Excercise 2

a) I show that $(C_1 \times C_2)^B \cong C_1^B \times C_2^B$. Consider the following diagram:

$$
\begin{tikzcd}
& C_1^B \times C_2^B \times B \arrow[dl, "\pi_1 \times 1_B"'] \arrow[dr, "\pi_2 \times 1_B"] \arrow[d, dotted, leftrightarrow, "\star"] & \\
C_1^B \times B \arrow[d, "\varepsilon"] & (C_1 \times C_2)^B \times B \arrow[l, dotted, "\bar{f}_1 \times 1_B"] \arrow[r, dotted, "\bar{f}_2 \times 1_B"] \arrow[d, "\varepsilon"] & C_2^B \times B \arrow[d, "\varepsilon"] \\
C_1 & C_1 \times C_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"] & C_2
\end{tikzcd}
$$

The solid lines are the evaluation functions for the exponentials (note: the three ε are all different) and the canonical projections for products.

By the universal property of the exponential, there is exactly one $\bar{f}_1$ and $\bar{f}_2$ such that $\bar{f}_1 \times 1_B$ and $\bar{f}_2 \times 1_B$ make the diagram commute at the dashed lines (f_1 is $\pi_1 \circ \varepsilon$ here). Now, by the universal property of the product it is exactly $\bar{f}_1 \times \bar{f}_2 \times 1_B$ in the upwards direction at $\star$ that makes the diagram commute (requiring identity on the B part).

For the other direction note that by the universal property of the product there is exactly one morphism from $C_1^B \times C_2^B \times B$ to $C_1 \times C_2$ that makes the diagram commute and this gives us (by the universal property of the exponential) the unique morphism $f : C_1^B \times C_2^B \to (C_1 \times C_2)^B$ such that $f \times 1_B$ in the downward dirction of $\star$ that makes the diagram commute.

Together we have found two unique morphisms from at $\star$ that don't touch the B part. Those must therefore be isomorphisms.

b) Consider the following diagram:

$$
\begin{tikzcd}
(C^B)^A \times (A \times B) \arrow[r, leftrightarrow, "\cong"] \arrow[d, dotted, "F_1"] & ((C^B)^A \times A) \times B \arrow[r, "\varepsilon \times 1_B"] & C^B \times B \arrow[d, "\varepsilon"] \\
C^{A \times B} \times (A \times B) \arrow[r, leftrightarrow, "\cong"] \arrow[rr, bend right, "\varepsilon"'] & (C^{A \times B} \times A) \times B \arrow[u, dotted, "F_3"] \arrow[ur, dotted, "F_2"] & C
\end{tikzcd}
$$

It uses the cannonical isomorphisms (using associativity of products) and the respective evaluation functions of exponentials.

We can construct the following morphisms uniquely:

- f_1 such that $F_1 = f_1 \times 1_{A \times B}$ makes the diagram commute (property of exponentials).
- f_2 such that $F_2 = f_2 \times 1_B$ makes the diagram commute (property of exponentials).
- f_3 such that $F_3 = f_3 \times 1_A \times 1_B$ makes the diagram commute (property of exponentials).

We have now constructed $f_1 : (C^B)^A \to C^{A \times B}$ and f_3 in the other direction uniquely. These two functions must therefore be isomorphisms.

Excercise 3

We know that the transpose of a function is unique, i.e. if we find a function fullfilling the property of the transpose, we know that it must be the transpose. That being said:

- The transpose of ε is 1_{B^A}, since the following commutes:

$$
\begin{tikzcd}
B^A \times A \arrow[r, "\varepsilon"] & B \\
B^A \times A \arrow[u, "1_{B^A} \times 1_A"] \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
$$

- The transpose of $1_{A\times B}$ in **Sets** is:

$$f : a \mapsto (f_a : b \mapsto (a, b))$$

because this makes the diagram commute:

$$\begin{tikzcd}
(A \times B)^B \times B \arrow[r, "\varepsilon"] & A \times B \\
A \times B \arrow[u, "f \times 1_B"] \arrow[ur, "1_{A\times B}"] &
\end{tikzcd}$$

- The transpose of $\varepsilon \circ \tau$ in **Sets** is:

$$f : a \mapsto (i_a : g \mapsto g(a))$$

(i.e. f is the function that maps a to the function i_a that inserts a into whatever function it gets as its paramter) because this makes the diagram commute:

$$\begin{tikzcd}
B^{(B^A)} \times B^A \arrow[r, "\varepsilon"] & B \\
A \times B^A \arrow[u, "f \times 1_{B^A}"] \arrow[r, "\tau"] & B^A \times A \arrow[u, "\varepsilon"']
\end{tikzcd}$$