MA 102 (Ordinary Differential Equations)

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Tutorial Sheet No. 4 **Date:** April 12, 2018

Systems of Linear Differential Equations.

(1) Find a general solution to the given equation for $x > 0$.
(a) $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$.
(b) $x^2y'' - 5xy' + 8y = 2x^3$.

Solution: (a) With the substitution $x = e^t$, the equation reduced to a linear ODE with constant coefficients as

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 0.$$

The AE is $r^3 - 6r^2 + 11r - 6 = 0 \Longrightarrow r = 1,\ 2,\ 3$. The GS is $y = c_1e^t + c_2e^{2t} + c_3e^{3t}$. Writing in the original variable $y = c_1x + c_2x^2 + c_3x^3$.

(b) With $x = e^t$, the differential equation get transformed into

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = 2e^{3t}.$$

The roots of AE is $r = 2,\ 4$. Thus, $y_h(x) = c_1e^{2t} + c_2e^{4t}$. The PS is $y_p = -2e^{3t}$. Hence, the GS is

$$y = c_1e^{2t} + c_2e^{4t} - 2e^{3t} = c_1x^2 + c_2x^4 - 2x^3.$$

(2) Given that $y = x$ is a solution of $x^2y'' + xy' - y = 0,\ x \neq 0$, find the general solution of $x^2y'' + xy' - y = x,\ x \neq 0$.
Solution: Proceed as in Q.11 of Tut-3 and obtain the GS (y_h) of $x^2y'' + xy' - y = 0$. Then use VPM to find y_p. The GS of the given equation is then obtained as $y(x) = y_h(x) + y_p(x)$.

(3) Rewrite the given scalar equation as a first-order system in normal form. Express the system in the matrix form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$.
(a) $y''(t) - 3y'(t) - 11y(t) = \sin t$; (b) $y^{(4)}(t) + y(t) = t^2$.

Solution: (a) Set $x_1(t) = y(t),\ x_2(t) = y'(t)$. Then $x_1'(t) = x_2(t)$ and $x_2'(t) = 11x_1(t) + 3x_2(t) + \sin(t)$. With $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$, we have

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 11 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sin t \end{bmatrix}.$$

(b) Set $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = y''(t)$ and $x_4(t) = y'''(t)$. Then $x_1'(t) = x_2(t)$, $x_2'(t) = x_3(t)$, $x_3'(t) = x_4(t)$ and $x_4'(t) = -x_1(t) + t^2$. Thus,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}.$$

(4) Determine over what interval we are assured that there is a unique solution to the following initial value problem.

(a) $\mathbf{x}'(t) = \begin{bmatrix} \cos t & \sqrt{t} \\ t^3 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \tan t \\ e^t \end{bmatrix}, \quad \mathbf{x}(2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$

(b) $\mathbf{x}'(t) = \begin{bmatrix} t^2 & 1+3t \\ 1 & \sin t \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Solution: (a) $A(t)$ is continuous in $[0,\infty)$ and $\mathbf{f}(t)$ is continuous in $(\pi/2, 3\pi/2)$. A unique solution to IVP exists on the interval $(\pi/2, 3\pi/2)$, which contains 2.

(b) $A(t)$ is continuous in $\mathbb{R}$ and $\mathbf{f}(t)$ is continuous on $\mathbb{R}$. A unique solution to IVP exists on $\mathbb{R}$.

(5) The vector functions $\mathbf{x}_1 = [e^{-t}, 2e^{-t}, e^{-t}]^T$, $\mathbf{x}_2 = [e^t, 0, e^t]^T$, $\mathbf{x}_3 = [e^{3t}, -e^{3t}, 2e^{3t}]^T$ are solutions to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.

Solution: $W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(t) = \begin{vmatrix} e^{-t} & e^t & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^t & 2e^{3t} \end{vmatrix} = -2e^{3t} \neq 0$. Therefore, the given set of vectors is linearly independent and so forms a fundamental solution set. A fundamental matrix is given by

$$\Phi(t) = \begin{bmatrix} e^{-t} & e^t & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^t & 2e^{3t} \end{bmatrix},$$

and a general solution of the system is

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} = c_1 \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} + c_3 \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{bmatrix}.$$

(6) Let $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ be two fundamental matrices for the same system $\mathbf{x}'(t) = A\mathbf{x}$. Then, there exists a constant matrix $\mathbf{C}$ such that $\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{C}$.

Solution: Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be columns of $\mathbf{X}(t)$ and let $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ be columns of $\mathbf{Y}(t)$. Since $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)\}$ is a fundamental solution set and $\mathbf{x}_j(t)$, $j = 1, \ldots, n$ are solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$, there exists constants $c_{1j}, c_{2j}, \ldots, c_{nj}$ such that

$$\mathbf{x}_j(t) = c_{1j}\mathbf{y}_1(t) + c_{2j}\mathbf{y}_2(t) + \ldots + c_{nj}\mathbf{y}_n(t), \quad j = 1, \ldots, n,$$

and which is equivalent to $\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{C}$, where $\mathbf{C} = [c_{ij}]$.

(7) Find a fundamental matrix of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ by computing e^{At}.

$$(a)\ A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \ (b)\ A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \ (c)\ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ (d)\ A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

Solution: (a) Since $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = S + N$. Note that $SN = NS$ and $N^2 = \mathbf{0}$. Thus

$$\Phi(t) = e^{At} = e^{St}\, e^{Nt} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} \left\{ I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right\} = e^{3t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

(b) $e^{At} = e^{2t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

(c) Note that

$$\begin{aligned} \Phi(t) = e^{At} &= I + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{t}{1!} + \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 1^2 \end{bmatrix} \frac{t^2}{2!} + \cdots + \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} \frac{t^k}{k!} + \cdots. \\ &= \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \end{aligned}$$

(d) $A = -2I + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = S + N$, where S and N commute. Thus,

$$\Phi(t) = e^{At} = e^{-2t}[I + Nt + N^2t^2/2] = e^{-2t}\begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}.$$

(8) Solve the initial value problems $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for the matrix

(a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, (c) $A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$,

(d) $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Solution: (a) $\lambda_1 = \lambda_2 = 2$. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = S + N$, where S and N commute and $N^2 = \mathbf{0}$. Thus,

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}e^{Nt} = e^{2t}\begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}\mathbf{x}_0.$$

(b) The eigenvalues are $\lambda_1 = 1+i$, $\lambda_2 = 1-i$ and $\lambda_3 = -2$. The corresponding eigenvectors $\mathbf{w} = [1-i, -1, 0]^T = [1, -1, 0] + i[-1, 0, 0]^T = \mathbf{u} + i\mathbf{v}$ and $\mathbf{v}_3 = [0, 0, 1]^T$. Thus, $P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the solution is

$$\mathbf{x}(t) = P\begin{bmatrix} e^t\cos t & -e^t\sin t & 0 \\ e^t\sin t & e^t\cos t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}P^{-1}\mathbf{x}_0.$$

(c) $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 1$. We now compute the generalized eigenvectors $\mathbf{v}_1 = [1, 0, 0]^T$, $\mathbf{v}_2 = [0, 1, 0]^T$ satisfying $(A - \lambda_1 I)\mathbf{v}_2 = \mathbf{0}$, and $\mathbf{v}_3 = [0, 2, 1]^T$. Now, determine

$$S = P\,\mathrm{diag}[-1, -1, 1]\,P^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \; N = A - S = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

S and N commute, $N^2 = \mathbf{0}$. Thus,

$$\begin{aligned} \mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{St}[I + Nt]\mathbf{x}_0 &= P\,\mathrm{diag}[e^{-t}, e^{-t}, e^t]\,P^{-1}[I + Nt]\mathbf{x}_0 \\ &= \begin{bmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & 2(e^t - e^{-t}) \\ 0 & 0 & e^t \end{bmatrix}\mathbf{x}_0 \end{aligned}$$

(d) $\lambda_1 = \lambda_2 = 1+i$ and the eigenvectors $\mathbf{w}_1 = [i, 1, 0, 0]^T = [0, 1, 0, 0]^T + i[1, 0, 0, 0] = \mathbf{u}_1 + i\mathbf{v}_1$, $\mathbf{w}_2 = [0, 0, i, 1]^T = [0, 0, 0, 1]^T + i[0, 0, 1, 0]^T = \mathbf{u}_2 + i\mathbf{v}_2$. The matrix $P = I$, $A = S = \mathrm{diag}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Thus,

$$\mathbf{x}(t) = e^t\begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix}\mathbf{x}_0.$$

(9) Let $\mathbf{x}(t)$ be a nontrivial solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $A + A^T$ is positive definite. Prove that $\|\mathbf{x}(t)\|$ is an increasing function of t. (Here, $\|\cdot\|$ denotes the Euclidean norm.)

Solution: We have

$$\begin{aligned}
\frac{d}{dt}\|\mathbf{x}(t)\|^2 &= \frac{d}{dt} < \mathbf{x}(t), \mathbf{x}(t) > \\
&= < \mathbf{x}'(t), \mathbf{x}(t) > + < \mathbf{x}(t), \mathbf{x}'(t) > \\
&= < A\mathbf{x}(t), \mathbf{x}(t) > + < \mathbf{x}(t), A\mathbf{x}(t) > \\
&= < A\mathbf{x}(t), \mathbf{x}(t) > + < A^T\mathbf{x}(t), \mathbf{x}(t) > \\
&= < (A + A^T)\mathbf{x}(t), \mathbf{x}(t) > > 0.
\end{aligned}$$

(10) Let A be a real 3×3 matrix such that $A^T = -A$. Let $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ be a real solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Prove that
(a) $\|\mathbf{x}(t)\|$ is independent of t.
(b) If $\mathbf{v} \in Ker(A)$ then $\mathbf{x}(t) \cdot \mathbf{v}$ is independent of t.

Solution: (a) We have

$$\begin{aligned}
\frac{d}{dt}\|\mathbf{x}(t)\|^2 &= \frac{d}{dt} < \mathbf{x}(t), \mathbf{x}(t) >= \frac{d}{dt}(\mathbf{x}(t).\mathbf{x}(t)) = 2\mathbf{x}(t).\mathbf{x}'(t) \\
&= 2 < \mathbf{x}(t), \mathbf{x}'(t) >= 2\mathbf{x}(t) \cdot A\mathbf{x}(t) = 2A^T\mathbf{x}(t) \cdot \mathbf{x}(t) \\
&= -2A\mathbf{x}(t) \cdot \mathbf{x}(t) = -2\mathbf{x}(t) \cdot A\mathbf{x}(t) = -\frac{d}{dt}\|\mathbf{x}(t)\|^2 \\
&\Longrightarrow \frac{d}{dt}\|\mathbf{x}(t)\|^2 = 0,
\end{aligned}$$

hence $\|\mathbf{x}(t)\|$ is constant.

(b) Note that

$$\begin{aligned}
\frac{d}{dt}(\mathbf{x}(t) \cdot \mathbf{v}) &= \frac{d\mathbf{x}}{dt} \cdot \mathbf{v} = A\mathbf{x}(t) \cdot \mathbf{v} \\
&= \mathbf{x}(t) \cdot A^T\mathbf{v} = -\mathbf{x}(t) \cdot A\mathbf{v} = 0.
\end{aligned}$$

(11) Solve the nonhomogeneous linear system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ with initial condition $\mathbf{x}(0) = [1, 0]^T$, where (a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}$; ($b$) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} t \\ 1 + 2t \end{bmatrix}$.

Solution: (a) The fundamental matrix

$$\Phi(t) = e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \text{ and } e^{-At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

The solution to the corresponding homogeneous system

$$\mathbf{x}_h(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}(0) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \ (\Phi^{-1}(0) = I)$$

A particular solution is given by

$$\mathbf{x}_p(t) = \Phi(t) \int_0^t \Phi^{-1}(s)\mathbf{f}(s)\, ds = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \int_0^t \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix} \begin{bmatrix} 2e^s \\ 4e^s \end{bmatrix} ds.$$

Thus, the unique solution is given by $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$.

(b) In this case

$$\Phi(t) = e^{3t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \Phi^{-}(t) = e^{-3t} \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

Then compute $\mathbf{x}_h(t)$ and $\mathbf{x}_p(t)$ as in (a).

(12) Show that $\Phi(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}$ is a fundamental matrix solution of the nonautonomous linear system $\mathbf{x}'(t) = A(t)\mathbf{x}$ with $A(t) = \begin{bmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{bmatrix}$. Find the inverse of $\Phi(t)$ and

solve $\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(0) = \mathbf{x}_0$ with $A(t)$ as given above and $\mathbf{f}(t) = [1,\ \ e^{-2t}]^T$.

Solution: $\Phi(t)$ satisfies

$$\Phi'(t) = \begin{bmatrix} -2e^{-2t}\cos t - e^{-2t}\sin t & -\cos t \\ -2e^{-2t}\sin t + e^{-2t}\cos t & -\sin t \end{bmatrix} = A(t)\Phi(t).$$

Now

$$\Phi^{-1}(t) = e^{2t}\begin{bmatrix} \cos t & -\sin t \\ -e^{2t}\sin t & e^{-2t}\cos t \end{bmatrix},\ \Phi^{-1}(0) = I.$$

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\int_0^t e^{2s}\begin{bmatrix} \cos s & -\sin s \\ -e^{2s}\sin s & e^{-2s}\cos s \end{bmatrix}\begin{bmatrix} 1 \\ e^{-2s} \end{bmatrix} ds.$$

(13) Find all critical points of each of the following plane autonomous systems:
(a) $x_1'(t) = -x_1 + x_2,\ \ x_2'(t) = x_1 - x_2;$ (b) $x_1'(t) = x_1^2 + x_2^2 - 6,\ \ x_2'(t) = x_1^2 - x_2.$
(c) $x_1'(t) = x_1^2 e^{x_2},\ \ x_2'(t) = x_2(e^{x_1} - 1).$

Solution: (a) All points on the line $x_1 = x_2$.
(b) If $x_2 = -3$, then $x_1^2 = -3$. So, there are no real solutions. If $x_2 = 2$, then $x_1 = \pm\sqrt{2}$. So, critical points are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.
(c) The critical points are $(0, x_2)$, x_2 arbitrary.

(14) Determine the nature of critical point $(0,0)$ of each of the linear autonomous systems $\mathbf{x}'(t) = A\mathbf{x}(t)$, and determine whether or not the critical point is stable.
(a) $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution: (a) The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. The origin is an unstable saddle point.

(b) The eigenvalues are $\lambda = 2 \pm i$. The origin is an unstable focus point.

(c) The eigenvalues are $\lambda = \pm i$. So, the origin is a center.