Mathematical Proofs

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Abstract

These notes provide a quick background on Mathematical Proofs.

1 Introduction

2 Well-Ordering Principle (WoP)

The well-ordering principle states that any non-empty set of natural numbers contains a least element. It is *equivalent* to the principle of mathematical induction that we'll encounter in the next section.

Definition 1 (Well-Ordering Principle)

Let $S \subseteq \mathbb{N}$ be any non-empty subset of natural numbers. Then, S contains a smallest element.

Note that the well-ordering principle is *not true* if we replace $\mathbb{N}$ by $\mathbb{Z}$ or $\mathbb{Q}$. Let's start with a simple example of the use of WoP.

Theorem 1

For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof. Suppose the statement of the theorem does not hold. Then, let $S = \{n \in \mathbb{N} : \sum_{i=1}^{n} i \neq n(n+1)/2\}$. By our assumption, S is non-empty. By the well-ordering principle, S contains a smallest element. Let n be the smallest element in S.

Then,

$$\sum_{i=1}^{n} i \neq \frac{n(n+1)}{2}$$

However, since n is the smallest element in S, it follows that

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$$

However,

$$\begin{aligned}\sum_{i=1}^{n} i = \sum_{i=1}^{n-1} i + n \\ &= \frac{(n-1)n}{2} + n \\ &= \frac{n(n-1)}{2}\end{aligned}$$

contradicting our assumption.

□

Theorem 2 (Division algorithm)

For any $a \in \mathbb{Z}$ and positive $n \in \mathbb{N}$, there exist unique $q, r \in \mathbb{Z}$, $0 \leq r < n$ such that:

$$a = qn + r$$

Proof. Let $S = \{t \in \mathbb{N} : a - qn \geq 0\}$. We first show that S is non-empty. Set $q = -|a|$. Then, $a - (-|a|)n = a + |a|n \geq 0$, since $n \geq 1$. By the well-ordering-principle therefore, there is a least element r in S. By definition $r \geq 0$. We claim that $0 \leq r < n$. Suppose $r = a - qn \geq n$, then,

$$\begin{aligned} r - n & \geq 0 \\ \Rightarrow a - qn - n \geq 0 \\ \Rightarrow a - q(n+1) \geq 0\end{aligned}$$

But, this implies $a - q(n+1) \in S$, but $a - q(n+1) < r$, contradicting our assumption that r is the smallest element in S. Hence, $0 \leq s < n$.

To show that q, r are unique. Suppose there exist q', r' different from q, r such that $a = q'n + r'$, then

$$\begin{aligned} qn + r = q'n + r', \text{ hence,} \\ (q - q')n + (r - r') = 0\end{aligned}$$

If $r = r'$, this implies $q = q'$ as $n > 0$. Hence, $r \neq r'$. If $q = q'$, this is impossible, so both $q \neq q'$ and $r \neq r'$. Assume wlog $r > r'$. Then, $0 < r - r' < n$. The proof follows. □

Theorem 3

Any $n \in \mathbb{N}$ has a unique representation as a product of primes.

Proof. Let $S = \{n \in \mathbb{N} : n$ does not have a representation as a product of primes$\}$. Suppose the statement of the theorem is not true. Then, S is non-empty and therefore, there is a smallest number $n \in S$. Since n is not prime, by definition $n = rs$, for $2 \leq r, s < n$. However, since n was the smallest element in S, it follows that both r and s have a representation as a product of primes. Let $r = p_1 \dots p_t$ and $s = q_1 \dots q_\ell$. Then,

$$n = p_1 \dots p_t q_1 \dots q_t$$

Suppose there is a different representation of n as a product of primes. For ease of notation, let $n = p_1 \dots p_t = q_1 \dots q_\ell$. Let the primes $p_1 \dots p_t$ be ordered so that for $i < j,\ p_i \leq p_j$. Similarly, for $i < j,\ q_i \leq q_j$. Let j be the smallest index where $p_j \neq q_j$. Then,

$$p_{j+1} \dots p_t = \frac{q_j \dots q_\ell}{p_j}$$

This implies p_j divides some number in the numerator, but this contradicts the assumption that q_is are primes.

□

3 Mathematical Induction

The principle of mathematical induction is the least intuitive of the proof techniques. The *principle of mathematical induction* is an axiom in Peano's axioms of the natural number system. While we don't need Peano's axioms in this course, here are the axioms nevertheless:

1. There is a least element of $\mathbb{N}$ called 0.
2. Every natural number a has a *successor*, denoted $s(a)$.
3. There is no natural number whose successor is 0.
4. Distinct natural numbers have distinct successors, i.e., for $a \neq b \Rightarrow s(a) \neq s(b)$.
5. If a subset S of natural numbers contains 0 and also has a property that whenever $a \in S \Rightarrow s(a) \in S$. Then, S is equal to $\mathbb{N}$. More succinctly, we can write the principle of mathematical induction as

$$\forall S \subseteq \mathbb{N}\, ((0 \in S) \wedge (\forall a \in \mathbb{N} a \in S \rightarrow s(a) \in S))\, S = \mathbb{N}$$

The last axiom is the one that justifies the principle of mathematical induction. Another way to see the principle of mathematical induction is the following: We want to prove that a predicate holds for all natural numbers. That is, we want to prove

$$P(0) \wedge P(1) \wedge P(2) \wedge \dots$$

Thus, if we prove $P(0)$. Then, we we prove $P(i) \rightarrow P(i+1)$ *for an arbitrary* i, i.e., the proof that $P(i) \rightarrow P(i+1)$ should not depend on i. Then, we have proved (via Modus Ponens):

$$\frac{P(0); P(0) \to P(1)}{P(1)}$$

and again via Modus Ponens,

$$\frac{P(1); P(1) \to P(2)}{P(2)}$$

and so on, and therefore $P(i)$ is true for all $i \in \mathbb{N}$.
An outline of a proof by induction consists therefore, of the following steps:

1. **Base case:** Prove that $P(0)$ is true.
2. **Inductive Step:**
   - $\forall i \in \mathbb{N}, P(i) \to P(i+1)$

The sentence $P(i)$ is called the *inductive hypothesis*. It is also good, especially when we are doing *structural induction* to explicitly state the inductive hypothesis, i.e., $P(i)$ is true.

Without further ado, let's start with an example:

Theorem 4

Let us prove that for every positive $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof. We prove by induction on n. *The statement* $P(k)$ *is* $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
Base case: For $n = 1$, the LHS is 1 and so is the RHS by direct calculation.

Inductive hypothesis (I.H): Suppose $P(n)$ is true, i.e., $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Inductive Step: We want to prove that $P(n+1)$ is true, i.e.,

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

which we proceed to show now.

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) && [\because \text{I.H}] \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= P(n+1)
\end{aligned}$$

□

We could have started the induction at $n = 0$ as $\sum_{i=1}^{0} i = 0$. Thus, we can restate the statement of the theorem as:

$$\forall n \in N; \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Theorem 5

For any $n \in \mathbb{N}$, the sum of the first n odd natural numbers is n^2, i.e., $\forall n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i-1) = n^2$

Proof. We prove by induction on n.
Base Case: $n = 1$. The $\sum_{i=1}^{1} 1 = 1$, and the theorem holds.
Inductive Hypothesis: $\sum_{i=1}^{n} 2i - 1 = n^2$
Inductive Step:

$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1) &= \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 \\
&= n^2 + 2(n+1) - 1 && [\because I.H] \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

□

Exercise 1

Prove by induction that the sum of the powers of 2 upto 2^n is $2^{n+1} - 1$.

Exercise 2

The Fibonacci numbers $F_0 = 1, F_1 = 1$, $F_2 = F_0 + F_1$, and in general $F_n = F_{n-1} + F_{n-2}$. Prove that

1. $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.
2. The n^{th} Fibonacci number $F_n < 2^n$
3. $\sum_{i=0}^{n} F_i = F_{n+2} + 1$

Sometimes, we don't know the right hand side of the inequality. We need to guess it by trying small values of n, and then we can prove the equality via induction. Here are a few.

Exercise 3

Determine a closed-form expression for $\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)}$, for any natural number n and prove the result via induction.

Exercise 4

Consider the following sequence of numbers: $G_0 = 0$, $G_1 = 2$, and $G_n = G_{n-1} + G_{n-2}$. What is the analogue of the result in Exercise 1?

3.1 Strong Induction

In many cases, instead of assuming $P(n)$ to prove $P(n+1)$ in the inductive step, it is more convenient to assume $P(0) \land P(1) \land \ldots P(n)$ and then prove

$$P(0) \land P(1) \land P(2) \land \ldots P(n) \to P(n+1)$$

Note that we use the word *convenient*, i.e., any statement that can be proved via the principle of mathematical induction can be proved via strong induction. However, in many cases, it is more convenient to use strong induction as it leads to a cleaner proof.

Definition 2 (Strong Induction Principle)

Let P be a predicate such that $P(0)$ is true, and for any natural number n, $P(0) \land P(1) \land P(2) \ldots P(n)$ implies $P(n+1)$. Then, P is true for all $n \in \mathbb{N}$. More succinctly,

$$(P(0) \land (\forall n \in \mathbb{N}\ P(0) \land P(1) \land \ldots P(n) \to P(n+1)) \to \forall n \in \mathbb{N}\ P(n)$$

Let us start with an example:

Theorem 6

Every positive integer greater than 1 can be expressed uniquely as a product of primes.

Proof. We prove by induction on n.

Base Case: Since 2 is prime, the theorem holds.

Inductive Hypothesis: Assume that for all $2 \leq n' < n$, n' can be expressed uniquely as a product of primes. *It is good to write out explicitly the inductive hypothesis instead of saying something generic like: Assume that for all* $2 \leq n' < n$ *the statement holds*

Inductive Step: Suppose n is prime, then we are done. Suppose n is not prime, by definition, there are numbers r and s, $1 < r, s < n$ s.t.

$$n = rs$$

By the (strong) inductive hypothesis, r has a unique representation as a product of primes. Let

$$r = p_{i_1} \dots p_{i_t} \tag{1}$$

Similarly, since $s < n$, by the (strong) inductive hypothesis, s has a unique representation as a product of primes. Let

$$s = p_{j_1} \dots p_{j_\ell} \tag{2}$$

Hence, from Eqn (1) and (2),

$$n = p_{i_1} \dots p_{i_t} p_{j_1} \dots p_{j_\ell}$$

and hence n can be represented as a product of primes. For simplicity of notation, let $n = p_1 \dots p_a$ where $p_1 \leq p_2 \dots p_a$. Suppose n had a different representation as a product of primes: $n = q_1 q_2 \dots q_b$. Let the q_is be written such that $q_i \leq q_j$ for $i \leq j$. Let ℓ be the first index where $q_\ell \neq p_\ell$. Then,

$$\begin{aligned} q_\ell q_{\ell+1} \dots q_b &= p_\ell p_{\ell+1} \dots p_a, \text{ hence,} \\ q_{\ell+1} \dots q_b &= \frac{p_\ell p_{\ell+1} \dots p_a}{q_\ell} \end{aligned}$$

But, since the primes are labeled in non-decreasing order, it follows that $q_\ell | p_j$ for some $j \in \{\ell, \dots, a\}$, contradicting the fact that $p_\ell, \dots, p_a$ are primes. Therefore, the representation is unique. $\square$

Theorem 7

Every positive natural number has a unique representation as a sum of distinct powers of 2.

Proof. We prove by induction on n.

Base case: For $n = 1$, $1 = 2^0$, and this representation is unique as $2^k > 1$ for any $k \geq 1$.

Inductive Hypothesis: For all $n' \leq n$, n' has a unique representation as a sum of distinct powers of 2 *Note the difference between this hypothesis and the hypothesis for mathematical induction*

Inductive Step: Let k be the largest power of 2 such that $2^k \leq n$. Since $n' = n - 2^k < n$, by the inductive hypothesis, n' has a unique representation as a sum of distinct powers of 2. Let $n' = 2^{i_1} + 2^{i_2} + \dots 2^{i_\ell}$. We claim that $k \notin \{i_1, \dots, i_\ell\}$. Suppose not. Then, $n \geq 2^k + 2^k = 2^{k+1}$ contradicting our assumption that k is the largest power of 2 that divides n.

$$2^{i_1} + \dots 2^{i_\ell} + 2^k = n$$

To show uniqueness, we claim that 2^k must appear in any representation of n: no larger power of 2 appears in a representation of n. Since no power of 2 appears more than once in a representation and $\sum_{i=1}^{k-1} 2^i = 2^k - 1 < 2^k < n$, it follows that 2^k appears in any representation of n. By the inductive hypothesis, n' has a unique representation as a sum of distinct powers of 2 and therefore, it follows that n has a unique representation as distinct powers of 2. $\square$

In a country there are only $5c$ and $3c$ coins, show that any number $n \geq 8$ can be made with the two denomenations. That is, let $P(n)$ be the predicate that we can make change for c $n + 8$.

Proof. Base Case: $P(0)$ is true as one $5c$ and one $3c$ make up $8c$.

Inductive Hypothesis: Suppose for every $n' < n$, $P(n')$ is true, that is, we can make change for $n' + 8$ with $5c$ and $3c$ coins.

Inductive Step: Here, we need to divide into cases depending on the value of n.

Case 1: $n = 1$, that is we want to make 9. We can make this with 3 $3c$ coins.

Case 2: $n = 2$, that is we want to make 10. We can make this with 2 $5c$ coins.

Case 3: $n \geq 3$, that is we want to make ≥ 11 with the coins. By the inductive hypothesis, we can make $n - 3$, as $n - 3 \geq 8$. Therefore, we can make n by adding an additional $3c$. □

We can unstack n boxes using exactly $n(n-1)/2$ moves. If you split a stack into a and b you get ab for that move.

Proof. Any move splits the boxes into two sets a, b such that $a + b = n$. Therefore,

$$\begin{aligned} & ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\ &= ab + \frac{a^2 - a + b^2 - b}{2} \\ &= ab + \frac{a^2 + b^2 - (a+b)}{2} \\ &= \frac{n(n-1)}{2} \end{aligned}$$

□

Exercise 5

Prove that every natural number has a unique binary representation.

For certain problems, it is *rather unintuitively* easier to prove a *stronger* statement. That is, we will strengthen the inductive hypothesis. Here is one example.

For $n \in \mathbb{N}$, consider a $2^n \times 2^n$ grid of cells. Show that it is always possible to tile the grid with "L" shaped tiles such that we leave the middle cell empty. Note that we can rotate the "L" shaped tile any way we like. This is shown in Figure 3 such that we leave the middle tile empty.

First attempt:

Proof. We prove by induction on n.

Base case: If $n = 1$, we use no tiles and the middle tile is empty.

Inductive Hypothesis: A grid of size $2^{n-1} \times 2^{n-1}$ admits a tiling with "L" shaped tiles leaving the middle cell empty.

Inductive Step: We divide the $2^n \times 2^n$ grid into four grids of size $2^{n-1} \times 2^{n-1}$. By the inductive hypothesis, we can tile each of the four smaller grid cells leaving the middle cell empty. See Figure 2.

Proof attempt 2 Let us try to prove the following: Given a grid of size $2^n \times 2^n$, and *any cell* c, we can tile the grid with "L" shaped tiles leaving the cell c empty.

The base case from above goes through. So we start with the inductive hypothesis.

Inductive Hypothesis: For a $2^{n-1} \times 2^{n-1}$ grid and any cell c in this grid, there is a tiling with "L" shaped tiles leaving the cell c empty.

Inductive Step: Given a $2^n \times 2^n$ grid and a cell c that is to be left empty. Partition the grid into four smaller cells of size $2^{n-1} \times 2^{n-1}$. The empty cell c lies in one of the four smaller problems. By the inductive hypothesis then, we can tile the four smaller cells such that we leave an empty "L" shape in the middle into which we can fit one more tile. □

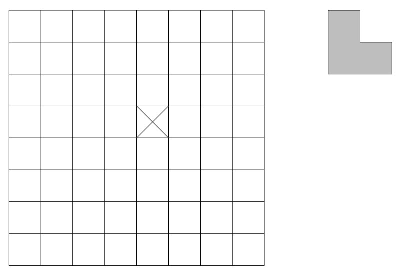


Figure 1: The figure shows that we want to tile the grid with the 'L'-shaped tile leaving the cell marked 'X' empty

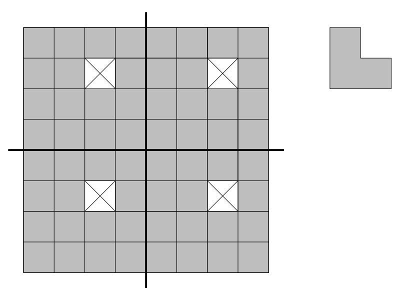


Figure 2: Figure showing the problem with the first attempt.

3.2 Errors in Induction

The following two examples highlight certain pitfalls in the use of induction. Try to identify the errors in the "proofs" below.

Consider the following *fake theorem*:

Fake Theorem: Let a be a positive number. For all positive integers n we have $a^{n-1} = 1$. Here is the "proof". If $n = 1$, $a^{n-1} = a^{1-1} = a^0 = 1$. By induction, assuming the theorem is true for

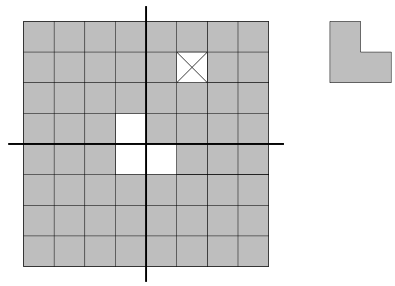


Figure 3: Figure showing the inductive step with a space for a new “L” shaped tile to fit leaving the cell marked “X” empty.

$1, 2 \ldots, n$, we have

$$a^{(n+1)-1} = a^n = \frac{a^{n-1}a^{n-1}}{a^{(n-1)-1}} = \frac{1 \times 1}{1} = 1$$

Fake Theorem: All students are wearing the same colored shirt.

Proof. By induction on n, the number of students. If $n = 1$, the statement is true. Suppose the statement is true for $n' < n$ students. Given n students, Partition the students into two groups:

$$s_1, s_2 \ldots, s_{n-1}, \text{ and } s_2, \ldots, s_n$$

Then, by the inductive hypothesis, in each group, the student's clothes are of the same color. But, s_2 has the same color in both and therefore, all students have clothes of the same color. □

Fake Theorem:

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(n-1)n} = \frac{3}{2} - \frac{1}{n}$$

Proof. By induction on n. For $n = 1$, we have $\frac{3}{2} - 1/n = \frac{1}{1 \times 2}$, and assuming true for n, we have

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} &= \frac{3}{2} - \frac{1}{n} + \frac{1}{n(n+1)} \\ &= \frac{3}{2} - \frac{1}{n}\left(1 - \frac{1}{n+1}\right) \\ &= \frac{3}{2} - \frac{1}{n+1} \end{aligned}$$

□

However, for $n = 6$, we have $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$, but $\frac{3}{2} - \frac{1}{6} = \frac{4}{3}$.

Fake Theorem:

$$n = \sqrt{1 + (n-1)\sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{\ldots}}}}}$$

Proof. We assume that the expression on the right converges to a finite value (we can show this, but skip it here).

Base Case: For $n = 1$, the RHS is $\sqrt{1 + 0(\dots)} = 1$, and the statement is correct.

Inductive Hypothesis: Assume that for n, we have

$$n = \sqrt{1 + (n-1)\sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{\dots}}}}}$$

Inductive Step:

$$n^2 = 1 + (n-1)\sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{\dots}}}}$$

Rearranging the terms, we get

$$\frac{n^2 - 1}{n - 1} = \sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{\dots}}}}$$

Hence,

$$n + 1 = \sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{\dots}}}}$$

□

4 Structural Induction

So far we have looked at induction based on a single value n. But induction is very useful even in settings where we are dealing with "objects" other than integers - for example, partial orders, graphs, etc. that are countable. For example, we may want to prove a statement $P(m, n)$ based on two integers m and n. We now show that induction works more generally in these settings, and is usually called *structural induction*.

This is extremely useful in proving theorems about *recursively defined structures*.

4.1 Well-ordered sets

Definition 3 (Well-ordered set)

A set S is well-ordered if for all non-empty $T \subseteq S$, T is well-ordered.

Exercise 6

Which of the following sets are well-ordered:

1. $\emptyset$
2. $\{0\} \cup \{\mathbb{Q}_>\}$, where $\mathbb{Q}_>$ are the positive rational numbers.
3. $6\mathbb{N} = \{n \in \mathbb{N} : n = 6k \text{ for } k \in \mathbb{N}\}$
4. $[1, 2]$.

needs to be completed....

4.2 Equivalence of Induction and WoP

to be written...

5 More examples of using Induction

Theorem 8

Let $\mathcal{L}$ be a set of n lines in *general position* in $\mathbb{R}^2$, i.e., no two lines are parallel, and no three lines meet at a point. The number of regions formed by the lines in $\mathcal{L}$ is $(n*(n+1)/2) + 1$.

Proof. We prove by induction on n.

Base Case: If $\mathcal{L} = \emptyset$, $n = 0$ and there is only one region $\mathbb{R}^2$.

Inductive hypothesis: Suppose for any arrangement of n lines in general position, the number of regions formed is $n(n+1)/2 + 1$.

Inductive Step: Given *any* arrangement $\mathcal{L}_{n+1}$ of $n+1$ lines in general position, let $\mathcal{L}_n$ be the arrangement obtained by removing one of the lines ℓ in $\mathcal{L}_{n+1}$ arbitrary. The resulting arrangement has n lines and is in general position. Therefore, by the inductive hypothesis, the number of regions formed by $\mathcal{L}_n$ is $n(n+1)/2 + 1$.

If we re-introduce ℓ, since none of the lines are parallel, ℓ intersects all the n lines in $\mathcal{L}_n$, and every pair of lines intersects exactly once. Each time ℓ intersects a line, it crosses a region, splitting it into two. Hence, ℓ crosses $n+1$ regions, and hence this is the number of new regions formed. Consequently, the number of regions formed by $\mathcal{L}_{n+1}$ is

$$\begin{aligned}\frac{n(n+1)}{2} + 1 + (n+1) &= \frac{n^2 + n + 2 + 2n + 2}{2} \\ &= \frac{n^2 + 3n + 4}{2} \\ &= \frac{(n+1)(n+2)}{2} + 1\end{aligned}$$

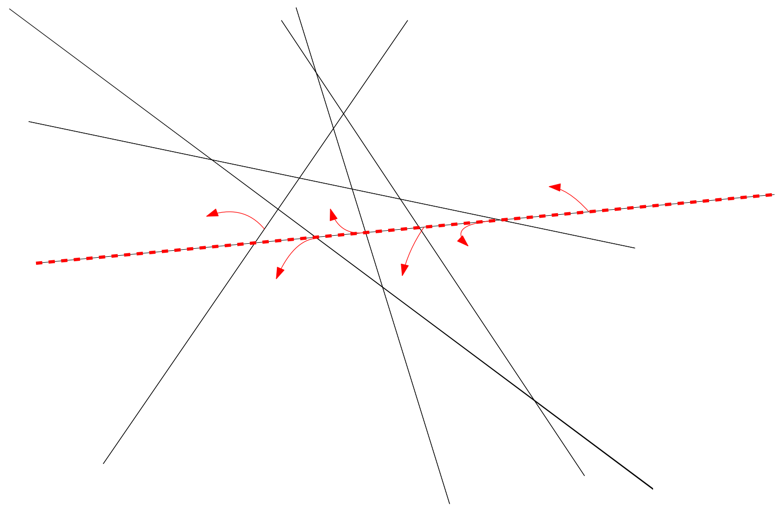


Figure 4: The figure above illustrates the proof. The $(n+1)^{st}$ line added is shown dashed. The arrows point to the regions corresponding the lines lines crossed in the arrangement of n lines.

Figure 4 illustrates the counting argument above.

$\square$

Here is a different proof of the same result:

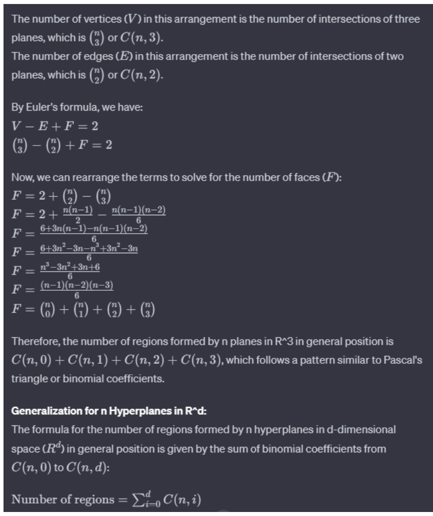
Alternate proof. Let us assume that none of the lines are horizontal. This is easily achieved by rotating the plane slightly. Note that some of the regions have a *lowest point* and some unbounded regions are unbounded below, and therefore do not have a lowest point. Let us consider only the regions of the former type, that we call the *first type*, and the later we'll call the *second type*.

Observe that each vertex in the intersection of the lines is the lowest point of some region of the first type. Since no lines are parallel, there are $n(n-1)/2$ such points and therefore, an equal number of regions of the first type. We are left with the problem of counting the number of regions of the second type. For this, draw a horizontal line below the intersection points of all the lines. This line intersects each of the n lines exactly once, and hence is adjacent to $n+1$ regions that are all of the second type. Therefore, the total number of regions follows. Figure 5 is an illustration of the proof. $\square$

We can re-write the number of regions in another way: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{0} = 1$, for any $n \geq 0$. Consider the regions that This suggests the following generalization.

Exercise 7

Given an arrangement of n planes in $\mathbb{R}^3$ in *general position*, i.e., no three plane meet at a line, no four planes meet at a line and no two planes are parallel. Show that the number of regions formed by the planes is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$. What is the generalization for n hyperplanes in $\mathbb{R}^d$? Can you prove this result?



The number of vertices (V) in this arrangement is the number of intersections of three planes, which is $\binom{n}{3}$ or $C(n,3)$.
The number of edges (E) in this arrangement is the number of intersections of two planes, which is $\binom{n}{2}$ or $C(n,2)$.

By Euler's formula, we have:
$V - E + F = 2$
$\binom{n}{3} - \binom{n}{2} + F = 2$

Now, we can rearrange the terms to solve for the number of faces (F):
$F = 2 + \binom{n}{2} - \binom{n}{3}$
$F = 2 + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{6}$
$F = \frac{6+3n(n-1)-n(n-1)(n-2)}{6}$
$F = \frac{6+3n^2-3n-n^3+3n^2-3n}{6}$
$F = \frac{n^3-3n^2+3n+6}{6}$
$F = \frac{(n-1)(n-2)(n-3)}{6}$
$F = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$

Therefore, the number of regions formed by n planes in R^3 in general position is $C(n,0)+C(n,1)+C(n,2)+C(n,3)$, which follows a pattern similar to Pascal's triangle or binomial coefficients.

Generalization for n Hyperplanes in R^d:
The formula for the number of regions formed by n hyperplanes in d-dimensional space (R^d) in general position is given by the sum of binomial coefficients from $C(n,0)$ to $C(n,d)$:

Number of regions $= \sum_{i=0}^{d} C(n,i)$

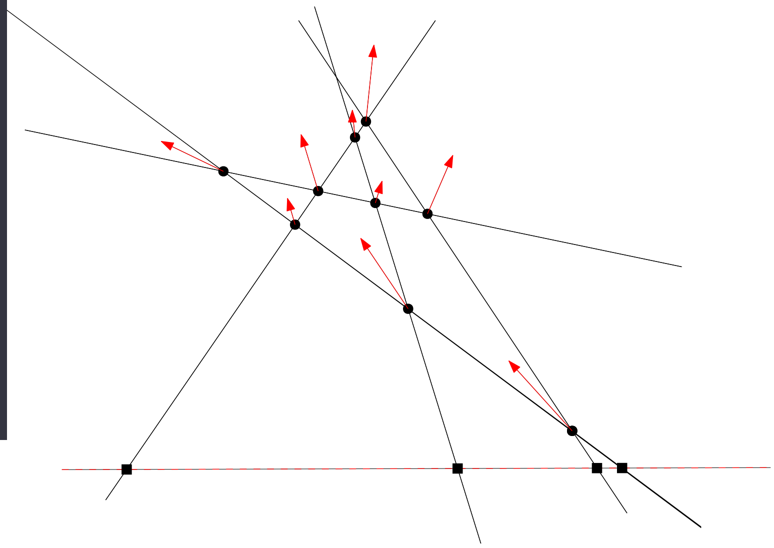


Figure 5: The figure above illustrates the alternate proof. The arrows show the region associated with each vertex. The dotted line with squares shows the count of the regions without a lowest point.

Exercise 8

Consider a set of n circles drawn in the euclidean plane that are pairwise intersecting. If we remove the circles, the plane partitions into *connected pieces*. Show we can assign one of two colors to each connected piece such that connected pieces that share an arc of a circle get distinct colors.

Exercise 9

Consider a set of n lines in the plane in general position. (i) Prove that one of the regions they form is a triangle. (ii) Prove that they form at least $n-2$ triangles. Is the general position assumption necessary in the claims of these statements?

Exercise 10

Consider a simple polygon drawn in the plane so that its vertices are *integral*, i.e., both the coordinates are integers. A simple polygon is one whose edges do not cross each other. Let p be the number of integral points on the boundary, and let q be the number of integral points inside the polygon. Prove that the area of the polygon is $p/2+q+1$ *[Caveat: This is harder than the other problems].*

6 Cutting cubes

Given a $3 \times 3 \times 3$ block of wood, we want to cut it into *unit cubes*, i.e., $1 \times 1 \times 1$ blocks to make a rubiks cube. We want to do this with the fewest cuts. In each cut, we can collect the blocks obtained in previous cuttings, put them together and cut the whole thing with one cut. For example, if the first cut cuts the cube into one cuboid of sides $1 \times 3 \times 3$ and another of dimensions $2 \times 3 \times 3$. The next time we can put these blocks together to form a cube and make one cut to obtain four blocks. Figure **??** shows this example. What is the minimum number of cuts required?

In such problems, we want to argue in two ways: Obtain a *lower bound*, i.e., show that any cutting requires *at least* k cuts, and obtain an *upper bound* to show that we can *always* solve the problem with at most ℓ cuts for some $k \leq \ell$. If $k = \ell$, then we have obtained an optimal solution.

Observe that any cut at most doubles the number of cubes. Therefore, after k cuts, the number of pieces we have obtained is at most 2^k. Since we are required to produce 27 unit cubes, we require that $2^k \geq 27$, which means that $k \geq 5$. Therefore, we require at least 5 cuts.

Now let us obtain a lower bound: Assume that the cube is axis-parallel, i.e., its sides are parallel to the xy, yz and xz planes. We make 2 cuts parallel to the yz plane, 2 cuts parallel to the xy plane, and finally two cuts parallel to the xz plane for a total of 6 cuts and we obtain a partition of the cube into unit cubes.

So the real question is whether the answer is 5 or 6. Could there be a clever way that can obtain the blocks with just 5 cuts? Here is an argument that shows that we require at least 6 cuts, and therefore the procedure above that obtained unit cubes with 6 cuts is optimal: Consider the middle cube. This cube has 6 faces and we require a separate cut to obtain each face of this cube. Therefore, any cutting procedure requires at least 6 cuts.

This was a cute puzzle, but let us now generalize to d dimensions. We are given a d-dimensional *hyper-cuboid* of size $(n_1, n_2, \ldots, n_d)$. How many cuts do you think are required to cut such a hyper-cuboid into unit *hypercubes*?

Here, we illustrate an important technique. Whenever we are asked to solve a problem in d-dimensions, think of solving the problem in small dimensions where it is somewhat easier and then try to *generalize* the result to d-dimensions via induction. So, let us consider the problem in one dimension. Here, we are given an $1 \times n$ block that we want to chop into unit cubes. A natural way to do this is to make a cut along the middle to obtain two blocks of size $n/2$ (for simplicity, I am assuming that n is a power of 2). We then put these blocks together and make a single cut to obtain 4 blocks of size $n/4$, and so on. This should remind you of *binary search* or balanced *binary search trees*, and it is easy to see that we require $\log_2 n$ cuts. If n is not a power of 2, there is a power of 2 that is at most $2n$. We can extend our input block into one that is a power of 2, and therefore we require at most $\log_2(2n) \leq \lceil \log_2 n \rceil$ cuts, where $\lceil x \rceil$ is the *ceiling function* that returns the smallest integer that is at least as large as x (similarly $\lfloor x \rfloor$ is the *floor function* that returns the largest integer that is at most x). Suppose we are given an $n_1 \times n_2$ block. By the method above, in $\lceil \log_2 n_1 \rceil$ cuts, we can obtain n_1 blocks of size $1 \times n_2$ each. We can then make $\lceil \log_2 n_2 \rceil$ cuts to the n_1 blocks together to obtain $n_1 n_2$ unit blocks. So we have used at most $\lceil \log_2 n_1 \rceil + \lceil \log_2 n_2$ cuts. Let us now complete the proof via induction.

Lemma 1. *Given any collection of $n \times 1$ blocks, we can obtain n unit cubes by making $\lceil \log_2(n+1) \rceil$ cuts.*

Proof. We prove by induction on n.

Base case: If $n = 1$, then no cuts are required, and the statement says that we require at most $\lceil \log_2(1 + 1) \rceil = 1$.

Inductive hypothesis: For any $n' < n$ and any collection of $n' \times 1$ blocks, we require at most $\lceil \log_2(n' + 1) \rceil$ cuts to obtain unit cubes.

Inductive step: Given an of $n \times 1$ blocks, we cut it into two blocks of sizes $\lfloor n/2 \rfloor$, and $\lceil n/2 \rceil$ with one cut. By the inductive hypothesis, we require $\lceil \log_2 \lceil n/2 \rceil \rceil$ cuts for the two blocks together. Therefore, the total number of cuts required for an $n \times 1$ block is $1 + \lceil \log_2 \lceil n/2 \rceil \rceil \leq 1 + \lceil \log_2(n + 1) - 1 \rceil \leq \lceil \log_2(n + 1) \rceil$. If we are given more than one such block, we can cut them in parallel and therefore we do not increase the number of cuts. □

Theorem 9

An $n_1 \times \dots n_d$ hypercuboid in d dimensions can be cut into unit hypercubes with at most $\sum_{i=1}^{d} \lceil \log_2(n_i + 1) \rceil$ cuts.

Proof. We prove by induction on the dimension d.

Base case: If $d = 1$, and any collection of $n \times 1$ cuboids, the proof follows from Lemma 1.

Inductive hypothesis: Assume that for $d - 1$ dimensions, and any collection of hypercuboids of dimension $(n'_1, \dots, n'_{d-1})$ we can obtain unit hypercuboids with at most $\sum_{i=1}^{d-1} \lceil \log_2(n'_i + 1) \rceil$ cuts.

Inductive Step: Given a d dimensional hypercuboid of dimensions $(n_1, \dots, n_d)$, we can obtain n_d hypercuboids of dimension $d - 1$ by making $\lceil \log_2(n_d + 1) \rceil$ cuts by Lemma 1. Therefore, by the inductive hypothesis, we obtain □

7 Sperner Lemma

In this section, we will look at a beautiful problem with a remarkably elementary proof but with deep connections to other parts of mathematics. The problem is the following: Consider a triangle T in the plane. A *triangulation* of T is the partition of T into triangles. Note that we can partition T into an arbitrary number of triangles. Let the vertices of T be labeled $1, 2$ and 3. Figure **??** shows a triangulation along with a labeling of the vertices.

A *sperner labelling* is an assignment of one of the labels $\{1, 2, 3\}$ to the vertices of the triangulation that satisfies the following rules: A vertex on an edge of T can only get one of the labels on the end-points of the edge. For example, all vertices on the edge $1, 2$ of T can only receive labels 1 or 2. Similarly, the vertices on the edge $1, 3$ can only receive labels 1 or 3. There is no restriction on the labels assigned to the vertices inside T. They can receive any one of $1, 2$ or 3.

The question is the following: For any triangulation of T and any spener labeling of this triangulation, is there always a small triangle (i.e., the empty triangle) whose vertices receive all three labels, i.e., its three vertices are labeled $1, 2$ and 3. Such a triangle is called a *full triangle*. In fact, we can show something stronger - that is that there are always an *odd* number of full triangles. Go ahead and try it, and try to avoid constructing a full triangle.

As before, if the problem at hand is too complicated, let us consider the problem in a smaller dimension. What could be the 1-dimensional version of this problem? In the one-dimensional version, we have a line segment - one end of which is labeled 1, and the other end is labeled 2.

A *triangulation* of this line segment is a partitioning of the line segment into shortest segments by adding additional vertices. A *degenerate* triangle in this setting is a segment between two consecutive vertices.Now, a sperner labeling implies we can assign either the label 1 or the label 2 to each vertex. Then, we'll show that there are an odd number segments whose end-points are labeled $1, 2$, i.e., an odd number of *full triangles*. Figure 6 shows the one dimensional version of the problem. This question should remind you of the *intermediate value theorem* from calculus. Indeed, we can use this result to prove the intermediate value theorem!

Figure 6: A one dimensional version of the problem with a sperner labeling.

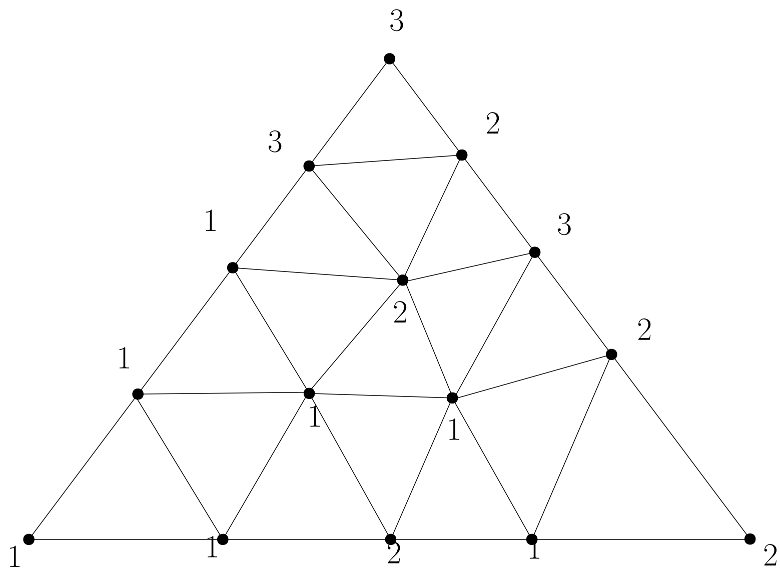


Figure 7: A sperner labeling of a triangulation of T.

But, for practice, let's use induction to prove this result.

Lemma 2. *Given a line segment L whose end-points are labeled 1 and 2, for any triangulation T of L and any sperner labelling of T, there are an odd number of full triangles.*

Proof. We prove by induction on the number n of points added in the triangulation.

Base Case: $n = 0$. Then, L itself is full triangle, and the statement of the lemma holds.

Inductive Hypothesis: For any triangulation T of L with n points, there are an odd number of full triangles.

Inductive Step: Consider a triangulation of L with $n + 1$ points. Remove an arbitrary interior point p. The resulting triangulation has n points and the induced labelling is sperner. Therefore, there are an odd number of full triangles. Let us add p back to L with its original label. Suppose the two consecutive points adjacent to p - the *neighbors* of p. had different labels, then removing p yielded a full triangle in the smaller triangulation. Adding p back, we retain one full triangle. If the two end-points had the same label, there are two cases. Either p had the same label its neighbors, in which case the segment induced by the neighbors of p did not induce a full triangle in the smaller triangulation, and therefore adding p back did not change the number of full triangles, and hence we still have an odd number of full triangles. On the other hand, suppose p had a label distinct from its neighbors, it creates two new full triangles, and therefore we again have an odd number of full triangles. □

Now, we are ready to prove the main theorem.

Theorem 10

Let T be a triangle. Then, for any triangulation of T and a sperner labeling of this triangulation, there are an odd number of full triangles.

We will give two proofs of this result.

First Proof: Fix a triangulation and an arbitrary sperner labeling. Imagine the triangles in the triangulation as a set of *rooms*. Let the edges labeled $\{1, 2\}$ be the *doors* of the rooms. For each door in the interior of T place a rock on either side of the door. For each door on the boundary of T, place a single rock in the interior of T. Figure 8 shows the placement of the rocks.

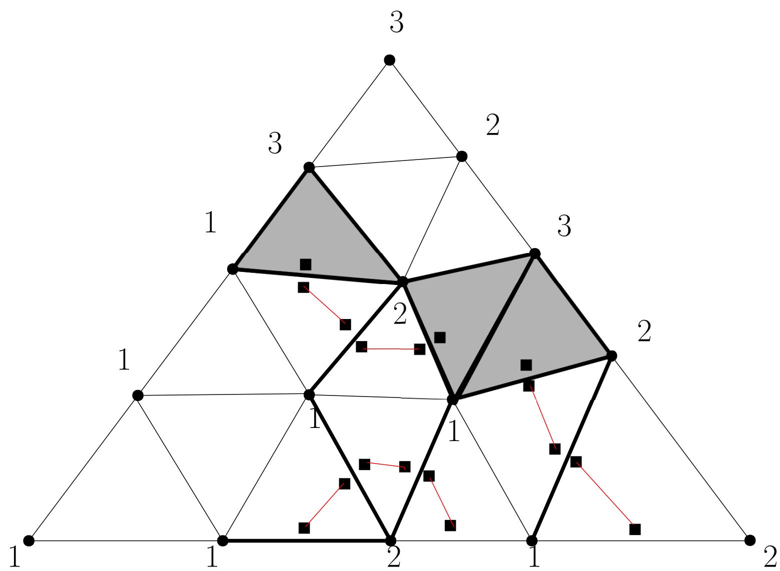


Figure 8: Each room has either 0 or 2 rocks. The only rooms with 1 rock are the full triangles.

It is immediately clear that the number of rocks placed is *odd*. This is because we can pair up the rocks placed adjacent to the interior doors. For the exterior doors, by Lemma 2 there are an odd number of doors, and hence an odd number of rocks placed beside these doors. Hence, the number of rocks placed is even.

Now, let us count the rocks another way *Counting the same thing in two different ways is a powerful trick in combinatorics*: Each room contains at most two rocks. Consider each non-full room in the interior. Since one side has labels $1, 2$, the third vertex of this room must be labeled 1 or 2. Therefore, each non-full room has *two* rocks placed inside them. There are an odd number of rocks. Therefore, there must exist an odd number of rooms with a single rock. Each such room is a full triangle. □

Second Proof: We place rocks as before - two at each internal door, and one at each external door. We are going to walk through a door as long as it has a rock beside it. When we walk, we will pick up the rock so that the door cannot be used again. Consider the walks that start from the exterior of T. Such a walk either exits T through another external door, or ends inside T. In the former case, we can *pair up* the two external doors encountered in the walk. In the later case, the walk must end in a full triangle - each room has either 0,1, or 2 stones. The only room with 1 stone are the full triangles. If a walk enters and exits a room, it must have picked up both rocks in the room. Therefore, if the walk gets stuck, it gets stuck in a room with exactly one stone, or in other words, a full triangle. By Lemma 2 there are an odd number external doors. Therefore, there must be an odd number of unpaired door. A walk from such a door must therefore end inside T. If the walk ends, it must do so in a room with a single door, in other words, a full triangle. Hence, we have an odd number of full triangles where a walk from the exterior of T ends in such a triangle.

We aren't done yet. There may still be additional rocks at internal doors. We claim that these will contribute an even number of full triangles. To see this, repeatedly do the following: Start with a room that contains a rock and start a walk. There are two possibilities: The room we started with is either a full triangle, or has two rocks. The walk can then terminate in one of two ways. Either the walk returns to the starting room, or ends in a full triangle. Let us analyze the cases. If the starting room was a full triangle, then the walk must end at a full triangle. Hence, we add 2 more full triangles. Now consider a walk that starts in a room with 2 rocks. There is one more rock in this room. The walk can cycle back to the starting room, in which case we add no more full triangles, or the walk ends at a full triangle. Now, consider the walk starting at the second rock in this room. This walk cannot cycle back to the starting room, and therefore must end at a full triangle. Hence, we add two additional full triangles. In all cases, we add an even number of full triangles in walks starting at an internal room. Figure 9 illustrates the proof

□

Exercise 11

Extend Sperner's lemma to 3 dimensions. In the three dimensional version of Sperner's lemma, we are given a tetrahedron T. The vertices are labeled with labels $1, 2, 3$ and 4. Given a *tetrahedralization* of T, a Sperner labeling is obtained by assigning a label of 1 or 2 on the edge 12, 1 or 3 on the edge 13 and so on. On the face 123 we allow only the labels $1, 2$ or 3. Inside the tetrahedron, there is no restriction. We want to show that there is a small tetrahedron with labels $1, 2, 3, 4$.

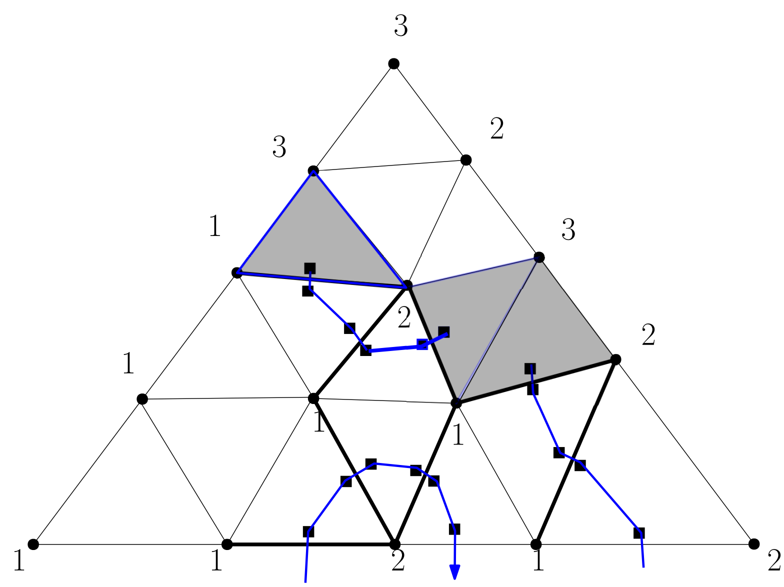


Figure 9: Each walk ends in a full triangle or exits the triangle.