

PART A

Ordinary Differential Equations (ODEs)

Chap. 1 First-Order ODEs

Sec. 1.1 Basic Concepts. Modeling

To get a good start into this chapter and this section, quickly **review your basic calculus**. Take a look at the front matter of the textbook and see a review of the main differentiation and integration formulas. Also, Appendix 3, pp. A63–A66, has useful formulas for such functions as exponential function, logarithm, sine and cosine, etc. The beauty of ordinary differential equations is that the subject is quite systematic and has different methods for different types of ordinary differential equations, as you shall learn. Let us discuss some Examples of Sec. 1.1, pp. 4–7.

Example 2, p. 5. Solution by Calculus. Solution Curves. To solve the first-order ordinary differential equation (ODE)

$$y' = \cos x$$

means that we are looking for a function whose derivative is $\cos x$. Your first answer might be that the desired function is $\sin x$, because $(\sin x)' = \cos x$. But your answer would be incomplete because also $(\sin x + 2)' = \cos x$, since the derivative of 2 and of any constant is 0. Hence the complete answer is $y = \sin x + c$, where c is an arbitrary constant. As you vary the constants you get an infinite family of solutions. Some of these solutions are shown in **Fig. 3**. The lesson here is that you should never forget your constants!

Example 4, pp. 6–7. Initial Value Problem. In an initial value problem (IVP) for a first-order ODE we are given an ODE, here $y' = 3y$, and an initial value condition $y(0) = 5.7$. For such a problem, the first step is to solve the ODE. Here we obtain $y(x) = ce^{3x}$ as shown in **Example 3**, p. 5. Since we also have an initial condition, we must substitute that condition into our solution and get $y(0) = ce^{3 \cdot 0} = ce^0 = c \cdot 1 = c = 5.7$. Hence the complete solution is $y(x) = 5.7e^{3x}$. The lesson here is that for an initial value problem you get a unique solution, also known as a particular solution.

Modeling means that you interpret a physical problem, set up an appropriate mathematical model, and then try to solve the mathematical formula. Finally, you have to interpret your answer. Examples 3 (exponential growth, exponential decay) and 5 (radioactivity) are examples of modeling problems. Take a close look at **Example 5**, p. 7, because it outlines all the steps of modeling.

Problem Set 1.1. Page 8

3. **Calculus.** From Example 3, replacing the independent variable t by x we know that $y' = 0.2y$ has a solution $y = 0.2ce^{0.2x}$. Thus by analogy, $y' = y$ has a solution

$$1 \cdot ce^{1 \cdot x} = ce^x,$$

where c is an arbitrary constant.

Another approach (to be discussed in details in Sec. 1.3) is to write the ODE as

$$\frac{dy}{dx} = y,$$

and then by algebra obtain

$$dy = y dx, \quad \text{so that} \quad \frac{1}{y} dy = dx.$$

Integrate both sides, and then apply exponential functions on both sides to obtain the same solution as above

$$\int \frac{1}{y} dy = \int dx, \quad \ln |y| = x + c, \quad e^{\ln |y|} = e^{x+c}, \quad y = e^x \cdot e^c = c^* e^x, \\ \text{(where } c^* = e^c \text{ is a constant).}$$

The technique used is called **separation of variables** because we separated the variables, so that y appeared on one side of the equation and x on the other side before we integrated.

7. **Solve by integration.** Integrating $y' = \cosh 5.13x$ we obtain (chain rule!) $y = \int \cosh 5.13x dx = \frac{1}{5.13}(\sinh 5.13x) + c$. Check: Differentiate your answer:

$$\left(\frac{1}{5.13}(\sinh 5.13x) + c \right)' = \frac{1}{5.13}(\cosh 5.13x) \cdot 5.13 = \cosh 5.13x, \text{ which is correct.}$$

11. **Initial value problem (IVP).** (a) Differentiation of $y = (x + c)e^x$ by product rule and definition of y gives

$$y' = e^x + (x + c)e^x = e^x + y.$$

But this looks precisely like the given ODE $y' = e^x + y$. Hence we have shown that indeed $y = (x + c)e^x$ is a solution of the given ODE. (b) Substitute the initial value condition into the solution to give $y(0) = (0 + c)e^0 = c \cdot 1 = \frac{1}{2}$. Hence $c = \frac{1}{2}$ so that the answer to the IVP is

$$y = (x + \frac{1}{2})e^x.$$

- (c) The graph intersects the x -axis at $x = 0.5$ and shoots exponentially upward.

- 19. Modeling: Free Fall.** $y'' = g = \text{const}$ is the model of the problem, an ODE of second order. Integrate on both sides of the ODE with respect to t and obtain the velocity $v = y' = gt + c_1$ (c_1 arbitrary). Integrate once more to obtain the distance fallen $y = \frac{1}{2}gt^2 + c_1t + c_2$ (c_2 arbitrary). To do these steps, we used calculus. From the last equation we obtain $y = \frac{1}{2}gt^2$ by imposing the initial conditions $y(0) = 0$ and $y'(0) = 0$, arising from the stone starting at rest at our choice of origin, that is the initial position is $y = 0$ with initial velocity 0. From this we have $y(0) = c_2 = 0$ and $v(0) = y'(0) = c_1 = 0$.

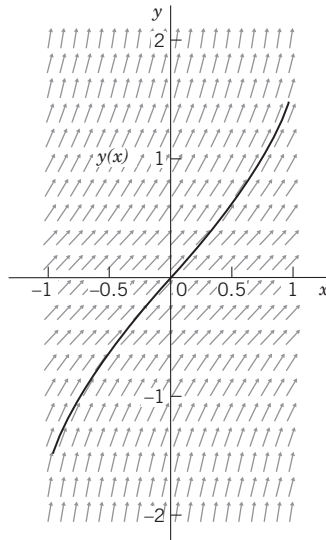
Sec. 1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method

Problem Set 1.2. Page 11

- 1. Direction field, verification of solution.** You may verify by differentiation that the general solution is $y = \tan(x + c)$ and the particular solution satisfying $y(\frac{1}{4}\pi) = 1$ is $y = \tan x$. Indeed, for the particular solution you obtain

$$y' = \frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = 1 + \tan^2 x = 1 + y^2$$

and for the general solution the corresponding formula with x replaced by $x + c$.



Sec. 1.2 Prob. 1. Direction Field

- 15. Initial value problem. Parachutist.** In this section the usual notation is (1), that is, $y' = f(x, y)$, and the direction field lies in the xy -plane. In Prob. 15 the ODE is $v = f(t, v) = g - bv^2/m$, where v suggests velocity. Hence the direction field lies in the tv -plane. With $m = 1$ and $b = 1$ the ODE becomes $v' = g - v^2$. To find the limiting velocity we find the velocity for which the acceleration equals zero. This occurs when $g - v^2 = 9.80 - v^2 = 0$ or $v = 3.13$ (approximately). For $v < 3.13$ you have $v' > 0$ (increasing curves) and for $v > 3.13$ you have $v' < 0$ (decreasing curves). Note that the isoclines are the horizontal parallel straight lines $g - v^2 = \text{const}$, thus $v = \text{const}$.

Sec. 1.3 Separable ODEs. Modeling**Problem Set 1.3. Page 18**

- 1. CAUTION! Constant of integration.** It is important to introduce the constant of integration immediately, in order to avoid getting the wrong answer. For instance, let

$$y' = y. \quad \text{Then} \quad \ln |y| = x + c, \quad y = c^* e^x \quad (c^* = e^c),$$

which is the correct way to do it (the same as in Prob. 3 of Sec. 1.1 above) whereas introducing the constant of integration later yields

$$y' = y, \quad \ln |y| = x, \quad y = e^x + C$$

which is not a solution of $y' = y$ when $C \neq 0$.

- 5. General solution.** Separating variables, we have $y \, dy = -36x \, dx$. By integration,

$$\frac{1}{2}y^2 = -18x^2 + \tilde{c}, \quad y^2 = 2\tilde{c} - 36x^2, \quad y = \pm\sqrt{c - 36x^2} \quad (c = 2\tilde{c}).$$

With the plus sign of the square root we get the upper half and with the minus sign the lower half of the ellipses in the answer on p. A4 in Appendix 2 of the textbook.

For $y = 0$ (the x -axis) these ellipses have a vertical tangent, so that at points of the x -axis the derivative y' does not exist (is infinite).

- 17. Initial value problem.** Using the extended method (8)–(10), let $u = y/x$. Then by product rule $y' = u + xu'$. Now

$$y' = \frac{y + 3x^4 \cos^2(y/x)}{x} = \frac{y}{x} + 3x^3 \cos^2\left(\frac{y}{x}\right) = u + 3x^3 \cos^2 u = u + x(3x^2 \cos^2 u)$$

so that $u' = 3x^2 \cos^2 u$.

Separating variables, the last equation becomes

$$\frac{du}{\cos^2 u} = 3x^2 dx.$$

Integrate both sides, on the left with respect to u and on the right with respect to x , as justified in the text then solve for u and express the intermediate result in terms of x and y

$$\tan u = x^3 + c, \quad u = \frac{y}{x} = \arctan(x^3 + c), \quad y = xu = x \arctan(x^3 + c).$$

Substituting the initial condition into the last equation, we have

$$y(1) = 1 \arctan(1^3 + c) = 0, \quad \text{hence} \quad c = -1.$$

Together we obtain the answer

$$y = x \arctan(x^3 - 1).$$

- 23. Modeling. Boyle–Mariotte’s law for ideal gases.** From the given information on the rate of change of the volume

$$\frac{dV}{dP} = -\frac{V}{P}.$$

Separating variables and integrating gives

$$\frac{dV}{V} = -\frac{dP}{P}, \quad \int \frac{1}{V} dV = -\int \frac{1}{P} dP, \quad \ln |V| = -\ln |P| + c.$$

Applying exponents to both sides and simplifying

$$e^{\ln |V|} = e^{-\ln |P| + c} = e^{-\ln |P|} \cdot e^c = \frac{1}{e^{\ln |P|}} \cdot e^c = \frac{1}{|P|} e^c.$$

Hence we obtain for nonnegative V and P the desired law (with $c^* = e^c$, a constant)

$$V \cdot P = c^*.$$

Sec. 1.4 Exact ODEs. Integrating Factors

Use (6) or (6*), on p. 22, only if inspection fails. Use only one of the two formulas, namely, that in which the integration is simpler. For integrating factors try both Theorems 1 and 2, on p. 25. Usually only one of them (or sometimes neither) will work. There is no completely systematic method for integrating factors, but these two theorems will help in many cases. Thus this section is slightly more difficult.

Problem Set 1.4. Page 26

1. Exact ODE. We proceed as in Example 1 of Sec. 1.4. We can write the given ODE as

$$M dx + N dy = 0 \quad \text{where } M = 2xy \text{ and } N = x^2.$$

Next we compute $\frac{\partial M}{\partial y} = 2x$ (where, when taking this partial derivative, we treat x as if it were a constant) and $\frac{\partial N}{\partial x} = 2x$ (we treat y as if it were a constant). (See Appendix A3.2 for a review of partial derivatives.) This shows that the ODE is exact by (5) of Sec. 1.4. From (6) we obtain by integration

$$u = \int M dx + k(y) = \int 2xy dx + k(y) = x^2 y + k(y).$$

To find $k(y)$ we differentiate this formula with respect to y and use (4b) to obtain

$$\frac{\partial u}{\partial y} = x^2 + \frac{dk}{dy} = N = x^2.$$

From this we see that

$$\frac{dk}{dy} = 0, \quad k = \text{const.}$$

The last equation was obtained by integration. Insert this into the equation for u , compare with (3) of Sec. 1.4, and obtain $u = x^2 y + c^*$. Because u is a constant, we have

$$x^2 y = c, \quad \text{hence} \quad y = c/x^2.$$