DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI

Even Semester of the Academic year 2014 - 2015

MA 102 Mathematics II

Problem Sheet 1: Introduction to Differential Equations, Direction Fields, Definitions and Terminologies, Initial value problems, Existence and Uniqueness theorems.

Instructor: Dr. J. C. Kalita and Dr. Shyamashree Upadhyaya

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1. Determine the order and degree of the following differential equations. Also, state whether they are linear or nonlinear:

   (a) $\dfrac{d^4y}{dx^4} + 19\left(\dfrac{dy}{dx}\right)^2 = 11y.$

   (b) $\dfrac{d^2y}{dx^2} + x\ sin\ y = 0.$

   (c) $\dfrac{d^2y}{dx^2} + y\ sin\ x = 0.$

   (d) $\left(1 + \dfrac{dy}{dx}\right)^{\frac{1}{2}} = x\dfrac{d^2y}{dx^2}.$

   (e) $\left[1 + (\dfrac{dy}{dx})^2\right]^3 = a^2\left(\dfrac{d^2y}{dx^2}\right)^2.$

2. For each of the following families of curves, find a differential equation (of least order) for which each member of the family is a solution.

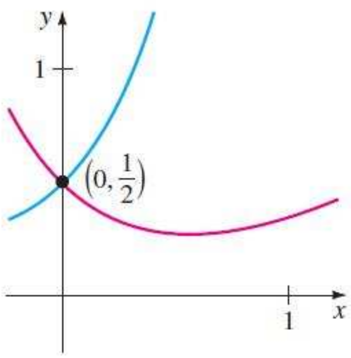
   (a) $\{y = c_1e^x + c_2e^{-3x} : c_1, c_2 \in \mathbb{R}\}.$

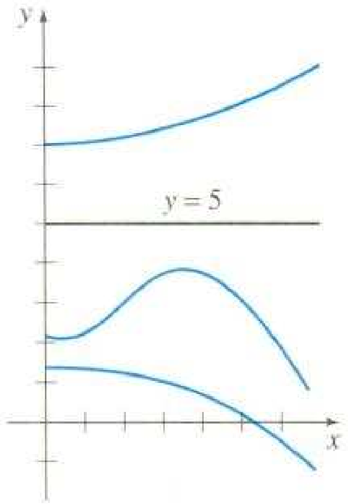
   (b) $\{y = x\ sin(x + c) : c \in \mathbb{R}\}.$

   (c) All lines of slope m tangent to the parabola $y^2 = 4x$.

3. Sketch the direction field of the first order differential equation $y' = xy$ and an approximation to the solution curve through the point $(1, 2)$.

4. Consider the initial value problem $y' = x - 2y,\ y(0) = \dfrac{1}{2}$. Determine which of the curves shown below is the only plausible solution curve? Explain your reasoning.





5. (a) Verify that $y = 5$, $-\infty < x < \infty$ which is shown in the above figure as the horizontal black line is a solution of $y' = 5 - y$.
   (b) Why aren't the curves shown in color in the figure plausible solution curves?
   (c) Sketch several plausible solution curves in the regions $y > 5$ and $y < 5$.

6. Consider the differential equation $y' = y^2 + 4$.
   (a) Explain why there exist no constant solutions of the DE.
   (b) Describe the graph of the solution $y = \phi(x)$. For example, can a solution curve have any relative extrema?
   (c) Explain why $y = 0$ is the y-coordinate of a point of inflection of the solution curve?
   (d) Sketch the graph of a solution $y = \phi(x)$ of the DE whose shape is suggested by (a)-(c).

7. Suppose $y = \phi(x)$ is a solution of the differential equation $\dfrac{dy}{dx} = y(a - by)$, where a and b are positive constants.
   (a) By inspection, find two constant solutions of the equation.
   (b) Only using the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. On which $y = \phi(x)$ is decreasing?
   (c) Using only the differential equation, explain why $y = \dfrac{a}{2b}$ is the y-coordinate of a point of inflection of the graph of the nonconstant solution $y = \phi(x)$.
   (d) On the same coordinate axes, sketch the graphs of the two constants solutions found in part (a). These constant solutions partition the xy-plane into three regions. In each region, sketch the graph of a nonconstant solution $y = \phi(x)$ whose shape is suggested by the results in (b) and (c).

8. (a) Verify that $y = -\dfrac{1}{x+c}$ is a one parameter solution of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\dfrac{\partial f}{\partial y} = 2y$ are continuous everywhere, the region R in the existence and uniqueness theorem can be taken as the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of the definition for the solution of each initial-value problem.

(c) Find a solution from the family in part (a) that satisfies $y' = y^2,\ y(0) = y_0,\ y_0 \neq 0$. Explain why the largest interval I of definition for the solution is either $-\infty < x < \dfrac{1}{y_0}$ or $\dfrac{1}{y_0} < x < \infty$.

(d) Determine the largest interval I of definition for the solution of the first order initial value problem $y' = y^2,\ y(0) = 0$. What type of solution is this?

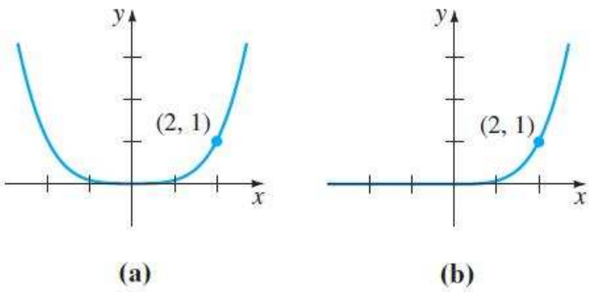
9. (a) Verify that $3x^2 - y^2 = c$ is a one parameter solution of the differential equation $y\dfrac{dy}{dx} = 3x$.

(b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all the explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval of definition of each explicit solution. The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions satisfies $y(-2) = 3$?

(c) Use the solutions in part (a) to find the implicit solution of the initial value problem $y\dfrac{dy}{dx} = 3x,\ y(2) = -4$. Then sketch by hand, the graph of the explicit solution of this problem and also the interval I of definition of the solution.

(d) Are there any explicit solutions of $y\dfrac{dy}{dx} = 3x$ that pass through the origin?

10. The functions $y(x) = \dfrac{1}{16}x^4,\ -\infty < x < \infty$ and $y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$ have the same domain but are clearly different. See the figures (a) and (b) below. Show that both functions are solutions of the initial value problem $y' = xy^{\frac{1}{2}},\ y(2) = 1$ on the interval $(-\infty < x < \infty)$. Resolve the apparent contradiction between this fact and the fact that **there exist some interval centered at** 2 **on which the initial value problem** $y' = xy^{\frac{1}{2}},\ y(2) = 1$ **has a unique solution**.



Definition: A function $f(x, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \in \mathbb{R}^2$ if a constant $L > 0$ exist with

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|,$$

whenever (x, y_1) and (x, y_2) are in D. The constant L is called the **Lipschitz constant** for f.

One can also frame the existence and uniqueness theorem which can be restated as follows

Theorem: Suppose that $D = \{(x, y) |\ a \leq x \leq b \text{ and } -\infty < y < \infty\}$ and that $f(x, y)$ is continuous in D. If f satisfies **Lipschitz condition** on D in the variable y, then the initial value problem

$$\frac{dy}{dx} = f(x, y),\ a \leq x \leq b, \quad y(a) = y_0,$$

has a unique solution $y(x)$ for $a \leq x \leq b$.

11. Using the above theorem, show that the differential equation $\dfrac{dy}{dx} = y^{\frac{2}{3}},\ y(0) = y_0$ has a unique solution at some interval I.