Graph Theory

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Abstract

These notes provide a quick background on graph theory.

1 Introduction

Graphs are ubiquitous in computer science and allied fields. In these notes, we study basic topics in graph theory. A graph G is a pair of sets (V, E), where V is an arbitrary set, and $E \subseteq V \times V$, i.e., E is a subset of the cartesian product of V with itself. E is called the set of *edges* of the graph. Note that we have not restricted V in any way. So, V can be finite, countably infinite, or even uncountably infinite. A graph G is said to be finite if V is finite, and is infinite otherwise. However, we will focus only on the setting where V is finite, as that is the most common kind of graphs that we encounter. So, unless stated otherwise, we assume that the graphs in these notes are finite.

The set $E \subseteq V \times V$. A standard way to represent a graph is by a drawing in the plane. The vertices are represented by points in the plane with labels, and the edges are drawn by directed arcs, where for an edge (i, j), we represent it by a directed arc with tail at i and head at j. An edge (i, i) is called a *self-loop*, and is drawn by an arc from i to i. Figure 1 shows a graph.

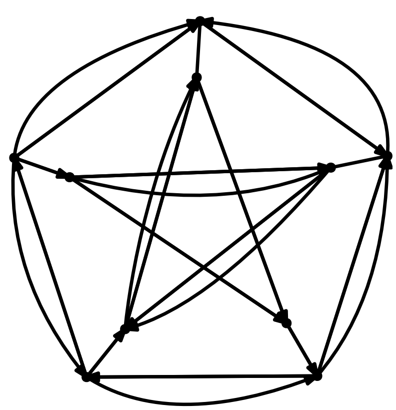


Figure 1: A drawing of a graph.

The graphs we just described are called *directed graphs with self-loops* and typically in graph theory we sometimes avoid self-loops. Graphs with directed arcs are also called *digraphs*. In graph theory, we typically restrict the notion of graphs even further - we assume that the set E is symmetric and not reflexive. If the relation E is symmetric, if $(i,j) \in E \Rightarrow (j,i) \in E$. Therefore, we can ignore directions and replace E by a set of *unordered pairs*, i.e., sets of size 2. If $\{i,j\}$ is an edge, then i and j are said to be *adjacent*. Figure 2 shows an example of a graph. We usually reserve the term *arc* for directed edges and use the term *edge* for the unordered pairs (or undirected edges).

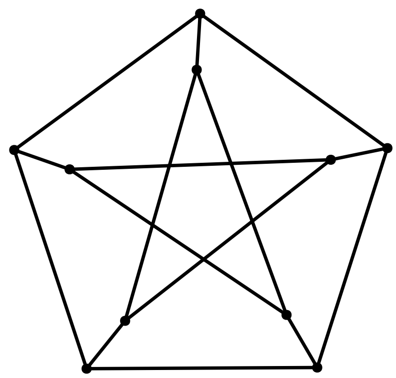


Figure 2: A drawing of an undirected graph.

Finally, sometimes it is useful in applications to allow E to be a *multi-set*. Hence, we may have many arcs or edges between pairs of vertices. Such a graph is called a multi-graph. Figure **??**. shows an example of a multi-graph. We use the term *simple graph* to distinguish the case that E is a set and not a multi-set.

To summarize, a graph can have directions on the edges. In this case, these objects are called digraphs. The graph have an edge or arc from a vertex to itself - a self-loop, or there can be multiple edges or arcs between the same pair of nodes. These are called multi-graphs. When we use the term *graph* in these notes, we assume that the graph is finite, simple, without self-loops and is undirected.

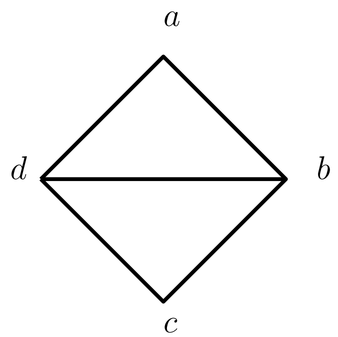
An alternate view to represent a graph is via a *matrix*. This is useful in many cases as in many cases, linear algebraic properties of the matrix can tell us about the combinatorial structure of the graph. An *adjacency matrix* A of a graph is a square matrix whose rows and columns are indexed by V. If $|V| = n$, then A is an $n \times n$ matrix. For an undirected graph,

$$A[i,j] = \begin{cases} 1, & \{i,j\} \in E \\ 0, & \{i,j\} \notin E \end{cases}$$

We can similarly, define an adjacency matrix for a directed graph $D = (V, A)$ as follows.

$$A[i,j] = \begin{cases} 1, & (i,j) \in E \\ -1, & (j,i) \in E \\ 0, & (i,j) \notin E \text{ and } (j,i) \notin E \end{cases}$$

Figure 3 shows an example of a graph and its adjacency matrix.



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & \left[\begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}\right] & & \end{array}$$

Figure 3: A graph and its adjacency matrix

2 Important Graphs and Basic Notions

In this section, we introduce some important classes of graphs that have special names. Recall that the term graph refers to finite, simple, undirected graphs without self-loops. A graph is a *clique* if there this an edge between every pair of vertices in the graph. A clique on n vertices is denoted K_n. Figure 4 shows examples of cliques. Note that a clique has the maximum possible number of edges, namely $\binom{n}{2}$.

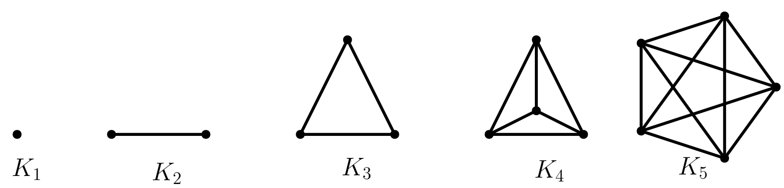


Figure 4: The graphs $K_1, K_2, \ldots, K_5$

A graph G is a *path* if the vertices can be linearly ordered $v_0, v_1, \ldots, v_n$ s.t. the edges are between vertices $\{v_i, v_{i+1}\}$, $i = 0, \ldots, n-1$. These graphs are denoted P_n. If $v_0 = v_n$, we obtain a cycle. A cycle on n vertices is denoted C_n. Figure 5 shows examples of paths and cycles on n vertices.

A graph $G = (V, E)$ is *bipartite* if the vertices in the graph can be partitioned into two disjoint sets $V_1 \sqcup V_2$, i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$ s.t. there are no edges between two vertices $u, v \in V_1$ or between vertices $u, v \in V_2$. A bipartite graph $G = (V_1 \cup V_2, E)$ with bipartition $V_1 \sqcup V_2$ with $|V_1| = m$ and $|V_2| = n$ is called a *complete bipartite graph* if for all $u \in V_1, v \in V_2$, $\{u, v\} \in E$. A complete bipartite graph is denoted $K_{m,n}$. Note that K_{mn} has mn edges. Figure 6 shows examples of complete bipartite graphs.

For a graph $G = (V, E)$ a *path* in G between two vertices u, v is a set of distinct vertices $u = u_0, u_1, \ldots, u_k = v$ s.t. $\{u_i, u_{i+1}\}$ is an edge for $i = 0, \ldots, k-1$. If $u = v$ and all vertices $u_1, \ldots, u_{k-1}$ are distinct, then such a path is a *cycle* in G.

A graph $G = (V, E)$ is *connected* if for all pairs $u, v \in V$, there is a path between u and v in G. If this is not the case, then G is said to be disconnected. Note that connectivity is an *equivalence relation.* A vertex u is connected to itself. If u is connected to v, then v is connected to u, and

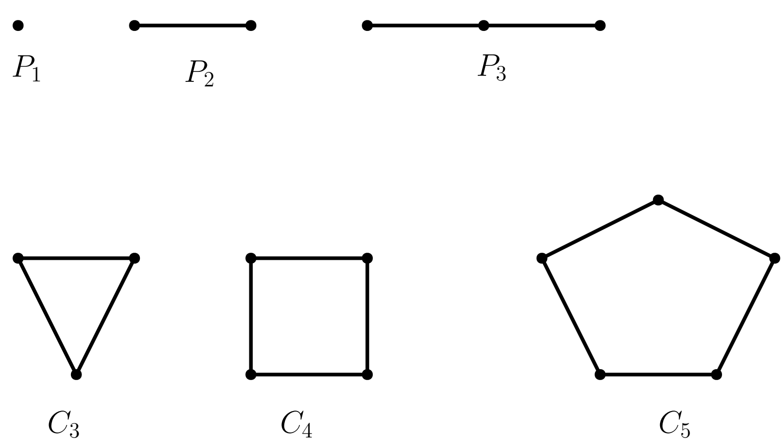
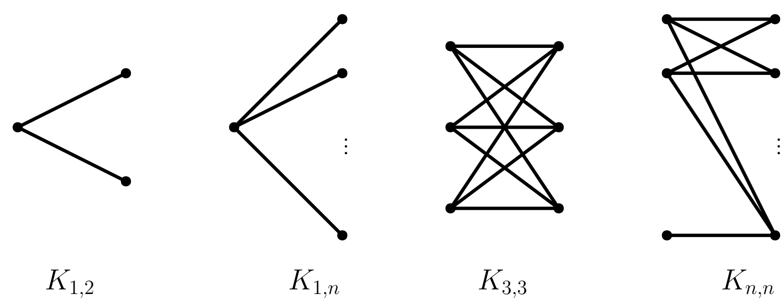


Figure 5: The graphs P_1, P_2, P_3 and C_3, C_4, C_5

Complete Graph



Complete Bipartite Graph

Figure 6: The graphs $K_{1,2}$, $K_{1,n}$, $K_{3,3}$, $K_{n,n}$

finally, if u is connected to v and v is connected to w, then u is connected to w via a path. The equivalence classes defined by connectivity are called the *connected components* of G. Figure 7 shows an example of a disconnected graph.

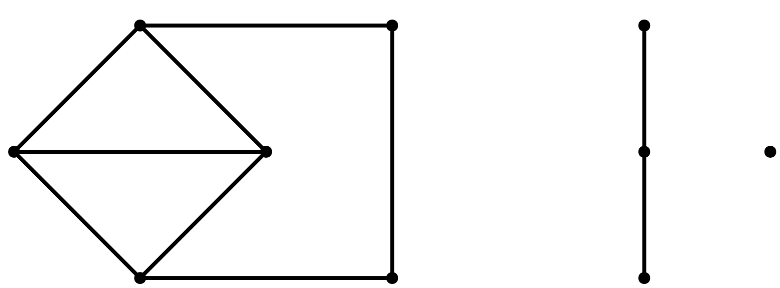


Figure 7: A graph with three connected components.

For a digraph $D = (V, A)$, a path is defined in the natural way. However, connectivity is not a symmetric relation. For vertices $u, v \in V$, there could be a path from u to v, but it is possible that there is no path from v to u. Such a digraph is called *strongly connected* if there is a path between every pair of vertices. Figure 8 shows an example of a strongly connected digraph.

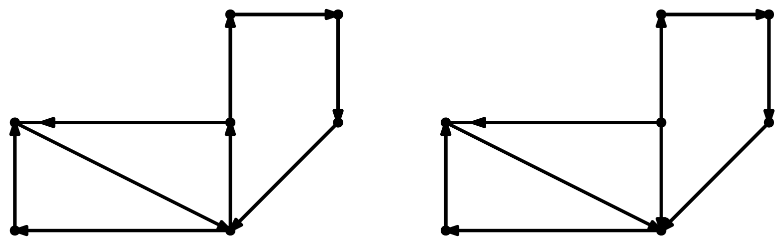


Figure 8: The graph on the left is strongly connected, but the graph on the right is not.

Exercise 1

For a vertex set V of size n, how many graphs can we have? How many digraphs?

$2^{\binom{n}{2}}$ $2^{2\binom{n}{2}}$

Exercise 2

Given a vertex set U of size m and V of size n, how distinct bipartite graphs are there with vertex set $U \cup V$? $2^{m \times n}$

Exercise 3

Let $V = \{1, \ldots, n\}$ be a vertex set of size n. How many distinct graphs can we form that are paths on n vertices? What about cycles? (n-1)!

n!

For a graph $G = (V, E)$, a *subgraph* is a graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq \{\{u, v\} \in E : u \in V' \text{ and } v \in V'\}$. That is, a subgraph is a graph that is obtained as follows: We select a subset of the vertices of G. Then, among the edges of E between pairs of vertices in V', we select a subset of the edges. This new graph is called a subgraph. An *induced subgraph* is a graph $G = (V', E')$, where we only select the subset $V' \subseteq V$. The edges $E' \subseteq E$ are all edges between pairs of vertices in E. That is, $E' = \{\{u, v\} : u \in V' \text{ and } v \in V'\}$. Figure 9 shows a subgraph and induced subgraph of a graph.

For a graph $G = (V, E)$, the *complement* of G is the graph $\overline{G} = (V, \overline{E})$ obtained by making each pair of non-adjacent vertices adjacent, and adjacent vertices non-adjacent. Note that $\overline{G}$ is also a simple graph. Figure 10 shows a graph and its complement.

In a graph $G = (V, E)$, the *degree* of a vertex $v \in V$, denoted $deg(v)$ is the number of edges incident on it. That is, $deg(v) = |\{\{u, v\} \in E|$. In a directed graph, we have two notions of degree, namely the *in-degree*, the number of arcs entering a vertex, or the number of arcs whose head is at the vertex, denoted $d_{in}(v)$, or $d_+(v)$. The *out-degree* is the number of arcs leaving a vertex, or the number of arcs whose tail is at the vertex. This is denoted $d_{out}(v)$ or $d_-(v)$. In a multigraph, we count multiplicity of arcs. That is, if there are two arcs (i, j) in G, then it contributes 2 to the degree. A self-loop contributes one to the in-degree and one to the out-degree.

We are ready to prove our first lemma about graphs.

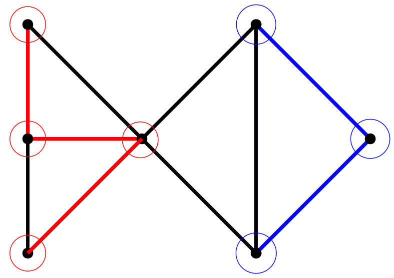


Figure 9: The shows a subgraph in red and an induced subgraph in blue.

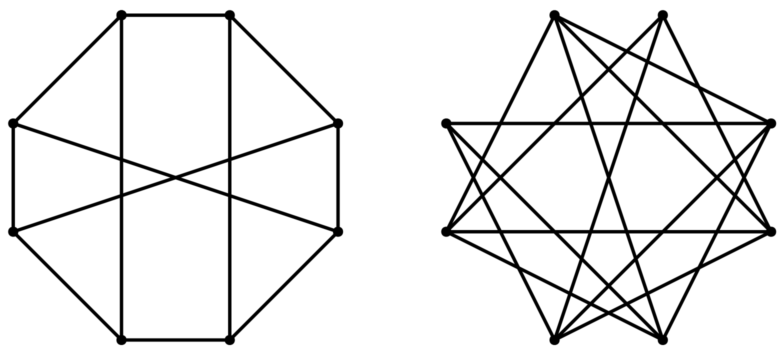


Figure 10: The shows a graph and its complement.

Lemma 1 (Handshake Lemma)**.** *Let* $G = (V, E)$ *be a graph. Then*

$$\sum_{v \in V} deg(v) = 2|E|$$

Proof. We can count the edges by adding up the edges incident on each vertex. But, this way, each edge is counted twice and the result follows. □

Exercise 4

Prove that the number of odd degree vertices in a graph must be even.

Exercise 5

Prove that either a graph or its complement is connected.

3 Isomorphism

A basic notion that we see throughout mathematics is *isomorphism*. The term *iso* means *same* and *morphism* means a transformation. The term *iso* for *same* is also used in natural language.

Some such terms you may have encountered are *isotopes*, *isotherm*, *isolines*, etc. Two graphs are *isomorphic* if they are *same* in some sense. More formally, we say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $\phi : V_1 \to V_2$ s.t. $\{u, v\} \in E_1 \Leftrightarrow \{\phi(u), \phi(v)\} \in E_2$. That is, there is an edge between two vertices in the graph G_1 if and only if there is an edge between their images in E_2. An isomorphism from a graph G_1 to itself is called an *automorphism*. Every graph has a trivial automorphism, namely the identity map.

Note that an isomorphism is always a bijection from V_1 to V_2. However, not all bijections are isomorphisms. We want to ensure that the adjacency relation between vertices is preserved.

Given an isomorphism between two graphs, it is easy to check that it is indeed an isomorphism. However, it is a challenging question whether we can design a fast algorithm to decide if two input graphs are isomorphic. Consider the set of all graphs on n vertices. Note that isomorphism is an equivalence relation. We now address the question: How many equivalence classes can we have if the set of graphs on n vertices related by isomorphism? Alternatively, we can ask: how many graphs on n vertices are there that are pairwise non-isomorphic? Note that if there are k equivalence classes, we can have at most k graphs that are pairwise non-isomorphic as each graph must come from a different equivalence class. It is difficult to obtain an exact answer to this question. However, we can obtain good upper and lower bounds. We have already seen in the previous section that the number of graphs on n vertices in $2^{\binom{n}{2}}$. Therefore, a trivial upper bound on the number of pairwise non-isomorphic graphs is $2^{\binom{n}{2}}$.

Consider a fixed graph G on n vertices. We know from the chapter on counting that there are $n!$ bijections between two sets of size n. Therefore, G must be isomorphic to at most $n!$ other graphs. Since G was chosen arbitrarily, this implies, each equivalence class has size at most $n!$. Since there are $2^{\binom{n}{2}}$ graphs, the number of pairwise non-isomorphic graphs is at least $2^{\binom{n}{2}} n!$. Hence, the number of non-isomorphic graphs is at least $2^{\binom{n}{2}/n!}$ and at most $2^{\binom{n}{2}}$. How far apart are these bounds? We can get an estimate by taking logs.

$$\log 2^{\binom{n}{2}} = \binom{n}{2} = \frac{n^2}{2}\left(1 - \frac{1}{n}\right)$$

$$\begin{aligned}
\log \frac{2^{\binom{n}{2}}}{n!} &= \frac{n^2}{2}\left(1 - \frac{1}{n}\right) - \log \frac{n^n}{2^{n/2}} \\
&\geq \frac{n^2}{2}\left(1 - \frac{1}{n}\right) - n \log n - \frac{n}{2} \\
&= \frac{n^2}{2}\left(1 - \frac{1}{n} - \frac{\log n}{n} - \frac{1}{2n}\right)
\end{aligned}$$

Here, we use the simple upper bound that $n! \leq n^n/2^{n/2}$. As $n \to \infty$, the lower and upper bounds nearly coincide. Therefore, the number of non-isomorphic graphs is *close to* $2^{\binom{n}{2}}$.

4 Graphic Sequences

Given a graph $G = (V, E)$, we construct an n-dimensional vector of non-decreasing natural numbers $(d_1, \ldots, d_n)$, where d_i is the degree of vertex i. Here, we order the vertices of G in non-decreasing order of their degree. If such a vector corresponds to a graph, then it is called a *graphic sequence*. Given a graph, we can easily construct a graphic sequence. What about the other direction? Given a vector $\mathbf{v} = (d_1, \ldots, d_n)$ s.t. for all $i < j$, $d_i \leq d_j$, is $\mathbf{v}$ a graphic sequence of some graph? In this section, we study this question. There are some natural requirements? It is easy to check that the maximum degree in a graph is at most $n-1$. Therefore, $d_n \leq n-1$. It is easy to construct sequences that satisfy this requirement but are not graphic sequences. From the exercises in the previous section, we know that the number of vertices of odd degree in a graph must be even. This is another necessary condition for a vector to be a graphic sequence. But, even this is not sufficient. Try to construct a sequence $(d_1, \ldots, d_n)$ such that $d_n \leq n-1$ and the number of odd entries is even, but the sequence is not graphic.

We now obtain a condition for a sequence to be graphic.

Havel-Hakimi Theorem

Theorem 1

Let $\mathbf{v} = (d_1, \ldots, d_n)$ be a sequence of natural numbers s.t. for $i < j$, $d_i \leq d_j$. Let $\mathbf{v}' = (d'_1, \ldots, d'_{n-1})$ be the sequence defined as follows:

$$d'_i = \begin{cases} d_i, & i < n - d_n \\ d_i - 1, & i \geq n - d_n \end{cases}$$

Then $\mathbf{v}$ is a graphic sequence if and only if $\mathbf{v}'$ is a graphic sequence.

Proof. One direction is easy. Suppose $\mathbf{v}'$ is a graphic sequence. There is a graph G' whose degree sequence is equal to $\mathbf{v}'$. Since ee obtain $\mathbf{v}$ by adding 1 to the last d_n entries and appending d_n to $\mathbf{v}$, the graph G obtained by adding a vertex v_n and connecting it to the last d_n vertices in $\mathbf{v}'$ has degree sequence $\mathbf{v}$. Therefore, $\mathbf{v}$ is graphic.

To show the other direction requires some work. Suppose $\mathbf{v}$ is graphic. There is a graph G whose degree sequence is $\mathbf{v}$. However, the vertex v_n with degree d_n may not be adjacent to the last d_n vertices in $\mathbf{v}$. Therefore, deleting G does not necessarily yield the graphic sequence $\mathbf{v}'$.

In order to show that $\mathbf{v}'$ is graphic, we proceed as follows. Let $\mathcal{G}_{\mathbf{v}}$ denote the set of all graphs whose degree sequence is $\mathbf{v}$. In this set, we will show that there is one *nice* graph, whose maximum degree vertex is adjacent to the d_n vertices of highest degree. To that end, let us order the graphs in $\mathcal{G}_{\mathbf{v}}$ as follows: For a graph $G \in \mathcal{G}_{\mathbf{v}}$, let $\tau(j)$ be the highest index of a vertex in its degree sequence such that v_n is *not adjacent* to v_j. That is, v_n is adjacent to $v_{n-1}, \ldots, v_{j+1}$. For two graphs $G_1, G_2 \in \mathcal{G}_{\mathbf{v}}$, $G_1 \leq G_2$ if $\tau(G_1) \leq \tau(G_2)$. Let G the graph in $\mathcal{G}_{\mathbf{v}}$ that is minimal in this order. Now, we analyze the structure of G. Consider the degree sequence of G. Let j be the highest index in the degree sequence of G s.t. v_n is not adjacent to v_j. If $j \leq d_n$, then v_n is adjacent to d_n vertices with highest index in its degree sequence. Deleting v_n therefore results in a graph G' whose degree sequence is $\mathbf{v}'$. So suppose $j > d_n$. This implies, v_n is adjacent to a vertex v_i that appears before j in the degree sequence. Now, $d_i \leq d_j$, and v_i is adjacent to v_n. Therefore, there is a vertex v_k, s.t. $d_k < d_j$ and v_k is adjacent to v_j but not v_i. Consider the graph G' obtained by adding the edges $\{v_n, v_j\}$ and $\{v_i, v_k\}$, and removing the edges $\{v_i, v_n\}$ and $\{v_j, v_k\}$. The degree sequence remains unchanged, but in G', the highest index of a non-adjacent vertex in the degree sequence is smaller than j, contradicting the choice of G. Figure 11 shows the argument used in this proof. □

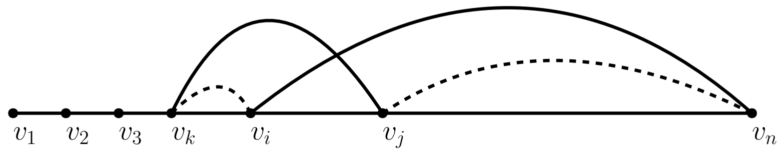


Figure 11: The shows an illustration of the proof of Theorem 1.

Exercise 6

Where was the assumption $d_1 \leq d_2 \leq \ldots \leq d_n$ used in the proof? Show that the statement is not true if we omit this assumption.

Exercise 7

Find two non-isomorphic graphs with the smallest number of vertices with the same degree sequence.

Exercise 8

Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or 6 vertices of degree 5.

Exercise 9

Decide for which $n \geq 2$ there exist graphs whose vector consists of n distinct numbers. For which n are there are graphs whose vectors have $n-1$ distinct numbers?

Exercise 10

(*) Prove that $(d_1, \ldots, d_n)$ is a graphic sequence if and only if

1. $\sum_{j=1}^{n} d_i \equiv 0 \mod 2$, and
2. $\sum_{j=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ for all $1 \leq k \leq n$.

5 Euler Tours

6 Trees

7 Planar Graphs

At the start of this chapter we said that graphs are visualized by drawing them in the plane. The vertices are represented by points and edges by continuous curves between points. However, if the graph is *complicated*, the edges cross each other and makes visualization difficult. The class of graphs that can be drawn in the plane with their edges begin internally disjoint are called planar graphs. In this section, we study planar graphs and their properties.

Let us start with the notion of drawing formally, and before we do that we want to introduce the notion of an arc. An arc α is the image of an injective map from the closed interval $[0, 1]$ to $\mathbb{R}^2$. That is $\alpha = \gamma([0, 1])$, where $\gamma : [0, 1] \to \mathbb{R}^2$ is a continuous, injective function. The fact that the function is injective implies that no two points of $[0, 1]$ map to the same point in $\mathbb{R}^2$. In other words, the curve does not cross itself. Continuity of course, means what you think it means - one can draw the arc from one end to the other without lifting the pen off of the paper. The points $\gamma(0)$ and $\gamma(1)$ are called the *end-points* of the arc.

Definition 1 (Drawing)

By a drawing of a graph $G = (V, E)$ we mean an assignment as follows: Let $f : V \to \mathbb{R}^2$ be an injective function. For each edge $e = \{u, v\}$, assign an arc $\alpha(e)$ with end-points u and v. Further, we assume that no point $f(v)$ lies in the interior of any arc $\alpha(e)$.

A graph together with its drawing is called a *topologoical graph*. If the arcs are pairwise internally disjoint, i.e., they do not share any vertices other than their end-points then the drawing is called planar. A graph is *planar* if it admits a planar drawing. A planar graph along with its topological drawing is called a *plane graph*. Figure **??** shows two different drawings of the same plane graph.

A set $R \subseteq \mathbb{R}^2$ is *connected*, or *path-connected* if any two points in R have an arc that is contained entirely in R. Let $G = (V, E)$ be a plane graph. Removing the points corresponding to the vertices, and the arcs corresponding to the edges, the plane $\mathbb{R}^2$ breaks into disjoint path-connected sets. Each path-connected component is called a *face* of the drawing. Since G is finite, there must be an *unbounded face*. All other faces are called *bounded faces*. Note that these are notions specific to a plane drawing and not to the planar graph itself - which may have many different drawings, and a different set of faces in each drawing.

Another way to view planar graphs that is particularly appealing is the drawing of planar graphs on a sphere. In this case, we don't have to deal with one face being unbounded - all faces are bounded. To see that any planar graph can be drawn on a sphere, and that for any drawing of a finite graph on a sphere, we can obtain a plane embedding, we rely on a *stereographic projection.* Figure **??** shows an example of a stereographic projection.

We need not restrict our attention to drawing graphs in the plane. We can consider drawing graphs on other surfaces. For example, we can define drawings of graphs on a *torus*, or even a *Möbius strip* or other more complicated surfaces. Figure **??** shows examples of some surfaces.

Graphs can be classified according to the surfaces on which they can be drawn. Again note that by drawing on a surface, we mean that the vertices and edges are mapped injectively to points and arcs on the surface, and such that the arcs are pairwise internally disjoint.

We will see that the graph K_5 cannot be drawn in the plane, but we can obtain a drawing of K_5 on a torus. A sphere with g *handles* is said to have *genus* g. It is easy to see that any graph can be drawn on a surface with sufficiently large genus - by making a handle for an edge that crosses other edges. The smallest genus of a surface on which a graph can be drawn is called the genus of the graph.

7.1 Cycles in planar graphs

A basic result we need while dealing with planar graphs is the infamous *Jordan curve theorem*, whose statement is intuitively obvious, but whose proof is not! A *jordan curve* is a closed curve without self-intersections. That is, it is an image of a function $\phi : [0,1] \to \mathbb{R}^2$ that is injective when restricted to $(0,1)$, and $\phi(0) = \phi(1)$.

Theorem 2 (Jordan curve theorem)

Any jordan curve c divides the plane into exactly two path-connected components - the interior of c and the exterior of c.

odd cuts -> outside
even cuts -> Inside

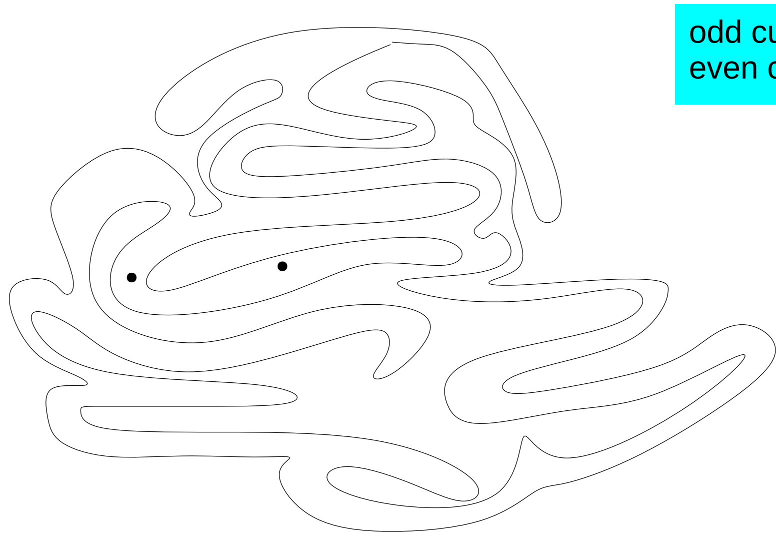


Figure 12: A jordan curve and a point in its interior and a point in its exterior

Assuming the Jordan curve theorem, we can prove that some graphs are non-planar.

Theorem 3

K_5 is non-planar.

Proof. We prove by contradiction. Let $b_1, \ldots, b_5$ be the five points corresponding to the vertices of K_5 in some drawing. The arc connected b_i and b_j will be denoted $\alpha(i,j)$. Since the vertices corresponding to points b_1, b_2 and b_3 form a cycle in K_5, it follows that there are arcs $\alpha_1(b_1, b_2)$, $\alpha_2(b_2, b_3)$ and $\alpha_3(b_3, b_1)$. The union of the arcs α_1, α_2 and α_3 is Jordan curve c, and thus breaks $\mathbb{R}^2$ into the interior and exterior. There are 2 choices for each of the points b_4 and b_5. Each of them can be in the interior of c or in the exterior of c. Suppose one of them, say b_4 is inside c and b_5 is in the exterior of c. Then arc corresponding to the edge between vertices corresponding to b_4 and b_5 must cross c, which is impossible in any drawing. Therefore, either both b_4 and b_5 are in the interior of c, or they are both in the exterior of c. The cases are symmetric, so suppose they are in the interior of c.

Since there are arcs $\alpha_4(b_4, b_1)$, $\alpha_5(b_4, b_2)$ and $\alpha_6(b_4, b_3)$, it follows that we obtain three additional Jordan curves c_1, c_2 and c_3, where c_1 is defined by α_4, α_5 and α_1, and so on. Now, b_5 is in the interior of one of the Jordan curves c_1, c_2 or c_3. Whichever Jordan curve c_i, b_5 is in, one of the points b_1, b_2 or b_3 is in the exterior of the curve. This implies the arc corresponding to this edge must cross another arc in its interior. Therefore, there is no such drawing. □

For a planar graph G, if $e_1, \ldots, e_k$ are edges in a cycle, then then the arcs $\alpha(e_1), \ldots, \alpha(e_k)$ form a Jordan curve c in any drawing of G. By the Jordan curve theorem, each face of G either lies inside c, or outside c. We call this Jordan curve a cycle too. Thus a cycle may refer to a graph-theoretic cycle, or the Jordan curve in the drawing. For some topological graphs, each face is the interior or exterior of some cycle in G. But, this is not always true. For example a tree is a planar graph, and there is only one face.

We show that if we exclude planar graphs that are not *2-connected*, then the property above is satisfied. We already know when a graph is connected. 2-connectivity is a stronger notion. We say that a graph is 2-connected if removing any vertex leaves the graph connected. For example, if G were a cycle, then removing a vertex, we still have a path between any pair of vertices.

Theorem 4

A graph is 2-connected if and only if it can be obtained by starting with a triangle and applying a sequence of edge-subdivisions and edge-additions.

Theorem 5

Let G be a 2-connected planar graph. Then, every face in any planar drawing of G is a region of some cycle in G.

Proof. □

Proof. We prove by induction on n. If G is a triangle, then we are done. The statement follows from the Jordan curve theorem. Let $G = (V, E)$ be a connected topological planar graph with at least 4 vertices. Then, by Proposition **??** there is either an edge $e \in E(G)$ s.t. $G \setminus \{e\}$ is 2-connected, or there are a 2-connected graph $G' = (V', E')$ and an edge $e \in E'$ s.t. $G = G\%e$, where $G\%e$ is the edge-subdivision of G. Since G is topological planar, so is G'. Each face of G' is a region of some cycle.

Consider the case $G' = G - e$, where $e = \{u, v\}$. The vertices u and v are connected by an arc $\alpha(e)$. Hence, they lie on a face F of G'. If F' denotes the face bounding $\alpha(e)$, then adding $\alpha(e)$ divides F into two faces f_1 and f_2. The faces F_1 and F_2 are faces bound by cycles. Hence, the faces are bound by cycles.

If $G' = G\%e$, then, each face of G' is a region of some cycle. Then, G has the same property. This follows immediately from edge-subdivision. □

7.2 A combinatorial Characterization of Planar graphs

Theorem 6

A graph is planar if and only if it has no subgraph isomorphic to a subdivision of $K_{3,3}$ or K_5.

We will not prove this theorem. But, remarkably, this gives us a combinatorial characterization of planar graphs without any appeal to drawings.

7.3 Euler's formula

Theorem 7

For any drawing of a connected planar graph $G = (V, E)$, $|V| - |E| + |F| = 2$.

Proof. We prove by induction on the number of edges of G. If $|E| = \emptyset$, then $|V| = 1$ and $|F| = 1$, and the formula holds. So suppose $|E| \geq 1$. We distinguish two cases: If G is a tree, then we have shown that $|V| = |E| + 1$, and $|F| = 1$. Therefore, the formula holds. We therefore only need to consider planar graphs that contain cycles. Let $e \in E$ s.t. e is contained in a cycle. Then, $G' = G - e$ is connected and planar. Therefore, $|V(G')| - |E(G')| + |F(G')| = 2$. Adding e back increases the number of edges by 1 and the number of faces by 1. Therefore, we obtain $|V(G)| - |E(G')| - 1 + |F(G')| + 1 = 2$. □

Theorem 8

A planar graph with at least 3 vertices has at most $3n - 6$ edges. Equality holds for any maximal planar graph.

Proof. If G is not maximal, we can keep adding additional edges until it becomes maximal. In a maximal planar graph, all faces including the outer face must be triangles. If G is disconnected, we can connect them by adding an edge. If G is connected but not 2-connected, it has some vertex v whose removal disconnects the graph creating components $V_1, \ldots, V_k$, $k \geq 2$. Choose two edges e, e' connecting v to distinct components V_i and V_j s.t. e, e' are drawn next to each other. Hence, in a maximal planar graph with at least 3 vertices is 2-connected. Hence, each face is bounded by a cycle. If a bounding face contains at least 4 vertices, we can add a diagonal.

Therefore, $3|F| = 2|E|$. Thus, from Euler's formula we have

$$
\begin{aligned}
|V| - |E| + |F| &= 2 \\
|V| - |E| + \frac{2}{3}|E| &= 2 \\
|V| - \frac{1}{3}|E| &= 2 \\
|E| &= 3|V| - 6
\end{aligned}
$$

□

Exercise 11

Show that a planar bipartite graph has at most $2n - 4$ edges.

7.4 Platonic Solids

A platonic solid, or a *regular polytope* is a three-dimensional bounded, convex polyhedron with a finite number of faces, all of whose faces are congruent regular polygons, and such that at each vertex of the polytope an equal number of polygons meet. The ancient Greeks were fascinated with the highly symmetric nature of regular polytopes and Kepler even suggested that between the six planets - Mercury, Venus, Earth, Mars, Juipter and Saturn there lie regular polytopes. A possible reason why Kepler suggested this is that there are only 5 platonic solids in three dimensions. That's it. In this section, we will prove this result. Before we do that however, let us list the platonic solids. Some may be familiar to you, but the others may not. Corresponding to each platonic solid, we can associate a planar graph - imagine the edges of the polytope made of an elastic material. Imagine blowing a balloon that lies inside the polytope. The vertices and edges of the polytope form a graph embedded in the sphere. As we saw earlier, a planar graph can be equivalently been seen as embedded in the plane or on the sphere. The platonic solids and their corresponding graphs are shown in Figure **??**.

Theorem 9

There are only five platonic solids.

Proof. The planar graph corresponding to each regular polytope has the following properties: each vertex has the same degree, and each face has the same number of sides. Let m, n and f denote respectively, the number of edges, vertices and faces in the graph, let d denote the degree of each vertex and let k denote the number of sides of each face. Since each vertex has equal degree, by the Handshake Lemma,

In geometry, a **Platonic solid** is a **convex, regular polyhedron** in three-dimensional Euclidean space. Being a regular polyhedron means that the faces are congruent (identical in shape and size) regular polygons (all angles congruent and all edges congruent), and the same number of faces meet at each vertex. There are only five such polyhedra:

1. **Tetrahedron**: 4 faces, 4 vertices, and 6 edges.
2. **Cube**: 6 faces, 8 vertices, and 12 edges.
3. **Octahedron**: 8 faces, 6 vertices, and 12 edges.
4. **Dodecahedron**: 12 faces, 20 vertices, and 30 edges.
5. **Icosahedron**: 20 faces, 12 vertices, and 30 edges [1,2].

$$\sum_{v \in V} deg(v) = dn = 2m$$

Since each face has k sides, and each edge lies on two faces, we have

$$kf = 2m$$

Plugging in these in Euler's formula we obtain

$$\begin{aligned} n - m + f &= \frac{2m}{d} - m + \frac{2m}{k} \\ &= m\left(\frac{2}{d} - 1 + \frac{2}{k}\right) \\ &= 2 \end{aligned}$$

Dividing both sides by m, we obtain

$$\begin{aligned} \frac{2}{d} - 1 + \frac{2}{k} &= \frac{2}{m} && \text{[subtracting 1 and dividing by 2]} \\ \frac{1}{d} + \frac{1}{k} &= \frac{1}{2} + \frac{1}{m} \end{aligned}$$

Note that RHS is larger than $1/2$. Further, if both d and k were at least 4, then $1/d+1/k \leq 1/2$. Therefore, at least one of them must be at most 3. Hence, $\min\{d,k\} = 3$. Suppose $d = 3$. If $d \geq 6$, then,

$$\frac{1}{3} + \frac{1}{k} = \frac{1}{3} + \frac{1}{6} \leq \frac{1}{2}$$

Therefore, $d \leq 5$. By a symmetric argument, if $k = 3$, the only allowable values are $3, 4$ and 5. This gives the following possible values as shown in Figure **??**, and this in turn implies there are only 5 platnoic solids. □

8 Coloring planar graphs

A coloring of a graph $G = (V,E)$ is a function $\phi : V \to [k]$, where $[k] = \{1,\ldots,k\}$ such that $\phi(u) \neq \phi(v)$ for all $\{u,v\} \in E$. The smallest k s.t. there exists a coloring is called the *chromatic number* of a graph, denoted $\chi(G)$. A set $U \subseteq V$ is an *independent set*, or a *stable set* if the vertices in U are pairwise non-adjacent. Note that for $i = 1,\ldots,k$, $\phi^{-1}(i) = \{u \in V : \phi(u) = i\}$ forms an independent set. The study of the chromatic number of a graph constitutes a very active area of research in graph theory. Figure 13, 14 shows a coloring of C_4 and C_5. It is easy to check that $\chi(C_4) = 2$, but $\chi(C_5) = 3$. In general $\chi(C_{2k+1}) = 3$ for $k \in \mathbb{N}$, and $\chi(C_{2k}) = 2$. Let us look at some other examples. Consider the chromatic numbers of the following graphs: K_3, K_4, K_5. Since each vertex in K_n is adjacent to all other vertices, we have that $\chi(K_n) = n$. Figure fig:k345 shows a coloring of K_3, K_4 and K_5. A graph is *bipartite* if the vertices of G can be partitioned into two disjoint sets $X \cup Y$ s.t. all vertices in X are pairwise disjoint, and all vertices in Y are pairwise disjoint. It is easy to check that $\chi(G) = 2$ for G, a bipartite graph. In fact, bipartite graphs are precisely the class of graphs with chromatic number 2.

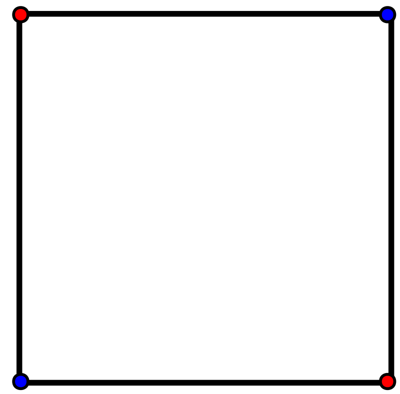


Figure 13: Coloring C_4

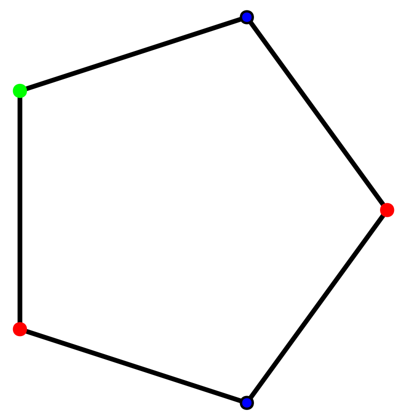


Figure 14: Coloring C_5

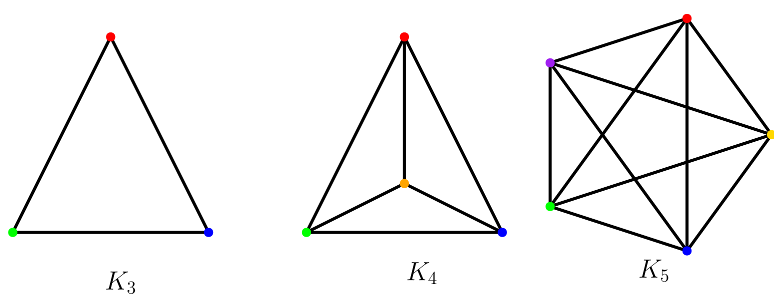


Figure 15: Coloring K_3, K_4 and K_5

Theorem 10

A graph $G = (X \cup Y, E)$ is bipartite if and only if $\chi(G) = 2$.

Proof. If G is bipartite, since the vertices in X and the vertices in Y are both independent sets, we can color X with color 1 and Y with color 2. On the other hand, if G is 2-colorable, let $\phi : V \to [2]$ be a 2-coloring. Then,$\phi^{-1}(1)$ and $\phi^{-1}(2)$ forms a partition of V s.t. $\phi^{-1}(1)$ is an independent set, and $\phi^{-1}(2)$ is an independent set. □

It is easy to check that trees are bipartite graphs.

Exercise 12

Let $T = (V, E)$ be a tree, then show that T is a bipartite graph.

Given a graph, how do we obtain a lower bound on the chromatic number of a graph? For a graph $G = (V, E)$, a set $S \subseteq V$ s.t. all pairs of vertices in S are adjacent is called a *clique* in G. The *clique number* of G, denoted $\omega(G)$ is the largest size of a clique in G. Note that if G has a clique of size k, then the chromatic number of G is at least k as the vertices in the clique require at least k colors. Note that for $k \in \mathbb{N}$, $\omega(C_{2k+1}) = 2$, while $\chi(C_{2k+1}) = 3$. Therefore, odd cycles provide an example of a graph where $\chi(G) > \omega(G)$. But, how much larger can $\chi(G)$ can be compared to $\omega(G)$? It might seem that a large clique is the only obstacle to having a small chromatic number. However, the following remarkable theorem was proved by Erdös.

Theorem 11 (Erdös)

For any $k \in \mathbb{N}$, there exist graphs with $\omega(G) = 2$ and $\chi(G) \geq k$.

While we won't see the proof of this theorem, there are explicit constructions of such graphs. For example, the Mysielski graph provides such an example. Even more remarkably, Erdös and Fuchs proved the following theorem:

Theorem 12 (Erdös-Fuchs)

For any $g, k \in \mathbb{N}$, $g \geq 3$, there exist graphs with $\chi(G) \geq k$ and $girth(G) \geq g$.

The *girth* of a graph G is the length of the smallest cycle in G. This is quite remarkable since if the girth is g, then from any vertex v in the graph, the subgraph induced by the vertices at distance at most $g - 1$ from v, the graph is a tree, which is 2-colorable. Even then, there exist graphs with arbitrarily large chromatic number. So the chromatic number is a rather ill-behaved function.

Let $G = (V, E)$ be a graph and let $\Delta = \max_{v \in V} deg(v)$. That is, Δ is the maximum degree of a vertex in G. For any graph, we can bound its chromatic number in terms of Δ.

Theorem 13

For any graph $G = (V, E)$ with maximum degree Δ,. $\chi(G) \leq \Delta + 1$.

Proof. Let us order the vertices of G in linear order. Let $v_1, \ldots, v_n$ be this order. We color the vertices with colors $\{1, 2, \ldots, \Delta + 1\}$ in this order so that having colored $v_1, \ldots, v_{i-1}$ we color v_i with the *smallest* available color. It is immediate that the coloring produced is a legal coloring and it uses at most $\Delta + 1$ colors. The only thing we need to show is that when we arrive at a vertex, there is an available color. To see this, when we arrive at vertex v_i, the only vertices that have been colored are the vertices $v_1, \ldots, v_{i-1}$, and among these vertices, v_i has at most Δ neighbors. Therefore, there is always an available color for v_i. □

A graph is d-degenerate if there is an ordering $v_1, \ldots, v_n$ of the vertices of G s.t. each vertex v_i has at most d neighbors in $\{v_1, \ldots, v_{i-1}\}$. Note that by the arguments in the proof of the theorem above, if G is d-degenerate, $\chi(G) \leq d + 1$. Some graphs may have vertices of large degree and yet have small degeneracy.

Theorem 14

Any planar graph is 5-degenerate.

Proof. Let G be a planar graph. We have seen that $m \leq 3n - 6$. By the handshake lemma, we have $\sum_{v \in V} deg(v) = 2m \leq 6n - 12$. Thus, $m \leq 6 - 12/m$. Since m is an integer, there is a vertex of degree at most 5. Let v_n be this vertex. We remove v_n and put it at the end of a list. We repeat this process in the remaining graph. Let v_{n-1} be the next vertex of degree ≤ 5 found. We put v_{n-1} in front of v_n and repeat th process with the remaining vertices. Let $v_1, \ldots, v_n$ be the ordering. Then, v_i has at most 5 neighbors in $v_1, \ldots, v_{i-1}$ for any i. □

Corollary 1. *Any planar graph can be colored with at most 6 colors.*

We now improve this bound.

Theorem 15

Let $G = (V, E)$ be a planar graph. Then, $\chi(G) \leq 5$.

Proof. If $\Delta(G) \leq 4$, then $\chi(G) \leq 5$. Therefore, we can assume that there is a vertex of degree 5. Let ϕ be a coloring of G with the smallest number of colors. Suppose ϕ uses 6 colors. Then, there is a vertex v of degree 5 s.t. all its neighbors have different colors. Consider a plane embedding of G. Let a, b, c, d and e be the neighbors of v in cyclic order in the embedding. Without loss of generality, let the colors of a, b, c, d and e under ϕ be respectively $1, 2, 3, 4$ and 5. Since G is planar, it does not have K_5 as a complete subgraph. Therefore, one of the pairs among a, b, c, d and e is non-adjcent. Let a and c be non-adjacent. Consider the induced subgraph graph G_{13} of vertices colored 1 or 3. Let V_a be the set of vertices of G_{13} in the connected component of a and let V_c be the connected component of c in G_{13}. If $a \notin V_c$, then consider the coloring ϕ' obtained by swapping the colors of the vertices in V_a - vertices colored 1 are now colored 3, and vertices colored 3 are colored 1. Note that this is a legal coloring. In ϕ' two of v's neighbors have the same color and therefore v can be colored with one of 5 colors. So suppose $a \in V_c$. Then, there is a path P in V_a from a to c consisting of vertices colored 1 or 3. Adding the vertex v and the edges va and vc yields a cycle C, the edges and vertices of which form a Jordan curve. Since a and c are non-adjacent in G, the cycle C contains at least one of the neighbors of v in its interior, say b and another in its exterior, say d. Now consider the graph G_{24}, the induced subgraph of vertices colored 2 or 4. It follows that V_b and V_d are disconnected components since ay path between them must go across C, which is impossible. Now the previous argument implies we can obtain a new coloring ϕ'' swapping colors 2 and 4 so that in ϕ'' v has 2 neighbors with the same color, and hence v can be colored with one of 5 colors. $\square$

We can improve the 5 to 4. This is the famous 4-color theorem.

Theorem 16 (Appel-Haken)

Every planar graph can be colored with 4 colors.

Unfortunately, this proof is computer-assisted, requiring a check of thousands of cases. The number of cases has been reduced to less than 700 by Robertson, Seymour and Thomas, and so if one is extremely patient, it is possible to check the proof by hand.

Surprisingly, the chromatic number of graphs embedded in a surface of higher genus are easier. For a graph embedded in a surface of genus g, its chromatic number is at most $\frac{7+\sqrt{1+48g}}{2}$.