DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
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MA 102 Mathematics II
Problem Sheet 4: Method of undetermined coefficients, annihilator approach (operator method), variation of parameters.

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1. Solve the given differential equations by the method of undetermined coefficients.
   (a) $y'' - 8y' + 20y = 100x^2 - 26xe^x$.
   (b) $y'' + 4y = (x^2 - 3)\sin 2x$.
   (c) $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$.
   (d) $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$.
   (e) $y^{(4)} - y'' = 4x + 2xe^{-x}$.

2. Solve the given initial value problems.
   (a) $y'' + 4y' + 4y = (3 + x)e^{-2x},\ y(0) = 2,\ y'(0) = 5$.
   (b) $\dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t,\ x(0) = 0,\ x'(0) = 0$.
   (c) $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x},\ y(0) = \dfrac{1}{2},\ y'(0) = \dfrac{5}{2},\ y''(0) = -\dfrac{9}{2}$.

3. Solve the given boundary value problems.
   (a) $y'' + y = x^2 + 1,\ y(0) = 5,\ y(1) = 0$.
   (b) $y'' - 2y' + 2y = 2x - 2,\ y(0) = 0,\ y(\pi) = \pi$.

4. Solve the following initial value problem in which the input function $g(x)$ is discontinuous. [Hint: Solve each problem on two intervals, and then find a solution such that y and y' are continuous at $x = \dfrac{\pi}{2}$.]
   $y'' + 4y = g(x),\ y(0) = 1,\ y'(0) = 2$, where
$$g(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 0, & x > \frac{\pi}{2} \end{cases}$$

5. Consider the differential equation $ay'' + by' + cy = e^{kx}$, where a, b, c and k are constants. The auxiliary equation of the associated homogeneous equation is $am^2 + bm + c = 0$.
   (a) If k is not a root of the auxiliary equation, show that we can find a particular solution of the form $y_p = Ae^{kx}$, where $A = \dfrac{1}{ak^2 + bk + c}$.
   (b) If k is a root of the auxiliary equation of multiplicity one, show that we can find a particular solution of the form $y_p = Axe^{kx}$, where $A = \dfrac{1}{2ak + b}$. Explain how we know that $k \neq -\dfrac{b}{2a}$.

(c) If k is a root of the auxiliary equation of multiplicity two, show that we can find a particular solution of the form $y_p = Ax^2e^{kx}$, where $A = \dfrac{1}{2a}$.

6. Solve the given differential equations by the method of undetermined coefficients (annihilator approach).

   (a) $y'' - 2y' + y = x^3 + 4x$.

   (b) $y'' + 6y' + 8y = 3e^{-2x} + 2x$.

   (c) $y'' + y' + \dfrac{1}{4}y = e^x(\sin 3x - \cos 3x)$.

   (d) $y''' - y'' + y' - y = xe^x - e^{-x} + 7$.

   (e) $y^{(4)} - 2y''' + y'' = e^x + 1$.

7. Solve the following initial value problems by the method of undetermined coefficients (annihilator approach).

   (a) $y'' - 5y' = x - 2,\ y(0) = 0,\ y'(0) = 2$.

   (b) $y'' + 5y' - 6y = 10e^{2x},\ y(0) = 1,\ y'(0) = 1$.

   (c) $y'' + y' = 8\cos 2x - 4\sin x,\ y(\frac{\pi}{2}) = -1,\ y'(\frac{\pi}{2}) = 0$.

   (d) $y^{(4)} - y''' = x + e^x,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0$.

8. Suppose that L is a linear differential operator that factors but has variable coefficients. Do the factors of L commute? Defend your answer.

9. Solve each of the given differential equations by the method of variation of parameters.

   (a) $y'' + y = \sec\theta\tan\theta$.

   (b) $y'' - 4y = \dfrac{e^{2x}}{x}$.

   (c) $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$.

   (d) $y'' + 3y' + 2y = \sin e^x$.

   (e) $4y'' - 4y' + y = e^{\frac{x}{2}}\sqrt{1 - x^2}$.

10. Solve each of the given differential equations by the method of variation of parameters, subject to initial conditions $y(0) = 1,\ y'(0) = 0$.

    (a) $2y'' + y' - y = x + 1$.

    (b) $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$.

11. Discuss why the interval of definition of the general solution $y = c_1\cos(\ln x) + c_2\sin(\ln x)$ of the differential equation $x^2y'' + xy' + y = \sec(\ln x)$ is not $(0, \infty)$.

12. Solve the following initial value problems which are in Cauchy-Euler form.

    (a) $x^2y'' - 5xy' + 8y = 0,\ y(2) = 32,\ y'(2) = 0$.

(b) $xy'' + y' = x, \ y(1) = 1, \ y'(1) = -\frac{1}{2}$.

13. Solve the following problems by the substitution $x = e^t$ to transform the given Cauchy-Euler equations to differential equations with constant coefficients.

    (a) $x^2y'' - 4xy' + 6y = \ln x^2$.

    (b) $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$.

14. How would you solve the equation $(x+2)^2y'' + (x+2)y' + y = 0$? Carry out your ideas. State an interval over which the solution is defined.

15. Can a Cauchy-Euler equation of lowest order with real coefficients can be found if it is known that 2 and $1-i$ are the roots of the auxiliary equations?

16. The initial conditions $y(0) = y_0$ and $y'(0) = y_1$ apply to each of the following differential equations:

$$x^2y'' = 0, \ x^2y'' - 2xy' + 2y = 0 \text{ and } x^2y'' - 4xy' + 6y = 0.$$

    For what values of y_0 and y_1 does each initial value problem have a solution?

17. Suppose that $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x} dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x) dx$$

    with $P(x) = p_0 + p_1x + p_2x^2 + ... + p_kx^k$, where $p_k \neq 0$.

    (a) Show that $y = e^{\alpha x}u$, where

$$u' + \alpha u = P(x). \tag{1}$$

    (b) Show that (1) has a particular solution of the form

$$u_p = A_0 + A_1x + A_2x^2 + ... + A_kx^k,$$

    where A_k, A_{k-1},....,A_0 can be computed successively by equating coefficients of x^k, x^{k-1}, ..., 1 on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

    (c) Conclude that

$$\int e^{\alpha x} P(x) dx = \left(A_0 + A_1x + A_2x^2 + ... + A_kx^k\right) e^{\alpha x} + c,$$

    where c is a constant of integration.

18. Use the method in the above exercise to evaluate
    (a) $\int e^{-x}(-1 + x^2)dx$ and (b) $\int e^{3x}(-14 + 30x + 27x^2)dx$.

19. Solve each of the given differential equations by systematic elimination.

    (a) $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$, $\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$.
    (b) $Dx + D^2y = e^{3t}$, $(D+1)x + (D-1)y = 4e^{3t}$.
    (c) $Dx + z = e^t$, $(D-1)x + Dy + Dz = 0$, $x + 2y + Dz = e^t$.
    (d) $\dfrac{dx}{dt} = -x + z$, $\dfrac{dy}{dt} = -y + z$, $\dfrac{dz}{dt} = -x + y$.

20. Discuss the system $Dx - 2Dy = t^2$, $(D+1)x - 2(D+1)y = 1$.

21. Solve each of the given non-linear differential equations by reduction of order.

    (a) $y'' + (y')^2 + 1 = 0$.
    (b) $x^2y'' + (y')^2 = 0$.
    (c) $y'' + 2y(y')^3 = 0$.
    (d) $y^2y'' = y'$.

22. Find a a solution of the initial value problem

$$y'' + yy' = 0,\ y(0) = 1,\ y'(0) = -1.$$

    Find an interval of validity for the solution of the problem.

23. Find two solutions of the initial-value problem

$$(y'')^2 + (y')^2 = 1,\ y\left(\frac{\pi}{2}\right) = \frac{1}{2},\ y'\left(\frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}.$$

24. In calculus, the curvature of a curve that is defined by $y = f(x)$ is defined as

$$\kappa = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}}.$$

    Find a function $y = f(x)$ for which $\kappa = 1$. [**Hint:** For simplicity, ignore constants of integration.]

25. In the following problems, verify that y_1 and y_2 are solutions of the given differential equation but that $y = c_1y_1 + c_2y_2$ is, in general, not a solution.

$$\text{(a) } (y'')^2 = y^2;\ y_1 = e^x,\ y_2 = \cos x, \quad \text{(b) } yy'' = \frac{1}{2}(y')^2;\ y_1 = 1,\ y_2 = x^2.$$

26. It is easy to verify that $\sin x$, $\cos x$, e^x and e^{-x} are four solutions of the nonlinear equation $(y'')^2 - y^2 = 0$. Without attempting to solve the differential equation, discuss how these explicit solutions can be found by using knowledge about linear equations. Without attempting to verify, discuss why the two special linear combinations $y = c_1e^x + c_2e^{-x}$ and $y = c_3\cos x + c_4\sin x$ must satisfy the differential equation.

27. Discuss how the method of reduction of order can be applied to the third order differential equation $y''' = \sqrt{1 + (y'')^2}$. Carry out your ideas and solve the equation.