Differentiation

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1 Introduction

Isaac Newton was not a pleasant man. His relations with other academics were notorious, with most of his later life spent embroiled in heated disputes.... A serious dispute arose with the German philosopher Gottfried Leibniz. Both Leibniz and Newton had independently developed a branch of mathematics called calculus, which underlies most of modern physics... Following the death of Leibniz, Newton is reported to have declared that he had taken great satisfaction in "breaking Leibniz's heart". —Stephen Hawking, A Brief History of Time, 1988......

The concept of derivative is the main theme of differential calculus, one of the major discoveries in mathematics, and in science in general. Differentiation is the process of finding the best local linear approximation of a function. The idea of the derivative comes from the intuitive concepts of velocity or rate of change, which are thought of as instantaneous or infinitesimal versions of the basic difference quotient $\frac{(f(x)-f(x_0))}{(x-x_0)}$, where f is a real-valued function defined in a neighborhood of x_0. A geometric way to describe the notion of derivative is the slope of the tangent line at some particular point on the graph of a function.This means that at least locally (that is, in a small neighborhood of any point), the graph of a smooth function may be approximated with a straight line. Our goal in this chapter is to carry out this analysis by making the intuitive approach mathematically rigorous.

Definition 1.1. A real valued function $f(x)$ defined on a domain D, containing a neighborhood of x_0, is said to be differentiable at x_0 if

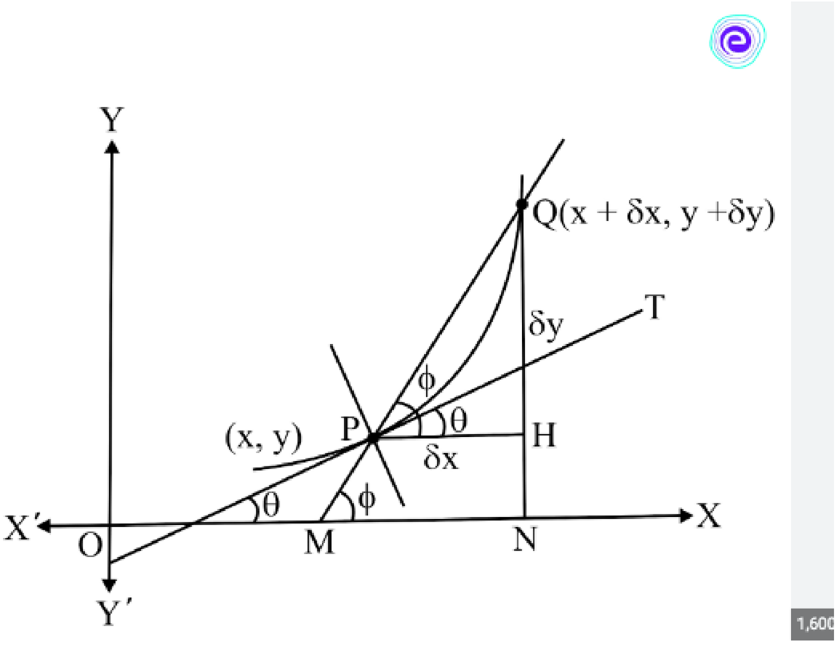
$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \tag{1}$$

$$\text{Or } \lim_{x\to x_0} \frac{f(x)-f(x_0)}{(x-x_0)} \tag{2}$$

exists. This limit is called the derivative of f at x_0, denoted by $f'(x_0)$, $\frac{df(x)}{dx}$.

Definition 1.2. A real valued function $f(x)$ defined on D is said to be differentiable if it is differentiable at every point in D.

Geometric Interpretation of Derivatives



Example 1.1. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x$. *Then*

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{x_0 + h - x_0}{h} = 1$$

Now x_0 *is any arbitary point in* $\mathbb{R}$. *So* f *is differentiable in* $\mathbb{R}$.

Example 1.2. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = c$ *where* $c \in \mathbb{R}$. *Then*

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0$$

Now x_0 *is any arbitary point in* $\mathbb{R}$. *So* f *is differentiable in* $\mathbb{R}$.

Example 1.3. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = |x|$. *Then at* $x = 0$

$$\frac{f(0 + h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

Now $\lim_{x\to 0} \frac{|x|}{x}$ *does not exist. (Take* $x_n = \frac{1}{n}$ *and* $y_n = \frac{-1}{n}$ *for all* $n \in \mathbb{N}$.*) Now* x_0 *is any arbitary point in* $\mathbb{R}$. *So* f *is differentiable in* $\mathbb{R}$.

Theorem 1.1. *Suppose* $f(x)$ *defined on a neighborhood of* x_0, *If* $f(x)$ *is differentiable at* x_0, *then it is continuous at* x_0.

Proof. Suppose x is a point in the neighborhood of x_0,(where f is defined). So for $x \neq x_0$, we may write

$$f(x) = (x - x_0)\frac{f(x) - f(x_0)}{(x - x_0)} + f(x_0)$$

Now taking limit $x \to x_0$, we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (x - x_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} + \lim_{x \to x_0} f(x_0)$$

Since $\lim_{x \to x_0}(x - x_0) = 0$ and $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{(x-x_0)} = f'(x_0)$ as f is differentiable at x_0. We have $\lim_{x \to x_0} f(x) = f(x_0)$. □

Remark 1.1. But the converse is not true i.e every continuous function is not differentiable. Example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is not differentiable at $x = 0$.

1.1 Algebra on Differentiable functions

Theorem 1.2. *Let f, g be differentiable at $c \in (a, b)$. Then $f \pm g$, fg and $\frac{f}{g}$ $(g(c) \neq 0)$ is also differentiable at c and*

$$1. (f \pm g)'(c) = f'(c) \pm g'(c)$$

$$2. (fg)'(c) = f'(c)g(c) + g'(c)f(c)$$

$$3. \left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)} \quad \textit{provided} \;\; g(c) \neq 0$$

Proof. Given

$$\lim_{x \to x_0} \frac{f(x) - f(c)}{(x - c)} = f'(c) \tag{3}$$

and

$$\lim_{x \to x_0} \frac{g(x) - g(c)}{(x - c)} = g'(c) \tag{4}$$

1. We have

$$\begin{aligned} \frac{(f \pm g)(x) - (f \pm g)(c)}{x - c} &= \frac{(f(x) \pm g(x)) - (f(c) \pm g(c))}{x - c} \\ &= \frac{(f(x) - f(c)) \pm (g(x) - g(c))}{x - c} \\ &= \frac{f(x) - f(c)}{x - c} \pm \frac{g(x) - g(c)}{x - c} \end{aligned}$$

Now taking the limit $x \to c$, we get the

$$\lim_{x \to c} \frac{(f \pm g)(x) - (f \pm g)(c)}{x - c} = = f'(c) \pm g'(c) \quad \text{by (\textbf{??}) and (4)}$$

2. We give the proof for product formula: First note that

$$
\begin{aligned}
\frac{(fg)(x)-(fg)(c)}{x-c} &= \frac{f(x)g(x)-f(c)g(c)}{x-c} \\
&= \frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c} \\
&= \frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c} \\
&= g(x)\frac{f(x)-f(c)}{x-c}+f(c)\frac{g(x)-g(c)}{x-c}
\end{aligned}
$$

Now taking the limit $x \to c$, we get the

$$
\begin{aligned}
\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} &= \lim_{x\to c} g(x)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}+f(c)\lim_{x\to c}\frac{g(x)-g(c)}{x-c} \\
&= g(c)f'(c)+f(c)g'(c) \quad \text{by (3) and (4)}
\end{aligned}
$$

and since $g(c) \neq 0$ and g is continuous, we get $g(x) \neq 0$ in a small interval around c say $(c-\delta, c+\delta)$, so that $\frac{1}{g(x)}$ is defined in $(c-\delta, c+\delta)$. Therefore

$$
\begin{aligned}
\frac{f}{g}(x)-\frac{f}{g}(c) &= \frac{f(x)}{g(x)}-\frac{f(c)}{g(c)} \\
&= \frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)} \\
&= \frac{f(x)g(c)-g(c)f(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)}
\end{aligned}
$$

Hence

$$
\frac{\frac{f}{g}(x)-\frac{f}{g}(c)}{x-c} = \{g(c)\frac{f(x)-f(c)}{x-c}-f(c)\frac{g(x)-g(c)}{x-c}\}\frac{1}{g(x)g(c)}
$$

Now taking the limit $x \to c$, we get the

$$
\begin{aligned}
\lim_{x\to c}\frac{\frac{f}{g}(x)-\frac{f}{g}(c)}{x-c} &= \left(g(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}-f(c)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\right)\lim_{x\to c}\frac{1}{g(x)g(c)} \\
&= \frac{g(c)f'(c)-f(c)g'(c)}{g^2(c)} \quad \text{by (3) and (4)}
\end{aligned}
$$

and since $g(c) \neq 0$ and g is continuous, we get $g(x) \neq 0$ in a small interval around c so $\lim_{x\to c}\frac{1}{g(x)} = \frac{1}{g(c)}$

□

Remark 1.2. Consider the function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x|x|$. Product rule not applicable to g but g is differentiable at $x = 0$.

Theorem 1.3. (Chain Rule): *Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then thee composition function $h(x) := (g \circ f)(x) = g(f(x))$ is differentiable at c and*

$$h'(c) = f'(c)g'(f(c))$$

Proof. Required to prove $\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{(g\circ f)(x)-(g\circ f)(c)}{x-c} = \lim_{x\to c} \frac{g(f(x))-g(f(c))}{x-c}$ exists and equal to $f'(c)g'(f(c))$. Define the function $F(y)$ as

$$F(y) = \left\{ \begin{array}{cc} \frac{(g(y)-g(f(c))}{y-f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c) \end{array} \right. .$$

Then the function F is continuous at $y = f(c)$ and $g(y)-g(f(c)) = F(y)(y-f(c))$. So

$$\frac{g(f(x)) - g(f(c))}{x-c} = \frac{F(f(x))(f(x)-f(c))}{x-c}$$

As limit $x \to c$, $f(x) \to f(c)$ which implies $F(f(x)) \to F(f(c))$ (the function F is continuous at $y = f(c)$). So $\lim_{x\to c} F(f(x)) = F(f(c))$.

$$\begin{aligned} \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c} &= \lim_{x\to c} F(f(x)) \lim_{x\to c} \frac{(f(x)-f(c))}{x-c} \\ \Rightarrow \quad \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c} &= F(f(c))f'(c) \\ \Rightarrow \quad \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c} &= g'(f(c))f'(c) \end{aligned}$$

□

Example 1.4. *Consider $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = (3x+1)^2$. Here $g(y) = y^2$ and $f(x) = 3x+1$. Then $g'(y) = 2y \ \forall \ y \in \mathbb{R}$ and $f'(x) = 3 \ \forall \ x \in \mathbb{R}$. Take any $c \in \mathbb{R}$.*

$$\begin{aligned} h'(c) &= f'(c)g'(f(c)) \\ &= 3.2f(c) \\ &= 6(3c+1) \end{aligned}$$

Since c is arbitary. $h'(x) = 6(3x+1) \ \forall \ x \in \mathbb{R}$.

Definition 1.3. Local extremum: A point $x = c$ is called local maximum of $f(x)$, if there exists $\delta > 0$ such that

$$0 < |x-c| < \delta \quad \Rightarrow \quad f(c) \geq f(x).$$

Similarly, one can define local minimum: $x = b$ is a local minimum of $f(x)$ if there exists

$$0 < |x-c| < \delta \quad \Rightarrow \quad f(c) \leq f(x).$$

Theorem 1.4. *Let $f(x)$ be a differentiable function on (a,b) and let $c \in (a,b)$ is a local maximum of f. Then $f'(c) = 0$.*

Proof. Let $\delta > 0$ be as in the above definition. Then

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(c) \geq f(x).$$

So

$$x \in (c, c+\delta) \quad \Rightarrow \quad \frac{f(x) - f(c)}{x - c} \leq 0 \quad \Rightarrow \quad f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

Again

$$x \in (c - \delta, c) \quad \Rightarrow \quad \frac{f(x) - f(c)}{x - c} \geq 0 \quad f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

□

From the above two conditions we can conclude $f'(c) = 0$.

Theorem 1.5. *Let $f(x)$ be a differentiable function on (a, b) and let $c \in (a, b)$ is a local minimum of f. Then $f'(c) = 0$.*

Proof. Same as the above. □

Remark 1.3. Is the converse true? Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. So $f'(0) = 0$. Then??

Remark 1.4. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. f has both an absolute maximum and minimum on the compact interval $[a, b]$. Theorem 1.4 and Theorem 1.5 implies these extrema must occur at points of the set

$$C := \{x \in (a, b) : f'(x) = 0\} \cup \{x \in [a, b] : f'(x) \text{ does not exist}\}.$$

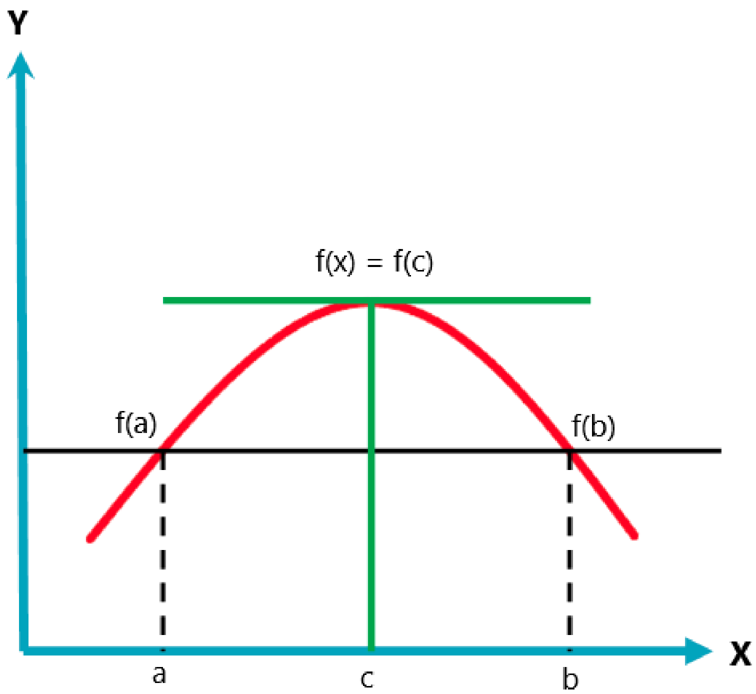
The elements of C are often called the critical points or critical numbers of f on $[a, b]$. To find the maximum and minimum values of f on $[a, b]$, it suffices to find its maximum and minimum on the smaller set C, which is often finite.

2 Differentiable Functions

Differentiation becomes most useful when a function has a derivative at each point of an interval.

The fundamental theorem about differentiable functions is the Mean Value Theorem. Following is its simplest form.

Theorem 2.1. ***Rolle's Theorem*** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differeniable on the open interval (a, b) and $f(a) = 0 = f(b)$, then there is at least one $c \in (a, b)$ such that $f'(c) = 0$.*



Proof. Since [a, b] is closed and bounded, implies the existence of $x_m, x_M \in [a,b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a,b]$. If $f(x_m) = f(x_M)$, then f is constant on $[a,b]$ and any $c \in (a,b)$ satisfies the lemma. Otherwise, either $f(x_m) < 0$ (since f achieves minimum at x_m, so $f(x_m) \leq f(a) = 0$) or $f(x_M) > 0$ (since f achieves maximum at x_M, so $f(x_M) \geq f(b) = 0$). If $f(x_m) < 0$, then $x_m \in (a,b)$ and Theorem 1.5 implies $f'(x_m) = 0$. If $f(x_M) > 0$, then $x_M \in (a,b)$ and Theorem 1.5 implies $f'(x_M) = 0$. □

Rolle's Theorem is just a stepping-stone on the path to the Mean Value Theorem. Two versions of the Mean Value Theorem follow. The first is a version more general than the one given in most calculus courses.

Theorem 2.2. *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on the open interval (a,b), and $f(a) = f(b)$, then there exists at least one $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Consider the function $g : [a,b] \to \mathbb{R}$ such that $g(x) = f(x) - f(a)$. Then g is continuous on $[a,b]$, differentiable on the open interval (a,b) with $g'(x) = f'(x)$, and $g(a) = g(b) = 0$. By Rolle's Theorem there exists at least one c in the open interval (a,b) such that $g'(c) = 0$ which implies $f'(c) = 0$. □

This version of Rolle's theorem is used to prove the mean value theorem, of which Rolle's theorem is indeed a special case. It is also the basis for the proof of Taylor's theorem.

Remark 2.1. If differentiability fails at an interior point of the interval, the conclusion of Rolle's theorem may not hold. Consider the absolute value function

$$f(x) = |x|, \qquad x \in [-1, 1]$$

Then $f(-1) = f(1)$, but there is no c between -1 and 1 for which the $f'(c) = 0$ is zero. This is because that function, although continuous, is not differentiable at $x = 0$. Note that the derivative of f changes its sign at $x = 0$, but without attaining the value 0. The theorem cannot be applied to this function because it does not satisfy the condition that the function must be differentiable for every x in the open interval. However, when the differentiability requirement is dropped from Rolle's theorem, f will still have a critical number in the open interval (a, b), but it may not yield a horizontal tangent (as in the case of the absolute value represented in the graph).

Theorem 2.3. (Cauchy Mean Value Theorem 1) *If $f : [a, b] \to \mathbb{R}$ and If $g : [a, b] \to \mathbb{R}$ are both continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one $c \in (a, b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

Let $h : [a, b] \to \mathbb{R}$ be a function defined by

$$h(x) = (g(b) - g(a))(f(a) - f(x)) + (f(b) - f(a))(g(x) - g(a))$$

Because of the assumptions on f and g, h is continuous on $[a, b]$ and differentiable on (a, b) with $h(a) = h(b) = 0$. Theorem 2.1 implies there is at least one $c \in (a, b)$ such that $h'(c) = 0$. Then

$$\begin{aligned} 0 = h'(c) &= -(g(b) - g(a))f'(c) + (f(b) - f(a))g'(c) \\ &\Rightarrow \quad (g(b) - g(a))f'(c) = (f(b) - f(a))g'(c) \end{aligned}$$

Hence we are done.

Theorem 2.4. (Cauchy Mean Value Theorem 2) *If $f : [a, b] \to \mathbb{R}$ and If $g : [a, b] \to \mathbb{R}$ are both continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists c in (a, b) such that*

$$\frac{(f(b) - f(a))}{(g(b) - g(a))} = \frac{f'(c)}{g'(c)}$$

Proof. As in the proof of the Mean Value Theorem, we introduce a function to which Rolle's Theorem will apply. First we note that since $g'(x) \neq 0$ for all x in (a, b), it follows from Rolle's Theorem that $g(a) \neq g(b)$. For x in $[a, b]$, we now define

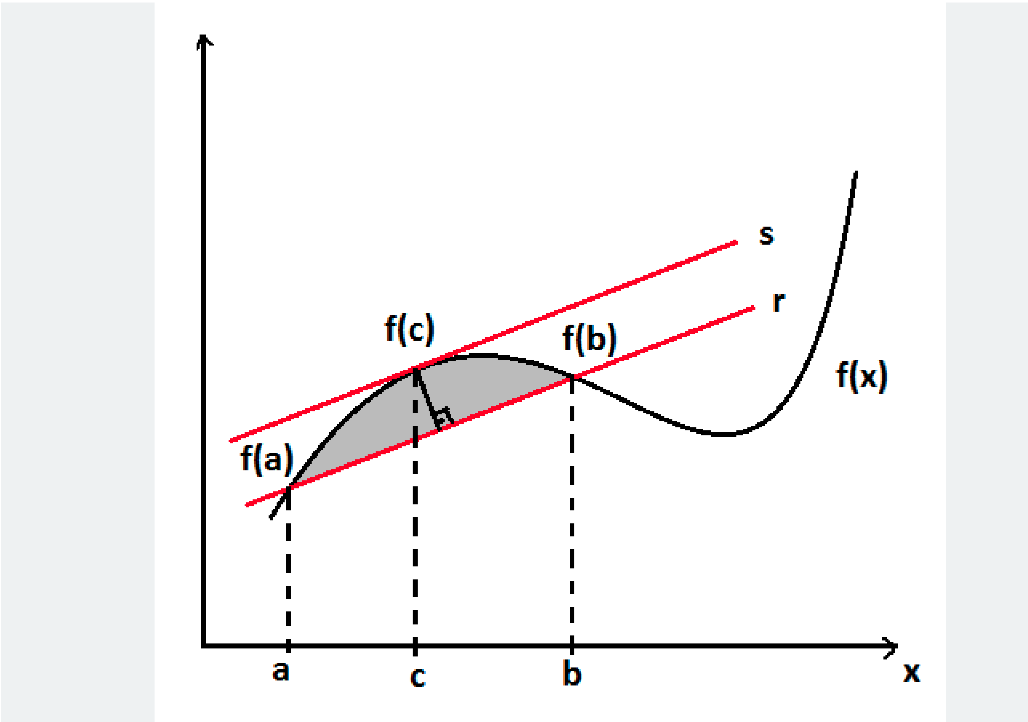
$$h(x) = (f(x) - f(a)) + \frac{(f(b) - f(a))}{(g(b) - g(a))}(g(x) - g(a))$$

Now apply Roll's Theorem. □

Corollary 2.5. Lagrange's Mean Value theorem *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on the open interval (a, b), then there exists at least one $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof. Take $g(x) = x$ in Theorem 2.3. □

Remark 2.2. Geometric interpretation



Lagrange's mean value theorem has a simple geometrical meaning. The chord passing through the points of the graph corresponding to the ends of the segment a and b has the slope equal to

$$k = \tan \alpha = \frac{f(b) - f(a)}{b - a}$$

Then there is a point $x = c \in (a, b)$ inside the interval where the tangent to the graph is parallel to the chord (joining A $(a, f(a))$ and B $(b,\ f(b))$).

Remark 2.3. If the derivative $f'(x)$ is zero at all points of the interval (a, b), then the function is constant on this interval. Indeed, for any two points x_1 and x_2 in the interval $[a, b]$, then there exists a point $c \in (x_1, x_2)$ such that

$$0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$f(x_2) = f(x_1)$$

As x_1 and x_2 were any two arbitary points in the interval $[a, b]$. So f is constant.

Remark 2.4. Show that the converse of the above is not true. That means f may have local maximum and minimum at $x = c$ but that its derivative has both positive and negative values in $(c-\delta, c)$ as well as $(c, c+\delta)$ for every $\delta > 0$.

Problem 2.1. Suppose f is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$, and $f(0) = 0$. Prove $x \leq f(x) \leq 2x$ for all $x \geq 0$.

Proof. Apply Lagrange's mean value theorem, for all $x \neq 0$

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(\zeta_x)$$

Now given $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$. In particular, putting $x = \zeta_x$, we have $1 \leq f'(\zeta_x) \leq 2$. Hence

$$1 \leq \frac{f(x)}{x} = f'(\zeta_x) \leq 2 \forall\ x \neq 0$$
$$x \leq f(x) \leq 2x \quad \forall\ x \neq 0$$

Since the above inequality holds for $x = 0$, so we are done. $\square$

Problem 2.2. Let f be continuous on $[a, b]$ and differentiable at every point in (a, b). Suppose there exists $c \in \mathbb{R}$ such that $f'(x) = c\ \forall\ x \in (a, b)$. Then there exists $A \in \mathbb{R}$ such that

$$f(x) = cx + A \quad \forall\ x \in (a, b)$$

In particular, $f'(x) = 0 \quad \forall\ x \in (a, b)$, then f is a constant function.
To see this consider $x_0 \in (a, b)$. Then for any $x \in [a, b]$, there exists η_x lying between x_0 and x (by MVT) such that

$$f(x) - f(x_0) = f'(\eta_x)(x - x_0) = c(x - x_0)$$

. Hence, $f(x) = f(x_0) + c(x - x_0)$. Thus, $f(x) = cx + A$ with $A = f(x_0) - cx_0$.

Problem 2.3. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).
1) Suppose $f'(x) \neq 0\ \forall\ x \in (a, b)$, then f is one-one (i.e $f(x) \neq f(y)$ whenever $x \neq y$.)
2) If $f'(x) \geq 0$ (resp. $f'(x) > 0$) $\forall\ x \in [a, b]$, then f is increasing (resp strictly increasing) on $[a, b]$. (We have a similar result for decreasing functions.)

1) Suppose, $x, y \in [a, b]$ and $x \neq y$. We can apply Mean value theorem, to conclude that there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

But $f'(c) \neq 0$ which implies $f(x) \neq f(y)$.

2)Let x_1, x_2 be any two points in $[a, b]$. Then applying MVT, there exists $c \in (x_1, x_2)$ such that

$$0 \leq f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If $x_1 \leq x_2$ then from the above we can conclude $f(x_1) \leq f(x_2)$.

Problem 2.4. a)Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$.

b) Using the above prove that $\sin x \leq x \ \forall \ x \geq 0$.

Proof. a) Let x be any real number with $x \geq 0$,

$$\begin{aligned} h(x) &= g(x) - f(x) \\ h'(x) &= g'(x) - f'(x) \geq 0 \text{ by hypothesis} \end{aligned}$$

So h is increasing function (by 2) of Problem 2.2), so $h(x) \geq h(0) = 0 \ \forall \ x \geq 0$ implying $g(x) \geq f(x) \ \forall \ x \geq 0$.
b) Here $g(x) = x$ and $f(x) = \sin x$. Then $f(0) = g(0) = 0$ and $f'(x) = \cos x \leq g'(x) = 1$ for all $x \in \mathbb{R}$. $\square$

Problem 2.5. Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. Suppose f' exists $f'(x) \leq 1 \ \forall \ x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a, a)$.
Let $g(x) = f(x) - x$ on $[-a, a]$. Note that $g'(x) = f'(x) - 1 \leq 1 - 1 = 0$ on $(-a, a)$. Therefore, g is decreasing. Since $g(a) = g(-a) = 0$, we have $g(x) = 0$ for every $x \in (-a, a)$.

Problem 2.6. Use the mean value theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
Consider $f(t) = \sin t$ for all $t \in \mathbb{R}$. Let $x, y \in \mathbb{R}$. By the mean value theorem, there exists $c \in (x, y)$, such that $\sin x - \sin y = (x - y)\cos c$. Hence $|\sin x - \sin y| \leq |x - y|$.

Similarly, using the mean value theorem to prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Problem 2.7. Use the mean value theorem to prove that $e^x \geq 1 + x$ for all $x > 0$.
Consider $f(t) = e^x$. Let $x, 0 \in \mathbb{R}$. By the mean value theorem, there exists $c \in (0, x)$, such that

$$\frac{e^x - 1}{x} = \frac{e^x - e^0}{x - 0} = e^c$$

. Hence using $e^c > 1$ we get the result.

Similarly, using the mean value theorem to prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Problem 2.8. Let f be twice differentiable on $[0,2]$. Show that if $f(0) = 0$, $f(1) = 2$ and $f(2) = 4$, then there is $x_0 \in (0,2)$ such that $f^{(2)}(x_0) = 0$.
By the mean value theorem there exist $x_1 \in (0,1)$ and $x_2 \in (1,2)$ such that

(5)
$$f'(x_1) = \frac{f(1) - f(0)}{1 - 0} = 2 - 0 = 2 \quad \text{and} \quad f'(x_2) = \frac{f(2) - f(1)}{2 - 1} = (4 - 2) = 2$$

Apply Rolle's theorem to f' on $[x_1, x_2]$. So there exists $x_0 \in (x_1, x_2) \subset (0, 2)$ such that $f^{(2)}(x_0) = 0$.

Problem 2.9. Using the above result, show that the following function is uniformly continuous.

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 \quad x = 0 \end{cases}$$

Theorem 2.6. (Intermediate Value Theorem for Derivatives/ Darboux Theorem) *Assume $f : (a,b) \to \mathbb{R}$ be a differentiable function. Whenever $a < x_1 < x_2 < b$ and y lies between $f'(x_1)$ and $f'(x_2)$ (that is $f'(x_1) < y < f'(x_2)$ or $f'(x_2) < y < f'(x_1)$), there exists c in (x_1, x_2) such that $y = f'(c)$.*

Problem 2.10. Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.
(a) Show $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$. (b) Show $f^(x) = \frac{1}{7}$ for some $x \in (0,2)$.

Proof. By the mean value theorem there exist $x_1 \in (0,1)$ and $x_2 \in (1,2)$ such that

$$f'(x_1) = \frac{f(1) - f(0)}{1 - 0} = 1 - 0 = 1 \quad \text{and} \quad f'(x_2) = \frac{f(2) - f(1)}{2 - 1} = (1 - 1) = 0.$$

Now $0 < x_1 < x_2 < 2$ and $\frac{1}{2}, \frac{1}{7}$ are the two points lying between $f'(x_1) = 1$ and $f'(x_2) = 0$. Hence by intermediate value theorem, there exist $y_1 \in (0,2)$ and $y_2 \in (0,2)$ such that $f'(y_1) = \frac{1}{2}$ and $f'(y_2) = \frac{1}{7}$. $\square$

Theorem 2.7. (First Derivative Test) *Let f be continuous on the interval $I := [a,b]$ and let c be an interior point of I. Assume that f is differentiable on (a,c) and (c,b). Then:*
a) If there is a neighborhood $(c-\delta, c+\delta)$ such that $f'(x) \geq 0$ for all x such that $c - \delta < x < c$ and $f'(x) \leq 0$ for all x such that $c < x < c+\delta$, then f has a local maximum at c.
b) If there is a neighborhood $(c-\delta, c+\delta)$ such that $f'(x) \geq 0$ for all x such that $c < x < c+\delta$ and $f'(x) \leq 0$ for all x such that $c - \delta < x < c$, then f has a local minimum at c.

Proof. If $x \in (c-\delta, c)$, then it follows from the Mean Value Theorem that there exists a point $c_x \in (c-\delta, c)$ such that $f(c) - f(x) = f'(c_x)(c-x)$. Since $f'(x) \geq 0$ for all x such that $c - \delta < x < c$, so $f'(c_x) \geq 0$ we infer $f(c) \geq f(x)$ for all x such that $c - \delta < x < c$. Similarly, $f(c) \geq f(x)$ for all x such that $c < x < c + \delta$. Therefore, $f(c) \geq f(x)$ for all $x \in (c - \delta, c + \delta)$, so that f has a relative maximum at c.
(b) The proof is similar. □

Remark 2.5. The converse of the above theorem is not true. For example $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^4 + x^4 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Now f has an absolute minimum at $x = 0$ (why?). $f'(x) = 8x^3 + 4x^3 \sin(1/x) - x^2 \cos(1/x)$. And $f'(0) = 0$. But $f'(x)$ takes both positive as well as negative values around every neighbourhood of $x = 0$ (check).

3 Taylor's Theorem

Remark 3.1. In the last few lectures, we discussed the mean value theorem (which relates a function and its derivative) and its applications. We will now discuss a result called Taylor's Theorem which relates a function, its derivative and its higher derivatives. We will see that Taylor's Theorem is an extension of the mean value theorem. Though Taylor's Theorem has applications in numerical methods, inequalities and local maximum and minimum, it deals with approximation of functions by polynomials. To understand this type of approximation let us start with the linear approximation or tangent line approximation.

Let f be a k times differentiable function on an interval $[a, b]$ of $\mathbb{R}$. We want to approximate this function by a polynomial $P_n(x)$ such that $P_n(c) = f(c)$ at a point c. Moreover, if the derivatives of f and P_n also equal at c then we see that this approximation becomes more accurate in a neighbourhood of c. So the best coefficients of the polynomial can be calculated using the relation $f^{(k)}(c) = P_n^{(k)}(c)$ for $k = 0, 1, 2, ..., n$. The best is in the sense that if $f(x)$ itself is a polynomial of degree less than or equal to n, then both f and P_n are equal. This implies that the polynomial is $\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$. Then we write $f(x) = P_n(x) + R_n(x)$ in a neighbourhood of c. From this, we also expect the $R_n(x) \to 0$ as $x \to c$. We have the following theorem known as Taylor's theorem:

Definition 3.1. Linear Approximation: Let f be a function, differentiable at $x_0 \in \mathbb{R}$. Then the linear polynomial

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

is the natural linear approximation to $f(x)$ near x_0. Geometrically, this is clear because we approximate the curve near $(x_0, f(x_0))$ by the tangent line at $(x_0, f(x_0))$. The following result provides an estimation of the size of the error $E_1(x) = f(x) - P_1(x)$.

Theorem 3.1. (Taylor's Theorem) *If f and its derivative of order m are continuous and $f^{(m+1)}(x)$ exists in a neighbourhood of a. Then there exists $c_x \in (a,x)$ or $c_x \in (x,a)$ such that*

$$f(x) = f(a) + f'(a)(x-a) + f^{(2)}(a)\frac{(x-a)^2}{2!} + \dots\dots\dots + f^{(m)}(a)\frac{(x-a)^m}{m!} + R_m(x)$$

where $R_m(x) = f^{(m+1)}(c_x)\frac{(x-a)^{(m+1)}}{(m+1)!}$

Proof. Let $x_0 \in (a-\delta, a+\delta) \setminus \{a\}$ be any arbitrary point. Define the function F defined in a a neighbourhood of a as follows:

(6)

$$F(y) = f(x_0) - f(y) - f'(y)(x_0 - y) - f^{(2)}(y)\frac{(x_0-y)^2}{2!} + \dots\dots\dots - f^{(m)}(y)\frac{(x-y)^m}{m!}$$

Hence $F(x_0) = 0$ and

(7)

$$F(a) = f(x_0) - f(a) - f'(a)(x_0 - a) - f^{(2)}(a)\frac{(x_0-a)^2}{2!} + \dots\dots\dots - f^{(m)}(a)\frac{(x_0-a)^m}{m!}$$

We will show $F(a) = f^{(m+1)}(c_{x_0})\frac{(x_0-a)^{(m+1)}}{(m+1)!}$ for some $c_{x_0} \in (a, x_0)$ or (x_0, a).

$$F'(y) = -f'(y) - f^{(2)}(y)(x_0-y) + f'(y) - f^{(3)}(y)\frac{(x-y)^2}{2!} + f^{(2)}(y)(x_0-y) + \dots\dots\dots\dots\dots$$
$$\dots\dots\dots - f^{(m+1)}(y)\frac{(x_0-y)^m}{m!}$$

which implies

$$F'(y) = -f^{(m+1)}(y)\frac{(x_0-y)^m}{m!} \tag{8}$$

Now we consider another function g defined in a a neighbourhood of a as follows:

$$g(y) = F(y) - \frac{(x_0-y)^{m+1}}{(x_0-a)^{m+1}}F(a)$$

Now

$$g'(y) = F'(y) + (m+1)\frac{(x_0-y)^m}{(x_0-a)^{m+1}}F(a) \tag{9}$$

We have $g(a) = 0$ and $g(x_0) = F(x_0) = 0$. Therefore by Roll's theorem, there exists $c_{x_0} \in (a, x_0)$ or (x_0, a) such that $g'(c_{x_0}) = 0$. So by (9)

$$0 = g'(c_{x_0}) = F'(c_{x_0}) + (m+1)\frac{(x-c_{x_0})^m}{(x_0-a)^{m+1}}F(a)$$

$$F'(c_{x_0}) = -(m+1)\frac{(x_0-c_{x_0})^m}{(x_0-a)^{m+1}}F(a) \tag{10}$$

On the other hand by (8),

$$F'(c_{x_0}) = -f^{(m+1)}(c_{x_0})\frac{(x-c_{x_0})^m}{m!} \tag{11}$$

So equating (10) and (11), we have

$$(m+1)\frac{(x_0-c_{x_0})^m}{(x_0-a)^{m+1}}F(a) = f^{(m+1)}(c_x)\frac{(x_0-c_{x_0})^m}{m!}$$

$$F(a) = f^{(m+1)}(c_{x_0})\frac{(x_0-a)^{m+1}}{(m+1)!} \tag{12}$$

Again equating (7) and(12)

$$f(x_0) - f(a) - f'(a)(x_0-a) - f^{(2)}(a)\frac{(x_0-a)^2}{2!} + \dots\dots\dots - f^{(m)}(a)\frac{(x-a)^m}{m!} = f^{(m+1)}(c_{x_0})\frac{(x_0-a)^{m+1}}{(m+1)!}$$

$$f(x_0) = f(a) + f'(a)(x_0-a) + f^{(2)}(a)\frac{(x_0-a)^2}{2!} + \dots\dots\dots + f^{(m)}(a)\frac{(x_0-a)^m}{m!} + f^{(m+1)}(c_x)\frac{(x_0-a)^{m+1}}{(m+1)!}$$

Since x_0 is arbitrary, hence the result. □

Remark 3.2. In case of $a = 0$, the formula obtained in Taylor's theorem is known as Maclaurin's formula.

Remark 3.3. Suppose f is infinitely differentiable at a and if the remainder term in the Taylor's formula, $R_n(x) \to 0$ as $n \to \infty$. Then we say that the Taylor series converges at the point x. So we may formally write

$$f(x) = \sum_{m=1}^{\infty} f^{(m)}(a)\frac{(x-a)^m}{m!}$$

For fixed x the above infinite sum is a series of real numbers. One can check the convergence of such series by the convergence tests. This series is called Taylor series of $f(x)$ about the point a.

Problem 3.1. Show that $1-\frac{1}{2}x^2 \leq \cos x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Take $f(x) = \cos x$ and $a = 0$ in Taylor's Theorem. Then there exists $c \in (0, x)$

$$\cos x = \cos 0 + (x-0)(-\sin 0) + \frac{(x-0)^2}{2!}(-\cos 0) + \frac{(x-0)^3}{3!}(\sin 0) + \frac{(x-0)^4}{4!}(\cos c)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{x^4}{4!}(\cos c)$$

Now $\cos y \geq 0$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So we can conclude

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{x^4}{4!}(\cos c) \geq 1 - \frac{1}{2}x^2$$

Theorem 3.2. *Let f be a real valued function that is differentiable $2n$ times in a neighbourhood of a and $f^{(2n)}$ is continuous at $x = a$. Then*
1. If $f^{(k)}(a) = 0$ for $k = 1, 2, \dots 2n-1$ and $f^{(2n)}(a) > 0$, then 'a' is a point of local minimum of $f(x)$.
2. If $f^{(k)}(a) = 0$ for $k = 1, 2, \dots 2n-1$ and $f^{(2n)}(a) < 0$, then 'a' is a point of local maximum of $f(x)$.
3. $f^{(k)}(a) = 0$ for $k = 1, 2, \dots 2n-2$ and $f^{(2n-1)}(a) \neq 0$, then 'a' is called a point of inflection. i.e., f has neither local maximum nor local minimum at $x = a$.

Proof. Since $f^{(2n)}(a) > 0$ and $f^{(2n)}$ is continuous at $x = a$. The there exists $\delta_1 > 0$ such that $f^{(2n)}(y) > 0$ for all $y \in (a - \delta_1, a + \delta_1)$. And by Theorem 3.1, for any $x \in (a - \delta_1, a + \delta_1)$ there exists $c_x \in (a, x)$ or (x, a),

$$\begin{aligned}
f(x) = f(a) + f'(a)(x-a) + f^{(2)}(a)\frac{(x-a)^2}{2!} + \dots\dots\dots + f^{(2n-1)}(a)\frac{(x-a)^{2n-1}}{(2n-1)!} \\
+ f^{(2n)}(c_x)\frac{(x-a)^{2n}}{(2n)!}
\end{aligned}$$

$$f(x) = f(a) + f^{(2n)}(c_x)\frac{(x-a)^{2n}}{(2n)!}$$

$$f(x) - f(a) = f^{(2n)}(c_x)\frac{(x-a)^{2n}}{(2n)!}$$

So for all $x \in (a - \delta_1, a + \delta_1)$, $(x - a)^{2n} \geq 0$ and $f^{(2n)}(c_x) \geq 0$ (as $f^{(2n)}(y) > 0$ for all $y \in (a - \delta_1, a + \delta_1)$. So $f(x) - f(a) \geq 0$ implying $f(x) \geq f(a)$. So 'a' is a point of local minimum of $f(x)$.

2) Can be proved similarly.

3)Since $f^{(2n-1)}(a) \neq 0$, either $f^{(2n-1)}(a) > 0$ or $f^{(2n-1)}(a) < 0$. With out loss of generality, let $f^{(2n-1)}(a) > 0$. As $f^{(2n-1)}(x)$ is continuous at $x = a$. The there exists $\delta_1 > 0$ such that $f^{(2n-1)}(y) > 0$ for all $y \in (a - \delta_1, a + \delta_1)$. And by Theorem 3.1, for any $x \in (a - \delta_1, a + \delta_1)$ there exists $c_x \in (a, x)$ or (x, a),

$$\begin{aligned}
f(x) = f(a) + f'(a)(x-a) + f^{(2)}(a)\frac{(x-a)^2}{2!} + \dots\dots\dots + f^{(2n-2)}(a)\frac{(x-a)^{2n-2}}{(2n-2)!} \\
+ f^{(2n-1)}(c_x)\frac{(x-a)^{2n-1}}{(2n-1)!}
\end{aligned}$$

$$f(x) = f(a) + f^{(2n-1)}(c_x)\frac{(x-a)^{2n-1}}{(2n-1)!}$$

$$f(x) - f(a) = f^{(2n-1)}(c_x)\frac{(x-a)^{2n-1}}{(2n-1)!}$$

Case 1) If $x \in (a - \delta_1, a)$, then $(x - a)^{2n-1} \leq 0$ and $f^{(2n-1)}(c_x) \geq 0$ (as $f^{(2n-1)}(y) > 0$ for all $y \in (a - \delta_1, a + \delta_1)$). So $f(x) - f(a) \leq 0$ implies $f(x) \leq f(a)$. So $\forall\ x \in (a - \delta_1, a)$, we have $f(x) \leq f(a)$

Case 2) If $x \in (a, a + \delta_1)$, then $(x - a)^{2n-1} \geq 0$ and $f^{(2n-1)}(c_x) \geq 0$ (as $f^{(2n-1)}(y) > 0$ for all $y \in (a - \delta_1, a + \delta_1)$). So $f(x) - f(a) \geq 0$ implies $f(x) \geq f(a)$. So $\forall\ x \in (a, a + \delta_1)$, we have $f(x) \geq f(a)$.

Hence $f(x) \leq f(a)\ \forall\ x \in (a - \delta_1, a)$ and at the same time $f(x) \geq f(a)\ \forall\ x \in (a, a + \delta_1)$. So f has neither local maximum nor local minimum at $x = a$. □

Problem 3.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$. Now $f'(x) = 4x^3$

4 L'Hospital Rule

Remark 4.1. In mathematics, more specifically calculus, L'Hôpital's rule or L'Hospital's rule (French: [lopital]) provides a technique to evaluate limits of indeterminate forms. Application (or repeated application) of the rule often converts an indeterminate form to an expression that can be easily evaluated by substitution. The rule is named after the 17th-century French mathematician Guillaume de l'Hôpital. Although the rule is often attributed to L'Hôpital, the theorem was first introduced to him in 1694 by the Swiss mathematician Johann Bernoulli.

According to the properties of limits when $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ both exist, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

as long as $\lim_{x\to x_0} g(x) \neq 0$. But, it is easy to find examples where both $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$ and $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists (for example take $f(x) = ax$ and $g(x) = x$ then $\lim_{x\to 0} \frac{f(x)}{g(x)} = a$ but $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$), as well as similar examples where $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ fails to exist. Because of this, such a limit problem is said to be in the indeterminate form and called $\frac{0}{0}$ form. The following theorem allows us to determine many such limits.

There are several versions or forms of L'Hospital rule.

4.1 A Preliminary Result

Theorem 4.1. A Preliminary Result *Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions on $[a, b]$ and differentiable at every point in (a, b) Let $x_0 \in (a, b)$ and $f(x_0) = 0 = g(x_0)$ and $g'(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq (a, b)$. Then*

$$\frac{f'(x_0)}{g'(x_0)} = \lim_{x\to x_0} \frac{f(x)}{g(x)} \tag{13}$$

Proof. Since $g'(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$, that will ensure $g(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Suppose $g(y) = 0$ for some $y \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. Then by Lagrange's Mean value Theorem, there exists a $c \in (y, x_0)$ or (x_0, y) such that $0 = g(y) = g(y) - g(x_0) = (y - x_0)g'(c)$. Since $y \neq x_0$ that will imply $g'(c) = 0$ and that will contradict the fact $g'(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. We note that

$$\begin{aligned}
\frac{f'(x_0)}{g'(x_0)} &= \frac{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}} \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \qquad \text{as } g'(x_0) \neq 0 \\
&= \lim_{x\to x_0} \frac{f(x)}{g(x)} \qquad \text{as } f(x_0) = g(x_0) = 0
\end{aligned}$$

□

Remark 4.2. The condition $g'(x) \neq 0$ for all $x \in (x_0-\delta, x_0+\delta)$ can be replaced by that f' and g' are continuous functions and $g'(x_0) \neq 0$.

Remark 4.3. The condition $f(x_0) = g(x_0) = 0$ is essential in the previous result. For example,

$$\lim_{x\to 0} \frac{x+17}{2x+3} = \frac{17}{3}$$

But

$$\frac{f'(x_0)}{g'(x_0)} = \frac{1}{2}$$

Example 4.1. *Prove that the* $\lim_{x\to 0} \frac{e^x-1}{x^2+x} = 1$.
Here $f(x) = e^x - 1$ *and* $g(x) = x^2 + x$ *and* $f(0) = g(0) = 0$. *Also* $f'(0) = 1$ *and* $g'(0) = 1$. *Infact,* $g'(x) = 1 \neq 0$ *for all* x. *Hence by preliminary result*

$$\lim_{x\to x_0} \frac{e^x-1}{x^2+x} = \frac{f'(x_0)}{g'(x_0)} = \frac{1}{1} = 1$$

4.2 L'Hospital Rule I

Theorem 4.2. (L'Hospital Rule I that is $[\frac{0}{0}]$ form) *Let us suppose* $-\infty \leq a < b \leq +\infty$ *and let* f, g *are two differentiable functions on* (a, b) *such that* $g'(x) \neq 0$ *for all* $x \in (a, b)$. *Suppose* $\lim_{x\to a+} f(x) = 0 = \lim_{x\to a+} g(x)$.
1) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ *implies* $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.
2) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$ *implies* $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.
The same result holds for

- *the left-hand limit* $\lim_{x\to b-}$.
- *the two-sided limit* $\lim_{x\to x_0}$ *where* $x_0 \in (a, b)$ *and if* f *and* g *are differentiable except possibly at* $x_0 \in (a, b)$.

Proof. Let us prove Case 1). Case 1) If $L \in \mathbb{R}$ and if $\varepsilon > 0$ be given, then there exists $\delta > 0$ such that

$$L - \frac{\varepsilon}{2} < \frac{f'(u)}{g'(u)} < L + \frac{\varepsilon}{2} \ \forall\ u \in (a, a+\delta) \tag{14}$$

If $a < x < y < a + \delta$, then Rolle's Theorem implies that $g(y) \neq g(x)$. Further, by the Cauchy Mean Value version 2, there exists $u_{x,y} \in (x, y) \subset (a, a+\delta)$ such that

$$\frac{f(y) - f(x)}{g(x) - g(y)} = \frac{f'(u_{x,y})}{g'(u_{x,y})} \tag{15}$$

Since $u_{x,y} \in (a, a+\delta)$ then it satisfies (14). Hence (14) and (15) together implies

$$L - \frac{\varepsilon}{2} < \frac{f'(u_{x,y})}{g'(u_{x,y})} = \frac{f(y) - f(x)}{g(y) - g(x)} < L + \frac{\varepsilon}{2} \;\forall\; x, y \in (a, a+\delta)$$

Giving

$$L - \frac{\varepsilon}{2} < \frac{f(y) - f(x)}{g(y) - g(x)} < L + \frac{\varepsilon}{2} \;\forall\; x, y \in (a, a+\delta) \tag{16}$$

As x is any arbitrary points in (a, b) taking limit $x \to a+$, we have

$$\lim_{x \to a+} \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(y) - \lim_{x \to a+} f(x)}{g(y) - \lim_{x \to a+} g(x)} = \frac{f(y)}{g(y)},$$

as $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$. (given)

Hence using ordered property of limit in (16), we find that

$$L - \varepsilon < L - \frac{\varepsilon}{2} <\leq \lim_{x \to a+} \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(y)}{g(y)} \leq L + \frac{\varepsilon}{2} < L + \varepsilon \;\forall\; y \in (a, a+\delta)$$

Hence For a given $\varepsilon > 0$, we have found $\delta > 0$ such that

$$\left| \frac{f(y)}{g(y)} - 0 \right| < \varepsilon \quad y \in (a, a+\delta)$$

Hence the result.

Case 2) is Homework.

$\square$

Example 4.2. *Prove that* $\lim_{x \to 0} \frac{2x - \sin 2x}{x^2 \sin x} = 4/3$.
Consider $F : (-\pi/2, \pi/2) \to \mathbb{R}$ *defined by* $F(x) = \frac{2x - \sin 2x}{x^2 \sin x}$. *So* $\lim_{x \to 0} F(x)$ *has the indeterminate form* $\frac{0}{0}$. *Required to find* $\lim_{x \to 0} F(x)$.
Now take $f(x) = 2x - \sin 2x$ *and* $g(x) = x^2 \sin x$. *And* $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$, *so* $\lim_{x \to 0} F(x) = \lim_{x \to 0} \frac{f(x)}{g(x)} = [\frac{0}{0}]$. *Although* $g'(x) = 2x \sin x + x^2 \cos x \neq 0$ *in* $(-\pi/2, \pi/2) \setminus \{0\}$ *but* $\lim_{x \to 0} f'(x) = \lim_{x \to 0}(2 - 2\cos 2x) = 0 = \lim_{x \to 0} g'(x) = \lim_{x \to 0}(2x \sin x + x^2 \cos x) = 0$. *So* $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = [\frac{0}{0}]$. *Therefore we can not apply L'Hospital I directly.*
Now, let $f'(x) = f_1(x)$ *and* $g'(x) = g_1(x)$, *then* $\lim_{x \to 0} f_1(x) = 0 = \lim_{x \to 0} g_1(x)$ *and* $g_1'(x) = g''(x) = 2\sin x + 4x \cos x - x^2 \cos x \neq 0$ *in* $(-\pi/2, \pi/2) \setminus \{0\}$ *(Please check). Now* $\lim_{x \to 0} f_1'(x) = \lim_{x \to 0}(4 \sin 2x) = 0 = \lim_{x \to 0} g_1'(x) = \lim_{x \to 0} 2\sin x + 4x \cos x - x^2 \cos x = 0$. *So* $\lim_{x \to 0} \frac{f_1'(x)}{g_1'(x)} = [\frac{0}{0}]$. *Therefore we can not apply L'Hospital I directly.*
Now, let $f_1'(x) = f_2(x)$ *and* $g_1'(x) = g_2(x)$, *then* $\lim_{x \to 0} f_2(x) = 0 = \lim_{x \to 0} g_2(x)$ *and* $g_2'(x) = g'''(x) = 4\cos x - 4x \sin x + x^2 \sin x \neq 0$ *in* $(-\pi/2, \pi/2) \setminus \{0\}$ *(Please check). Now* $\lim_{x \to 0} f_2'(x) = \lim_{x \to 0}(8 \cos 2x) = 8$ *and* $\lim_{x \to 0} g_2'(x) = 6$. *So*

$$\lim_{x \to 0} \frac{f_2'(x)}{g_2'(x)} = \frac{\lim_{x \to 0} f_2'(x)}{\lim_{x \to 0} g_2'(x)} = \frac{4}{3}$$

. *Therefore we can apply L'Hospital I directly to obtain*

$$\lim_{x\to 0}\frac{f_1'(x)}{g_1'(x)} = \lim_{x\to 0}\frac{f_2(x)}{g_2(x)} = \lim_{x\to 0}\frac{f_2'(x)}{g_2'(x)} = \frac{4}{3}$$

Again we apply L'Hospital I directly to f' *and* g' *to obtain*

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f_1(x)}{g_1(x)} = \lim_{x\to 0}\frac{f_1'(x)}{g_1'(x)} = \frac{4}{3}$$

Since $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = 4/3$. *Hence again by L'Hospital I to* f *and* g *we obtain*

$$\lim_{x\to 0}\frac{2x-\sin 2x}{x^2\sin x} = \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{4}{3}$$

4.3 L'Hospital Rule II

Theorem 4.3. (L'Hospital Rule II that is $[\frac{0}{0}]$ form) *Let us suppose* $-\infty \leq a < b \leq +\infty$ *and let* f, g *are two differentiable functions on* (a, b) *such that* $g'(x) \neq 0$ *for all* $x \in (a, b)$. *Suppose* $\lim_{x\to a+}|f(x)| = \infty = \lim_{x\to a+}|g(x)|$.
1) If $\lim_{x\to a+}\frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ *implies* $\lim_{x\to a+}\frac{f(x)}{g(x)} = L$.
2) If $\lim_{x\to a+}\frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$ *implies* $\lim_{x\to a+}\frac{f(x)}{g(x)} = L$.
The same result holds for

- *the left-hand limit* $\lim_{x\to b-}$.
- *the two-sided limit* $\lim_{x\to x_0}$ *where* $x_0 \in (a, b)$ *and if* f *and* g *are differentiable except possibly at* $x_0 \in (a, b)$.

Example 4.3. *Consider the limit* $\lim_{x\to\frac{\pi}{2}-}\frac{\tan x}{\tan 3x}$.
According to L'Hopital's rule, we differentiate both the numerator and denominator a few times until the indeterminate form disappears:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x} = [\frac{\infty}{\infty}]$$

Consider $F : (0, \pi/2) \to \mathbb{R}$ *defined by* $F(x) = \frac{\tan x}{\tan 3x}$.
Required to find $\lim_{x\to\frac{\pi}{2}-}F(x)$.
Now take $f(x) = \tan x$ *and* $g(x) = \tan 3x$. *And* $\lim_{x\to\frac{\pi}{2}-}f(x) = \infty = \lim_{x\to\frac{\pi}{2}-}g(x)$.
Although $g'(x) = 3\sec^2 3x \neq 0$ *in* $(0, \pi/2)$ *but* $\lim_{x\to\frac{\pi}{2}-}f'(x) = \lim_{x\to\frac{\pi}{2}-}\sec^2 x = \infty = \lim_{x\to\frac{\pi}{2}-}g'(x) = \lim_{x\to\frac{\pi}{2}-}3\sec^2 3x = \infty$. *So* $\lim_{x\to\frac{\pi}{2}-}\frac{f'(x)}{g'(x)} = [\frac{\infty}{\infty}]$.
Therefore we can **not** *apply L'Hospital II directly to* f *and* g.
Now

$$\frac{f'(x)}{g'(x)} = \frac{\sec^2 x}{3\sec^2 3x} = \frac{\cos^2 3x}{3\cos^2 x}$$

Now, let $f_1(x) = \cos^2 3x$ *and* $g_1(x) = 3\cos^2 x$, *then* $\lim_{x\to\frac{\pi}{2}-}f_1(x) = 0 = \lim_{x\to\frac{\pi}{2}-}g_1(x)$ *and* $g_1'(x) = -6\sin x\cos x = -3\sin 2x \neq 0$ *in* $(0, \pi/2)$. *Also*

$f_1'(x) = -6\cos 3x \sin 3x = -3\sin 6x$. But $\lim_{x\to\frac{\pi}{2}^-} \frac{f_1'(x)}{g_1'(x)} = [\frac{0}{0}]$. Therefore we can **not** *apply L'Hospital I directly to f_1 and g_1.*
Now, let $f_2(x) = f_1'(x)$ and $g_1'(x) = g_2(x)$, then $\lim_{x\to\frac{\pi}{2}^-} f_2(x) = 0 = \lim_{x\to\frac{\pi}{2}^-} g_2(x)$ and $g_2'(x) = -6\cos 2x \neq 0$ in $(0, \pi/2)$. Also $f_2'(x) = -18\cos 3x$. But $\lim_{x\to\frac{\pi}{2}^-} \frac{f_2'(x)}{g_2'(x)} = \frac{\lim_{x\to\frac{\pi}{2}^-} f_2'(x)}{\lim_{x\to\frac{\pi}{2}^-} g_2'(x)} = [\frac{0}{-1}] = 0$. Therefore we can apply L'Hospital I directly to f_2 and g_2.

$$\lim_{x\to\frac{\pi}{2}^-} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{f_2(x)}{g_2(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{f_2'(x)}{g_2'(x)} = 0$$

Again we can apply L'Hospital 5, to f_1 and g_1

$$\lim_{x\to\frac{\pi}{2}^-} \frac{f'(x)}{g'(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{f_1(x)}{g_1(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{f_1'(x)}{g_1'(x)} = 0$$

Again we can apply L'Hospital I, to f and g

$$\lim_{x\to\frac{\pi}{2}^-} \frac{f(x)}{g(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{f'(x)}{g'(x)} = 0$$

Example 4.4. *Calculate the limit of $\lim_{x\to\infty} \frac{x^2}{2^x}$.*
Here $f(x) = x^2$ and $g(x) = 2^x$. And $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. And $g'(x) = 2^x \log 2 \neq 0$ for all x. Now $f'(x) = 2x$, so $\lim_{x\to\infty} f'(x) = \infty = \lim_{x\to\infty} g'(x)$. So

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = [\frac{\infty}{\infty}] \text{ form,}$$

So we canot apply L'Hospital II to f and g. Let $f_1(x) = f'(x) = 2x$ and $g_1(x) = g'(x) = 2^x \log x$. Again $g_1'(x) = g''(x) = 2^x(\log 2)^2 \neq 0$ for all x. And

$$\lim_{x\to\infty} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} = \lim_{x\to\infty} \frac{2}{2^x(\log 2)^2} = \frac{2}{(\log 2)^2} \lim_{x\to\infty} \frac{1}{2^x} = 0,$$

As the above limit exists, so by L'Hospital II to $f' = f_1$ and $g' = g_1$, we have

$$\lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty} \frac{f_1'(x)}{g_1'(x)} = 0$$

that is

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} = 0$$

Since $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists so we can apply L'Hospital II to f and g to obtain

$$\lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} = 0$$

Example 4.5. *Consider the limit* $\lim_{x\to\infty} \frac{\ln x}{x}$.
Here $f(x) = \ln x$ *and* $g(x) = x$. *And And* $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. *But* $f'(x)(x) = \frac{1}{x}$ *and* $g'(x) = 1 \neq 0$ *for all* x. *Here So*

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1} = \frac{0}{1} = 0.$$

So $\lim_{x\to\infty} \frac{f''(x)}{g''(x)}$ *exists so by L'Hospital's rule II applying to* f *and* g, *we have*

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0.$$

Example 4.6. *Consider the limit* $\lim_{x\to\infty} \frac{x^n}{e^x}$ *where* $n \in \mathbb{N}$.
The function has an indeterminate form of type $\frac{\infty}{\infty}$. *So by L'Hospital Rule II* n *times.*

4.4 Other Indeterminate Form

Remark 4.4. Other Indeterminate Form Indeterminate forms such as $1^\infty, \infty^0, 0^0$ can be reduced to the previously considered cases by algebraic manipulations and the use of the logarithmic and exponential functions. Instead of formulating these variations as theorems, we illustrated the pertinent techniques by means of examples.

Example 4.7. *Consider* $F : (0, \pi/2) \to \mathbb{R}$ *defined by* $F(x) = \frac{1}{x} - \frac{1}{\sin x}$. *So it has the indeterminate form* $\infty - \infty$.
Then we can rewrite $F(x) = \frac{\sin x - x}{x \sin x}$. *Required to find* $\lim_{x\to 0^+} F(x)$.
Now take $f(x) = \sin x - x$ *and* $g(x) = x \sin x$. *And* $\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^+} g(x)$, *so* $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = [\frac{0}{0}]$. *Although* $g'(x) = \sin x + x \cos x \neq 0$ *in* $(0, \pi/2)$ *but* $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (\cos x - 1) = 0$ *and* $\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} (\sin x - x \cos x) = 0$. *So* $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = [\frac{0}{0}]$. *Therefore we can not apply L'Hospital I directly.*
Now, let $f'(x) = f_1(x)$ *and* $g'(x) = g_1(x)$, *then* $\lim_{x\to 0^+} f_1(x) = 0 = \lim_{x\to 0^+} g_1(x)$ *and* $g_1'(x) = g''(x) = 2\cos x + x \sin x \neq 0$ *in* $(0, \pi/2)$. *Surprisingly,* $\lim_{x\to 0^+} \frac{f_1'(x)}{g_1'(x)}$ *exists as*

$$\lim_{x\to 0^+} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x + x \sin x} = \frac{-\lim_{x\to 0^+} \sin x}{\lim_{x\to 0^+} (2\cos x + x \sin x)} = \frac{0}{2} = 0.$$

So by L'Hospital Rule I, to f_1 *and* g_1

$$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{f_1(x)}{g_1(x)} = \lim_{x\to 0^+} \frac{f_1'(x)}{g_1'(x)} = 0$$

So $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)}$ *exists so again by applying L'Hospital Rule I to the function* f *and* g*,*

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = 0$$

Example 4.8. $F : (1, \infty) \to \mathbb{R}$ *defined by* $F(x) = (1 + \frac{1}{x})^x$*.*
Let us find $\lim_{x\to\infty} F(x)$*. It is clear* $\lim_{x\to\infty} F(x) = [1^\infty]$*.*
We note that

$$(1 + \frac{1}{x})^x = e^{x\log(1+\frac{1}{x})}.$$

Let $\mathcal{F}(x) = x\log(1+\frac{1}{x})$*. Then* $\lim_{x\to\infty} \mathcal{F}(x) = \lim_{x\to\infty} x\log(1+\frac{1}{x}) = \lim_{y\to 0+} \frac{\log(1+y)}{y}$*.*
Or if we take $f(y) = \log(1 + y)$ *and* $g(x) = y$*, then* $\lim_{y\to 0+} f(y) = 0 = \lim_{y\to 0+} g(y)$ *and* $g'(y) = 1 \neq 0$ *for all* $y \in (0, 1)$ *and*

$$\lim_{y\to 0+} \frac{f'(y)}{g'(y)} = \lim_{y\to 0+} \frac{1}{(1 + y)} = 1$$

exists. Applying L'Hospital Rule I,

$$\lim_{x\to\infty} x\log(1 + \frac{1}{x}) = \lim_{y\to 0+} \frac{f(y)}{g(y)} = \lim_{y\to 0+} \frac{f'(y)}{g'(y)} = 1,$$

Now $G(y) = e^y$ *and* $F(x) = x\log(1+\frac{1}{x})$ *are continuous functions so is* $e^{x\log(1+\frac{1}{x})}$*.*
(Composition of two continuous function is a continuous function), we infer that $\lim_{x\to\infty}(1 + \frac{1}{x})^x = e^1 = e$

Example 4.9. *Calculate the limit of* $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$*.*
Direct substitution leads to the indeterminate form of type 1^∞*. Let* $y = (\sin x)^{\tan x}$*.*
Take logarithms of both sides

$$\log y = \tan x \log \sin x$$

Applying L'Hospital Rule

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \log y &= \lim_{x\to\frac{\pi}{2}} \tan x \log \sin x = \lim_{x\to\frac{\pi}{2}} \frac{\log \sin x}{\cot x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \lim_{x\to\frac{\pi}{2}} \frac{(\log \sin x)'}{(\cot x)'} \\ &= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin^2 x}} = \lim_{x\to\frac{\pi}{2}} (\sin x \cos x) = 0\end{aligned}$$

The final answer is

$$\lim_{x\to\frac{\pi}{2}} y = \lim_{x\to\frac{\pi}{2}} e^{\log y} = e^0 = 1$$

Procedure is same as Example 4.3, Example 4.4 and Example 4.5

Problem 4.1. Evaluate the limit $\lim_{x\to\infty} x^{\sin(1/x)}$.

Proof. Now $\lim_{x\to\infty} x^{\sin(1/x)} = [\infty^0]$. We take $y = x^{\sin(1/x)}$ and evaluate the limit of $\log y$ using L'Hospital's rule

$$\begin{aligned}
\lim_{x\to\infty} \log y = \lim_{x\to\infty} \sin(1/x)\log x = \lim_{x\to\infty} \frac{\log x}{\frac{1}{\sin(1/x)}} = \left[\frac{\infty}{\infty}\right] \\
= -\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{-\cos(1/x)}{\sin^2(1/x)x^2}} = -\lim_{x\to\infty} \frac{\sin^2(1/x)}{\frac{1}{x}\cos(1/x)} = \left[\frac{0}{0}\right] = -\lim_{x\to\infty} \frac{-2\frac{1}{x^2}\sin(1/x)\cos(1/x)}{-\frac{1}{x^2}\cos(1/x) - \frac{1}{x^3}\sin(1/x)} \\
= -\lim_{x\to\infty} \frac{2\sin(1/x)\cos(1/x)}{\cos(1/x) + \frac{1}{x}\sin(1/x)} = 0
\end{aligned}$$

Now

$$\lim_{x\to\infty} y = e^{\lim_{x\to\infty}\log y} = e^0 = 1$$

Procedure is same as Example 4.3, Example 4.4 and Example 4.5 □

Problem 4.2. Let f be differentiable on some interval (c, ∞) and suppose $\lim_{x\to\infty}[f(x) + f'(x)] = L$, where L is finite. Prove $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.

Proof. We can write $f(x)$ as $\frac{f(x)e^x}{e^x}$. Consider $f_1(x) = f(x)e^x$ and $g_1(x) = e^x$. As $\lim_{x\to\infty}[f(x) + f'(x)] = L$, so $\lim_{x\to\infty} f(x)$ must be finite (check). Keep in mind $\lim_{x\to\infty} e^x = \infty$. We know $\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} g_1(x) = \infty$. So we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = \left[\frac{\infty}{\infty}\right]$$

We observe $g_1'(x) = e^x \neq 0$ for all x. And

$$\lim_{x\to\infty} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to\infty} \frac{e^x[f(x) + f'(x)]}{e^x} = \lim_{x\to\infty}[f(x) + f'(x)] = L$$

By L'Hospital II, to f_1 and g_1

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty} \frac{f_1'(x)}{g_1'(x)} = L. \tag{17}$$

So we can conclude

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}[f(x) + f'(x) - f(x)] = \lim_{x\to\infty}[f(x) + f'(x)] - \lim_{x\to\infty} f(x) = L - L = 0$$

□