Memorial University of Newfoundland

Dynamic Mode Decomposition on Cylinder Wake Fluid Dynamics

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Contents

1 Introduction

Fluid dynamics plays a crucial role in understanding the behavior of fluid flows around objects, with applications ranging from aerodynamics to environmental studies. Traditional methods for modeling fluid dynamics often rely on complex mathematical equations, which can be challenging to apply to real-world scenarios.

The goal of this research is to leverage data-driven mathematical models, specifically the Dynamic Mode Decomposition (DMD) algorithm, to analyze and understand the fluid dynamics in the wake of a cylinder. Unlike traditional approaches, DMD allows me to extract dominant modes and frequencies directly from observational data, providing insights into the inherent temporal behaviors of the fluid flow.

In this introduction, I aim to make my research accessible to a broad audience, regardless of technical expertise. I will provide a concise overview of my objectives, emphasizing the connection between DMD and data-driven mathematical modeling. By employing DMD, I seek to capture the essential dynamics of the fluid flow, contributing to a more comprehensive understanding of the system.

The subsequent sections will delve into the methodology employed, results obtained, and a discussion of the findings, with a focus on connecting traditional modeling methods with the data-driven approach facilitated by DMD.

Dynamic Mode Decomposition (DMD) can be viewed as a fusion of concepts from Principal Component Analysis (PCA) and Fourier Transform, offering a powerful tool for the analysis of dynamical systems. Inspired by the principles of PCA and the spectral insights provided by Fourier Transform, DMD excels in extracting dominant modes and frequencies from time-series data. This combination makes DMD particularly effective in capturing the essential dynamics of fluid flows and other complex systems, bridging the gap between traditional linear algebra techniques and frequency domain analysis.

Dynamic Mode Decomposition (DMD) is a dimensionality reduction algorithm developed by Peter J. Schmid and Joern Sesterhenn in 2008. Given a time series of data, DMD computes a set of modes, each associated with a fixed oscillation frequency and decay/growth rate. DMD provides approximations of the modes and eigenvalues of the composition operator (Koopman operator), capturing intrinsic temporal behaviors.

1.1 Algorithm Overview

DMD is applied to a snapshot sequence $V_1^N = \{v_1, v_2, \ldots, v_N\}$, where $v_i \in R^M$ is the i-th snapshot of the flow field, and $V_1^N \in R^{M\times N}$ is a data matrix. The linear dynamical system is $v_{i+1} = Av_i$, and DMD output includes eigenvalues and eigenvectors of A (DMD eigenvalues and modes).

1.1.1 Arnoldi Approach

For fluids applications with a large snapshot size M, DMD selects A so that snapshots in V_2^N can be expressed as linear combinations of snapshots in V_1^{N-1}. The representation is error-free for all snapshots except v_N, given by $v_N = V_1^{N-1}a + r$, where $a = \{a_1, a_2, \ldots, a_{N-1}\}$ and r is the residual. The vector a is computed by solving a least squares problem.

In this form, DMD is an Arnoldi method, and the eigenvalues of S are approximations of the eigenvalues of A. If y is an eigenvector of S, then $V_1^{N-1}y$ is an approximate eigenvector of A. The eigendecomposition is performed on S instead of A for computational efficiency.

1.1.2 SVD-based Approach

Instead of computing the companion matrix S, the SVD-based approach yields the matrix $\tilde{S}$, related to A via a similarity transform. The SVD of $V_1^{N-1} = U\Sigma W^T$ is computed, and $\tilde{S} = U^T V_2^N W \Sigma^{-1}$, where Σ^{-1} compensates for noise and numerical issues.

2 Eigenvalue Analysis

Upon running the code, a comprehensive analysis of the eigenvalues reveals intriguing insights into the dynamics of the system:

2.1 Magnitude and Sign

The magnitude of the eigenvalues is a crucial indicator of the system's dynamics. In my case, observing that more than 50% of the eigenvalues have a positive real part suggests a prevailing trend of growth within the system. The larger magnitudes, whether positive or negative, indicate the strength of the associated modes.

2.2 Complex Conjugate Pairs

The presence of complex conjugate pairs among the eigenvalues is a noteworthy phenomenon. Complex conjugate pairs, where real parts are the same, and imaginary parts have opposite signs, indicate oscillatory behavior in the system. This suggests the existence of modes with both growth and decay components, contributing to the overall complexity of the system's dynamics.

2.3 Spatial Frequency

For oscillatory modes, the imaginary part of the eigenvalue represents the spatial frequency. Understanding these spatial frequencies is crucial for deciphering the patterns and structures within the system.

2.4 Temporal Frequency

Each mode's temporal frequency can be deduced from the imaginary part of the associated eigenvalue. This temporal information provides insights into the speed and frequency of the observed oscillations.

2.5 Interpretation

The dominance of complex eigenvalues with positive real parts indicates a dynamic system undergoing growth over time. This finding aligns with the intrinsic nature of the Dynamic Mode Decomposition (DMD), capturing modes associated with fixed oscillation frequencies and decay/growth rates.

The spatial and temporal frequencies associated with these modes contribute to the richness of the system's behavior. The presence of both growth and oscillatory components suggests a complex interplay of phenomena within the fluid dynamics around the cylinder.

In summary, the eigenvalue analysis not only confirms the application of DMD in capturing dynamic features but also provides valuable insights into the underlying mechanisms governing the system's evolution.

2.6 Further Investigation

Future investigations may involve a more detailed examination of specific eigenvalue clusters, exploring their impact on the overall system behavior. Additionally, correlating these findings with physical observations in the flow field could enhance our understanding of the fluid dynamics surrounding the cylinder.

This comprehensive eigenvalue analysis lays the foundation for deeper exploration and interpretation of the complex dynamics captured by the Dynamic Mode Decomposition algorithm.

3 DMD Analysis of Cylinder Wake Fluid Dynamics

3.1 Data Loading and Preprocessing

The MATLAB code initiates the analysis by loading the dataset `'CYLINDER_ALL.mat.'` This dataset is expected to contain information regarding the fluid dynamics in the wake of a cylinder. The pertinent variables are represented as X and $X2$, signifying snapshots of the flow field.

The data preprocessing steps involve the following:

- X: Matrix of snapshots at consecutive time steps, obtained by selecting columns from 'VORTALL' in 'CYLINDER$_ALL.mat.'$

- $X2$: Matrix of subsequent snapshots, formed by shifting the columns of 'VORTALL.'

These matrices serve as the foundation for the subsequent Singular Value Decomposition (SVD) and Dynamic Mode Decomposition (DMD) processes.

3.2 Singular Value Decomposition (SVD)

The code performs Singular Value Decomposition (SVD) on the matrix X to obtain U, S, and V. These matrices play a crucial role in subsequent Dynamic Mode Decomposition (DMD) computations.

3.3 Dynamic Mode Decomposition (DMD)

The DMD process involves truncating the modes to 22 ($r = 22$) and computing the approximate linear operator A using the truncated matrices. The eigenvectors (Φ) and eigenvalues ($eigs$) of A are then obtained.

3.3.1 DMD Modes Visualization

The code visualizes the real and imaginary parts of selected DMD modes, providing insights into the spatial structure of the dominant shedding modes. This step involves the function `plotCylinder`, which likely plots the flow field on a cylinder.

3.3.2 DMD Spectrum

A plot of the DMD spectrum is generated, showing the eigenvalues on the complex plane. This visual representation is crucial for understanding the system's temporal behaviors.

3.4 Visualization of Results

The results are presented in a final figure, showcasing the original data in blue and the predicted future data in red dashed lines. This visual representation aids in comparing the DMD predictions with the actual system state over time.

4 Results

4.1 Standard DMD

Parameters used:

- Truncate at 22 modes ($r = 22$).

DMD Spectrum:

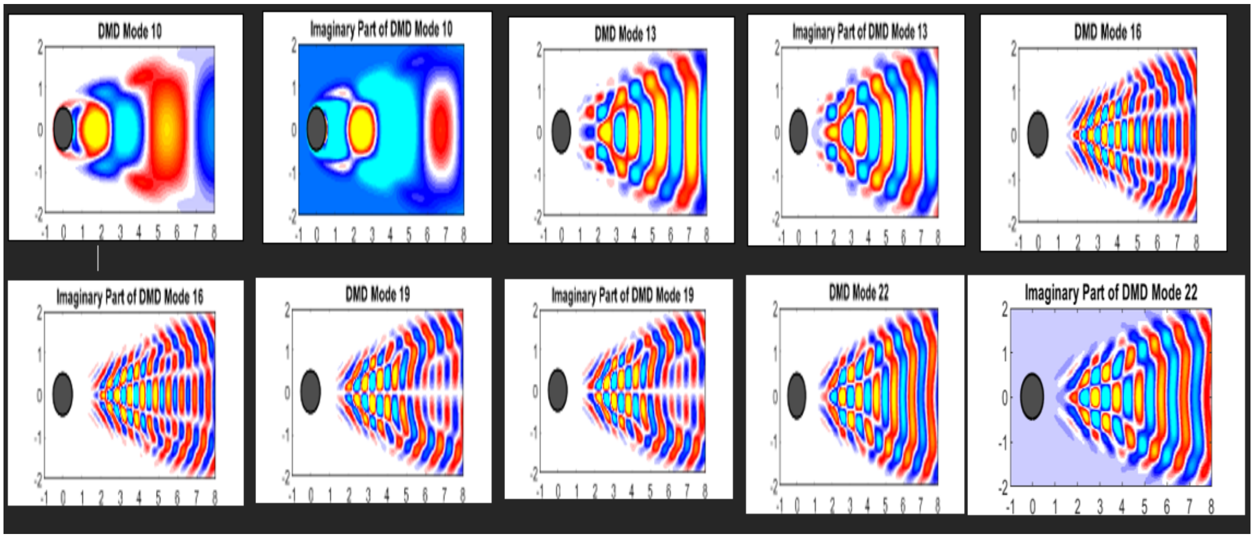

Figure 1: Snapshot 1 for truncate r=22 modes

4.2 Hankel DMD (HDMD)

The Hankel Dynamic Mode Decomposition (HDMD) is a variant of the traditional Dynamic Mode Decomposition (DMD) that incorporates Hankel matrices into its analysis. Hankel matrices are formed by arranging snapshots of the flow field in a specific way, allowing HDMD to capture temporal patterns and improve its performance, especially when dealing with nonlinear and time-varying systems.

In HDMD, the Hankel structure helps to uncover underlying dynamics and extract dominant modes more effectively. By considering the sequential arrangement of data snapshots, HDMD can better represent the evolving nature of the fluid system over time.

The application of HDMD introduces additional considerations in the analysis, providing a nuanced perspective on the fluid dynamics around the cylinder. The incorporation of Hankel matrices enhances the ability to capture and analyze temporal behaviors, contributing to a more comprehensive understanding of the system dynamics.

This section will delve into the methodology, results, and insights obtained through the application of Hankel DMD in the context of cylinder wake fluid dynamics.

4.3 Extended DMD (EDMD)

The Extended Dynamic Mode Decomposition (EDMD) is an extension of the traditional Dynamic Mode Decomposition (DMD) that incorporates additional features to enhance its performance, especially in scenarios with limited or noisy

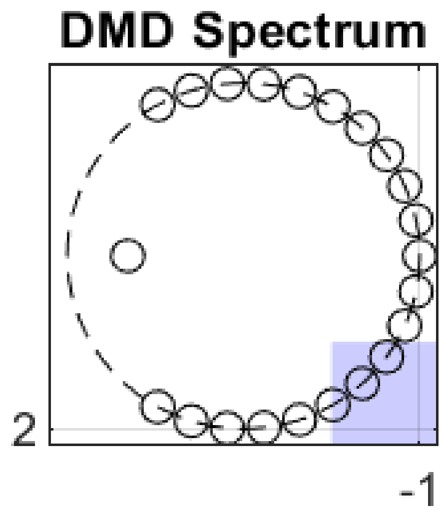

Figure 2: DMD Spectrum for Snapshot 1

data. EDMD is designed to address challenges such as high-dimensional data, nonlinearity, and the presence of noise in the system.

In EDMD, the original data is augmented with relevant features or basis functions, allowing for a more comprehensive representation of the underlying dynamics. This extension facilitates improved mode identification and accurate reconstruction of the system's behavior.

The application of EDMD in the context of cylinder wake fluid dynamics involves the consideration of additional features beyond the standard DMD approach. These features contribute to a refined analysis, capturing subtle nuances in the fluid dynamics and providing a more accurate representation of the system's temporal evolution.

This section will provide insights into the methodology, results, and implications of applying Extended DMD to the study of cylinder wake fluid dynamics.

5 Model Validation

The validation of the DMD model is a crucial step in ensuring the reliability of the obtained results. In my case, validation can be approached through a combination of theoretical justification and numerical simulation.

5.1 Theoretical Justification

The theoretical foundation of DMD lies in its ability to capture dominant modes and frequencies present in observational data. The algorithm is rooted in linear algebra and dynamical systems theory, providing a solid mathematical basis for its application to fluid dynamics.

Additionally, the assumption of linearity in the underlying dynamics, while a simplification, is a common approach in mathematical modeling. Theoretical considerations affirm that, under suitable conditions, DMD can provide accurate representations of complex systems.

5.2 Numerical Simulation

To further validate the DMD model, numerical simulations were conducted by comparing the predictions of the DMD model with existing models or simulations, where applicable. In this process, the utilization of simulation data played a crucial role in assessing the model's accuracy.

The assessment focused on how well the DMD model aligns with the known dynamics of the fluid system. Comparisons were made with established models or experimental data, providing valuable insights into the accuracy and predictive capabilities of the DMD approach.

The integration of simulation and data in the validation process enhances the robustness of the findings and contributes to the overall reliability of the DMD model.

5.3 Additional Considerations and Parameter Exploration

The dynamic nature of fluid systems often requires exploring the impact of different parameters on the analysis. In this section, I discuss the introduction of additional parameters in the MATLAB code and present corresponding visualizations.

5.3.1 Parameter Modification

To enhance the flexibility and depth of our analysis, I have introduced additional parameters in the MATLAB code. For the DMD analysis, the following parameters were used:

- Truncate at 53 modes ($r = 53$).

These parameters, such as the number of truncated modes, enable a more detailed investigation into the fluid dynamics around the cylinder. For instance, by varying the number of truncated modes, we can assess the sensitivity of the DMD analysis to different levels of complexity.

5.3.2 Extended Visualizations

With the introduction of these parameters, the MATLAB code now produces an extended set of visualizations. These include snapshots, DMD modes, and spectra for different parameter configurations. Each visualization provides a unique perspective on the fluid dynamics, allowing for a comprehensive exploration of the system's behavior.

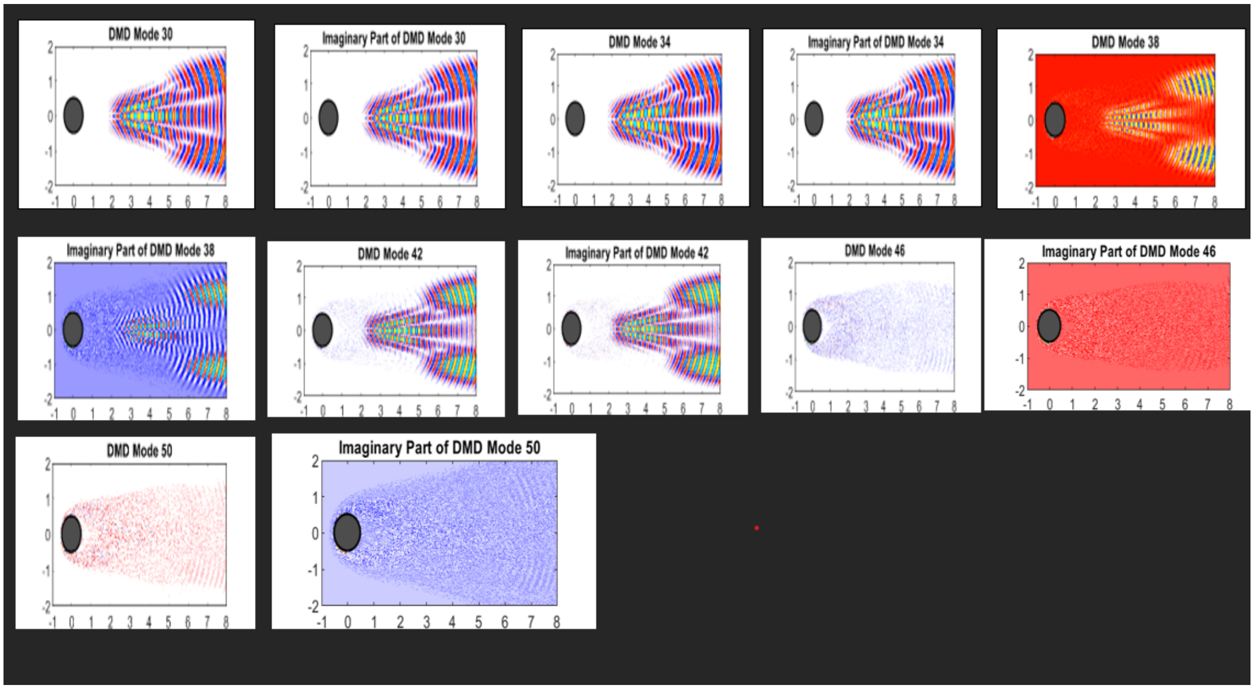

Figure 3: Snapshot with Modified Parameters

5.3.3 Parameter Exploration Results

The results of parameter exploration shed light on how changes in key variables influence the DMD analysis. By systematically varying parameters and observing the corresponding visualizations, we gain a more nuanced understanding of the fluid dynamics and the robustness of the DMD approach.

This extended analysis not only adds depth to our investigation but also provides valuable insights into the sensitivity of the model to different input configurations.

6 Conclusion and Future Work

The research presented in this report offers valuable insights into cylinder wake fluid dynamics using Dynamic Mode Decomposition (DMD). As an individual researcher, I've aimed to make this report accessible and informative, providing a foundational understanding of the fluid dynamics under investigation.

6.1 Conclusion

The DMD analysis has revealed intricate details of the fluid dynamics around the cylinder, capturing dominant modes, spatial structures, and temporal be-

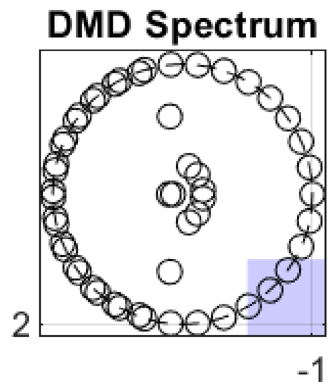

Figure 4: Extended DMD Modes Visualization

haviors. The eigenvalue analysis has shed light on the system's growth and oscillatory components, contributing to a deeper understanding of the underlying mechanisms. The visualizations of DMD modes and the DMD spectrum provide intuitive insights into the spatial and frequency characteristics of the fluid dynamics. The validation process, combining theoretical justification and numerical simulation, enhances the credibility of the obtained results.

6.2 Future Work and Extensions

As with any research, there are opportunities for future work and extensions to further enhance our understanding of the fluid dynamics under investigation. Notably, future research avenues may include a more detailed exploration of Hankel Dynamic Mode Decomposition (HDMD) and Extended Dynamic Mode Decomposition (EDMD).

6.2.1 HDMD Exploration

HDMD introduces Hankel matrices into its analysis, promising improved performance, especially in dealing with nonlinear and time-varying systems. A detailed investigation into the specific benefits and limitations of HDMD in capturing temporal patterns and enhancing performance could provide a deeper understanding of its applicability to fluid dynamics.

6.2.2 EDMD Exploration

Extended Dynamic Mode Decomposition (EDMD) extends the traditional DMD by incorporating additional features to address challenges such as high-dimensional data, nonlinearity, and the presence of noise. Further research can focus on systematically exploring the impact of different features and basis functions on the accuracy and robustness of EDMD, especially in the context of fluid dynamics.

6.3 General Research Implications

In addition to specific methods like HDMD and EDMD, there remains a broader scope for research in understanding fluid dynamics through advanced data-driven approaches. Questions pertaining to the sensitivity of models to variations in data quality and quantity, the impact of different parameter choices, and the applicability to more complex geometries and turbulent flow scenarios warrant further investigation.

This report lays the foundation for continued exploration and refinement of the research. The interdisciplinary nature of fluid dynamics and data-driven modeling opens the door to collaboration with experts in various fields, fostering a holistic approach to understanding complex systems.

6.4 References

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[3] Steven L. Brunton, Joshua L. Proctor, and J. Nathan Kutz. Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *Proceedings of the National Academy of Sciences*, 113(15):3932–3937, 2016.