DS Cheat Sheet

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1 Relations and Functions

Definition (Principle of Mathematical Induction(PMI)**.** Let $X \subseteq \mathbb{N}$, and suppose the following conditions hold for X:

1. $0 \in X$
2. If $n \in X$, then $n + 1 \in X$ also

then $X = \mathbb{N}$

Definition (Principle of Mathematical Induction - Strong Version (SPMI))**.** Let $X \subseteq \mathbb{N}$, and suppose the following conditions hold for X:

1. $0 \in X$
2. If $0, 1, .., n \in X$, then $n + 1 \in X$ also

then $X = \mathbb{N}$

Definition (Well-Ordering Priciple(WOP))**.** If X is a non-empty subset of $\mathbb{N}$, then X possesses a smallest or least element. More precisely,

$$\exists \alpha \in X, \alpha \leq x, \forall x \in X$$

Proposition 1. the following three statements are equivalent

- PMI
- SPMI
- WOP

Proposition 2. Let $f : X \to Y$ and $g : Y \to Z$. Then

- If f and g are injective, then $g \circ f$ is also injective
- If f and g are surjective, then $g \circ f$ is also surjective
- If f and g are bijective, then $g \circ f$ is also bijective
- For any function $f : X \to Y, \exists$ a set Z, an injective function $h : Z \to Y$, and a surjective function $g : X \to Z$ such that $f = g \circ h$.

In other words, every function can be decomposed as the composition of a surjective function followed by an injective function

Definition. A partition of a set X is a family of non-empty subsets of X, which are pair-wise disjoint and whose union is all of X. The subsets in a partition are called the parts of the partition.

Proposition 3. If R is a equivalence relation on X, then R induces a ***partition*** of X, the parts of the partition being the***equivalence classes***, i.e. the equivalence classes are pair-wise disjoint subsets whose union is the whole set X. Conversely, given any partition of a set X, there exists a corresponding equivalence relation R on X.

Definition. Given any set X, the relation $\Delta_X = \{(x, x) : x \in X\}$ is the smallest possible reflexive relation on X - it is called the *diagonal* on set X

2 Posets

2.1 Introduction and Isomorphism

Definition. A relation R is said to be **antisymmetric** if whenever xRy and yRx both hold, then $x = y$. Alternatively, R is **antisymmetric** if $R \cap R^{-1} \subseteq \Delta_X$
A relation R on a set X is said to be a partial ordering if it is *reflexive, antisymmetric* and *transitive.*
A pair $< X, \leq >$, where $\leq$ is a partial ordering on the set X is called a **poset**

Definition. Let $< X, \leq >$ be a poset. We say that an element $x \in X$ is an **immediate predecessor** of an element $y \in X$ if $x < y$ but there is no $t \in X$ such that $x < t < y$. Symbol - $\Diamond$

Proposition 4. Let $< X, \leq >$ be a **finite** poset. Then for any $x, y \in X, x < y$ holds **iff** there exists elements $x_1, x_2, ...x_k \in X$ such that $x \Diamond x_1 \Diamond x_2 \Diamond ..... \Diamond x_k \Diamond y$, $k \geq 0$

Definition. Let $< X, \leq >$ be a poset. Then an element $a \in X$ is said to be a **minimal** element if there is no element $x \in X$ such that $x < a$. Similarly, an element $a \in X$ is said to be a maximal if there is no $x \in X$ such that $a < x$

Proposition 5. Every finite poset $< X, \leq >$ has at least one minimal element.

Definition. An element a of a $< X, \leq >$ is said to be a least or minimum element if $a \leq x$ holds $\forall x \in X$, and if it exists, must be unique.

Proposition 6. Every partial ordering on a finite set X has a **linear extension**. In other words, if $< X, \leq >$ is a finite poset, then there exists a linear (total) ordering "#" on X such that $\forall x, y \in X$, $x \leq y \implies x \# y$

Definition. Two posets $< X, R >$ and $< Y, S >$ are said to be **isomorphic** if there exists a bijection $f : X \to Y$ (called an **isomorphism**) such that xRz iff $f(x)Sf(z)$ $\forall x, y \in X$

Definition. If $< X, R >$ and $< Y, S >$ are posets, a mapping $f : X \to Y$ is said to be an **embedding** if

1. f is injective and
2. $f(x)Sf(z)$ iff xRz $\forall x, z \in X$

Proposition 7. For every poset $< X, R >$, there exists an embedding into the poset $< \mathbb{P}(X), \subseteq >$

Let $f : X \to \mathbb{P}(x)$ and $f(x) = \{z \in X : z \leq x\}$

2.2 Chains and Anti Chains

Definition. Let P be a fixed but arbitrary poset $< X, \leq >$. Elements $x, y \in X$ are said to be **comparable** if either $x \leq y$ or $y \leq X$; else, they are said to be **incomparable**
A set $A \subseteq X$ is said to be independent or an antichain in P if the elements of A are mutually incomparable. A set $A \subseteq X$ is said to be a chain in P if the elements of A are mutually comparable.

Definition. The **independence number** of P, denoted by $\alpha(P)$ is the size of the largest antichain in P;
i.e. $\alpha(P) = max\{|A|$: A is an antichain in $P\}$.
Similarly we wil use $\omega(P)$ for the size of the largest chain in P.

Proposition 8 (important). For every finite poset $P =< X, \leq >$, we have $\alpha(P)\omega(P) \geq |X|$

Erdős-Szekeres's Theorem. *An arbitrarily finite real sequence* $< x_k >$ *of length at least* $n^2 + 1$ *contains a monotone subsequence of length at least* n

Corollary 1.1. An infinite real sequence $< x_k >$ contains a monotone subsequence of any arbitrary finite length

Dilworth Decomposition's Theorem. *Let* $P =< X, \leq >$ *be a finite poset. Then* X *can be decomposed into the disjoint union of* $\alpha(P)$ *chains, and cannot be decomposed into any smaller number of chains. i.e. we get a* ***partition*** *of* X *into* $\alpha(P)$ *chains, which is the best possible*

2.3 Lattices

Definition.
- Let $P =< X, \leq >$ be a poset, and let A be a non-empty subset of X. An element $u \in X$ is said to be an **upper bound** for A if $x \leq u$ $\forall x \in A$.
- u need not be an element of A.
- An element $u \in X$ is said to be a **least upper bound**(lub) or **supremum**(sup) for A if u is an upper bound for A.
- $sup(A)$, if it exists at all, has to be unique

We can define **lower bounds**, and the **greatest lower bound**(glb) or **infimum**(inf) for any non-empty subset A of X

Definition. A poset $< X, \leq >$ in which every *pair* of elements has both a supremum and an infimum is called a **lattice**.

Proposition 9. For any a, b, c, d in a lattice $< X, \leq >$, we have

1. $a \leq a \vee b$ and $a \wedge b \leq a$
2. If $a \leq b$ and $c \leq d$, then $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$

Definition. Dual of a a lattice $< X, R >$ is $< X, R^{-1} >$. Statements about join in the original lattice become statements about meet in the dual lattice,and vice-versa.

Definition. A lattice $L =< X, \leq >$ is said to be **distributive** if

1. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
2. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Proposition 11. If either of the two above statements holds, then the other one also holds

Proposition 12. Let $L =< X, \leq >$ be a lattice with 0 and 1. Then, the following holds $\forall a \in X$

- $a \vee 1 = 1$ and $a \wedge 1 = a$
- $a \vee 0 = a$ and $a \wedge 0 = 0$

Definition. An element u is said to be a **complement** of an element $x \in X$ if $x \vee u = 1$ and $x \wedge u = 0$

Proposition 13. In a distributive lattice, if an element has a complement, the complement must be unique

Definition. A lattice $L =< X, \leq >$ is said to be **complemented** if every element $x \in X$ has a complement.

Definition. A complemented and distributive lattice $L =< X, \leq >$ os also called a **Boolean lattice**

Uniqueness of Boolean Lattice's Theorem. *Let $B =< X, \leq >$ be a* ***finite*** *boolean lattice with associated boolean algebra $< X, \vee, \wedge, \overline{X} >$. Then there exists a finite set S such that B is isomorphic to $< \mathbb{P}(S), \subseteq >$ with associated boolean lattice $< \mathbb{P}(S), \cup, \cap, \overline{X} >$. In particular,$|X| = 2^n$, where $n = |S|$*

3 Combinatorial Counting

3.1 Permutations

Proposition 14. Let X be a finite set with $|X| = n, n \in \mathbb{N}$, and Y be a finite set with $|Y| = m, m \in \mathbb{Z}^+$. Then the number of all possible functions(or mappings) $f : X \to Y$ is m^n

Corollary 14.1. Let X be a finite set with $|X| = n, n \in \mathbb{N}$. Then the number of subsets of X is 2^n, i.e. $|\mathbb{P}(X)| = 2^n$

Corollary 14.2. Let X be a finite set with $|X| = n, n \in \mathbb{Z}^+$. Then the number of odd-sized subsets of X is exactly equal to the number of even-sized subsets, i.e. 2^{n-1} in both cases

Proposition 15. The number of injective functions(mappings) from an $n - element$ set to an $m - element$ set, $m, n \in \mathbb{N}$ is exactly $m(m-1)(m-2).....(m-(n-1)) = \prod_{i=0}^{n-1}(m-i)$

Definition. A bijective function from a finite set X to itself is called a permutation

Proposition 16. The number of permutations of an $n - element$ set, $n \in \mathbb{N}$, is exactly $n(n-1)....2.1$

Definition. $M_n = \{1, 2, 3, ...n\}$ The set of all permutations on M_n is denoted by S_n(occasionally Σ_n, and is usually referred to as the ***symmetric group on n symbols***. $S_n = n!$

Proposition 17. The set S_n satisfies for following properties for $\pi, \sigma, \tau \in S_n$

- $\pi \circ \sigma \in S_n$
- $(\pi \circ \sigma) \circ \tau = \pi \circ (\sigma \circ \tau)$
- $\pi \circ \iota = \iota \circ \pi = \pi$, where ι indicates the identity permutation
- If $\pi \in S_n$, then $\pi^{-1} \in S_n$ and $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = \iota$

Definition. Given a permutation $\sigma \in S_n$, let $M(\sigma) = \{x \in M_n : \sigma(x) \neq x\}$ i.e. the set of symbols **moved** by σ. Similarly, let $Fix(\sigma) = \{x \in M_n : \sigma(x) = x\}$, namely, the set of symbols fixed by σ. We say that two permutations σ and π are disjoint if $M(\sigma) \cap M(\pi) = \phi$

Definition. The cycle $(x_1, x_2, ...x_t)$ is the permutation which takes x_1 to x_2, x_2 to x_3, ... x_{t-1} to x_t, x_t to x_1, and leaves all other symbols fixed. It is referred to as a cycle of length t or a $t - cycle$

Proposition 18. Any permutation $\sigma \in S_n$ can be decomposed as the product of pairwise disjoint cycles. Furthermore, this decomposition is unique up to rearranging the cycles and the cyclic order within the cycles.

Definition. Let $\pi \in S_n$ and let π be decomposed as the product of disjoint cycles of lengths $n_1 \geq n_2 \geq .... \geq n_k \geq 1$. Then $n_1 + n_2 + ... + n_k = n$ and we say that the **cycle type** of π is $(n_1, n_2, ...n_k)$. If the cycle type of π is $(k, 1, ..., 1$, then we refer to π as a $k - cycle$

Definition. A permutation $\tau \in S_n$ is called a **transposition** provided

- $\exists i, j \in M_n, i \neq j$ such that $\tau(i) = j$ and $\tau(j) = i$
- $\forall k \in M_n, \tau(k) = k, k \neq i, j$

In other words, transposition is a 2-cycle

Proposition 19. Every permutation in S_n can be expressed as the product(composition) of transpositions.

Definition. Let $\pi \in S_n$ and let $i, j \in M_n$ with $i < j$. The pair (i, j) is called an **inversion** in π if $\pi(i) > \pi(j)$

Lemma 20.1 (important)**.** Let (cd) be a transposition in S_n with $c < d$. Then the number of inversions in (cd) is $2(d-c-1)+1$ i.e. an odd number of inversions.

Lemma 20.2 (important)**.** If the identity permutation is written as a product of transpositions, then the number of transpositions must be even.

Proposition 20 (important)**.** Let $\pi \in S_n$ and suppose π is decomposed into transpositions as: $\pi = \tau_1 \cdot \tau_2 \cdot ...... \cdot \tau_a$ and $\pi = \sigma_1 \cdot \sigma_2 \cdot ...... \cdot \sigma_b$. Then a and b have the same ***parity*** i.e. either both are odd or both are even.

Definition. Let $\pi \in S_n$. Then π is said to be **odd/even** provided it can be written as a product of even or odd number of transpositions resp.

Proposition 21. For $n \geq 2$, the number of odd permutations is equal to the number of even permutations

Definition. For a permutation $\pi \in S_n$, it's **sign** or **signature** is defined by $sgn(\pi) = 1$ if and only if π is even, and $sng(\pi) = -1$ if and only if π is odd. In particular, the signature of a transposition is (-1); for a $k - cycle$ $\pi = (a_1 a_2 ... a_k)$, $sgn(\pi) = (-1)^{k-1}$

Theorem 4 (Determinant Formula). If $A = [a_{ij}]$ is a square $n \times n$ matrix, then $det(A) = \sum\limits_{\sigma \in S_n} sgn(\sigma) a_{1\sigma_{(1)}} a_{2\sigma_{(2)}} .... a_{n\sigma_{(n)}}$

3.2 Combinations

Definition. Let $n \geq k$ be non-negative integers. The **binomial coefficient** $B(n, k)$ is a function of the variables n, k defined by the formula

$$B(n, k) = \frac{n(n-1)...(n-(k-1))}{k!} = \frac{\prod_0^{k-1}(n-i)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

Definition. Let X be a set and let k be a non-negative integer. The notation $B(X, k)$ denotes the family (set) of all $k - element$ subsets of X.

Proposition 26. Let X be a finite set. Then, $|B(X,k)| = B(|X|,k)$

Proposition 27.
- $B(n,k) = B(n,n-k)$
- **[Pascal's identity]** $B(n-1,k-1) + B(n-1,k) = B(n,k)$
- **[Binomial theorem]** For any $n \in \mathbb{N}, (1+x)^n = \sum_{k=0}^{n} B(n,k)x^k$

Definition. Given $n, k_1, k_2, ..k_m \in \mathbb{Z}^+$ with $n = k_1 + k_2 + ... + k_m$, the **multinomial coefficient** $B(n, k_1, k_2, ...k_m) = n!/(k_1!k_2!....k_m)!$

Proposition 28.
- Suppose we have objects of m different kinds, with k_i indistinguishable objects of the i^{th} kind, and $n = k_1 + k_2 + ... + k_m$, then the number of distinct arguments is given by the multinomial coefficient.
- **(Multinomial Theorem)** For arbitrary $x_1, x_2, ..., x_m \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, we have

$$(x_1 + x_2 + ..x_m) = \sum_{k_1+k_2+...k_m=n} B(n, k_1, k_2, ..., k_m){x_1}^{k_1}{x_2}^{k_2}...{x_m}^{k_m}$$

Proposition 29. **(Piegonhole Principle)** If we have n boxes and we place more than n objects into them, then at least one box contains more than one object.

4 Finite and Infinite Sets

Definition.
- The empty set ϕ has 0 elements.
- If $n \in \mathbb{Z}^+$, a set S has n elements if there exists a bijection on S onto $M_n = \{1, 2, 3, ..., n\}$
- A set S is infinite if it's not finite

Definition. A set S is infinite if there exists a proper subset T of S such that there is a bijection on S onto T. A set S is finite if there exists no proper subset $T \subseteq S$ such that there is a bijection on S onto T, i.e. if S is not infinite.

Proposition 22.
- If a set A has m elements and a set B has n elements, with $A \cap B = \phi$, then the set $A \cup B$ has $m+n$ elements.
- If A is a set with m elements and $C \subseteq A$ is a set with 1 element, then $A - C$ has $m - 1$ elements
- If C is a infinite set and B is a finite set, then $C - B$ is an infinite set

Proposition 23. Suppose S and T are sets with $T \subseteq S$, then :-
- If S is finite, then so is T
- If T is infinite, then so is S

4.1 Countable and Uncountable Sets

Definition.
- A set S is **Denumerable** if there exists a bijection from S onto $\mathbb{N}$.
- A set S is **Countable** if it is either finite or denumerable.
- A set S is **uncountable** if it not countable.
- A set S is **Countably infinite** if it is denumerable, i.e. if it is countable but not finite.

Proposition 24. Suppose S and T are sets with $T \subseteq S$, then:-
- If S is countable, then so is T
- If T is uncountable, then so is S

Proposition 25. The following are equivalent :-
- S is a countable set.
- There exists a surjection from $\mathbb{N}$ onto S.
- There exists an injection from S into $\mathbb{N}$

Cantor's Theorem. *If A is any set, there is no surjection on A onto $\mathbb{P}(A)$.*

Corollary 5.1. $\mathbb{P}(\mathbb{N})$ is uncountable

Definition. Two sets S and T are said to be **equipotent** or have the **same cardinatity** if there is a bijection on S onto T. In view of Cantor's theorem, we can say that we have a never-ending progression of sets of increasing cardinality.

Corollary 5.2. The open interval $(0, 1)$ is uncountable.

Corollary 5.3. The open interval $(0, 1)$ and the set of real numbers $\mathbb{R}$ have the same cardinality.

Corollary 5.4. The open interval $(0, 1)$ and $\mathbb{P}(\mathbb{N})$ have the same cardinality.

5 Logic

Definition. A **proposition** is a declarative statement that is either TRUE or FALSE **(but not both).**

Symbol	Definition
p	Assertion
$\neg p$	Negation(not)
$p \vee q$	Disjunction(or)
$p \wedge q$	Conjuction(and)
$p \rightarrow q$	Implication(if.. then)
$p \leftrightarrow q$	Bi-implication(iff)
No standard symbol	Exclusive(xor)
$q \rightarrow p$	Converse
$\neg p \rightarrow \neg q$	Inverse
$\neg q \rightarrow \neg p$	Contra-positive
$\forall$	For all quantifier
$\exists$	Exists quantifier

Table 1: Logic Symbols and their Definition.

Definition. When two compound statements always have the same truth value, they are called **equivalent**. In other words, the rows/columns in the corresponding truth tables have the same truth values.

Definition.

- Tautology - always true
- Contradiction - always false
- Contingency - neither a Tautology or Contradiction

Definition. The compound propositions A and B are called logically equivalent if $A \leftrightarrow B$ is a tautology, or $A \equiv B$

Definition.

- An **argument** in propositional logic is a sequence of propositions. We start an argument with some propositions, the **premises** or **hypotheses**. The final proposition in the sequence is called the **conclusion**.
- An argument is valid if the truth of all its premises implies that the conclusion is true. We can always use truth table to find this out

Definition. **Predicate logic** is concerned with expressions of the form $P(x)$, known as **propositional functions**, in which P is a predicate or property and x is a variable. The number of variables in a propositional function need not be one, but it has to be finite.

Definition. When a quantifier is used on a variable x in a propositional function, that variable is said to be **bound**. Variables which are not bound are said to **free**.

Definition. Statements involving predicates and quantifiers are **logically** equivalent iff they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

If $S = \forall x P(x)$ then $\neg S = \exists x \neg P(x)$, and if $S = \exists x P(x)$ then $\neg S = \forall x \neg P(x)$

Name	Tautology	Rule
Modus Ponens	$((p \to q) \land (p)) \to q$	$p \to q$ p q
Modus Tollens	$((p \to q) \land (\neg q)) \to \neg p$	$p \to q$ $\neg q$ $\neg p$
Hypothetical Syllogism	$((p \to q) \land (q \to r)) \to (p \to r)$	$p \to q$ $q \to r$ $p \to r$
Disjunctive Syllogism	$((p \lor q) \land (\neg p)) \to (q)$	$p \lor q$ $\neg p$ q
Addition	$p \to (p \lor q)$ $q \to (p \lor q)$	p q $p \lor q$ $p \lor q$
Simplification	$(p \land q) \to p$ $(p \land q) \to p$	$p \land q$ $p \land q$ p q
Resolution	$((p \land q) \lor (\neg p \land r)) \to (q \land r)$	$p \land q$ $\neg p \land r$ $q \land r$

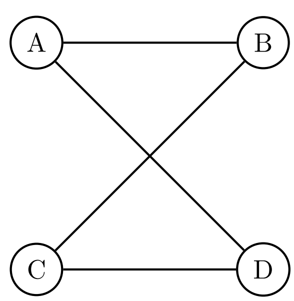
6 Graph Theory

Definition. A graph G is a triplet consisting of :-

- A vertex set $V(G)$
- An edge set $E(G)$, **disjoint from the vertex set**
- A relation between an edge and a pair of vertices (not necessarily distinct); these vertices are referred to as the endpoints or end-vertices of the edge
- General notation : $|V(G)| = n, |E(G)| = m$

Some definitions :-

- **Loop**: An edge whose endpoints are equal
- **Multiple edges**: Edges have the same pair of endpoints
- **Simple Graph**: No loops or multiple edges
- **Null graph**: A graph whose vertex set and edge set are empty
- **Adjacent/Neighbours**: Two vertices are adjacent and are neighbors if they are the endpoints of an edge. For example in the graph below, A and B are neighbours and A and D are neighbours.
- **Degree**: number of edges incident(one of the vertices is v_k) upon v_k . It is denoted as $d(v_k)$ – usually used only for loop-less graphs.



Proposition 30. Let G be a (loop-less) graph. Then, the sum of the degrees of the vertices is twice the number of edges, i.e. $\sum_{v \in V(G)} d(v) = 2|E(G)|$

- **Adjacency Matrix**: the $n \times n$ matrix in which entry a_{ij} is the number of edges in G with endpoints $\{v_i, v_j\}$.
- **Complement of G**: The complement G' of a simple graph G is a simple graph with $V(G') = V(G)$ and $E(G') = \{uv : uv \notin E(G)\}$

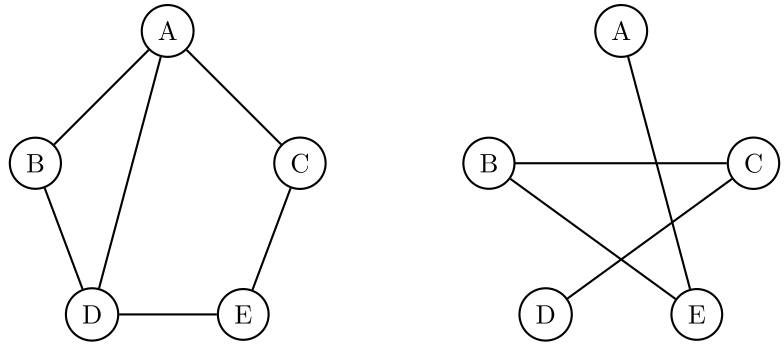


Figure 1: Complement of graphs

- **Subgraph**: A subgraph of a graph G is a graph H such that: $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and the assignment of endpoints to edges in H is the same as in G
- **Isomorphism**: An isomorphism from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Notation is $G \cong H$. It 's an **equivalence relation**
- **Some standard notations for graphs**:
  - K_n : complete graph of n vertices
  - P_n : Paths with n vertices
  - C_n : Cycles with n vertices
  - $K_{r,s}$ Complete bipartite graphs with classes or partite sets of order r,s, etc.
- **Automorphism**: A permutation of $V(G)$ that is an isomorphism from G to G.

6.1 Paths and Cycles

- **Path**: a sequence of distinct vertices such that two consecutive vertices are adjacent. Example - (a, b, c, d, e) is a path
- **Cycle**: A closed path. Example - (a, b, c, d)
- **Walk**: A sequence, $(v_0, e_1, v_1, e_2, ..., e_k, v_k)$ of vertices and edges such that $e_i = v_{i-1} - v_i$ for all i.
  - *Trail*: A walk with no repeated edge
  - *Path*: A walk with no repeated vertex.
  - u, v-*walk* or u, v-*trail*: first vertex u and last vertex v; these are its endpoints.
  - walk is **closed** if it has length at least one and its endpoints are equal
    - Cycle is a closed trail in which "first = last" is the only vertex repetition
    - A loop is a cycle of length one

Proposition 31. Every u, v-walk contains a u, v-path

- **Connected**: A graph G is connected if there exists at least one path between any two distinct vertices (for all pairs of vertices). Otherwise, G is **Disconnected**
- **Components** : maximal connected subgraphs of G
- A component is **trivial** if it has no edges; otherwise it is nontrivial
- **Isolated vertex**: A vertex of degree 0.
- **Complete Graph**: A simple graph where all vertices are pairwise adjacent. Denoted by K_n
- **Clique**: A set of pairwise adjacent vertices in a graph(a complete subgraph)
- **Independent Set**: A set of pairwise non-adjacent vertices
- **Bipartite Graph**: The vertices can be partitioned into two sets such that each set is independent
- **Simple Bipartite Graph** Simple bipartite graph such that two vertices are adjacent if and only if they are in different classes. Denoted by $K_{r,s}$
- **Maximal Path** A path P in graph G which is not contained ini any longer path.

Proposition 32 (Characterization of Bipartite Graphs)**.** A graph with at least two vertices is bipartite if and only if it has no odd cycle.

Proposition 33. If every vertex of a graph G has degree at least 2, then G contains a cycle

Definition. A graph is **Eulerian** if it has a closed trail passing through all the edges exactly once. For convenience, a graph consisting of trivial components is regarded as Eulerian.

Theorem 6. A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.

Corollary 6.1. Let G be a connected graph with exactly two vertices of odd degree, say u and v. Then G has an Eulerian trail that starts at u and ends at v.

Definition. A decomposition of a graph is a list of subgraphs such that every edge of G belongs to exactly one of the subgraphs in the list.

Corollary 6.2. Every connected nontrivial even graph decomposes into cycles.

- **Order(**$n(G)$**:** number of vertices in G.
- An n-vertex graph is a graph of order n.
- **Size(**$e(G)$**:** number of edges in G.
- **Degree**$(d(v)$**:** number of edges incident to v, except that each loop at v counts twice.
- **Maximum Degree** $= \Delta(G)$, **Minimum Degree** $= \delta(G)$
- **Regular:** $\delta(G) = \Delta(G)$
- G is $k-$regular if common degree is k
- **Neighbourhood**$N_g(v)$**:** set of vertices adjacent to v.

Proposition 34. If $k > 0$, then a $k-$regular bipartite graph has the same number of vertices in each partite set.

Proposition E. very graph with n vertices and k edges has at least $n - k$ components

Definition. Degree Sequence/Score of a graph is the list of vertex degrees, usually written in non-increasing order, as $d_1 \geq d_2 \geq .. \geq d_n$

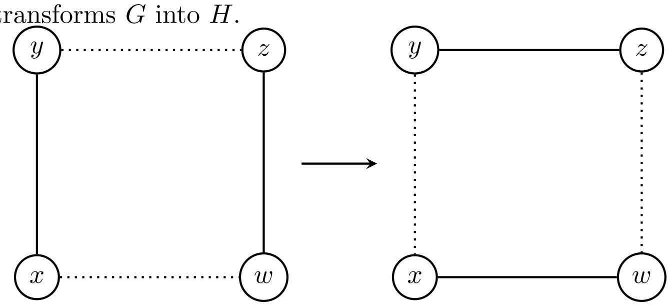
Definition. A graphic sequence is a list of nonnegative numbers that is the degree sequence (score) of some simple graph.

A simple graph realizes d means: A simple graph with degree sequence d.

Havel-Hakimi's Theorem. *For $n > 1$, an integer list d of size n is graphic* ***if and only if*** *d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1- element graphic sequence is $d_1 = 0$.*

Definition. A 2-switch is the replacement of a pair of edges xy and zw in a simple graph by the edges yz and wx, given that yz and wx did not appear in the graph originally.'

Corollary 7.1. If G and H are two simple graphs with vertex set V, then $d_G(v) = d_H(v)$ for every $v \in V$ if and only if there is a sequence of 2-switches that transforms G into H.



7 General Problem Solving Techniques

- To prove a set is a subset of another set(for example $A \subseteq B$), take some element $x \in A$ and prove $x \in B$

- Induction questions regarding posets often involve considering $k = n + 1$, removing some element a, and constructing a new set $Y = X \setminus \{a\}$, applying IH on this set and then adding back a to prove for $n + 1$

- Characteristic equation comes in handy. For some subset $A \subseteq X$,

$$f_A(a) = \begin{cases} 1 & a \in A \\ 0 & otherwise \end{cases}$$

- Mappings are useful when talking about inclusion and exclusion. Let $A \subseteq M_n$, where $M_n = \{1, 2, ..., n\}$. Then you can create a bijection $f : A \to V^n$, $V = \{0, 1\}$. $f(A) = V_A, V_A = (v_1, v_2, ...v_n)$ where $v_i = 1$ if $i \in A$, and 0 otherwise.

- Use Pigeonhole Principle in problems which require you to prove that there is at least one element which satisfies certain condition, or at least 2 things are same. Figuring out what the holes and the pigeons are is half the battle

- TONCAS(The Obvious Necessary Condition Also Suffices) comes up a lot in GT proofs. Use it
- Solution to Q4 of midsem (only written the first step, can do the rest by following the pattern)

$$
\begin{aligned}
\sigma &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot (45)(15)(13)(23)(43) \\
&= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{bmatrix} \cdot (15)(13)(23)(43)
\end{aligned}
$$

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