Assignment 5

October 13, 2023

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.

   Proof. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$. Then by Algebra of limits of sequences $\lim_{n\to\infty} 2x_n = 0$. As $\lim_{x\to 0} f(x) = L$, so this will imply $\lim_{n\to\infty} f(x_n) = L$ as well as $\lim_{n\to\infty} f(2x_n) = L$. But $f(2x_n) = 2f(x_n)$ by the given properties of f (Given $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. In particular, if $x = y$ that will imply $f(2x) = 2f(x)$ for all $x \in \mathbb{R}$.) Hence

   $$L = \lim_{n\to\infty} f(2x_n) = 2 \lim_{n\to\infty} f(x_n) = 2L,$$

   Hence $2L = L$ implies $L = 0$.
   Let $c \neq 0$, and let $\{x_n\}$ be any sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = c$. Then $\lim_{n\to\infty}(x_n - c) = 0$, hence $0 = \lim_{n\to\infty} f(x_n - c) = \lim_{n\to\infty}(f(x_n) - f(c))$(as $f(x-y) = f(x) - f(y)$) and so we will have $0 = \lim_{n\to\infty} f(x_n) - f(c)$ which imply $\lim_{n\to\infty} f(x_n) = f(c)$. □

2. Consider two functions f, g.
   (a) Show that if both $\lim_{x\to c} f(x)$ and $\lim_{x\to c}(f(x) + g(x))$ exist, then $\lim_{x\to c} g(x)$ exists.
   (b) If both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} f(x)g(x)$ exist, does it follow that $\lim_{x\to c} g(x)$ exists?

   Proof. a) Consider $g = (f+g)-f$. So we have $\lim_{x\to c} g(x) = \lim_{x\to c}(f(x)+g(x)) - \lim_{x\to c} f(x)$ (by Algebra of limit of the functions).
   b) If $\lim_{x\to c} f(x) \neq 0$ then $\lim_{x\to c} g(x) = \frac{\lim_{x\to c} f(x)g(x)}{\lim_{x\to c} f(x)}$ (by Algebra of limit of the functions) since $\lim_{x\to c} f(x) \neq 0$. □

3. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.

   Proof. Hint: Consider the function $f, g : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} -1 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 \text{ if } x \leq 0 \\ -1 \text{ if } x > 0 \end{cases}$$

   then limit of f and g do not exist at $x = 0$. But $f + g$ and fg have limits at $x = 0$. □

4. Show that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$.

Proof. Given $\lim_{x\to\infty} xf(x) = L$. Let $\{x_n\}$ be a sequence converging to ∞. Then $\lim_{n\to\infty} x_n f(x_n) = L$ as $\lim_{x\to\infty} xf(x) = L$. Let $X_n = x_n f(x_n)$ and $Y_n = \frac{1}{x_n}$. Now $\lim_{n\to\infty} X_n = L$ and $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \frac{1}{x_n} = 0$. So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} X_n \times \lim_{n\to\infty} Y_n = L \times 0 = 0$. So for any arbitrary sequence $\{x_n\}$ converging to ∞, $\lim_{n\to\infty} f(x_n) = 0$ concluding $\lim_{x\to\infty} f(x) = 0$. $\square$

5. Let $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2}$ for $x \neq 0$. Find the limit of $\lim_{x\to 0} f(x)$.

Proof. We have $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2} = \frac{3x^2}{x^2(\sqrt{1+3x^2}+1)}$ for $x \neq 0$. Hence $f(x) = \frac{3}{(\sqrt{1+3x^2}+1)}$ for $x \neq 0$. Then apply the algebra of limits, $\lim_{x\to 0} f(x) = \frac{\lim_{x\to 0} 3}{\lim_{x\to 0}(\sqrt{1+3x^2}+1)} = \frac{3}{2}$. $\square$

6. Let $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$, then $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

Proof. Let $F(x) = \frac{f(x)}{x^2}$ when $x \neq 0$ and $G(x) = x$ which gives $F(x)G(x) = \frac{f(x)}{x}$ when $x \neq 0$. Hence $\lim_{x\to 0} F(x) = 5$ (given) and $\lim_{x\to 0} G(x) = 0$. So by Algebra of limits of functions $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} F(x)G(x) = \lim_{x\to 0} F(x) \times \lim_{x\to 0} G(x) = 5 \times 0 = 0$. $\square$

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 0} f(x^2)$ exists but $\lim_{x\to 0} f(x)$ does not exist.

Proof. Hint: Example $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$$

Then $\lim_{x\to 0} f(x)$ does not exist. Now $x^2 \geq 0$ so $f(x^2) = 1$ for all $x \in \mathbb{R}$ giving $\lim_{x\to 0} f(x^2) = 1$ $\square$

8. Let $f : (0, \infty) \to \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = L$ iff $\lim_{x\to 0+} f(\frac{1}{x}) = L$.

Proof. Given $\lim_{x\to\infty} f(x) = L$. Required to prove $\lim_{x\to 0+} f(\frac{1}{x}) = L$. Consider $g(x) = f(\frac{1}{x})$ Let $\{x_n\}$ be any sequence of positive real numbers converging to 0. Then the sequence $\{y_n\}$ be the sequence of positive real numbers converging to ∞ where $y_n = \frac{1}{x_n}$ for $n \geq 1$. As $\lim_{x\to\infty} f(x) = L$, so $\lim_{n\to\infty} f(y_n) = L$ which implies $\lim_{n\to\infty} f(\frac{1}{x_n}) = \lim_{n\to\infty} g(x_n) = L$. Hence $\lim_{x\to 0+} f(\frac{1}{x} = \lim_{x\to 0+} g(x) = L$ as $\{x_n\}$ is any arbitrary sequence of positive real numbers converging to 0.

Similarly, we can prove the converse. $\square$

9. Consider the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = x^{\frac{3}{2}}$. Prove that the function is continuous at $c \in [0, 2]$.

Proof. Let $\varepsilon > 0$ be any given real number. When $c = 0$, we have

$$|f(x) - 0| = x^{3/2} < \varepsilon$$

whenever $0 < |x - 0| < \delta = \varepsilon^{2/3}$.
When $c \neq 0$. Then

$$\begin{aligned}
|f(x) - f(c)| &= |x^{\frac{3}{2}} - c^{\frac{3}{2}}| = |x^{\frac{1}{2}} - c^{\frac{1}{2}}||x + \sqrt{xc} + c| \\
&\leq |x^{\frac{1}{2}} - c^{\frac{1}{2}}|(2 + \sqrt{4} + 2) \quad \text{since} \quad x, c \leq 2 \\
&= 6|x^{\frac{1}{2}} - c^{\frac{1}{2}}| \\
&= \frac{6}{|x^{\frac{1}{2}} + c^{\frac{1}{2}}|}|x - c| \\
&\leq \frac{3}{c^{\frac{1}{2}}}|x - c| \quad \text{since } x \geq 0, \quad x^{\frac{1}{2}} + c^{\frac{1}{2}} \geq c^{\frac{1}{2}} \\
&< \varepsilon,
\end{aligned}$$

whenever $0 < |x - c| < \delta = \frac{\varepsilon c^{\frac{1}{2}}}{3}$. $\square$