box-muller-transform

May 13, 2021

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns; sns.set()
```

1 Box-Muller transform

Problem Given the random variable X following the standard normal distribution $\mathcal{N}(0, 1)$, generate values of X.

Method

To generate 2 values, do:

1. Sample u_1 and u_2 independently from Uniform$([0, 1])$.
2. Compute

$$z_1 = \sqrt{-2 \ln u1} \cos(2\pi u_2)$$

$$z_2 = \sqrt{-2 \ln u1} \sin(2\pi u_2)$$

[3]:
```
def box_muller(mu=0, sigma=1, N=1):
    # Number of samples needed
    M = N // 2 + 1

    # Sample from the uniform distribution
    u = np.random.rand(2, M)

    # Compute the values
    R = np.sqrt(-2 * np.log(u[0]))
    O = 2 * np.pi * u[1]
    z = np.hstack([
        R * np.cos(O),
        R * np.sin(O)
    ])[:N]

    # Scale and ranslate
    z = z * sigma + mu

    if N == 1:
        return z[0]
```

```
    return z
```

Proof of correctness

Let $R = \sqrt{-2 \ln U_1}$ where $U_1 \sim \text{Uniform}([0, 1])$. Its CDF is

$$F_R(r) = P(R \leq r) = P(\sqrt{-2 \ln U_1} \leq r) = P\left[U_1 \geq \exp\left(-\frac{r^2}{2}\right)\right] = 1 - \exp\left(-\frac{r^2}{2}\right).$$

Therefore its PDF is

$$f_R(r) = \frac{dF_R}{dr}(r) = r \exp\left(-\frac{r^2}{2}\right), r \geq 0.$$

Let $\Theta = 2\pi U_2$ where $U_2 \sim \text{Uniform}(0, 1)$, then $\Theta \sim \text{Uniform}(0, 2\pi)$ and its PDF is

$$f_\Theta(\theta) = \frac{1}{2\pi}, \theta \in [0, 2\pi].$$

Since U_1 and U_2 are independent, R and Θ also are. Therefore

$$f_{R,\Theta}(r, \theta) = f_R(r) f_\Theta(\theta) = \frac{1}{2\pi} r \exp\left(-\frac{r^2}{2}\right).$$

Let g be the transformation function to the polar coordinate, $z = (z_1, z_2) = g(r, \theta) = (r \cos\theta, r \sin\theta)$ or $r = \sqrt{z_1^2 + z_2^2}$ and $\theta = \arctan z_2/z_1$.

We have

$$f_{Z_1,Z_2}(z_1, z_2) = f_{Z_1,Z_2}(g(r, \theta)) = \frac{f_{R,\Theta}(r, \theta)}{|\det \frac{\partial g}{\partial z}|} = \frac{f_{R,\Theta}(r, \theta)}{r} = \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right)$$

which leads to

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_1^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) = f_{Z_1}(z_1) f_{Z_2}(z_2).$$

So each of Z_1, Z_2 is normally-distributed.

Note

1. Change of variable

$$f_X(x) = f_Y(g(x)) \left|\det \frac{\partial g}{\partial x}\right|$$

2. For the polar coordinates,

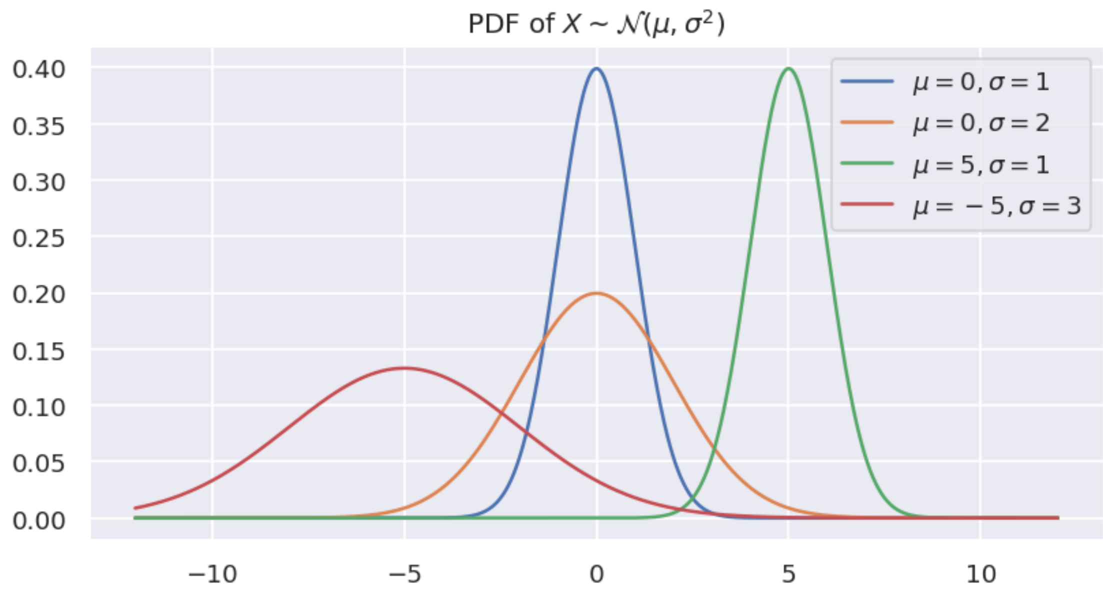
$$\left|\det \frac{\partial g}{\partial z}\right| = \left|\frac{\partial r \cos\theta}{\partial r} \frac{\partial r \sin\theta}{\partial \theta} - \frac{\partial r \sin\theta}{\partial r} \frac{\partial r \cos\theta}{\partial \theta}\right| = |r \cos^2\theta + r \sin^2\theta| = |r| = r, r \geq 0$$

2 Demo

[4]:
```
def normal_pdf(x, mu, sigma):
    return np.exp(-(x - mu) ** 2 / (2 * sigma ** 2)) / (np.sqrt(2 * np.pi) *␣
 ↪sigma)
```

[5]:
```
mus = [0, 0, 5, -5]
sigmas = [1, 2, 1, 3]

fig = plt.figure(figsize=(8, 4), dpi=120)
x = np.linspace(start=-12, stop=12, num=1000)
for mu, sigma in zip(mus, sigmas):
    f = normal_pdf(x, mu, sigma)
    plt.plot(x, f, label=f'$\mu = {mu}, \sigma = {sigma}$')
plt.title('PDF of $X \sim \mathcal{N}(\mu, \sigma^2)$')
plt.legend()
plt.show()
```



[6]:
```
mu1 = 0
sigma1 = 1
N1 = 5000

X1 = box_muller(mu1, sigma1, N1)
```

[7]:
```
mu2 = -5
sigma2 = 3
N2 = 5000

X2 = box_muller(mu2, sigma2, N2)
```

[9]:
```
fig = plt.figure(figsize=(8, 4), dpi=120)

# Generated data density distribution
sns.histplot(X1, stat='density', label='Generated Data 1', color='blue')

# True PDF f(x)
x = np.linspace(start=-12, stop=12, num=1000)
f = normal_pdf(x, mu1, sigma1)
plt.plot(x, f, label='True PDF 1', color='blue')

# Generated data density distribution
sns.histplot(X2, stat='density', label='Generated Data 2', color='orange')

# True PDF f(x)
x = np.linspace(start=-12, stop=12, num=1000)
f = normal_pdf(x, mu2, sigma2)
plt.plot(x, f, label='True PDF 2', color='orange')

plt.ylim(0, 0.5)

plt.legend()
plt.show()
```