Ordinary Differential Equations

(MA102 Mathematics II)

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Books

Texts/References:

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4. E. L. Ince, Ordinary Differential Equations, Dover Publications, 1958.

Differential Equations : Formal definition

Definition

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a **differential equation**.

Examples

(i) $\frac{dy}{dx} = x.$

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = sin\ x.$

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Classification by type: Ordinary and Partial

Definition

If a differential equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, then it is called an **ordinary differential equation** (ODE).

Examples

(i) $\frac{dy}{dx} = x$, (ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = sin\ x$.

Definition

If a differential equation contains partial derivatives of one or more dependent variables with respect to two or more independent variables, then it is called a **partial differential equation** (PDE).

Examples

(i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
(ii) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = xy^2$.

Order and Degree of an ODE

In this course, we will deal with ODE s only.

Definition

The **order** of a differential equation (ODE or PDE) is the order of the highest derivative in the equation.

Example

The order of $\frac{d^3y}{dx^3} + 5x\frac{dy}{dx} = ye^x$ is 3.

Definition

The **degree** of a differential equation is the power of the highest order derivative occuring in the differential equation (after rationalizing the differential equation as far as the derivatives are concerned).

Example

$1 + (\frac{d^2y}{dx^2})^{\frac{1}{2}} = \frac{dy}{dx}$. After rationalizing, we get $(\frac{dy}{dx} - 1)^2 = \frac{d^2y}{dx^2}$.
This equation has order 2 and degree 1. Note that the degree is not 2!

General form of ODEs

The general form of a n-th order ODE is given by

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

where F is a real valued function of the $n+2$ many variables $x, y, y', \ldots, y^{(n)}$.
If the equation $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ can be expressed in the form

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

then this new equation is called the **normal form** of the previous equation.

Linear and non-linear ODEs

Definition

The differential equation $F(x,y,y',y'',\ldots,y^{(n)})=0$ is said to be **linear** if F is linear in $y,y',\ldots,y^{(n)}$. In other words, an n-th order linear differential equation can be written as

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_n(x)y=g(x).$$

In a linear differential equation $F(x,y,y',y'',\ldots,y^{(n)})=0$,

- *The dependent variable and its derivatives occur in first degree. For instance, expressions like* $y^2, xy^2, (\frac{dy}{dx})^2$ *should not appear in the differentail equation.*
- *No product of the dependent variable and its derivatives should occur in the differential equation. For instance, expressions like* $y\frac{dy}{dx}$ *should not appear in the differentail equation.*
- *No transcendental functions of the dependent variable and its derivatives should occur in the differential equation. For instance, expressions like* $sin\ y, e^y, log\ y$ *should not appear in the equation.*

Solution of an ODE

Definition

A function ϕ defined on an interval I and possessing at least n derivatives, which when substituted into an n-th order ODE reduces the equation to an identity, is said to be a **solution** of the ODE on the interval I.
In other words, a solution of an n-th order ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ is a function ϕ that possesses at least n derivatives and for which

$$F(x, \phi(x), \ldots, \phi^{(n)}(x)) = 0$$

for all $x \in I$.

Particular solution and Singular Solution

Definition

Any solution to an n-th order ODE involving n arbitrary constants is called a **general solution** of the ODE.

For example, $y = x^2 + c$ is a general solution of the ODE $\frac{dy}{dx} = 2x$.

Definition

If we assign some particular values to these n arbitrary constants, then the resulting solution is called a **particular solution** of the ODE.

For example, $y = x^2$ is a particular solution of the ODE $\frac{dy}{dx} = 2x$. This solution is obtained from the general solution $y = x^2 + c$ by assigning the value 0 to the constant c.

Definition

If a solution to an ODE cannot be obtained from a general solution, then it is called a **singular solution**.

Explicit and Implicit solutions

Consider the n-th order ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

Definition

A solution of the above ODE in which the dependent variable y is expressed solely in terms of the independent variable x and constants, is said to be an **explicit solution** of the ODE.

Definition

The relation $G(x, y) = 0$ is said to be an **implicit solution** of the above ODE if it gives rise to at least one explicit solution of the above ODE.

Example

$y = x^2 + c$ is an explicit solution of the ODE $\frac{dy}{dx} = 2x$. And $x^2 + y^2 = 1$ is an implicit solution of the ODE $y\frac{dy}{dx} + x = 0$ on the interval $[-1, 1]$ because the equation $x^2 + y^2 = 1$ gives rise to 2 explicit solutions of the ODE $y\frac{dy}{dx} + x = 0$ on the interval $[-1, 1]$, namely, $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.

Initial value problem (IVP)

A problem to the type

$$\textit{On some interval } I \textit{ containing } x_0$$

$$\text{Find a solution y(x) of the ODE}$$

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

$$\textit{subject to the conditions}$$

$$y(x_0) = a_0, y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are arbitrarily specified real constants, is called an **Initial value problem** or **IVP**. The conditions $y(x_0) = a_0, y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$ are called **initial conditions**.

The above problem is also called an n-**th order initial value problem**.

A well-posed IVP

An IVP is said to be well-posed if

- it has a solution,
- the solution is unique and,
- the solution is continuously depends on the initial data y_0 and f.

Peano Theorem

Theorem **(Peano's Theorem)**:

Let $R\ :\ |x - x_0| \leq a,\ |y - y_0| \leq b$ be a rectangle. If $f(x,y) \in C(R)$ then the IVP

$$y'(x) = f(x,y), \quad y(x_0) = y_0$$

has at least one solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\{a, \frac{b}{K}\}, \quad K = \max_{(x,y)\in R} |f(x,y)|.$$

Does every IVP has a solution?

Consider the IVP

$$x\frac{dy}{dx} = 4y \; subject \; to \; y(0) = 1$$

It can be easily verified that $y = cx^4$ is a general solution of the ODE $x\frac{dy}{dx} = 4y$, but this general solution does not satisfy the initial condition $y(0) = 1$. Hence the above IVP has **no solution**.

Question: If a IVP has a solution, is it unique?

Answer: No! Consider the following IVP:

Solve $\frac{dy}{dx} = \sqrt{y}$ subject to $y(0) = 0$. Observe that $y \equiv 0$ and $y = \{ \begin{array}{ccc} \frac{x^2}{4} & if & x \geq 0 \\ 0 & if & x < 0 \end{array}$ are two different solutions of the same IVP!

An IVP having a unique solution

Consier the IVP

$$\frac{dy}{dx} = 2x \; subject \; to \; y(0) = 1$$

The general solution of the ODE $\frac{dy}{dx} = 2x$ is given by $y = x^2 + c$. Putting, the initial condition $y(0) = 1$, we get $c = 1$. Hence $y = x^2 + 1$ is a *unique* solution of the above IVP.
Two fundamental questions: (i) When does an IVP has a solution?
(ii) If an IVP has a solution, is it unique or when is it unique?

These 2 questions are answered by the **existence and uniqueness theorem**, which we will state now.

Lipschitz Condition

Definition

Let $f(x, y)$ be a real valued function defined on a bounded domain D. We say that f satisfies Lipschitz condition in the (second) variable y with a Lipschitz constant K if

$$|f(x, y_1) - f(x, y_2)| \leq K\,|y_1 - y_2| \qquad \text{for any } (x, y_1) \text{ and } (x, y_2) \text{ in } D\,.$$

Example: Let $D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 2 \text{ and } -10 < y < 10\}$.
Let $f(x, y) = x^2 \sin(y)$ for $(x, y) \in D$.
Then, for any (x, y_1) and (x, y_2) in D we have

$$|f(x, y_1) - f(x, y_2)| = x^2\,|\sin(y_1) - \sin(y_2)| \leq x^2\,|y_1 - y_2| \leq 4\,|y_1 - y_2|\,.$$

Therefore f satisfies Lipschitz condition on the bounded domain D in the variable y.

Existence and Uniqueness of Solutions to IVPs: Picard Theorem

Let $R = \{(x,\ y) \in \mathbb{R}^2\ :\ |x - x_0| \leq a,\ |y - y_0| \leq b,\ a > 0,\ b > 0\}$ be a closed and bounded rectangle containing the point $(x_0,\ y_0)$.
The initial value problem (⊛) is given by

ODE $\quad y' = f(x,\ y)\quad$ on R ,

Initial Condition: $\quad y(x_0) = y_0$.

Theorem

Picard Theorem: If $f(x, y)$ is continuous and satisfies Lipschitz condition in the variable y with Lipschitz constant K on the closed rectangle R, then there exists a unique solution $y(x)$ to the IVP (⊛) on the interval $|x - x_0| \leq h$ where $h = \min(a,\ b/M)$ and $|f(x, y)| \leq M$ for all $(x, y) \in R$.

Example: IVP with a unique solution

$$\text{ODE:} \qquad y' = 4y \qquad \text{for } x \in \mathbb{R}\,,$$

$$\text{Initial Condition:} \qquad y(0) = 1\,.$$

The solution to this IVP is $y(x) = e^{4x}$ for $x \in \mathbb{R}$.
Further it is a unique solution to this IVP.

Example: IVP having infinitely many solutions

ODE: $y' = 3\, y^{2/3}$ for $x \in \mathbb{R}$,

Initial Condition: $y(0) = 0$.

The solutions to this IVP are:

$$y_c(x) = \begin{cases} 0 & \text{if} \quad x \leq c \\ (x-c)^3 & \text{if} \quad x \geq c \end{cases} \qquad \text{where } c \geq 0 \ .$$

For each real number $c \geq 0$, we have a solution $y_c(x)$ to the IVP.
Therefore, this IVP has infinitely many solutions.

Theorem

Theorem **(Continuous dependence on initial data)**:

Let $f, \frac{\partial f}{\partial y} \in C(R)$ and $(x_0, y_0) \in R$. Let $\phi_1(x)$ be the solution of

$$y' = f(x, y), y(x_0) = y_0,$$

and let $\phi_2(x)$ be the solution of

$$y' = f(x, y), y(x_0) = \hat{y}_0,$$

in R for $|x - x_0| \leq h$.

Then, for $|x - x_0| \leq h$, we have

$$|\phi_1(x) - \phi_2(x)| \leq |y_0 - \hat{y}_0| e^{Lh},$$

where $|(\partial f / \partial y)(x, y)| \leq L$ for all $(x, y) \in R$.

Moreover, as $\hat{y}_0 \to y_0$, $\phi(x, \hat{y}_0) \to \phi(x, y_0)$ uniformly on $[x_0 - h, x_0 + h]$.

Theorem

Theorem(**Continuous dependence on f**):

Let $f_1(x,y)$, $f_2(x,y)$, $\frac{\partial f}{\partial y} \in C(R)$, and $(x_0, y_0) \in R$. Let $\phi(x)$ is the solution of

$$y' = f_1(x,y), \ y(x_0) = y_0,$$

and $\psi(x)$ is the solution of

$$y' = f_2(x,y), \ \ y(x_0) = y_0.$$

Assume that both $\phi(x)$, $\psi(x)$ exist on $[x_0 - h, x_0 + h]$. Then, for $|x - x_0| \leq h$, we have

$$|\phi(x) - \psi(x)| \leq \epsilon h e^{Lh}$$

whenever $|f_1(x,y) - f_2(x,y) \leq \epsilon$ for all $(x,y) \in R$, where $|(\partial f/\partial y)(x,y)| \leq L$ for all $(x,y) \in R$.

Moreover, as $f_1 \to f_2$, $\phi(x) \to \psi(x)$ uniformly on $[x_0 - h, x_0 + h]$.

Method to find the solution to the IVP provided by Picard Theorem (Method of Successive Approximations)

Step 1: Set $y_0(x) = y_0$ for all $x \in \mathbb{R}$.
Step 2: Iterative Step
Compute

$$y_n(x) = y_0(x) + \int_{x_0}^{x} f(s,\ y_{n-1}(s))\, ds \qquad \text{for } n = 1,\ 2,\ \cdots$$

This $y_n(x)$ is called the Picard Successive Approximation or Picard Iterate.
Step 2: Limit of Iterates

$$y(x) := \lim_{n\to\infty} y_n(x) \qquad \text{for } \ x \in I = [x_0 - h,\ x_0 + h]\ .$$

Under the hypothesis of Picard's theorem, $\{y_n(x)\}$ converges uniformly on the interval I and the limit function $y(x)$ is the unique solution of the given IVP ($\circledast$) in I.

Worked out Example

Solve $y' = 2y$ with $y(0) = 1$ by the method of successive approximations.
Here $f(x, y) = 2y$ for $(x, y) \in \mathbb{R}^2$.

Step 1: Initial Approximation

Set $y_0(x) = y_0 = 1$ for all $x \in \mathbb{R}$.

Step 2: Computing Successive Approximations

$$y_1(x) = y_0(x) + \int_{x_0}^{x} f(s,\, y_0(s))\, ds = 1 + \int_0^x 2\, ds = 1 + 2x\,.$$

$$y_2(x) = y_0(x) + \int_{x_0}^{x} f(s,\, y_1(s))\, ds = 1 + \int_0^x 2(1 + 2s)\, ds = 1 + 2x + 2x^2\,.$$

$$y_3(x) = y_0(x) + \int_{x_0}^{x} f(s,\, y_2(s))\, ds = 1 + \int_0^x 2(1 + 2s + 2s^2)\, ds = 1 + 2x + 2x^2 + \frac{4x^3}{3}\,.$$

Step 3: Limit of Successive Approximations (if possible compute)

$$y_n(x) = \sum^{n} \frac{(2x)^k}{k!} \to e^{2x} =: y(x) \text{ for } x \in \mathbb{R} \text{ as } n \to \infty\,.$$

Corollary to Picard Theorem

Let $R = \{(x,\ y) \in \mathbb{R}^2\ :\ |x - x_0| \leq a,\ |y - y_0| \leq b,\ a > 0,\ b > 0\}$ be a closed and bounded rectangle containing the point $(x_0,\ y_0)$.
The initial value problem (⊛) is given by

ODE $\quad y' = f(x,\ y)$ on R ,

Initial Condition: $\quad y(x_0) = y_0$.

Corollary to Picard Theorem: If $f(x, y)$ is continuous and $\dfrac{\partial f}{\partial y}$ is continuous on the closed rectangle R, then there exists a unique solution $y(x)$ to the IVP (⊛) on the interval $|x - x_0| \leq h$ where $h = \min(a,\ b/M)$ and $|f(x, y)| \leq M$ for all $(x, y) \in R$.

If $\dfrac{\partial f}{\partial y}$ is continuous on R then by applying the MVT for the variable y, it follows that f satisfies Lipschitz condition in the variable y with Lipschitz constant K on R and then Picard theorem completes the remaining proof

Direction field

Definition

Suppose we have the differential equation

$$\frac{dy}{dx} = f(x, y).$$

Then the **direction field** corresponding to this differential equation will be a collection of line segments drawn at various points (x, y) of a rectangular grid (xy-plane) such that at each point (x, y), the slope of the line segment equals the value $f(x, y)$.

To see how to plot an approximate solution to a differential equation of the form $\frac{dy}{dx} = f(x, y)$ passing through a point (x_0, y_0), please look at the black board.

Interpretation

interpret the existence and uniqueness theorem for the equations:

$$(a)\ \ y' = -2y; \qquad (b)\ y' = -y/x.$$

Interpretation contd...

For (a), choose a starting point x_0 and initial value $y(x_0) = y_0$. Since $f(x, y) = -2y \in C^1$ for all x, y, we can enclose (x_0, y_0) in a rectangle R and conclude that the IVP has one and only one solution curve passing through (x_0, y_0).

For (b), $f(x, y) = -y/x$ does not meet the continuity conditions when $x = 0$. However, for any $x_0 \neq 0$ and any initial value $y(x_0) = y_0$, we can enclose (x_0, y_0) in a rectangle of continuity that excludes the y-axis. Thus, we can be assured of a unique solution curve passing through (x_0, y_0).

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First order ODE s

We will now discuss different methods of solutions of first order ODEs. The first type of such ODEs that we will consider is the following:

Definition

Separable variables: A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called **separable** or to have **separable variables**.

Such ODEs can be solved by direct integration:
Write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)dx$ and then integrate both sides!

Example

$e^x \frac{dy}{dx} = e^{-y} + e^{-2x-y}$
This equation can be rewritten as $\frac{dy}{dx} = e^{-x}e^{-y} + e^{-3x-y}$, which is the same as $\frac{dy}{dx} = e^{-y}(e^{-x} + e^{-3x})$. This equation is now in separable variable form.

Losing a solution while separating variables

Some care should be exercised in separating the variables, since the variable divisors could be zero at certain points. Specifically, if r is a zero of the function $h(y)$, then substituting $y = r$ in the ODE $\frac{dy}{dx} = g(x)h(y)$ makes both sides of the equation zero; in other words, $y = r$ is a constant solution of the ODE $\frac{dy}{dx} = g(x)h(y)$. But after variables are separated, the left hand side of the equation $\frac{dy}{h(y)} = g(x)dx$ becomes undefined at r. As a consequence, $y = r$ might not show up in the family of solutions that is obtained after integrating the equation $\frac{dy}{h(y)} = g(x)dx$.
Recall that such solutions are called Singular solutions of the given ODE.

Example

Observe that the constant solution $y \equiv 0$ is lost while solving the IVP $\frac{dy}{dx} = xy; y(0) = 0$ by separable variables method.

First order linear ODEs

Recall that a **first order linear ODE** has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{1}$$

Definition

A first order linear ODE (of the above form (1)) is called **homogeneous** if $g(x) = 0$ and **non-homogeneous** otherwise.

Definition

By dividing both sides of equation (1) by the leading coefficient $a_1(x)$, we obtain a more useful form of the above first order linear ODE, called the **standard form**, given by

$$\frac{dy}{dx} + P(x)y = f(x). \tag{2}$$

Equation (2) is called the **standard form** of a first order linear ODE.

Theorem

Theorem

Existence and Uniqueness: *Suppose* $a_1(x), a_0(x), g(x) \in C((a,b))$ *and* $a_1(x) \neq 0$ *on* (a,b) *and* $x_0 \in (a,b)$. *Then for any* $y_0 \in \mathbb{R}$, *there exists a unique solution* $y(x) \in C^1((a,b))$ *to the IVP*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x); y(x_0) = y_0.$$

Solving a first order linear ODE

Steps for solving a first order linear ODE:

(1) Transform the given first order linear ODE into a first order linear ODE in standard form $\frac{dy}{dx} + P(x)y = f(x)$.

(2) Multiply both sides of the equation (in the standard form) by $e^{\int P(x)dx}$*. Then the resulting equation becomes*

$$\frac{d}{dx}[ye^{\int P(x)dx}] = f(x)e^{\int P(x)dx} \qquad (3).$$

(3) Integrate both sides of equation (3) *to get the solution.*

Example

Solve $x\frac{dy}{dx} - 4y = x^6e^x$.

The standrad form of this ODE is $\frac{dy}{dx} + (\frac{-4}{x})y = x^5e^x$. Then multiply both sides of this equation by $e^{\int \frac{-4}{x}dx}$ and integrate.

Differential of a function of 2 variables

Definition

Differential of a function of 2 variables: If $f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential df is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

In the special case when $f(x, y) = c$, where c is a constant, we have $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Therefore, we have $df = 0$, or in other words,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

So given a one-parameter family of functions $f(x, y) = c$, we can generate a first order ODE by computing the differential on both sides of the equation $f(x, y) = c$.

Exact differential equation

Definition

A differential expression $M(x,y)dx + N(x,y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined on R. A first order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact equation** if the expression on the left hand side is an exact differential.

Example: 1) $x^2y^3dx + x^3y^2dy = 0$ is an exact equation since $x^2y^3dx + x^3y^2dy = d(\frac{x^3y^3}{3})$.
2) $ydx + xdy = 0$ is an exact equation since $ydx + xdy = d(xy)$.
3) $\frac{ydx-xdy}{y^2} = 0$ is an exact equation since $\frac{ydx-xdy}{y^2} = d(\frac{x}{y})$.

Criterion for an exact differential

Theorem

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition for $M(x,y)dx + N(x,y)dy$ to be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example

Solve the ODE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$.
This equation can be expressed as $M(x,y)dx + N(x,y)dy = 0$ where $M(x,y) = 3x^2 + 4xy$ and $N(x,y) = 2x^2 + 2y$. It is easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$. Hence the given ODE is exact.
We have to find a function f such that $\frac{\partial f}{\partial x} = M = 3x^2 + 4xy$ and $\frac{\partial f}{\partial y} = N = 2x^2 + 2y$. Now $\frac{\partial f}{\partial x} = 3x^2 + 4xy \Rightarrow f(x,y) = \int (3x^2 + 4xy)dx = x^3 + 2x^2y + \phi(y)$ for some function $\phi(y)$ of y.
Again $\frac{\partial f}{\partial y} = 2x^2 + 2y$ and $f(x,y) = x^3 + 2x^2y + \phi(y)$ together imply that
$2x^2 + \phi'(y) = 2x^2 + 2y \Rightarrow \phi(y) = y^2 + c_1$ for some constant c_1. Hence the solution is
$f(x,y) = c$ or $x^3 + 2x^2y + y^2 + c_1 = c$.

Converting a first order non-exact DE to exact DE

Consider the following example:

Example

The first order DE $ydx - xdy = 0$ is clearly not exact. But observe that if we multiply both sides of this DE by $\frac{1}{y^2}$, the resulting ODE becomes $\frac{dx}{y} - \frac{x}{y^2}dy = 0$ which is exact!

Definition

It is sometimes possible that even though the original first order DE $M(x,y)dx + N(x,y)dy = 0$ is not exact, but we can multiply both sides of this DE by some function (say, $\mu(x,y)$) so that the resulting DE $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ becomes exact. Such a function/factor $\mu(x,y)$is known as an **integrating factor** for the original DE $M(x,y)dx + N(x,y)dy = 0$.

***Remark**: It is possible that we LOSE or GAIN solutions while multiplying a ODE by an integrating factor.*

How to find an integrating factor?

We will now list down some rules for finding integrating factors, but before that, we need the following definition:

Definition

A function $f(x,y)$ is said to be **homogeneous** of **degree** n if $f(tx,ty) = t^n f(x,y)$ for all (x,y) and for all $t \in \mathbb{R}$.

Example

1) $f(x,y) = x^2 + y^2$ is homogeneous of degree 2.
2) $f(x,y) = tan^{-1}(\frac{y}{x})$ is homogeneous of degree 0.
3) $f(x,y) = \frac{x(x^2+y^2)}{y^2}$ is homogeneous of degree 1.
4) $f(x,y) = x^2 + xy + 1$ is NOT homogeneous.

How to find an integrating factor? contd...

Definition

A first order DE of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be **homogeneous** if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

NOTE: Here the word “homogeneousÂ¨ does not mean the same as it did for first order linear equation $a_1(x)y' + a_0(x)y = g(x)$ when $g(x) = 0$.

Some rules for finding an integrating factor: Consider the DE

$$M(x,y)dx + N(x,y)dy = 0. \qquad (*)$$

Rule 1: If $(*)$ is a homogeneous DE with $M(x,y)x + N(x,y)y \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor for $(*)$.

How to find an integrating factor? contd...

Rule 2: If $M(x, y) = f_1(xy)y$ and $N(x, y) = f_2(xy)x$ and $Mx - Ny \neq 0$, where f_1 and f_2 are functions of the product xy, then $\frac{1}{Mx-Ny}$ is an integrating factor for $(*)$.

Rule 3: If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for $(*)$.

Rule 4: If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = F(y)$ (function of y-alone), then $e^{-\int F(y)dy}$ is an integrating factor for $(*)$.

Proof of Rule 3

Proof.

Let $f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$. To show: $\mu(x) := e^{\int f(x)dx}$ is an integrating factor. That is, to show $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$.
Since μ is a function of x alone, we have $\frac{\partial}{\partial y}(\mu M) = \mu \frac{\partial M}{\partial y}$. Also $\frac{\partial}{\partial x}(\mu N) = \mu'(x)N + \mu(x)\frac{\partial N}{\partial x}$.
So we must have:
$\mu(x)[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = \mu'(x)N$, or equivalently we must have,
$\frac{\mu'(x)}{\mu(x)} = f(x)$,
which is anyways true since $\mu(x) := e^{\int f(x)dx}$. □

The proof of Rule 4 is similar. The proof of Rule 2 is an exercise.

Another rule for finding an I.F.

- If a differential equation is in the special form

$$y(Ax^p y^q + Bx^r y^s)dx + x(Cx^p y^q + Dx^r y^s)dy = 0,$$

where A, B, C, D are constants, then an I.F. has the form $\mu(x, y) = x^a y^b$, where a and b are suitably chosen constants.

Solution by substitution

Often the first step of solving a differential equation consists of transforming it into another differential equation by means of a ***substitution****.*
For example, suppose we wish to transform the first order differential equation $\frac{dy}{dx} = f(x,y)$ *by the substitution* $y = g(x,u)$*, where* u *is regarded as a function of the variable* x*. If* g *possesses first partial derivatives, then the chain rule*

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx}$$

gives $\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$*. The original differential equation* $\frac{dy}{dx} = f(x,y)$ *now becomes* $g_x(x,u) + g_u(x,u)\frac{du}{dx} = f(x, g(x,u))$*. This equation is of the form* $\frac{du}{dx} = F(x,u)$*, for some function* F*. If we can determine a solution* $u = \phi(x)$ *of this last equation, then a solution of the original differential equation will be* $y = g(x, \phi(x))$*.*

Use of substitution : Homogeneous equations

Recall: A first order differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be **homogeneous** if both M and N are homogeneous functions of the same degree.
Such equations can be solved by the substitution : $y = vx$.

Example

Solve $x^2 y dx + (x^3 + y^3)dy = 0$.
Solution: The given differential equation can be rewritten as $\frac{dy}{dx} = \frac{x^2 y}{x^3+y^3}$.
Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Putting this in the above equation, we get $v + x\frac{dv}{dx} = \frac{v}{1+v^3}$. Or in other words, $(\frac{1+v^3}{v^4})dv = -\frac{dx}{x}$, which is now in separable variables form.

DE reducible to homogeneous DE

For solving differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

use the substitution

- $x = X + h$ and $y = Y + k$, if $\frac{a}{a'} \neq \frac{b}{b'}$, where h and k are constants to be determined.
- $z = ax + by$, if $\frac{a}{a'} = \frac{b}{b'}$.

Example

Solve $\frac{dy}{dx} = \frac{x+y-4}{3x+3y-5}$.
Observe that this DE is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $\frac{a}{a'} = \frac{b}{b'}$.
Use the substitution $z = x + y$. Then we have $\frac{dz}{dx} = 1 + \frac{dy}{dx}$. Putting these in the given DE, we get $\frac{dz}{dx} - 1 = \frac{z-4}{3z-5}$, or in other words, $\frac{3z-5}{4z-9}dz = dx$. This equation is now in separable variables form.

DE reducible to homogeneous DE, contd...

Example

Solve $\frac{dy}{dx} = \frac{x+y-4}{x-y-6}$.

Observe that this DE is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $1 = \frac{a}{a'} \neq \frac{b}{b'} = -1$.

Put $x = X + h$ and $y = Y + k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and

$$\frac{dY}{dX} = \frac{X + Y + (h + k - 4)}{X - Y + (h - k - 6)}. \qquad (*)$$

If h and k are such that $h + k - 4 = 0$ and $h - k - 6 = 0$, then $(*)$ becomes

$$\frac{dY}{dX} = \frac{X + Y}{X - Y}$$

which is a homogeneous DE. We can easily solve the system

$$h + k = 4$$

$$h - k = 6$$

of linear equations to detremine the constants h and k!

Reduction to separable variables form

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C),$$

where A, B, C are real constants with $B \neq 0$ can always be reduced to a differential equation with separable variables by means of the substitution $u = Ax + By + C$.

Observe that since $B \neq 0$, we get $\frac{u}{B} = \frac{A}{B}x + y + \frac{C}{B}$, or in other words, $y = \frac{u}{B} - \frac{A}{B}x - \frac{C}{B}$. This implies that $\frac{dy}{dx} = \frac{1}{B}(\frac{du}{dx}) - \frac{A}{B}$. Hence we have $\frac{1}{B}(\frac{du}{dx}) - \frac{A}{B} = f(u)$, that is, $\frac{du}{dx} = A + Bf(u)$. Or in other words, we have $\frac{du}{A+Bf(u)} = dx$, which is now in separable variables form.

Equations reducible to linear DE: Bernoullii's DE

Definition

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1}$$

where n is any real number, is called **Bernoullii's differential equation**.

Note that when $n = 0\ or\ 1$, Bernoullii's DE is a linear DE.
Method of solution: Multiply by y^{-n} throughout the DE (1) to get

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x). \tag{2}$$

Use the substitution $z = y^{1-n}$. Then $\frac{dz}{dx} = (1-n)\frac{1}{y^n}\frac{dy}{dx}$. Substituting in equation (2), we get $\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$, which is a linear DE.

Example of Bernoullii's DE

Example

Solve the Bernoullis DE $\frac{dy}{dx} + y = xy^3$.
Multiplying the above equation throughout by y^{-3}, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x.$$

Putting $z = \frac{1}{y^2}$, we get $\frac{dz}{dx} - 2z = -2x$, which is a linear DE.
The integrating factor for this linear DE will be $= e^{-\int 2dx} = e^{-2x}$. Therefore, the solution is $z = e^{2x}[-2\int xe^{-2x}dx + c] = x + \frac{1}{2} + ce^{2x}$. Putting back $z = \frac{1}{y^2}$ in this, we get the final solution $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$.

Ricatti's DE

The differential equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

is known as **Ricatti's differential equation**. A Ricatti's equation can be solved by method of substitution, provided we know a paticular solution y_1 of the equation.
Putting $y = y_1 + u$ in the Ricatti's DE, we get

$$\frac{dy_1}{dx} + \frac{du}{dx} = P(x) + Q(x)[y_1 + u] + R(x)[y_1^2 + u^2 + 2uy_1].$$

But we know that y_1 is a particular solution of the given Ricatti's DE. So we have $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$. Therefore the above equation reduces to

$$\frac{du}{dx} = Q(x)u + R(x)(u^2 + 2uy_1)$$

or, $\frac{du}{dx} - [Q(x) + 2y_1(x)R(x)]u = R(x)u^2$, which is Bernoulli's DE.

Orthogonal Trajectories

Orthogonal Trajectories

Suppose

$$\frac{dy}{dx} = f(x, y)$$

represents the DE of the family of curves. Then, the slope of any orthogonal trajectory is given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \quad \text{or} \quad -\frac{dx}{dy} = f(x, y),$$

which is the DE of the orthogonal trajectories.

Example: Consider the family of circles $x^2 + y^2 = c^2$. Differentiate w.r.t x to obtain $x + y\frac{dy}{dx} = 0$. The differential equation of the orthogonal trajectories is $x + y\left(-\frac{dx}{dy}\right) = 0$. Separating variable and integrating we obtain $y = c\,x$ as the equation of the orthogonal trajectories.

MA 102
Mathematics II
Lecture 3

4 March, 2015

Initial value problem (IVP)

A problem to the type

$$\textit{On some interval } I \textit{ containing } x_0$$

$$\text{Find a solution y(x) of the ODE}$$

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

$$\textit{subject to the conditions}$$

$$y(x_0) = a_0, y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are arbitrarily specified real constants, is called an **Initial value problem** or **IVP**. The conditions $y(x_0) = a_0, y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$ are called **initial conditions**. The above problem is also called an n-**th order initial value problem**.

First and second order IVPs

For example,

$$Solve\ \frac{dy}{dx} = f(x, y)$$

$$subject\ to\ y(x_0) = a_0$$

is a *first order IVP* and

$$Solve\ \frac{d^2y}{dx^2} = f(x, y, y')$$

$$subject\ to\ y(x_0) = a_0\ and\ y'(x_0) = a_1$$

is a *second order IVP*.

Geometric Interpretaion

The following picture gives a geometric interpretation of the IVP

$$Solve\ \frac{dy}{dx} = f(x,y)$$
$$subject\ to\ y(x_0) = a_0$$

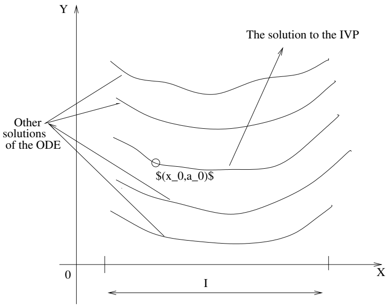

Figure: Solution of the above first order IVP

Geometric interpretaion contd...

Exercise: Try to think, what will be the geometric interpretation of the second order IVP

$$Solve\ \frac{d^2y}{dx^2} = f(x, y, y')$$

$$subject\ to\ y(x_0) = a_0\ and\ y'(x_0) = a_1$$

Does every IVP has a solution?

Consider the IVP

$$x\frac{dy}{dx} = 4y \; subject \; to \; y(0) = 1$$

It can be easily verified that $y = cx^4$ is a general solution of the ODE $x\frac{dy}{dx} = 4y$, but this general solution does not satisfy the initial condition $y(0) = 1$. Hence the above IVP has **no solution**.

Question: If a IVP has a solution, is it unique?

Answer: No! Consider the following IVP:

Solve $\frac{dy}{dx} = \sqrt{y}$ subject to $y(0) = 0$. Observe that $y \equiv 0$ and $y = \{ \begin{array}{ccc} \frac{x^2}{4} & if & x \geq 0 \\ 0 & if & x < 0 \end{array}$ are two different solutions of the same IVP!

An IVP having a unique solution

Consier the IVP

$$\frac{dy}{dx} = 2x \; subject \; to \; y(0) = 1$$

The general solution of the ODE $\frac{dy}{dx} = 2x$ is given by $y = x^2 + c$. Putting, the initial condition $y(0) = 1$, we get $c = 1$. Hence $y = x^2 + 1$ is a *unique* solution of the above IVP.

Two fundamental questions: (i) When does an IVP has a solution?
(ii) If an IVP has a solution, is it unique or when is it unique?

These 2 questions are answered by the **existence and uniqueness theorem**, which we now state for first order IVPs.

Existence and Uniqueness theorem

Theorem

Consider the IVP

$$\frac{dy}{dx} = f(x,y) \text{ subject to } y(x_0) = y_0. \quad (*)$$

(a) If f is continuous on an open rectangle

$$R := \{(x,y) | a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then there exists an open subinterval I of (a,b) containing x_0 on which the above IVP $(*)$ has at least one solution.
(b) If both f and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists an open subinterval I of (a,b) containing x_0 on which $(*)$ has a unique solution.

Remark: (i) Part (a) is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0, but provides no information about how to find the solution or how to determine the open subinterval on which it exists. Moreover, (a) provides no information on the number of solutions that the IVP $(*)$ can have.

Remarks on the existence and uniqueness theorem

Remark: (ii) Part (b) is a *uniqueness theorem*. It guarantees that the IVP $(*)$ has a unique solution on some open subinterval I of (a, b) that contains x_0.
Example: Consider the IVP

$$\frac{dy}{dx} = x^2 + y^2 \ subject \ to \ y(0) = 0$$

Observe that by the existence and uniqueness theorem, the above IVP has a unique solution on some interval (a, b) containing the point 0.
Remark: (iii) It is quite possible that the IVP $(*)$ has more than one solutions on an interval larger than I or larger than (a, b). For instance, look at exercise (2) in the next to next slide.

Remark on the existence and uniqueness theorem

Remark: If $(a, b) \neq (-\infty, \infty)$, then the IVP $(*)$ may have more than one solutions on a larger interval that contains (a, b).
For example, it may happen that $b < \infty$ and all solutions of the IVP $(*)$ have the same values on $(a, b]$. But two solutions y_1 and y_2 may exist on some interval (a, b_1) where $b_1 > b$ such that

$$y_1(x) = y_2(x) \; \forall x \in (a, b]$$

$$But \; y_1(x) \neq y_2(x) \; \forall x \in (b, b_1).$$

So the graphs of the two solutions y_1 and y_2 will "branch off" in different directions at $x = b$.
This implies that $y_1(b) = y_2(b)$ (call their common value $\overline{y}$) and y_1, y_2 are both solutions of the IVP

$$y' = f(x, y), \; y(b) = \overline{y} \qquad (**)$$

that differ on every open interval that contains b. Therefore, f or f_y must have a discontinuity at some point in each open rectangle that contains the point $(b, \overline{y})$; because if this were not so, the IVP $(**)$ would have a unique solution on some open interval containing b.
We can do a similar analysis of the case where $a > -\infty$.

Some exercises ...

Exercises : 1) Consider the IVP

$$y' = \frac{10}{3}xy^{2/5}, y(x_0) = y_0$$

For what points (x_0, y_0) does the above IVP has a solution? And for what points (x_0, y_0) does this IVP has a unique solution on some open interval containing x_0?
2) Observe that it follows from exercise (1) that the IVP

$$y' = \frac{10}{3}xy^{2/5}, y(0) = -1$$

has a unique solution on some open interval that contains the point 0. Find a solution and determine the largest open interval (a, b) on which it is unique. [Hint: You will see that there are two solutions of this IVP on $(-\infty, \infty)$ which coincide on the interval $(-1, 1)$, but on no larger open interval.]

Remark on the existence and uniqueness theorem

Remark: Note that the conditions given in the statement of the existence and uniqueness theorem are only sufficient conditions, not necessary. In other words, if the IVP $(*)$ in the statement of the existence and uniqueness theorem has a solution (or a unique solution) on some open interval containing x_0, then it need not always be true that $f(x, y)$ (or $\frac{\partial f}{\partial y}$) is continuous on an open rectangle containing the point (x_0, y_0). If the conditions stated in the hypothesis of the existence and uniqueness theorem are not satisfied, then anything can happen: the IVP $(*)$ may have a unique solution or several solutions or no solution at all!

A problem to keep in mind: *Suppose that the first order ODE $\frac{dy}{dx} = f(x, y)$ possesses a one parameter family of solutions and $f(x, y)$ satisfies the hypothesis of the existence and uniqueness theorem on some open rectangle R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R?*

Ordinary Differential Equations

(MA102 Mathematics II)

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Reduction of order

Consider the following second order linear homogeneous DE given by

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \qquad (1)$$

We can construct a second solution y_2 of equation (1) on an interval I from a known non-trivial solution y_1. We seek a second solution $y_2(x)$ of (1) so that y_1 and y_2 are linearly independent on I. Recall that if y_1 and y_2 are linearly independent, then their ratio $\frac{y_2}{y_1}$ is non-constant on I. That is, $y_2(x) = u(x)y_1(x)$ for some non-constant function $u(x)$. The idea is to find the function $u(x)$ by substituting $y_2(x) = u(x)y_1(x)$ into the given DE. This method is called **reduction of order** since after substituting $y_2(x) = u(x)y_1(x)$ into the given DE, we have to solve a linear first order DE to find $u(x)$.

Example: A 2nd solution by reduction of order

Example

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, we will use reduction of order to find a second solution y_2.

Solution: If $y_2(x) = u(x)y_1(x) = u(x)e^x$, then the product rule gives

$$y_2' = ue^x + e^x u', \; y_2'' = ue^x + 2e^x u' + e^x u'',$$

$$\Rightarrow y_2'' - y_2 = e^x(u'' + 2u') = 0.$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, then this linear 2nd order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Solving this, we get $w = c_1 e^{-2x}$ or $u' = c_1 e^{-2x}$. Integrating, we get $u = -\frac{c_1}{2} e^{-2x} + c_2$. Thus

$$y_2(x) = u(x)y_1(x) = u(x)e^x = -\frac{c_1}{2} e^{-x} + c_2 e^x.$$

By picking $c_2 = 0$ and $c_1 = -2$ we obtain the desired second solution $y_2 = e^{-x}$. Because $W(e^x, e^{-x}) \neq 0$ for every x, the solutions y_1 and y_2 are linearly independent on $(-\infty, \infty)$.

General case

Suppose we divide equation (1) by $a_2(x)$ in order to put equation (1) in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \qquad (2)$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Let us suppose further that $y_1(x)$ is a known solution of (2) on I and that $y_1(x) \neq 0$ for every $x \in I$.
A second solution of equation (2) is

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx.$$

Proof

Let $Ly := y'' + P(x)y' + Q(x)y$. Suppose $y_2(x) = u(x)y_1(x)$ and $L(y_2) = 0$. Then we have

$$u(y_1'' + Py_1' + Qy_1) + u''y_1 + u'(2y_1' + Py_1) = 0$$

Since $L(y_1) = 0$, we have

$$u''y_1 + u'(2y_1' + Py_1) = 0 \Rightarrow \frac{u''}{u'} = -2\frac{y_1'}{y_1} - P.$$

Integrating, we get

$$u' = \frac{1}{y_1^2} e^{-\int P(x)dx} \Rightarrow u(x) = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx.$$

Thus the second solution is $y_2(x) = u(x)y_1(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$.
Exercise: Verify that the function $y_2(x)$ defined above satisfies equation (2) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero. Show that $W(y_1, y_2) = e^{-\int P(x)dx}$.

Homogeneous Linear equations with constant coefficients

Aim: To find a basis for $Ker(L)$. That is, to find a set of fundamental solution to the homogeneous equation $L(y) = 0$, where

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

and $a_n \neq 0$, $a_{n-1}, \ldots, a_0$ are real constants.
For $y = e^{rx}$, we find

$$\begin{aligned} L(e^{rx}) &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} \\ &= e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0) = e^{rx} P(r), \end{aligned}$$

where $P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$.
Thus $L(e^{rx}) = 0$ provided r is a root of the auxiliary equation

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0.$$

Case I

Case I (Distinct real roots): Let $r_1, \ldots, r_n$ be real and distinct roots. The n solutions are given by

$$y_1(x) = e^{r_1 x},\ y_2(x) = e^{r_2 x}, \ldots, y_n(x) = e^{r_n x}.$$

We need to show

$$c_1 e^{r_1 x} + \cdots + c_n e^{r_n x} = 0 \Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

$P(r)$ can be factored as

$$P(r) = a_n(r - r_1)(r - r_2)\cdots(r - r_n).$$

Writing the operator $L(y)$ as

$$L = P(D) = a_n(D - r_1)\cdots(D - r_n).$$

Now, construct the polynomial $P_k(r)$ by deleting the factor $(r - r_k)$ from $P(r)$. Then

$$L_k := P_k(D) = a_n(D - r_1)\cdots(D - r_{k-1})(D - r_{k+1})\cdots(D - r_n).$$

Case I contd...

By linearity

$$L_k(\sum_{i=1}^{n} c_i e^{r_i x}) = L_k(0) \Rightarrow c_1 L_k(e^{r_1 x}) + \cdots + c_n L_k(e^{r_n x}) = 0.$$

Since $L_k = P_k(D)$, we find that $L_k(e^{rx}) = e^{rx} P_k(r)$ for all r. Thus

$$\sum_{i=1}^{n} c_i e^{r_i x} P_k(r_i) = 0 \Longrightarrow c_k e^{r_k x} P_k(r_k) = 0,$$

as $P_k(r_i) = 0$ for $i \neq k$. Since r_k is not a root of $P_k(r)$, then $P_k(r_k) \neq 0$. This yields $c_k = 0$. As k is arbitrary, we have

$$c_1 = c_2 = \cdots = c_n = 0.$$

Theorem: If $P(r) = 0$ has n distinct roots $r_1, r_2, \ldots, r_n$. Then the general solution of $L(y) = 0$ is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example

Example: Consider $y'' - 3y' + 2y = 0$. The auxiliary equation $P(r) = r^2 - 3r + 2 = 0$ has two roots $r_1 = 1, \quad r_2 = 2$. The general solution is $y(x) = C_1 e^x + C_2 e^{2x}$..

Case II-Repeated real roots

Theorem: If $P(r) = 0$ has the real root r_1 occurring m times and the remaining roots $r_{m+1}, r_{m+2}, \ldots, r_n$ are distinct, then the general solution of $L(y) = 0$ is

$$\begin{aligned} y(x) &= (C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1})e^{r_1 x} \\ &\quad + C_{m+1} e^{r_{m+1} x} + \cdots + C_n e^{r_n x}, \end{aligned}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example

Example: Consider $y^{(4)} - 8y'' + 16y = 0$. In this case, $r_1 = r_2 = 2$ and $r_3 = r_4 = -2$. The general solution is

$$y = (C_1 + C_2 x)e^{2x} + (C_3 + C_4 x)e^{-2x}.$$

Case III-Complex roots

Theorem

If $P(r) = 0$ has non-repeated complex roots $\alpha + i\beta$ and $\alpha - i\beta$, then the corresponding part of the general solution is

$$y(x) = e^{\alpha x}(C_1 cos(\beta x) + C_2 sin(\beta x)).$$

If $\alpha + i\beta$ and $\alpha - i\beta$ are each repeated roots of multiplicity m, then the corresponding part of the general solution is

$$y(x) = e^{\alpha x}[(C_1 + C_2 x + \cdots + C_m x^{m-1}) cos(\beta x)$$

$$+(C_{m+1} + C_{m+2} x + \cdots + C_{2m} x^{m-1}) sin(\beta x)],$$

where $C_1, C_2, \ldots, C_{2m}$ are arbitrary constants.

Example

Consider $y'''' - 2y''' + 2y'' - 2y' + y = 0$. Here $r_1 = r_2 = 1$, $r_3 = i$, $r_4 = -i$. The general solution is

$$y(x) = (C_1 + C_2 x)e^x + (C_3 cos\ x + C_4 sin\ x).$$

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A non-trivial example

Remark: There was a mistake in the last slide of the last lecture.
Example: Recall that we considered the IVP

$$\frac{dy}{dx} + y = f(x); y(0) = 0, where\ f(x) = \left\{ \begin{array}{lll} 1 & if & 0 \leq x \leq 1 \\ 0 & if & x > 1. \end{array} \right.$$

Observe that this IVP has discontinuous non-homogeneous term and the (**continuous**) function $y(x)$ defined by

$$y(x) = \left\{ \begin{array}{lll} 1 - e^{-x} & if & 0 \leq x \leq 1 \\ (e-1)e^{-x} & if & x > 1. \end{array} \right.$$

satisfies the above IVP separately on the intervals $[0, 1]$ and $(1, \infty)$.

Exercise: Think why is it technically incorrect to say that the above function $y(x)$ is a solution to the IVP on the interval $[0, \infty)$.

Differential of a function of 2 variables

Definition

Differential of a function of 2 variables: If $f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential df is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

In the special case when $f(x, y) = c$, where c is a constant, we have $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Therefore, we have $df = 0$, or in other words,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.$$

So given a one-parameter family of functions $f(x, y) = c$, we can generate a first order ODE by computing the differential on both sides of the equation $f(x, y) = c$.

Exact differential equation

Definition

A differential expression $M(x,y)dx + N(x,y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined on R. A first order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact equation** if the expression on the left hand side is an exact differential.

Example: 1) $x^2y^3dx + x^3y^2dy = 0$ is an exact equation since $x^2y^3dx + x^3y^2dy = d(\frac{x^3y^3}{3})$.
2) $ydx + xdy = 0$ is an exact equation since $ydx + xdy = d(xy)$.
3) $\frac{ydx - xdy}{y^2} = 0$ is an exact equation since $\frac{ydx - xdy}{y^2} = d(\frac{x}{y})$.

Criterion for an exact differential

Theorem

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition for $M(x, y)dx + N(x, y)dy$ to be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example

Solve the ODE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$.
This equation can be expressed as $M(x, y)dx + N(x, y)dy = 0$ where $M(x, y) = 3x^2 + 4xy$ and $N(x, y) = 2x^2 + 2y$. It is easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$. Hence the given ODE is exact.
We have to find a function f such that $\frac{\partial f}{\partial x} = M = 3x^2 + 4xy$ and $\frac{\partial f}{\partial y} = N = 2x^2 + 2y$. Now
$\frac{\partial f}{\partial x} = 3x^2 + 4xy \Rightarrow f(x, y) = \int(3x^2 + 4xy)dx = x^3 + 2x^2y + \phi(y)$ for some function $\phi(y)$ of y. Again $\frac{\partial f}{\partial y} = 2x^2 + 2y$ and $f(x, y) = x^3 + 2x^2y + \phi(y)$ together imply that $2x^2 + \phi'(y) = 2x^2 + 2y \Rightarrow \phi(y) = y^2 + c_1$ for some constant c_1. Hence the solution is $f(x, y) = c$ or $x^3 + 2x^2y + y^2 + c_1 = c$.

Converting a first order non-exact DE to exact DE

Consider the following example:

Example

The first order DE $ydx - xdy = 0$ is clearly not exact. But observe that if we multiply both sides of this DE by $\frac{1}{y^2}$, the resulting ODE becomes $\frac{dx}{y} - \frac{x}{y^2}dy = 0$ which is exact!

Definition

It is sometimes possible that even though the original first order DE $M(x, y)dx + N(x, y)dy = 0$ is not exact, but we can multiply both sides of this DE by some function (say, $\mu(x, y)$) so that the resulting DE $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$ becomes exact. Such a function/factor $\mu(x, y)$is known as an **integrating factor** for the original DE $M(x, y)dx + N(x, y)dy = 0$.

How to find an integrating factor?

If a first order DE has one integrating factor, then it has infinitely many integrating factors. [For a proof, see tutorial problem sheet.]

We will now list down some rules for finding integrating factors, but before that, we need the following definition:

Definition

A function $f(x, y)$ is said to be **homogeneous** of **degree** n if $f(tx, ty) = t^n f(x, y)$ for all (x, y) and for all $t \in \mathbb{R}$.

Example

1) $f(x, y) = x^2 + y^2$ is homogeneous of degree 2.
2) $f(x, y) = tan^{-1}(\frac{y}{x})$ is homogeneous of degree 0.
3) $f(x, y) = \frac{x(x^2+y^2)}{y^2}$ is homogeneous of degree 1.
4) $f(x, y) = x^2 + xy + 1$ is NOT homogeneous.

How to find an integrating factor? contd...

Definition

A first order DE of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous** if both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.
NOTE: Here the word “homogeneousÂ¨ does not mean the same as it did for first order linear equation $a_1(x)y' + a_0(x)y = g(x)$ when $g(x) = 0$.

Some rules for finding an integrating factor: Consider the DE

$$M(x, y)dx + N(x, y)dy = 0. \qquad (*)$$

Rule 1: If $(*)$ is a homogeneous DE with $M(x, y)x + N(x, y)y \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor for $(*)$.

How to find an integrating factor? contd...

Rule 2: If $M(x,y) = f_1(xy)y$ and $N(x,y) = f_2(xy)x$ and $Mx - Ny \neq 0$, where f_1 and f_2 are functions of the product xy, then $\frac{1}{Mx-Ny}$ is an integrating factor for $(*)$.

Rule 3: If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for $(*)$.

Rule 4: If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = F(y)$ (function of y-alone), then $e^{-\int F(y)dy}$ is an integrating factor for $(*)$.

Proof of Rule 1

Proof.

Observe that $Mdx + Ndy = \frac{1}{2}[(Mx + Ny)(\frac{dx}{x} + \frac{dy}{y}) - (Mx - Ny)(\frac{dx}{x} - \frac{dy}{y})]$
$= \frac{1}{2}[(Mx + Ny)d(log\ xy) + (Mx - Ny)d(log\frac{x}{y})]$.
Since we have assumed that $Mx + Ny \neq 0$, therefore we can divide both sides of the above equation by $Mx + Ny$ to get
$\frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}[d\ log(xy) + (\frac{Mx-Ny}{Mx+Ny})d\ log(\frac{x}{y})]$. Since the DE $(*)$ is homogeneous, therefore $\frac{Mx-Ny}{Mx+Ny}$ is a function (say, $f(\frac{x}{y})$) of $\frac{x}{y}$. Hence we have
$\frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}[d\ log(xy) + f(\frac{x}{y})d\ log(\frac{x}{y})]$. Since $\frac{x}{y} = e^{log\frac{x}{y}}$, therefore
$\frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}[d\ log(xy) + F(log(\frac{x}{y}))d(log\frac{x}{y})]$ for some function F. Hence
$\frac{Mdx+Ndy}{Mx+Ny} = d[\frac{1}{2}log(xy) + \frac{1}{2}\int F(log\frac{x}{y})d(log\frac{x}{y})]$,
which is an exact differential. This completes the proof. $\square$

Proof of Rule 3

Proof.

Let $f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$. To show: $\mu(x) := e^{\int f(x)dx}$ is an integrating factor. That is, to show $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$.

Since μ is a function of x alone, we have $\frac{\partial}{\partial y}(\mu M) = \mu \frac{\partial M}{\partial y}$. Also $\frac{\partial}{\partial x}(\mu N) = \mu'(x)N + \mu(x)\frac{\partial N}{\partial x}$. So we must have:
$\mu(x)[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = \mu'(x)N$, or equivalently we must have,
$\frac{\mu'(x)}{\mu(x)} = f(x)$,
which is anyways true since $\mu(x) := e^{\int f(x)dx}$. $\square$

The proof of Rule 4 is similar. The proof of Rule 2 is an exercise.

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Solution by substitution

Often the first step of solving a differential equation consists of transforming it into another differential equation by means of a ***substitution****.*
For example, suppose we wish to transform the first order differential equation $\frac{dy}{dx} = f(x,y)$ *by the substitution* $y = g(x,u)$*, where* u *is regarded as a function of the variable* x*. If* g *possesses first partial derivatives, then the chain rule*

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx}$$

gives $\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$*. The original differential equation* $\frac{dy}{dx} = f(x,y)$ *now becomes* $g_x(x,u) + g_u(x,u)\frac{du}{dx} = f(x, g(x,u))$*. This equation is of the form* $\frac{du}{dx} = F(x,u)$*, for some function* F*. If we can determine a solution* $u = \phi(x)$ *of this last equation, then a solution of the original differential equation will be* $y = g(x, \phi(x))$*.*

Use of substitution : Homogeneous equations

Recall: A first order differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be **homogeneous** if both M and N are homogeneous functions of the same degree.
Such equations can be solved by the substitution : $y = vx$.

Example

Solve $x^2ydx + (x^3 + y^3)dy = 0$.
Solution: The given differential equation can be rewritten as $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$.
Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Putting this in the above equation, we get $v + x\frac{dv}{dx} = \frac{v}{1+v^3}$. Or in other words, $(\frac{1+v^3}{v^4})dv = -\frac{dx}{x}$, which is now in separable variables form.

DE reducible to homogeneous DE

For solving differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

use the substitution

- $x = X + h$ and $y = Y + k$, if $\frac{a}{a'} \neq \frac{b}{b'}$, where h and k are constants to be determined.
- $z = ax + by$, if $\frac{a}{a'} = \frac{b}{b'}$.

Example

Solve $\frac{dy}{dx} = \frac{x+y-4}{3x+3y-5}$.
Observe that this DE is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $\frac{a}{a'} = \frac{b}{b'}$.
Use the substitution $z = x + y$. Then we have $\frac{dz}{dx} = 1 + \frac{dy}{dx}$. Putting these in the given DE, we get $\frac{dz}{dx} - 1 = \frac{z-4}{3z-5}$, or in other words, $\frac{3z-5}{4z-9} dz = dx$. This equation is now in separable variables form.

DE reducible to homogeneous DE, contd...

Example

Solve $\frac{dy}{dx} = \frac{x+y-4}{x-y-6}$.
Observe that this DE is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $1 = \frac{a}{a'} \neq \frac{b}{b'} = -1$.
Put $x = X + h$ and $y = Y + k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and

$$\frac{dY}{dX} = \frac{X + Y + (h + k - 4)}{X - Y + (h - k - 6)}. \qquad (*)$$

If h and k are such that $h + k - 4 = 0$ and $h - k - 6 = 0$, then $(*)$ becomes

$$\frac{dY}{dX} = \frac{X + Y}{X - Y}$$

which is a homogeneous DE. We can easily solve the system

$$h + k = 4$$

$$h - k = 6$$

of linear equations to detremine the constants h and k!

Reduction to separable variables form

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C),$$

where A, B, C are real constants with $B \neq 0$ can always be reduced to a differential equation with separable variables by means of the substitution $u = Ax + By + C$.

Observe that since $B \neq 0$, we get $\frac{u}{B} = \frac{A}{B}x + y + \frac{C}{B}$, or in other words, $y = \frac{u}{B} - \frac{A}{B}x - \frac{C}{B}$. This implies that $\frac{dy}{dx} = \frac{1}{B}(\frac{du}{dx}) - \frac{A}{B}$. Hence we have $\frac{1}{B}(\frac{du}{dx}) - \frac{A}{B} = f(u)$, that is, $\frac{du}{dx} = A + Bf(u)$. Or in other words, we have $\frac{du}{A+Bf(u)} = dx$, which is now in separable variables form.

Equations reducible to linear DE: Bernoullii's DE

Definition

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1}$$

where n is any real number, is called **Bernoullii's differential equation**.

Note that when $n = 0 \; or \; 1$, Bernoullii's DE is a linear DE.
<u>**Method of solution:**</u> Multiply by y^{-n} throughout the DE (1) to get

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x). \tag{2}$$

Use the substitution $z = y^{1-n}$. Then $\frac{dz}{dx} = (1-n)\frac{1}{y^n}\frac{dy}{dx}$. Substituting in equation (2), we get $\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$, which is a linear DE.

Example of Bernoullii's DE

Example

Solve the Bernoullis DE $\frac{dy}{dx} + y = xy^3$.
Multiplying the above equation throughout by y^{-3}, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x.$$

Putting $z = \frac{1}{y^2}$, we get $\frac{dz}{dx} - 2z = -2x$, which is a linear DE.
The integrating factor for this linear DE will be $= e^{-\int 2dx} = e^{-2x}$. Therefore, the solution is $z = e^{2x}[-2\int xe^{-2x}dx + c] = x + \frac{1}{2} + ce^{2x}$. Putting back $z = \frac{1}{y^2}$ in this, we get the final solution $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$.

Ordinary Differential Equations

(MA102 Mathematics II)

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Operator method for finding y_p

Writing $Ly = g$ as $P(D)y = g(x)$, where

$$L = P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0.$$

With each $P(D)$, associate a polynomial

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$$

called the auxiliary polynomial of $P(D)$.

If $P(r)$ can be factored as product of n linear factors, say

$$P(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n),$$

then the corresponding factorization of $P(D)$ has the form

$$P(D) = a_n (D - r_1)(D - r_2) \cdots (D - r_n),$$

where $r_1, r_2, \ldots, r_n$ are the roots of $P(r) = 0$.

Inverse operator

Note that

- $Dy_p(x) = g(x) \Rightarrow y_p(x) = \int g(x)dx$. It is natural to define
$$\frac{1}{D}g(x) := \int g(x)dx.$$
- $(D-r)y_p = g(x)$, where r is a constant. Formally, we write
$$y_p = \frac{1}{D-r}g(x).$$
The solution of $(D-r)y_p = g(x)$ is
$$y_p(x) = e^{rx}\int e^{-rx}g(x)dx.$$
Thus, we define
$$\frac{1}{D-r}g(x) := e^{rx}\int e^{-rx}g(x)dx.$$
Operator like $\frac{1}{D}$, $\frac{1}{D-r}$ are called inverse operators.

Successive integrations

Let $\frac{1}{P(D)}$ be the inverse of the operator $P(D)$. Then the particular solution to $P(D)y = g(x)$ is given by

$$y_p(x) = \frac{1}{P(D)} g(x),$$

where $g(x)$ is of the special functions of the type x^n, $e^{\alpha x}$, $\sin(\alpha x)$, $\cos(\alpha x)$ and a finite number of combinations of these terms.

Method 1:(Successive integrations)
If $P(D) = (D - r_1)(D - r_2)\cdots(D - r_n)$, then

$$\begin{aligned} y_p(x) &= \frac{1}{P(D)} g(x) = \frac{1}{(D - r_1)(D - r_2)\cdots(D - r_n)} g(x) \\ &= \frac{1}{(D - r_1)} \frac{1}{(D - r_2)} \cdots \frac{1}{(D - r_n)} g(x). \end{aligned}$$

An example

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

Here $P(D)y = (D-1)(D-2)y = xe^x$. The particular solution y_p is

$$\begin{aligned} y_p(x) &= \frac{1}{D-1}\frac{1}{D-2}xe^x \\ &= \frac{1}{D-1}\left[e^{2x}\int e^{-2x}xe^x dx\right] = \frac{1}{D-1}\left[-(1+x)e^x\right] \\ &= -e^x\int e^{-x}(1+x)e^x dx = -\frac{1}{2}(1+x)^2e^x. \end{aligned}$$

Note: The successive integrations are likely to become complicated and time-consuming.

Partial fractions

Method 2:(Partial fractions)
If the factors of $P(D)$ are distinct, we can decompose operator $\frac{1}{P(D)}$ into partial fractions as

$$y_p = \frac{1}{P(D)} g(x) = \left[\frac{A_1}{(D-r_1)} + \frac{A_1}{(D-r_2)} + \cdots + \frac{A_n}{(D-r_n)} \right] g(x),$$

for suitable constants A_i's.

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

$$\begin{aligned} y_p(x) &= \frac{1}{(D-1)(D-2)} = \left[\frac{1}{D-2} - \frac{1}{D-1} \right] xe^x \\ &= \frac{1}{D-2} xe^x - \frac{1}{D-1} xe^x \\ &= e^{2x} \int e^{-2x} xe^x dx - e^x \int e^{-x} xe^x dx \\ &= -(1 + x + \frac{1}{2}x^2)e^x. \end{aligned}$$

Series Expansion

Method 3:(Series expansions)
If $g(x) = x^n$, expand the inverse operator $\frac{1}{P(D)}$ in a power series in D so that

$$y_p(x) = \frac{1}{P(D)} g(x) = (a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n) g(x),$$

where $(a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n)$ is the expansion of $\frac{1}{P(D)}$ to $n+1$ terms as $D^k x^n = 0$ if $k > n$.
Example: Find y_p of $y''' - 3y'' + 2y = x^4 + 2x + 5$.

$$\frac{1}{1 - 2D^2 + D^3} = 1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \cdots .$$

$$\begin{aligned} y_p(x) &= \frac{1}{1 - 2D^2 + D^3}(x^4 + 2x + 5) \\ &= (1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \cdots)(x^4 + 2x + 5) \\ &= (x^4 + 2x + 5) + 2(12x^2) - (24x) + 4(24) \\ &= x^4 + 24x^2 - 22x + 101. \end{aligned}$$

The rule for exponentials

Method 4: If $g(x) = e^{\alpha x}$, α a constant, then

$$(D - r)e^{\alpha x} = (\alpha - r)e^{\alpha x}.$$

Operating both sides of the above identity by $(\alpha - r)^{-1}(D - r)^{-1}$, we obtain

$$\frac{1}{(D - r)}e^{\alpha x} = \frac{1}{(\alpha - r)}e^{\alpha x},$$

provided $\alpha \neq r$. Similarly, if $P(D) = (D - r_1) \cdots (D - r_n)$ then

The rule for exponentials

$$\frac{1}{P(D)}e^{\alpha x} = \frac{1}{(D-r_1)\ldots(D-r_n)}e^{\alpha x}$$

$$= \frac{1}{(\alpha - r_1)\cdots(\alpha - r_n)}e^{\alpha x},$$

provided $r_1, \ldots, r_2$ are distinct from α.

- If $P(D)$ is a polynomial in D such that $P(\alpha) \neq 0$, then
$$\frac{1}{P(D)}e^{\alpha x} = \frac{e^{\alpha x}}{P(\alpha)}.$$

An example

Example: Find a particular solution of

$$y''' - y'' + y' + y = 3e^{-2x}.$$

$$\begin{aligned} y_p &= \frac{1}{P(D)} 3e^{-2x} \\ &= \frac{3e^{-2x}}{P(-2)} \\ &= \frac{3e^{-2x}}{(-2)^3 - (-2)^2 - 2 + 1} \\ &= -\frac{3}{13} e^{-2x}. \end{aligned}$$

The exponential shift rule

Before we state the exponential shift rule, we need the following result:

Result: $P(D)\{e^{ax}V(x)\} = e^{ax}P(D+a)V(x)$.
Proof: $D^n\{e^{ax}V(x)\} = \Sigma_{k=0}^{n} \; {}^nC_k D^k e^{ax} D^{n-k}V(x)$
$= e^{ax}[\Sigma_{k=0}^{n} \; {}^nC_k a^k D^{n-k}]V(x) = e^{ax}(D+a)^n V(x)$

Now comes the exponential shift rule:

The rule: $\frac{1}{P(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{P(D+a)}V(x)$.
Proof: Observe that $P(D)[e^{ax}\frac{1}{P(D+a)}V(x)] = P(D)[e^{ax}\phi(x)]$
where $\phi(x) = \frac{1}{P(D+a)}V(x)$.
$P(D)[e^{ax}\phi(x)] = e^{ax}P(D+a)\phi(x)$ *[Since* $P(D)[e^{ax}V(x)] = e^{ax}P(D+a)V(x)$*]*
$= e^{ax}P(D+a)\frac{1}{P(D+a)}V(x) = e^{ax}V(x)$.
So, $\frac{1}{P(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{P(D+a)}V(x)$.

A particular case

In particular, taking $a = r_1 = r_2 = \cdots = r_n = r$ and $V(x) = 1$, we get

$$\frac{1}{(D-r)^k} e^{rx} = e^{rx} \frac{1}{(D)^k}.1 = e^{rx}.\frac{x^k}{k!}.$$

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 3D + 2} xe^x = e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} x \\
&= e^x \frac{1}{D^2 - D} x = -e^x \frac{1}{D} \frac{1}{1-D} x \\
&= -e^x \left[\frac{1}{D} + 1 + D + D^2 + \cdots \right] x \\
&= -e^x (\frac{1}{2} x^2 + x + 1).
\end{aligned}$$

When $g(x)$ is Sine or Cosine function

Let $g(x) = Sin(\alpha x)$. If $P(D)$ is an even

polynomial in D, write $P(D) = F(D^2)$ so that $F(D^2)$ is a polynomial in D^2. Then

$$F(D^2)\sin(\alpha x) = F(-\alpha^2)\sin(\alpha x),$$

and hence

$$\frac{1}{P(D)}\sin(\alpha x) = \frac{\sin(\alpha x)}{F(-\alpha^2)}.$$

If $P(D)$ is not even, but it has an even polynomial factor $G(D)$, i.e., $P(D) = G(D)H(D)$. Then

$$\begin{aligned}\frac{1}{P(D)}\sin(\alpha x) &= \frac{1}{G(D)H(D)}\sin(\alpha x)\\ &= \frac{H(-D)}{G(D)H(D)H(-D)}\sin(\alpha x).\end{aligned}$$

Sine or Cosine contd...

Now $G(D)H(D)H(-D)$ is an even polynomial in D and writing $G(D)H(D)H(-D) = K(D^2)$, we have

$$\begin{aligned}\frac{1}{P(D)}\sin(\alpha x) &= \frac{H(-D)}{K(D^2)}\sin(\alpha x)\\ &= \frac{H(-D)}{K(-\alpha^2)}\sin(\alpha x)\end{aligned}$$

provided that $K(-\alpha^2) \neq 0$. Similarly, $\frac{1}{P(D)}\cos(\alpha x)$ can be determined provided that $K(-\alpha^2) \neq 0$.

An example

Example: Find a particular solution of
$3y'' + 2y' - 8y = 5\cos x$.

$$\begin{aligned}
y_p &= \frac{5}{3D^2 + 2D - 8} \cos x \\
&= \frac{5(3D^2 - 2D - 8)}{(3D^2 + 2D - 8)(3D^2 - 2D - 8)} \cos x \\
&= \frac{5(3D^2 - 2D - 8)}{9D^4 - 52D^2 + 64} \cos x \\
&= \frac{5(3D^2 - 2D - 8)}{9 + 52 + 64} \cos x \\
&= \frac{1}{25}\{(3D^2 - 8) - 2D\} \cos x \\
&= \frac{1}{25}(2\sin x - 11\cos x).
\end{aligned}$$

Some results that are needed

Result 1: $F(D)e^{ax} = F(a)e^{ax}$.
Proof: $F(D)e^{ax} = (a_0D^n + a_1D^{n-1} + \cdots + a_n)e^{ax} = (a_0a^n + a_1a^{n-1} + \cdots + a_n)e^{ax} = F(a)e^{ax}$.

Result 2: $F(D)\{e^{ax}V(x)\} = e^{ax}F(D+a)V(x)$.
Proof: $D^n\{e^{ax}V(x)\} = \Sigma_{k=0}^{n} \, {}^nC_k D^k e^{ax} D^{n-k}V(x)$
$= e^{ax}[\Sigma_{k=0}^{n} \, {}^nC_k a^k D^{n-k}]V(x) = e^{ax}(D+a)^n V(x)$

Result 3: $F(D^2)Sin(ax+b) = F(-a^2)Sin(ax+b)$.
Proof: $F(D^2)Sin(ax+b) = (a_0D^{2n} + a_1D^{2n-2} + \cdots + a_n)Sin(ax+b)$
$= [a_0(-a^2)^n + a_1(-a^2)^{n-1} + \cdots + a_n]Sin(ax+b) = F(-a^2)Sin(ax+b)$.
Similarly, $F(D^2)Cos(ax+b) = F(-a^2)Cos(ax+b)$.

Inverse Transform

We have $F(D)y = f(x)$. So $y = \frac{1}{F(D)}f(x)$.
$\frac{1}{F(D)}$ is called the INVERSE TRANSFORM of $F(D)$.

Example

$D = \frac{d}{dx}, \frac{1}{D} = \int$.

Result 1: $\frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(a)}$ *provided* $F(a) \neq 0$.
Proof: We know that $F(D)e^{ax} = F(a)e^{ax}$. *Applying* $\frac{1}{F(D)}$ *on both sides, we get*
$e^{ax} = F(a)\frac{1}{F(D)}e^{ax}$.
i.e., $\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$ *provided* $F(a) \neq 0$.

Results contd...

Result 2: $\frac{1}{F(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{F(D+a)}V(x)$.
Proof: Observe that $F(D)[e^{ax}\frac{1}{F(D+a)}V(x)] = F(D)[e^{ax}\phi(x)]$
where $\phi(x) = \frac{1}{F(D+a)}V(x)$.
$F(D)[e^{ax}\phi(x)] = e^{ax}F(D+a)\phi(x)$ *[Since* $F(D)[e^{ax}V(x)] = e^{ax}F(D+a)V(x)$*]*
$= e^{ax}F(D+a)\frac{1}{F(D+a)}V(x) = e^{ax}V(x)$.
So, $\frac{1}{F(D)}[e^{ax}V(x)] = e^{ax}\frac{1}{F(D+a)}V(x)$.

Note: The formula $\frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(a)}$ fails if $F(a) = 0$. In this case, we can use the previous result (Result 2) as a working tool.

Example

$\frac{1}{D^2-2D+1}e^x = \frac{1}{(D-1)^2}e^x.1$
$= e^x\frac{1}{((D+1)-1)^2}.1 = e^x\frac{1}{D^2}.1 = e^x.\frac{x^2}{2}$.

Examples contd...

Example

To compute $\frac{1}{D^2+2}x^2e^{3x}$.

$\frac{1}{D^2+2}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2+2}x^2 = e^{3x}\frac{1}{D^2+6D+11}x^2$

$= \frac{e^{3x}}{11}[1 + \frac{D^2+6D}{11}]^{-1}x^2$

$= \frac{e^{3x}}{11}[1 - \frac{D^2+6D}{11} + \frac{(D^2+6D)^2}{121} - \ldots]x^2$

$= \frac{e^{3x}}{11}(x^2 - \frac{12}{11}x + \frac{50}{121})$.

Results contd ...

Result 3: $\frac{1}{F(D^2)} Sin(ax+b) = \frac{1}{F(-a^2)} Sin(ax+b)$ *provided* $F(-a^2) \neq 0$.
Proof: We know that $F(D^2) Sin(ax+b) = F(-a^2) Sin(ax+b)$. *Applying* $\frac{1}{F(D^2)}$ *on both sides, we get the required result.*
Result 4: $\frac{1}{F(D^2)} Cos(ax+b) = \frac{1}{F(-a^2)} Cos(ax+b)$ *provided* $F(-a^2) \neq 0$.
Proof: Similar.

Example

$\frac{1}{D^2+1} Sin(2x) = \frac{Sin(2x)}{-3}$.

Example contd ...

Example

Compute $\frac{1}{D^2+1}Sin\ x$. (Observe that we cannot apply result 3 here)

$\frac{1}{D^2+1}Sin\ x = Imaginary\ part\ of\ \frac{1}{D^2+1}e^{ix}$.

=Imaginary part of $e^{ix}\frac{1}{(D+i)^2+1}.1$

=Imaginary part of $e^{ix}\frac{1}{D^2+2iD}.1$

=Imaginary part of $e^{ix}\frac{1}{D}[\frac{1}{D+2i}].1$

=Imaginary part of $e^{ix}\frac{1}{D}\frac{1}{2i}[1+\frac{D}{2i}]^{-1}.1$

=Imaginary part of $e^{ix}\frac{1}{D}\frac{1}{2i}[1-\frac{D}{2i}+\cdots].1$

=Imaginary part of $e^{ix}\frac{1}{D}\frac{1}{2i}$

=Imaginary part of $e^{ix}\frac{x}{2i}$

$= \frac{-x}{2}Cos\ x$.

Example contd ...

Example

$\frac{1}{D^2+a^2} Sin(ax)$ = Imaginary part of $\frac{1}{D^2+a^2} e^{iax}$
=Imaginary part of $\frac{1}{(D-ia)(D+ia)} e^{iax}$
=Imaginary part of $\frac{1}{2ia} \frac{1}{D-ia} e^{iax}$
=Imaginary part of $\frac{1}{2ia} e^{iax} \frac{1}{D+ia-ia} .1$
=Imaginary part of $\frac{1}{2ia} e^{iax} x$
$= \frac{-x}{2a} Cos(ax)$

Results contd ...

Result 5: $\frac{1}{F(D)}[xV(x)] = x\frac{1}{F(D)}V + \{\frac{d}{dD}\frac{1}{F(D)}\}V.$
Proof: Omitted.

Example

$$\frac{1}{D^2+4}xSin(x) = x\frac{1}{D^2+4}Sin(x) + \{\frac{d}{dD}(\frac{1}{D^2+4})\}Sin(x)$$
$$= \frac{x}{3}Sin(x) - \frac{2D}{(D^2+4)^2}Sin(x)$$
$$= \frac{x}{3}Sin(x) - 2D\frac{1}{(-1^2+4)^2}Sin(x)$$
$$= \frac{x}{3}Sin(x) - \frac{2D}{9}Sin(x)$$
$$= \frac{x}{3}Sin(x) - \frac{2}{9}Cos(x).$$