Adiabatic speed of sound and heat capacity in a system with multiple conserved charges

Volodymyr Vovchenko

January 26, 2026

1 Setup and Notation

Consider a relativistic system with multiple conserved charges, such as baryon number B, electric charge Q, strangeness S, etc. The system is in equilibrium at temperature T and chemical potentials $\mu_1, \ldots, \mu_k$, where the index i labels the conserved charges. The system is then fully characterized by its *pressure* P, which is a function of the thermodynamic variables: described by an equation of state

$$P \;=\; P\big(T,\; \mu_1, \ldots, \mu_k\big), \tag{1}$$

where

- T is the temperature,
- μ_i (for $i = 1, \ldots, k$) are chemical potentials associated with k different conserved charge densities n_i,

Derivatives of the pressure read:

$$s \;\equiv\; \left(\frac{\partial P}{\partial T}\right)_{\mu_1,\ldots,\mu_k}, \quad n_i \;\equiv\; \left(\frac{\partial P}{\partial \mu_i}\right)_{T,\mu_{j\neq i}}, \tag{2}$$

so that s is the *entropy density* and n_i are the *charge densities.* The *energy density* ϵ satisfies the standard Gibbs relation for a relativistic fluid:

$$\epsilon \;+\; P \;=\; T\, s \;+\; \sum_{i=1}^{k} \mu_i\, n_i. \tag{3}$$

The energy differential $d\epsilon$ is then given by

$$d\epsilon \;=\; T\, ds \;+\; \sum_{i=1}^{k} \mu_i\, dn_i, \tag{4}$$

while the pressure differential is

$$dP \;=\; s\, dT \;+\; \sum_{i=1}^{k} n_i\, d\mu_i. \tag{5}$$

2 Speed of Sound at Fixed Entropy-per-Charge Ratios

The speed of sound is defined as the adiabatic derivative of pressure with respect to energy density. However, in a system with multiple conserved charges, we must specify the conditions under which we take this derivative. The most natural choice is to hold the ratios s/n_i fixed, since these are the *physical* quantities that remain constant during a small perturbation. In other words, we consider a small perturbation of the system where the entropy density s and all charge densities n_i vary, but the ratios s/n_i remain constant.

More specifically, we have

$$\frac{ds}{s} \;=\; \frac{dn_1}{n_1} \;=\; \frac{dn_2}{n_2} \;=\; \ldots \;=\; \frac{dn_k}{n_k} \quad\Longrightarrow\quad dn_i \;=\; \frac{n_i}{s}\, ds \quad (i = 1, \ldots, k). \tag{6}$$

Effectively, out of the $(k+1)$ densities $\{s, n_1, \ldots, n_k\}$, there is only one independent direction of variation once we impose k constraints $s/n_i = \text{const}$. We take the variation in entropy density s to be the independent variable, and then all other densities are determined by the constraint $s/n_i = \text{const}$.

The relativistic speed of sound squared, c_s^2, is then defined as the adiabatic derivative of pressure with respect to energy density, under the constraint that s/n_i remains constant:

$$c_s^2 \;=\; \frac{\partial P}{\partial \epsilon}\bigg|_{\{s/n_i\}} . \tag{7}$$

We may choose the entropy density as our independent variable under the constraints $dn_i = \frac{n_i}{s}\, ds$ for $i = 1, \ldots, k$, and then form

$$c_s^2 = \frac{\frac{dP}{ds}\big|_{\{s/n_i\}}}{\frac{d\epsilon}{ds}\big|_{\{s/n_i\}}} . \tag{8}$$

2.1 Denominator $d\epsilon/ds$

From $d\epsilon = T\, ds + \sum_i \mu_i\, dn_i$ [Eq. (4)], imposing $dn_i = \frac{n_i}{s}\, ds$, we get

$$\frac{d\epsilon}{ds}\bigg|_{\{s/n_i\}} \;=\; T \;+\; \sum_{i=1}^{k} \mu_i \frac{n_i}{s} \;=\; \frac{1}{s}\big(T\, s + \sum_i \mu_i\, n_i\big) \;=\; \frac{\epsilon + P}{s}.$$

2.2 Numerator dP/ds

Similarly, P can be viewed as a function of $(s, n_1, \ldots, n_k)$. The total differential of P is

$$dP \;=\; \big(\tfrac{\partial P}{\partial s}\big)_{n_i}\, ds \;+\; \sum_{i=1}^{k} \big(\tfrac{\partial P}{\partial n_i}\big)_{s, n_{j\neq i}}\, dn_i.$$

Hence under $dn_i = \frac{n_i}{s}\, ds$, we have

$$\frac{dP}{ds}\bigg|_{\{s/n_i\}} \;=\; \big(\tfrac{\partial P}{\partial s}\big)_{n_i} \;+\; \sum_{i=1}^{k} \big(\tfrac{\partial P}{\partial n_i}\big)_{s, n_{j\neq i}} \frac{n_i}{s}.$$

Therefore

$$c_s^2 = \frac{\frac{dP}{ds}\big|_{\{s/n_i\}}}{\frac{d\epsilon}{ds}\big|_{\{s/n_i\}}} \;=\; \frac{\big(\frac{\partial P}{\partial s}\big)_{n_i} s \;+\; \sum_{i=1}^{k} \big(\frac{\partial P}{\partial n_i}\big)_{s, n_{j\neq i}}\, n_i}{(\epsilon + P)} \tag{9}$$

Let us introduce a short-hand notation for entropy and charge densities:

$$\mathbf{x} \;=\; (s, n_1, \ldots, n_k),$$

such that $x_0 = s$, $x_i = n_i$ for $i = 1, \ldots, k$. Then we can write

$$c_s^2 \;=\; \frac{\sum_{i=0}^{k} \big(\frac{\partial P}{\partial x_i}\big)_{x_{j\neq i}}\, x_i}{(\epsilon + P)}. \tag{10}$$

2.3 Grand-canonical form

Equation (10) expresses the sound velocity in terms of pressure derivatives with respect to entropy and charge density. However, it is more natural to express the speed of sound in terms of the *grand-canonical* variables T and μ_i. Let us introduce a short-hand notation for the grand-canonical variables T and μ_i:

$$\mathbf{y} \;=\; (T, \mu_1, \ldots, \mu_k), \tag{11}$$

such that $y_0 = T$, $y_i = \mu_i$ for $i = 1, \ldots, k$. Taking into account that

$$\left(\frac{\partial P}{\partial T}\right)_{\mu_i} = s, \quad \left(\frac{\partial P}{\partial \mu_i}\right)_{T, \mu_{j\neq i}} = n_i,$$

one has

$$\left(\frac{\partial P}{\partial y_i}\right)_{y_{j\neq i}} = x_i, \quad i = 0, 1, \ldots, k. \tag{12}$$

Using the chain rule, we can express the derivatives of pressure with respect to entropy and charge density in terms of the grand-canonical variables. First, we write

$$\begin{aligned}\left(\frac{\partial P}{\partial x_i}\right)_{x_{j\neq i}} &= \sum_{j=0}^{k}\left(\frac{\partial P}{\partial y_j}\right)_{y_{l\neq j}}\left(\frac{\partial y_j}{\partial x_i}\right)_{x_{l\neq i}} \\ &= \sum_{j=0}^{k} x_j \left(\frac{\partial y_j}{\partial x_i}\right)_{x_{l\neq i}}.\end{aligned} \tag{13}$$

The speed of sound then reads

$$c_s^2 \;=\; \frac{\displaystyle\sum_{i,j=0}^{k} x_j \left(\frac{\partial y_j}{\partial x_i}\right)_{x_{l\neq i}} x_i}{(\epsilon + P)} \tag{14}$$

Let us define the Jacobian matrix $\Pi_{ij} = \left(\frac{\partial x_i}{\partial y_j}\right)_{y_{l\neq j}}$. We have

$$\begin{aligned}\Pi &= \begin{pmatrix} \left(\frac{\partial s}{\partial T}\right)_{\mu_l} & \left(\frac{\partial n_i}{\partial T}\right)_{\mu_l} \\ \left(\frac{\partial s}{\partial \mu_i}\right)_{T,\mu_{l\neq i}} & \left(\frac{\partial n_i}{\partial n_j}\right)_{T,\mu_{l\neq j}} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{\partial^2 P}{\partial T^2}\right)_{\mu_l} & \left(\frac{\partial^2 P}{\partial T\partial \mu_i}\right)_{\mu_l} \\ \left(\frac{\partial^2 P}{\partial \mu_i \partial T}\right)_{T,\mu_{l\neq i}} & \left(\frac{\partial^2 P}{\partial \mu_i \partial \mu_j}\right)_{T,\mu_{l\neq j}}, \end{pmatrix}\end{aligned} \tag{15}$$

i.e. Π is the Hessian matrix of the pressure with respect to the grand-canonical variables T and μ_i. The derivatives $\left(\frac{\partial y_j}{\partial x_i}\right)_{x_{l\neq i}}$ in Eq. (14) can be expressed in terms of the inverse of the Jacobian matrix Π^{-1}:

$$\left(\frac{\partial y_j}{\partial x_i}\right)_{x_{l\neq i}} = \Pi^{-1}_{ji}. \tag{16}$$

The speed of sound squared can then be expressed as

$$c_s^2 \;=\; \frac{\displaystyle\sum_{i,j=0}^{k} x_i \Pi^{-1}_{ij} x_j}{(\epsilon + P)}\;, \tag{17}$$

which corresponds entirely to the grand-canonical variables T and μ_i.

3 Heat Capacity at Constant Densities

The heat capacity at constant volume (and constant charge densities) is defined as:

$$C_V = T\left(\frac{\partial s}{\partial T}\right)_{n_i}. \tag{18}$$

We want to express this in terms of the grand-canonical variables and the Hessian matrix Π.

3.1 Derivation

Since $\mathbf{x} = \mathbf{x}(\mathbf{y})$, we have:

$$d\mathbf{x} = \Pi \, d\mathbf{y} \tag{19}$$

or equivalently:

$$d\mathbf{y} = \Pi^{-1} d\mathbf{x}. \tag{20}$$

Explicitly, the differential of temperature at constant charge densities ($dn_i = 0$ for $i = 1, \ldots, k$) is:

$$dT = \sum_{j=0}^{k} \Pi^{-1}_{0j} \, dx_j = \Pi^{-1}_{00} \, ds, \tag{21}$$

where we used $dx_0 = ds$ and $dx_i = dn_i = 0$ for $i \geq 1$.

Therefore:

$$\left(\frac{\partial T}{\partial s} \right)_{n_i} = \Pi^{-1}_{00} \tag{22}$$

and consequently:

$$\left(\frac{\partial s}{\partial T} \right)_{n_i} = \frac{1}{\Pi^{-1}_{00}}. \tag{23}$$

The heat capacity at constant densities is then:

$$C_V = \frac{T}{\Pi^{-1}_{00}} \tag{24}$$

3.2 Explicit Form Using Cofactors

The inverse matrix element Π^{-1}_{00} can be written in terms of the cofactor:

$$\Pi^{-1}_{00} = \frac{C_{00}}{\det \Pi}, \tag{25}$$

where C_{00} is the cofactor of the $(0, 0)$ element, i.e., the determinant of the $k \times k$ submatrix obtained by removing the first row and column. This submatrix is precisely the susceptibility matrix $\chi_{ij} = \partial^2 P / \partial \mu_i \partial \mu_j$:

$$C_{00} = \det \chi, \quad \text{where } \chi_{ij} = \left(\frac{\partial^2 P}{\partial \mu_i \partial \mu_j} \right)_T. \tag{26}$$

Thus:

$$C_V = T \frac{\det \Pi}{\det \chi} \tag{27}$$

4 Partial cases

4.1 Zero chemical potential

In the case of zero chemical potentials, the densities are vanishing $n_i = 0$ and $dn_i = 0$. In this case only the first term in Eq. (14) survives, and the speed of sound squared is given by

$$c_s^2 \; = \; \frac{\left.\frac{dT}{ds}\right|_{\mu_i = 0} s^2}{(\epsilon + P)} \; . \tag{28}$$

Taking into account that $\epsilon + P = Ts$ at zero chemical potetials, we obtain

$$\begin{aligned} c_s^2(T) &= \frac{s(T)}{T s'(T)} \\ &= \frac{s(T)}{\epsilon'(T)}, \end{aligned} \tag{29}$$

where $'$ denotes the derivative with respect to T.

For the heat capacity, at $\mu_i = 0$ the susceptibility matrix χ decouples from the entropy sector (mixed derivatives $\partial_T \partial_{\mu_i} P$ vanish by symmetry), and we have:

$$C_V = T\,\partial_T^2 P = T\left(\frac{\partial s}{\partial T}\right)_{\mu=0} . \tag{30}$$

4.2 One conserved charge B

In the case of one conserved charge B, the Jacobian matrix Π is a 2×2 matrix, and its inverse can be easily obtained:

$$\Pi^{-1} = \frac{1}{\det \Pi}\begin{pmatrix} \partial^2_{\mu_B} P & -\partial_T \partial_{\mu_B} P \\ -\partial_T \partial_{\mu_B} P & \partial_T^2 P \end{pmatrix} \tag{31}$$

where $\det \Pi = \partial_T^2 P \partial^2_{\mu_B} P - (\partial_T \partial_{\mu_B} P)^2$.

The speed of sound squared can then be expressed as

$$c_s^2 \;=\; \frac{1}{(\epsilon+P)} \frac{n_B^2 \partial_T^2 P - 2 s n_B\, \partial_T \partial_{\mu_B} P + s^2 \partial^2_{\mu_B} P}{\partial_T^2 P \partial^2_{\mu_B} P - (\partial_T \partial_{\mu_B} P)^2} . \tag{32}$$

This result coincides with one from [1, 2].

The heat capacity at constant baryon density is:

$$\begin{aligned} C_V &= T \frac{\det \Pi}{\chi_2^B} \\ &= T\left[\partial_T^2 P - \frac{(\partial_T \partial_{\mu_B} P)^2}{\chi_2^B}\right] . \end{aligned} \tag{33}$$

4.3 Zero temperature limit

In the zero temperature limit, the entropy density vanishes $s = 0$. Thus, $ds = 0$, and the speed of sound squared is given by

$$c_s^2 \;=\; \frac{\sum_{i,j=1}^{k} n_i \chi^{-1}_{ij} n_j}{(\epsilon+P)} , \tag{34}$$

where $\chi_{ij} = \left(\frac{\partial^2 P}{\partial \mu_i \partial \mu_j}\right)_{T=0}$ is the susceptibility matrix.

For a single conserved charge B one has

$$\begin{aligned} c_s^2 &= \frac{n_B^2}{(\epsilon+P)\chi_2^B} \\ &= \frac{n_B}{\mu_B \chi_2^B} . \end{aligned} \tag{35}$$

For two conserved charges B and Q, for example, in asymmetric nuclear matter, the speed of sound squared is given by

$$c_s^2 = \frac{n_B^2 \chi_2^Q + n_Q^2 \chi_2^B - 2 n_B n_Q \chi_{11}^{BQ}}{(\epsilon+P)[\chi_2^B \chi_2^Q - (\chi_{11}^{BQ})^2].} , \tag{36}$$

For instance, in beta-equilibrium neutron star matter one has $n_Q = 0$ (including leptons) and

$$c_s^2 = \frac{n_B}{\mu_B \left[\chi_2^B - \frac{(\chi_{11}^{BQ})^2}{\chi_2^Q}\right].} , \tag{37}$$

For the heat capacity, the third law of thermodynamics requires $C_V \to 0$ as $T \to 0$. From Eq. (24), this follows from the explicit factor of T:

$$C_V = \frac{T}{\Pi^{-1}_{00}} \to 0 \quad \text{as} \quad T \to 0. \tag{38}$$

References

[1] S. Floerchinger and M. Martinez, Phys. Rev. C **92**, 064906 (2015), arXiv:1507.05569 [nucl-th] .

[2] P. Parotto, M. Bluhm, D. Mroczek, M. Nahrgang, J. Noronha-Hostler, K. Rajagopal, C. Ratti, T. Schäfer, and M. Stephanov, Phys. Rev. C **101**, 034901 (2020), arXiv:1805.05249 [hep-ph] .