Appendix A
Families of Random Variables

A.1 Discrete Random Variables

Bernoulli (p)

For $0 \le p \le 1$,

$$P_X(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = 1 - p + pe^s$$

$$E[X] = p$$

$$\text{Var}[X] = p(1-p)$$

Binomial (n, p)

For a positive integer n and $0 \le p \le 1$,

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \phi_X(s) = \left(1 - p + pe^s\right)^n$$

$$E[X] = np$$

$$\text{Var}[X] = np(1-p)$$

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Discrete Uniform (k, l)

For integers k and l such that $k < l$,

$$P_X(x) = \begin{cases} 1/(l-k+1) & x = k, k+1, \ldots, l \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{e^{sk} - e^{s(l+1)}}{(l-k+1)(1-e^s)}$$

$$E[X] = \frac{k+l}{2}$$

$$\mathrm{Var}[X] = \frac{(l-k)(l-k+2)}{12}$$

Geometric (p)

For $0 < p \leq 1$,

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$E[X] = 1/p$$

$$\mathrm{Var}[X] = (1-p)/p^2$$

Multinomial

For integer $n > 0$, $p_i \geq 0$ for $i = 1, \ldots, n$, and $p_1 + \cdots + p_n = 1$,

$$P_{X_1,\ldots,X_r}(x_1, \ldots, x_r) = \binom{n}{x_1, \ldots, x_r} p_1^{x_1} \cdots p_r^{x_r}$$

$$E[X_i] = np_i$$

$$\mathrm{Var}[X_i] = np_i(1-p_i)$$

Pascal (k, p)

For positive integer k, and $0 < p < 1$,

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} \qquad \phi_X(s) = \left(\frac{pe^s}{1-(1-p)e^s}\right)^k$$

$$E[X] = k/p$$

$$\mathrm{Var}[X] = k(1-p)/p^2$$

Poisson (α)

For $\alpha > 0$,

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = e^{\alpha(e^s - 1)}$$

$$E[X] = \alpha$$
$$\mathrm{Var}[X] = \alpha$$

Zipf (n, α)

For positive integer $n > 0$ and constant $\alpha \geq 1$,

$$P_X(x) = \begin{cases} \dfrac{c(n, \alpha)}{x^\alpha} & x = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where

$$c(n, \alpha) = \left(\sum_{k=1}^{n} \frac{1}{k^\alpha} \right)^{-1}$$

A.2 Continuous Random Variables

Beta (i, j)

For positive integers i and j, the beta function is defined as

$$\beta(i, j) = \frac{(i + j - 1)!}{(i - 1)!(j - 1)!}$$

For a $\beta(i, j)$ random variable X,

$$f_X(x) = \begin{cases} \beta(i, j) x^{i-1} (1 - x)^{j-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{i}{i + j}$$

$$\mathrm{Var}[X] = \frac{ij}{(i + j)^2 (i + j + 1)}$$

Cauchy (a, b)

For constants $a > 0$ and $-\infty < b < \infty$,

$$f_X(x) = \frac{1}{\pi} \frac{a}{a^2 + (x-b)^2} \qquad \phi_X(s) = e^{bs - a|s|}$$

Note that $E[X]$ is undefined since $\int_{-\infty}^{\infty} x f_X(x)\, dx$ is undefined. Since the PDF is symmetric about $x = b$, the mean can be defined, in the sense of a principal value, to be b.

$$E[X] \equiv b$$
$$\text{Var}[X] = \infty$$

Erlang (n, λ)

For $\lambda > 0$, and a positive integer n,

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^n$$

$$E[X] = n/\lambda$$
$$\text{Var}[X] = n/\lambda^2$$

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$E[X] = 1/\lambda$$
$$\text{Var}[X] = 1/\lambda^2$$

Gamma (a, b)

For $a > -1$ and $b > 0$,

$$f_X(x) = \begin{cases} \dfrac{x^a e^{-x/b}}{a! b^{a+1}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{1}{(1 - bs)^{a+1}}$$

$$E[X] = (a+1)b$$
$$\text{Var}[X] = (a+1)b^2$$

Gaussian (μ, σ)

For constants $\sigma > 0$, $-\infty < \mu < \infty$,

$$f_X(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \qquad \phi_X(s) = e^{s\mu + s^2\sigma^2/2}$$

$$E[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2$$

Laplace (a, b)

For constants $a > 0$ and $-\infty < b < \infty$,

$$f_X(x) = \frac{a}{2} e^{-a|x-b|} \qquad \phi_X(s) = \frac{a^2 e^{bs}}{a^2 - s^2}$$

$$E[X] = b$$

$$\mathrm{Var}[X] = 2/a^2$$

Log-normal (a, b, σ)

For constants $-\infty < a < \infty$, $-\infty < b < \infty$, and $\sigma > 0$,

$$f_X(x) = \begin{cases} \dfrac{e^{-(\ln(x-a)-b)^2/2\sigma^2}}{\sqrt{2\pi}\sigma(x-a)} & x > a \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = a + e^{b+\sigma^2/2}$$

$$\mathrm{Var}[X] = e^{2b+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

Maxwell (a)

For $a > 0$,

$$f_X(x) = \begin{cases} \sqrt{2/\pi}\, a^3 x^2 e^{-a^2x^2/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sqrt{\frac{8}{a^2\pi}}$$

$$\mathrm{Var}[X] = \frac{3\pi - 8}{\pi a^2}$$

Pareto (α, μ)

For $\alpha > 0$ and $\mu > 0$,

$$f_X(x) = \begin{cases} (\alpha/\mu)\,(x/\mu)^{-(\alpha+1)} & x \geq \mu \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{\alpha\mu}{\alpha - 1} \qquad (\alpha > 1)$$

$$\operatorname{Var}[X] = \frac{\alpha\mu^2}{(\alpha - 2)(\alpha - 1)^2} \qquad (\alpha > 2)$$

Rayleigh (a)

For $a > 0$,

$$f_X(x) = \begin{cases} a^2 x e^{-a^2x^2/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sqrt{\frac{\pi}{2a^2}}$$

$$\operatorname{Var}[X] = \frac{2 - \pi/2}{a^2}$$

Uniform (a, b)

For constants $a < b$,

$$f_X(x) = \begin{cases} \dfrac{1}{b - a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{e^{bs} - e^{as}}{s(b - a)}$$

$$E[X] = \frac{a + b}{2}$$

$$\operatorname{Var}[X] = \frac{(b - a)^2}{12}$$

Appendix B
A Few Math Facts

This text assumes that the reader knows a variety of mathematical facts. Often these facts go unstated. For example, we use many properties of limits, derivatives, and integrals. Generally, we have omitted comment or reference to mathematical techniques typically employed by engineering students.

However, when we employ math techniques that a student may have forgotten, the result can be confusion. It becomes difficult to separate the math facts from the probability facts. To decrease the likelihood of this event, we have summarized certain key mathematical facts. In the text, we have noted when we use these facts. If any of these facts are unfamiliar, we encourage the reader to consult with a textbook in that area.

Trigonometric Identities

Math Fact B.1 Half Angle Formulas

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \qquad \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \sin 2A = 2 \sin A \cos A$$

Math Fact B.2 Products of Sinusoids

$$\sin A \sin B = \frac{1}{2}\big[\cos(A-B) - \cos(A+B)\big]$$

$$\cos A \cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$$

$$\sin A \cos B = \frac{1}{2}\big[\sin(A+B) + \sin(A-B)\big]$$

Math Fact B.3 The Euler Formula

The Euler formula $e^{j\theta} = \cos\theta + j\sin\theta$ is the source of the identities

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Sequences and Series

Math Fact B.4 **Finite Geometric Series**
The finite geometric series is

$$\sum_{i=0}^{n} q^i = 1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q}.$$

To see this, multiply left and right sides by $(1 - q)$ to obtain

$$(1 - q)\sum_{i=0}^{n} q^i = (1 - q)(1 + q + q^2 + \cdots + q^n) = 1 - q^{n+1}.$$

Math Fact B.5 **Infinite Geometric Series**
When $|q| < 1$,

$$\sum_{i=0}^{\infty} q^i = \lim_{n\to\infty} \sum_{i=0}^{n} q^i = \lim_{n\to\infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q}.$$

Math Fact B.6

$$\sum_{i=1}^{n} i q^i = \frac{q\,(1 - q^n[1 + n(1 - q)])}{(1 - q)^2}.$$

Math Fact B.7 If $|q| < 1$,

$$\sum_{i=1}^{\infty} i q^i = \frac{q}{(1 - q)^2}.$$

Math Fact B.8

$$\sum_{j=1}^{n} j = \frac{n(n + 1)}{2}.$$

Math Fact B.9

$$\sum_{j=1}^{n} j^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Calculus

Math Fact B.10 Integration by Parts

The integration by parts formula is

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du.$$

Math Fact B.11 Gamma Function

The gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

If $z = n$, a positive integer, then $\Gamma(n) = (n-1)!$. Also note that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \sqrt{\pi}/2$, and $\Gamma(5/2) = 3\sqrt{\pi}/4$.

Math Fact B.12 Leibniz's Rule

The function

$$R(\alpha) = \int_{a(\alpha)}^{b(\alpha)} r(\alpha, x)\,dx$$

has derivative

$$\frac{dR(\alpha)}{d\alpha} = -r\big(\alpha, a(\alpha)\big)\frac{da(\alpha)}{d\alpha} + r\big(\alpha, b(\alpha)\big)\frac{db(\alpha)}{d\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial r(\alpha, x)}{\partial\alpha}\,dx.$$

In the special case when $a(\alpha) = a$ and $b(\alpha) = b$ are constants,

$$R(\alpha) = \int_a^b r(\alpha, x)\,dx,$$

and Leibniz's rule simplifies to

$$\frac{dR(\alpha)}{d\alpha} = \int_a^b \frac{\partial r(\alpha, x)}{\partial\alpha}\,dx.$$

Math Fact B.13 Change-of-Variable Theorem

Let $\mathbf{x} = T(\mathbf{y})$ be a continuously differentiable transformation from $\mathcal{U}^n$ to $\mathcal{R}^n$. Let R be a set in $\mathcal{U}^n$ having a boundary consisting of finitely many smooth sets. Suppose that R and its boundary are contained in the interior of the domain of T, T is one-to-one of R, and $\det(()T')$, the Jacobian determinant of T, is nonzero on R. Then, if $f(\mathbf{x})$ is bounded and continuous on $T(R)$,

$$\int_{T(R)} f(\mathbf{x})dV_{\mathbf{x}} = \int_R f(T(\mathbf{y}))|\det(T)'|\,dV_{\mathbf{y}}.$$

Vectors and Matrices

Math Fact B.14 Vector/Matrix Definitions

(a) Vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\mathbf{x}'\mathbf{y} = 0$.

(b) A number λ is an *eigenvalue* of a matrix $\mathbf{A}$ if there exists a vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is an *eigenvector* of matrix $\mathbf{A}$.

(c) A matrix $\mathbf{A}$ is *symmetric* if $\mathbf{A} = \mathbf{A}'$.

(d) A square matrix $\mathbf{A}$ is *unitary* if $\mathbf{A}'\mathbf{A}$ equals the identity matrix $\mathbf{I}$.

(e) A real symmetric matrix $\mathbf{A}$ is *positive definite* if $\mathbf{x}'\mathbf{Ax} > 0$ for every nonzero vector $\mathbf{x}$.

(f) A real symmetric matrix $\mathbf{A}$ is *positive semidefinite* if $\mathbf{x}'\mathbf{Ax} \geq 0$ for every nonzero vector $\mathbf{x}$.

(g) A set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is *orthonormal* if $\mathbf{x}'_i\mathbf{x}_j = 1$ if $i = j$ and otherwise equals zero.

(h) A matrix $\mathbf{U}$ is *unitary* if its columns $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ are orthonormal.

Math Fact B.15 Real Symmetric Matrices

A real symmetric matrix $\mathbf{A}$ has the following properties:

(a) All eigenvalues of $\mathbf{A}$ are real.

(b) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors of $\mathbf{A}$ corresponding to eigenvalues $\lambda_1 \neq \lambda_2$, then $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal vectors.

(c) $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{UDU}'$ where $\mathbf{D}$ is a diagonal matrix and $\mathbf{U}$ is a unitary matrix with columns that are n orthonormal eigenvectors of $\mathbf{A}$.

Math Fact B.16 Positive Definite Matrices

For a real symmetric matrix $\mathbf{A}$, the following statements are equivalent:

(a) $\mathbf{A}$ is a *positive definite* matrix.

(b) $\mathbf{x}'\mathbf{Ax} > 0$ for all nonzero vectors $\mathbf{x}$.

(c) Each eigenvalue λ of $\mathbf{A}$ satisfies $\lambda > 0$.

(d) There exists a nonsingular matrix $\mathbf{W}$ such that $\mathbf{A} = \mathbf{WW}'$.

Math Fact B.17 Positive Semidefinite Matrices

For a real symmetric matrix $\mathbf{A}$, the following statements are equivalent:

(a) $\mathbf{A}$ is a positive semi-definite matrix.

(b) $\mathbf{x}'\mathbf{Ax} \geq 0$ for all vectors $\mathbf{x}$.

(c) Each eigenvalue λ of $\mathbf{A}$ satisfies $\lambda \geq 0$.

(d) There exists a matrix $\mathbf{W}$ such that $\mathbf{A} = \mathbf{WW}'$.