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b-matrix
for Echo Planar Imaging

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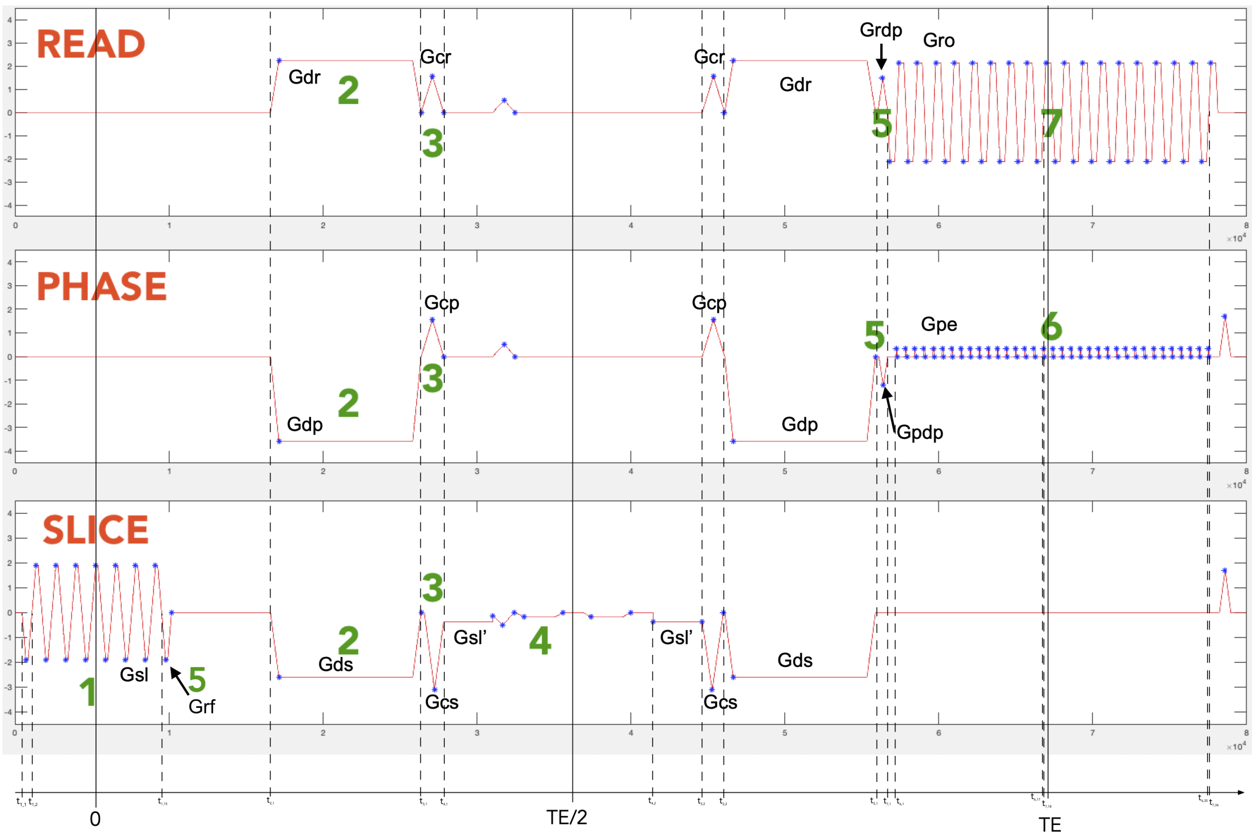
1 Pulse sequence



2 General analytic form

$$
\begin{aligned}
b_{ij} = \gamma^2 \Big[& G_{1i}G_{1j}\tau_{11} + (G_{1i}G_{2j}+G_{2i}G_{1j})\tau_{12} + (G_{1i}G_{3j}+G_{3i}G_{1j})\tau_{13} + \\
& + (G_{1i}G_{4j}+G_{4i}G_{1j})\tau_{14} + (G_{1i}G_{5j}+G_{5i}G_{1j})\tau_{15} + (G_{1i}G_{6mj}+G_{6mi}G_{1j})\tau_{16m} + \\
& + (G_{1i}G_{71j}+G_{71i}G_{1j})\tau_{171} + G_{2i}G_{2j}\tau_{22} + (G_{2i}G_{3j}+G_{3i}G_{2j})\tau_{23} + \\
& + (G_{2i}G_{4j}+G_{4i}G_{2j})\tau_{24} + (G_{2i}G_{5j}+G_{5i}G_{2j})\tau_{25} + (G_{2i}G_{6mj}+G_{6mi}G_{2j})\tau_{26m} + \\
& + (G_{2i}G_{71j}+G_{71i}G_{2j})\tau_{271} + G_{3i}G_{3j}\tau_{33} + (G_{3i}G_{4j}+G_{4i}G_{3j})\tau_{34} + \\
& + (G_{3i}G_{5j}+G_{5i}G_{3j})\tau_{35} + (G_{3i}G_{6mj}+G_{6mi}G_{3j})\tau_{36m} + (G_{3i}G_{71j}+G_{71i}G_{3j})\tau_{371} + \\
& + G_{4i}G_{4j}\tau_{44} + (G_{4i}G_{5j}+G_{5i}G_{4j})\tau_{45} + (G_{4i}G_{6mj}+G_{6mi}G_{4j})\tau_{46m} + \\
& + (G_{4i}G_{71j}+G_{71i}G_{4j})\tau_{471} + G_{5i}G_{5j}\tau_{55} + (G_{5i}G_{6mj}+G_{6mi}G_{5j})\tau_{56m} + (G_{5i}G_{71j}+G_{71i}G_{5j})\tau_{571} \\
& + G_{6mi}G_{6mj}\tau_{6m6m} + (G_{6i}G_{71j}+G_{71i}G_{6j})\tau_{671} + G_{71i}G_{71j}\tau_{7171} + \\
& + (G_{7mi}G_{7(m+1)j}+G_{7(m+1)i}G_{7mj})\tau_{7m7(m+1)} \Big]
\end{aligned} \tag{1}
$$

3 Diagonal terms

For the read direction we have: $G_1 = 0$; $G_2 = G_{dr}$; $G_3 = G_{cr}$; $G_4 = 0$; $G_5 = G_{rdp}$; $G_6 = 0$; $G_7 = G_{ro}$ which gives the following for the diagonal readout term

$$\begin{aligned} b_{rr} = \gamma^2 \Big(& G_{dr}^2 \tau_{22} + 2G_{dr}G_{cr}\tau_{23} + 2G_{dr}G_{rdp}\tau_{25rp} + 2G_{dr}G_{ro}\tau_{271} + G_{cr}^2\tau_{33} + 2G_{cr}G_{rdp}\tau_{35rp} + \\ & + 2G_{cr}G_{ro}\tau_{371} + G_{rdp}^2\tau_{55rp} + 2G_{rdp}G_{ro}\tau_{5rp71} + G_{ro}^2(\tau_{7171} + \tau_{7m7(m+1)})\Big) \end{aligned} \tag{2}$$

taking into account refocused gradient pairs we get:

$$b_{rr} = \gamma^2 \Big(G_{dr}^2\tau_{22} + 2G_{dr}G_{cr}\tau_{23} + G_{cr}^2\tau_{33} + G_{rdp}^2\tau_{55rp} + 2G_{rdp}G_{ro}\tau_{5rp71} + G_{ro}^2(\tau_{7171} + \tau_{7m7(m+1)})\Big) \tag{3}$$

For the phase direction we have: $G_1 = 0$; $G_2 = G_{dp}$; $G_3 = G_{cp}$; $G_4 = 0$; $G_5 = G_{pdp}$; $G_6 = G_{pe}$; $G_7 = 0$ which gives the following for the diagonal phase-encoding

$$\begin{aligned} b_{pp} = \gamma^2 (& G_{dp}^2\tau_{22} + 2G_{dp}G_{cp}\tau_{23} + 2G_{dp}G_{pdp}\tau_{25rp} + 2G_{dp}G_{pe}\tau_{26m} + G_{cp}^2\tau_{33} + 2G_{cp}G_{pdp}\tau_{35rp} \\ & + G_{cp}G_{pe}\tau_{36m} + G_{pdp}^2\tau_{55rp} + 2G_{pdp}G_{pe}\tau_{5rp6m} + G_{pe}^2\tau_{6m6m}) \end{aligned} \tag{4}$$

taking into account refocused gradient pairs we get:

$$b_{pp} = \gamma^2 (G_{dp}^2\tau_{22} + 2G_{dp}G_{cp}\tau_{23} + G_{cp}^2\tau_{33} + G_{pdp}^2\tau_{55rp} + 2G_{pdp}G_{pe}\tau_{5rp6m} + G_{pe}^2\tau_{6m6m}) \tag{5}$$

For the slice direction we have: $G_1 = G_{sl}$; $G_2 = G_{ds}$; $G_3 = G_{cs}$; $G_4 = G'_{sl}$; $G_5 = G_{rf}$; $G_6 = 0$; $G_7 = 0$ which gives the following for the diagonal slice-select:

$$\begin{aligned} b_{ss} = \gamma^2 \bigg(& \frac{14}{3}G_{sl}^2\tau_{11} + G_{sl}G_{ds}\tau_{12} + G_{sl}G_{cs}\tau_{13} + G_{sl}G'_{sl}\tau_{14} + G_{sl}G_{rf}\tau_{15s} + G_{ds}^2\tau_{22} + 2G_{ds}G_{cs}\tau_{23} + \\ & + G_{ds}G'_{sl}\tau_{24} + G_{ds}G_{rf}\tau_{25s} + G_{cs}^2\tau_{33} + G_{cs}G'_{sl}\tau_{34} + 2G_{cs}G_{rf}\tau_{35} + G'^2_{sl}\tau_{44} + G'_{sl}G_{rf}\tau_{45s} + \\ & + G_{rf}^2\tau_{55s}\bigg) \end{aligned} \tag{6}$$

with refocused gradient pairs one arrives to:

$$\begin{aligned} b_{ss} = \gamma^2 \Big(& \frac{14}{3}G_{sl}^2\tau_{11} + G_{ds}^2\tau_{22} + 2G_{ds}G_{cs}\tau_{23} + G_{ds}G'_{sl}\tau_{24} + G_{ds}G_{rf}\tau_{25s} + G_{cs}^2\tau_{33} + G_{cs}G'_{sl}\tau_{34} + \\ & + 2G_{cs}G_{rf}\tau_{35s} + \frac{1}{4}G'^2_{sl}\tau_{44} + G'_{sl}G_{rf}\tau_{45s} + \frac{1}{4}G_{rf}^2\tau_{55s}\Big) \end{aligned} \tag{7}$$

4 Off-diagonal terms

Summarizing all gradients together in one place:

$$\begin{array}{lll} G_{1r} = 0 & G_{1p} = 0 & G_{1s} = G_{sl} \\ G_{2r} = G_{dr} & G_{2p} = G_{dp} & G_{2s} = G_{ds} \\ G_{3r} = G_{cr} & G_{3p} = G_{cp} & G_{3s} = G_{cs} \\ G_{4r} = 0 & G_{4p} = 0 & G_{4s} = G'_{sl} \\ G_{5r} = G_{rdp} & G_{5p} = G_{pdp} & G_{5s} = G_{rf} \\ G_{6r} = 0 & G_{6p} = G_{pe} & G_{6s} = 0 \\ G_{7r} = G_{ro} & G_{7p} = 0 & G_{7s} = 0 \end{array}$$

Then starting from the read-out/phase term plugging the gradients G_r and G_p into equation Eq. 1:

$$\begin{aligned} b_{rp} = b_{pr} = \gamma^2 \Big(& G_{dr}G_{dp}\tau_{22} + (G_{dr}G_{cp} + G_{cr}G_{dp})\tau_{23} + (G_{dr}G_{pdp} + G_{rdp}G_{dp})\tau_{25rp} + G_{dr}G_{pe}\tau_{26m} + \\ & + G_{ro}G_{dp}\tau_{271} + G_{cr}G_{cp}\tau_{33} + (G_{cr}G_{pdp} + G_{rdp}G_{cp})\tau_{35rp} + G_{cr}G_{pe}\tau_{36m} + G_{ro}G_{cp}\tau_{371} + \\ & + G_{rdp}G_{pdp}\tau_{55rp} + G_{rdp}G_{pe}\tau_{5rp6m} + G_{ro}G_{pdp}\tau_{5rp71} + G_{pe}G_{ro}\tau_{6m71}\Big) \end{aligned} \tag{8}$$

taking into account refocused gradient pairs we get:

$$b_{rp} = b_{pr} = \gamma^2 \Big(G_{dr}G_{dp}\tau_{22} + (G_{dr}G_{cp} + G_{cr}G_{dp})\tau_{23} + G_{cr}G_{cp}\tau_{33} + G_{rdp}G_{pdp}\tau_{55rp} + G_{rdp}G_{pe}\tau_{5rp6m} + \\ + G_{ro}G_{pdp}\tau_{5rp71} + G_{pe}G_{ro}\tau_{6m71} \Big) \tag{9}$$

the read-out/slice select term gives:

$$b_{rs} = b_{sr} = \gamma^2 \Big(G_{dr}G_{sl}\tau_{12} + G_{cr}G_{sl}\tau_{13} + G_{rdp}G_{sl}\tau_{15s} + G_{ro}G_{sl}\tau_{171} + G_{dr}G_{ds}\tau_{22} + \\ + (G_{dr}G_{cs} + G_{cr}G_{ds})\tau_{23} + \frac{1}{2}G_{dr}G'_{sl}\tau_{24} + (\frac{1}{2}G_{dr}G_{rf} + G_{rdp}G_{ds})\tau_{25s} + G_{ds}G_{ro}\tau_{271} + \\ + G_{cr}G_{cs}\tau_{33} + \frac{1}{2}G_{cr}G'_{sl}\tau_{34} + (G_{cr}G_{rf} + \frac{1}{2}G_{rdp}G_{cs})\tau_{35s} + G_{rdp}G'_{sl}\tau_{45s} + \\ + G_{ro}G'_{sl}\tau_{471} + \frac{1}{2}G_{rdp}G_{rf}\tau_{55rp} + + \frac{1}{2}G_{ro}G_{rf}\tau_{5s71} \Big) \tag{10}$$

again by taking into account refocused gradient pairs the expression simplifies to:

$$b_{rs} = b_{sr} = \gamma^2 \Big((G_{dr}G_{cs} + G_{cr}G_{ds})\tau_{23} + \frac{1}{2}G_{dr}G'_{sl}\tau_{24} + \frac{1}{2}(G_{dr}G_{rf})\tau_{25s} + G_{cr}G_{cs}\tau_{33} \\ + \frac{1}{2}G_{cr}G'_{sl}\tau_{34} + \frac{1}{2}G_{cr}G_{rf}\tau_{35s} + \frac{1}{2}G_{rdp}G_{rf}\tau_{55rp} + \frac{1}{2}G_{ro}G_{rf}\tau_{5s71} \Big) \tag{11}$$

and the last term the phase/slice select:

$$b_{sp} = b_{ps} = \gamma^2 \Big(G_{sl}G_{dp}\tau_{12} + G_{sl}G_{cp}\tau_{13} + G_{sl}G_{pdp}\tau_{15rp} + G_{sl}G_{pe}\tau_{16m} + G_{dp}G_{ds}\tau_{22} + \\ + (G_{ds}G_{cp} + G_{ds}G_{cp})\tau_{23} + \frac{1}{2}G_{dp}G'_{sl}\tau_{24} + (\frac{1}{2}G_{dp}G_{rf}\tau_{25s} + G_{ds}G_{pdp}\tau_{25rp}) + G_{pe}G_{ds}\tau_{26m} + \\ + G_{cp}G_{cs}\tau_{33} + \frac{1}{2}G_{cp}G'_{sl}\tau_{34} + (\frac{1}{2}G_{cp}G_{rf} + G_{pdp}G_{cs})\tau_{35s} + G_{pe}G_{cs}\tau_{36m} + G_{pdp}G'_{sl}\tau_{45s} + \\ + G_{pe}G'_{sl}\tau_{46m} + \frac{1}{2}G_{pdp}G_{rf}\tau_{55rp} + \frac{1}{2}G_{pe}G_{rf}\tau_{5s6m} \Big) \tag{12}$$

with refocused gradient pairs gives final expression:

$$b_{sp} = b_{ps} = \gamma^2 \Big(G_{dp}G_{ds}\tau_{22} + (G_{ds}G_{cp} + G_{dp}G_{cs})\tau_{23} + \frac{1}{2}G_{dp}G'_{sl}\tau_{24} + \frac{1}{2}G_{dp}G_{rf}\tau_{25s} + G_{cp}G_{cs}\tau_{33} + \\ + \frac{1}{2}G_{cp}G'_{sl}\tau_{34} + \frac{1}{2}G_{cp}G_{rf}\tau_{35s} + \frac{1}{2}G_{pdp}G_{rf}\tau_{55rp} + \frac{1}{2}G_{pe}G_{rf}\tau_{5s6m} \Big) \tag{13}$$

5 Timing multipliers

$$\tau_{11} = \delta_1^3 \tag{14}$$

$$\tau_{22} = \delta_2^2\left(\Delta_2 - \frac{1}{3}\delta_2\right) + \frac{1}{30}\epsilon_2^3 - \frac{1}{6}\delta_2\epsilon_2^2, \qquad \Delta_2 = t_{22} - t_{21} \tag{15}$$

$$\tau_{23} = \delta_2\delta_3\Delta_3, \qquad \Delta_3 = t_{32} - t_{31} \tag{16}$$

$$\tau_{24} = \delta_2\delta_4\Delta_4, \qquad \Delta_4 = t_{42} - t_{41} \tag{17}$$

$$\tau_{25rp} = \frac{1}{2}\delta_{5rp}\delta_2\Delta_2, \qquad \Delta_2 = t_{22} - t_{21} \tag{18}$$

$$\tau_{25s} = \frac{1}{2}\delta_{5s}\delta_2\Delta_2, \qquad \Delta_2 = t_{22} - t_{21} \tag{19}$$

$$\tau_{33} = \delta_3^2\left(\Delta_3 - \frac{1}{3}\delta_3\right) + \frac{1}{30}\epsilon_3^3 - \frac{1}{6}\delta_3\epsilon_3^2, \qquad \Delta_3 = t_{32} - t_{31} \tag{20}$$

$$\tau_{34} = \delta_3\delta_4\Delta_4, \qquad \Delta_4 = t_{42} - t_{41} \tag{21}$$

$$\tau_{35rp} = \frac{1}{2}\delta_{5rp}\delta_3\Delta_3, \qquad \Delta_3 = t_{32} - t_{31} \tag{22}$$

$$\tau_{35s} = \frac{1}{2}\delta_{5s}\delta_3\Delta_3, \qquad \Delta_3 = t_{32} - t_{31} \tag{23}$$

$$\tau_{44} = \delta_4^2\left(\Delta_4 - \frac{1}{3}\delta_4\right) + \frac{1}{30}\epsilon_4^3 - \frac{1}{6}\delta_4\epsilon_4^2, \qquad \Delta_4 = t_{42} - t_{41} \tag{24}$$

$$\tau_{45rp} = \frac{1}{2}\delta_{5rp}\delta_4\Delta_4, \qquad \Delta_4 = t_{42} - t_{41} \tag{25}$$

$$\tau_{45s} = \frac{1}{2}\delta_{5s}\delta_4\Delta_4, \qquad \Delta_4 = t_{42} - t_{41} \tag{26}$$

$$\tau_{55rp} = \delta_{5rp}^2\left(\Delta_{5rp} - \frac{1}{3}\delta_{5rp}\right) + \frac{1}{30}\epsilon_{5rp}^3 - \frac{1}{6}\delta_{5rp}\epsilon_{5rp}^2, \qquad \Delta_{5rp} = \mathrm{TE} - t_{5rp} \tag{27}$$

$$\tau_{55s} = \delta_{5s}^2\left(\Delta_{5s} - \frac{1}{3}\delta_{5s}\right) + \frac{1}{30}\epsilon_{5s}^3 - \frac{1}{6}\delta_{5s}\epsilon_{5s}^2, \qquad \Delta_{5s} = \mathrm{TE} - t_{5s} \tag{28}$$

$$\tau_{5rp6m} = \epsilon_{5rp}\delta_{5rp}\sum_{m=1}^{\mathrm{res}/2}\left(\Delta_{6m} - \epsilon_6\right), \qquad \Delta_{6m} = \mathrm{TE} - t_{6i} \tag{29}$$

$$\tau_{5s6m} = -\epsilon_{5s}\delta_{rf}\sum_{m=1}^{\mathrm{res}/2}\left(\Delta_{6m} - \epsilon_6\right), \qquad \Delta_{6m} = \mathrm{TE} - t_{6i} \tag{30}$$

$$\tau_{5rp71} = \delta_5\left[\delta_7\left(\Delta_{75} - \frac{1}{4}\delta_7\right) + \frac{1}{12}\epsilon_7^2 - \frac{1}{2}\delta_7\epsilon_7\right], \qquad \Delta_{75} = \Delta_{71} = \mathrm{TE} - t_{71} \tag{31}$$

$$\tau_{5s71} = \delta_5\left[\delta_7\left(\Delta_{75} - \frac{1}{4}\delta_7\right) + \frac{1}{12}\epsilon_7^2 - \frac{1}{2}\delta_7\epsilon_7\right], \qquad \Delta_{75} = \Delta_{71} = \mathrm{TE} - t_{71} \tag{32}$$

$$\tau_{6m6m} = \epsilon_6^2 \sum_{m=1}^{\mathrm{res}/2} \left[(2m-1)\Delta_{6m} - (m\frac{67}{30} - 1)\epsilon_6 \right], \qquad \Delta_{6m} = \mathrm{TE} - t_{6i} \tag{33}$$

$$\tau_{6m71} = \frac{1}{4}\epsilon_6 \left(\delta_7 \Delta_{71} - \frac{1}{60}\epsilon_7^2 \right), \qquad \Delta_{71} = \Delta_{75} = \mathrm{TE} - t_{71} \tag{34}$$

$$\tau_{7171} = \frac{1}{4} \left[\delta_7^2 \left(\Delta_{71} - \frac{1}{3}\delta_7 \right) + \frac{1}{30}\epsilon_7^3 - \frac{1}{2}\delta_7^2 \epsilon_7 \right], \qquad \Delta_{71} = \Delta_{75} = \mathrm{TE} - t_{71} \tag{35}$$

$$\tau_{7m7(m+1)} = \left(\frac{\mathrm{res}}{2} - 1 \right) \left[\frac{1}{12}\delta_7^3 + \frac{1}{60}\epsilon_7 + \frac{1}{4}\delta_7^2 \epsilon_7 - \frac{1}{12}\delta_7 \epsilon_7^2 \right], \qquad \mathrm{res} = 34 \tag{36}$$