DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
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MA 102 Mathematics II
Problem Sheet 5: Power series solution of differential equation, Legendre's and Bessel's equations.

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1. Find radius of convergence and interval of convergence for each of the following power series.
   (a)$\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n}x^n$, (b)$\displaystyle\sum_{n=0}^{\infty}\frac{100^n}{n!}(x+7)^n$, (c)$\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{10^k}(x-5)^k$ and (d)$\displaystyle\sum_{k=0}^{\infty}k!(x-7)^k$.

2. Suppose there is an integer M such that $b_m \neq 0$ for all $m \geq M$ and
$$\lim_{m\to\infty}\left|\frac{b_{m+1}}{b_m}\right| = L,$$
where $0 \leq L \leq \infty$. Show that the radius of convergence of $\displaystyle\sum_{m=0}^{\infty} b_m(x-x_0)^{2m}$ is $R = \dfrac{1}{\sqrt{L}}$, which should be interpreted to mean that $R = 0$ if $L = \infty$ and $R = \infty$ if $L = 0$.

   [Hint: Apply the following theorem to the series $\displaystyle\sum_{m=0}^{\infty} b_m z^m$ and then let $z = (x-x_0)^2$.
   Theorem: Suppose there is an integer N such that $a_n \neq 0$ for all $n \geq N$ and
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$
where $0 \leq L \leq \infty$. Then the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = \dfrac{1}{L}$, which should be interpreted to mean that $R = 0$ if $L = \infty$ and $R = \infty$ if $L = 0$.]

   What would be the radius of convergence of the series $\displaystyle\sum_{m=0}^{\infty} b_m(x-x_0)^{km}$?

3. Find two linearly independent power series solutions for each of the following differential equations about the ordinary point $x = 0$.
   (a) $(x-1)y'' + y' = 0$.
   (b) $(x+2)y'' + xy' - y = 0$.
   (c) $(x^2-1)y'' + 4xy' + 2y = 0$.
   (d) $(x^2+1)y'' - 6y = 0$.
   (e) $y'' + (\sin x)y = 0$.
   (f) $y'' + e^x y' - y = 0$.

4. Discuss how power series method can be used to solve the nonhomogeneous equations such as $y'' - xy = 1$ and $y'' - 4xy' - 4y = e^x$. Carry out your ideas by solving both the equations.

5. Is $x = 0$ an ordinary or a singular point of the differential equation $xy'' + (\sin x)y = 0$? Defend your answer with sound mathematics.

6. The differential equation $xy'' + y' + xy = 0$ has a solution $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$. This solution is analytic at $x = 0$ although $x = 0$ is a singular point of the given differential equation. It is known that this equation possesses a second solution that is not analytic at $x = 0$. Make up a linear second order equation with a singular point at $x = 0$ that has two solutions which are analytic at $x = 0$. Do not think profound thoughts.

7. Show that the coefficients a_n in any solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ of

$$\{1 + \alpha(x - x_0)^2\}y'' + \beta(x - x_0)y' + \gamma y = 0$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)} a_n, \; n \geq 0,$$

where $p(n) = \alpha n(n-1) + \beta n + \gamma$. Moreover the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)} a_{2m}, \; m \geq 0,$$

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1}, \; m \geq 0,$$

where a_0 and a_1 are arbitrary.

8. Using the results of the above example, compute $a_0, a_1, ..., a_7$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1 + 2x^2)y'' + 10xy' + 8y = 0, \; y(0) = 2, \; y'(0) = -3.$$

9. (a) Find a power series in x for the general solution of

$$(1 + x^2)y'' + 4xy' + 2y = 0. \tag{1}$$

(b) Use (a) and the formula

$$\frac{1}{1-r} = 1 + r + r^2 + ... + r^n + ... \ (-1 < r < 1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (1) on $(-1, 1)$.

(c) Show that the expression obtained in (b) is actually the general solution of (1) on $(-\infty, \infty)$.

10. Check for regular or irregular singular points for the following differential equations.

    (a) $(x^2 - 4)^2 y'' + (x - 2)y' + y = 0$.

    (b) $x^2(x + 1)^2 y'' + (x^2 - 1)y' + 2y = 0$.

    (c) $(1 - x^2)y'' - 2xy' + 30y = 0$.

    (d) $x^3 y'' - 2xy' + 5y = 0$.

    (e) $(x^2 + 9)y'' - 3xy' + (1 - x)y = 0$.

11. In each of the following problems, show that the given differential equation has a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ($x > 0$) corresponding to the larger root. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root as well.

    (a) $3x^2 y'' + 2xy' + x^2 y = 0$.

    (b) $xy'' + (1 - x)y' - y = 0$.

    (c) $x^2 y'' - x(x + 3)y' + (x + 3)y = 0$.

    (d) $x^2 y'' + (x^2 + \frac{1}{4})y = 0$.

12. The Chebychev equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

    (a) Show that $x = 1$ and $x = -1$ are regular singular points, and find the exponents at each of these singularities.

    (b) Find two linearly independent solutions about $x = 1$.

13. In the following problems, show that the indicial roots differ by an integer. Use the method of Frobenius to obtain two linearly independent series solutions about the regular singular point $x_0 = 0$. Form the general solution on $(0, \infty)$.

    (a) $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$.

    (b) $x(x - 1)y'' + 3y' - 2y = 0$.

    (c) $xy'' + (1 - x)y' - y = 0$.

(d) $xy'' + y = 0$.

(e) $xy'' - xy' + y = 0$.

14. Making use of the fact that $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, show that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

15. Verify that $y = x^n J_n(x)$ is a particular solution of

$$xy'' + (1-2n)y' + xy = 0,\ x > 0.$$

16. Derive the formula $xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x)$ and use this result to show that $\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$.

17. Making use of either of the formulas $\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu}J_{\nu+1}(x)$ and $\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$, show that (a) $\int_0^x rJ_0(r)dr = xJ_1(x)$ and (b) $J'_0(x) = J_{-1}(x) = -J_1(x)$.

18. Discuss how $J'_0(x) = J_{-1}(x) = -J_1(x)$ in the above problem and Rolle's theorem from Calculus can be used to demonstrate that the zeros of $J_0(x)$ and $J_1(x)$ interlace– that is between any consecutive zeros of $J_0(x)$, there exists a zero of $J_1(x)$. For reference, see the following figure and table.

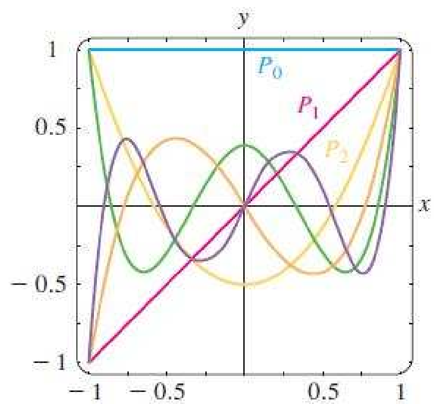

FIGURE Legendre polynomials for $n = 0, 1, 2, 3, 4, 5$

TABLE Zeros of J_0, J_1, Y_0, and Y_1

$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
2.4048	0.0000	0.8936	2.1971
5.5201	3.8317	3.9577	5.4297
8.6537	7.0156	7.0861	8.5960
11.7915	10.1735	10.2223	11.7492
14.9309	13.3237	13.3611	14.8974

19. Show that $y = x^{\frac{1}{2}}W\left(\frac{2}{3}\alpha x^{\frac{3}{2}}\right)$ is a solution of the Airy's differential equation $y'' + \alpha^2 xy = 0,\ x > 0$ whenever W is a solution of the Bessel's equation $t^2W'' + tW' + \left(t^2 - \frac{1}{9}\right)W = 0,\ t > 0$. **Hint:** After differentiating, substituting, and simplifying, let $t = \frac{2}{3}\alpha x^{\frac{3}{2}}$.

20. We know that when $n = 1$, $y_1(x) = x$ is a solution to Legendre's differential equation $(1 - x^2)y'' - 2xy' + 2y = 0$. Show that a second linearly independent solution on the interval $-1 < x < 1$ is

$$y_2(x) = \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1.$$

21. (a) Use binomial series to formally show that

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

(b) Use the results in part (a) to show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.