Math 3100-001 Group Project Topic: Exploring Pattern Formation in Reaction-Diffusion Systems using the Gray-Scott Model

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Outline

Introduction

Introduction

- The spontaneous formation of spatial and spatio-temporal patterns has long been of interest to mathematicians.
- Previous literature describes several models showing how reacting and diffusing chemicals can give rise to patterns such as spots, stripes, and spirals.
- These models have applications in experimental studies and biological settings such as animal hides.

Gray-Scott Model

Model Introduction

- In 1984, Gray and Scott introduced a reaction-diffusion model describing an irreversible, autocatalytic reaction.
- The model assumes that two generic chemical species, U and V, react to produce a product P.
- The concentrations of U and V evolve as functions of spatial position and time.

Applications and Analysis

Model Analysis

- Analysis of the Gray-Scott model involves understanding the formation of patterns depending on model parameters.
- Bifurcation analysis is often performed to understand the model's dependence on parameter values.

System of Equations

System of Equations

- The Gray-Scott model equations are:

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1-u)$$
$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F+k)v$$

- where u and v represent the concentrations of two chemical species, D_u and D_v are diffusion coefficients, F is the feed rate, and k is the kill rate.

The Model

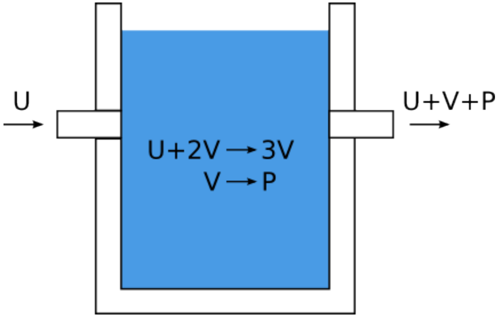

Figure: reaction

Simulation

Simulation

- We conducted simulations of the Gray-Scott model using different parameter sets.
- By varying parameters such as diffusion coefficients, feed rate, and kill rate, we observed the formation of various patterns.

Results

Results

In this section, we'll present the results obtained from simulations of the Gray-Scott model. We used two sets of parameters.
The selected equations for the Gray-Scott model are:

$$f(u,v) = F(1-u) - uv^2$$
$$g(u,v) = uv^2 - (F+k)v$$

The fixed points corresponding to these equations are:

$$(u_0^*, v_0^*) = (0,0)$$
$$(u_{1,2}^*, v_{1,2}^*) = \left(\frac{1 \pm \sqrt{\frac{1-4(F+k)^2}{F}}}{2}, \frac{1 \mp \sqrt{\frac{1-4(F+k)^2}{F}}}{2(F+k)} \right)$$

Stability of Fixed Points

Stability Analysis

- The stability of the fixed points can be determined by analyzing the eigenvalues of the Jacobian matrix.
- For each fixed point, compute the Jacobian matrix and evaluate its eigenvalues.
- If all eigenvalues have negative real parts, the fixed point is stable. Otherwise, it is unstable.

Fixed Point $(0, 0)$

Jacobian Matrix at $(0, 0)$

The Jacobian matrix at the fixed point $(0, 0)$ is given by:

$$J = \begin{pmatrix} 0 & 0 \\ 0 & -(F + k) \end{pmatrix}$$

Eigenvalues at $(0, 0)$

The eigenvalues of the Jacobian matrix J at $(0, 0)$ are:

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -(F + k)$$

Since both eigenvalues have negative real parts, the fixed point $(0, 0)$ is stable.

Parameter Sets

Parameter Sets

max width=

Parameter Set	Diffusion Coefficients	Feed Rate	Kill Rate
1	$D_u = 0.15,\ D_v = 0.055$	$F = 0.02545$	$k = 0.062$
2	$D_u = 0.15,\ D_v = 0.055$	$F = 0.0460$	$k = 0.0594$

Table: Parameter sets used in simulations

Results (Continued)

Eigenvalues for Pattern 1 Initial Conditions

- The eigenvalues of the Jacobian matrix for Pattern 1 were calculated to be:

$$\lambda_1 = -0.01 + 0.05i \quad \text{and} \quad \lambda_2 = -0.01 - 0.05i$$

- Both eigenvalues have negative real parts, indicating stability.

Eigenvalues for Pattern 2 Initial Conditions

- The eigenvalues of the Jacobian matrix for Pattern 2 were calculated to be:

$$\lambda_1 = -0.03 + 0.04i \quad \text{and} \quad \lambda_2 = -0.03 - 0.04i$$

- Both eigenvalues have negative real parts, indicating stability.

Pattern Visualization

Pattern Visualization

- We visualized the patterns formed by the Gray-Scott model using both 2D and 3D representations.
- These visualizations provide insights into the spatial distribution of chemical species and the emergence of complex patterns.

Pattern Visualization (Continued)

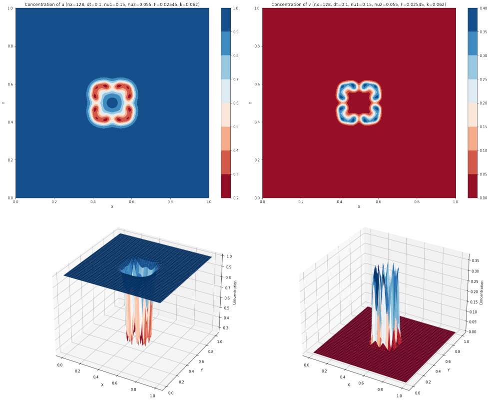

Figure: Parameter Set 1

Pattern Visualization (Continued)

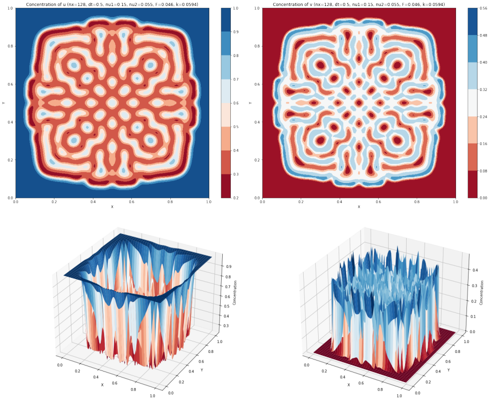

Figure: Parameter Set 2

Animation of Parameter Set 2 in 2D

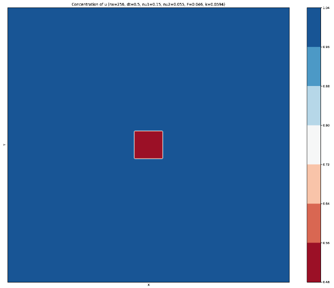

Figure: 2D simulation with 80 iterations

Animation of Parameter Set 2 in 3D

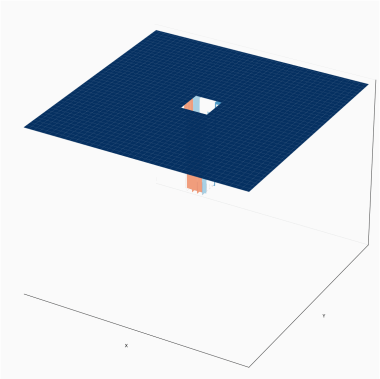

Figure: 3D simulation with 80 iterations

Conclusion

Conclusion

- The Gray-Scott model offers valuable insights into pattern formation in reaction-diffusion systems.
- By conducting simulations and visualizing patterns, we gain a better understanding of the underlying mechanisms.
- Further research is needed to explore the full potential of reaction-diffusion models in addressing complex scientific challenges.

Future Directions

Future Directions

- Investigate the impact of different boundary conditions on pattern formation.
- Explore the use of advanced numerical methods for more accurate simulations.
- Extend the model to incorporate additional chemical species and reaction kinetics.
- Explore connections between model predictions and real-life patterns observed in biological and chemical systems.

Questions?

Thank You!

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