MTH 204 : Lecture 16

Definition: A function $f(x)$ is called analytic at $x=x_0$ if it can be represented by a power series in powers of $x-x_0$ with positive radius of convergence.

Frobenius Method: Let $b(x)$ and $c(x)$ be any functions that are analytic at $x=0$ (i.e. they have no singularities at $x=0$)

Then the ODE
$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

has atleast one solution that can be represented in the form
$$y = x^r \sum_{m=0}^{\infty} a_m x^m \quad (a_0 \neq 0)$$

where r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$)
The ODE also has a second solution (such that these two solutions are linearly independent) that may be similar to the previous one (with different r and coefficients) or may contain a logarithmic term.

Ex: Bessel's Equation:
$$y'' + \frac{1}{x}y' + \frac{x^2-\nu^2}{x^2}y = 0$$

(Here $b(x) = 1$, $c(x) = x^2 - \nu^2$)

Ex: Hypergeometric equation:

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

(where a, b, c are constants)

Regular and Singular points:

A regular point of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

is a point x_0 at which $p(x)$ and $q(x)$ are analytic.

Similarly a regular point of the ODE

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

is a point x_0 at which $\tilde{h}, \tilde{p}, \tilde{q}$ are analytic and $\tilde{h}(x_0) \neq 0$ (so that we can divide by $\tilde{h}$)

- If x_0 is not a regular point, it is called a singular point.

Indicial Equation:

Consider the ODE: $\quad y'' + \dfrac{b(x)}{x}y' + \dfrac{c(x)}{x^2}y = 0$

Multiplying by x^2 gives $\quad x^2y'' + x\,b(x)y' + c(x)y = 0$

If $b(x)$ and $c(x)$ are not polynomials, we expand $b(x)$ and $C(x)$ in power series

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots\cdots$$
$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots\cdots$$

If $y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + \cdots\cdots)$

is a solution, then

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1} \sum_{m=0}^{\infty} (m+r) a_m x^m$$
$$= x^{r-1} (r a_0 + (r+1) a_1 x \cdots\cdots)$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$
$$= x^{r-2} \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^m$$
$$= x^{r-2} [r(r-1) a_0 + (r+1) r a_1 x + \cdots\cdots]$$

Now substituting in $x^2 y'' + x b y' + c y = 0$

we obtain

$$\left.\begin{aligned} & x^r [r(r-1) a_0 + \cdots\cdots] + (b_0 + b_1 x + \cdots) x^r (r a_0 + \cdots\cdots) \\ & + (c_0 + c_1 x + \cdots\cdots) x^r (a_0 + a_1 x + \cdots) = 0 \end{aligned}\right\} \cdots (1)$$

Equating coefficient of x^r,

$$r(r-1) a_0 + b_0 r a_0 + c_0 a_0 = 0$$

$$\Rightarrow \quad a_0\left[r(r-1)+b_0 r+c_0\right]=0$$

$$\Rightarrow \quad \boxed{r(r-1)+b_0 r+c_0=0}$$

This quadratic equation is called the indicial equation and provides the r of one of the solutions and determines the form of the other solution.

So, one of the solutions of the basis is :

$$y_1 = x^{r_1}\left(a_0 + a_1 x + \cdots\right)$$

The other is :

- Distinct roots (including complex roots) not differing by an integer $1, 2, \ldots$

$$y_2 = x^{r_2}\left(A_0 + A_1 x + \cdots\right)$$

with coefficients obtained successively from ① with $r = r_1$ and $r = r_2$ respectively.

- <u>A Double root</u>

$$y_2 = y_1 \log(x) + x^{r_1}\left(A_1 x + A_2 x^2 + \cdots\right)$$

$$\left(r_2 = r_1 = \tfrac{1}{2}(1-b_0)\right)$$

- Roots differing by an integer $1, 2, \ldots$ $(r_1 > r_2)$

$$y_2 = k y_1 \log(x) + x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

(k may turn out to be zero)

Ex: Euler-Cauchy Equation:

$$x^2y'' + b_0 x y' + c_0 y = 0 \quad (b_0, c_0 \text{ are constants})$$

Substituting $y = x^r$, we get $r(r-1)x^r + b_0 r x^r + c_0 x^r = 0$

The indicial equation is

$$r(r-1) + b_0 r + c_0 = 0$$

This will give two values of r : r_1 & r_2 (say).

If $r_1 \neq r_2$, a basis of solutions : $y_1 = x^{r_1}$

$y_2 = x^{r_2}$

If $r_1 = r_2$, a basis of solutions: $y_1 = x^{r_1}$

$y_2 = x^{r_1}\log(x)$

Ex: $$x(x-1)y'' + (3x-1)y' + y = 0$$

We rewrite it as $$y'' + \frac{3x-1}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

$$\Rightarrow y'' + \frac{\frac{3x-1}{x-1}}{x}y' + \frac{\frac{x}{x-1}}{x^2} = 0$$

So, $b(x) = \dfrac{3x-1}{x-1}$ and $c(x) = \dfrac{x}{x-1}$

and these functions are analytic around $x=0$
and we can apply Frobenius method.

Let us substitution the Frobenius solution
$$y = x^r \sum_{m=0}^{\infty} a_m x^m \quad \text{into the ODE}$$
$$x(x-1)y'' + (3x-1)y' + y = 0$$
$$\left(\text{i.e. } x^2 y'' - x y'' + 3x y' - y' + y = 0\right)$$

Then
$$x^2\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right)$$
$$-x\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right)$$
$$+3x\left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right) - \left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right)$$
$$+\left(x^r\sum_{m=0}^{\infty}a_m x^m\right) = 0$$
$$\Rightarrow \quad x^r\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m - x^{r-1}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m$$
$$+3x^r\sum_{m=0}^{\infty}(m+r)a_m x^m - x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m$$
$$+x^r\sum_{m=0}^{\infty}a_m x^m = 0 \qquad \cdots\cdots (1)$$

The smallest power is x^{r-1} and its coefficient gives the indicial equation

$$-r(r-1)a_0 - ra_0 = 0$$
$$\Rightarrow -r^2 = 0$$

So, we have a double root at $r=0$

First solution

Substituting in ①

$$\sum_{m=0}^{\infty} m(m-1)a_m x^m - x^{-1}\sum_{m=0}^{\infty} m(m-1)a_m x^m$$
$$+ 3\sum_{m=0}^{\infty} m a_m x^m - x^{-1}\sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[m(m-1)+3m+1\right]a_m x^m - \sum_{m=0}^{\infty}\left[m(m-1)+m\right]a_m x^{m-1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}(m^2+2m+1)a_m x^m - \sum_{m=0}^{\infty} m^2 a_m x^{m-1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}(m^2+2m+1)a_m x^m - \sum_{m'=-1}^{\infty}(m'+1)^2 a_{m'+1} x^{m'} = 0$$

(Let $m-1 = m'$)

$$\Rightarrow \sum_{m=0}^{\infty}(m^2+2m+1)a_m x^m - \sum_{m'=0}^{\infty}(m'+1)^2 a_{m'+1} x^{m'} = 0$$

(for $m'=-1$, the term is zero)

$$\Rightarrow \sum_{m=0}^{\infty}(m^2+2m+1)a_m x^m - \sum_{m=0}^{\infty}(m+1)^2 a_{m+1} x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}(m+1)^2\left[a_m - a_{m+1}\right]x^m = 0$$

$$\Rightarrow (m+1)^2\left[a_m - a_{m+1}\right] = 0$$

$$\Rightarrow a_m = a_{m+1}$$

$$\Rightarrow a_0 = a_1 = a_2 = \cdots$$

Let us choose $a_0 = 1$

Then the first solution is :

$$y_1 = \sum_{m=0}^{\infty} x^m = \boxed{\frac{1}{1-x}} \quad \text{for } |x|<1$$

Second solution :

We apply reduction of order

Recall: $y'' + py' + qy = 0$

Then $U = \frac{1}{y_1^2} e^{-\int p\,dx}$ $\qquad u = \int U\,dx$

and then $y_2 = u y_1$.

We have $y'' + \dfrac{\frac{3x-1}{x-1}}{x} y' + \dfrac{\frac{x}{x-1}}{x^2} y = 0$

$$-\int \frac{3x-1}{x(x-1)}\,dx = -\int\left(\frac{2}{x-1} + \frac{1}{x}\right)dx$$

$$= -2\log|x-1| - \log|x| = \log\frac{1}{x(x-1)^2}$$

$$v = \frac{1}{\left(\frac{1}{1-x}\right)^2} e^{\log\frac{1}{x(x-1)^2}}$$

$$= (1-x)^2 \frac{1}{x(x-1)^2} = \frac{1}{x}$$

$$u = \int \frac{1}{x}\,dx = \log(x) \Rightarrow y_2 = u\,y_1$$

$$= \frac{\log(x)}{1-x}$$

$$\Rightarrow \boxed{y_2 = \frac{\log(x)}{(1-x)}}$$

Hence the general solution of $x(x-1)y'' + (3x-1)y' + y = 0$ is :

$$\boxed{y = c_1\frac{1}{1-x} + c_2\frac{\log(x)}{1-x}}$$

Ex: $(x^2-x)y'' - xy' + y = 0$.

Substituting the Frobenius Solution
$$y = x^r \sum_{m=0}^{\infty} a_m x^m \quad \text{in the ODE}$$

we get
$$(x^2-x)\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right)$$
$$- x\left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right) + \left(x^r \sum_{m=0}^{\infty} a_m x^m\right) = 0$$

$$\Rightarrow x^r \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m - x^{r-1}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m$$
$$- x^r \sum_{m=0}^{\infty}(m+r)a_m x^m + x^r \sum_{m=0}^{\infty} a_m x^m = 0 \quad \cdots\cdots (1)$$

The smallest power is $r-1$ whose coefficient is $\quad -r(r-1)a_0 = 0 \quad \Rightarrow r_1 = 1,\ r_2 = 0$

We have two roots, whose difference is an integer.

First solution:

Substituting $r = 1$ in (1) we obtain

$$x \sum_{m=0}^{\infty} (m+1) m a_m x^m - \sum_{m=0}^{\infty} (m+1) m a_m x^m - x . \sum_{m=0}^{\infty} (m+1) a_m x^m$$
$$+ x \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+1) m a_m x^{m+1} - \sum_{m=1}^{\infty} (m+1) m a_m x^m - \sum_{m=0}^{\infty} (m+1) a_m x^{m+1}$$
$$+ \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+1) m a_m x^{m+1} - \sum_{m'=0}^{\infty} (m'+2)(m'+1) a_{m'+1} x^{m'+1}$$
$$(\text{Let } m = m'+1)$$
$$- \sum_{m=0}^{\infty} (m+1) a_m x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+1) m a_m x^{m+1} - \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+1} x^{m+1}$$
$$- \sum_{m=0}^{\infty} (m+1) a_m x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \Big(\big[(m+1) m - (m+1) + 1 \big] a_m - (m+2)(m+1) a_{m+1} \Big) x^{m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[m^2 a_m - (m+2)(m+1)a_{m+1}\right]x^{m+1} = 0$$

$$\Rightarrow \quad m^2 a_m - (m+2)(m+1)a_{m+1} = 0$$

$$\Rightarrow \quad a_{m+1} = \frac{m^2}{(m+2)(m+1)} a_m$$

If we choose $a_0 = 1$, then

$$a_1 = \frac{0^2}{(0+2)(0+1)} a_0 = 0$$

Then $a_2 = a_3 = \cdots = 0$

So, $y_1 = x^{r_1} a_0 = x^1 \cdot 1 = \boxed{x}$

Second Solution:

Let us apply a reduction of order:

$$y_2 = u y_1 = u x$$

$$y_2' = u + u'x$$

$$y_2'' = u' + u' + u''x = u''x + 2u'$$

substituting in the ODE

$$(x^2 - x)(u''x + 2u') - x(xu' + u) + xu = 0$$

$$\Rightarrow (x^2-x)u''x + (x^2-x)2u' - x^2u' - u\cancel{x} + x\cancel{u} = 0$$

$$\Rightarrow (x^2-x)u'' + (x-1)2u' - xu' = 0$$

$$\Rightarrow (x^2-x)u'' + (x-2)u' = 0$$

$$\Rightarrow \frac{u''}{u'} = -\frac{(x-2)}{(x^2-x)} = -\left[\frac{2}{x} - \frac{1}{x-1}\right]$$

$$\Rightarrow \frac{u''}{u'} = -\frac{2}{x} + \frac{1}{x-1}$$

$$\Rightarrow \log|u'| = -2\log|x| + \log|x-1|$$

$$= \log\left|\frac{x-1}{x^2}\right|$$

$$\Rightarrow u' = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$$

$$\Rightarrow u = \log(x) + \frac{1}{x}$$

So, $y_2 = uy_1 = \left[\log(x) + \frac{1}{x}\right]x$

$$= \boxed{x\log(x) + 1}$$

Hence the general solution is

$$y = c_1 x + c_2\left[x\log(x) + 1\right]$$

Bessel Equation:

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \ , \quad \nu \geq 0$$

We can transform it into

$$y'' + \frac{1}{x} y' + \frac{x^2 - \nu^2}{x^2} y = 0$$

The functions 1 and $x^2 - \nu^2$ are analytic at $x = 0$ and we can use Frobenius solution

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r} \quad (a_0 \neq 0)$$

Substituting y'', y' and y in the Bessel Equation we get

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \quad \ldots (1)$$

The smallest power is r and its coefficient gives the indicial equation

$$r(r-1) a_0 + r a_0 - \nu^2 a_0 = 0$$

$$\Rightarrow \left[r^2 - \cancel{r} + \cancel{r} - \nu^2\right] a_0 = 0 \Rightarrow (r^2 - \nu^2) a_0 = 0$$

$$\Rightarrow \quad r_1, r_2 = \pm \nu$$

Substituting $r = \nu$ in equation (1) we get

$$x^\nu \sum_{m=0}^{\infty} (m+\nu)(m+\nu-1) a_m x^m + x^\nu \sum_{m=0}^{\infty} (m+\nu) a_m x^m$$

$$+ \; x^{\nu+2} \sum_{m=0}^{\infty} a_m x^m - \nu^2 x^\nu \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(m+\nu)(m+\nu-1) + (m+\nu) - \nu^2\right] a_m x^{m+\nu}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+\nu+2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[m^2 + m\nu - \cancel{m} + m\nu + \cancel{\nu^2} - \cancel{\nu} + \cancel{m} + \cancel{\nu} - \cancel{\nu^2}\right] a_m x^{m+\nu}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+\nu+2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} m(m+2\nu) a_m x^{m+\nu} + \sum_{m=0}^{\infty} a_m x^{m+\nu+2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} m(m+2\nu) a_m x^{m+\nu} + \sum_{m'=2}^{\infty} a_{m'-2} x^{m'+\nu} = 0$$

$$(m' = m+2)$$

$$\Rightarrow (1+2\nu) a_1 x^{1+\nu} + \sum_{m=2}^{\infty} m(m+2\nu) a_m x^{m+\nu} + \sum_{m=2}^{\infty} a_{m-2} x^{m+\nu} = 0$$

$$\Rightarrow (1+2\nu)a_1 x^{1+\nu} + \sum_{m=0}^{\infty}\left[m(m+2\nu)a_m + a_{m-2}\right]x^{m+\nu} = 0$$

$$\Rightarrow (1+2\nu)a_1 = 0 \Rightarrow a_1 = 0$$

and $\quad m(m+2\nu)a_m + a_{m-2} = 0$

$$\Rightarrow a_m = -\frac{1}{m(2\nu+m)}a_{m-2}$$

Now if m is odd, $\quad 0 = a_1 = a_3 = a_5 = \cdots$

If m is even, we can write $m = 2k$

and $\quad a_{2k} = -\dfrac{1}{2k(2\nu+2k)}a_{2k-2} = -\dfrac{1}{2^2 k(\nu+k)}a_{2k-2}$

for $k = 1, 2, \ldots$

Thus $\quad a_2 = -\dfrac{1}{2^2(\nu+1)}a_0$

$$a_4 = -\frac{1}{2^2\, 2(\nu+2)}a_2 = -\frac{1}{2^2\, 2(\nu+2)}\left(-\frac{1}{2^2(\nu+1)}a_0\right)$$

$$= \frac{1}{2^4\, 2!(\nu+1)(\nu+2)}a_0$$

In general, $\quad a_{2k} = \dfrac{(-1)^m}{2^{2k}k!(\nu+1)(\nu+2)\cdots(\nu+k)}a_0$

The first solution of Bessel equation is

$$y_1 = x^\nu \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}k!(\nu+1)(\nu+2)\cdots(\nu+k)}a_0 x^{2k}$$

If ν is an integer let $\nu = n$

Let us choose $a_0 = \frac{1}{2^n n!}$

Then $$y_1 = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (n+1)(n+2) \cdots (n+k)} \frac{1}{2^n n!} x^{2k}$$

$$\Rightarrow \quad y_1 = \boxed{J_n = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n} k! \, (n+k)!} x^{2k}}$$

- This is the Bessel's function of the first kind and order n (The series converges for all x by ratio test)
- For large x, they fullfill

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$
(asymptotically equal)

The Gamma Function:

Define Gamma function as:
$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x \, dt \qquad (\text{for } x > -1)$$

Integrating by parts we get
$$\Gamma(x+1) = -e^{-t} t^x \Big|_0^{\infty} + \int_0^{\infty} x t^{x-1} e^{-t} \, dt$$

$$\Rightarrow \Gamma(x+1) = 0 + x\int_0^\infty e^{-t}t^{x-1}dt = x\Gamma(x)$$

$$\Rightarrow \boxed{\Gamma(x+1) = x\Gamma(x)}$$

Now substituting $x=0$ in the definition

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1 \qquad \Rightarrow \boxed{\Gamma(1) = 1}$$

Thus for any nonnegative integer n

$$\boxed{\Gamma(n+1) = n!} \qquad n = 0, 1, 2, \ldots$$

Hence the gamma function generalizes the factorial function to arbitrary positive ν.

Another important result is $\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$

Bessel function of first kind $J_\nu(x)$:

The first solution of Bessel equation is:

$$y_1 = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}k!(\nu+1)(\nu+2)\ldots(\nu+k)} a_0 x^{2k}$$

If ν is not an integer, let $a_0 = \dfrac{1}{2^\nu \Gamma(\nu+1)}$

Then $$y_1 = x^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (\nu+1)(\nu+2)\cdots(\nu+k)} \frac{1}{2^{\nu}\Gamma(\nu+1)} x^{2k}$$

$$\Rightarrow \quad y_1 = \boxed{J_{\nu}(x) = x^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k! \, \Gamma(\nu+k+1)} x^{2k}}$$

- $J_{\nu}(x)$ is called the Bessel function of the first kind of order n. The series converges for all x by ratio test.

- An interesting result is: $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$

  and $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$

- If ν is not an integer, then the general solution is: $\boxed{y = c_1 J_{\nu} + c_2 J_{-\nu}}$

  where $$J_{-\nu} = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-\nu} k! \, \Gamma(-\nu+k+1)} x^{2k}$$

- If ν is an integer, there is a problem because $J_{-n} = (-1)^n J_n$

that is, there is a linear dependence between the two solutions. (This will be resolved by

Bessel function of second kind.)

• Some Useful properties:

Derivatives: (1) $\left[x^{\nu} J_{\nu}(x)\right]' = x^{\nu} J_{\nu-1}(x)$

(2) $\left[x^{-\nu} J_{\nu}(x)\right]' = -x^{-\nu} J_{\nu+1}(x)$

Recursion: (3) $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x)$

(4) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x)$

Bessel Equation:

Let us find a second solution of Bessel equation in case ν is an integer.

First we assume $\nu = n = 0$

Then the ODE becomes: $x^2 y'' + xy' + x^2 y = 0$

$$\Rightarrow \quad xy'' + y' + xy = 0$$

Then the indicial equation becomes $r^2 = 0$

i.e. it has a double root at $r = 0$.

The first solution is $y_1 = J_0$

The second solution according to Frobenius method must be of the form:

$$y_2 = y_1 \log(x) + x^r \sum_{m=1}^{\infty} A_m x^m$$

$$\Rightarrow y_2 = J_0 \log(x) + \sum_{m=1}^{\infty} A_m x^m \quad (r=0)$$

$$\Rightarrow y_2' = J_0' \log(x) + J_0 \frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' \log(x) + 2 J_0' \frac{1}{x} - J_0 \frac{1}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Substituting in the ODE $xy'' + y' + xy = 0$

we get

$$x\left(J_0'' \log(x) + 2J_0' \frac{1}{x} - J_0 \frac{1}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}\right)$$

$$+ \left(J_0' \log(x) + J_0 \frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}\right)$$

$$+ x\left(J_0 \log(x) + \sum_{m=1}^{\infty} A_m x^m\right) = 0$$

$$\Rightarrow (xJ_0'' + J_0' + xJ_0) \log(x) + 2J_0' + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-1}$$

$$+ \sum_{m=1}^{\infty} m A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$\Rightarrow \quad 2J_0' + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \quad \ldots\ldots (1)$$

Now $$J_n = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n}\, k!\, (n+k)!} x^{2k}$$

In particular $$J_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} (m!)^2} x^{2m}$$

$$J_0' = \sum_{m=1}^{\infty} \frac{(-1)^m \quad 2m}{2^{2m} (m!)^2} x^{2m-1} = \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-1} m!\, (m-1)!} x^{2m-1}$$

Substituting in (1) we get

$$2\left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-1} m!\, (m-1)!} x^{2m-1}\right) + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1}$$

$$= 0$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-2} m!\, (m-1)!} x^{2m-1} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1}$$

$$+ \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$\Rightarrow \left(-x + \frac{1}{2^2\, 2!} x^3 - \ldots\right) + \left(A_1 + 4A_2 x + \cdots\right)$$

$$+ \left(A_1 x^2 + A_2 x^3 + \cdots\right) = 0$$

Now equating powers of x^0 from both sides we get $A_1 = 0$

Now consider even powers x^{2s}

First series has no even power.

In the second series $m-1 = 2s \Rightarrow m = (2s+1)$

and the term is $(2s+1)^2 A_{2s+1} x^{2s}$

In the third series $m+1 = 2s \Rightarrow m = 2s-1$

and the term is $A_{2s-1} x^{2s}$

Hence $(2s+1)^2 A_{2s+1} + A_{2s-1} = 0$ for $s = 1, 2, \ldots$

$$\Rightarrow A_{2s+1} = -\frac{1}{(2s+1)^2} A_{2s-1}$$

Since $A_1 = 0$, we get $0 = A_3 = A_5 = \cdots$

Now Consider the coefficients of x^{2s+1}.

For $s=0$, we get $-1 + 4A_2 = 0 \Rightarrow A_2 = \frac{1}{4}$

For other values of s, in the first series $2s+1 = 2m-1$

$\Rightarrow m = s+1$

in the second series $2s+1 = m-1 \Rightarrow m = 2s+2$

in the third series $2s+1 = m+1 \Rightarrow m = 2$

Therefore $\dfrac{(-1)^{s+1}}{2^{2s}(s+1)!\,s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$

For $s=1$, we get $\frac{1}{8} + 16A_4 + A_2 = 0$

$$\Rightarrow A_4 = -\frac{3}{128}$$

In general $A_{2m} = \dfrac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)$

for $m = 1, 2, \ldots\ldots$

Let us use the notation: $h_1 = 1$, $h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$

for $m = 2, 3, \ldots$

Then $y_2(x) = J_0(x)\log(x) + \displaystyle\sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}$

$$= J_0(x)\log(x) + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \cdots\cdots$$

Since J_0 and y_2 are linearly independent functions they form a basis of solutions for $x > 0$.
It is customary to use a different basis

where y_2 will be replaced by an independent particular solution $Y_0 = a(y_2 + bJ_0)$ where $a(\neq 0)$

and b are constants chosen as $a_0 = \frac{2}{\pi}$
and $b = \gamma - \ln 2$

where γ is Euler's constant

$$\gamma = \lim_{s \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s\right) \approx .5772$$

Thus $$Y_0(x) = \frac{2}{\pi}\left[y_2 + (\gamma - \log(2)) J_0(x)\right]$$

$$= \frac{2}{\pi}\left[J_0(x)\left(\log\left(\frac{x}{2}\right) + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}\right]$$

and $J_0(x)$ and $Y_0(x)$ forms a basis of solutions.

- $Y_0(x)$ is called the Bessel function of the second kind of order zero. (or Neumann's function of order zero)

- For small $x > 0$, $Y_0(x)$ behaves like $\log(x)$ and $Y_0(x) \longrightarrow -\infty$ as $x \longrightarrow 0$

- In general,

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right]$$

and $\quad Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$

This function is called the Bessel function of second kind of order ν (or Neumann's function of order ν).

- J_ν and Y_ν are linearly independent for all ν (and $x > 0$)

- Therefore a general solution of Bessel Equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \quad (\nu \geq 0)$$

can be given by $\boxed{y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)}$

Note: Sometimes we need solutions of Bessel Equation that are complex for real values of x. In that case the solutions:

$$H_\nu^{(1)}(x) = J_\nu(x) + i Y_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - i Y_\nu(x)$$

are used.

These linearly independent functions are called Bessel functions of third kind of order ν <u>or</u> first and second Hankel functions of order ν.

- For large x, they satisfy

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

- There are related recurrence formulas and formulas for derivatives of $J_\nu(x)$ and $Y_\nu(x)$

<u>Modified Bessel Equation:</u>

Bessel Equation: $x^2y'' + xy' + (x^2-\nu^2)y = 0$

Modified Bessel Equation: $x^2y'' + xy' - (x^2+\nu^2)y = 0$

- The general solution of the Modified Bessel Equation is of the form :

$$y = c_1 I_\nu + c_2 K_\nu$$

where I_ν is the modified Bessel function of first kind

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = x^\nu \sum_{k=0}^{\infty} \frac{1}{2^{2k+\nu}\, k!\, \Gamma(k+\nu+1)} x^{2k}$$

$$\left(\text{Compare it to: } J_\nu(x) = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k! \, \Gamma(\nu+k+1)} x^{2k}\right)$$

and K_ν is the modified Bessel function of second kind:

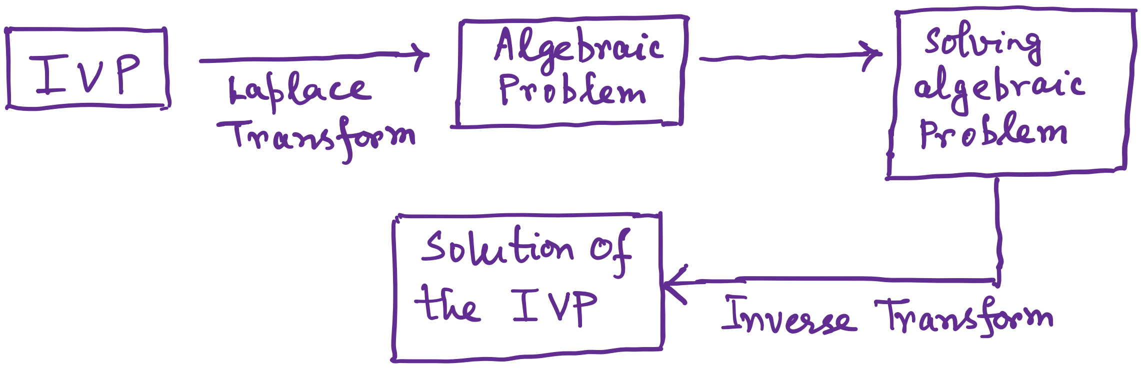
$$K_\nu(x) = \frac{\pi}{2} \cdot \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$$

$$\left(\text{Compare it to } \quad Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}\right)$$

MTH 204 : Lecture 17

Laplace Transforms :

The process of solving an ODE using the Laplace transform method consists of three steps.



- The type of mathematics that converts problems of calculus to algebraic problems is known as Operational Calculus.
- Advantages of the Method :

(1) IVP can be solved without first determining a general solution. Nonhomogeneous ODEs are solved without first solving the corresponding homogeneous ODE.

(2) The use of unit step function (Heaviside

function and Dirac's Delta) make the method powerful for problems with inputs that have discontinuities or represent short impulses or complicated periodic functions.

Laplace Transform :

If $f(t)$ is a function defined for all $t \geq 0$, then its Laplace transform is defined by

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt \quad \text{------} (1)$$

We assume that $f(t)$ is such that the above integral exists (that is, has some finite value)

- The above is an improper integral and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t)\, dt$$

(provided the limit exists)

- The Laplace transform is an integral transform

$$F(s) = \int_0^\infty k(s,t) f(t)\, dt$$

with kernel $K(s,t) = e^{-st}$ and it transforms a function in one space to a function is another space.

In general the kernel is a function of the (two) variables in the two spaces.

- The given function $f(t)$ in (1) is called the inverse transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$.

  We write $\quad f(t) = \mathcal{L}^{-1}(F) \quad \ldots\ldots\ldots(2)$

From (1) and (2), we get $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and

$\mathcal{L}(\mathcal{L}^{-1}(F)) = F$

Notation: • original functions are denoted by lowercase letters and their transforms by the same letters in capital.

- Original functions depend on t and their transform on s

Ex: Calculate the Laplace transform of
(a) $f(t) = 1$ and (b) $f(t) = e^{at}$ when $t \geq 0$

(a) $F(s) = \int_0^\infty e^{-st}\, 1\, dt = \left[\frac{e^{-st}}{-s} \right]_0^\infty = \boxed{\frac{1}{s}}$

(The above is really $\lim_{T\to\infty} \int_0^T e^{-st} \times 1\, dt$

$= \lim_{T\to\infty} \left[\frac{e^{-st}}{-s} \right]_0^T$)

(b) $F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$

$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty$

The above exists when $s-a > 0$ i.e. $s > a$

Hence $F(s) = \mathcal{L}(e^{at})(s) = \frac{1}{s-a}$ for $s > a$

Linearity of Laplace Transform:

The Laplace transform is a linear operation: that is, for any functions $f(t)$ and $g(t)$

whose transforms exist and for any constants a and b, the transform of $af(t)+bg(t)$ exists and

$$\mathcal{L}\{a f(t)+b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

<u>Proof:</u> $\mathcal{L}\{a f(t)+b g(t)\}$

$$= \int_0^\infty e^{-st}\left[a f(t)+b g(t)\right] dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

<u>Ex:</u> Find the Laplace transform of $\cosh(at)$ and $\sinh(at)$

Since $\cosh(at) = \dfrac{e^{at}+e^{-at}}{2}$ and $\sinh(at) = \dfrac{e^{at}-e^{-at}}{2}$

$$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{at})+\frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2} \text{ for } s>|a|$$

$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$$

for $s > |a|$

Ex: Calculate the Laplace Transforms of $\cos(\omega t)$ and $\sin(\omega t)$

Let us denote the Laplace transform of $\cos(\omega t)$ by L_c and the Laplace transform of $\sin(\omega t)$ by L_s

Now $L_c = \int_0^\infty e^{-st}\cos(\omega t)\,dt$

$$= \left.\frac{e^{-st}}{-s}\cos(\omega t)\right|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)\,dt$$

$$\Rightarrow \quad L_c = \frac{1}{s} - \frac{\omega}{s}L_s \qquad \ldots\ldots (1)$$

Now $L_s = \int_0^\infty e^{-st}\sin(\omega t)\,dt$

$$= -\left.\frac{e^{-st}}{s}\sin(\omega t)\right|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)\,dt$$

$$= 0 + \frac{\omega}{s}L_c$$

$$\Rightarrow \quad L_s = \frac{\omega}{s} L_c \quad \cdots\cdots (2)$$

Solving (1) and (2), $L_c = \frac{1}{s} - \frac{\omega^2}{s^2} L_c$

$$\Rightarrow \left(1 + \frac{\omega^2}{s^2}\right) L_c = \frac{1}{s}$$

$$\Rightarrow \frac{(s^2+\omega^2)}{s^2} L_c = \frac{1}{s} \Rightarrow \boxed{L_c = \frac{s}{s^2+\omega^2}}$$

Then (2) $\Rightarrow$

$$L_s = \frac{\omega}{s} \frac{s}{s^2+\omega^2}$$

$$\Rightarrow \boxed{L_s = \frac{\omega}{s^2+\omega^2}}$$

Ex: Calculate the Laplace transform of t^{n+1}

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1} dt = -\frac{e^{-st}}{s} t^{n+1}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty e^{-st} t^n dt$$

$$= 0 + \frac{n+1}{s} \mathcal{L}(t^n) = \frac{n+1}{s} \mathcal{L}(t^n)$$

$$= \frac{(n+1)}{s} \frac{n}{s} \mathcal{L}(t^{n-1})$$

$$= \frac{(n+1)}{s} \frac{n}{s} \cdots \frac{2}{s} \frac{1}{s} \mathcal{L}(t^0)$$

$$= \frac{(n+1)}{s} \cdot \frac{n}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} \cdot \frac{1}{s}$$

$$= \boxed{\frac{(n+1)!}{s^{n+2}}}$$

- **Some functions $f(t)$ and their Laplace Transform $\mathcal{L}(f)$**

(1) $f(t) = 1 \Rightarrow \mathcal{L}(f) = \frac{1}{s}$

(2) $f(t) = t \Rightarrow \mathcal{L}(f) = \frac{1}{s^2}$

(3) $f(t) = t^2 \Rightarrow \mathcal{L}(f) = \frac{2!}{s^3}$

(4) $f(t) = t^n \Rightarrow \mathcal{L}(f) = \frac{n!}{s^{n+1}}$
$(n = 0, 1, \ldots)$

(5) $f(t) = t^a \Rightarrow \mathcal{L}(f) = \frac{\Gamma(a+1)}{s^{a+1}}$
$(a > 0)$

(6) $f(t) = e^{at} \Rightarrow \mathcal{L}(f) = \frac{1}{s-a}$

(7) $f(t) = \cos \omega t \Rightarrow \mathcal{L}(f) = \frac{s}{s^2+\omega^2}$

(8) $f(t) = \sin \omega t \Rightarrow \mathcal{L}(f) = \frac{\omega}{s^2+\omega^2}$

(9) $f(t) = \cosh(at) \Rightarrow \mathcal{L}(f) = \dfrac{s}{s^2 - a^2}$

(10) $f(t) = \sinh(at) \Rightarrow \mathcal{L}(f) = \dfrac{a}{s^2 - a^2}$

(11) $f(t) = e^{at}\cos(\omega t) \Rightarrow \mathcal{L}(f) = \dfrac{(s-a)}{(s-a)^2 + \omega^2}$

(12) $f(t) = e^{at}\sin(\omega t) \Rightarrow \mathcal{L}(f) = \dfrac{\omega}{(s-a)^2 + \omega^2}$

Note:

(5) $\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt$

Let $st = x$
Then $t = \dfrac{x}{s}$
$dt = \dfrac{dx}{ds}$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx \quad (\text{for } s > 0)$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}$$

s-Shifting:

First Shifting Theorem:

If $f(t)$ has Laplace transform $F(s)$,
(where $s > k$ for some k)

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n,\ n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\ n=1,2,3,\ldots$	$\dfrac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\ n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^nf(t),\ n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\displaystyle\int_s^\infty F(u)\,du$	32.	$\displaystyle\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

then $e^{at}f(t)$ has the transform $F(s-a)$ (where $s-a > k$)

Thus $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

and $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Proof: $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$$= \int_0^\infty e^{-st}\left(e^{at}f(t)\right)dt = \mathcal{L}\{e^{at}f(t)\}$$

(If $F(s)$ exists for some $s>k$, the first integral exists for $s-a>k$)

Now by taking the inverse Laplace transform

we get $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Ex: Since $\mathcal{L}\{\cos(\omega t)\} = \dfrac{s}{s^2+\omega^2}$

and $\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{s^2+\omega^2}$

applying the shifting theorem,

$$\mathcal{L}\{e^{at}\cos(\omega t)\} = \boxed{\frac{(s-a)}{(s-a)^2+\omega^2}}$$

and $\mathcal{L}\{e^{at}\sin(\omega t)\} = \dfrac{\omega}{(s-a)^2+\omega^2}$

Ex: Find f if $\mathcal{L}(f) = \dfrac{3s-137}{s^2+2s+401}$

$$\mathcal{L}^{-1}\left\{\frac{3s-137}{s^2+2s+401}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+400}\right\} - \mathcal{L}^{-1}\left\{\frac{140}{(s+1)^2+400}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+20^2}\right\} - 7\,\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}$$

$$= \boxed{3e^{-t}\cos(20t) - 7e^{-t}\sin(20t)}$$

Existence and Uniqueness of Laplace Transform:

If $f(t)$ is defined and is piecewise continuous on every finite interval on $t \geq 0$

and $$|f(t)| \leq M e^{kt}$$

for some constants M and K

This is called the growth condition: It means it doesnot grow too fast then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$ and if it exists, it uniquely determined.

Conversely if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points. Thus inverse of a given transform is essentially unique.

In particular if two continuous functions have the same transform, they are completely identical.

<u>Proof for existence:</u>

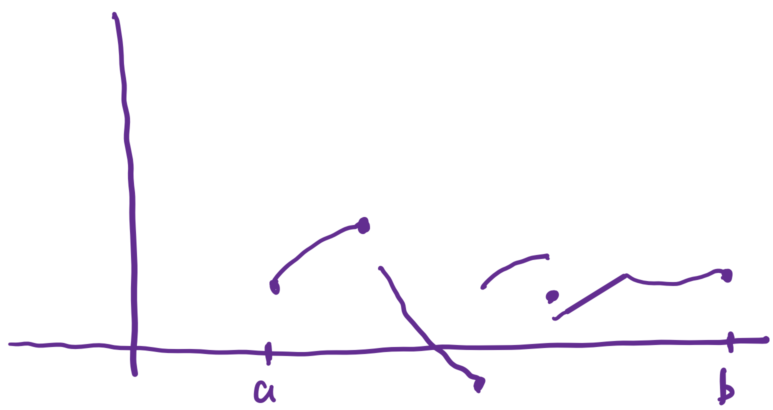
$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\,dt \right| \leq \int_0^\infty |e^{-st}|\,|f(t)|\,dt$$

$$\leq \int_0^\infty M e^{kt} e^{-st}\,dt = M \int_0^\infty e^{-(s-k)t}\,dt$$

$$= M\left[-\frac{e^{-(s-k)}}{(s-k)} \right]_0^\infty = \frac{M}{s-k} < \infty$$

(for $s > k$).

<u>Note:</u> $f(t)$ is said to be piecewise continuous on $[a, b]$ if this interval can be divided into finitely many subintervals in each of which f is continuous and has finite limit as t approaches either endpoint of such subinterval from the interior. This then gives finite jumps as the only possible discontinuities.



$$\boxed{\text{MTH 204 : Lecture 18}}$$

Laplace Transforms of Derivatives and Integrals

Laplace Transform of first derivative:

If $f(t)$ is continuous for all $t \geq 0$ and satisfies the growth condition and $f'(t)$ is piecewise continuous on every finite interval on the semiaxis $t \geq 0$, then

$$\boxed{\mathcal{L}(f') = s\mathcal{L}(f) - f(0)}$$

Proof: $\mathcal{L}(f')(s) = \int_0^\infty e^{-st} f'(t)\,dt$

$$= e^{-st} f(t) \Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}(f)(s)$$

(The first expression is 0 at the upper limit when $s > k$)

Hence $\boxed{\mathcal{L}(f') = s\mathcal{L}(f) - f(0)}$

Laplace Transform of Second derivative:

If $f(t)$ and $f'(t)$ are continuous for all $t \geq 0$, satisfies the growth condition and f'' is piecewise continuous on every finite interval on the semiaxis $t \geq 0$, then

$$\boxed{\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)}$$

<u>Proof:</u> Applying the previous result

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0)$$
$$= s(s\mathcal{L}\{f\} - f(0)) - f'(0)$$
$$\Rightarrow \boxed{\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)}$$

<u>Laplace Transform of the Derivative $f^{(n)}$ of any order:</u>

If $f, f', \ldots, f^{(n-1)}$ are continuous for all $t \geq 0$ and satisfy the growth condition and $f^{(n)}$ is piecewise continuous on every finite interval on the semiaxis $t \geq 0$, then

$$\boxed{\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$$

<u>Ex:</u> Let $f(t) = t\sin(\omega t)$ Find $\mathcal{L}\{f\}$

$$f(t) = t\sin(\omega t) \Rightarrow f(0) = 0$$
$$f'(t) = \sin(\omega t) + t\omega\cos(\omega t) \Rightarrow f'(0) = 0$$
$$f''(t) = \omega\cos(\omega t) + \omega\cos(\omega t) - t\omega^2\sin(\omega t)$$
$$\Rightarrow f''(t) = 2\omega\cos(\omega t) - \omega^2 t\sin(\omega t)$$

Now $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0) = s^2\mathcal{L}\{f\}$

$$\mathcal{L}\{f''\} = \mathcal{L}\{2\omega\cos(\omega t) - \omega^2 t\sin(\omega t)\}$$

$$= 2\omega\frac{s}{s^2+\omega^2} - \omega^2\mathcal{L}\{f\}$$

$$\Rightarrow \quad s^2\mathcal{L}\{f\} = \frac{2\omega s}{s^2+\omega^2} - \omega^2\mathcal{L}\{f\}$$

$$\Rightarrow \quad (s^2+\omega^2)\mathcal{L}\{f\} = \frac{2\omega s}{s^2+\omega^2}$$

$$\Rightarrow \quad \boxed{\mathcal{L}\{f\} = \frac{2\omega s}{(s^2+\omega^2)^2}}$$

<u>Ex:</u> <u>Another way of calculating $\mathcal{L}\{\cos(\omega t)\}$ and $\mathcal{L}\{\sin(\omega t)\}$:</u>

Let $f(t) = \cos(\omega t)$

$f(0) = 1$, $f'(t) = -\omega\sin(\omega t) \Rightarrow f'(0) = 0$

$f''(t) = -\omega^2\cos(\omega t)$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}\{-\omega^2\cos(\omega t)\} = s^2\mathcal{L}\{\cos(\omega t)\} - s\times 1 - 0$$

$$\Rightarrow \quad -\omega^2\mathcal{L}\{\cos(\omega t)\} = s^2\mathcal{L}\{\cos(\omega t)\} - s$$

$$\Rightarrow \quad (s^2+\omega^2)\mathcal{L}\{\cos(\omega t)\} = s$$

$$\Rightarrow \quad \boxed{\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}}$$

Similarly let $g(t) = \sin(\omega t)$

$g(0)=0$, $g'(t) = \omega\cos(\omega t) \Rightarrow g'(0) = \omega$

$g''(t) = -\omega^2 \sin(\omega t)$

Now $\mathcal{L}\{g''\} = s^2\mathcal{L}\{g\} - sg(0) - g'(0)$

$$\Rightarrow \mathcal{L}\{-\omega^2\sin(\omega t)\} = s^2\mathcal{L}\{\sin(\omega t)\} - s\times 0 - \omega$$

$$\Rightarrow -\omega^2\mathcal{L}\{\sin(\omega t)\} = s^2\mathcal{L}\{\sin(\omega t)\} - \omega$$

$$\Rightarrow \quad (s^2+\omega^2)\mathcal{L}\{\sin(\omega t)\} = \omega$$

$$\Rightarrow \boxed{\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{(s^2+\omega^2)}}$$

Laplace Transform of Integral:

If $f(t)$ is a piecewise continuous function for all $t \geq 0$ and satisfies the growth condition then for $s > k > 0$ and $t > 0$ we have

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s) \quad \left(= \frac{1}{s}\mathcal{L}\{f\}(s)\right)$$

$$\left(\text{Therefore} \quad \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}\right)$$

Proof: Since $f(t)$ is piecewise continuous, it integral is continuous and it satisfies the growth condition because

$$\left|\int_0^t f(\tau)\,d\tau\right| \leq \int_0^t |f(\tau)|\,d\tau \leq \int_0^t Me^{k\tau}\,d\tau$$

$$= M\frac{(e^{kt}-1)}{k} \leq \frac{Me^{kt}}{k}$$

Also the derivative of the integral is $f(t)$ except at points of discontinuity of f.

Applying the first derivative theorem to the integral of f we get

$$\mathcal{L}\left\{\left(\int_0^t f(\tau)\,d\tau\right)'\right\}(s) = s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} - \int_0^0 f(\tau)\,d\tau$$

$$\Rightarrow \mathcal{L}\{f(t)\}(s) = s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f\}(s)$$

Ex: Find the inverse Laplace Transform of $\dfrac{1}{s(s^2+\omega^2)}$ and $\dfrac{1}{s^2(s^2+\omega^2)}$

Since $\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{s^2+\omega^2}$

we have $\mathcal{L}\left\{\dfrac{\sin(\omega t)}{\omega}\right\} = \dfrac{1}{s^2+\omega^2}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau = \left[-\frac{\cos(\omega\tau)}{\omega^2}\right]_0^t$$

$$= \boxed{\frac{[1-\cos(\omega t)]}{\omega^2}}$$

Now since $\mathcal{L}\left\{\dfrac{[1-\cos(\omega t)]}{\omega^2}\right\} = \dfrac{1}{s(s^2+\omega^2)}$, $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+\omega^2)}\right\} = \displaystyle\int_0^t \frac{[1-\cos\omega\tau]}{\omega^2}\,d\tau$

$$= \left[\frac{\tau}{\omega^2} - \frac{\sin(\omega\tau)}{\omega^2}\right]_0^t = \boxed{\left[\frac{t}{\omega^2} - \frac{\sin(\omega t)}{\omega^2}\right]}$$

Differential Equations, Initial Value Problems:

Given the IVP : $y'' + ay' + by = r(t)$, $y(0)=K_0$, $y'(0)=K_1$

where a are b are constants.

$r(t)$: Input (driving force)

$y(t)$: Output (response to the input)

Let us transform the equation

$$\{s^2Y - sy(0) - y'(0)\} + a\{sY - y(0)\} + bY = R(s)$$

$$\Rightarrow \quad (s^2+as+b)Y(s) = (s+a)y(0) + y'(0) + R(s)$$

$$\Rightarrow \quad Y(s) = \frac{(s+a)y(0) + y'(0) + R(s)}{(s^2+as+b)}$$

Let us define the Transfer function as

$$Q(s) = \frac{1}{s^2+as+b} = \frac{1}{(s+\frac{1}{2}a)^2 + b^2 - \frac{1}{4}a^2}$$

Then we have $Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$

If the system is originally at rest, then

$y(0) = 0$, $y'(0) = 0$ and $\boxed{Y(s) = Q(s)R(s)}$

<u>Note:</u> Transfer function $Q = \dfrac{Y}{R} = \left\{\dfrac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{Input})}\right\}$

This explains the name Q.

Q depends neither on $r(t)$ nor on the initial conditions (but only on a and b)

To recover y we need to perform $\mathcal{L}^{-1}$ on Y

Thus $\quad y = \mathcal{L}^{-1}\{Y\}$

Ex: Solve $y'' - y = t$, $y(0) = 1$, $y'(0) = 1$

Now taking Laplace transform, we get

$$\{s^2 Y - s\,y(0) - y'(0)\} - Y = \frac{1}{s^2}$$

$$\Rightarrow \quad (s^2-1)Y = \frac{1}{s^2} + s + 1$$

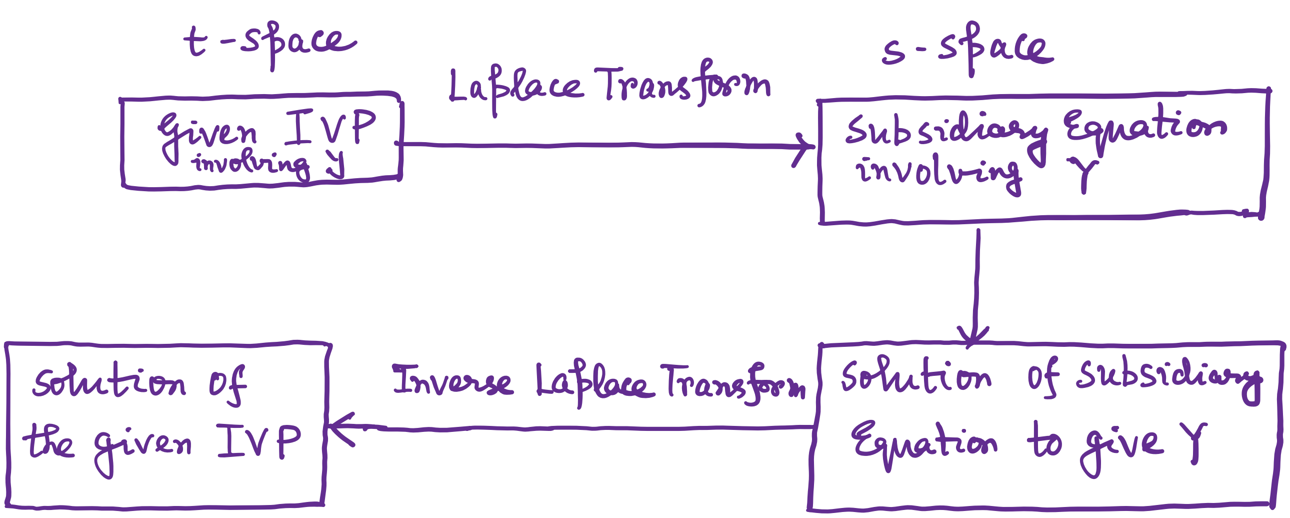
$$\Rightarrow \quad Y = \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1}$$

$$\Rightarrow \quad Y = \frac{1}{s^2-1} - \frac{1}{s^2} + \frac{1}{s-1}$$

Now $y = \mathcal{L}^{-1}\{Y\} = \sinh(t) - t + e^t$

$$= \boxed{e^t + \sinh(t) - t}$$

Summary of the method:



Ex: $y'' + y' + 9y = 0$, $y(0) = 0.16$, $y'(0) = 0$

Taking Laplace Transform we get

$$\{s^2 Y - s y(0) - y'(0)\} + \{sY - y(0)\} + 9Y = 0$$

$$\Rightarrow \quad s^2 Y - .16\,s + sY - .16 + 9Y = 0$$

$$\Rightarrow (s^2 + s + 9)Y = .16s + .16$$

$$\Rightarrow \quad Y = \frac{.16(s+1)}{s^2+s+9} = \frac{.16(s+\frac{1}{2}) + .08}{(s+\frac{1}{2})^2 + \frac{35}{4}}$$

Hence by the first shifting theorem, we obtain

$$y(t) = \mathcal{L}^{-1}\{Y\} = e^{-t/2}\left(.16\cos\sqrt{\frac{35}{4}}\,t + \frac{.08}{\sqrt{\frac{35}{4}}}\sin\sqrt{\frac{35}{4}}\,t\right)$$

Ex: $y'' + y = 2t$, $y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, $y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$

Problem: The initial values are not given at $t = 0$

Let $x = t - \frac{\pi}{4}$ (So $t = \frac{\pi}{4} \Rightarrow x = 0$)

$$\Rightarrow \quad t = x + \frac{\pi}{4}$$

Substituting $t = x + \frac{\pi}{4}$ in y, we get a function of x, let us call it $\tilde{y}(x) = y(t)$

Now $\quad y' = \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{dy}{dx} = \dfrac{d\tilde{y}}{dx}$

and $y'' = \frac{d(y')}{dt} = \frac{d\left(\frac{d\tilde{y}}{dx}\right)}{dt} = \frac{d\left(\frac{d\tilde{y}}{dx}\right)}{dx}\frac{dx}{dt} = \frac{d\left(\frac{d\tilde{y}}{dx}\right)}{dx} = \frac{d^2\tilde{y}}{dx^2}$

So, the IVP becomes

$$\frac{d^2\tilde{y}}{dx^2} + \tilde{y} = 2\left(x + \frac{\pi}{4}\right), \quad \tilde{y}(0) = \frac{\pi}{2}, \quad \left.\frac{d\tilde{y}}{dx}\right|_{x=0} = 2 - \sqrt{2}$$

Taking Laplace Transform we get

$$\left\{s^2\tilde{Y} - s\tilde{y}(0) - \left.\frac{d\tilde{y}}{dx}\right|_{x=0}\right\} + \tilde{Y} = 2\left(\frac{1}{s^2} + \frac{\pi}{4}\frac{1}{s}\right)$$

$$\Rightarrow (s^2+1)\tilde{Y} = s\frac{\pi}{2} + 2 - \sqrt{2} + \frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s}$$

$$\Rightarrow \tilde{Y} = \frac{2}{s^2(s^2+1)} + \frac{\frac{\pi}{2}}{s(s^2+1)} + \frac{\frac{\pi}{2}s}{(s^2+1)} + \frac{(2-\sqrt{2})}{s^2+1}$$

Hence

$$\tilde{y} = \mathcal{L}^{-1}\{\tilde{Y}\} = 2(x - \sin x) + \frac{\pi}{2}(1 - \cos x) + \frac{\pi}{2}\cos x + (2-\sqrt{2})\sin x$$

$$\Rightarrow \tilde{y} = 2x + \frac{\pi}{2} - \sqrt{2}\sin x$$

$$\Rightarrow y = 2\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2} - \sqrt{2}\sin\left(t - \frac{\pi}{4}\right)$$

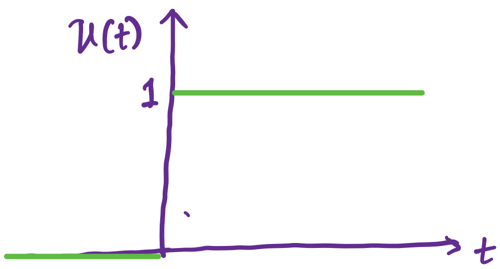
$$= 2t - \frac{\pi}{2} + \frac{\pi}{2} - \sqrt{2}\left(\sin t\cos\frac{\pi}{4} - \cos t\sin\frac{\pi}{4}\right)$$

$$= 2t - \sqrt{2} \times \frac{1}{\sqrt{2}}(\sin t - \cos t)$$

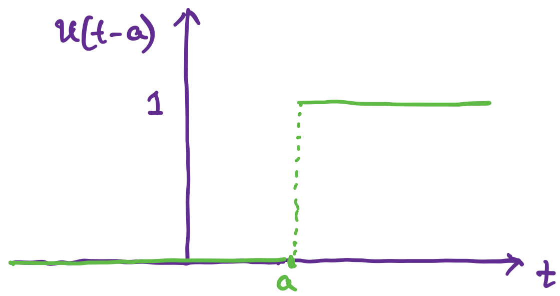
$$\Rightarrow \boxed{y = 2t - \sin t + \cos t}$$

Unit step function (Heaviside function):

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases} \quad \text{where } a \in \mathbb{R}$$



Unit Step function $u(t)$

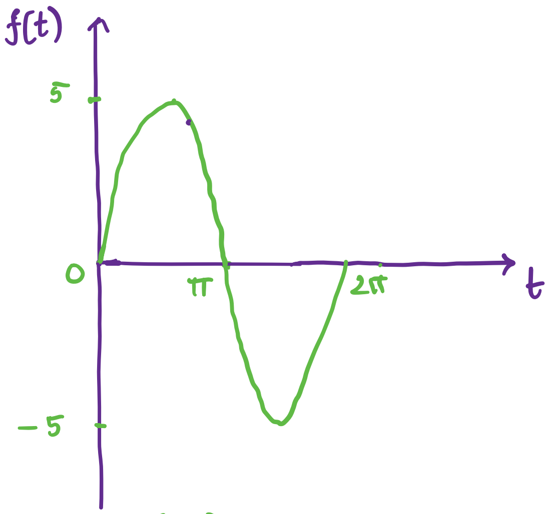


Unit step function $u(t-a)$

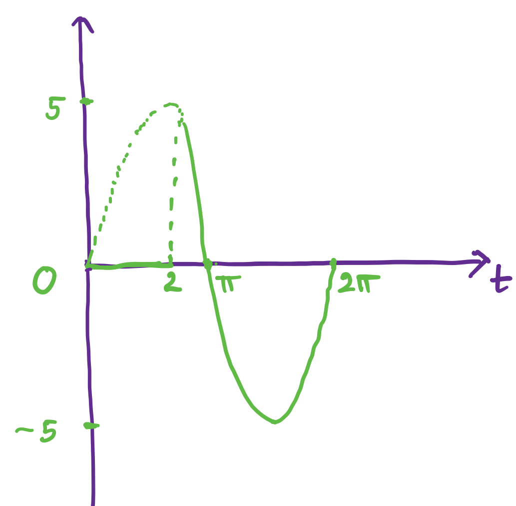
$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt = \int_a^\infty e^{-st}\,dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_a^\infty = \frac{e^{-as}}{s}$$

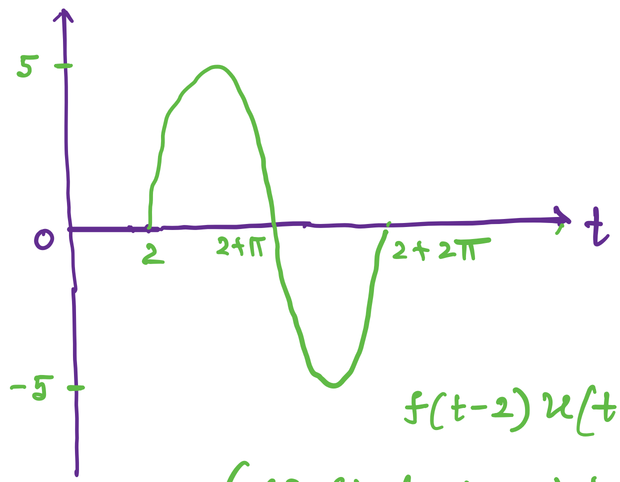
Ex:



$f(t) = 5\sin t$
(Given function)



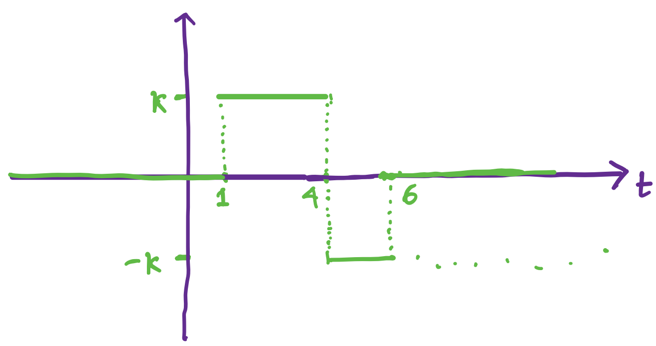
$f(t)\,u(t-2)$
(switching off and on)



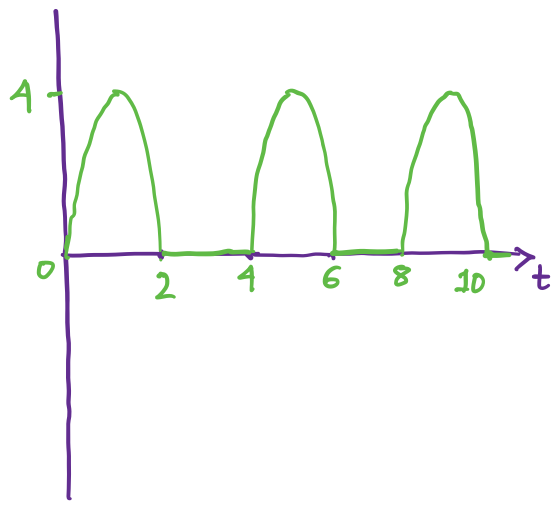
$f(t-2)\,u(t-2)$

(Shift (or translate) $f(t)$ by 2)

Ex:



$k\left[u(t-1) - 2u(t-4) + u(t-6)\right]$



$4\sin\left(\frac{\pi}{2}t\right)\left[u(t) - u(t-2) + u(t-4) - \ldots\right]$

Use of Unit Step functions

Time Shifting (t-Shifting) Theorem:

If $f(t)$ has the Laplace transform $F(s)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

Proof: Note that $f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$

So, $\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$

$= \int_a^\infty e^{-st} f(t-a)\,dt$ Let $x = t-a$ then $dx = dt$

$= \int_0^\infty e^{-s(x+a)} f(x)\,dx = e^{-as}\int_0^\infty e^{-sx} f(x)\,dx = \boxed{e^{-as}F(s)}$

$= \boxed{e^{-as}\mathcal{L}\{f\}(s)}$

Now $f(t)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t) & \text{if } t>a \end{cases}$

So, $\mathcal{L}\{f(t)u(t-a)\}(s) = \int_0^\infty e^{-st} f(t)u(t-a)\,dt$

$= \int_a^\infty e^{-st} f(t)\,dt$ Let $x = t-a$, then $dx = dt$

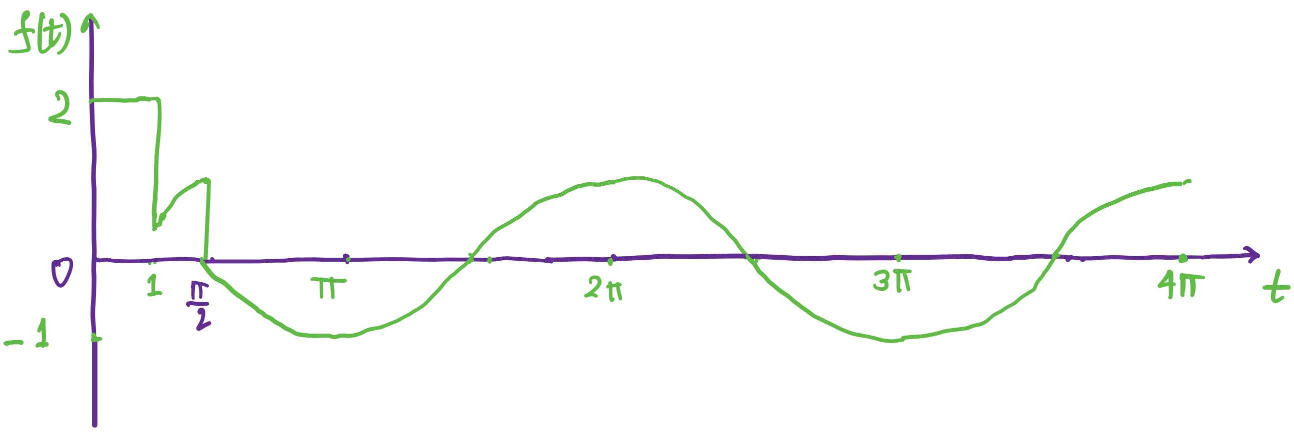
$= \int_0^\infty e^{-s(x+a)} f(x+a)\,dx = e^{-as}\int_0^\infty e^{-sx} f(x+a)\,dx$

$= e^{-as}\mathcal{L}\{f(x+a)\}(s) = e^{-as}\mathcal{L}\{f(t+a)\}(s)$

MTH 204: Lecture 19

Ex: Compute the Laplace Transform of the function:

$$f(t) = \begin{cases} 2 & \text{for } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{for } 1 < t < \frac{\pi}{2} \\ \cos(t) & \text{for } t > \frac{\pi}{2} \end{cases}$$



Now
$$f(t) = 2\left[1 - u(t-1)\right] + \frac{1}{2}t^2\left[u(t-1) - u\left(t-\frac{\pi}{2}\right)\right] + \cos(t)\, u\left(t-\frac{\pi}{2}\right)$$

Now $\mathcal{L}\left\{2\left[1-u(t-1)\right]\right\} = 2\left(\frac{1}{s} - \frac{e^{s}}{s}\right) = 2\left(\frac{1-e^{s}}{s}\right)$

$$\mathcal{L}\left[\frac{1}{2}t^2 u(t-1)\right] = \mathcal{L}\left[\left\{\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right\} u(t-1)\right]$$

$$= \left[\frac{2}{2s^3} + \frac{1}{s^2} + \frac{1}{2s}\right] e^{-s}$$

$$= \left[\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right] e^{-s}$$

$$\mathcal{L}\left[\frac{1}{2}t^2 u\left(t-\frac{\pi}{2}\right)\right] = \mathcal{L}\left[\left\{\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2 + \frac{\pi}{2}\left(t-\frac{\pi}{2}\right) + \frac{\pi^2}{8}\right\} \times u\left(t-\frac{\pi}{2}\right)\right]$$

$$= \left[\frac{1}{2}\,\frac{2}{s^3} + \frac{\pi}{2}\,\frac{1}{s^2} + \frac{\pi^2}{8}\,\frac{1}{s}\right] e^{-\frac{\pi}{2}s}$$

$$= \left[\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right] e^{-\frac{\pi}{2}s}$$

$$\mathcal{L}\left\{\cos(t)\, u\left(t-\frac{\pi}{2}\right)\right\} = \mathcal{L}\left\{-\sin\left(t-\frac{\pi}{2}\right) u\left(t-\frac{\pi}{2}\right)\right\}$$

$$= -\frac{1}{s^2+1} e^{-\frac{\pi}{2}s}$$

Combining

$$\mathcal{L}\{f\} = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right) e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right) e^{-\frac{\pi}{2}s} - \frac{1}{s^2+1} e^{-\frac{\pi}{2}s}$$

Note: Another way to calculate

$$\mathcal{L}\left\{\frac{1}{2}t^2 u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\}$$

$$= e^{-s}\mathcal{L}\left\{ \tfrac{1}{2}t^2 + t + \tfrac{1}{2} \right\}$$

$$= e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s} \right)$$

$$\mathcal{L}\left\{ \tfrac{1}{2}t^2 u\left(t - \tfrac{\pi}{2}\right) \right\} = e^{-\frac{\pi}{2}s}\,\mathcal{L}\left\{ \tfrac{1}{2}\left(t + \tfrac{\pi}{2}\right)^2 \right\}$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\left\{ \tfrac{1}{2}t^2 + \tfrac{\pi}{2}t + \tfrac{\pi^2}{8} \right\}$$

$$= e^{-\frac{\pi}{2}s}\left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s} \right)$$

and $\mathcal{L}\left\{ \cos(t)\, u\left(t - \frac{\pi}{2}\right) \right\} = e^{-\frac{\pi}{2}s}\mathcal{L}\left\{ \cos\left(t + \frac{\pi}{2}\right) \right\}$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\left\{ (-\sin t) \right\} = -\frac{e^{-\frac{\pi}{2}s}}{s^2 + 1}$$

Combining we get the expression for $\mathcal{L}\{f\}$

<u>Ex:</u> Find the inverse transform of

$$F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-2s}}{s^2 + \pi^2} + \frac{e^{-3s}}{(s+2)^2}$$

Note that $\mathcal{L}^{-1}\left(\dfrac{1}{s^2 + \pi^2} \right) = \dfrac{\sin \pi t}{\pi}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \quad \text{and so} \quad \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t}$$

(By first shifting theorem)

So, $$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+\pi^2}\right\} = \frac{\sin \pi(t-1)}{\pi}\, u(t-1)$$

(second shifting theorem)

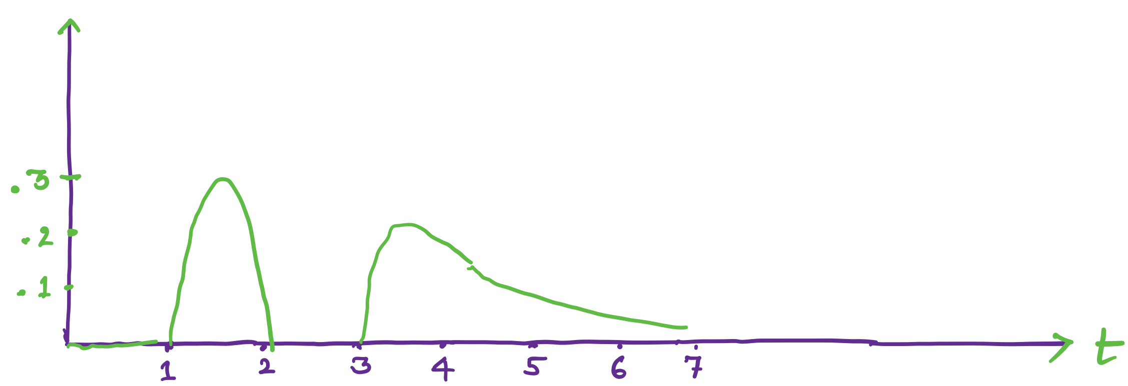
$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+\pi^2}\right\} = \frac{\sin \pi(t-2)}{\pi}\, u(t-2)$$

(second shifting theorem)

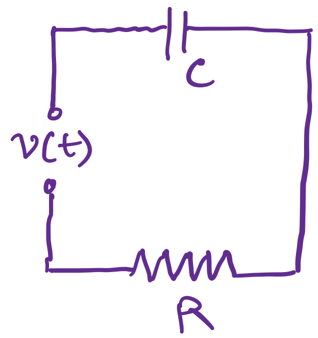
$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+2)^2}\right\} = (t-3)\, e^{-2(t-3)}\, u(t-3)$$

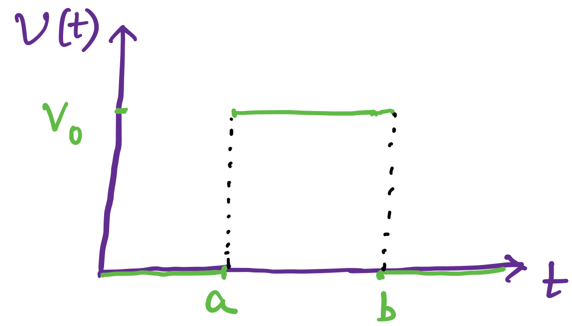
(second shifting theorem)

Hence $$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{\pi}\sin(\pi(t-1))\, u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))\, u(t-2) + (t-3)\, e^{-2(t-3)}\, u(t-3)$$



Ex: Find the current $i(t)$ in the RC-Circuit if a single rectangular wave with voltage V_0 is applied.





$$R\,i(t) + \frac{q(t)}{C} = R\,i(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = V_0\left[u(t-a) - u(t-b)\right]$$

Using Laplace transform

$$RI + \frac{1}{C}\,\frac{1}{s}\,I = V_0\left(\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right)$$

$$\Rightarrow I\left(R + \frac{1}{Cs}\right) = V_0\,\frac{1}{s}\left(e^{-as} - e^{-bs}\right)$$

$$\Rightarrow I = \frac{C\cancel{s}}{RCs+1}\,V_0\,\frac{1}{\cancel{s}}\left(e^{-as} - e^{-bs}\right)$$

$$\Rightarrow I = \frac{\cancel{C}V_0}{R\cancel{C}\left(s + \frac{1}{RC}\right)}\left(e^{-as} - e^{-bs}\right)$$

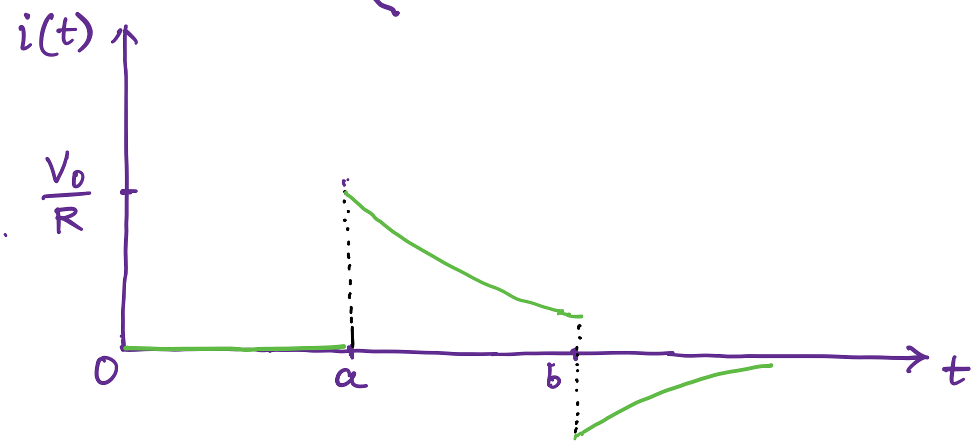
$$\Rightarrow \quad I = \frac{V_0}{R} \frac{1}{s + \frac{1}{RC}} \left(e^{-as} - e^{-bs} \right)$$

Now $\mathcal{L}^{-1}\left\{ \frac{V_0}{R} \cdot \frac{1}{s + \frac{1}{RC}} \right\} = \frac{V_0}{R} e^{-\frac{t}{RC}}$

Hence $i = \mathcal{L}^{-1}\left\{ \frac{V_0}{R} \frac{1}{s + \frac{1}{RC}} \left(e^{-as} - e^{-bs} \right) \right\}$

$$= \frac{V_0}{R} \left\{ e^{-\frac{t-a}{RC}} u(t-a) - e^{-\frac{(t-b)}{RC}} u(t-b) \right\}$$

(Using t-shifting theorem)



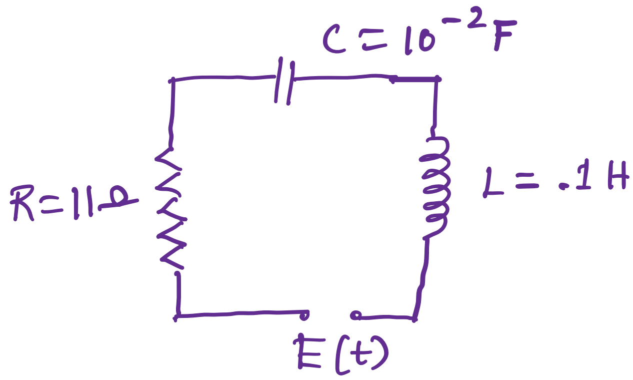
Now $i(t) = 0$ if $t < a$

$= k_1 e^{-t/RC}$ if $a < t < b$

$= (k_1 - k_2) e^{-t/RC}$ for $t > b$

where $k_1 = V_0 e^{a/RC}$, $k_2 = V_0 e^{b/RC}$ (Note that $k_1 < k_2$)

Ex:



$$E(t) = 100 \sin(400t)[u(t) - u(t-2\pi)]$$

(or we can write $E(t) = 100 \sin(400t)[1 - u(t-2\pi)]$ for $t > 0$)

Find the response (the current) of the RLC circuit where $E(t)$ is sinusoidal acting for a short time interval only:

say $E(t) = 100 \sin(400t)$ if $0 < t < 2\pi$

$= 0$ if $t > 2\pi$

Current and charge are initially zero.

Now $L i'(t) + R i(t) + \frac{1}{C} \int_0^t i(\tau) d\tau$

$= E(t)$ (Also $i(0) = 0$, $i'(0) = 0$)

Taking Laplace transform:

$$LsI + RI + \frac{1}{Cs} I$$

$$= 100 \frac{400}{s^2 + 400^2} \left(1 - e^{-2\pi s}\right)$$

$$\Rightarrow \left(Ls + R + \frac{1}{Cs}\right)I = 100 \times \frac{400s}{s^2+400^2}\left(\frac{1}{s} - \frac{e^{-2\pi s}}{s}\right)$$

$$\Rightarrow I = \frac{Cs}{LCs^2 + RCs + 1} \times 100 \times \frac{400s}{s^2+400^2}\left(\frac{1}{s} - \frac{e^{-2\pi s}}{s}\right)$$

$$\Rightarrow I = \frac{.01}{.001s^2 + .11s + 1} \cdot \frac{4\times10^4 s}{s^2+400^2}\left(1 - e^{-2\pi s}\right)$$

$$\Rightarrow I = \frac{1}{.001s^2 + .11s + 1} \frac{4\times10^2 s}{s^2+400^2}\left(1 - e^{-2\pi s}\right)$$

$$\Rightarrow I = \frac{1}{(.001)} \frac{1}{(s^2 + 110s + 1000)} \frac{4\times10^2 s}{s^2+400^2}\left(1 - e^{-2\pi s}\right)$$

$$\Rightarrow I = \frac{1}{(s+10)(s+100)} \frac{4\times10^5 s}{s^2+400^2}\left(1 - e^{-2\pi s}\right)$$

Let us factorize

$$\frac{4\times10^5 s}{(s+10)(s+100)(s^2+400^2)} = \frac{A}{s+10} + \frac{B}{s+100} + \frac{Ds+K}{s^2+400^2}$$

$$\Rightarrow 4\times10^5 s = A(s+100)(s^2+400^2) + B(s+10)(s^2+400^2)$$
$$+ (Ds+K)(s+10)(s+100)$$

Letting $s=-10$, $-4\times10^6 = A(90)(10^2+400^2) \Rightarrow A = -.2776$

$s=-100 \Rightarrow \quad -4\times10^7 = B(-90)(100^2+400^2) \Rightarrow B = 2.6144$

Equating the coefficient of s^3, $A+B+D=0 \Rightarrow D = -2.3368$

Equating the coefficient of s^2, $A(100)+B(10)+100D+K=0$

$\Rightarrow K = 258.66 = .6467\times400$

$\Rightarrow K = 258.66 = .6467\times400$

Hence

$$\frac{1}{(s+10)(s+100)}\times\frac{4\times10^5 s}{(s^2+400^2)} = -\frac{.2776}{(s+10)} + \frac{2.6144}{(s+100)} - \frac{(2.3368)s}{(s^2+400^2)} + \frac{.6467\times400}{(s^2+400^2)}$$

$$\Rightarrow \frac{1}{(s+10)(s+100)}\times\frac{4\times10^5}{s^2+400^2} = -\frac{.2776}{(s+10)} + \frac{2.6144}{(s+100)} - \frac{(2.3368)s}{(s^2+400^2)} + \frac{.6467\times400}{(s^2+400^2)}$$

Hence $i(t) = i_1(t) - i_2(t)$

where $i_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+10)(s+100)}\;\frac{4\times10^5 s}{s^2+400^2}\right\}$

$$= -.2776e^{-10t} + 2.6144e^{-100t} - 2.3368\cos(400t) + .6467\sin(400t) \quad \text{for } 0<t<2\pi$$

The second term in I differs from the first term by a factor $e^{-2\pi s}$.

By the t-shifting theorem $i_2(t) = 0$ for $0 < t < 2\pi$

Since $\cos\{400(t-2\pi)\} = \cos(400t)$ and
$\sin\{400(t-2\pi)\} = \sin(400t)$,

by the t-shifting theorem

for $t > 2\pi$

$$i_2(t) = -.2776\, e^{-10(t-2\pi)} + 2.6144\, e^{-100(t-2\pi)} - 2.3368 \cos 400t + .6467 \sin 400t$$

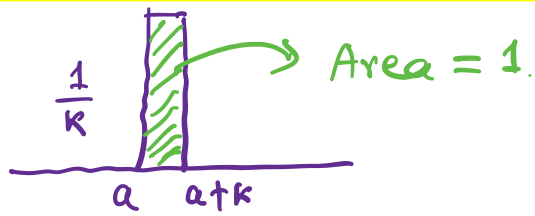
Hence for $t > 2\pi$

$$i(t) = -.2776\left(e^{-10t} - e^{-10(t-2\pi)}\right) + 2.6144\left(e^{-100t} - e^{-100(t-2\pi)}\right)$$

Short Impulses. Dirac Delta Function

Dirac Delta function represent short impulses and they can be seen as the limit of the function

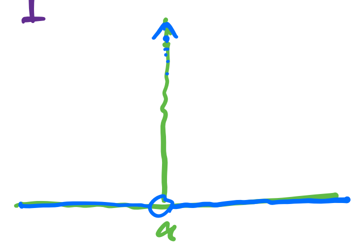
$$f_k(t-a) = \begin{cases} \frac{1}{k} & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$



Note that for any k, the area under f_k is always 1.

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = \frac{1}{k}(k+\cancel{a}-\cancel{a})$$
$$= 1$$

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$



Note: Technically δ is not a function
It is a generalized function (distribution)
It is zero everywhere except at a

point whose value is ∞ and its integral is 1.

It has the shifting property

$$\int_{-\infty}^{\infty} g(t)\,\delta(t-a)\,dt = g(a)$$

Laplace Transform of $\delta(t-a)$:

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$

$$\mathcal{L}\{f_k(t-a)\} = \mathcal{L}\left\{\frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]\right\}$$

$$= \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right]$$

$$= e^{-as}\,\frac{\left(1-e^{-ks}\right)}{ks}$$

Taking limit as $k \to 0$ and Using L'Hospital Rule

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$$

$$\left(\begin{aligned} &\lim_{k\to 0} \frac{1-e^{-ks}}{ks} \\ &= \lim_{k\to 0} \frac{\cancel{s}e^{-ks}}{\cancel{s}} = \lim_{k\to 0} e^{-ks} \\ &= 1 \end{aligned}\right)$$

Ex: Hammer blow on a mass spring system:

What is the impulse response of a mass-spring system to a blow at $t=1$?

$$y'' + 3y' + 2y = \delta(t-1)$$

$$\left(y(0) = 0 \;,\; y'(0) = 0\right)$$

Taking the Laplace transform:

$$s^2 Y + 3sY + 2Y = e^{-s}$$

$$\Rightarrow (s^2+3s+2)Y = e^{-s}$$

$$\Rightarrow \quad Y = \frac{1}{(s^2+3s+2)} e^{-s}$$

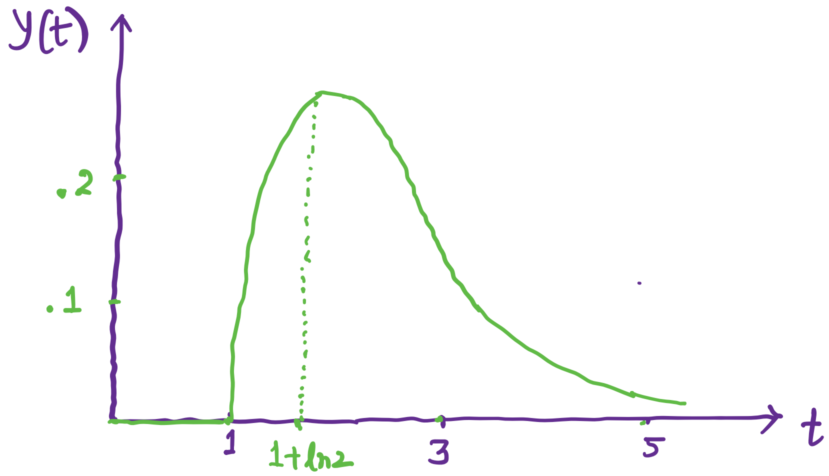
$$\Rightarrow \; Y = \frac{1}{(s+1)(s+2)} e^{-s}$$

$$= \left[\frac{1}{s+1} - \frac{1}{s+2}\right] e^{-s}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left\{\left[\frac{1}{s+1} - \frac{1}{s+2}\right] e^{-s}\right\}$$

$$= \left[e^{-(t-1)} - e^{-2(t-1)}\right] u(t-1)$$

$$\left(\begin{array}{l} \text{Since } \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}, \text{ by } t\text{-shifting} \\ \text{theorem } \mathcal{L}^{-1}\left\{\frac{1}{s+1} e^{-s}\right\} = e^{-(t-1)} u(t-1) \\ \text{Similarly since } \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}, \text{ by} \\ t\text{-shifting theorem } \mathcal{L}^{-1}\left\{\frac{1}{s+2} e^{-s}\right\} = e^{-2(t-1)} u(t-1) \end{array} \right)$$



Mass-spring System Under a Square Wave

The response of the damped mass-spring system under a square wave is modeled by $y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2)$

$y(0) = 0, \quad y'(0) = 0$

Taking Laplace Transform, we get the subsidiary equation as:

$$s^2 Y + 3sY + 2Y = \frac{1}{s} e^{-s} - \frac{1}{s} e^{-2s}$$

$$\Rightarrow \quad Y = \frac{(e^{-s} - e^{-2s})}{s(s^2+3s+2)}$$

$$\Rightarrow \quad Y = \frac{1}{s(s+1)(s+2)} \left(e^{-s} - e^{-2s}\right)$$

$$\Rightarrow \quad Y = \left[\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right]\left(e^{-s} - e^{-2s}\right)$$

Now $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, $\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$ $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$

Then $y = \mathcal{L}^{-1}\left\{\left[\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right]\left(e^{-s} - e^{-2s}\right)\right\}$

$$= \frac{1}{2} u(t-1) - e^{-(t-1)} u(t-1) + \frac{1}{2} e^{-2(t-1)} u(t-1)$$

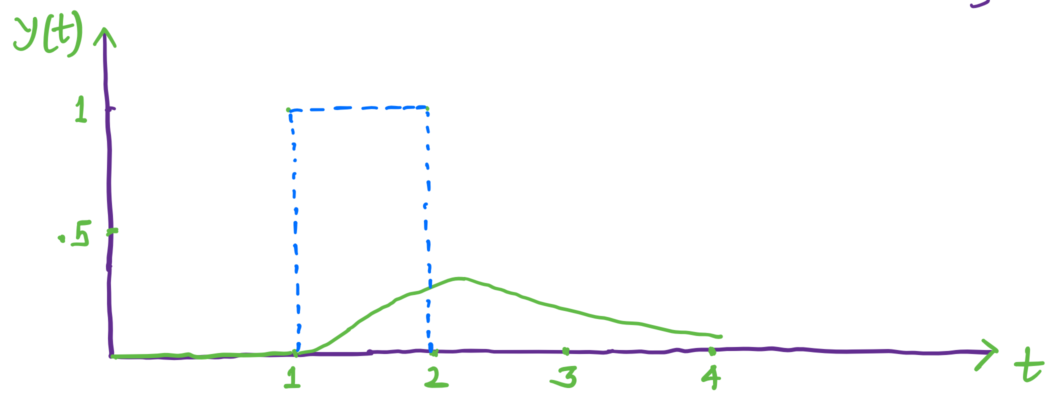
$$- \frac{1}{2} u(t-2) + e^{-(t-2)} u(t-2) - \frac{1}{2} e^{-2(t-2)} u(t-2)$$

Therefore $\quad y(t) = 0 \qquad$ if $0<t<1$

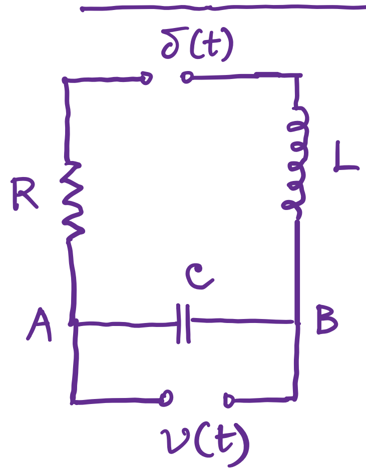
$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} \qquad$ if $1<t<2$

$= -e^{-(t-1)} + e^{-(t-2)} + \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2(t-2)}$

if $t>2$



Ex: Four terminal RLC-Network



Find the output voltage response if $R = 20\Omega$, $L = 1H$, $C = 10^{-4}F$.

The input is $\delta(t)$ (a unit impulse at time $t=0$)

The current and charge are zero at time $t=0$

If $i(t)$ is the current at time t, then

$$Li' + Ri + \frac{q}{c} = \delta(t)$$

(or $\quad Li' + Ri + \frac{1}{c}\int_0^t i(t)\,dt = \delta(t)$)

Note that $i(t) = q' = \frac{dq}{dt} = c\frac{dv}{dt}$ $\left(\frac{q}{c} = v\right)$

Hence $LCv'' + RCv' + v = \delta(t)$

$\Rightarrow v'' + \frac{R}{L}v' + \frac{1}{LC}v = \frac{1}{LC}\delta(t)$

Note that $v(0) = 0$, $v'(0) = 0$

Taking Laplace Transform of both sides

$$s^2\mathcal{L}(v) + \frac{R}{L}s\mathcal{L}(v) + \frac{1}{LC}\mathcal{L}(v) = \frac{1}{LC}$$

$\left(\text{Since } \mathcal{L}(\delta(t)) = e^{0\times t} = 1\right)$

Now $R = 20\,\Omega$, $C = 10^{-4}F$, $L = 1H$

$\Rightarrow s^2\mathcal{L}(v) + 20s\mathcal{L}(v) + 10^4\mathcal{L}(v) = 10^4$

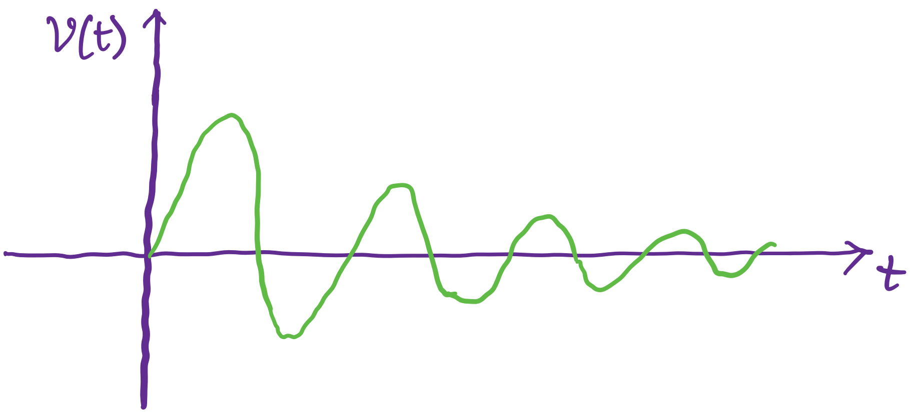
$$\Rightarrow \mathcal{L}(v) = \frac{10^4}{s^2 + 20s + 10^4}$$

$$\Rightarrow \mathcal{L}(v) = \frac{10^4}{(s+10)^2 + 9900}$$

$$\Rightarrow \mathcal{L}(v) = \frac{10^4\left(\sqrt{9900}\right)}{\sqrt{9900}\left\{(s+10)^2 + \left(\sqrt{9900}\right)^2\right\}}$$

Taking the inverse Laplace Transform, we get $\boxed{v(t) = \frac{10^4}{\sqrt{9900}} \sin\left(\sqrt{9900}\, t\right) \times e^{-10t}}$

or $\boxed{v(t) = 100.5\, e^{-10t} \sin(99.5t)}$



Partial Fractions :

In many cases, the solution Y of a subsidiary equation usually appears as a quotient of polynomials

$$Y(s) = \frac{F(s)}{G(s)}$$

Depending on the factorization of $G(s)$,

we have a partial fraction representation leading to a sum of expressions whose inverse we can obtain.
These representations are sometimes called Heaviside Expansions.

- An unrepeated factor $(s-a)$ in $G(s)$
  $\longrightarrow$ Single partial fraction $\frac{A_1}{s-a}$
  $\longrightarrow$ Inverse: $A_1 e^{-at}$

- Repeated real factor $(s-a)^2 \rightarrow \frac{A_2}{(s-a)^2} + \frac{A_1}{(s-a)}$
  $\longrightarrow$ Inverse : $(A_1 + A_2 t) e^{-at}$

- Repeated real factor $(s-a)^3 \rightarrow \frac{A_3}{(s-a)^3} + \frac{A_2}{(s-a)^2} + \frac{A_1}{(s-a)}$
  $\rightarrow$ Inverse : $\left(A_1 + A_2 t + \frac{1}{2} A_3 t^2\right) e^{at}$

- Unrepeated complex factors $(s-a)(s-\bar{a})$
  where $a = \alpha + i\beta$

$$\longrightarrow \quad \frac{A_1 s + A_2}{(s-\alpha)^2 + \beta^2} = \frac{A_1(s-\alpha) + A_2 + A_1\alpha}{(s-\alpha)^2 + \beta^2}$$

$\longrightarrow$ Inverse: $e^{-\alpha t}\left\{A_1 \cos(\beta t) + \frac{A_2 + A_1\alpha}{\beta}\sin(\beta t)\right\}$

Ex: $y'' + 2y' + 2y = 10\sin(2t)\left(u(t) - u(t-\pi)\right)$

$y(0) = 1,\ y'(0) = -5$

Taking Laplace Transform:

$$(s^2 Y - s + 5) + 2(sY - 1) + 2Y$$

$$= 10\,\frac{2}{s^2+4}\left(1 - e^{-\pi s}\right)$$

$$\Rightarrow (s^2 + 2s + 2)Y = (s-3) + \frac{20}{s^2+4}\left(1 - e^{-\pi s}\right)$$

$$\Rightarrow Y = \frac{(s-3)}{(s^2+2s+2)} + \frac{20}{(s^2+2s+2)(s^2+4)}\left(1 - e^{-\pi s}\right)$$

Now $\mathcal{L}^{-1}\left(\frac{(s-3)}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left\{\frac{(s+1)-4}{(s+1)^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos(t) - 4e^{-t}\sin t$$
$$= e^{-t}\left[\cos(t) - 4\sin(t)\right]$$

Now let $\dfrac{20}{(s^2+2s+2)(s^2+4)} = \dfrac{As+B}{(s+1)^2+1} + \dfrac{Ms+N}{s^2+4}$

Then $20 = (As+B)(s^2+4) + (Ms+N)(s^2+2s+2)$

$\Rightarrow 20 = (A+M)s^3 + (B+N+2M)s^2 + (4A+2M+2N)s + (4B+2N)$

$\Rightarrow A+M=0$, $B+N+2M=0$, $4A+2M+2N=0$
and $4B+2N=20$

$\Rightarrow M=-A$, $N=10-2B \Rightarrow B+10-2B-2A=0$

$\Rightarrow 2A+B=10$ and $4A-2A+20-4B=0$

$\Rightarrow 2A-4B=-20$
and $2A+B=10$

Hence $-5B=-30 \Rightarrow \boxed{B=6}$

Now $2A=10-6 \Rightarrow \boxed{A=2}$

Then $\boxed{M=-2}$ and $\boxed{N=-2}$

Therefore

$$\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{2s+6}{(s+1)^2+1} - 2\,\frac{(s+1)}{s^2+4}$$

$$= \frac{2(s+1)+4}{(s+1)^2+1} - 2\,\frac{s}{s^2+4} - 2\,\frac{1}{s^2+4}$$

$$= 2\,\frac{(s+1)}{(s+1)^2+1} + 4\,\frac{1}{(s+1)^2+1} - 2\,\frac{s}{s^2+2^2} - \frac{2}{s^2+2^2}$$

So, $\mathcal{L}^{-1}\left\{\frac{20}{(s^2+2s+2)(s^2+4)}\right\}$

$$= e^{-t}\left[2\cos t + 4\sin t\right] - 2\cos(2t) - \sin(2t)$$

Combining, we can say that for $0 < t < \pi$

$$\boxed{y(t) = 3e^{-t}\cos t - 2\cos(2t) - \sin(2t)}$$

and for $t > \pi$

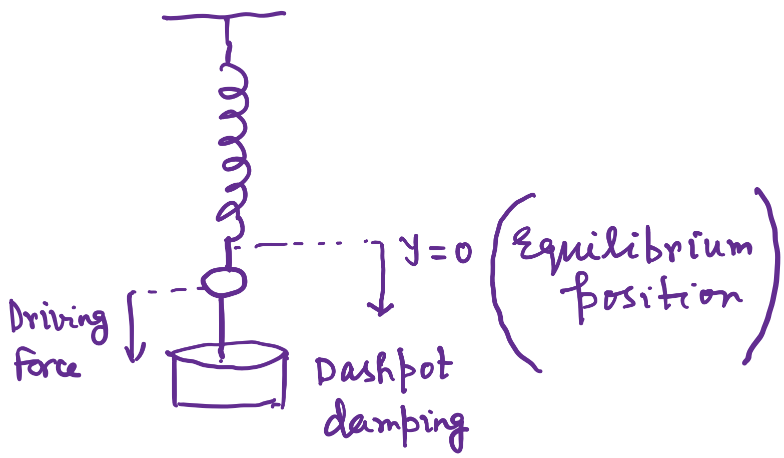
$$y(t) = 3e^{-t}\cos t - 2\cos(2t) - \sin(2t)$$
$$- e^{-(t-\pi)}\left[2\cos(t-\pi) + 4\sin(t-\pi)\right]$$
$$+ 2\cos\left[2(t-\pi)\right] + \sin\left[2(t-\pi)\right]$$

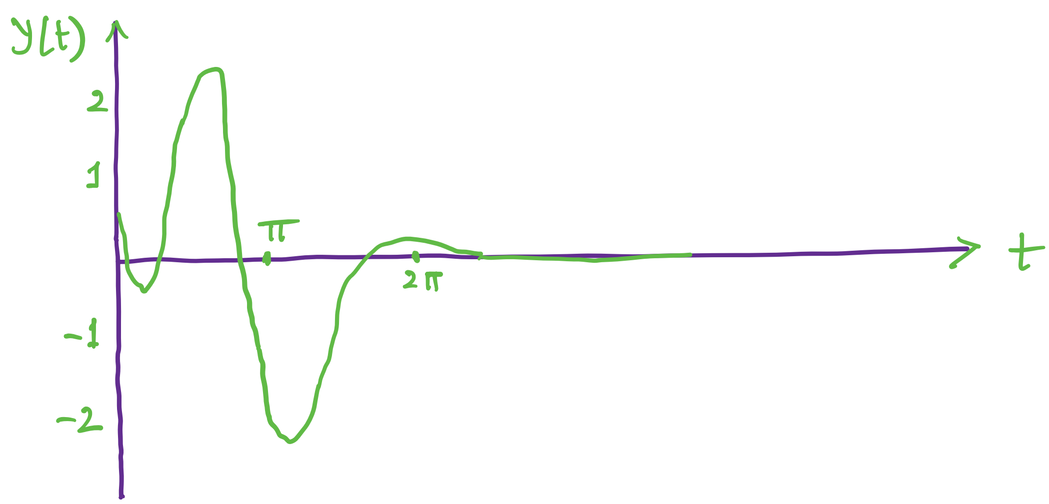
$$= 3e^{-t}\cos t - 2\cos(2t) - \sin(2t)$$
$$+ \; e^{-(t-\pi)}\left[2\cos t + 4\sin t\right]$$
$$+ \; 2\cos(2t) + \sin(2t)$$

$$\Rightarrow \boxed{y(t) = e^{-t}\left[(3+2e^{\pi})\cos t + 4e^{\pi}\sin t\right]} \text{ for } t > \pi$$

Note:

In this case differential equation for damped forced vibration is considered.





MTH 204 : Lecture 20

Convolution:

If two functions f and g satisfy the assumptions of the existence theorem so that their transforms F and G exist, then

$$\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

where the convolution of f and g (denoted by $f*g$) is defined by

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

Proof:

$$F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-sp} g(p)\,dp\right)$$

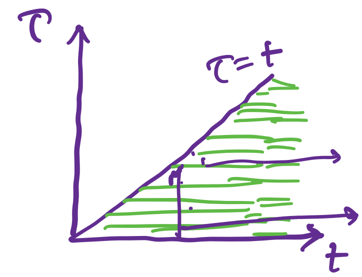
$$= \left\{\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right\}\left\{\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right\}$$

Let $t = p+\tau$
$p = t-\tau$

$$= \left\{\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right\}\left\{e^{s\tau}\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right\}$$

$$= \int_0^\infty e^{-s\tau} e^{s\tau} f(\tau)\left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right)d\tau$$

$$= \int_0^\infty f(\tau)\left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right) d\tau$$



$$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau) g(t-\tau)\,d\tau\right) dt$$ (changing the order of integration)

$$= \mathcal{L}\left\{ \int_0^t f(\tau) g(t-\tau)\,d\tau \right\}$$

$$= \mathcal{L}\left\{ (f * g)(t) \right\}$$

<u>Ex:</u> Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = e^{at} * 1$$ (since $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $\mathcal{L}(1) = \frac{1}{s}$)

$$= \int_0^t e^{a\tau}\, 1\, d\tau = \left.\frac{e^{a\tau}}{a}\right|_0^t$$

$$= \frac{1}{a}\left(e^{at} - 1\right)$$

<u>Ex:</u> Find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)}\,\frac{1}{(s^2+\omega^2)}\right\} = \left(\frac{1}{\omega}\sin\omega t\right) * \left(\frac{1}{\omega}\sin\omega t\right)$$

$$= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin\{\omega(t-\tau)\}\,d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [-\cos\omega t + \cos(2\omega\tau - \omega t)]\,d\tau$$

$$= \frac{1}{2\omega^2}\left[-t\cos\omega t + \left.\frac{\sin\omega(2\tau - t)}{2\omega}\right|_0^t\right]$$

$$= \frac{1}{2\omega^2}\left[-t\cos\omega t + \frac{\sin\omega t}{2\omega} - \frac{(-\sin\omega t)}{2\omega}\right]$$

$$= \frac{1}{2\omega^2}\left[-t\cos(\omega t) + \frac{\sin(\omega t)}{\omega}\right]$$

Ex: $y'' + \omega_0^2 y = K\sin(\omega_0 t)$, $y(0) = y'(0) = 0$

$\left(\omega_0^2 = \frac{K}{m},\ K = \text{spring constant}\right)$

Taking Laplace Transform

$$s^2 Y + \omega_0^2 Y = \frac{k\,\omega_0}{s^2 + \omega_0^2}$$

$$\Rightarrow Y = \frac{k\,\omega_0}{(s^2 + \omega_0^2)^2}$$

This is a case of resonance. The input signal is of the same frequency as the natural frequency of the system.

By the previous example,

$$y(t) = K\omega_0 \frac{1}{2\omega_0^2}\left[-t\cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{\omega_0}\right]$$

$$= \frac{K}{2\omega_0}\left[-t\cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{\omega_0}\right]$$

Properties of Convolution:

(1) Commutativity: $f * g = g * f$

(2) Associativity: $(f * g) * h = f * (g * h)$

(3) Distributivity: $f * (g_1 + g_2) = f * g_1 + f * g_2$

(4) Zero element: $f * 0 = 0 * f = 0$

Note that $f * 1 \neq f$ in general

and $(f*f)(t)$ may not be a nonnegative function.

Application to Nonhomogeneous Linear ODEs

Let us consider $y'' + ay' + by = r(t)$

where a, b are constants

Taking Laplace transform we get

$$s^2 Y(s) - s\,y(0) - y'(0) + a\left(sY(s) - y(0)\right) + bY(s) = R(s)$$

$$\Rightarrow (s^2 + as + b)Y(s) = (s+a)\,y(0) + y'(0) + R(s)$$

$$\Rightarrow Y(s) = \frac{[(s+a)y(0) + y'(0)]}{(s^2+as+b)} + \frac{R(s)}{(s^2+as+b)}$$

Now if $Q(s) = \dfrac{1}{s^2+as+b}$

then $Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$

If $y(0) = 0$, $y'(0) = 0$, (Response of system at rest)

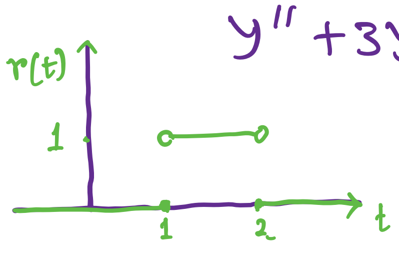
$$Y(s) = Q(s)R(s)$$

$$\Rightarrow y(t) = \int_0^t q(t-\tau)\,r(\tau)\,d\tau.$$

(i.e. $y(t) = q(t) * r(t)$)

Ex: Response of a Damped vibrating system to a single square wave:

$$y'' + 3y' + 2y = r(t) \qquad y(0) = y'(0) = 0$$



$$r(t) = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

So, $r(t) = u(t-1) - u(t-2)$

Then $Q(s) = \dfrac{1}{s^2+3s+2} = \dfrac{1}{(s+1)(s+2)}$

$\Rightarrow Q(s) = \dfrac{1}{s+1} - \dfrac{1}{s+2}$

$\Rightarrow q(t) = e^{-t} - e^{-2t}$

So, $y(t) = \left(e^{-t} - e^{-2t}\right) * r(t)$

$\Rightarrow y(t) = \left(e^{-t} - e^{-2t}\right) * \left[u(t-1) - u(t-2)\right]$

If we take the indefinite integral of

$$\int q(t-\tau)\times 1 \, d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \, d\tau\right]$$

$$= e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)}$$

Now if $t < 1$, $r(t) = 0$ and so $y(t) = 0$

Now if $1 < t < 2$,

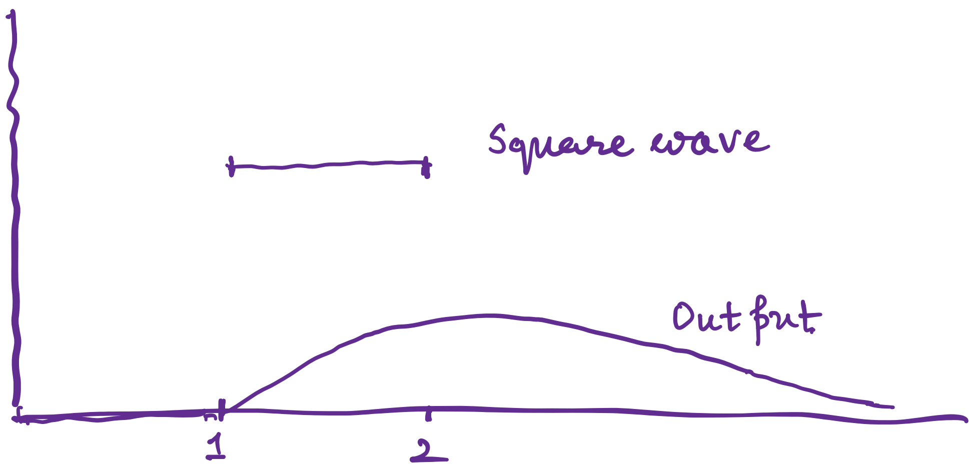
$$y(t) = \int_1^t q(t-\tau)\times 1 \, d\tau = \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)}\right]_1^t$$

$$= 1 - \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

If $t>2$, $y(t) = \int_1^2 r(t-\tau)\,1\,d\tau$

$$= e^{-(t-2)} - \frac{1}{2} e^{-2(t-2)} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$



A Voterra Integral Equation of the second kind :

$$\boxed{y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t}$$

It can be written as

$$y(t) - y * \sin t = t$$

Taking Laplace Transform we get

$$Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s)\frac{(s^2+1-1)}{s^2+1} = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\Rightarrow \quad y(t) = t + \frac{t^3}{6} \quad \left(\begin{array}{c}\text{By taking} \\ \text{inverse Laplace} \\ \text{Transform}\end{array}\right)$$

<u>Ex:</u>

$$y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh t$$

We can write it as

$$y(t) - (1+t) * y(t) = 1 - \sinh t$$

Taking Laplace Transform:

$$Y(s) - \left(\frac{1}{s} + \frac{1}{s^2}\right) Y(s) = \frac{1}{s} - \frac{1}{s^2-1}$$

$$\Rightarrow \quad Y(s)\left(\frac{s^2-s-1}{s^2}\right) = \frac{s^2-s-1}{s(s^2-1)}$$

$$\Rightarrow \quad Y(s) = \frac{s^2}{s(s^2-1)}$$

$$\Rightarrow \quad Y(s) = \frac{s}{(s^2-1)}$$

$$\Rightarrow \quad y(t) = \cosh t$$

Differentiation of Transforms :

If f satisfies the assumptions of the existence theorem:
then $\mathcal{L}\{t f(t)\} = -F'(s)$ where $F(s) = \mathcal{L}\{f(t)\}$
Hence $\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$

Proof: $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$$F'(s) = \frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\,dt\right\}$$

$$= \int_0^\infty \frac{d}{ds}\left\{e^{-ts}\right\} f(t)\,dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left(-t f(t)\right) dt$$

$$= \mathcal{L}\{-t f(t)\}$$

So, $\mathcal{L}\{t f(t)\} = -F'(s)$

Hence $\mathcal{L}^{-1}(F'(s)) = -t f(t)$

Ex: $\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{s^2+\omega^2} = F(s)$

$$\mathcal{L}\{-t\sin(\omega t)\} = F'(s) = \frac{-2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}\{-t\sin(\omega t)\} = -\frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}\{t\sin(\omega t)\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2\omega s}{(s^2+\omega^2)^2}\right\} = t\sin(\omega t)$$

Integration of Transforms:

If f satisfies the assumptions of the existence theorem and the limit of $\frac{f(t)}{t}$ exists when t approaches 0 from the right then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$

Hence, $\mathcal{L}^{-1}\left\{\int_s^\infty F(x)\,dx\right\} = \frac{f(t)}{t}$

We know $F(x) = \int_0^\infty e^{-xt} f(t)\,dt$

$$\int_s^\infty F(x)\,dx = \int_s^\infty \left\{\int_0^\infty e^{-xt} f(t)\,dt\right\} dx$$

$$= \int_0^\infty \left\{\int_s^\infty e^{-xt} f(t)\,dx\right\} dt$$

$$= \int_0^\infty f(t) \left\{\int_s^\infty e^{-xt}\,dx\right\} dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-xt}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \frac{e^{-st}}{t}\,dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\}.$$

$$\mathcal{L}^{-1}\left\{\int_s^{\infty} F(x)\,dx\right\} = \frac{f(t)}{t}$$

Ex: Calculate $\mathcal{L}^{-1}\left\{\log\frac{(s^2+w^2)}{s^2}\right\}$

Let us define $G = F' = \frac{d\left\{\log\left(\frac{s^2+w^2}{s^2}\right)\right\}}{ds}$

$$= \frac{2s}{s^2+w^2} - \frac{2s}{s^2}$$

$$= \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$

Then $g = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$

$$= 2\cos(\omega t) - 2 = 2[\cos(\omega t) - 1].$$

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left\{-\int_s^{\infty} G(s)\,ds\right\}$$

$$= -\frac{g(t)}{t} = \boxed{\frac{-2(\cos(\omega t)-1)}{t}}$$

ODEs with Variable Coefficients

Let $\mathcal{L}(y) = Y$

$$\mathcal{L}\{y'\} = sY - y(0)$$
$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}\{ty'\} = -\frac{d\{sY - y(0)\}}{ds}$$
$$= -[Y + sY'] = -Y - sY'$$

$$\mathcal{L}\{ty''\} = -\frac{d\{s^2Y - sy(0) - y'(0)\}}{ds}$$
$$= -[2sY + s^2Y' - y(0)]$$
$$= -2sY - s^2Y' + y(0)$$

Ex: **Laguerre's Equation:**

$$ty'' + (1-t)y' + ny = 0$$

Taking Laplace Transform,

$$-2sY - s^2Y' + y(0) + sY - Y(0) + Y + sY' + nY = 0$$

$$\Rightarrow \quad (s-s^2)Y' + (n+1-s)Y = 0$$

$$\Rightarrow \quad Y' = -\frac{(n+1-s)}{(s-s^2)}Y$$

$$\Rightarrow \quad \frac{Y'}{Y} = \frac{(n+1-s)}{s(s-1)}$$

$$\Rightarrow \quad \frac{dY}{Y} = \left(\frac{n}{s-1} - \frac{n+1}{s}\right)ds$$

$$\Rightarrow \quad \log Y = n\log(s-1) - (n+1)\log(s)$$

$$\Rightarrow \quad \log Y = \log\left\{\frac{(s-1)^n}{s^{n+1}}\right\}$$

$$\Rightarrow \quad \boxed{Y = \frac{(s-1)^n}{s^{n+1}}}$$

Note that, $\mathcal{L}\{t^n e^{-t}\} = \dfrac{n!}{(s+1)^{n+1}}$ (since $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$)

$$\Rightarrow \mathcal{L}\left\{\frac{d^n(t^n e^{-t})}{dt^n}\right\} = \frac{n!\, s^n}{(s+1)^n}$$ (derivatives upto order $(n-1)$ are zero at 0)

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{s^n}{(s+1)^{n+1}}\right\} = \frac{1}{n!}\frac{d^n(t^n e^{-t})}{dt^n} \quad \cdots\cdots (1)$$

Let $L_0 = 1$, $\quad L_n(t) = \dfrac{e^t}{n!}\dfrac{d^n(t^n e^{-t})}{dt^n}$ for $n = 1, 2, \ldots$

These are polynomials because exponential terms cancel if we perform the indicated differentiations. They are called Laguerre Polynomials.

From (1) $\mathcal{L}\left\{\dfrac{e^t}{n!}\dfrac{d^n(t^n e^{-t})}{dt^n}\right\} = \dfrac{(s-1)^n}{s^{n+1}}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{(s-1)^n}{s^{n+1}}\right\} = \frac{e^t}{n!}\frac{d^n(t^n e^{-t})}{dt^n} \Rightarrow \boxed{\mathcal{L}^{-1}(Y) = L_n(t)} \text{ i.e. } \boxed{y = L_n(t)}$$

Note: $L_n(t) = \dfrac{e^t}{n!}\dfrac{d^n(t^n e^{-t})}{dt^n} = \dfrac{1}{n!}(D-1)^n(t^n)$

Systems of ODEs:

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t) \end{aligned}\right\}$$

Taking the Laplace Transform of both sides,

$$\left.\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s) \end{aligned}\right\}$$

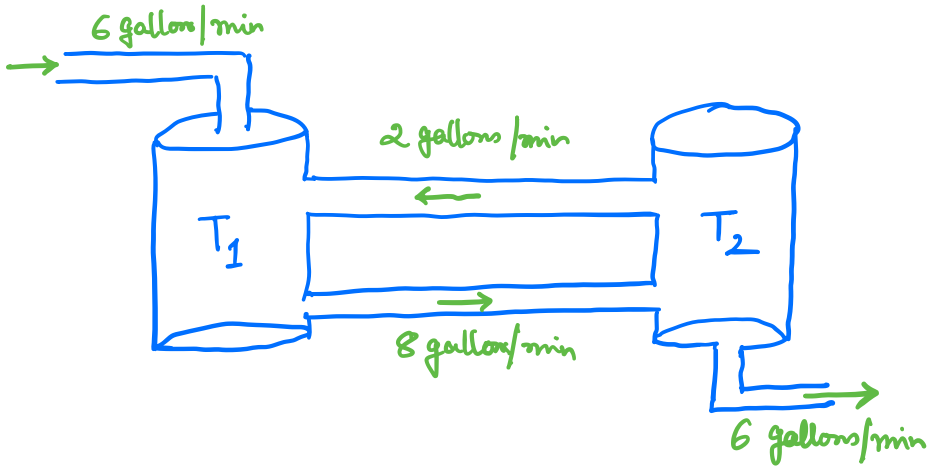
where $Y_i = \mathcal{L}(y_i)$, $G_i = \mathcal{L}(g_i)$ for $i = 1, 2$

Therefore
$$\left.\begin{aligned} (a_{11}-s)Y_1 + a_{12}Y_2 &= -y_1(0) - G_1(s) \\ a_{21}Y_1 + (a_{22}-s)Y_2 &= -y_2(0) - G_2(s) \end{aligned}\right\}$$

Solving the system algebraically for $Y_1(s)$ and $Y_2(s)$ and then taking the inverse Laplace Transform we obtain the solution as: $y_1 = \mathcal{L}^{-1}(Y_1)$ and $y_2 = \mathcal{L}^{-1}(Y_2)$

Note: Setting $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ and $A = [a_{ij}]_{2\times 2}$, we can write the system of ODEs as: $y' = Ay + g$ and we obtain $(A - sI)Y = -y(0) - G$

Ex:



Tank T_1 initially contains 100 gallons of pure water. Tank T_2 initially contains 100 gallons of water in which 150 pounds of salt are dissolved. The inflow into T_1 is 2 gallons/min from T_2 and 6 gallons/min containing 6 pounds of salt from the outside. The inflow into T_2 is 8 gallons/min from T_1. The outflow from T_2 is $2+6=8$ gallons/min. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2 respectively.

- Time rate of change = Inflow/min − Outflow/min for the two tanks.

$$\left.\begin{aligned} y_1' &= -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6 \\ y_2' &= \frac{8}{100}y_1 - \frac{8}{100}y_2 \end{aligned}\right\} \quad \begin{aligned} y_1(0)&=0 \\ y_2(0)&=150 \end{aligned}$$

Taking Laplace Transform:

$$\left.\begin{aligned} (-.08-s)Y_1 + .02Y_2 &= -\frac{6}{s} \\ .08Y_1 + (-.08-s)Y_2 &= -150 \end{aligned}\right\} \quad \left(\begin{aligned} &\text{using } y_1(0)=0 \\ &\text{and } y_2(0)=150 \end{aligned}\right)$$

$$\Rightarrow \begin{pmatrix} -.08-s & .02 \\ .08 & -.08-s \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{s} \\ -150 \end{pmatrix}$$

$$\Rightarrow \quad Y_1 = \frac{\begin{vmatrix} -\frac{6}{s} & .02 \\ -150 & -.08-s \end{vmatrix}}{\begin{vmatrix} -.08-s & .02 \\ .08 & -.08-s \end{vmatrix}}$$

$$= \frac{-\frac{6}{s}(-.08-s) + 150(.02)}{(-.08-s)^2 - (.08)(.02)} = \frac{9 + \frac{.48}{s}}{s^2 + .16s + .0064 - .0016}$$

$$= \frac{9s + .48}{s(s^2 + .16s + .0048)} = \frac{9s + .48}{s(s+.12)(s+.04)}$$

$$\text{and} \quad Y_2 = \frac{\begin{vmatrix} -.08-s & -\frac{6}{s} \\ .08 & -150 \end{vmatrix}}{\begin{vmatrix} -.08-s & .02 \\ .08 & -.08-s \end{vmatrix}} = \frac{12 + 150s + \frac{.48}{s}}{(s+.12)(s+.04)}$$

$$= \frac{150s^2 + 12s + .48}{s(s+.12)(s+.04)}$$

Using Partial fraction decomposition:

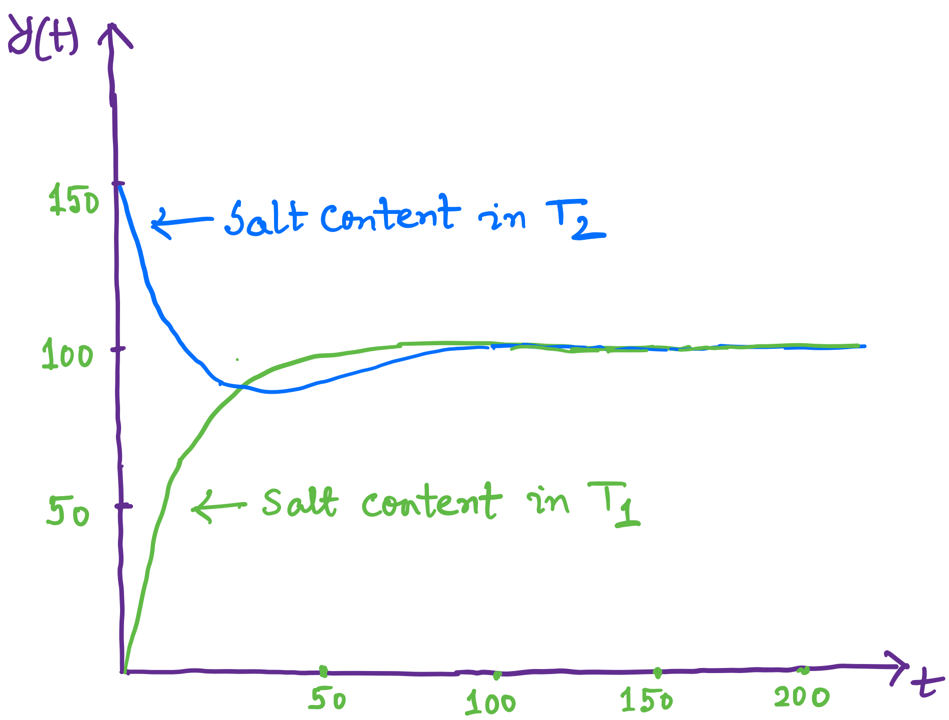
$$Y_1 = \frac{100}{s} - \frac{62.5}{s+.12} - \frac{37.5}{s+.04}$$

and $$Y_2 = \frac{100}{s} + \frac{125}{s+.12} - \frac{75}{s+.04}$$

By taking inverse Laplace Transform:

$$y_1 = 100 - 62.5\, e^{-.12t} - 37.5\, e^{-.04t}$$

and $$y_2 = 100 + 125\, e^{-.12t} - 75 e^{-.04t}$$



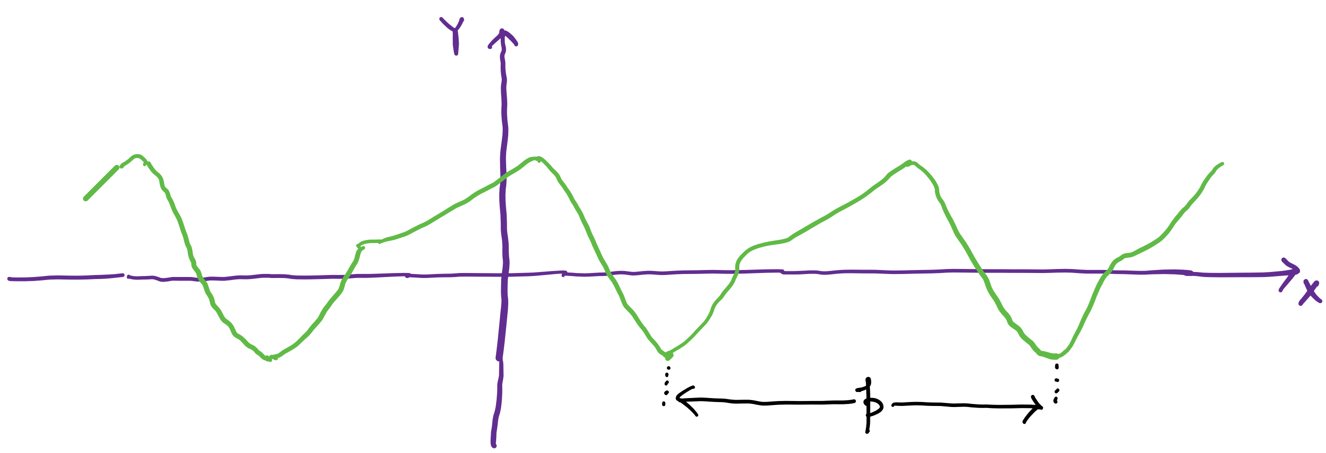
MTH 204 : Lecture 21

Fourier Series:

Trigonometric Series: It is a linear combination of $\{1, \cos x, \sin x, \cos(2x), \sin(2x), \ldots\}$ with coefficients $a_0, a_1, b_1, a_2, b_2, \ldots$ respectively.

Periodic Function: A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of $f(x)$ such that

$$f(x+p) = f(x) \quad \text{for all } x \text{ in the domain of } f.$$



- The graph of a periodic function can be obtained by periodic repetition of its graph of any interval of length p.

- The smallest positive period is called the Fundamental period.
  If $f(x)$ has a period p, it also has a period $2p, 3p, \ldots, np$ for any integer $n = 1, 2, \ldots$

$$f(x+2p) = f([x+p]+p) = f(x+p) = f(x)$$

and $f(x+np) = f(x)$ for all x in the domain of f

- If $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b also has the period p.

- $\sin x$, $\cos x$, $\tan x$, $\cot x$ are some familiar periodic function.
  Examples of functions that are not periodic : $\{x, x^2, x^3, e^x, \cosh x, \sinh x, \ln x$ etc.

- $f(x) = \tan x$ is a periodic function that is not defined for all real x but undefined at countably many points $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots$

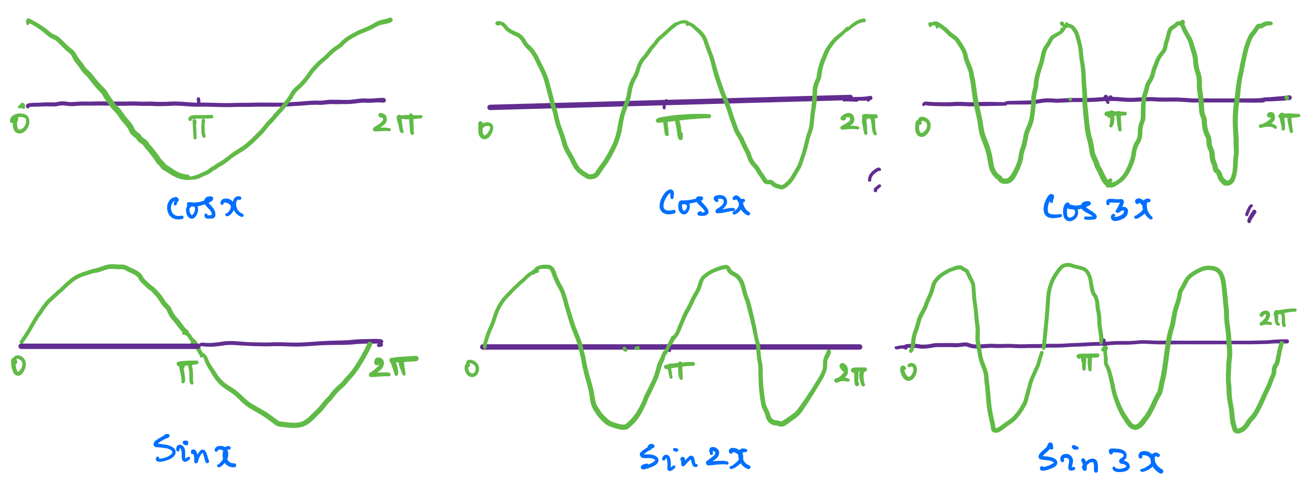
- In Fourier Series we consider representation of various functions of period 2π in terms of simple functions

$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx,$

All these function have period 2π.

(1 is of course periodic with any period)

These functions form "Trigonometric System".



$\sin 2x, \cos 2x$ have fundamental period π. $\sin nx, \cos nx$ have fundamental period $\frac{2\pi}{n}$.

- A series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0, a_1, b_1, a_2, b_2, \dots$ are constants is called a Trigonometric Series.

Each term has period 2π. Hence if the coefficients $a_0, a_1, b_1, a_2, b_2, \ldots$ are such that the series converges, its sum will be a function of period 2π.

- Next let $f(x)$ be a function of period 2π and is such that it can be represented by a series $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

(i.e. the series converges and has sum $f(x)$)

then we can write

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

and the series $f(x)$ is called the Fourier series representation of $f(x)$.

The coefficients of the series are called Fourier Coefficients of $f(x)$

and they are given by the Euler's Formulas:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{for } n=1,2,\dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{for } n=1,2,\dots$$

(Proof will be given later)

Note: There are functions $f(x)$ such that the series

$$a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

doesnot converge or doesnot converge to $f(x)$ (i.e. the series has a sum different from $f(x)$)

In that case we will call the series $a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ Fourier series corresponding to $f(x)$ (not the representation of $f(x)$) and write

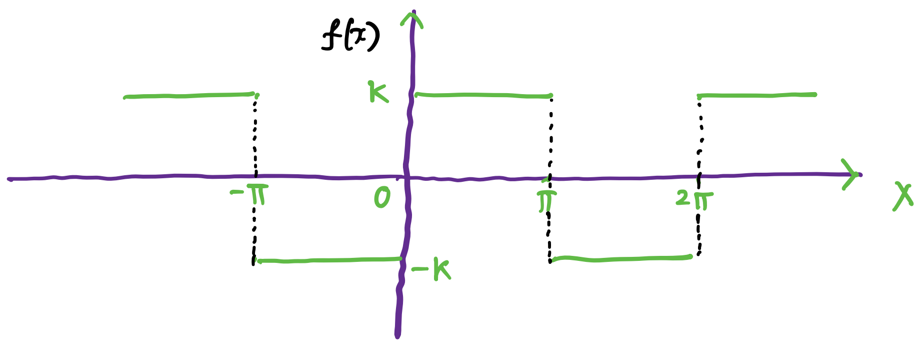
$$f \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Ex:

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$

(Periodic rectangular wave)



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-K)\,dx + \int_{0}^{\pi}(K)\,dx\right]$$

$$= \frac{1}{2\pi}\left[-K\pi + K\pi\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\cos(nx)\,dx + \int_{0}^{\pi} K\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\,\frac{\sin(nx)}{n}\Bigg|_{-\pi}^{0} + K\,\frac{\sin nx}{n}\Bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}(0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-k)\,\sin(nx)\,dx + \int_{0}^{\pi} (k)\,\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\;k\,\frac{\cos nx}{n}\Bigg|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\Bigg|_{0}^{\pi}\right]$$

$$= \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0\right]$$

$$= \frac{k}{n\pi}\left[2\cos 0 - 2\cos n\pi\right]$$

$$= \frac{2k}{n\pi}\left[1 - (-1)^n\right]$$

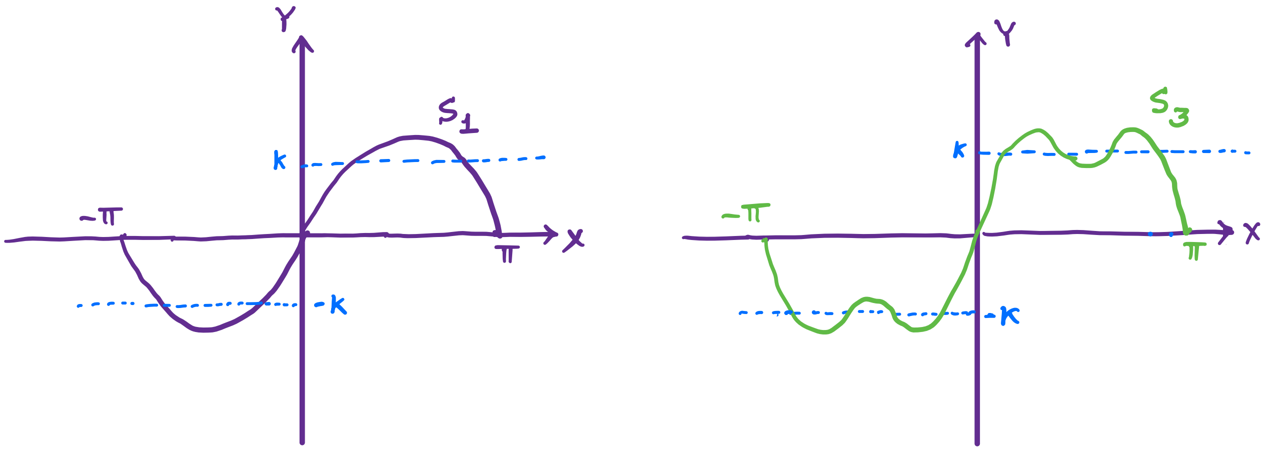
Hence $b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$, $b_4 = 0$,

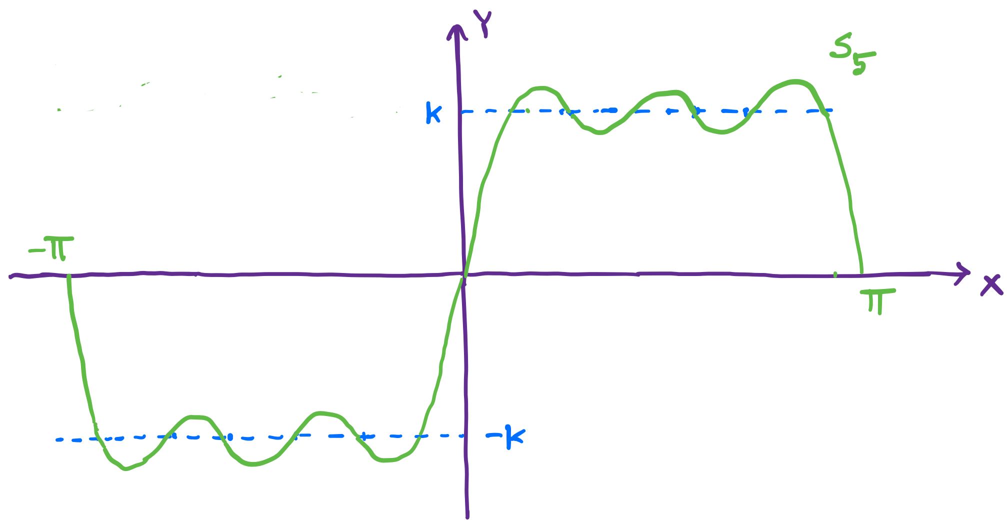
$b_5 = \frac{4k}{5\pi}$ . . .

So, $$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left[1-(-1)^n\right]\sin nx}$$

Now $S_1 = \frac{4k}{\pi}\sin x$, $S_3 = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin(3x)$

$$S_5 = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin(3x) + \frac{4k}{5\pi}\sin(5x)$$





Note: • In the above example $f(x)$ is discontinuous at $x=0$. However the Fourier series is 0 at $x=0$

Note that $f(0+) = k$, $f(0-) = -k$

and $\frac{f(0+)+f(0-)}{2} = 0 =$ Value of the Fourier series at $x=0$

• Now $f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

Let us substitute $x = \frac{\pi}{2}$

Then $$K = \frac{4K}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right)$$

$$\Rightarrow K = \frac{4K}{\pi}\left(1 + \frac{1}{3}(-1) + \frac{1}{5}(1) - \cdots\right)$$

$$\Rightarrow 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

$$\Rightarrow \boxed{1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}}$$

Orthogonality :

Let V be the vector space of 2π-periodic functions. Then V is an inner product space with respect to the inner product :

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\,g(x)\,dx$$

(It is the same inner product as in $L^2[-\pi, \pi]$)

In this space $\{1, \cos x, \sin x, \cos(2x), \sin(2x) \ldots\ldots\}$ is an "orthogonal basis".

$$\langle 1, \cos(nx)\rangle = \int_{-\pi}^{\pi} \cos(nx)\,dx = \frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi} = 0$$

$$\langle 1, \sin(nx)\rangle = \int_{-\pi}^{\pi} \sin(nx) = -\frac{\cos(nx)}{n}\Big|_{-\pi}^{\pi} = 0$$

$$\langle \sin(nx), \cos(mx)\rangle = \int_{-\pi}^{\pi} \sin(nx)\cos(nx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin((n+m)x) + \sin((n-m)x)\right]dx = 0$$

$$\langle \sin(nx), \sin(mx)\rangle = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos((n-m)x) - \cos((n+m)x)\right]dx$$

$$\left.\begin{aligned} &= 0 \quad \text{if } n \neq m \\ \text{and } &= \frac{1}{2}\int_{-\pi}^{\pi}\left[1-\cos((2m)x)\right]dx = \pi \quad \text{if } n = m \end{aligned}\right\}$$

$$\langle \cos(nx), \cos(mx)\rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos((n+m)x) + \cos((n-m)x)\right]dx$$

$$\left.\begin{aligned} &= 0 \quad \text{if } n \neq m \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos((2m)x) + 1\right]dx = \pi \quad \text{if } n = m \end{aligned}\right\}$$

Note: The above set is not orthonormal.

$$\langle 1, 1\rangle = \|1\|^2 = 2\pi$$

$$\langle \cos(nx), \cos(nx)\rangle = \|\cos(nx)\|^2 = \pi$$

$$\langle \sin(nx), \sin(mx)\rangle = \|\sin(nx)\|^2 = \pi$$

Proof of Euler Formulas:

Suppose the function $f(x)$ has a Fourier series representation as:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

Then $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)\right] dx$

$$= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos(nx)\,dx + b_n \int_{-\pi}^{\pi} \sin(nx)\,dx\right) dx$$

$$= a_0 \times 2\pi + 0 = 2\pi a_0$$

So, $\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx} = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle}$

Now $\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)\right]\cos(mx)\,dx$

$$= a_0 \int_{-\pi}^{\pi} \cos(mx)\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx + b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx\right]$$

$$= 0 + a_m \times \pi + 0 = a_m \pi$$

$\Rightarrow \boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx} = \frac{\langle f(x), \cos(mx)\rangle}{\langle \cos(mx), \cos(mx)\rangle}$

$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)\right]\sin(mx)\,dx$

$$= a_0 \int_{-\pi}^{\pi} \sin(mx)\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx + b_n \int_{-\pi}^{\pi} \sin(nx)\sin mx\,dx\right]$$

$$= 0 + 0 + b_m \times \pi = b_m \pi$$

$$\Rightarrow \boxed{b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx} = \frac{\langle f(x), \sin(mx) \rangle}{\langle \sin(mx), \sin(mx) \rangle}$$

Representation by a Fourier Series:

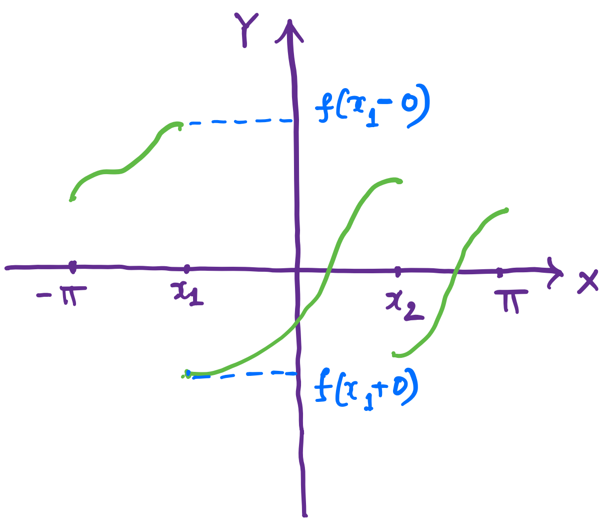
Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $[-\pi, \pi]$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0 i.e. $\dfrac{f(x_0+0) + f(x_0-0)}{2}$

Here $f(x_0+0) = \lim\limits_{\substack{h \to 0 \\ h > 0}} f(x_0+h)$

$$f(x_0-h) = \lim_{\substack{h\to 0\\ h>0}} f(x_0-h)$$

$$f'(x_0+0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(x_0+h)-f(x_0+0)}{h}$$

$$f'(x_0-0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(x_0-h)-f(x_0-0)}{-h}$$



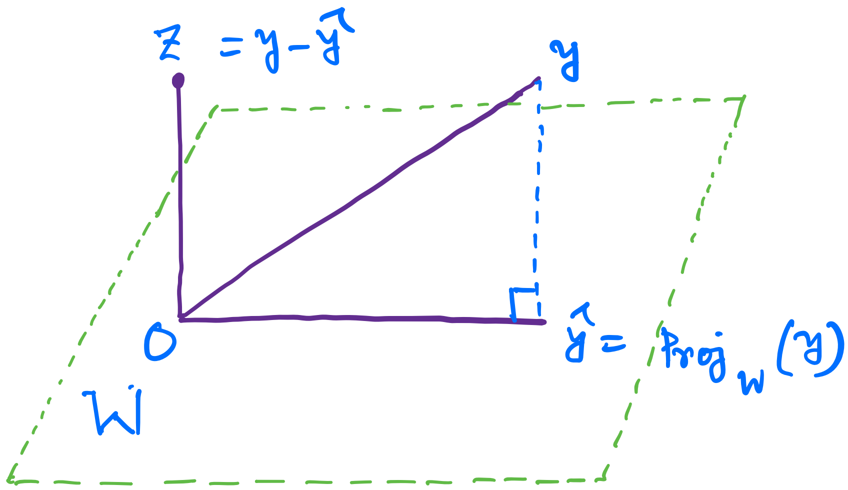
- Sum of the fourier series is $f(x)$ where f is continuous
- At x_1, the sum is $\dfrac{f(x_1+0)+f(x_1-0)}{2}$
- At x_2, the sum is $\dfrac{f(x_2+0)+f(x_2-0)}{2}$

Orthogonal Decomposition:

Let W be a vector subspace of a vector space V. Then any vector $y \in V$ can be

written uniquely as $y = \hat{y} + z$
where $\hat{y} \in W$ and $z \in W^{\perp}$.

If $\{u_1, u_2, \ldots, u_p\}$ (or $\{u_1, u_2, \ldots\}$ if W is infinite dimensional) is an orthogonal basis of W,
then
$$\hat{y} = \frac{\langle y, u_1 \rangle}{\| u_1 \|^2} u_1 + \frac{\langle y, u_2 \rangle}{\| u_2 \|^2} u_2$$
$$+ \cdots\cdots + \frac{\langle y, u_p \rangle}{\| u_p \|^2} u_p$$
(a series in case W is infinite dimensional)



Fourier Series can be viewed as an Orthogonal projection.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$\left.\begin{aligned}
\text{where } a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\langle f(x), 1\rangle}{\|1\|^2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{\langle f(x), \cos(nx)\rangle}{\|\cos(nx)\|^2} \\
\text{and } b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{\langle f(x), \sin(nx)\rangle}{\|\sin(nx)\|^2}
\end{aligned}\right\}$$

MTH 204: Lecture 22

Periodic Functions with arbitrary period:

Suppose $f(x)$ is periodic function with period $p = 2L$ (i.e. $f(x+2L) = f(x)$)

Then we can introduce a new variable v such that $v = \frac{2\pi}{p}x = \frac{2\pi}{2L}x = \frac{\pi}{L}x$

$$\Rightarrow \quad x = \frac{L}{\pi}v$$

Then the function $f(x) = f\left(\frac{L}{\pi}v\right)$ as a function of v is a periodic function with period 2π

($v = \pm\pi$ corresponds to $x = \pm L$)

Thus $f(x) = f\left(\frac{L}{\pi}v\right)$ has a fourier series of the form:

$$f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

where $\quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad \text{for } n=1,2,\dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad \text{for } n=1,2,\dots$$

Check: If $f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nv) + b_n\sin(nv)\right)$

then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)dv = \frac{1}{2\pi}\int_{-L}^{L} f(x)\frac{\pi}{L}dx$

$$\left(\begin{array}{l} v = \frac{\pi}{L}x \Rightarrow dv = \frac{\pi}{L}dx \\ v = \pm\pi \Rightarrow x = \pm L \end{array}\right) \quad = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos(nv)\,dv \\ &= \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\frac{\pi}{L}dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \end{aligned}\right\}$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin(nv)\,dv$

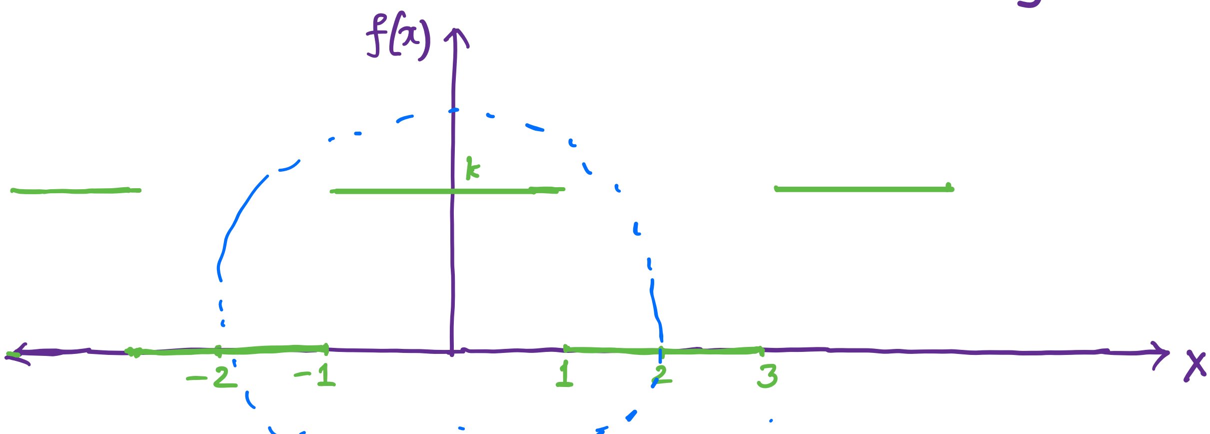
$$= \frac{1}{\pi}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\frac{\pi}{L}dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \Bigg)$$

- The set $\left\{1, \cos\left(\frac{\pi}{L}x\right), \sin\left(\frac{\pi}{L}x\right), \cos\left(\frac{2\pi}{L}x\right), \sin\left(\frac{2\pi}{L}x\right), \ldots, \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right), \ldots\right\}$ forms an orthogonal basis of the vector space of 2L-periodic functions.

Ex:

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$



Period $p = 2L = 4 \Rightarrow L = 2$

Now $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx$

$$= \frac{1}{4}k(2) = \boxed{\frac{k}{2}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\cdot\frac{2}{n\pi}\,k\left[\sin\left(\frac{n\pi}{2}x\right)\right]_{-1}^{1}$$

$$= \boxed{\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2k}{n\pi}(-1)^m & \text{if } n = 2m+1 \end{cases}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi}{2}x\right)dx = \frac{2k}{2n\pi}\left[-\cos\frac{n\pi}{2}x\right]_{-1}^{1}$$

$$= \boxed{0}$$

So, $$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty}\left(\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi}{L}x\right)$$

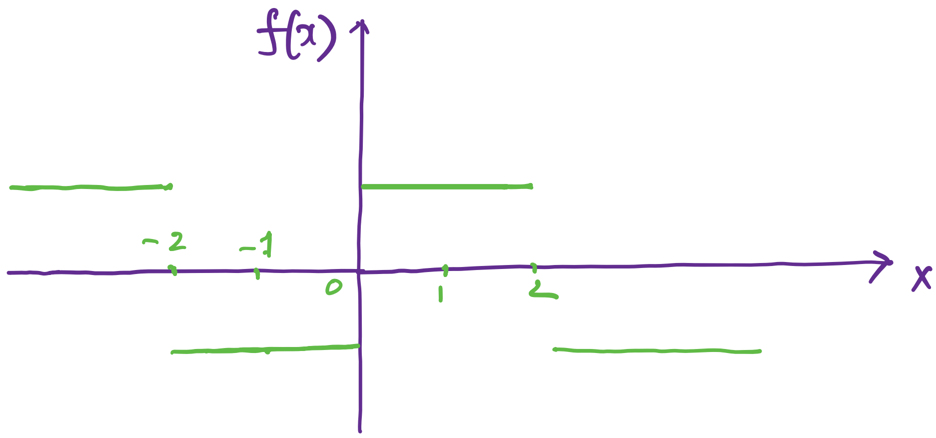
$$= \frac{k}{2} + \frac{2k}{\pi}\left[\cos\left(\frac{\pi}{2}x\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}x\right) - \cdots\right]$$

$$= \frac{k}{2} + \sum_{m=0}^{\infty}\frac{2k}{(2m+1)\pi}(-1)^m\cos\left(\frac{(2m+1)\pi}{2}x\right)$$

Since the function is even, it has a cosine only series.

<u>Ex:</u> <u>Change of Scale</u>:

Let
$$f(x) = -k \qquad -2 < x < 0$$
$$= k \qquad 0 < x < 2$$
and $f(x+4) = f(x)$



The period $p = 2L = 4 \Rightarrow L = 2$

Now from a previous example: If $g(x)$ is a periodic function with period 2π

$$g(v) = \begin{cases} -k & \text{if } -\pi < v < 0 \\ k & \text{if } 0 < v < \pi \end{cases}$$

$$= \sum_{n=1}^{\infty} \frac{2k}{n\pi}\left[1-(-1)^n\right] \sin nv$$

$$= \frac{4k}{\pi}\sin v + \frac{4\pi}{3\pi}\sin(3v) + \frac{4\pi}{5\pi}\sin(5v) + \cdots\cdots$$

Now let $v = \frac{\pi}{2}x$

So, $x = \frac{2v}{\pi}$ and $v = \pm\pi \Rightarrow x = \pm 2$

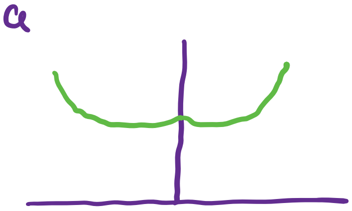
Thus $f(x) = g(v) = g\left(\frac{\pi}{2}x\right)$

$$= \frac{4k}{\pi}\sin\left(\frac{\pi}{2}x\right) + \frac{4k}{3\pi}\sin\left(\frac{3\pi}{2}x\right) + \frac{4k}{5\pi}\sin\left(\frac{5\pi}{2}x\right) + \cdots\cdots$$

Even and Odd functions:

- If $f(x)$ is an even function
  i.e. $f(-x) = f(x)$

its Fourier series reduces to a Fourier cosine series

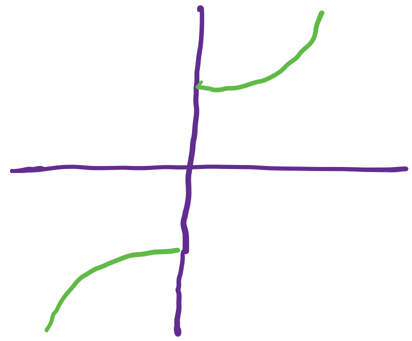


$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

where $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$

and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots$

• If $f(x)$ is an odd function that is $f(-x) = -f(x)$, then its fourier series reduces to a Fourier sine series



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)$$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

(We have used $\int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx$ if g is even

and $\int_{-L}^{L} h(x)\,dx = 0$ if h is odd)

Theorem: (Sum and scalar multiple)

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

In otherwords we can say that:

The process of obtaining Fourier Series is Linear.

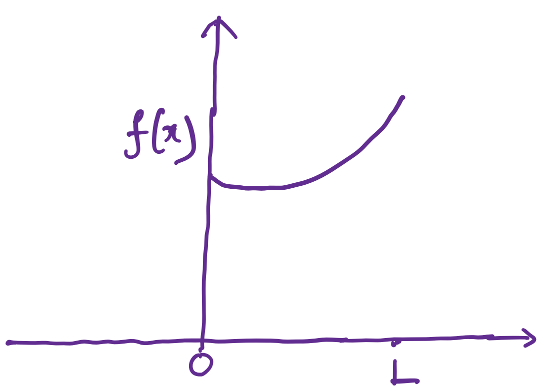
Let FS be an operator that assigns to each function, its Fourier series.

Then $FS(f_1+f_2) = FS(f_1)+FS(f_2)$

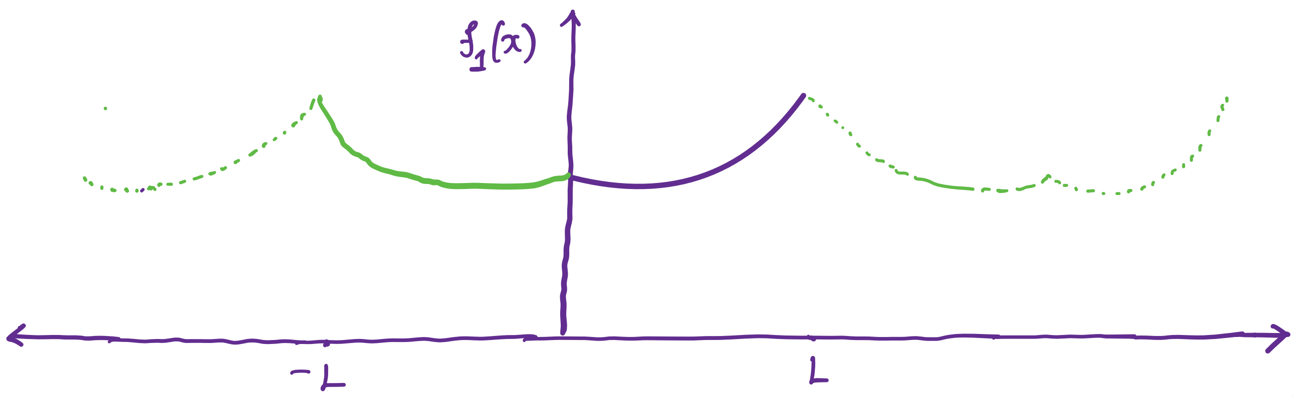
$$FS(cf) = c\,FS(f)$$

Half-range Expansion

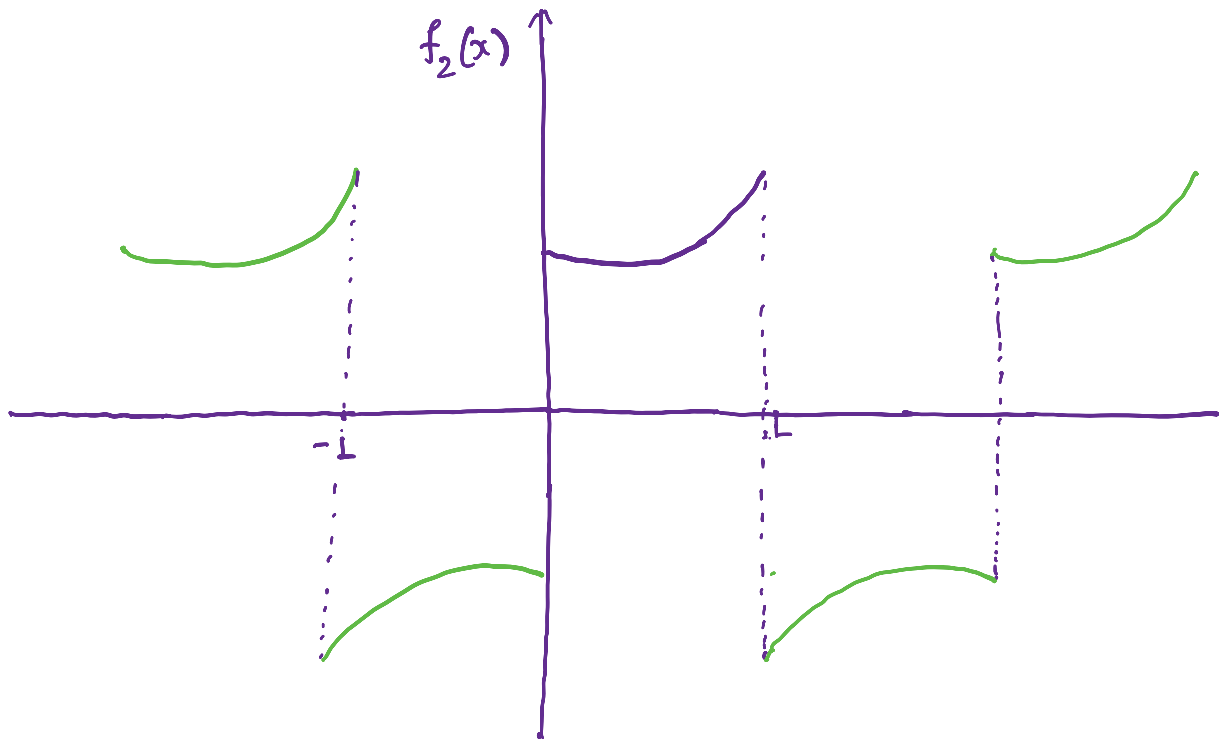
If only half of the range $[0, L]$ is of interest, we may extend, we may extend the function in an odd or even way and then use the simplified Fourier series expression for odd and even function.



(The given function $f(x)$)



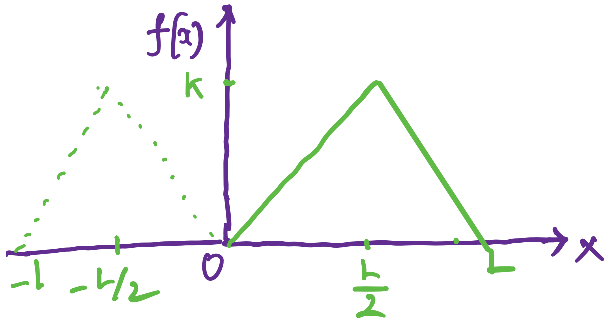
($f(x)$ continued as an even periodic function of period $2L$)



($f(x)$ continued as an odd periodic function of period $2L$)

Ex: $f(x) = \frac{2k}{L}x$ for $0 < x < \frac{L}{2}$

$= \frac{2k}{L}(L-x)$ for $\frac{L}{2} < x < L$



Even extension:

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\cdot\frac{1}{2}Lk = \frac{k}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\left(\frac{n\pi}{L}x\right) + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx\right]$$

$$= \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right] \quad \left(\text{using integration by parts}\right)$$

So, $\tilde{f}_e(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{L}x\right) + \cdots\right)$

Odd extension:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\sin\left(\frac{n\pi}{L}x\right)dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi}{L}x\right)dx\right]$$

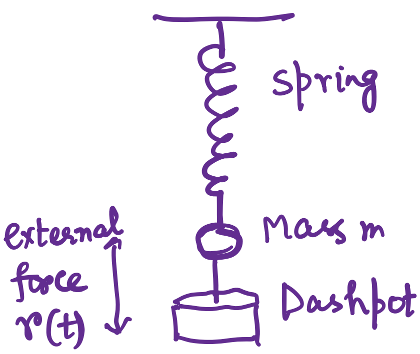
$$= \frac{8k}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \quad \text{(using integration by parts)}$$

Therefore

$$\tilde{f}_0(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2} \sin\left(\frac{\pi}{L}x\right) - \frac{1}{3^2} \sin\left(\frac{3\pi}{L}x\right) + \frac{1}{5^2} \sin\left(\frac{5\pi}{L}x\right) - \cdots \right)$$

Forced oscillations.

Example: undamped string :

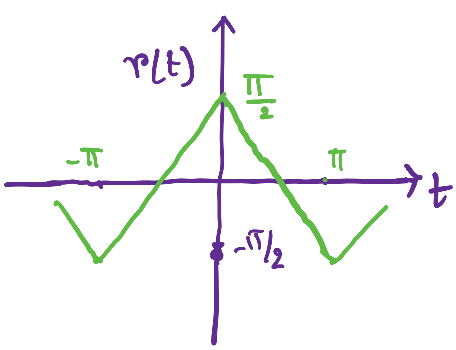


$$my'' + cy' + ky = r(t)$$

Given: $m = 1 \text{ (gm)}$

$c = .05 \text{ (gm/sec)}$

$k = 25 \text{ (gm/sec}^2\text{)}$



Then $y'' + .05y' + 25y = r(t)$

Given:

$$r(t) = t + \frac{\pi}{2} \quad \text{for } -\pi < t < 0$$
$$= -t + \frac{\pi}{2} \quad \text{for } 0 < t < \pi$$

We expand the driving force in its fourier series

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\cos(5t) + \cdots\cdots\right)$$

Consider the ODE: $y'' + .05y' + 25y = \frac{4}{\pi n^2}\cos(nt)$

The steady state solution is of the form $y_n = A_n\cos(nt) + B_n\sin(nt)$

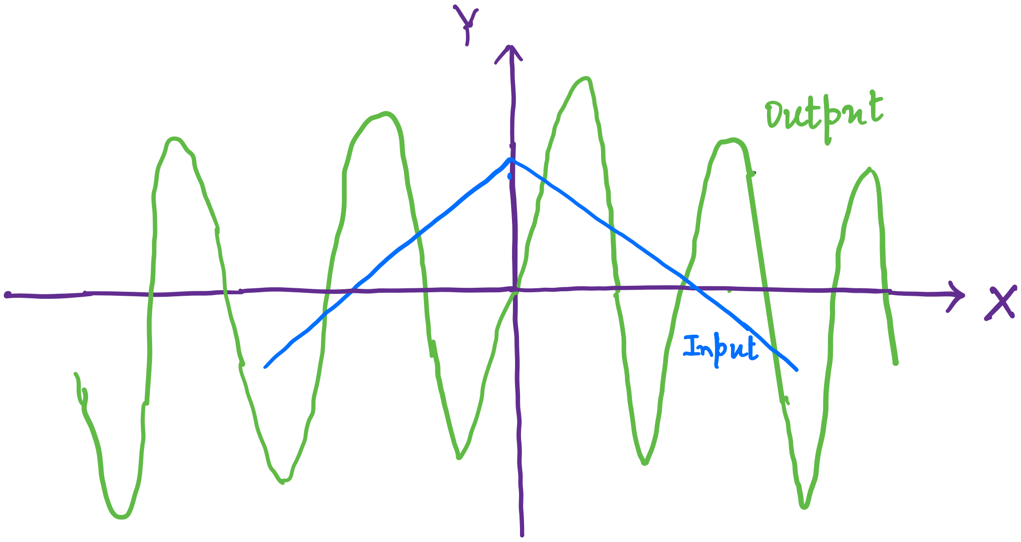
Substituting in the equation we get $A_n = \frac{4(25-n^2)}{n^2\pi D_n}$

and $B_n = \frac{0.2}{n\pi D_n}$ where $D_n = (25-n^2)^2 + (.05n)^2$

The solution is:

$$\boxed{y_n = \frac{4(25-n^2)}{n^2\pi D_n}\cos(nt) + \frac{.2}{n\pi D_n}\sin(nt)}$$

The steady state solution of the problem for the given $r(t)$ is: $\boxed{y = y_1 + y_3 + y_5 + \cdots\cdots}$



Approximation by Trigonometric Polynomials

Let $f(x)$ be a function on the interval $[-\pi, \pi]$ that can be represented on this interval by a Fourier series.

Then the Nth partial sum of the Fourier series

$$a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$

is an approximation of the given $f(x)$.

If we choose an arbitrary N and keep it fixed, then we can ask for the "best" approximation of f by a Trigonometric Polynomial of the same degree N that is by a function of the form:

$$F(x) = A_0 + \sum_{n=1}^{N} [A_n \cos(nx) + B_n \sin(nx)]$$

(N is fixed)

We choose the error of approximation as:

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx$$

This is called the square error of F relative to the function f on $[-\pi, \pi]$.

Clearly $E \geq 0$.

Now, $E = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} f F dx + \int_{-\pi}^{\pi} F^2 dx$

Now $\int_{-\pi}^{\pi} F^2 dx = \int_{-\pi}^{\pi} \left[A_0 + \sum_{n=1}^{N}\left[A_n \cos(nx) + B_n \sin(nx)\right]\right]^2 dx$

$= 2\pi A_0^2 + A_1^2 \pi + A_2^2 \pi + \cdots + A_N^2 \pi$
$+ B_1^2 \pi + B_2^2 \pi + \cdots + B_N^2 \pi + 0$

(By orthogonality relation)

$= \pi\left(2A_0^2 + A_1^2 + A_2^2 + \cdots + A_N^2 + B_1^2 + B_2^2 + \cdots + B_N^2\right)$

Now $\int_{-\pi}^{\pi} f F dx = \int_{-\pi}^{\pi} f\left[A_0 + \sum_{n=1}^{N}\left[A_n \cos(nx) + B_n \sin(nx)\right]\right] dx$

$= A_0 \int_{-\pi}^{\pi} f dx + \sum_{n=1}^{N}\left[A_n \int_{-\pi}^{\pi} f \cos(nx) dx + B_n \int_{-\pi}^{\pi} f \sin(nx) dx\right]$

$= \pi\left(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N\right)$

Therefore $E = \int_{-\pi}^{\pi} f^2 dx - 2\pi\left[2A_0 a_0 + \sum_{n=1}^{N}(A_n a_n + B_n b_n)\right]$

$+ \pi\left[2A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right]$

Now we choose $A_n = a_n$ and $B_n = b_n$ for $n = 0, 1, \ldots, N$

(i.e. we choose the nth partial sum of the Fourier series as our approximating polynomial)

We call the error for this particular Trigonometric polynomial as E^*.

Then $$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]$$

So, $$E - E^* = \pi\left[2(A_0 - a_0)^2 + \sum_{n=1}^{N}\left[(A_n - a_n)^2 + (B_n - b_n)^2\right]\right] \geq 0$$

Thus $E \geq E^*$

and $E = E^*$ if and only if $A_0 = a_0, \ldots, B_N = b_N$

Therefore we have the following result:

<u>Minimum Square Error:</u>

The square error of F (with fixed N) relative to f on $[-\pi, \pi]$ is minimum if and only if the coefficients of F are the Fourier coefficients of f.

The minimum error is given by

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]$$

Note:

- Since $E^* \geq 0$ we get

$$2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, dx$$

Since the above holds for every N, we obtain the Bessel Inequality:

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, dx$$

for the Fourier Coefficients of any function f for which the integral on the right exists.

- It can also be shown that for such a function f (for which $\int_{-\pi}^{\pi} f^2 dx$ exists), equality holds in the above inequality and it becomes Parseval's identity:

$$\boxed{2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, dx}$$

MTH 204 : Lecture 23

Sturm-Liouville Problems:

Consider a second order ODE of the form

$$[p(x)\,y']' + [q(x) + \lambda r(x)]\,y = 0 \quad \ldots\ldots\ (1)$$

on some interval $a \le x \le b$, satisfying conditions of the form

$$(2) \ \cdots \begin{cases} k_1 y + k_2 y' = 0 & \text{at } x = a \quad \ldots\ 2(a) \\ \ell_2 y + \ell_2 y' = 0 & \text{at } x = b \quad \ldots\ 2(b) \end{cases}$$

Here λ is a parameter and k_1, k_2, ℓ_1, ℓ_2 are given real constants. Furthermore, atleast one of each constant in each condition of (2) must be different from zero.

Equation (1) is called Sturm-Liouville equation.

Together with 2(a) and 2(b), it is known as Sturm-Liouville Problem.

It is an example of Boundary Value

Problem.

Note: A boundary value problem consists of an ODE and given boundary conditions referring to the two boundary points (end points) $x=a$ and $x=b$ of a given interval $[a,b]$

Eigen Values, Eigen Functions:

clearly $y \equiv 0$ is a solution of the problem (1) (Trivial solution) for any λ.

Would like to find eigen functions $y(x)$, that is solutions of (1) satisfying (2) without being identically zero.

We call a number λ for which an eigen function exists an eigen value of the Sturm-Liouville equations.

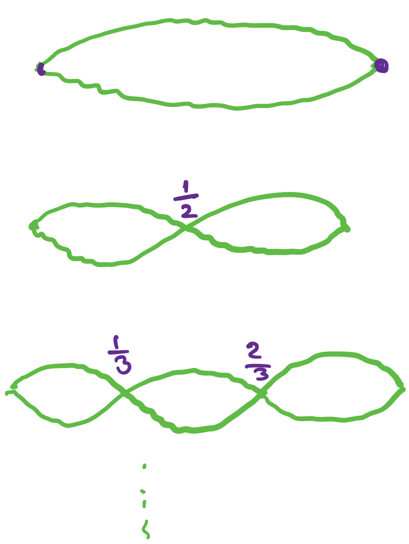
Note:

(1) Under rather general conditions on the

functions p, q, r in (1), the Sturm-Liouville problem (1), (2) has infinitely many eigenvalues.

(2) If p, q, r and p' in (1) are real-valued and continuous on $[a,b]$ and r is positive throughout the interval (or negative throughout the interval), then all eigenvalues of the Sturm-Liouville problem (1), (2) are real.

<u>Ex:</u> <u>Vibrating String:</u>



$$y'' + \lambda y = 0, \quad y(0)=0, \quad y(\pi)=0.$$

We can reformulate the problem as a Sturm-Liouville problem as:

$$(1y')' + (0 + \lambda 1)y = 0$$

$$1y(0) + 0y'(0) = 0$$

$$1y(\pi) + 0y'(\pi) = 0$$

(Here $r(x) \equiv 1$)

- If $\lambda = -\nu^2$ is negative, the general solution is $$y = c_1 e^{\nu x} + c_2 e^{-\nu x}$$

From the boundary conditions, we get $c_1 = c_2 = 0$

If $\lambda = 0$, the general solution is
$$y = (c_1 + c_2 x)$$
and again from the boundary conditions
$c_1 = c_2 = 0$

If $\lambda = \nu^2$ is positive, the general solution is
$$y = c_1 \cos(\nu x) + c_2 \sin(\nu x)$$
From the boundary conditions $c_1 = 0$
and $y(\pi) = c_2 \sin(\nu \pi) = 0$
$$\Rightarrow \quad \nu = \pm 1, \pm 2, \ldots$$

Therefore the functions
$$\boxed{y_\nu = \sin(\nu x) \quad , \quad \nu = \sqrt{\lambda} = 1, 2, 3, \ldots}$$

are eigen functions of the ODE and their associated eigenvalue is $\lambda = \nu^2$

Note that the solution of the problem is precisely the Trigonometric System of the Fourier series.

Orthogonal Functions

Orthogonality:

Let us define the inner product of two functions y_m and y_n with respect to the weight function $r(x)$ $(r(x) > 0)$ in the interval $[a, b]$ as :

$$\langle f, g \rangle_r = \int_a^b r(x) f(x) g(x)\, dx$$

The norm of a function is

$$\|f\|_r = \sqrt{\langle f, f \rangle_r}$$

Two functions f and g are called orthogonal if $\langle f, g \rangle_r = 0$

(with respect to the weight function w)

A set of functions $\{y_1, y_2, \ldots\}$ is called orthonormal if

$$\langle y_m, y_n \rangle_r = \delta_{mn} \begin{array}{l} = 0 \text{ if } m \neq n \\ = 1 \text{ if } m = n \end{array} \Bigg\}$$

Ex: (continued from previous example):

The set of functions $y_\nu = \sin(\nu x)$

where $\nu = \sqrt{\lambda} = 1, 2, \ldots, 3$

are orthogonal in the interval $[0, \pi]$.

$$\langle y_{\nu_1}, y_{\nu_2} \rangle = \int_0^\pi \sin(\nu_1 x) \sin(\nu_2 x)\, dx$$

$$= \frac{1}{2} \int_0^\pi \cos\left[(\nu_1 - \nu_2) x\right] dx - \frac{1}{2} \int_0^\pi \cos\left[(\nu_1 + \nu_2) x\right] dx$$

$$= \frac{1}{2} \frac{\sin\left[(\nu_1 - \nu_2) x\right]}{(\nu_1 - \nu_2)} \Bigg|_0^\pi - \frac{1}{2} \frac{\sin\left[(\nu_1 + \nu_2) x\right]}{(\nu_1 + \nu_2)} \Bigg|_0^\pi$$

$$= 0$$

but they are not orthonormal because

$$\| y_\nu \|^2 = \int_0^\pi \sin^2(\nu x)\, dx = \int_0^\pi \left[\frac{1 - \cos(2\nu x)}{2}\right] dx$$

$$= \frac{\pi}{2}$$

The set of functions $\tilde{y}_\nu = \sqrt{\frac{2}{\pi}} \sin(\nu x)$

$\nu = \sqrt{\lambda} = 1, 2, 3, \ldots$

is orthonormal.

Orthogonality of Eigen functions in case of Sturm-Liouville problems:

If p, q, r and p' are real valued and continuous in the interval $[a, b]$ and $r > 0$.

Let the function y_m and y_n be eigenfunctions associated to different eigenvalues λ_m and λ_n, then

$$\langle y_m, y_n \rangle_r = 0$$

(i.e. y_m and y_n are orthogonal on $[a, b]$ with respect to the weight function r)

- If $p(a)=0$, then 2(a) can be dropped from the problem.

  If $p(b)=0$, then 2(b) can be dropped from the problem.

  (It is then required that y and y' remain bounded at such a point and the problem is called <u>singular</u> as opposed to a regular problem in which (2) is used.)

- If $p(a)=p(b)$ then (2) can be replaced by the "periodic boundary conditions"

$$\left. y(a)=y(b)\ , \quad y'(a)=y'(b) \right\} \cdots\cdots (3)$$

- The boundary value problem consisting of the Sturm-Liouville equation (1) and the periodic boundary condition is called periodic-Sturm-Liouville problem.

Mixed Dirichlet-Neumann boundary Condition:

$$k_1 y(a) + k_2 y'(a) = \alpha$$
$$l_1 y(b) + l_2 y'(b) = \beta$$

If $\alpha = \beta = 0$, the boundary conditions are said to be homogeneous.

If $k_2 = l_2 = 0$, they are called Dirichlet boundary conditions.

If $k_1 = l_1 = 0$, they are called Neumann boundary conditions.

Singular Sturm-Liouville problem

A Sturm-Liouville problem

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$

is called singular in any one of the following cases.

- $p(a) = 0$, Boundary condition at a is dropped, Boundary condition at b is homogeneous mixed.

- $p(b)=0$, Boundary condition at b is dropped, Boundary condition at a is homogeneous ; mixed.

- $p(a)=p(b)=0$ and there is no Boundary condition.

- The interval $[a,b]$ is infinite.

Otherwise the problem is called regular.

Ex: <u>Legendre's Equation and Polynomials.</u>

Legendre Equation:

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

is a Sturm-Liouville problem

$$\left[(1-x^2)y'\right]'+n(n+1)y=0 \qquad [-1,1].$$

i.e. of the form $\left[p(x)y'\right]'+\left[q(x)+\lambda r(x)\right]y=0$

with $p = 1-x^2$, $q=0$, $r=1$

Note that

$p(-1) = p(1) = 0$ and so the Sturm-Liouville problem is singular and we donot need boundary conditions.

- The Legendre polynomial $P_n(x)$ is a non-trivial solution of the problem associated to the eigen value $\lambda = n(n+1)$
- By the previous theorem, Legendre polynomials are orthogonal in the interval $[-1, 1]$.

Generalized Fourier Series

Let the set $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ be orthogonal with respect to the weight function r in an interval $[a, b]$.

Let f be a function that we want to expand in this orthogonal basis

$$f = \sum_{m=0}^{\infty} a_m y_m(x)$$

To find the Fourier Coefficients a_m, we compute the inner product of f with y_n.

$$\langle f, y_n \rangle_r = \left\langle \sum_{m=0}^{\infty} a_m y_m(x), y_n \right\rangle_r$$

$$= \sum_{m=0}^{\infty} a_m \langle y_m(x), y_n(x) \rangle_r$$

$$= a_n \| y_n \|_r^2$$

So, $$a_n = \frac{\langle f, y_n \rangle_r}{\| y_n \|_r^2} = \frac{\int_a^b r f y_n \, dx}{\int_a^b r y_n^2 \, dx}$$

<u>Fourier-Legendre Series</u>

Legendre polynomials $P_m(x)$ are orthogonal in $[-1, 1]$ with respect to $r(x) = 1$.

In this interval we can perform an eigen function expansion of the form

$$f = \sum_{m=0}^{\infty} \frac{\langle f, P_m \rangle}{\| P_m \|^2} P_m(x)$$

It can be shown that

$$\| P_m \|^2 = \frac{2}{2m+1}$$

Ex: $f = \sin(\pi x)$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} \sin(\pi x) P_m(x)\, dx$$

Then $f = .95493 P_1 - 1.15824 P_3$
$+ .21929 P_5 - .01664 P_7$
$+ .00068 P_9 - .00002 P_{11} + \cdots$

- Similarly look at Fourier-Bessel series

Ex: Expand $f(x) = \frac{1}{1-x^2}$ in terms of Fourier-Bessel series.

Fourier Transform:

If f is a absolutely integrable (i.e. $\int_{-\infty}^{\infty} |f|dx < \infty$) on $\mathbb{R}$ and piecewise continuous on every finite interval then the Fourier Transform of f is defined by

$$\hat{f}(s) = \mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$$

(Notation)

. Now $$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(st) - i\sin(st)]\, f(t)\, dt$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(st) f(t)\, dt}_{\text{(Fourier Cosine Transform)}} - i \underbrace{\int_{-\infty}^{\infty} \sin(st) f(t)\, dt}_{\text{(Fourier Sine Transform)}}$$

Inverse Fourier Transform:

Suppose $\mathcal{F}(f)(s) = \hat{f}(s)$

Then $\hat{f}(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ist} f(t)\,dt$

Then $f(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \hat{f}(s)\, e^{ist}\, ds$

This is called the inverse Fourier Transform of f i.e. $\mathcal{F}^{-1}(\hat{f}) = f$

if $\mathcal{F}(f) = \hat{f}$

Existence of the Fourier Transform:

If $f(x)$ is absolutely integrable on $\mathbb{R}$ and piecewise continuous on every finite interval, then the Fourier Transform $\hat{f}$ of f exists.

Linearity of Fourier Transform:

The Fourier Transform is a linear operation: that is, for any functions f and g whose Fourier Transforms exist and constants a and b, the Fourier Transform of $af+bg$ exists and

$$\mathcal{F}(af+bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$$

Proof:

$$\mathcal{F}(af+bg)(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af(t)+bg(t))e^{-ist}\,dt$$

$$= a \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt + b \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)\,e^{-ist}\,dt$$

$$= a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$$

Ex: Let $f(x) = \left.\begin{array}{ll} 1 & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{array}\right\}$

Compute the Fourier Transform.

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist} f(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ist} \times 1\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-ist}}{(-is)}\Bigg|_{-1}^{1} = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{is} - e^{-is}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{(\cos s + i\sin s) - (\cos s - i\sin s)}{is}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2i\sin s}{is} = \boxed{\sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}}$$

<u>Ex:</u> Let $f(x) = e^{-ax}$ if $x > 0$ where $a>0$
$= 0$ otherwise

Compute the Fourier Transform:

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ist} e^{-at}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+is)t}\,dt = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+is)t}}{-(a+is)} \right|_{0}^{\infty}$$

$$= \boxed{\frac{1}{\sqrt{2\pi}\,(a+is)}}$$

Fourier Transform of the Derivative of $f(x)$:

Let f be continuous on $\mathbb{R}$ and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. Furthermore let $f'(x)$ be absolutely integrable on $\mathbb{R}$.

Then $\mathcal{F}(f')(s) = is\,\mathcal{F}(f)(s)$

<u>Proof:</u> $$\mathcal{F}(f')(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t)\, e^{-ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(t)\, e^{-ist} \Big|_{-\infty}^{\infty} - (-is) \int_{-\infty}^{\infty} f(t)\, e^{-ist}\,dt \right]$$

Since $f(t) \to 0$ as $|t| \to \infty$, we get

$$\boxed{\mathcal{F}(f')(s) = is\,\mathcal{F}(f)(s)}$$

- continuing in this way: $\mathcal{F}(f'')(s) = -s^2 \mathcal{F}(f)(s)$

$\vdots$

and $\boxed{\mathcal{F}(f^{(n)})(s) = (is)^n \mathcal{F}(f)(s)}$

Ex: Find the Fourier Transform of xe^{-x^2}.

$$\mathcal{F}(xe^{-x^2})(s) = \mathcal{F}\left(-\tfrac{1}{2}\left(e^{-x^2}\right)'\right)(s)$$

$$= -\tfrac{1}{2}\mathcal{F}\left(\left(e^{-x^2}\right)'\right)(s)$$

$$= -\tfrac{1}{2}\, is\, \mathcal{F}\left(e^{-x^2}\right)(s)$$

$$= -\frac{is}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist} e^{-t^2}\,dt$$

$$= \frac{-is}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(t^2 + 2\frac{is}{2}t + \left(\frac{is}{2}\right)^2\right)} e^{\left(\frac{is}{2}\right)^2}\,dt$$

$$= \frac{-is}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(t+\frac{is}{2}\right)^2} e^{-\frac{s^2}{4}}\,dt$$

$$= e^{-s^2/4} \frac{(-is)}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(t+\frac{is}{2}\right)^2} dt$$

$$= e^{-s^2/4} \quad \frac{-is}{2\sqrt{2\pi}} \times \sqrt{\pi}$$

$$= \boxed{\frac{-is}{2\sqrt{2}} \, e^{-\frac{s^2}{4}}}$$

- In the above we have used the following result: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

<u>Proof:</u> Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

Then $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$; then $r^2 = x^2 + y^2$

and $dx\,dy = r\,dr\,d\theta$

Then $I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta$

$$= \int_0^{2\pi} \left[-\frac{e^{-r^2}}{2} \right]_0^{\infty} d\theta = \frac{1}{2} \times 2\pi = \pi$$

So, $\boxed{I = \sqrt{\pi}}$

__Ex:__ Consider the wave Equation

$$u_{tt} = c^2 u_{xx}$$

($u = u(x,t)$ is a function of x and t)

Taking the Fourier Transform of the equation in variable x

$$\frac{\partial^2 [\mathcal{F}(u)(s,t)]}{\partial t^2} = c^2(-s^2)\,\mathcal{F}(u)(s,t)$$

$$\Rightarrow (\hat{u})_{tt} = -c^2 s^2 \hat{u}$$ ($\hat{u}$ is the Fourier Transform in variable x)

Let $\hat{u} = y$

Then we have $y'' + c^2 s^2 y = 0$

$$\Rightarrow \hat{u}(s,t) = y = A(s)\cos(cst) + B(s)\sin(cst)$$

Taking the inverse Fourier Transform,

we will get

$$u(x,t) = \mathcal{F}^{-1}(\hat{u})(x,t).$$

Fourier Transform of the integral

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\,d\tau\right\} = \frac{\mathcal{F}(f)}{i\omega} + c\,\delta(f)$$

where c is a value such that

$$\int_{-\infty}^{t}(f(\tau)-c)\,d\tau = 0$$

It is normally referred to as the DC or average value.

Fourier Transform of the Convolution:

Convolution: The convolution $f*g$ of functions f and g is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\,g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)\,g(p)\,dp$$

Convolution Theorem:

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded and absolutely integrable on $\mathbb{R}$.

Then $\quad \mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$

Proof:

$$\mathcal{F}(f*g)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(p)\, g(t-p)\, dp \right) e^{-ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(t-p)\, e^{-ist}\, dt \right) f(p)\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(q)\, e^{-is(p+q)}\, dq \right) f(p)\, dp$$

$$(\text{Let } t-p = q)$$

$$= \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} f(p)\, dp \right\} \left\{ \int_{-\infty}^{\infty} g(q)\, e^{-isq}\, dq \right\}$$

$$= \sqrt{2\pi} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} f(p)\, dp \right\} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isq} g(q)\, dq \right\}$$

$$= \sqrt{2\pi}\, \mathcal{F}(f)(s)\, \mathcal{F}(g)(s)$$

$$= \sqrt{2\pi} \left(\mathcal{F}(f)\, \mathcal{F}(g) \right)(s)$$

Hence $\boxed{\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g)}$

Calculation of the convolution:

Since $\mathcal{F}(f*g)(s) = \sqrt{2\pi}\, \mathcal{F}(f)(s)\, \mathcal{F}(g)(s)$, taking inverse Fourier Transform, we get

$$(f*g)(t) = \mathcal{F}^{-1}\left(\sqrt{2\pi}\, \hat{f}(s)\, \hat{g}(s) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\, \hat{f}(s)\, \hat{g}(s)\, e^{ist}\, dt$$

So, $\boxed{(f*g)(t) = \int_{-\infty}^{\infty} \hat{f}(s)\, \hat{g}(s)\, e^{ist}\, dt}$ (This formula is useful in solving PDE)

MTH 204: Lecture 24

Partial Differential Equation (PDE):

A PDE is an equation involving one or more partial derivatives of an (unknown) function, call it u, that depends on two or more variables.

Often time is one variable and one or several variables with space (spatial variables) are involved.

- The order of the highest derivative is called the order of the PDE.
- A PDE is linear if each term in the PDE is of first degree in the unknown function or its partial derivatives. Otherwise we call it nonlinear. Ex: The equation $uu_x + u_y = 0$ is nonlinear.
- We call a linear PDE homogeneous if each of its terms contains either the function or one of its partial derivatives (i.e. if there is no term which doesn't include the function or its derivatives)

Otherwise it is called non homogeneous.

Examples of PDE :

(1) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$: 1-dimensional wave equation

: It can model the vibrating string (and the vibrating membrane)

(2) $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$: 1-dimensional heat equation

: It can model temperature in a bar or wire.

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$: 2-dimensional Laplace Equation

: can model Electrostatic Potential.

(4) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$: 2-dimensional Poisson Equation

(5) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$: 2-dimensional Wave Equation

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$: 3-dimensional Laplace Equation

- <u>Note:</u> Here c is a positive constant. t is time, x, y, z are cartesian coordinates and dimension is the number of these coordinates in the equation.

- All the above equations are linear. The equation (4) (with f not identically zero) is nonhomogeneous. The other equations are homogeneous.

- A solution of a PDE in some region R of the space of independent variables is a function that has all the partial derivatives (appearing in the PDE) in some domain D containing R and satisfies the PDE everywhere in R.

- In general the set of solutions of a PDE can be very large and one needs some constraints (boundary conditions or initial conditions) to restrict the solution to have physical meaning.

For example, the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

is satisfied by
$$u = x^2 - y^2$$
$$u = e^x \cos(y)$$
$$u = \sin(x) \cosh(y)$$
$$u = \log(x^2 + y^2)$$

- <u>Fundamental Theorem on Superposition:</u>

If u_1 and u_2 are solutions of a homogeneous linear PDE in some region R, then
$$u = c_1 u_1 + c_2 u_2$$
with any constants c_1 and c_2 is also a solution of that PDE in the region R.

<u>Ex:</u> Find solutions depending on x and y of
$$u_{xx} - u = 0$$

- Since no y-derivatives occur, the above is like $u'' - u = 0$

The general solution is $u = Ae^{x} + Be^{-x}$

Here A and B may be functions of y and so, $\boxed{u = A(y)e^{x} + B(y)e^{-x}}$

Ex: Find solutions depending on x and y of $\quad u_{xy} = -u_x$

- Setting $v = u_x$ we get

$$v_y = -v \Rightarrow \frac{dv}{v} = -dy$$

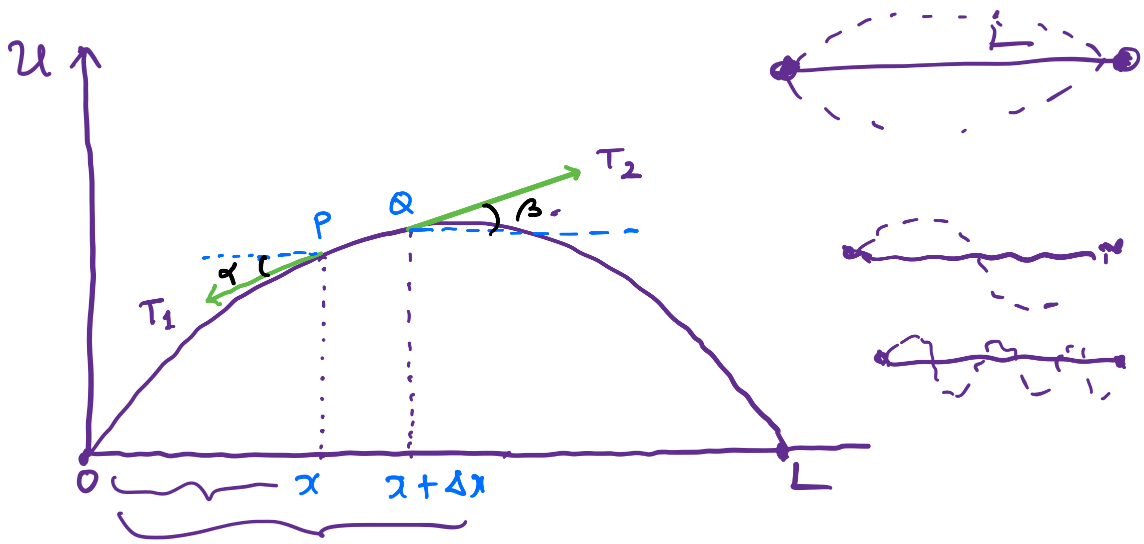
$$\Rightarrow \ln|v| = -y + c_1(x) \Rightarrow v = c_2(x)e^{-y}$$

Integrating with respect to x,

$$u = \int c_2(x)e^{-y}dx = f(x)e^{-y} + g(y)$$

$$\Rightarrow u = f(x)e^{-y} + g(y) \quad \text{where } f(x) = \int c_2(x)\,dx$$

Vibrating String : Wave Equation



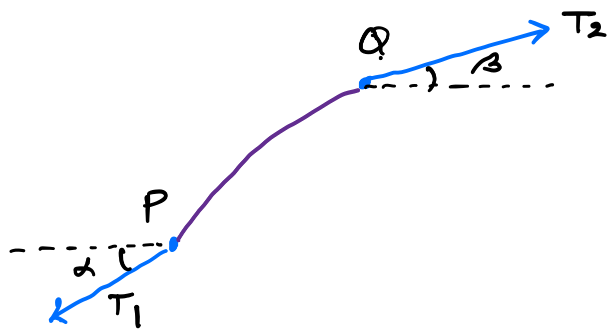
Physical assumptions:

- The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.

- The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.

- The string performs small transverse

motions in a vertical plane, that is, every particle of the string moves strictly vertically and so the deflection and the slope at every point of the string always remain small in absolute value.



Since the string offers no resistance to bending, the tension is tangential to the curve at each point.

Let T_1 and T_2 be tension at the points P and Q.

Since the points move vertically (and not horizontally), the horizontal tension must cancel at every point.

Horizontally $\quad T_2 \cos\beta - T_1 \cos\alpha = 0$

$$\Rightarrow T_2 \cos\beta = T_1 \cos\alpha = T \text{ (constant)}$$

- Vertically the difference of the forces translates into an acceleration.

So $\quad T_2 \sin\beta - T_1 \sin\alpha = (\rho \Delta x)\, u_{tt}$

where ρ = mass density of the string

Δx = distance between $P = x$ and $Q = x + \Delta x$

Therefore
$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{(\rho \Delta x)}{T} u_{tt}$$

$$\Rightarrow \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{(\rho \Delta x)}{T} u_{tt}$$

$$\Rightarrow \frac{\tan\beta - \tan\alpha}{\Delta x} = \frac{\rho}{T} u_{tt}$$

$$\Rightarrow \frac{\left.\left(\frac{\partial u}{\partial x}\right)\right|_{x+\Delta x} - \left.\left(\frac{\partial u}{\partial x}\right)\right|_{x}}{\Delta x} = \frac{\rho}{T} u_{tt}$$

Taking limit as $\Delta x \longrightarrow 0$

we have
$$u_{xx} = \frac{\rho}{T} u_{tt}$$

$$\Rightarrow u_{tt} = \frac{T}{\rho} u_{xx}$$

$$\Rightarrow \boxed{u_{tt} = c^2 u_{xx}}$$

This is the 1D wave equation and c is the propagation speed.

- Thus the model of the vibrating string consists of the 1D wave equation

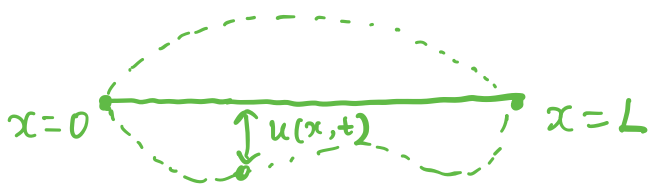
$$u_{tt} = c^2 u_{xx}$$

with some boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for all } t \geq 0$$

and some initial conditions on the initial shape and velocity of the string

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } 0 \leq x \leq L$$



The solution has three steps.

(1) Separating variables

(2) Satisfying the boundary conditions

(3) Satisfying the initial conditions. (using Fourier series)

Step1 : Let us look for a solution of the form:

$$u(x,t) = F(x)G(t)$$

Then $u_{tt} = FG_{tt}$, $u_{xx} = GF_{xx}$

So, the PDE becomes $FG_{tt} = c^2 GF_{xx}$

$$\Rightarrow \quad \frac{1}{c^2}\frac{G_{tt}}{G} = \frac{F_{xx}}{F}$$

The left hand side depends only on t, while the right hand side depends only on x.

Hence $\dfrac{1}{c^2}\dfrac{G_{tt}}{G} = \dfrac{F_{xx}}{F} = k$ (constant)

$$\Rightarrow \quad \left.\begin{aligned} F_{xx} - kF = 0 \\ G_{tt} - c^2 kG = 0 \end{aligned}\right\}$$

Step2:

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0$$
$$u(L,t) = F(L)G(t) = 0$$

Now $G(t)$ can't be zero because it will give a solution $u = 0$ which is of no interest.

Hence $F(0) = F(L) = 0$

Now consider the ODE for F :

$$F_{xx} - kF = 0$$

- If $k=0$, then the general solution is

$$F = ax + b$$

and two boundary conditions will make $a=0$, $b=0$ which is again of no interest.

- If $k = \mu^2 > 0$, then the general solution is :

$$F = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

The two boundary conditions will make $c_1 = 0$, $c_2 = 0$ which is of no interest.

- If $k = -\mu^2 < 0$ then the general solution is

$$F = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Using the two boundary conditions

$$0 = c_1 \times 1 + c_2 \times 0 \Rightarrow c_1 = 0$$

and $\quad 0 = c_2 \sin(\mu L)$

$$\Rightarrow \quad \sin(\mu L) = 0 \Rightarrow \mu L = n\pi$$

$$\Rightarrow \mu = \frac{n\pi}{L}$$

Thus there are infinitely many solutions of the form $F(x) = F_n(x) = \sin\left(\frac{n\pi}{L}x\right)$

We now solve $G_{tt} - c^2 k G = 0$

for $k = -\left(\frac{n\pi}{L}\right)^2$

Define $\lambda_n = c\mu = \frac{c\pi}{L}n$

Then $G_{tt} + \lambda_n^2 G = 0$

The general solution is

$$G(t) = a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)$$

Hence the solutions of the wave equation satisfying the boundary conditions can be written as:

$$u_n(x,t) = \left[a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)\right] \sin\left(\frac{n\pi}{L}x\right)$$

for $n = 1, 2, \ldots$

These functions are called eigen functions and $\lambda_n = \frac{c\pi}{L} n$ are called

eigen values of the vibrating string.
The set $\{\lambda_1, \lambda_2, \dots\}$ is called the spectrum.

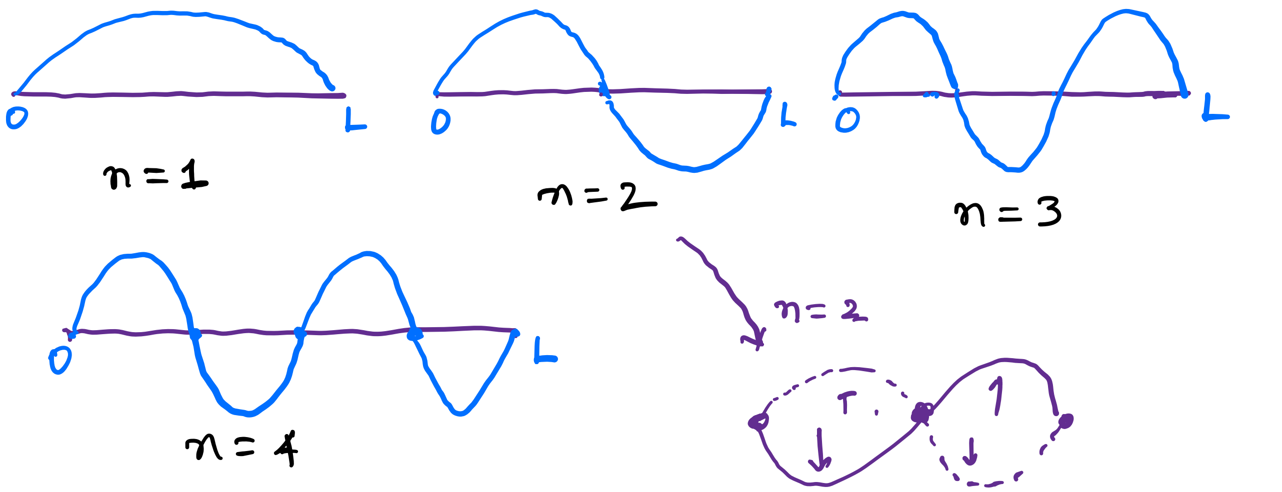
So, $u_n(x,t) = \left[a_n \cos\left(\frac{c\pi n}{L} t\right) + b_n \sin\left(\frac{c\pi n}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right)$

for $n = 1, 2, \dots$

- Each u_n represents a harmonic motion having the frequency $\frac{\lambda_n}{2\pi} = \frac{c\pi n}{2\pi} = \frac{cn}{2L}$ (cycles/unit time)

This motion is called nth normal mode of the string.
The first normal mode is called the fundamental mode $(n=1)$ and the others are called overtones.
Now $\sin\left(\frac{n\pi x}{L}\right) = 0$ at $x = \frac{L}{n}, \frac{2L}{n}, \dots, \frac{n-1}{n} L$,

the nth normal mode has $(n-1)$ nodes, i.e. the points of the string that do not move (in addition to the fixed end points)



Note that $c = \sqrt{\frac{T}{\rho}}$ so that tuning an instrument amounts to changing T and ultimately c. The other two variables to control are ρ and L.

Step 3:

The general solution of the vibrating string is:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)\right)$$

For the initial shape condition we have

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

If we expand $f(x)$ in its fourier series assuming we make an odd extension of it and make it of period $2L$, then f can be expressed as

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(\nu) \sin\left(\frac{n\pi}{L}\nu\right) d\nu \right) \sin\left(\frac{n\pi}{L}x\right)$$

Comparing, $\boxed{a_n = \frac{2}{L} \int_0^L f(\nu) \sin\left(\frac{n\pi}{L}\nu\right) d\nu}$

Now for initial speed, we take the derivative of the general solution:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left(-a_n \lambda_n \sin(\lambda_n t) + b_n \lambda_n \cos(\lambda_n t)\right)$$

Now $u_t(x,0) = \sum_{n=1}^{\infty} b_n \lambda_n \sin\left(\frac{n\pi}{L}x\right) = g(x)$

If we expand $g(x)$ in its fourier series assuming we make an odd extension of it and make it of period $2L$, then g can be expressed as

$$g(x) = \sum_{n=1}^{\infty} \left\{ \frac{2}{L} \int_0^L g(\nu) \sin\left(\frac{n\pi}{L}\nu\right) d\nu \right\} \sin\left(\frac{n\pi}{L}x\right)$$

Comparing, $\boxed{b_n = \frac{2}{\lambda_n L} \int_0^L g(\nu) \sin\left(\frac{n\pi}{L}\nu\right) d\nu}$

Finally the particular solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)\right)$$

where $$a_n = \frac{2}{L}\int_0^L f(v)\sin\left(\frac{n\pi}{L}v\right)dv$$

$$b_n = \frac{2}{\lambda_n L}\int_0^L g(v)\sin\left(\frac{n\pi}{L}v\right)dv$$

$$\lambda_n = c\mu = \frac{c\pi}{L}n$$

- We may reformulate the solution as:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{c\pi}{L}nt\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{c\pi}{L}nt\right)$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{2}\left[\sin\left(\frac{n\pi}{L}(x-ct)\right) + \sin\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{2}\left[\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\sin\left(\frac{n\pi}{2}(x-ct)\right)+\frac{b_n}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right)\right]$$

$$+ \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\sin\left(\frac{n\pi}{2}(x+ct)\right)-\frac{b_n}{2}\left(\cos\frac{n\pi}{L}(x+ct)\right)\right]$$

- If we consider the case where the initial velocity $g(x)\equiv 0$, then $b_n=0 \ \forall n$

and so the solution $u(x,t)$ reduces to

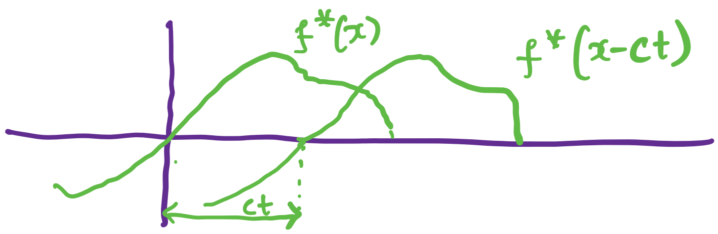
$$u(x,t)=\sum_{n=1}^{\infty}\frac{a_n}{2}\sin\left(\frac{n\pi}{L}(x-ct)\right)$$

$$+\sum_{n=1}^{\infty}\frac{a_n}{2}\sin\left(\frac{n\pi}{L}(x+ct)\right)$$

Thus $u(x,t)=\frac{1}{2}\left[f^*(x-ct)+f^*(x+ct)\right]$

$$\left(f^*(\xi)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi}{L}\xi\right)\right)$$

So, u is sum of two travelling waves.



Ex:

$$f(x) = \begin{cases} \dfrac{2k}{L} x & \text{for } 0 < x < \dfrac{L}{2} \\ = \dfrac{2k}{L}(L-x) & \text{for } \dfrac{L}{2} < x < L \end{cases}$$

and $g(x) = 0$

Therefore $b_n = 0$ for all n.

$f(x)$ Can be expanded (Half Range Expansion)

Then

$$u(x,t) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{c\pi}{L}t\right) - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right)\cos\left(\frac{3c\pi}{L}t\right) + \cdots\right)$$

(• Please see the graphs in the book)

$$\boxed{\text{MTH 204: Lecture 25}}$$

D'Alembert's Solution of the Wave Equation:

The wave equation is $u_{tt} = c^2 u_{xx}$ $\quad \left(c^2 = \frac{T}{\rho}\right)$

Let us introduce the variables

$$v = x + ct \quad , \quad w = x - ct$$

Then $u_x = u_v v_x + u_w w_x = u_v(1) + u_w(1)$

$$\Rightarrow \quad u_x = u_v + u_w$$

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x$$
$$= (u_{vv} + u_{wv}) v_x + (u_{vw} + u_{ww}) w_x$$
$$= u_{vv} + 2u_{wv} + u_{ww}$$

Now $\quad u_t = u_v v_t + u_w w_t = u_v(c) + u_w(-c)$

$$\Rightarrow \quad u_t = c(u_v - u_w)$$

$$u_{tt} = c(u_v - u_w)_t = c\left[(u_v - u_w)_v v_t + (u_v - u_w)_w w_t\right]$$
$$= c\left[(u_{vv} - u_{wv})c + (u_{vw} - u_{ww})(-c)\right]$$
$$= c^2\left[u_{vv} - 2u_{wv} + u_{ww}\right]$$

$u_{tt} = c^2 u_{xx}$

Hence the PDE becomes

$$c^2\left[u_{vv} - 2u_{wv} + u_{ww}\right] = c^2\left[u_{vv} + 2u_{wv} + u_{ww}\right]$$

$$\Rightarrow -u_{wv} = u_{wv} \Rightarrow u_{wv} = 0$$

Integrating with respect to v we get

$$u_w = h(w)$$

Integrating with respect to w we get

$$u = \int h(w)\,dw = \psi(w) + \phi(v)$$

In terms of x and t we have

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

where ϕ and ψ are two (possibly different) travelling waves.

This is known as d'Alembert's solution of the wave equation.

Initial Conditions:

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

We now impose the initial conditions

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x)$$

Now $u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$

$$u(x,0) = \phi(x) + \psi(x) = f(x)$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$$

Then $\displaystyle\int_{x_0}^{x} \phi'(s)\,ds - \int_{x_0}^{x} \psi'(s)\,ds = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$

$$\Rightarrow \quad \phi(x) - \phi(x_0) - \psi(x) + \psi(x_0) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

$$\Rightarrow \quad \phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

Adding this to the equation $\phi(x) + \psi(x) = f(x)$ we get,

$$2\phi(x) = \left[\phi(x_0) - \psi(x_0)\right] + f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

$$\Rightarrow 2\phi(x) = k(x_0) + f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

where $k(x_0) = \phi(x_0) - \psi(x_0)$

Therefore $\displaystyle\phi(x) = \frac{1}{2}\left[k(x_0) + f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds\right]$

Similarly subtracting we get

$$2\psi(x) = -k(x_0) + f(x) - \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

$$\Rightarrow \quad \psi(x) = \frac{1}{2}\left[-k(x_0) + f(x) - \frac{1}{c}\int_{x_0}^{x} g(s)\,ds\right]$$

Now the solution of the wave equation is:

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}\left[k(x_0) + f(x+ct) + \frac{1}{c}\int_{x_0}^{x+ct} g(s)\,ds\right]$$

$$+ \frac{1}{2}\left[-k(x_0) + f(x-ct) - \frac{1}{c}\int_{x_0}^{x-ct} g(s)\,ds\right]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

If the initial velocity is zero, this reduces to

$$u(x,t) = \boxed{\frac{1}{2}\left[f(x+ct) + f(x-ct)\right]}$$

Method of characteristics:

The idea of d'Alembert's solution is a special case of method of characteristics that deals with the problem:

$$A u_{xx} + 2B u_{xy} + C u_{yy} = F(x, y, u, u_x, u_y)$$

The Equation is quasilinear (i.e. linear in highest derivatives)
Depending on the discriminant $AC-B^2$, we can classify:

Type	Defining Condition	Example
Hyperbolic	$AC-B^2<0$	Wave Equation
Parabolic	$AC-B^2=0$	Heat Equation
Elliptic	$AC-B^2>0$	Laplace Equation

A, B, C may be functions of x and y.
So, the problem may be of mixed type
that is different type in different regions

Ex: Consider the 1D wave equation:

$$u_{tt} = c^2 u_{xx}$$

Making a change of variable $y = ct$,
We have $u_{tt} = c^2 u_{yy}$
So, the wave equation becomes $c^2 u_{yy} = c^2 u_{xx}$

$$\Rightarrow \boxed{u_{xx} - u_{yy} = 0} \quad (A=1,\ B=0,\ C=-1)$$

Now, $AC - B^2 = (1)(-1) - 0^2 = -1 < 0$
equation is hyperbolic.

Ex: Consider the 1D heat equation

$$u_t = c^2 u_{xx}$$

Now making the change of variable $y = c^2 t$,

we get $u_t = c^2 u_y$

Then the equation becomes $u_{xx} = u_y$ $(A=1, B=0, C=0)$

Then $AC - B^2 = 1(0) - 0^2 = 0$

So, the equation is parabolic.

Transformation to Normal form:

The characteristic Equation of the PDE is the ODE $Ay'^2 - 2By' + C = 0$

where $y' = \frac{dy}{dx}$

- The normal forms of the PDE and the corresponding transformations depend on the type of PDE.

They are obtained by solving the characteristic equation.

The solutions of the characteristic equation are called the characteristics of the PDE and are written in the form $\psi(x,y) = \text{Constant}$, $\Phi(x,y) = \text{Constant}$

Then the transformations giving new variables v, w instead of x, y and the normal forms of the PDE can be given:

Type	New Variables	Normal Form
Hyperbolic	$v = \Phi$, $w = \psi$	$u_{vw} = F_1$
Parabolic	$v = x$ $\quad w = \Phi = \psi$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \psi)$, $w = \frac{1}{2i}(\Phi - \psi)$	$u_{vv} + u_{ww} = F_3$

D'Alembert solution

$$u_{tt} - c^2 u_{xx} = 0$$

With the change of variable $y = ct$, the above PDE is transformed into

$$u_{xx} - u_{yy} = 0 \qquad (A=1,\ C=-1,\ B=0)$$

The characteristic equation is

$$(y')^2 - 1 = 0$$

$$\Rightarrow (y'+1)(y'-1) = 0$$

$$\Rightarrow \quad y' + 1 = 0 \Rightarrow y' = -1 \Rightarrow y = -x + C_1$$

$$\Rightarrow x + y = C_1$$

$$\Rightarrow \Phi(x,y) = x + y = C_1$$

and $y' - 1 = 0 \Rightarrow y' = 1 \Rightarrow y = x + C_2'$

$$\Rightarrow \quad x - y = C_2 \quad (C_2 = -C_2')$$

$$\Rightarrow \Psi(x,y) = x - y = C_2$$

Since the equation is hyperbolic, the change of variable is

$$v = \Phi(x,y) = x + y = x + ct$$

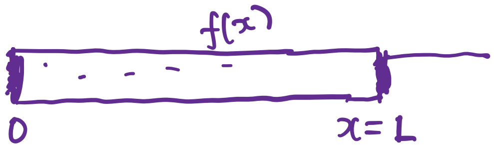
$$w = \Psi(x,y) = x - y = x - ct$$

and the associated normal form is:

$$u_{vw} = 0$$

MTH 204 : Lecture 26

1D Heat Equation:



(Bar under consideration)

The PDE is $u_t = c^2 u_{xx}$

with the boundary conditions:

$u(0,t) = 0$, $u(L,t) = 0$ for all $t \geq 0$

(i.e. the two ends of the bar are kept at temperature 0)

with the initial condition $u(x,0) = f(x)$ (given)

(By ② $f(0) = 0$, $f(L) = 0$)

The solution has three steps: (a) Separating variables
(b) satisfying the boundary conditions
(c) satisfying the initial condition

(a) Let us try a solution of the form

$$u(x,t) = F(x)\,G(t)$$

Then $u_t = F G_t$, $u_x = F_x G$, $u_{xx} = F_{xx} G$

Substituting we get $F G_t = c^2 F_{xx} G$

$$\Rightarrow \quad \frac{G_t}{c^2 G} = \frac{F_{xx}}{F}$$

The left hand side depends only on t and the right hand side depends only on x. So, it must be

$$\frac{G_t}{c^2 G} = \frac{F_{xx}}{F} = \text{constant } k$$

- We can show that for $k=0$ or $k>0$, the only solution $u = FG$ satisfying the boundary conditions is $u=0$

- Therefore $\dfrac{G_t}{c^2 G} = \dfrac{F_{xx}}{F} = -p^2$

This gives us the two equation

$$\left.\begin{aligned} F_{xx} + p^2 F &= 0 \\ G_t + c^2 p^2 G &= 0 \end{aligned}\right\}$$

Now $F_{xx} + p^2 F = 0 \Rightarrow F = A_1 \cos(px) + B_1 \sin(px)$

Now $G_t + c^2p^2G \Rightarrow \dfrac{G_t}{G} = -c^2p^2$

$\Rightarrow \ln|G| = -c^2p^2t + c_1 \Rightarrow G = c_2 e^{-c^2p^2t}$

Hence $\boxed{u(x,t) = [A\cos(px) + B\sin(px)]\, e^{-c^2p^2t}}$

(b) Boundary conditions:

$u(0,t) = 0 \Rightarrow A e^{-c^2p^2t} = 0 \Rightarrow A = 0$

$u(L,t) = 0 \Rightarrow B(\sin pL)\, e^{-c^2p^2t} = 0$

$\Rightarrow B\sin(pL) = 0 \Rightarrow pL = n\pi$

Let us define $\lambda_n = c\,\dfrac{n\pi}{L}$

Then the eigen functions of the problem

$$\boxed{u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}}$$

and $\lambda_n = c\,\dfrac{n\pi}{L}$ is the corresponding eigen values.

The solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \boxed{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}}$$

Now the initial condition will give

$$u(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

So, B_n must be the coefficients of the sine Fourier series

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

for $n = 1, 2, \ldots$

Because of the exponential factor, all the terms in the solution approaches zero as t approaches infinity. The rate of decay increases with n.

Example (Sinusoidal Initial temperature):

Find the temperature $u(x,t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100\sin(\pi x/80)$ °C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C?

Physical data for Copper: density $8.92\ g/cm^3$

Specific heat $0.092\ cal/(g\ ^\circ c)$

Thermal conductivity $0.95\ cal/(cm\ sec\ ^\circ c)$

Now from the problem $f(x) = 100 \sin\left(\frac{\pi}{80} x\right)$

$\Rightarrow B_1 = 100$, $B_n = 0$ for $n = 2, 3, \ldots$

Now $\lambda_1^2 = \frac{c^2 \pi^2}{L^2}$

$$c^2 = \frac{K}{\sigma \rho} = \frac{.95}{.092 \times 8.92} = 1.158 \left(\frac{cm^2}{s}\right)$$

$$\lambda_1^2 = c^2 \frac{\pi^2}{L^2} = 1.158 \left(\frac{cm^2}{s}\right) \cdot \frac{\pi^2}{(80)^2 (cm^2)}$$

$$= 1.785 \times 10^{-3} (s^{-1})$$

Thus we get the solution as:

$$u(x,t) = 100 \sin\left(\frac{\pi}{80} x\right) e^{-1.785 \times 10^{-3} t}$$

To calculate the time for the maximum temperature to drop to $50^\circ c$,

$$100\, e^{-1.785\times 10^{-3} t} = 50$$

$$\Rightarrow \quad t = \frac{\log(0.5)}{-1.785\times 10^{-3}} = 388 \text{ second}$$

= 6 minute 28 seconds

$\simeq$ 6.5 minutes.

Ex: Solve the same problem with initial temperature $f(x) = 100 \sin\left(\frac{3\pi}{80}x\right)$ $^\circ c$ and the other data as before.

- Instead of $n=1$, we now have $n=3$

$$B_3 = 100, \quad B_n = 0 \text{ for } n = 1, 2, 4, 5, \dots$$

$$\lambda_3^2 = 3^2 \lambda_1^2 = 9 \times 1.785\times 10^{-3} = 1.607\times 10^{-2}$$

So, the solution is

$$u(x,t) = 100 \sin\left(\frac{3\pi x}{80}\right) e^{-1.607\times 10^{-2} t}$$

For the maximum temperature to drop to $50^\circ c$,

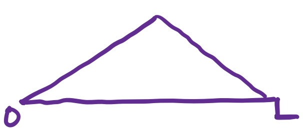
$$100\, e^{-1.607\times 10^{-2} t} = 50$$

$$\Rightarrow \quad t = \frac{\log(.5)}{-1.607\times 10^{-2}} = 43 \text{ seconds.}$$

(Approximately 9 times as fast as the previous example)

Ex: Triangular Initial temperature in a Bar:

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is



$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{L}{2} \\ L-x & \text{if } \frac{L}{2} \le x \le L \end{cases}$$

- Here we will get

$$B_n = \frac{2}{L}\left(\int_0^{L/2} x \sin\frac{n\pi}{L}x\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx\right)$$

Then $B_n = 0$ if n is even

and $B_n = \dfrac{4L}{n^2\pi^2}$ (for $n = 1, 5, 9, \ldots$)

and $B_n = -\dfrac{4L}{n^2\pi^2}$ (for $n = 3, 7, 11, \ldots$)

Hence, the solution is:

$$u(x,t)=\frac{4L}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)e^{-\left(\frac{c\pi}{L}\right)^2 t}-\frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)e^{-\left(\frac{3c\pi}{L}\right)^2 t}+\cdots\cdots\right]$$

The temperature decreasing with increasing t because of heat loss due to the cooling of the ends.

Ex: Bar with insulated ends:

Find a solution of the problem where the boundary condition is replaced by the condition that both ends of the bar are insulated.

- The rate of heat flow is proportional to the gradient of the temperature.

Thus if the ends $x=0$ and $x=L$ of the bar are insulated so the no heat can flow through the ends, we have

$$u_x(0,t)=0\,,\quad u_x(L,t)=0 \quad \text{for all } t\geq 0$$

The equation and the initial conditions

remain the same. $\left.\begin{array}{l} u_t = c^2 u_{xx} \\ u(x,0) = f(x) \end{array}\right\}$

Since the equation has not changed, the solution is still of the form

$$u(x,t) = \left[A\cos(px) + B\sin(px)\right] e^{-c^2p^2t}$$

Now $u_x(x,t) = F_x(x)\, G(t)$

$$= \left[-Ap\sin(px) + Bp\cos(px)\right] e^{-c^2p^2t}$$

Now $u_x(0,t) = 0 \Rightarrow Bp = 0$

If $p = 0$, the number of solutions is limited.

So, we choose $B = 0$

Now $u_x(L,t) = 0 \Rightarrow -Ap\sin(pL) = 0$

$$\Rightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L}$$

Let $p_n = \frac{n\pi}{L}$

Then we have the eigen functions

$$u_n(x,t) = A_n\cos(p_n x)\, e^{-c^2p_n^2t} = A_n\cos\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t}$$

where $\lambda_n = \dfrac{cn\pi}{L}$

Note that now $n = 0, 1, 2, \cdots$
instead of $n = 1, 2, \cdots$.
i.e. we can have the solution $u_0 = A_0$
(Thus 0 can be an eigen value)

$$u_n(x,t) = A_n \cos(p_n x)\, e^{-c^2 p_n^2 t}$$

$$= A_n \cos\left(\frac{n\pi}{L} x\right) e^{-\lambda_n^2 t}$$

where $\lambda_n = \dfrac{cn\pi}{L}$

The particular solution must be of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) e^{-\lambda_n^2 t}$$

Now the initial condition gives:

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right)$$

So, the coefficients A_n are the coefficients of the Fourier cosine series of $f(x)$.

$$A_0 = \frac{1}{L} \int_0^L f(x)\,dx \quad , \quad A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

Steady two dimensional Heat Problems:

Consider the two dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

For steady state (i.e. time independent)

$$\frac{\partial u}{\partial t} = 0$$

and so heat equation becomes

$$\nabla^2 u = 0$$

This equation will be considered in some region R of the xy-plane and a boundary condition on the boundary curve C of R is given.

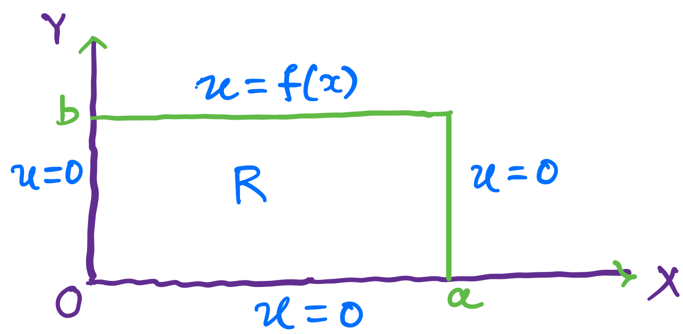
Thus this is a BVP (Boundary value problem). This will will be called:

(1) Dirichet Problem : If u is prescribed on C

(2) Neumann Problem : If normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C

(3) Robin Problem : If u is prescribed on some portion of C and u_n on the rest of C (Mixed Boundary Condition)

Dirichlet Problem in a Rectangle R



Let us solve the problem by separation of variables:

$$\text{Let } u(x,y) = F(x)G(y)$$

Then $\nabla^2 u = 0 \Rightarrow F_{xx}G + FG_{yy} = 0$

$$\Rightarrow \quad \frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = -K$$

$$\Rightarrow \quad F_{xx} + KF = 0 \;\Rightarrow\; F = A\cos(\sqrt{K}\,x) + B\sin(\sqrt{K}\,x)$$

Now $F(0) = 0 \Rightarrow A(1) = 0 \Rightarrow A = 0$

$F(a) = 0 \Rightarrow B\sin(\sqrt{K}\,a) = 0 \Rightarrow \sqrt{K}\,a = n\pi$

$$\Rightarrow K = \left(\frac{n\pi}{a}\right)^2$$

The non zero solutions are $F_n(x) = \sin\left(\frac{n\pi}{a}x\right)$

for $n = 1, 2, \ldots.$

Now $\quad G_{yy} - KG = 0$

$$\Rightarrow \quad G_n = A_n e^{\sqrt{K}\,y} + B_n e^{-\sqrt{K}\,y} \quad \text{where } K = \left(\frac{n\pi}{a}\right)^2$$

$$\Rightarrow \quad G_n = A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}$$

Now the boundary conditions

$$G_n(0) = 0 \Rightarrow A_n + B_n = 0 \Rightarrow B_n = -A_n$$

Then $\quad G_n = A_n e^{\frac{n\pi}{a}y} - A_n e^{-\frac{n\pi}{a}y}$

$$= 2A_n \sinh\left(\frac{n\pi}{a}y\right)$$

Therefore the eigen functions are

$$u_n(x,y) = F_n(x)G_n(y) = A_n^* \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

The solution is $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1} A_n^* \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$

Now $\quad u(x,b) = f(x)$

$$\Rightarrow \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right) = f(x)$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty}\left[A_n^* \sinh\left(\frac{n\pi}{a}b\right)\right]\sin\left(\frac{n\pi}{a}x\right)$$

Hence $A_n^* \sinh\left(\frac{n\pi}{a}b\right)$ must be the Fourier coefficients b_n of $f(x)$

So, $b_n = A_n^* \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$

Therefore the solution of the problem is:

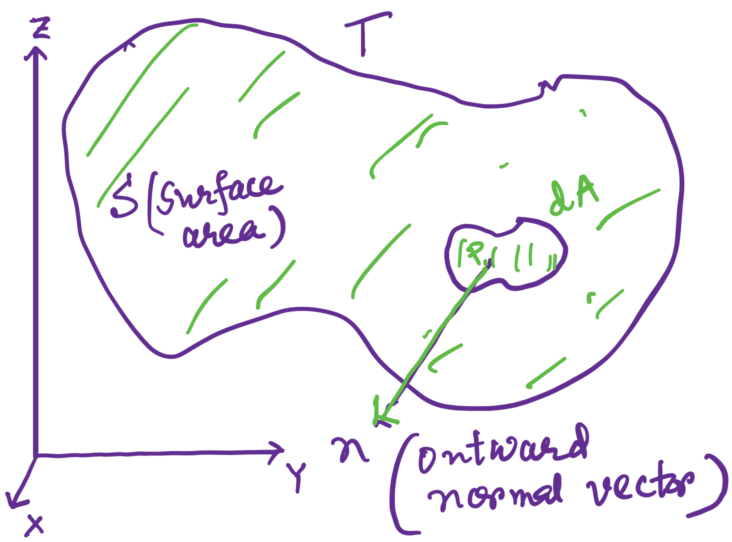
$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

where $$A_n^* = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

Addendum to Lecture 26

Heat Equation:

- Specific heat σ, density ρ of the material of the body are constant.
- Heat flows in the direction of decreasing temperature.



Let $u(x,y,z,t)$ is the temperature at a point (x,y,z) and time t.

Note that

$$\text{grad } u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

- Then velocity of heat flow in the body:

$$v = -K \text{ grad } u$$ where K = Constant = Thermal conductivity of the material

- $v \cdot n$ is the component of v in the direction of n.
- The amount of heat leaving T or entering T per unit time at some point P of S through a

a small portion ΔS of area ΔA is $|v.n \Delta A|$

Hence total amount of heat that flows across S

from T $= \iint_S v.n\, dA = -k\iint_S (\text{grad } u).n\, dA$

$$= -k \iiint_T \text{div}(\text{grad } u)\, dx\, dy\, dz \quad \left(\begin{array}{c}\text{By Gauss's} \\ \text{divergence} \\ \text{Theorem}\end{array}\right)$$

$$= -k \iiint_T \nabla^2 u\, dx\, dy\, dz \quad \ldots\ldots\ldots (1)$$

$$\left(\text{Here} \quad \text{div}(\text{grad } u) = \text{div}\left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla^2 u\right)$$

On the other hand, the total amount of heat in T : $H = \iiint_T \sigma \rho\, u(x,y,z,t)\, dx\, dy\, dz$

Now $$-\frac{\partial H}{\partial t} = \iiint_T \sigma \rho \frac{\partial u}{\partial t}\, dx\, dy\, dz \quad \ldots\ldots (2)$$

(time rate of decrease of H)

From (1) and (2)

$$K \nabla^2 u = \sigma \rho \frac{\partial u}{\partial t}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{K}{\sigma \rho} \nabla^2 u$$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = c^2 \nabla^2 u} \quad \text{where } c^2 = \frac{K}{\sigma \rho}$$

<u>This is the Heat Equation.</u>

<u>1D Heat Equation:</u> $\quad \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

<u>2D Heat Equation:</u> $\quad \dfrac{\partial u}{\partial t} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$

MTH 204 : Lecture 27

1 dimensional Heat Equation:

$$u_t = c^2 u_{xx} .$$

- Let us assume that the bar is very long (like a wire) and it goes from $-\infty$ to ∞.
- We don't have boundary conditions but only the initial condition

$$u(x,0) = f(x) \quad (-\infty < x < \infty)$$

We use separation of variables

$$u(x,t) = F(x)\,G(t)$$

Then $\quad FG_t = c^2 F_{xx}\, G$

$$\Rightarrow \frac{G_t}{c^2 G} = \frac{F_{xx}}{F} = -p^2$$

Then $\quad F_{xx} + p^2 F = 0 \Rightarrow F = A\cos(px) + B\sin(px)$

and $\quad G_t + c^2p^2 G = 0 \Rightarrow G = e^{-c^2p^2t}$

Hence a solution is:

$$u(x,t) = \left[A\cos(px) + B\sin(px)\right] e^{-c^2p^2t}$$

Here we chose the separation constant to be negative $-p^2$ because positive values of the constant would lead to an increasing exponential function in the solution which has no physical meaning.

Since $f(x)$ is not assumed to be periodic, it is natural to use Fourier integrals. The constants A, B are also function of p.

Thus the eigen functions are

$$u_p(x,t) = \left[A(p)\cos(px) + B(p)\sin(px)\right] e^{-c^2p^2t}$$

and since the heat equation is linear and homogeneous, the function

$$u(x,t) = \int_0^\infty u_p(x,t)\,dp$$

$$= \int_0^\infty \left[A(p)\cos(px) + B(p)\sin(px)\right] e^{-c^2p^2t}\,dp$$

is a solution of the heat equation provided this integral exists and can be differentiated twice with respect to x and once with respect to t.

The initial condition $u(x,0) = f(x)$ will imply

$$f(x) = \int_0^\infty \left[A(p)\cos(px) + B(p)\sin(px)\right] dp$$

This is the Fourier Integral and $A(p)$ and $B(p)$ are given by

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)\, dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)\, dv$$

Now $u(x,0) = \int_0^\infty \left[A(p)\cos(px) + B(p)\sin(px)\right] dp$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\, dv\right] dp$$

Similarly $u(x,t) = \int_0^\infty \left[A(p)\cos(px) + B(p)\sin(px)\right] e^{-c^2p^2t}\, dp$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\, dv\right] e^{-c^2p^2t}\, dp$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\, e^{-c^2p^2t}\, dv\right] dp$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} \cos(px - pv)\, e^{-c^2p^2t}\, dp\right] dv$$

To evaluate the inner integral, let us substitute

$$s = cp\sqrt{t} \Rightarrow p = \frac{s}{c\sqrt{t}} \Rightarrow dp = \frac{ds}{c\sqrt{t}}$$

Then $\int_0^{\infty} \cos(px - pv)\, e^{-c^2p^2t}\, dp$

$$= \int_0^{\infty} \cos\left(\frac{s}{c\sqrt{t}}(x - v)\right) e^{-s^2} \frac{ds}{c\sqrt{t}}$$

$$= \frac{1}{c\sqrt{t}} \int_0^{\infty} \cos(2bs)\, e^{-s^2}\, ds \quad \left(\text{where } b = \frac{1}{2}\,\frac{x - v}{c\sqrt{t}}\right)$$

$$= \frac{1}{c\sqrt{t}}\,\frac{\sqrt{\pi}}{2}\, e^{-b^2}$$

$$= \frac{\sqrt{\pi}}{2c\sqrt{t}}\, e^{-\frac{(x-v)^2}{4c^2t}}$$

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} \cos(px - pv)\, e^{-c^2p^2t}\, dp\right] dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\frac{\sqrt{\pi}}{2c\sqrt{t}}\, e^{-\frac{(x-v)^2}{4c^2t}}\right] dv$$

$$= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v)\, e^{-\frac{(x-v)^2}{4c^2 t}}\, dv$$

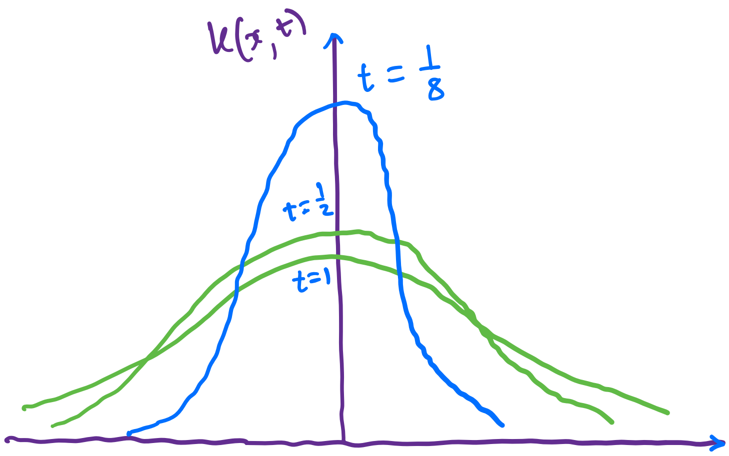
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(x + 2cz\sqrt{t}\right) e^{-z^2}\, dz$$

where $z = \dfrac{v-x}{2c\sqrt{t}}$

<u>Ex:</u> Find the temperature in the infinite bar if

$$f(x) = \left.\begin{matrix} T_0 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{matrix}\right\}$$

Then $u(x,t) = \dfrac{T_0}{2c\sqrt{\pi t}} \displaystyle\int_{-1}^{1} e^{-\frac{(x-v)^2}{4c^2 t}}\, dv$



Example with Fourier Transform :

Let us solve the same problem using Fourier Transform.

$$u_t = c^2 u_{xx}$$

Taking Fourier Transform w.r.t. x ,

$$\mathcal{F}_x(u_t) = c^2 \mathcal{F}_x(u_{xx})$$

Considering u as only a function of x (not of (x,t))

$$\mathcal{F}_x(u_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t \, e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} u \, e^{-i\omega x} dx \right\}$$

$$= \frac{\partial}{\partial t} \mathcal{F}_x(u) = \frac{\partial \hat{u}(\omega, t)}{\partial t} = \hat{u}_t$$

Then the PDE becomes

$$u_t = c^2 u_{xx} \Rightarrow \hat{u}_t = -c^2 \omega^2 \hat{u}$$

$$\int \frac{d\hat{u}}{\hat{u}} = -\int c^2 \omega^2 \, dt$$

$$\log \hat{u} = -c^2 \omega^2 t^2 + C(\omega)$$

$$\Rightarrow \quad \hat{u}(\omega, t) = C(\omega)\, e^{-c^2 \omega^2 t}$$

$u(x,0) = f(x)$
$\hat{u}(\omega,0) = \hat{f}(\omega)$

The initial condition makes

$$\hat{u}(\omega, 0) = \mathcal{F}_x\{f(x)\} = \hat{f}(\omega) = C(\omega)$$

So, by taking inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-c^2 \omega^2 t}\, e^{+i\omega x}\, d\omega$$

<u>Example with convolution:</u>

As in the previous example

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-c^2 \omega^2 t}\, e^{i\omega x}\, d\omega$$

$$\Rightarrow \quad u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\omega) \left(\frac{1}{\sqrt{2\pi}} e^{-c^2 \omega^2 t} \right) e^{i\omega x}\, d\omega$$

Then $u(x,t) = (f*g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}(\omega)\, e^{i\omega x}\, d\omega$

where $\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}$

Hence

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp \quad \left(\text{By definition of Convolution}\right)$$

Now we use the fact that

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

Let $c^2 t = \frac{1}{4a}$.

Then $\mathcal{F}\left(e^{-\frac{x^2}{4c^2t}}\right) = \sqrt{2c^2t}\, e^{-c^2\omega^2 t}$

$$= \sqrt{2c^2t}\, \sqrt{2\pi}\, \hat{g}(\omega)$$

Hence $g(x) = \mathcal{F}^{-1}(\hat{g}(\omega)) = \frac{1}{\sqrt{2c^2t}\,\sqrt{2\pi}} e^{-\frac{x^2}{4c^2t}}$

So, $g(x-p) = \frac{1}{2c\sqrt{\pi t}} e^{-\frac{(x-p)^2}{4c^2t}}$

Therefore $u(x,t) = (f*g)(x) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p)\, e^{-\frac{(x-p)^2}{4c^2t}}\, dp$