Exercise 1

Let $f \times g : P \to X$ be the product of f and g in the slice category (note: This is not the normal product!), with projections $\pi_f : P \to A$ and $\pi_g : P \to B$ such that $f \circ \pi_f = g \circ \pi_g = f \times g$ (this is what it means to be a morphism in the slice category).

Because of this equality we immediately know that π_f and π_g factorise uniquely over $A \times_X B$ via a morphism $i_1 : P \to A \times_X B$ (this uses the property of the pullback).

On the other hand p_1 and p_2 are morphisms from $f \circ p_1 = g \circ p_2$ to f and g in the slice category respectively, therefore they factorise uniquely over the product $f \times g$ via some unique morphism $i_2 : A \times_X B \to P$. By uniqueness, we have that i_1 and i_2 must be inverse to one another and therefore pullbacks are exactly products in the slice category.

$$\begin{tikzcd}
 & A \times_X B \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
 & P \arrow[dl, "\pi_f"'] \arrow[d, "f \times g"] \arrow[dr, "\pi_g"] & \\
A \arrow[r, "f"] & X & B \arrow[l, "g"']
\end{tikzcd}$$

Exercise 7

Let L be the limit of D with (with morphisms $l_i : L \to C_i$).
Let L' be the limit of $Hom(C, -) \circ D$ (with morphisms $l'_i : L' \to Hom(C, C_i)$).

We have to show that $Hom(C, L) \cong L'$.

The unique morphism from $Hom(C, L)$ to L' exists since L' is a limit and mapping a cone via a functor yields a cone. The other direction is the interesting one:

Let $x \in L'$ be an element of L' (note that we are in **Sets**). The morphisms l'_i map this x to morphisms $l'_i(x) : C \to C_i$. Because L was a limit over D, we know that there is a factorisation through L, i.e. there exists a unique $\xi : C \to L$ such that $l'_i(x) = l_i \circ \xi$. In other words, we indentified the unique element of $Hom(C, L)$ that behaves under l_i in the same way that x behaves under l'_i. Since $x \in L'$ was chosen arbitrarily, we conclude that there is a unique map from $L' \to Hom(C, L)$ that makes everything commute. This completes the contruction of the isomorphism and $Hom(C, L)$ is therefore also a limit of $Hom(C, -) \circ D$.