SEQUENCES

1. Sequences of Real Numbers

We begin our study of analysis with sequences. There are several reasons for starting here. First, sequences are the simplest way to introduce limits, the central idea of calculus. Second, sequences are a direct route to the topology of the real numbers. The combination of limits and topology provides the tools to finally prove the theorems you have already used in your calculus courses.

In everyday usage of the English language the words "sequence" and "series" are synonyms and they are used to suggest a succession of things or events arranged in some order. In mathematics these words have some special technical meanings. The word "sequence" is employed as in the common use of the term to convey the idea of a set of things arranged in order but the word "series" is used in a somewhat different sense. The concept of sequence will be discussed in this chapter.

Suppose for each positive integer n, we are given a real number x_n. Then, the list of numbers,

$$x_1, x_2, ..., x_n, ...$$

is called a sequence. More precisely, we define a sequence as follows:

Definition 1.1. : A function $f\colon \mathbb{N} \to \mathbb{R}$ is called a sequence of real numbers. We write $f(n) = x_n$, then the sequence is denoted by $(x_1,\ x_2,...,x_n,....)$ or (x_n) or simply by $\{x_n\}$. We call x_n the nth term of the sequence or the value of the sequence at n.

Example 1.1. *Few Examples of Sequences:*

(1) $\{x_n\}$ *with* $x_n = 1$ *for all* $n \in \mathbb{N}$– *a constant sequence with value* 1 *throughout.*
(2) $\{x_n\}$ *with* $x_n = n$ *for all* $n \in \mathbb{N}$.
(3) $\{x_n\}$ *with* $x_n = 1/n$ *for all* $n \in \mathbb{N}$.
(4) $\{x_n\}$ *with* $x_n = \sqrt{n}$ *for all* $n \in \mathbb{N}$.
(5) $\{x_n\}$ *with* $x_n = n/(n+1)$ *for all* $n \in \mathbb{N}$.
(6) $\{x_n\}$ *with* $x_n = (-1)^n$ *for all* $n \in \mathbb{N}$ *– the sequence takes values* 1 *and* -1 *alternately.*
(7) *The celebrated* **Fibonacci sequence** $F = (f_n)$*, is given by the inductive definition*

$$f_1 := 1,\ f_2 = 1,\ f_{n+1} = f_n + f_{n-1}\ (n \geq 2)$$

Thus each term past the second is the sum of its two immediate predecessors. The first ten terms of F *are seen to be* $\{1; 1; 2; 3; 5; 8; 13; 21; 34; 55; ...\}$.

2. Convergence of a Sequence

We could say that a given sequence $\{x_n\}$ has a limiting value of L as n tends to ∞ when the terms $\{x_n\}$ eventually get microscopically close to the number L. For instance, the sequence $\{\frac{1}{n^5}\}$ seems to have a limiting value of 0. The sequence $.3, .33, .333, .3333, \ldots$ seems to have a limiting value of 1/3. Simple as this may seem, an approach to limits based on such hopeful impressions is only the beginning.

To go further we must ask **quantitative** questions. For example, how far do you have to take $\frac{1}{n^5}$ to be sure that it approximates $p = 0$ with 8 decimal places of accuracy? Let's see what the answer could be. We need to know how far to go with n before we hit $\frac{1}{n^5} < \frac{1}{10^8}$. In other words, how far should we go before we obtain $10^{8/5} < n$? Since $10^{8/5} = 39.8$, it seems pretty clear that we have to wait until $n > 39.8$. Once $n = 40$ and beyond, we can be sure that $\frac{1}{n^5}$ approximates 0 with 8 decimal places of accuracy. If we wanted 16 decimal places of accuracy we would wait until n had gone beyond $10^{16/5} = 1584.9$, in other words until n hit 1585. If we wanted still more accuracy, say 80 decimal places we would wait quite a bit more, until in fact n got past $10^{80/5} = 10^{16}$. No matter how much accuracy we specify, the limit can be approximated to satisfy that accuracy if we wait long enough.

This quantitative approach brings us to a central idea in calculus. The idea is that a sequence $\{x_n\}$ has a limit L provided x_n can be brought as close to L as we like by simply going far enough out in the sequence. In the tradition of calculus the symbol used to specify an arbitrary amount of closeness is the Greek letter ε, called epsilon. You should get used to thinking of the letter ε to represent an arbitrary, yet very small positive number. Here is the formal and very important definition of limit of a sequence.

Definition 2.1. Limit of a sequence A sequence $\{x_n\}$ in $\mathbb{R}$ is said to converge to a real number L if for every $\varepsilon > 0$, there exists positive integer N_ε (in general depending on ε) such that

$$|x_n - L| < \varepsilon \qquad \forall\ n \geq N_\varepsilon$$

and in that case, the number L is called a limit of the sequence $\{x_n\}$, and $\{x_n\}$ is called a convergent sequence.

Notations 2.1. We write $L = \lim_{n\to\infty} x_n$ or $x_n \to L$ as $n \to \infty$.

Note that

$$|x_n - L| < \varepsilon \qquad \forall\ n \geq N$$

if and only if

$$L - \varepsilon < x_n < L + \varepsilon \quad \forall\ n \geq N.$$

Thus, $x_n \to L$ as $n \to \infty$ if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in (L - \varepsilon, L + \varepsilon)\ \forall n \geq N$. Thus, $x_n \to L$ if and only if for every $\varepsilon > 0$, x_n belongs to the open interval $(L - \varepsilon, L + \varepsilon)$ for all n after some finite stage, and this finite stage may vary according as ε varies.

Note The notation N_ε is used to emphasize that the choice of N depends on the value of ε. However, it is often convenient to write N instead of N_ε. In most cases, a "small" value of ε will usually require a "large" value of N to guarantee that the distance $|x_n - L|$ between x_n and L is less than ε for all $n \geq N_\varepsilon$.

Remark 2.1. Suppose (x_n) is a sequence and $M \in \mathbb{R}$. Then to show that $\{x_n\}$ does not converge to M, we should be able to find an $\varepsilon_0 > 0$ such that infinitely may x_n's are outside the interval $(M - \varepsilon_0, M + \varepsilon_0)$.

Remark 2.2. The definition of the limit of a sequence of real numbers is used to verify that a proposed value L is indeed the limit. It does not provide a means for initially determining what that value of x might be. Later results will contribute to this end, but quite often it is necessary in practice to arrive at a conjectured value of the limit by direct calculation of a number of terms of the sequence. Computers can be helpful in this respect, but since they can calculate only a finite number of terms of a sequence, such computations do not in any way constitute a proof of the value of the limit.

The following examples illustrate how the definition is applied to prove that a sequence has a particular limit. In each case, a positive ε is given and we are required to find a N_ε, depending on ε, as required by the definition.

Example 2.1. $\lim_{n\to\infty} \frac{1}{n} = 0$.

For the sake of illustrating how to use the definition to justify the above statement, let us provide the details of the proofs:
Let $x_n = 1/n$ *for all* $n \in \mathbb{N}$,*and let* $\varepsilon > 0$ *be given.*
We have to identify (or show existence) an $N_\varepsilon \in \mathbb{N}$ *such that* $|x_n - L| = |1/n - 0| = 1/n < \varepsilon \; \forall \; n \geq N$.
By Archimedean property, for $x = \varepsilon$ *and* $y = 1$, *there exists* $N_\varepsilon \in \mathbb{N}$ *such that* $N_\varepsilon \varepsilon > 1$ *that is* $N_\varepsilon > \frac{1}{\varepsilon}$.
In particular, we can choose $N = [\frac{1}{\varepsilon}] + 1 > \frac{1}{\varepsilon}$ *which in turn implies* $n\varepsilon > N_\varepsilon \varepsilon > 1 \; \forall n \geq N_\varepsilon$. *Hence we have*

$$1/n < \varepsilon \qquad\qquad \forall \; n \geq N$$

Hence, $\{1/n\}$ *converges to* 0.

For instance, if we take $\varepsilon = .002$. *We have decided above that a suitable cut-off point is*

$$\left[\frac{1}{.002}\right] + 1 = 501$$

In other words, starting with the 501*th term of the sequence, you know that from then on the sequence will be less than the distance* $.002$ *away from the limit* 0. *Try checking with your calculator that the distance between the limit* 0 *and the terms*

$$\frac{1}{501}, \frac{1}{502}, \ldots$$

is indeed less than $.002$.

Example 2.2. $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$.
Let $x_n = \frac{1}{n^2+1}$ *for all* $n \in \mathbb{N}$,*and let* $\varepsilon > 0$ *be given.*
We have to identify an $N \in \mathbb{N}$ *such that* $|x_n - L| = |\frac{1}{n^2+1} - 0| = \frac{1}{n^2+1} < \varepsilon \ \forall\ n \geq N$.
To find $N \in \mathbb{N}$, *we first notice that*

$$\frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} \ \forall\ n \in \mathbb{N}.$$

Just like the previous example: By Archimedean property, for $x = \varepsilon$ *and* $y = 1$, *there exists* $N_\varepsilon \in \mathbb{N}$ *such that* $N_\varepsilon \varepsilon > 1$ *which in turn implies* $n\varepsilon > N_\varepsilon \varepsilon > 1 \ \forall n \geq N_\varepsilon$. *Hence we have*

$$|x_n - L| = |\frac{1}{n^2+1} - 0| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} < \varepsilon \ \forall\ n \geq N_\varepsilon.$$

Hence, we have shown that the limit of the sequence is zero.

Example 2.3. $\lim_{n\to\infty} \frac{3n+1}{n+1} = 3$.
Let $x_n = \frac{3n+1}{n+1}$ *for all* $n \in \mathbb{N}$,*and let* $\varepsilon > 0$ *be given.*
We have to identify an $N \in \mathbb{N}$ *such that* $|x_n - L| = |\frac{3n+1}{n+1} - 3| = \frac{2}{n+1} < \varepsilon \ \forall\ n \geq N$.
To find $N_1 \in \mathbb{N}$, *we first notice that*

$$\frac{2}{n+1} < \frac{2}{n} \ \forall\ n \in \mathbb{N} \tag{1}$$

Just like the previous example: By Archimedean property, for $x = \varepsilon$ *and* $y = 2$, *there exists* $N \in \mathbb{N}$ *such that* $N\varepsilon > 2$ *which in turn implies* $n\varepsilon > N\varepsilon > 2$ $\forall n \geq N$ *which implies*

$$\frac{2}{n} < \frac{2}{N} < \varepsilon \ \forall n \geq N. \tag{2}$$

So (1) *together with* (2) *gives*

$$\frac{2}{n+1} < \frac{2}{n} < \frac{2}{N} < \varepsilon \ \forall n \geq N$$

. *Hence we have*

$$|x_n - L| = \left|\frac{3n+1}{n+1} - 3\right| = \frac{2}{n+1} < \varepsilon \ \forall\ n \geq N$$

Hence, we have shown that the limit of the sequence is 3.

Definition 2.2. Eventually constant sequence: A sequence $\{x_n\}$ is said to be eventually constant if there exists $k \in \mathbb{N}$ such that $x_{k+n} = x_k \ \forall n \geq 1$. And it is easy to see that every eventually constant sequence converges.

Definition 2.3. Sequence diverging to infinity: Let $\{x_n\}$ be a sequence of real numbers. We say that the sequence $\{x_n\}$ approaches infinity or diverges to positive infinity, if for any real number $G > 0$ (however large), there exists $N \in \mathbb{N}$ such that

$$x_n \geq G \quad \forall n \geq N$$

If $\{x_n\}$ approaches infinity, then we write $x_n \to +\infty$ as $n \to \infty$.

A similar definition is given for the sequences diverging to $-\infty$. Let $\{x_n\}$ be a sequence of real numbers. We say that the sequence $\{x_n\}$ approaches infinity or diverges to negative infinity, if for any real number $g < 0$ (however small), there exists $N \in \mathbb{N}$ such that

$$x_n \leq g \quad \forall n \geq N$$

If $\{x_n\}$ approaches negative infinity, then we write $x_n \to -\infty$ as $n \to \infty$.

Example 2.4. *(i) The sequence $\{\log(1/n)\}$ diverges to $-\infty$. In order to prove this, for any $M > 0$, we must produce a $N \in \mathbb{N}$ such that*

$$\log(1/n) < -M, \quad n \geq N.$$

But this is equivalent to saying that $n > e^M$, $\forall n \geq N$. Choose $N \geq e^M$. Then, for this choice of N,

$$\log(1/n) < -M, \quad n \geq N.$$

hence $\{\log(1/n)\}$ diverges to $-\infty$.

Example 2.5. *Let r be a positive real number.*
Then $\lim_{n\to\infty} r^n = \left\{ \begin{array}{cc} 0 & 0 < r < 1 \\ \infty & r \geq 1 \end{array} \right.$.

Case 1) $0 < r < 1$. Let $0 < \varepsilon < 1$ be given. So we need to prove for the given ε, there exists a natural number N_ε (depending on ε) such that $|r^n - 0| < \varepsilon$ for all $n \geq N_\varepsilon$. Now

$$r^n = |r^n - 0| < \varepsilon \;\Rightarrow\; n \log r < \log \varepsilon$$

We know log is negative in the interval $(0, 1)$. So we can write from the above

$$-n \log r > -\log \varepsilon \;\Rightarrow\; n \log \frac{1}{r} > \log \frac{1}{\varepsilon} \;\Rightarrow\; n > \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{r}}$$

We choose $N_\varepsilon = [\frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{r}}] + 1$. Then for this choice of $N_\varepsilon \in \mathbb{N}$ we have

$$|r^n - 0| < \varepsilon \qquad\qquad \forall\, n \geq N_\varepsilon$$

.

Case 2)Let G (however large) be a given real number, then $r^n > G$ for all $n \geq [\frac{\log G}{\log r}] + 1$. Hence the sequence diverges to infinity.

Definition 2.4. Oscillatory Sequences If a sequence $\{x_n\}$ does not converge to a value in $\mathbb{R}$ and also does not diverge to ∞ or $-\infty$, we say that $\{x_n\}$ oscillates.

Example 2.6. *Consider the sequence $\{(-1)^n\}$. It is a bounded sequence, so it is not divergent. It is intuitively clear that this sequence does not have a limit or it does not approach to any real number. We now prove this by definition. Assume to the contrary, that there exists an $L \in \mathbb{R}$ such that the sequence $(-1)^n$ converges to L.*
Then for $\varepsilon = 1/2$, there exists an $N \in \mathbb{N}$ such that

$$|(-1)^n - L| < 1/2, \qquad \forall n \geq N. \tag{3}$$

For n even, (3) says

$$|-1 - L| < 1/2 \qquad \forall n \geq N \tag{4}$$

While for n odd (3) says

$$|1 - L| < 1/2 \qquad \forall n \geq N \tag{5}$$

hence by (4) and (5) we have $2 = |1+1| \leq |1-L| + |1+L| < 1$ which is a contradiction. Hence it is an oscillatory sequence.

Theorem 2.2. Uniqueness of Limits *Limit of a convergent sequence is unique. That, is if $x_n \to L$ and also $x_n \to M$ as $n \to \infty$, then $L = M$.*

Proof. Now $x_n \to L$ as $n \to \infty$ and $x_n \to M$ as $n \to \infty$ are given. For any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon/2 \qquad \forall\ n \geq N_1$$

. Similarly, for the given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|x_n - M| < \varepsilon/2 \qquad \forall\ n \geq N_2$$

Now we know

$$|L - M| = |(L - x_n) + (x_n - M)| \leq |x_n - L| + |x_n - M|$$

. Then it follows that

$$|L - M| \leq |x_n - L| + |x_n - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\forall n \geq N := max(N_1, N_2)$. Since $|L - M| < \varepsilon$ is true for all $\varepsilon > 0$, it follows that $L = M$.

□

2.1. **Boundedness of a sequence.**

Definition 2.5. A sequence x_n is said to be bounded above, if there exists $M \in \mathbb{R}$ such that $x_n \leq M\ \forall\ \ n \in \mathbb{N}$. Similarly, we say that a sequence $\{x_n\}$ is bounded below, if there exists $m \in \mathbb{R}$ such that $x_n \geq m\ \forall\ \ n \in \mathbb{N}$. Thus a sequence x_n is said to be bounded if it is both bounded above and below.

Lemma 2.3. *Every convergent sequence is bounded.*

Proof. Let $\{x_n\}$ be a convergent sequence and $L = \lim_{n\to\infty} x_n$. Let $\varepsilon = 1$. Then there exists such that $N \in \mathbb{N}$ such that

$$|x_n - L| < 1 \qquad \forall\ n \geq N$$

. Further,

$$|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < 1 + |L| \qquad \forall n \geq N$$

. Let $M = max\{|x_1|, |x_2|, ..., |x_{N-1}|, 1 + |L|\}$. Then $|x_n| \leq M\ \forall n \in \mathbb{N}$ which implies $-M \leq x_n \leq M\ \forall n \in \mathbb{N}$. Hence x_n is bounded.

□

Remark 2.3. But the converse of the above theorem is not true. Consider the sequence $\{x_n\}$ with $x_n = (-1)^n$ for all $n \in \mathbb{N}$ – the sequence takes values 1 and -1 alternately. So the sequence is bounded above by 1 and bounded below by -1 but the sequence is not convergent.

Corollary 2.4. *If $\{x_n\}$ is an unbounded sequence, then $\{x_n\}$ is divergent.*

Proof. Follows from lemma 2.3. □

Remark 2.4. Let $\{x_n\}$ be a convergent sequence and $L = \lim_{n\to\infty} x_n$. Then the sequence $\{|x_n|\}$ is convergent and $|L| = \lim_{n\to\infty} |x_n|$ because of the fact $||x_n| - |L|| \leq |x_n - L|$.

2.2. **Operations on Convergent Sequences.**

Theorem 2.5. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$. Then*

$$i)\ \lim_{n\to\infty} (x_n + y_n) = L + M.$$

$$ii)\ \lim_{n\to\infty} (x_n - y_n) = L - M.$$

$$iii)\ \lim_{n\to\infty} (x_n y_n) = LM.$$

$$iv)\ \lim_{n\to\infty} c x_n = cL \qquad c \in \mathbb{R}.$$

$$v)\ \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \left(\frac{L}{M}\right) \qquad M \neq 0.$$

Proof. i) Let $\varepsilon > 0$ be given. Since $\{x_n\}$ converges to L there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon/2 \qquad \forall\ n \geq N_1.$$

Similarly, for the given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|y_n - M| < \varepsilon/2 \qquad \forall\ n \geq N_2.$$

Thus $\forall n \geq N = max\{N_1, N_2\}$

$$|(x_n + y_n) - (L + M)| \leq |x_n - L| + |y_n - M| < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

(ii) Easy to prove. Hence left as an exercise to the students.

(iii) Let $\varepsilon > 0$ be given. Since $\{x_n\}$ is a convergent sequence, it is bounded by M_1 (say). That is $|x_n| \leq |M_1| \quad \forall n \geq 1$.

Also as $\{x_n\}$ converges to L there exists$N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{\varepsilon}{2M} \qquad \forall\ n \geq N_1.$$

Similarly, for the given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|y_n - M| < \frac{\varepsilon}{2M_1} \qquad \forall\ n \geq N_2.$$

Then $\forall n \geq N = max\{N_1, N_2\}$ we have

$$\begin{aligned}
|x_n y_n - LM| &= |x_n y_n - x_n M + x_n M - LM| \leq |x_n(y_n - M)| + |M(x_n - L)| \\
&\leq |x_n||(y_n - M)| + |M||(x_n - L)| \\
&< M_1 \frac{\varepsilon}{2M_1} + |M| \frac{\varepsilon}{2M} \\
&< \varepsilon
\end{aligned}$$

(v) In order to prove this, it is enough to prove that if $\lim_{n\to\infty} y_n = M$, $M \neq 0$, then $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{M}$. Let $\varepsilon_0 > 0$ be given. As $\{y_n\}$ converges to M, for $\varepsilon = \frac{M^2\varepsilon_0}{2}$ there exists $N_1 \in \mathbb{N}$ such that

$$|y_n - M| < \frac{M^2\varepsilon_0}{2} \qquad \forall\ n \geq N_1.$$

Also if we choose $\varepsilon = \frac{|M|}{2}$, then for this choice of ε there exists $N_2 \in \mathbb{N}$ such that

$$|y_n - M| < \frac{|M|}{2} \qquad \forall\ n \geq N_2$$

Now we know $|M| - |y_n| \leq |y_n - M|$ (by triangle inequality $|a| - |b| \leq |a - b|$). Hence

$$|M| - |y_n| \leq |y_n - M| < \frac{|M|}{2} \ \Rightarrow\ \frac{|M|}{2} < |y_n| \qquad \forall\ n \geq N_2$$

which in turn implies $|y_n| \geq \frac{|M|}{2}, \forall n \geq N_2$ Now we choose $N = max\{N_1, N_2\}$. Then $\forall n \geq N$,

$$\left| \frac{1}{y_n} - \frac{1}{M} \right| = \frac{|y_n - M|}{|M||y_n|} < \frac{2M^2\varepsilon_0}{2M^2} < \varepsilon_0.$$

As $\varepsilon_0 > 0$ is arbitary, so $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{M}$.

□

Remark 2.5. But Convergence of $\{x_n+y_n\}$ does not imply the convergence of $\{x_n\}$ and $\{y_n\}$. For example take $x_n = (-1)^n$ and $y_n = (-1)^{n-1}$.

Example 2.7. *Prove that* $\lim_{n\to\infty} s_n = \frac{1}{4}$ *where*

$$s_n = \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}.$$

We have

$$\begin{aligned} s_n &= \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}. \\ &= \frac{1 + \frac{6}{n} + \frac{7}{n^3}}{4 + \frac{3}{n} - \frac{4}{n^3}} \end{aligned}$$

Now $\lim_{n\to\infty} \frac{1}{n} = 0$. *By Theorem 2.5 (iii),* $\lim_{n\to\infty} \frac{1}{n^3} = 0$. *Again by 2.5 (i) and (iv) we have*

$$\lim_{n\to\infty}(1 + \frac{6}{n} + \frac{7}{n^3}) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{6}{n} + \lim_{n\to\infty} \frac{7}{n^3} = 1 + 0 + 0 = 1$$

And by 2.5 (i), (ii) and (Iv), we have

$$\lim_{n\to\infty}(4 + \frac{3}{n} - \frac{4}{n^3}) = \lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{3}{n} - \lim_{n\to\infty} \frac{4}{n^3} = 4 + 0 -) = 4$$

.

hence by 2.5 (v), we can say $\lim_{n\to\infty} s_n = \frac{1}{4}$.

Corollary 2.6. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences.*

(1) *If* $\{x_n\}$ *and* $\{y_n\}$ *both diverge to infinity, then so is* $\{x_n + y_n\}$ *and* $\{x_n y_n\}$.
(2) *If* $\{x_n\}$ *diverges to infinity, and* $\{y_n\}$ *converges, then* $\{x_n + y_n\}$ *diverges to infinity.*

Proof. (1) Let $G > 0$ be an arbitary large number. Since $\{x_n\}$ and $\{y_n\}$ both diverge to infinity, so there exists one N_1 (depending on G) such that $x_n > \frac{G}{2}$ for all $n \geq N_1$ and there exists one N_2 (depending on G) such that $y_n > \frac{G}{2}$ for all $n \geq N_2$. We choose $N = max(N_1, N_2)$. So for all $n \geq N$ we gave $x_n + y_n > \frac{G}{2} + \frac{G}{2} = G$. So $\{x_n + y_n\}$ diverges.

□

2.3. **Ordered Properties of limits.**

Theorem 2.7. *Let* $\{x_n\}$, $\{y_n\}$ *be two convergent sequences and* $x_n \leq y_n$ $\forall\ n \geq N_0$. *If* $\lim_{n\to\infty} x_n = L$ *and* $\lim_{n\to\infty} y_n = M$. *Then* $L \leq M$.

Proof. Let $\varepsilon > 0$ be given. Since $\{x_n\}$ converges to L, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \qquad \forall\ n \geq N_1. \tag{6}$$

Similarly, for the given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|y_n - M| < \varepsilon \qquad \forall\ n \geq N_2. \tag{7}$$

Choose $N = max\{N_1, N_2, N_0\}$. Then $\forall\ n \geq N$

$$\begin{aligned} 0 \leq y_n - x_n &= y_n - M - (x_n - L) + (M - L) \\ &\leq |y_n - M| + |x_n - L| + (M - L) \\ &< \varepsilon + \varepsilon + (M - L) \end{aligned}$$

So from the last step we obtain $0 < 2\varepsilon + (M - L)$ or $L < M + 2\varepsilon$. Since ε is arbitary, we have $L \leq M$. □

Remark 2.6. If $x_n = 0$ for all n i.e the constant sequence 0 in Theorem 2.7, then $L = 0$ and $M \geq 0$.

Example 2.8. *Let $\{a_n\}$ be a sequence of non negative terms such that $\lim_{n\to\infty} a_n = M$. Then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{M}$ whenever $a_n \geq 0\ \forall\ n \in \mathbb{N}$.*

As $\{a_n\}$ is a sequence of non negative terms, this implies $M \geq 0$ by Theorem 2.7 (choosing $y_n = a_n$ for all n and $x_n = 0$ for all n i.e the constant sequence 0.
Case 1) $M = 0$.
Given $\varepsilon > 0$. As $\{a_n\}$ converges to 0, for the given $\varepsilon^2 > 0$, there exists $N_{\varepsilon^2} \in \mathbb{N}$ such that

$$\begin{aligned} a_n = |a_n - 0| < \varepsilon^2\ \forall\ n \geq N_{\varepsilon^2} \\ \Rightarrow\ \sqrt{a_n} = |\sqrt{a_n} - 0| < \varepsilon\ \forall\ n \geq N_{\varepsilon^2} \end{aligned}$$

Case 2) $M > 0$.
Given $\varepsilon > 0$. As $\{a_n\}$ converges to M, for the given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|a_n - M| < \varepsilon\sqrt{M} \qquad \forall\ n \geq N_{\varepsilon\sqrt{M}}.$$

As $a_n \geq 0$ and $M > 0$ so $\sqrt{a_n} + \sqrt{M} \geq \sqrt{M}$, we have from the above expression

$$|\sqrt{a_n} - \sqrt{M}| = \frac{|a_n - M|}{\sqrt{a_n} + \sqrt{M}} < \frac{|a_n - M|}{\sqrt{M}} \leq \frac{\varepsilon\sqrt{M}}{\sqrt{M}} = \varepsilon \quad \forall\ n \geq N_{\varepsilon\sqrt{M}}$$

So $\{a_n\}$ converges to $\sqrt{M}$.

Corollary 2.8. *Let $\{x_n\}$, $\{y_n\}$ be two sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$. Then*

(1) *If $\lim_{n\to\infty} x_n = \infty$ then $\lim_{n\to\infty} y_n = \infty$.*

(2) *If $\lim_{n\to\infty} y_n = -\infty$ then $\lim_{n\to\infty} x_n = -\infty$.*

Proof. Proof is an exercise. □

Corollary 2.9. *Let* $\{x_n\}$ *be a convergent sequence such that* $\lim_{n\to\infty} x_n = L$ *and* α *and* β *be two real numbers such that* $\alpha \leq x_n \leq \beta\ \forall n \in \mathbb{N}$. *Then we have* $\alpha \leq L \leq \beta$.

Proof. Proof follows from Theorem (2.7). □

Theorem 2.10. Sandwich theorem for sequences *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be three convergent sequences such that* $x_n \leq y_n \leq z_n\ \forall n \in \mathbb{N}$. *If* $\lim_{n\to\infty} x_n = L$ *and* $\lim_{n\to\infty} z_n = L$ *then* $\lim_{n\to\infty} y_n = L$.

Proof. Since $\{x_n\}$ converges to L there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \qquad \forall\ n \geq N_1. \tag{8}$$

Similarly, for the given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|z_n - L| < \varepsilon \qquad \forall\ n \geq N_2. \tag{9}$$

Choose $N = max\{N_1, N_2\}$. Then by (8) we have

$$L - \varepsilon < x_n \quad \forall\ n \geq N$$

and by (9), we have

$$z_n < L + \varepsilon \quad \forall\ n \geq N$$

. Thus

$$L - \varepsilon < x_n \leq y_n \leq z_n \leq L + \varepsilon.$$

Thus $|y_n - L| < +\varepsilon\ \ \forall\ n \geq N$. Hence the proof.

□

Example 2.9. *We have* $\lim_{n\to\infty} n^{1/n} = 1$.

Write $n^{1/n} = 1 + x_n$. *Then, we have by Binomial expansion,*

$$\begin{aligned} n &= (1 + x_n)^n \\ &\geq x_n^2 \binom{n}{2} \\ &> x_n^2 \frac{n(n-1)}{2} \quad \forall n \geq 2 \end{aligned}$$

So $x_n < \sqrt{\frac{2}{n-1}}$. *Now,* $\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$. *Also* $x_n > 0$, *so by Sandwich theorem* $\lim_{n\to\infty} x_n = 0$ *Hence* $\lim_{n\to\infty} n^{1/n} = 1 + 0$.

Example 2.10. *Let* $\{x_n\}$ *be a sequence of real numbers such that* $x_n > 0$ *for all* n *and* $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$. *Then*
1. if $\lambda < 1$ *then* $\lim_{n\to\infty} x_n = 0$,
2. if $\lambda > 1$ *then* $\lim_{n\to\infty} x_n = \infty$.

Proof. 1. Since $\lambda < 1$, we can find $\varepsilon_1 > 0$ (for example $\varepsilon_1 = \frac{1-\lambda}{2}$) such that $\lambda + \varepsilon_1 < 1$. As $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$. So for this $\varepsilon_1 > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|\frac{x_{n+1}}{x_n} - \lambda| < \varepsilon_1 \qquad \forall\ n \geq N_{\varepsilon_1}$$
$$\frac{x_{n+1}}{x_n} < \lambda + \varepsilon_1 = r \text{ (say)} \qquad \forall\ n \geq N_{\varepsilon_1}$$

Now for $n > N_{\varepsilon_1}$, we have

$$0 < x_n < rx_{n-1} < r^2 x_{n-2} < \ldots\ldots < r^{n-N_{\varepsilon_1}} x_{n-(n-N_{\varepsilon_1})} = r^n \frac{x_{N_{\varepsilon_1}}}{r^{N_{\varepsilon_1}}} = C_{N_{\varepsilon_1}} r^n$$

where $C_{N_{\varepsilon_1}} = \frac{x_{N_{\varepsilon_1}}}{r^{N_{\varepsilon_1}}}$. As $r = \lambda + \varepsilon_1 < 1$, so $\lim_{n\to\infty} r^n = 0$ (by example 2.2). Hence by Theorem 2.7 or by Theorem 2.10, $\lim_{n\to\infty} x_n = 0$.

2. Since $\lambda > 1$, then we can find $\varepsilon_1 > 0$ (for example $\varepsilon_1 = \frac{\lambda-1}{2}$) such that $\lambda - \varepsilon_1 > 1$. As $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$. So for this $\varepsilon_1 > 0$ there exists $N_{\varepsilon_1} \in \mathbb{N}$ such that

$$|\frac{x_{n+1}}{x_n} - \lambda| < \varepsilon_1 \qquad \forall\ n \geq N_{\varepsilon_1}$$
$$u := \lambda - \varepsilon_1 < \frac{x_{n+1}}{x_n} \qquad \forall\ n \geq N_{\varepsilon_1}$$

Now for $n > N_{\varepsilon_1}$, we have

$$x_n > ux_{n-1} > u^2 x_{n-2} > \ldots\ldots > u^{n-N_{\varepsilon_1}} x_{n-(n-N_{\varepsilon_1})} = u^n \frac{x_{N_{\varepsilon_1}}}{u^{N_{\varepsilon_1}}} = C_{N_{\varepsilon_1}} u^n$$

where $C_{N_{\varepsilon_1}} = \frac{x_{N_{\varepsilon_1}}}{u^{N_{\varepsilon_1}}}$. As $u = \lambda - \varepsilon_1 > 1$, so $\lim_{n\to\infty} u^n = \infty$ (by example 2.2). Hence by Corollary 2.8, $\lim_{n\to\infty} x_n = \infty$.

For example Let $x_n = \frac{n}{2^n}$. Then $\lambda = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} (1 + \frac{1}{n}).\frac{1}{2} = \frac{1}{2} < 1$. So $\{\frac{n}{2^n}\}$ converges.
If $\lambda = 1$ then we cannot make any conclusion. For example, consider the two sequences $\{n\}$ (diverges) and $\{\frac{1}{n}\}$ (converges to 0) separately. $\square$

3. Monotone Sequences

One of the problems with using the definition of convergence to prove a given sequence converges is the limit of the sequence must be known in order to verify that the sequence converges. This gives rise in the best cases to a "chicken and egg" problem of somehow determining the limit before you even know the sequence converges. In the worst case, there is no nice representation of the limit to use, so you don't even have a "target" to shoot at. The next few sections are ultimately concerned with removing this deficiency from Definition 2.5, but some interesting side-issues are explored along the way. Not surprisingly, we begin with the simplest case.

Definition 3.1. Monotone sequence A sequence $\{a_n\}$ of real numbers is called a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is called a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence that is monotonically increasing or monotonically decreasing is called a monotone sequence.

Example 3.1. (1) *The sequences* $\{1 - \frac{1}{n}\}$, $\{n\}$ *are monotonically increasing sequences.*
(2) *The sequences* $\{\frac{1}{n}\}$, $\{\frac{1}{n^2}\}$ *are monotonically decreasing sequences.*
(3) *The sequences* $\{(-1)^n\}$, $\{\cos(n\pi)\}$, $\{(-1)^n n\}$ *are not monotonic. sequences.*

Theorem 3.1. *(i) A monotonically increasing sequence which is bounded above is convergent and it converges to its supremum.*
(ii) A monotonically decreasing sequence which is bounded below is convergent and it converges to its infimum.

Proof. (i) Let $\{a_n\}$ be a monotonically increasing, bounded above sequence and $M = Sup_{n\geq 1} a_n$. We claim that M is the limit of the sequence $\{a_n\}$. Indeed, let $\varepsilon > 0$ be given. Since $M - \varepsilon$ is not an upper bound for $\{a_n\}$, there exists $N \in \mathbb{N}$ such that $a_N > M - \varepsilon$. As the sequence is monotonically increasing, we have $M - \varepsilon < a_N \leq a_n$ for all $n \geq N$. Also it is clear that $a_n \leq M$ for all $n \in \mathbb{N}$. Thus,

$$M - \varepsilon < a_n \leq M + \varepsilon \quad \forall n \geq N$$

Hence the proof.
(ii) Similar to the previous one.

□

Example 3.2. *The sequence* $\{\frac{1}{n}\}$ *is monotonically decreasing sequences. And from the 1st chapter, we know the set* $S = \{\frac{1}{n};\ n \in \mathbb{N}\}$ *has infimum* 0*. That is why* $\lim_{n\to\infty} \frac{1}{n} = 0$ *by the above Theorem.*

Example 3.3. *The sequence* $\{\frac{n}{n+1}\}$ *is monotonically increasing sequences. Then for all* $n \in \mathbb{N}$,

$$x_n = \frac{n}{n+1} = \frac{(n+1)}{(n+2)} \frac{n(n+2)}{(n+1)^2} = \frac{(n+1)}{(n+2)} \frac{(n^2+2n)}{(n^2+2n+1)} \leq \frac{n+1}{n+2} = x_{n+1}$$

And from the 1st chapter, we know the set $S = \{\frac{n}{n+1};\ n \in \mathbb{N}\}$ *has Supremum* 1*. That is why* $\lim_{n\to\infty} \frac{n}{n+1} = 1$ *by the above Theorem.*

Corollary 3.2. *(i) A monotonically increasing sequence which is not bounded above diverges to* ∞*.*
(ii) A monotonically decreasing sequence which is not bounded below diverges to $-\infty$*.*

Proof. (i) Let $\{a_n\}$ be a monotonically increasing, not a bounded above sequence. For any real number G (however large) there exists $n_0 \in \mathbb{N}$ such that $G < a_{n_0}$. Now $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Hence $G < a_{n_0} < a_n\ \forall n \geq n_0$. So $\{a_n\}$ diverges to infinity.
(ii) Similar to the previous one. □

Example 3.4. *$\{n\}$ is monotonically increasing sequence. And from the 1st chapter, we know the set $S = \{n;\ n \in \mathbb{N}\}$ is not bounded above. That is why $\{n\}$ diverges to infinity by the above Theorem.*

Example 3.5. *Suppose $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. Prove that the sequence converges and it converges to the limit 2.*
Obviously $x_n > 0$. We use induction to show $x_n < 2$. As $x_1 = \sqrt{2} < 2$. Suppose $x_k < 2$ for some $k \in \mathbb{N}$. Then $x_{k+1} = \sqrt{2 + x_k} < \sqrt{2+2} = 2$ ($x_k < 2$ implies $x_k + 2 < 2 + 2 = 4$). So by induction $x_n < 2$ for all $n \in \mathbb{N}$. Next we prove $\{x_n\}$ is a monotonically increasing sequence that is $x_n \leq x_{n+1}$ for all $n \geq 1$. Suppose it is not. That means there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} \leq x_{n_0}$ which implies

$$\sqrt{2 + x_{n_0}} \leq x_{n_0} \ \Rightarrow\ 2 + x_{n_0} \leq x_{n_0}^2 \ \Rightarrow 2 \leq x_{n_0}(x_{n_0} - 1)$$

But $x_{n_0}(x_{n_0} - 1) < 2.1$ (as $x_n < 2$ for all $n \in \mathbb{N}$). So we obtain

$$2 \leq x_{n_0}(x_{n_0} - 1) < 2 \ \Rightarrow\ 2 < 2$$

which is a contradiction. Hence $x_n \leq x_{n+1}$ for all $n \geq 1$. This implies $\{x_n\}$ is a monotonically increasing sequence. And also bounded above by 2. So by the Theorem 3.1, the sequence converges.
Suppose $\lim_{n\to\infty} x_n = l$. As $x_{n+1} = \sqrt{2 + x_n}$ this will imply

$$\begin{aligned} l &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} \sqrt{2 + x_n} \\ &= \sqrt{2 + \lim_{n\to\infty} x_n} \ \textit{(by Theorem 2.5 and example 2.5)} \\ &= \sqrt{2 + l} \end{aligned}$$

Or equivalently, $l^2 = l + 2$. Solving we will get $l = 2$ or -1. But $\{x_n\}$ is a sequence of positive terms so by ordered properties of limit $l = \lim_{n\to\infty} x_n \geq 0$. So l can not take the value -1. Therefore $l = 2$.

Example 3.6. *Let $a > 0$ and $x_1 = 1$ and define the recursive sequence*

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$$

Show that this sequence converges and it converges to $\sqrt{a}$.

First, note that $x_n > 0$ for all n (prove it) so that the sequence is bounded below.

Next, let's see if the sequence is monotone decreasing, in which case it would have to converge to some limit. Compute

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{a}{x_n}) = \frac{1}{2}\frac{(x_n^2 - a)}{x_n}$$

Now let's take a look at $(x_n^2 - a)$*:*

$$\begin{aligned}(x_n^2 - a) &= \frac{1}{4}(x_{n-1} + \frac{a}{x_{n-1}})^2 - a \\ &= \frac{1}{4}(x_{n-1} - \frac{a}{x_{n-1}})^2 \text{ \textit{check}} \\ &\geq 0\end{aligned}$$

But that means that $x_n - x_{n+1} \geq 0$*, or equivalently* $x_n \geq x_{n+1}$*. Hence, the sequence is monotone decreasing and bounded below by* 0 *so it must converge. We now know that* $\lim_{n\to\infty} x_n = L$*. To find that limit, we could try the following:*

$$\begin{aligned}L &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} \frac{1}{2}(x_n + \frac{a}{x_n}) \\ &= \frac{1}{2}(\lim_{n\to\infty} x_n + \lim_{n\to\infty} \frac{a}{x_n}) \text{ \textit{(by Theorem 2.5 and example 2.5)}} \\ &= (L + \frac{a}{L})\end{aligned}$$

or equivalently $L^2 = a \;\Rightarrow L \pm \sqrt{a}$*. But* $\{x_n\}$ *is a sequence of positive terms which means that the limit* L *is indeed the square root of* a*, as required.*

4. Subsequences

So far we have learned the basic definitions of a sequence (a function from the natural numbers to the reals), the concept of convergence, and we have extended that concept to one which does not pre-suppose the unknown limit of a sequence (Cauchy sequence). Unfortunately, however, not all sequences converge. We will now introduce some techniques for dealing with those sequences. The firrst is to change the sequence into a convergent one (extract subsequences).

Definition 4.1. Subsequence: Let $\{a_n\}$ be a sequence and $\{n_1, n_2, ...\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}$ is called a subsequence of $\{a_n\}$. In other words, let $\{a_n\}$ be a sequence and $\sigma : \mathbb{N} \to \mathbb{N}$ be a function such that $m < n$ implies $\sigma(m) < \sigma(n)$; i.e., $\sigma(n)$ is a strictly increasing sequence of natural numbers. Then $b_n = a_{\sigma(n)}$ is a subsequence of $\{a_n\}$.

The idea here is that the subsequence $\{b_n\}$ is a new sequence formed from an old sequence $\{a_n\}$ by possibly leaving terms out of $\{a_n\}$. In other words, all the terms of $\{b_n\}$ must also appear in $\{a_n\}$, and they must appear in the same order.

Example 4.1. *Suppose $\{a_n\}$ is a sequence and $\{1, 2, ...\}$ be a sequence of positive integers then $\{a_n\}$ is a subsequence of itself.*

Example 4.2. *Let $\{2, 4, 8, ...\}$ be a sequence of positive integers and $\{a_n\}$ be a sequence. Then the subsequence $\{a_{n_i}\}$ looks like*

$$a_2, a_4, a_8, ..., a_{2n}, ...$$

The subsequence has every second term of the original sequence.

Example 4.3. *Let $\{3, 6, 9, ...\}$ be a sequence of positive integers and $\{a_n\}$ be a sequence. Then the subsequence $\{a_{n_i}\}$ looks like*

$$a_3, a_6, a_9, ..., a_{3n}, ...$$

The subsequence has every third term of the original sequence.

Example 4.4. *For example, $\{\frac{1}{k^2}\}$ and $\{\frac{1}{2^k}\}$ are the subsequences of the sequence $\{\frac{1}{n}\}$ where $n_k = k^2$ (for the subsequence $\{\frac{1}{k^2}\}$) and $n_k = 2^k$ (for the subsequence $\{\frac{1}{2^k}\}$).*

Definition 4.2. Subsequential Limit Let $\{a_n\}$ be a sequence. Let $\{a_{n_j}\}$ a subsequence of $\{a_n\}$. Suppose that $\{a_{n_j}\}$ converges to a limit l that is $\lim_{r\to\infty} a_{n_r} = l$. Then l is called a subsequential limit of $\{a_n\}$.

Example 4.5. *For example, $\{-1, -1, \cdots\}$ and $\{1, 1, \cdots\}$ are the subsequences of the sequence $\{(-1)^n\}$. And -1 and 1 are the limits of the subsequences $\{-1, -1, \cdots\}$ and $\{1, 1, \cdots\}$ respectively. But $\{(-1)^n\}$ does not converge.*

Theorem 4.1. *The sequence of real numbers $\{a_n\}$ is convergent to L, iff every subsequence of $\{a_n\}$ is also convergent to L.*

Proof. Let us assume $\{a_n\}$ is convergent to L. Required to prove every subsequence of $\{a_n\}$ is also convergent to L. Let $\{n_i\}$ be a sequence of positive integers such that $\{a_{n_i}\}$ forms a subsequence of $\{a_n\}$. Let $\varepsilon > 0$ be given. As $\{a_n\}$ converges to L, then there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon, \quad \forall n \geq N.$$

Choose $M \in \mathbb{N}$ such that $n_i \geq N$ for $i \geq M$ (why we can choose such an M?). Then

$$|a_{n_i} - L| < \varepsilon, \quad \forall i \geq M.$$

Hence the proof.

Conversely, suppose every subsequence of $\{a_n\}$ is also convergent to L. Since $\{a_n\}$ is a subsequence of itself, it is obvious that $\{a_n\}$ is also convergent to L. $\square$

Example 4.6. *For example,* $\{\frac{1}{k^2}\}$ *and* $\{\frac{1}{2^k}\}$ *are the subsequences of the sequence* $\{\frac{1}{n}\}$*. Now we know* $\lim_{n\to\infty}\frac{1}{n}=0$ *So* $\lim_{n\to\infty}\frac{1}{n^2}=0$ *as well as* $\lim_{n\to\infty}\frac{1}{2^n}=0$ *by the above Theorem.*

Remark 4.1. It's also apparent that when $a_n \to \infty$ or $a_n \to -\infty$ then all its subsequences diverge in the same way by Theorem 4.1.

Remark 4.2. In some cases you do not need to show that every subsequence of $\{a_n\}$ is convergent to prove that the original sequence $\{a_n\}$ is convergent. For example

Example 4.7. *If a monotonic sequence has a convergent subsequence then the monotonic sequence is convergent. (H.W)*
Show that the original sequence is bounded above and then use Theorem 3.1.

Theorem 4.2. *Every sequence has a monotone subsequence.*

Proof. Let $\{x_n\}$ be a sequence of real numbers, and call the m-th term of the sequence a "peak" if for all $n \geq m$ we have that $x_m \geq x_n$. In other words, we will call the m-th term a peak if every successive term is less than or equal in size. We note that an increasing sequence will have no peaks as each successive term is larger than the previous, while a decreasing sequence will have infinitely many peaks as each successive term is smaller than the previous. Now consider the follow two cases.
Case 1: Suppose that the sequence $\{x_n\}$ has infinitely many peaks. Form a subsequence with these peaks denoted $x_{n_1}, x_{n_2}, ..., x_{n_k}, ...$ Since each of these terms is a peak and $n_1 < n_2 < .... < n_k < ....$, we have that $x_{n_1} \geq x_{n_2} \geq .... \geq x_{n_k} \geq ....$ and so $\{x_{n_k}\}$ is a monotonic decreasing sub sequence of $\{x_n\}$.

Case 2 : Suppose that the sequence $\{x_n\}$ only has a finite number of peaks denoted $x_{n_1}, x_{n_2}, ..., x_{n_k}$. Let $s_1 = n_k + 1$, that is let s_1 be the first index after the last peak in the sequence. Then we have that s_1 is not a peak, and so then there exists an s_2 such that $x_{s_1} < x_{s_2}$ and $s_1 \leq s_2$. Also, s_2 is not a peak and so there exists an s_3 such that $s_2 < s_3$ and $x_{s_2} \leq x_{s_3}$. Continue on and crease a subsequence $x_{s_1} \leq x_{s_2} \leq ... \leq x_{s_n} \leq ....$, and so $\{x_{s_m}\}$ is a monotonic increasing subsequence of $\{x_n\}$. Thus we have shown that whether $\{x_n\}$ has infinitely many or finitely many peaks, then $\{x_n\}$ regardless has a monotonic subsequence.

□

5. **Bolzano - Weierstrass Theorem**

A fundamental tool used in the analysis of the real line is the well-known Bolzano-Weierstrass Theorem. The Bolzano–Weierstrass theorem is named after mathematicians Bernard Bolzano and Karl Weierstrass. It was actually first proved by Bolzano in 1817 as a lemma in the proof of the intermediate value theorem. Some fifty years later the result was identified as significant in its own right, and proved again by Weierstrass. It has since become an essential theorem of analysis.

Theorem 5.1. Bolzano - Weierstrass Theorem. *Every bounded sequence has a convergent subsequence.*

Proof. **Method 2** Let $\{x_n\}$ be a bounded non-constant sequence. By Theorem 4.2, $\{x_n\}$ has a monotone subsequence say $\{x_{n_k}\}$. Now $\{x_n\}$ is bounded so is $\{x_{n_k}\}$. So we have $\{x_{n_k}\}$ is a bounded monotone sequence, so it converges to its Supremum or infimum (depending on whether the sequence is monotonically increasing or decreasing) by Theorem 3.1. □

6. **Cauchy Sequences**

Often the biggest problem with showing that a sequence converges using the techniques we have seen so far is we must know ahead of time to what it converges. This is the "chicken and egg" problem mentioned above. An escape from this dilemma is provided by Cauchy sequences.

Definition 6.1. A sequence $\{a_n\}$ is called a Cauchy sequence if for any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (depending on $\varepsilon > 0$) such that

$$|a_n - a_m| < \varepsilon \quad \forall\ n, m \geq N$$

This definition is a bit more subtle than it might at first appear. It sort of says that all the terms of the sequence are close together from some point onward. The emphasis is on all the terms from some point onward.

Example 6.1. *The sequence $\{\frac{1}{n}\}$ is a Cauchy sequence. If $\varepsilon > 0$ is given, we choose a natural number $N_\varepsilon = N$ such that $N > \frac{2}{\varepsilon}$ (By Archimedean Property). Then if $m, n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \frac{\varepsilon}{2}$ and similarly $\frac{1}{m} \leq \frac{1}{N} < \frac{\varepsilon}{2}$. Therefore, it follows that if $m, n \geq N$, then*

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\{\frac{1}{n}\}$ is a Cauchy sequence.

Example 6.2. *The sequence $\{(1 + (-1)^n)\}$ is not a Cauchy sequence.*
The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every N there exist at least one $n_N > N$ and at least one $m_N > N$ such that $|x_{n_N} - x_{m_N}| > \varepsilon_0$.
For the terms $x_n := (1 + (-1)^n)$, we observe that if n is even, then $x_n = 2$ and $x_{n+1} = 0$. If we take $\varepsilon_0 = 2$, then for any N we can choose an even number $n > N$ and let $m := n + 1$ to get

$$|x_n - x_{n+1}| = 2 = \varepsilon_0$$

We conclude that it is not a Cauchy sequence.

Remark 6.1. Remark We emphasize that to prove a sequence $\{x_n\}$ is a Cauchy sequence, we may not assume a relationship between m and n, since the required inequality $|x_n - x_m| < \varepsilon$ must hold for all $n, m \geq N_\varepsilon$. But to prove a sequence is not a Cauchy sequence, we may specify a relation between n and m as long as arbitrarily large values of n and m can be chosen so that $|x_n - x_m| > \varepsilon_0$.

Lemma 6.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Let $\{a_n\}$ be a sequence such that $\{a_n\}$ converges to L (say). Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|a_k - L| < \varepsilon \quad \forall k \geq N.$$

Thus if $n, m \geq N$, we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \varepsilon + \varepsilon = 2\varepsilon$$

Thus $\{a_n\}$ is Cauchy. □

Lemma 6.2. *If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded.*

Proof. Since $\{a_n\}$ forms a Cauchy sequence, for $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1, \quad \forall n, m \geq N.$$

Hence if $n \geq N$, then

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N|, \quad \forall n \geq N.$$

Let $M = max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1+|a_N|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$is bounded.

□

Lemma 6.3. *If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is convergent.*

Proof. Let $\{a_{n_k}\}$ be a monotone subsequence of the Cauchy sequence $\{a_n\}$ by Theorem 4.2. Then $\{a_{n_k}\}$ is a bounded, monotone subsequence (as by previous Theorem every Cauchy sequence is bounded). Hence $\{a_{n_k}\}$ converges to L(say). Now we claim that the sequence $\{a_n\}$ itself converges to L. Let $\varepsilon > 0$. Choose N_1, N_2 such that

$$n, k \geq N_1 \Longrightarrow |a_n - a_{n_k}| < \varepsilon/2$$
$$k \geq N_2 \Longrightarrow |a_{n_k} - L| < \varepsilon/2.$$

(Remember $n_k \geq k$ for any $k \in \mathbb{N}$). Hence

$$n, n_k \geq max\{N_1, N_2\} \Longrightarrow |a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

Therefore, by Lemmas 6.1 and 6.3 we have the following Criterion:

Cauchy's Criterion for convergence: A sequence $\{a_n\}$ converges if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall\, m, n \geq N.$$

Example 6.3. *Suppose $0 < \alpha < 1$ and $\{x_n\}$ is a sequence satisfying the contractive condition: $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n|$ for $n = 1, 2, 3, ...$ Then $\{x_n\}$ satisfies the Cauchy criterion.*

To show $\{x_n\}$ is a Cauchy sequence, let $\varepsilon > 0$ be given. Since $\alpha^n \to 0$ as $n \to \infty$ we can choose $N \in \mathbb{N}$ so that

$$\frac{\alpha^{m-1}}{1-\alpha}|x_2 - x_1| < \varepsilon \quad \forall n \geq N$$

Note that for any $n \in \mathbb{N}$,

$$|x_{k+2} - x_{k+1}| \leq \alpha|x_{k+1} - x_k| \leq \alpha^2|x_k - x_{k-1}| \leq .... \leq \alpha^k|x_2 - x_1|$$

Fo $n > m \geq N$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - .... + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + ... + |x_{m+1} - x_m| \\
&\leq (\alpha^{n-2} + \alpha^{n-3} + ... + \alpha^{m-1})|x_2 - x_1| \\
&\leq \alpha^{m-1}\frac{(1-\alpha^{n-m})}{1-\alpha}|x_2 - x_1| \\
&\leq \frac{\alpha^{m-1}}{1-\alpha}|x_2 - x_1| \\
&< \varepsilon
\end{aligned}$$

Thus $\{x_n\}$ satisfies the Cauchy criterion. Therefore it converges as every Cauchy sequence converges (by Lemma 6.3).

Example 6.4. *Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n}$. Then*

$$|x_{n+2} - x_{n+1}| = \frac{1}{(2 + x_{n+1})(2 + x_n)}|x_n - x_{n+1}| < \frac{1}{4}|x_n - x_{n+1}|.$$

Therefore $\{x_n\}$ satisfies the contractive condition with $\alpha = 1/4$ and hence it satisfies the Cauchy criterion by previous example. Therefore it converges as every Cauchy sequence converges (by Lemma 6.3).

7. **Lower and Upper Limit of a sequence**

There are an uncountable number of strictly increasing functions $\sigma : \mathbb{N} \to \mathbb{N}$ so every sequence $\{a_n\}$ has an uncountable number of subsequences. If $\{a_n\}$ converges, then Theorem 4.1 shows all of these subsequences converge to the same limit. It's also apparent that when $a_n \to \infty$ or $a_n \to -\infty$ then all its subsequences diverge in the same way. When $\{a_n\}$ does not converge or diverge to ∞ the situation is a bit more difficult because some subsequences may converge and others may diverge. The limit inferior and limit superior of a sequence can be thought of as limiting (i.e., eventual and extreme) bounds on the sequence.

Definition 7.1. Let $\{a_n\}$ be a bounded sequence. For each $j \geq 1$ consider the set

$$E_j := \{a_j, a_{j+1}, ...\} = \{a_n \mid n \geq j\}$$

it is clear from the construction of the set

$$E_1 \supseteq E_2 \supset E_3 \supset E_3 \supseteq ....E_j \supset E_{j+1} \supseteq ....$$

For each $j \geq 1$ denote by $\beta_j := SupE_j = Sup\{a_j, a_{j+1}, ...\} = Sup\{a_n \mid n \geq j\}$ is in $(-\infty, \infty)$.
Then one can easily see that

$$\beta_1 \geq \beta_2 \geq .... \geq \beta_j \geq \beta_{j+1} \geq ......$$

Hence $\{\beta_k\}$ is a decreasing sequence which is bounded from below, so by Theorem 3.1, $\lim_{k\to\infty} \beta_k$ exists and it is the infimum of the set i.e $\lim_{k\to\infty} \beta_k = Inf_{k\geq 1}\beta_k$. We call this limit as **limit superior** of the sequence $\{a_n\}$. In notation we write

$$\overline{\lim}\ a_n \ or\ \lim sup\ a_n$$

Similarly, for each $j \geq 1$ denote by $\alpha_j := InfE_j = Inf\{a_j, a_{j+1}, ...\} = Inf\{a_n \mid n \geq j\}$ is in $(-\infty, \infty)$.

Then one can easily see that

$$\alpha_1 \leq \alpha_2 \leq .... \leq \alpha_j \leq \alpha_{j+1} \leq ......$$

Hence $\{\alpha_k\}$ is an increasing sequence which is bounded from above, so by Theorem 3.1, $\lim_{k\to\infty} \alpha_k$ exists and it is the infimum of the set. We call this limit as **limit inferior** of the sequence $\{a_n\}$. In notation we write

$$\underline{\lim}\ a_n \ or\ \lim inf\ a_n$$

Remark 7.1. Now we know $\alpha_j = InfE_j \leq SupE_j = \beta_j$. So by Theorem 2.7, $\lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} \beta_n$ Giving

$$\underline{\lim}\ a_n \leq \overline{\lim}\ a_n.$$

Example 7.1. *If* $a_n = -n$, *then* $E_j = \{-j, -j-1, ...\}$. *So* $SupE_j = \beta_j = -j$. *Hence* $\overline{\lim}\ a_n = -\infty$. *Also* $\underline{\lim}\ a_n = -\infty$.

Example 7.2. *Consider next the sequence* $1, -1, 1, -2, 1, -3, 1, -4, ...$ *Then* $SupE_j = \beta_j = 1$ *for all* $j \geq 1$. *Hence* $\overline{\lim}\ a_n = 1$. *Also* $\underline{\lim}\ a_n = -\infty$.

Example 7.3. *Consier the sequence* $a_n = \sin\frac{n\pi}{3}$. *Then by listing the elements of* $\{a_n\}$ *it is easy to see that* $\beta_j = \frac{\sqrt{3}}{2}$ *and* $\alpha_j = -\frac{\sqrt{3}}{2}$. *Therefore*

$$\underline{\lim}\ a_n = -\frac{\sqrt{3}}{2}\ and\ \overline{\lim}\ a_n = \frac{\sqrt{3}}{2}.$$

Example 7.4. *Consider the sequence* $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, ...\}$. *Then for large* $k \in \mathbb{N}$.
For k *odd,* $E_k = \{\frac{1}{k+1}, \frac{k+1}{k+2}, \frac{1}{k+3}, \frac{k+3}{k+4} \cdots\}$ *and*
For k *even* $E_k = \{\frac{k}{k+1}, \frac{1}{k+2}, \frac{k+2}{k+3}, \frac{k+3}{k+4} \cdots\}$. *So in any case*

$$\frac{k}{k+1} = 1 - \frac{1}{k+1} < \beta_k = sup\{a_m, m \geq k\} < 1$$

$$0 < \alpha_k = inf\{a_m, m \geq k\} \leq \frac{1}{k}$$

Then by sandwich theorem, we see that $\lim sup\ a_n = 1$ *and* $\lim inf\ a_n = 0$.

Theorem 7.1. *If* $\{a_n\}$ *be a convergent sequence of real numbers. Then*

$$\overline{\lim}\ a_n = \underline{\lim}\ a_n = \lim_{n\to\infty} a_n.$$

Proof. Let $L = \lim_{n\to\infty} a_n$. Then given $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon\ \forall\ n \geq N_\varepsilon$$

$$L - \varepsilon < a_n < L + \varepsilon\ \forall\ n \geq N_\varepsilon$$

So For all $n \geq N_\varepsilon$, $L + \varepsilon$ is an upper bound of the set $\{a_n, a_{n+1}, a_{n+2}, ...\}$ and $L - \varepsilon$ is not an upper bound of the set $\{a_n, a_{n+1}, a_{n+2}, ...\}$.

$$L - \varepsilon < \beta_n = Sup\{a_n, a_{n+1}, a_{n+2}, ...\} < L + +\varepsilon \quad \text{for all} n \geq N_\varepsilon$$

So by Theorem 2.7,

$$L - \varepsilon \leq \lim_{n\to\infty} \beta_n \leq L + \varepsilon.$$

But $\lim_{n\to\infty} \beta_n = \overline{\lim}\ a_n$. Thus

$$L - \varepsilon \leq \overline{\lim}\ a_n \leq L + \varepsilon.$$

As ε is arbitary, this implies $\overline{\lim}\ a_n = L$.
Similarly, we can prove $\underline{\lim}\ a_n = \lim_{n\to\infty} a_n$. □

Theorem 7.2. *If* $\{a_n\}$ *is a bounded sequence and if* $\lim sup\ a_n = \lim inf a_n = L$, *for* $L \in \mathbb{R}$, *then* $\{a_n\}$ *is a convergent sequence and* $\lim_{n\to\infty} a_n = L$.

Proof. Given $\varepsilon > 0$ We notice that

$$inf_{j\geq 1}\beta_j = \lim sup\ a_n = L$$

where $\beta_j = Sup\{a_j, a_{j+1}, ...\}$. As $inf_{j\geq 1}\beta_j = L < L + \varepsilon$, then by the property of infimum, there exists $N_1 \geq 1$ (depending on $\varepsilon > 0$) such that

$$L \leq \beta_{N_1} < L + \varepsilon$$

which in turn implies

$$\beta_{N_1} = Sup\{a_{N_1}, a_{N_1+1}, ...\} < L + \varepsilon$$

So we can write

$$a_n < L + \varepsilon \text{ for all } n \geq N_1. \tag{10}$$

Similarly, using the property of limit infimum, we can say there exists $N_2 \geq 1$ (depending on $\varepsilon > 0$) such that

$$L - \varepsilon < a_n \text{ for all } n \geq N_2. \tag{11}$$

Next we choose $N = max\{N_1, N_2\}$, then by (10) and (11), we have

$$L - \varepsilon < a_n < L + \varepsilon \text{ for all } n \geq N.$$

Hence $\lim_{n\to\infty} a_n = L$. □

Theorem 7.3. *If $\{a_n\}$ is a bounded sequence, then there exists subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ such that*

$$\lim sup\ a_n = \lim a_{n_k} \text{ and } \lim inf a_n = \lim a_{m_k}.$$

Proof. Since $\{a_n\}$ is bounded, $\lim sup\ a_n = \beta$ exists. Then from the definition, $inf_{j\geq 1}\beta_j = \lim sup\ a_n = \beta$. Then by the property of Infimum for each $k \geq 1$, there exists one $j_0 \geq 1$ (j_0 depends on k) such that

$$\beta \leq \beta_{j_0} < \beta + \frac{1}{k}$$

$$\alpha - \frac{1}{k} < \alpha < \beta_{j_0} = Sup\{a_{j_0}, a_{j_0+1}, a_{j_0+2}, ...a_{j_0+n}, ...\} < \alpha + \frac{1}{k}$$

Since $\beta - \frac{1}{k} < \beta_{j_0} = Sup\{a_{j_0}, a_{j_0+1}, ...\} = SupE_{j_0}$, therefore by property of the supremum there exists $a_{j_0+m} \in E_{j_0}$ for some $m \in \mathbb{N}$. Let us rename a_{j_0+m} as a_{n_k}. Hence

$$\beta - \frac{1}{k} < a_{n_k} < \beta + \frac{1}{k}$$

. Therefore, $a_{n_k} \to \beta$ as $k \to \infty$. Similarly, one can obtain $\{a_{m_k}\}$. □

Theorem 7.4. *If there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim a_{n_k} = t$. Then $t \leq \beta = \lim sup\ a_n$.*

Proof. Suppose NOT. Then we can choose $\varepsilon > 0$ such that $t - \beta > \varepsilon$. As $inf_{j\geq 1}\beta_j = \lim sup\ a_n = \beta < t - \varepsilon$. Then by the property of infimum, there exists one $j_0 \geq 1$ such that

$$\beta \leq \beta_{j_0} < t - \varepsilon$$

$$\beta \leq \beta_{j_0} = Sup\{a_{j_0}, a_{j_0+1}, ...\} < t - \varepsilon$$

So for all $n \geq j_0$, we have

$$a_n < t - \varepsilon$$

There fore $|a_n - t| > \varepsilon$ for all $n \geq j_0$ contradicting the fact $\lim a_{n_k} = t$. So our assumption is wrong. □

Corollary 7.5. *If there exists a subsequence $\{a_{m_k}\}$ of $\{a_n\}$ such that $\lim a_{m_k} = s$. Then $s \geq \alpha = \lim inf\ a_n$.*

Proof. Same as the above. □

Remark 7.2. From the above two theorems we can say that the limsup is the supremum of all limits of subsequences of a sequence. Similarly the lim inf is the infimum of all limits of subsequences of a sequence.
In case of unbounded sequences, either limsup or limsup or both can approach infinity. Even in this case, one can show the existence of subsequences that approach infinity.