Limit and Continuity

October 31, 2023

1 Introduction

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

2 Limit of a function

Definition 2.1. Let $(a, b) \subset \mathbb{R}$ and $f(x)$ be a real valued function defined on (a, b) except possibly at $c \in (a, b)$. We say that the **limit of $f(x)$ at c is L or L is a limit of $f(x)$ as x approaches to c and write** $\lim_{x\to c} f(x) = L$ **if**, for every sequence $\{x_n\}$ in (a, b) and $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$, we have $f(x_n) \to L$ as $n \to \infty$ or $\lim_{n\to\infty} f(x_n) = L$.

Example 2.1. *Consider the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x$. Let $\{x_n\}$ be any sequence converging to $1/2$ that is $\lim_{n\to\infty} x_n = 1/2$. As $f(x_n) = x_n$ so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = 1/2$. Hence $\lim_{x\to 1/2} f(x) = 1/2$. (It is important to note that $\lim_{x\to 1/2} f(x) = 1/2 = f(1/2)$.)*

Example 2.2. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined by $f(x) = x \sin \frac{1}{x}$. Then $\lim_{x\to 0} f(x)$ exists.*
Let $\{x_n\}$ be a sequence in $\mathbb{R} \setminus \{0\}$ converging to 0 that is $\lim_{n\to\infty} x_n = 0$. Now $\lim_{n\to\infty} x_n = 0$ implies $\lim_{n\to\infty} |x_n| = 0$

$$0 \leq |x_n \sin \frac{1}{x_n}| \leq |x_n|$$
$$\Rightarrow 0 \leq |f(x_n)| \leq |x_n|$$

As $\lim_{n\to\infty} |x_n| = 0$, we have $\lim_{n\to\infty} |f(x_n)| = 0$ by Sandwich Theorem, which in turn implies $\lim_{n\to\infty} f(x_n) = 0$. As $\{x_n\}$ is arbitary sequence converging to 0, so $\lim_{x\to 0} f(x) = 0$.

Remark 2.1. Limit does not Exist If we can find two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$ but $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits that is $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$. Then it is obvious that Limit does not Exist.

Example 2.3. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined by $f(x) = [x]$. Find $\lim_{x\to 1} f(x)$.*
Choose $x_n = 1 + \frac{1}{n+1}$ and $y_n = 1 - \frac{1}{n+1}$. Then $x_n, y_n \to 1$ as $n \to \infty$ and $f(x_n) = [x_n] = 1$, $f(y_n) = [y_n] = 0$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = 0$. Hence the limit does not exist.

Example 2.4. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined by $f(x) = \sin \frac{1}{x^2}$. Then $\lim_{x\to 0} f(x)$ does not exist.*
Choose $x_n = \frac{1}{\sqrt{2\pi n}}$ and $y_n = \frac{1}{\sqrt{2\pi n + \frac{\pi}{2}}}$. Then $x_n, y_n \to 0$ as $n \to \infty$ and $f(x_n) = \sin 2\pi n = 0$, $f(y_n) = \sin(2\pi n + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(x_n) = 0$ and $\lim_{n\to\infty} f(y_n) = 1$. Hence the limit does not exist.

Remark 2.2. Let us look at some important facts:

1. By a real valued function f, we mean a function $f : A \to \mathbb{R}$ where A is an interval of the form (a, b) or $(a, b]$ or $[a, b)$ or $(-\infty, a)$ or $(-\infty, a]$ or (a, ∞) or $[a, \infty)$ or $(-\infty, \infty) = \mathbb{R}$.

2. Since the limit of a sequence is unique (if it exists) is unique, by definition it follows that the limit of a function (if exists is unique).

   Theorem 2.1. *If limit exists, then it is unique.*

   Proof. Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} f(x) = M$. Let $\{x_n\}$ be a sequence with $x_n \to x_0$ as $n \to \infty$. Then by uniqueness of limit of a sequence $\lim_{n\to\infty} f(x_n)$ must be unique. Hence $L = M$.

   □

3. The function f may not be defined at c. But $\lim_{x\to c} f(x)$ may exists. For example: consider example 2.3

4. Even if $f(c)$ is defined $f(c)$ may not be equal to L. For example: $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

   Let $\{x_n\}$ be a sequence in $\mathbb{R}$ converging to 2. That is $\lim_{n\to\infty} x_n = 2$. Then by Algebra of limits $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = 4$.
   Hence we can see $\lim_{x\to 2} f(x) = 4 \neq f(2) = 1$.

Definition 2.2. One sided limits:
Let $f(x)$ is areal valued function defined on (c, b). The **right hand limit of** $f(x)$ **at** $x = c$ is L, if for any sequence $\{x_n\}$ in (c, b) (that is $x_n \in (c, b)$ for all $n \in \mathbb{N}$) with $\lim_{n\to\infty} x_n = c$ implies $\lim_{n\to\infty} f(x_n) = L$. Notation: We write $\lim_{x\to c^+} f(x) = L$.

Let $f(x)$ is areal valued function defined on (a, c). The **left hand limit of** $f(x)$ **at** $x = c$ is L, if for any sequence $\{x_n\}$ in (a, c) (that is $x_n \in (a, c)$ for all $n \in \mathbb{N}$)with $\lim_{n\to\infty} x_n = c$ implies $\lim_{n\to\infty} f(x_n) = L$. Notation: We write $\lim_{x\to c^-} f(x) = L$.

Example 2.5. *Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x$. Let $\{x_n\}$ be any sequence in $(0, 1)$ converging to 0 that is $\lim_{n\to\infty} x_n = 0$ then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = 0$. So $\lim_{x\to 0^+} f(x) = 0$.*

Definition 2.3. Limits at infinity :
Let $f : (a, \infty) \to \mathbb{R}$ be a function. We say $f(x)$ has limit L as x approaches ∞ and write $\lim_{x\to\infty} f(x) = L$, if for any sequence $\{x_n\} \in (a, \infty)$ (that is (that is $x_n \in (a, \infty)$ for all $n \in \mathbb{N}$) with $\lim_{n\to\infty} x_n = \infty$ implies $\lim_{n\to\infty} f(x_n) = L$. Notation: We write $\lim_{x\to\infty} f(x) = L$.
Similarly, one can define limit as x approaches $-\infty$.

Example 2.6. *(i)Let $f : (1, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$. Let $\{x_n\}$ be any sequence in $(1, \infty)$ with $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{x_n} = 0$. So $\lim_{x\to\infty} f(x) = 0$.*
(ii) Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} \frac{1}{x^2} = 0$. same as above.
(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sin x$. Then $\lim_{x\to\infty} \sin x$ does not exist.
Choose $x_n = n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n, y_n \to \infty$ as $n \to \infty$ and $\sin x_n = 0$, $\sin y_n = 1$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(x_n) = 0$ and $\lim_{n\to\infty} f(y_n) = 1$. Hence the limit does not exist.

Definition 2.4. (Infinite Limit):
Let $f : A - \{c\} \to \mathbb{R}$ be a function. We say that the function $f(x)$ approaches ∞ as x approaches to c , if for any sequence $\{x_n\} \in A - \{c\}$ with $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = \infty$. Notation: We write $\lim_{x\to c} f(x) = \infty$.
Similarly, one can define for $-\infty$. Also one can define one sided limit of $f(x)$ approaching ∞ or $-\infty$.

Example 2.7. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x^2}$. We can easily show that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.*

Example 2.8. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$. Choose $x_n = \frac{1}{n}$ and $y_n = \frac{-1}{n}$. Then $x_n, y_n \to 0$ as $n \to \infty$ and $f(x_n) = n$, $f(y_n) = -n$ for all $n \in \mathbb{N}$. So $f(x_n) \to \infty$ as $n \to \infty$ and $f(y_n) \to -\infty$ as $n \to \infty$. Hence $\lim_{n\to\infty} f(x_n) = +\infty$ and $\lim_{n\to\infty} f(y_n) = -\infty$. Hence the limit does not exist.*

Theorem 2.2. *Let f be a real-valued function defined on (a, b) and $c \in (a, b)$. Then $\lim_{x\to c} f(x) = L$ iff $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$.*

Proof. Let $\lim_{x\to c} f(x) = L$. RTP $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$.
Let $\{x_n\}$ be any sequence in (a, c) such that $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty} f(x_n) = L$, so $f(x_n) \to L$ as $n \to \infty$. So $\lim_{x\to c^-} f(x) = L$.
Similarly, $\lim_{x\to c^-} f(x) = L$.
Conversely, let $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$. RTP $\lim_{x\to c} f(x) = L$.
Let $\{x_n\}$ be any sequence in (a, b) such that $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. RTP $\lim_{n\to\infty} f(x_n) = L$ that is for any given $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that

$$|f(x_n) - L| < \varepsilon \quad \forall\, n \geq N_\varepsilon$$

.

1. If $x_n > c$ for only finite values of n, then there exists some $N_1 \in \mathbb{N}$ such that $x_n < c$ for all $n \geq N_1$. Hence $x_n \to L$ as $n \to \infty$ or $\lim_{n\to c} f(x_n) = L$ (as $\lim_{n\to c^-} f(x_n) = L$).

2. If $x_n < c$ for only finite values of n, then there exists some $N_2 \in \mathbb{N}$ such that $x_n > c$ for all $n \geq N_2$. Hence $f(x_n) \to L$ as $n \to \infty$ or $\lim_{n\to c} f(x_n) = L$ (as $\lim_{n\to c^+} f(x_n) = L$).

3. If $x_n > c$ and $x_n < c$ for infinitely many values of n. Take $\{r_1, r_2, \cdots\}$ such that $x_{r_n} < c$ for all $n \in \mathbb{N}$ (take those elements x_n's from the sequence which is $< c$ and rename them as $\{x_{r_1}, x_{r_2} \cdots\}$) and $\{j_1, j_2, \cdots\}$ such that $x_{j_n} > c$ take those elements x_n's from the sequence which is $> c$ and rename them as $\{x_{j_1}, x_{j_2} \cdots\}$)for all $n \in \mathbb{N}$.

   $$\{x_1, x_2, \cdots\} = \{x_{r_1}, x_{r_2} \cdots\} \cup \{x_{j_1}, x_{j_2} \cdots\}$$

   Now $\{x_{r_n}\}$ and $\{x_{j_n}\}$ are subsequences of $\{x_n\}$. Since $\lim_{n\to\infty} x_n = c$, so $\lim_{n\to\infty} x_{r_n} = c$ and $\lim_{n\to\infty} x_{j_n} = c$.
   Now $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$ are given, this will imply $\lim_{n\to\infty} f(x_{r_n}) = L$ and $\lim_{n\to\infty} f(x_{j_n}) = L$.
   Given $\varepsilon > 0$.
   Since $\lim_{n\to\infty} f(x_{r_n}) = L$. So for this given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

   $$|f(x_{r_n}) - L| < \varepsilon \quad \forall\, n \geq N_1 \tag{1}$$

   Since $\lim_{n\to\infty} f(x_{j_n}) = L$. So for this given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

   $$|f(x_{j_n}) - L| < \varepsilon \quad \forall\, n \geq N_2 \tag{2}$$

   Take $M = \max\{r_{N_1}, j_{N_2}\}$. Then by (1) and (2)

   $$|f(x_n) - L| < \varepsilon \quad \forall\, n \geq M$$

   As $\varepsilon > 0$ is arbitarily fixed so $\lim_{n\to\infty} f(x_n) = L$.
   And as $\{x_n\}$ is arbitarily fixed sequence in (a, b) such that $x_n \neq c$ for all $n \geq N$ and $x_n \to c$or $\lim_{n\to\infty} x_n = c$ so $\lim_{x\to c} f(x) = L$.

$\square$

Example 2.9. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = \left\{ \begin{array}{ll} \frac{1}{x} & x > 0 \\ 1 & x \leq 0 \end{array} \right.$.
In this case, we see that $\lim_{x \to 0^+} f(x) = \infty$ *but* $\lim_{x \to 0^-} f(x) = 1$*. Hence* $\lim_{x \to 0} f(x)$ *does not exist.*

Theorem 2.3. *Let* f *be a real-valued function such that* $\lim_{x \to c} f(x) = L$ *where* $L > 0$*. Then there exists an open interval* $(c - \delta, c + \delta)$ *containing* c *such that* $f(x) > 0$ *for all* $x \in (c - \delta, c + \delta) \setminus \{c\}$*.*
Similarly, Let f *be a real-valued function such that* $\lim_{x \to c} f(x) = L$ *where* $L < 0$*. Then there exists an open interval* $(c - \delta, c + \delta)$ *containing* c *such that* $f(x) < 0$ *for all* $x \in (c - \delta, c + \delta) \setminus \{c\}$*.*

Proof. We claim that, there exists an $N \in \mathbb{N}$ such that $f(x) > 0$ for all $x \in (c - \frac{1}{N}, c + \frac{1}{N}) \setminus \{c\}$.
On the contrary assume that there is no such $N \in \mathbb{N}$ such that $f(x) > 0$ for all $x \in (c - \frac{1}{N}, c + \frac{1}{N}) \setminus \{c\}$ holds.
Then for all $n \in \mathbb{N}$, there exists $z_n \in (c - \frac{1}{n}, c + \frac{1}{n}) \setminus \{c\}$ such that $f(z_n) \leq 0$ for all $n \in \mathbb{N}$. Hence we obtain a sequence $z_n \neq c$ for all $n \in \mathbb{N}$ such that $z_n \to c$ as $n \to \infty$ or $\lim_{n \to \infty} z_n = c$.
Now $\lim_{x \to c} f(x) = L$ implies $f(z_n) \to L$ as $n \to \infty$ or $\lim_{n \to \infty} f(z_n) = L$. But $f(z_n) \leq 0$ for all $n \in \mathbb{N}$. Hence by ordered properties of limit $\lim_{n\infty} f(z_n) = L \leq 0$ which is a contradiction to the given fact $L > 0$. So our assumption is wrong. $\square$

Example 2.10. *Consider the function* f *defined by*

$$f(x) = \left\{ \begin{array}{l} x \text{ if } x \in (0,1) \cup (1,2) \\ 0 \text{ if } x = 1 \end{array} \right. \tag{3}$$

Then $f(x)$ *is defined for* $(0, 2)$*. Here* $\lim_{x \to 1} f(x) = 1 > 0$*. Here* $f(x) > 0$ *for all* $x \in (0, 2) \setminus \{1\}$*. But* $f(1) = 0$*.*

Theorem 2.4. *Let* f *be a real-valued function such that* $\lim_{x \to c^-} f(x) = L$ *where* $L > 0$*. Then there exists an open interval* $(c - \delta, c)$ *containing* c *such that* $f(x) > 0$ *for all* $x \in (c - \delta, c) \setminus \{c\}$*.*
Similarly, Let f *be a real-valued function such that* $\lim_{x \to c^-} f(x) = L$ *where* $L < 0$*. Then there exists an open interval* $(c - \delta, c)$ *containing* c *such that* $f(x) < 0$ *for all* $x \in (c - \delta, c) \setminus \{c\}$*.*

Theorem 2.5. *Let* $f, g, h : A \to \mathbb{R}$ *be three functions such that* $\lim_{x \to c} f(x) = L$ *and* $\lim_{x \to c} g(x) = M$*, then*

1. $\lim_{x \to c} (f(x) \pm g(x)) = L \pm M$.

2. $\lim_{x\to c} cf(x) = cL$ *for some* $c \in \mathbb{R}$.

3. $\lim_{x\to c}(fg)(x) = LM$.

4. $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$ *whenever* $M \neq 0$ *(Here* $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ *whenever* $g(x) \neq 0$*).*

5. *If* $M \neq 0$*, then* $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$*. (Here* $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ *whenever* $g(x) \neq 0$*).*

6. **(Ordered Properties)***:* $f(x) \leq g(x)$ *for all* x *in an open interval* $(c-\delta, c+\delta)$ *containing* c*. Then* $L \leq M$*.*

7. **(Sandwich)***: Suppose that* $h(x)$ *satisfies* $f(x) \leq h(x) \leq g(x)$ *in an interval* $(c-\delta, c+\delta)$ *containing* c*, and* $L = M$*. Then* $\lim_{x\to c} h(x) = L$*.*

Proof.

1. Let $\{x_n\}$ be any sequence in A such that $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$ that is $f(x_n) \to L$ and $g(x_n) \to M$ as $n \to \infty$ hence by Algebra of sequences, $\lim_{n\to\infty}(f \pm g)(x_n) = \lim_{n\to\infty}(f(x_n) \pm g(x_n)) = L \pm M$or $(f \pm g)(x_n) = f(x_n) \pm g(x_n) \to L \pm M$ as $n \to \infty$. As $\{x_n\}$ is any sequence in A converging to c, so $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$.

2. Let $\{x_n\}$ be any sequence in A such that $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty} f(x_n) = L$ that is $f(x_n) \to L$ as $n \to \infty$, hence by Algebra of sequences, $\lim_{n\to\infty} cf(x_n) = cL$ or $cf(x_n) \to cL$ as $n \to \infty$. As $\{x_n\}$ is any sequence in A converging to c, so $\lim_{x\to c} cf(x) = cL$.

3. Let $\{x_n\}$ be any sequence in A such that $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$ that is $f(x_n) \to L$ and $g(x_n) \to M$ as $n \to \infty$ hence by Algebra of sequences, $\lim_{n\to\infty}(fg)(x_n) = \lim_{n\to\infty} f(x_n)g(x_n) = LM$ or $(fg)(x_n) = f(x_n)g(x_n) \to LM$ as $n \to \infty$. Hence $\lim_{x\to c}(fg)(x) = LM$.

4. Since $M \neq 0$, there exists a $\delta > 0$ such that $g(x) \neq 0$ for all $x \in (c-\delta, c+\delta) \setminus \{c\}$. Thus $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is well-defined on $x \in (c-\delta, c+\delta) \setminus \{c\}$. Let $\{x_n\}$ be any sequence in $(c-\delta, c+\delta) \setminus \{c\}$ such that $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$ that is $f(x_n) \to L$ and $g(x_n) \to M$ as $n \to \infty$ hence by Algebra of sequences,
$$\left(\frac{f}{g}\right)(x_n) = \frac{f(x_n)}{g(x_n)} \to \frac{L}{M} \text{ as } n \to \infty$$
$$\Rightarrow \lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{L}{M}.$$

5. Let $\{x_n\}$ be any sequence in $I \setminus \{c\}$ (where $I = (c-\delta, c+\delta)$ contained in A) such that $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Now it is given $f(x) \leq g(x)$ for

all $x \in I$. As x_n's are in I, we have

$$f(x_n) \leq g(x_n) \; \forall \; n \; \in \mathbb{N}$$
$$\Rightarrow \lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} g(x_n) \text{ by Ordered property of limits}$$
$$\Rightarrow L \leq M.$$

6. Suppose that $h(x)$ satisfies $f(x) \leq h(x) \leq g(x)$ in an interval (say $I = (c-\delta, c+\delta)$ contained in A) containing c and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$. Let $\{x_n\}$ be any sequence in $I \setminus \{c\}$ such that $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$. Now $f(x) \leq h(x) \leq g(x)$ for all $x \in I$. As x_n's are in I, we have

$$f(x_n) \leq h(x_n) \leq g(x_n) \;\Rightarrow\; \lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} h(x_n) \leq \lim_{n\to\infty} g(x_n)$$
$$\text{(by Sandwich theorem of limit of Sequences)}$$
$$L \leq \lim_{n\to\infty} h(x_n) \leq L \;\Rightarrow\; \lim_{n\to\infty} h(x_n) = L.$$

□

Example 2.11. *Show that* $\lim_{x\to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$.

Example 2.12. *If* $f : [0,\infty) \to \mathbb{R}$ *defined by* $f(x) = x^j$ *where* $j \geq 1$. *Then using induction and Algebra of limits one can prove that* $\lim_{x\to c} f(x) = c^j$ *for any* $c \in \mathbb{R}$.

Remark 2.3. Suppose $f(x)$ is bounded in an interval containing x_0 and $\lim_{x\to x_0} g(x) = 0$. Then $\text{llim}_{x\to x_0} f(x)g(x) = 0$. For example

Example 2.13. *(i)* $\lim_{x\to 0} |x| \sin \frac{1}{x} = 0$.
(ii) $\lim_{x\to 0} |x| ln(1 + |x|) = 0$.

Remark 2.4. Suppose $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$. If D_1 and D_2 are the domains of f and g respectively, and if $f(x) \in D_2 - \{b\}$ for every $x \in D_1 - \{a\}$, then $\lim_{x\to a} g(f(x)) = c$.
It is enough to prove that for any sequence $\{x_n\}$ in $D_1 - \{a\}$. such that $x_n \to a$, as $n \to \infty$, we have $\lim_{x\to a} g(f(x_n)) \to c$ as $n \to \infty$. So, let $\{x_n\}$ be in $D_1 - \{a\}$ such that $x_n \to a$, as $n \to \infty$. Since $\lim_{x\to a} f(x) = b$, so, by sequential definition of the limit, $\lim_{n\to\infty} f(x_n) = b$. Let $y_n = f(x_n)$, for $n \in \mathbb{N}$. By the assumption, $y_n \in D_2 - \{b\}$ for all $n \in \mathbb{N}$ and $= \lim_{n\to\infty} f(x_n) = b$. Since $\lim_{y\to b} g(y) = c$. so by sequential definition of the limit $\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(f(x_n)) = c$ which completes the proof.

Equivalent definition of limit

Theorem 2.6. Equivalent Definition of Limit *Let f be a real valued function defined on $\mathbb{R}$. Then $\lim_{x\to c} f(x) = L$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending upon ε) such that*

$$0 < |x - c| < \delta(\varepsilon) = \delta \quad \textit{implies } |f(x) - L| < \varepsilon. \tag{4}$$

Proof. Given $\lim_{x\to c} f(x) = L$. RTP for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta(\varepsilon) = \delta$ (depending upon ε) implies $|f(x) - L| < \varepsilon$.
Suppose that the condition does not hold. i.e, there exists $\varepsilon_0 > 0$ (for which no δ will work) such that for any $\delta > 0$, there is a x_δ such that

$$|x_\delta - x_0| < \delta, \quad \text{but} \quad |f(x_\delta) - L| \geq \varepsilon_0.$$

Then take $\delta = \frac{1}{n}$ and pick x_n such that $|x_n - x_0| < \frac{1}{n}$, then $x_n \to x_0$ as $n \to \infty$ but $|f(x_n) - L| \geq \varepsilon_0$. which contradicts $\lim_{x\to c} f(x) = L$.So the condition must hold.
Conversely, Given $\varepsilon > 0$ then there exists a real number $\delta > 0$ (depending on ε) such that

$$0 < |x - x_0| < \delta \quad \Rightarrow |f(x) - L| < \varepsilon \tag{5}$$

Let $\{x_n\}$ be a sequence in D converging to x_0. Then there exists N such that $|x_n - x_0| < \delta$ for $n \geq N$. Then by the definition, or by equation (5)

$$|f(x_n) - L| < \varepsilon. \ \forall \ \ n \geq N \ \ i.e., \ \ f(x_n) \to L \ \ as \ \ n \to \infty$$

. ☐

Lemma 2.7. *If $\lim_{x\to c} f(x) = L$, then there exists a deleted neighbourhood D_δ of c (means $(c-\delta, c+\delta) - \{c\}$) and $M > 0$ such that $|f(x)| \leq M$ for all $x \in D_\delta$.*

Proof. Given $\lim_{x\to c} f(x) = L$. We choose $\varepsilon = 1$. Then for this choice of ε, there exists a $\delta > 0$ such that for all $x \in D$

$$\begin{aligned} 0 < |x - c| < \delta \quad &\Rightarrow \quad |f(x) - L| < 1 \\ x \in (c - \delta, c + \delta) - \{c\} \quad &\Rightarrow \quad |f(x) - L| < 1 \end{aligned}$$

Let $D_\delta = (c - \delta, c + \delta) - \{c\}$

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + |L| \text{ for all } x \in D_\delta$$

So f is bounded in D_δ. ☐

.

Example 2.14.
- *Let $f(x) = x$. Let $a \in \mathbb{R}$.*

$$|f(x) - a| = |x - a| < \varepsilon,$$

it follows that for any $\varepsilon > 0$, $|f(x) - a| < \varepsilon$ whenever $0 < |x - a| < \delta := \varepsilon$. Hence, $\lim_{x\to a} f(x) = a$.

- *Let* $f(x) = \frac{3}{2} - 1$ *be a function. Then* $\lim_{x\to 1}(\frac{3}{2}x - 1) = \frac{1}{2}$.
  Let $\varepsilon > 0$. *We have to find* $\delta_\varepsilon > 0$ *(depending on* $\varepsilon > 0$*) such that (4) holds with* $L = 1/2$. *Working backwards,*

$$|(\frac{3}{2}x - 1) - \frac{1}{2}| < \varepsilon \Rightarrow \frac{3}{2}|x - 1| \Rightarrow |x - 1| < \frac{2}{3}\varepsilon = \delta$$

  . *So for given* $\varepsilon > 0$, *there exists* $\delta = \frac{2}{3}\varepsilon$ *such that*

$$0 < |x - 1| < \delta \quad \Rightarrow |f(x) - \frac{1}{2}| < \varepsilon.$$

Example 2.15. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$, *show that* $\lim_{x\to c} x^2 = c^2$ *where* $c \in \mathbb{R}$. *We want to make the difference*

$$|f(x) - c^2| = |x^2 - c^2|$$

less than the preassigned $\epsilon > 0$ *by taking* x *sufficiently close to* c. *To do so, we note that* $x^2 - c^2 = (x + c)(x - c)$. *Moreover, if* $|x - c| < 1$, *then*

(6)
$|x| \leq |c| + 1$ *then* $|x^2 - c^2| = |(x + c)(x - c)| \leq (|x| + |c|)|x - c| \leq (2|c| + 1)|x - c|$

Moreover this last term will be less than $\varepsilon > 0$ *provided we take* $lx - cl < \frac{\varepsilon}{(2|c|+1)}$. *Consequently, if we choose*

$$\delta(\varepsilon) := \min\{1, \frac{\varepsilon}{(2|c| + 1)}\}$$

then if $0 < |x - c| < \delta(\varepsilon)$, *it will follow first that* $|x - c| < 1$ *so that* (6) *is valid, and therefore, since* $|x - c| < \frac{\varepsilon}{(2|c|+1)}$ *that*

$$|x^2 - c^2| = |(x + c)(x - c)| \leq (|x| + |c|)|x - c| \leq (2|c| + 1)|x - c| < \varepsilon$$

Since we have shown that there exists an $\delta(\varepsilon) > 0$ *for an arbitrary choice of* $\varepsilon > 0$, *we infer that* $\lim_{x\to c} x^2 = c^2$.

Example 2.16. *The function* $f : (0, 1) \to \mathbb{R}$ *defined by* $f(x) = \frac{1}{x}$. *Show that* $\lim_{x\to c} f(x) = \frac{1}{c} = f(c)$ *where* $c \in (0, 1)$. **Using** $\varepsilon - \delta$ **definition:** *Let* c *be any number in* $(0, 1)$. *Let* $\varepsilon > 0$ *be given. Define* $\delta = min\{\frac{c}{2}, \frac{c^2\varepsilon}{2}\}$. *Assume* $|x - c| < \delta$. *Then*

$$|x - c| < \delta < \frac{c}{2} \;\;\Rightarrow\;\; -\frac{c}{2} < x - c < \frac{c}{2} \;\;\Rightarrow\;\; 0 < \frac{c}{2} < x.$$

Therefore, therefore $|x - c| < \delta$ *implies*

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{c}| = |\frac{c - x}{cx}| < \frac{1}{c}\frac{1}{x}|x - c| < \frac{1}{c}\frac{2}{c}|x - c| < \frac{2}{c^2}\delta \leq \frac{2}{c^2}\frac{c^2\varepsilon}{2} = \varepsilon$$

and $\lim_{x\to c} f(x) = \frac{1}{c} = f(c)$ *where* $c \in (0, 1)$. **Using sequential definition:** *Let* c *be any number in* $(0, 1)$. *Suppose* $\{x_n\}$ *be any sequence such that* $x_n \to c \neq 0$ *as* $n \to \infty$, *then* $\frac{1}{x_n} \to \frac{1}{c}$, *as* $n \to \infty$ *(by property of a sequence).*

Corollary 2.8. *If* $\lim_{x\to c} f(x) = L$*, then* $\lim_{x\to c} |f(x)| = |L|$ *.*

Proof. Given $\varepsilon > 0$, there exists δ (depending on ε) such that for all $x \in D$ with

$$0 < |x - c| < \delta \ \Rightarrow \ |f(x) - L| < \varepsilon. \tag{7}$$

Now $||f(x)| - |L|| \leq |f(x) - f(c)|$ as $||a| - |b|| \leq |a - b| = |b - a|$. Hence by (8), we can write for all $x \in D$ with

$$0 < |x - c| < \delta \ \Rightarrow \ ||f(x)| - |L|| \leq |f(x) - L| < \varepsilon.$$

As $\varepsilon > 0$, is arbitary, so $\lim_{x\to c} |f(x)| = |L|$. $\square$

Example 2.17. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by setting* $f(x) := x$ *if* x *is rational, and* $f(x) = 0$ *if* x *is irrational.*
(a) Show that f *has a limit at* $x = 0$*.*
(b) Use a sequential argument to show that if $a \neq 0$*, then* f *does not have a limit at* a*.*

Proof. Let $a \neq 0$ and $a \in \mathbb{Q}$ and $a \neq 0$ then $f(a) = a$. Then there exists a sequence $\{x_n\}$ of rationals and a sequence $\{y_n\}$ of irrationals both converging to a. Then $f(x_n) = x_n$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = a$. And $f(y_n) = 0$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(y_n) = 0$. So $x_n, y_n \to a$ as $n \to \infty$ and $\lim_{n\to\infty} f(y_n) = 0$ and $\lim_{n\to\infty} f(x_n) = a \neq 0$. Hence the limit does not exist. Similarly, the case $a \neq 0$ and $a \notin \mathbb{Q}$.
What about $a = 0$? Let $\{x_n\}$ be any sequence such that $\lim_{n\to\infty} x_n = 0$. RTP $\lim_{n\to\infty} f(x_n) = 0$. Given $\varepsilon > 0$.
As $\lim_{n\to\infty} x_n = 0$. For this $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$|x_n - 0| < \varepsilon \ \forall n \geq N$$

Now let $m \geq N$. If $x_m \in \mathbb{Q}$, then

$$|f(x_m) - 0| = |x_m - 0| < \varepsilon$$

If If $x_m \in \mathbb{R} \setminus \mathbb{Q}$, then

$$|f(x_m) - 0| = |0 - 0| < \varepsilon$$

trivially. So in any case

$$|f(x_m) - 0| < \varepsilon \quad \forall m \geq N$$

As ε is arbitary, so $\lim_{n\to\infty} f(x_n) = 0$. As $\{x_n\}$ is any sequence converging to 0 so $\lim_{x\to 0} f(x) = 0$. $\square$

3 Continuity

Most of the calculus involves the study of continuous functions. In this section we study continuous and uniformly continuous functions.

Definition 3.1. Continuity at a point Let $f : A \to \mathbb{R}$ be a function and $c \in A$. We say f is continuous at c if every sequence $\{x_n\}$ in A and $x_n \neq c$ for all $n \geq N$ and $x_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = c$, we have $f(x_n) \to f(c)$ as $n \to \infty$ or $\lim_{n\to\infty} f(x_n) = f(c)$. or in terms of limit we can write f is continuous at c iff $\lim_{x\to x_0} f(x) = f(c)$. The set of all points at which f is continuous is denoted $C(f)$.

Theorem 3.1. Equivalent Definition of Continuity *The above definition is equivalent to: we say f is continuous at c iff for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ (depending on $\varepsilon > 0$) such that $0 < |x - c| < \delta(\varepsilon) = \delta$ (depending upon ε) implies $|f(x) - f(c)| < \varepsilon$.*

Proof. The proof is similar to Theorem 2.7. $\square$

Definition 3.2. Continuous function A function is said to be continuous if it is continuous at every point of its domain.

Example 3.1. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$. Let $a \in \mathbb{R}\setminus\{0\}$. Let $\{x_n\}$ be a sequence in $\mathbb{R}\setminus\{0\}$ converging to a that is $\lim_{n\to\infty} x_n = a$. Then by Algebra of sequence, we have $f(x_n) = \frac{1}{x_n} \to \frac{1}{a} = f(a)$ as $n \to \infty$ or $\lim_{n\to\infty} f(x_n) = f(a)$. So f is continuous at a. As a is any arbitary point at $\mathbb{R} \setminus \{0\}$, so f is continuous at every point in $\mathbb{R} \setminus \{0\}$.*

Example 3.2. *Examples 2.1, 2.2, 2.13, 2.14, and 2.15 are continuous functions.*

Remark 3.1. Function is not continuous at a given point If there exists a sequence $\{y_n\}$ in A such that $y_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} y_n = c$, but $f(x_n) \nrightarrow f(c)$ as $n \to \infty$ or $\lim_{n\to\infty} f(x_n) \neq f(c)$, then it is obvious that the Function is not continuous at c.

Example 3.3. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Case 1*: Let $a < 0$, then $f(a) = -1$. Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n = a < 0$. Then there exists an $N \in \mathbb{N}$ such that $x_n < 0$ for all $n \geq N$.*

$f(x_n) = -1$ for all $n \geq N$. So $\lim_{n\to\infty} f(x_n) = -1 = f(a)$. Thus f is continuous at $a < 0$.
Case 2: *Let $a > 0$, then $f(a) = 1$. Similarly one can prove f is continuous at $a > 0$.*
Case 3: *Let $a = 0$. Then $f(0) = 1$. Let $x_n = -\frac{1}{n}$ for all $n \in \mathbb{N}$. Then $f(x_n) = -1$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} f(x_n) = -1 \neq 1 = f(0)$. Hence f is not continuous at $a = 0$.*

Example 3.4. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Proof is same as Example 2.16.

Here are some examples of checking the continuity using ϵ-δ definition.

Example 3.5. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by defined by $f(x) = x^2$, show that the function is continuous.*
Let $c \in \mathbb{R}$. RTP $\lim_{x\to c} f(x) = f(c)$ that is $\lim_{x\to c} x^2 = c^2$.
The proof will be same as Example 2.14. As c is arbitary real number in $\mathbb{R}$. So the function is continuous in $\mathbb{R}$.

Example 3.6. *The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Show that $\lim_{x\to c} f(x) = \frac{1}{c} = f(c)$ where $c \in (0, 1)$. Same as Example 2.15.*

Example 3.7. *Let us consider $f(x) = \{ \begin{array}{cc} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{array}$.*
Given $\varepsilon > 0$. We need to check whether f is continuous at $x = 0$.
Let us choose $\delta = \sqrt{\varepsilon}$. Then for $|x| = |x - 0| < \sqrt{\varepsilon} = \delta$ we have

$$|f(x) - f(0)| = |f(x)| \leq x^2 < \varepsilon$$

Hence f is continuous at $x = 0$.

Example 3.8. *Let $k \in \mathbb{N}$. Then the function f defined by $f(x) = x^{1/k}$, for $x \geq 0$ is continuous at every $x_0 \geq 0$.*
Using sequential definition it is easy to prove.
Let $\varepsilon > 0$ be given. First, consider the point $x_0 = 0$. Then we have

$$|f(x) - f(x_0)| = x^{1/k} < \varepsilon \text{ whenever } |x| < \varepsilon^k.$$

Thus, f is continuous at $x_0 = 0$. Next, assume that $x_0 > 0$.

Let $y = x^{1/k}$ and $y_0 = x_0^{1/k}$. Since

$$y^k - y_0^k = (y - y_0)(y^{k-1} + y^{k-2}y_0 + ... + yy_0^{k-2} + y_0^{k-1}),$$

so that

$$x - x_0 = (x^{1/k} - x_0^{1/k})(y^{k-1} + y^{k-2}y_0 + ... + yy_0^{k-2} + y_0^{k-1})$$

Hence,

$$|x^{1/k} - x_0^{1/k}| = \frac{|x - x_0|}{y^{k-1} + y^{k-2}y_0 + ... + yy_0^{k-2} + y_0^{k-1}} \leq \frac{|x - x_0|}{y_0^{k-1}}$$

Thus,

$$|x^{1/k} - x_0^{1/k}| < \varepsilon \text{ whenever } |x - x_0| < \varepsilon y_0^{k-1} = \varepsilon x_0^{1-1/k} = \delta$$

Thus, f is continuous at every $x_0 > 0$.

Example 3.9. *If $f : \mathbb{R} \to \mathbb{R}$ and there is an $\alpha > 0$ such that $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in \mathbb{R}$, then show that f is continuous.*
Let $x_0 \in \mathbb{R}$. Given $\varepsilon > 0$. Required to prove f is continuous at x_0. We choose $\delta = \frac{\varepsilon}{\alpha}$ (given $\alpha > 0$). Then for all $x \in \mathbb{R}$ with

$$0 < |x - x_0| < \delta = \frac{\varepsilon}{\alpha} \quad \Rightarrow |f(x) - f(x_0)| <\leq \alpha|x - x_0| < \alpha\frac{\varepsilon}{\alpha} = \varepsilon.$$

So f is continuous at x_0. Since x_0 is arbitary. So f is continuous for all $x \in \mathbb{R}$.

Example 3.10. *Also see Example 2.14 and Example 2.15.*

Theorem 3.2. *Let $f(x)$ be a continuous function on $[a, b]$ and let $f(c) > 0$ for some $c \in (a, b)$, Then there exists $\delta > 0$ such that $f(x) > 0$ in $(c - \delta, c + \delta) \subset (a, b)$.*

Proof. The proof is similar to Theorem 2.3 of previous section. □

Theorem 3.3. *Suppose f and g are continuous at x_0. Then*
(i) $f \pm g$ (defined by$(f \pm g)(x) = f(x) \pm g(x)$) is also continuous at x_0.
(ii) fg (defined by $fg(x) = f(x)g(x)$) is continuous at x_0.
(iii) $\frac{f}{g}$ (defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$) is continuous at x_0 if $g(x_0) \neq 0$.
(iv) Let $c \in \mathbb{R}$. Then the function cf (defined by $(cf)(x) = cf(x)$) is continuous at x_0.

Proof. The **proofs can be done using Sequential definition just like Theorem 2.5 with $L = f(x_0)$ and $y = g(x_0)$.**

$(fg)(x_n) = f(x_n)g(x_n) \to f(x_0)g(x_0)$ as $n \to \infty$. Hence

□

Example 3.11. *If $f(x)$ is a polynomial, say $f(x) = a_0 + a_1x + ... + a_kx^k$, then f is continuous on $\mathbb{R}$.*
Let $j \in \mathbb{N}$. Consider the function $g_j(x) = x^j$ (see example 2.12). Let $x_0 \in \mathbb{R}$. Let $\{x_n\}$ be any sequence converging to x_0. Then by multiplicative property of the limit of a sequence, the sequence $\{x_n^j\}$ converges to x_0^j as $n \to \infty$. So by sequential definition of continuity, we can say $g_j(x)$ is continuous at x_0. Since x_0 is an arbitrary point in $\mathbb{R}$. We can say g_j is continuous on $\mathbb{R}$. Thus, $f(x)$ is continuous at every point in $\mathbb{R}$ by (i) and (iv) of Theorem 3.3.

Remark 3.2. Composition of two continuous functions is continuous
That is Let $A, B \subseteq \mathbb{R}$ Let $h : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two continuous functions. If $h(A) \subseteq B$, then the composite function go $g \circ h$ is continuous on A.
(For proof see remark 2.6)

Example 3.12. *$f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^{j/2}$ where $j \geq 1$ is an odd integer is a continuous function.*
We know $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = \sqrt{x}$ is a continuous function and $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^j$ is also continuous. It is easy to see $h(\mathbb{R}) \subseteq \mathbb{R}$. Hence $f(x) = g \circ h(x) = x^{j/2}$ is continuous by remark 3.2.

Example 3.13. *Determine where the function below is not continuous.*

$$h(t) = \frac{4t + 10}{t^2 - 2t - 15}$$

(Rational functions are continuous everywhere except where we have division by zero). Let $f(t) = 4t + 10$ and $g(t) = t^2 - 2t - 15$. f, g are polynomials so continuous on $\mathbb{R}$. But $h(t)$ will be continuous at those points in $\mathbb{R}$, where $g(t) \neq 0$. So all that we need to is determine where the denominator ($g(t)$) is zero. That's easy enough to determine by setting the denominator equal to zero and solving.

$$t^2 - 2t - 15 = (t - 5)(t + 3) = 0$$

So h is not continuous at $t = 5$ and $t = -3$.

Corollary 3.4. *If $f(x)$ is continuous at c, then $|f|$ is also continuous at c.*

Proof. Given $\varepsilon > 0$, there exists δ (depending on ε) such that for all $x \in D$ with

$$0 < |x - c| < \delta \;\Rightarrow\; |f(x) - f(c)| < \varepsilon. \tag{8}$$

Now $||f(x)| - |f(c)|| \leq |f(x) - f(c)|$ as $||a| - |b|| \leq |a - b| = |b - a|$. Hence by (8), we can write for all $x \in D$ with

$$0 < |x - c| < \delta \;\Rightarrow\; ||f(x)| - |f(c)|| \leq |f(x) - f(c)| < \varepsilon.$$

So $|f|$ is also continuous at c. □

Example 3.14. *If f, g are continuous at c, then $max(f, g)$ and $min(f, g)$ are continuous at c.*
We can write

$$max(f,g) = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2} \text{ and } min(f,g) = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$$

. Now f and g are continuous at c. So $\frac{f(x)+g(x)}{2}$ and $\frac{|f(x)-g(x)|}{2}$ are continuous at c by (i) of Theorem 3.3. Again by (i) of Theorem 3.3, we can conclude that $max(f, g)$ and $min(f, g)$ are continuous at c.

Theorem 3.5. *Continuous functions on closed, bounded interval is bounded.*

Proof. Let $f(x)$ be continuous on $[a, b]$ and suppose it is not bounded. Then we can find a sequence $\{x_n\} \subset [a, b]$ such that $|f(x_n)| > n$. But $\{x_n\}$ is a bounded sequence and hence by Bolzano Weierstrass theorem there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to c in $[a, b]$. Since f is continuous on $[a, b]$, we have $f(x_{n_k}) \to f(c)$ as $k \to \infty$, a contradiction to the fact $|f(x_{n_k})| > n_k$ implying $|f(x_{n_k})| \to \infty$ as $k \to \infty$. Hence our assumption is wrong. □

Theorem 3.6. *Let $f(x)$ be a continuous function on closed, bounded interval $[a, b]$. Then supremum and infimum of functions are achieved in $[a, b]$.*

Proof. So Theorem 3.5 implies f is bounded in $[a, b]$ and the completeness of $\mathbb{R}$ implies that supremum as well as infimum of f is finite. Let $sup\{f(x) : \ \forall x \ \in [a, b]\} = M$. For any $n \geq 1$, we take $\varepsilon = \frac{1}{n}$, then there exists $x_n \in [a, b]$ be such that $M - \frac{1}{n} < f(x_n) \leq M$. Then $\{x_n\} \in [a, b]$ is bounded and hence by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converging to some $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$ as $k \to \infty$ or $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$. Now $M - \frac{1}{n_k} < f(x_{n_k}) \leq M$. Taking limits $f(x_0) = M = sup\{f(x) : \ \forall x \ \in [a, b]\}$ (by order property of limits). The attainment of minimum can be proved by noting that $-f$ is also continuous and $minf = -sup(-f)$. □

Remark 3.3. Closed and boundedness of the interval is important in the above theorem. Consider the examples
(i) $f(x) = \frac{1}{x}$ on $(0, 1)$.
(ii) $f(x) = x$ on $\mathbb{R}$.

Theorem 3.7. *Let $f(x)$ be a continuous function on a domain D containing $[a, b]$ and let $f(a)f(b) < 0$ for some a, b. Then there exits $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\} \subset [a, b]$. Then S is bounded. Let $c = sup\ S$. We claim that $f(c) = 0$.
Then there exists some $m \in \mathbb{N}$, such that for all $n \geq m$, we have $x_n = c + \frac{1}{n} > c$ with $x_n \in [a, b]$ but $x_n \notin S$ and $x_n \to c$ as $n \to \infty$. Therefore, $f(c) =$

$\lim_{n\to\infty} f(x_n) \geq 0$ as f is continuous. On the other hand, note that $c - \frac{1}{n} < c$. Therefore, there exists a point $y_n \in S$ such that $c - \frac{1}{n} < y_n \leq c$. Then note $y_n \to c$ as $n \to \infty$. Therefore, $f(c) = \lim_{n\to\infty} f(y_n) \leq 0$ as f is continuous. Hence $f(c) = 0$. □

Theorem 3.8. Intermediate value theorem: *Let $f(x)$ be a continuous function on a domain D containing $[a, b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = y$.*

Proof. Consider $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - y$. As f is continuous in $[a, b]$, so g is continuous in $[a, b]$. Then $g(a) = f(a) - y < 0$ and $g(b) = f(b) - y > 0$, then $g(a)g(b) < 0$. Now we apply Theorem 3.7 to conclude that there exists $c \in (a, b)$ such that $g(c) = 0$ i.e $f(c) = y$.

□

Remark 3.4. A continuous function assumes all values between its Supremum and Infimum.

Example 3.15. Application 1*: (Fixed point): Let $f(x)$ be a continuous function from $[0, 1]$ into $[0, 1]$. Then show that there is a point $c \in [0, 1]$ such that $f(c) = c$.*
Consider the function $g : [0, 1] \to [0, 1]$ such that $g(x) = f(x) - x$ and apply Intermediate Value Theorem.

Example 3.16. Application 2*: Root finding: To find the solutions of $f(x) = 0$,*
Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval $[-1, 2]$.
What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

somewhere between -1 and 2. In other words, we want to show that there is a number c such that $-1 < c < 2$ and $p(c) = 0$.
So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function $p(x)$ is continuous and $p(-1)p(2) < 0$ and then we will be done.
Now polynomial functions are continuous on $\mathbb{R}$, by example 3.2.

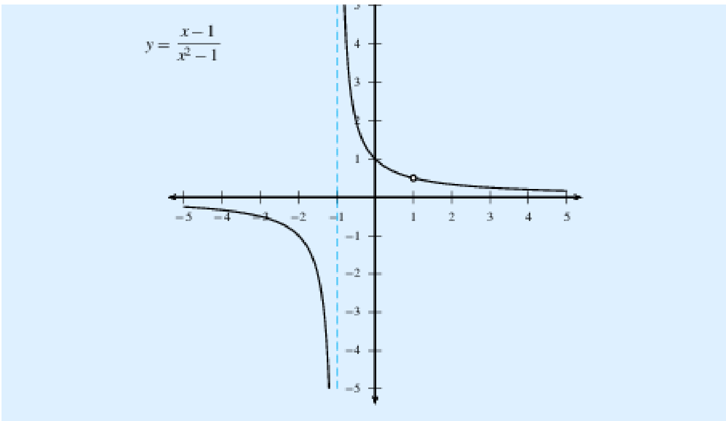
$$p(-1) = 8 \textit{ and } p(2) = -19$$
$$\textit{so } p(-1)p(2) < 0$$

We can conclude that there is a number c such that $-1 < c < 2$ and $p(c) = 0$ by Intermediate Value Theorem.

Proofs of Theorem 3.7 and Theorem 3.8 will be discussed in next class.

3.1 Types of discontinuities

Definition 3.3. Removable discontinuity: we say that $f(x)$ has a removable discontinuity at $x = a$ if
1) $f(x)$ is defined everywhere in a domain D containing a except at $x = a$ and limit exists at $x = a$ i.e $\lim_{x\to a} f(x)$ exists
OR 2) $f(x)$ is defined also at $x = a$ and limit is NOT equal to function value at $x = a$ i.e $\lim_{x\to a} f(x) \neq f(a)$.
These functions can be extended as continuous by defining the value of f as the limit value at $x = a$.



Example 3.17.

Example 3.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by* $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. *Here* $\lim_{x\to 0} \frac{\sin x}{x} = 1$. *Although $f(0) = 0$ is given but f can be defined to be 1 at $f(0)$. So 0 is the removable singularity of f.*

Consider the function $f(x) = \frac{x^2-2x-15}{x-5}$. f is not defined at $x = 5$ but $\lim_{x\to 5} \frac{x^2-2x-15}{x-5} = \lim_{x\to 5}(x + 3) = 8$. Condition 1 of the definition is satisfied.

Example 3.19. *Removable discontinuities can be "filled in" if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make f(x) continuous you could redefine it as:*

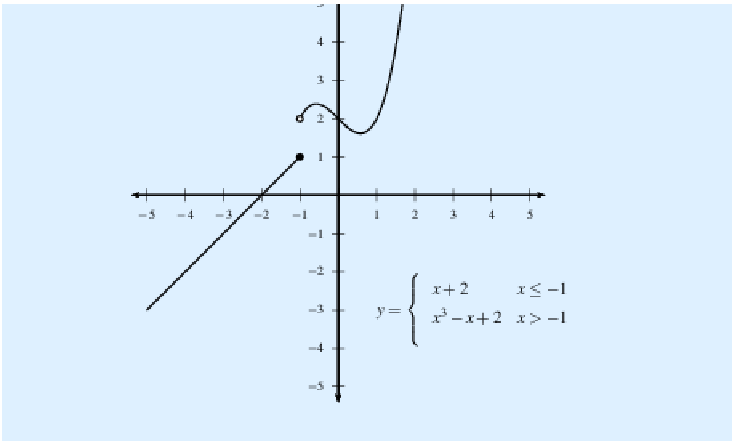
Definition 3.4. Jump discontinuity/ discontinuities of first kind: The left and right limits of $f(x)$ exist but not equal that is $\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x)$. This type of discontinuity is also called discontinuities of the first kind.

Example 3.20. *$f : \mathbb{R} \to \mathbb{R}$ be defined by* $f(x) = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases}$.

Easy to see that left and right limits at 0 are different.

Example 3.21. *$f : \mathbb{R} \to \mathbb{R}$ be defined by* $f(x) = \begin{cases} x^2 - 4 & x < 1 \\ -1 & x = 1 \\ -\frac{1}{2}x + 1 & x > 1 \end{cases}$.

Easy to see that left and right limits at 0 are different.



Example 3.22.

Example 3.23. *Let*

$$g(x) = \begin{cases} x+1, & x < 2 \\ -x, & x \geq 2 \end{cases}$$

Easy to see that left and right limits at 2 are different.

Definition 3.5. Discontinuity of second kind: If either $\lim_{x\to c^+} f(x)$ or $\lim_{x\to c^-} f(x)$ does not exist, then we say f has discontinuity of the second kind.

Example 3.24. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}.$$

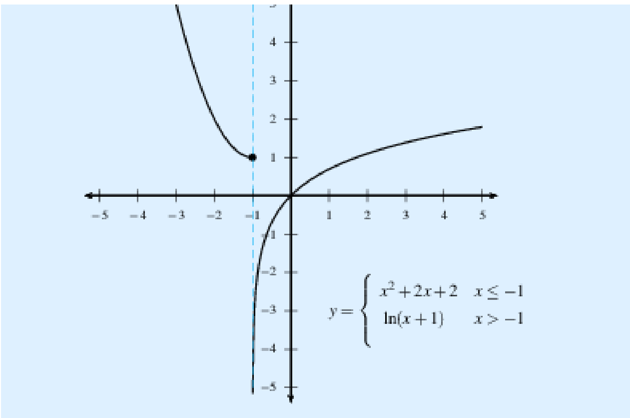
Then f does not have left or right limit any point c. Indeed, if $c \in \mathbb{Q}$, then $x_n = c + \frac{1}{n} \in \mathbb{Q}$ and $y_n = c + \frac{\pi}{n} \notin \mathbb{Q}$. Both $\{x_n\}$ and $\{y_n\}$ both converge to c from the right hand side. But $\lim_{n\to\infty} f(x_n) = 0$ and $\lim_{n\to\infty} f(y_n) = 1$. So $\lim_{x\to c^+} f(x)$ does not exist. If $c \notin \mathbb{Q}$, then there exist sequences $x_n \in (c, c + 1/n) \cap \mathbb{Q}$ and $y_n = c + \frac{1}{n} \notin \mathbb{Q}$.

Definition 3.6. Infinite discontinuity: If either $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$, then we have an infinite discontinuity, also called an asymptotic discontinuity. (See the example below, with $a = -1$.) In an infinite discontinuity, the left- and right-hand limits are infinite; they may be both positive infinity, both negative infinity, or one positive infinity and one negative infinity.

Example 3.25. *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$\begin{cases} f(x) & = \frac{1}{x} \ x \neq 0, \\ & = 1 \ x = 0. \end{cases}$$

has infinite discontinuity at $x = 0$.



Example 3.26.