Compressible Flow

lagrangian perspective

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  - Equations of state for materials (gases, plasmas, liquids, solids)

solution approaches
- separation of variables
- method of characteristics
- self similar solutions

Blasius

Classification of Partial Differential Equations (from Computational Fluid Dynamics by T. J. Chung)

Partial differential equations (PDE's) are classified in one of three categories:

1. elliptic → 2 second-order derivative terms / diffusion
2. parabolic → 1 second-order derivative term / diffusion
3. hyperbolic → 2 second-order derivative terms / opposite sign / diffusion

PDE → multiple ODE's

The solution of the equations (solution method, form of the solution, etc) all depend on how it is classified. In fluid mechanics, the behavior depends on the flow speed. In general, the flow is a mixture of elliptic and hyperbolic, although special cases can be parabolic as well. For inviscid Euler equations, it depends on the flow velocity and Mach number.

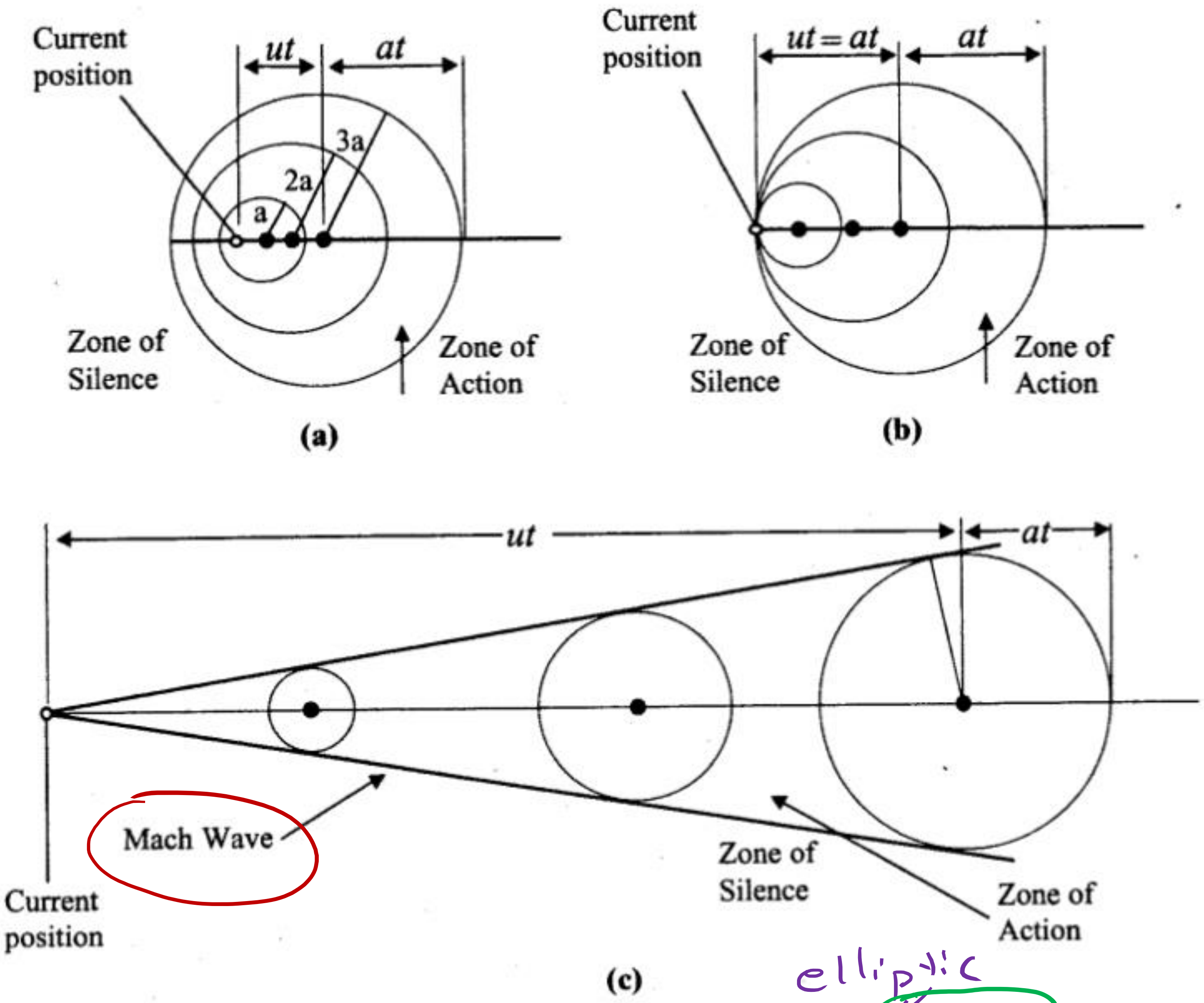

Figure 2.1.1 Subsonic, sonic, and supersonic flows. (a) Subsonic ($u < a$, $M < 1$). (b) Sonic ($u = a$, $M = 1$). (c) Supersonic ($u > a$, $M > 1$).

elliptic

hyperbolic

velocity = u
moving dot
generating sound
traveling at the sonic velocity = a

$M = \frac{u}{a}$

Mach number

"Sonic boom"

Objective: Figure out when equations are elliptic, parabolic, or hyperbolic. For sake of generality, consider

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \quad (27)$$

general PDE

2 independent variables x, y

1 dependent variable u

where the coefficients A-G could be constant, or general functions of all the variables. To assure the continuity of the first derivative, let's consider the first derivatives variables, so we have u, u_x, and u_y. The latter two are first order derivatives using the notation introduced in chapter 1. Let's write out the exact differentials of these latter two variables

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy = \frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial x \partial y}dy \quad (28)$$

and

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy = \frac{\partial^2 u}{\partial x \partial y}dx + \frac{\partial^2 u}{\partial y^2}dy \quad (29)$$

We can put these 3 PDE's in matrix form, giving

$$\left[u_x = \frac{\partial u}{\partial x} \quad u_y = \frac{\partial u}{\partial y} \right]$$

type of PDE

elliptic, parabolic, hyperbolic } values of A, B, C, D, E, F, G

$$\begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix}\begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} H \\ du_x \\ du_y \end{bmatrix} \quad (30)$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$$

$$u_{yy} = \frac{\partial^2 y}{\partial y^2}$$

where

$$H = -(D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G) \quad (31)$$

To determine the type of PDE, the determinant of the coefficient matrix in Eq. 30 is equal to 0. This gives the equation

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \tag{32}$$

matrix

which yields

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{33}$$

quadratic equation / unknown $\left(\frac{dy}{dx}\right)$

determinant matrix $= 0$

Solving this quadratic equation yields the equations of the characteristics in physical space,

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{34}$$

solution of the quadratic eqn.

The sign of the argument inside the square root

1. Elliptic if $B^2 - 4AC < 0$. The characteristics do not exist
2. Parabolic if $B^2 - 4AC = 0$. One set of characteristics exist
3. Hyperbolic if $B^2 - 4AC > 0$. Two sets of characteristics exist

lines/planes/solution space for the PDE

PDE → multiple ODE's

Examples

Examples

(a) Elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1, \quad B = 0, \quad C = 1$$

$$B^2 - 4AC = -4 < 0$$

(b) Parabolic equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (\alpha > 0)$$

$$A = -\alpha, \quad B = 0, \quad C = 0$$

$$B^2 - 4AC = 0$$

(c) Hyperbolic equation

1-D First Order Wave Equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (a > 0)$$

1-D Second Order Wave Equation

Differentiating (2.1.11) with respect to x and t,

$$\frac{\partial^2 u}{\partial t \partial x} + a \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial^2 u}{\partial t \partial x} = 0$$

Combining (2.1.12a) and (2.1.12b) yields

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$$

where

$$A = 1, \quad B = 0, \quad C = -a^2$$

$$B^2 - 4AC = 4a^2 > 0$$

6.1 Introduction to differential form of governing equations

There is a plethora of forms of the conservation equations. In general, they all mean about the same thing. The analysis of problems is as follows

1. Determine a model of a fluid. We chose a control volume to represent a fluid element.
2. Apply physics principles to obtain the equations embodying those principles. We use conservation of mass, momentum, and energy.
3. Use the equations to solve the problem of interest. Basically, we apply conservation equations, throw away terms we decide are not important, and then simplify in order to make the equations solvable. We will often use the integral forms because the problems are simple and are described in one spatial dimension.

An alternative is to resort to differential analysis, and use computers to solve the equations. To do this, we take the integral form and apply it to a differential control volume, and make the approximation that the differential form applies at that point in the flow. We make use of the following vector identities (Divergence theory, aka Gauss's theorem):

$$\oiint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_{\mathcal{V}} (\nabla\cdot\mathbf{A})\, d\mathcal{V} \tag{35}$$

$$\oiint_S \mathbf{\Phi}\, d\mathbf{S} = \iiint \mathcal{V}(\nabla\mathbf{\Phi})\, d\mathcal{V} \tag{36}$$

where A and Φ are vector and scalar functions, respectively, of space and time. They read 'the flux of the property through the surface is equal to the divergence of that property inside the volume'.

6.2 *Differential Equations in Conservation Form*

Our approach is as follows. Start with the integral form of the conservation equation from chapter 2, use the identity above to get rid of the flux terms to replace with dV term, then you can combine all terms within the integral.

6.2.1 Continuity Equation

Continuity is

$i_x\ i_y\ i_z$ = unit vector

$$-\oiint_S \rho \mathbf{V}\cdot d\mathbf{S} = \oiiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} \tag{37}$$

using our vector identity, this becomes

$\vec{V}$ = velocity vector

ρ = density

$$\oiiint_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{V})\right] d\mathcal{V} = 0 \tag{38}$$

For the arbitrary case of any volume, the only way this equation is satisfied is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{V}) = 0 \tag{39}$$

$\vec{V} = u\vec{i_x} + v\vec{i_y} + w\vec{i_z}$

$\vec{\nabla}\cdot(\rho\vec{V}) = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$

This is the differential form of continuity equation.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$