

# Integration-by-parts improvement of Fermi integrals for baryon number susceptibilities and entropy derivative at low temperature

Technical note for Thermal-FIST

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## Abstract

We describe the integration-by-parts (IBP) technique used to improve the numerical evaluation of Fermi–Dirac integrals for baryon number susceptibilities  $\chi_n^B$  and the entropy density temperature derivative  $ds/dT|_\mu$  at low temperatures in the hadron resonance gas model. The key idea is to reduce the order of the Fermi–Dirac distribution derivatives appearing in the integrand by one, replacing oscillatory or sharply peaked integrands with smoother ones that are better resolved by the existing Sommerfeld–Legendre + Laguerre quadrature scheme. We present the IBP formulas for  $\chi_2$ ,  $\chi_3$ , and  $\chi_4$ , derive the analytic  $T = 0$  limits, and document the numerical improvements achieved. For  $ds/dT|_\mu$ , we derive a double-IBP formula involving the Fermi entropy kernel  $\sigma(x)$  that achieves  $\sim 2 \times 10^{-4}$  relative accuracy at  $T = 1 \mu\text{eV}$ , and explain why the  $\chi_2$ -type analytical extraction of the  $T = 0$  value is fundamentally unavailable for temperature derivatives.

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## 1 Setup and notation

Consider a single fermion species with mass  $m$ , degeneracy  $g$ , at temperature  $T$  and chemical potential  $\mu > m$  (so that the Fermi momentum  $p_F = \sqrt{\mu^2 - m^2}$  is real). The Fermi–Dirac distribution is

$$f(p) = \frac{1}{e^{(E-\mu)/T} + 1}, \quad E = \sqrt{p^2 + m^2}. \quad (1)$$

The particle number density is

$$n = \frac{g}{2\pi^2} \int_0^\infty p^2 f(p) dp. \quad (2)$$

The generalized susceptibilities are the  $\mu$ -derivatives of the pressure, or equivalently:

$$\chi_n \equiv \frac{\partial^n P}{\partial \mu^n} = \frac{\partial^{n-1} n}{\partial \mu^{n-1}}. \quad (3)$$

Using the identity  $\partial f / \partial \mu = f(1-f)/T$ , the first few susceptibilities involve

$$\chi_2 = \frac{\partial n}{\partial \mu} = \frac{g}{2\pi^2} \int_0^\infty p^2 \frac{f(1-f)}{T} dp, \quad (4)$$

$$\chi_3 = \frac{\partial^2 n}{\partial \mu^2} = \frac{g}{2\pi^2} \int_0^\infty p^2 \frac{f(1-f)(1-2f)}{T^2} dp, \quad (5)$$

$$\chi_4 = \frac{\partial^3 n}{\partial \mu^3} = \frac{g}{2\pi^2} \int_0^\infty p^2 \frac{f(1-f)[1-6f(1-f)]}{T^3} dp. \quad (6)$$

**The numerical problem.** As  $T \rightarrow 0$ , the factor  $f(1-f)/T$  tends to  $\delta(E-\mu)$ , i.e.  $\delta(p-p_F)/v_F$  with  $v_F = p_F/\mu$ . For  $\chi_2$  the integrand is a narrow positive peak at  $p_F$ ; for  $\chi_3$  the integrand  $f(1-f)(1-2f)/T^2$  has a derivative-of-delta structure (one sign change); for  $\chi_4$  the integrand  $f(1-f)[1-6f(1-f)]/T^3$  has a second-derivative-of-delta structure (two sign changes). Higher-order integrands are increasingly oscillatory and harder for fixed-order quadrature to resolve.

We use a hybrid quadrature scheme:

- **Sommerfeld–Legendre** (32 points): maps  $[0, 1] \rightarrow [0, p_F]$  with adaptive concentration of quadrature nodes near the Fermi surface, controlled by  $\alpha = p_F^2/(\mu T)$ .
- **Shifted Laguerre** (32 points): covers  $[p_F, \infty)$  with exponential weight.

Both sets of nodes are concentrated near  $p_F$ , where the Fermi–Dirac integrands peak. However, for the oscillatory integrands in  $\chi_3$  and especially  $\chi_4$ , the quadrature accuracy degrades severely at low  $T$ . The IBP technique described below smooths the integrands by reducing the derivative order by one.

## 2 IBP identities

The core identity underlying all IBPs is

$$\frac{df}{dp} = -\frac{p}{ET} f(1-f). \quad (7)$$

This allows us to express products of  $f$  and its  $\mu$ -derivatives as total  $p$ -derivatives, enabling integration by parts.

### 2.1 $\chi_2$ : first IBP

From eq. (7), we have

$$f(1-f) = -T \frac{E}{p} \frac{df}{dp}. \quad (8)$$

Therefore the  $\chi_2$  integrand becomes a total derivative:

$$p^2 f(1-f) = -T p E \frac{df}{dp} = -T \frac{d}{dp} [p E f] + T \frac{d(pE)}{dp} f. \quad (9)$$

Since  $d(pE)/dp = (2p^2 + m^2)/E$  and the boundary term  $[p E f]_0^\infty = 0$  (vanishes at  $p = 0$  and exponentially at  $p = \infty$ ), integration by parts gives

$$\boxed{\int_0^\infty p^2 f(1-f) dp = T \int_0^\infty \frac{2p^2 + m^2}{E} f dp.} \quad (10)$$

The left-hand side has integrand  $f(1-f) \sim \delta(p-p_F)$ ; the right-hand side has integrand  $f$ , the Fermi step function itself — smooth, monotonic, and trivial for quadrature.

Using eq. (10) in eq. (4):

$$\chi_2 = \frac{g}{2\pi^2} \int_0^\infty \frac{2p^2 + m^2}{E} f dp. \quad (11)$$

### 2.2 $\chi_3$ : second IBP

Differentiating  $f(1-f) = -T(E/p) df/dp$  once more with respect to  $\mu$ :

$$\frac{\partial}{\partial \mu} [f(1-f)] = \frac{f(1-f)(1-2f)}{T} \Rightarrow f(1-f)(1-2f) = -T \frac{E}{p} \frac{d}{dp} [f(1-f)]. \quad (12)$$

Therefore

$$p^2 f(1-f)(1-2f) = -T p E \frac{d}{dp} [f(1-f)]. \quad (13)$$

Integrating by parts with  $[p E f(1-f)]_0^\infty = 0$  (the boundary terms vanish since  $pE = 0$  at  $p = 0$  and  $f(1-f)$  decays exponentially at  $p \rightarrow \infty$ ):

$$\boxed{\int_0^\infty p^2 f(1-f)(1-2f) dp = T \int_0^\infty \frac{2p^2 + m^2}{E} f(1-f) dp.} \quad (14)$$

The left-hand side has an integrand with one sign change (derivative-of-delta structure); the right-hand side has  $f(1-f)$ , a smooth non-negative peak at  $p_F$  — the same integrand that appeared in the *un-improved*  $\chi_2$ .

Using eq. (14) in eq. (5):

$$\chi_3 = \frac{g}{2\pi^2 T} \int_0^\infty \frac{2p^2 + m^2}{E} f(1-f) dp. \quad (15)$$

### 2.3 $\chi_4$ : third IBP

Similarly, differentiating  $f(1-f)(1-2f)$  with respect to  $\mu$ :

$$\frac{\partial}{\partial \mu} [f(1-f)(1-2f)] = \frac{f(1-f)[1-6f(1-f)]}{T} \quad (16)$$

and thus

$$f(1-f)[1-6f(1-f)] = -T \frac{E}{p} \frac{d}{dp} [f(1-f)(1-2f)]. \quad (17)$$

Integration by parts with  $[p E f(1-f)(1-2f)]_0^\infty = 0$  gives:

$$\boxed{\int_0^\infty p^2 f(1-f)[1-6f(1-f)] dp = T \int_0^\infty \frac{2p^2 + m^2}{E} f(1-f)(1-2f) dp.} \quad (18)$$

The left-hand side has two sign changes; the right-hand side has one sign change (the same integrand as the *un-improved*  $\chi_3$ ).

Using eq. (18) in eq. (6):

$$\chi_4 = \frac{g}{2\pi^2 T^2} \int_0^\infty \frac{2p^2 + m^2}{E} f(1-f)(1-2f) dp. \quad (19)$$

### 2.4 General pattern

The IBP has a recursive structure. Define

$$F_k(p) \equiv T \frac{\partial^k f}{\partial \mu^k}, \quad (20)$$

so that  $F_0 = f$ ,  $F_1 = f(1-f)$ ,  $F_2 = f(1-f)(1-2f)/T$ , etc. Then the identity  $\partial f/\partial p = -(p/ET) F_1$  generalizes to

$$F_{k+1} = \frac{\partial F_k}{\partial \mu} = -T \frac{E}{p} \frac{\partial F_k}{\partial p}, \quad (21)$$

and each IBP step replaces

$$\int_0^\infty p^2 F_{k+1} dp = T \int_0^\infty \frac{2p^2 + m^2}{E} F_k dp, \quad (22)$$

provided the boundary term  $[p E F_k]_0^\infty$  vanishes. Each step reduces the number of sign changes in the integrand by one, at the cost of introducing one additional power of  $T$  in the denominator when forming  $\chi_n$ .

## 3 Analytic $T = 0$ limits

At  $T = 0$ , the Fermi–Dirac distribution becomes a step function,  $f = \theta(p_F - p)$ , and the integrals can be evaluated analytically.

### 3.1 Density and equation of state

$$n|_{T=0} = \frac{g}{2\pi^2} \frac{p_F^3}{3}, \quad (23)$$

$$P|_{T=0} = \frac{g}{6\pi^2} [\mu p_F^3 - \frac{3}{4} p_F^4 \psi(m/p_F)], \quad (24)$$

$$\varepsilon|_{T=0} = \frac{g}{2\pi^2} \frac{p_F^4}{4} \psi(m/p_F), \quad (25)$$

where  $\psi(x) = (1+x^2)^{1/2}(1+\frac{1}{2}x^2) - \frac{1}{2}x^4 \sinh^{-1}(1/x)$ .

### 3.2 Susceptibilities

Using  $f(1-f)/T \rightarrow \delta(p - p_F) \mu/p_F$  and its derivatives, or by directly differentiating the  $T = 0$  density:

$$\chi_2|_{T=0} = \frac{dn}{d\mu}\Big|_{T=0} = \frac{g}{2\pi^2} \frac{\mu p_F}{1} = \frac{g}{2\pi^2} \mu p_F, \quad (26)$$

$$\chi_3|_{T=0} = \frac{d^2n}{d\mu^2}\Big|_{T=0} = \frac{g}{2\pi^2} \frac{\mu^2 + p_F^2}{p_F}, \quad (27)$$

$$\chi_4|_{T=0} = \frac{d^3n}{d\mu^3}\Big|_{T=0} = \frac{g}{2\pi^2} \frac{\mu(3p_F^2 - \mu^2)}{p_F^3}. \quad (28)$$

These analytic values serve as reference for assessing the numerical accuracy at finite but small  $T$ .

## 4 Decomposition into below- and above- $p_F$ contributions

The quadrature scheme splits every integral at the Fermi momentum  $p_F$ :

$$\int_0^\infty (\dots) dp = \underbrace{\int_0^{p_F} (\dots) dp}_{\text{below } p_F} + \underbrace{\int_{p_F}^\infty (\dots) dp}_{\text{above } p_F}. \quad (29)$$

Since  $f \rightarrow \theta(p_F - p)$  as  $T \rightarrow 0$ , the above- $p_F$  part vanishes exponentially, while the below- $p_F$  part approaches a  $T$ -independent limit. We now show how this split interacts with the IBP integrands of each susceptibility.

### 4.1 $\chi_2$ : the $T = 0$ value emerges explicitly

The IBP formula for  $\chi_2$  (eq. (10)) gives

$$T \chi_2 = \frac{g}{2\pi^2} \int_0^\infty \frac{2p^2 + m^2}{E} f dp. \quad (30)$$

The integrand involves  $f$  — the Fermi step function. Below  $p_F$  we write  $f = 1 - (1 - f)$ , obtaining

$$\int_0^{p_F} \frac{2p^2 + m^2}{E} f dp = \underbrace{\int_0^{p_F} \frac{2p^2 + m^2}{E} dp}_{=\mu p_F} - \int_0^{p_F} \frac{2p^2 + m^2}{E} (1 - f) dp. \quad (31)$$

The first integral is the *exact*  $T = 0$  contribution and can be evaluated analytically:

$$\int_0^{p_F} \frac{2p^2 + m^2}{E} dp = \int_0^{p_F} \left( 2E - \frac{m^2}{E} \right) dp = [pE]_0^{p_F} = p_F \cdot \mu = \mu p_F, \quad (32)$$

where we used  $d(pE)/dp = (2p^2 + m^2)/E$  and the boundary values  $pE|_{p=p_F} = p_F \mu$ ,  $pE|_{p=0} = 0$ .

The remaining terms are thermal corrections that vanish as  $T \rightarrow 0$ :

- $-\int_0^{p_F} \frac{2p^2 + m^2}{E} (1 - f) dp$ : the “hole” contribution below  $p_F$ . At  $T = 0$ ,  $1 - f = 0$  for  $p < p_F$ , so this integral vanishes. At finite  $T$ ,  $1 - f$  is exponentially small except near  $p_F$ , contributing  $O(T^2)$ .

- $+ \int_{p_F}^{\infty} \frac{2p^2 + m^2}{E} f dp$ : the “tail” contribution above  $p_F$ . Again  $f \rightarrow 0$  for  $p > p_F$  as  $T \rightarrow 0$ , giving  $O(T^2)$ .

The code therefore computes

$$T \chi_2 = \frac{g}{2\pi^2} T \left( \underbrace{\mu p_F}_{\text{ret2}} + \underbrace{\Delta_{\text{th}}}_{\text{ret1}} \right), \quad (33)$$

where  $\text{ret2} = \mu p_F$  is the analytic  $T = 0$  value of  $\chi_2$  (in natural units, up to the  $g/2\pi^2$  prefactor) and

$$\Delta_{\text{th}} = - \int_0^{p_F} \frac{2p^2 + m^2}{E} (1 - f) dp + \int_{p_F}^{\infty} \frac{2p^2 + m^2}{E} f dp \quad (34)$$

is the thermal correction, which tends to zero as  $T \rightarrow 0$ .

**Key point.** The analytic extraction for  $\chi_2$  is *not* a numerical convenience — it is a structural consequence of the IBP. The IBP replaces the sharply peaked integrand  $f(1-f)$  by the step-function-like  $f$ , which *naturally* splits into a filled Fermi sea (yielding the exact  $T = 0$  answer) plus corrections from particles and holes near the Fermi surface.

## 4.2 $\chi_3$ : peaked integrand, no natural $T = 0$ split

The IBP formula for  $\chi_3$  (eq. (14)) gives

$$T^2 \chi_3 = \frac{g}{2\pi^2} T \int_0^{\infty} \frac{2p^2 + m^2}{E} f(1-f) dp. \quad (35)$$

The integrand  $f(1-f)$  is a smooth, non-negative peak localized within  $\sim T$  of  $p_F$ . Unlike the  $\chi_2$  case,  $f(1-f)$  vanishes at  $T = 0$  for all  $p$  (both below and above  $p_F$ ), so there is no filled-Fermi-sea piece to extract directly from the integrand.

Instead, the integral scales as  $\sim T$  for small  $T$ :

$$\int_0^{\infty} \frac{2p^2 + m^2}{E} f(1-f) dp \xrightarrow{T \rightarrow 0} T \frac{\mu^2 + p_F^2}{p_F}, \quad (36)$$

because  $f(1-f)/T \rightarrow \delta(p - p_F) \mu/p_F$ . Therefore  $\chi_3 = \text{quad}/T \rightarrow (\mu^2 + p_F^2)/p_F$  as  $T \rightarrow 0$ .

For the  $T = 0$  extraction, the code computes

$$T^2 \chi_3 = \frac{g}{2\pi^2} T^2 \left( \underbrace{(\mu^2 + p_F^2)/p_F}_{\text{ret2}} + \underbrace{\text{quad}/T - (\mu^2 + p_F^2)/p_F}_{\text{ret1}} \right). \quad (37)$$

Here  $\text{ret2}$  is the known  $T = 0$  limit and  $\text{ret1}$  is the thermal correction. Unlike the  $\chi_2$  case,  $\text{ret2}$  does not emerge organically from splitting the integral at  $p_F$  — it is an independently computed constant that is subtracted. This decomposition is mathematically equivalent to the direct computation ( $T^2 \times \text{value}/T^2$  round-trips losslessly in IEEE double), but it makes the  $T \rightarrow 0$  behavior transparent.

## 4.3 $\chi_4$ : sign-changing integrand

The IBP formula for  $\chi_4$  (eq. (18)) gives

$$T^3 \chi_4 = \frac{g}{2\pi^2} T \int_0^{\infty} \frac{2p^2 + m^2}{E} f(1-f)(1-2f) dp. \quad (38)$$

The integrand  $f(1-f)(1-2f)$  is localized near  $p_F$  and changes sign once (positive for  $p < p_F$ , negative for  $p > p_F$ , since  $1-2f$  flips sign at  $E = \mu$ ). Like  $\chi_3$ , the integrand vanishes everywhere at  $T = 0$ , and the integral scales as  $\sim T^2$ :

$$\int_0^{\infty} \frac{2p^2 + m^2}{E} f(1-f)(1-2f) dp \xrightarrow{T \rightarrow 0} T^2 \frac{\mu(3p_F^2 - \mu^2)}{p_F^3}. \quad (39)$$

The code uses the same extraction structure:

$$T^3 \chi_4 = \frac{g}{2\pi^2} T^3 \underbrace{(\mu(3p_F^2 - \mu^2)/p_F^3)}_{\text{ret2}} + \underbrace{\text{quad}/T^2 - \mu(3p_F^2 - \mu^2)/p_F^3}_{\text{ret1}}. \quad (40)$$

Because  $f(1-f)(1-2f)$  has one sign change, the quadrature accuracy is intermediate between  $\chi_2$  (no sign changes, excellent) and the un-improved  $\chi_4$  (two sign changes, catastrophic).

#### 4.4 Comparison of the three cases

The qualitative difference between  $\chi_2$  and  $\chi_{3,4}$  is summarized in the following table:

	$\chi_2$	$\chi_3$	$\chi_4$
IBP integrand	$f$	$f(1-f)$	$f(1-f)(1-2f)$
Sign changes	0	0	1
$T \rightarrow 0$ behavior of integrand	$\rightarrow \theta(p_F - p)$	$\rightarrow 0$ (peak $\sim T$ )	$\rightarrow 0$ (peak $\sim T$ )
$T=0$ from integral splitting?	<b>Yes</b>	No	No
How $T=0$ enters	$\int_0^{p_F} \frac{2p^2+m^2}{E} dp = \mu p_F$	$\text{quad}/T \rightarrow \text{const}$	$\text{quad}/T^2 \rightarrow \text{const}$

For  $\chi_2$ , the IBP is special: the integrand  $f$  has a non-vanishing  $T = 0$  limit (the filled Fermi sea), and splitting  $f = 1 - (1 - f)$  below  $p_F$  yields the analytic answer *organically*. For  $\chi_3$  and  $\chi_4$ , the integrands are purely thermal ( $\rightarrow 0$  everywhere as  $T \rightarrow 0$ ), so the  $T = 0$  values emerge only through the scaling of the integral with powers of  $T$ , and the analytic extraction is a separate subtraction rather than a structural decomposition.

## 5 Implementation details

### 5.1 Quadrature scheme

The integration domain  $[0, \infty)$  is split at the Fermi momentum  $p_F$ :

**Below  $p_F$ : Sommerfeld–Legendre quadrature.** A nonlinear mapping  $s \in [0, 1] \rightarrow p \in [0, p_F]$  concentrates quadrature nodes near the Fermi surface:

$$p(s) = p_F \left( 1 - \frac{u(s)}{\alpha} \right), \quad u(s) = -\ln(1 - \beta(1-s)), \quad \alpha = \frac{p_F^2}{\mu T}, \quad \beta = 1 - e^{-\alpha}. \quad (41)$$

This is applied with 32-point Gauss–Legendre nodes and weights on  $[0, 1]$ . For  $\alpha \ll 1$  (i.e.  $T \gg p_F^2/\mu$ , or equivalently  $T \gg$  Fermi energy), the mapping reduces to uniform:  $p = s p_F$ .

**Above  $p_F$ : shifted Laguerre quadrature.** The substitution  $p = p_F + T t$  transforms the Laguerre integration variable  $t \in [0, \infty)$  with 32-point Gauss–Laguerre nodes and weights. The Jacobian  $dp = T dt$  contributes a factor  $T$ .

## 5.2 Below- $p_F$ forms

For  $p < p_F$ , we have  $E < \mu$  so  $x = (E - \mu)/T < 0$ . The Fermi–Dirac products are evaluated using  $e^x$  (which is  $< 1$ ):

$$f = \frac{1}{e^x + 1}, \quad (42)$$

$$f(1-f) = \frac{e^x}{(1+e^x)^2}, \quad (43)$$

$$f(1-f)(1-2f) = \frac{e^x}{(1+e^x)^2} \left(1 - \frac{2}{e^x + 1}\right) = \frac{e^x(e^x - 1)}{(1+e^x)^3}. \quad (44)$$

These forms are numerically stable for all  $x < 0$ .

For  $\chi_2$ , the code computes

$$-\int_0^{p_F} \frac{2p^2 + m^2}{E} \frac{1}{e^{-(E-\mu)/T} + 1} dp, \quad (45)$$

where the negative sign and  $e^{-x}$  in the denominator implement the “hole” integrand  $-(1-f) = -(e^{-x}/(1+e^{-x})) = -1/(e^{-(E-\mu)/T} + 1)$ . This equals  $-\int_0^{p_F} (2p^2 + m^2)/E \cdot (1-f) dp$ , which when combined with the analytic  $\mu p_F$  from  $\int_0^{p_F} (2p^2 + m^2)/E dp$  gives the total below- $p_F$  contribution.

## 5.3 Above- $p_F$ forms

For  $p > p_F$ , we have  $x = (E - \mu)/T > 0$ . The substitution  $p = p_F + Tt$  maps the integration to Laguerre variables  $t \in [0, \infty)$ , and the Fermi–Dirac products are evaluated using  $e^{-x}$  (which is  $< 1$ ):

$$f = \frac{e^{-x}}{1 + e^{-x}}, \quad (46)$$

$$f(1-f) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad (47)$$

$$f(1-f)(1-2f) = \frac{e^{-x}(1 - e^{-x})}{(1 + e^{-x})^3}. \quad (48)$$

In terms of the scaled variable  $t$ , the momentum is  $p = Tt$ , the energy-over- $T$  is  $E/T = \sqrt{t^2 + (m/T)^2}$ , and  $x = E/T - \mu/T$ . The kinematic factor becomes

$$\frac{2p^2 + m^2}{E} dp = T^2 \frac{2t^2 + (m/T)^2}{E/T} dt. \quad (49)$$

The extra factor of  $T^2$  (one from  $dp = T dt$ , one from  $2p^2/E = 2T^2t^2/(ET)$ , but the  $m^2/E$  term contributes less) appears in the code as the explicit  $T*T$  multiplying the Laguerre contribution.

## 5.4 Analytic $T = 0$ extraction

The code stores each susceptibility in the form

$$T^k \chi_{k+1} = \frac{g}{2\pi^2} T^k \left( \underbrace{\chi_{k+1}^{(0)}}_{\text{ret2}} + \underbrace{\Delta\chi_{k+1}}_{\text{ret1}} \right), \quad (50)$$

where  $\chi_{k+1}^{(0)}$  is the analytic  $T = 0$  value from section 3 and  $\Delta\chi_{k+1} = \text{quad}/T^k - \chi_{k+1}^{(0)}$  is the thermal correction.

- For  $\chi_2$  ( $k = 1$ ): the extraction is structural. The below- $p_F$  quadrature computes the hole contribution  $-\int_0^{p_F} (2p^2 + m^2)/E \cdot (1-f) dp$ , the above- $p_F$  quadrature computes the tail  $+\int_{p_F}^\infty (2p^2 + m^2)/E \cdot f dp$ , and  $\text{ret2} = \mu p_F$  is the exact Fermi-sea integral. The sum  $\text{ret1} + \text{ret2}$  equals the full  $\chi_2$  value to machine precision.
- For  $\chi_3$  ( $k = 2$ ): both quadratures compute  $\int (2p^2 + m^2)/E \cdot f(1-f) dp$  directly (no  $f = 1 - (1-f)$  decomposition, since  $f(1-f) \rightarrow 0$  below  $p_F$ ). The analytic value  $\text{ret2} = (\mu^2 + p_F^2)/p_F$  is subtracted after dividing the total quadrature by  $T$ .
- For  $\chi_4$  ( $k = 3$ ): similarly, both quadratures compute  $\int (2p^2 + m^2)/E \cdot f(1-f)(1-2f) dp$  directly, and  $\text{ret2} = \mu(3p_F^2 - \mu^2)/p_F^3$  is subtracted after dividing by  $T^2$ .

Since  $T^k \times \text{value}/T^k$  round-trips without precision loss in IEEE-754 double arithmetic (no overflow or underflow for  $T \gtrsim 10^{-6}$  GeV), this decomposition is numerically equivalent to the direct computation in all cases.

## 5.5 Summary of implemented formulas

The code computes  $T^k d^k n/d\mu^k$  for  $k = 1, 2, 3$  using the IBP-improved integrands:

Quantity	Code returns	IBP integrand	$T=0$ value of $d^k n/d\mu^k$
$\chi_2 = dn/d\mu$	$T dn/d\mu$	$(2p^2 + m^2)/E \cdot f$	$\mu p_F$
$\chi_3 = d^2n/d\mu^2$	$T^2 d^2n/d\mu^2$	$(2p^2 + m^2)/E \cdot f(1-f)$	$(\mu^2 + p_F^2)/p_F$
$\chi_4 = d^3n/d\mu^3$	$T^3 d^3n/d\mu^3$	$(2p^2 + m^2)/E \cdot f(1-f)(1-2f)$	$\mu(3p_F^2 - \mu^2)/p_F^3$

All integrands include the common prefactor  $g/(2\pi^2)$  and an overall factor of  $T$  from the IBP (absorbed into the  $T^k$  prefactor). The density ( $n$ ,  $k = 0$ ), pressure ( $P$ ), and energy density ( $\varepsilon$ ) do not use IBP — their integrands ( $p^2 f$ ,  $p^4 f/E$ ,  $p^2 E f$ ) are already smooth.

## 6 Numerical results

We test the IBP improvements using the `ZeroTemperatureComparison` example at  $\mu_B = 1.2$  GeV with the PDG2025 particle list, comparing the Ideal HRG, excluded-volume diagonal (EV), quantum van der Waals (QvdW), and real gas (CS+VDW) models. All models use quantum statistics with 32+32 quadrature points.

Tables 1 and 2 show relative deviations of the finite- $T$  results from the exact  $T = 0$  values.

Table 1: Relative deviation  $(\chi_3(T) - \chi_3(0))/\chi_3(0)$  at various temperatures.

Model	Old (Laguerre only)		New (IBP)	
	$T=0.001$ MeV	$T=1$ MeV	$T=0.001$ MeV	$T=1$ MeV
Ideal HRG	-1.000	-1.000	$-1.4 \times 10^{-4}$	$4.3 \times 10^{-2}$
EV-Diagonal	0.294	0.294	$3.5 \times 10^{-9}$	$-1.9 \times 10^{-5}$
QvdW	0.202	0.202	$2.9 \times 10^{-9}$	$2.1 \times 10^{-4}$
RealGas (CS+VDW)	0.879	0.879	$2.4 \times 10^{-8}$	$-3.6 \times 10^{-4}$

Table 2: Relative deviation  $(\chi_4(T) - \chi_4(0))/\chi_4(0)$  at various temperatures.

Model	Old (Laguerre only)		New (IBP)	
	$T=0.001 \text{ MeV}$	$T=1 \text{ MeV}$	$T=0.001 \text{ MeV}$	$T=1 \text{ MeV}$
Ideal HRG	-1.000	-1.000	1.46	$-8.7 \times 10^{-2}$
EV-Diagonal	0.846	0.846	$1.6 \times 10^{-4}$	$9.7 \times 10^{-5}$
QvdW	0.505	0.505	$6.8 \times 10^{-5}$	$3.3 \times 10^{-4}$
RealGas (CS+VDW)	3.905	3.904	$1.7 \times 10^{-3}$	$4.2 \times 10^{-4}$

### Key observations.

1.  **$\chi_3$  (one IBP step):** The old code produced 20–100% errors for all models below  $T \sim 1 \text{ MeV}$ , with the Ideal HRG returning essentially zero (the Laguerre-only quadrature completely misses the Fermi surface peak). After IBP, the integrand  $f(1-f)$  is a smooth, non-negative peak, and the errors drop to  $10^{-9}$ – $10^{-4}$  at  $T = 0.001 \text{ MeV}$ .
2.  **$\chi_4$  (one IBP step):** The old code had 50–390% errors. After IBP, the integrand  $f(1-f)(1-2f)$  still has one sign change, so the improvement is less dramatic but still substantial: errors drop to  $10^{-5}$ – $10^{-3}$  for interacting models. The Ideal HRG at  $T = 0.001 \text{ MeV}$  remains challenging (146% error) because many species with large Fermi momenta contribute, and the sign-changing integrand is harder to resolve per species. By  $T = 0.1 \text{ MeV}$  the Ideal HRG error is already below 0.4%.
3. **The old errors are flat in  $T$ :** For the interacting models with the old code, the relative deviations are essentially constant from  $T = 0.001 \text{ MeV}$  to  $T = 0.1 \text{ MeV}$ . This confirms the errors are systematic quadrature failures (the Laguerre nodes do not resolve the Fermi surface), not physical effects.
4. **A second IBP for  $\chi_4$  is not possible.** One might attempt to apply the IBP once more to replace  $f(1-f)(1-2f)$  with  $f(1-f)$ . However, the boundary term  $[pE \cdot f(1-f)]_0^\infty$  does vanish, but the resulting integrand involves  $d[(2p^2+m^2)/E]/dp = p(p^2+2m^2)/E^3$ , which when combined with  $pE \cdot f(1-f)$  produces terms proportional to  $(2p^2+m^2)/p$  that diverge at  $p = 0$ . This makes the second IBP inapplicable.

## 7 IBP for the entropy derivative $ds/dT|_\mu$

The entropy density temperature derivative at fixed chemical potential is

$$\frac{ds}{dT} \Big|_\mu = \frac{g}{2\pi^2} \frac{1}{T} \int_0^\infty p^2 x^2 f(1-f) dp, \quad x \equiv \frac{E-\mu}{T}. \quad (51)$$

At  $T = 0$ , the Sommerfeld expansion gives

$$\frac{ds}{dT} \Big|_{T=0} = \frac{g}{6} \mu p_F, \quad (52)$$

which has the same structure  $\sim \mu p_F$  as the  $T = 0$  value of  $\chi_2$ . However, as we show below, the numerical treatment of  $ds/dT$  is fundamentally more challenging than  $\chi_2$ .

## 7.1 The numerical problem

The integrand in eq. (51) is  $\propto x^2 f(1-f)/T$ . The factor  $x^2 = (E - \mu)^2/T^2$  vanishes at the Fermi surface ( $E = \mu$ ), while  $f(1-f)/T \rightarrow \delta(p - p_F) \mu/p_F$ . The product gives a peaked integrand  $\propto T$  near  $p_F$ . The integral therefore scales as  $T^2$ , and dividing by  $T$  yields the finite  $T = 0$  limit.

In the original implementation, the code computed this integral directly and subtracted the analytic  $T = 0$  value, i.e.

$$\frac{ds}{dT} = \frac{g}{2\pi^2} \left( \frac{\pi^2}{3} \mu p_F + \Delta_{\text{th}} \right), \quad (53)$$

where  $\Delta_{\text{th}} = \text{quad}/T - (\pi^2/3) \mu p_F$  is the thermal correction. At low  $T$ , both  $\text{quad}/T$  and  $(\pi^2/3) \mu p_F$  are large and nearly equal, so their difference suffers from cancellation. This led to  $\sim 0.35\%$  relative error already at  $T = 1 \mu\text{eV}$ .

## 7.2 First IBP: $f \cdot G$ form

Using  $f(1-f) = -(TE/p) df/dp$  as in the  $\chi_2$  case, we integrate by parts on the momentum integral. The relevant identity is

$$p^2 \frac{(E - \mu)^2}{T^2} f(1-f) = -\frac{1}{T} p E (E - \mu)^2 \frac{df}{dp}. \quad (54)$$

Integrating by parts with  $u = pE(E - \mu)^2$  and  $dv = df$ , and noting that the boundary terms vanish:

$$\boxed{\int_0^\infty p^2 \frac{(E - \mu)^2}{T^2} f(1-f) dp = \frac{1}{T} \int_0^\infty f(p) G(p) dp}, \quad (55)$$

where

$$G(p) = \frac{d}{dp} [pE(E - \mu)^2] = \varepsilon \left[ \frac{(2p^2 + m^2)\varepsilon}{E} + 2p^2 \right], \quad \varepsilon \equiv E - \mu. \quad (56)$$

So that

$$\frac{ds}{dT} \Big|_\mu = \frac{g}{2\pi^2 T^2} \int_0^\infty f(p) G(p) dp. \quad (57)$$

**Difference from  $\chi_2$ .** For  $\chi_2$ , the IBP converts  $\int p^2 f(1-f) dp = T \int (2E - m^2/E) f dp$ , where the integrand  $(2E - m^2/E)$  is nonzero at  $p_F$ :

$$\frac{2p_F^2 + m^2}{\mu} = \frac{2p_F^2 + m^2}{\mu} \neq 0. \quad (58)$$

The filled Fermi sea therefore contributes  $\int_0^{p_F} (2p^2 + m^2)/E dp = \mu p_F \neq 0$ , which is the exact  $T = 0$  value.

For  $ds/dT$ , the integrand  $G(p)$  vanishes at the Fermi surface because  $\varepsilon(p_F) = 0$ :

$$G(p_F) = 0. \quad (59)$$

Consequently, the Fermi-sea integral vanishes:

$$\int_0^{p_F} G(p) dp = [pE(E - \mu)^2]_0^{p_F} = p_F \mu \cdot 0 - 0 = 0. \quad (60)$$

There is *no*  $T = 0$  piece to extract analytically — the entire integral is a thermal contribution of order  $T^2$ . The result is  $ds/dT = (g/2\pi^2 T^2) \times O(T^2)$ , which is finite but computed entirely from the quadrature. This achieves  $\sim 4 \times 10^{-4}$  relative error at  $T = 1 \mu\text{eV}$ .

### 7.3 Double IBP: $\sigma \cdot H$ form

To improve accuracy further, we perform a second integration by parts, this time converting from  $f$  to the Fermi entropy kernel.

**The entropy kernel.** Define the Fermi entropy function

$$\sigma(x) = -f \ln f - (1-f) \ln(1-f) = \ln\left(1 + e^{-|x|}\right) + \frac{|x|}{e^{|x|} + 1}, \quad (61)$$

which satisfies

$$\sigma'(x) \equiv \frac{d\sigma}{dx} = -x f(1-f). \quad (62)$$

In momentum space:

$$\frac{d\sigma}{dp} = \sigma'(x) \frac{dx}{dp} = \sigma'(x) \frac{p}{TE} = -\frac{p(E-\mu)}{T^2 E} f(1-f). \quad (63)$$

**Derivation.** Start from eq. (51):

$$\left. \frac{ds}{dT} \right|_\mu = \frac{g}{2\pi^2 T} \int_0^\infty p^2 x^2 f(1-f) dp. \quad (64)$$

Using  $\sigma'(x) = -x f(1-f)$ , we have  $x^2 f(1-f) = -x \sigma'(x)$ . The momentum derivative of  $\sigma$  is  $d\sigma/dp = \sigma'(x) \cdot p/(TE)$ , so  $\sigma'(x) = (TE/p) d\sigma/dp$ . Substituting:

$$p^2 x^2 f(1-f) = -p^2 x \cdot \frac{TE}{p} \frac{d\sigma}{dp} = -pE(E-\mu) \frac{d\sigma}{dp}. \quad (65)$$

Integrating by parts with  $u = pE(E-\mu)$  and  $dv = d\sigma$ , using  $[pE(E-\mu) \sigma]_0^\infty = 0$  (since  $p=0$  at the lower limit and  $\sigma \rightarrow 0$  exponentially at the upper limit):

$$\int_0^\infty p^2 x^2 f(1-f) dp = \int_0^\infty \sigma(x) H(p) dp,$$

(66)

where

$$H(p) = \frac{d}{dp} [pE(E-\mu)] = \frac{(E-\mu)(2p^2 + m^2)}{E} + p^2. \quad (67)$$

The final formula is

$$\left. \frac{ds}{dT} \right|_\mu = \frac{g}{2\pi^2 T} \int_0^\infty \sigma(x) H(p) dp. \quad (68)$$

#### Key properties.

- The prefactor is  $1/T$  (not  $1/T^2$  as in the single-IBP form).
- $\sigma(x)$  is intrinsically smooth and *zero* at  $T=0$  (where  $f=0$  or  $f=1$ ), just like the entropy density integrand  $p^2 \sigma$ .
- $H(p_F) = p_F^2 \neq 0$ . The Sommerfeld expansion gives  $\int_0^\infty \sigma H dp \approx (\pi^2/3) T (\mu/p_F) H(p_F) = (\pi^2/3) \mu p_F T$ , so  $ds/dT \approx (g/2\pi^2 T) \times (\pi^2/3) \mu p_F T = (g/6) \mu p_F$ , matching eq. (52).
- Because  $\sigma=0$  at  $T=0$ , the quadrature evaluates the thermal contribution *directly*, with no analytical subtraction needed and no cancellation.

This is the same  $\sigma$  kernel used in the entropy density  $s = (g/2\pi^2) \int p^2 \sigma dp$ , so the quadrature accuracy is comparable to that of  $s$  itself:  $\sim 2 \times 10^{-4}$  relative error at  $T = 1 \text{ }\mu\text{eV}$ .

## 7.4 Can we do better? Obstruction to $\chi_2$ -level accuracy

Since eq. (68) has only a  $1/T$  prefactor (not  $1/T^2$ ), one might hope that a further IBP could convert the  $\sigma$  kernel into plain  $f$ , producing a  $\chi_2$ -type formula where the  $T = 0$  value emerges as a Fermi-sea integral.

**Attempt via  $\sigma = \ln(1 + e^{-x}) + xf$ .** One can verify the identity

$$\sigma(x) = \ln(1 + e^{-x}) + x f(x). \quad (69)$$

Substituting into eq. (66):

$$\int_0^\infty \sigma H dp = \int_0^\infty \ln(1 + e^{-x}) H dp + \frac{1}{T} \int_0^\infty (E - \mu) f H dp. \quad (70)$$

The second term contains plain  $f$  multiplied by  $H$ , which looks promising. However, at  $T = 0$ :

$$\frac{1}{T} \int_0^{p_F} (E - \mu) H dp = \frac{1}{T} \underbrace{[p E (E - \mu)^2]_0^{p_F}}_{=0} - \frac{1}{T} \int_0^{p_F} p^2 (E - \mu) dp = -\frac{1}{T} \int_0^{p_F} p^2 (E - \mu) dp. \quad (71)$$

This *diverges* as  $1/T$ . Meanwhile, the first term also diverges as  $1/T$  because  $\ln(1 + e^{-x}) \rightarrow (\mu - E)/T$  for  $p < p_F$  at  $T = 0$ . The two  $1/T$  divergences cancel to give the finite  $\sigma$  integral, but using them separately reintroduces catastrophic cancellation — exactly the problem we sought to avoid.

**Attempt via  $d\sigma/dp$ .** One could also try IBP on  $\int \sigma H dp$  with  $u = \sigma$  and  $dv = H dp$ . But  $d\sigma/dp = -p(E - \mu)f(1-f)/(T^2 E)$  from eq. (63), which reintroduces  $f(1-f)$  and  $1/T^2$  — going backwards.

**Fundamental reason.** For  $\chi_2$ , the IBP converts  $f(1-f) \rightarrow f$ , and the plain- $f$  integrand has a *nonzero* Fermi-sea integral  $\int_0^{p_F} (2p^2 + m^2)/E dp = \mu p_F$ . This works because  $\chi_2 = dn/d\mu$  is a chemical potential derivative — its  $T = 0$  value is a property of the filled Fermi sea.

In contrast,  $ds/dT|_\mu$  is a *temperature derivative*. At  $T = 0$ , the entropy vanishes ( $s = 0$ ) and  $ds/dT|_{T=0}$  is the first nonzero term in  $s(T) \approx (ds/dT)T$ . Its value  $(g/6)\mu p_F$  arises entirely from the *thermal smearing* of the Fermi surface, not from the filled sea. Any IBP that converts the integrand to plain  $f$  necessarily produces a weight function that vanishes at  $p_F$  (because the temperature derivative of all quantities is zero inside the Fermi sea at  $T = 0$ ), making  $\int_0^{p_F} (\text{weight}) \cdot f dp = 0$ .

Therefore, no IBP decomposition can extract the  $T = 0$  value of  $ds/dT$  as an analytical Fermi-sea integral in the way  $\chi_2$  extracts  $\mu p_F$ . The  $\sigma \cdot H$  form (eq. (68)) achieves the best accuracy available from direct quadrature of the thermal contribution.

## 7.5 Numerical results

Table 3 shows the relative deviation of  $ds/dT$  from its  $T = 0$  value for the Ideal HRG at  $\mu_B = 1.2 \text{ GeV}$ .

The double-IBP  $\sigma \cdot H$  form achieves  $\sim 16 \times$  improvement over the original at  $T = 1 \mu\text{eV}$ . The accuracy matches that of the entropy density  $s$  itself (which uses the same  $\sigma$  kernel), confirming that the quadrature resolution of  $\sigma$ -weighted integrals is the limiting factor. The interacting models (EV, QvdW, RealGas) achieve  $\sim 1.8 \times 10^{-4}$  at  $T = 1 \mu\text{eV}$ .

Table 3: Relative deviation  $(ds/dT(T) - ds/dT(0))/(ds/dT(0))$  at low temperatures for the Ideal HRG at  $\mu_B = 1.2 \text{ GeV}$ .

	<b>Original</b>	<b>Single IBP (<math>f \cdot G</math>)</b>	<b>Double IBP (<math>\sigma \cdot H</math>)</b>
$T = 0.001 \text{ MeV}$	$-3.5 \times 10^{-3}$	$-4.3 \times 10^{-4}$	$-2.2 \times 10^{-4}$
$T = 0.01 \text{ MeV}$	$-3.5 \times 10^{-3}$	$-4.4 \times 10^{-4}$	$-2.2 \times 10^{-4}$
$T = 0.1 \text{ MeV}$	$-3.7 \times 10^{-3}$	$-5.7 \times 10^{-4}$	$-3.4 \times 10^{-4}$
$T = 1 \text{ MeV}$	$-8.4 \times 10^{-3}$	$-8.4 \times 10^{-3}$	$-8.1 \times 10^{-3}$

## 8 Summary

The IBP improvement consists of one integration-by-parts step for each susceptibility  $\chi_n$  ( $n \geq 2$ ), using the identity

$$\int_0^\infty p^2 F_{k+1}(p) dp = T \int_0^\infty \frac{2p^2 + m^2}{E} F_k(p) dp, \quad (72)$$

where  $F_k = T \partial^k f / \partial \mu^k$ . Each step replaces an integrand with  $k$  sign changes by one with  $k-1$  sign changes, dramatically improving quadrature convergence at low  $T$ . The improvement is most pronounced for  $\chi_3$  (from  $\sim 100\%$  to  $\sim 10^{-9}$  relative error) and significant for  $\chi_4$  (from  $\sim 400\%$  to  $\sim 10^{-3}$  for interacting models).

For  $\chi_2$ , the IBP was already implemented earlier and replaces the  $f(1-f)$  peak by the smooth Fermi step  $f$ , yielding excellent precision at all temperatures.

For  $ds/dT|_\mu$ , a *double* IBP replaces the  $x^2 f(1-f)$  integrand by the entropy kernel  $\sigma(x) = -f \ln f - (1-f) \ln(1-f)$ :

$$\left. \frac{ds}{dT} \right|_\mu = \frac{g}{2\pi^2 T} \int_0^\infty \sigma(x) \frac{d}{dp} [pE(E-\mu)] dp. \quad (73)$$

Since  $\sigma$  is intrinsically zero at  $T = 0$ , the quadrature evaluates the thermal contribution directly with no cancellation, achieving  $\sim 2 \times 10^{-4}$  relative error at  $T = 1 \text{ } \mu\text{eV}$  — a  $16\times$  improvement. Unlike  $\chi_2$ , the  $T = 0$  value of  $ds/dT$  cannot be extracted as a Fermi-sea integral because temperature derivatives vanish identically inside the filled sea; the leading Sommerfeld coefficient arises entirely from the thermal smearing at the Fermi surface.