Exercise 2

(a) **Rel** is isomorphic to **Rel**^{op} by the "trivial" isomorphism F:

$$F(S) = S, \qquad F(R) = \{(b, a) \mid (a, b) \in R\}$$

It is easy to check that this maps identity morphisms to identity morphisms and if $R_1 \subset A \times B$ and $R_2 \subset B \times C$ are relations then:

$$\begin{aligned}
F(R_2 \circ R_1) &= \left\{(c, a) \,\middle|\, (a, c) \in R_2 \circ R_1\right\} \\
&= \left\{(c, a) \,\middle|\, \exists b : (a, b) \in R_1 \wedge (b, c) \in R_2\right\} \\
&= \left\{(c, a) \,\middle|\, \exists b : (b, a) \in F(R_1) \wedge (c, b) \in F(R_2)\right\} \\
&= F(R_1) \circ F(R_2) \\
&= F(R_2) \circ^{op} F(R_1)
\end{aligned}$$

It is easy to see that F is self inverse (and the inverse is also a Funktor).

(b) **Sets** is not isomorphic to **Sets**^{op}. Assume otherwise, then there are Funktors F : **Sets** $\to$ **Sets**^{op} and F' : **Sets**^{op} $\to$ **Sets** that are inverse to one another. In particular $F'(F(\emptyset)) = \emptyset$. Assume there is a morphism $f : X \to F(\emptyset)$ in **Sets**^{op}. Then $F'(f) : F'(X) \to \emptyset$ and $F'(X) = \emptyset$ by the properties of the empty set. Since F' must be monic ("injective") we conclude $X = F(\emptyset)$. In otherwords, there must be a magical object X in **Sets**^{op} whose only incoming morphisms come from itself. This corresponds to a set whose only outgoing morphisms are to itself. This does not exist.

(c) They cannot be isomorphic since this would correspond to a bijection between a set and its powerset which never exists (well known theorem by Canthor).

Exercise 3

(a) Let $f : A \to B$ be an isomorphism in **Sets**, meaning there is $f^{-1} : B \to A$ such that $f' \circ f = id_A$ and $f \circ f' = id_B$. The former tells us that f is injective, the latter tells us that f is surjective. Therefore it is a bijection.
On the other hand, if f is a bijection, then we immediately know that such an f^{-1} exists.

(b) Let $f : M_1 \to M_2$ be an isomorphism. Since the forgetful functor is a functor, the morphism f is also an isomorphism on the underlying set and therefore a bijection. By definition it is a homomorphism.
Conversely, if $f : M_1 \to M_2$ is a homomorphism and bjective, then the inverse

function f^{-1} is also a monoid homomorphism since it maps the unit element to the unit element (which is unique) and we have:

$$f^{-1}(a \cdot b) = f^{-1}(f(f^{-1}(a)) \cdot f(f^{-1}(b))) = f^{-1}(f(f^{-1}(a) \cdot f^{-1}(b))) = f^{-1}(a) \cdot f^{-1}(b)$$