HINTS FOR SELECTED EXERCISES

Reader: Do not look at these hints unless you are stymied. However, after putting a considerable amount of thought into a problem, sometimes just a little hint is all that is needed. Many of the exercises call for proofs, and there is usually no single approach that is correct, so even if you have a totally different argument, yours may be correct. Very few of the following hints give much detail, and some may seem downright cryptic at first. Somewhat more detail is presented for the earlier material.

Section 1.1

1. (a) $\{5, 11, 17\}$
3. Show that if $A \subseteq B$, then $A = A \cap B$. Next show that if $A = A \cap B$, then $A \subseteq B$.
4. Show that if $x \in A \backslash (B \cap C)$, then $x \in (A \backslash B) \cup (A \backslash C)$. Next show that if $y \in (A \backslash B) \cup (A \backslash C)$, then $y \in A \backslash (B \cap C)$. Since the sets $A \backslash (B \cap C)$ and $A \backslash (B \cap C)$ contain the same elements, they are equal.
7. (a) $A_1 \cap A_2 = \{6, 12, 18, 24, \ldots\} = \{6k : k \in \mathbb{N}\} = A_5$.
   (b) $\cup A_n = \mathbb{N} \backslash \{1\}$ and $\cap A_n = \emptyset$.
9. No. For example, both (0, 1) and (0, -1) belong to C.
11. (a) $f(E) = [2, 3]$, so $h(E) = g(f(E)) = g([2, 3]) = [4, 9]$.
    (b) $g^{-1}(G) = [-2, 2]$, so $h^{-1}(G) = [-4, 0]$.
15. If $x \in f^{-1}(G) \cap f^{-1}(H)$, then $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$, so that $f(x) \in G$ and $f(x) \in H$. Then $f(x) \in G \cap H$, and hence $x \in f^{-1}(G \cap H)$. This shows that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$.
17. One possibility is $f(x) := (x - a)/(b - a)$.
21. If $g(f(x_1)) = g(f(x_2))$, then $f(x_1) = f(x_2)$, so that $x_1 = x_2$, which implies that $g \circ f$ is injective. If $w \in C$, there exists $y \in B$ such that $g(y) = w$, and there exists $x \in A$ such that $f(x) = y$. Then $g(f(x)) = w$, so that $g \circ f$ is surjective. Thus $g \circ f$ is a bijection.
22. (a) If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, which implies $x_1 = x_2$, since $g \circ f$ is injective. Thus f is injective.

Section 1.2

1. Note that $1/(1 \cdot 2) = 1/(1 + 1)$. Also $k/(k + 1) + 1/[(k + 1)(k + 2)] = (k + 1)/(k + 2)$.
2. $\left[\frac{1}{2}k(k+1)\right]^2 + (k+1)^3 = \left[\frac{1}{2}(k+1)(k+2)\right]^2$.
4. $\frac{1}{3}\left(4k^3 - k\right) + (2k+1)^2 = \frac{1}{3}\left[4(k+1)^3 - (k+1)\right]$.
6. $(k+1)^3 + 5(k+1) = \left(k^3 + 5k\right) + 3k(k+1) + 6$ and $k(k+1)$ is always even.
8. $5^{k+1} - 4(k+1) - 1 = 5 \cdot 5^k - 4k - 5 = \left(5^k - 4k - 1\right) + 4\left(5^k - 1\right)$.
13. If $k < 2^k$, then $k + 1 < 2^k + 1 < 2^k + 2^k = 2\left(2^k\right) = 2^{k+1}$.
16. It is true for $n = 1$ and $n \geq 5$, but false for $n = 2, 3, 4$.
18. $\sqrt{k} + 1/\sqrt{k+1} = \left(\sqrt{k}\sqrt{k+1} + 1\right)/\sqrt{k+1} > (k+1)/\sqrt{k+1} = \sqrt{k+1}$.

Section 1.3

1. Use Exercise 1.1.21 (= Exercise 21 of Section 1.1).
2. Part (b) Let f be a bijection of $\mathbb{N}_m$ onto A and let $C = \{f(k)\}$ for some $k \in \mathbb{N}_m$. Define g on $\mathbb{N}_{m-1}$ by $g(i) := f(i)$ for $i = 1, \ldots, k-1$, and $g(i) := f(i+1)$ for $i = k, \ldots, m-1$. Then g is a bijection of $\mathbb{N}_{m-1}$ onto $A \backslash C$.
3. (a) There are $6 = 3 \cdot 2 \cdot 1$ different injections of S into T.
   (b) There are 3 surjections that map a into 1, and there are 3 other surjections that map a into 2.
7. If T_1 is denumerable, take $T_2 = \mathbb{N}$. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}$, then (by Exercise 1.1.21) $g \circ f$ is a bijection of T_1 onto $\mathbb{N}$, so that T_1 is denumerable.
9. If $S \cap T = \emptyset$ and $f : \mathbb{N} \to S$, $g : \mathbb{N} \to T$ are bijections onto S and T, respectively, let $h(n) := f((n+1)/2)$ if n is odd and $h(n) := g(n/2)$ if n is even.
11. (a) $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ has $2^2 = 4$ elements.
   (c) $\mathcal{P}(\{1, 2, 3, 4\})$ has $2^4 = 16$ elements.
12. Let $S_{n+1} := \{x_1, \ldots, x_n, x_{n+1}\} = S_n \cup \{x_{n+1}\}$ have $n+1$ elements. Then a subset of S_{n+1} either (i) contains x_{n+1}, or (ii) does not contain x_{n+1}. There are a total of $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of S_{n+1}.
13. For each $m \in \mathbb{N}$, the collection of all subsets of $\mathbb{N}_m$ is finite. Note that $\mathcal{F}(\mathbb{N}) = \bigcup_{m=1}^{\infty} \mathcal{P}(\mathbb{N}_m)$.

Section 2.1

1. (a) Justify the steps in: $b = 0 + b = (-a + a) + b = -a + (a + b) = -a + 0 = -a$.
   (c) Apply (a) to the equation $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$.
2. (a) $-(a+b) = (-1)(a+b) = (-1)a + (-1)b = (-a) + (-b)$.
   (c) Note that $(-a)(-(1/a)) = a(1/a) = 1$.
3. (a) $3/2$ (b) $0, 2$
   (c) $2, -2$ (d) $1, -2$
6. Note that if $q \in \mathbb{Z}$ and if $3q^2$ is even, then q^2 is even, so that q is even.
7. If $p \in \mathbb{N}$, then there are three possibilities: for some $m \in \mathbb{N} \cup \{0\}$, (i) $p = 3m$, (ii) $p = 3m + 1$, or (iii) $p = 3m + 2$.
10. (a) If $c = d$, then 2.1.7(b) implies $a + c < b + d$. If $c < d$, then $a + c < b + c < b + d$.
13. If $a \neq 0$, then 2.1.8(a) implies that $a^2 > 0$; since $b^2 \geq 0$, it follows that $a^2 + b^2 > 0$.
15. (a) If $0 < a < b$, then 2.1.7(c) implies that $0 < a^2 < ab < b^2$. Then by Example 2.1.13(a), we infer that $a = \sqrt{a^2} < \sqrt{ab} < \sqrt{b^2} = b$.
16. (a) $\{x : x > 4 \text{ or } x < -1\}$. (b) $\{x : 1 < x < 2 \text{ or } -2 < x < -1\}$.
   (c) $\{x : -1 < x < 0 \text{ or } x > 1\}$. (d) $\{x : x < 0 \text{ or } x > 1\}$.
19. The inequality is equivalent to $0 \leq a^2 - 2ab + b^2 = (a-b)^2$.
20. (a) Use 2.1.7(c).
21. (a) Let $S := \{n \in \mathbb{N} : 0 < n < 1\}$. If S is not empty, the Well-Ordering Property of $\mathbb{N}$ implies there is a least element m in S. However, $0 < m < 1$ implies that $0 < m^2 < m$, and since m^2 is also in S, this is a contradiction of the fact that m is the least element of S.
22. (a) Let $x := c - 1 > 0$ and apply Bernoulli's Inequality 2.1.13(c).
24. (a) If $m > n$, then $k := m - n \in \mathbb{N}$, and $c^k \geq c > 1$, which implies that $c^m > c^n$. Conversely, the hypotheses that $c^m > c^n$ and $m \leq n$ lead to a contradiction.

25. Let $b := c^{1/mn}$ and show that $b > 1$. Exercise 24(a) implies that $c^{1/n} = b^m > b^n = c^{1/m}$ if and only if $m > n$.

26. Fix $m \in \mathbb{N}$ and use Mathematical Induction to prove that $a^{m+n} = a^m a^n$ and $(a^m)^n = a^{mn}$ for all $n \in \mathbb{N}$. Then, for a given $n \in \mathbb{N}$, prove that the equalities are valid for all $m \in \mathbb{N}$.

Section 2.2

1. (a) If $a \geq 0$, then $|a| = a = \sqrt{a^2}$; if $a < 0$, then $|a| = -a = \sqrt{a^2}$.
   (b) It suffices to show that $|1/b| = 1/|b|$ for $b \neq 0$ (why?). Consider the cases $b > 0$ and $b < 0$.

3. If $x \leq y \leq z$, then $|x - y| + |y - z| = (y - x) + (z - y) = z - x = |z - x|$. To establish the converse, show that $y < x$ and $y > z$ are impossible. For example, if $y < x \leq z$, it follows from what we have shown and the given relationship that $|x - y| = 0$, so that $y = x$, a contradiction.

6. (a) $-2 \leq x \leq 9/2$ (b) $-2 \leq x \leq 2$.

7. $x = 4$ or $x = -3$.

10. (a) $x < 0$ (b) $-3/2 < x < 1/2$.

12. $\{x : -3 < x < -5/2 \text{ or } 3/2 < x < 2\}$.

13. $\{x : 1 < x < 4\}$.

14. (a) $\{(x, y) : y = \pm x\}$. (c) The hyperbolas $y = 2/x$ and $y = -2/x$.

15. (a) If $y \geq 0$, then $-y \leq x \leq y$ and we get the region in the upper half-plane on or between the lines $y = x$ and $y = -x$.

18. (a) Suppose that $a \leq b$.

19. If $a \leq b \leq c$, then $\text{mid}\{a, b, c\} = b = \min\{b, c, c\} = \min\{\max\{a, b\}, \max\{b, c\}, \max\{c, a\}\}$. The other cases are similar.

Section 2.3

1. Since $0 \leq x$ for all $x \in S_1$, then $u = 0$ is a lower bound of S_1. If $v > 0$, then v is not a lower bound of S_1 because $v/2 \in S_1$ and $v/2 < v$. Therefore $\inf S_1 = 0$.

3. Since $1/n \leq 1$ for all $n \in \mathbb{N}$, then 1 is an upper bound for S_3.

4. $\sup S_4 = 2$ and $\inf S_4 = 1/2$.

7. Let $u \in S$ be an upper bound of S. If v is another upper bound of S, then $u \leq v$. Hence $u = \sup S$.

10. Let $u := \sup A$, $v := \sup B$, and $w := \sup\{u, v\}$. Then w is an upper bound of $A \cup B$, because if $x \in A$, then $x \leq u \leq w$, and if $x \in B$, then $x \leq v \leq w$. If z is any upper bound of $A \cup B$, then z is an upper bound of A and of B, so that $u \leq z$ and $v \leq z$. Hence $w \leq z$. Therefore, $w = \sup(A \cup B)$.

12. Consider two cases: $u \geq s^*$ and $u < s^*$.

Section 2.4

1. Since $1 - 1/n < 1$ for all $n \in \mathbb{N}$, 1 is an upper bound. To show that 1 is the supremum, it must be shown that for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $1 - 1/n > 1 - \varepsilon$, which is equivalent to $1/n < \varepsilon$. Apply the Archimedean Property 2.4.3 or 2.4.5.

2. $\inf S = -1$ and $\sup S = 1$.

4. (a) Let $u := \sup S$ and $a > 0$. Then $x \leq u$ for all $x \in S$, whence $ax \leq au$ for all $x \in S$, whence it follows that au is an upper bound of aS. If v is another upper bound of aS, then $ax \leq v$ for

all $x \in S$, whence $x \leq v/a$ for all $x \in S$, showing that v/a is an upper bound for S so that $u \leq v/a$, from which we conclude that $au \leq v$. Therefore $au = \sup(aS)$.

6. Let $u := \sup f(X)$. Then $f(x) \leq u$ for all $x \in X$, so that $a + f(x) \leq a + u$ for all $x \in X$, whence $\sup\{a + f(x) : x \in X\} \leq a + u$. If $w < a + u$, then $w - a < u$, so that there exists $x_w \in X$ with $w - a < f(x_w)$, whence $w < a + f(x_w)$, and thus w is not an upper bound for $\{a + f(x) : x \in X\}$.

8. If $u := \sup f(X)$ and $v := \sup g(X)$, then $f(x) \leq u$ and $g(x) \leq v$ for all $x \in X$, whence $f(x) + g(x) \leq u + v$ for all $x \in X$.

10. (a) $f(x) = 1$ for $x \in X$. (b) $g(y) = 0$ for $y \in Y$.

12. Let $S := \{h(x, y) : x \in X, y \in Y\}$. We have $h(x, y) \leq F(x)$ for all $x \in X$, $y \in Y$ so that $\sup S \leq \sup\{F(x) : x \in X\}$. If $w < \sup\{F(x) : x \in X\}$, then there exists $x_0 \in X$ with $w < F(x_0) = \sup\{h(x_0, y) : y \in Y\}$, whence there exists $y_0 \in Y$ with $w < h(x_0, y_0)$. Thus w is not an upper bound of S, and so $w < \sup S$. Since this is true for any w such that $w < \sup\{F(x) : x \in X\}$, we conclude that $\sup\{F(x) : x \in X\} \leq \sup S$.

14. Note that $n < 2^n$ (whence $1/2^n < 1/n$) for any $n \in \mathbb{N}$.

15. Let $S_3 := \{s \in \mathbb{R} : 0 \leq s,\ s^2 < 3\}$. Show that S_3 is nonempty and bounded by 3 and let $y := \sup S_3$. If $y^2 < 3$ and $1/n < (3 - y^2)/(2y + 1)$ show that $y + 1/n \in S_3$. If $y^2 > 3$ and $1/m < (y^2 - 3)/2y$ show that $y - 1/m \in S_3$. Therefore $y^2 = 3$.

18. If $x < 0 < y$, then we can take $r = 0$. If $x < y < 0$, we apply 2.4.8 to obtain a rational number between $-y$ and $-x$.

Section 2.5

2. S has an upper bound b and a lower bound a if and only if S is contained in the interval $[a,\ b]$.

4. Because z is neither a lower bound nor an upper bound of S.

5. If $z \in \mathbb{R}$, then z is not a lower bound of S so there exists $x_z \in S$ such that $x_z \leq z$. Similarly, there exists $y_z \in S$ such that $z \leq y_z$.

8. If $x > 0$, then there exists $n \in \mathbb{N}$ with $1/n < x$, so that $x \notin J_n$. If $y \leq 0$, then $y \notin J_1$.

10. Let $\eta := \inf\{b_n : n \in \mathbb{N}\}$; we claim that $a_n \leq \eta$ for all n. Fix $n \in \mathbb{N}$; we will show that a_n is a lower bound for the set $\{b_k : k \in \mathbb{N}\}$. We consider two cases. (i) If $n \leq k$, then since $I_n \supseteq I_k$, we have $a_n \leq a_k \leq b_k$. (ii) If $k < n$, then since $I_k \supseteq I_n$, we have $a_n \leq b_n \leq b_k$. Therefore $a_n \leq b_k$ for all $k \in \mathbb{N}$, so that a_n is a lower bound for $\{b_k : k \in \mathbb{N}\}$ and so $a_n \leq \eta$. In particular, this shows that $\eta \in [a_n,\ b_n]$ for all n, so that $\eta \in \cap I_n$.

12. $\frac{3}{8} = (.011000\ldots)_2 = (.010111\ldots)_2$. $\frac{7}{16} = (.0111000\ldots)_2 = (.0110111\ldots)_2$.

13. (a) $\frac{1}{3} \approx (.0101)_2$. (b) $\frac{1}{3} = (.010101\ldots)_2$, the block 01 repeats.

16. $\frac{1}{7} = .142\,857\ldots$, the block repeats. $\frac{2}{19} = .105\,263\,157\,894\,736\,842\ldots$, the block repeats.

17. $1.25\,137\ldots137\ldots = 31253/24975$, $35.14653\ldots653\ldots = 3511139/99900$.

Section 3.1

1. (a) 0, 2, 0, 2, 0 (c) 1/2, 1/6, 1/12, 1/20, 1/30

3. (a) 1, 4, 13, 40, 121 (c) 1, 2, 3, 5, 4.

5. (a) We have $0 < n/(n^2 + 1) < n/n^2 = 1/n$. Given $\varepsilon > 0$, let $K(\varepsilon) \geq 1/\varepsilon$.
   (c) We have $|(3n + 1)/(2n + 5) - 3/2| = 13/(4n + 10) < 13/4n$. Given $\varepsilon > 0$, let $K(\varepsilon) \geq 13/4\varepsilon$.

6. (a) $1/\sqrt{n + 7} < 1/\sqrt{n}$ (b) $|2n/(n + 2) - 2| = 4/(n + 2) < 4/n$
   (c) $\sqrt{n}/(n + 1) < 1/\sqrt{n}$ (d) $\left|(-1)^n n/(n^2 + 1)\right| \leq 1/n$.

9. $0 < \sqrt{x_n} < \varepsilon \Longleftrightarrow 0 < x_n < \varepsilon^2$.

11. $|1/n - 1/(n+1)| = 1/n(n+1) < 1/n^2 \le 1/n$.

14. Let $b := 1/(1+a)$ where $a > 0$. Since $(1+a)^n > \frac{1}{2}n(n-1)a^2$, we have that $0 < nb^n \le n/\left[\frac{1}{2}n(n-1)a^2\right] \le 2/[(n-1)a^2]$. Thus $\lim(nb^n) = 0$.

16. If $n > 4$, then $0 < n^2/n! < n/(n-2)(n-1) < 1/(n-3)$.

Section 3.2

1. (a) $\lim(x_n) = 1$ (c) $x_n \ge n/2$, so the sequence diverges.

3. $Y = (X+Y) - X$.

6. (a) 4 (b) 0 (c) 1 (d) 0.

8. In (3) the exponent k is fixed, but in $(1+1/n)^n$ the exponent varies.

9. $\lim(y_n) = 0$ and $\lim(\sqrt{n}y_n) = \frac{1}{2}$.

12. b.

14. (a) 1 (b) 1

16. (a) $L = a$ (b) $L = b/2$ (c) $L = 1/b$ (d) $L = 8/9$.

19. (a) Converges to 0 (c) Converges to 0.

21. (a) (1) (b) (n).

22. Yes. (Why?)

23. From Exercise 2.2.18, $u_n = \frac{1}{2}(x_n + y_n + |x_n - y_n|)$.

24. Use Exercises 2.2.18(b), 2.2.19, and the preceding exercise.

Section 3.3

1. (x_n) is a bounded decreasing sequence. The limit is 4.

2. The limit is 1.

3. The limit is 2.

4. The limit is 2.

5. (y_n) is increasing. The limit is $y = \frac{1}{2}(1 + \sqrt{1+4p})$.

7. (x_n) is increasing.

10. Note $y_n = 1/(n+1) + 1/(n+2) + \cdots + 1/2n < 1/(n+1) + 1/(n+1) + \cdots + 1/(n+1) = n/(n+1) < 1$.

12. (a) e (b) e^2 (c) e (d) $1/e$.

13. Note that if $n \ge 2$, then $0 \le s_n - \sqrt{2} \le s_n^2 - 2$.

14. Note that $0 \le s_n - \sqrt{5} \le \left(s_n^2 - 5\right)/\sqrt{5} \le \left(s_n^2 - 5\right)/2$.

15. $e_2 = 2.25$, $e_4 = 2.441\,406$, $e_8 = 2.565\,785$, $e_{16} = 2.637\,928$.

16. $e_{50} = 2.691\,588$, $e_{100} = 2.704\,814$, $e_{1000} = 2.716\,924$.

Section 3.4

1. For example $x_{2n-1} := 2n - 1$ and $x_{2n} := 1/2n$.

3. $L = \frac{1}{2}\left(1 + \sqrt{5}\right)$.

7. (a) e (b) $e^{1/2}$ (c) e^2 (d) e^2.

8. (a) 1 (b) $e^{3/2}$.

12. Choose $n_1 \geq 1$ so that $|x_{n_1}| > 1$, then choose $n_2 > n_1$, so that $|x_{n_2}| > 2$, and, in general, choose $n_k > n_{k-1}$ so that $|x_{n_k}| > k$.

13. $(x_{2n-1}) = (-1, -1/3, -1/5, \ldots)$.

14. Choose $n_1 \geq 1$ so that $x_{n_1} \geq s - 1$, then choose $n_2 > n_1$ so that $x_{n_2} > s - 1/2$, and, in general, choose $n_k > n_{k-1}$ so that $x_{n_k} > s - 1/k$.

Section 3.5

1. For example, $((-1)^n)$.

3. (a) Note that $\left|(-1)^n - (-1)^{n+1}\right| = 2$ for all $n \in \mathbb{N}$.
   (c) Take $m = 2n$, so $x_m - x_n = x_{2n} - x_n = \ln 2n - \ln n = \ln 2$ for all n.

5. $\lim\left(\sqrt{n+1} - \sqrt{n}\right) = 0$. But, if $m = 4n$, then $\sqrt{4n} - \sqrt{n} = \sqrt{n}$ for all n.

8. Let $u := \sup\{x_n : n \in \mathbb{N}\}$. If $\varepsilon > 0$, let H be such that $u - \varepsilon < x_H \leq u$. If $m \geq n \geq H$, then $u - \varepsilon < x_n \leq x_m \leq u$ so that $|x_m - x_n| < \varepsilon$.

10. $\lim(x_n) = (1/3)x_1 + (2/3)x_2$.

12. The limit is $\sqrt{2} - 1$.

13. The limit is $1 + \sqrt{2}$.

14. Four iterations give $r = 0.201\,64$ to 5 places.

Section 3.6

1. If $\{x_n : n \in \mathbb{N}\}$ is not bounded above, choose $n_{k+1} > n_k$ such that $x_{n_k} \geq k$ for $k \in \mathbb{N}$.

3. Note that $|x_n - 0| < \varepsilon$ if and only if $1/x_n > 1/\varepsilon$.

4. (a) $[\sqrt{n} > a] \Longleftrightarrow [n > a^2]$
   (c) $\sqrt{n-1} \geq \sqrt{n/2}$ when $n \geq 2$.

8. (a) $n < (n^2 + 2)^{1/2}$.
   (c) Since $n < (n^2 + 1)^{1/2}$, then $n^{1/2} < (n^2 + 1)^{1/2}/n^{1/2}$.

9. (a) Since $x_n/y_n \to \infty$, there exists K_1 such that if $n \geq K_1$, then $x_n \geq y_n$. Now apply Theorem 3.6.4(a).

Section 3.7

1. The partial sums of $\sum b_n$ are a subsequence of the partial sums of $\sum a_n$.

3. (a) Since $1/(n+1)(n+2) = 1/(n+1) - 1/(n+2)$, the series is telescoping.

6. (a) $4/35$.

9. (a) The sequence $(\cos n)$ does not converge to 0.
   (b) Since $\left|(\cos n)/n^2\right| \leq 1/n^2$, the convergence of $\sum(\cos n)/n^2$ follows from Example 3.7.6 (c) and Theorem 3.7.7.

10. The "even" sequence (s_{2n}) is decreasing, the "odd" sequence (s_{2n+1}) is increasing, and $-1 \leq s_n \leq 0$. Also $0 \leq s_{2n} - s_{2n+1} = 1/\sqrt{2n+1}$.

12. $\sum 1/n^2$ is convergent, but $\sum 1/n$ is not.

14. Show that $b_k \geq a_1/k$ for $k \in \mathbb{N}$, whence $b_1 + \cdots + b_n \geq a_1(1 + \cdots + 1/n)$.

15. Evidently $2a(4) \leq a(3) + a(4)$ and $2^2 a(8) \leq a(5) + \cdots + a(8)$, etc. Also $a(2) + a(3) \leq 2a(2)$ and $a(4) + \cdots + a(7) \leq 2^2 a\left(2^2\right)$, etc. The stated inequality follows by addition. Now apply the Comparison Test 3.7.7.

17. (a) The terms are decreasing and $2^n/2^n \ln(2^n) = 1/(n \ln 2)$. Since $\sum 1/n$ diverges, so does $\sum 1/(n \ln n)$.

18. (a) The terms are decreasing and $2^n/2^n(\ln 2^n)^c = (1/n^c) \cdot (1/\ln 2)^c$. Now use the fact that $\sum(1/n^c)$ converges when $c > 1$.

Section 4.1

1. (a–c) If $|x-1| \le 1$, then $|x+1| \le 3$ so that $\left|x^2-1\right| \le 3|x-1|$. Thus, $|x-1| < 1/6$ assures that $\left|x^2-1\right| < 1/2$, etc.
   (d) If $|x-1| < 1$, then $\left|x^3-1\right| \le 7|x-1|$.
2. (a) Since $|\sqrt{x}-2| = |x-4|/(\sqrt{x}+2) \le \frac{1}{2}|x-4|$, then $|x-4| < 1$ implies that we have $|\sqrt{x}-2| < \frac{1}{2}$.
   (b) If $|x-4| < 2 \times 10^{-2} = .02$, then $|\sqrt{x}-2| < .01$.
5. If $0 < x < a$, then $0 < x+c < a+c < 2a$, so that $\left|x^2-c^2\right| = |x+c||x-c| \le 2a|x-c|$. Given $\varepsilon > 0$, take $\delta := \varepsilon/2a$.
8. If $c \neq 0$, show that $|\sqrt{x}-\sqrt{c}| \le (1/\sqrt{c})|x-c|$, so we can take $\delta := \varepsilon\sqrt{c}$. If $c = 0$, we can take $\delta := \varepsilon^2$.
9. (a) If $|x-2| < 1/2$ show that $|1/(1-x)+1| = |(x-2)/(x-1)| \le 2|x-2|$. Thus we can take $\delta := \inf\{1/2, \varepsilon/2\}$.
   (c) If $x \neq 0$, then $\left|x^2/|x| - 0\right| = |x|$. Take $\delta := \varepsilon$.
10. (a) If $|x-2| < 1$, then $\left|x^2+4x-12\right| = |x+6||x-2| < 9|x-2|$. We may take $\delta := \inf\{1, \varepsilon/9\}$.
    (b) If $|x+1| < 1/4$, then $|(x+5)/(3x+2)-4| = 7|x+1|/|2x+3| < 14|x+1|$, and we may take $\delta := \inf\{1/4, \varepsilon/14\}$.
12. (a) Let $x_n := 1/n$. (c) Let $x_n := 1/n$ and $y_n := -1/n$.
14. (b) If $f(x) := \operatorname{sgn}(x)$, then $\lim\limits_{x\to 0}(f(x))^2 = 1$, but $\lim\limits_{x\to 0} f(x)$ does not exist.
15. (a) Since $|f(x)-0| \le |x|$, we have $\lim\limits_{x\to 0} f(x) = 0$.
    (b) If $c \neq 0$ is rational, let (x_n) be a sequence of irrational numbers that converges to c; then $f(c) = c \neq 0 = \lim(f(x_n))$. What if c is irrational?
17. The restriction of sgn to [0, 1] has a limit at 0.

Section 4.2

1. (a) 10 (b) -3 (c) 1/12 (d) 1/2.
2. (a) 1 (b) 4 (c) 2 (d) 1/2.
3. Multiply the numerator and denominator by $\sqrt{1+2x}+\sqrt{1+3x}$.
4. Consider $x_n := 1/2\pi n$ and $\cos(1/x_n) = 1$. Use the Squeeze Theorem 4.2.7.
8. If $|x| \le 1$, $k \in \mathbb{N}$, then $|x^k| = |x|^k \le 1$, whence $-x^2 \le x^{k+2} \le x^2$.
11. (a) No limit (b) 0 (c) No limit (d) 0.

Section 4.3

2. Let $f(x) := \sin(1/x)$ for $x < 0$ and $f(x) := 0$ for $x > 0$.
3. Given $\alpha > 0$, if $0 < x < 1/\alpha^2$, then $\sqrt{x} < 1/\alpha$, and so $f(x) > \alpha$.
5. (a) If $\alpha > 1$ and $1 < x < \alpha/(\alpha-1)$, then $\alpha < x/(x-1)$, hence we have $\lim\limits_{x\to 1+} x/(x-1) = \infty$.
   (c) Since $(x+2)/\sqrt{x} > 2/\sqrt{x}$, the limit is ∞.
   (e) If $x > 0$, then $1/\sqrt{x} < \left(\sqrt{x+1}\right)/x$, so the right-hand limit is ∞.
   (g) 1 (h) -1.
8. Note that $|f(x)-L| < \varepsilon$ for $x > K$ if and only if $|f(1/z)-L| < \varepsilon$ for $0 < z < 1/K$.
9. There exists $\alpha > 0$ such that $|xf(x)-L| < 1$ whenever $x > \alpha$. Hence $|f(x)| < (|L|+1)/x$ for $x > \alpha$.

12. No. If $h(x) := f(x) - g(x)$, then $\lim_{x\to\infty} h(x) = 0$ and we have $f(x)/g(x) = 1 + h(x)/g(x) \to 1$.

13. Suppose that $|f(x) - L| < \varepsilon$ for $x > K$, and that $g(y) > K$ for $y > H$. Then $|f \circ g(y) - L| < \varepsilon$ for $y > H$.

Section 5.1

4. (a) Continuous if $x \neq 0, \ \pm 1, \ \pm 2, \ldots$ (b) Continuous if $x \neq \pm 1, \ \pm 2, \ldots$
(c) Continuous if $\sin x \neq 0, 1$ (d) Continuous if $x \neq 0, \pm 1, \ \pm 1/2, \ldots$.

7. Let $\varepsilon := f(c)/2$, and let $\delta > 0$ be such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$, which implies that $f(x) > f(c) - \varepsilon = f(c)/2 > 0$.

8. Since f is continuous at x, we have $f(x) = \lim(f(x_n)) = 0$. Thus $x \in S$.

10. Note that $\big||x| - |c|\big| \leq |x - c|$.

13. Since $|g(x) - 6| \leq \sup\{|2x - 6|, |x - 3|\} = 2|x - 3|$, g is continuous at $x = 3$. If $c \neq 3$, let (x_n) be a sequence of rational numbers converging to c and let (y_n) be a sequence of irrational numbers converging to c. Then $\lim(g(x_n)) \neq \lim(g(y_n))$.

Section 5.2

1. (a) Continuous on $\mathbb{R}$ (c) Continuous for $x \neq 0$.

2. Use 5.2.1(a) and Induction, or use 5.2.7 with $g(x) := x^n$.

4. Continuous at every noninteger.

7. Let $f(x) := 1$ if x is rational, and $f(x) := -1$ if x is irrational.

12. First show that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$; then note that $f(x - x_0) = f(x) - f(x_0)$. Consequently f is continuous at the point x_0 if and only if it is continuous at 0. Thus, if f is continuous at x_0, then it is continuous at 0, and hence everywhere.

13. First show that $f(0) = 0$ and (by Induction) that $f(x) = cx$ for $x \in \mathbb{N}$, and hence also for $x \in \mathbb{Z}$. Next show that $f(x) = cx$ for $x \in \mathbb{Q}$. Finally, if $x \notin \mathbb{Q}$, let $x = \lim(r_n)$ for some sequence in $\mathbb{Q}$.

15. If $f(x) \geq g(x)$, then both expressions give $h(x) = f(x)$; and if $f(x) \leq g(x)$, then $h(x) = g(x)$ in both cases.

Section 5.3

1. Apply either the Boundedness Theorem 5.3.2 to $1/f$, or the Maximum-Minimum Theorem 5.3.4 to conclude that $\inf f(I) > 0$.

3. Choose a sequence (x_n) such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)| \leq \left(\frac{1}{2}\right)^n |f(x_1)|$. Apply the Bolzano-Weierstrass Theorem to obtain a convergent subsequence.

4. Suppose that p has odd degree n and that the coefficient a_n of x^n is positive. By 4.3.16, $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$.

5. In the intervals $[1.035, \ 1.040]$ and $[-7.026, -7.025]$.

7. In the interval $[0.7390, \ 0.7391]$.

8. In the interval $[1.4687, \ 1.4765]$.

9. (a) 1 (b) 6.

10. $1/2^n < 10^{-5}$ implies that $n > (5 \ln 10)/\ln 2 \approx 16.61$. Take $n = 17$.

11. If $f(w) < 0$, then it follows from Theorem 4.2.9 that there exists a δ-neighborhood $V_\delta(w)$ such that $f(x) < 0$ for all $x \in V_\delta(w)$.

14. Apply Theorem 4.2.9 to $\beta - f(x)$.
15. If $0 < a < b \le \infty$, then $f((a,\, b)) = (a^2, b^2)$; if $-\infty \le a < b < 0$, then $f((a,\, b)) = (b^2, a^2)$. If $a < 0 < b$, then $f((a,\, b))$ is not an open interval, but equals $[0,\, c)$ where $c := \sup\{a^2, b^2\}$. Images of closed intervals are treated similarly.
16. For example, if $a < 0 < b$ and $c := \inf\{1/(a^2+1),\ 1/(b^2+1)\}$, then $g((a,\, b)) = (c,\, 1]$. If $0 < a < b$, then $g((a,b)) = (1/(b^2+1),\ 1/(a^2+1))$. Also $g([-1,\, 1]) = [1/2, 1]$. If $a < b$, then $h((a,\, b)) = (a^3,\, b^3)$ and $h((a,\, b]) = (a^3, b^3]$.
17. Yes. Use the Density Theorem 2.4.8.
19. Consider $g(x) := 1/x$ for $x \in J := (0,\, 1)$.

Section 5.4

1. Since $1/x - 1/u = (u-x)/xu$, it follows that $|1/x - 1/u| \le (1/a^2)|x-u|$ for $x, u \in [a,\, \infty)$.
3. (a) Let $x_n := n + 1/n,\ u_n := n$.
   (b) Let $x_n := 1/2n\pi,\ u_n := 1/(2n\pi + \pi/2)$.
6. If M is a bound for both f and g on A, show that $|f(x)g(x) - f(u)g(u)| \le M|f(x) - f(u)| + M|g(x) - g(u)|$ for all $x, u \in A$.
8. Given $\varepsilon > 0$ there exists $\delta_f > 0$ such that $|y - v| < \delta_f$ implies $|f(y) - f(v)| < \varepsilon$. Now choose $\delta_g > 0$ so that $|x - u| < \delta_g$ implies $|g(x) - g(u)| < \delta_f$.
11. If $|g(x) - g(0)| \le K|x - 0|$ for all $x \in [0,\, 1]$, then $\sqrt{x} \le Kx$ for $x \in [0,\, 1]$. But if $x_n := 1/n^2$, then K must satisfy $n \le K$ for all $n \in \mathbb{N}$, which is impossible.
14. Since f is bounded on $[0,\, p]$, it follows that it is bounded on $\mathbb{R}$. Since f is continuous on $J := [-1,\, p+1]$, it is uniformly continuous on J. Now show that this implies that f is uniformly continuous on $\mathbb{R}$.

Section 5.5

1. (a) The δ-intervals are $\left[-\frac{1}{4}, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{3}{4}\right]$, and $\left[\frac{3}{8}, \frac{9}{8}\right]$.
   (b) The third δ-interval does not contain $\left[\frac{1}{2},\, 1\right]$.
2. (a) Yes. (b) Yes.
3. No. The first δ_2-interval is $\left[-\frac{1}{10}, \frac{1}{10}\right]$ and does not contain $\left[0,\, \frac{1}{4}\right]$.
4. (b) If $t \in \left(\frac{1}{2}, 1\right)$ then $[t - \delta(t), t + \delta(t)] = \left[-\frac{1}{2} + \frac{3}{2}t, \frac{1}{2} + \frac{1}{2}t\right] \subset \left(\frac{1}{4},\, 1\right)$.
6. We could have two subintervals having c as a tag with one of them not contained in the δ-interval around c.
7. If $\dot{\mathcal{P}} := \{([a,\, x_1], t_1), \ldots ([x_{k-1}, c], t_k), ([c,\, x_{k+1}], t_{k+1}), \ldots, ([x_n, b], t_n)\}$ is δ^*-fine, then $\dot{\mathcal{P}}' := \{([a, x_1], t_1), \ldots, ([x_{k-1}, c], t_k)\}$ is a δ'-fine partition of $[a, c]$ and $\dot{\mathcal{P}}'' := \{([c,\, x_{k+1}], t_{k+1}), \ldots, ([x_n, b], t_n)\}$ is a δ''-fine partition of $[c, b]$.
9. The hypothesis that f is locally bounded presents us with a gauge δ. If $\{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ-fine partition of $[a,\, b]$ and M_i is a bound for $|f|$ on $[x_{i-1}, x_i]$ let $M := \sup\{M_i : i = 1, \ldots, n\}$.

Section 5.6

1. If $x \in [a,\, b]$, then $f(a) \le f(x)$.
4. If $0 \le f(x_1) \le f(x_2)$ and $0 \le g(x_1) \le g(x_2)$, then $f(x_1)g(x_1) \le f(x_2)g(x_1) \le f(x_2)g(x_2)$.
6. If f is continuous at c, then $\lim(f(x_n)) = f(c)$, since $c = \lim(x_n)$. Conversely, since $0 \le j_f(c) \le f(x_{2n}) - f(x_{2n+1})$, it follows that $j_f(c) = 0$, so f is continuous at c.

7. Apply Exercises 2.4.4, 2.4.5, and the Principle of the Iterated Infima (analogous to the result in Exercise 2.4.12).

8. Let $x_1 \in I$ be such that $y = f(x_1)$ and $x_2 \in I$ be such that $y = g(x_2)$. If $x_2 \leq x_1$, then $y = g(y_2) < f(x_2) \leq f(x_1) = y$, a contradiction.

11. Note that f^{-1} is continuous at every point of its domain $[0, 1] \cup (2, 3]$.

14. Let $y := x^{1/n}$ and $z := x^{1/q}$ so that $y^n = x = z^q$, whence (by Exercise 2.1.26) $y^{np} = x^p = z^{qp}$. Since $np = mq$, show that $\left(x^{1/n}\right)^m = \left(x^{1/q}\right)^p$ or $x^{m/n} = x^{p/q}$. Now consider the case where $m, p \in \mathbb{Z}$.

15. Use the preceding exercise and Exercise 2.1.26.

Section 6.1

1. (a) $f'(x) = \lim_{h \to 0}[(x+h)^3 - x^3]/h = \lim_{h \to 0}\left(3x^2 + 3xh + h^2\right) = 3x^2$,

   (c) $h'(x) = \lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$.

4. Note that $|f(x)/x| \leq |x|$ for $x \in \mathbb{R}$.

5. (a) $f'(x) = (1 - x^2)/(1 + x^2)^2$ (b) $g'(x) = (x-1)/\sqrt{5 - 2x + x^2}$
   (c) $h'(x) = mkx^{k-1}\left(\cos x^k\right)\left(\sin x^k\right)^{m-1}$ (d) $k'(x) = 2x \sec^2(x^2)$.

6. The function f' is continuous for $n \geq 2$ and is differentiable for $n \geq 3$.

8. (a) $f'(x) = 2$ for $x > 0$, $f'(x) = 0$ for $-1 < x < 0$, and $f'(x) = -2$ for $x < -1$,
   (c) $h'(x) = 2|x|$ for all $x \in \mathbb{R}$.

10. If $x \neq 0$, then $g'(x) = 2x \sin(1/x^2) - (2/x)\cos(1/x^2)$. Moreover, $g'(0) = \lim_{h \to 0} \sin\left(1/h^2\right) = 0$. Consider $x_n := 1/\sqrt{2n\pi}$.

11. (a) $f'(x) = 2/(2x + 3)$ (b) $g'(x) = 6(L(x^2))^2/x$
    (c) $h'(x) = 1/x$ (d) $k'(x) = 1/(xL(x))$.

14. $1/h'(0) = 1/2$, $1/h'(1) = 1/5$, and $1/h'(-1) = 1/5$.

16. $D[\text{Arctan}\, y] = 1/D[\tan x] = 1/\sec^2 x = 1/(1 + y^2)$.

Section 6.2

1. (a) Increasing on $[3/2, \infty)$, decreasing on $(-\infty, 3/2]$,
   (c) Increasing on $(-\infty, -1]$ and $[1, \infty)$.

2. (a) Relative minimum at $x = 1$; relative maximum at $x = -1$,
   (c) Relative maximum at $x = 2/3$.

3. (a) Relative minima at $x = \pm 1$; relative maxima at $x = 0, \pm 4$,
   (c) Relative minima at $x = -2, 3$; relative maximum at $x = 2$.

6. If $x < y$ there exists c in (x, y) such that $|\sin x - \sin y| = |\cos c||y - x|$.

9. $f(x) = x^4(2 + \sin(1/x)) > 0$ for $x \neq 0$, so f has an absolute minimum at $x = 0$. Show that $f'(1/2n\pi) < 0$ for $n \geq 2$ and $f'(2/(4n+1)\pi) > 0$ for $n \geq 1$.

10. $g'(0) = \lim_{x \to 0}(1 + 2x \sin(1/x)) = 1 + 0 = 1$, and if $x \neq 0$, then $g'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x)$. Now show that $g'(1/2n\pi) < 0$ and that we have $g'(2/(4n+1)\pi) > 0$ for $n \in \mathbb{N}$.

14. Apply Darboux's Theorem 6.2.12.

17. Apply the Mean Value Theorem to the function $g - f$ on $[0, x]$.

20. (a, b) Apply the Mean Value Theorem.
    (c) Apply Darboux's Theorem to the results of (a) and (b).

Section 6.3

1. $A = B(\lim_{x\to c} f(x)/g(x)) = 0$.
4. Note that $f'(0) = 0$, but that $f'(x)$ does not exist if $x \neq 0$.
7. (a) 1 (b) 1 (c) 0 (d) 1/3.
8. (a) 1 (b) ∞ (c) 0 (d) 0.
9. (a) 0 (b) 0 (c) 0 (d) 0.
10. (a) 1 (b) 1 (c) e^3 (d) 0.
11. (a) 1 (b) 1 (c) 1 (d) 0.

Section 6.4

1. $f^{(2n-1)}(x) = (-1)^n a^{2n-1} \sin ax$ and $f^{(2n)}(x) = (-1)^n a^{2n} \cos ax$ for $n \in \mathbb{N}$.
4. Apply Taylor's Theorem to $f(x) := \sqrt{1+x}$ at $x_0 := 0$ and note that $R_1(x) < 0$ and $R_2(x) > 0$ for $x > 0$.
5. $1.095 < \sqrt{1.2} < 1.1$ and $1.375 < \sqrt{2} < 1.5$.
6. $R_2(0.2) < 0.0005$ and $R_2(1) < 0.0625$.
11. With $n = 4$, ln $1.5 = 0.40$; with $n = 7$, ln $1.5 = 0.405$.
17. Apply Taylor's Theorem to f at $x_0 = c$ to show that $f(x) \geq f(c) + f'(c)(x - c)$.
19. Since $f(2) < 0$ and $f(2.2) > 0$, there is a zero of f in [2.0, 2.2]. The value of x_4 is approximately 2.094 551 5.
20. $r_1 \approx 1.452\,626\,88$ and $r_2 \approx -1.164\,035\,14$.
21. $r \approx 1.324\,717\,96$.
22. $r_1 \approx 0.158\,594\,34$ and $r_2 \approx 3.146\,193\,22$.
23. $r_1 \approx 0.5$ and $r_2 \approx 0.809\,016\,99$.
24. $r \approx 0.739\,085\,13$.

Section 7.1

1. (a) $||\mathcal{P}_1|| = 2$ (b) $||\mathcal{P}_2|| = 2$ (c) $||\mathcal{P}_3|| = 1.4$ (d) $||\mathcal{P}_4|| = 2$.
2. (a) $0^2 \cdot 1 + 1^2 \cdot 1 + 2^2 \cdot 2 = 0 + 1 + 8 = 9$
   (b) 37 (c) 13 (d) 33.
5. (a) If $u \in [x_{i-1}, x_i]$, then $x_{i-1} \leq u$ so that $c_1 \leq t_i \leq x_i \leq x_{i-1} + ||\dot{\mathcal{P}}||$ whence $c_1 - ||\dot{\mathcal{P}}|| \leq x_{i-1} \leq u$. Also $u \leq x_i$ so that $x_i - ||\dot{\mathcal{P}}|| \leq x_{i-1} \leq t_i \leq c_2$, whence $u \leq x_i \leq c_2 + ||\dot{\mathcal{P}}||$.
10. g is not bounded. Take rational tags.
12. Let $\mathcal{P}_n$ be the partition of [0, 1] into n equal parts. If $\dot{\mathcal{P}}_n$ is this partition with rational tags, then $S(f; \dot{\mathcal{P}}_n) = 1$, while if $\dot{\mathcal{Q}}_n$ is this partition with irrational tags, then $S(f; \dot{\mathcal{Q}}_n) = 0$.
13. If $||\dot{\mathcal{P}}|| < \delta_\varepsilon := \varepsilon/4\alpha$, then the union of the subintervals in $\dot{\mathcal{P}}$ with tags in $[c, d]$ contains the interval $[c + \delta_\varepsilon, d - \delta_\varepsilon]$ and is contained in $[c - \delta_\varepsilon, d + \delta_\varepsilon]$. Therefore $\alpha(d - c - 2\delta_\varepsilon) \leq S(\varphi; \dot{\mathcal{P}}) \leq \alpha(d - c + 2\delta_\varepsilon)$, whence $\left|S(\varphi : \dot{\mathcal{P}}) - \alpha(d - c)\right| \leq 2\alpha\delta_\varepsilon < \varepsilon$.
14. (b) In fact, $\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right) \cdot (x_i - x_{i-1}) = x_i^3 - x_{i-1}^3$.
    (c) The terms in $S\left(Q : \dot{\mathcal{Q}}\right)$ telescope.
15. Let $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a tagged partition of $[a, b]$ and let $\dot{\mathcal{Q}} := \{([x_{i-1} + c, x_i + c], t_i + c)\}_{i=1}^n$ so that $\dot{\mathcal{Q}}$ is a tagged partition of $[a + c, b + c]$ and $||\dot{\mathcal{Q}}|| = ||\dot{\mathcal{P}}||$. Moreover, $S(g;\, \dot{\mathcal{Q}}) = S(f; \dot{\mathcal{P}})$ so that $\left|S(g;\, \dot{\mathcal{Q}}) - \int_a^b f\right| = \left|S(f; \dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon$ when $||\dot{\mathcal{Q}}|| < \delta_\varepsilon$.

Section 7.2

2. If the tags are all rational, then $S(h;\ \dot{\mathcal{P}}) \geq 1$, while if the tags are all irrational, then $S(h;\ \dot{\mathcal{P}}) = 0$.

3. Let $\dot{\mathcal{P}}_n$ be the partition of [0, 1] into n equal subintervals with $t_1 = 1/n$ and $\dot{\mathcal{Q}}_n$ be the same subintervals tagged by irrational points.

5. If $c_1, \ldots, c_n$ are the distinct values taken by φ, then $\varphi^{-1}(c_j)$ is the union of a finite collection $\{J_{j1}, \ldots, J_{jr_j}\}$ of disjoint subintervals of $[a, b]$. We can write $\varphi = \sum_{j=1}^{n}\sum_{k=1}^{r_j} c_j\varphi_{J_{jk}}$.

6. Not necessarily.

8. If $f(c) > 0$ for some $c \in (a, b)$, there exists $\delta > 0$ such that $f(x) > \frac{1}{2}f(c)$ for $|x - c| \leq \delta$. Then $\int_a^b f \geq \int_{c-\delta}^{c+\delta} f \geq (2\delta)\frac{1}{2}f(c) > 0$. If c is an endpoint, a similar argument applies.

10. Use Bolzano's Theorem 5.3.7.

12. Indeed, $|g(x)| \leq 1$ and is continuous on every interval $[c, 1]$ where $0 < c < 1$. The preceding exercise applies.

13. Let $f(x) := 1/x$ for $x \in (0, 1]$ and $f(0) := 0$.

16. Let $m := \inf f(x)$ and $M := \sup f$. By Theorem 7.1.5(c), we have $m(b - a) \leq \int_a^b f \leq M(b - a)$. By Bolzano's Theorem 5.3.7, there exists $c \in [a, b]$ such that $f(c) = (\int_a^b f)/(b - a)$.

19. (a) Let $\dot{\mathcal{P}}_n$ be a sequence of tagged partitions of $[0, a]$ with $||\dot{\mathcal{P}}_n|| \to 0$ and let $\dot{\mathcal{P}}_n^*$ be the corresponding "symmetric" partition of $[-a, a]$. Show that $S(f;\ \dot{\mathcal{P}}_n^*) = 2S(f; \dot{\mathcal{P}}_n) \to 2\int_0^a f$.

20. Note that $x \mapsto f(x^2)$ is an even continuous function.

Section 7.3

1. Suppose that $E := \{a = c_0 < c_1 < \cdots < c_m = b\}$ contains the points in $[a, b]$ where the derivative $F'(x)$ either does not exist, or does not equal $f(x)$. Then $f \in \mathcal{R}[c_{i-1}, c_i]$ and $\int_{c_{i-1}}^{c_i} f = F(c_i) - F(c_{i-1})$. Exercise 7.2.14 and Corollary 7.2.11 imply that $f \in \mathcal{R}[a, b]$ and that $\int_a^b f = \sum_{i=1}^{m} (F(c_i) - F(c_{i-1})) = F(b) - F(a)$.

2. $E = \emptyset$.

3. Let $E := \{-1,\ 1\}$. If $x \notin E$, $G'(x) = g(x)$.

4. Indeed, $B'(x) = |x|$ for all x.

6. $F_c = F_a - \int_a^c f$.

7. Let h be Thomae's function. There is no function $H : [0,\ 1] \to \mathbb{R}$ such that $H'(x) = h(x)$ for x in some nondegenerate open interval; otherwise Darboux's Theorem 6.2.12 would be contradicted on this interval.

9. (a) $G(x) = F(x) - F(c)$, (b) $H(x) = F(b) - F(x)$, (c) $S(x) = F(\sin x) - F(x)$.

10. Use Theorem 7.3.6 and the Chain Rule 6.1.6.

11. (a) $F'(x) = 2x(1 + x^6)^{-1}$ (b) $F'(x) = (1 + x^2)^{1/2} - 2x(1 + x^4)^{1/2}$.

15. $g'(x) = f(x + c) - f(x - c)$.

18. (a) Take $\varphi(t) = 1 + t^2$ to get $\frac{1}{3}(2^{3/2} - 1)$.
    (b) Take $\varphi(t) = 1 + t^3$ to get $\frac{4}{3}$.
    (c) Take $\varphi(t) = 1 + \sqrt{t}$ to get $\frac{4}{3}(3^{3/2} - 2^{3/2})$.
    (d) Take $\varphi(t) = t^{1/2}$ to get $2(\sin 2 - \sin 1)$.

19. In (a) – (c) $\varphi'(0)$ does not exist. For (a), integrate over $[c, 4]$ and let $c \to 0+$. For (c), the integrand is even so the integral equals $2\int_0^1 (1 + t)^{1/2} dt$.

20. (b) $\bigcup_n Z_n$ is contained in $\bigcup_{n,k} J_k^n$ and the sum of the lengths of these intervals is $\leq \sum_n \varepsilon/2^n = \varepsilon$.
21. (a) The Product Theorem 7.3.16 applies.
    (b) We have $\mp 2t \int_a^b fg \leq t^2 \int_a^b f^2 + \int_a^b g^2$.
    (c) Let $t \to \infty$ in (b).
    (d) If $\int_a^b f^2 \neq 0$, let $t = \left(\int_a^b g^2 / \int_a^b f^2\right)^{1/2}$ in (b).
22. Note that $\operatorname{sgn} \circ h$ is Dirichlet's function, which is not Riemann integrable.

Section 7.4

2. Show that if $\mathcal{P}$ is any partition, then $L(f;\, \mathcal{P}) = U(f;\, \mathcal{P}) = c(b-a)$.
4. If $k \geq 0$, then $\inf\{kf(x) : x \in I_j\} = k \inf\{f(x) : x \in I_j\}$.
6. Consider the partition $\mathcal{P}_\varepsilon := (0,\, 1-\varepsilon/2,\, 1+\varepsilon/2,\, 2)$.
9. See Exercise 2.4.8.
11. If $|f(x)| \leq M$ for $x \in [a,\, b]$ and $\varepsilon > 0$, let $\mathcal{P}$ be a partition such that the total length of the subintervals that contain any of the given points is less than ε/M. Then $U(f;\, \mathcal{P}) - L(f;\, \mathcal{P}) < \varepsilon$ so that Theorem 7.4.8 applies. Also $0 \leq U(f;\, \mathcal{P}) \leq \varepsilon$, so that $U(f) = 0$.

Section 7.5

1. Use (4) with $n = 4$, $a = 1$, $b = 2$, $h = 1/4$. Here $1/4 \leq f''(c) \leq 2$, so $T_4 \approx 0.697\,02$.
3. $T_4 \approx 0.782\,79$.
4. The index n must satisfy $2/12n^2 < 10^{-6}$; hence $n > 1000/\sqrt{6} \approx 408.25$.
5. $S_4 \approx 0.785\,39$.
6. The index n must satisfy $96/180n^4 < 10^{-6}$; hence $n \geq 28$.
12. The integral is equal to the area of one quarter of the unit circle. The derivatives of h are unbounded on $[0, 1]$. Since $h''(x) \leq 0$, the inequality is $T_n(h) < \pi/4 < M_n(h)$. See Exercise 8.
13. Interpret K as an area. Show that $h''(x) = -(1-x^2)^{3/2}$ and that $h^{(4)}(x) = -3(1+4x^2)(1-x^2)^{-7/2}$. To eight decimal places, $\pi = 3.141\,592\,65$.
14. Approximately 3.653 484 49.
15. Approximately 4.821 159 32.
16. Approximately 0.835 648 85.
17. Approximately 1.851 937 05.
18. 1.
19. Approximately 1.198 140 23.
20. Approximately 0.904 524 24.

Section 8.1

1. Note that $0 \leq f_n(x) \leq x/n \to 0$ as $n \to \infty$.
3. If $x > 0$, then $|f_n(x) - 1| < 1/(nx)$.
5. If $x > 0$ then $|f_n(x)| \leq 1/(nx) \to 0$.
7. If $x > 0$, then $0 < e^{-x} < 1$.
9. If $x > 0$, then $0 \leq x^2 e^{-nx} = x^2(e^{-x})^n \to 0$, since $0 < e^{-x} < 1$.
10. If $x \in \mathbb{Z}$, the limit equals 1. If $x \notin \mathbb{Z}$, the limit equals 0.
11. If $x \in [0,\, a]$, then $|f_n(x)| \leq a/n$. However, $f_n(n) = 1/2$.
14. If $x \in [0,\, b]$, then $|f_n(x)| \leq b^n$. However, $f_n(2^{-1/n}) = 1/3$.
15. If $x \in [a,\, \infty)$, then $|f_n(x)| \leq 1/(na)$. However, $f_n(1/n) = \frac{1}{2}\sin 1 > 0$.
18. The maximum of f_n on $[0,\, \infty)$ is at $x = 1/n$, so $\|f_n\|_{[0,\, \infty)} = 1/(ne)$.

20. If n is sufficiently large, $||f_n||_{[a,\infty)} = n^2a^2/e^{na}$. However, $||f_n||_{[0,\infty)} = 4/e^2$.

23. Let M be a bound for $(f_n(x))$ and $(g_n(x))$ on A, whence also $|f(x)| \le M$. The Triangle Inequality gives $|f_n(x)g_n(x) - f(x)g(x)| \le M(|f_n(x) - f(x)| + |g_n(x) - g(x)|)$ for $x \in A$.

Section 8.2

1. The limit function is $f(x) := 0$ for $0 \le x < 1$, $f(1) := 1/2$, and $f(x) := 1$ for $1 < x \le 2$.

4. If $\varepsilon > 0$ is given, let K be such that if $n \ge K$, then $||f_n - f||_I < \varepsilon/2$. Then $|f_n(x_n) - f(x_0)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| \le \varepsilon/2 + |f(x_n) - f(x_0)|$. Since f is continuous (by Theorem 8.2.2) and $x_n \to x_0$, then $|f(x_n) - f(x_0)| < \varepsilon/2$ for $n \ge K'$, so that $|f_n(x_n) - f(x_0)| < \varepsilon$ for $n \ge \max\{K, K'\}$.

6. Here $f(0) = 1$ and $f(x) = 0$ for $x \in (0, 1]$. The convergence is not uniform on $[0, 1]$.

7. Given $\varepsilon := 1$, there exists $K > 0$ such that if $n \ge K$ and $x \in A$, then $|f_n(x) - f(x)| < 1$, so that $|f_n(x)| \le |f_K(x)| + 1$ for all $x \in A$. Let $M := \max\{||f_1||_A, \ldots, ||f_{K-1}||_A, ||f_K||_A + 1\}$.

8. $f_n(1/\sqrt{n}) = \sqrt{n}/2$.

10. Here (g_n) converges uniformly to the zero function. The sequence (g_n') does not converge uniformly.

11. Use the Fundamental Theorem 7.3.1 and Theorem 8.2.4.

13. If $a > 0$, then $||f_n||_{[a,\pi]} \le 1/(na)$ and Theorem 8.2.4 applies.

15. Here $||g_n||_{[0,\,1]} \le 1$ for all n. Now apply Theorem 8.2.5.

20. Let $f_n(x) := x^n$ on $[0, 1)$.

Section 8.3

1. Let $A := x > 0$ and let $m \to \infty$ in (5). For the upper estimate on e, take $x = 1$ and $n = 3$ to obtain $\left|e - 2\frac{2}{3}\right| < 1/12$, so $e < 2\frac{3}{4}$.

2. Note that if $n \ge 9$, then $2/(n+1)! < 6 \times 10^{-7} < 5 \times 10^{-6}$. Hence $e \approx 2.71828$.

3. Evidently $E_n(x) \le e^x$ for $x \ge 0$. To obtain the other inequality, apply Taylor's Theorem 6.4.1 to $[0, a]$.

5. Note that $0 \le t^n/(1+t) \le t^n$ for $t \in [0, x]$.

6. $\ln 1.1 \simeq 0.0953$ and $\ln 1.4 \approx 0.3365$. Take $n > 19{,}999$.

7. $\ln 2 \approx 0.6931$.

10. $L'(1) = \lim[L(1 + 1/n) - L(1)]/(1/n) = \lim L((1 + 1/n)^n) = L(\lim(1 + 1/n)^n) = L(e) = 1$.

11. (c) $(xy)^\alpha = E(\alpha L(xy)) = E(\alpha L(x) + \alpha L(y)) = E(\alpha L(x)) \cdot E(\alpha L(y)) = x^\alpha \cdot y^\alpha$.

12. (b) $(x^\alpha)^\beta = E(\beta L(x^\alpha)) = E(\beta\alpha L(x)) = x^{\alpha\beta}$, and similarly for $(x^\beta)^\alpha$.

15. Use 8.3.14 and 8.3.9(vii).

17. Indeed, we have $\log_a x = (\ln x)/(\ln a) = [(\ln x)/(\ln b)] \cdot [(\ln b)/(\ln a)]$ if $a \ne 1$, $b \ne 1$. Now take $a = 10$, $b = e$.

Section 8.4

1. If $n > 2|x|$, then $|\cos x - C_n(x)| \le (16/15)|x|^{2n}/(2n)!$, so $\cos(0.2) \approx 0.980\,067$, $\cos 1 \approx 0.549\,302$. Similarly, $\sin(0.2) \approx 0.198\,669$ and $\sin 1 \approx 0.841\,471$.

4. We integrate 8.4.8(x) twice on $[0, x]$. Note that the polynomial on the left has a zero in the interval [1.56,1.57], so $1.56 \le \pi/2$.

5. Exercise 8.4.4 shows that $C_4(x) \le \cos x \le C_3(x)$ for all $x \in \mathbb{R}$. Integrating several times, we get $S_4(x) \le \sin x \le S_5(x)$ for all $x > 0$. Show that $S_4(3.05) > 0$ and $S_5(3.15) < 0$. (This procedure can be sharpened.)

6. If $|x| \le A$ and $m > n > 2A$, then $|c_m(x) - c_n(x)| < (16/15)A^{2n}/(2n)!$, whence the convergence of (c_n) to c is uniform on each interval $[-A, A]$.

7. $D[(c(x))^2 - (s(x))^2] = 0$ for all $x \in \mathbb{R}$. For uniqueness, argue as in 8.4.4.

8. Let $g(x) := f(0)c(x) + f'(0)s(x)$ for $x \in \mathbb{R}$, so that $g''(x) = g(x)$, $g(0) = f(0)$ and $g'(0) = f'(0)$. Therefore $h(x) := f(x) - g(x)$ has the property that $h''(x) = h(x)$ for all $x \in \mathbb{R}$ and $h(0) = 0$, $h'(0) = 0$. Thus $g(x) = f(x)$ for all $x \in \mathbb{R}$, so that $f(x) = f(0)c(x) + f'(0)s(x)$.

9. If $\varphi(x) := c(-x)$, show that $\varphi''(x) = \varphi(x)$ and $\varphi(0) = 1$, $\varphi'(0) = 0$, so that $\varphi(x) = c(x)$ for all $x \in \mathbb{R}$. Therefore c is even.

Section 9.1

1. Let s_n be the nth partial sum of $\sum_1^\infty a_n$, let t_n be the nth partial sum of $\sum_1^\infty |a_n|$, and suppose that $a_n \ge 0$ for $n > P$. If $m > n > P$, show that $t_m - t_n = s_m - s_n$. Now apply the Cauchy Criterion.

3. Take positive terms until the partial sum exceeds 1, then take negative terms until the partial sum is less than 1, then take positive terms until the partial sum exceeds 2, etc.

5. Yes.

6. If $n \ge 2$, then $s_n = -\ln 2 - \ln n + \ln(n+1)$. Yes.

9. We have $s_{2n} - s_n \ge n a_{2n} = \frac{1}{2}(2n a_{2n})$, and $s_{2n+1} - s_n \ge \frac{1}{2}(2n+1)a_{2n+1}$. Consequently $\lim(n a_n) = 0$.

11. Indeed, if $|n^2 a_n| \le M$ for n, then $|a_n| \le M/n^2$.

13. (a) Rationalize to obtain $\sum x_n$ where $x_n := \left[\sqrt{n}\left(\sqrt{n+1} + \sqrt{n}\right)\right]^{-1}$ and note that $x_n \approx y_n := 1/(2n)$. Now apply the Limit Comparison Test 3.7.8.
    (b) Rationalize and compare with $\sum 1/n^{3/2}$.

14. If $\sum a_n$ is absolutely convergent, the partial sums of $\sum |a_n|$ are bounded, say by M. Evidently the absolute value of the partial sums of any subseries of a_n are also bounded by M.
    Conversely, if every subseries of $\sum a_n$ is convergent, then the subseries consisting of the strictly positive (and strictly negative) terms are absolutely convergent, whence it follows that $\sum a_n$ is absolutely convergent.

Section 9.2

1. (a) Convergent; compare with $\sum 1/n^2$. (c) Divergent; note that $2^{1/n} \to 1$.

2. (a) Divergent; apply 9.2.1 with $b_n := 1/n$.
   (c) Convergent; use 9.2.4 and note that $(n/(n+1))^n \to 1/e < 1$.

3. (a) $(\ln n)^p < n$ for large n, by L'Hospital's Rule.
   (c) Convergent; note that $(\ln n)^{\ln n} > n^2$ for large n.
   (e) Divergent; apply 9.2.6 or Exercise 3.7.15.

4. (a) Convergent (b) Divergent (c) Divergent
   (d) Convergent; note that $(\ln n)\exp\left(-n^{1/2}\right) < n\exp\left(-n^{1/2}\right) < 1/n^2$ for large n, by L'Hospital's Rule.
   (e) Divergent (f) Divergent.

6. Apply the Integral Test 9.2.6.

7. (a, b) Convergent (c) Divergent (d) Convergent.

9. If $m > n \ge K$, then $|s_m - s_n| \le |x_{n+1}| + \cdots + |x_m| < r^{n+1}/(1-r)$. Now let $m \to \infty$.

12. (a) A crude estimate of the remainder is given by $s - s_4 < \int_5^\infty x^{-2}dx = 1/5$. Similarly $s - s_{10} < 1/11$and $s - s_n < 1/(n+1)$, so that 999 terms suffice to get $s - s_{999} < 1/1000$.
(d) If $n \geq 4$, then $x_{n+1}/x_n \leq 5/8$ so (by Exercise 10) $|s - s_4| \leq 5/12$. If $n \geq 10$, then $x_{n+1}/x_n \leq 11/20$ so that $|s - s_{10}| \leq (10/2^{10})(11/9) < 0.012$. If $n = 14$, then $|s - s_{14}| < 0.000\,99$.

13. (b) Here $\sum_{n+1}^{\infty} < \int_n^\infty x^{-3/2}dx = 2/\sqrt{n}$, so $|s - s_{10}| < 0.633$ and $|s - s_n| < 0.001$ when $n > 4 \times 10^6$.
(c) If $n \geq 4$, then $|s - s_n| \leq (0.694)x_n$ so that $|s - s_4| < 0.065$. If $n \geq 10$, then $|s - s_n| \leq (0.628)x_n$ so that $|s - s_{10}| < 0.000\,023$.

14. Note that (s_{3n}) is not bounded.

16. Note that, for an integer with n digits, there are 9 ways of picking the first digit and 10 ways of picking each of the other $n-1$ digits. There is one value of m_k from 1 to 9, there is one value from 10 to 19, one from 20 to 29, etc.

18. Here $\lim(n(1 - x_{n+1}/x_n)) = (c - a - b) + 1$, so the series is convergent if $c > a + b$ and is divergent if $c < a + b$.

Section 9.3

1. (a) Absolutely convergent (b) Conditionally convergent
(c) Divergent (d) Conditionally convergent.

2. Show by induction that $s_2 < s_4 < s_6 < \cdots < s_5 < s_3 < s_1$. Hence the limit lies between s_n and s_{n+1} so that $|s - s_n| < |s_{n+1} - s_n| = z_{n+1}$.

5. Use Dirichlet's Test with $(y_n) := (+1, -1, -1, +, +1, -1, -1, \ldots)$. Or group the terms in pairs (after the first) and use the Alternating Series Test

7. If $f(x) := (\ln x)^p/x^q$, then $f'(x) < 0$ for x sufficiently large. L'Hospital's Rule shows that the terms in the alternating series approach 0.

8. (a) Convergent (b) Divergent (c) Divergent (d) Divergent.

11. Dirichlet's Test does not apply (directly, at least), since the partial sums of the series generated by $(1, -1, -1, 1, 1, 1, \ldots)$ are not bounded.

15. (a) Use Abel's Test with $x_n := 1/n$.
(b) Use the Cauchy Inequality with $x_n := \sqrt{a_n}$, $y_n := 1/n$, to get $\sum \sqrt{a_n}/n \leq (\sum a_n)^{1/2} (\sum 1/n^2)^{1/2}$, establishing convergence.
(d) Let $a_n := [n(\ln n)^2]^{-1}$, which converges by the Integral Test. However, $b_n := [\sqrt{n}\ln n]^{-1}$, which diverges.

Section 9.4

1. (a) Take $M_n := 1/n^2$ in the Weierstrass M-Test.
(c) Since $|\sin y| \leq |y|$, the series converges for all x. But it is not uniformly convergent on $\mathbb{R}$. If $a > 0$, the series is uniformly convergent for $|x| \leq a$.
(d) If $0 \leq x \leq 1$, the series is divergent. If $1 < x < \infty$, the series is convergent. It is uniformly convergent on $[a, \infty)$ for $a > 1$. However, it is not uniformly convergent on $(1, \infty)$.

4. If $\rho = \infty$, then the sequence $\left(|a_n|^{1/n}\right)$ is not bounded. Hence if $|x_0| > 0$, then there are infinitely many $k \in \mathbb{N}$ with $|a_k|^{1/k} > 1/|x_0|$ so that $\left|a_k x_0^k\right| > 1$. Thus the series is not convergent when $x_0 \neq 0$.

5. Suppose that $L := \lim(|a_n|/|a_{n+1}|)$ exists and that $0 < L < \infty$. It follows from the Ratio Test that $\sum a_n x^n$ converges for $|x| < L$ and diverges for $|x| > L$. The Cauchy-Hadamard Theorem implies that $L = R$.

6. (a) $R = \infty$ (b) $R = \infty$
(c) $R = 1/e$ (d) 1
(e) $R = 4$ (f) $R = 1$.

8. Use $\lim(n^{1/n}) = 1$.

10. By the Uniqueness Theorem 9.4.13, $a_n = (-1)^n a_n$ for all n.

12. If $n \in \mathbb{N}$, there exists a polynomial P_n such that $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$ for $x \neq 0$.

13. Let $g(x) := 0$ for $x \geq 0$ and $g(x) := e^{-1/x^2}$ for $x < 0$. Show that $g^{(n)}(0) = 0$ for all n.

16. Substitute $-y$ for x in Exercise 15 and integrate from $y = 0$ to $y = x$ for $|x| < 1$, which is justified by Theorem 9.4.11.

19. $\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/n!(2n+1)$ for $x \in \mathbb{R}$.

20. Apply Exercise 14 and $\int_0^{\pi/2} (\sin x)^{2n} dx = \dfrac{\pi}{2} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$.

Section 10.1

1. (a) Since $t_i - \delta(t_i) \leq x_{i-1}$ and $x_i \leq t_i + \delta(t_i)$, then $0 \leq x_i - x_{i-1} \leq 2\delta(t_i)$.
(b) Apply (a) to each subinterval.

2. (b) Consider the tagged partition $\{([0, 1], 1), ([1, 2], 1), ([2, 3], 3), ([3, 4], 3)\}$.

3. (a) If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ and if t_k is a tag for both subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, we must have $t_k = x_k$. We replace these two subintervals by the subinterval $[x_{k-1}, x_{k+1}]$ with the tag t_k, keeping the δ-fineness property.
(b) No.
(c) If $t_k \in (x_{k-1}, x_k)$, then we replace $[x_{k-1}, x_k]$ by the two intervals $[x_{k-1}, t_k]$ and $[t_k, x_k]$ both tagged by t_k, keeping the δ-fineness property.

4. If $x_{k-1} \leq 1 \leq x_k$ and if t_k is the tag for $[x_{k-1}, x_k]$, then we cannot have $t_k > 1$, since then $t_k - \delta(t_k) = \frac{1}{2}(t_k + 1) > 1$. Similarly, we cannot have $t_k < 1$, since then $t_k + \delta(t_k) = \frac{1}{2}(t_k + 1) < 1$. Therefore $t_k = 1$.

5. (a) Let $\delta(t) := \frac{1}{2}\min\{|t-1|, |t-2|, |t-3|\}$ if $t \neq 1, 2, 3$ and $\delta(t) := 1$ for $t = 1, 2, 3$.
(b) Let $\delta_2(t) := \min\{\delta(t), \delta_1(t)\}$, where δ is as in part (a).

7. (a) $F_1(x) := (2/3)x^{3/2} + 2x^{1/2}$,
(b) $F_2(x) := (2/3)(1-x)^{3/2} - 2(1-x)^{1/2}$,
(c) $F_3(x) := (2/3)x^{3/2}(\ln x - 2/3)$ for $x \in (0, 1]$ and $F_3(0) := 0$,
(d) $F_4(x) := 2x^{1/2}(\ln x - 2)$ for $x \in (0, 1]$ and $F_4(0) := 0$,
(e) $F_5(x) := -\sqrt{1-x^2} + \operatorname{Arcsin} x$,
(f) $F_6(x) := \operatorname{Arcsin}(x - 1)$.

8. The tagged partition $\dot{\mathcal{P}}_z$ need not be δ_ε-fine, since the value $\delta_\varepsilon(z)$ may be much smaller than $\delta_\varepsilon(x_j)$.

9. If f were integrable, then $\int_0^1 f \geq \int_0^1 s_n = 1/2 + 1/3 + \cdots + 1/(n+1)$.

10. We enumerate the nonzero rational numbers as $r_k = m_k/n_k$ and define $\delta_\varepsilon(m_k/n_k) := \varepsilon/(n_k 2^{k+1})$ and $\delta_\varepsilon(x) := 1$ otherwise.

12. The function M is not continuous on $[-2, 2]$.

13. L_1 is continuous and $L_1'(x) = l_1(x)$ for $x \neq 0$, so Theorem 10.1.9 applies.

15. We have $C_1'(x) = (3/2)x^{1/2}\cos(1/x) + x^{-1/2}\sin(1/x)$ for $x > 0$. Since the first term in C_1' has a continuous extension to $[0, 1]$, it is integrable.

16. We have $C_2'(x) = \cos(1/x) + (1/x)\sin(1/x)$ for $x > 0$. By the analogue of Exercise 7.2.12, the first term belongs to $\mathcal{R}[0, 1]$.

17. (a) Take $\varphi(t) := t^2 + t - 2$ so $E_\varphi = \emptyset$ to get 6.
(b) Take $\varphi(t) := \sqrt{t}$ so $E_\varphi = \{0\}$ to get $2(2 + \ln 3)$.

(c) Take $\varphi(t) := \sqrt{t-1}$ so $E_\varphi = \{1\}$ to get 2 Arctan 2.
(d) Take $\varphi(t) :=$ Arcsin t so $E_\varphi = \{1\}$ to get $\frac{1}{4}\pi$.

19. (a) In fact $f(x) := F'(x) = \cos(\pi/x) + (\pi/x)\sin(\pi/x)$ for $x > 0$. We set $f(0) := 0$, $F'(0) := 0$. Note that f is continuous on $(0, 1]$.
(b) $F(a_k) = 0$ and $F(b_k) = (-1)^k/k$. Apply Theorem 10.1.9.
(c) If $|f| \in \mathcal{R}^*[0, 1]$, then $\sum_{k=1}^{n} 1/k \le \sum_{k=1}^{n} \int_{a_k}^{b_k} |f| \le \int_0^1 |f|$ for all $n \in \mathbb{N}$.

20. Indeed, $\operatorname{sgn}(f(x)) = (-1)^k = m(x)$ on $[a_k, b_k]$ so $m(x)\cdot f(x) = |m(x)f(x)|$ for $x \in [0, 1]$. Since the restrictions of m and $|m|$ to every interval $[c, 1]$ for $0 < c < 1$ are step functions, they belong to $\mathcal{R}[c, 1]$. By Exercise 7.2.11, m and $|m|$ belong to $\mathcal{R}[0, 1]$ and $\int_0^1 m = \sum_{k=1}^{\infty} (-1)^k/k(2k+1)$ and $\int_0^1 |m| = \sum_{k=1}^{\infty} 1/k(2k+1)$.

21. Indeed, $\varphi(x) = \Phi'(x) = |\cos(\pi/x)| + (\pi/x)\sin(\pi/x)\cdot \operatorname{sgn}(\cos(\pi/x))$ for $x \notin E$ by Example 6.1.7(c). Evidently φ is not bounded near 0. If $x \in [a_k, b_k]$, then $\varphi(x) = |\cos(\pi/x)| + (\pi/x)|\sin(\pi/x)|$ so that $\int_{a_k}^{b_k} |\varphi| = \Phi(b_k) - \Phi(a_k) = 1/k$, whence $|\varphi| \notin \mathcal{R}^*[0, 1]$.

22. Here $\psi(x) = \Psi'(x) = 2x|\cos(\pi/x)| + \pi\sin(\pi/x)\cdot\operatorname{sgn}(\cos(\pi/x))$ for $x \notin \{0\} \cup E_1$ by Example 6.1.7(b). Since ψ is bounded, Exercise 7.2.11 applies. We cannot apply Theorem 7.3.1 to evaluate $\int_0^b \psi$ since E is not finite, but Theorem 10.1.9 applies and $\psi \in \mathcal{R}[0, 1]$. Corollary 7.3.15 implies that $|\psi| \in \mathcal{R}[0, 1]$.

23. If $p \ge 0$, then $mp \le fp \le Mp$, where m and M denote the infimum and the supremum of f on $[a, b]$, so that $m\int_a^b p \le \int_a^b fp \le M\int_a^b p$. If $\int_a^b p = 0$, the result is trivial; otherwise, the conclusion follows from Bolzano's Intermediate Value Theorem 5.3.7.

24. By the Multiplication Theorem 10.1.14, $fg \in \mathcal{R}^*[a, b]$. If g is increasing, then $g(a)f \le fg \le g(b)f$ so that $g(a)\int_a^b f \le \int_a^b fg \le g(b)\int_a^b f$. Let $K(x) := g(a)\int_a^x f + g(b)\int_x^b f$, so that K is continuous and takes all values between $K(b)$ and $K(a)$.

Section 10.2

2. (a) If $G(x) := 3x^{1/3}$ for $x \in [0, 1]$ then $\int_c^1 g = G(1) - G(c) \to G(1) = 3$.
(b) We have $\int_c^1 (1/x)dx = \ln c$, which does not have a limit in $\mathbb{R}$ as $c \to 0$.

3. Here $\int_0^c (1-x)^{-1/2}dx = 2 - 2(1-c)^{1/2} \to 2$ as $c \to 1-$.

5. Because of continuity, $g_1 \in \mathcal{R}^*[c, 1]$ for all $c \in (0, 1)$. If $\omega(x) := x^{-1/2}$, then $|g_1(x)| \le \omega(x)$ for all $x \in [0, 1]$. The "left version" of the preceding exercise implies that $g_1 \in \mathcal{R}^*[0, 1]$ and the above inequality and the Comparison Test 10.2.4 imply that $g_1 \in \mathcal{L}[0, 1]$.

6. (a) The function is bounded on [0, 1] (use l'Hospital) and continuous in (0, 1).
(c) If $x \in \left(0, \frac{1}{2}\right]$, the integrand is dominated by $\left|\left(\ln\frac{1}{2}\right)\ln x\right|$. If $x \in \left[\frac{1}{2}, 1\right)$, the integrand is dominated by $\left|\left(\ln\frac{1}{2}\right)\ln(1-x)\right|$.

7. (a) Convergent (b, c) Divergent (d, e) Convergent (f) Divergent.

10. By the Multiplication Theorem 10.1.14, $fg \in \mathcal{R}^*[a, b]$. Since $|f(x)g(x)| \le B|f(x)|$, then $fg \in \mathcal{L}[a, b]$ and $||fg|| \le B||f||$.

11. (a) Let $f(x) := (-1)^k 2^k/k$ for $x \in [c_{k-1}, c_k)$ and $f(1) := 0$, where the c_k are as in Example 10.2.2(a). Then $f^+ := \max\{f, 0\} \notin \mathcal{R}^*[0, 1]$.
(b) Use the first formula in the proof of Theorem 10.2.7.

13. (ii) If $f(x) = g(x)$ for all $x \in [a, b]$, then $\operatorname{dist}(f, g) = \int_a^b |f-g| = 0$.
(iii) $\operatorname{dist}(f, g) = \int_a^b |f-g| = \int_a^b |g-f| = \operatorname{dist}(g, f)$.
(iv) $\operatorname{dist}(f, h) = \int_a^b |f-h| \le \int_a^b |f-g| + \int_a^b |g-h| = \operatorname{dist}(f, g) + \operatorname{dist}(g, h)$.

16. If (f_n) converges to f in $\mathcal{L}[a, b]$, given $\varepsilon > 0$ there exists $K(\varepsilon/2)$ such that if $m, n \ge K(\varepsilon/2)$ then $||f_m - f|| < \varepsilon/2$ and $||f_n - f|| < \varepsilon/2$. Therefore $||f_m - f_n|| \le ||f_m - f|| + ||f - f_n|| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus we may take $H(\varepsilon) := K(\varepsilon/2)$.

18. If $m > n$, then $||g_m - g_n|| \le 1/n + 1/m \to 0$. One can take $g := \text{sgn}$.

19. No.

20. We can take k to be the 0-function.

Section 10.3

1. Let $b \ge \max\{a,\ 1/\delta(\infty)\}$. If $\dot{\mathcal{P}}$ is a δ-fine partition of $[a, b]$, show that $\dot{\mathcal{P}}$ is a δ-fine subpartition of $[a, \infty)$.

3. If $f \in \mathcal{L}[a, \infty)$, apply the preceding exercise to $|f|$. Conversely, if $\int_p^q |f| < \varepsilon$ for $q > p \ge K(\varepsilon)$, then $\left|\int_a^q f - \int_a^p f\right| \le \int_p^q |f| < \varepsilon$ so both $\lim_\gamma \int_a^\gamma f$ and $\lim_\gamma \int_a^\gamma |f|$ exist; therefore $f, |f| \in \mathcal{R}^*[a, \infty)$ and so $f \in \mathcal{L}[a, \infty)$.

5. If $f, g \in \mathcal{L}[a, \infty)$, then $f, |f|, g$, and $|g|$ belong to $\mathcal{R}^*[a, \infty)$, so Example 10.3.3(a) implies that $f + g$ and $|f| + |g|$ belong to $\mathcal{R}^*[a, \infty)$ and that $\int_a^\infty (|f| + |g|) = \int_a^\infty |f| + \int_a^\infty |g|$. Since $|f + g| \le |f| + |g|$, it follows that $\int_a^\gamma |f + g| \le \int_a^\gamma |f| + \int_a^\gamma |g| \le \int_a^\infty |f| + \int_a^\infty |g|$, whence $||f + g|| \le ||f|| + ||g||$.

6. Indeed, $\int_1^\gamma (1/x)\, dx = \ln \gamma$, which does not have a limit as $\gamma \to \infty$. Or, use Exercise 2 and the fact that $\int_p^{2p} (1/x) dx = \ln 2 > 0$ for all $p \ge 1$.

8. If $\gamma > 0$, then $\int_0^\gamma \cos x\, dx = \sin \gamma$, which does not have a limit as $\gamma \to \infty$.

9. (a) We have $\int_0^\gamma e^{-sx} dx = (1/s)(1 - e^{-s\gamma}) \to 1/s$.
   (b) Let $G(x) := -(1/s)e^{-sx}$ for $x \in [0, \infty)$, so G is continuous on $[0, \infty)$ and $G(x) \to 0$ as $x \to \infty$. By the Fundamental Theorem 10.3.5, we have $\int_0^\infty g = -G(0) = 1/s$.

12. (a) If $x \ge e$, then $(\ln x)/x \ge 1/x$.
   (b) Integrate by parts on $[1, \gamma]$ and then let $\gamma \to \infty$.

13. (a) $|\sin x| \ge 1/\sqrt{2} > 1/2$ and $1/x > 1/(n + 1)\pi$ for $x \in (n\pi + \pi/4,\ n\pi + 3\pi/4)$.
   (b) If $\gamma > (n + 1)\pi$, then $\int_0^\gamma |D| \ge (1/4)(1/1 + 1/2 + \cdots + 1/(n + 1))$.

15. Let $u = \varphi(x) = x^2$. Now apply Exercise 14.

16. (a) Convergent (b, c) Divergent (d) Convergent (e) Divergent
   (f) Convergent.

17. (a) If $f_1(x) := \sin x$, then $f_1 \notin \mathcal{R}^*[0, \infty)$. In Exercise 14, take $f_2(x) := x^{-1/2} \sin x$ and $\varphi_2(x) := 1/\sqrt{x}$.
   (c) Take $f(x) := x^{-1/2}\sin x$, and $\varphi(x) := (x + 1)/x$.

18. (a) $f(x) := \sin x$ is in $\mathcal{R}^*[0, \gamma]$, and $F(x) := \int_0^x \sin t\, dt = 1 - \cos x$ is bounded on $[0, \infty)$, and $\varphi(x) := 1/x$ decreases monotonely to 0.
   (c) $F(x) := \int_0^x \cos t\, dt = \sin x$ is bounded on $[0, \infty)$ and $\varphi(x) := x^{-1/2}$ decreases monotonely to 0.

19. Let $u = \varphi(x) := x^2$.

20. (a) If $\gamma > 0$, then $\int_0^\gamma e^{-x} dx = 1 - e^{-\gamma} \to 1$, so $e^{-x} \in \mathcal{R}^*[0, \infty)$. Similarly $e^{-|x|} = e^x \in \mathcal{R}^*(-\infty, 0]$.
   (c) $0 \le e^{-x^2} \le e^{-x}$ for $|x| \ge 1$, so $e^{-x^2} \in \mathcal{R}^*[0, \infty)$. Similarly on $(-\infty, 0]$.

Section 10.4

1. (a) Converges to 0 at $x = 0$, to 1 on (0, 1]. Not uniform. Bounded by 1. Increasing. Limit $= 1$.
   (c) Converges to 1 on [0, 1), to $\frac{1}{2}$ at $x = 1$. Not uniform. Bounded by 1. Increasing. Limit $= 1$.

2. (a) Converges to $\sqrt{x}$ on [0, 1]. Uniform. Bounded by 1. Increasing. Limit $= 2/3$.
   (c) Converges to $\frac{1}{2}$ at $x = 1$, to 0 on (1, 2]. Not uniform. Bounded by 1. Decreasing. Limit $= 0$.

3. (a) Converges to 1 at $x = 0$, to 0 on $(0, 1]$. Not uniform. Bounded by 1. Decreasing. Limit $= 0$.
(c) Converges to 0. Not uniform. Bounded by $1/e$. Not monotone. Limit $= 0$.
(e) Converges to 0. Not uniform. Bounded by $1/\sqrt{2e}$. Not monotone. Limit $= 0$.

4. (a) The Dominated Convergence Theorem applies.
(b) $f_k(x) \to 0$ for $x \in [0, 1)$, but $(f_k(1))$ is not bounded. No obvious dominating function. Integrate by parts and use (a). The result shows that the Dominated Convergence Theorem does not apply.

6. Suppose that $(f_k(c))$ converges for some $c \in [a, b]$. By the Fundamental Theorem, $f_k(x) - f_k(c) = \int_c^x f_k'$. By the Dominated Convergence Theorem, $\int_c^x f_k' \to \int_c^x g$, whence $(f_x(x))$ converges for all $x \in [a, b]$. Note that if $f_k(x) := (-1)^k$, then $(f_k(x))$ does not converge for any $x \in [a, b]$.

7. Indeed, $g(x) := \sup\{f_k(x) : k \in \mathbb{N}\}$ equals $1/k$ on $(k-1, k]$, so that $\int_0^n g = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Hence $g \notin \mathcal{R}^*[0, \infty)$.

10. (a) If $a > 0$, then $|(e^{-tx}\sin x)/x| \le e^{-ax}$ for $t \in J_a := (a, \infty)$. If $t_k \in J_a$ and $t_k \to t_0 \in J_a$, then the argument in 10.4.6(d) shows that E is continuous at t_0. Also, if $t_k \ge 1$, then $|(e^{-t_k x}\sin x)/x| \le e^{-x}$ and the Dominated Convergence Theorem implies that $E(t_k) \to 0$. Thus $E(t) \to 0$ as $t \to \infty$.
(b) It follows as in 10.4.6(e) that $E'(t_0) = -\int_0^\infty e^{-t_0 x}\sin x\,dx = -1/\left(t_0^2 + 1\right)$.
(c) By 10.1.9, $E(s) - E(t) = \int_t^s E'(t)dt = -\int_t^s (t^2+1)^{-1}\,dt = \operatorname{Arctan} t - \operatorname{Arctan} s$ for $s, t > 0$. But $E(s) \to 0$ and $\operatorname{Arctan} s \to \pi/2$ as $s \to \infty$.
(d) We do not know that E is continuous as $t \to 0+$.

12. Fix $x \in I$. As in 10.4.6(e), if $t, t_0 \in [a, b]$, there exists t_x between t, t_0 such that $f(t, x) - f(t_0, x) = (t - t_0)\frac{\partial f}{\partial t}(t_x, x)$. Therefore $\alpha(x) \le [f(t, x) - f(t_0, x)]/(t - t_0) \le \omega(x)$ when $t \ne t_0$. Now argue as before and use the Dominated Convergence Theorem 10.4.5.

13. (a) If (s_k) is a sequence of step functions converging to f a.e., and (t_k) is a sequence of step functions converging to g a.e., Theorem 10.4.9(a) and Exercise 2.2.18 imply that $(\max\{s_k, t_k\})$ is a sequence of step functions that converges to $\max\{f, g\}$ a.e. Similarly, for $\min\{f, g\}$.

14. (a) Since $f_k \in \mathcal{M}[a, b]$ is bounded, it belongs to $\mathcal{R}^*[a, b]$. The Dominated Convergence Theorem implies that $f \in \mathcal{R}^*[a, b]$. The Measurability Theorem 10.4.11 now implies that $f \in \mathcal{M}[a, b]$.
(b) Since $t \mapsto \operatorname{Arctan} t$ is continuous, Theorem 10.4.9(b) implies that $f_k := \operatorname{Arctan} \circ g_k \in \mathcal{M}[a, b]$. Further, $|f_k(x)| \le \frac{1}{2}\pi$ for $x \in [a, b]$.
(c) If $g_k \to g$ a.e., it follows from the continuity of Arctan that $f_k \to f$ a.e. Parts (a, b) imply that $f \in \mathcal{M}[a, b]$ and Theorem 10.4.9(b) applied to $\varphi = \tan$ implies that $g = \tan \circ f \in \mathcal{M}[a, b]$.

15. (a) Since $\mathbf{1}_E$ is bounded, it is in $\mathcal{R}^*[a, b]$ if and only if it is in $\mathcal{M}[a, b]$.
(c) $\mathbf{1}_{E'} = 1 - \mathbf{1}_E$.
(d) $\mathbf{1}_{E \cup F}(x) = \max\{\mathbf{1}_E(x), \mathbf{1}_F(x)\}$ and $\mathbf{1}_{E \cap F}(x) = \min\{\mathbf{1}_E(x), \mathbf{1}_F(x)\}$. Further, $E \backslash F = E \cap F'$.
(e) If (E_k) is an increasing sequence in $\mathbb{M}[a, b]$, then $(\mathbf{1}_{E_k})$ is an increasing sequence in $\mathcal{M}[a, b]$ with $\mathbf{1}_E(x) = \lim \mathbf{1}_{E_k}(x)$, and we can apply Theorem 10.4.9(c). Similarly, $(\mathbf{1}_{F_k})$ is a decreasing sequence in $\mathcal{M}[a, b]$ and $\mathbf{1}_F(x) = \lim \mathbf{1}_{F_k}(x)$.
(f) Let $A_n := \cup_{k=1}^n E_k$, so that (A_n) is an increasing sequence in $\mathbb{M}[a, b]$ with $\cup_{n=1}^\infty A_n = E$, so (e) applies. Similarly, if $B_n := \cap_{k=1}^n F_k$, then (B_n) is a decreasing sequence in $\mathbb{M}[a, b]$ with $\cap_{n=1}^\infty B_n = F$.

16. (a) $m(\emptyset) = \int_a^b 0 = 0$ and $0 \le \mathbf{1}_E \le 1$ implies $0 \le m(E) = \int_a^b \mathbf{1}_E \le b - a$.
(b) Since $\mathbf{1}_{[c, d]}$ is a step function, then $m([c, d]) = d - c$.
(c) Since $\mathbf{1}_{E'} = 1 - \mathbf{1}_E$, we have $m(E') = \int_a^b (1 - \mathbf{1}_E) = (b - a) - m(E)$.
(d) Note that $\mathbf{1}_{E \cup F} + \mathbf{1}_{E \cap F} = \mathbf{1}_E + \mathbf{1}_F$.

(f) If (E_k) is increasing in $\mathbb{M}[a, b]$ to E, then $(\mathbf{1}_{E_k})$ is increasing in $\mathcal{M}[a,b]$ to $\mathbf{1}_E$. The Monotone Convergence Theorem 10.4.4 applies.

(g) If (C_k) is pairwise disjoint and $E_n := \bigcup_{k=1}^{n} C_k$ for $n \in \mathbb{N}$, then $m(E_n) = m(C_1) + \cdots + m(C_n)$. Since $\bigcup_{k=1}^{\infty} C_k = \bigcup_{n=1}^{\infty} E_n$ and (E_n) is increasing, (f) implies that $m\big(\bigcup_{k=1}^{\infty} C_k\big) = \lim_n m(E_n) = \lim_n \sum_{k=1}^{n} m(C_k) = \sum_{n=1}^{\infty} m(C_k)$.

Section 11.1

1. If $|x-u| < \inf\{x,\ 1-x\}$, then $u < x + (1-x) = 1$ and $u > x - x = 0$, so that $0 < u < 1$.
3. Since the union of two open sets is open, then $G_1 \cup \cdots \cup G_k \cup G_{k+1} = (G_1 \cup \cdots \cup G_k) \cup G_{k+1}$ is open.
5. The complement of $\mathbb{N}$ is the union $(-\infty,\ 1) \cup (1, 2) \cup \cdots$ of open intervals.
7. Corollary 2.4.9 implies that every neighborhood of x in $\mathbb{Q}$ contains a point not in $\mathbb{Q}$.
10. x is a boundary point of $A \Longleftrightarrow$ every neighborhood V of x contains points in A and points in $\mathcal{C}(a) \Longleftrightarrow x$ is a boundary point of $\mathcal{C}(a)$.
12. The sets F and $\mathcal{C}(F)$ have the same boundary points. Therefore F contains all of its boundary points $\Longleftrightarrow \mathcal{C}(F)$ does not contain any of its boundary points $\Longleftrightarrow \mathcal{C}(F)$ is open.
13. $x \in A^\circ \Longleftrightarrow x$ belongs to an open set $V \subseteq A \Longleftrightarrow x$ is an interior point of A.
15. Since A^- is the intersection of all closed sets containing A, then by 11.1.5(a) it is a closed set containing A. Since $\mathcal{C}(A^-)$ is open, then $z \in \mathcal{C}(A^-) \Longleftrightarrow z$ has a neighborhood $V_\varepsilon(z)$ in $\mathcal{C}(A^-) \Longleftrightarrow z$ is neither an interior point nor a boundary point of A.
19. If $G \neq \emptyset$ is open and $x \in G$, then there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq G$, whence it follows that $a := x - \varepsilon$ is in A_x.
21. If $a_x < y < x$ then since $a_x := \inf A_x$ there exists $a' \in A_x$ such that $a_x < a' \le y$. Therefore $(y, x] \subseteq (a', x] \subseteq G$ and $y \in G$.
23. If $x \in \mathbb{F}$ and $n \in \mathbb{N}$, the interval I_n in F_n containing x has length $1/3^n$. Let y_n be an endpoint of I_n with $y_n \neq x$. Then $y_n \in \mathbb{F}$ (why?) and $y_n \to x$.
24. As in the preceding exercise, take z_n to be the midpoint of I_n. Then $z_n \notin \mathbb{F}$ (why?) and $z_n \to x$.

Section 11.2

1. Let $G_n := (1 + 1/n,\ 3)$ for $n \in \mathbb{N}$.
3. Let $G_n := (1/2n, 2)$ for $n \in \mathbb{N}$.
5. If $\mathcal{G}_1$ is an open cover of K_1 and $\mathcal{G}_2$ is an open cover of K_2, then $\mathcal{G}_1 \cup \mathcal{G}_2$ is an open cover of $K_1 \cup K_2$.
7. Let $K_n := [0,\ n]$ for $n \in \mathbb{N}$.
10. Since $K \neq \emptyset$ is bounded, it follows that $\inf K$ exists in $\mathbb{R}$. If $K_n := \{k \in K : k \le (\inf K) + 1/n\}$, then K_n is closed and bounded, hence compact. By the preceding exercise $\cap K_n \neq \emptyset$, but if $x_0 \in \cap K_n$, then $x_0 \in K$ and it is readily seen that $x_0 = \inf K$. [Alternatively, use Theorem 11.2.6.]
12. Let $\emptyset \neq K \subseteq \mathbb{R}$ be compact and let $c \in \mathbb{R}$. If $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\sup\{|c-x| : x \in K\} - 1/n < |c - x_n|$. Now apply the Bolzano-Weierstrass Theorem.
15. Let $F_1 := \{n : n \in \mathbb{N}\}$ and $F_2 := \{n + 1/n : n \in \mathbb{N}, n \ge 2\}$.

Section 11.3

1. (a) If $a < b \le 0$, then $f^{-1}(I) = \emptyset$. If $a < 0 < b$, then $f^{-1}(I) = \big(-\sqrt{b}, \sqrt{b}\big)$. If $0 \le a < b$ then $f^{-1}(I) = \big(-\sqrt{b}, -\sqrt{a}\big) \cup \big(\sqrt{a}, \sqrt{b}\big)$.

3. $f^{-1}(G) = f^{-1}([0, \varepsilon)) = [1, 1+\varepsilon^2) = (0,\ 1+\varepsilon^2) \cap I$.
4. Let $G := (1/2,\ 3/2)$. Let $F := [-1/2,\ 1/2]$.
8. Let f be the Dirichlet Discontinuous Function.
9. First note that if $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then we have $x \in f^{-1}(\mathbb{R}\backslash A) \iff f(x) \in \mathbb{R}\backslash A \iff f(x) \notin A \iff x \notin f^{-1}(A) \iff x \in \mathbb{R}\backslash f^{-1}(A)$; therefore, $f^{-1}(\mathbb{R}\backslash A) = \mathbb{R}\backslash f^{-1}(A)$. Now use the fact that a set $F \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R}\backslash F$ is open, together with Corollary 11.3.3.

Section 11.4

1. If $P_i := (x_i, y_i)$ for $i = 1, 2, 3$, then $d_1(P_1,\ P_2) \le (|x_1 - x_3| + |x_3 - x_2|) + (|y_1 - y_3| + |y_3 - y_2|) = d_1(P_1,\ P_3) + d_1(P_3,\ P_2)$. Thus d_1 satisfies the Triangle Inequality.
2. Since $|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| \le d_\infty(f, h) + d_\infty(h, g)$ for all $x \in [0, 1]$, it follows that $d_\infty(f, g) \le d_\infty(f, h) + d_\infty(h, g)$ and d_∞ satisfies the Triangle Inequality.
3. We have $s \ne t$ if and only if $d(s, t) = 1$. If $s \ne t$, the value of $d(s, u) + d(u, t)$ is either 1 or 2 depending on whether u equals s or t, or neither.
4. Since $d_\infty(P_n, P) = \sup\{|x_n - x|, |y_n - y|\}$, if $d_\infty(P_n, P) \to 0$ then it follows that both $|x_n - x| \to 0$ and $|y_n - y| \to 0$, whence $x_n \to x$ and $y_n \to y$. Conversely, if $x_n \to x$ and $y_n \to y$, then $|x_n - x| \to 0$ and $|y_n - y| \to 0$, whence $d_\infty(P_n, P) \to 0$.
6. If a sequence (x_n) in S converges to x relative to the discrete metric d, then $d(x_n, x) \to 0$, which implies that $x_n = x$ for all sufficiently large n. The converse is trivial.
7. Show that a set consisting of a single point is open. Then it follows that every set is an open set, so that every set is also a closed set. (Why?)
10. Let $G \subseteq S_2$ be open in (S_2, d_2) and let $x \in f^{-1}(G)$ so that $f(x) \in G$. Then there exists an ε-neighborhood $V_\varepsilon(f(x)) \subseteq G$. Since f is continuous at x, there exists a δ-neighborhood $V_\delta(x)$ such that $f(V_\delta(x)) \subseteq V_\varepsilon(f(x))$. Since $x \in f^{-1}(G)$ is arbitrary, we conclude that $f^{-1}(G)$ is open in $(S_1,\ d_1)$. The proof of the converse is similar.
11. Let $\mathcal{G} = \{G_\alpha\}$ be a cover of $f(S) \subseteq \mathbb{R}$ by open sets in $\mathbb{R}$. It follows from 11.4.11 that each set $f^{-1}(G_\alpha)$ is open in $(S,\ d)$. Therefore, the collection $\{f^{-1}(G_\alpha)\}$ is an open cover of S. Since $(S,\ d)$ is compact, a finite subcollection $\{f^{-1}(G_{\alpha_1}), \ldots, f^{-1}(G_{\alpha_N})\}$ covers S, whence it follows that the sets $\{G_{\alpha_1}, \ldots, G_{\alpha_N}\}$ must form a finite subcover of $\mathcal{G}$ for $f(S)$. Since $\mathcal{G}$ was an arbitrary open cover of $f(S)$, we conclude that $f(S)$ is compact.

APPENDIX A

LOGIC AND PROOFS

Natural science is concerned with collecting facts and organizing these facts into a coherent body of knowledge so that one can understand nature. Originally much of science was concerned with observation, the collection of information, and its classification. This classification gradually led to the formation of various ''theories'' that helped the investigators to remember the individual facts and to be able to explain and sometimes predict natural phenomena. The ultimate aim of most scientists is to be able to organize their science into a coherent collection of general principles and theories so that these principles will enable them both to understand nature and to make predictions of the outcome of future experiments. Thus they want to be able to develop a system of general principles (or axioms) for their science that will enable them to *deduce* the individual facts and consequences from these general laws.

Mathematics is different from the other sciences; by its very nature, it is a deductive science. That is not to say that mathematicians do not collect facts and make observations concerning their investigations. In fact, many mathematicians spend a large amount of time performing calculations of special instances of the phenomena they are studying in the hopes that they will discover ''unifying principles.'' (The great Gauss did a vast amount of calculation and studied much numerical data before he was able to formulate a conjecture concerning the distribution of prime numbers.) However, even after these principles and conjectures are formulated, the work is far from over, for mathematicians are not satisfied until conjectures have been derived (i.e., proved) from the axioms of mathematics, from the definitions of the terms, and from results (or theorems) that have previously been proved. Thus, a mathematical statement is not a theorem until it has been carefully derived from axioms, definitions, and previously proved theorems.

A few words about the axioms (i.e., postulates, assumptions, etc.) of mathematics are in order. There are a few axioms that apply to all of mathematics—the ''axioms of set theory''—and there are specific axioms within different areas of mathematics. Sometimes these axioms are stated formally, and sometimes they are built into definitions. For example, we list properties in Chapter 2 that we assume the real number system possesses; they are really a set of axioms. As another example, the definition of a ''group'' in abstract algebra is basically a set of axioms that we assume a set of elements to possess, and the study of group theory is an investigation of the consequences of these axioms.

Students studying real analysis for the first time usually do not have much experience in understanding (not to mention constructing) proofs. In fact, one of the main purposes of this course (and this book) is to help the reader gain experience in the type of critical thought that is used in this deductive process. The purpose of this appendix is to help the reader gain insight about the techniques of proof.

Statements and Their Combinations

All mathematical proofs and arguments are based on **statements**, which are declarative sentences or meaningful strings of symbols that can be classified as being true or false. It is not necessary that we know whether a given statement is actually true or false, but it must be one or the other, and it cannot be both. (This is the *Principle of the Excluded Middle*.) For example, the sentence "Chickens are pretty" is a matter of opinion and not a statement in the sense of logic. Consider the following sentences:

- It rained in Kuala Lumpur on June 2, 1988.
- Thomas Jefferson was shorter than John Adams.
- There are infinitely many twin primes.
- This sentence is false.

The first three are statements: the first is true, the second is false, and the third is either true or false, but we are not sure which at this time. The fourth sentence is not a statement; it can be neither true nor false since it leads to contradictory conclusions.

Some statements (such as "$1 + 1 = 2$") are always true; they are called **tautologies**. Some statements (such as "$2 = 3$") are always false; they are called **contradictions** or **falsities**. Some statements (such as "$x^2 = 1$") are sometimes true and sometimes false (e.g., true when $x = 1$ and false when $x = 3$). Or course, for the statement to be completely clear, it is necessary that the proper context has been established and the meaning of the symbols has been properly defined (e.g., we need to know that we are referring to integer arithmetic in the preceding examples).

Two statements P and Q are said to be **logically equivalent** if P is true exactly when Q is true (and hence P is false exactly when Q is false). In this case we often write $P \equiv Q$. For example, we write

$$(x \text{ is Abraham Lincoln}) \equiv (x \text{ is the 16th president of the United States}).$$

There are several different ways of forming new statements from given ones by using logical connectives.

If P is a statement, then its **negation** is the statement denoted by

$$\text{not } P,$$

which is true when P is false, and is false when P is true. (A common notation for the negation of P is $\neg P$.) A little thought shows that

$$P \equiv \text{not}(\text{not } P).$$

This is the *Principle of Double Negation*.

If P and Q are statements, then their **conjunction** is the statement denoted by

$$P \text{ and } Q,$$

which is true when both P and Q are true, and is false otherwise. (A standard notation for the conjunction of P and Q is $P \wedge Q$.) It is evident that

$$(P \text{ and } Q) \equiv (Q \text{ and } P).$$

Similarly, the **disjunction** of P and Q is the statement denoted by

$$P \text{ or } Q$$

which is true when at least one of P and Q is true, and false only when they are both false. In legal documents "or" is often denoted by "and/or" to make it clear that this disjunction is also true when *both* P and Q are true. (A standard notation for the disjunction of P and Q is $P \vee Q$.) It is also evident that

$$(P \text{ or } Q) \equiv (Q \text{ or } P).$$

To contrast disjunctive and conjunctive statements, note that the statement "$2 < \sqrt{2}$ and $\sqrt{2} < 3$" is false, but the statement "$2 < \sqrt{2}$ or $\sqrt{2} < 3$" is true (since $\sqrt{2}$ is approximately equal to 1.4142 . . .).

Some thought shows that negation, conjunction, and disjunction are related by *De Morgan's Laws:*

$$\text{not } (P \text{ and } Q) \equiv (\text{not } P) \text{ or } (\text{not } Q),$$
$$\text{not } (P \text{ or } Q) \equiv (\text{not } P) \text{ and } (\text{not } Q).$$

The first of these equivalencies can be illustrated by considering the statements

$$P : x = 2, \quad Q : y \in A.$$

The statement (P and Q) is true when both ($x = 2$) and ($y \in A$) are true, and it is false when at least one of ($x = 2$) and ($y \in A$) is false; that is, the statement not(P and Q) is true when at least one of the statements ($x \neq 2$) and ($y \notin A$) holds.

Implications

A very important way of forming a new statement from given ones is the **implication** (or **conditional**) statement, denoted by

$$(P \Rightarrow Q), \qquad (\text{if } P \text{ then } Q), \qquad \text{or} \qquad (P \text{ implies } Q).$$

Here P is called the **hypothesis,** and Q is called the **conclusion** of the implication. To help understand the truth values of the implication, consider the statement

If I win the lottery today, then I'll buy Sam a car.

Clearly this statement is false if I win the lottery and don't buy Sam a car. What if I don't win the lottery today? Under this circumstance, I haven't made any promise about buying anyone a car, and since the condition of winning the lottery did not materialize, my failing to buy Sam a car should not be considered as breaking a promise. Thus the implication is regarded as true when the hypothesis is not satisfied.

In mathematical arguments, we are very much interested in implications when the hypothesis is true, but not much interested in them when the hypothesis is false. The accepted procedure is to take the statement $P \Rightarrow Q$ to be false only when P is true and Q is false; in all other cases the statement $P \Rightarrow Q$ is true. (Consequently, if P is false, then we agree to take the statement $P \Rightarrow Q$ to be true whether or not Q is true or false. That may seem strange to the reader, but it turns out to be convenient in practice and consistent with the other rules of logic.)

We observe that the definition of $P \Rightarrow Q$ is logically equivalent to

$$\text{not } (P \text{ and } (\text{not } Q)),$$

because this statement is false only when P is true and Q is false, and it is true in all other cases. It also follows from the first De Morgan Law and the Principle of Double Negation that $P \Rightarrow Q$ is logically equivalent to the statement

$$(\text{not } P) \text{ or } Q,$$

since this statement is true unless both (not P) and Q are false; that is, unless P is true and Q is false.

Contraposive and Converse

As an exercise, the reader should show that the implication $P \Rightarrow Q$ is logically equivalent to the implication

$$(\text{not } Q) \Rightarrow (\text{not } P),$$

which is called the **contrapositive** of the implication $P \Rightarrow Q$. For example, if $P \Rightarrow Q$ is the implication

$$\text{If I am in Chicago, then I am in Illinois,}$$

then the contrapositive (not Q) $\Rightarrow$ (not P) is the implication

$$\text{If I am not in Illinois, then I am not in Chicago.}$$

The equivalence of these two statements is apparent after a bit of thought. In attempting to establish an implication, it is sometimes easier to establish the contrapositive, which is logically equivalent to it. (This will be discussed in more detail later.)

If an implication $P \Rightarrow Q$ is given, then one can also form the statement

$$Q \Rightarrow P,$$

which is called the **converse** of $P \Rightarrow Q$. The reader must guard against confusing the converse of an implication with its contrapositive, since they are quite different statements. While the contrapositive is logically equivalent to the given implication, the converse is not. For example, the converse of the statement

$$\text{If I am in Chicago, then I am in Illinois,}$$

is the statement

$$\text{If I am in Illinois, then I am in Chicago.}$$

Since it is possible to be in Illinois but not in Chicago, these two statements are evidently *not* logically equivalent.

There is one final way of forming statements that we will mention. It is the **double implication** (or the **biconditional**) statement, which is denoted by

$$P \Longleftrightarrow Q \qquad \text{or} \qquad P \text{ if and only if } Q,$$

and which is defined by

$$(P \Rightarrow Q) \text{ and } (Q \Rightarrow P).$$

It is a straightforward exercise to show that $P \Longleftrightarrow Q$ is true precisely when P and Q are both true, or both false.

Context and Quantifiers

In any form of communication, it is important that the individuals have an appropriate context in mind. Statements such as "I saw Mary today" may not be particularly informative if the hearer knows several persons named Mary. Similarly, if one goes into the middle of a mathematical lecture and sees the equation $x^2 = 1$ on the blackboard, it is useful for the viewer to know what the writer means by the letter x and the symbol 1. Is x an integer? A function? A matrix? A subgroup of a given group? Does 1 denote a natural number? The identity function? The identity matrix? The trivial subgroup of a group?

Often the context is well understood by the conversants, but it is always a good idea to establish it at the start of a discussion. For example, many mathematical statements involve one or more variables whose values usually affect the truth or the falsity of the statement, so we should always make clear what the possible values of the variables are.

Very often mathematical statements involve expressions such as "for all," "for every," "for some," "there exists," "there are," and so on. For example, we may have the statements

$$\text{For any integer } x,\ x^2 = 1$$

and

$$\text{There exists an integer } x \text{ such that } x^2 = 1.$$

Clearly the first statement is false, as is seen by taking $x = 3$; however, the second statement is true since we can take either $x = 1$ or $x = -1$.

If the context has been established that we are talking about integers, then the above statements can safely be abbreviated as

$$\text{For any } x,\ x^2 = 1$$

and

$$\text{There exists an } x \text{ such that } x^2 = 1.$$

The first statement involves the **universal quantifier** "for every," and is making a statement (here false) about *all* integers. The second statement involves the **existential quantifier** "there exists," and is making a statement (here true) about *at least one* integer.

These two quantifiers occur so often that mathematicians often use the symbol $\forall$ to stand for the universal quantifier, and the symbol $\exists$ to stand for the existential quantifier. That is,

$$\begin{array}{ll} \forall \ \text{denotes} & \text{``for every,''} \\ \exists \ \text{denotes} & \text{``there exists.''} \end{array}$$

While we do not use these symbols in this book, it is important for the reader to know how to read formulas in which they appear. For example, the statement

(i) $$(\forall x)(\exists y)(x + y = 0)$$

(understood for integers) can be read

For every integer x, there exists
an integer y such that $x + y = 0$.

Similarly, the statement

(ii) $$(\exists y)(\forall x)(x + y = 0)$$

can be read

There exists an integer y, such that
for every integer x, then $x + y = 0$.

These two statements are very different; for example, the first one is true and the second one is false. The moral is that the *order* of the appearance of the two different types of quantifiers is *very important*. It must also be stressed that if several variables appear in a mathematical expression with quantifiers, the values of the later variables should be assumed to depend on all of the values of the variables that are mentioned earlier. Thus in the (true) statement (i) above, the value of y depends on that of x; here if $x = 2$, then $y = -2$, while if $x = 3$, then $y = -3$.

It is important that the reader understand how to negate a statement that involves quantifiers. In principle, the method is simple.

(a) To show that it is false that every element x in some set possesses a certain property $\mathcal{P}$, it is enough to produce a single **counter-example** (that is, a particular element in the set that does not possess this property); and

(b) To show that it is false that there exists an element y in some set that satisfies a certain property $\mathcal{P}$, we need to show that every element y in the set fails to have that property.

Therefore, in the process of forming a negation,

$$\text{not } (\forall x)\mathcal{P} \quad \text{becomes} \quad (\exists\, x) \text{ not } \mathcal{P}$$

and similarly,

$$\text{not } (\exists\, y)\mathcal{P} \quad \text{becomes} \quad (\forall y) \text{ not } \mathcal{P}.$$

When several quantifiers are involved, these changes are repeatedly used. Thus the negation of the (true) statement (i) given previously becomes in succession

$$\begin{aligned}
&\text{not } (\forall\, x)(\exists\, y)(x + y = 0),\\
&(\exists\, x) \text{ not } (\exists\, y)(x + y = 0),\\
&(\exists\, x)(\forall\, y) \text{ not } (x + y = 0),\\
&(\exists\, x)(\forall\, y)(x + y \neq 0).
\end{aligned}$$

The last statement can be rendered in words as:

There exists an integer x, such that
for every integer y, then $x + y \neq 0$.

(This statement is, of course, false.)

Similarly, the negation of the (false) statement (ii) given previously becomes in succession

$$\begin{aligned}
&\text{not } (\exists\, y)(\forall\, x)(x + y = 0),\\
&(\forall\, y) \text{ not } (\forall\, x)(x + y = 0),\\
&(\forall\, y)(\exists\, x) \text{ not } (x + y = 0),\\
&(\forall\, y)(\exists\, x)(x + y \neq 0).
\end{aligned}$$

The last statement is rendered in words as

For every integer y, there exists
an integer x such that $x + y \neq 0$.

Note that this statement is true, and that the value (or values) of x that make $x + y \neq 0$ depends on y, in general.

Similarly, the statement

For every $\delta > 0$, the interval $(-\delta,\, \delta)$
contains a point belonging to the set A,

can be seen to have the negation

There exists $\delta > 0$ such that the interval
$(-\delta,\, \delta)$ does not contain any point in A.

The first statement can be symbolized

$$(\forall\, \delta > 0)(\exists\, y \in A)(y \in (-\delta,\, \delta)),$$

and its negation can be symbolized by

$$(\exists\, \delta > 0)(\forall\, y \in A)(y \notin (-\delta,\, \delta)),$$

or by

$$(\exists\, \delta > 0)(A \cap (-\delta,\, \delta) = \emptyset).$$

It is the strong opinion of the authors that, while the use of this type of symbolism is often convenient, it is *not* a substitute for thought. Indeed, the readers should ordinarily reason for themselves what the negation of a statement is and not rely slavishly on symbolism. While good notation and symbolism can often be a useful aid to thought, it can never be an adequate replacement for thought and understanding.

Direct Proofs

Let P and Q be statements. The assertion that the hypothesis P of the implication $P \Rightarrow Q$ implies the conclusion Q (or that $P \Rightarrow Q$ is a theorem) is the assertion that whenever the hypothesis P is true, then Q is true.

The construction of a **direct proof** of $P \Rightarrow Q$ involves the construction of a string of statements $R_1, R_2, \ldots, R_n$ such that

$$P \Rightarrow R_1, \quad R_1 \Rightarrow R_2, \quad \ldots, \quad R_n \Rightarrow Q.$$

(The *Law of the Syllogism* states that if $R_1 \Rightarrow R_2$ and $R_2 \Rightarrow R_3$ are true, then $R_1 \Rightarrow R_3$ is true.) This construction is usually not an easy task; it may take insight, intuition, and considerable effort. Often it also requires experience and luck.

In constructing a direct proof, one often works forward from P and backward from Q. We are interested in logical consequences of P; that is, statements $Q_1, \cdots, Q_k$ such that $P \Rightarrow Q_i$. And we might also examine statements $P_1, \ldots, P_r$ such that $P_j \Rightarrow Q$. If we can work forward from P and backward from Q so the string "connects" somewhere in the middle, then we have a proof. Often in the process of trying to establish $P \Rightarrow Q$ one finds that one must strengthen the hypothesis (i.e., add assumptions to P) or weaken the conclusion (that is, replace Q by a nonequivalent consequence of Q).

Most students are familiar with "direct" proofs of the type described above, but we will give one elementary example here. Let us prove the following theorem.

A.1 Theorem *The square of an odd integer is also an odd integer.*

If we let n stand for an integer, then the hypothesis is:

$$P : n \text{ is an odd integer.}$$

The conclusion of the theorem is:

$$Q : n^2 \text{ is an odd integer.}$$

We need the definition of odd integer, so we introduce the statement

$$R_1 : n = 2k - 1 \text{ for some integer } k.$$

Then we have $P \Rightarrow R_1$. We want to deduce the statement $n^2 = 2m - 1$ for some integer m, since this would imply Q. We can obtain this statement by using algebra:

$$\begin{aligned} R_2 &: n^2 = (2k-1)^2 = 4k^2 - 4k + 1, \\ R_3 &: n^2 = \left(4k^2 - 4k + 2\right) - 1, \\ R_4 &: n^2 = 2\left(2k^2 - 2k + 1\right) - 1. \end{aligned}$$

If we let $m = 2k^2 - 2k + 1$, then m is an integer (why?), and we have deduced the statement

$$R_5 : n^2 = 2m - 1.$$

Thus we have $P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow R_3 \Rightarrow R_4 \Rightarrow R_5 \Rightarrow Q$, and the theorem is proved.

Of course, this is a clumsy way to present a proof. Normally, the formal logic is suppressed and the argument is given in a more conversational style with complete English sentences. We can rewrite the preceding proof as follows.

Proof of A.1 Theorem. If n is an odd integer, then $n = 2k - 1$ for some integer k. Then the square of n is given by $n^2 = 4k^2 - 4k + 1 = 2\left(2k^2 - 2k + 1\right) - 1$. If we let $m = 2k^2 - 2k + 1$, then m is an integer (why?) and $n^2 = 2m - 1$. Therefore, n^2 is an odd integer. Q.E.D.

At this stage, we see that we may want to make a preliminary argument to prove that $2k^2 - 2k + 1$ is an integer whenever k is an integer. In this case, we could state and prove this fact as a **Lemma**, which is ordinarily a preliminary result that is needed to prove a theorem, but has little interest by itself.

Incidentally, the letters Q.E.D. stand for *quod erat demonstrandum*, which is Latin for "which was to be demonstrated."

Indirect Proofs

There are basically two types of indirect proofs: (i) contrapositive proofs, and (ii) proofs by contradiction. Both types start with the assumption that the conclusion Q is false, in other words, that the statement "not Q" is true.

(i) **Contrapositive proofs.** Instead of proving $P \Rightarrow Q$, we may prove its logically equivalent contrapositive: not $Q \Rightarrow$ not P.

Consider the following theorem.

A.2 Theorem *If n is an integer and n^2 is even, then n is even.*

The negation of "$Q : n$ is even" is the statement "not $Q : n$ is odd." The hypothesis "$P : n^2$ is even" has a similar negation, so that the contrapositive is the implication: If n is odd, then n^2 is odd. But this is exactly Theorem A.1, which was proved above. Therefore this provides a proof of Theorem A.2.

The contrapositive proof is often convenient when the universal quantifier is involved, for the contrapositive form will then involve the existential quantifier. The following theorem is an example of this situation.

A.3 Theorem *Let $a \geq 0$ be a real number. If, for every $\varepsilon > 0$, we have $0 \leq a < \varepsilon$, then $a = 0$.*

Proof. If $a = 0$ is false, then since $a \geq 0$, we must have $a > 0$. In this case, if we choose $\varepsilon_0 = \frac{1}{2}a$, then we have $\varepsilon_0 > 0$ and $\varepsilon_0 < a$, so that the hypothesis $0 \leq a < \varepsilon$ for all $\varepsilon > 0$ is false. Q.E.D.

Here is one more example of a contrapositive proof.

A.4 Theorem *If m, n are natural numbers such that $m + n \geq 20$, then either $m \geq 10$ or $n \geq 10$.*

Proof. If the conclusion is false, then we have both $m < 10$ and $n < 10$. (Recall De Morgan's Law.) Then addition gives us $m + n < 10 + 10 = 20$, so that the hypothesis is false. Q.E.D.

Proof by contradiction. This method of proof employs the fact that if C is a contradiction (i.e., a statement that is always false, such as "$1 = 0$"), then the two statements

$$(P \text{ and } (\text{not } Q)) \Rightarrow C, \qquad P \Rightarrow Q$$

are logically equivalent. Thus we establish $P \Rightarrow Q$ by showing that the statement (P and (not Q)) implies a contradiction. Q.E.D.

A.5 Theorem *Let $a > 0$ be a real number. If $a > 0$, then $1/a > 0$.*

Proof. We suppose that the statement $a > 0$ is true and that the statement $1/a > 0$ is false. Therefore, $1/a \leq 0$. But since $a > 0$ is true, it follows from the order properties of $\mathbb{R}$ that $a(1/a) \leq 0$. Since $1 = a(1/a)$, we deduce that $1 \leq 0$. However, this conclusion contradicts the known result that $1 > 0$. Q.E.D.

There are several classic proofs by contradiction (also known as *reductio ad absurdum*) in the mathematical literature. One is the proof that there is no rational number r that satisfies $r^2 = 2$. (This is Theorem 2.1.4 in the text.) Another is the proof of the infinitude of primes, found in Euclid's *Elements.* Recall that a natural number p is prime if its only integer divisors are 1 and p itself. We will assume the basic results that each prime number is greater than 1 and each natural number greater than 1 is either prime or divisible by a prime.

A.6 Theorem (Euclid's *Elements*, Book IX, Proposition 20.) *There are infinitely many prime numbers.*

Proof. If we suppose by way of contradiction that there are finitely many prime numbers, then we may assume that $S = \{p_1, \ldots, p_n\}$ is the set of *all* prime numbers. We let $m = p_1 \cdots p_n$, the product of all the primes, and we let $q = m + 1$. Since $q > p_i$ for all i, we see that q is not in S, and therefore q is not prime. Then there exists a prime p that is a divisor of q. Since p is prime, then $p = p_j$ for some j, so that p is a divisor of m. But if p divides both m and $q = m + 1$, then p divides the difference $q - m = 1$. However, this is impossible, so we have obtained a contradiction. Q.E.D.

APPENDIX B

FINITE AND COUNTABLE SETS

We will establish the results that were stated in Section 1.3 without proof. The reader should refer to that section for the definitions.

The first result is sometimes called the "Pigeonhole Principle." It may be interpreted as saying that if m pigeons are put into n pigeonholes and if $m > n$, then at least two pigeons must share one of the pigeonholes. This is a frequently used result in combinatorial analysis. It yields many useful consequences.

B.1 Theorem *Let $m, n \in \mathbb{N}$ with $m > n$. Then there does not exist an injection from $\mathbb{N}_m$ into $\mathbb{N}_n$.*

Proof. We will prove this by induction on n.

If $n = 1$ and if g is any map of $\mathbb{N}_m (m > 1)$ into $\mathbb{N}_1$, then it is clear that $g(1) = \cdots = g(m) = 1$, so that g is not injective.

Assume that $k > 1$ is such that if $m > k$, there is no injection from $\mathbb{N}_m$ into $\mathbb{N}_k$. We will show that if $m > k + 1$, there is no function $h : \mathbb{N}_m \to \mathbb{N}_{k+1}$ that is an injection.

Case 1: If the range $h(\mathbb{N}_m) \subseteq \mathbb{N}_k \subset \mathbb{N}_{k+1}$, then the induction hypothesis implies that h is not an injection of $\mathbb{N}_m$ into $\mathbb{N}_k$, and therefore into $\mathbb{N}_{k+1}$.

Case 2: Suppose that $h(\mathbb{N}_m)$ is not contained in $\mathbb{N}_k$. If more than one element in $\mathbb{N}_m$ is mapped into $k + 1$, then h is not an injection. Therefore, we may assume that a single $p \in \mathbb{N}_m$ is mapped into $k + 1$ by h. We now define $h_1 : \mathbb{N}_{m-1} \to \mathbb{N}_k$ by

$$h_1(q) := \begin{cases} h(q) & \text{if } q = 1, \ldots, p-1, \\ h(q+1) & \text{if } q = p, \ldots, m-1. \end{cases}$$

Since the induction hypothesis implies that h_1 is not an injection into $\mathbb{N}_k$, it is easily seen that h is not an injection into $\mathbb{N}_{k+1}$. Q.E.D.

We now show that a finite set determines a unique number in $\mathbb{N}$.

1.3.2 Uniqueness Theorem *If S is a finite set, then the number of elements in S is a unique number in $\mathbb{N}$.*

Proof. If the set S has m elements, there exists a bijection f_1 of $\mathbb{N}_m$ onto S. If S also has n elements, there exists a bijection f_2 of $\mathbb{N}_n$ onto S. If $m > n$, then (by Exercise 21 of Section 1.1) $f_2^{-1} \circ f_1$, is a bijection of $\mathbb{N}_m$ onto $\mathbb{N}_n$, which contradicts Theorem B.1. If $n > m$, then $f_1^{-1} \circ f_2$ is a bijection of $\mathbb{N}_n$ onto $\mathbb{N}_m$, which contradicts Theorem B.1. Therefore we have $m = n$. Q.E.D.

B.2 Theorem *If $n \in \mathbb{N}$, there does not exist an injection from $\mathbb{N}$ into $\mathbb{N}_n$.*

Proof. Assume that $f : \mathbb{N} \to \mathbb{N}_n$ is an injection, and let $m := n + 1$. Then the restriction of f to $\mathbb{N}_m \subset \mathbb{N}$ is also an injection into $\mathbb{N}_n$. But this contradicts Theorem B.1. Q.E.D.

1.3.3 Theorem *The set $\mathbb{N}$ of natural numbers is an infinite set.*

Proof. If $\mathbb{N}$ is a finite set, there exists some $n \in \mathbb{N}$ and a bijection f of $\mathbb{N}_n$ onto $\mathbb{N}$. In this case the inverse function f^{-1} is a bijection (and hence an injection) of $\mathbb{N}$ onto $\mathbb{N}_n$. But this contradicts Theorem B.2. Q.E.D.

We will next establish Theorem 1.3.8. In connection with the array displayed in Figure 1.3.1, the function h was defined by $h(m,n) = \psi(m+n-2) + m$, where $\psi(k) = 1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$. We now prove that the function h is a bijection.

1.3.8 Theorem *The set $\mathbb{N} \times \mathbb{N}$ is denumerable.*

Proof. We will show that the function h is a bijection.

(a) We first show that h is injective. If $(m,n) \neq (m',n')$, then either (i) $m+n \neq m'+n'$, or (ii) $m+n = m'+n'$ and $m \neq m'$.

In case (i), we may suppose $m+n < m'+n'$. Then, using formula (1), the fact that ψ is increasing, and $m' > 0$, we have

$$\begin{aligned} h(m,n) &= \psi(m+n-2) + m \leq \psi(m+n-2) + (m+n-1) \\ &= \psi(m+n-1) \leq \psi(m'+n'-2) \\ &< \psi(m'+n'-2) + m' = h(m',n'). \end{aligned}$$

In case (ii), if $m+n = m'+n'$ and $m \neq m'$, then

$$h(m,n) - m = \psi(m+n-2) = \psi(m'+n'-2) = h(m',n') - m',$$

whence $h(m,n) \neq h(m',n')$.

(b) Next we show that h is surjective.

Clearly $h(1,1) = 1$. If $p \in \mathbb{N}$ with $p \geq 2$, we will find a pair $(m_p, n_p) \in \mathbb{N} \times \mathbb{N}$ with $h(m_p, n_p) = p$. Since $p < \psi(p)$, then the set $E_p := \{k \in \mathbb{N} : p \leq \psi(k)\}$ is nonempty.

Using the Well-Ordering Property 1.2.1, we let $k_p > 1$ be the least element in E_p. (This means that p lies in the k_pth diagonal.) Since $p \geq 2$, it follows from equation (1) that

$$\psi(k_p - 1) < p \leq \psi(k_p) = \psi(k_p - 1) + k_p.$$

Let $m_p := p - \psi(k_p - 1)$ so that $1 \leq m_p \leq k_p$, and let $n_p := k_p - m_p + 1$ so that $1 \leq n_p \leq k_p$ and $m_p + n_p - 1 = k_n$. Therefore,

$$h(m_p, n_p) = \psi(m_p + n_p - 2) + m_p = \psi(k_p - 1) + m_p = p.$$

Thus h is a bijection and $\mathbb{N} \times \mathbb{N}$ is denumerable. Q.E.D.

The next result is crucial in proving Theorems 1.3.9 and 1.3.10.

B.3 Theorem *If $A \subseteq \mathbb{N}$ and A is infinite, there exists a function $\varphi : \mathbb{N} \to A$ such that $\varphi(n+1) > \varphi(n) \geq n$ for all $n \in \mathbb{N}$. Moreover, φ is a bijection of $\mathbb{N}$ onto A.*

Proof. Since A is infinite, it is not empty. We will use the Well-Ordering Property 1.2.1 of $\mathbb{N}$ to give a recursive definition of φ.

Since $A \neq 0$, there is a least element of A, which we define to be $\varphi(1)$; therefore, $\varphi(1) \geq 1$.

Since A is infinite, the set $A_1 := A \backslash \{\varphi(1)\}$ is not empty, and we define $\varphi(2)$ to be least element of A_1. Therefore $\varphi(2) > \varphi(1) \geq 1$, so that $\varphi(2) \geq 2$.

Suppose that φ has been defined to satisfy $\varphi(n+1) > \varphi(n) \geq n$ for $n = 1, \ldots, k-1$, whence $\varphi(k) > \varphi(k-1) \geq k-1$ so that $\varphi(k) \geq k$. Since the set A is infinite, the set

$$A_k := A \backslash \{\varphi(1), \ldots, \varphi(k)\}$$

is not empty and we define $\varphi(k+1)$ to be the least element in A_k. Therefore $\varphi(k+1) > \varphi(k)$, and since $\varphi(k) \geq k$, we also have $\varphi(k+1) \geq k+1$. Therefore, φ is defined on all of $\mathbb{N}$.

We claim that φ is an injection. If $m > n$, then $m = n + r$ for some $r \in \mathbb{N}$. If $r = 1$, then $\varphi(m) = \varphi(n+1) > \varphi(n)$. Suppose that $\varphi(n+k) > \varphi(n)$; we will show that $\varphi(n+(k+1)) > \varphi(n)$. Indeed, this follows from the fact that $\varphi(n+(k+1)) = \varphi((n+k)+1) > \varphi(n+k) > \varphi(n)$. Since $\varphi(m) > \varphi(n)$ whenever $m > n$, it follows that φ is an injection.

We claim that φ is a surjection of $\mathbb{N}$ onto A. If not, the set $\tilde{A} := A \backslash \varphi(\mathbb{N})$ is not empty, and we let p be the least element in $\tilde{A}$. We claim that p belongs to the set $\{\varphi(1), \ldots, \varphi(p)\}$. Indeed, if this is not true, then

$$p \in A \backslash \{\varphi(1), \ldots, \varphi(p)\} = A_p,$$

so that $\varphi(p+1)$, being the least element in A_p, must satisfy $\varphi(p+1) \leq p$. But this contradicts the fact that $\varphi(p+1) > \varphi(p) \geq p$. Therefore $\tilde{A}$ is empty and φ is a surjection onto A. Q.E.D.

B.4 Theorem *If $A \subseteq \mathbb{N}$, then A is countable.*

Proof. If A is finite, then it is countable, so it suffices to consider the case that A is infinite. In this case, Theorem B.3 implies that there exists a bijection φ of $\mathbb{N}$ onto A, so that A is denumerable and, therefore, countable. Q.E.D.

1.3.9 Theorem *Suppose that S and T are sets and that $T \subseteq S$.*

(a) *If S is a countable set, then T is a countable set.*

(b) *If T is an uncountable set, then S is an uncountable set.*

Proof. (a) If S is a finite set, it follows from Theorem 1.3.5(a) that T is finite, and therefore countable. If S is denumerable, then there exists a bijection ψ of S onto $\mathbb{N}$. Since $\psi(S) \subseteq \mathbb{N}$, Theorem B.4 implies that $\psi(S)$ is countable. Since the restriction of ψ to T is a bijection onto $\psi(T)$ and $\psi(T) \subseteq \mathbb{N}$ is countable, it follows that T is also countable.

(b) This assertion is the contrapositive of the assertion in (a). Q.E.D.

APPENDIX C

THE RIEMANN AND LEBESGUE CRITERIA

We will give here proofs of the Riemann and Lebesgue Criteria for a function to be Riemann integrable. First we will give the Riemann Criterion, which is interesting in itself, and also leads to the more incisive Lebesgue Criterion.

C.1 Riemann Integrability Criterion *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the following assertions are equivalent:*

(a) $f \in \mathcal{R}[a, b]$.

(b) *For every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ such that if $\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2$ are any tagged partitions having the same subintervals as $\mathcal{P}_\varepsilon$, then*

$$\left|S(f; \dot{\mathcal{P}}_1) - S(f; \dot{\mathcal{P}}_2)\right| < \varepsilon. \tag{1}$$

(c) *For every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon = \{I_i\}_{i=1}^n = \{[x_{i-1}, x_i]\}_{i=1}^n$ such that if $m_i := \inf\{f(x) : x \in I_i\}$ and $M_i := \sup\{f(x) : x \in I_i\}$ then*

$$\sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < 2\varepsilon. \tag{2}$$

Proof. (a) $\Rightarrow$ (b) Given $\varepsilon > 0$, let $\eta_\varepsilon > 0$ be as in the Cauchy Criterion 7.2.1, and let $\mathcal{P}_\varepsilon$ be any partition with $||\mathcal{P}_\varepsilon|| < \eta_\varepsilon$. Then if $\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2$ are any tagged partitions with the same subintervals as $\mathcal{P}_\varepsilon$, then $||\dot{\mathcal{P}}_1|| < \eta_\varepsilon$ and $||\dot{\mathcal{P}}_2|| < \eta_\varepsilon$ and so (1) holds.

(b) $\Rightarrow$ (c) Given $\varepsilon > 0$, let $\mathcal{P}_\varepsilon = \{I_i\}_{i=1}^n$ be a partition as in (b) and let m_i and M_i be as in the statement of (c). Since m_i is an infimum and M_i is a supremum, there exist points u_i and v_i in I_i with

$$f(u_i) < m_i + \frac{\varepsilon}{2(b-a)} \qquad \text{and} \qquad M_i - \frac{\varepsilon}{2(b-a)} < f(v_i),$$

so that we have

$$M_i - m_i < f(v_i) - f(u_i) + \frac{\varepsilon}{(b-a)} \qquad \text{for} \quad i = 1, \ldots, n.$$

If we multiply these inequalities by $(x_i - x_{i-1})$ and sum, we obtain

$$\sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} (f(v_i) - f(u_i))(x_i - x_{i-1}) + \varepsilon.$$

We let $\dot{\mathcal{Q}}_1 := \{(I_i, u_i)\}_{i=1}^n$ and $\dot{\mathcal{Q}}_2 := \{(I_i, v_i)\}_{i=1}^n$, so that these tagged partitions have the same subintervals as $\mathcal{P}_\varepsilon$ does. Also, the sum on the right side equals $S(f; \dot{\mathcal{Q}}_2) - S(f; \dot{\mathcal{Q}}_1)$. Hence it follows from (1) that inequality (2) holds.

(c) $\Rightarrow$ (a) Define the step functions α_ε and ω_ε on $[a, b]$ by

$$\alpha_\varepsilon(x) := m_i \qquad \text{and} \qquad \omega_\varepsilon(x) := M_i \qquad \text{for} \quad x \in (x_{i-1}, x_i),$$

and $\alpha_\varepsilon(x_i) := f(x_i) =: \omega_\varepsilon(x_i)$ for $i = 0, 1, \ldots, n$; then $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ for $x \in [a, b]$. Since α_ε and ω_ε are step functions, they are Riemann integrable and

$$\int_a^b \alpha_\varepsilon = \sum_{i=1}^n m_i(x_i - x_{i-1}) \qquad \text{and} \qquad \int_a^b \omega_\varepsilon = \sum_{i=1}^n M_i(x_i - x_{i-1}).$$

Therefore it follows that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}).$$

If we apply (2), we have that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the Squeeze Theorem implies that $f \in \mathcal{R}[a, b]$. Q.E.D.

We have already seen that every continuous function on $[a, b]$ is Riemann integrable. We also saw in Example 7.1.7 that Thomae's function is Riemann integrable. Since Thomae's function has a countable set of points of discontinuity, it is evident that continuity is not a necessary condition for Riemann integrability. Indeed, it is reasonable to ask "how discontinuous" a function may be, yet still be Riemann integrable. The Riemann Criterion throws some light on that question in showing that sums of the form (2) must be arbitrarily small. Since the terms $(M_i - m_i)(x_i - x_{i-1})$ in this sum are all ≥ 0, it follows that each of these terms must be small. Such a term will be small if (i) the difference $M_i - m_i$ is small (which will be the case if the function is continuous on the interval $[x_{i-1}, x_i]$) or if (ii) an interval where the difference $M_i - m_i$ is not small has small length.

The Lebesgue Criterion, which we will discuss next, makes these ideas more precise. But first it is convenient to have the notion of the oscillation of a function.

C.2 Definition Let $f : A \to \mathbb{R}$ be a bounded function. If $S \subseteq A \subseteq \mathbb{R}$, we define the **oscillation of** f **on** S to be

$$W(f; S) := \sup\{|f(x) - f(y)| : x, y \in S\}. \tag{3}$$

It is easily seen that we can also write

$$\begin{aligned} W(f; S) &= \sup\{f(x) - f(y) : x, y \in S\} \\ &= \sup\{f(x) : x \in S\} - \inf\{f(x) : x \in S\}. \end{aligned}$$

It is also trivial that if $S \subseteq T \subseteq A$, then

$$0 \le W(f; S) \le W(f; T) \le 2 \cdot \sup\{|f(x)| : x \in A\}.$$

If $r > 0$, we recall that the r-neighborhood of $c \in A$ is the set

$$V_r(c) := \{x \in A : |x - c| < r\}.$$

C.3 Definition If $c \in A$, we define the **oscillation of** f **at** c by

$$w(f; c) := \inf\{W(f; V_r(c)) : r > 0\} = \lim_{r \to 0+} W(f; V_r(c)). \tag{4}$$

Since $r \mapsto W(f; V_r(c))$ is an increasing function for $r > 0$, this right-hand limit exists and equals the indicated infimum.

C.4 Lemma *If $f : A \to \mathbb{R}$ is bounded and $c \in A$, then f is continuous at c if and only if the oscillation $w(f; c) = 0$.*

Proof. $(\Rightarrow)$ If f is continuous at c, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in V_r(c)$, then $|f(x) - f(c)| < \varepsilon/2$. Therefore, if $x, y \in V_r(c)$, we have $|f(x) - f(y)| < \varepsilon$, whence $0 \leq w(f; c) \leq W(f; V_r(c)) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that $w(f; c) = 0$.

$(\Leftarrow)$ If $w(f; c) = 0$ and $\varepsilon > 0$, there exists $s > 0$ with $W(f; V_s(c)) < \varepsilon$. Thus, if $|x - c| < s$ then $|f(x) - f(c)| < \varepsilon$, and f is continuous at c. Q.E.D.

We will now give the details of the proof of the Lebesgue Integrability Criterion. First we recall the statement of the theorem.

Lebesgue's Integrability Criterion *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on [a, b].*

Proof. $(\Rightarrow)$ Let $\varepsilon > 0$ be given and, for each $k \in \mathbb{N}$, let $H_k := \{x \in [a, b] : w(f; x) > 1/2^k\}$. We will show that H_k is contained in the union of a finite number of intervals having total length $< \varepsilon/2^k$.

By the Riemann Criterion, there is a partition $\mathcal{P}_k = \{[x_{i-1}^k, x_i^k]\}_{i=1}^{n(k)}$ such that if m_i^k (respectively, M_i^k) is the infimum (respectively, supremum) of f on the interval $[x_{i-1}^k, x_i^k]$, then

$$\sum_{i=1}^{n(k)} (M_i^k - m_i^k)(x_i^k - x_{i-1}^k) < \varepsilon/4^k.$$

If $x \in H_k \cap (x_{i-1}^k, x_i^k)$, there exists $r > 0$ such that $V_r(x) \subseteq (x_{i-1}^k, x_i^k)$, whence

$$1/2^k \leq w(f; x) \leq W(f; V_r(x)) \leq M_i^k - m_i^k.$$

If we denote a summation over those i with $H_k \cap (x_{i-1}^k, x_i^k) \neq \emptyset$ by $\sum'$, then

$$(1/2^k) \sum{}' (x_i^k - x_{i-1}^k) \leq \sum_{i=1}^{n(k)} (M_i^k - m_i^k)(x_i^k - x_{i-1}^k) \leq \varepsilon/4^k,$$

whence it follows that

$$\sum{}' (x_i^k - x_{i-1}^k) \leq \varepsilon/2^k.$$

Since H_k differs from the union of sets $H_k \cap (x_i^k - x_{i-1}^k)$ by at most a finite number of the partition points, we conclude that H_k is contained in the union of a finite number of intervals with total length $< \varepsilon/2^k$.

Finally, since $D := \{x \in [a, b] : w(f; x) > 0\} = \bigcup_{k=1}^{\infty} H_k$, it follows that the set D of points of discontinuity of $f \in \mathcal{R}[a, b]$ is a null set.

$(\Leftarrow)$ Let $|f(x)| \leq M$ for $x \in [a, b]$ and suppose that the set D of points of discontinuity of f is a null set. Then, given $\varepsilon > 0$ there exists a countable set $\{J_k\}_{k=1}^{\infty}$ of open intervals with $D \subseteq \bigcup_{k=1}^{\infty} J_k$ and $\sum_{k=1}^{\infty} l(J_k) < \varepsilon/2M$. Following R. A. Gordon, we will define a gauge on [a, b] that will be useful.

(i) If $t \notin D$, then f is continuous at t and there exists $\delta(t) > 0$ such that if $x \in V_{\delta(t)}(t)$ then $|f(x) - f(t)| < \varepsilon/2$, whence

$$0 \leq M_t - m_t := \sup\{f(x) : x \in V_{\delta(t)}(t)\} - \inf\{f(x) : x \in V_{\delta(t)}(t)\} \leq \varepsilon.$$

(ii) If $t \in D$, we choose $\delta(t) > 0$ such that $V_{\delta(t)}(t) \subseteq J_k$ for some k. For these values of t, we have $0 \leq M_t - m_t \leq 2M$.

Thus we have defined a gauge δ on $[a, b]$. If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ-fine partition of $[a, b]$, we divide the indices i into two disjoint sets

$$S_c := \{i : t_i \notin D\} \quad \text{and} \quad S_d := \{i : t_i \in D\}.$$

If $\dot{\mathcal{P}}$ is δ-fine, we have $[x_{i-1}, x_i] \subseteq V_{\delta(t_i)}(t_i)$, whence it follows that $M_i - m_i \leq M_{t_i} - m_{t_i}$. Consequently, if $i \in S_c$ then $M_i - m_i \leq \varepsilon$, while if $i \in S_d$ we have $M_i - m_i \leq 2M$. However, the collection of intervals $[x_{i-1}, x_i]$ with $i \in S_d$ are contained in the union of the intervals $\{J_k\}$ whose total length is $< \varepsilon/2M$. Therefore

$$\begin{aligned}
&\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i\in S_c}(M_i - m_i)(x_i - x_{i-1}) + \sum_{i\in S_d}(M_i - m_i)(x_i - x_{i-1}) \\
&\leq \sum_{i\in S_c} \varepsilon(x_i - x_{i-1}) + \sum_{i\in S_d} 2M(x_i - x_{i-1}) \\
&\leq \varepsilon(b-a) + 2M \cdot (\varepsilon/2M) \leq \varepsilon(b - a + 1).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}[a, b]$. Q.E.D.

APPENDIX D

APPROXIMATE INTEGRATION

We will supply here the proofs of Theorems 7.5.3, 7.5.6, and 7.5.8. We will not repeat the statement of these results, and we will use the notations introduced in Section 7.5 and refer to numbered equations there. It will be seen that some important results from Chapters 5 and 6 are used in these proofs.

Proof of Theorem 7.5.3. If $k = 1, 2, \ldots, n$, let $a_k := a + (k-1)h$ and let $\varphi_k : [0, h] \to \mathbb{R}$ be defined by

$$\varphi_k(t) := \tfrac{1}{2}t[f(a_k) + f(a_k + t)] - \int_{a_k}^{a_k+t} f(x)dx$$

for $t \in [0, h]$. Note that $\varphi_k(0) = 0$ and that (by Theorem 7.3.6)

$$\begin{aligned}\varphi_k'(t) &= \tfrac{1}{2}[f(a_k) + f(a_k + t)] + \tfrac{1}{2}tf'(a_k + t) - f(a_k + t)\\ &= \tfrac{1}{2}[f(a_k) - f(a_k + t)] + \tfrac{1}{2}tf'(a_k + t).\end{aligned}$$

Consequently $\varphi_k'(0) = 0$ and

$$\begin{aligned}\varphi_k''(t) &= -\tfrac{1}{2}f'(a_k + t) + \tfrac{1}{2}f'(a_k + t) + \tfrac{1}{2}tf''(a_k + t)\\ &= \tfrac{1}{2}tf''(a_k + t).\end{aligned}$$

Now let A, B be defined by

$$A := \inf\{f''(x) : x \in [a, b]\}, \quad B := \sup\{f''(x) : x \in [a, b]\}$$

so that we have $\frac{1}{2}At \le \varphi_k''(t) \le \frac{1}{2}Bt$ for $t \in [0, h], k = 1, 2, \ldots, n$. Integrating and applying Theorem 7.3.1, we obtain (since $\varphi_k'(0) = 0$) that $\frac{1}{4}At^2 \le \varphi_k'(t) \le \frac{1}{4}Bt^2$ for $t \in [0, h]$, $k = 1, 2, \ldots, n$. Integrating again and taking $t = h$, we obtain (since $\varphi_k(0) = 0$) that

$$\tfrac{1}{12}Ah^3 \le \varphi_k(h) \le \tfrac{1}{12}Bh^3$$

for $k = 1, 2, \ldots, n$. If we add these inequalities and note that

$$\sum_{k=1}^{n} \varphi_k(h) = T_k(f) - \int_a^b f(x)dx,$$

we conclude that $\frac{1}{12}Ah^3n \le T_n(f) - \int_a^b f(x)dx \le \frac{1}{12}Bh^3n$. Since $h = (b-a)/n$, we have

$$\tfrac{1}{12}A(b-a)h^2 \le T_n(f) - \int_a^b f(x)dx \le \tfrac{1}{12}B(b-a)h^2.$$

Since f'' is continuous on $[a, b]$, it follows from the definitions of A and B and Bolzano's Intermediate Value Theorem 5.3.7 that there exists a point c in $[a, b]$ such that equation (4) in Section 7.5 holds. Q.E.D.

Proof of Theorem 7.5.6. If $k = 1, 2, \ldots, n$, let $c_k := a + (k - \frac{1}{2})h$, and $\psi_k : \left[0, \frac{1}{2}h\right] \to \mathbb{R}$ be defined by

$$\psi_k(t) := \int_{c_k - t}^{c_k + t} f(x)dx - f(c_k)2t$$

for $t \in \left[0, \frac{1}{2}h\right]$. Note that $\psi_k(0) = 0$ and that since

$$\psi_k(t) := \int_{c_k}^{c_k + t} f(x)dx - \int_{c_k}^{c_k - t} f(x)dx - f(c_k)2t,$$

we have

$$\begin{aligned} \psi_k'(t) &= f(c_k + t) - f(c_k - t)(-1) - 2f(c_k) \\ &= [f(c_k + t) + f(c_k - t)] - 2f(c_k). \end{aligned}$$

Consequently $\psi_k'(0) = 0$ and

$$\begin{aligned} \psi_k''(t) &= f'(c_k + t) + f'(c_k - t)(-1) \\ &= f'(c_k + t) - f'(c_k - t). \end{aligned}$$

By the Mean Value Theorem 6.2.4, there exists a point $c_{k,t}$ with $|c_k - c_{k,t}| < t$ such that $\psi_k''(t) = 2tf''(c_{k,t})$. If we let A and B be as in the proof of Theorem 7.5.3, we have $2tA \le \psi_k''(t) \le 2tB$ for $t \in [0, h/2], k = 1, 2, \ldots, n$. It follows as before that

$$\tfrac{1}{3}At^3 \le \psi_k(t) \le \tfrac{1}{3}Bt^3$$

for all $t \in \left[0, \frac{1}{2}h\right]$, $k = 1, 2, \ldots, n$. If we put $t = \frac{1}{2}h$, we get

$$\frac{1}{24}Ah^3 \le \psi_k(\tfrac{1}{2}h) \le \frac{1}{24}Bh^3.$$

If we add these inequalities and note that

$$\sum_{k=1}^{n} \psi_k(\tfrac{1}{2}h) = \int_a^b f(x)dx - M_n(f),$$

we conclude that

$$\frac{1}{24}Ah^3n \le \int_a^b f(x)dx - M_n(f) \le \frac{1}{24}Bh^3n.$$

If we use the fact that $h = (b - a)/n$ and apply Bolzano's Intermediate Value Theorem 5.3.7 to f'' on $[a, b]$ we conclude that there exists a point $\gamma \in [a, b]$ such that (7) in Section 7.5 holds. Q.E.D.

Proof of Theorem 7.5.8. If $k = 0, 1, 2, \ldots, \frac{1}{2}n - 1$, let $c_k := a + (2k + 1)h$, and let $\varphi_k : [0, h] \to \mathbb{R}$ be defined by

$$\varphi_k(t) := \tfrac{1}{3}t[f(c_k - t) + 4f(c_k) + f(c_k + t)] - \int_{c_k - t}^{c_k + t} f(x)dx.$$

Evidently $\varphi_k(0) = 0$ and

$$\varphi_k'(t) = \tfrac{1}{3}t[-f'(c_k - t) + f'(c_k + t)] - \tfrac{2}{3}[f(c_k - t) - 2f(c_k) + f(c_k + t)],$$

so that $\varphi_k'(0) = 0$ and

$$\varphi_k''(t) = \tfrac{1}{3}t[f''(c_k - t) + f''(c_k + t)] - \tfrac{1}{3}[-f'(c_k - t) + f'(c_k + t)],$$

so that $\varphi_k''(0) = 0$ and

$$\varphi_k'''(t) = \tfrac{1}{3}t[f'''(c_k + t) - f'''(c_k - t)].$$

Hence it follows from the Mean Value Theorem 6.2.4 that there is a $\gamma_{k,t}$ with $|c_k - \gamma_{k,t}| \le t$ such that $\varphi_k'''(t) = \frac{2}{3}t^2 f^{(4)}(\gamma_{k,t})$. If we let A and B be defined by

$$A := \inf\{f^{(4)}(x) : x \in [a, b]\} \qquad \text{and} \qquad B := \sup\{f^{(4)}(x) : x \in [a, b]\},$$

then we have

$$\tfrac{2}{3}At^2 \le \varphi_k'''(t) \le \tfrac{2}{3}Bt^2$$

for $t \in [0, h]$, $k = 0, 1, \ldots, \frac{1}{2}n - 1$. After three integrations, this inequality becomes

$$\frac{1}{90}At^5 \le \varphi_k(t) \le \frac{1}{90}Bt^5$$

for all $t \in [0, h]$, $k = 0, 1, \ldots, \frac{1}{2}n - 1$. If we put $t = h$, we get

$$\frac{1}{90}Ah^5 \le \varphi_k(h) \le \frac{1}{90}Bh^5$$

for $k = 0, 1, \ldots, \frac{1}{2}n - 1$. If we add these $\frac{1}{2}n$ inequalities and note that

$$\sum_{k=0}^{\frac{1}{2}n-1} \varphi_k(h) = S_n(f) - \int_a^b f(x)dx,$$

we conclude that

$$\frac{1}{90}Ah^5\frac{n}{2} \le S_n(f) - \int_a^b f(x)dx \le \frac{1}{90}Bh^5\frac{n}{2}.$$

Since $h = (b - a)/n$, it follows from Bolzano's Intermediate Value Theorem 5.3.7 (applied to $f^{(4)}$) that there exists a point $c \in [a, b]$ such that the relation (10) in Section 7.5 holds. Q.E.D.

APPENDIX E

TWO EXAMPLES

In this appendix we will give an example of a continuous function that has a derivative at no point and of a continuous curve in $\mathbb{R}^2$ whose range contains the entire unit square of $\mathbb{R}^2$. Both proofs use the Weierstrass *M*-Test 9.4.6.

A Continuous Nowhere Differentiable Function

The example we will give is a modification of one due to B. L. van der Waerden in 1930. Let $f_0 : \mathbb{R} \to \mathbb{R}$ be defined by $f_0(x) := \text{dist}(x, \mathbb{Z}) = \inf\{|x - k| : k \in \mathbb{Z}\}$, so that f_0 is a continuous "sawtooth" function whose graph consists of lines with slope ± 1 on the intervals $[k/2, (k+1)/2]$, $k \in \mathbb{Z}$. For each $m \in \mathbb{N}$, let $f_m(x) := (1/4^m) f_0(4^m x)$, so that f_m is also a continuous sawtooth function whose graph consists of lines with slope ± 1 and with $0 \leq f_m(x) \leq 1/(2 \cdot 4^m)$. (See Figure E.1.)

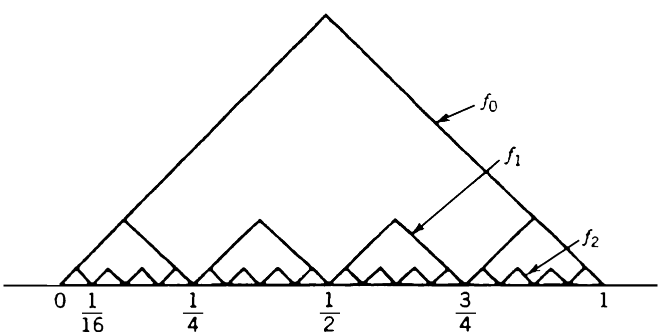

Figure E.1 Graphs of f_0, f_1, and f_2.

We now define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \sum_{m=0}^{\infty} f_m(x)$. The Weierstrass *M*-Test implies that the series is uniformly convergent on $\mathbb{R}$; hence g is continuous on $\mathbb{R}$. We will now show that g is *not* differentiable at any point of $\mathbb{R}$.

Fix $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, let $h_n := \pm 1/4^{n+1}$, with the sign chosen so that both $4^n x$ and $4^n(x + h_n)$ lie in the same interval $[k/2, (k+1)/2]$. Since f_0 has slope ± 1 on this interval, then

$$\varepsilon_n := \frac{f_n(x + h_n) - f_n(x)}{h_n} = \frac{f_0(4^n x + 4^n h_n) - f_0(4^n x)}{4^n h_n} = \pm 1.$$

In fact if $m < n$, then the graph of f_m also has slope ± 1 on the interval between x and $x + h_n$ and so

$$\varepsilon_m := \frac{f_m(x + h_n) - f_m(x)}{h_n} = \pm 1 \qquad \text{for} \quad m < n.$$

On the other hand, if $m > n$, then $4^m(x + h_n) - 4^m x = \pm 4^{m-n-1}$ is an integer, and since f_0 has period equal to 1, it follows that

$$f_m(x + h_n) - f_m(x) = 0.$$

Consequently, we have

$$\frac{g(x + h_n) - g(x)}{h_n} = \sum_{m=0}^{n} \frac{f_m(x + h_n) - f_m(x)}{h_n} = \sum_{m=0}^{n} \varepsilon_m,$$

whence the difference quotient $(g(x + h_n) - g(x))/h_n$ is an odd integer if n is even, and an even integer if n is odd. Therefore, the limit

$$\lim_{h \to 0} \frac{g(x + h) - g(x)}{h}$$

does not exist, so g is not differentiable at the arbitrary point $x \in \mathbb{R}$.

A Space-Filling Curve

We will now give an example of a space-filling curve that was constructed by I. J. Schoenberg in 1938. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the continuous, even function with period 2 given by

$$\varphi(t) := \begin{cases} 0 & \text{for} \quad 0 \le t \le 1/3, \\ 3t - 1 & \text{for} \quad 1/3 < t < 2/3, \\ 1 & \text{for} \quad 2/3 \le t \le 1. \end{cases}$$

(See Figure E.2.) For $t \in [0,\ 1]$, we define the functions

$$f(t) := \sum_{k=0}^{\infty} \frac{\varphi\left(3^{2k} t\right)}{2^{k+1}} \qquad \text{and} \qquad g(t) := \sum_{k=0}^{\infty} \frac{\varphi\left(3^{2k+1} t\right)}{2^{k+1}}.$$

Since $0 \le \varphi(x) \le 1$ and is continuous, the Weierstrass M-Test implies that f and g are continuous on [0, 1]; moreover, $0 \le f(t) \le 1$ and $0 \le g(t) \le 1$. We will now show that an arbitrary point (x_0, y_0) in $[0, 1] \times [0, 1]$ is the image under (f, g) of some point $t_0 \in [0, 1]$. Indeed, let x_0 and y_0 have the binary (= base 2) expansions:

$$x_0 = \frac{a_0}{2} + \frac{a_2}{2^2} + \frac{a_4}{2^3} + \cdots \qquad \text{and} \qquad y_0 = \frac{a_1}{2} + \frac{a_3}{2^2} + \frac{a_5}{2^3} + \cdots,$$

where each a_k equals 0 or 1. It will be shown that $x_0 = f(t_0)$ and $y_0 = g(t_0)$, where t_0 has the ternary (= base 3) expansion

$$t_0 = \sum_{k=0}^{\infty} \frac{2a_k}{3^{k+1}} = \frac{2a_0}{3} + \frac{2a_1}{3^2} + \frac{2a_2}{3^3} + \frac{2a_3}{3^4} + \cdots.$$

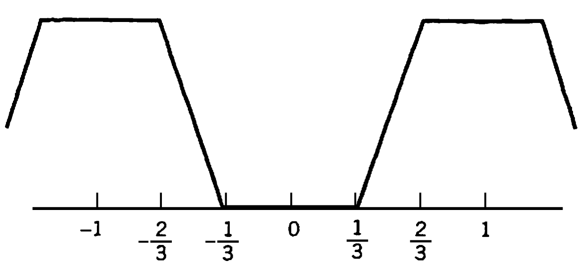

Figure E.2 Graph of φ

First, we note that the above formula does yield a number in [0, 1]. We also note that if $a_0 = 0$, then $0 \leq t_0 \leq 1/3$ so that $\varphi(t_0) = 0$, and if $a_0 = 1$, then $2/3 \leq t_0 \leq 1$ so that $\varphi(t_0) = 1$; therefore, in both cases $\varphi(a_0) = a_0$. Similarly, it is seen that for each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that

$$3^n t_0 = 2m_n + \frac{2a_n}{3} + \frac{2a_{n+1}}{3^2} + \cdots,$$

whence it follows from the fact that φ has period 2 that $\varphi(3^n t_0) = a_n$. Finally, we conclude that

$$f(t_0) = \sum_{k=0}^{\infty} \frac{\varphi\left(3^{2k} t_0\right)}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{a_{2k}}{2^{k+1}} = x_0,$$

and

$$g(t_0) = \sum_{k=0}^{\infty} \frac{\varphi\left(3^{2k+1} t_0\right)}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2^{k+1}} = y_0.$$

Therefore $x_0 = f(t_0)$ and $y_0 = g(t_0)$ as claimed.