Table of Integrals

1 $\int u\,dv = uv - \int v\,du$

2 $\int u^n du = \frac{1}{n+1}u^{n+1} + C$

3 $\int \frac{du}{u} = \ln|u| + C$

4 $\int e^u du = e^u + C$

5 $\int a^u du = \frac{1}{\ln(a)}a^u + C$

6 $\int \mathrm{sen}(u)\,du = -\cos(u) + C$

7 $\int \cos(u)\,du = \mathrm{sen}(u) + C$

8 $\int \sec^2(u)\,du = \mathrm{tg}(u) + C$

9 $\int \mathrm{cossec}^2(u)\,du = -\mathrm{cotg}(u) + C$

10 $\int \sec(u)\,\mathrm{tg}(u)\,du = \sec(u) + C$

11 $\int \frac{\mathrm{cotg}(u)}{\mathrm{sen}(u)}\,du = -\frac{1}{\mathrm{sen}(u)} + C$

12 $\int \mathrm{tg}(u)\,du = \ln|\sec(u)| + C$

13 $\int \mathrm{cotg}(u)\,du = \ln|\mathrm{sen}(u)| + C$

14 $\int \sec(u)\,du = \ln|\sec(u) + \mathrm{tg}(u)| + C$

15 $\int \frac{du}{\mathrm{sen}(u)} = \ln\left|\frac{1}{\mathrm{sen}(u)} - \frac{\cos(u)}{\mathrm{sen}(u)}\right| + C$

16 $\int \frac{du}{\sqrt{a^2-u^2}} = \mathrm{arc\,sen}(\frac{u}{a}) + C$

17 $\int \frac{du}{a^2+u^2} = \frac{1}{a}\mathrm{arc\,tg}(\frac{u}{a}) + C$

18 $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\mathrm{arc\,sec}(\frac{u}{a}) + C$

19 $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20 $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

21 $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

22 $\int u^2\sqrt{a^2+u^2}\,du = \frac{(a^2u+2u^3)\sqrt{a^2+u^2}}{8} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

23 $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

24 $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

25 $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

26 $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$

27 $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

28 $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

29 $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

30 $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\mathrm{arc\,sen}(\frac{u}{a}) + C$

31 $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\mathrm{arc\,sen}(\frac{u}{a}) + C$

32 $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

33 $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \mathrm{arc\,sen}(\frac{u}{a}) + C$

34 $\int \frac{u^2du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\mathrm{arc\,sen}(\frac{u}{a}) + C$

35 $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2-u^2}+a}{u}\right| + C$

36 $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

37 $\int (a^2+u^2)^{3/2}\,du = -\frac{(2u^3-5a^2u)\sqrt{a^2-u^2}}{8} + \frac{3a^4}{8}\mathrm{arc\,sen}(\frac{u}{a}) + C$

38 $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

39 $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

40 $\int u^2\sqrt{u^2-a^2}\,du = -\frac{(2u^3-a^2u)\sqrt{u^2-a^2}}{8} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

41 $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\,\mathrm{arc\,cos}(\frac{a}{|u|}) + C$

42 $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

43 $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

44 $\int \frac{u^2du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

45 $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

46 $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

47 $\int \frac{udu}{a+bu} = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right) + C$

48 $\int \frac{u^2du}{a+bu} = \frac{\left[(a+bu)^2-4a(a+bu)+2a^2\ln|a+bu|\right]}{2b^3} + C$

49 $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

50 $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

51 $\int \frac{udu}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

52 $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

53 $\int \frac{u^2du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right) + C$

54 $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

55 $\int \frac{udu}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

56 $\int \frac{u^2du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

57 $\int \frac{du}{u\sqrt{a+bu}}\,du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + c$, se $a > 0$

58 $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

59 $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

60 $\int u^n\sqrt{a+bu}\,du = \frac{2\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]}{b(2n+3)}$

61 $\int \frac{u^n du}{\sqrt{a+du}} = \frac{2u^n\sqrt{a+bu}}{b(2n-1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}du}{\sqrt{a+bu}}$

62 $\int \frac{u^{-n}du}{\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{u^{-n+1}du}{\sqrt{a+bu}}$

63 $\int \mathrm{sen}^2(u)du = \frac{1}{2}u - \frac{1}{4}\mathrm{sen}(2u) + C$

64 $\int \cos^2(u)du = \frac{1}{2}u + \frac{1}{4}\mathrm{sen}(2u) + C$

65 $\int \mathrm{tg}^2(u)du = \mathrm{tg}(u) - u + C$

66 $\int \mathrm{cotg}^2(u)du = -\mathrm{cotg}(u) - u + C$

67 $\int \mathrm{sen}^3(u)du = -\frac{\left[2+\mathrm{sen}^2(u)\right]\cos(u)}{3} + C$

68 $\int \cos^3 u\,du = \frac{\left[2+\cos^2(u)\right]\mathrm{sen}(u)}{3} + C$

69 $\int \mathrm{tg}^3(u)du = \frac{\mathrm{tg}^2(u)}{2} + \ln|\cos(u)| + C$

70 $\int \mathrm{cotg}^3(u)du = -\frac{\mathrm{cotg}^2(u)}{2} - \ln|\mathrm{sen}(u)| + C$

71 $\int \sec^3(u)du = -\frac{\sec(u)\mathrm{tg}(u)}{2} - \frac{\ln|\mathrm{sen}(u)+\mathrm{tg}(u)|}{2} + C$

72 $\int \frac{du}{\mathrm{sen}^3(u)} = -\frac{\mathrm{cotg}(u)}{2\mathrm{sen}(u)} + \frac{\ln|\mathrm{cossec}(u)-\mathrm{cotg}(u)|}{2} + C$

73 $\int \mathrm{sen}^n(u)du = -\frac{\mathrm{sen}^{n-1}(u)\cos(u)}{n} + \frac{n-1}{n}\int \mathrm{sen}^{n-2}(u)du$

74 $\int \cos^n(u)du = \frac{\cos^{n-1}(u)\mathrm{sen}(u)}{n} + \frac{n-1}{n}\int \cos^{n-2}(u)du$

75 $\int \mathrm{tg}^n(u)du = \frac{\mathrm{tg}^{n-1}(u)}{n-1} - \int \mathrm{tg}^{n-2}(u)du$

76 $\int \mathrm{cotg}^n(u)\,du = -\frac{\mathrm{cotg}^{n-1}(u)}{n-1} - \int \mathrm{cotg}^{n-2}(u)\,du$

77 $\int \sec^n(u)\,du = \frac{\mathrm{tg}(u)\sec^{n-2}(u)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(u)\,du$

78 $\int \frac{du}{\mathrm{sen}^n(u)} = -\frac{\mathrm{cotg}(u)}{(n-1)\mathrm{sen}^{n-2}(u)} + \frac{n-2}{n-1}\int \frac{du}{\mathrm{sen}^{n-2}(u)}$

79 $\int \mathrm{sen}(au)\,\mathrm{sen}(bu)du = \frac{\mathrm{sen}(a-b)u}{2(a-b)} - \frac{\mathrm{sen}(a+b)u}{2(a+b)} + C$

80 $\int \cos(au)\cos(bu)du = \frac{\mathrm{sen}(a-b)u}{2(a-b)} + \frac{\mathrm{sen}(a+b)u}{2(a+b)} + C$

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n,\ n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\ n=1,2,3,\ldots$	$\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\ n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^nf(t),\ n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\displaystyle\int_s^{\infty}F(u)\,du$	32.	$\displaystyle\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$ $= \int_0^t f(t-\tau) g(\tau)\, d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s)$ $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref.* [A9] *in Appendix 1.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!}t^{n-1}e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}t^{k-1}e^{at}$	
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\sin \omega t$	6.1
14	$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$	
15	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$	
16	$\dfrac{s}{s^2-a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}e^{at}\sinh \omega t$	
18	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos \omega t$	
19	$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1-\cos \omega t)$	6.2
20	$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$	

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
21	$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6
22	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega} \sin \omega t$	6.6
23	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	6.6
24	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ $\quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$	
25	$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$	
27	$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2}t\right)$	I 5.5
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	J 5.4
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1+2at)$	
33	$\dfrac{1}{(s^2-a^2)^k}$ $\quad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	I 5.5
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	J 5.4
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}}$ $\quad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	

(*continued*)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
40	$\dfrac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	γ 5.5
41	$\ln \dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \dfrac{s^2+\omega^2}{s^2}$	$\dfrac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \dfrac{s^2-a^2}{s^2}$	$\dfrac{2}{t}(1 - \cosh at)$	
44	$\arctan \dfrac{\omega}{s}$	$\dfrac{1}{t}\sin \omega t$	
45	$\dfrac{1}{s}\operatorname{arccot} s$	$\mathrm{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. State the Laplace transforms of a few simple functions from memory.
2. What are the steps of solving an ODE by the Laplace transform?
3. In what cases of solving ODEs is the present method preferable to that in Chap. 2?
4. What property of the Laplace transform is crucial in solving ODEs?
5. Is $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$? $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
6. When and how do you use the unit step function and Dirac's delta?
7. If you know $f(t) = \mathcal{L}^{-1}\{F(s)\}$, how would you find $\mathcal{L}^{-1}\{F(s)/s^2\}$?
8. Explain the use of the two shifting theorems from memory.
9. Can a discontinuous function have a Laplace transform? Give reason.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

11–19 LAPLACE TRANSFORMS

Find the transform, indicating the method used and showing the details.

11. $5 \cosh 2t - 3 \sinh t$

12. $e^{-t}(\cos 4t - 2 \sin 4t)$

13. $\sin^2 (\frac{1}{2}\pi t)$

14. $16t^2 u(t - \frac{1}{4})$

15. $e^{t/2}u(t - 3)$

16. $u(t - 2\pi) \sin t$

17. $t \cos t + \sin t$

18. $(\sin \omega t) * (\cos \omega t)$

19. $12t * e^{-3t}$

20–28 INVERSE LAPLACE TRANSFORM

Find the inverse transform, indicating the method used and showing the details:

20. $\dfrac{7.5}{s^2 - 2s - 8}$

21. $\dfrac{s+1}{s^2} e^{-s}$

22. $\dfrac{\frac{1}{16}}{s^2 + s + \frac{1}{2}}$

23. $\dfrac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}$

24. $\dfrac{s^2 - 6.25}{(s^2 + 6.25)^2}$

25. $\dfrac{6(s+1)}{s^4}$

26. $\dfrac{2s - 10}{s^3} e^{-5s}$

27. $\dfrac{3s + 4}{s^2 + 4s + 5}$

28. $\dfrac{3s}{s^2 - 2s + 2}$

29–37 ODEs AND SYSTEMS

Solve by the Laplace transform, showing the details and graphing the solution:

29. $y'' + 4y' + 5y = 50t, \quad y(0) = 5, \quad y'(0) = -5$

30. $y'' + 16y = 4\delta(t - \pi), \quad y(0) = -1, \quad y'(0) = 0$

DAY ELEVEN

Limits, Continuity and Differentiability

Learning & Revision for the Day

- Limits
- Important Results on Limit
- Methods to Evaluate Limits
- Continuity
- Differentiability

Limits

Let $y = f(x)$ be a function of x. If the value of $f(x)$ tend to a definite number as x tends to a, then the number so obtained is called the **limit** of $f(x)$ at $x = a$ and we write it as $\lim\limits_{x \to a} f(x)$.

- If $f(x)$ approaches to l_1 as x approaches to 'a' from left, then l_1 is called the **left hand limit** of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0)$ or $\lim\limits_{x \to a^-} f(x)$ or $\lim\limits_{h \to 0} f(a-h)$
- Similarly, **right hand limit** can be expressed as
  $$f(a+0) \text{ or } \lim_{x \to a^+} f(x) \text{ or } \lim_{h \to 0} f(a+h)$$
- $\lim\limits_{x \to a} f(x)$ exists iff $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist and equal.

Fundamental Theorems on Limits

If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$ (where, l and m are real numbers), then

(i) $\lim\limits_{x \to a} \{f(x) + g(x)\} = l + m$ [sum rule]

(ii) $\lim\limits_{x \to a} \{f(x) - g(x)\} = l - m$ [difference rule]

(iii) $\lim\limits_{x \to a} \{f(x) \cdot g(x)\} = l \cdot m$ [product rule]

(iv) $\lim\limits_{x \to a} k \cdot f(x) = k \cdot l$ [constant multiple rule]

(v) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}, m \neq 0$ [quotient rule]

(vi) If $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$

(vii) $\lim\limits_{x \to a} |f(x)| = \left|\lim\limits_{x \to a} f(x)\right|$

(viii) $\lim\limits_{x \to a} \log\{f(x)\} = \log\{\lim\limits_{x \to a} f(x)\}$, provided $\lim\limits_{x \to a} f(x) > 0$

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)—
  (Without referring Explanations)

- Accuracy Level ($z/y \times 100$)—
- Prep Level ($z/x \times 100$)—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

(ix) If $f(x) \le g(x)$, $\forall x$, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$

(x) $\lim_{x \to a} [f(x)]^{g(x)} = \{\lim_{x \to a} f(x)\}^{\lim_{x \to a} g(x)}$

(xi) $\lim_{x \to a} f\{g(x)\} = f\{\lim_{x \to a} g(x)\} = f(m)$ provided f is continuous at $\lim_{x \to a} g(x) = m$.

(xii) **Sandwich Theorem** If $f(x) \le g(x) \le h(x) \forall x \in (\alpha, \beta) - \{a\}$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$, then $\lim_{x \to a} g(x) = l$ where $a \in (\alpha, \beta)$

Important Results on Limit

Some important results on limits are given below

1. Algebraic Limits

(i) $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, $n \in Q$, $a > 0$

(ii) $\lim_{x \to \infty} \dfrac{1}{x^n} = 0$, $n \in N$

(iii) If m, n are positive integers and a_0, b_0 are non-zero real numbers, then

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n} = \begin{cases} \dfrac{a_0}{b_0} & \text{if } m = n \\ 0 & \text{if } m < n \\ \infty & \text{if } m > n, a_0 b_0 > 0 \\ -\infty & \text{if } m > n, a_0 b_0 < 0 \end{cases}$$

(iv) $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x} = n$ (v) $\lim_{x \to 0} \dfrac{(1+x)^m - 1}{(1+x)^n - 1} = \dfrac{m}{n}$

2. Trigonometric Limits

(i) $\lim_{x \to 0} \dfrac{\sin x}{x} = 1 = \lim_{x \to 0} \dfrac{x}{\sin x}$

(ii) $\lim_{x \to 0} \dfrac{\tan x}{x} = 1 = \lim_{x \to 0} \dfrac{x}{\tan x}$

(iii) $\lim_{x \to 0} \dfrac{\sin^{-1} x}{x} = 1 = \lim_{x \to 0} \dfrac{x}{\sin^{-1} x}$

(iv) $\lim_{x \to 0} \dfrac{\tan^{-1} x}{x} = 1 = \lim_{x \to 0} \dfrac{x}{\tan^{-1} x}$

(v) $\lim_{x \to 0} \dfrac{\sin x^\circ}{x} = \dfrac{\pi}{180}$ (vi) $\lim_{x \to \infty} \dfrac{\sin x}{x} = \lim_{x \to \infty} \dfrac{\cos x}{x} = 0$

(vii) $\lim_{x \to \infty} \sin x$ or $\lim_{x \to \infty} \cos x$ oscillates between -1 to 1.

(viii) $\lim_{x \to 0} \dfrac{\sin^P mx}{\sin^P nx} = \left(\dfrac{m}{n}\right)^P$

(ix) $\lim_{x \to 0} \dfrac{\tan^P mx}{\tan^P nx} = \left(\dfrac{m}{n}\right)^P$

(x) $\lim_{x \to 0} \dfrac{1 - \cos mx}{1 - \cos nx} = \dfrac{m^2}{n^2}$; $\lim_{x \to 0} \dfrac{\cos ax - \cos bx}{\cos cx - \cos dx} = \dfrac{a^2 - b^2}{c^2 - d^2}$

(xi) $\lim_{x \to 0} \dfrac{\cos mx - \cos nx}{x^2} = \dfrac{n^2 - m^2}{2}$

3. Logarithmic Limits

(i) $\lim_{x \to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$; $a > 0, \ne 1$

(ii) In particular, $\lim_{x \to 0} \dfrac{\log_e(1+x)}{x} = 1$ and $\lim_{x \to 0} \dfrac{\log_e(1-x)}{x} = -1$

4. Expotential Limits

(i) $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$, $a > 0$

(ii) In particular, $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$ and $\lim_{x \to 0} \dfrac{e^{\lambda x} - 1}{x} = \lambda$

(iii) $\lim_{x \to \infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{Does not exist}, & a < 0 \end{cases}$

5. 1^∞ Form Limits

(i) If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$

(ii) If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} \{f(x) - 1\} g(x)}$

In General Cases

(i) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (ii) $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

(iii) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^\lambda$ (iv) $\lim_{x \to \infty} \left(1 + \dfrac{\lambda}{x}\right)^x = e^\lambda$

(v) $\lim_{x \to 0} (1 + ax)^{b/x} = \lim_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^{bx} = e^{ab}$

Methods To Evaluate Limits

To find $\lim_{x \to a} f(x)$, we substitute $x = a$ in the function.

If $f(a)$ is finite, then $\lim_{x \to a} f(x) = f(a)$.

If $f(a)$ leads to one of the following form $\dfrac{0}{0}; \dfrac{\infty}{\infty}; \infty - \infty; 0 \times \infty; 1^\infty$, 0 and ∞^0 (called indeterminate forms), then $\lim_{x \to a} f(x)$ can be evaluated by using following methods

(i) **Factorization Method** This method is particularly used when on substituting the value of x, the expression take the form 0/0.

(ii) **Rationalization Method** This method is particularly used when either the numerator or the denominator or both involved square roots and on substituting the value of x, the expression take the form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$.

NOTE To evaluate $x \to \infty$ type limits write the given expression in the form N/D and then divide both N and D by highest power of x occurring in both N and D to get a meaningful form.

L'Hospital's Rule

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

(ii) both are continuous at $x = a$.

(iii) both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g(a) \neq 0$.

Above rule is also applicable, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

If $f'(x), g'(x)$ satisfy all the conditions embeded in L'Hospital's rule, then we can repeat the application of this rule on

$$\frac{f'(x)}{g'(x)} \text{ to get } \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}.$$

Sometimes, following expansions are useful in evaluating limits.

- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + (-1 < x \le 1)$
- $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots (-1 < x < 1)$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
- $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \ldots$

Continuity

If the graph of a function has no break (or gap), then it is **continuous**. A function which is not continuous is called a **discontinuous** function. e.g. x^2 and e^x are continuous while $\frac{1}{x}$ and $[x]$, where $[\cdot]$ denotes the greatest integer function, are discontinuous.

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if and only if it satisfies the following conditions

(i) $f(a)$ exists, where ('a' lies in the domain of f

(ii) $\lim_{x\to a} f(x)$ exist, i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

or LHL = RHL

(iii) $\lim_{x\to a} f(x) = f(a)$, $f(x)$ is said to be

left continuous at $x = a$, if $\lim_{x\to a^-} f(x) = f(a)$

right continuous at $x = a$, if $\lim_{x\to a^+} f(x) = f(a)$

Continuity of a Function in an Interval

A function $f(x)$ is said to be continuous in (a,b) if it is continuous at every point of the interval (a,b). A function $f(x)$ is said to be continuous in $[a,b]$, if $f(x)$ is continuous in (a,b). Also, in addition $f(x)$ is continuous at $x = a$ from right and continuous at $x = b$ from left.

Results on Continuous Functions

(i) Sum, difference product and quotient of two continuous functions are always a continuous function. However, $r(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

(ii) Every polynomial is continuous at each point of real line.

(iii) Every rational function is continuous at each point where its denominator is different from zero.

(iv) Logarithmic functions, exponential functions, trigonometric functions, inverse circular functions and modulus function are continuous in their domain.

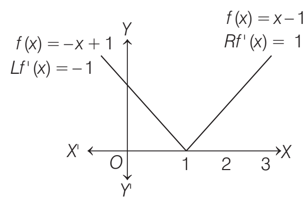
(v) $[x]$ is discontinuous when x is an integer.

(vi) If $g(x)$ is continuous at $x = a$ and f is continuous at $g(a)$, then fog is continuous at $x = a$.

(vii) $f(x)$ is a continuous function defined on $[a,b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there is atleast one value of x for which $f(x)$ vanishes, i.e. $f(a) > 0$, $f(b) < 0 \Rightarrow \exists\, c \in (a,b)$ such that $f(c) = 0$.

Differentiability

The function $f(x)$ is differentiable at a point P iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point, i.e. the function is not differentiable at those points where graph of the function has holes or sharp edges. Let us consider the function $f(x) = |x-1|$. It is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.



(graph of $f(x)$ describe differentiability)

Differentiability of a Function at a Point

A function f is said to be differentiable at $x = c$, if **left hand** and **right hand** derivatives at c exist and are equal.

- **Right hand derivative** of $f(x)$ at $x = a$ denoted by $f'(a+0)$ or $f'(a^+)$ is $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

- **Left hand derivative** of $f(x)$ at $x = a$ denoted by $f'(a-0)$ or $f'(a^-)$ is $\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$.
- Thus, f is said to be **differentiable** at $x = a$, if $f'(a+0) = f'(a-0) =$ finite.
- The common limit is called the **derivative** of $f(x)$ at $x = a$ denoted by $f'(a)$. i.e. $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

Results on Differentiability

(i) Every polynomial, constant and exponential function is differentiable at each $x \in R$.

(ii) The logarithmic, trigonometric and inverse trigonometric function are differentiable in their domain.

(iii) The sum, difference, product and quotient of two differentiable functions is differentiable.

(iv) Every differentiable function is continuous but converse may or may not be true.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 $\lim_{x\to 0} |x|^{[\cos x]}$, where [.] is the greatest integer function, is

(a) 1 (b) 0
(c) Does not exist (d) None of these

2 Let $f : R \to [0, \infty)$ be such that $\lim_{x\to 5} f(x)$ exists and $\lim_{x\to 5} \frac{[f(x)]^2 - 9}{\sqrt{|x-5|}} = 0$. Then, $\lim_{x\to 5} f(x)$ is equal to

(a) 3 (b) 0 (c) 1 (d) 2

3 If $\lim_{x\to\infty} \frac{2}{x}\left[\frac{x}{5}\right] = \frac{m}{n}$ (where [·] denotes greatest integer function), then $m+n$ (where m, n are relatively prime) is

(a) 2 (b) 7 (c) 5 (d) 6

4 The value of the constant α and β such that $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1} - \alpha x - \beta\right) = 0$ are respectively

(a) (1, 1) (b) (–1, 1) (c) (1, – 1) (d) (0, 1)

5 $\lim_{n\to\infty} \frac{3\cdot 2^{n+1} - 4\cdot 5^{n+1}}{5\cdot 2^n + 7\cdot 5^n}$ is equal to

(a) 0 (b) $\frac{3}{5}$ (c) $-\frac{4}{7}$ (d) $-\frac{20}{7}$

6 $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $\log 2$ (d) e^4

7 The value of $\lim_{n\to\infty} \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)$ is

(a) 1 (b) $\frac{\sin x}{x}$ (c) $\frac{x}{\sin x}$ (d) None of these

8 If f is periodic with period T and $f(x) > 0\,\forall x \in R$, then $\lim_{n\to\infty} n\left(\frac{f(x+T)+2f(x+2T)+\ldots+nf(x+nT)}{f(x+T)+4f(x+4T)+\ldots+n^2 f(x+n^2T)}\right)$ is equal to

(a) 2 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) None of these

9 $\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to → **JEE Mains 2015, 13**

(a) 4 (b) 3 (c) 2 (d) $\frac{1}{2}$

10 If $\lim_{x\to 0} \frac{[(a-n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n+\frac{1}{n}$

11 $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to → **JEE Mains 2014**

(a) $\frac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π

12 If $f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim_{x\to 0} \frac{f(x)}{x^2}$ is equal to

(a) 3 (b) –1 (c) 0 (d) 1

13 The limit of the following is

$$\lim_{x\to 3} \frac{\sqrt{1-\cos(x^2-10x+21)}}{(x-3)}$$

(a) $-(2)^{3/2}$ (b) $(2)^{1/2}$ (c) $(2)^{-3/2}$ (d) 3

14 The value of $\lim_{x\to 0} \frac{1}{x}\left[\tan^{-1}\left(\frac{x+1}{2x+1}\right) - \frac{\pi}{4}\right]$ is → **JEE Mains 2013**

(a) 1 (b) $-\frac{1}{2}$
(c) 2 (d) 0

15 $\lim_{x\to\pi/2} \frac{\cot x - \cos x}{(\pi-2x)^3}$ equals → **JEE Mains 2017**

(a) $\frac{1}{24}$ (b) $\frac{1}{16}$
(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

DAY TWELVE

Differentiation

Learning & Revision for the Day

- Derivative (Differential Coefficient)
- Geometrical Meaning of Derivative at a point
- Methods of Differentiation
- Second Order Derivative
- Differentiation of a Determinant

Derivative (Differential Coefficient)

The rate of change of a quantity y with respect to another quantity x is called the **derivative or differential coefficient** of y with respect to x. The process of finding derivative of a function called **differentiation**.

Geometrical Meaning of Derivative at a Point

Geometrically derivative of a function at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $P\{c, f(c)\}$.

Slope of tangent at $P = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \left\{\dfrac{df(x)}{dx}\right\}_{x=c}$ or $f'(c)$.

Derivative of Some Standard Functions

- $\dfrac{d}{dx}(\text{constant}) = 0$
- $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$
- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$, for $x > 0$
- $\dfrac{d}{dx}(a^x) = a^x \log a$, for $a > 0$
- $\dfrac{d}{dx} x^n = nx^{n-1}$
- $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{n}{x^{n+1}}$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
- $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\dfrac{d}{dx}(e^x) = e^x$
- $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log a}$, for $x > 0$, $a > 0$, $a \neq 1$

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)— *(Without referring Explanations)*

- Accuracy Level ($z/y \times 100$)—
- Prep Level ($z/x \times 100$)—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, for $x \in R$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for $x \in R$

Methods of Differentiation

(i) If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\{f(x) \pm g(x)\}$

$$= f'(x) \pm g'(x)$$

(ii) If $y = c \cdot f(x)$, where c is any constant, then $\frac{dy}{dx} = \frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$. [Scalar multiple rule]

(iii) If $y = f(x) \cdot g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\{f(x) \cdot g(x)\}$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x) \quad \text{[Product rule]}$$

(iv) If $y = \frac{f(x)}{g(x)}$, then $\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\{g(x)\}^2}$

$g(x) \neq 0$

(v) If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) \quad \text{[Chain rule]}$$

This rule can be extended as follows. If $y = f(u), u = g(v)$ and $v = h(x)$, then $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$,

(vi) $\frac{d}{dx}(f\ \{g(x)\}) = f'(g(x)) \cdot g'(x)$

(vii) If given function cannot be expressed in the form $y = f(x)$ but can be expressed in the form $f(x, y) = 0$, then to find derivatives of each term of $f(x, y) = 0$ w.r.t x. [differentiation of implicit function]

(viii) If y is the product or the quotient of a number of complicated functions or if it is of the form $(f(x))^{g(x)}$, then the derivative of y can be found by first taking log on both sides and then differentiating it. [logarithmic differentiation rule]

When $y = (f(x))^{g(x)}$, then $\frac{dy}{dx} = (f(x))^{g(x)}$

$$\left[\frac{g(x)}{f(x)} \cdot f'(x) + \log f(x) \cdot g'(x)\right]$$

(ix) If $x = \phi(t)$ and $y = \Psi(t)$, where t is parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{[Parametric differentiation rule]}$$

(x) If $u = f(x)$ and $v = g(x)$, then the differentiation of u with respect to v is $\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$.

[Differentiation of a function w.r.t another function]

Differentiation Using Substitution

In order to find differential coefficients of complicated expressions, some substitution are very helpful, which are listed below

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
(iii)	$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iv)	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(v)	$\sqrt{\frac{a^2+x^2}{a^2-x^2}}$ or $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos 2\theta$
(vi)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$
(vii)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
(viii)	$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(ix)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

Usually this is done in case of inverse trigonometric functions.

Second Order Derivative

If $y = f(x)$, then $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the **second order derivative** of y w.r.t x. It is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ or y'' or y_2.

Differentiation of a Determinant

If $y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}$, then $\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix}$

$$+ \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$$

DAY FIFTEEN

Indefinite Integrals

Learning & Revision for the Day

- Integral as an Anti-derivative
- Fundamental Integration Formulae
- Methods of Integration

Integral as an Anti-derivative

A function $\phi(x)$ is called a **primitive** or **anti-derivative** of a function $f(x)$, if $\phi'(x) = f(x)$. If $f_1(x)$ and $f_2(x)$ are two anti-derivatives of $f(x)$, then $f_1(x)$ and $f_2(x)$ differ by a constant. The collection of all its anti-derivatives is called **indefinite integral** of $f(x)$ and is denoted by $\int f(x)\,dx$.

Thus, $\frac{d}{dx}\{\phi(x) + C\} = f(x) \quad \Rightarrow \quad \int f(x)\,dx = \phi(x) + C$

where, $\phi(x)$ is an anti-derivative of $f(x)$, $f(x)$ is the **integrand** and C is an arbitrary constant known as the **constant of integration**. Anti-derivative of odd function is always even and of even function is always odd.

Properties of Indefinite Integrals

- $\int \{f(x) \pm g(x)\}\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$, where k is any non-zero real number.
- $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\,dx = k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \ldots + k_n \int f_n(x)\,dx$, where $k_1, k_2, \ldots k_n$ are non-zero real numbers.

Fundamental Integration Formulae

There are some important fundamental formulae, which are given below

1. Algebraic Formulae

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

(ii) $\int (ax+b)^n dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + C,\ n \neq -1$

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)— *(Without referring Explanations)*

- Accuracy Level ($z/y \times 100$)—
- Prep Level ($z/x \times 100$)—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

(iii) $\int \frac{1}{x}\,dx = \log|x| + C$

(iv) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}(\log|ax+b|) + C$

(v) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(vi) $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(vii) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(viii) $\int \frac{-1}{a^2 + x^2}\,dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(ix) $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log|x + \sqrt{x^2 - a^2}| + C$

(x) $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log|x + \sqrt{x^2 + a^2}| + C$

(xi) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xii) $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xiii) $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xiv) $\int \frac{-1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

(xv) $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\log|x + \sqrt{x^2 - a^2}| + C$

(xvii) $\int \sqrt{x^2 + a^2}\,dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2\log|x + \sqrt{x^2 + a^2}| + C$

2. Trigonometric Formulae

(i) $\int \sin x\,dx = -\cos x + C$

(ii) $\int \cos x\,dx = \sin x + C$

(iii) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(iv) $\int \cot x\,dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(v) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(vi) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

(vii) $\int \sec^2 x\,dx = \tan x + C$

(viii) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(ix) $\int \sec x \cdot \tan x\,dx = \sec x + C$

(x) $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + C$

3. Exponential Formulae

(i) $\int e^x dx = e^x + C$

(ii) $\int e^{(ax+b)}\,dx = \frac{1}{a}\cdot e^{(ax+b)} + C$

(iii) $\int a^x\,dx = \frac{a^x}{\log_e a} + C,\ a > 0 \text{ and } a \neq 1$

(iv) $\int a^{(bx+c)}dx = \frac{1}{b}\cdot\frac{a^{(bx+c)}}{\log_e a} + C,\ a > 0 \text{ and } a \neq 1$

Methods of Integration

Following methods are used for integration

1. Integration by Substitutions

The method of reducing a given integral into one of the standard integrals, by a proper substitution, is called **method of substitution**.

To evaluate an integral of the form $\int f\{g(x)\}\cdot g'(x)dx$, we substitute $g(x) = t$, so that $g'(x)dx = dt$ and given integral reduces to $\int f(t)\,dt$.

NOTE
- $\int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + C$
- If $\int f(x)\,dx = \phi(x)$, then $\int f(ax+b)\,dx = \frac{1}{a}\phi(ax+b) + C$

(i) To evaluate integrals of the form

$$\int \frac{dx}{ax^2 + bx + c} \text{ or } \int \frac{dx}{\sqrt{ax^2 + bx + c}} \text{ or}$$

$$\int \sqrt{ax^2 + bx + c}\,dx$$

We write, $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4c}$$

This process reduces the integral to one of following forms

$$= \int \frac{dX}{X^2 - A^2}, \int \frac{dX}{X^2 + A^2} \text{ or } \int \frac{dX}{A^2 - X^2},$$

$$\int \frac{dX}{\sqrt{A^2 - X^2}}, \int \frac{dX}{\sqrt{X^2 - A^2}}, \int \frac{dX}{\sqrt{X^2 + A^2}}$$

or $\int \sqrt{A^2 - X^2}\,dX, \int \sqrt{X^2 - A^2}\,dX, \int \sqrt{A^2 + X^2}\,dX$

(ii) To evaluate integrals of the form

$$\int \frac{(px+q)}{ax^2 + bx + c}\,dx \text{ or } \int \frac{(px+q)}{\sqrt{ax^2 + bx + c}}\,dx$$

or $\int (px+q)\sqrt{ax^2 + bx + c}\,dx$

We put $px + q = A$ {differentiation of $(ax^2 + bx + c)$} + B, where A and B can be found by comparing the coefficients of like powers of x on the two sides.

2. Integration using Trigonometric Identities

In this method, we have to evaluate integrals of the form

- $\int \sin mx \cdot \cos nx\, dx$ or $\int \sin mx \cdot \sin nx\, dx$ or $\int \cos mx \cdot \cos nx\, dx$ or $\int \cos mx \cdot \sin nx\, dx$

In this method, we use the following trigonometrical identities

(i) $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$
(ii) $2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$
(iii) $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$
(iv) $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$
(v) $2\sin A \cdot \cos A = \sin 2A$
(vi) $\cos^2 A = \left(\frac{1+\cos 2A}{2}\right)$
(vii) $\sin^2 A = \left(\frac{1-\cos 2A}{2}\right)$
(viii) $\cos^2 A - \sin^2 A = \cos 2A$
(ix) $\sin^2 A + \cos^2 A = 1$

3. Integration of Different Types of Functions

- To evaluate integrals of the form $\int \sin^p x \cos^q x\, dx$

Where $p, q \in Q$, we use the following rules :

(i) If p is odd, then put $\cos x = t$
(ii) If q is odd, then put $\sin x = t$
(iii) If both p,q are odd, then put either $\sin x = t$ or $\cos x = t$
(iv) If both p,q are even, then use trigonometric identities only.
(v) If p,q are rational numbers and $\left(\frac{p+q-2}{2}\right)$ is a negative integer, then put $\cot x = t$ or $\tan x = t$ as required.

- To evaluate integrals of the form $\int \frac{dx}{a+b\cos^2 x}$ or $\int \frac{dx}{a+b\sin^2 x}$ or $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$, $\int \frac{dx}{(a\sin x + b\cos x)^2}$ or $\int \frac{dx}{a+b\sin^2 x + c\cos^2 x}$

(i) Divide both the numerator and denominator by $\cos^2 x$.
(ii) Replace $\sec^2 x$ by $1+\tan^2 x$ in the denominator, if any.
(iii) Put $\tan x = t$, so that $\sec^2 x\, dx = dt$

- To evaluate integrals of the form $\int \frac{1}{a\sin x + b\cos x}dx$ or $\int \frac{1}{a+b\sin x}dx$ or $\int \frac{1}{a+b\cos x}dx$ or $\int \frac{1}{a\sin x + b\cos x + c}dx$

(i) Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$
(ii) Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$ and put $\tan\frac{x}{2} = t$.

- To evaluate integral of form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$, we write $a\sin x + b\cos x = A\frac{d}{dx}(c\sin x + d\cos x) + B(c\sin x + d\cos x)$

Where A and B can be found by equating the coefficient of $\sin x$ and $\cos x$ on both sides.

To evaluate integral of the form $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$.

We write $a\sin x + b\cos x + c = A\frac{d}{dx}(p\sin x + q\cos x + r) + B(p\sin x + q\cos x + r) + C$

Where A, B and C can be found by equating the coefficient of $\sin x$, $\cos x$ and the constant term.

- To evaluate integrals of the form $\int \frac{x^2+1}{x^4+kx^2+1}dx$ or $\int \frac{x^2-1}{x^4+kx^2+1}dx$

We divide the numerator and denominator by x^2 and make perfect square in denominator as $\left(x \pm \frac{1}{x}\right)^2$ and then put $x + \frac{1}{x} = t$ or $x - \frac{1}{x} = t$ as required.

- Substitution for Some Irrational Integrand

(i) $\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$, $x = a\cos 2\theta$
(ii) $\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}, \frac{1}{\sqrt{x(a+x)}}$, $x = a\tan^2\theta$ or $x = a\cot^2\theta$
(iii) $\sqrt{\frac{x}{a-x}}, \sqrt{\frac{a-x}{x}}, \sqrt{x(a-x)}, \frac{1}{\sqrt{x(a-x)}}$, $x = a\sin^2\theta$ or $x = a\cos^2\theta$

(iv) $\sqrt{\dfrac{x}{x-a}}, \sqrt{\dfrac{x-a}{x}}, \sqrt{x(x-a)}, \dfrac{1}{\sqrt{x(x-a)}}$, $x = a\sec^2\theta$

(v) $\int \dfrac{dx}{(x-\alpha)(\beta-x)}, \int \sqrt{\left(\dfrac{x-\alpha}{\beta-x}\right)}\,dx$

$\int \sqrt{(x-\alpha)(\beta-x)}\,dx$, put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(vi) $\int \dfrac{dx}{(px+q)\sqrt{ax+b}}$, put $ax+b = t^2$

(vii) $\int \dfrac{dx}{(ax^2+bx+c)\sqrt{px+q}}$, put $px+q = t^2$

(viii) $\int \dfrac{dx}{(px+q)\sqrt{ax^2+bx+c}}$, put $px+q = \dfrac{1}{t}$

(ix) $\int \dfrac{dx}{(px^2+q)\sqrt{(ax^2+b)}}$ first put $x = \dfrac{1}{t}$

and then $a + bt^2 = z^2$

4. Integration by Parts

(i) If u and v are two functions of x, then

$$\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$$

We use the following preference in order to select the first function

I $\rightarrow$ Inverse function
L $\rightarrow$ Logarithmic function
A $\rightarrow$ Algebraic function
T $\rightarrow$ Trigonometric function
E $\rightarrow$ Exponential function

(ii) If one of the function is not directly integrable, then we take it as the first function.

(iii) If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easily integrable.

(iv) If only one which is not directly integrable, function is there e.g. $\int \log x\,dx$, then 1 (unity) is taken as second function.

Some more Special Integrals Based on Integration by Parts

(i) $\int e^x\{f(x) + f'(x)\}dx = f(x)e^x + C$

(ii) $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}\{a\sin(bx+c) - b\cos(bx+c)\} + k$

(iii) $\int e^{ax}\cos(bx+c)dx = \dfrac{e^{ax}}{a^2+b^2}\{a\cos(bx+c) + b\sin(bx+c)\} + k$

Here, c and k are integration constant.

5. Integration by Partial Fractions

To evaluate the integral of the form $\int \dfrac{P(x)}{Q(x)}dx$, where $P(x), Q(x)$ are polynomial in x with degree of $P(x) <$ degree of $Q(x)$ and $Q(x) \neq 0$, we use the method of partial fraction.

The partial fractions depend on the nature of the factors of $Q(x)$.

(i) According to nature of factors of $Q(x)$, corresponding form of partial fraction is given below:
If $Q(x) = (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)$, then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \ldots + \frac{A_n}{(x-a_n)},$$

where the constants $A_1, A_2, \ldots, A_n$ can be determined by equating the coefficients of like power of x or by substituting $x = a_1, a_2, \ldots, a_n$.

(ii) If $Q(x) = (x-a)^k\,(x-a_1)(x-a_2)\ldots(x-a_r)$, then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + \ldots + \frac{B_r}{(x-a_r)}$$

where the constants $A_1, A_2, \ldots, A_k, B_1, B_2 \ldots, B_r$ can be obtained by equating the coefficients of like power of x.

(iii) If some of the factors in $Q(x)$ are quadratic and non-repeating, corresponding to each quadratic factor ax^2+bx+c (non-factorisable), we assume the partial fraction of the type $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of like powers of x.

(iv) If some of the factors in $Q(x)$ are quadratic and repeating, for every quadratic repeating factor of the type $(ax^2+bx+c)^k$ where ax^2+bx+c cannot be further factorise, we assume

$$\frac{A_1x + A_2}{ax^2+bx+c} + \frac{A_3x + A_4}{(ax^2+bx+c)^2} + \ldots + \frac{A_{2k-1}x + A_{2k}}{(ax^2+bx+c)^k}$$

If degree of $P(x) >$ degree of $Q(x)$, then we first divide $P(x)$ by $Q(x)$ so that $\dfrac{P(x)}{Q(x)}$ is expressed in the form of $T(x) + \dfrac{P_1(x)}{Q(x)}$, where $T(x)$ is a polynomial in x and $\dfrac{P_1(x)}{Q(x)}$ is a proper rational function (i.e. degree of $P_1(x) <$ degree of $Q(x)$)

DAY SIXTEEN

Definite Integrals

Learning & Revision for the Day

- Concept of Definite Integrals
- Leibnitz Theorem
- Walli's Formula
- Inequalities in Definite Integrals
- Definite Integration as the Limit of a sum

Concept of Definite Integrals

Let $\phi(x)$ be an anti-derivative of a function $f(x)$ defined on $[a,b]$ i.e. $\frac{d}{dx}[\phi(x)] = f(x)$. Then, definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is defined as $[\phi(b) - \phi(a)]$ i.e. $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$. The numbers a and b are called the limits of integration, where a is called **lower limit** and b is **upper limit**.

NOTE
- Every definite integral has a unique value.
- The above definition is nothing but the statement of second fundamental theorem of integral calculus.

PRED MIRROR
Your Personal Preparation Indicator

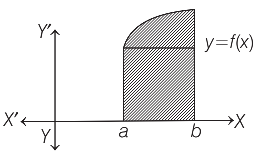
- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)— *(Without referring Explanations)*

- Accuracy Level $(z/y \times 100)$—
- Prep Level $(z/x \times 100)$—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

Geometrical Interpretation of Definite Integral

In general, $\int_a^b f(x)dx$ represents an algebraic sum of areas of the region bounded by the curve $y = f(x)$, the X-axis, and the ordinates $x = a$ and $x = b$ as show in the following figure.



Evaluation of Definite Integrals by Substitution

When the variable in a definite integral is changed due to substitution, then the limits of the integral will accordingly be changed.

For example, to evaluate definite integral of the form $\int_a^b f[g(x)]\cdot g'(x)\,dx$, we use the following steps

***Step* I** Substitute $g(x)=t$ so that $g'(x)\,dx=dt$

***Step* II** Find the limits of integration in new system of variable. Here, the lower limit is $g(a)$, the upper limit is $g(b)$ and the integral is now $\int_{g(a)}^{g(b)} f(t)\,dt$.

***Step*III** Evaluate the integral, so obtained by usual method.

Properties of Definite Integrals

(i) $\int_\alpha^\alpha f(x)dx=0$

(ii) $\int_\alpha^\beta f(x)\,dx=\int_\alpha^\beta f(t)\,dt$

(iii) $\int_\alpha^\beta f(x)\,dx=-\int_\beta^\alpha f(x)\,dx$

(iv) $\int_\alpha^\beta f(x)\,dx=\int_\alpha^{c_1} f(x)\,dx+\int_{c_1}^{c_2} f(x)\,dx+\ldots+\int_{c_n}^\beta f(x)\,dx$

where, $\alpha<c_1<c_2<\ldots<c_n<\beta$

(v) (a) $\int_\alpha^\beta f(x)\,dx=\int_\alpha^\beta f(\alpha+\beta-x)\,dx$

(b) $\int_0^\alpha f(x)\,dx=\int_0^\alpha f(\alpha-x)\,dx$

(vi) $\int_{-\alpha}^\alpha f(x)\,dx$

$$=\begin{cases}2\int_0^\alpha f(x)\,dx, \text{ if } f(-x)=f(x) \\ \qquad\qquad \text{i.e. } f(x) \text{ is an even function} \\ 0, \qquad \text{if } f(-x)=-f(x) \\ \qquad\qquad \text{i.e. } f(x) \text{ is an odd function}\end{cases}$$

(vii) $\int_0^{2\alpha} f(x)dx=\int_0^\alpha f(x)dx+\int_0^\alpha f(2\alpha-x)dx$

(viii) $\int_0^{2\alpha} f(x)\,dx=\begin{cases}2\int_0^\alpha f(x)\,dx, & \text{if } f(2\alpha-x)=f(x)\\ 0, & \text{if } f(2\alpha-x)=-f(x)\end{cases}$

(ix) $\int_\alpha^\beta f(x)\,dx=(\beta-\alpha)\int_0^1 f[(\beta-\alpha)x+\alpha]dx$

(x) If $f(x)$ is a periodic function with period T, then

(a) $\int_\alpha^{\alpha+nT} f(x)\,dx=n\int_0^T f(x)\,dx, n\in I$

(b) $\int_{\alpha T}^{\beta T} f(x)\,dx=(\beta-\alpha)\int_0^T f(x)\,dx, \quad \alpha,\beta\in I$

(c) $\int_{\alpha+nT}^{\beta+nT} f(x)\,dx=\int_\alpha^\beta f(x)\,dx, n\in I$

(xi) Some important integrals, which can be obtained with the help of above properties.

(a) $\int_0^{\pi/2}\log\sin x dx=\int_0^{\pi/2}\log\cos x dx=\frac{\pi}{2}\log\left(\frac{1}{2}\right).3$

(b) $\int_0^{\pi/4}\log(1+\tan x)dx=\frac{\pi}{8}\log 2.$

(xii) If a function $f(x)$ is discontinuous at points $x_1, x_2,\ldots,x_n$ in (a,b), then we can define sub-intervals $(a,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n),(x_n,b)$ such that $f(x)$ is continuous in each of these sub-intervals and

$$\int_a^b f(x)dx=\int_a^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+\ldots+\int_{x_{n-1}}^{x_n} f(x)dx+\int_{x_n}^b f(x)dx.$$

Leibnitz Theorem

If function $\phi(x)$ and $\psi(x)$ are defined on $[\alpha,\beta]$ and differentiable on $[\alpha,\beta]$ and $f(t)$ is continuous on $[\psi(\alpha),\phi(\beta)]$, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right]=\left\{\frac{d}{dt}\psi(x)\right\}f(\psi(x))-\left\{\frac{d}{dx}\{\phi(x)\}\right\}\{f(\phi(x))\}$$

Walli's Formula

This is a special type of integral formula whose limits from 0 to $\pi/2$ and integral is either integral power of $\cos x$ or $\sin x$ or $\cos x\sin x$.

$$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx$$

$$=\begin{cases}\dfrac{(n-1)(n-3)(n-5)\ldots 5\cdot3\cdot1}{n(n-2)(n-4)\ldots6\cdot4\cdot2}\times\dfrac{\pi}{2}, & \text{if } n=2m \text{ (even)}\\[2ex] \dfrac{(n-1)(n-3)(n-5)\ldots 6\cdot4\cdot2}{n(n-2)(n-4)\ldots5\cdot3\cdot1}, & \text{if } n=2m+1 \text{ (odd)}\end{cases}$$

where, n is positive integer.

$$\int_0^{\pi/2}\sin^m x\cdot\cos^n x\,dx$$

$$=\begin{cases}\dfrac{(m-1)(m-3)\ldots(2\text{ or }1).(n-1)(n-3)\ldots(2\text{ or }1)}{(m+n)(m+n-2)\ldots(2\text{ or }1)}\dfrac{\pi}{2}, \\ \text{when both } m \text{ and } n \text{ are even positive integers}\\ \dfrac{(m-1)(m-3)\ldots(2\text{ or }1)\cdot(n-1)(n-3)\ldots(2\text{ or }1)}{(m+n)(m+n-2)\ldots(2\text{ or }1)}, \\ \text{when either } m \text{ or } n \text{ or both are odd} \\ \text{positive integers}\end{cases}$$

Inequalities in Definite Integrals

(i) If $f(x)\ge g(x)$ on $[\alpha,\beta]$, then $\int_\alpha^\beta f(x)\,dx\ge\int_\alpha^\beta g(x)\,dx$

(ii) If $f(x)\ge 0$ in the interval $[\alpha,\beta]$, then $\int_\alpha^\beta f(x)\,dx\ge 0$

(iii) If $f(x), g(x)$ and $h(x)$ are continuous on $[a,b]$ such that $g(x)\le f(x)\le h(x)$, then $\int_a^b g(x)\,dx\le\int_a^b f(x)dx\le\int_a^b h(x)dx$

(iv) If f is continuous on $[\alpha, \beta]$ and $l \le f(x) \le M, \forall$ $x \in [\alpha, \beta]$, then $l(\beta-\alpha) \le \int_\alpha^\beta f(x)\,dx \le M(\beta-\alpha)$

(v) If f is continuous on $[\alpha, \beta]$, then $\left|\int_\alpha^\beta f(x)\,dx\right| \le \int_\alpha^\beta |f(x)|\,dx$

(vi) If f is continuous on $[\alpha, \beta]$ and $|f(x)| \le k, \forall\ x \in [\alpha, \beta]$, then $\left|\int_\alpha^\beta f(x)\,dx\right| \le k(\beta-\alpha)$

Definite Integration as the Limit of a Sum

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh)$

where, $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

The converse is also true, i.e. if we have an infinite series of the above form, it can be expressed as definite integral.

Some Particular Cases

(i) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$ or $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

(ii) $\lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$

where, $\alpha = \lim_{n\to\infty} \frac{r}{h} = 0$ $(\because r = 1)$

and $\beta = \lim_{n\to\infty} \frac{r}{h} = p$ $(\because r = pn)$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to → **JEE Mains 2017**

(a) -2 (b) 2 (c) 4 (d) -1

2 If $f(x)$ is continuous function, then

(a) $\int_{-2}^{2} f(x)\,dx = \int_0^2 [f(x) - f(-x)]\,dx$

(b) $\int_{-3}^{5} 2f(x)\,dx = \int_6^{10} f(x-1)\,dx$

(c) $\int_{-3}^{5} f(x)\,dx = \int_{-4}^{4} f(x-1)\,dx$

(d) $\int_{-3}^{5} f(x)\,dx = \int_{-2}^{6} f(x-1)\,dx$

3 $\int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}]\,dx$ is equal to

(a) $\sqrt{2}\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{\sqrt{2}}$ (d) 2π

4 $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) dx$ is equal to

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/2$ (d) π

5 If $I_{1(n)} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)}{\sin x}\,dx$ and $I_{2(n)} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x}\,dx$, $n \in N$, then

(a) $I_{2(n+1)} - I_{2(n)} = I_{1(n)}$ (b) $I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$
(c) $I_{2(n+1)} + I_{1(n)} = I_{2(n)}$ (d) $I_{2(n+1)} + I_{1(n+1)} = I_{2(n)}$

6 $\int_{-1}^{1} \{x^2 + x - 3\}\,dx$, where $\{x\}$ denotes the fractional part of x, is equal to

(a) $\frac{1}{3}(1+3\sqrt{5})$ (b) $\frac{1}{6}(1+3\sqrt{5})$

(c) $\frac{1}{3}(3\sqrt{5}-1)$ (d) $\frac{1}{6}(3\sqrt{5}-1)$

7 $\int_0^2 [x^2]\,dx$ is equal to

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$
(c) $\sqrt{2}-1$ (d) $-\sqrt{2}-\sqrt{3}+5$

8 If $\int_{-2}^{x} |2t|\,dt = f(x)$, then for any $x \ge 0$, $f(x)$ is equal to

(a) $4+x^2$ (b) $4-x^2$
(c) $\frac{1}{2}(4+x^2)$ (d) $\frac{1}{4}(4-x^2)$

9 $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx$ is equal to

(a) 1 (b) 0
(c) -1 (d) None of these

10 Let a, b and c be non-zero real numbers such that $\int_0^3 (3ax^2 + 2bx + c)\,dx = \int_1^3 (3ax^2 + 2bx + c)\,dx$, then

(a) $a+b+c=3$ (b) $a+b+c=1$
(c) $a+b+c=0$ (d) $a+b+c=2$

11 The value of $\int_1^a [x] f'(x)\,dx$, $a > 1$, where $[x]$ denotes the greatest integer not exceeding x, is

(a) $[a]f(a) - \{f(1) + f(2) + \ldots + f([a])\}$
(b) $[a]f([a]) - \{f(1) + f(2) + \ldots + f(a)\}$
(c) $af([a]) - \{f(1) + f(2) + \ldots + f(a)\}$
(d) $af(a) - \{f(1) + f(2) + \ldots + f([a])\}$

12 The correct evaluation of $\int_0^{\pi/2} \left|\sin\left(x - \frac{\pi}{4}\right)\right| dx$ is

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$
(c) $-2+\sqrt{2}$ (d) 0

DAY TWENTY

Trigonometric Functions and Equations

Learning & Revision for the Day

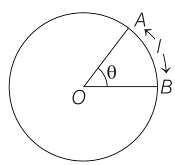
- Angle on Circular System
- Trigonometric Functions
- Trigonometric Identities
- Trigonometric Ratios/Functions of Acute Angles
- Trigonometric Ratios of Compound and Multiple Angles
- Transformation Formulae
- Conditional Identities
- Maximum and Minimum Values
- Trigonometric Equations
- Summation of Some Trigonometric Series

Angle on Circular System

If the angle subtended by an arc of length l at the centre of a circle of radius r is θ, then $\theta = \frac{l}{r}$

If the length of arc is equal to the radius of the circle, then the angle subtended at the centre of the circle will be one radian. One radian is denoted by 1^c and $1^c = 57°16'22''$ approximately.

(Figure shows the angle whose measure are one radian.)

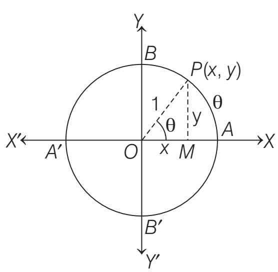


$$2\pi^C = 360°, 1^C = \frac{180°}{\pi}, 1° = \left(\frac{\pi}{180}\right)^C$$

Trigonometric Functions

Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B' as shown in the figure

Also, let $P(x, y)$ be any on the circle with $\angle AOP = \theta$ radian, i.e. length of arc $AP = \theta$



PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)—
  (Without referring Explanations)

- Accuracy Level $(z/y \times 100)$—
- Prep Level $(z/x \times 100)$—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

Then, the six trigonometric functions, can be defined as

(i) $\cos\theta = \dfrac{OM}{OP} = x$

(ii) $\sin\theta = \dfrac{PM}{OP} = y$

(iii) $\sec\theta = \dfrac{OP}{OM} = \dfrac{1}{x}$, $x \neq 0$

(iv) $\text{cosec}\,\theta = \dfrac{OP}{PM} = \dfrac{1}{y}$, $y \neq 0$

(v) $\tan\theta = \dfrac{PM}{OM} = \dfrac{y}{x}$, $x \neq 0$

(vi) $\cot\theta = \dfrac{OM}{PM} = \dfrac{x}{y}$, $y \neq 0$

Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions which is true for all those angles for which the functions are defined is called trigonometric identity.

Some identities are given below

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sec^2\theta - \tan^2\theta = 1$

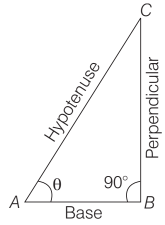
(iii) $\text{cosec}^2\theta - \cot^2\theta = 1$

Trigonometric Ratios/Functions of Acute Angles

The ratios of the sides of a triangles with respect to its acute angles are called trigonometric ratios or T-ratios.

In a right angled ΔABC, if $\angle CAB = \theta$, then

1. $\sin\theta = \dfrac{BC}{AC} = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$
2. $\cos\theta = \dfrac{AB}{AC} = \dfrac{\text{Base}}{\text{Hypotenuse}}$
3. $\tan\theta = \dfrac{BC}{AB} = \dfrac{\text{Perpendicular}}{\text{Base}}$
4. $\text{cosec}\,\theta = \dfrac{1}{\sin\theta} = \dfrac{AC}{BC} = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}}$
5. $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{AC}{AB} = \dfrac{\text{Hypotenuse}}{\text{Base}}$
6. $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{AB}{BC} = \dfrac{\text{Base}}{\text{Perpendicular}}$



Sign for Trigonometric Ratios in four Quadrants

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\,\theta$
I. $(0, 90°)$	+	+	+	+	+	+
II. $(90°, 180°)$	+	−	−	−	−	+
III. $(180°, 270°)$	−	−	+	+	−	−
IV. $(270°, 360°)$	−	+	−	−	+	−

Trigonometric ratios of some useful angles between 0°and 90°

Angle	**0°/0**	**15°/$\frac{\pi}{12}$**	**18°/$\frac{\pi}{10}$**	**22.5°/$\frac{\pi}{8}$**	**30°/$\frac{\pi}{6}$**	**36°/$\frac{\pi}{5}$**	**45°/$\frac{\pi}{4}$**	**54°/$\frac{3\pi}{10}$**	**60°/$\frac{\pi}{3}$**	**67.5°/$\frac{3\pi}{8}$**	**72°/$\frac{2\pi}{5}$**	**75°/$\frac{5\pi}{12}$**	**90°/$\frac{\pi}{2}$**
$\sin\theta$	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	1
$\cos\theta$	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	0
$\tan\theta$	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	1	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{3}$	$\sqrt{2}+1$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	∞
$\cot\theta$	∞	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\sqrt{2}+1$	$\sqrt{3}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	1	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{1}{\sqrt{3}}$	$\sqrt{2}-1$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	0
$\sec\theta$	1	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{5}-1$	$\sqrt{2}$	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	2	$\sqrt{4+2\sqrt{2}}$	$\sqrt{5}+1$	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	∞
$\text{cosec}\,\theta$	∞	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	$\sqrt{5}+1$	$\sqrt{4+2\sqrt{2}}$	2	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{2}$	$\sqrt{5}-1$	$\frac{2}{\sqrt{3}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	1

Trigonometric ratios of allied angles

θ	$\sin\theta$	$\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$90^\circ-\theta$	$\cos\theta$	$\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$90^\circ+\theta$	$\cos\theta$	$\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$180^\circ-\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$-\tan\theta$	$-\cot\theta$
$180^\circ+\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$-\cos\theta$	$-\sec\theta$	$\tan\theta$	$\cot\theta$
$270^\circ-\theta$	$-\cos\theta$	$-\sec\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cot\theta$	$\tan\theta$
$270^\circ+\theta$	$-\cos\theta$	$-\sec\theta$	$\sin\theta$	$\operatorname{cosec}\theta$	$-\cot\theta$	$-\tan\theta$
$360^\circ-\theta$	$-\sin\theta$	$-\operatorname{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$

Trigonometric Ratios of Compound and Multiple Angles

Compound Angles

(i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(ii) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

(iii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(iv) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

(v) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(vi) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(vii) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

(viii) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

(ix) (a) $\dfrac{1+\tan A}{1-\tan A}=\tan\left(\dfrac{\pi}{4}+A\right)$

(b) $\dfrac{1-\tan A}{1+\tan A}=\tan\left(\dfrac{\pi}{4}-A\right)$

Multiple Angles

(i) $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A=\cos^2 A-\sin^2 A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$=2\cos^2 A-1=1-2\sin^2 A$

(iii) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

(iv) $\sin 3A=3\sin A-4\sin^3 A$

(v) $\cos 3A=4\cos^3 A-3\cos A$

(vi) $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Transformation Formulae

(i) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(ii) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(iii) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(iv) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

(v) $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(vi) $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(vii) $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(viii) $\cos C-\cos D=-2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(ix) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

(x) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$

(xi) $\cos A\cos 2A\cos 4A\cos 8A\ldots\cos 2^{n-1}A$

$=\dfrac{1}{2^n\sin A}\sin(2^n A)$

Conditional Identities

If $A+B+C=180^\circ$, then

(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(ii) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(iii) $\sin A+\sin B+\sin C=4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iv) $\cos A+\cos B+\cos C=1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(v) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(vi) $\tan\dfrac{A}{2}\tan\dfrac{B}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}+\tan\dfrac{C}{2}\tan\dfrac{A}{2}=1$

Maximum and Minimum Values

1. $-1\le\sin x\le 1, |\sin x|\le 1$ 2. $-1\le\cos x\le 1, |\cos x|\le 1$
3. $|\sec x|\ge 1, |\operatorname{cosec} x|\ge 1$
4. $\tan x, \cot x$ take all real values

NOTE
- Maximum value of $a\cos\theta \pm b\sin\theta = \sqrt{a^2 + b^2}$
- Minimum value of $a\cos\theta \pm b\sin\theta = -\sqrt{a^2 + b^2}$
- Maximum value of $a\cos\theta \pm b\sin\theta + c = c + \sqrt{a^2 + b^2}$
- Minimum value of $a\cos\theta \pm b\sin\theta + c = c - \sqrt{a^2 + b^2}$

Trigonometric Equations

An equation involving one or more trigonometrical ratios of unknown angle is called a **trigonometrical equation**.

e.g. $\sin\theta + \cos^2\theta = 0$

Principal Solution

The value of the unknown angle (say θ) which satisfies the trigonometric equation is known as principal solution, if $0 \le \theta < 2\pi$.

General Solution

Since, trigonometrical functions are periodic function, solution of trigonometric equation can be generalised with the help of the periodicity of the trigonometrical functions.

The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Some general trigonometric equations and their solutions

Equations	Solutions	Equations	Solutions
$\sin x = \sin\alpha \left(-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}\right)$	$x = n\pi + (-1)^n \alpha$, $n \in I$	$\sin^2 x = \sin^2 \alpha$	$x = n\pi \pm \alpha,\ n \in I$
$\cos x = \cos\alpha\ (0 \le \alpha \le \pi)$	$x = 2n\pi \pm \alpha$, $n \in I$	$\cos^2 x = \cos^2 \alpha$	
$\tan x = \tan\alpha \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right)$	$x = n\pi + \alpha,\ n \in I$	$\tan^2 x = \tan^2 \alpha$	
$\sin x = 0$	$x = n\pi,\ n \in I$	$\sin x = 1$	$x = (4n+1)\pi/2,\ n \in I$
$\cos x = 0$	$x = (2n+1)\pi/2,\ n \in I$	$\cos x = 1$	$x = 2n\pi,\ n \in I$
$\tan x = 0$	$x = n\pi,\ n \in I$	$\cos x = -1$	$x = (2n+1)\pi,\ n \in I$
		$\sin x = \sin\alpha$ and $\cos x = \cos\alpha$	$x = 2n\pi + \alpha,\ n \in I$

Summation of Some Trigonometric Series

(i) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots$ to n terms

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \sin\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}$$

(ii) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots$ to n terms

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \cos\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}$$

DAY TWENTY ONE

Properties of Triangle, Height and Distances

Learning & Revision for the Day

- Properties Related to Triangle
- Some Important Theorems
- Circles Connected with Triangle
- Angle of Elevation and Depression

Properties Related to Triangle

In any ΔABC,

(i) perimeter, $2s = a + b + c$

(ii) sum of all angles of a triangle is 180°, i.e. $\angle A + \angle B + \angle C = 180°$

(iii) $a + b > c, b + c > a, c + a > b$

(iv) $|a - b| < c, |b - c| < a, |c - a| < b$

(v) $a > 0, b > 0, c > 0$

Relations between the Sides and Angles of Triangle

For a triangle ΔABC with sides a, b, c and opposite angles are respectively A, B and C, then

(i) **Sine Rule** In any ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(ii) **Cosine Rule**

(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)—
(Without referring Explanations)

- Accuracy Level ($z/y \times 100$)—
- Prep Level ($z/x \times 100$)—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

(iii) **Projection Rule**

(a) $a = b\cos C + c\cos B$ (b) $b = c\cos A + a\cos C$

(c) $c = a\cos B + b\cos A$

(iv) **Napier's Analogy**

(a) $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$ (b) $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(c) $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

(v) **Half Angle of Triangle**

(a) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ (b) $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$

(c) $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$ (d) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

(e) $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}}$ (f) $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

(g) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$ (h) $\tan\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

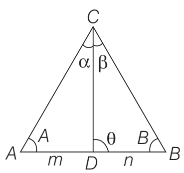
(i) $\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

(vi) Area of a Triangle $\Delta = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B = \dfrac{1}{2}ab\sin C$
$= \sqrt{s(s-a)(s-b)(s-c)}$

Some Important Theorems

1. *m*-*n* Theorem (Trigonometric Theorem)

If in a ΔABC, D divides AB in the ratio $m:n$, then (shown as in given figure)

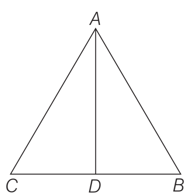


(i) $(m+n)\cot\theta = n\cot A - m\cot B$

(ii) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

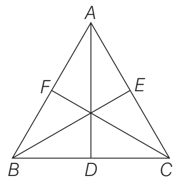
2. Appolonius Theorem

If in ΔABC, AD is median, then



$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

The length of medians AD, BE and CF of a ΔABC are (shown as in given figure)



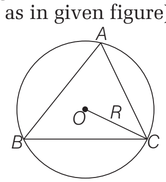
$AD = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, $BE = \dfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$

and $CF = \dfrac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

Circles Connected with Triangle

1. Circumcircle

The circle passing through the vertices of the ΔABC is called the circumcircle. (shown as in given figure)



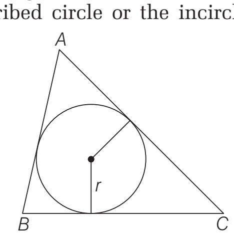
Its radius R is called the circumradius,

and $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = \dfrac{abc}{4\Delta}$

NOTE
- The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of the triangle.
- The mid-point of the hypotenuse of a right angled triangle is the circumcentre of the triangle.
- Distance of circumcentre from the side AC is $R\cos B$.
- Radius of circumcircle of a n-sided regular polygon with each side a is $R = \dfrac{a}{2}\text{cosec}\dfrac{\pi}{n}$.

2. Incircle

The circle touching the three sides of the triangle internally is called the inscribed circle or the incircle of the triangle. Its



radius r is called inradius of the circle.

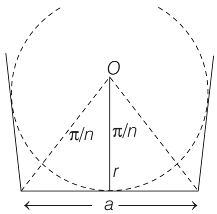
(i) $r = \dfrac{\Delta}{s}$

(ii) $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

(iii) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

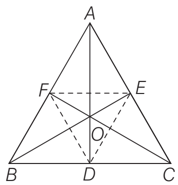
(iv) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

NOTE Radius of incircle of a n-sided regular polygon with each side a is $r = \dfrac{a}{2}\cot\dfrac{\pi}{n}$. (shown as in given figure)



3. Orthocentre and Pedal Triangle

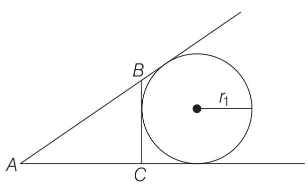
- The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called orthocentre. (shown as in given figure)



- The ΔDEF formed by joining the feet of the altitudes is called the pedal triangle.
- Orthocentre of the triangle is the incentre of the pedal triangle.
- Distance of the orthocentre of the triangle from the angular points are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.

4. Escribed Circle

The circle touching BC and the two sides AB and AC produced of ΔABC, is called the escribed circle opposite to A. Its radius is denoted by r_1. (shown as in given figure)



Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to angles B and C, respectively.

r_1, r_2 and r_3 are called the exradius of ΔABC.

(i) $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(ii) $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2} = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$

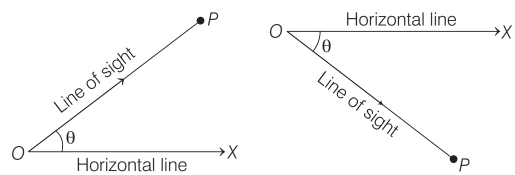
(iii) $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

NOTE
- $r_1 + r_2 + r_3 = 4R + r$
- $r_1r_2 + r_2r_3 + r_3r_1 = \dfrac{r_1r_2r_3}{r}$
- $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$
- $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

Angle of Elevation and Depression

Let O be the observer's eye and OX be the horizontal line through O. (shown as in following figures)

If a object P is at a higher level than eye, then $\angle POX$ is called the **angle of elevation**.

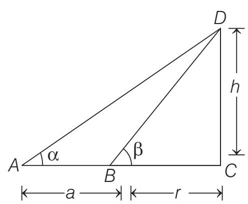


If a object P is at a lower level than eye, then $\angle POX$ is called the **angle of depression**.

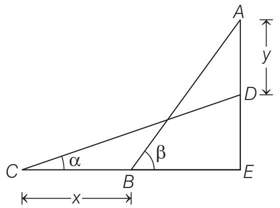
Important Results on Heights and Distances

Results shown by the following figures.

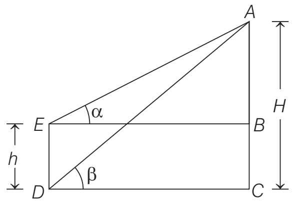
(i) $a = h(\cot\alpha - \cot\beta)$



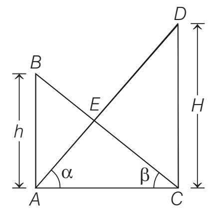
(ii) If $AB = CD$, then $x = y \tan\left(\frac{\alpha+\beta}{2}\right)$



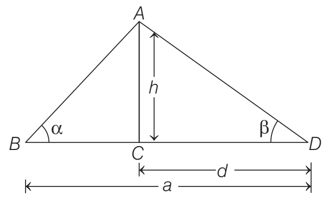
(iii) $h = \dfrac{H \sin(\beta - \alpha)}{\cos\alpha \sin\beta}$ and $H = \dfrac{h\cot\alpha}{\cot\alpha - \cot\beta}$



(iv) $H = \dfrac{h\cot\beta}{\cot\alpha}$



(v) $a = h(\cot\alpha + \cot\beta)$, $h = a\sin\alpha\sin\beta\,\text{cosec}(\alpha+\beta)$
and $d = h\cot\beta = a\sin\alpha\cos\beta\,\text{cosec}(\alpha+\beta)$



DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If a, b and c are sides of a triangle, then
(a) $\sqrt{a}+\sqrt{b}>\sqrt{c}$
(b) $\left|\sqrt{a}-\sqrt{b}\right|>\sqrt{c}$ (if c is smallest)
(c) $\sqrt{a}+\sqrt{b}<\sqrt{c}$
(d) None of the above

2 If in a ΔABC, $A = 30°$, $B = 45°$ and $a = 1$, then the values of b and c are respectively
(a) $\sqrt{2}, \frac{\sqrt{3}+1}{\sqrt{2}}$ (b) $\sqrt{2}, \frac{\sqrt{3}-1}{\sqrt{2}}$
(c) $\sqrt{3}, \frac{\sqrt{3}-1}{\sqrt{2}}$ (d) $\sqrt{2}, \frac{\sqrt{3}+2}{\sqrt{2}}$

3 If $A = 75°$, $B = 45°$, then $b + c\sqrt{2}$ is equal to
(a) $2a$ (b) $2a + 1$ (c) $3a$ (d) $2a - 1$

4 If in a ΔABC, $2b^2 = a^2 + c^2$, then $\frac{\sin 3B}{\sin B}$ is equal to
(a) $\frac{c^2-a^2}{2ca}$ (b) $\frac{c^2-a^2}{ca}$
(c) $\left(\frac{c^2-a^2}{ca}\right)^2$ (d) $\left(\frac{c^2-a^2}{2ca}\right)^2$

5 The sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0<\alpha<\frac{\pi}{2}$. Then, the greatest angle of the triangle is
(a) 60° (b) 90° (c) 120° (d) 150°

6 In a ΔABC, $(c + a + b)(a + b - c) = ab$. The measure of $\angle C$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$ (d) None of these

7 In ΔABC, if $\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{2}$, then a, b and c are in
(a) AP (b) GP
(c) HP (d) None of these

8 In a ΔABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is
(a) $\frac{16}{9}$ (b) $\frac{16}{7}$ (c) $\frac{11}{7}$ (d) $\frac{7}{16}$

9 In any ΔABC, $4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)$ is equal to
(a) $\frac{r}{R}$ (b) $\frac{2r}{R}$
(c) $\frac{3r}{R}$ (d) None of these

10 In a ΔABC, R = circumradius and r = inradius. The value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is
(a) $\frac{R}{r}$ (b) $\frac{R}{2r}$ (c) $\frac{r}{R}$ (d) $\frac{2r}{R}$

DAY TWENTY TWO

Inverse Trigonometric Function

Learning & Revision for the Day

- Inverse Trigonometric Function
- Properties of Inverse Trigonometric Function

Inverse Trigonometric Function

Trigonometric functions are not one-one and onto on their natural domains and ranges, so their inverse do not exists in the whole domain. If we restrict their domain and range, then their inverse may exists.

$y = f(x) = \sin x$. Then, its inverse is $x = \sin^{-1} y$.

NOTE
- $\sin^{-1} y \neq (\sin y)^{-1}$
- $\sin^{-1} y \neq \sin\left(\frac{1}{y}\right)$

The value of an inverse trigonometric functions which lies in its principal value branch is called the principal value of that inverse trigonometric function.

Domain and range of inverse trigonometric functions

Function	Domain	Range (Principal Value Branch)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$R-(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)—
  (Without referring Explanations)

- Accuracy Level ($z/y \times 100$)—
- Prep Level ($z/x \times 100$)—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

Properties of Inverse Trigonometric Functions

1. (i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$; $(-1 \le x \le 1)$

   (ii) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$; $x \in R$

   (iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$; $(x \le -1 \text{ or } x \ge 1)$

2. (i) $\sin^{-1}(-x) = -\sin^{-1} x$; $(-1 \le x \le 1)$

   (ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$; $(-1 \le x \le 1)$

   (iii) $\tan^{-1}(-x) = -\tan^{-1}(x)$; $(-\infty < x < \infty)$

   (iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$; $(-\infty < x < \infty)$

   (v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$; $x \le -1 \text{ or } x \ge 1)$

   (vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$; $(x \le -1 \text{ or } x \ge 1)$

3. (i) $\sin^{-1}(\sin x)$ is a periodic function with period 2π.

$$\sin^{-1}(\sin x) = \begin{cases} x, & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \pi - x, & x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ x - 2\pi, & x \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \\ 3\pi - x, & x \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right] \end{cases}$$

   (ii) $\cos^{-1}(\cos x)$ is a periodic function with period 2π.

$$\cos^{-1}(\cos x) = \begin{cases} x, & x \in [0, \pi] \\ 2\pi - x, & x \in [\pi, 2\pi] \\ x - 2\pi, & x \in [2\pi, 3\pi] \\ 4\pi - x, & x \in [3\pi, 4\pi] \end{cases}$$

   (iii) $\tan^{-1}(\tan x)$ is a periodic function with period π.

$$\tan^{-1}(\tan x) = \begin{cases} x, & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x - \pi, & x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ x - 2\pi, & x \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \\ x - 3\pi, & x \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right] \end{cases}$$

   (iv) $\cot^{-1}(\cot x)$ is a periodic function with period π.
   $\cot^{-1}(\cot x) = x$; $0 < x < \pi$

   (v) $\sec^{-1}(\sec x)$ is a periodic function with period 2π.
   $\sec^{-1}(\sec x) = x$; $0 \le x < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < x \le \pi$

   (vi) $\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is a periodic function with period 2π.
   $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$; $-\dfrac{\pi}{2} \le x < 0$ or $0 < x \le \dfrac{\pi}{2}$

4. (i) $\sin^{-1}\left(\dfrac{1}{x}\right) = \operatorname{cosec}^{-1} x$, if $x \in (-\infty, -1] \cup [1, \infty)$

   (ii) $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1} x$, if $x \in (-\infty, -1] \cup [1, \infty)$

   (iii) $\tan^{-1}\left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{if } x > 0 \\ -\pi + \cot^{-1} x, & \text{if } x < 0 \end{cases}$

5. (i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosex}^{-1}\left(\frac{1}{x}\right), \text{ if } x \in (0, 1)$$

   (ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), \text{ if } x \in (0, 1)$$

   (iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$

$$= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}(\sqrt{1+x^2}), \text{ if } x \in (0, \infty)$$

6. (i) $\sin^{-1} x + \sin^{-1} y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}); \ |x|, |y| \le 1 \text{ and} \\ \quad x^2 + y^2 \le 1 \text{ or } (xy < 0 \text{ and } x^2 + y^2 > 1) \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}); \ 0 < x, y \le 1 \\ \quad \text{and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}); \\ -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

   (ii) $\sin^{-1} x - \sin^{-1} y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}); \ |x|, |y| \le 1 \\ \text{and } x^2 + y^2 \le 1 \text{ or } (xy > 0 \text{ and } x^2 + y^2 > 1) \\ \pi - \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}); \\ 0 < x \le 1, -1 \le y < 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}); \\ -1 \le x < 0, 0 < y \le 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

   (iii) $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \ |x|, |y| \le 1 \\ \quad \text{and } x + y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \\ \quad |x|, |y| \le 1 \text{ and } x + y \le 0 \end{cases}$$

   (iv) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy + \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \ |x|, |y| \le 1 \\ \text{and } x \le y \\ -\cos^{-1}\{xy + \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \\ -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

(v) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x > 0, y > 0, xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x < 0, y < 0, xy > 1 \end{cases}$$

(vi) $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy < -1, x > 0, y < 0 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy < -1, x < 0, y > 0 \end{cases}$$

7. (i) $2\sin^{-1} x$

$$= \begin{cases} \sin^{-1}\{2x\sqrt{(1-x^2)}\}; & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}); & \dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}); & -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2-1); & 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1); & -1 \le x < 0 \end{cases}$

(iii) $2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & x < -1 \end{cases}$

(iv) $2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & x < -1 \end{cases}$

(v) $2\tan^{-1} x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & -\infty < x \le 0 \end{cases}$

NOTE
- If $\sin^{-1} x + \sin^{-1} y = \theta$, then $\cos^{-1} x + \cos^{-1} y = \pi - \theta$
- If $\cos^{-1} x + \cos^{-1} y = \theta$, then $\sin^{-1} x + \sin^{-1} y = \pi - \theta$

8. (i) $3\sin^{-1} x = \begin{cases} \sin^{-1}(3x - 4x^3), & \text{if } \dfrac{-1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), & \text{if } \dfrac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), & \text{if } -1 \le x < \dfrac{-1}{2} \end{cases}$

(ii) $3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3-3x), & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), & \text{if } \dfrac{-1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), & \text{if } -1 \le x \le \dfrac{1}{2} \end{cases}$

(iii) $3\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x < \dfrac{-1}{\sqrt{3}} \end{cases}$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 The principal value of $\sin^{-1}\left(\cos\dfrac{33\pi}{5}\right)$ is → **NCERT Exemplar**

(a) $\dfrac{3\pi}{5}$ (b) $\dfrac{7\pi}{5}$ (c) $\dfrac{\pi}{10}$ (d) $-\dfrac{\pi}{10}$

2 If $\sum_{i=1}^{20} \sin^{-1} x_i = 10\pi$, then $\sum_{i=1}^{20} x_i$ is equal to

(a) 20 (b) 10 (c) 0 (d) None of these

3 The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is → **NCERT Exemplar**

(a) [1, 2] (b) [– 1, 1]
(c) [0, 1] (d) None of these

4 The value of $\cos(2\cos^{-1} x + \sin^{-1} x)$ at $x = \dfrac{1}{5}$ is

(a) 1 (b) 3 (c) 0 (d) $-\dfrac{2\sqrt{6}}{5}$

5 The value of $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$ is

(a) $\dfrac{1}{\sqrt{x^2+2}}$ (b) $\sqrt{\dfrac{x^2+2}{x^2+1}}$ (c) $\sqrt{\dfrac{x^2+1}{x^2+2}}$ (d) $\dfrac{1}{\sqrt{x^2+1}}$

6 The equation $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ has → **NCERT Exemplar**

(a) no solution
(b) unique solution
(c) infinite number of solutions
(d) two solutions

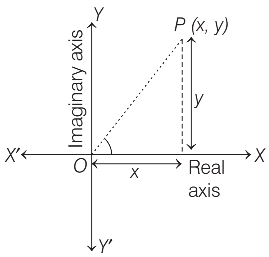
Complex Numbers

Learning & Revision for the Day

- Complex Numbers and Its Representation
- Algebra and Equality of Complex Numbers
- Conjugate and Modulus of a Complex Number
- Argument or Amplitude of a Complex Number
- Different forms of a Complex Number
- Concept of Rotation
- Square Root of a Complex Number
- De-Moivre's Theorem
- Cube Roots of Unity
- *n*th Roots of Unity
- Applications of Complex Numbers in Geometry

Complex Numbers and Its Representation

- A number in the form of $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, is called a **complex number**. The real numbers x and y are respectively called **real** and **imaginary** parts of complex number z.
  i.e. $x = \text{Re}(z)$, $y = \text{Im}(z)$ and the symbol i is called ***iota***.



Argand diagram

- A complex number $z = x + iy$ is said to be purely real if $y = 0$ and purely imaginary if $x = 0$.
- **Integral power of iota** (i)
  (i) $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$
  (ii) If n is an integer, then $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$ and $i^{4n+3} = -i$
  (iii) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$
- The complex number $z = x + iy$ can be represented by a point P in a plane called **argand plane** or **Gaussian plane** or **complex plane**. The coordinates of P are referred to the rectangular axes XOX' and YOY' which are called **real** and **imaginary axes**, respectively.

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)— *(Without referring Explanations)*

- Accuracy Level ($z/y \times 100$)—
- Prep Level ($z/x \times 100$)—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be

Algebra and Equality of Complex Numbers

If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are two complex numbers, then

(i) $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

(ii) $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

(iii) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

(iv) $\dfrac{z_1}{z_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i\,(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

(v) z_1 and z_2 are said to be equal if $x_1 = x_2$ and $y_1 = y_2$.

NOTE • Complex numbers does not possess any inequality, e.g. $3 + 2i > 1 + 2i$ does not make any sense.

Conjugate and Modulus of a Complex Number

- If $z = x + iy$ is a complex number, then **conjugate** of z is denoted by $\bar{z}$ and is obtained by replacing i by $-i$.
  i.e. $\bar{z} = x - iy$
- If $z = x + iy$, then **modulus** or **magnitude** of z is denoted by $|z|$ and is given by $|z| = \sqrt{x^2 + y^2}$

Results on Conjugate and Modulus

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}\,(z),\ z - \bar{z} = 2\,i\,\text{Im}(z)$

(iii) $z = \bar{z} \Leftrightarrow z$ is purely real.

(iv) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.

(v) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(vi) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(vii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, if $z_2 \neq 0$

(viii) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where a_i, b_i, c_i; $(i = 1, 2, 3)$ are complex numbers.

(ix) $|z| = 0 \Leftrightarrow z = 0$

(x) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(xi) $-|z| \le \text{Re}(z),\ \text{Im}(z) \le |z|$

(xii) $|z_1 z_2| = |z_1|\,|z_2|$

(xiii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $|z_2| \neq 0$

(xiv) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm \bar{z}_1 z_2 \pm z_1 \bar{z}_2$
$= |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}\,(z_1 \bar{z}_2)$

(xv) $|z^n| = |z|^n$, $n \in N$

(xvi) **Reciprocal of a complex number** For non-zero complex number $z = x + iy$, the reciprocal is given by $z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$.

(xvii) **Triangle Inequality**

(a) $|z_1 + z_2| \le |z_1| + |z_2|$ (b) $|z_1 + z_2| \ge ||z_1| - |z_2||$

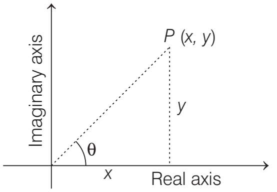
(c) $|z_1 - z_2| \le |z_1| + |z_2|$ (d) $|z_1 - z_2| \ge ||z_1| - |z_2||$

Argument or Amplitude of a Complex Number

Let $z = x + iy$ be a complex number, represented by a point $P(x, y)$ in the argand plane. Then, the angle θ which OP makes with the positive direction of Real axis (X-axis) is called the argument or amplitude of z and it is denoted by **arg (z)** or **amp (z)**.

The argument of z, is given by $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

- The value of argument θ which satisfies the inequality $-\pi < \theta \le \pi$, is called principal value of argument.
- The principal value of arg(z) is θ, $\pi - \theta$, $-\pi + \theta$ or $-\theta$ according as z lies in the 1st, 2nd, 3rd or 4th quadrants respectively, where $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$.



- Argument of z is not unique. General value of argument of z is $2n\pi + \theta$.

Results on Argument

If z, z_1 and z_2 are complex numbers, then

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(iv) The general value of $\arg(\bar{z})$ is $2n\pi - \arg(z)$.

(v) If z is purely imaginary then $\arg(z) = \pm\dfrac{\pi}{2}$.

(vi) If z is purely real then $\arg(z) = 0$ or π.

(vii) If $|z_1 + z_2| = |z_1 - z_2|$, then $\arg\left(\dfrac{z_1}{z_2}\right)$ or $\arg(z_1) - \arg(z_2) = \dfrac{\pi}{2}$

(viii) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$

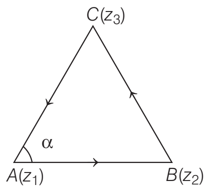
Different forms of a Complex Number

- **Polar or Trigonometrical Form** of $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$.

  If we use the general value of the argument θ, then the polar form of z is $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$, where n is an integer.
- **Euler's form** of $z = x + iy$ is $z = re^{i\theta}$, where $r = |z|, \theta = \arg(z)$ and $e^{i\theta} = \cos\theta + i\sin\theta$.

Concept of Rotation

Let z_1, z_2, z_3 be the vertices of ΔABC as shown in figure, then $\alpha = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$ and $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$



NOTE • Always mark the direction of arrow in anti-clockwise sense and keep that complex number in the numerator on which the arrow goes.

Square Root of a Complex Number

- If $z = a + ib$, then
$$\sqrt{z} = \sqrt{a + ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{|z| + a} + i\sqrt{|z| - a}\right]$$
- If $z = a - ib$, then $\sqrt{z} = \sqrt{a - ib} = \pm \dfrac{1}{\sqrt{2}}\left[\sqrt{|z| + a} - i\sqrt{|z| - a}\right]$

De-Moivre's Theorem

- If n is any integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- If n is any rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.
- If n is any positive integer, then
$$(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)$$
where, $k = 0, 1, 2, \ldots n - 1$

Cube Root of Unity

Cube roots of unity are $1, \omega, \omega^2$

where, $\omega = \dfrac{-1 + \sqrt{3}i}{2}$ and $\omega^2 = \dfrac{-1 - \sqrt{3}i}{2}$

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $1 + \omega^n + \omega^{2n} = \begin{cases} 0 & \text{if } n \neq 3m, \quad m \in N \\ 3 & \text{if } n = 3m, \quad m \in N \end{cases}$

*n*th Roots of Unity

By nth root of unity we mean any complex number z which satisfies the equation $z^n = 1$.

(i) The nth roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, where $\alpha = e^{\frac{i2\pi}{n}}$

(ii) $1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} = 0$

(iii) $1 \cdot \alpha \cdot \alpha^2 \ldots \alpha^{n-1} = [-1]^{n-1}$

Applications of Complex Numbers in Geometry

1. Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_2 - z_1|$.
2. Let point $P(z)$ divides the line segment joining $A(z_1)$ and $B(z_2)$ in the ratio $m : n$. Then,
   (i) for internal division, $z = \dfrac{mz_2 + nz_1}{m + n}$
   (ii) for external division, $z = \dfrac{mz_2 - nz_1}{m - n}$
3. Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$, then centroid $G(z)$ of the ΔABC is given by z
$$= \frac{1}{3}(z_1 + z_2 + z_3)$$
Area of ΔABC is given by $\Delta = \dfrac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$
4. For an equilateral triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$, $z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$
5. The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.
6. (i) An equation of the circle with centre at z_0 and radius r, is $|z - z_0| = r$
   (ii) $|z - z_0| < r$ represents the interior of circle and $|z - z_0| > r$ represents the exterior of circle.
   (iii) General equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where b is real number, with centre is $-a$ and radius is $\sqrt{a\bar{a} - b}$.
7. If z_1 and z_2 are two fixed points and $k > 0$, $k \neq 1$ is a real number, then $\dfrac{|z - z_1|}{|z - z_2|} = k$ represents a circle.

   For $k = 1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.
8. If end points of diameter of a circle are $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on the circle, then equation of circle in diameter form is
$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$

DAY ONE

Sets, Relations and Functions

Learning & Revision for the Day

- Sets
- Venn Diagram
- Operations on Sets
- Law of Algebra of Sets
- Cartesian Product of Sets
- Relations
- Composition of Relations
- Functions or Mapping
- Composition of Functions

Sets

- A **set** is a well-defined class or collection of the objects.
- Sets are usually denoted by the symbol $A, B, C, \dots$ and its elements are denoted by $a, b, c, \dots$ etc.
- If a is an element of a set A, then we write $a \in A$ and if not then we write $a \notin A$.

Representations of Sets

There are two methods of representing a set :

- In **roster method**, a set is described by listing elements, separated by commas, within curly braces$\{\neq\}$. e.g. A set of vowels of English alphabet may be described as $\{a, e, i, o, u\}$.
- In **set-builder method**, a set is described by a property $P(x)$, which is possessed by all its elements x. In such a case the set is written as $\{x : P(x) \text{ holds}\}$ or $\{x \mid P(x) \text{ holds}\}$, which is read as the set of all x such that $P(x)$ holds. e.g. The set $P = \{0, 1, 4, 9, 16, \dots\}$ can be written as $P = \{x^2 \mid x \in Z\}$.

Types of Sets

- The set which contains no element at all is called the **null set** (empty set or void set) and it is denoted by the symbol 'ϕ' or '$\{\}$' and if it contains a single element, then it is called **singleton set**.
- A set in which the process of counting of elements definitely comes to an end, is called a **finite set,** otherwise it is an **infinite set.**
- Two sets A and B are said to be **equal set** iff every element of A is an element of B and also every element of B is an element of A. i.e. $A = B$, if $x \in A \Leftrightarrow x \in B$.

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)—
  (Without referring Explanations)

- Accuracy Level $(z/y \times 100)$—
- Prep Level $(z/x \times 100)$—

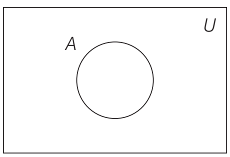
In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

- **Equivalent sets** have the same number of elements but not exactly the same elements.
- A set that contains all sets in a given context is called **universal set** (U).
- Let A and B be two sets. If every element of A is an element of B, then A is called a **subset** of B, i.e. $A \subseteq B$.
- If A is a subset of B and $A \neq B$, then A is a **proper subset** of B. i.e. $A \subset B$.
- The null set ϕ is a subset of every set and every set is a subset of itself i.e. $\phi \subset A$ and $A \subseteq A$ for every set A. They are called **improper subsets** of A.
- If S is any set, then the set of all the subsets of S is called the **power set** of S and it is denoted by $P(S)$. Power set of a given set is always non-empty. If A has n elements, then $P(A)$ has 2^n elements.

NOTE
- The set $\{\phi\}$ is not a null set. It is a set containing one element ϕ.
- Whenever we have to show that two sets A and B are equal show that $A \subseteq B$ and $B \subseteq A$.
- If a set A has m elements, then the number m is called **cardinal number** of set A and it is denoted by $n(A)$. Thus, $n(A) = m$.

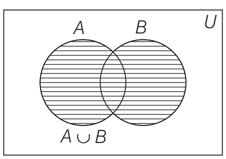
Venn Diagram

The combination of rectangles and circles is called **Venn Euler diagram** or Venn diagram. In Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle on some closed geometrical figure. Where, A is the set and U is the universal set.

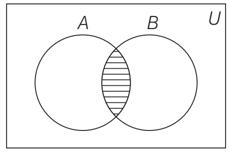


Operations on Sets

- The **union** of sets A and B is the set of all elements which are in set A or in B or in both A and B.
i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

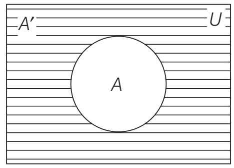


- The **intersection** of A and B is the set of all those elements that belong to both A and B.
i.e. $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

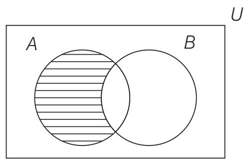


$A \cap B$

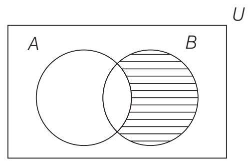
- If $A \cap B = \phi$, then A and B are called **disjoint sets**.
- Let U be an universal set and A be a set such that $A \subset U$. Then, **complement of A** with respect to U is denoted by A' or A^c or $\overline{A}$ or $U - A$. It is defined as the set of all those elements of U which are not in A.



- The **difference** $A - B$ is the set of all those elements of A which does not belong to B.
i.e. $A - B = \{x : x \in A \text{ and } x \notin B\}$
and $B - A = \{x : x \in B \text{ and } x \notin A\}$.

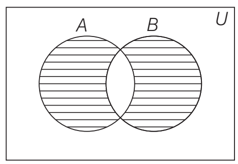


$A - B$



$B - A$

- The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \,\Delta\, B$.
i.e. $A \,\Delta\, B = (A - B) \cup (B - A)$



$A \,\Delta\, B$

Law of Algebra of Sets

If A, B and C are any three sets, then

1. **Idempotent Laws**
(i) $A \cup A = A$ (ii) $A \cap A = A$
2. **Identity Laws**
(i) $A \cup \phi = A$ (ii) $A \cap U = A$
3. **Distributive Laws**
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. **De-Morgan's Laws**
   (i) $(A \cup B)' = A' \cap B'$
   (ii) $(A \cap B)' = A' \cup B'$
   (iii) $A - (B \cap C) = (A - B) \cup (A - C)$
   (iv) $A - (B \cup C) = (A - B) \cap (A - C)$

5. **Associative Laws**
   (i) $(A \cup B) \cup C = A \cup (B \cup C)$
   (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

6. **Commutative Laws**
   (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$
   (iii) $A \Delta B = B \Delta A$

Important Results on Operation of Sets

1. $A - B = A \cap B'$
2. $B - A = B \cap A'$
3. $A - B = A \Leftrightarrow A \cap B = \phi$
4. $(A - B) \cup B = A \cup B$
5. $(A - B) \cap B = \phi$
6. $A \subseteq B \Leftrightarrow B' \subseteq A'$
7. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
8. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
9. $n(A \cup B) = n(A) + n(B)$
   $\Leftrightarrow$ A and B are disjoint sets.
10. $n(A - B) = n(A) - n(A \cap B)$
11. $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
12. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
    $- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
13. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
14. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Cartesian Product of Sets

Let A and B be any two non-empty sets. Then the cartesian product $A \times B$, is defined as set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.
i.e.

- $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$
- $B \times A = \{(b, a) : b \in B \text{ and } a \in A\}$
  and $A \times A = \{(a, b) : a, b \in A\}$.
- $A \times B = \phi$, if either A or B is an empty set.
- If $n(A) = p$ and $n(B) = q$, then
  $n(A \times B) = n(A) \cdot n(B) = pq$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Relations

- Let A and B be two non-empty sets, then **relation** R from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$.
- If $(a, b) \in R$, then we say a is related to b by the relation R and we write it as aRb.
- Domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$.
- If $n(A) = p$ and $n(B) = q$, then the total number of relations from A to B is 2^{pq}.

Types of Relations

- Let A be any non-empty set and R be a relation on A. Then,
  (i) R is said to be **reflexive** iff $(a, a) \in R, \forall a \in A$.
  (ii) R is said to be **symmetric** iff
  $(a, b) \in R$
  $\Rightarrow$ $(b, a) \in R, \forall a, b \in A$
  (iii) R is said to be a **transitive** iff $(a, b) \in R$ and $(b, c) \in R$
  $\Rightarrow$ $(a, c) \in R, \forall a, b, c \in A$
  i.e. aRb and bRc $\Rightarrow$ $aRc, \forall a, b, c \in A$.
- The relation $I_A = \{(a, a) : a \in A\}$ on A is called the **identity relation** on A.
- R is said to be an **equivalence relation** iff
  (i) it is reflexive i.e. $(a, a) \in R, \forall a \in A$.
  (ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
  (iii) it is transitive
  i.e. $(a, b) \in R$ and $(b, c) \in R$
  $\Rightarrow$ $(a, c) \in R, \forall a, b, c \in A$

Inverse Relation

Let R be a relation from set A to set B, then the **inverse of** $\boldsymbol{R}$, denoted by R^{-1}, is defined by
$R^{-1} = \{(b, a) : (a, b) \in R\}$. Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.

NOTE
- The intersection of two equivalence relations on a set is an equivalence relation on the set.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation A.

Composition of Relations

Let R and S be two relations from set A to B and B to C respectively, then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. This relation is called the **composition of** $\boldsymbol{R}$ **and** $\boldsymbol{S}$.

$$RoS \neq SoR$$

Functions or Mapping

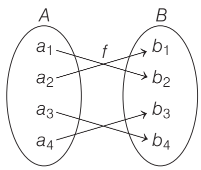
- If A and B are two non-empty sets, then a rule f which associates each $x \in A$, to a unique member $y \in B$, is called a function from A to B and it is denoted by $f : A \to B$.
- The set A is called the **domain** of $f(D_f)$ and set B is called the **codomain** of $f(C_f)$.
- The set consisting of all the f-images of the elements of the domain A, called the range of $f(R_f)$.

NOTE
- A relation will be a function, if no two distinct ordered pairs have the same first element.
- Every function is a relation but every relation is not necessarily a function.
- The number of functions from a finite set A into finite set B is $\{n(B)\}^{n(A)}$.

Different Types of Functions

Let f be a function from A to B, i.e. $f : A \to B$. Then,

f is said to be **one-one function** or injective function, if different elements of A have different images in B.

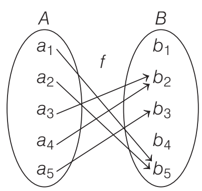


Methods to Check One-One Function

Method I If $f(x) = f(y) \Rightarrow x = y$, then f is one-one.

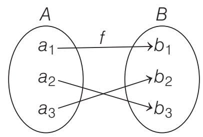
Method II A function is one-one iff no line parallel to X-axis meets the graph of function at more than one point.

- The number of one-one function that can be defined from a finite set A into finite set B is $\begin{cases} {}^{n(B)}P_{n(A)}, & \text{if } n(B) \ge n(A) \\ 0, & \text{otherwise} \end{cases}$.
- f is said to be a **many-one function**, if two or more elements of set A have the same image in B.



i.e. $f : A \to B$ is a many-one function, if it is not a one-one function.

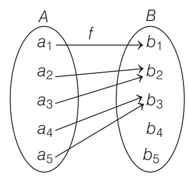
- f is said to be **onto function** or **surjective function**, if each element of B has its pre-image in A.



Method to Check Onto Function

Find the range of $f(x)$ and show that range of $f(x) =$ codomain of $f(x)$.

- Any polynomial function of odd degree is always onto.
- The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 2 elements $= 2^n - 2$.
- If $n(A) \ge n(B)$, then number of onto function is 0.
- If A has m elements and B has n elements, where $m < n$, then number of onto functions from A to B is $n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \dots$, $m < n$.
- f is said to be an **into function**, if there exists atleast one element in B having no pre-image in A. i.e. $f : A \to B$ is an into function, if it is not an onto function.



- f is said to be a **bijective function**, if it is one-one as well as onto.

NOTE
- If $f : A \to B$ is a bijective, then A and B have the same number of elements.
- If $n(A) = n(B) = m$, then number of bijective map from A to B is $m!$.

Composition of Functions

Let $f : A \to B$ and $g : B \to C$ are two functions. Then, the composition of f and g, denoted by

$gof : A \to C$, is defined as,

$gof(x) = g[f(x)]$, $\forall x \in A$.

NOTE
- gof is defined only if $f(x)$ is an element of domain of g.
- Generally, $gof \ne fog$.

DAY SIX

Determinants

Learning & Revision for the Day

- Determinants
- Properties of Determinants
- Cyclic Determinants
- Area of Triangle by using Determinants
- Minors and Cofactors
- Adjoint of a Matrix
- Inverse of a Matrix
- Solution of System of Linear Equations in Two and Three Variables

Determinants

Every square matrix A can be associated with a number or an expression which is called its determinant and it is denoted by det (A) or $|A|$ or Δ.

If $A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then det $(A)=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

- If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$
- If $A=\begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$, then $|A|=\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$

$$=a\begin{vmatrix} q & r \\ v & w \end{vmatrix}-b\begin{vmatrix} p & r \\ u & w \end{vmatrix}+c\begin{vmatrix} p & q \\ u & v \end{vmatrix} \quad \text{[expanding along } R_1]$$

$$=a(qw-vr)-b(pw-ur)+c(pv-uq)$$

There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2, R_3) and three columns (C_1, C_2, C_3).

NOTE
- Rule to put + or – sign in the expansion of determinant of order 3. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$
- A square matrix A is said to be singular, if $|A|=0$ and non-singular, if $|A|\neq 0$.

PRED MIRROR
Your Personal Preparation Indicator

- No. of Questions in Exercises (x)—
- No. of Questions Attempted (y)—
- No. of Correct Questions (z)—
  (Without referring Explanations)

- Accuracy Level $(z/y \times 100)$—
- Prep Level $(z/x \times 100)$—

In order to expect good rank in JEE, your Accuracy Level should be above 85 & Prep Level should be above 75.

Properties of Determinants

(i) If each element of a row (column) is zero, then $\Delta = 0$.

(ii) If two rows (columns) are proportional, then $\Delta = 0$.

(iii) $|A^T| = |A|$, where A^T is a transpose of a matrix.

(iv) If any two rows (columns) are interchanged, then Δ becomes $-\Delta$.

(v) If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant

(vi) $\det(kA) = k^n \det(A)$, if A is of order $n \times n$.

(vii) If each element of a row (column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants i.e.

$$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

(viii) If a scalar multiple of any row (column) is added to another row (column), then Δ is unchanged

i.e. $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p+ka & q+kb & r+kc \\ u & v & w \end{vmatrix}$, which is obtained by the operation $R_2 \to R_2 + kR_1$

Product of Determinants

If $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $|B| = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$, then

$$|A| \times |B| = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix} = |AB|$$

[multiplying row by row]

We can multiply rows by columns or columns by rows or columns by columns

NOTE
- $|AB| = |A|\,|B| = |BA| = |A^TB| = |AB^T| = |A^TB^T|$
- $|A^n| = |A|^n, n \in Z^+$

Cyclic Determinants

In a cyclic determinant, the elements of row (or column) are arranged in a systematic order and the value of a determinant is also in systematic order.

(i) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(ii) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

(iii) $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

(iv) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= -(a^3+b^3+c^3-3abc)$

(v) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc\,(a-b)(b-c)(c-a)$

Area of Triangle by using Determinants

If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of ΔABC, then

Area of $\Delta ABC = \dfrac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

$= \dfrac{1}{2}\,|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$

If these three points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ and *vice-versa*.

Minors and Cofactors

The **minor** M_{ij} of the element a_{ij} is the determinant obtained by deleting the ith row and jth column of Δ.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

then $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ etc.

The **cofactor** C_{ij} of the element a_{ij} is $(-1)^{i+j} M_{ij}$.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ etc.

The sum of product of the elements of any row (or column) with their corresponding cofactors is equal to the value of determinant.

i.e. $\Delta = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

$= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$

$= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$

But if elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

Adjoint of a Matrix

If $A = [a_{ij}]_{nxn}$, then adjoint of A, denoted by adj (A), is defined as $[C_{ij}]^T_{nxn}$, where C_{ij} is the cofactor of a_{ij}.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

NOTE
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then adj $(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Properties of Adjoint of a Matrix

Let A be a square matrix of order n, then

(i) $(\text{adj } A)A = A(\text{adj } A) = |A| \cdot I_n$

(ii) $|\text{adj } A| = |A|^{n-1}$, if $|A| \neq 0$

(iii) adj $(AB) = (\text{adj } B)(\text{adj } A)$

(iv) $\text{adj}(A^T) = (\text{adj } A)^T$

(v) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, if $|A| \neq 0$

(vi) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, if $|A| \neq 0$

Inverse of a Matrix

Let A be any non-singular (i.e. $|A| \neq 0$) square matrix, then inverse of A can be obtained by following two ways.

1. Using determinants

In this, $A^{-1} = \dfrac{1}{|A|}$ adj (A)

2. Using Elementary operations

In this, first write $A = IA$ (for applying row operations) or $A = AI$ (for applying column operations) and then reduce A of LHS to I, by applying elementary operations simultaneously on A of LHS and I of RHS. If it reduces to $I = PA$ or $I = AP$, then $P = A^{-1}$.

Properties of Inverse of a Matrix

(i) A square matrix is invertible if and only if it is non-singular.

(ii) If $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $A^{-1} = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})$ provided $\lambda_i \neq 0 \forall, i = 1, 2, \ldots n$.

Solution of System of Linear Equations in Two and Three Variables

Let system of linear equations in three variables be

$$a_1 x + b_1 y + c_1 z = d_1,\ a_2 x + b_2 y + c_2 z = d_2$$
and $$a_3 x + b_3 y + c_3 z = d_3.$$

Now, we have two methods to solve these equations.

1. Matrix Method

In this method we first write the above system of equations in matrix form as shown below.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ or } AX = B$$

where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Case I When system of equations is non-homogeneous (i.e. when $B \neq 0$).

- If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.
- If $|A| = 0$ and $(\text{adj } A)$, $B \neq 0$, then the system of equations is inconsistent and has no solution.
- If $|A| = 0$ and $(\text{adj } A) \cdot B = O$, then the system of equations may be either consistent or inconsistent according as the system have infinitely many solutions or no solution.

Case II When system of equations is homogeneous (i.e. when $B = 0$).

- If $|A| \neq 0$, then system of equations has only trivial solution, namely $x = 0$, $y = 0$ and $z = 0$.
- If $|A| = 0$, then system of equations has non-trivial solution, which will be infinite in numbers.

2. Cramer's Rule Method

In this method we first determine

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Case I When system of equations is non-homogeneous

- If $D \neq 0$, then it is consistent with unique solution given by $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, $z = \dfrac{D_3}{D}$.
- If $D = 0$ and atleast one of D_1, D_2 and D_3 is non-zero, then it is inconsistent (no solution).
- If $D = D_1 = D_2 = D_3 = 0$, then it may be consistent or inconsistent according as the system have infinitely many solutions or no solution.

Case II When system of equations is homogeneous

- If $D \neq 0$, then $x = y = z = 0$ is the only solution, i.e. the trivial solution.
- If $D = 0$, then it has infinitely many solutions.

Above methods can be used, in a similar way, for the solution of system of linear equations in two variables.