Exercise 7

Let P_1 and P_2 be projectives, $e : E \to X$ any epimorphism and $f : P_1 + P_2 \to X$. We need to show that f factorises through e.

We start with the solid lines in the following diagram.

$$
\begin{tikzcd}
 & & E \arrow[dd, "e"] \\
P_1 \arrow[dr, "i_1"'] \arrow[urr, dashed, "\bar{f}_1"] & P_2 \arrow[d, "i_2"'] \arrow[ur, dashed, "\bar{f}_2"] & \\
 & P_1 + P_2 \arrow[r, "f"] \arrow[uur, dashed, "\bar{f}"] & X
\end{tikzcd}
$$

Because P_1 is a projection, the function $f \circ i_1$ factorises over E (call the function $\bar{f}_1$). The same holds for P_2. Now we can use the universal property of the coproduct and construct $\bar{f}$ such that $\bar{f} \circ i_1 = \bar{f}_1$ and $\bar{f} \circ i_2 = \bar{f}_2$

Now all paths starting from P_1 and P_2 commute. Using the universal property of the coproduct again, we see that $e \circ \bar{f}$ and f both go from $P_1 + P_2$ to X and must therefore be the same (unique) morphism that makes the diagram comute and we are done since we have found $e \circ \bar{f} = f$.

Exercise 15

Let $f, g : X \to Y$ be two morphisms in **Top**. I claim the coequaliser of f and g is the projection π to $Y/\sim$ where $\sim$ is the smallest equivalence relation containing $\{(f(x), g(x)) \mid x \in X\}$.

This means, that $Y/\sim$ is the space of equivalence classes of $\sim$ and the open sets are exactly those sets S such that $\pi^{-1}(S)$ is open in Y, where π denotes the projection $x \mapsto [x]_\sim$. It is simple to check that this indeed constitutes a valid topology.

Now for the universal property: Assume there is a morphism $c : Y \to Z$ such that $c \circ f = c \circ g$. This means that c respects $\sim$, namely $y_1 \sim y_2 \Rightarrow c(y_1) = c(y_2)$. Therefore $c^* : Y/\sim \to Z$ given by $[x]_\sim \mapsto c(x)$ is a well-defined function such that $c^* \circ \pi = c$. It is also continuous since, if $O \subset Z$ is open, then so is $c^{*-1}(O) \subset Y$ because this just means that $\pi^{-1}(c^{*-1}(O)) = c^{-1}(O)$ must be open which is true because c is continuous. It is obvious, that we could not have defined c^* differently, so the factorisation over $Y/\sim$ is unique as we want it to be.