local static temperature, and Mach number are 0.9 atm, 250 K, and 0.7, respectively. At this point, calculate $T_o$, $P_o$, $T^*$, $P^*$, $a^*$, and the flow velocity.

$P = 0.9$ atm
$T = 250$ K
$M = 0.7$

determine

$T_o$ $P_o$ $T^*$ $P^*$ $a$ $V$

assumptions
- isentropic flow -
- one-dimensional -
- steady -
- ideal gas - $\gamma = 1.4$ -

table A-1
$M = 0.7$
$P_o/p = 1.387$
$T_o/T = 1.098$

$P_o = 1.26 \times 10^5$ N/m²
$= 1.248$ atm

$T_o = 274.5$ °K

$$P^* = P\left(\frac{P_0}{P}\right)\left(\frac{P^*}{P_0}\right) = 0.659\ \text{atm} = 6.66\times10^4\ N/m^2$$

$$T^* = T\left(\frac{T_0}{T}\right)\left(\frac{T^*}{T_0}\right) = 228.8\ ^\circ K$$

$$\frac{P^*}{P_0} = \frac{1}{1.893} \qquad \frac{T^*}{T_0} = \frac{1}{1.2}$$

$$a^* = \left[\gamma R T^*\right]^{\frac{1}{2}} = 303.2\ m/sec.$$

$$V = Ma = M\left[\gamma R T\right]^{\frac{1}{2}}$$

$$= 0.7\underbrace{\left[(1.4)\left(287\ \frac{J}{kg\ ^\circ K}\right)(250\ ^\circ K)\right]^{\frac{1}{2}}}_{317\ m/sec.} = 222\ m/sec.$$

## 3.6 NORMAL SHOCK RELATIONS

one-dimensional flow

steady flow

Let us now apply the previous information to the practical problem of a normal shock wave. As discussed in Sec. 3.1, normal shocks occur frequently as part of many supersonic flowfields. By definition, a normal shock wave is perpendicular to the flow, as sketched in Fig. 3.3. The shock wave is a very thin region (the shock thickness is usually on the order of a few molecular mean free paths, typically $10^{-5}$ cm for air at standard conditions). The flow is supersonic ahead of the wave, and subsonic behind it, as noted in Fig. 3.3. Furthermore, the static pressure, temperature, and density increase across the shock, whereas the velocity decreases, all of which we will demonstrate shortly.

across shock waves → increase of entropy → non-isentropic

categories / shock waves

→ normal shock waves strongest

→ oblique shock waves weakest

Mach wave

important area

- control of shock waves -

adverse pressure gradients

static pressure rises / across shock waves / opposes flow direction

constant inlet Mach number

strength shock wave

static pressure rise

Nature establishes shock waves in a supersonic flow as a solution to a perplexing problem having to do with the propagation of disturbances in the flow. To obtain some preliminary physical feel for the creation of such shock waves, consider a flat-faced cylinder mounted in a flow, as sketched in Fig. 3.7. Recall that the flow consists of individual molecules, some of which impact on the face of the cylinder. There is in general a change in molecular energy and momentum due to impact with the cylinder, which is seen as an obstruction by the molecules. Therefore, just as in our example of the creation of a sound wave in Sec. 3.3, the random motion of the molecules communicates this change in energy and momentum to other regions of the flow. The presence of the body tries to be propagated everywhere, including directly upstream, by sound waves. In Fig. 3.7*a*, the incoming stream is subsonic, $V_\infty < a_\infty$, and the sound waves can work their way upstream and forewarn the flow about the presence of the body. In this fashion, as shown in Fig. 3.7*a*, the flow streamlines begin to change and the flow properties begin to compensate for the body *far* upstream (theoretically, an infinite distance upstream). In contrast, if the flow is supersonic, then $V_\infty > a_\infty$, and the sound waves can no longer propagate upstream. Instead, they tend to coalesce a short distance ahead of the body. In so doing, their coalescence forms a thin shock wave, as shown in Fig. 3.7*b*. Ahead of the shock wave, the flow has no idea of the presence of the body. Immediately behind the normal shock, however, the flow is subsonic, and hence the streamlines quickly compensate for the obstruction. Although the picture shown in Fig. 3.7*b* is only one of many situations in which nature creates shock waves, the physical mechanism discussed above is quite general.

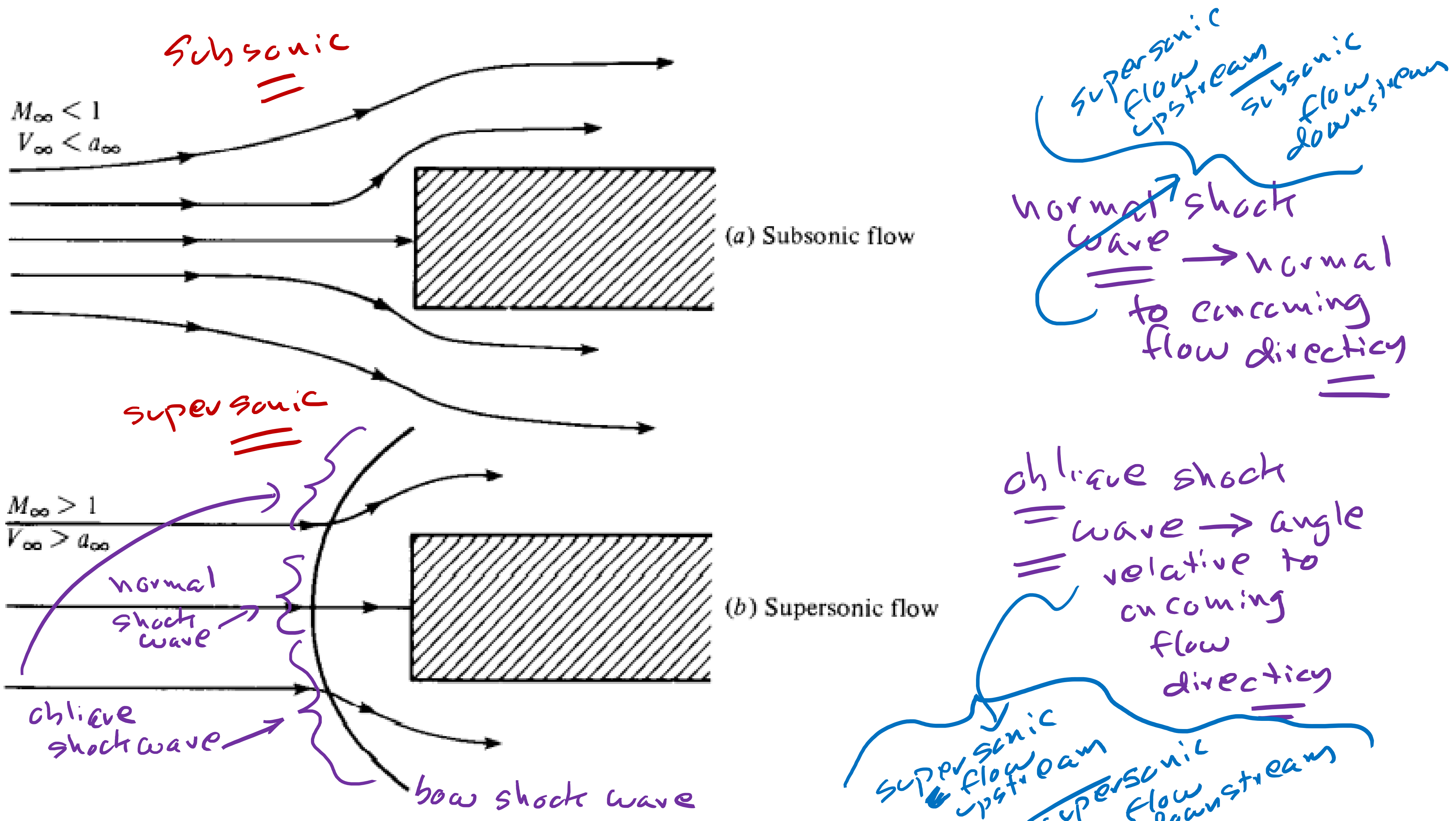
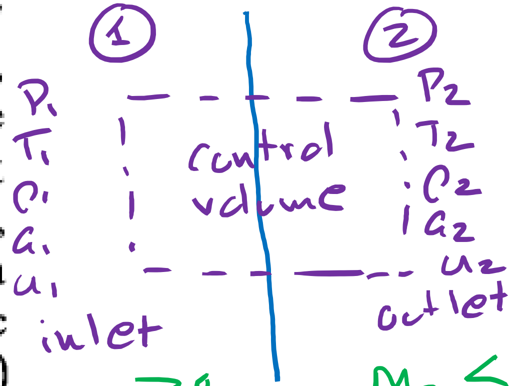



**Figure 3.7** Comparison between subsonic and supersonic streamlines for flow over a flat-faced cylinder or slab.

$a^* = \text{constant}$
use to relate $a_1 u_1$ and $a_2 u_2$

To begin a quantitative analysis of changes across a normal shock wave, consider again Fig. 3.3. Here, the normal shock is assumed to be a discontinuity across which the flow properties suddenly change. For purposes of discussion, assume that all conditions are known ahead of the shock (region 1), and that we want to solve for all conditions behind the shock (region 2). There is no heat added or taken away from the flow as it traverses the shock wave (for example, we are not putting the shock in a refrigerator, nor are we irradiating it with a laser); hence the flow across the shock wave is *adiabatic*. Therefore, the basic normal shock equations are obtained directly from Eqs. (3.2), (3.5), and (3.9) (with $q = 0$) as




$M_1 > 1$ normal shock wave $M_2 < 1$

FIG. 3.3

- assumptions -
- adiabatic -
- steady -
- one-dimensional -
- ideal gas -
- const. spec. heats -

analysis w.r.t. shock wave

$$\rho_1 u_1 = \rho_2 u_2 \qquad \text{(continuity)} \qquad (3.38)$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \qquad \text{(momentum)} \qquad (3.39)$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \qquad \text{(energy)} \qquad (3.40)$$

$M = \frac{V}{a} = \frac{u_1}{a}$

$h$ = static enthalpy

Equations (3.38) through (3.40) are general—they apply no matter what type of gas is being considered. Also, in general they must be solved numerically for the properties behind the shock wave, as will be discussed in Chap. 14 for the cases of thermally perfect and chemically reacting gases. However, for a calorically perfect gas, we can immediately add the thermodynamic relations

$c_p = \frac{\gamma R}{\gamma - 1}$

$\gamma = \frac{c_p}{c_v}$

$$p = \rho R T \qquad (3.41)$$

absolute static temperature

and

$$h = c_p T \qquad (3.42)$$

$p_2 > p_1$ $\rho_2 > \rho_1$
$T_2 > T_1$ $u_2 < u_1$

Equations (3.38) through (3.42) constitute five equations with five unknowns: $\rho_2$, $u_2$, $p_2$, $h_2$, and $T_2$. Hence, they can be solved algebraically, as follows.

First, divide Eq. (3.39) by (3.38):

adiabatic $\rightarrow$ $a^* = \underline{\underline{\text{constant}}}$

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1 \tag{3.43}$$

Recalling that $a = \sqrt{\gamma p/\rho}$, Eq. (3.43) becomes

$a = \sqrt{\gamma RT}$

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \tag{3.44}$$

Equation (3.44) is a combination of the continuity and momentum equations. The energy equation (3.40), can be utilized in one of its alternative forms, namely, Eq. (3.26), which yields

$a^* =$ characteristic $\underline{\underline{\text{sonic velocity}}}$

$a_1\ u_1\ \gamma \rightarrow a^*$

$a_2\ u_2\ \gamma \rightarrow a^*$

$$a_1^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_1^2 \tag{3.45}$$

and

$$a_2^2 = \frac{\gamma+1}{2}a^{*2} - \frac{\gamma-1}{2}u_2^2 \tag{3.46}$$

Since the flow is adiabatic across the shock wave, $a^*$ in Eqs. (3.45) and (3.46) is

$a^*$ is the same constant value (see Sec. 3.5). Substituting Eqs. (3.45) and (3.46) into (3.44), we obtain

$$\frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_1} - \frac{\gamma-1}{2\gamma}u_1 - \frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_2} + \frac{\gamma-1}{2\gamma}u_2 = u_2 - u_1$$

or

$$\frac{\gamma+1}{2\gamma u_1 u_2}(u_2 - u_1)a^{*2} + \frac{\gamma-1}{2\gamma}(u_2 - u_1) = u_2 - u_1$$

Dividing by $(u_2 - u_1)$,

$$\frac{\gamma+1}{2\gamma u_1 u_2}a^{*2} + \frac{\gamma-1}{2\gamma} = 1$$

Solving for $a^*$, this gives

$$\boxed{a^{*2} = u_1 u_2} \tag{3.47}$$

the Prandtl relation

Equation (3.47) is called the *Prandtl relation*, and is a useful intermediate relation for normal shocks. For example, from this simple equation we obtain directly

$M > 1 \quad M^* > 1$
$M < 1 \quad M^* < 1$
$M = 1 \quad M^* = 1$

$$1 = \frac{u_1}{a^*}\frac{u_2}{a^*} = M_1^* M_2^*$$

$M^*$ = characteristic Mach number

or

$$\boxed{M_2^* = \frac{1}{M_1^*}} \tag{3.48}$$

Based on our previous physical discussion, the flow ahead of a shock wave must be supersonic, i.e., $M_1 > 1$. From Sec. 3.5, this implies $M_1^* > 1$. Thus, from Eq. (3.48), $M_2^* < 1$ and thus $M_2 < 1$. Hence, the *Mach number behind the normal shock is always subsonic*. This is a general result, not just limited to a calorically perfect gas.

upstream normal shock wave → supersonic flow / required for shock wave to be present

downstream normal shock wave → ~~[illegible]~~ subsonic flow

Recall Eq. (3.37), which, solved for $M^*$, gives

$$M^{*2} = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2} \tag{3.49}$$

Substitute Eq. (3.49) into (3.48):

$$\frac{(\gamma + 1)M_2^2}{2 + (\gamma - 1)M_2^2} = \left[\frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}\right]^{-1} \tag{3.50}$$

Solving Eq. (3.50) for $M_2^2$,

$$\boxed{M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}} \tag{3.51}$$

$M_2 = M_2(M_1, \gamma)$

Equation (3.51) demonstrates that, for a calorically perfect gas with a constant

always $M_1 \geq 1$, $M_2 \leq 1$

if $M_1 \sim 1 = 1.001$ Mach wave / normal & oblique Mach waves

normal shock wave $\left\{ \begin{array}{l} \text{adiabatic} \\ \text{steady} \\ \text{one-dimensional} \\ \text{ideal gas} \\ \text{const. spec. heats} \end{array} \right.$