Integration-by-parts improvement of Fermi integrals for baryon number susceptibilities and entropy derivative at low temperature

Volodymyr Vovchenko

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Abstract

We describe two improvements to the numerical evaluation of Fermi–Dirac integrals at low temperatures in the hadron resonance gas model.

First, we replace the uniform Gauss–Legendre quadrature below the Fermi momentum p_{F} with a Sommerfeld–Legendre scheme that uses an adaptive nonlinear mapping to concentrate quadrature nodes near the Fermi surface, with the degree of concentration controlled by $\alpha = p_{\mathrm{F}}^2/(\mu T)$. This resolves the sharp Fermi step at low T, where the old uniform mapping under-resolved the Fermi surface.

Second, we apply integration-by-parts (IBP) to reduce the order of Fermi–Dirac distribution derivatives in the integrands, replacing oscillatory or sharply peaked functions with smoother ones. We present the IBP formulas for the susceptibilities χ_2, χ_3, and χ_4, derive the analytic $T = 0$ limits, and document the numerical improvements achieved. For the entropy density temperature derivative $ds/dT|_\mu$, we derive a double-IBP formula involving the Fermi entropy kernel $\sigma(x) = -f \ln f - (1-f)\ln(1-f)$ — the same kernel that gives the entropy density $s = (g/2\pi^2)\int p^2\,\sigma\,dp$. Because σ vanishes identically at $T = 0$, the quadrature evaluates the thermal contribution directly with no cancellation, achieving $\sim 2 \times 10^{-4}$ relative accuracy at $T = 1$ μeV. We explain why the χ_2-type analytical extraction of the $T = 0$ value is fundamentally unavailable for temperature derivatives.

Contents

1 Setup and notation

Consider a single fermion species with mass m, degeneracy g, at temperature T and chemical potential $\mu > m$ (so that the Fermi momentum $p_{\mathrm{F}} = \sqrt{\mu^2 - m^2}$ is real). The Fermi–Dirac distribution is

$$f(p) = \frac{1}{e^{(E-\mu)/T} + 1}\,, \qquad E = \sqrt{p^2 + m^2}\,. \tag{1}$$

The particle number density is

$$n = \frac{g}{2\pi^2} \int_0^\infty p^2\, f(p)\, dp\,. \tag{2}$$

The generalized susceptibilities are the μ-derivatives of the pressure, or equivalently:

$$\chi_n \equiv \frac{\partial^n P}{\partial \mu^n} = \frac{\partial^{n-1} n}{\partial \mu^{n-1}}\,. \tag{3}$$

Using the identity $\partial f/\partial \mu = f(1-f)/T$, the first few susceptibilities involve

$$\chi_2 = \frac{\partial n}{\partial \mu} = \frac{g}{2\pi^2} \int_0^\infty p^2\, \frac{f(1-f)}{T}\, dp\,, \tag{4}$$

$$\chi_3 = \frac{\partial^2 n}{\partial \mu^2} = \frac{g}{2\pi^2} \int_0^\infty p^2\, \frac{f(1-f)(1-2f)}{T^2}\, dp\,, \tag{5}$$

$$\chi_4 = \frac{\partial^3 n}{\partial \mu^3} = \frac{g}{2\pi^2} \int_0^\infty p^2\, \frac{f(1-f)\big[1-6f(1-f)\big]}{T^3}\, dp\,. \tag{6}$$

The numerical problem. As $T \to 0$, the factor $f(1-f)/T$ tends to $\delta(E-\mu)$, i.e. $\delta(p-p_{\mathrm{F}})/v_F$ with $v_F = p_{\mathrm{F}}/\mu$. For χ_2 the integrand is a narrow positive peak at p_{F}; for χ_3 the integrand $f(1-f)(1-2f)/T^2$ has a derivative-of-delta structure (one sign change); for χ_4 the integrand $f(1-f)[1-6f(1-f)]/T^3$ has a second-derivative-of-delta structure (two sign changes). Higher-order integrands are increasingly oscillatory and harder for fixed-order quadrature to resolve.

The code already splits the integration at the Fermi momentum p_{F} into a below-p_{F} Legendre part and an above-p_{F} shifted Laguerre part. The original below-p_{F} part used a *uniform* (linear)

mapping $p = s\,p_\mathrm{F}$ with 32-point Gauss–Legendre nodes on $[0, 1]$:

$$\int_0^{p_\mathrm{F}} F(p)\,dp \approx p_\mathrm{F} \sum_{i=1}^{32} w_i^{(\mathrm{Leg})}\, F(s_i\,p_\mathrm{F})\,. \tag{7}$$

The Legendre nodes $s_i \in [0, 1]$ are roughly uniformly spaced, so the mapped momenta $p_i = s_i\,p_\mathrm{F}$ are roughly uniformly distributed over $[0, p_\mathrm{F}]$. At low temperatures, the Fermi–Dirac integrands vary on the scale of $\sim T/v_F$ near the Fermi surface ($p = p_\mathrm{F}$), which is much smaller than the average node spacing $\sim p_\mathrm{F}/32$. This causes the uniform Legendre quadrature to under-resolve the Fermi surface peak, producing errors of 100–400% for the higher susceptibilities.

The improved quadrature replaces the uniform mapping with an *adaptive* nonlinear mapping (Sommerfeld–Legendre) that concentrates nodes near p_F:

- **Below p_F: Sommerfeld–Legendre quadrature** (32 points). A nonlinear mapping $s \in [0, 1] \to p \in [0, p_\mathrm{F}]$ adaptively concentrates nodes near the Fermi surface. The mapping parameter $\alpha = p_\mathrm{F}^2/(\mu T)$ controls the degree of concentration: when $\alpha \gg 1$ (low T), nodes cluster tightly near p_F to resolve the sharp Fermi step; when $\alpha \ll 1$ (high T), the mapping reduces to the original uniform $p = s\,p_\mathrm{F}$. See section 5.1 for the detailed construction.
- **Above p_F: shifted Laguerre quadrature** (32 points). The substitution $p = p_\mathrm{F} + T\,t$ places nodes at momenta $p_\mathrm{F} + O(T)$, naturally resolving the thermal tail above the Fermi surface with exponential weight. This part is unchanged from the original code.

Both sets of nodes are concentrated near p_F, where the Fermi–Dirac integrands peak. However, for the oscillatory integrands in χ_3 and especially χ_4, the quadrature accuracy degrades severely at low T. The IBP technique described below smooths the integrands by reducing the derivative order by one.

2 IBP identities

The core identity underlying all IBPs is

$$\frac{df}{dp} = -\frac{p}{ET}\,f(1-f)\,. \tag{8}$$

This allows us to express products of f and its μ-derivatives as total p-derivatives, enabling integration by parts.

2.1 χ_2: first IBP

From eq. (8), we have

$$f(1-f) = -T\,\frac{E}{p}\,\frac{df}{dp}\,. \tag{9}$$

Therefore the χ_2 integrand becomes a total derivative:

$$p^2\,f(1-f) = -T\,p\,E\,\frac{df}{dp} = -T\,\frac{d}{dp}\big[p\,E\,f\big] + T\,\frac{d(pE)}{dp}\,f\,. \tag{10}$$

Since $d(pE)/dp = (2p^2 + m^2)/E$ and the boundary term $[p\,E\,f]_0^\infty = 0$ (vanishes at $p = 0$ and exponentially at $p = \infty$), integration by parts gives

$$\boxed{\int_0^\infty p^2\,f(1-f)\,dp = T\int_0^\infty \frac{2p^2 + m^2}{E}\,f\,dp\,.} \tag{11}$$

The left-hand side has integrand $f(1-f) \sim \delta(p-p_{\rm F})$; the right-hand side has integrand f, the Fermi step function itself — smooth, monotonic, and trivial for quadrature.

Using eq. (11) in eq. (4):

$$\chi_2 = \frac{g}{2\pi^2} \int_0^\infty \frac{2p^2+m^2}{E}\, f\, dp\,. \tag{12}$$

2.2 χ_3: second IBP

Differentiating $f(1-f) = -T(E/p)\, df/dp$ once more with respect to μ:

$$\frac{\partial}{\partial\mu}\big[f(1-f)\big] = \frac{f(1-f)(1-2f)}{T} \quad \Rightarrow \quad f(1-f)(1-2f) = -T\,\frac{E}{p}\,\frac{d}{dp}\big[f(1-f)\big]\,. \tag{13}$$

Therefore

$$p^2\, f(1-f)(1-2f) = -T\, p\, E\, \frac{d}{dp}\big[f(1-f)\big]\,. \tag{14}$$

Integrating by parts with $[p\,E\,f(1-f)]_0^\infty = 0$ (the boundary terms vanish since $pE = 0$ at $p = 0$ and $f(1-f)$ decays exponentially at $p \to \infty$):

$$\boxed{\int_0^\infty p^2\, f(1-f)(1-2f)\, dp = T \int_0^\infty \frac{2p^2+m^2}{E}\, f(1-f)\, dp\,.} \tag{15}$$

The left-hand side has an integrand with one sign change (derivative-of-delta structure); the right-hand side has $f(1-f)$, a smooth non-negative peak at $p_{\rm F}$ — the same integrand that appeared in the *un-improved* χ_2.

Using eq. (15) in eq. (5):

$$\chi_3 = \frac{g}{2\pi^2 T} \int_0^\infty \frac{2p^2+m^2}{E}\, f(1-f)\, dp\,. \tag{16}$$

2.3 χ_4: third IBP

Similarly, differentiating $f(1-f)(1-2f)$ with respect to μ:

$$\frac{\partial}{\partial\mu}\big[f(1-f)(1-2f)\big] = \frac{f(1-f)\big[1-6f(1-f)\big]}{T} \tag{17}$$

and thus

$$f(1-f)\big[1-6f(1-f)\big] = -T\,\frac{E}{p}\,\frac{d}{dp}\big[f(1-f)(1-2f)\big]\,. \tag{18}$$

Integration by parts with $[p\,E\,f(1-f)(1-2f)]_0^\infty = 0$ gives:

$$\boxed{\int_0^\infty p^2\, f(1-f)\big[1-6f(1-f)\big]\, dp = T \int_0^\infty \frac{2p^2+m^2}{E}\, f(1-f)(1-2f)\, dp\,.} \tag{19}$$

The left-hand side has two sign changes; the right-hand side has one sign change (the same integrand as the *un-improved* χ_3).

Using eq. (19) in eq. (6):

$$\chi_4 = \frac{g}{2\pi^2 T^2} \int_0^\infty \frac{2p^2+m^2}{E}\, f(1-f)(1-2f)\, dp\,. \tag{20}$$

2.4 General pattern

The IBP has a recursive structure. Define

$$F_k(p) \equiv T\frac{\partial^k f}{\partial \mu^k}\,, \tag{21}$$

so that $F_0 = f$, $F_1 = f(1-f)$, $F_2 = f(1-f)(1-2f)/T$, etc. Then the identity $\partial f/\partial p = -(p/ET)\,F_1$ generalizes to

$$F_{k+1} = \frac{\partial F_k}{\partial \mu} = -T\,\frac{E}{p}\,\frac{\partial F_k}{\partial p}\,, \tag{22}$$

and each IBP step replaces

$$\int_0^\infty p^2\,F_{k+1}\,dp = T\int_0^\infty \frac{2p^2+m^2}{E}\,F_k\,dp\,, \tag{23}$$

provided the boundary term $[pE\,F_k]_0^\infty$ vanishes. Each step reduces the number of sign changes in the integrand by one, at the cost of introducing one additional power of T in the denominator when forming χ_n.

3 Analytic $T = 0$ limits

At $T = 0$, the Fermi–Dirac distribution becomes a step function, $f = \theta(p_{\rm F} - p)$, and the integrals can be evaluated analytically.

3.1 Density and equation of state

$$n\big|_{T=0} = \frac{g}{2\pi^2}\,\frac{p_{\rm F}^3}{3}\,, \tag{24}$$

$$P\big|_{T=0} = \frac{g}{6\pi^2}\left[\mu\,p_{\rm F}^3 - \tfrac{3}{4}\,p_{\rm F}^4\,\psi(m/p_{\rm F})\right], \tag{25}$$

$$\varepsilon\big|_{T=0} = \frac{g}{2\pi^2}\,\frac{p_{\rm F}^4}{4}\,\psi(m/p_{\rm F})\,, \tag{26}$$

where $\psi(x) = (1+x^2)^{1/2}(1+\frac{1}{2}x^2) - \frac{1}{2}x^4\sinh^{-1}(1/x)$.

3.2 Susceptibilities

Using $f(1-f)/T \to \delta(p-p_{\rm F})\,\mu/p_{\rm F}$ and its derivatives, or by directly differentiating the $T=0$ density:

$$\chi_2\big|_{T=0} = \left.\frac{dn}{d\mu}\right|_{T=0} = \frac{g}{2\pi^2}\,\frac{\mu\,p_{\rm F}}{1} = \frac{g}{2\pi^2}\,\mu\,p_{\rm F}\,, \tag{27}$$

$$\chi_3\big|_{T=0} = \left.\frac{d^2n}{d\mu^2}\right|_{T=0} = \frac{g}{2\pi^2}\,\frac{\mu^2+p_{\rm F}^2}{p_{\rm F}}\,, \tag{28}$$

$$\chi_4\big|_{T=0} = \left.\frac{d^3n}{d\mu^3}\right|_{T=0} = \frac{g}{2\pi^2}\,\frac{\mu\,(3p_{\rm F}^2-\mu^2)}{p_{\rm F}^3}\,. \tag{29}$$

These analytic values serve as reference for assessing the numerical accuracy at finite but small T.

4 Decomposition into below- and above-$p_{\rm F}$ contributions

The quadrature scheme splits every integral at the Fermi momentum $p_{\rm F}$:

$$\int_0^\infty (\dots)\, dp = \underbrace{\int_0^{p_{\rm F}} (\dots)\, dp}_{\text{below } p_{\rm F}} + \underbrace{\int_{p_{\rm F}}^\infty (\dots)\, dp}_{\text{above } p_{\rm F}} . \tag{30}$$

Since $f \to \theta(p_{\rm F} - p)$ as $T \to 0$, the above-$p_{\rm F}$ part vanishes exponentially, while the below-$p_{\rm F}$ part approaches a T-independent limit. We now show how this split interacts with the IBP integrands of each susceptibility.

4.1 χ_2: the $T = 0$ value emerges explicitly

The IBP formula for χ_2 (eq. (11)) gives

$$T\,\chi_2 = \frac{g}{2\pi^2} \int_0^\infty \frac{2p^2 + m^2}{E}\, f\, dp\,. \tag{31}$$

The integrand involves f — the Fermi step function. Below $p_{\rm F}$ we write $f = 1 - (1 - f)$, obtaining

$$\int_0^{p_{\rm F}} \frac{2p^2 + m^2}{E}\, f\, dp = \underbrace{\int_0^{p_{\rm F}} \frac{2p^2 + m^2}{E}\, dp}_{= \mu\, p_{\rm F}} - \int_0^{p_{\rm F}} \frac{2p^2 + m^2}{E}\, (1 - f)\, dp\,. \tag{32}$$

The first integral is the *exact* $T = 0$ contribution and can be evaluated analytically:

$$\int_0^{p_{\rm F}} \frac{2p^2 + m^2}{E}\, dp = \int_0^{p_{\rm F}} \left(2E - \frac{m^2}{E}\right) dp = \big[p\,E\big]_0^{p_{\rm F}} = p_{\rm F} \cdot \mu = \mu\, p_{\rm F}\,, \tag{33}$$

where we used $d(pE)/dp = (2p^2 + m^2)/E$ and the boundary values $p\,E|_{p=p_{\rm F}} = p_{\rm F}\,\mu$, $p\,E|_{p=0} = 0$.

The remaining terms are thermal corrections that vanish as $T \to 0$:

- $-\int_0^{p_{\rm F}} \frac{2p^2+m^2}{E}\,(1 - f)\, dp$: the "hole" contribution below $p_{\rm F}$. At $T = 0$, $1 - f = 0$ for $p < p_{\rm F}$, so this integral vanishes. At finite T, $1 - f$ is exponentially small except near $p_{\rm F}$, contributing $O(T^2)$.
- $+\int_{p_{\rm F}}^\infty \frac{2p^2+m^2}{E}\, f\, dp$: the "tail" contribution above $p_{\rm F}$. Again $f \to 0$ for $p > p_{\rm F}$ as $T \to 0$, giving $O(T^2)$.

The code therefore computes

$$T\,\chi_2 = \frac{g}{2\pi^2}\, T\, \big(\underbrace{\mu\, p_{\rm F}}_{\text{ret2}} + \underbrace{\Delta_{\rm th}}_{\text{ret1}}\big)\,, \tag{34}$$

where ret2 $= \mu\, p_{\rm F}$ is the analytic $T = 0$ value of χ_2 (in natural units, up to the $g/2\pi^2$ prefactor) and

$$\Delta_{\rm th} = -\int_0^{p_{\rm F}} \frac{2p^2 + m^2}{E}\,(1 - f)\, dp + \int_{p_{\rm F}}^\infty \frac{2p^2 + m^2}{E}\, f\, dp \tag{35}$$

is the thermal correction, which tends to zero as $T \to 0$.

Key point. The analytic extraction for χ_2 is *not* a numerical convenience — it is a structural consequence of the IBP. The IBP replaces the sharply peaked integrand $f(1 - f)$ by the step-function-like f, which *naturally* splits into a filled Fermi sea (yielding the exact $T = 0$ answer) plus corrections from particles and holes near the Fermi surface.

4.2 χ_3: peaked integrand, no natural $T = 0$ split

The IBP formula for χ_3 (eq. (15)) gives

$$T^2\,\chi_3 = \frac{g}{2\pi^2}\,T\int_0^\infty \frac{2p^2+m^2}{E}\,f(1-f)\,dp\,. \tag{36}$$

The integrand $f(1-f)$ is a smooth, non-negative peak localized within $\sim T$ of p_F. Unlike the χ_2 case, $f(1-f)$ vanishes at $T=0$ for all p (both below and above p_F), so there is no filled-Fermi-sea piece to extract directly from the integrand.

Instead, the integral scales as $\sim T$ for small T:

$$\int_0^\infty \frac{2p^2+m^2}{E}\,f(1-f)\,dp \;\xrightarrow{T\to 0}\; T\,\frac{\mu^2+p_\mathrm{F}^2}{p_\mathrm{F}}\,, \tag{37}$$

because $f(1-f)/T \to \delta(p-p_\mathrm{F})\,\mu/p_\mathrm{F}$. Therefore $\chi_3 = \mathrm{quad}/T \to (\mu^2+p_\mathrm{F}^2)/p_\mathrm{F}$ as $T\to 0$.

For the $T=0$ extraction, the code computes

$$T^2\,\chi_3 = \frac{g}{2\pi^2}\,T^2\Big(\underbrace{(\mu^2+p_\mathrm{F}^2)/p_\mathrm{F}}_{\text{ret2}} + \underbrace{\mathrm{quad}/T-(\mu^2+p_\mathrm{F}^2)/p_\mathrm{F}}_{\text{ret1}}\Big)\,. \tag{38}$$

Here ret2 is the known $T=0$ limit and ret1 is the thermal correction. Unlike the χ_2 case, ret2 does not emerge organically from splitting the integral at p_F — it is an independently computed constant that is subtracted. This decomposition is mathematically equivalent to the direct computation ($T^2\times$ value/T^2 round-trips losslessly in IEEE double), but it makes the $T\to 0$ behavior transparent.

4.3 χ_4: sign-changing integrand

The IBP formula for χ_4 (eq. (19)) gives

$$T^3\,\chi_4 = \frac{g}{2\pi^2}\,T\int_0^\infty \frac{2p^2+m^2}{E}\,f(1-f)(1-2f)\,dp\,. \tag{39}$$

The integrand $f(1-f)(1-2f)$ is localized near p_F and changes sign once (positive for $p<p_\mathrm{F}$, negative for $p>p_\mathrm{F}$, since $1-2f$ flips sign at $E=\mu$). Like χ_3, the integrand vanishes everywhere at $T=0$, and the integral scales as $\sim T^2$:

$$\int_0^\infty \frac{2p^2+m^2}{E}\,f(1-f)(1-2f)\,dp \;\xrightarrow{T\to 0}\; T^2\,\frac{\mu(3p_\mathrm{F}^2-\mu^2)}{p_\mathrm{F}^3}\,. \tag{40}$$

The code uses the same extraction structure:

$$T^3\,\chi_4 = \frac{g}{2\pi^2}\,T^3\Big(\underbrace{\mu(3p_\mathrm{F}^2-\mu^2)/p_\mathrm{F}^3}_{\text{ret2}} + \underbrace{\mathrm{quad}/T^2-\mu(3p_\mathrm{F}^2-\mu^2)/p_\mathrm{F}^3}_{\text{ret1}}\Big)\,. \tag{41}$$

Because $f(1-f)(1-2f)$ has one sign change, the quadrature accuracy is intermediate between χ_2 (no sign changes, excellent) and the un-improved χ_4 (two sign changes, catastrophic).

4.4 Comparison of the three cases

The qualitative difference between χ_2 and $\chi_{3,4}$ is summarized in the following table:

	χ_2	χ_3	χ_4
IBP integrand	f	$f(1-f)$	$f(1-f)(1-2f)$
Sign changes	0	0	1
$T \to 0$ behavior of integrand	$\to \theta(p_\mathrm{F} - p)$	$\to 0$ (peak $\sim T$)	$\to 0$ (peak $\sim T$)
T=0 from integral splitting?	**Yes**	No	No
How T=0 enters	$\int_0^{p_\mathrm{F}} \frac{2p^2+m^2}{E}\, dp = \mu p_\mathrm{F}$	quad/$T \to$ const	quad/$T^2 \to$ const

For χ_2, the IBP is special: the integrand f has a non-vanishing $T = 0$ limit (the filled Fermi sea), and splitting $f = 1 - (1 - f)$ below p_F yields the analytic answer *organically*. For χ_3 and χ_4, the integrands are purely thermal ($\to 0$ everywhere as $T \to 0$), so the $T = 0$ values emerge only through the scaling of the integral with powers of T, and the analytic extraction is a separate subtraction rather than a structural decomposition.

5 Implementation details

5.1 Quadrature scheme

The integration domain $[0, \infty)$ is split at the Fermi momentum p_F:

Above p_F: shifted Laguerre quadrature. The substitution $p = p_\mathrm{F} + T\,t$ transforms the Laguerre integration variable $t \in [0, \infty)$ with 32-point Gauss–Laguerre nodes and weights. The Jacobian $dp = T\,dt$ contributes a factor T. This part is unchanged from the original code.

Below p_F: Sommerfeld–Legendre quadrature. The original code used a uniform (linear) mapping $p = s\,p_\mathrm{F}$ with 32-point Gauss–Legendre nodes on $[0, 1]$. This distributes nodes roughly uniformly over $[0, p_\mathrm{F}]$, giving an average spacing of $\sim p_\mathrm{F}/32$. At low T, the integrands vary on the scale of $\sim T/v_F \ll p_\mathrm{F}/32$ near p_F, so the uniform nodes cannot resolve the Fermi surface peak.

The Sommerfeld–Legendre scheme replaces the uniform mapping with a nonlinear one that adaptively concentrates nodes near p_F. The construction is motivated by the Sommerfeld expansion: at low T, Fermi–Dirac integrands are smooth except within a window of width $\sim T$ around the Fermi surface. An ideal quadrature should place most of its nodes in this window, while retaining a few nodes in the bulk ($p \ll p_\mathrm{F}$) where the integrand varies slowly.

The mapping is defined by

$$p(s) = p_\mathrm{F}\left(1 - \frac{u(s)}{\alpha}\right), \qquad u(s) = -\ln\big(1 - \beta\,(1 - s)\big), \qquad \alpha = \frac{p_\mathrm{F}^2}{\mu T}, \qquad \beta = 1 - e^{-\alpha}\,. \quad (42)$$

This is applied with 32-point Gauss–Legendre nodes $\{s_i\}$ and weights $\{w_i\}$ on $[0, 1]$. The Jacobian is

$$\frac{dp}{ds} = \frac{p_\mathrm{F}}{\alpha} \cdot \frac{\beta}{1 - \beta\,(1 - s)}\,. \quad (43)$$

Limiting behavior. The parameter $\alpha = p_\mathrm{F}^2/(\mu T) \approx E_F/T$ (where $E_F = p_\mathrm{F}^2/(2\mu)$ is the non-relativistic Fermi energy) controls the degree of node concentration:

- **Low temperature** ($\alpha \gg 1$, i.e. $T \ll E_F$): $\beta \to 1$ and $u(s) \approx -\ln(s)$, so $p(s) \approx p_\mathrm{F}(1 + \ln(s)/\alpha)$. As s varies from 0 to 1, the mapped momentum sweeps from $p = 0$ to $p = p_\mathrm{F}$, but with strong logarithmic compression near $p = p_\mathrm{F}$ (where $s \to 1$). For the standard 32-point Legendre nodes, which are roughly uniformly spaced in $[0, 1]$, this places

the majority of mapped momenta within $\sim p_{\rm F}/\alpha \sim \mu T/p_{\rm F} \sim T/v_F$ of the Fermi surface — matching the thermal width of the Fermi step.

- **High temperature** ($\alpha \ll 1$, i.e. $T \gg E_F$): $\beta \to \alpha$ and $u(s) \to \alpha\,(1-s)$, so $p(s) \to s\,p_{\rm F}$ — the uniform (linear) mapping. The Fermi step is broad and there is no need for node concentration.

Comparison with uniform Legendre. In the old scheme, the below-$p_{\rm F}$ integration used the uniform mapping $p = s\,p_{\rm F}$ with 32-point Gauss–Legendre:

$$\int_0^{p_{\rm F}} F(p)\,dp \approx p_{\rm F} \sum_{i=1}^{32} w_i^{(\rm Leg)}\, F(s_i\,p_{\rm F})\,. \tag{44}$$

The 32 Legendre nodes are roughly uniformly spaced in $[0,1]$, giving an average spacing of $\Delta p \sim p_{\rm F}/32$ in momentum. For a nucleon ($m \approx 0.938\,\text{GeV}$) at $\mu_B = 1.2\,\text{GeV}$ and $T = 0.001\,\text{MeV}$:

$$p_{\rm F} \approx 0.73\,\text{GeV}, \qquad \Delta p \sim p_{\rm F}/32 \approx 0.023\,\text{GeV}, \qquad T/v_F \sim \mu T/p_{\rm F} \approx 1.6\times 10^{-6}\,\text{GeV}\,. \tag{45}$$

The thermal width T/v_F is $\sim 10^4$ times smaller than the node spacing. Only 1–2 nodes happen to fall near the Fermi surface, which is completely insufficient for resolving the peaked or oscillatory integrands of χ_3 and χ_4.

In the Sommerfeld–Legendre scheme, the adaptive mapping concentrates ~ 20 of the 32 nodes within $\sim T/v_F$ of the Fermi surface at low T, while the remaining nodes cover the bulk. Combined with the 32 shifted Laguerre nodes above $p_{\rm F}$, the Fermi surface peak is resolved by ~ 50 closely spaced nodes — regardless of how small T is.

5.2 Below-$p_{\rm F}$ forms

For $p < p_{\rm F}$, we have $E < \mu$ so $x = (E-\mu)/T < 0$. The Fermi–Dirac products are evaluated using e^x (which is < 1):

$$f = \frac{1}{e^x+1}\,, \tag{46}$$

$$f(1-f) = \frac{e^x}{(1+e^x)^2}\,, \tag{47}$$

$$f(1-f)(1-2f) = \frac{e^x}{(1+e^x)^2}\left(1-\frac{2}{e^x+1}\right) = \frac{e^x(e^x-1)}{(1+e^x)^3}\,. \tag{48}$$

These forms are numerically stable for all $x < 0$.

For χ_2, the code computes

$$-\int_0^{p_{\rm F}} \frac{2p^2+m^2}{E}\,\frac{1}{e^{-(E-\mu)/T}+1}\,dp\,, \tag{49}$$

where the negative sign and e^{-x} in the denominator implement the "hole" integrand $-(1-f) = -(e^{-x}/(1+e^{-x})) = -1/(e^{-(E-\mu)/T}+1)$. This equals $-\int_0^{p_{\rm F}}(2p^2+m^2)/E\cdot(1-f)\,dp$, which when combined with the analytic $\mu\,p_{\rm F}$ from $\int_0^{p_{\rm F}}(2p^2+m^2)/E\,dp$ gives the total below-$p_{\rm F}$ contribution.

5.3 Above-$p_{\rm F}$ forms

For $p > p_{\rm F}$, we have $x = (E-\mu)/T > 0$. The substitution $p = p_{\rm F} + T\,t$ maps the integration to Laguerre variables $t \in [0,\infty)$, and the Fermi–Dirac products are evaluated using e^{-x} (which is

< 1):

$$f = \frac{e^{-x}}{1+e^{-x}}\,, \tag{50}$$

$$f(1-f) = \frac{e^{-x}}{(1+e^{-x})^2}\,, \tag{51}$$

$$f(1-f)(1-2f) = \frac{e^{-x}(1-e^{-x})}{(1+e^{-x})^3}\,. \tag{52}$$

In terms of the scaled variable t, the momentum is $p = T\,t$, the energy-over-T is $E/T = \sqrt{t^2+(m/T)^2}$, and $x = E/T - \mu/T$. The kinematic factor becomes

$$\frac{2p^2+m^2}{E}\,dp = T^2\,\frac{2t^2+(m/T)^2}{E/T}\,dt\,. \tag{53}$$

The extra factor of T^2 (one from $dp = T\,dt$, one from $2p^2/E = 2T^2t^2/(ET)$, but the m^2/E term contributes less) appears in the code as the explicit `T*T` multiplying the Laguerre contribution.

5.4 Analytic $T = 0$ extraction

The code stores each susceptibility in the form

$$T^k\,\chi_{k+1} = \frac{g}{2\pi^2}\,T^k\big(\underbrace{\chi^{(0)}_{k+1}}_{\text{ret2}} + \underbrace{\Delta\chi_{k+1}}_{\text{ret1}}\big)\,, \tag{54}$$

where $\chi^{(0)}_{k+1}$ is the analytic $T = 0$ value from section 3 and $\Delta\chi_{k+1} = \text{quad}/T^k - \chi^{(0)}_{k+1}$ is the thermal correction.

- For χ_2 ($k = 1$): the extraction is structural. The below-p_F quadrature computes the hole contribution $-\int_0^{p_\text{F}}(2p^2+m^2)/E\cdot(1-f)\,dp$, the above-$p_\text{F}$ quadrature computes the tail $+\int_{p_\text{F}}^{\infty}(2p^2+m^2)/E\cdot f\,dp$, and ret2 $= \mu\,p_\text{F}$ is the exact Fermi-sea integral. The sum ret1 + ret2 equals the full χ_2 value to machine precision.
- For χ_3 ($k = 2$): both quadratures compute $\int(2p^2+m^2)/E\cdot f(1-f)\,dp$ directly (no $f = 1-(1-f)$ decomposition, since $f(1-f) \to 0$ below p_F). The analytic value ret2 $= (\mu^2+p_\text{F}^2)/p_\text{F}$ is subtracted after dividing the total quadrature by T.
- For χ_4 ($k = 3$): similarly, both quadratures compute $\int(2p^2+m^2)/E\cdot f(1-f)(1-2f)\,dp$ directly, and ret2 $= \mu(3p_\text{F}^2-\mu^2)/p_\text{F}^3$ is subtracted after dividing by T^2.

Since $T^k \times \text{value}/T^k$ round-trips without precision loss in IEEE-754 double arithmetic (no overflow or underflow for $T \gtrsim 10^{-6}$ GeV), this decomposition is numerically equivalent to the direct computation in all cases.

5.5 Summary of implemented formulas

The code computes $T^k\,d^kn/d\mu^k$ for $k = 1, 2, 3$ using the IBP-improved integrands:

Quantity	**Code returns**	**IBP integrand**	$T{=}0$ **value of** $d^kn/d\mu^k$
$\chi_2 = dn/d\mu$	$T\,dn/d\mu$	$(2p^2{+}m^2)/E\cdot f$	$\mu\,p_\text{F}$
$\chi_3 = d^2n/d\mu^2$	$T^2\,d^2n/d\mu^2$	$(2p^2{+}m^2)/E\cdot f(1{-}f)$	$(\mu^2{+}p_\text{F}^2)/p_\text{F}$
$\chi_4 = d^3n/d\mu^3$	$T^3\,d^3n/d\mu^3$	$(2p^2{+}m^2)/E\cdot f(1{-}f)(1{-}2f)$	$\mu(3p_\text{F}^2{-}\mu^2)/p_\text{F}^3$

All integrands include the common prefactor $g/(2\pi^2)$ and an overall factor of T from the IBP (absorbed into the T^k prefactor). The density (n, $k = 0$), pressure (P), and energy density (ε) do not use IBP — their integrands ($p^2 f$, $p^4 f/E$, $p^2 E f$) are already smooth.

6 Numerical results

We test the IBP improvements using the `ZeroTemperatureComparison` example at $\mu_B = 1.2\,\text{GeV}$ with the PDG2025 particle list, comparing the Ideal HRG, excluded-volume diagonal (EV), quantum van der Waals (QvdW), and real gas (CS+VDW) models. All models use quantum statistics with 32+32 quadrature points.

Tables 1 and 2 show relative deviations of the finite-T results from the exact $T = 0$ values.

Table 1: Relative deviation $(\chi_3(T) - \chi_3(0))/\chi_3(0)$ at various temperatures.

	Old (Laguerre only)		**New (IBP)**	
Model	T=0.001 MeV	T=1 MeV	T=0.001 MeV	T=1 MeV
Ideal HRG	-1.000	-1.000	-1.4×10^{-4}	4.3×10^{-2}
EV-Diagonal	0.294	0.294	3.5×10^{-9}	-1.9×10^{-5}
QvdW	0.202	0.202	2.9×10^{-9}	2.1×10^{-4}
RealGas (CS+VDW)	0.879	0.879	2.4×10^{-8}	-3.6×10^{-4}

Table 2: Relative deviation $(\chi_4(T) - \chi_4(0))/\chi_4(0)$ at various temperatures.

	Old (Laguerre only)		**New (IBP)**	
Model	T=0.001 MeV	T=1 MeV	T=0.001 MeV	T=1 MeV
Ideal HRG	-1.000	-1.000	1.46	-8.7×10^{-2}
EV-Diagonal	0.846	0.846	1.6×10^{-4}	9.7×10^{-5}
QvdW	0.505	0.505	6.8×10^{-5}	3.3×10^{-4}
RealGas (CS+VDW)	3.905	3.904	1.7×10^{-3}	4.2×10^{-4}

Key observations.

1. χ_3 **(one IBP step):** The old code produced 20–100% errors for all models below $T \sim 1\,\text{MeV}$, with the Ideal HRG returning essentially zero (the Laguerre-only quadrature completely misses the Fermi surface peak). After IBP, the integrand $f(1-f)$ is a smooth, non-negative peak, and the errors drop to 10^{-9}–10^{-4} at $T = 0.001\,\text{MeV}$.

2. χ_4 **(one IBP step):** The old code had 50–390% errors. After IBP, the integrand $f(1-f)(1-2f)$ still has one sign change, so the improvement is less dramatic but still substantial: errors drop to 10^{-5}–10^{-3} for interacting models. The Ideal HRG at $T = 0.001\,\text{MeV}$ remains challenging (146% error) because many species with large Fermi momenta contribute, and the sign-changing integrand is harder to resolve per species. By $T = 0.1\,\text{MeV}$ the Ideal HRG error is already below 0.4%.

3. **The old errors are flat in T:** For the interacting models with the old code, the relative deviations are essentially constant from $T = 0.001\,\text{MeV}$ to $T = 0.1\,\text{MeV}$. This confirms

the errors are systematic quadrature failures (the Laguerre nodes do not resolve the Fermi surface), not physical effects.

4. **A second IBP for χ_4 is not possible.** One might attempt to apply the IBP once more to replace $f(1-f)(1-2f)$ with $f(1-f)$. However, the boundary term $[pE \cdot f(1-f)]_0^\infty$ does vanish, but the resulting integrand involves $d[(2p^2+m^2)/E]/dp = p(p^2+2m^2)/E^3$, which when combined with $pE \cdot f(1-f)$ produces terms proportional to $(2p^2+m^2)/p$ that diverge at $p = 0$. This makes the second IBP inapplicable.

7 IBP for the entropy derivative $ds/dT|_\mu$

The entropy density temperature derivative at fixed chemical potential is

$$\left.\frac{ds}{dT}\right|_\mu = \frac{g}{2\pi^2}\,\frac{1}{T}\int_0^\infty p^2\,x^2\,f(1-f)\,dp\,, \qquad x \equiv \frac{E-\mu}{T}\,. \tag{55}$$

At $T = 0$, the Sommerfeld expansion gives

$$\left.\frac{ds}{dT}\right|_{T=0} = \frac{g}{6}\,\mu\,p_{\rm F}\,, \tag{56}$$

which has the same structure $\sim \mu\,p_{\rm F}$ as the $T = 0$ value of χ_2. However, as we show below, the numerical treatment of ds/dT is fundamentally more challenging than χ_2.

7.1 The numerical problem

The integrand in eq. (55) is $\propto x^2 f(1-f)/T$. The factor $x^2 = (E-\mu)^2/T^2$ vanishes at the Fermi surface ($E = \mu$), while $f(1-f)/T \to \delta(p-p_{\rm F})\,\mu/p_{\rm F}$. The product gives a peaked integrand $\propto T$ near $p_{\rm F}$. The integral therefore scales as T^2, and dividing by T yields the finite $T = 0$ limit.

In the original implementation, the code computed this integral directly and subtracted the analytic $T = 0$ value, i.e.

$$\frac{ds}{dT} = \frac{g}{2\pi^2}\left(\frac{\pi^2}{3}\,\mu\,p_{\rm F} + \Delta_{\rm th}\right), \tag{57}$$

where $\Delta_{\rm th} = \text{quad}/T - (\pi^2/3)\,\mu\,p_{\rm F}$ is the thermal correction. At low T, both quad$/T$ and $(\pi^2/3)\,\mu\,p_{\rm F}$ are large and nearly equal, so their difference suffers from cancellation. This led to $\sim 0.35\%$ relative error already at $T = 1$ μeV.

7.2 First IBP: $f \cdot G$ form

Using $f(1-f) = -(TE/p)\,df/dp$ as in the χ_2 case, we integrate by parts on the momentum integral. The relevant identity is

$$p^2\,\frac{(E-\mu)^2}{T^2}\,f(1-f) = -\frac{1}{T}\,p\,E\,(E-\mu)^2\,\frac{df}{dp}\,. \tag{58}$$

Integrating by parts with $u = pE(E-\mu)^2$ and $dv = df$, and noting that the boundary terms vanish:

$$\boxed{\int_0^\infty p^2\,\frac{(E-\mu)^2}{T^2}\,f(1-f)\,dp = \frac{1}{T}\int_0^\infty f(p)\,G(p)\,dp\,,} \tag{59}$$

where

$$G(p) = \frac{d}{dp}\big[pE(E-\mu)^2\big] = \varepsilon\left[\frac{(2p^2+m^2)\,\varepsilon}{E} + 2p^2\right], \qquad \varepsilon \equiv E-\mu\,. \tag{60}$$

So that

$$\left.\frac{ds}{dT}\right|_\mu = \frac{g}{2\pi^2\,T^2}\int_0^\infty f(p)\,G(p)\,dp\,. \tag{61}$$

Difference from χ_2. For χ_2, the IBP converts $\int p^2 f(1-f)\,dp = T\int (2E - m^2/E)\,f\,dp$, where the integrand $(2E - m^2/E)$ is nonzero at p_F:

$$\frac{2p_\mathrm{F}^2 + m^2}{\mu} = \frac{2p_\mathrm{F}^2 + m^2}{\mu} \neq 0\,. \tag{62}$$

The filled Fermi sea therefore contributes $\int_0^{p_\mathrm{F}} (2p^2 + m^2)/E\,dp = \mu\, p_\mathrm{F} \neq 0$, which is the exact $T = 0$ value.

For ds/dT, the integrand $G(p)$ vanishes at the Fermi surface because $\varepsilon(p_\mathrm{F}) = 0$:

$$G(p_\mathrm{F}) = 0\,. \tag{63}$$

Consequently, the Fermi-sea integral vanishes:

$$\int_0^{p_\mathrm{F}} G(p)\,dp = \big[pE(E-\mu)^2\big]_0^{p_\mathrm{F}} = p_\mathrm{F}\mu \cdot 0 - 0 = 0\,. \tag{64}$$

There is *no* $T = 0$ piece to extract analytically — the entire integral is a thermal contribution of order T^2. The result is $ds/dT = (g/2\pi^2 T^2) \times O(T^2)$, which is finite but computed entirely from the quadrature. This achieves $\sim 4 \times 10^{-4}$ relative error at $T = 1$ μeV.

7.3 Double IBP: $\sigma \cdot H$ form

To improve accuracy further, we perform a second integration by parts, this time converting from f to the Fermi entropy kernel.

The entropy kernel. Define the Fermi entropy function

$$\sigma(x) = -f \ln f - (1-f)\ln(1-f) = \ln\Big(1 + e^{-|x|}\Big) + \frac{|x|}{e^{|x|} + 1}\,, \tag{65}$$

which satisfies

$$\sigma'(x) \equiv \frac{d\sigma}{dx} = -x\, f(1-f)\,. \tag{66}$$

In momentum space:

$$\frac{d\sigma}{dp} = \sigma'(x)\,\frac{dx}{dp} = \sigma'(x)\,\frac{p}{TE} = -\frac{p\,(E-\mu)}{T^2 E}\,f(1-f)\,. \tag{67}$$

Derivation. Start from eq. (55):

$$\left.\frac{ds}{dT}\right|_\mu = \frac{g}{2\pi^2 T}\int_0^\infty p^2\, x^2\, f(1-f)\,dp\,. \tag{68}$$

Using $\sigma'(x) = -x\,f(1-f)$, we have $x^2 f(1-f) = -x\,\sigma'(x)$. The momentum derivative of σ is $d\sigma/dp = \sigma'(x)\cdot p/(TE)$, so $\sigma'(x) = (TE/p)\,d\sigma/dp$. Substituting:

$$p^2\, x^2\, f(1-f) = -p^2 x \cdot \frac{TE}{p}\,\frac{d\sigma}{dp} = -pE(E-\mu)\,\frac{d\sigma}{dp}\,. \tag{69}$$

Integrating by parts with $u = pE(E-\mu)$ and $dv = d\sigma$, using $[pE(E-\mu)\,\sigma]_0^\infty = 0$ (since $p = 0$ at the lower limit and $\sigma \to 0$ exponentially at the upper limit):

$$\boxed{\int_0^\infty p^2\, x^2\, f(1-f)\,dp = \int_0^\infty \sigma(x)\,H(p)\,dp\,,} \tag{70}$$

where

$$H(p) = \frac{d}{dp}\big[pE(E-\mu)\big] = \frac{(E-\mu)(2p^2 + m^2)}{E} + p^2\,. \tag{71}$$

The final formula is

$$\left.\frac{ds}{dT}\right|_\mu = \frac{g}{2\pi^2\, T}\int_0^\infty \sigma(x)\,H(p)\,dp\,. \tag{72}$$

Key properties.

- The prefactor is $1/T$ (not $1/T^2$ as in the single-IBP form).
- $\sigma(x)$ is intrinsically smooth and *zero* at $T = 0$ (where $f = 0$ or $f = 1$), just like the entropy density integrand $p^2\,\sigma$.
- $H(p_\mathrm{F}) = p_\mathrm{F}^2 \neq 0$. The Sommerfeld expansion gives $\int_0^\infty \sigma\, H\, dp \approx (\pi^2/3)\, T\, (\mu/p_\mathrm{F})\, H(p_\mathrm{F}) = (\pi^2/3)\,\mu\, p_\mathrm{F}\, T$, so $ds/dT \approx (g/2\pi^2 T) \times (\pi^2/3)\,\mu\, p_\mathrm{F}\, T = (g/6)\,\mu\, p_\mathrm{F}$, matching eq. (56).
- Because $\sigma = 0$ at $T = 0$, the quadrature evaluates the thermal contribution *directly*, with no analytical subtraction needed and no cancellation.

7.4 The σ representation: entropy and its derivatives

The Fermi entropy kernel $\sigma(x)$ defined in eq. (65) is the natural integrand for all entropy-related quantities. The entropy density itself is

$$s = \frac{g}{2\pi^2} \int_0^\infty p^2\, \sigma\!\left(\frac{E-\mu}{T}\right) dp\,, \tag{73}$$

where $\sigma(x) = -f \ln f - (1-f)\ln(1-f)$ is the entropy per mode in a Fermi gas. This representation has several advantages over the textbook form $s = (\varepsilon + P - \mu n)/T$:

1. **No cancellation at low T.** The textbook formula involves subtracting three large, individually $O(p_\mathrm{F}^4)$ quantities (ε, P, μn) to obtain a result of order $O(T)$ — a severe cancellation. In the σ form, the integrand is directly of order T (since $\sigma \to 0$ for $|x| \to \infty$, i.e. deep inside or far outside the Fermi sea), so no cancellation occurs.
2. **Intrinsically non-negative.** $\sigma(x) \geq 0$ for all x, so the integrand is non-negative everywhere. This prevents the sign-change-induced quadrature failures that plague the χ_4 integrand.
3. **Smooth and peaked.** $\sigma(x)$ has a maximum $\sigma(0) = \ln 2$ at the Fermi surface ($E = \mu$) and decays exponentially for $|x| \gg 1$. The peak width is $\Delta x \sim O(1)$, corresponding to $\Delta p \sim T/v_F$ in momentum space. The Sommerfeld–Legendre + Laguerre quadrature resolves this peak well.
4. **$T = 0$ limit is trivial.** At $T = 0$, every mode is either fully occupied ($f = 1$, $\sigma = 0$) or fully empty ($f = 0$, $\sigma = 0$), so $\sigma = 0$ everywhere and $s = 0$. There is no finite contribution to subtract analytically.

These same properties carry over to ds/dT through the double-IBP $\sigma \cdot H$ form (eq. (72)). The weight function $H(p) = d[pE(E-\mu)]/dp$ modifies the shape but not the sign or smoothness of the integrand. The key point is that σ encodes the *thermal* contribution directly, avoiding both the cancellation inherent in $(\varepsilon + P - \mu n)/T$ and the oscillations inherent in $x^2 f(1-f)$.

Implementation note. The function $\sigma(x)$ is computed via the numerically stable form

$$\sigma(x) = \ln\!\left(1 + e^{-|x|}\right) + \frac{|x|}{e^{|x|}+1}\,, \tag{74}$$

which avoids overflow for large $|x|$ and evaluates correctly to $\sigma(0) = \ln 2$ at the Fermi surface. In the code, the entropy density (eq. (73)) and ds/dT (eq. (72)) share the same `FermiEntropySigma` function, ensuring consistency between s and its temperature derivative.

7.5 Can we do better? Obstruction to χ_2-level accuracy

Since eq. (72) has only a $1/T$ prefactor (not $1/T^2$), one might hope that a further IBP could convert the σ kernel into plain f, producing a χ_2-type formula where the $T = 0$ value emerges as a Fermi-sea integral.

Attempt via $\sigma = \ln(1 + e^{-x}) + xf$. One can verify the identity

$$\sigma(x) = \ln\bigl(1 + e^{-x}\bigr) + x\, f(x)\,. \tag{75}$$

Substituting into eq. (70):

$$\int_0^\infty \sigma\, H\, dp = \int_0^\infty \ln\bigl(1 + e^{-x}\bigr)\, H\, dp + \frac{1}{T}\int_0^\infty (E - \mu)\, f\, H\, dp\,. \tag{76}$$

The second term contains plain f multiplied by H, which looks promising. However, at $T = 0$:

$$\frac{1}{T}\int_0^{p_\mathrm{F}} (E - \mu)\, H\, dp = \frac{1}{T}\underbrace{\bigl[pE(E - \mu)^2\bigr]_0^{p_\mathrm{F}}}_{=\,0} - \frac{1}{T}\int_0^{p_\mathrm{F}} p^2(E - \mu)\, dp = -\frac{1}{T}\int_0^{p_\mathrm{F}} p^2(E - \mu)\, dp\,. \tag{77}$$

This *diverges* as $1/T$. Meanwhile, the first term also diverges as $1/T$ because $\ln(1 + e^{-x}) \to (\mu - E)/T$ for $p < p_\mathrm{F}$ at $T = 0$. The two $1/T$ divergences cancel to give the finite σ integral, but using them separately reintroduces catastrophic cancellation — exactly the problem we sought to avoid.

Attempt via $d\sigma/dp$. One could also try IBP on $\int \sigma\, H\, dp$ with $u = \sigma$ and $dv = H\, dp$. But $d\sigma/dp = -p(E - \mu)f(1{-}f)/(T^2E)$ from eq. (67), which reintroduces $f(1{-}f)$ and $1/T^2$ — going backwards.

Fundamental reason. For χ_2, the IBP converts $f(1{-}f) \to f$, and the plain-f integrand has a *nonzero* Fermi-sea integral $\int_0^{p_\mathrm{F}} (2p^2 + m^2)/E\, dp = \mu\, p_\mathrm{F}$. This works because $\chi_2 = dn/d\mu$ is a chemical potential derivative — its $T = 0$ value is a property of the filled Fermi sea.

In contrast, $ds/dT|_\mu$ is a *temperature derivative*. At $T = 0$, the entropy vanishes ($s = 0$) and $ds/dT|_{T=0}$ is the first nonzero term in $s(T) \approx (ds/dT)\, T$. Its value $(g/6)\, \mu\, p_\mathrm{F}$ arises entirely from the *thermal smearing* of the Fermi surface, not from the filled sea. Any IBP that converts the integrand to plain f necessarily produces a weight function that vanishes at p_F (because the temperature derivative of all quantities is zero inside the Fermi sea at $T = 0$), making $\int_0^{p_\mathrm{F}} (\text{weight}) \cdot f\, dp = 0$.

Therefore, no IBP decomposition can extract the $T = 0$ value of ds/dT as an analytical Fermi-sea integral in the way χ_2 extracts $\mu\, p_\mathrm{F}$. The $\sigma \cdot H$ form (eq. (72)) achieves the best accuracy available from direct quadrature of the thermal contribution.

7.6 Numerical results

Table 3 shows the relative deviation of ds/dT from its $T = 0$ value for the Ideal HRG at $\mu_B = 1.2\,\mathrm{GeV}$.

The double-IBP $\sigma \cdot H$ form achieves $\sim 16\times$ improvement over the original at $T = 1\ \mu\mathrm{eV}$. The accuracy matches that of the entropy density s itself (which uses the same σ kernel), confirming that the quadrature resolution of σ-weighted integrals is the limiting factor. The interacting models (EV, QvdW, RealGas) achieve $\sim 1.8 \times 10^{-4}$ at $T = 1\ \mu\mathrm{eV}$.

Table 3: Relative deviation $(ds/dT(T) - ds/dT(0))/(ds/dT(0))$ at low temperatures for the Ideal HRG at $\mu_B = 1.2\,\mathrm{GeV}$.

	Original	**Single IBP** $(f \cdot G)$	**Double IBP** $(\sigma \cdot H)$
$T = 0.001\,\mathrm{MeV}$	-3.5×10^{-3}	-4.3×10^{-4}	-2.2×10^{-4}
$T = 0.01\,\mathrm{MeV}$	-3.5×10^{-3}	-4.4×10^{-4}	-2.2×10^{-4}
$T = 0.1\,\mathrm{MeV}$	-3.7×10^{-3}	-5.7×10^{-4}	-3.4×10^{-4}
$T = 1\,\mathrm{MeV}$	-8.4×10^{-3}	-8.4×10^{-3}	-8.1×10^{-3}

8 Summary

The IBP improvement consists of one integration-by-parts step for each susceptibility χ_n ($n \geq 2$), using the identity

$$\int_0^\infty p^2\, F_{k+1}(p)\, dp = T \int_0^\infty \frac{2p^2 + m^2}{E}\, F_k(p)\, dp\,, \tag{78}$$

where $F_k = T\,\partial^k f/\partial\mu^k$. Each step replaces an integrand with k sign changes by one with $k-1$ sign changes, dramatically improving quadrature convergence at low T. The improvement is most pronounced for χ_3 (from $\sim 100\%$ to $\sim 10^{-9}$ relative error) and significant for χ_4 (from $\sim 400\%$ to $\sim 10^{-3}$ for interacting models).

For χ_2, the IBP was already implemented earlier and replaces the $f(1-f)$ peak by the smooth Fermi step f, yielding excellent precision at all temperatures.

For $ds/dT|_\mu$, a *double* IBP replaces the $x^2 f(1-f)$ integrand by the entropy kernel $\sigma(x) = -f\ln f - (1-f)\ln(1-f)$:

$$\left.\frac{ds}{dT}\right|_\mu = \frac{g}{2\pi^2\, T} \int_0^\infty \sigma(x)\, \frac{d}{dp}\big[pE(E-\mu)\big]\, dp\,. \tag{79}$$

Since σ is intrinsically zero at $T = 0$, the quadrature evaluates the thermal contribution directly with no cancellation, achieving $\sim 2 \times 10^{-4}$ relative error at $T = 1$ μeV — a $16\times$ improvement. Unlike χ_2, the $T = 0$ value of ds/dT cannot be extracted as a Fermi-sea integral because temperature derivatives vanish identically inside the filled sea; the leading Sommerfeld coefficient arises entirely from the thermal smearing at the Fermi surface.