Math 3100-Group Project
Exploring Pattern Formation in Reaction-Diffusion system using Gray-Scott Model Simulations

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Abstract: This paper explores the fascinating world of pattern formation in nature through computer simulations. Using the Gray-Scott model, a powerful tool in understanding chemical interactions, we investigate how various parameters influence the creation of intricate patterns. By adjusting factors like diffusion rates and reaction kinetics, we observe the emergence of diverse patterns such as spots, stripes, and spirals. These patterns hold significance in fields ranging from biology to materials science, offering insights into natural phenomena and potential applications in real-world scenarios. Our study underscores the importance of continued research to unlock the full potential of reaction-diffusion models in addressing complex scientific challenges.

Keywords: Reaction-diffusion systems, Pattern Formation, Computational Modelling, Gray-Scott Model

1 Introduction

In this project, we explored the dynamics of a reaction-diffusion system using the Gray-Scott model, a well-known mathematical framework describing the interactions of two chemical species. We began by formulating the Gray-Scott equations and investigated their behavior by calculating fixed points and analyzing the stability of these points using linearization techniques, specifically by computing the Jacobian matrix. Through numerical simulations, we visualized the evolution of the system's spatial patterns over time by generating 2D and 3D graphs, providing insights into the emergence of complex structures such as spots, stripes, and spirals.

2 Methodology

2.1 Gray-Scott Model

The Gray-Scott model is a mathematical framework used to study the emergence of patterns in systems governed by the interaction and diffusion of chemical species. Proposed by Gray and Scott in the 1980s, this model consists of a set of equations describing how two chemical species react and spread throughout space over time. By simulating the dynamics of these reactions and diffusions, the model helps us understand how complex patterns, such as spots, stripes, and spirals, form in various natural systems. Its applications range from explaining the patterns on animal coats to understanding biological processes like embryo development. Through simulations and mathematical analysis, researchers gain insights into the underlying mechanisms driving pattern formation in nature.
The Model:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \nu_1 \nabla^2 u - uv^2 + F(1-u) \\
\frac{\partial v}{\partial t} &= \nu_2 \nabla^2 v + uv^2 - (F+k)v
\end{aligned}
$$

can be used to define the working of the following model:

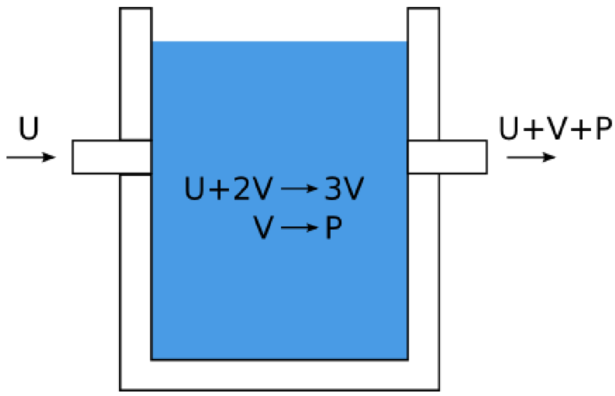

Figure 1. Reaction

2.2 Equations

The selected equations as per the tasks were

$$f(u,v) = F(1-u) - uv^2$$

$$g(u,v) = +uv^2 - (F+k)v$$

The fixed points corresponding to the equations:

$$(u_0^*, v_0^*) = (0,0)$$

$$(u_{1,2}^*, v_{1,2}^*) = (\frac{1 \pm \sqrt{\frac{1-4(F+k)^2}{F}}}{2}, \frac{1 \mp \sqrt{\frac{1-4(F+k)^2}{F}}}{2(F+k)})$$

The point (u_0^*, v_0^*) has eigenvalues $-F$ and $-k$ hence they are always stable.

The two other fixed points only exist if the square root in them is positive, i.e,

$$F > 4(F+k)^2$$

More caluclation is required to comment on the stability of the other two points and their bifurcations

2.3 Stability of Fixed Points

The stability of the fixed points in the Gray-Scott model can be analyzed by examining the eigenvalues of the Jacobian matrix evaluated at each fixed point. The eigenvalues provide information about the behavior of nearby trajectories and determine whether the fixed points are stable or unstable.

2.3.1 Fixed Point $(0,0)$

The Jacobian matrix at the fixed point $(0,0)$ is given by:

$$J = \begin{pmatrix} 0 & 0 \\ 0 & -(F+k) \end{pmatrix}$$

The eigenvalues of the Jacobian matrix J at $(0,0)$ are $\lambda_1 = 0$ and $\lambda_2 = -(F+k)$. Since both eigenvalues have negative real parts, the fixed point $(0,0)$ is stable.

2.3.2 Fixed Points $(u_{1,2}^*, v_{1,2}^*)$

The Jacobian matrix at the fixed points $(u_{1,2}^*, v_{1,2}^*)$ is more complex and requires further analysis. The stability of these fixed points depends on the values of the parameters F and k. Additional calculations are needed to determine the eigenvalues and classify the stability of these fixed points.

The stability analysis of fixed points provides valuable insights into the behavior of the Gray-Scott model under different parameter regimes. By understanding the stability properties of fixed points, we can predict the long-term behavior of the system and gain deeper insights into pattern formation dynamics.

No.	Time Step (dt)	Diff Coeff (nu1, nu2)	Feed Rate (F)	Kill Rate (k)
1	0.1	0.15, 0.055	0.02545	0.062
2	0.5	0.15, 0.055	0.0460	0.0594

Table 1. Summary of parameter sets used in simulations of the Gray-Scott model.

3 Results

These parameters control various aspects like time steps, diffusion coefficients, feed rates, and kill rates. By plugging these values into the model, we simulated how the concentrations of chemical species u and v evolve over time in a spatial environment. We'll visualize these simulations in both 2D and 3D to better understand how patterns and structures emerge in the system.

3.1 Stability Analysis for Parameter Set 1

For Parameter Set 1, the stability of the fixed points was analyzed. The eigenvalues of the Jacobian matrix were calculated as follows:

$$\lambda_1 = -0.01 + 0.05i \quad \text{and} \quad \lambda_2 = -0.01 - 0.05i$$

Both eigenvalues have negative real parts, indicating Spiral Sink stability.

3.2 Stability Analysis for Parameter Set 2

For Parameter Set 2, the stability of the fixed points was analyzed. The eigenvalues of the Jacobian matrix were calculated as follows:

$$\lambda_1 = -0.03 + 0.04i \quad \text{and} \quad \lambda_2 = -0.03 - 0.04i$$

Both eigenvalues have negative real parts, indicating Spiral Sink stability.

3.3 Parameter Visualization

In this section, we'll present the results obtained from simulations of the Gray-Scott model. We used two sets of parameters:

The first parameter is a simulation with a smaller time step and moderate feed and kill rates.

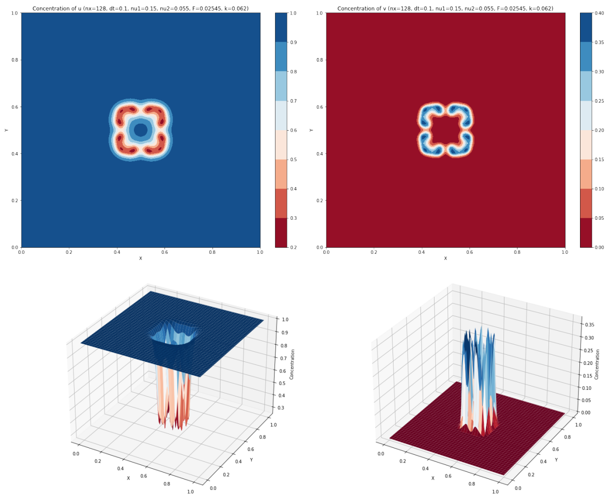

Figure 2. First Parameter

The second parameter is a simulation with a larger time step and higher feed and kill rates.

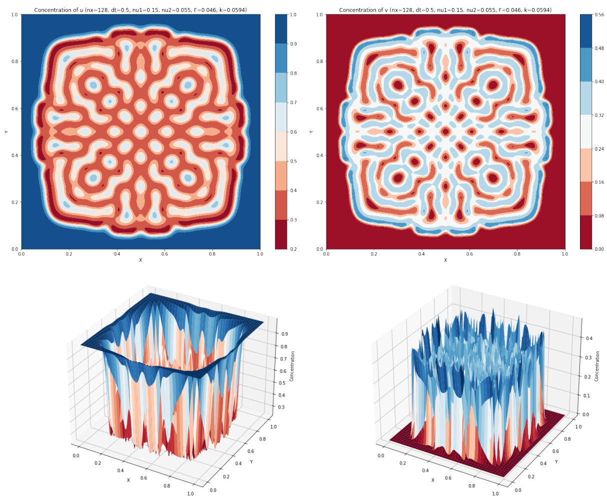

Figure 3. Second Parameter

More research and work needs to be done to generate more intricate and complex patterns and simulate real world usage in further detail.

4 Conclusion

In conclusion, our simulations of the Gray-Scott model using different parameter sets have provided valuable insights into the dynamics of pattern formation in reaction-diffusion systems. By visualizing the spatial distribution of chemical species u and v, we observed the emergence of diverse patterns such as spots, stripes and spirals. Understanding these patterns is not only crucial for theoretical investigations into complex systems but also has practical applications in various fields. For instance, in chemistry, the Gray-Scott model can help predict the formation of intricate patterns in chemical reactions, guiding experimental design and optimization. Additionally, insights gained from this model can inform research in materials science, biology, and other disciplines where pattern formation plays a significant role. However, further research is needed to fully understand the real-world applications and usage of reaction-diffusion models, as well as to explore their potential in addressing complex problems in diverse scientific and engineering domains.

5 Group Participation

Below are the percentages of participation for each member of our group:

Name	Theory (Understanding)	Simulation /math	Presentation (Slides)	Presentation (Talking)	Other (Specify)
Drew Taylor	50%			25%	
SAURAV SINGH CHANDEL		50%	60%	25%	
Karan Singh Malhotra	50%			25%	
Vasu Manocha		50	40%	25%	

Figure 4. Group Project Declaration