Module 6: Lecture 16
Time of Interception; Miss Distance

Keywords. Pure Pursuit, Time of Interception; Miss Distance

7.1.4 Time of interception

Is it possible to obtain the time at which interception occurs when the initial state is within the capture region? For this, let us consider (7.6).

$$\begin{aligned} & (V_R + V_M)^2 + V_\theta^2 = V_T^2 \\ \Rightarrow\quad & V_R^2 + V_M^2 + 2V_M V_R + V_\theta^2 = V_T^2 \\ \Rightarrow\quad & \dot{R}^2 + R\ddot{R} + 2V_M \dot{R} = V_T^2 - V_M^2 \end{aligned} \tag{7.11}$$

Note that we have used the relations $V_R = \dot{R}$ and $V_\theta^2 = R\dot{V}_R$ from (7.9) in the above equation. Integrating (7.11), we obtain

$$R\left(V_R + 2V_M\right) = \left(V_T^2 - V_M^2\right)t + b \tag{7.12}$$

where,

$$b = R_0\left(V_{R0} + 2V_M\right) \tag{7.13}$$

Interception occurs at $t = t_f$ when $R = 0$. Substituting this in (7.12), we get,

$$t_f = \frac{R_0\left(V_{R0} + 2V_M\right)}{V_M^2 - V_T^2} \tag{7.14}$$

Note that when $V_M > V_T$, i.e., when the initial condition lies inside the capture region, we have $t_f > 0$ and finite automatically. However, we should also note that when $V_M < V_T$, even then $t_f > 0$ provided that

$$V_{R0} < -2V_M$$

Does it mean that interception occurs at this value of t_f? Actually this is not so. Observe that the equation for the final time was obtained by setting the LHS of (7.12) to zero, with the implicit assumption that the LHS becomes zero when $R = 0$. But note that the LHS can also become zero when $V_R = -2V_M$. This condition never arises when

$V_T < V_M$ but does occur at point X marked in Figure 7.2 when $V_T > V_M$ and the initial condition lies below the line $V_R = -2V_M$. Thus, when $V_T > V_M$ and $V_{R0} < -2V_M$, (7.14) only gives the time at which the (V_θ, V_R) point crosses the point X.

7.1.5 Lateral acceleration history

In the above analysis we have studied the capture region and the time of interception for a perfectly implemented pursuit guidance law. But what about the latax history of the missile following the pursuit guidance law? For this we need to look at the rate at which the missile velocity vector has to turn in order to satisfy the requirements of pure pursuit. If α_M denotes the velocity direction of the missile then in the case of pure pursuit,

$$\begin{aligned} \alpha_M = \theta \qquad & \Rightarrow \;\; \dot{\alpha_M} = \dot{\theta} \\ & \Rightarrow \;\; V_M \dot{\alpha_M} = V_M \dot{\theta} \;\; \Rightarrow \;\; a_M = V_M \dot{\theta} \\ & \Rightarrow a_M = \frac{V_M V_T}{R} \sin(\alpha_T - \theta) \end{aligned} \tag{7.15}$$

In principle this expression can be used to implement the pure pursuit guidance law, provided that initially the missile points directly towards the target. So, if we know R and θ then we can immediately obtain the lateral acceleration a_M of the missile. To obtain R as a function of θ we must then go back to (7.5) the explicit expression for which is obtained by solving (7.3) as,

$$\begin{aligned} \frac{1}{R}\, dR &= \{\cot(\alpha_T - \theta) - \nu\, \text{cosec}(\alpha_T - \theta)\}\, d\theta \\ \Rightarrow \quad R = K \frac{\left\{\tan\left(\frac{\alpha_T - \theta}{2}\right)\right\}^\nu}{\sin(\alpha_T - \theta)} &= K \frac{\{\sin(\alpha_T - \theta)\}^{\nu - 1}}{\{1 + \cos(\alpha_T - \theta)\}^\nu} \end{aligned} \tag{7.16}$$

where, K is a constant obtained from the initial conditions as,

$$K = R_0 \frac{\sin(\alpha_T - \theta_0)}{\left\{\tan\left(\frac{\alpha_T - \theta_0}{2}\right)\right\}^\nu} = R_0 \frac{\{1 + \cos(\alpha_T - \theta_0)\}^\nu}{\{\sin(\alpha_T - \theta_0)\}^{\nu - 1}} \tag{7.17}$$

If we substitute R from the above into (7.15) then we obtain the latax as a function of θ as:

$$a_M = \frac{V_M V_T}{K} \frac{\sin^2(\alpha_T - \theta)}{\left\{\tan\left(\frac{\alpha_T - \theta}{2}\right)\right\}^\nu} \tag{7.18}$$

This does not give us the latax history directly since we do not have any explicit expression that gives a_M as a function of time. However, there is an indirect relationship through another function that relates R and θ with t as follows: Consider (7.12), from which we get,

$$t = \frac{b - R(V_R + 2V_M)}{V_M^2 - V_T^2} \tag{7.19}$$

Substituting from (7.13) and (7.1),

$$t = \frac{R_0\{V_T \cos(\alpha_T - \theta_0) + V_M\} - R\{V_T \cos(\alpha_T - \theta) + V_M\}}{V_M^2 - V_T^2} \tag{7.20}$$

The equations (7.16) and (7.20) together can be used to obtain R and θ as functions of time. Substituting these values in (7.18) we can obtain a_M as a function of time.

Note that at interception the engagement geometry is a tail-chase one, and so when the initial condition lies inside the capture region and interception is guaranteed, as $t \to t_f$ we have $\theta \to \alpha_T$. So the terminal value of a_M can be obtained from (7.18) by evaluating,

$$\lim_{t \to t_f} a_M = \lim_{\theta \to \alpha_T} a_M \tag{7.21}$$

This yields the following results: As $t \to t_f$, if

$$\begin{aligned} 1 < \nu < 2 \quad & a_M \to 0 \\ \nu = 2 \quad & a_M \to \frac{4V_M V_T}{K} \\ \nu > 2 \quad & a_M \to \infty \end{aligned} \tag{7.22}$$

The above result is illustrated in Figure 7.4.

7.1.6 Miss-distance

In the case when $V_T > V_M$, interception does not take place and the corresponding miss-distance R_{miss}, which occurs at t_{miss}, is obtained as follows:

From Figure 7.2, the point at which miss-distance occurs, we have

$$\begin{aligned} V_R = 0 \qquad & \Rightarrow V_T \cos\left(\alpha_{T_{\text{miss}}} - \theta_{\text{miss}}\right) = V_M \\ & \Rightarrow \alpha_{T_{\text{miss}}} - \theta_{\text{miss}} = \cos^{-1} \nu \end{aligned}$$

From (7.16), we get,

$$R_{\text{miss}} = K \frac{\left\{\tan\left(\frac{\cos^{-1}\nu}{2}\right)\right\}^{\nu}}{\sin(\cos^{-1}\nu)} = K \frac{\{\sin(\cos^{-1}\nu)\}^{\nu-1}}{(1+\nu)^{\nu}} \tag{7.23}$$

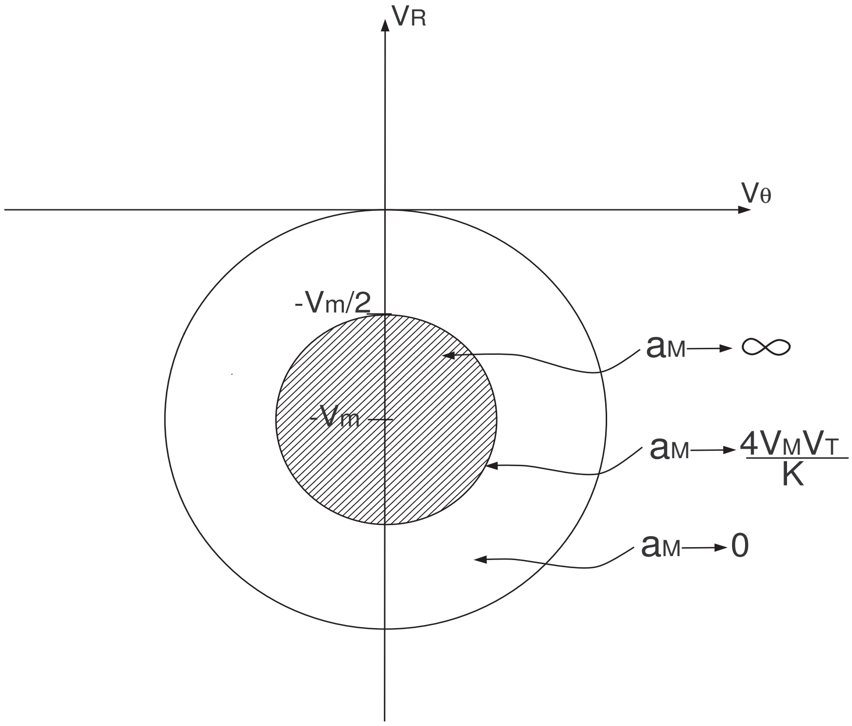


Figure 7.4: *Limits of latax in the pure pursuit guidance law*

where, K is given by (7.17).

Now, from (7.12), we can write

$$2V_M R_{\text{miss}} = \left(V_T^2 - V_M^2\right) t_{\text{miss}} + b \tag{7.24}$$

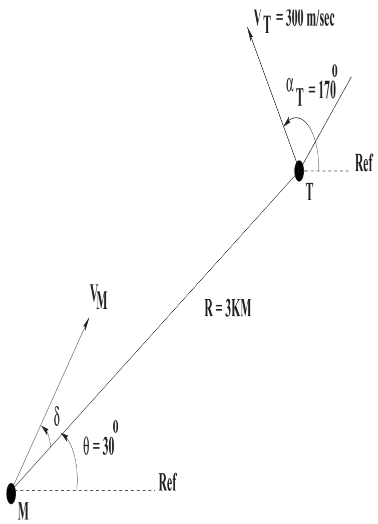
So,

$$t_{\text{miss}} = \frac{2V_M R_{\text{miss}} - b}{V_T^2 - V_M^2} \tag{7.25}$$

Questions

1. Show that the trajectory in the relative velocity space when the missile uses a pure pursuit guidance law right from the beginning of the engagement, is a circle. Determine the direction of movement of a point on this trajectory.

2. Obtain the capture region of the pure pursuit guidance law in the relative velocity space.



3. **PART A: Pure Pursuit,** $\delta = 0^o$**;** $\alpha_T \in [0^o,\ 180^o]$

   A1. Integrate the equations of motion and plot R, θ, V_θ, V_R, and a_M against time for $\nu = 0.9, 1.5, 3$. Note that $\nu = V_M/V_T$.

   A2. Also plot the missile and target trajectories for each case.

A3. Check if interception occurs. If yes, then find t_f. If no then find $t_{\mathbf{miss}}$ and $R_{\mathbf{miss}}$.

A4. Plot (V_θ, V_R) in (V_θ, V_R)-space.

A5. Interpret the simulation results in the light of the analytical results discussed in the notes.

PART B: Implementation *This part will be discussed in the next lectures, but the question is given here for completeness.*

C1. Use the implementation of pure pursuit as

$$a_M = V_M\dot{\theta} - k(\alpha_M - \theta)$$

when, in the above figure, the initial missile velocity vector lags the LOS by 10^o.

Plot R, θ, V_θ, V_R, α_M and a_M against time for $\nu = 0.9, 1.5, 3; k = 1, 5, 10$.

Plot (V_θ, V_R) in (V_θ, V_R)-space.

Plot the missile and target trajectories for each case.