Sets, Functions and Relations

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Abstract

These notes provide a quick background on Sets, Functions and Relations.

1 Introduction

2 Set Theory

A flock of sheep, a school of fish, etc. are examples of a collection of things. A set is, therefore a *collection of objects*. Like a point is not defined in geometry, we will not *define* sets. The development of Set Theory as a universal language for mathematics was developed in the early 19^{th} century, initiated by Georg Cantor. Just as Euclidean geometry is developed from a small list of *axioms*, we can similarly develop Set Theory from a small collection of *self evident* axioms. However, doing this out of scope of these notes and we will only allude to the axiomatic notions. A set contains a collection of *members*. There is apriori no restriction on the kind of members a set can contain. Therefore, a set can contain other sets as members. In mathematics, we frequently encounter sets that contain other sets as members. For example, a line is a collection of points. The set of all lines in the plane is a natural collection of sets of sets.

The principal concept of set theory is that of *belonging*, denoted $x \in A$, for an element x to belong to set A. We usually denote the members of a set by small letters, and use capital letters for sets. However, sometimes a set may contain other sets, and we may occasionally use gaudier symbols such as $\mathcal{S}$ to denote that $\mathcal{S}$ contains sets as members. A primitive relation between sets is that of equality. We say $A = B$ if they contain the same collection of elements, or $A \neq B$ to denote that the members of A are different from the members of B. In axiomatic set theory, this is captured by the **Axiom of extension**, which says that two sets are equal if and only if they contain the same elements.

$$\textbf{Axiom of Extension: } \forall A, B\ \forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A = B)$$

If every element of a set A are also members of a set B, then we say that A is a *subset* of B, denoted $A \subseteq B$, or that $B \supseteq A$ (B is a *superset* of A). It follows that $A \subseteq A$ and if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. We will explore these notions when we consider relations in Section 3. If A and B are such that $A \subseteq B$ and $B \subseteq A$, then they have the same members, and by the axiom of extension, it follows that $A = B$. In other words, if A and B are sets, then a necessary and sufficient condition that $A = B$ is that $A \subseteq B$ and $B \subseteq A$. Correspondingly, all proofs that $A = B$

will involve showing $A \subseteq B$ and $B \subseteq A$. To show that $A \neq B$ therefore, it is sufficient to establish the existence of an element $x \in A$ s.t. $x \notin B$.

Observe that $\in$ and $\subseteq$ are conceptually very different things. For any set $A \subseteq A$ is always true. What about $A \in A$? We will soon see that allowing $A \subseteq A$ leads to substantial difficulties.

In defining subsets, we usually specify it explicitly as $A = \{a, b, c\}$, or implicitly, for example $B = \{x \in A : S(x)\}$, where $S(x)$ is a condition (for example, we could let$S(x)$ could be a sentence in first-order logic). We can use this to establish that there is an *empty set*. Let $S(x)$ be a sentence that is universally false. Thus, $B = \{x : S(x)\}$. Since $S(x)$ is universally false, B contains no elements. We use $\emptyset$ to denote a set containing no elements. We now show that the empty set is unique.

Proposition 1. *If ϕ and ϕ' are empty sets, then $\emptyset = \emptyset'$.*

Proof. Since $\forall x(x \in \emptyset \Leftrightarrow x \in \emptyset')$, it follows that $\phi = \phi'$. □

We can now argue that $\emptyset \subset A$ for any set A.

Proposition 2. *$\forall A, \emptyset \subseteq A$.*

Proof. $\forall x(x \in \emptyset \rightarrow x \in A)$ is true. Thus, $\emptyset \subseteq A$. □

We should immediately highlight the fact that $\emptyset$ is distinct from $\{\emptyset\}$. The former is the empty set, but the latter set is not empty. It contains a single element, namely the empty set! The difference is akin to having nothing to eat versus having an empty tiffin box. Similarly, $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$ are all distinct. Since $\{\emptyset\} \in \{\{\emptyset\}\}$, but $\{\emptyset\} \notin \emptyset$, it follows that $\emptyset \neq \{\emptyset\}$. However, $\emptyset \subseteq \{\emptyset\}$. We can similarly show that $\{\emptyset\} \neq \{\{\emptyset\}\}$.

2.1 Operations on Sets

We use the notation $A \cup B$ to denote the elements contained either in A or in B, and $A \cap B$ to denote the collection of elements that belong to both sets A and B. Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Some further notation: We use $A \setminus B$ to denote the collection of elements that belong to A but not to B. Thus,

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

where we use $x \notin B$ to denote the sentence $\neg(x \in B)$. One last piece of notation. We denote the *symmetric difference* between two sets, i.e., $(A \setminus B) \cup (B \setminus A)$ by $A \Delta B$. Thus,

$$A \Delta B = \{x : x \in (A \setminus B) \cup (B \setminus A)\}$$

Here are some easily proved identities of unions and intersections:

$$
\begin{aligned}
A \cup \emptyset &= A \\
A \cup B &= B \cup A \text{ (commutativity)} \\
A \cup (B \cup C) &= (A \cup B) \cup C \text{ (associativity)} \\
A \cup A &= A \text{ (idempotence)} \\
A \subset B &\Leftrightarrow A \cap B = B
\end{aligned}
$$

$$
\begin{aligned}
A \cap \emptyset &= \emptyset \\
A \cap B &= B \cap A \text{ (commutativity)} \\
A \cap (B \cap C) &= (A \cap B) \cap C \text{ (associativity)} \\
A \cap A &= A \text{ (idempotence)} \\
A \subset B &\Leftrightarrow A \cap B = A
\end{aligned}
$$

Exercise 1

Prove the identities about intersections and unions defined above.

We now show two useful identities between unions and intersections, called the *distributive laws*

$$
\begin{aligned}
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C)
\end{aligned}
$$

Exercise 2

Prove the distributive laws defined above.

Exercise 3

Show that a necessary and sufficient condition that $(A \cap B) \cup C = A \cap (B \cup C)$ is that $C \subset A$. Observe that this has nothing to do with the set B.

Consider a collection of sets that are all subsets of a *universal set* $\mathcal{U}$. We now define the *complementation* of a set denoted A^c and sometimes denoted $\overline{A}$ or A', which is the set $\mathcal{U} \setminus A$. Thus,

$$A^c = \mathcal{U} \setminus A$$

Here are some basic facts about complementation:

$$\begin{aligned}(A^c)^c &= A\\ \emptyset^c &= \mathcal{U},\\ \mathcal{U}^c &= \emptyset\\ A\cap A^c &= \emptyset\\ A\cup A^c &= \mathcal{U}\\ A\subset B &\Leftrightarrow B^c\subset A^c\end{aligned}$$

We can now define what are called *DeMorgan's laws*

$$\begin{aligned}(A\cup B)^c &= A^c\cap B^c\\ (A\cap B)^c &= A^c\cup B^c\end{aligned}$$

Exercise 4

Show the following:

$$\begin{aligned}A\setminus B &= A\cap B^c\\ A\setminus(A\setminus B) &= A\cap B\\ A\cap(B\setminus C) &= (A\cap B)\setminus(A\cap C)\\ A\cap B &\subset (A\cap C)\cup(B\cap C^c)\\ (A\cup C)\cap(B\cup C^c) &\subset A\cup B\\ A\subset B &\Leftrightarrow A\setminus B=\emptyset\end{aligned}$$

2.2 Paradoxes in Set Theory

A little after the development of Set Theory by Cantor and others, Gottlob Frege, a German logician tried to develop a *logical foundation of mathematics* - the language of first-order logic that we studied is broadly due to Frege. As Frege was completing his book, the *Grundlagen der Arithmetik*, he received an inocuous letter from a young British logician who, while admiring the clarity of Frege's work highlighted however, a niggling problem - as stated earlier, we allowed sets to contain other sets, and in doing so, we did not preclude a set from containing itself as a member. Russell discovered by being so liberal with the notion of a set leads to paradoxes in set theory. Since Frege hoped to develop tools to put all of mathematics on a sound foundation, Russell's letter essentially laid waste Frege's life-work. While he wrote a note about this paradox in his book, he never really recovered from this blow. He became a bitter old man, anti-democratic, and an anti-semite. This issue is in the too liberal notion of defining a set. If we define a set as $\{x : x \text{ satisfies } P(x)\}$, where $P(x)$ is *any* property of x. Anyhow, here is the paradox.

Call a set *extraordinary* if it contains itself as a member. Otherwise call a set *ordinary*. Now, consider a set A consisting of all the *ordinary sets*. Is A ordinary or extraordinary? Suppose A

is ordinary. Then, A must contain itself as A contains all sets that do not contain themselves as a member. But, this implies that A contains itself and is therefore extraordinary. On the other hand, suppose A were extraordinary, then it must contain itself. But, contradicts the condition that A contains just the ordinary sets.

More colorfully, the paradox is usually presented as the *Barber's paradox*. In a certain town, a *barber* is one who shaves those and all those who do not shave themselves. Now, what about the barber himself? If he shaves himself, he is not a barber as a barber is one shaves those that do not shave themselves. On the other hand, if he does not shave himself, he must shave himself as he shaves all those who do not shave themselves.

Russell and Whitehead resolved to develop a system that does not succumb to this paradox. They developed the notion of *class*, where a set belongs to a class and can only contain elements that belong to a class below. This avoids the paradox of Russell but it fell out of favor in the development of mathematics as it was deemed too unwieldy. However, this notion of classes are related to *types* as in programming languages, and have connections to functional programming.

A second fix came from Zermelo and Frankel who developed an axiomatic set theory. Today, all of mathematics is based on the Zermelo-Frankel axiom system, called **ZF**. There is an extension of the ZF system, namely the *axiom of choice*, and most of mathematics today is built on **ZFC**, or the Zermelo-Frankel axiom system with the axiom of choice. We will briefly mention the axiom of choice towards the end of this section.

3 Relations

As stated in the previous section, the elements of a set are enclosed in *curly braces*, e.g., $A = \{1, 2, 3\}$. The curly braces denote that there is no *ordering* on the elements in a set; in other words, the elements of a set form an *unordered collection*. Hence, $\{1, 2, 3\}, \{3, 2, 1\}, \{1, 3, 2\}$, etc. denote the same set. On the other hand, an *ordered* collection of elements is denoted by *paranthesis*. Thus, $(1, 2, 3)$ is distinct from $(3, 2, 1)$ or $(1, 3, 2)$.

Definition 1 (Cartesian Product)

Let A and B be sets. We define the *cartesian product* of sets A and B to be all the *ordered pairs* of elements (a, b) where $a \in A$ and $b \in B$. In other words,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Simiarly, Given sets $A_1, A_2, \ldots, A_k$, the cartesian product of these sets is the set

$$A_1 \times \ldots \times A_k = \{(a_1, \ldots, a_k) : a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$$

If either $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

The best example of a cartesian product is of course, the cartesian plane $\mathbb{R} \times \mathbb{R}$. We use A^2 to denote $A \times A$.

Exercise 5

$$(A \cup B) \times X = (A \cup X) \times (B \cup X)$$
$$(A \cap B) \times X = (A \cap X) \times (B \cap X)$$
$$(A \setminus B) \times X = (A \times X) \setminus (B \times X)$$

Using ordered pairs, we can define the notion of a relation. Given two sets A and B, a relation R between A and B is a subset of $A \times B$, i.e., $R \subseteq A \times B$. A relation is thus, just a collection of ordered pairs of elements where the first element is from A, and the second, from B. Since R is a subset, we could define $R = \{(a, b) : a \in A, b \in B \text{ and } P(a, b)\}$, where $P(a, b)$ is a predicate with a, b being free variables.

Definition 2 (Arity)

If $R \subseteq A_1 \times \ldots A_k$, then R is said to be of *arity* k, or that R is a k-ary relation.

A 0-ary relation or a *nullary* relation is a relation with no attributes. There are only two 0-ary relations - the empty relation, and the relation that contains the unique 0-tuple. If $R \subseteq A$ for a set A, then R is said to be a *unary* relation. For example, $R = \{n \in \mathbb{N} : n \text{ is prime}\}$. If $R \subseteq A \times B$, R is called a *binary relation*. Familiar binary relations are the usual arithmetic operations: $+, -, \times, \div$. For example, $\mathbb{Z} + \mathbb{Z} \subseteq \mathbb{Z}$ is the binary relation of addition, but this is equal to $\mathbb{Z}$. We obtain an interesting relation if we restrict our attention to pairs of numbers whose sum is positive. For example $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b > 0\}$. If $R \subseteq X \times Y \times Z$, then R is said to be *ternary* relation. An example of a ternary relation is *betweenness*. For example, $R = \{(a, b, c) : a \in \mathbb{N}, b \in \mathbb{N}, c \in \mathbb{N} \text{ and } a < b < c\}$.

If $(a, b) \in R$, we denote it by aRb. For a relation, we define the *domain* of a relation by $\{x : \exists\, y\ Ry\}$ and the *range* by $\{y : \exists\, x\ xRy\}$.

3.1 Representing relations

We start with a representation of binary relations, which are the most common. Let $R \subseteq A \times B$. One way to represent binary relations is via an *adjacency matrix* whose rows are indexed by A and columns are indexed by B. The $(i, j)^{th}$ entry of the matrix is 1 if and only if iRj. Otherwise, the $(i, j)^{th}$ entry is 0. For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

represents the relation $R = \{(1, 2), (2, 4), (3, 2), (4, 2), (4, 4)\}$, where $A, B = \{1, 2, 3, 4\}$. The same relation can also be represented as a *directed graph* as shown in Figure 1, if $A = B$.

A third representation is to place the elements of set A on the left vertically, and the elements of set B on the right vertically, and add a directed edge from an element $a \in A$ to an element $b \in B$ if and only if $(a, b) \in R$. Figure 2 shows the same relation above in this representation.

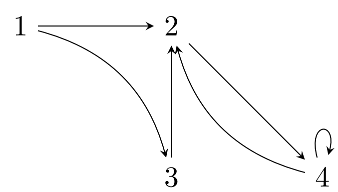


Figure 1: The relation $R \subseteq A \times A$, where $A = \{1, 2, 3, 4\}$ above represented as a directed graph (figure produced by chat-gpt3.5)

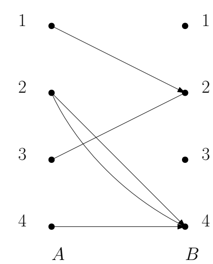


Figure 2: A representation of the relation above as a bipartite graph

3.2 Composition and Inverse

Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two relations. The *composition* of the relations R and S is the relation $T \subseteq X \times Z$, where $xTz \Leftrightarrow \exists y \in Y : xRy \wedge yRz$. The relation T is denoted $T = R \circ S$. Figure 3 shows the composition of two relations.

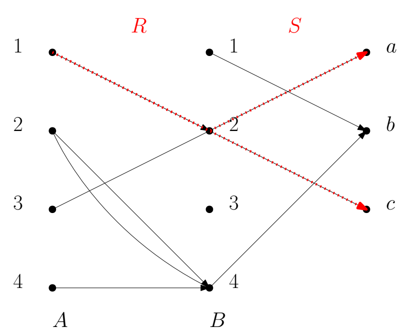


Figure 3: Composition of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$. The figure shows that $(1, a)$ and $(1, c)$ are in T.

For example, in the relation above $R = \{(1, 2), (2, 4), (3, 2), (4, 2), (4, 4)\}$ and $S = \{(1, b), (2, a), (2, c), (4, b)\}$. Thus, $T = R \circ S = \{(1, a), (1, c), (2, b), (3, a), (3, c), (4, a), (4, c), (4, b)\}$.

Exercise 6

Construct an example of relation $R \subseteq X \times X$ and $S \subseteq X \times X$ s.t. $R \circ S \neq S \circ R$.

Definition 3 (Inverse)

For a relation $R \subseteq (X, Y)$, the *inverse relation* is the relation $R^{-1} \subseteq (Y, X)$, where $R^{-1} = \{(y, x) : (x, y) \in R\}$.

3.3 Properties of relations

The most important kind of relation is a relation on $X \times X$ for a set X. We classify these relations into four types.

For a relation $R \subseteq A \times A$, we say that R is

1. *reflexive* if $\forall a \in A, aRa$
2. *symmetric* if $\forall a, b \in A$, $aRb \Leftrightarrow bRa$, i.e., $R = R^{-1}$.
3. *transitive* if $\forall\ a, b, c \in A, aRb \wedge bRc \to aRc$

Exercise 7

For each of the properties above, find a natural relation that does not satisfy one of the properties, but satisfies the other two.

A relation is *anti-symmetric* if $\forall a, b \in A, aRb \Leftrightarrow \neg(bRa)$. We sometimes call such a relation *weakly anti-symmetric* while a *strongly anti-symmetric relation* will also disallow reflexivity, i.e., $\nexists x : xRx$.

Let $\Delta_X = \{(x, x) : x \in X\}$. This denotes the smallest reflexive relation on a set X. The relation Δ_X is called the *diagonal* (on the set X). An alternative way to state that R is anti-symmetric is that $R \cap R^{-1} = \Delta_X$.

Definition 4 (Equivalence Relation)

A relation $R \subseteq X \times X$ that is reflexive, symmetric and transitive is called an equivalence relation.

Definition 5 (Order Relation)

A relation R on X is called an *ordering* or an *order relation* if R is reflexive, anti-symmetric and transitive.

Definition 6 (Linear Order)

A relation R is a linear order if R is an ordering relation and moreover, $R \cup R^{-1} = X$, i.e., for each pair $x, y \in X$ either xRy or yRx.

Example 1

Consider the following relations defined on $\mathbb{N}$.

1. $R = \{(x, y) : x - y \text{ is an even number}\}$ is an equivalence relation, but not an ordering.
2. $R = \{(x, y) : x \text{ divides } y\}$ is an ordering, but not an equivalence relation nor a linear ordering.
3. $R = \{(x, y) : x \leq y\}$ is a linear ordering, and thus an ordering; but not an equivalence relation.

Exercise 8

For a set X what is the smallest and largest equivalence relation on $X \times X$?

There is an intimate connection between equivalence relations and *partitions*. Given a set X, a partition is a collection of disjoint non-empty subsets of X whose union is X. An equivalence relation defines a partition of the elements of X. Conversely, every partition defines an equivalence relation on the elements in X. If R is an equivalence relation, the set of elements in the same partition as an element x is called the *equivalence class* defined by x, denoted $R[x]$. That is, $R[x] = \{y \in X : xRy\}$. We now prove this in the following proposition.

Proposition 3. *For any equivalence relation R on X we have:*

1. *$R[x]$ is non-empty for any $x \in X$.*
2. *For any two elements $x, y \in X$, either $R[x] = R[y]$ or $R[x] \cap R[y] = \emptyset$.*
3. *The equivalence classes determine R uniquely.*

Proof. 1. The set $R[x]$ contains x and is therefore non-empty.

2. (a) Let $x, y \in X$. We distinguish two cases depending on whether xRy. Suppose xRy, then for any $z \in R[x]$,

$$\begin{aligned} &xRz && [\because z \in R[x]] \\ &zRx && [\because \text{ symmetry}] \\ &zRy && [\because \text{transitivity}] \\ &yRz && [\because \text{ symmetry}] \end{aligned}$$

Thus, $R[x] \subseteq R[y]$. By a symmetric argument, $R[y] \subseteq R[x]$.

(b) Suppose $\neg(xRy)$. We show $R[x] \cap R[y] = \emptyset$ via contradiction. Suppose there exists $z \in R[x] \cap R[y]$. Then,

$$\begin{aligned} &xRz && [\because z \in R[x]] \\ &yRz && [\because z \in R[y]] \\ &zRy && [\because \text{ symmetry}] \\ &xRy && [\because \text{ transitivity}] \end{aligned}$$

3. The relation is R is uniquely defined by xRy iff $y \in R[x]$.

□

Exercise 9

Prove that a relation R on X satisfies $R \circ R^{-1} = \Delta_X$ if and only if R is reflexive and anti-symmetric.

Exercise 10

Let R and S be arbitrary equivalence(order) relations on a set X. Which of the following are equivalence(order) relations? Justify your answer.

1. $R \cap S$
2. $R \cup S$
3. $R \setminus S$
4. $R \circ S$.

4 Functions

In school we have learned about functions that are described *albegraically*, for example $y = x^2$, etc. that can be graphed, or more generally $z = x^2 + y^2$ that can be graphed with some effort on a computer. However, this will not do. We need a more general notion of function. For us, given two sets X and Y, a function associates for each element $x \in X$ a *unique* element $y \in Y$.

Definition 7 (Function)

- If X and Y are sets, a *function* from X to Y associates to each element $x \in X$ a *unique* element $y \in Y$. The set X is called the *domain* and the set Y is called the *co-domain* of the function.
- A function is denoted $f : X \to Y$ to denote that f takes each element of X to a unique element of Y. We also write $f(x) = y$ for $x \in X$ and $y \in Y$ to denote that the function takes the element $x \in X$ to the element $y \in Y$.

- Sometimes we also write $f : x \mapsto y$ to denote the fact that f takes an element of x to an element of y.
- The set X is called the *domain* of f, and Y is called the *co-domain* of f. The set of elements in Y to which some element of X maps to it is called the *image* of f, denoted $im(f)$ is defined as follows: $im(f) = \{y \in Y : \exists\, x \in X : f(x) = y\}$ (The term *range* is sometimes used to refer to either the co-domain or the image of f. The definition is not standard).

Note that we assume that *each* element of X is assigned some element in Y, and not just a subset. If the assignment is not specified for some elements in X, such a function is called a *partial function*, but we won't deal with the notion of partial functions in this course. Note that this definition of function works equally well for finite as well as infinite sets. The important thing is the uniqueness of the element in y associated to each element in Y. The way to remember this is that in Figure 8 each element $x \in X$ has *exactly* one arrow emanating from it. Hence, a function is a special case of a relation, but not the other way around.

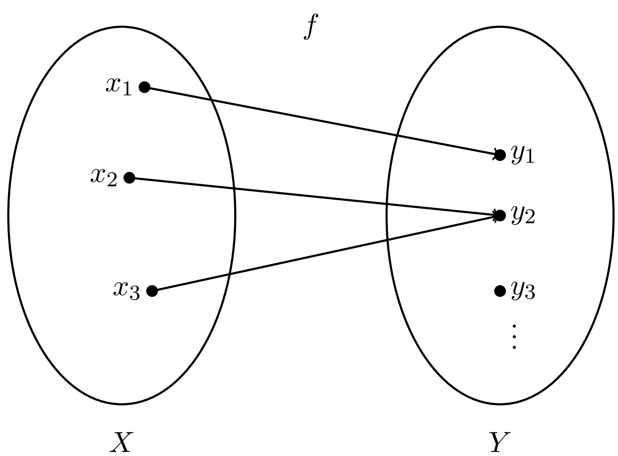


Figure 4: An pictorial description of a function showing that the elements (The tikz code was generated using chat-gpt 3.5 with minor modifications)

Exercise 11

Which of the following are definitions of functions? For all the examples below, $Y = \mathbb{R}$, the real line.

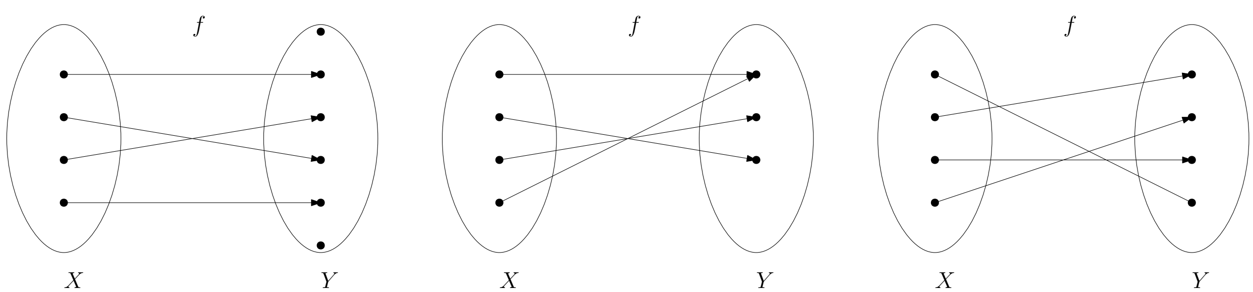
1. $f(x) = x^2 + 2$, $X = \mathbb{R}$.
2. $f(x) = sin(x)$, $X = \mathbb{R}$
3. $f(x) = \sqrt{x}$, $X = \mathbb{R}_{\geq 0}$, the non-negative real numbers.

4. $f(x) = \sqrt{-x^2+1}$, $X = [-1, 1]$, the real interval between -1 and 1.

4.1 Classification of functions

We broadly classify functions based on the *number of arrows pointing into elements in Y*. The table below gives the classification.

f	Name	Alternate name	Figure
$\forall y \in Y, \lvert\{x \in X : f(x) = y\}\rvert \leq 1$ Atmost 1 arrow into y $\forall$ $y \in Y$	Injection	$1-1$ (one-to-one)	9a
$\forall y \in Y, \lvert\{x \in X : f(x) = y\}\rvert \geq 1$ Atleast 1 arrow to each $y \in Y$	Surjection	onto	9b
$\forall y \in Y, \lvert\{x \in X : f(x) = y\}\rvert = 1$ $\forall y \in Y$, exactly one $x \in X : f(x) = y$	Bijection	$1-1$ and onto	9c



(a) Injective or $1-1$ fn. (b) Surjective or Onto function (c) Bijective or $1-1$ and onto fn.

Figure 5: Classification of functions

Definition 8 (Identity function)

A bijection $X \to X$ is called the *identity* function if it takes each element x to itself, i.e., $\forall x \in X f(x) = x$. We denote this function by $id : X \to X$.

Definition 9 (Inverse function)

If $f : X \to Y$ is a bijection, then we can define a function, namely the *inverse function* $f^{-1} : Y \to X$, where $f^{-1}(y)$ is the unique x s.t. $f(x) = y$. We can visualize the inverse function by reversing the arrows in the figure for the bijection. Note that in order to define an inverse function, the original function f must be a bijection. Further, if f is a bijection, then so is f^{-1}.

Since the existence of a bijection from X to Y implies an bijection from Y to X, we sometimes say that there is a bijection *between* X and Y ignoring the direction of the function. Another function that will be very useful for us is the *characteristic* function, denoted $\chi : X \to \{0, 1\}$.

Definition 10 (Characteristic function)

A function $\chi : X \to \{0, 1\}$ is called a characteristic function.

We use characteristic functions to denote the elements in any subset of X, namely the elements mapped to 1. If $X = \{1, 2, 3\}$, then $\chi(1) = 0, \chi(2) = 1, \chi(3) = 1$ denotes the subset $\{2, 3\}$.

Exercise 12

Let X and Y be finite sets.

1. If there is a surjection from X to Y what can we say about $|X|$ and $|Y|$?
2. If there is an injection from X to Y, what can we say about $|X|$ and $|Y|$?
3. If there is an injection from X to Y, what can we say about $|X|$ and $|Y|$?
4. If there is a bijection from X to Y, what can we say about $|X|$ and $|Y|$?

4.2 Function Composition

Let $f : X \to Y$ and $g : Y \to Z$. The composition of the functions f and g is the function $h : X \to Z$, where $h(x) = g(f(x))$. h is denoted $(g \circ f)(x)$ for each $x \in X$. Note that the notation is the *opposite* of that for composition of relations. This is unfortunate, but this is the convention used in most textbooks and we will follow this.

Exercise 13

Show that the composition of functions is associative but not commutative.

Exercise 14

1. If $g \circ f$ is an onto function, does g have to be onto? Does f have to be onto?
2. If $g \circ f$ is a one-to-one function, does g have to be one-to-one? Does f have to be one-to-one?

Exercise 15

Prove that the following two statements about a function $f : X \to Y$ are equivalent (X and Y are some arbitrary sets):

1. f is one-to-one.
2. For any set Z and any two distinct functions $g_1 : Z \to X$ and $g_2 : Z \to X$, the composed functions $f \circ g_1$ and $f \circ g_2$ are also distinct

5 Infinite Sets

In the previous exercise, you have probably observed that there is a bijection between two finite sets X and Y only if their cardinalities are equal. How do we measure the cardinality of infinite sets? This may sound like a silly question. Of course, an infinite set has cardinality ∞. But, infinity is not a number. However, does that mean that any two infinite sets have the same cardinality? Let us re-define what it means for two sets to have the same cardinality.

Two sets X, Y have the same cardinality if and only if there is a bijection between X and Y. If there is an injection from X to Y, then we say that $|X| \leq |Y|$. The existence of a bijection $f : X \to Y$ implies an injective function f from X to Y, and an injective function f^{-1} from Y to X (we will soon see that the existence of an injective function from X to Y, and an injective function from Y to X implies a bijection between X and Y). Observe that we are now making a different statement. We are saying that to show that X and Y have the same cardinality, it is sufficient to show the existence of a bijection between X and Y. This idea of establishing a bijection to show two sets have equal cardinality probably predates the notion of a number. Imagine two ancient pastorialists who measure their wealth by the number of goats they have (goats and dogs were the first animals to be domesticated), but they still don't have a concept of a number. So here's what they could do: They pair up their goats - one goat of pastorialist A with one goat of pastorialist B. If A has unpaired goats remaining after this exercise, it follows that A has more goats, and is hence richer than B. On the other hand, if B has more goats, it implies B is richer. But, neither has any unpaired goats then they must be equally wealthy.

Our definition therefore, of establishing a bijection to show two sets are equal is somehow more fundamental than counting the number of elements - which we cannot hope to do anyway for infinite sets. These notions seem trivial for finite sets, but when we deal with infinite sets, we'll see that we obtain startling and very counter-intuitive results. These results were first studied by Georg Cantor, who developed Set Theory, but the leading mathematicans of his day did not accept these notions.

Let us start with a rather surprising result. Let $\mathbb{N} = \{1, 2, \ldots, \}$, the set of natural numbers. Let $\mathbb{E} = \{2, 4, 6, \ldots\}$, the even numbers and $\mathbb{O} = \{1, 3, 5, \ldots\}$ the odd numbers. It is clear that $\mathbb{E} \subset \mathbb{N}$, and $\mathbb{O} \subset \mathbb{N}$. We will nevertheless establish a bijection between $\mathbb{E}$ and $\mathbb{N}$!

Consider the function $f : \mathbb{N} \to \mathbb{E}$, where $f : k \mapsto 2k$. That f is a bijection follows immediately: for each element $e \in \mathbb{E}$, there is a unique element $e/2 \in \mathbb{N}$ that maps to it. Therefore $|\mathbb{E}| = |\mathbb{N}|$. Figure 10 shows this mapping pictorially.

A couple of points are in order. Observe that while $\mathbb{E}$ is a proper subset of $\mathbb{N}$, we could still obtain a bijection from $\mathbb{N} \to \mathbb{E}$. This is certainly not possible if the sets are finite. Therefore, if a

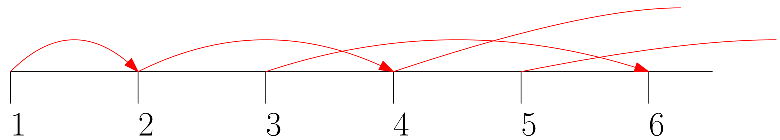


Figure 6: A bijection from $\mathbb{N}$ to $\mathbb{E}$.

set is infinite, there could exist a bijection to a proper subset.

Exercise 16

Show that if $B \subset A$ and there is a bijection $A \to B$, then B, and therefore A is necessarily infinite.

Solution: Suppose A were finite. Let $|A| = n$. Since $B \subset A$, there is an element in A that is not in B. Thus, $|B| < |A|$, but we know that if there is a bijection from A to B, we must have that $|A| = |B|$. Therefore, A cannot be finite.

We use this as the definition of infinite sets. A set is *infinite* if there is a bijection from a proper subset of the set to itself.

Exercise 17

Show that $|\mathbb{O}| = |\mathbb{N}|$.

Next, we show that the set $\mathbb{Z}$, the integers has the same cardinality as the natural numbers. We do this by establishing a bijection from $\mathbb{Z}$ to $\mathbb{N}$. To start with, instead of establishing a bijection from $\mathbb{Z}$ to $\mathbb{N}$, we establish a bijection from $\mathbb{Z}$ to $\mathbb{N} \cup \{0\}$. We now define the bijection. We map 0 to itself. We showed earlier that there is a bijection from the natural numbers to the positive natural numbers. We apply this bijection so that all the natural numbers are mapped to the even natural numbers. This leaves *spaces* - the odd natural numbers do not have anything mapping to them. What is left to be mapped are the negative integers. But, their cardinality is the same as that of the natural numbers and we can obtain a bijection between the negative integers and the odd natural numbers as we did in the exercise above. Combining these functions yields the desired bijection.

More formally, we define the following function.

$$f(k) = \begin{cases} 2k, & k \geq 0 \\ -2k - 1, & k < 0 \end{cases}$$

Figure 11 shows the mapping.

Exercise 18

We showed a bijection from $\mathbb{Z}$ to $\mathbb{N} \cup \{0\}$. Modify the construction above to obtain a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

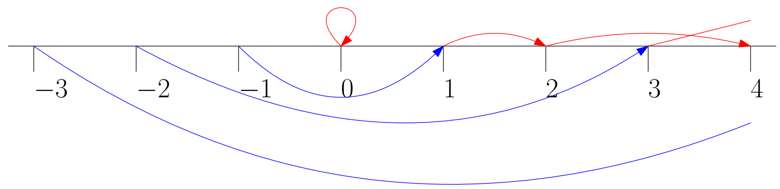


Figure 7: A bijection from $\mathbb{Z}$ to $\mathbb{N} \cup \{0\}$

Next, we will show a truly startling result. We will show that the set $\mathbb{Q}$ of rational numbers has the same cardinality as the set of natural numbers! But, before we do that we introduce a very useful tool in establishing bijections.

Theorem 1 (Schröder-Bernstein Theorem)

Let X, Y be two sets. If there is an injective function $f : X \to Y$ and an injective function $g : Y \to X$, then there is a bijection between X and Y.

Note that we do not have to explicitly show a bijection between X and Y. Just showing an injective function from X to Y, and an injective function from Y to X suffices to show the existence of a bijection (without actually showing what it is). We will not prove the Schröder-Bernstein theorem, but we will use it for the next result.

Recall that $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$. Before we establish a bijection bewteen $\mathbb{Q}$ and $\mathbb{N}$, we start by establishing a bijection between $\mathbb{N} \times \mathbb{N} = \{(p, q) : p \in \mathbb{N}, q \in \mathbb{N}\}$. That is, we establish a bijection between ordered pairs of natural numbers and the set of natural numbers.

Theorem 2

$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Proof. The map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, where $f : p \mapsto (p, 1)$ is clearly an injective function. Therefore, we only need to establish an injection g from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Consider the following function:

$$g(p, q) = \frac{1}{2}(p + q - 1)(p + q - 2) + p$$

First, note that $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. This is because, one of $(p + q - 1)$ or $(p + q - 2)$ is even and therefore the resulting value $\frac{1}{2}(p + q - 1)(p + q - 2) + p$ is an integer. To show that g is injective, consider the collection of subsets of $\mathbb{N} \times \mathbb{N}$ s.t. $p + q = r$, for each $r \in \mathbb{N}$. Note that this is a partition of $\mathbb{N} \times \mathbb{N}$. For a fixed $r \in \mathbb{N}$, the set of elements (p, q) s.t. $p + q = r$ satisfy $(i, r - i), i = 1, \ldots, r - 1$. Then, g maps the elements $\{(p, q) : p + q = r\}$ to $1/2(r - 1)(r - 2) + j$, $j = 1, \ldots, r - 1$. To see that $g(p, q)$ is an injection, observe that for $p + q = r$, $g(p, q) \in \{\frac{r^2}{2} - \frac{3}{2}r + 2, \frac{r^2}{2} - \frac{r}{2}\}$ and for $p + q = r + 1$, $g(p + q) \in \{\frac{r^2}{2} - \frac{r}{2} + 1, \frac{r^2}{2} + \frac{r}{2} - 1\}$. Therefore, g is an injection.

Since we have established an injection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and an injection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, by the Schröder-Bernstein theorem, there is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. $\square$

Exercise 19

Using the construction above to establish a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, show that there is a bijection between $\mathbb{Q}$ and $\mathbb{Z}$.

Note that even though between any two rationals no matter how close, there is always another rational, it is still the case that $|\mathbb{Q}| = |\mathbb{N}|$! So far, we have established that $|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}| = |\mathbb{E}| = |\mathbb{O}|$.

6 The cardinality of $\mathbb{R}$

We now show the most surprising result in this section; that $|\mathbb{R}| > |\mathbb{N}|$. We prove this via what is called Cantor's *diagonalization argument*, and this argument is central to both Gödel's proof of incompleteness and Turing's result on undecidability. We will use this diagonalization argument several times in the theory of computing.

Theorem 3

$|\mathbb{R}| > |\mathbb{N}|$

Proof. We will establish something stronger, namely that $|[0,1]| > |\mathbb{N}|$, i.e., that the cardinality of just the real numbers in the interval $[0,1]$ is larger than the cardinality of the natural numbers. First, note that any real number in $[0,1]$ has a representation as an infinite non-recurring decimal expansion of the form $0.x_1x_2\ldots$, where each $x_i \in \{0,\ldots,9\}$.

We prove the result by contradiction. Suppose $|[0,1]| = |\mathbb{N}|$. This means that there is a bijection between $[0,1]$ and $\mathbb{N}$, that is a function from $[0,1] \to \mathbb{R}$ that is both injective and surjective. In other words, we can write an infinitely large table that is indexed by the natural numbers. Since the function is injective, to each real number in $[0,1]$ there is at most one natural number mapped to it. Further, since it is a surjection, *all* the real numbers in $[0,1]$ are present in the table. For example the table could look like the one in Figure 1, where x^i_j is the j^{th} digit after the decimal of the number in $[0,1]$ that $i \in \mathbb{N}$ maps to.

$$\begin{array}{cccccccccc}
1 & \mapsto & 0. & x^1_1 & x^1_2 & x^1_3 & x^1_4 & x^1_5 & x^1_6 & \ldots \\
2 & \mapsto & 0. & x^2_1 & x^2_2 & x^2_3 & x^2_4 & x^2_5 & x^2_6 & \ldots \\
3 & \mapsto & 0. & x^3_1 & x^3_2 & x^3_3 & x^3_4 & x^3_5 & x^3_6 & \ldots \\
4 & \mapsto & 0. & x^4_1 & x^4_2 & x^4_3 & x^4_4 & x^4_5 & x^4_6 & \ldots \\
5 & \mapsto & 0. & x^5_1 & x^5_2 & x^5_3 & x^5_4 & x^5_5 & x^5_6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Table 1: A bijection from $\mathbb{N}$ to $[0,1]$

Now, construct a number $y = 0.\overline{x^1_1}\ \overline{x^2_2}\ \overline{x^3_3}\ \ldots$, where $\overline{x^i_i}$ is a digit that is distinct from x^i_i in the bijection in Figure 1. Since we have a bijection, y must appear in the table above in some row. But, this is impossible as y is distinct from every real number in the table. Therefore, y cannot appear in the table contradicting our assumption that there must be a bijection between

$\mathbb{N}$ and $[0,1]$. Since there is an injective map $f : \mathbb{N} \to [0,1]$ where $f : n \mapsto 1/n$, it follows that $|[0,1]| > |\mathbb{N}|$. $\square$

This is a truly shocking result, and mathematicians of the day did not accept it. What Cantor showed is that there are some infinite sets that are larger than other infinite sets.

The set $\mathbb{N}$, and thus $\mathbb{Z}$ and $\mathbb{Q}$ are called *countably infinite* sets, while $\mathbb{R}$ is called an *uncountably infinite* set. Some authors also use the term *denumerable* to denote countable sets (finite or countably infinite) and *non-denumerable* to denote uncountably infinite sets.

Definition 11 (Infinite Cardinals)

We require a new number to denote $|\mathbb{N}|$, and this is denoted $\aleph_0$ (where $\aleph$, pronounced *aleph* is a letter in the Hebrew language). Since $|\mathbb{R}| > |\mathbb{N}|$, we require a different symbol to denote its cardinality. We say $|\mathbb{R}|$ by $\aleph_1$. In particular, any set that admits a bijection to $\mathbb{N}$ has cardinality $\aleph_0$, and any set with cardinality that of $\mathbb{R}$ is denoted $\aleph_1$.

We will now see that there no need to stop at $\aleph_1$. We can define $\aleph_2, \aleph_3, \ldots$ for each natural number in $\mathbb{N}$, and for each $i \in \mathbb{N}$ we can construct sets whose cardinality is $\aleph_i$.

For a set S, we define its *power set*, denoted $\mathcal{P}(S)$, or sometimes 2^S as the collection of all its subsets. For example, if $S = \{1,2,3\}$, then $2^S = \mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Consider the set of all functions $\mathcal{F} = \{f : f : S \to \{0,1\}\}$. The set of functions $\mathcal{F}$ is in bijective correspondence with $\mathcal{P}(S)$. To see this, consider a function $f \in \mathcal{F}$, and let $f^{-1}(1) = \{x \in S : f(x) = 1\}$, i.e., the set of elements in S mapped to 1. This defines a subset of S. Thus, there is an injective function from $\mathcal{F}$ to $\mathcal{P}(S)$. To see the reverse direction, for each subset $T \subseteq S$, let $f_T : S \to \{0,1\}$ be defined as follows:

$$f_T(x) = \begin{cases} 0, & x \notin S \\ 1, & x \in S \end{cases}$$

For any two sets $T, T' \subseteq S$, it follows from our definition that $f_T \neq f_{T'}$, and hence this defines an injection from $\mathcal{P}(S) \to \mathcal{F}$. Hence, by Schröder-Bernstein theorem, there is a bijection between $\mathcal{P}(S)$ and $\mathcal{F}$.

Another way to visualize a function in $\mathcal{F}$ is via an *indicator string*, which is a boolean string indexed by the elements of S. The boolean strings are in bijective correspondence (i.e., there is a bijection) between $\mathcal{P}(S)$ and the set of boolean strings - for each subset $S' \subseteq S$ in $\mathcal{P}(S)$, construct a string that contains a 1 for exactly those indices corresponding to elements in S'. In the other direction, each string corresponds to a subset consisting of exactly those elements whose indices are 1 in the string. By the argument above, the binary strings are in bijective correspondence with $\mathcal{P}(S)$. Table 2 below gives the bijection for the power set of $\{1,2,3\}$ above.

In a similar vein, we obtain a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$. We will now show that $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$.

Theorem 4

$|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$

$\emptyset$	000
$\{1\}$	001
$\{2\}$	010
$\{3\}$	100
$\{1,2\}$	011
$\{2,3\}$	110
$\{1,3\}$	101
$\{1,2,3\}$	111

Table 2: Bijection between binary strings and the power set of the set $\{1, 2, 3\}$.

Proof. (Sketch) Our proof follows essentially from Cantor's diagonalization argument. Assume $|\mathcal{P}(\mathbb{N})| = |\mathbb{N}|$, then there is a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$. Now, by diagonalization we can construct a new string y that differs from all the binary strings in the bijection, contradicting the fact that we have a bijection. □

Exercise 20

Show that if $|A| = |B|$, then, $\mathcal{P}(A) = \mathcal{P}(B)$.

Exercise 21

Show that if $A \subseteq B$, and A is uncountably infinite, then so is B.

Exercise 22

Show that there is a $1-1$ correspondence between the set of subsets of $\mathbb{N}$ and the set of real numbers in $[0, 1]$.

Exercise 23

Show that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \aleph_1$.

Exercise 24

Show that for any set S, $|\mathcal{P}(S)| > |S|$.

From Exercise 28 above, it follows that $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$, and we denote $|\mathcal{P}(\mathbb{R})|$ by $\aleph_2$, and $|\mathcal{P}(\mathcal{P}(\mathbb{R}))|$ by $\aleph_3$, and so on.

6.1 Order

Recall that an ordering relation is a relation that is reflexive, transitive and anti-symmetric. Notions of orderings that first come to mind are $(\mathbb{N}, \leq)$ or $(\mathbb{Z}, \leq)$, i.e., the natural numbers or the integers ordered by the $\leq$ relation. Orderings are typically denoted by $\preceq$, and *strict ordering*, with $\prec$, that is $a \prec b$ if $a \preceq b$ and $a \neq b$.

Note that in an ordering relation, we do not necessarily have that for all x, y either (x, y) or (y, x) is in R. Thus, orderings relations are generally referred to as *partial orders*. If for each pair of elements $x, y \in X$ if (x, y) or (y, x) is in R, then R is a total order.

As an example, consider the ordering on $\mathbb{Z} \times \mathbb{Z}$, where $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. To see that this is an ordering relation we show the following:

Reflexivity: $(x, y) \preceq (x, y)$ holds by definition.

Anti-symmetry: For $(x_1, y_1) \neq (x_2, y_2)$, either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $(x_1, y_1) \preceq (x_2, y_2)$, then either $x_1 < x_2$ or $y_1 < y_2$. This implies $(x_2, y_2) \not\preceq (x_1, y_1)$.

Transitivity: For $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, if $(x_1, y_1) \preceq (x_2, y_2)$, then $x_1 \leq x_2$ and $y_1 \leq y_2$. Similarly, $(x_2, y_2) \preceq (x_3, y_3)$ implies $x_2 \leq x_3$ and $y_2 \leq y_3$. Therefore, $x_1 \leq x_3$ and $y_1 \leq y_3$, and $(x_1, y_1) \preceq (x_3, y_3)$.

Since $\preceq$ is reflexive, anti-symmetric and transitive, it follows that $(\mathbb{Z} \times \mathbb{Z}, \preceq)$ is an ordering relation. Note that the relation is not a linear order as $(3, 5)$ and $(5, 3)$ are not comparable in our ordering.

Another natural order on $\mathbb{Z}\times\mathbb{Z}$ is the *lexicographic order* defined as follows: For $(x_1, y_1), (x_2, y_2)$, $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 < x_2$, or $x_1 = x_2$ and $y_1 \leq y_2$. As before, one can show that this is indeed an ordering relation. Consider any pair $(x_1, y_1) \neq (x_2, y_2)$, where $x_1 \leq x_2$. If $x_1 < x_2$, then $(x_1, y_1) \preceq (x_2, y_2)$. However, if $x_1 = x_2$, then either $y_1 < y_2$ or $y_2 > y_1$, which yields, respectively, $(x_1, y_1) \preceq (x_2, y_2)$, or $(x_2, y_2) \preceq (x_1, y_1)$. Since any pair can be compared, the lexicographic ordering yields a *linear order* on $\mathbb{Z} \times \mathbb{Z}$.

We can generalize lexicographic orderings to more than two sets. Let $X_1, \ldots, X_k$ be k linearly ordered sets. Let $X = X_1 \times \ldots X_k$ be the cartesian product of the sets. Then $(x_1, x_2, \ldots, x_k) \preceq_{\text{lex}}$ $(y_1, \ldots, y_k)$ if and only if either $(x_1, \ldots, x_k) = (y_1, \ldots, y_k)$, or there is a $j \leq k$ s.t. $x_i = y_i$ for all $i < j$ and $x_j < y_j$.

Exercise 25

For $a, b \in \mathbb{N}$, let $a|b$ mean that a divides b, i.e., there exists a natural number c such that $b = ac$. Show that $|$ is a partial order on $\mathbb{N}$.

Exercise 26

For a set X, define the order: for $A, B \subseteq X$, $A \preceq B \Leftrightarrow A \subseteq B$. Show that $\preceq$ is a partial order.

6.1.1 Hasse diagrams

We can represent a partial order $\preceq$ on a set X by putting a point for each element in X and add a directed arrow from x to y if $x \preceq y$. However, such a representation will require too many arrows. For example, suppose X is a linearly ordered set, we will have to a directed arrow between every pair of elements. However, here is a simpler way to represent a linear order. For example, let $X = \{1, \ldots, 10\}$ be the first 10 natural numbers with the natural order on the elements. We can represent them as shown in Figure 4, where if $x \preceq y$, then x is drawn *below* y. Only some of the relations are represented by *undirected* segments. However, the order is implicit in the heights of the points connected by a segment. Since ordering relations are transitive, the other relations between elements are implicitly represented. Thus, $1 < 3$ follows since $1 < 2$ and $2 < 3$ in the figure.

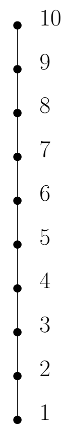


Figure 8: Linear order on $X = \{1, \ldots, 10\}$

Let us generalize this notion. For a pair of elements $x, y \in X$, x is an *immediate predecessor* of y in X if $x \preceq y$, and there is no $x, y \neq z \in X$ s.t. $x \preceq z \preceq y$. Let $x \lhd y$ denote the fact that x is an immediate predecessor of y. We claim that if we are given the immediate predecessor relation, then the underlying partial order $\preceq$ can be recovered uniquely.

Proposition 4. *Let $(X, \preceq)$ be a partial order on a finite set X, and let $(X, \lhd)$ be the immediate predecessor relation on the elements in X. Then, for any two elements $x, y \in X$, $x \preceq y$ if and only if there exists elements $x_1, \ldots, x_k$ s.t.*

$$x \lhd x_1 \lhd \ldots \lhd x_k \lhd y$$

By the proposition above, it is sufficient to draw the relation of immediate predecessors by arrows. If all arrows are drawn upwards, then we can forgo the arrows and connect points by a simple line segment, where if $x \lhd y$, y is drawn higher than x and there is a segment connecting x and y. Such a drawing of a partial order is called a Hasse diagram. Figures **??** and Figure **??** show Hasse diagrams for three partial orders.

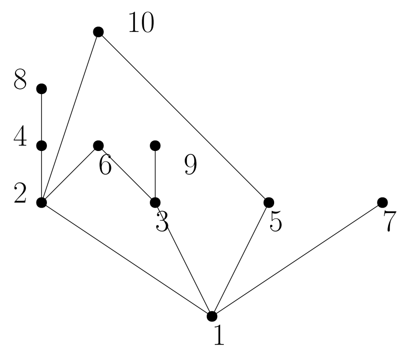


Figure 9: Partial order of elements $\{1, \ldots, 10\}$ ordered by the division relation

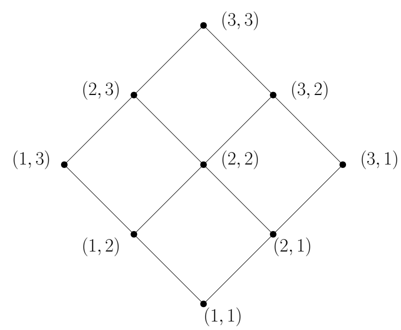


Figure 10: Partial order of the set $\{1, 2, 3\} \times \{1, 2, 3\}$ ordered by the relation $(a_1, b_1) \preceq (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$.

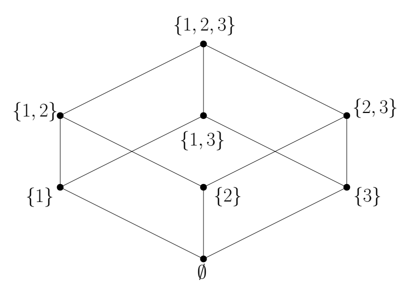


Figure 11: Partial order of the subsets of $\{1, 2, 3\}$ ordered by the subset relation.

Exercise 27

Let $(X, \leq)$ and $(Y, \preceq)$ be two partial orders. An *isomorphism* between $(X, \leq)$ and $(Y, \preceq)$ is a bijection $\phi : X \to Y$ s.t. $x \leq y \Leftrightarrow \phi(x) \preceq \phi(y)$.

1. Draw all non-isomorphic partial orders on 3-element sets.
2. Prove that two n-element linearly ordered sets are isomorphic.
3. Find two non-isomorphic orderings of $\mathbb{N}$.

4. Can you find uncountably many linear orders on $\mathbb{N}$?

Exercise 28

Prove that Proposition 4 does not hold for infinite sets.