Econ 7330, Problem Set 1

August 28, 2019

This is due on Friday, 8/30. Please hand it in no later than the beginning of section (9AM). Feel free to work in groups, but write your solutions individually.

1. Let A and B be arbitrary subsets of the set X, and define the complement of a set $A^c = \{x \in X | x \notin A\}$. Prove deMorgan's law: $(A \cup B)^c = (A^c \cap B^c)$

2. A set $X \subset Y \subset \mathbb{R}$ is said to be *dense* in Y if for any two points $y_1, y_2 \in Y$, $y_1 < y_2$, there exists a point $x \in X$ such that $y_1 < x < y_2$. Informally, this means that there is at least one point of X in between every two points of Y. Let $\mathbb{Q}$ be the set of rational numbers (i.e., numbers that can be written as $^m/_n$ where m and n are integers with no common factors besides 1). You will now prove that $\mathbb{Q}$ is dense in $\mathbb{R}$. I will step you through the proof.

   a) First, prove that if $y_1 > 0$ then there exists a positive integer n such that $ny_1 > y_2$. This is known as the *Archimedean Property of* $\mathbb{R}$, named after the famous Greek mathematician. (Hint: Prove this by contradiction.)

   b) Show that there exists an n such that $n(y_2 - y_1) > 1$.

   c) Show that there exist positive numbers m_1 and m_2 such that $-m_2 \leq ny_1 < m_1$.

   d) Show that there exists an integer $m \in [-m_2, m_1]$ such that $m - 1 \leq ny_1 < m$.

   e) Show that $ny_1 < m \leq 1 + ny_1 < ny_2$ and complete the proof.

3. In class, we proved that the sum of the first k integers is equal to $t_k = \frac{k(k+1)}{2}$. The numbers t_k are called triangular numbers because the first k integers can be stacked into a triangle. Similarly, we could stack the first k triangular numbers into a three dimensional tetrahedron. Define $h_k = \sum_{i=1}^{k} t_i$ as the k^{th} tetrahedral number. Prove that $h_k = \frac{k(k+1)(k+2)}{6}$.

4. Are the following valid examples of metrics? If yes, prove it. If no, provide a counterexample.

   a) For points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = \max\left\{(x_1 - x_2)^2, (y_1 - y_2)^2\right\}$

   b) For points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = \min\left\{(x_1 - x_2)^2, (y_1 - y_2)^2\right\}$

   c) $d(x, y) = 1$ if $x \neq y$,
   $d(x, y) = 0$ if $x = y$

5. Consider an infinite set X with the metric defined in problem 4(c).

   a) Which subsets of the resulting metric space are open? (Hint: For any subset of $A \subset X$, what is the neighborhood of radius $1/2$ around any point in A?)

   b) Which subsets are closed? (Hint: Using the hint above, what are the limit points of $A \subset X$?)

6. In class, we proved that the set $[0, 1] \subset \mathbb{R}$ was compact. Now I want you to prove that the set $(0, 1) \subset \mathbb{R}$ is not compact.

   a) Consider the collection of open sets $\{F_n\}_{n=1}^{\infty}$ where $F_n = (1/n+1, 1/n)$. Show that this is not an open cover of $(0, 1)$.

   b) Modify the collection in part (a) in such a way that it is an open cover.

   c) Prove that the open cover you made in part (b) does not contain a finite subcover.

7. Convergence.

   a) Prove that if the sequence $\{x_n\}$ converges, then the sequence $\{|x_n|\}$ also converges.

   b) Is the converse true? If yes, prove it. If no, provide a counterexample

8. Let (X, d) be a complete metric space, and let $T : X \to X$ be a function whose n^{th} iteration is a contraction. Prove that T has a unique fixed point.