Table of Integrals

1 $\int u\,dv = uv - \int v\,du$

2 $\int u^n du = \frac{1}{n+1}u^{n+i} + C$

3 $\int \frac{du}{u} = \ln|u| + C$

4 $\int e^u du = e^u + C$

5 $\int a^u du = \frac{1}{\ln(a)}a^u + C$

6 $\int \operatorname{sen}(u)\,du = -\cos(u) + C$

7 $\int \cos(u)\,du = \operatorname{sen}(u) + C$

8 $\int \sec^2(u)\,du = \operatorname{tg}(u) + C$

9 $\int \operatorname{cossec}^2(u)\,du = -\operatorname{cotg}(u) + C$

10 $\int \sec(u)\,\operatorname{tg}(u)\,du = \sec(u) + C$

11 $\int \frac{\operatorname{cotg}(u)}{\operatorname{sen}(u)}\,du = -\frac{1}{\operatorname{sen}(u)} + C$

12 $\int \operatorname{tg}(u)\,du = \ln|\sec(u)| + C$

13 $\int \operatorname{cotg}(u)\,du = \ln|\operatorname{sen}(u)| + C$

14 $\int \sec(u)\,du = \ln|\sec(u) + \operatorname{tg}(u)| + C$

15 $\int \frac{du}{\operatorname{sen}(u)} = \ln\left|\frac{1}{\operatorname{sen}(u)} - \frac{\cos(u)}{\operatorname{sen}(u)}\right| + C$

16 $\int \frac{du}{\sqrt{a^2-u^2}} = \operatorname{arc\,sen}(\frac{u}{a}) + C$

17 $\int \frac{du}{a^2+u^2} = \frac{1}{a}\operatorname{arc\,tg}(\frac{u}{a}) + C$

18 $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arc\,sec}(\frac{u}{a}) + C$

19 $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20 $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

21 $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

22 $\int u^2\sqrt{a^2+u^2}\,du = \frac{(a^2u+2u^3)\sqrt{a^2+u^2}}{8} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

23 $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

24 $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

25 $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

26 $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$

27 $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

28 $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

29 $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

30 $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\operatorname{arc\,sen}(\frac{u}{a}) + C$

31 $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\operatorname{arc\,sen}(\frac{u}{a}) + C$

32 $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

33 $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \operatorname{arc\,sen}(\frac{u}{a}) + C$

34 $\int \frac{u^2du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\operatorname{arc\,sen}(\frac{u}{a}) + C$

35 $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2-u^2}+a}{u}\right| + C$

36 $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

37 $\int (a^2+u^2)^{3/2}du = -\frac{(2u^3-5a^2u)\sqrt{a^2-u^2}}{8} + \frac{3a^4}{8}\operatorname{arc\,sen}(\frac{u}{a}) + C$

38 $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

39 $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

40 $\int u^2\sqrt{u^2-a^2}\,du = -\frac{(2u^3-a^2u)\sqrt{u^2-a^2}}{8} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

41 $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\operatorname{arc\,cos}(\frac{a}{|u|}) + C$

42 $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

43 $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

44 $\int \frac{u^2du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

45 $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

46 $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

47 $\int \frac{udu}{a+bu} = \frac{1}{b^2}\left(a+bu - a\ln|a+bu|\right) + C$

48 $\int \frac{u^2du}{a+bu} = \frac{\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right]}{2b^3} + C$

49 $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

50 $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

51 $\int \frac{udu}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

52 $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

53 $\int \frac{u^2du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

54 $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

55 $\int \frac{udu}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

56 $\int \frac{u^2du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

57 $\int \frac{du}{u\sqrt{a+bu}}\,du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + c, \text{ se } a>0$

58 $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int\frac{du}{u\sqrt{a+bu}}$

59 $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int\frac{du}{u\sqrt{a+bu}}$

60 $\int u^n\sqrt{a+bu}\,du = \frac{2\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]}{b(2n+3)}$

61 $\int \frac{u^n du}{\sqrt{a+du}} = \frac{2u^n\sqrt{a+bu}}{b(2n-1)} - \frac{2na}{b(2n+1)}\int\frac{u^{n-1}du}{\sqrt{a+bu}}$

62 $\int \frac{u^{-n}du}{\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int\frac{u^{-n+1}du}{\sqrt{a+bu}}$

63 $\int \operatorname{sen}^2(u)du = \frac{1}{2}u - \frac{1}{4}\operatorname{sen}(2u) + C$

64 $\int \cos^2(u)du = \frac{1}{2}u + \frac{1}{4}\operatorname{sen}(2u) + C$

65 $\int \operatorname{tg}^2(u)du = \operatorname{tg}(u) - u + C$

66 $\int \operatorname{cotg}^2(u)du = -\operatorname{cotg}(u) - u + C$

67 $\int \operatorname{sen}^3(u)du = -\frac{\left[2+\operatorname{sen}^2(u)\right]\cos(u)}{3} + C$

68 $\int \cos^3 u\,du = \frac{\left[2+\cos^2(u)\right]\operatorname{sen}(u)}{3} + C$

69 $\int \operatorname{tg}^3(u)du = \frac{\operatorname{tg}^2(u)}{2} + \ln|\cos(u)| + C$

70 $\int \operatorname{cotg}^3(u)du = -\frac{\operatorname{cotg}^2(u)}{2} - \ln|\operatorname{sen}(u)| + C$

71 $\int \sec^3(u)du = -\frac{\sec(u)\operatorname{tg}(u)}{2} - \frac{\ln|\operatorname{sen}(u)+\operatorname{tg}(u)|}{2} + C$

72 $\int \frac{du}{\operatorname{sen}^3(u)} = -\frac{\operatorname{cotg}(u)}{2\operatorname{sen}(u)} + \frac{\ln|\operatorname{cossec}(u)-\operatorname{cotg}(u)|}{2} + C$

73 $\int \operatorname{sen}^n(u)du = -\frac{\operatorname{sen}^{n-1}(u)\cos(u)}{n} + \frac{n-1}{n}\int\operatorname{sen}^{n-2}(u)du$

74 $\int \cos^n(u)du = \frac{\cos^{n-1}(u)\operatorname{sen}(u)}{n} + \frac{n-1}{n}\int\cos^{n-2}(u)du$

75 $\int \operatorname{tg}^n(u)du = \frac{\operatorname{tg}^{n-1}(u)}{n-1} - \int\operatorname{tg}^{n-2}(u)du$

76 $\int \operatorname{cotg}^n(u)\,du = -\frac{\operatorname{cotg}^{n-1}(u)}{n-1} - \int\operatorname{cotg}^{n-2}(u)\,du$

77 $\int \sec^n(u)\,du = \frac{\operatorname{tg}(u)\sec^{n-2}(u)}{n-1} + \frac{n-2}{n-1}\int\sec^{n-2}(u)\,du$

78 $\int \frac{du}{\operatorname{sen}^n(u)} = -\frac{\operatorname{cotg}(u)}{(n-1)\operatorname{sen}^{n-2}(u)} + \frac{n-2}{n-1}\int\frac{du}{\operatorname{sen}^{n-2}(u)}$

79 $\int \operatorname{sen}(au)\operatorname{sen}(bu)du = \frac{\operatorname{sen}(a-b)u}{2(a-b)} - \frac{\operatorname{sen}(a+b)u}{2(a+b)} + C$

80 $\int \cos(au)\cos(bu)du = \frac{\operatorname{sen}(a-b)u}{2(a-b)} + \frac{\operatorname{sen}(a+b)u}{2(a+b)} + C$

MTH 204 : Lecture 01

Modeling:

$$\begin{array}{l}
\boxed{\text{Physical System}} \\
\quad\downarrow \\
\boxed{\text{Mathematical Model}} \longrightarrow \text{ODEs/PDEs} \\
\quad\downarrow \\
\boxed{\text{Mathematical solution}} \\
\quad\downarrow \\
\boxed{\text{Physical Interpretation}}
\end{array}$$

A mathematical model is one or more equations that helps us to understand a physical process.

- A first order ODE is an equation of the form: $\boxed{F(y', y, x) = 0}$

Any function involving y', y and x

$x \rightarrow$ independent variable (time, distance ....)

$y \rightarrow$ dependent variable $y(x)$

$y' \longrightarrow$ First derivative of y i.e. $y' = \dfrac{dy}{dx}$

(ordinary derivative)

Example: $y' = \cos x$ (1st order)

2nd Order ODE: $F(y'', y', y, x) = 0$

3rd Order ODE: $G(y''', y'', y', y, x) = 0$

Examples: $y'' + 9y = e^{-2x}$ (2nd order)

$y'y''' - \frac{3}{2}(y')^2 = 0$ (3rd Order)

Order: Highest derivative of dependent variable present in the equation determines the order.

ODE: It is an equation involving

- One independent variable (x)
- One dependent variable (y)
- Derivatives of dependent variable (of different orders) w.r.t. the independent variable (ie. $y', y'', y''' \ldots$)
- Must contain a derivative term.

Ex: $y' = \cos x$

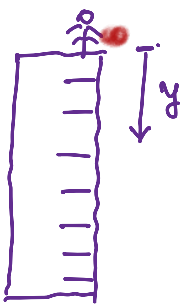
$\Leftrightarrow \quad y = \sin x + C \rightarrow$ Not a Differential Equation.

$y'' = -\sin x.$

Examples:

$$m y'' = mg$$

(1)



(Falling Stone)

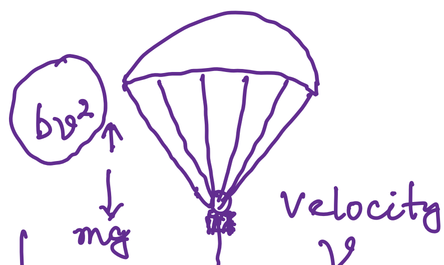
Independent variable: t

Dependent variable $= y$

$$\frac{d^2 y}{dt^2} = y'' = g = \text{constant}$$

$$\left(y = \frac{1}{2} g t^2 \right)$$

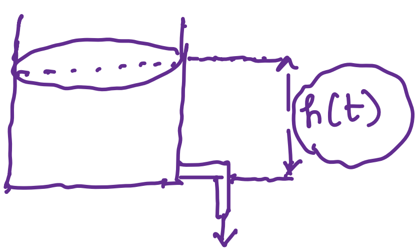
(2)



(Parachutist)

$$\boxed{m\frac{dv}{dt} = mv = mg - bv^2 \qquad b \to \text{"drag" constant}}$$

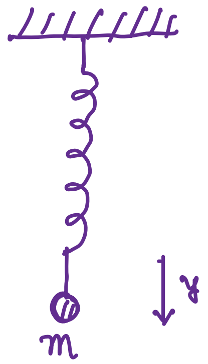
(3)



$$\underbrace{\text{Out flowing water}}$$

$$\boxed{\frac{dh}{dt} = h' = -k\sqrt{h}}$$

④



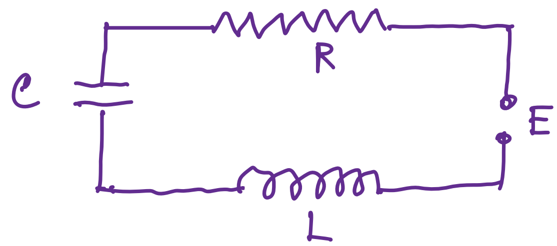
Displacement = y

$$\boxed{my'' + ky = 0}$$

(Vibrating mass on a spring)

K = Spring Constant.

⑤

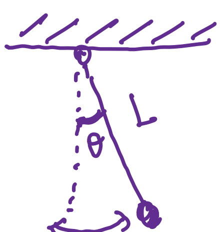


Current I in a RLC circuit

$$\boxed{LI'' + RI' + \frac{1}{C} I = E'}$$

independent variable = t (time)

⑥

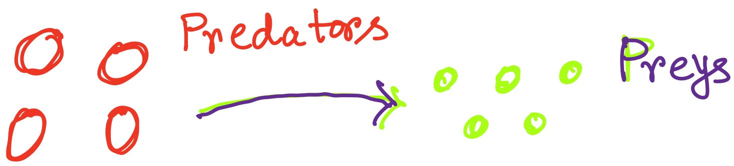


Pendulum

$$\boxed{L\theta'' + g \sin\theta = 0}$$

independent variable = t time)

⑦



(Lotka-Volterra predator-prey model)

$$y_1' = ay_1 - by_1y_2$$
$$y_2' = ky_1y_2 - ly_2$$

(independent variable $= t$)

(System of two first order ODEs)

ODEs	PDEs
$\underbrace{f(x), \frac{df}{dx}, \frac{d^2f}{dx^2}, \dots}$	$f(x,y), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \dots$ (Both x & y are independent variable)
• Highest derivative present is the order	• Highest partial derivative present in the equation determines the order

Definition:

Solution: A function $y = h(x)$ is a solution of a given ODE $F(x, y, y', \dots, y^{(n)}) = 0$ on some open interval $a < x < b$ (a, b) if $h(x)$ is defined and it has derivatives upto nth order $(h, h', \dots, h^{(n)})$ in the interval (a, b) and the equation is satisfied if $y, y', y'', \dots, y^{(n)}$ are replaced by $h, h', h'', \dots, h^{(n)}$

i.e. $\boxed{F(x, h, h', \dots, h^{(n)}) = 0}$ on $a < x < b$

Ex: Consider the ODE: $xy' = -y$

Then $y = \frac{c}{x}$ is a solution on $(0, \infty)$ or on $(-\infty, 0)$

because $y' = -\frac{c}{x^2} \Rightarrow xy' = x \times \left(-\frac{c}{x^2}\right)$

$$= -\frac{c}{x} = -y$$

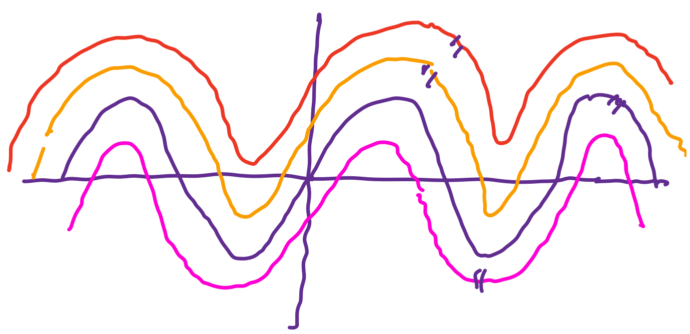
Thus $h(x) = \frac{c}{x}$

Ex: $y' = \cos x$

Then $y = \int \cos x \, dx$

$\Rightarrow y = \sin x + C$

on $(-\infty, \infty)$



Ex: $y' = ky$, k is a constant

$$\Rightarrow \frac{dy}{dx} = ky \Rightarrow \int \frac{dy}{y} = \int k \, dx$$

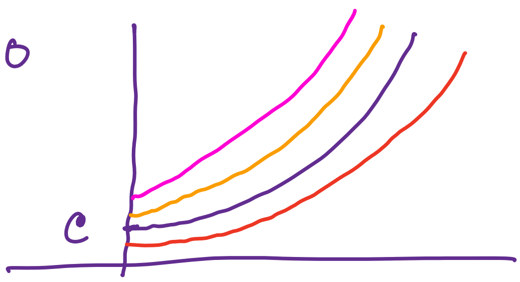
$\Rightarrow \ln|y| = kx + c_1$ where c_1 is a constant

$$\Rightarrow y = e^{kx + c_1} = e^{c_1} e^{kx}$$

$$\Rightarrow \boxed{y = c e^{kx}} \quad \text{where} \quad c = e^{c_1}$$

If $k>0$, it will give exponential growth
If $k<0$, it will give exponential decay

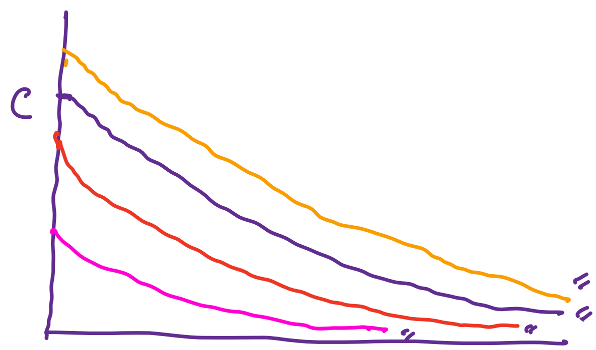
$k = 0.2 > 0$



(Exponential growth)

$k = -0.2 < 0$

(Exponential Decay)



General Solution: $\quad y = c e^{kx}$
c = arbitrary constant.

Particular solution:

$y = c e^{kx} \quad y(0) = 5 \Rightarrow c = 5 \quad y = 5 e^{kx}$

Initial Value Problem (IVP): $\left. \begin{array}{l} y' = f(x,y) \\ y(x_0) = y_0 \end{array} \right\}$

In the above case $\quad y' = ky, \quad y(0) = 5$

Ex: IVP : $y' = 3y$, $y(0) = 5.7$

General solution : $y = Ce^{3x}$

Now $y(0) = 5.7 \Rightarrow$ $5.7 = Ce^{3\times 0} \Rightarrow C = 5.7$

Hence $\boxed{y = 5.7\, e^{3x}}$ → Particular solution of the IVP

Modeling : Radioactivity :

Step 1 : Setting up a mathematical model

$y(t) \to$ quantity of a radioactive substance after time t

Then $\boxed{y' = -ky}$

$k \to$ depends on the substance

Unit of $k \to s^{-1}$ $k = 1.4 \times 10^{-11}\, s^{-1}$ for Ra^{226}_{88} (Radium)

Step 2 : Find solution of mathematical model

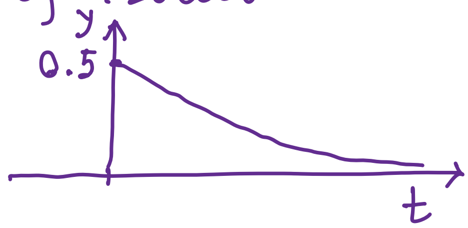
$y(t) = ce^{-kt}$

(given $y(0) = 0.5$ kg) Then $y(t) = 0.5\, e^{-kt}$

Step 3: Interpretation of solution

$k = 1.5$

$y(t) = 0.5\, e^{-1.5t}$



Half Life: $y(T) = \dfrac{y(0)}{2}$ Find T?

$.5\, e^{-1.5T} = \dfrac{.5}{2} \Rightarrow e^{-1.5T} = \dfrac{1}{2} \Rightarrow -1.5T = -\ln 2 \Rightarrow \dfrac{3}{2} T = \ln 2 \Rightarrow \boxed{T = \dfrac{2\ln 2}{3}}$

Geometric Meaning: $F(y', y, x) = 0$

$\longrightarrow$ Most general form of a first order ODE

$y' = f(x, y) \longrightarrow$ Explicit form of a first order ODE

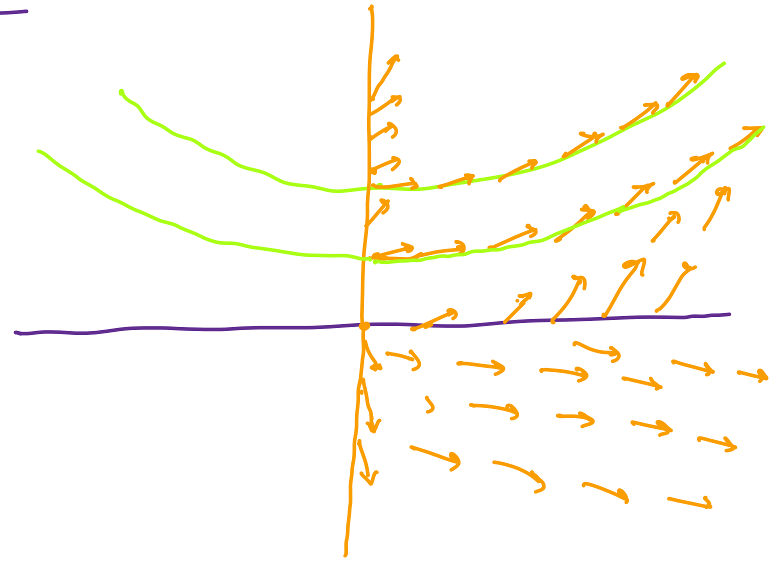
$\dfrac{dy}{dx}$ is given by $f(x, y)$

$\longrightarrow$ slope of the tangent at the point $(x, y(x))$ of the curve $y = y(x)$

Ex: $y' = x + y$ So, $f(x, y) = x + y$

At each point (x, y) of the plane $(\mathbb{R}^2)$ we can draw a small vector representing the slope $x + y$.

(x,y)	$x+y$
$(0,1)$	1
$(1,0)$	1
$(1,1)$	2
$(-1,0)$	-1
$(0,-1)$	-1
$(1,2)$	3
$(2,1)$	3



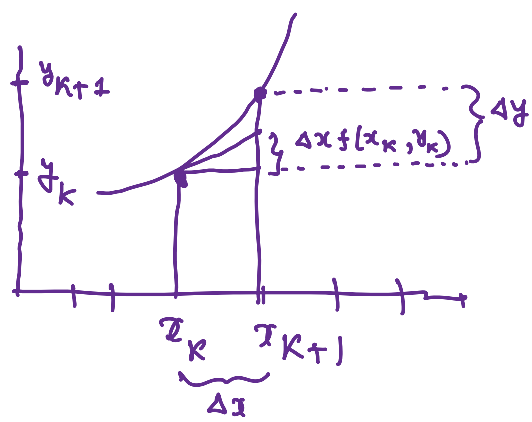
Direction field of $y' = x + y$

- Vectors in direction field represent tangent vectors of solution curves

Euler Method (Numerical).

$$y' = f(x,y)$$

$$\frac{\Delta y}{\Delta x} \simeq \frac{dy}{dx} = f(x,y)$$



Let $\Delta x = x_{k+1} - x_k$

and $\Delta y = y_{k+1} - y_k$

Then $\Delta y \simeq \Delta x f(x, y)$

$$\Rightarrow y_{k+1} - y_k \simeq \Delta x f(x, y)$$

$$\Rightarrow y_{k+1} \simeq y_k + \Delta x f(x_k, y_k)$$

Let $\Delta x = h$ (fixed)

$$
\begin{array}{l}
(x_0, y_0) \\
\downarrow \\
(x_1, y_1) \\
\downarrow \\
(x_2, y_2) \\
\downarrow \\
(x_3, y_3) \\
\vdots
\end{array}
\qquad
\left\{
\begin{array}{ll}
x_1 = x_0 + h, & y_1 = y_0 + h f(x_0, y_0) \\
x_2 = x_1 + h, & y_2 = y_1 + h f(x_1, y_1) \\
x_3 = x_2 + h, & y_3 = y_2 + h f(x_2, y_2) \\
\vdots & \vdots
\end{array}
\right.
$$

Separable ODEs :

A First order ODE is separable if it can be written as

$$g(y)\, y' = f(x)$$

(y on the left) (x on the right)

Then $g(y)\, y' = f(x)$

$$g(y)\, dy = f(x)\, dx$$

$$\Rightarrow \int g(y)\, dy = \int f(x)\, dx + c$$

<u>Ex:</u> $y' = 1 + y^2$

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{dy}{1+y^2} = \int dx \qquad \Rightarrow \tan^{-1} y = x + c$$

$$\Rightarrow \boxed{y = \tan(x+c)}$$

<u>Ex:</u> $y' = (x+1)\, e^{-x}\, y^2$

$$\int \frac{dy}{y^2} = \int (x+1)\, e^{-x}\, dx$$

$$\Rightarrow \quad -\frac{1}{y} = -(x+1)\, e^{-x} - e^{-x} + C_1$$

$$\Rightarrow -\frac{1}{y} = -x e^{-x} - 2e^{-x} + C_1$$

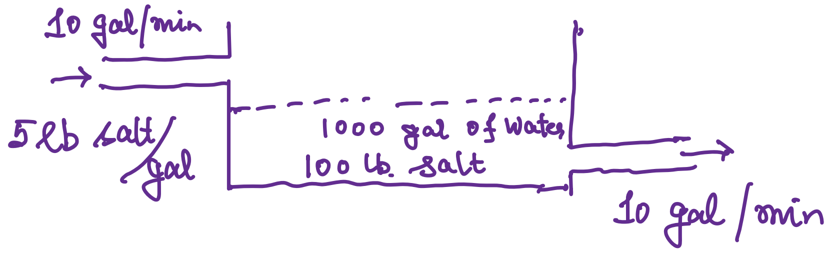
$$\Rightarrow \quad -\frac{1}{y} = -(x+2)e^{-x} - C$$

$$\Rightarrow \quad \frac{1}{y} = (x+2)e^{-x} + C$$

$$\Rightarrow \quad \boxed{y = \frac{1}{(x+2)e^{-x} + C}}$$

MTH 204 : Lecture 02

<u>Mixing Problem:</u> A tank contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.



To find the amount of salt at any time.

Let $y(t)$ be the amount of salt at time t.

Given: $y(0) = 100$

$$\frac{dy}{dt} = \underbrace{\text{salt inflow rate}} - \underbrace{\text{salt outflow rate}}$$

$$\Rightarrow \frac{dy}{dt} = 5 \times 10 \text{ lbs/min} - \frac{y}{1000} \times 10 \text{ lbs/min}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{5000 - y}{100}$$

$$\Rightarrow \quad \frac{dy}{y - 5000} = -\frac{dt}{100}$$

$$\Rightarrow \quad \ln|y - 5000| = -.01t + C_1$$

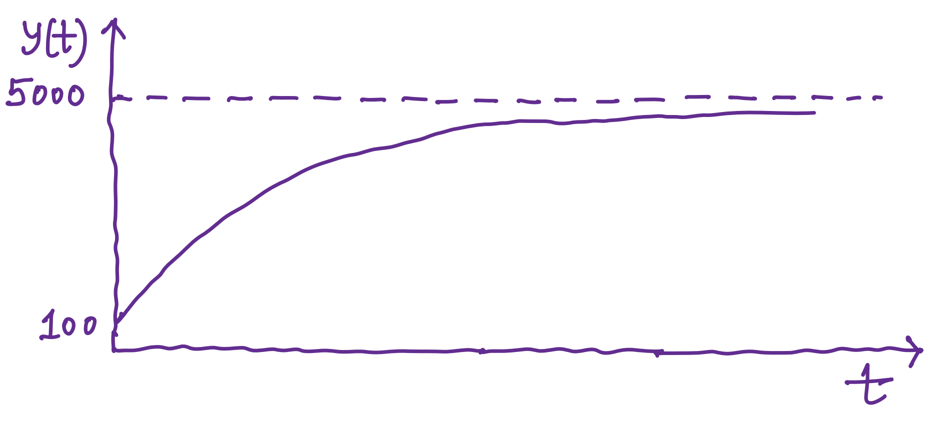
$$\Rightarrow \quad y - 5000 = C e^{-.01t}$$

$$y = 5000 + C e^{-.01t}$$

$$y(0) = 100 \Rightarrow \quad 100 = 5000 + C$$

$$\Rightarrow C = -4900$$

So, $\boxed{y(t) = 5000 - 4900 e^{-.01t}}$



Ex: (Newton's Law of Cooling):

Suppose that in the winter, the daytime temperature is maintained at 70°F (in a building). The Heating is turned off at 10 PM and turned on again at 6 AM.

On a certain day, the temperature inside the building at 2 AM was found to be 65°F. The outside temperature was 50°F at 10 PM and had dropped to 40°F by 2 AM. What was the temperature inside the building at 6 AM?

Physical information:

Newton's Law of Cooling states that the time rate of change of temperature T of a body is proportional to the difference between T and the temperature of the surrounding medium.

	10 PM	2 AM	6 AM
70° inside	50° (outside)	65° (inside)	40° (outside)

Let $T(t)$ be the temperature inside the building at time t (after 10 PM).

To get the temperature outside at any time t, we take an average

$$T_{out} = \frac{50+40}{2} = 45^\circ F.$$

By Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_{out})$$

$$\Rightarrow \quad \frac{dT}{dt} = k(T-45)$$

$$\Rightarrow \quad \frac{dT}{T-45} = k\,dt$$

Then $\quad T-45 = ce^{kt}$

$$\Rightarrow \quad T(t) = 45 + ce^{kt}$$

given: $T(0) = 70$ (at 10 PM) $\Rightarrow$ $70 = 45 + c$

$\Rightarrow c = 25$

$$\Rightarrow \quad T(t) = 45 + 25e^{kt}$$

when $T(4) = 65^\circ$ (at 2 AM)

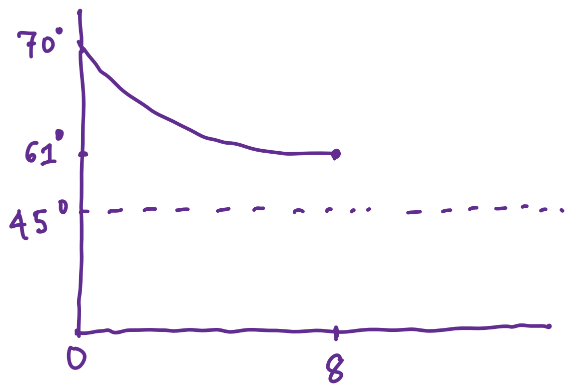
$$\Rightarrow \quad 65 = 45 + 25e^{k(4)}$$

$$\Rightarrow \quad 20 = 25e^{k(4)}$$

$$\Rightarrow \quad \frac{4}{5} = e^{k(4)} \Rightarrow k = \frac{1}{4}\ln\left(\frac{4}{5}\right)$$

$$\approx -.056$$

So, $\boxed{T(t) = 45 + 25e^{-.056t}}$



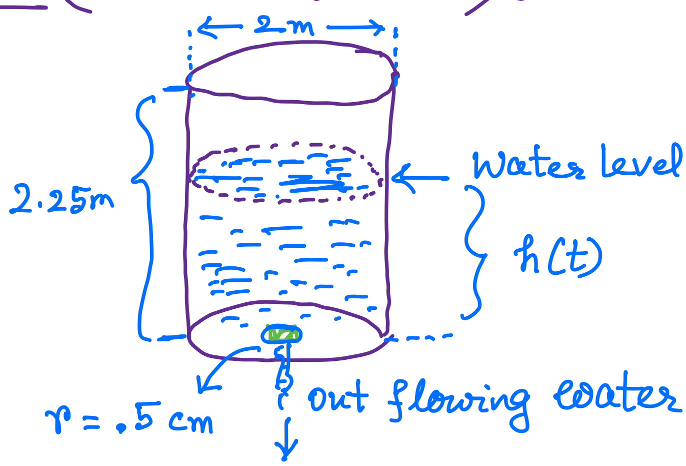
Want T at 6AM

i.e. at $t=8$

$$T(8) = 45 + 25e^{-.056 \times 8}$$

$$= 60.97 \approx 61^\circ F$$

<u>Ex:</u> (Torricelli's Law):



<u>Physical information</u>: Under the influence of gravity the outflowing water has velocity

$$v(t) = .6\sqrt{2g\,h(t)}$$

where $h(t)$ is the height of the water above the hole at time t

and $g = 980\ \text{cm/sec}^2 = 32.17\ \text{ft/sec}^2$ is the acceleration due to gravity at the surface of the earth.

Let ΔV be the volume of water outflowing through a hole in time Δt and let A be the surface area of the hole.

$$\left.\begin{aligned}
&\text{Then} \quad \Delta V = A v \Delta t = A v(t) \Delta t \\
&\text{Also,} \quad \Delta V \propto \Delta h \Rightarrow \Delta V = -B \Delta h \text{ where} \\
&\qquad\qquad B \text{ is the base surface area}
\end{aligned}\right\}$$

$$\Rightarrow \quad -B\Delta h = A v \Delta t$$

$$\Rightarrow \quad \frac{\Delta h}{\Delta t} = -\frac{A}{B} v$$

Taking limit as $\Delta t \to 0$ we get

$$\frac{dh}{dt} = -\frac{A}{B} v = -\frac{A}{B}(.6)\sqrt{2g\,h(t)}$$

$$\Rightarrow \quad \frac{dh}{dt} = D\sqrt{h} \quad \text{where } D = -\frac{A}{B}(.6)\sqrt{2g}$$

$$\Rightarrow \quad \frac{dh}{\sqrt{h}} = D\,dt$$

$$\int h^{-\frac{1}{2}}\,dh = \int D\,dt$$

$$\Rightarrow\ 2h^{\frac{1}{2}} = Dt + c_1$$

$$\Rightarrow\ \sqrt{h} = \frac{D}{2}t + c \qquad \left(c = \frac{c_1}{2}\right).$$

$$\Rightarrow\ \boxed{h = \left(\frac{D}{2}t + c\right)^2}$$

Now, $D = -\frac{A}{B} \cdot .6\sqrt{2g}$

$A = \pi(100)^2\ cm^2$

$B = \pi(.5)^2\ cm^2$

$\Big\}$ Then $D = -.000664$

$$h = (-.000332t + c)^2\ cm.$$

$$h(0) = 2.25\,m = 225\ cm$$

$$225 = c^2 \Rightarrow c = 15$$

So, $\boxed{h = (-.000332t + 15)^2\ cm}$

Reduction to Separable form:

For those ODEs that can be written as:

$$y' = f(y/x)$$

we make the change of variable

$$u = \frac{y}{x}$$

$$\Rightarrow y = ux \Rightarrow y' = u'x + u$$

Then $y' = f(y/x) \Rightarrow u'x + u = f(u)$

$$\Rightarrow u'x = f(u) - u$$

$$\Rightarrow x\frac{du}{dx} = f(u) - u$$

$$\Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x}$$

Now the equation is reduced to a separable form and can be integrated now.

Ex: $2xyy' = y^2 - x^2$

$$\Rightarrow y' = \frac{y^2}{2xy} - \frac{x^2}{2xy}$$

$$\Rightarrow y' = \frac{1}{2}\frac{y}{x} - \frac{1}{2}\frac{x}{y}$$

$$\Rightarrow y' = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$

Let $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = xu' + u.$

So, $xu' + u = \frac{1}{2}\left(u - \frac{1}{u}\right)$

$$\Rightarrow xu' = -u + \frac{1}{2}u - \frac{1}{2u}$$

$$\Rightarrow xu' = -\frac{1}{2}u - \frac{1}{2u} = -\frac{1}{2}\frac{(u^2+1)}{u}$$

$$\Rightarrow \int \frac{2u\,du}{u^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln(u^2+1) = -\ln|x| + C_1$$

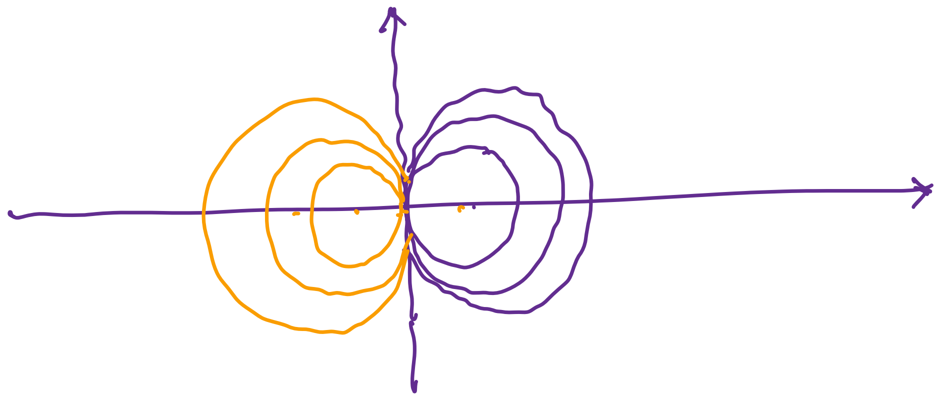
$$\Rightarrow u^2 + 1 = \frac{c}{x}$$

$$\Rightarrow \frac{y^2}{x^2} + 1 = \frac{c}{x}$$

$$\Rightarrow \frac{y^2+x^2}{x^2} = \frac{c}{x} \Rightarrow x^2 + y^2 = cx$$

$$\Rightarrow \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

Thus the general solution is the family of circles centered at $\left(\frac{c}{2}, 0\right)$ and radius $\frac{|c|}{2}$



General solution (family of circles)

Ex: $$xy' = y + 2x^3 \sin^2\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad y' = \frac{y}{x} + 2x^2 \sin^2\left(\frac{y}{x}\right)$$

Let $u = \frac{y}{x}$. Then $y = ux \Rightarrow y' = xu' + u$

So, $$xu' + \cancel{u} = \cancel{u} + 2x^2 \sin^2(u)$$

$$\Rightarrow \quad u' = \frac{2x^2}{x} \sin^2(u)$$

$$\Rightarrow \quad \frac{du}{\sin^2(u)} = 2x\,dx$$

$$\Rightarrow \int \operatorname{cosec}^2(u)\,du = \int 2x\,dx$$

$$\Rightarrow \quad -\cot u = x^2 + C$$

$$\Rightarrow \quad \boxed{-\cot\left(\frac{y}{x}\right) = x^2 + c}$$

$$\left(\text{or } \; y = x\cot^{-1}(-x^2 - c)\right)$$

Method of Exact ODEs:

Recall: If a function $u(x, y)$ has continuous partial derivatives, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

If $u(x,y) = C$ then $du = 0$

Example: $u = x + x^2y^3 = C$

Then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

$$\Rightarrow (1 + 2xy^3)\,dx + (3x^2y^2)\,dy = 0$$

$$\Rightarrow y' = -\frac{1+2xy^3}{3x^2y^2}$$

- A first order ODE
$M(x,y) + N(x,y)\,y' = 0$ $\left(\text{or } y' = \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}\right)$
can be rewritten as:
$$M(x,y)\,dx + N(x,y)\,dy = 0$$

This ODE is an Exact Differential Equation
if there is a differentiable function
(hence partial derivatives exist and are continuous)
such that

$$\frac{\partial u}{\partial x} = M(x,y), \quad \frac{\partial u}{\partial y} = N(x,y)$$

and hence $M(x,y)\,dx + N(x,y)\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du = 0$

$$\underbrace{\qquad\qquad\qquad}_{\text{(Exact differential)}}$$

Ex: $\cos(x+y)\,dx + \left[3y^2+2y+\cos(x+y)\right]dy = 0$

Consider $u(x,y) = \sin(x+y) + y^2 + y^3$

Then $\dfrac{\partial u}{\partial x} = \cos(x+y)$

$$\frac{\partial u}{\partial y} = \cos(x+y) + 2y + 3y^2$$

Then the differential equation becomes

$\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = 0 \Rightarrow du = 0 \Rightarrow \underbrace{u = C}_{\text{(Implicit solution of the ODE)}}$ (constant)

$\Rightarrow \boxed{\sin(x+y) + y^2 + y^3 = c}$ (Solution of the ODE)

Solution of Exact ODE:

$$M\,dx + N\,dy = du = 0$$

$\Leftrightarrow u = C$ (Implicit Solution of ODE)

i.e. Level curves of the function u are solutions of this ODE.

Question (1) : How to check if an ODE is exact?

Question (2) : Once it is exact, how to find its solution?

- An ODE $M\,dx + N\,dy = 0$ is exact iff $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Note: Thus if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y}$, we can conclude that it is not exact.

Question (2) Once we know that $M\,dx + N\,dy = 0$ is exact, we know that there exists a function $u(x,y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

To find u, start with $\frac{\partial u}{\partial x} = M(x,y)$

Then $u(x,y) = \int \frac{\partial u}{\partial x} dx = \int M(x,y)\,dx + g(y)$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y)$$

Now we know that $\frac{\partial u}{\partial y} = N$

$$\Rightarrow \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y) = N(x,y)$$

$$\Rightarrow g'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right)$$

Integrate to find $g(y)$

Then $\boxed{u(x,y) = \int M(x,y)\,dx + g(y)}$

Note: Can also be done by first integrating $N(x,y)$ with respect to y.

Ex: $\cos(x+y)\,dx + \left[3y^2+2y+\cos(x+y)\right]dy = 0$

- $M(x,y) = \cos(x+y)$, $N(x,y) = 3y^2+2y+\cos(x+y)$

$$\frac{\partial M}{\partial y} = -\sin(x+y), \quad \frac{\partial N}{\partial x} = -\sin(x+y)$$

So, the ODE is exact.

- So, there exists a function $u(x,y)$ such that $\frac{\partial u}{\partial x} = M(x,y) = \cos(x+y)$

$$u(x,y) = \int \cos(x+y)\,dx$$

$$\Rightarrow u(x,y) = \sin(x+y) + g(y)$$

Now $\dfrac{\partial u}{\partial y} = \cos(x+y) + g'(y)$

Hence $\dfrac{\partial u}{\partial y} = N(x,y) = \cos(x+y) + g'(y)$

$\Rightarrow 3y^2 + 2y + \cancel{\cos(x+y)} = \cancel{\cos(x+y)} + g'(y)$

$\Rightarrow g'(y) = 3y^2 + 2y \Rightarrow g(y) = y^3 + y^2 + C$

So, $\boxed{u(x,y) = \sin(x+y) + y^3 + y^2 + C}$

- can also be done by taking

$\dfrac{\partial u}{\partial y} = N(x,y) = 3y^2 + 2y + \cos(x+y)$

$\Rightarrow u(x,y) = \int N(x,y)\,dy = y^3 + y^2 + \sin(x+y) + f(x)$

$\Rightarrow \dfrac{\partial u}{\partial x} = \cos(x+y) + f'(x) \Rightarrow \cos(x+y) = \cos(x+y) + f'(x)$

$\Rightarrow f'(x) = 0 \Rightarrow f(x) = c$ (some constant)

Thus $\boxed{u(x,y) = y^3 + y^2 + \sin(x+y) + C}$

(Therefore the answers are same.)

Ex: $\left[\cos(y)\sinh(x) + 1\right]dx - \left[\sin(y)\cosh(x)\right]dy = 0$

Given: $y(1) = 2$

Note: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

So, $$\left.\begin{aligned} \frac{d[\sinh(x)]}{dx} &= \frac{e^x + e^{-x}}{2} = \cosh(x) \\ \text{and } \frac{d[\cosh(x)]}{dx} &= \frac{e^x - e^{-x}}{2} = \sinh(x) \end{aligned}\right\}$$

Now $M(x,y) = \cos y \sinh(x) + 1$

$N(x,y) = -[\sin y \cosh(x)]$

$$\frac{\partial M}{\partial y} = -\sin y \sinh(x) \text{ and } \frac{\partial N}{\partial x} = -\sin y \sinh(x)$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ The differential equation is exact.

Now $\dfrac{\partial u}{\partial x} = M(x,y) = \cos y \sinh(x) + 1$

Then integrating with respect to x,

$$u(x,y) = \cos y \cosh(x) + x + g(y)$$

Now $\dfrac{\partial u}{\partial y} = -\sin y \cosh(x) + g'(y) \quad = N(x,y)$

Therefore $-\sin y \cosh(x) + g'(y) = -\sin y \cosh(x)$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1 \text{ (Constant)}$$

Thus $u(x,y) = \cos y \cosh(x) + x + C_1$

So, the general solution of the ODE is $u(x,y) = C_2$

$$\Rightarrow \boxed{\cos y \cosh(x) + x = C} \quad (C = C_2 - C_1)$$

Now $y(1) = 2 \Rightarrow \cos 2 \cosh(1) + 1 = C$

So, the particular solution of the IVP is:

$$\boxed{\cos y + \cosh(x) + x = \cos 2 \cosh(1) + 1}$$

Ex: $-y\,dx + x\,dy = 0$

Here $M = -y$, $N = x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

So, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so the above ODE is not exact.

However it is separable. $-y\,dx + x\,dy = 0$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \ln|y| = \ln|x| + c_1 \Rightarrow \boxed{y = cx}$$

Note: Although the above ODE is not exact, it becomes exact if we multiply it by $\frac{1}{x^2}$

$$-y\,dx + x\,dy = 0 \Rightarrow -\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{y}{x} = c \Rightarrow \boxed{y = cx}$$

$$\boxed{\text{MTH 204 : Lecture 03}}$$

Ex: $\underbrace{-y}_{M}\,dx + \underbrace{x}_{N}\,dy = 0$ This ODE is not exact since $\frac{\partial(-y)}{\partial y} = -1$ and $\frac{\partial x}{\partial x} = 1$.

Now multiply both sides by $\frac{1}{x^2}$

$$\frac{1}{x^2}(-y\,dx) + \frac{1}{x^2}(x\,dy) = 0 \Rightarrow -\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$

Now $\frac{\partial(-y/x^2)}{\partial y} = -\frac{1}{x^2}$, $\frac{\partial\left(\frac{1}{x}\right)}{\partial x} = -\frac{1}{x^2}$

So it becomes an exact ODE

and can be written as $d\left(\frac{y}{x}\right) = 0$

$$\Rightarrow \frac{y}{x} = c$$

- Can also multiply by $\frac{1}{y^2}$

Then $\frac{1}{y^2}(-y\,dx + x\,dy) = -\frac{1}{y}dx + \frac{x}{y^2}dy = 0$

$$\Rightarrow d\left(\frac{x}{y}\right) = 0 \Rightarrow \frac{x}{y} = \text{Constant}$$

- Can also multiply by $\frac{1}{xy}$

Then $\frac{1}{xy}(-y\,dx + x\,dy) = -\frac{1}{x}dx + \frac{1}{y}dy = 0$

$$\Rightarrow \quad d\left(\ln \frac{y}{x}\right) = 0$$

- Can also multiply by $\frac{1}{x^2+y^2}$

Then $\frac{1}{x^2+y^2}(-y\,dx + x\,dy) = 0 \Rightarrow -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$

$$\Rightarrow \quad d\left(\tan^{-1}\frac{y}{x}\right) = 0$$

<u>Definition:</u> An integrating factor I.F. is a function $F(x,y)$ such that the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

becomes an exact ODE after multiplication:

i.e. $FM\,dx + FN\,dy = 0$ is an exact equation.

- IF of an ODE is not unique.
- A non exact ODE may not have an I.F.

<u>How to Find I.F. :</u>

Let $M\,dx + N\,dy = 0$

Multiplying by $F(x,y)$ we have

$$F(x,y)M(x,y)\,dx + F(x,y)N(x,y)\,dy = 0$$

This ODE will be exact if

$$\frac{\partial (F M)}{\partial y} = \frac{\partial (F N)}{\partial x}$$

$$M \frac{\partial F}{\partial y} + F \frac{\partial M}{\partial y} = N \frac{\partial F}{\partial x} + F \frac{\partial N}{\partial x}$$

- If $M\,dx + N\,dy = 0$ is already exact,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then F is an I.F. if $\boxed{M \frac{\partial F}{\partial y} = N \frac{\partial F}{\partial x}}$

<u>Otherwise</u>

after multiplication the ODE is exact if

$$M \frac{\partial F}{\partial y} + F \frac{\partial M}{\partial y} = N \frac{\partial F}{\partial x} + F \frac{\partial N}{\partial x}$$

- If the I.F. F depends only on x and not on y, then $\frac{\partial F}{\partial y} = 0$

So, $$F \frac{\partial M}{\partial y} = N \frac{dF}{dx} + F \frac{\partial N}{\partial x}$$

Dividing both sides by FN,

$$\frac{1}{N}\frac{\partial M}{\partial y} = \frac{1}{F}\frac{dF}{dx} + \frac{1}{N}\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\Rightarrow \int\frac{dF}{F} = \int\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$$

$$\Rightarrow \boxed{F = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}}$$

- If the I.F. F depends only on y and not on x (i.e. $\frac{\partial F}{\partial x} = 0$)

then $$M\frac{dF}{dy} + F\frac{\partial M}{\partial y} = F\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{1}{F}\frac{dF}{dy} + \frac{1}{M}\frac{\partial M}{\partial y} = \frac{1}{M}\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{1}{F}\frac{dF}{dy} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \int\frac{dF}{F} = \int\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy$$

$$\Rightarrow \boxed{F = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy}}$$

Ex:

$$\underbrace{\left(e^{x+y}+ye^{y}\right)}_{M}dx+\underbrace{\left(xe^{y}-1\right)}_{N}dy=0$$

Check for exactness:

$$\frac{\partial M}{\partial y}=\frac{\partial\left(e^{x+y}+ye^{y}\right)}{\partial y}=e^{x+y}+ye^{y}+e^{y}$$

$$\frac{\partial N}{\partial x}=\frac{\partial\left(xe^{y}-1\right)}{\partial x}=e^{y}\neq\frac{\partial M}{\partial y}$$

So, this ODE is not exact

Now
$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=e^{x+y}+ye^{y}+e^{y}-e^{y}$$
$$=e^{x+y}+ye^{y}$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-\cancel{\left(e^{x+y}+ye^{y}\right)}}{\cancel{\left(e^{x+y}+ye^{y}\right)}}=-1$$

So, the ODE has an I.F. depending on y only.

Hence I.F. $F=e^{\int\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy}$

$$=e^{-\int dy}=e^{-y}$$

Multiplying the original ODE by e^{-y},

$$(e^{x}+y)\,dx + (x-e^{-y})\,dy = 0$$

Check: $\frac{\partial(e^{x}+y)}{\partial y} = 1$, $\frac{\partial(x-e^{-y})}{\partial x} = 1$

So, this is an exact equation.

So, there exists $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = e^{x}+y \quad , \quad \frac{\partial u}{\partial y} = x-e^{-y}$$

$$u(x) = \int (e^{x}+y)\,dx = e^{x} + yx + g(y)$$

$$\frac{\partial u}{\partial y} = x + g'(y) = x - e^{-y}$$

$$\Rightarrow \quad g'(y) = -e^{-y} \quad \Rightarrow g(y) = e^{-y} + \text{constant}$$

So, the general solution of the ODE is

$$u(x,y) = e^{x} + yx + e^{-y} = C$$

Ex: $\underbrace{3(y+1)}_{M}\,dx \underbrace{-\,2x}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 3 \quad , \quad \frac{\partial N}{\partial x} = -2$$

So, this ODE is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3-(-2)}{-2x} = -\frac{5}{2x}$$

This is a function of x only.

$$\text{IF } F = e^{\int -\frac{5}{2x}\,dx} = e^{-5/2 \ln x} = e^{\ln(x^{-5/2})}$$

$$F = x^{-5/2}$$

Multiplying the ODE by $x^{-5/2}$ we get

$$3x^{-5/2}(y+1)\,dx - 2x^{-3/2}\,dy = 0$$

Check: $\dfrac{\partial\left[3x^{-5/2}(y+1)\right]}{\partial y} = 3x^{-5/2}$

$$\frac{\partial\left(-2x^{-3/2}\right)}{\partial x} = (-2)\left(-\frac{3}{2}\right)x^{-3/2-1} = 3x^{-5/2}$$

So, this new ODE is exact.

Thus there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = 3x^{-5/2}(y+1)$$

$$u(x,y) = \frac{3\,x^{-5/2+1}}{(-5/2+1)}(y+1) + g(y)$$

$$= \frac{3\,x^{-3/2}}{\left(-\frac{3}{2}\right)}(y+1) + g(y)$$

$$= -2x^{-3/2}(y+1) + g(y)$$

$$\frac{\partial u}{\partial y} = -2x^{-3/2} + g'(y) = -2x^{-3/2}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c_1$$

So, the solution is $u(x,y) = c_2$

$$\Rightarrow -2x^{-3/2}(y+1) + c_1 = c_2 \Rightarrow -2x^{-3/2}(y+1) = c_2 - c_1$$

$$\Rightarrow 2x^{-3/2}(y+1) = c_1 - c_2 \Rightarrow y+1 = \frac{(c_1-c_2)}{2}x^{3/2}$$

$$\Rightarrow y+1 = c\,x^{3/2} \quad \text{where } c = \frac{c_1-c_2}{2}$$

$$\Rightarrow \boxed{y = cx^{3/2} - 1}$$

Table for IF

- $$\frac{M_y - N_x}{N} = f(x) \Rightarrow \text{IF: } F = e^{\int f(x)\,dx}$$

- $$\frac{N_x - M_y}{M} = g(y) \Rightarrow \text{IF: } F = e^{\int g(y)\,dy}$$

(Some other cases)

- $$\frac{M_y - N_x}{yN - xM} = h(xy), \quad \text{I.F: } F = e^{\int h(z)\,dz}$$ where $z = xy$

⋮

Linear ODE: A first order ODE is called linear if it can be written as:

$$y' + p(x)\,y = r(x).$$

(It is a linear combination of y', y and 1
coefficients/scalars are functions of x only)

Note that $a(x)\,y' + b(x)\,y + c(x)1 = 0$

$$\Rightarrow \quad y' + \underbrace{\frac{b(x)}{a(x)}}_{p(x)} y = \underbrace{-\frac{c(x)}{a(x)}}_{r(x)}$$

Homogeneous Linear ODE:

A linear, first order ODE is said to be homogeneous if $r(x) = 0$ (In the interval in consideration) and hence $y' + p(x)y = 0$

Separation of Variables:

$$\frac{dy}{y} = -p(x)\,dx$$

$$\Rightarrow \quad \boxed{y = c\,e^{-\int p(x)\,dx}}$$

$y = 0$ is also a solution (Trivial solution)

Non homogeneous Linear ODE:

$$y' + p(x)y = r(x)$$

$$\Rightarrow (p(x)y - r(x))\,dx + dy = 0$$

$$\underbrace{\qquad\qquad}_{M} \qquad \underbrace{\quad}_{N=1}$$

Then $\dfrac{\partial M}{\partial y} = p(x)$, $\dfrac{\partial N}{\partial x} = 0$ (This is nonexact if $p(x) \neq 0$)

$$\frac{M_y - N_x}{N} = \frac{p(x) - 0}{1} = p(x)$$ (a function of x only)

I.F. $\quad F = e^{\int p(x)dx}$

Multiplying by the I.F.,

$$e^{\int p(x)dx} y' + e^{\int p(x)dx} p(x) y = e^{\int p(x)dx} r(x)$$

$$\Rightarrow \frac{d\left(y\, e^{\int p(x)dx}\right)}{dx} = r(x)\, e^{\int p(x)dx}$$

$$\Rightarrow y\, e^{\int p(x)dx} = \int r(x)\, e^{\int p(x)dx} dx + c$$

$$\Rightarrow \boxed{y = e^{-\int p(x)dx}\left[\int r(x)\, e^{\int p(x)dx} dx + c\right]}$$

Ex: $y' + y\tan x = \sin(2x)$, $\underbrace{y(0) = 1}$

$p(x) = \tan x$, $r(x) = \sin(2x)$

$\int p(x)dx = \int \tan x \, dx = \log|\sec x|$

$e^{\int p(x)dx} = |\sec x|$

$e^{-\int p(x)dx} = |\cos x|$

So, $y(x) = \cos x \left(\int \sec x \sin 2x \, dx + c \right)$

$= \cos x \left(\int 2\sin x \, dx + c \right)$

$= \cos x (-2\cos x + c)$

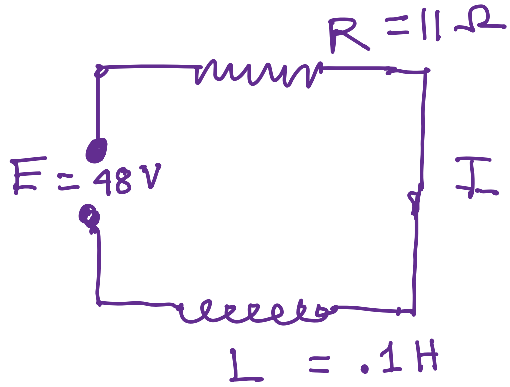
given $y(0) = 1 \Rightarrow 1 = \cos 0^\circ (-2\cos 0^\circ + c)$

$\Rightarrow 1 + 2 = c \Rightarrow c = 3$

So, $\boxed{y(x) = \cos x(-2\cos x + 3)}$

Ex:

Voltage drop across the resistor $= IR$



(LR circuit)

Voltage dross across inductor $= LI' = L\frac{dI}{dt}$

$$E = LI' + IR$$

$$\Rightarrow \boxed{I' + \frac{R}{L} I = \frac{E}{L}} \quad \text{(Linear ODE)}$$

$$p(t) = \frac{R}{L}$$

$$\int p(t)\, dt = \int \frac{R}{L} dt = \frac{R}{L} t$$

So, $$I(t) = e^{-\frac{R}{L}t} \left[\int e^{\frac{R}{L}t} \frac{E}{L} dt + c \right]$$

$$= e^{-\frac{R}{L}t} \left[\frac{E}{\cancel{L}} \cdot \frac{e^{\frac{R}{L}t}}{\frac{R}{\cancel{L}}} + c \right]$$

$$= e^{-\frac{R}{L}t} \left[\frac{E}{R} e^{R/L\, t} + c \right]$$

$$\Rightarrow I(t) = \frac{E}{R} + c\, e^{-R/L\, t}$$

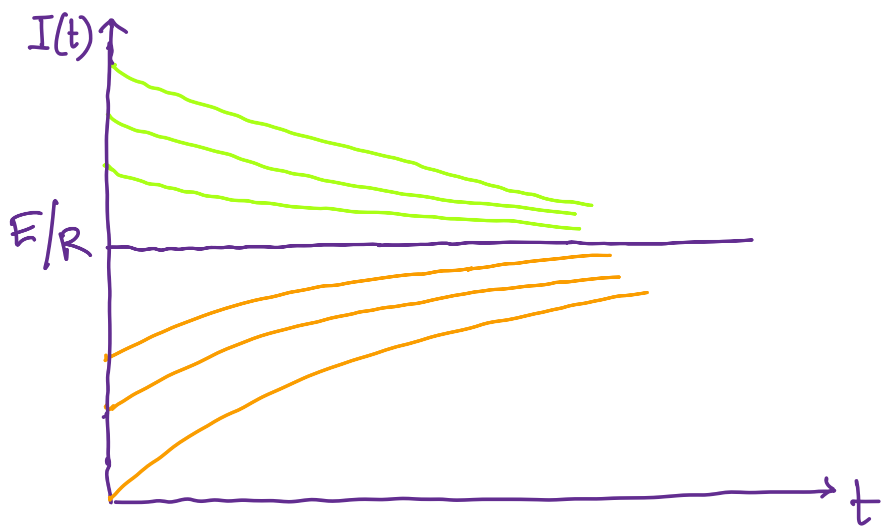
$$I(0) = 0 \Rightarrow 0 = \frac{E}{R} + c \Rightarrow c = -\frac{E}{R}$$

So, $\boxed{I(t) = \frac{E}{R} - \frac{E}{R} e^{-R/L\, t}}$

Given: $R = 11\,\Omega$, $L = .1\,H$, $E = 48\,V$

So, $I(t) = \frac{48}{11}\left(1 - e^{-110t}\right)$

As $t \to \infty$, $I(t) \longrightarrow \frac{48}{11} = \frac{E}{R}$



For different initial condition, we will get different particular solution.

MTH 204 : Lecture 04

Ex: Assume that the level of a certain hormone in the blood of a patient varies with time. Suppose that the time rate of change is the difference between a sinusoidal input of a 24-hour period from the thyroid gland and a continuous removal rate proportional to the level present. Set up a model for the hormone level in the blood and find its general solution. Find the particular solution satisfying a suitable initial condition.

Let $y(t)$ be the level of the hormone in blood at time t.

Then $\dfrac{dy}{dt} = \underbrace{\text{Input}}_{\substack{\text{sinusoidal input} \\ (A + B\cos(\omega t))}} - \underbrace{\text{Output}}_{\substack{\text{Proportional to } y(t) \\ (\text{i.e. } k\,y(t) \text{ where } k \text{ is a constant})}}$

So, $$\frac{dy}{dt} = A + B\cos(\omega t) - k\,y(t)$$

$$\Rightarrow \quad \frac{dy}{dt} + k\,y(t) = A + B\cos(\omega t)$$

$$\text{I.F.} = e^{\int k\,dt} = e^{kt}$$

Hence $$y(t) = e^{-kt}\left[\int \left(A + B\cos(\omega t)\right) e^{kt}\,dt + C\right]$$

$$= e^{-kt}\left[\frac{A e^{kt}}{k} + B\, e^{kt}\,\frac{k\cos(\omega t) + \omega\sin(\omega t)}{\omega^2 + k^2} + C\right]$$

because $\int e^{kt}\cos(\omega t)\,dt$

$$= \frac{e^{kt}}{k}\cos(\omega t) - \int -\omega\sin\omega t\,\frac{e^{kt}}{k}\,dt$$

$$= \frac{e^{kt}}{k}\cos\omega t + \frac{\omega}{k}\int \sin\omega t\, e^{kt}\,dt$$

$$= \frac{e^{kt}}{k}\cos\omega t + \frac{\omega}{k}\left[\frac{e^{kt}}{k}\sin\omega t - \int \omega\cos\omega t\,\frac{e^{kt}}{k}\,dt\right.$$

$$= \frac{e^{kt}}{k}\cos\omega t + \frac{\omega}{k^2}e^{kt}\sin\omega t - \frac{\omega^2}{k^2}\int e^{kt}\cos\omega t\,dt$$

$$\Rightarrow \left[1 + \frac{\omega^2}{k^2}\right]\int e^{kt}\cos(\omega t)\,dt$$

$$= \frac{1}{k^2}e^{kt}\left[k\cos(\omega t) + \omega\sin(\omega t)\right]$$

$$\Rightarrow \frac{(k^2+\omega^2)}{k^2}\int e^{kt}\cos(\omega t)\,dt = \frac{e^{kt}}{k^2}\left[k\cos(\omega t) + \omega\sin(\omega t)\right]$$

$$\Rightarrow I = \int e^{kt}\cos(\omega t)\,dt = \frac{e^{kt}}{(k^2+\omega^2)}\left[k\cos(\omega t) + \omega\sin(\omega t)\right]$$

Hence $\quad y(t) = e^{-kt}\left[\frac{A}{k}e^{kt} + \frac{B}{\omega^2+k^2}e^{kt}\left(k\cos(\omega t) + \omega\sin(\omega t)\right) + C\right]$

$$\Rightarrow \boxed{y(t) = \frac{A}{k} + \frac{B}{\omega^2+k^2}\left(k\cos(\omega t) + \omega\sin(\omega t)\right) + Ce^{-kt}}$$

Since the variation is every 24 hours,
frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{24} = \frac{\pi}{12}$

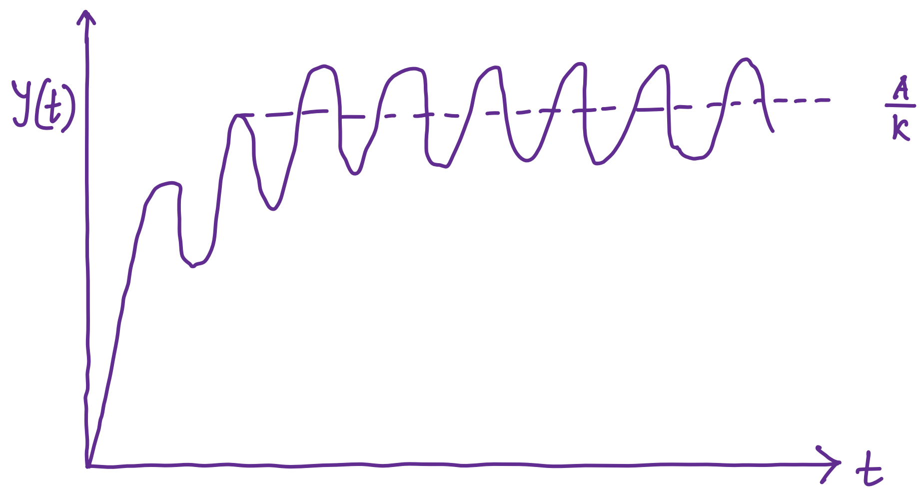
Then $y = \frac{A}{k} + \frac{B}{\frac{\pi^2}{12^2} + k^2}\left(k\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{12}\sin\left(\frac{\pi}{12}t\right)\right.$
$\left. + Ce^{-kt}\right)$

If we assume $y(0) = 0$, then

$$0 = \frac{A}{k} + \frac{Bk}{k^2+\frac{\pi^2}{12^2}} + C \Rightarrow C = -\left(\frac{A}{k} + \frac{Bk}{k^2+\frac{\pi^2}{12^2}}\right)$$

Then
$$y = \frac{A}{k}\left(1-e^{-kt}\right) + \frac{B}{k^2+\frac{\pi^2}{12^2}}\left(k\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{12}\sin\left(\frac{\pi}{12}t\right) - ke^{-kt}\right)$$

In particular if $A = B = 1$ and $k = .05$, then the solution will have the following graphical form



Reduction to Linear ODEs:

Bernoulli Equation: $y' + p(x)y = g(x)y^a$

Question:
For what values of a, a Bernoulli Equation is linear? : $a = 0$ and $a = 1$

Thus Bernoulli Equation is non-linear for $a \neq 0$ and $a \neq 1$

Let $u = y^{1-a}$

Then $u' = (1-a)y^{-a}y'$

$$\Rightarrow u' = (1-a)y^{-a}\left(-p(x)y + g(x)y^a\right)$$

$$\Rightarrow u' = -(1-a)p(x)y^{1-a} + (1-a)g(x)$$

$$\Rightarrow u' = (1-a)\left(g(x) - p(x)u\right)$$

$$\Rightarrow \boxed{u' + (1-a)p\,u = (1-a)g} \longrightarrow \text{Linear ODE}$$

Ex: <u>Logistic Equation</u>:

$$y' = Ay - By^2$$

$y=0$ is a solution.

Otherwise it is a Bernoulli Equation with $a=2$

Let $u = y^{1-2} = y^{-1}$

$$u' = (-1)y^{-1-1}y' = -\frac{1}{y^2}y'$$

$$\Rightarrow \quad u' = -\frac{1}{y^2}(Ay - By^2)$$

$$\Rightarrow u' = -Ay^{-1} + B$$

$$\Rightarrow u' + Au = B \quad \text{(Linear ODE)}$$

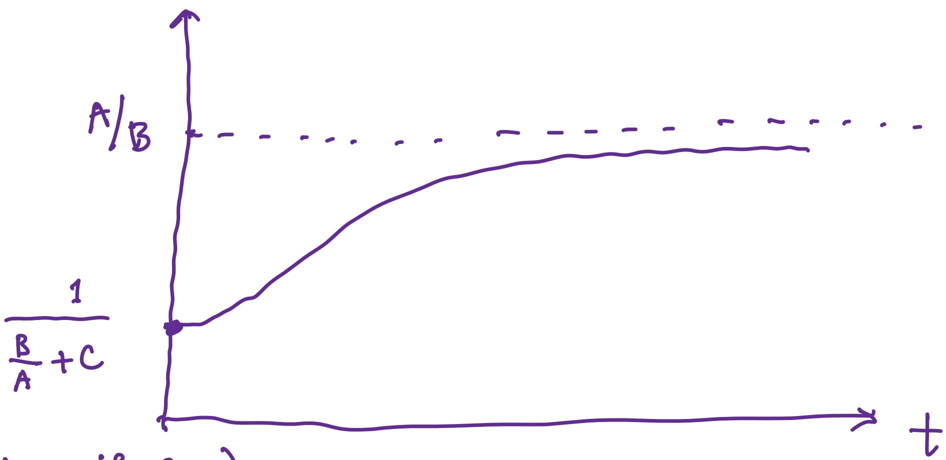
$$\text{I.F.} = e^{\int A\,dx} = e^{Ax}$$

So, $u = e^{-Ax}\left[\int Be^{Ax}\,dx + c\right]$

$$= e^{-Ax}\left[\frac{B}{A}e^{Ax} + c\right]$$

$$\Rightarrow \quad \frac{1}{y} = \frac{B}{A} + ce^{-Ax}$$

$$\Rightarrow \quad \boxed{y = \frac{1}{\frac{B}{A} + ce^{-Ax}}}$$



(In this graph $C > 0$)

Ex: Population Dynamics:

Let $y(t)$ be the population at time t.

Then the logistic Equation can be written as:

$$y' = Ay - By^2$$

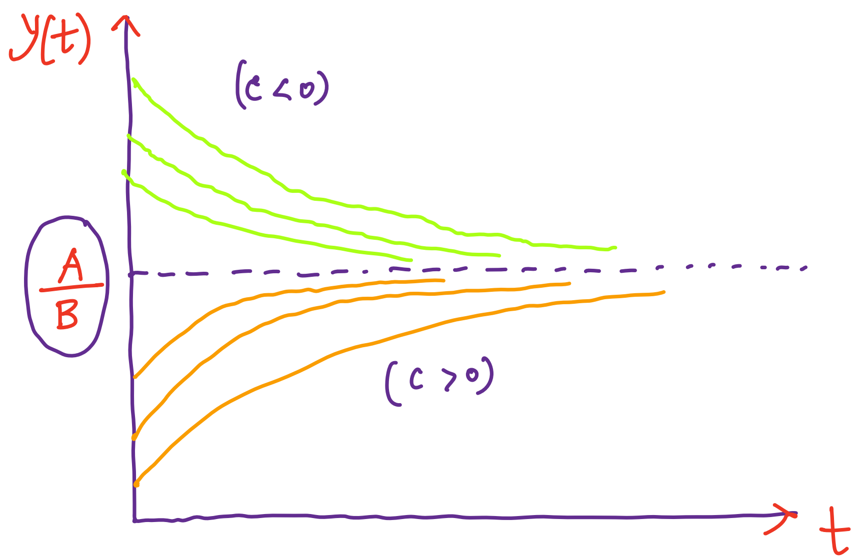
$$\Rightarrow y' = Ay\left(1 - \frac{B}{A}y\right) \qquad A, B > 0$$

If $y < \frac{A}{B}$, then $y' > 0 \Rightarrow$ Population grows

If $y > \frac{A}{B}$, then $y' < 0 \Rightarrow$ Population decreases.

$\frac{A}{B}$: "equilibrium solution" "stable"

$$\boxed{y(t) = \frac{1}{\frac{B}{A} + Ce^{-At}}}$$



Note: The term $-By^2$ acts as a 'braking term' that prevents the population from growing infinitely.

Note: For a small population

$$y' = Ay \quad (\text{Malthus Law}) \quad \left(\begin{array}{c}\text{Particular}\\ \text{case of logistic}\\ \text{Equation}\end{array}\right)$$

$$\Rightarrow y = \frac{1}{c\,e^{-At}} = \frac{1}{c}e^{At}$$

Autonomous ODE:

A general first order ODE $y' = f(x,y)$ in which $f(x,y)$ is just a function of y (i.e. x does not appear explicitly) is called an $\boxed{\text{Autonomous ODE}}$.

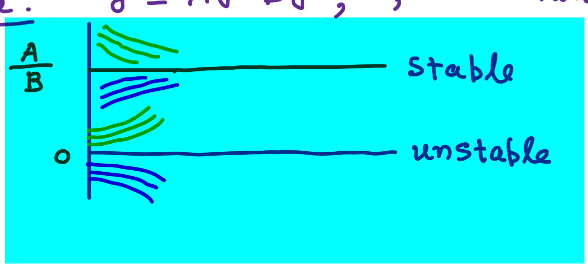
Thus it is of the form $\boxed{y' = f(y)}$

- Any solution of $f(y)=0$ is called ~~is called~~ a critical point or equilibrium point. (At these points $y'=0$ and so there is no change)

Note that a critical point gives a solution of the ODE since RHS $= f(y) = 0$ and LHS $= y' = 0$ (as y is a constant)

A critical point may be stable if solutions close to it for some t remain close to it for all further t or unstable if solutions initially close to it do not remain close as t increases.

Ex: $y' = Ay - By^2$, $A, B > 0$ Now $Ay - By^2 = 0 \Rightarrow y(A-By)=0 \Rightarrow y=0,\ y=\frac{A}{B}$

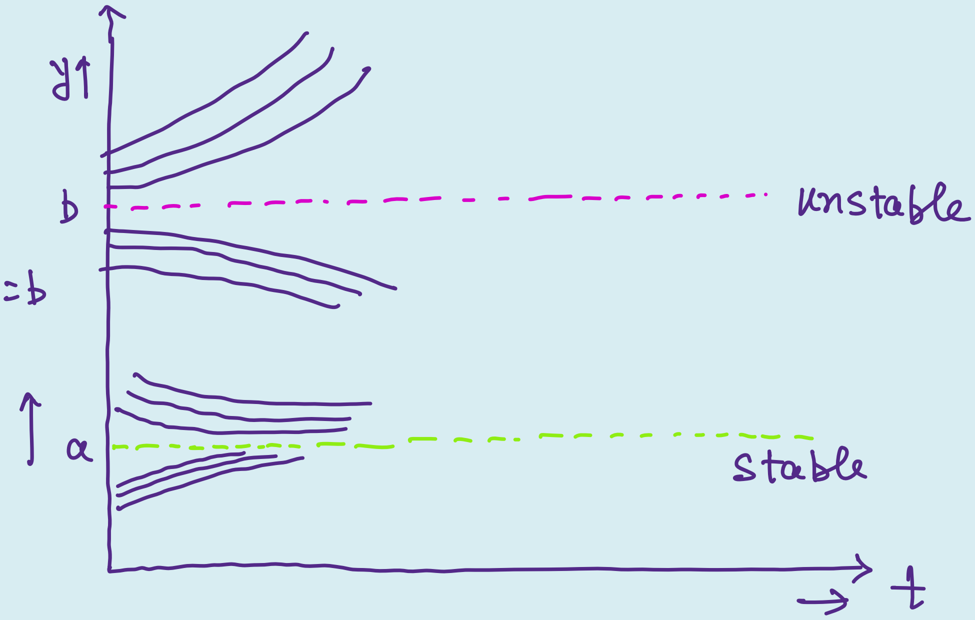


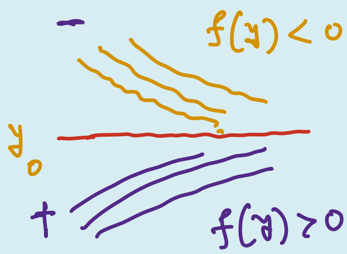
Suppose

$y' = f(y)$

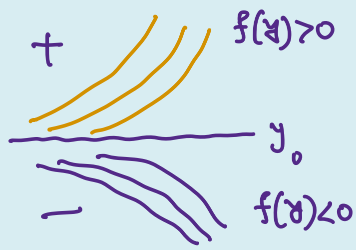
and $f(y) = 0$

at $y = a$ & $y = b$

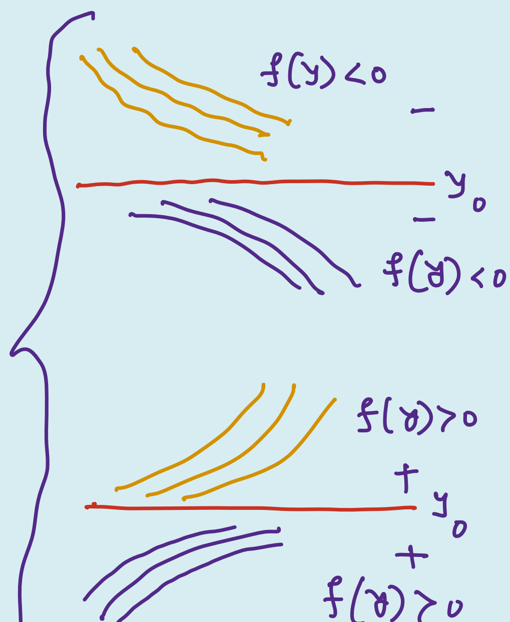




Stable



Unstable



Semi stable

$y' = f(y)$

$y' < 0$

$y' > 0$

(f(y) doesn't change sign across y_0)

Ex: $\dfrac{dy}{dt} = -(y-10)^2(y-4)$ (Autonomous ODE)

Here $f(y) = -(y-10)^2(y-4)$

The critical points are 4, 10

At 4, $f(y)$ changes sign from '+' to '−'
So, 4 is a stable critical point.
At 10, $f(y)$ doesn't change its sign (It is '−' to '−')
So, 10 is a semi stable critical point.

Ex: $\frac{dy}{dt} = (y^3 - 8)(e^y - 1)$ (Autonomous ODE)

Here $f(y) = (y^3 - 8)(e^y - 1)$

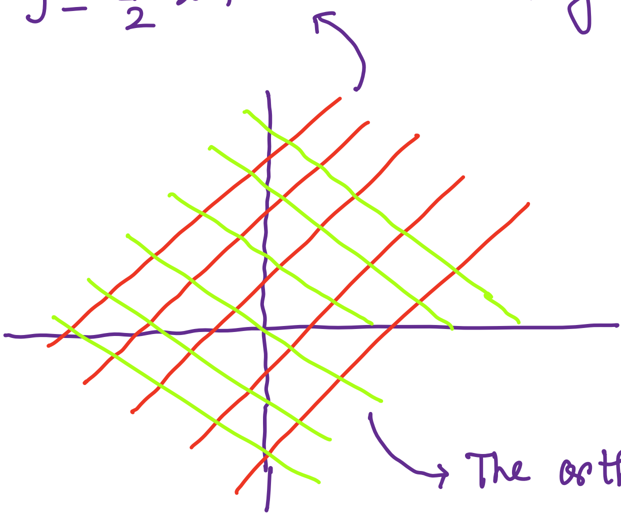
Equilibrium points are 0 and 2.
At 0, $f(y)$ changes its sign from '+' to '−'
So, 0 is a stable equilibrium point
At 2, $f(y)$ changes its sign from '−' to '+'
So, 2 is a unstable equilibrium point.

Orthogonal Trajectories:

Let us consider the family of curves that are given by $\boxed{G(x, y, c) = 0}$

Question:
What is the family of curves that are orthogonal to the above family?

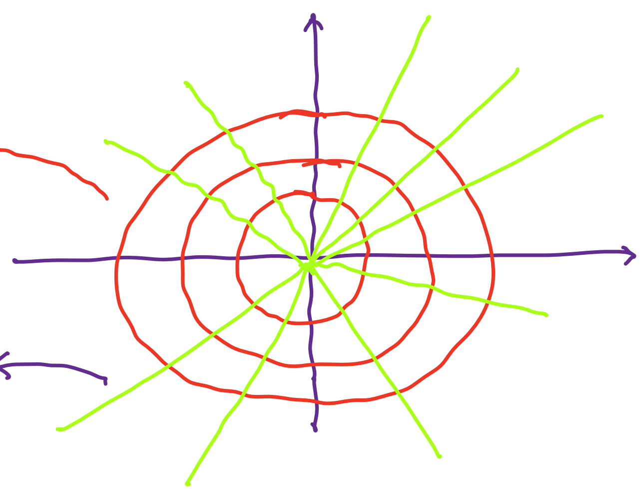
Ex: $y = \frac{1}{2}x + c$ is a given family of curves.



The orthogonal family is $\boxed{y = 2x + c_1}$

Ex: $\boxed{x^2 + y^2 = c^2}$

The orthogonal family is $\boxed{y = c_1 x}$



Ex: Find the family of curves which are orthogonal to $\frac{x^2}{2} + y^2 = c$

Let us find the first order ODE whose solutions are $\frac{x^2}{2} + y^2 = c$

$$\Rightarrow \quad \frac{2x}{2} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{2y}$$

The orthogonal family will have slope

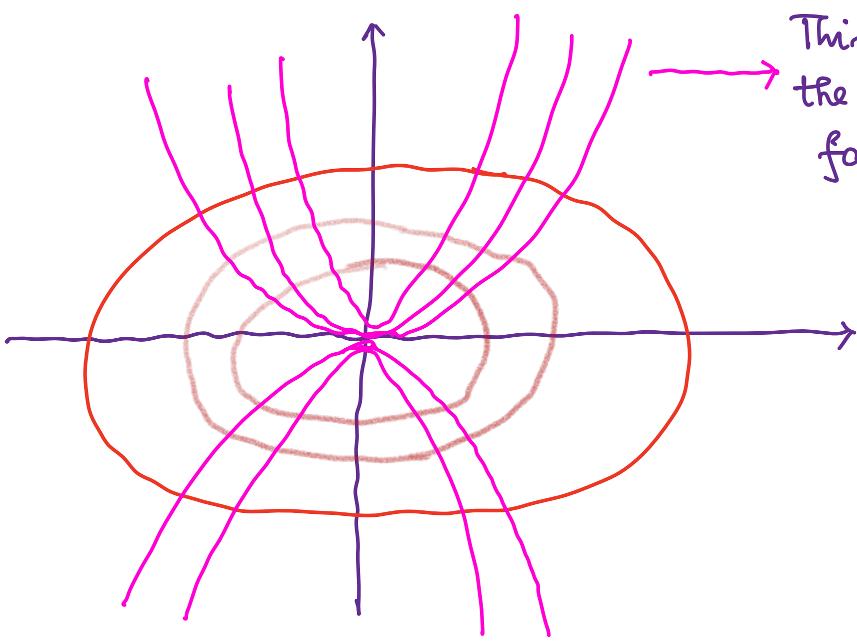
$$\frac{2y}{x}$$

So, $$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \quad \frac{dy}{y} = 2\frac{dx}{x}$$

$$\Rightarrow \quad \ln y = 2\ln x + \ln c$$

$$\Rightarrow \ln y = \ln(cx^2) \Rightarrow \boxed{y = cx^2}$$



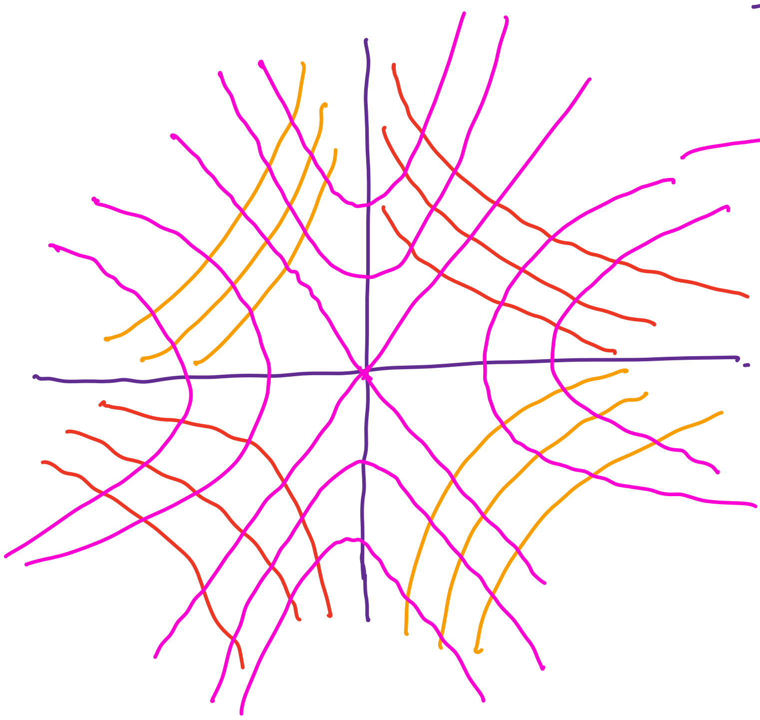
This is the orthogonal family of curves.

Ex: $xy = c$

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

The orthogonal family will have slope $\frac{x}{y}$; $\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$

$$\Rightarrow \boxed{y^2 = x^2 + c}$$



This is the orthogonal family of curves

Steps to find orthogonal trajectories:

(1) Find the first order ODE corresponding to the given family of curves by differentiating and eliminating the constant.

(2) Find y' in terms of x and y from the

obtained ODE, say $y' = f(x,y)$

(3) Step up a new ODE as $y' = -\frac{1}{f(x,y)}$

(4) Solve the ODE in step (3) to find the orthogonal family to the given family of curves.

Ex: $y = \sqrt{x+c} \Rightarrow y' = \frac{1}{2\sqrt{x+c}} = \frac{1}{2y}$

The orthogonal family will have slope $-2y$

$\Rightarrow \quad y' = -2y \Rightarrow \frac{dy}{y} = -2\,dx \Rightarrow \ln(y) = -2x + c_1$

$\Rightarrow \boxed{y = c\,e^{-2x}} \quad (c = e^{c_1})$

Existence and Uniqueness of solution of IVP:

Ex: Consider the IVP: $|y'| + |y| = 0$, $y(0) = 1$

Note that $y \equiv 0$ is the only possible solution of the ODE $|y'| + |y| = 0$

But the IVP: $|y'| + |y| = 0$, $y(0) = 1$ has no solution.

Ex: Consider $y' = 2x$, $y(0) = 1$

Then $y = \int 2x\,dx + C \Rightarrow y = x^2 + C$

Now $y(0)=1 \Rightarrow c=1$

Hence $y=x^2+1$ is the unique solution of the above IVP.

Ex: Consider $xy'=y-1$, $y(0)=1$

Then $\frac{dy}{y-1}=\frac{dx}{x} \Rightarrow \ln(y-1)=\ln x+\ln c$

$$\Rightarrow y-1=cx \Rightarrow y=1+cx$$

Now note that $y(0)=1$ for any c

Thus the above IVP has infinitely many solutions.

Existence Theorem:

Given the IVP $y=f(x,y)$, $y(x_0)=y_0$

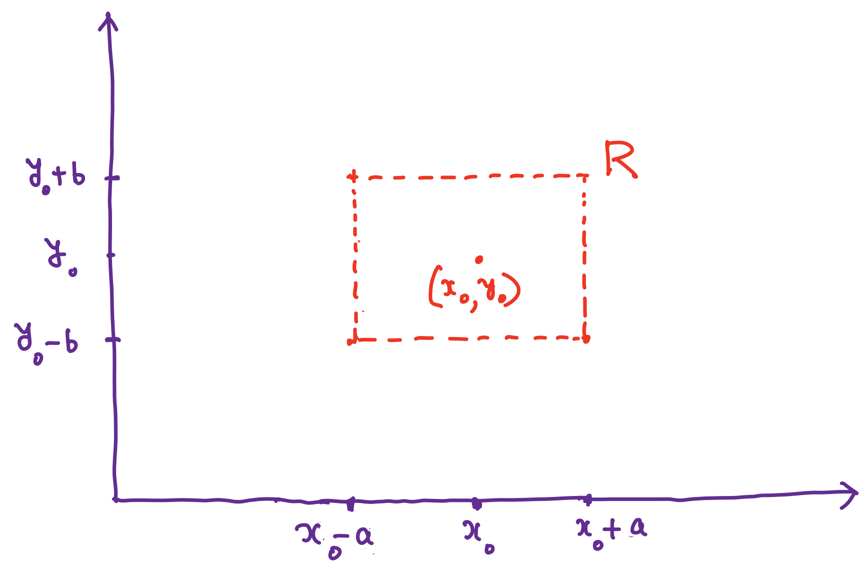
If $f(x,y)$ is continuous in a rectangle R

$$R=\{(x,y)\in\mathbb{R}^2 : |x-x_0|<a,\ |y-y_0|<b\}$$

and bounded in R (i.e. there exists $K>0$ such that $|f(x,y)|\le K \ \forall (x,y)\in R$),

then the IVP has atleast one solution $y(x)$.

This solution exists at least for all x in $|x-x_0|<\alpha$ where $\alpha=\min\{a, \frac{b}{K}\}$.



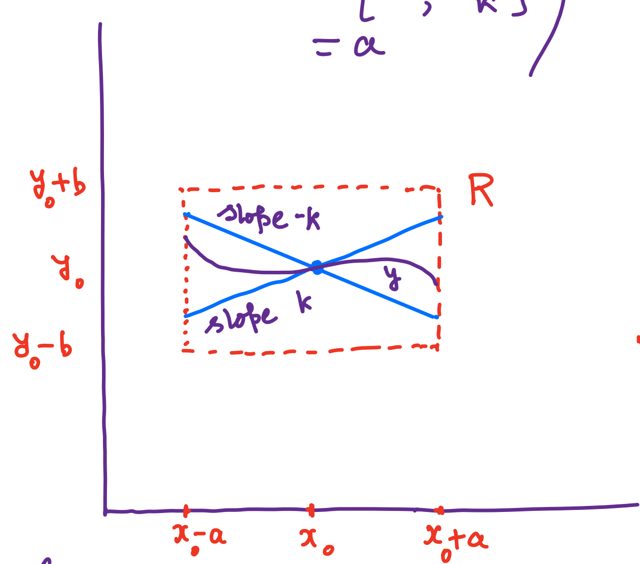
Note:

- $f(x,y)$ is bounded by K means

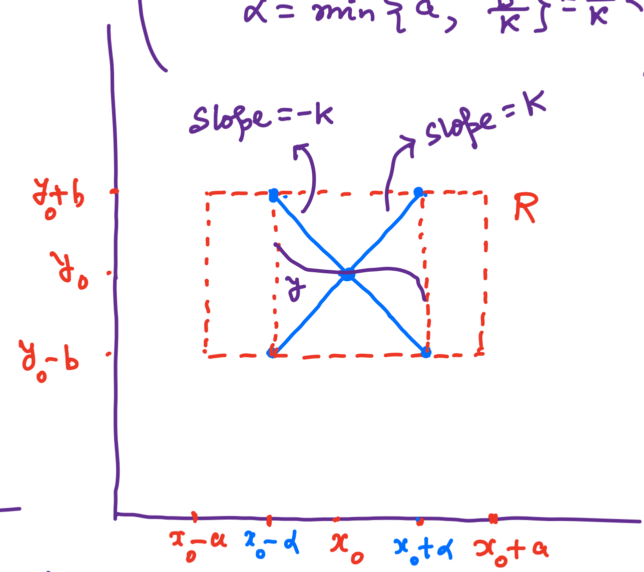
$$|y'| = |f(x,y)| \leq K$$

i.e. The slope of any solution is atleast $-K$ and atmost K.

(In this case $\alpha = \min\{a, \frac{b}{K}\} = a$)



(In this case $\alpha = \min\{a, \frac{b}{K}\} = \frac{b}{K} < a$)



(The second condition of the existence theorem)

Uniqueness Theorem:

- Given the IVP : $y' = f(x,y)$, $y(x_0) = y_0$
- Let the IVP meet the conditions of the Existence Theorem.

If $f_y = \frac{\partial f}{\partial y}$ is continuous in R and is bounded in R (i.e. there exists $M > 0$ such that $|f_y(x,y)| \leq M \quad \forall (x,y) \in R$), then the IVP has a unique solution $y(x)$.

This solution exists atleast for all x in $|x - x_0| < \alpha$ where $\alpha = \min\{a, \frac{b}{K}\}$.

Note: Proof of these theorems involve advanced analysis.

Addendum to Lecture 04

Existence and Uniqueness Theorem (Revisited):

Existence Theorem: Consider the IVP
$$y' = f(x,y), \quad y(x_0) = y_0 \quad \ldots\ldots (1)$$
Let the following two properties are satisfied.

(C1) There is a rectangle R centered at (x_0, y_0) of length a and width b such that $f(x,y)$ is continuous at all points of
$$R = \{(x,y) \in \mathbb{R}^2 : |x - x_0| < a, \ |y - y_0| < b\}$$

(C2) $f(x,y)$ is bounded in R i.e. there exists $K > 0$ such that $|f(x,y)| \leq K \ \forall (x,y) \in R$

$y = y(x)$

Then the IVP (1) has atleast one solution and the solution exists for all x s.t.
$$|x - x_0| < \alpha \quad \text{where} \quad \alpha = \min\{a, \tfrac{b}{K}\}$$

i.e. the graph of the solution lies inside a rectangle R_1 $(R_1 \subset R)$
where $R_1 = \{(x,y) : |x - x_0| < \min\{a, \frac{b}{K}\}, |y - y_0| < b\}$

Uniqueness Theorem: Consider the IVP (1) with conditions C1 and C2 and two more conditions that

(C3): $\frac{\partial f}{\partial y}$ is continuous at all points in R

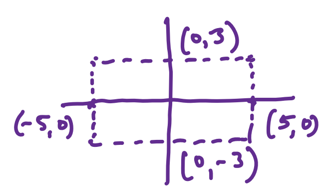
(C4): $\frac{\partial f}{\partial y}$ is bounded in R i.e. there exists $M > 0$ such that $\left|\frac{\partial f(x,y)}{\partial y}\right| \le M \ \forall (x,y) \in R$

then IVP (1) has a unique solution for all x s.t. $|x - x_0| < \alpha$ where $\alpha = \min\{a, b/K\}$

(i.e. the graph of the unique solution lies inside R_1)

Ex(1): $y' = 1 + y^2, \quad y(0) = 0$

$f(x,y) = 1 + y^2, \quad x_0 = 0, \ y_0 = 0$



Let $R = \{ (x,y) \mid |x| < 5 , |y| < 3 \}$

Then (C1) f is continuous everywhere & so in R

(C2) $|f(x,y)| \le |1 + y^2| \le 1 + |y|^2 < 1 + 3^2 = 10$

So, $f(x,y)$ is bounded in R with $K = 10$

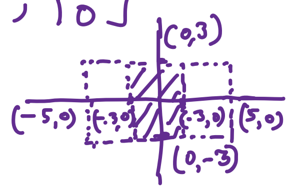
(C3) $\dfrac{\partial f}{\partial y} = 2y$ is continuous everywhere and so in R

(C4) $\left| \dfrac{\partial f}{\partial y} \right| \le |2y| \le 2|3| = 6$

So, $\dfrac{\partial f}{\partial y}$ is bounded in R with $M = 6$

Hence a unique solution exists for all x s.t. $|x| < \min\{ 5, \frac{3}{10} \}$

i.e. for $|x| < .3$



The graph of the solution lies in the rectangle $R_1 = \{ (x,y) : |x| < .3 , |y| < 3 \}$

Ex ②: $|y'| + |y| = 0$, $y(0) = 1$

This IVP has no solution

The only solution to $|y'| + |y| = 0$ is $y \equiv 0$ but this doesn't satisfy the condition $y(0) = 1$

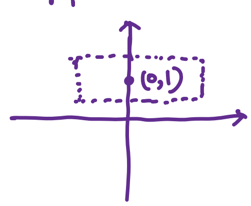
Now $|y'| + |y| = 0 \Rightarrow |y'| = -|y|$

$|y'| = -|y|$

$$\Rightarrow |y'| = \begin{cases} -y & \text{if } y \geq 0 \\ y & \text{if } y < 0 \end{cases}$$

$$\Rightarrow y' = \left.\begin{cases} -y & \text{if } y \geq 0,\ y' \geq 0 \\ y & \text{if } y < 0,\ y' \geq 0 \\ y & \text{if } y \geq 0,\ y' < 0 \\ -y & \text{if } y < 0,\ y' < 0 \end{cases}\right\} = \boxed{f(x, y)}$$

$f(x,y)$ is not defined at any point other than $(0,0)$ and so the conditions (C1) and (C2) are not satisfied. Hence, the existence theorem can't be applied here.



Ex ③: $xy' = y - 1 \qquad y(0) = 1$

$$\Rightarrow y' = \frac{y-1}{x} = f(x, y)$$

$$\Rightarrow \frac{dy}{y-1} = \frac{dx}{x} \Rightarrow \ln(y-1) = \ln x + \ln c \Rightarrow \boxed{y = 1 + cx}$$

So, there are infinitely many solutions.

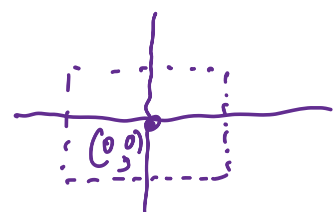
$f(x,y)$ is not defined on any point on the y-axis and hence in any rectangle centered at $(0,1)$ (it will contain points from y-axis)

So, the existence theorem can't be applied here.

However the IVP has infinitely many solutions

$$y(x) = 1 + cx \quad \text{for any constant } c$$

Ex ④ : $y' = \sqrt{|y|}$, $y(0) = 0$



$f(x,y) = \sqrt{|y|}$ continuous everywhere and bounded in any rectangle centered at $(0,0)$

So, by existence theorem, a solution exists.

Infact

$$y(x) = \begin{cases} x^2/4 & x \geq 0 \\ -x^2/4 & x < 0 \end{cases}$$

and $y \equiv 0$ are both solutions.

Note that $\frac{\partial f}{\partial y}$ doesnot exist at $(0,0)$

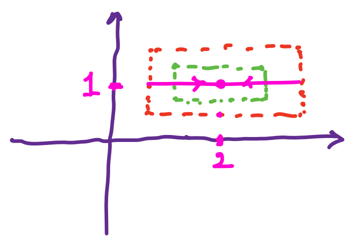
Hence the Uniqueness theorem can't be applied.

Here we are getting two solutions of the IVP.

Ex: $(x-2)y' = y$, $y(2)=1$

The above IVP can be written as $y' = \frac{y}{x-2}$

So, $f(x,y) = \frac{y}{x-2}$ $(x_0, y_0) = (2,1)$



Now $f(x,1) = \frac{1}{x-2}$

$\lim_{x \to 2^+} f(x,1) = \infty$, $\lim_{x \to 2^-} f(x,1) = -\infty$

$\Rightarrow$ $f(x,y)$ is unbounded in any rectangle centered at $(2,1)$

$\Rightarrow$ The existence theorem can't be applied here.

Actually from $(x-2)y' = y$

we get $\frac{dy}{y} = \frac{dx}{x-2} \Rightarrow \ln y = \ln(x-2) + \ln c$

$\Rightarrow y = c(x-2)$

But when $x = 2$, $y = 0$

Thus the initial condition $y(2) = 1$ is not satisfied.

Hence this IVP has no solution.

Ex: $y' = x + y$, $y(0) = 3$

Here $f(x,y) = x + y$

- $f(x,y)$ is continuous and bounded in any rectangle R centered at $(0,3)$ $R = \{(x,y) : |x-0| < a, |y-3| < b\}$
- $\frac{\partial f}{\partial y} = 1$ is continuous and bounded in R

Hence by the existence and uniqueness theorem, the IVP has a unique solution $y(x)$ for all x such that $|x - 0| < \alpha$

where $\alpha = \min\{a, \frac{b}{K}\}$ where $|f(x,y)| \leq K$
$\forall (x,y) \in R$

Here $|f(x,y)| = |x+y| \leq |x| + |y| < a + 3 + b = a + b + 3$
$\forall x, y \in R$

So, K can be taken as $a + b + 3$

So, $\quad \alpha = \min\left\{ a, \frac{b}{a+b+3} \right\}$

and the graph of the solution lies

in $R' = \{ (x,y) \in \mathbb{R}^2 : |x-0| < \alpha, |y-3| < b \} \subset R$

<u>Note:</u> $\quad y' = x + y \Rightarrow \frac{dy}{dx} - y = x$

$$\text{I.F.} = e^{-\int 1\,dx} = e^{-x}$$

Multiplying both sides by e^{-x} we get

$$(y e^{-x}) = \int x e^{-x} dx + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$\Rightarrow \quad y = -x - 1 + c e^{x}$$

Now $y(0) = 3 \Rightarrow 3 = -0 - 1 + c \Rightarrow c = 4$

Thus the unique solution of the IVP is

$$\boxed{y = -x - 1 + 4e^{x}}$$

MTH 204 : Lecture 05

Second Order ODEs:

Second order Linear ODE is of the form:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

It is homogeneous if $r(x) = 0$

Ex: • $y'' + 25y = e^{-x}\cos x$: Linear, Non-homogeneous

• $xy'' + y' + xy = 0$

$\Rightarrow y'' + \frac{1}{x}y' + y = 0$: Linear, homogeneous

• $y''y + (y')^2 = 0$: Non-linear

Principle of Superposition:

The linear combination of any two solutions of a second order linear homogeneous ODE is again a solution. That is, if y_1 and y_2 are solutions then $c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

Proof: Given: $y_1'' + p(x)y_1' + q(x)y_1 = 0$

and $y_2'' + p(x)y_2' + q(x)y_2 = 0$

Then $(c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2)$

$$= c_1 y_1'' + c_2 y_2'' + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2)$$
$$= c_1(y_1'' + p(x) y_1' + q(x) y_1) + c_2(y_2'' + p(x) y_2' + q(x) y_2)$$
$$= c_1 \cdot 0 + c_2 \cdot 0 = 0$$

Ex: $y'' + y = 0$

$y_1 = \cos x$, $y_2 = \sin x$ are solutions

$y_1' = -\sin x$, $y_2' = \cos x$

$y_1'' = -\cos x$, $y_2'' = -\sin x$

$y_1'' + y_1 = 0$, $y_2'' + y_2 = 0$

Then $y = c_1 \cos x + c_2 \sin x$ is also a solution.

Ex: The above doesn't work with non-linear ODEs.

$y'' y - x y' = 0$

$y_1 = x^2$, $y_2 = 1$

$y_2' = 0$, $y_2'' = 0$

$0 . y - x . 0 = 0$

$y_1'' = 2$, $y_1' = 2x$

$y_1'' y_1 - x y_1' = 2x^2 - x . 2x = 2x^2 - 2x^2 = 0$

Let $y = y_1 + y_2 = 1 + x^2$

$y' = 2x$, $y'' = 2$

Then $y'' y - x y' = 2(1+x^2) - x(2x) = 2 + 2x^2 - 2x^2$

$\Rightarrow y'' y - x y' = 2 \neq 0$

Ex: The above doesn't work with non homogeneous ODE :

$$y'' + y = 1$$

$y_1 = 1 + \cos x$, $y_2 = 1 + \sin x$

$y_1' = -\sin x$ $y_2' = \cos x$, $y_2'' = -\sin x$

$y_1'' = -\cos x$

$y_1'' + y_1 = -\cos x + 1 + \cos x = 1$ $y_2'' + y_2 = -\sin x + 1 + \sin x$
$= \underline{1}$

$y = y_1 + y_2 = 2 + \cos x + \sin x$

$y' = -\sin x + \cos x$, $y'' = -\cos x - \sin x$

Now $y'' + y = -\cos x - \sin x + 2 + \cos x + \sin x = 2 \neq 1$

So, $y = y_1 + y_2$ is not a solution.

Basis or Fundamental System of Solutions:

For a second order linear homogeneous ODE, two solutions y_1 and y_2 are said to form a basis or fundamental system of solutions if y_1 and y_2 are linearly independent

i.e. $c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 = c_2 = 0$

In other words, y_1 and y_2 shouldn't be a multiple of each other OR $\frac{y_1}{y_2}$ shouldn't be a constant.

General Solution: If y_1 and y_2 form a basis of solutions for a second order linear homogeneous ODE then $\boxed{y(x) = c_1 y_1(x) + c_2 \;_2(x)}$ is the general solution.

We get a particular solution by taking specific values of c_1 and c_2.

<u>Ex:</u> $y'' + y = 0$

$y_1 = \cos x$, $y_2 = \sin x$ are solutions of the ODE

$\frac{y_1}{y_2} = \frac{\cos x}{\sin x} = \cot x \longrightarrow$ not a constant

$\Rightarrow$ y_1 and y_2 are linearly independent

$\Rightarrow$ $y(x) = c_1 \cos x + c_2 \sin x$ is the general solution.

<u>Initial Value Problem (IVP):</u>

$$\left.\begin{array}{l} y'' + p(x) y' + q(x) y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = m_0 \end{array}\right\}$$

<u>Ex:</u> $y'' + y = 0$, $y(0) = 3$, $y'(0) = -0.5$

The general solution is $y(x) = c_1 \cos x + c_2 \sin x$

Then $y'(x) = -c_1 \sin x + c_2 \cos x$

$$\left.\begin{array}{l} y(0) = 3 \Rightarrow c_1 = 3 \\ y'(0) = -0.5 \Rightarrow c_2 = -.5 \end{array}\right\}$$

Hence the particular solution is $y(x) = 3\cos x - 0.5\sin x$

Ex: $(x^2-x)y'' - xy' + y = 0$

Clearly $y_1 = x$ is a solution since $y_1' = 1$, $y_1'' = 0$
and $(x^2-x)\times 0 - x \times 1 + x = 0$

Let us try to find another solution $y_2 = u y_1$
where u is a non-constant function.
(i.e. $\frac{y_2}{y_1} = u$)

Now $y_2' = u y_1' + u' y_1 = u'x + u$
$y_2'' = u''x + u' + u' = u''x + 2u'$

Then $(x^2-x)(u''x + 2u') - x(u'x + u) + ux = 0$

(Reduction of Order)

$$\Rightarrow (x^2-x)x u'' + (2x^2 - 2x - x^2)u' + (-x+x)u = 0$$
$$\Rightarrow x^2(x-1)u'' + (x^2-2x)u' = 0$$
$$\Rightarrow x^2(x-1)u'' + x(x-2)u' = 0$$
$$\Rightarrow x(x-1)u'' + (x-2)u' = 0$$

Let $u' = v$ Then $x(x-1)v' + (x-2)v = 0$

$$\Rightarrow \frac{dv}{v} = -\frac{(x-2)\,dx}{x(x-1)}$$

$$\Rightarrow \frac{dv}{v} = \left[\frac{1}{x-1} - \frac{2}{x}\right]dx$$

$$\Rightarrow \quad \ln v = \ln(x-1) - 2\ln x$$

(ignore the constant of integration)

$$\Rightarrow \quad \ln v = \ln\left(\frac{x-1}{x^2}\right)$$

$$\Rightarrow \quad u' = v = \frac{x-1}{x^2}$$

$$\Rightarrow \quad u' = \frac{1}{x} - \frac{1}{x^2}$$

$$\Rightarrow \quad u = \ln x + \frac{1}{x}$$

So, $y_2 = u y_1 = x\left(\ln x + \frac{1}{x}\right) = x\ln x + 1$

Thus $y_1 = x$ and $y_2 = x\ln x + 1$ and they are linearly independent. Hence they form a basis of solutions of the ODE and the general solution is: $y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 (x\ln x + 1)$

<u>Note:</u> $$-\frac{(x-2)}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$\Rightarrow \quad -\frac{(x-2)}{x(x-1)} = \frac{Ax - A + Bx}{x(x-1)}$$

$$\Rightarrow \quad \frac{-x+2}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$$

$$\Rightarrow \quad A + B = -1, \quad A = -2 \quad \Rightarrow \quad B = 1, \quad A = -2$$

Hence $$-\frac{(x-2)}{x(x-2)} = \frac{1}{x-1} - \frac{2}{x}$$

Reduction of Order:

Consider the ODE : $y'' + p(x)y' + q(x)y = 0$.

Let y_1 be a solution of it.

Design the second solution as: $y_2 = u y_1$

Then
$$y_2' = u' y_1 + u y_1'$$
$$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$
$$\Rightarrow y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

Then
$$\left(u'' y_1 + 2u' y_1' + u y_1''\right) + p(x)\left(u' y_1 + u y_1'\right) + q(x) u y_1 = 0$$

$$\Rightarrow u'' y_1 + \left(2y_1' + p(x) y_1\right) u' + \underbrace{\left(y_1'' + p(x) y_1' + q(x) y_1\right)}_{= 0} u = 0$$

$$\Rightarrow u'' y_1 + \left(2y_1' + p(x) y_1\right) u' = 0$$

Let $u' = V$

$$\Rightarrow y_1 V' + \left(2y_1' + p(x) y_1\right) V = 0$$

$$\Rightarrow \frac{dV}{V} = -\frac{2y_1' + p(x) y_1}{y_1} dx$$

$$\Rightarrow \frac{dV}{V} = -\left(\frac{2y_1'}{y_1} + p(x)\right) dx$$

$$\Rightarrow \ln V = -2\ln y_1 - \int p(x)\,dx$$

$$\Rightarrow V = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

Thus $u' = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$

$\Rightarrow u = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\, dx$

Hence $y_2 = u y_1 = \boxed{y_1 \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\, dx}$

Second Order Homogeneous ODEs with constant coefficients:

First we look at $y' + ky = 0$
and try if $y = e^{\lambda x}$ is a solution.

Then $\lambda e^{\lambda x} + k e^{\lambda x} = 0$

$\Rightarrow (\lambda + k) e^{\lambda x} = 0$

$\Rightarrow \lambda + k = 0 \Rightarrow \lambda = -k \rightarrow \lambda_1$

Then $y_1 = e^{\lambda_1 x} = e^{-kx}$

general solution: $y = c_1 y_1 = c_1 e^{-kx}$

Now we consider $y'' + ay' + by = 0$ where $a, b \in \mathbb{R}$

let us try if $y = e^{\lambda x}$ is a solution.

$y'' = \lambda^2 e^{\lambda x}, \quad y' = \lambda e^{\lambda x}$

Then $$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

$$\Rightarrow \boxed{\lambda^2 + a\lambda + b = 0} \rightarrow \boxed{\text{Characteristic Equation}}$$

Case 1: $a^2 - 4b > 0$ and so there are two distinct real roots of the characteristic equation

Let the roots be λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$.

Then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are two linearly independent solutions $\left(\frac{e^{\lambda_2 x}}{e^{\lambda_1 x}} = e^{(\lambda_2 - \lambda_1)x} \neq \text{Constant since } \lambda_2 - \lambda_1 \neq 0 \right)$

So, the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$= c_1 e^{\frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right)x} + c_2 e^{\frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)x}$$

Case 2: $a^2 - 4b = 0$ and so the real roots are equal (a double root)

So the root is $\frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} = \lambda$ (say)

So, $y_1 = e^{\lambda x} = e^{-\frac{a}{2}x}$ is one solution

Then by reduction of order, the second solution is given by $y_2(x) = e^{\lambda x} \int \frac{1}{(e^{\lambda x})^2} e^{-\int a\,dx}\,dx$

$$\Rightarrow \quad y_2(x) = e^{-\frac{a}{2}x} \int \frac{1}{\left(e^{-\frac{a}{2}}\right)^2} e^{-ax}\, dx$$

$$= e^{-\frac{a}{2}x} \int \frac{e^{-ax}}{e^{-ax}}\, dx = e^{-\frac{a}{2}x} x = x e^{-\frac{a}{2}x}$$

So, the general solution is

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

$$= e^{\lambda x}(c_1 + c_2 x) = \boxed{e^{-\frac{a}{2}x}(c_1 + c_2 x)}$$

<u>Case 3:</u> $a^2 - 4b < 0$ and so the roots are complex.

The two roots are $\lambda_1 = \dfrac{-a + \sqrt{a^2-4b}}{2} = -\dfrac{a}{2} + i\omega$

and $\lambda_2 = \dfrac{-a - \sqrt{a^2-4b}}{2} = -\dfrac{a}{2} - i\omega$

where $\omega = \dfrac{\sqrt{4b-a^2}}{2} \in \mathbb{R}$

$\left(\text{hence } \dfrac{\sqrt{a^2-4b}}{2} = i\omega\right)$

So, $y_1 = e^{\lambda_1 x} = e^{\left(-\frac{a}{2} + i\omega\right)x} = e^{-\frac{a}{2}x}\left(e^{i\omega x}\right)$

$$\Rightarrow y_1 = e^{-\frac{a}{2}x}\left(\cos(\omega x) + i \sin(\omega x)\right)$$

and $y_2 = e^{\lambda_2 x} = e^{\left(-\frac{a}{2} - i\omega\right)x} = e^{-\frac{a}{2}x}\left(e^{-i\omega x}\right)$

$$\Rightarrow y_2 = e^{-\frac{a}{2}x}\left(\cos(\omega x) - i \sin(\omega x)\right)$$

Now by using principle of superposition we can say $Y_1 = \frac{1}{2} y_1 + \frac{1}{2} y_2$ and $Y_2 = \frac{1}{2i} y_1 - \frac{1}{2i} y_2$ are also solutions.

So, $Y_1 = \frac{1}{2} y_1 + \frac{1}{2} y_2 = e^{-\frac{a}{2}x} \cos(\omega x)$ is a solution

and $Y_2 = \frac{1}{2i} y_1 - \frac{1}{2i} y_2 = e^{-\frac{a}{2}x} \sin(\omega x)$ is another solution.

Also $\frac{Y_1}{Y_2} = \cot(\omega x)$ is not a constant and

hence Y_1 and Y_2 are linearly independent solutions.

Therefore
$$y(x) = c_1 e^{-\frac{a}{2}x} \cos(\omega x) + c_2 e^{-\frac{a}{2}x} \sin(\omega x)$$
$$= \boxed{e^{-\frac{a}{2}x} \left(c_1 \cos(\omega x) + c_2 \sin(\omega x) \right)}$$

is the general solution.

MTH 204 : Lecture 06

Ex: $x^2y'' - 5xy' + 9y = 0 \qquad \left(\Rightarrow y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0\right)$

Now $y_1 = x^3$ is a solution of the above equation

Since $x^2y_1'' - 5xy_1' + 9y_1 = x^2 \times 6x - 5x \times 3x^2 + 9x^3$

$$= 6x^3 - 15x^3 + 9x^3 = 0$$

Now using method of reduction of order

$$u(x) = \int \frac{1}{x^6} e^{-\int -\frac{5}{x}dx} dx$$

$$= \int \frac{1}{x^6} e^{5\ln x} dx = \int \frac{x^5}{x^6} dx = \int \frac{1}{x} dx = \ln x$$

So, $y_2 = u(x)\, y_1 = (\ln x)x^3$ is another solution.

Hence the general solution is $y(x) = c_1x^3 + c_2(\ln x)x^3$

Ex: $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$

Characteristic Equation is: $\lambda^2 + \lambda - 2 = 0$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{(1)^2 - 4(-2)(1)}}{2} = \frac{-1 \pm 3}{2}$$

$$\Rightarrow \lambda = 1, -2$$

Hence the general solution is $y(x) = c_1e^x + c_2e^{-2x}$

Then $y'(x) = c_1e^x - 2c_2e^{-2x}$ Now $y(0) = 4 \Rightarrow c_1 + c_2 = 4$

and $y'(0) = -5 \Rightarrow c_1 - 2c_2 = -5$

$$\Rightarrow 3c_2 = 9 \Rightarrow c_2 = \boxed{3} \text{ and } c_1 = 4 - 3 = \boxed{1}$$

So, the particular solution of the IVP is

$$\boxed{y(x) = e^{x} + 3e^{-2x}}$$

Ex: $y'' + 6y' + 9y = 0$

The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$

$\Rightarrow (\lambda+3)^2 = 0 \Rightarrow -3$ is a double root of the characteristic equation $\Rightarrow$ The general solution is $\boxed{y(x) = e^{-3x}(c_1 + c_2 x)}$

Ex: $y'' + 0.4y' + 9.04y = 0, \ y(0) = 0, \ y'(0) = 3$

The characteristic equation is:

$$\lambda^2 + 0.4\lambda + 9.04 = 0$$

$$\Rightarrow \lambda = \frac{-0.4 \pm \sqrt{(-.4)^2 - 4 \times (9.04)}}{2} = \frac{-0.4 \pm \sqrt{.16 - 36.16}}{2}$$

$$= \frac{-0.4 \pm 6i}{2} = -0.2 \pm 3i$$

So, $\lambda_1 = -0.2 + 3i$, $\lambda_2 = -0.2 - 3i$

Thus the general solution is $y(x) = e^{-0.2x}(c_1 \cos(3x) + c_2 \sin(3x))$

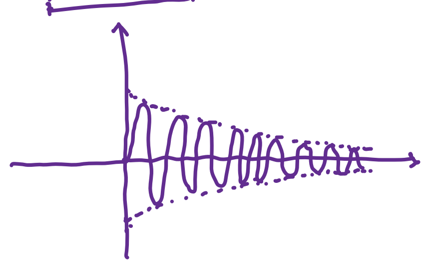
$y(0) = 0 \Rightarrow c_1(1) + c_2 \times 0 = 0 \Rightarrow \boxed{c_1 = 0}$

$y'(x) = -0.2\, e^{-0.2x}(c_2 \sin(3x)) + e^{-0.2x}\, 3c_2 \cos(3x))$

Now $y'(0) = 3 \Rightarrow 0 + (1)\, 3c_2 = 3 \Rightarrow \boxed{c_2 = 1}$

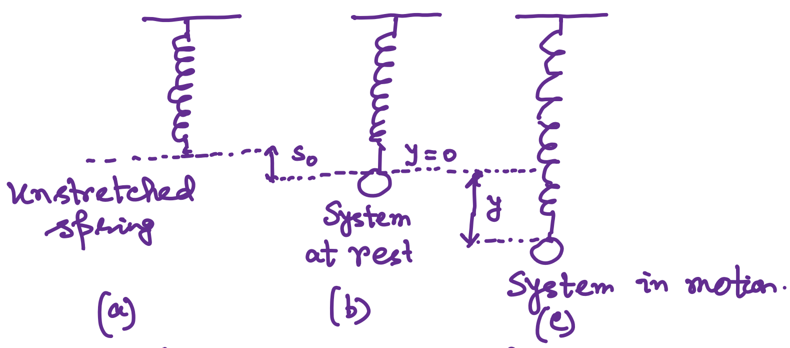
So, the solution of the IVP is

$$\boxed{y(x) = e^{-0.2x} \sin(3x)}$$



Free Oscillations (Undamped):

Free Oscillations of a mass-spring system



Mechanical mass-spring system

Let m be the mass of the ball.

Then Spring force $F = -ky$ $\quad k > 0$ (Hooke's Law)

(always opposes the motion) $\quad$ (k = spring Constant)

In (b), $\quad ks_0 = mg$

Now if a is the acceleration,

$$ma = -k(y + s_0) + mg = -ky$$

$$\Rightarrow \quad y'' = -\frac{k}{m} y \quad \Rightarrow \quad \boxed{y'' + \frac{k}{m} y = 0}$$

Characteristic Equation: $\quad \lambda^2 + \frac{k}{m} = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{k}{m}}$

$$\Rightarrow \lambda = \pm i\omega_0$$

The general solution is $y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$

This is called Harmonic Oscillation with

angular frequency $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

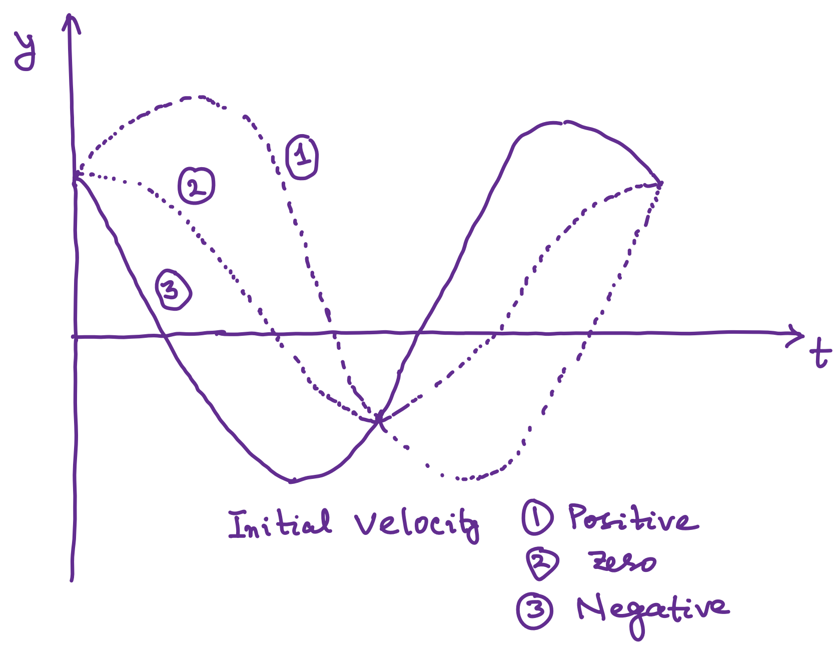
Natural Frequency $\boxed{f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$ (in Hz) and Oscillation period $\boxed{T_0 = \frac{1}{f_0} = 2\pi\sqrt{\frac{m}{k}}}$ (in s)

Now we can replace $c_1 = \cos\delta$ and $c_2 = C\sin\delta$, for two constants C and δ.

Then $y(t) = C\left(\cos\delta\cos(\omega_0 t) + \sin\delta\sin(\omega_0 t)\right)$

$\Rightarrow y(t) = C\cos(\omega_0 t - \delta)$

Where $\boxed{C = \sqrt{c_1^2 + c_2^2}}$ and $\boxed{\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right)}$



Typical harmonic oscillations with same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$ (Positive 1, zero 2, negative 3)

Ex: Weight $= 98\,N$, $S_0 = 1.09\,m$

How many cycles per minute will system execute?

$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Now $\quad ks_0 = mg \;\Rightarrow\; k \times 1.09 = 98 \;\Rightarrow\; k = 90$

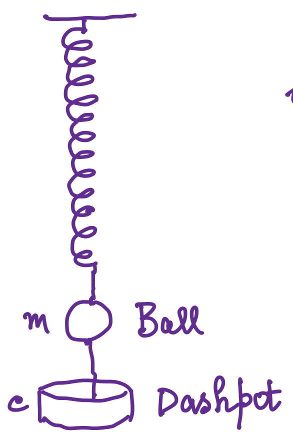
$m = 10$

Then $f_0 = \frac{1}{2\pi}\sqrt{\frac{90}{10}} = \frac{3}{2\pi} \simeq 0.48$ Hz $\left(\frac{1}{s}\right)$

In 1 minute, number of oscillations $= 60 \times .48 \approx 29$

Damped Oscillations:

Damped oscillations of a mass-spring System



There is a braking force present which can be modeled as $-cy'$ $(c>0)$ $\left(c = \text{damping constant}\right)$

Then $\quad my'' = -ky - cy'$

$\Rightarrow \; y'' = -\frac{k}{m}y - \frac{c}{m}y'$

$\Rightarrow \; \boxed{y'' + \frac{c}{m}y' + \frac{k}{m}y = 0}$

The characteristic equation is

$$\boxed{\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0}$$

$$\Rightarrow \quad \lambda_1 = \frac{-\frac{c}{m} + \sqrt{\frac{c^2}{m^2} - \frac{4k}{m}}}{2} = \frac{-c + \sqrt{c^2 - 4mk}}{2m}$$

$$=: -\alpha + \beta = -(\alpha - \beta)$$

and $\quad \lambda_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} =: -\alpha - \beta = -(\alpha + \beta)$

where $\quad \boxed{\alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{\sqrt{c^2 - 4mk}}{2m}}$

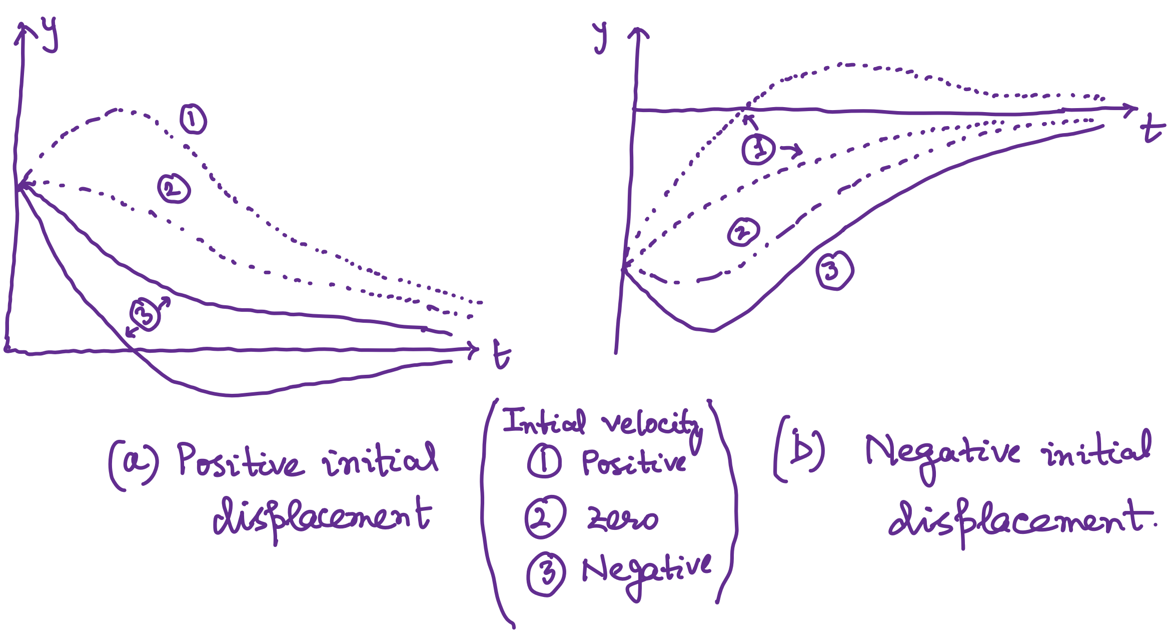
Now we need to consider three cases:

(1) $c^2 - 4mk > 0$ (over damping):

Then the roots of the characteristic equation are distinct and real.

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$

- Note that $\alpha > \beta \Rightarrow y(t) \to 0$ as $t \to \infty$
- There is no oscillation, mass will pass the mean position $(y=0)$ atmost once.



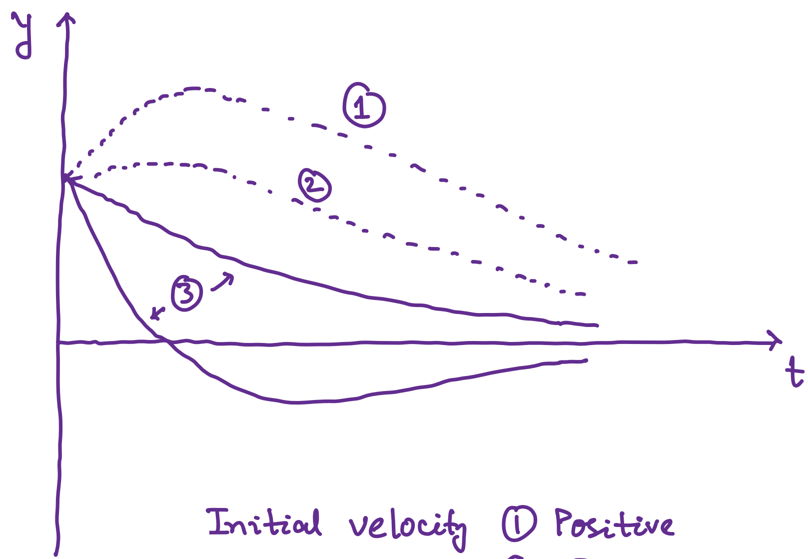
(2) $c^2 - 4mk = 0$ (Critical damping):

$$\Rightarrow \quad \alpha = \frac{c}{2m}, \quad \beta = 0$$

$$y(t) = (c_1 + c_2 t) e^{-\alpha t}$$

- $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$

- No Oscillation
- Crosses mean position ($y=0$) atmost once



Initial velocity ① Positive
② Zero
③ Negative

③ $c^2 - 4mk < 0$ (Under damping):

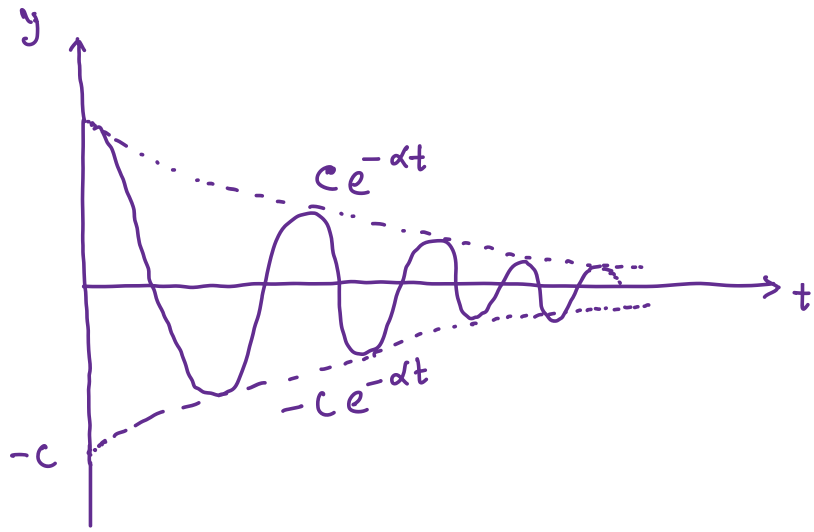
$$\beta = i \frac{1}{2m}\sqrt{4mk - c^2} = i\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = i\omega^* \text{ (say)}$$

Note that $\omega^* \longrightarrow \omega_0 = \sqrt{\frac{k}{m}}$ as $c \to 0$

The general solution is

$$y(t) = C e^{-\alpha t} \cos(\omega^* t - \delta)$$

- Note that $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$
- The motion is oscillatory.



Euler - Cauchy Equations:

They are equations of the form

$$x^2 y'' + a x y' + b y = 0$$

Let $y = x^m$ Then $y' = m x^{m-1}$, $y'' = m(m-1) x^{m-2}$

Then substituting we get

$$x^2\left(m(m-1)x^{m-2}\right) + ax\left(mx^{m-1}\right) + bx^m = 0$$

$$\Rightarrow m(m-1)x^m + amx^m + bx^m = 0$$

$$\Rightarrow x^m\left(m(m-1) + am + b\right) = 0$$

$$\Rightarrow x^m\left(m^2 + (a-1)m + b\right) = 0$$

Hence x^m is a solution of ODE iff m is a solution of $m^2 + (a-1)m + b = 0$

Case 1 The solutions are $\dfrac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$

$$= \frac{1-a}{2} \pm \sqrt{\frac{(1-a)^2}{4} - b}$$

So, general solution is $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

where $m_1 = \dfrac{1-a}{2} + \sqrt{\dfrac{(1-a)^2}{4} - b}$

$m_2 = \dfrac{1-a}{2} - \sqrt{\dfrac{(1-a)^2}{4} - b}$

Ex: $x^2 y'' + 1.5 x y' - 0.5 y = 0$

The equation for m is : ($a = 1.5$, $b = -0.5$) $\Rightarrow$

$$m^2 + .5m - .5 = 0$$

$$(m+1)(m-.5) = 0$$

$$\Rightarrow m_2 = -1 \text{ and } m_1 = .5$$

So, the general solution is $y = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$

$$\Rightarrow \boxed{y = c_1 \sqrt{x} + \frac{c_2}{x}}$$

(It must be $x > 0$ for this solution to exist.)

Case 2: A real double root:

- A real double root occurs when $\frac{1}{4}(1-a)^2 - b = 0$

$$\Rightarrow b = \frac{1}{4}(1-a)^2$$

Then $x^2y'' + axy' + by = 0 \Rightarrow x^2y'' + axy' + \frac{(1-a)^2}{4}y = 0$

$$\Rightarrow y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

In this case one solution is $\left(m = \frac{1-a}{2}\right)$

$$y_1 = x^{\frac{1-a}{2}}$$

By reduction of order

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int \frac{a}{x}dx} dx$$

$$= y_1 \int \frac{1}{y_1^2} e^{-a\ln x} dx$$

$$= y_1 \int \frac{1}{y_1^2} e^{\ln(x^{-a})} dx = y_1 \int \frac{1}{y_1^2} x^{-a} dx$$

$$= x^{\frac{1-a}{2}} \int \frac{x^{-a}}{\left(x^{\frac{1-a}{2}}\right)^2} dx$$

$$= x^{\frac{1-a}{2}} \int \frac{x^{-a}}{x^{1-a}} dx = x^{\frac{1-a}{2}} \int \frac{dx}{x}$$

$$= x^{\frac{1-a}{2}} \ln x$$

So, the general solution is

$$y = c_1 y_1 + c_2 y_2 = \boxed{(c_1 + c_2 \ln x)\, x^{\frac{1-a}{2}}}$$

(3) The two roots m_1 and m_2 are complex

Let $m_1 = \alpha + i\omega$, $m_2 = \alpha - i\omega$ (say)

Two independent solutions are

$$y_1 = x^{\alpha + i\omega} = x^{\alpha} x^{i\omega} = x^{\alpha} \left(e^{\ln x}\right)^{i\omega}$$

$$= x^{\alpha} e^{i\omega \ln x} = x^{\alpha} \left(\cos\omega(\ln x) + i \sin\omega(\ln x)\right)$$

$$y_2 = x^{\alpha - i\omega} = x^{\alpha} x^{-i\omega} = x^{\alpha} \left(e^{-\ln x}\right)^{i\omega}$$

$$= x^{\alpha} e^{-i\omega \ln x} = x^{\alpha}\left(\cos\omega(\ln x) - i\sin\omega(\ln x)\right)$$

Two other independent solutions are

$$y_1^* = \frac{y_1+y_2}{2} = x^{\alpha}\cos(\omega(\ln x))$$

$$y_2^* = \frac{y_1-y_2}{2i} = x^{\alpha}\sin(\omega(\ln x))$$

So, the general solution is

$$y = c_1 y_1^* + c_2 y_2^* = x^{\alpha}\left(c_1\cos(\omega(\ln x)) + c_2\sin(\omega(\ln x))\right)$$

Ex: $x^2 y'' - 5xy' + 9y = 0$

So, the equation for m is $(a=-5,\ b=9)$

$$m^2 + (-5-1)m + 9 = 0$$

$$\Rightarrow m - 6m + 9 = 0$$

$$\Rightarrow (m-3)^2 = 0$$

So, we have a real double root $m=3$

Hence the general solution is

$$\boxed{y(x) = (c_1 + c_2 \ln x)\, x^3}$$

Ex: $4x^2y'' + 5y = 0$

$\Rightarrow x^2y'' + \frac{5}{4}y = 0$

So, the equation for m is $m^2 + (0-1)m + \frac{5}{4} = 0$

$\left(a = 0,\ b = \frac{5}{4}\right)$ $\Rightarrow m^2 - m + \frac{5}{4} = 0$

$\Rightarrow 4m^2 - 4m + 5 = 0$

$$\Rightarrow m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(5)}}{(2)(4)} = \frac{4 \pm \sqrt{16-80}}{8}$$

$$= \frac{4 \pm 8i}{8} = \frac{1}{2} \pm i$$

So, the roots are complex and the general solution is $\boxed{y(x) = x^{1/2}\left(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))\right)}$

Ex: Find the electrostatic potential $v = v(r)$ between two concentric sphere of radius $r_1 = 5\,cm$ and $r_2 = 10\,cm$ kept at potential $v_1 = 110\,V$ and $v_2 = 0$

Physical information: $v(r)$ is a solution of $rv'' + 2v' = 0$ where $v' = \frac{dv}{dr}$

$$r v'' + 2v' = 0$$

Multiplying by r, we get

$$r^2 v'' + 2 r v' = 0$$

So, the equation is $\quad m^2 + (2-1)m = 0$

$$\Rightarrow \quad m^2 + m = 0 \quad \Rightarrow m = 0, -1$$

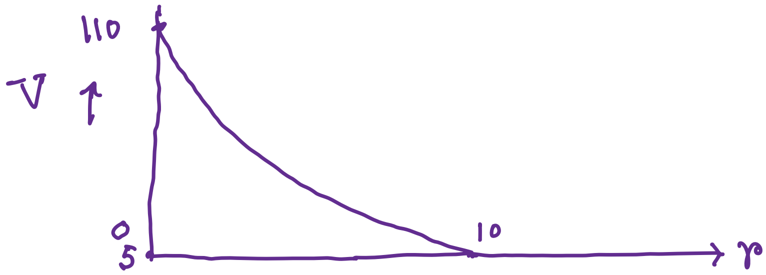
So, the general solution is

$$v = c_1 r^0 + c_2 r^{-1} = c_1 + \frac{c_2}{r}$$

$$\left.\begin{aligned} v(5) = 110 &\Rightarrow c_1 + \frac{c_2}{5} = 110 \\ v(10) = 0 &\Rightarrow c_1 + \frac{c_2}{10} = 0 \end{aligned}\right\}$$

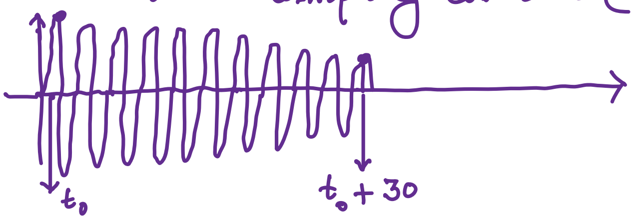
$$c_2 = 1100, \quad c_1 = -110$$

So, $\boxed{v = -110 + \dfrac{1100}{r}}$



MTH 204 : Lecture 07

Ex: Consider an underdamped motion of a body of mass $m = 0.5$ kg. If the time between two consecutive maxima is 3 sec. and the maximum amplitude decreases to half of its initial value after 10 cycles, what is the damping constant (c) of the system?



Here $m = 0.5$, $T = 3$.

Let maximum amplitude occurs at t_0.

Then $\frac{1}{2}$(max amplitude) occurs at $t_0 + 30$

(Since $T = 3$, time taken for 10 cycles is 30 sec)

Now the equation in this case is $y(t) = Ce^{-\alpha t}\cos(\omega^* t - \delta)$

Now maximum amplitude $= C e^{-\alpha t_0}$

$$\frac{1}{2}(\text{max amplitude}) = \frac{1}{2}Ce^{-\alpha t_0} = Ce^{-\alpha(t_0+30)}$$

$$\Rightarrow \frac{1}{2}Ce^{-\alpha t_0} = Ce^{-\alpha t_0}e^{-30\alpha} \Rightarrow e^{-30\alpha} = \frac{1}{2}$$

$$\Rightarrow 30\alpha = \ln 2 \Rightarrow \alpha = \frac{\ln 2}{30}$$

Now $\alpha = \frac{c}{2m} \Rightarrow \frac{\ln 2}{30} = \frac{c}{2m}$

$$\Rightarrow c = 2.m\frac{\ln 2}{30} = 2 \times 0.5 \frac{\ln 2}{30} = \boxed{\frac{\ln 2}{30}}$$

Non homogeneous ODEs (NH):

$$y'' + p(x)\, y' + q(x)\, y = r(x) \text{ is } \quad (NH)$$

A general solution of the NH ODE is of the form $y_h + y_p$ where y_h is the general solution of the homogeneous equation

$$y'' + p(x)\, y' + q(x) y = 0 \quad (H)$$

and y_p is a particular solution of (NH)

Theorem 1: If y_H is a solution of H and y_N is a solution of NH, then $y_H + y_N$ is a solution of NH.

Proof:

$$(y_H + y_N)'' + p(x)\,(y_H + y_N)' + q(x)\,(y_H + y_N)$$

$$= \left(y_H'' + p(x)\, y_H' + q(x)\, y_H\right) + \left(y_N'' + p(x)\, y_N' + q(x)\, y_N\right)$$

$$= r(x) + 0 = r(x)$$

Theorem 2: If y_1 and y_2 are two solutions of NH, then $y_1 - y_2$ is a solution of H.

Proof: $(y_1 - y_2)'' + p(x)\,(y_1 - y_2)' + q(x)\,(y_1 - y_2)$

$$= \left[y_1'' + p(x)y_1' + q(x)y_1\right] - \left[y_2'' + p(x)y_2' + q(x)y_2\right]$$

$$= r(x) - r(x) = 0$$

Method of Undetermined coefficients:

Ex: $y'' + y = 0.001x^2$, $y(0) = 0$, $y'(0) = 1.5$

H: $y'' + y = 0$

characteristic Equation: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

So, $y_h = c_1 \cos x + c_2 \sin x$

Let $y_p = Ax^2 + Bx + C$ (where A, B, C are constants)

$y_p' = 2Ax + B$

$y_p'' = 2A$

So, if y_p is a solution of the NH,

then $2A + Ax^2 + Bx + C = 0.001x^2$

$\Rightarrow Ax^2 + Bx + (2A + C) = 0.001x^2$

$\Rightarrow A = .001$, $B = 0$, $2A + C = 0$

$\Rightarrow C = -.002$

So, $y_p = .001x^2 - .002$

<u>check:</u> $y_p' = .002x$, $y_p'' = .002$

So, $y_p'' + y_p = \cancel{.002} + .001x^2 - \cancel{.002}$

$= .001x^2 =$ RHS of NH

So, the general solution of the NH

is: $y(x) = c_1 \cos x + c_2 \sin x + .001x^2 - .002$

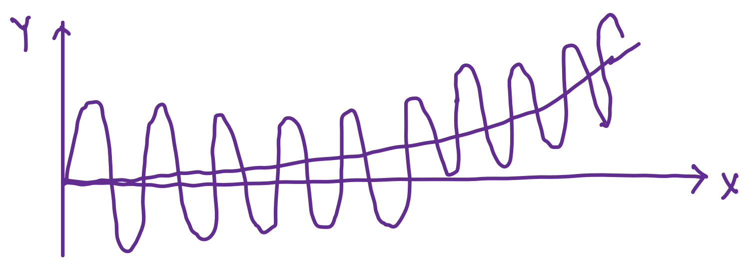
$y(0) = 0 \Rightarrow c_1 - .002 = 0 \Rightarrow c_1 = .002$

$y'(x) = -c_1 \sin x + c_2 \cos x + .002x$

$y'(0) = 1.5 \Rightarrow c_2 = 1.5$

So, the particular solution of the NH is

$y = .002 \cos x + 1.5 \sin x + .001x^2 - .002$



(2) $y'' + 3y' + 2.25y = -10e^{-1.5x}$, $y(0)=1$, $y'(0)=0$

<u>H:</u> characteristic Equation: $\lambda^2 + 3\lambda + 2.25 = 0$

$$\Rightarrow \quad (\lambda + 1.5)^2 = 0$$

$$\Rightarrow \quad \lambda = -1.5, -1.5$$

$$y_h = (c_1 + c_2 x)e^{-1.5x}$$

Note that $e^{-1.5x}$ is a solution of the homogeneous equation because
$2.25e^{-1.5x} - 4.5e^{-1.5x} + 2.25e^{-1.5x}$
$= 0$

Let $\quad y_p = Ax^2e^{-1.5x}$

$$y_p' = A\left(2x e^{-1.5x} - 1.5x^2e^{-1.5x}\right) = A(2x - 1.5x^2)e^{-1.5x}$$

$$y_p'' = A\left(2e^{-1.5x} - 3xe^{-1.5x} - 3xe^{-1.5x} + 2.25x^2e^{-1.5x}\right)$$

$$= A\left(2 - 6x + 2.25x^2\right)e^{-1.5x}$$

If y_p is a solution of NH,
then
$$A(2 - 6x + 2.25x^2)e^{-1.5x}$$
$$+ 3A(2x - 1.5x^2)e^{-1.5x}$$
$$+ 2.25Ax^2e^{-1.5x} = -10e^{-1.5x}$$

$$\Rightarrow \quad e^{-1.5x} A\left[2 - \cancel{6x} + 2.25\cancel{x^2} + \cancel{6x} - 4.5\cancel{x^2} + 2.\cancel{25x^2}\right] = -10e^{-1.5x}$$

$$\Rightarrow \quad 2A = -10 \Rightarrow A = -5$$

The general solution is :

$$y = e^{-1.5x}(c_1 + c_2 x) - 5x^2 e^{-1.5x}$$

$$= (c_1 + c_2 x - 5x^2)e^{-1.5x}$$

Given:

$y(0) = 1$, $y'(0) = 0$, $y(0) = 1 \Rightarrow c_1 = 1.$

$$y'(x) = (c_2 - 10x)e^{-1.5x} + (c_1 + c_2 x - 5x^2)(-1.5e^{-1.5x})$$

$$y'(0) = 0 \Rightarrow c_2 + c_1(-1.5) = 0$$

$$\Rightarrow c_2 = 1.5$$

So, $$y = (1 + 1.5x - 5x^2)e^{-1.5x}$$

Method of Undetermined Coefficient:

To solve $y'' + ay' + by = r(x)$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n=0,1,2,\ldots)$	$C_n x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$
$k\cos\omega x$, $k\sin\omega x$	$C_1\cos\omega x + C_2\sin\omega x$
$ke^{\alpha x}\cos\omega x$, $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(C_1\cos\omega x + C_2\sin\omega x)$

Modification: If the term in $r(x)$ is also a solution of H, then multiply by x or x^2 depending on if this solution corresponds to a single root or a double root of the characteristic polynomial.

Sum Rule: If $r(x)$ is a sum of functions, choose a sum of y_p's.

Example:

(1) If $r(x) = 10\sin 2x$, then $y_p = A\sin 2x + B\cos 2x$

(2) $r(x) = 8\cos x$ and it is a solution of H and it corresponds to a single root of the characteristic equation, then $y_p = x(A\cos x + B\sin x)$

(3) $r(x) = e^{2x}\sin 5x$ and it is a solution of H and it corresponds to a double root of the characteristic equation, then $y_p = x^2 e^{2x}(A\cos 5x + B\sin 5x)$

(4) $r(x) = \cos x + x^2 - 1$, then $y_p = A\cos x + B\sin x + Cx^2 + Dx + E$

(5) $r(x) = \cos x + x^2 - 1$ and $\cos x$ is a solution of H and it corresponds to a double root of the characteristic equation, then $y_p = x^2(A\cos x + B\sin x) + (Cx^2 + Dx + E)$

Ex: $y'' + 2y' + .75y = 2\cos x - .25\sin x + .09x$

$y(0) = 2.78, \quad y'(0) = -0.43$

For the general solution of the Homogeneous ODE, we get the characteristic equation as:

$$\lambda^2 + 2\lambda + .75 = 0$$

$$\Rightarrow \quad \lambda^2 + 2\lambda + \frac{3}{4} = 0 \Rightarrow \left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{3}{2}\right) = 0$$

$$\Rightarrow \lambda_1 = -\frac{1}{2},\ -\frac{3}{2}$$

So, the general solution of H: $y_h = c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x}$

Let $y_p = C\cos x + D\sin x + Ax + B$

Then $y_p' = -C\sin x + D_2\cos x + A$

$$y_p'' = -C\cos x - D\sin x$$

Since y_p is a solution of the NH,

$$-C\cos x - D\sin x - 2C\sin x + 2D\cos x + 2A + .75C\cos x$$
$$+ .75D\sin x + .75Ax + .75B$$
$$= 2\cos x - .25\sin x + .99x$$

$$\Rightarrow (-.25C + 2D)\cos x + (-2C - .25D)\sin x$$
$$+ .75Ax + (2A + .75B) = 2\cos x - .25\sin x + .09x$$

$$\Rightarrow \begin{aligned} -.25C + 2D &= 2 \\ -2C - .25D &= -.25 \end{aligned} \quad \Rightarrow \begin{aligned} -.5C + 4D &= 4 \\ -.5C - .625D &= -.625 \\ - \qquad + \qquad & \quad + \\ \hline 4.625D &= 4.625 \end{aligned} \Rightarrow \boxed{D=1}$$

Then $\boxed{C=0}$, Also $.75A = .09$ and $2A + .75B = 0$

$$\Rightarrow A = \frac{.09}{.75} = \boxed{.12} \text{ and } B = -\frac{2\times .12}{.75} = -\frac{.24}{.75} = \boxed{-.32}$$

Hence the general solution of the ODE is:

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x} + \sin x + .12x - .32$$

Now $y'(x) = -\frac{c}{2} e^{-\frac{1}{2}x} - \frac{3c_2}{2} e^{-\frac{3}{2}x} + \cos x + .12$

$y(0) = 2.78$ and $y'(0) = .43 \Rightarrow$ $c_1 + c_2 - .32 = 2.78$

and $-\frac{c_1}{2} - \frac{3c_2}{2} + 1 + .12 = -.43$

$\Rightarrow$ $c_1 + c_2 = 3.10$

$\Rightarrow$ $-c_1 - 3c_2 = -3.10$

$-2c_2 = 0 \Rightarrow c_2 = 0$ and hence $c_1 = 3.10$

Hence the particular solution of the Nonhomogeneous Equation

is: $\boxed{y(x) = 3.10\, e^{-\frac{1}{2}x} + \sin x + .12x - .32}$

Modification Rule:

If $r(x)$ is a product of functions in the table then the choice for y_p is also product of individual choices for each term in the product.

Ex: $r(x) = xe^{4x}$

Case 1: e^{4x} is not a solution of H : $y_p(x) = (Ax+B)\,e e^{4x} = (Acx+Bc)e^{4x}$

$= c_1 x e^{4x} + c_2 e^{4x}$

Case 2: e^{4x} is a solution of H: $y_p(x) = Ax^2e^{4x} + Bxe^{4x}$

Case 3: e^{4x} and xe^{4x} are both solutions of H : $y_p(x) = Ax^3e^{4x} + Bx^2e^{4x}$

Ex: $r(x) = (9x^2 - 103x)\cos x$

Case 1: $\cos x$ is not a solution of H: $y_p(x) = (Ax^2+Bx+C)(D\cos x + E\sin x)$

$= ADx^2\cos x + AEx^2\sin x + BDx\cos x + BEx\sin x + CD\cos x + CE\sin x$

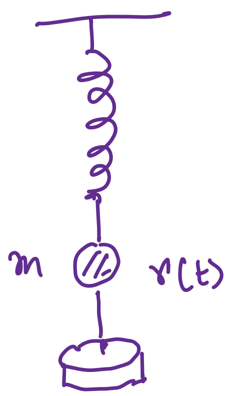
$= A_1x^2\cos x + B_1x^2\sin x + C_1x\cos x + D_1x\sin x + E_1\cos x + F_1\sin x$

Case 2 $\cos x$ is a solution of H :

$y_p(x) = A_1x^3\cos x + B_1x^3\sin x + C_1x^2\cos x + D_1x^2\sin x + E_1x\cos x + F_1x\sin x$

Forced Oscillation:

So far, we have considered vertical motion of a mass-spring system in the case of free motion:



That is no external force but only internal forces controlled the motion.

These forces are my'', damping force cy' $(c > 0)$ and spring force ky (restoring force)

The ODE for this model is the homogeneous equation: $my'' + cy' + ky = 0$

Now we include an external force $r(t)$ and the resulting motion is called a Forced motion. Then the ODE will be $my'' + cy' + ky = r(t)$

In this case $r(t)$ is known as input or driving force

and $y(t)$ is called the Output or response of the system to the driving force.

Let us assume $r(t) = F_0 \cos \omega t$ (Periodic driving force)
$(F_0 > 0, \omega > 0)$

Then the ODE is:

$$\boxed{m y'' + c y' + k y = F_0 \cos(\omega t)}$$

Consider the associated Homogeneous system: The roots of the characteristic equation are $\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$

Let y_h be the general solution of the homogeneous equation
We have seen different forms of y_h depending on the sign of $c^2 - 4mk$.

Now for the particular solution of the Non-homogeneous equation,

let $y_p = a \cos \omega t + b \sin \omega t$

Then $y_p' = -a\omega \sin \omega t + b\omega \cos \omega t$

and $y_p'' = -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t$

Substituting:

$$m\left(-a\omega^2\cos\omega t - b\omega^2\sin\omega t\right) + c\left(-a\omega\sin\omega t + b\omega\cos\omega t\right)$$
$$+ k\left(a\cos\omega t + b\sin\omega t\right) = F_0\cos\omega t$$

$$\Rightarrow \left(-ma\omega^2 + bc\omega + ka\right)\cos\omega t + \left(-mb\omega^2 - ca\omega + kb\right)\sin\omega t$$
$$= F_0\cos\omega t$$

$$\Rightarrow \left.\begin{aligned} -ma\omega^2 + bc\omega + ka &= F_0 \\ -mb\omega^2 - ca\omega + kb &= 0 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} (k - m\omega^2)a + (c\omega)b &= F_0 \\ (-c\omega)a + (k - m\omega^2)b &= 0 \end{aligned}\right\}$$

$$\Rightarrow \begin{pmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}$$

$$b = F_0 \frac{c\omega}{(k - m\omega^2)^2 + \omega^2 c^2}$$

Recall $\omega_0^2 = \frac{k}{m} \Rightarrow k = m\omega_0^2$

So, $$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2-\omega^2)^2 + \omega^2 c^2}$$

$$b = F_0 \frac{c\omega}{m^2(\omega_0^2-\omega^2)^2 + \omega^2 c^2}$$

So, $$y_p = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2-\omega^2)^2 + \omega^2 c^2} \cos\omega t + F_0 \frac{c\omega}{m^2(\omega_0^2-\omega^2)^2 + \omega^2 c^2} \sin\omega t$$

The general solution of the non-homogeneous equation

$$y(t) = y_h + y_p$$

Case 1: Undamped Forced Oscillation: $(c = 0)$

When $c = 0$

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos\omega t$$

(provided $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ i.e. $\cos\omega t$ is not a solution of the homogeneous equation $my'' + ky = 0$)

Also the general solution of the homogeneous equation: $y_h = C\cos(\omega_0 t - \delta)$

The general solution of the nonhomeogeous equation

is: $$y = c\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

(This is a superposition of two harmonic oscillations)

(This is valid as long as $\omega \neq \omega_0$ i.e. if $\cos \omega t$ is not a solution of the homogeneous equation.)

Let us choose c and δ in such a way that

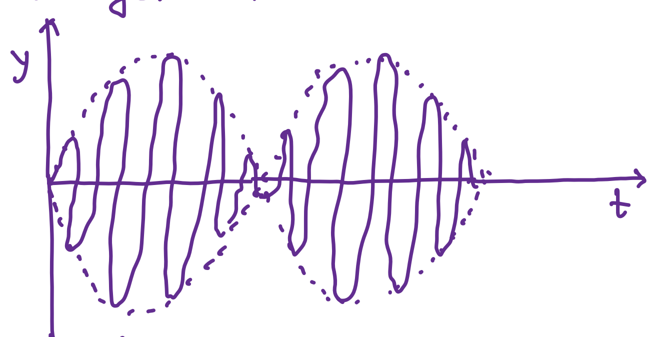
$$c = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad \delta = 0$$

Then

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left[\cos(\omega_0 t) + \cos(\omega t)\right]$$

$$= \boxed{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \cos\left(\frac{\omega_0 + \omega}{2} t\right) \cos\left(\frac{\omega_0 - \omega}{2} t\right)}$$

If $\omega_0 \approx \omega$, then we get a solution like:



(These solutions are called beats and this is what a musician hears when they tune their instruments)

(Forced undamped oscillation when the difference of the input and natural frequencies is small ("beats"))

$\boxed{\text{MTH 204: Lecture 08}}$

Case 2: **Undamped Forced Oscillation ($c=0$): Resonance**

If $\omega = \omega_0$ (i.e. $\cos \omega t$ is a solution of the homogeneous equation), the situation is called resonance.

In this case, the particular solution found earlier is no longer valid.

Now, the ODE is $\quad my'' + ky = F_0 \cos(\omega_0 t)$

$$\Rightarrow \quad y'' + \frac{k}{m} y = \frac{F_0}{m} \cos(\omega_0 t)$$

$$\Rightarrow \quad y'' + \omega_0^2 y = \frac{F_0}{m} \cos(\omega_0 t)$$

Let $y_p(t) = t\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right)$

$$y_p'(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) - At\omega_0 \sin(\omega_0 t) + B\omega_0 t\cos(\omega_0 t)$$

$$= (A + B\omega_0 t)\cos(\omega_0 t) + (B - At\omega_0)\sin(\omega_0 t)$$

$$y_p''(t) = B\omega_0 \cos(\omega_0 t) - (A\omega_0 + B\omega_0^2 t)\sin(\omega_0 t)$$

$$- A\omega_0 \sin(\omega_0 t) + (B\omega_0 - A\omega_0^2 t)\cos(\omega_0 t)$$

$$= (2B\omega_0 - At\omega_0^2)\cos \omega_0 t - (Bt\omega_0^2 + 2A\omega_0)\sin(\omega_0 t)$$

Now if $y_p(t)$ satisfies the above equation

we get $\quad (2B\omega_0 - At\omega_0^2)\cos(\omega_0 t) - (Bt\omega_0^2 + 2A\omega_0)\sin(\omega_0 t)$

$$+ \ \omega_0^2\left(tA\cos(\omega_0 t) + tB\sin(\omega_0 t)\right) = \frac{F_0}{m}\cos(\omega_0 t)$$

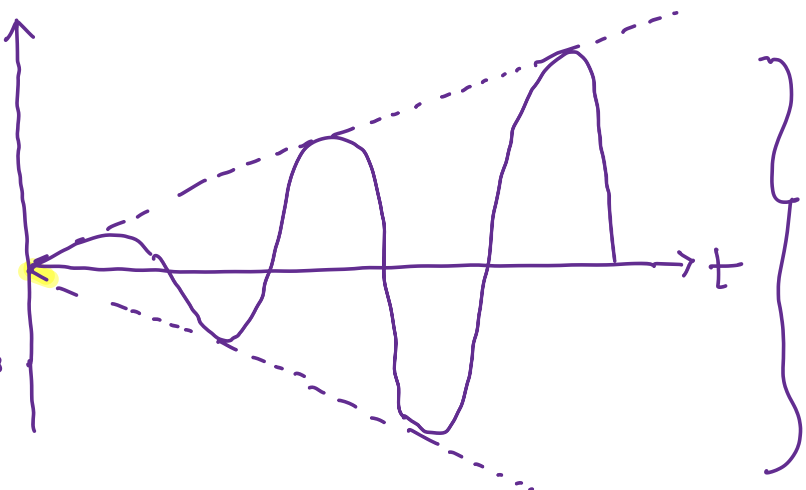
$$\Rightarrow \quad 2B\omega_0 \cos(\omega_0 t) - 2A\omega_0 \sin(\omega_0 t) = \frac{F_0}{m}\cos(\omega_0 t)$$

$$\Rightarrow \quad 2B\omega_0 = \frac{F_0}{m} \quad \text{and} \quad A = 0$$

$$\Rightarrow \quad B = \frac{F_0}{2m\omega_0} \quad \text{and} \quad A = 0$$

So, $$\boxed{y_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)}$$

This excitation of large oscillations by matching input and natural frequencies $(\omega = \omega_0)$ is called <u>Resonance</u>



<u>Tacoma Bridge resonance</u>

Because of the factor t, the amplitude of the vibration becomes larger and larger.

Thus with very little damping, system may undergo large vibrations that can destroy the system.

<u>Case: Damped Forced Oscillations (Practical Resonance):</u>

In practice, there is always some damping and the amplitude of the oscillation doesn't grow infinitely.

Let us find the maximum amplitude in this case.

The particular solution of $my'' + cy' + ky = F_0 \cos(\omega t)$ was $y_p = a\cos(\omega t) + b\sin(\omega t)$

where $a = \dfrac{F_0 \quad m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$ and $b = F_0 \dfrac{c\omega}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$

We can rewrite it as $y_p(t) = C_1 \cos(\omega t - \eta)$

where $C_1 = C_1(\omega) = \sqrt{a^2 + b^2} = F_0 \dfrac{1}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}$

and $\eta = \tan^{-1}\left(\dfrac{b}{a}\right)$

Now, for maximum amplitude $\dfrac{dC_1}{d\omega} = 0$

$$\Rightarrow F_0 \left(-\frac{1}{2}\left(m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c\right)^{-\frac{3}{2}}\right)\left[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2\right] = 0$$

$$\Rightarrow 2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2 = 0$$

$$\Rightarrow c^2 = 2m^2(\omega_0^2 - \omega^2)$$

$$\Rightarrow 2m^2\omega^2 = 2m^2\omega_0^2 - c^2$$

$$\Rightarrow \omega^2 = \omega_0^2 - \frac{c^2}{2m^2}$$

$$\Rightarrow \boxed{\omega_{max}^2 = \omega_0^2 - \frac{c^2}{2m^2}}$$

It can be shown that $\left.\dfrac{d^2C_1}{d\omega^2}\right|_{\omega_{max}} = -\dfrac{1}{2} \left.\dfrac{F_0 \quad 8m^2\omega^2}{\left[m^2(\omega_0^2-\omega^2)^2 + \omega^2c^2\right]^{3/2}}\right|_{\text{Evaluated at } \omega = \omega_{max}} < 0$

That is Practical resonance occurs at a frequency slightly smaller than the natural frequency.

Now maximum amplitude at ω_{max} can be calculated as

$$(C_1)_{max} = \frac{F_0}{\sqrt{m^2\left(\frac{c^2}{2m^2}\right)^2 + c^2\left(\omega_0^2 - \frac{c^2}{2m^2}\right)}} = \boxed{F_0 \frac{2m}{c\sqrt{4m^2\omega_0^2 - c^2}}}$$

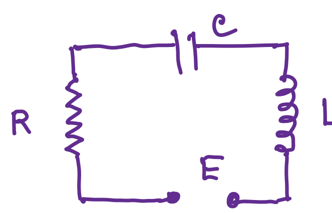
— × — × — × — × — × — × —

Note: The general solution of NH: $y = y_h + y_p$ is called Transient solution. When all the roots of the characteristic equation of the homogeneous ODE are negative or have negative real parts, $y_h \to 0$ as $x \to \infty$

and the transient solution $y = y_h + y_p \longrightarrow y_p$ (The steady state solution)

— × — × — × — × — × — × —

Electric Circuit:



$E(t) = E_0 \sin(\omega t)$

Let $I(t)$ be the current in the circuit at time t and $Q(t)$ be the total charge in capacitor at time t.

Then $I(t) = \frac{dQ}{dt} \Rightarrow Q(t) = \int I(t)\,dt$.

The ODE modeling the RLC circuit is:

$$LI'(t) + RI(t) + \frac{1}{C}\int I(t)\,dt = E(t) = E_0 \sin(\omega t)$$

$$\Rightarrow LI'' + RI' + \frac{1}{C} I = E'(t) = E_0 \omega \cos(\omega t)$$

$$\boxed{\Rightarrow LI'' + RI' + \frac{1}{C} I = E_0 \omega \cos(\omega t)}$$

(H): $LI'' + RI' + \frac{1}{C} I = 0$

The characteristic equation is: $\boxed{L\lambda^2 + R\lambda + \frac{1}{C} = 0}$

$$\Rightarrow \lambda = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

$$\Rightarrow \boxed{I_H(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

(Now $I_H(t) \to 0$ as $t \to \infty$
In actual circuit after a relatively short time.)

(NH): Let $I_p(t) = A\cos\omega t + B\sin\omega t$

$$I_p'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

and $I_p''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

Now if it is a solution of NH, then

$$L\left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) + R\left(-A\omega\sin\omega t + B\omega\cos\omega t\right) + \frac{1}{C}\left(A\cos\omega t + B\sin\omega t\right) = E_0\omega\cos\omega t$$

$$\Rightarrow \left[\left(-L\omega^2 + \frac{1}{C}\right)A + R\omega B\right]\cos\omega t + \left[-R\omega A + \left(-L\omega^2 + \frac{1}{C}\right)B\right]\sin\omega t = E_0\omega\cos\omega t$$

$$\Rightarrow \left(-L\omega^2 + \frac{1}{C}\right)A + R\omega B = E_0\omega$$

and $-R\omega A + \left(-L\omega^2 + \frac{1}{C}\right)B = 0$

$$\Rightarrow \begin{pmatrix} -L\omega^2 + \frac{1}{C} & R\omega \\ -R\omega & -L\omega^2 + \frac{1}{C} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} E_0\omega \\ 0 \end{pmatrix}$$

$$\Rightarrow \omega \begin{pmatrix} -L\omega + \frac{1}{C\omega} & R \\ -R & -L\omega + \frac{1}{C\omega} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \omega \begin{pmatrix} E_0 \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \frac{-E_0 S}{R^2 + S^2}, \quad B = \frac{E_0 R}{R^2 + S^2}$$

where S is the reactance, $S = L\omega - \frac{1}{C\omega}$

(The quantity $\sqrt{R^2+S^2}$ is called impedance or apparent resistance)

Thus $I_p(t) = (A\cos\omega t + B\sin\omega t)$

$$= \sqrt{A^2+B^2}\,\sin\left(\omega t + \tan^{-1}\frac{A}{B}\right)$$

$$\Rightarrow \boxed{I_p(t) = \frac{E_0}{\sqrt{R^2+S^2}}\sin\left(\omega t - \tan^{-1}\frac{S}{R}\right)}$$

Ex: Find the current $I(t)$ in an RLC circuit with $R = 11\,\Omega$, $L = 0.1\,H$, $C = 10^{-2}\,F$ which is connected to a source of $E(t)$

$$E(t) = 110\sin(60 \times 2\pi t) = 110\sin(377t) \quad \left(\text{Thus } \omega = 377\right)$$

In US, CANADA, 60 Hz = 60 cycles/sec

(In Europe, it is 220V and 50Hz)

Assume that current and capacitor charge are 0, when $t=0$ i.e. $I(0)=0$ and $C(0)=0$.

$$L I'' + R I' + \frac{1}{C} I = E_0 \omega \cos \omega t$$

<u>The Homogeneous Equation:</u>

$$.1 I'' + 11 I' + \frac{1}{10^{-2}} I = 0$$

$$\Rightarrow I'' + 110 I' + 1000 I = 0$$

<u>Characteristic Equation:</u> $\lambda^2 + 110\lambda + 1000 = 0$

$$\Rightarrow (\lambda + 10)(\lambda + 100) = 0$$

$$\Rightarrow \lambda = -10, -100$$

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}$$

$$\left.\begin{aligned} E_0 &= 110 \\ \omega &= 377 \end{aligned}\right\} \quad \boxed{S = L\omega - \frac{1}{C\omega}} = .1(377) - \frac{1}{10^{-2}\, 377}$$

$$= 37.7 - \frac{100}{377} \simeq 37.4.$$

$$I_p = \frac{110}{\sqrt{11^2 + (37.4)^2}} \sin\left(377t - \tan^{-1}\frac{37.4}{11}\right)$$

$$= 2.82 \sin(377t - 73.6)$$

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} + 2.82 \sin(377t - 73.6)$$

Given $I(0)=0 \Rightarrow c_1 + c_2 + 2.82 \sin(-73.6) = 0$

$\Rightarrow c_1 + c_2 - 2.71 = 0$ .... (1)

Also $Q(0)=0$: Now $LI'(t) + RI(t) + \frac{1}{C}Q(t) = E_0 \sin \omega t$

When $t=0$:

$$LI'(0) + RI(0) + \frac{1}{C}Q(0) = 0$$

$$\Rightarrow LI'(0) = 0 \Rightarrow I(0) = 0$$

Now $I'(t) = -10c_1 e^{-10t} - 100c_2 e^{-100t} + 2.82 \times 377 \times \cos(377t - 73.6)$

$I'(0)=0 \Rightarrow -10c_1 - 100c_2 + 2.82 \times 377 \cos(-73.6) = 0$

$\Rightarrow -10c_1 - 100c_2 + 300.17 = 0$ .... (2)

Now (2) $\Rightarrow$ $10c_1 + 100c_2 = 300.17$

and (1) $\times 10 \Rightarrow$ $\underline{-10c_1 \pm 10c_2 = -27.10}$

Subtracting we get $90c_2 = 273.07$

$$\Rightarrow c_2 = \frac{273.07}{90} = \boxed{3.034}$$

and $c_1 + c_2 = 2.71 \Rightarrow c_1 = 2.71 - 3.034 = \boxed{-0.324}$

Therefore $\boxed{I(t) = -0.324\, e^{-10t} + 3.034\, e^{-100t} + 2.82 \sin(377t - 73.6)}$

or, $I(t)$ can be written as:

$$I(t) = -0.324\, e^{-10t} + 3.034\, e^{-100t} + 2.82 \sin(377t) \times \cos(73.6) - 2.82 \cos(377t) \times \sin(73.6)$$

$$\Rightarrow \boxed{I(t) = -0.324\, e^{-10t} + 3.034\, e^{-100t} + .796 \sin(377t) - 2.71 \cos(377t)}$$

Method of Variation of Parameter:
(Lagrange)

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

p, q, r don't need to be constants but they are continuous function in an open interval I.
Let y_1 and y_2 be two independent solution of the associated homogeneous equation.
Let us assume that there is a particular solution of the NH equation of the form

$$y_p = u(x)\, y_1 + v(x)\, y_2$$

$$y_p' = u' y_1 + u y_1' + v' y_2 + v y_2'$$
$$= (u' y_1 + v' y_2) + (u y_1' + v y_2')$$

Let us impose the condition that $\boxed{u' y_1 + v' y_2 = 0}$
Then $y_p' = u y_1' + v y_2'$

$$\Rightarrow y_p'' = u y_1'' + u' y_1' + v y_2'' + v' y_2'$$
$$\Rightarrow y_p'' = u y_1'' + v y_2'' + u' y_1' + v' y_2'$$

Now $y_p'' + p(x) y_p' + q(x) y_p = r(x)$

$$\Rightarrow \left(u y_1'' + v y_2'' + u' y_1' + v' y_2'\right) + \left(u y_1' + v y_2'\right) p(x) + \left(u y_1 + v y_2\right) q(x) = r(x)$$

$$\Rightarrow \left[y_1'' + p(x) y_1' + q(x) y_1\right] u(x) + \left[y_2'' + p(x) y_2' + q(x) y_2\right] v(x) + u' y_1' + v' y_2' = r(x)$$

$$\Rightarrow \boxed{u' y_1' + v' y_2' = r(x)}$$

So,

$$\boxed{\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1' + v' y_2' &= r(x) \end{aligned}}$$

$$\Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ r(x) \end{pmatrix}$$

$$\Rightarrow u' = \frac{-r y_2}{W}, \quad v' = \frac{r y_1}{W}$$

where $W = y_1 y_2' - y_2 y_1'$
(Wronskian)

$$\Rightarrow \quad u = -\int \frac{r y_2}{W}\,dx$$

$$v = \int \frac{r y_1}{W}\,dx$$

So, $\boxed{y_p = -y_1 \int \frac{r y_2}{W}\,dx + y_2 \int \frac{r y_1}{W}\,dx}$

Hence

$$\boxed{y = c_1 y_1 + c_2 y_2 + (-y_1) \int \frac{r y_2}{W}\,dx + y_2 \int \frac{r y_1}{W}\,dx}$$

Ex: $\quad y'' + y = \frac{1}{\cos x}$

Note that

$y_1 = \cos x$ and $y_2 = \sin x$ are independent

Solutions of $\quad y'' + y = 0$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$y_p = -y_1 \int \frac{r y_2}{W} dx + y_2 \int \frac{r y_1}{W} dx$$

$$= -\cos x \int \frac{1}{\cos x} \cdot \frac{\sin x}{1} dx + \sin x \int \frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1} dx$$

$$= +\cos x \ln|\cos x| + x \sin x$$

So, the general solution is :

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

$$= \boxed{(c_1 + \ln|\cos x|) \cos x + (c_2 + x) \sin x}$$

MTH 204 : Lecture 09

Ex: $y'' + 4y = x\sin(2x) + 8$

(Note that here we need to use multification rule, modification rule and sum rule)

The characteristic Equation for the Homogeneous ODE

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

$$y_h(x) = c_1\cos(2x) + c_2\sin(2x)$$

$$y_p(x) = (A_1 x + B_1)\left(c_1\, x\cos 2x + c_2\, x\sin 2x\right) + E$$

$$\Rightarrow y_p(x) = A x^2\cos 2x + B x^2\sin 2x + C x\cos 2x + D x\sin 2x + E$$

Then
$$y_p'(x) = 2Ax\cos 2x - 2Ax^2\sin 2x + 2Bx\sin 2x + 2Bx^2\cos 2x$$
$$+ C\cos 2x - 2Cx\sin 2x + D\sin 2x + 2Dx\cos 2x$$

$$\Rightarrow y_p'(x) = 2Bx^2\cos 2x - 2Ax^2\sin 2x + (2A+2D)x\cos 2x$$
$$+ (2B-2C)x\sin 2x + C\cos 2x + D\sin 2x$$

$$\Rightarrow y_p''(x) = 4Bx\cos 2x - 4Bx^2\sin 2x - 4Ax\sin 2x - 4Ax^2\cos 2x$$
$$+ (2A+2D)\cos 2x - (4A+4D)x\sin 2x$$
$$+ (2B-2C)\sin 2x + (4B-4C)x\cos 2x$$
$$- 2C\sin 2x + 2D\cos 2x$$

Substituting and equating the coefficients :

$x^2\cos 2x$ $\quad -4A+4A=0$

$x^2\sin 2x$ $\quad -4B+4B=0$

$x\cos 2x$ $\quad 4B+4B-\cancel{4C}+\cancel{4C}=0 \Rightarrow 8B=0 \Rightarrow \boxed{B=0}$

$x\sin 2x$ $\quad -4A-4A-\cancel{4D}+\cancel{4D}=1 \Rightarrow -8A=1 \Rightarrow \boxed{A=-\frac{1}{8}}$

$\cos 2x$: $\quad 2A+2D+2D=0 \Rightarrow 2A+4D=0 \Rightarrow D=-\frac{1}{2}A$

$\Rightarrow \boxed{D=\frac{1}{16}}$

$\sin 2x$: $\quad 2B-2C-2C=0 \Rightarrow -4C=0 \Rightarrow C=0$

constant : $4E=8 \quad \Rightarrow \quad \boxed{E=2}$

So, $y_p(x) = -\frac{1}{8}x^2\cos 2x + \frac{1}{16}x\sin 2x + 2$

The general solution is

$$y(x) = c_1\cos 2x + c_2\sin 2x - \frac{1}{8}x^2\cos 2x + \frac{1}{16}x\sin 2x + 2$$

Existence and Uniqueness of Solutions:

Existence and Uniqueness Theorem for IVP:

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and $x_0 \in I$, then the initial value problem

$$y'' + p(x)y' + q(x)y = 0$$

with two initial conditions $y(x_0) = K_0$, $y'(x_0) = K_1$ with given K_0 and K_1

has a unique solution $y(x)$ on the interval I

We will also give a theorem about existence of general solution.

Linear Independence of solutions:

Consider the second order ODE: $y'' + p(x)y + q(x) = 0$ where $p(x)$ and $q(x)$ are continuous functions on an open interval I.

Two solutions y_1 and y_2 are called Linearly independent on I if the equation

$$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ on } I \Rightarrow c_1 = c_2 = 0$$

We call y_1 and y_2 linearly dependent on I if this equation also holds for constants c_1 and c_2 not both zero.

y_1 and y_2 are linearly dependent iff y_1 and y_2 are proportional on I

(That is, $y_1 = k y_2$ or $y_2 = l y_1$ $\forall x \in I$ where k, l are constants.)

<u>Theorem:</u> Let us consider the ODE

$$y'' + p(x) y' + q(x) y = 0$$ where

$p(x)$ and $q(x)$ are continuous functions on an open interval I.

Then two solutions y_1 and y_2 of the ODE on I are linearly dependent on I iff their Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$

W != 0 implies linearly Independent
W == 0 Implies linearly dependent

is zero at some $x_0 \in I$

Furthermore if $W = 0$ at an $x = x_0 \in I$, then $W = 0$ on I.

Hence, if there is an $x_1 \in I$ at which W is not 0, then y_1 and y_2 are linearly independent on I.

Proof:

$\Rightarrow$: Let y_1 and y_2 be linearly dependent on I

Then either $y_1 = k y_2$ or $y_2 = l y_1$ where k and l are constants.

Suppose $y_1 = k y_2$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = k y_2 y_2' - k y_2' y_2$$
$$= 0 \text{ on } I.$$

$$\Rightarrow W \equiv 0 \text{ on } I.$$

Similarly for $y_2 = l y_1$.

$\Leftarrow$: Conversely suppose $W(y_1, y_2) = 0$ for some $x_0 \in I$.

Let us consider the system of equation in unknown C_1 and C_2

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{aligned}\right\}$$

$$\Rightarrow \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $W(y_1(x_0), y_2(x_0)) = 0$, the above system has a non trivial solution $\{\tilde{c}_1, \tilde{c}_2\}$

Let $Y(x) = c_1 y_1(x) + c_2 y_2(x)$

By superposition principle $Y(x)$ is a solution of the ODE.

Moreover $Y(x_0) = 0$ and $Y'(x_0) = 0$

Thus $Y(x)$ is a solution of the IVP

$$\left.\begin{array}{l} y'' + p(x)y' + q(x)y = 0 \\ y(x_0) = 0 \text{ and } y'(x_0) = 0 \end{array}\right\}$$

But $y^* \equiv 0$ is also a solution of the above IVP.

So, by the uniqueness theorem

$$Y \equiv y^* \equiv 0$$

$\Rightarrow$ $c_1 y_1 + c_2 y_2 = 0$ on I

where c_1 and c_2 are not both zeros.

Hence y_1 and y_2 are linearly dependent.

If $W(y_1(x_0), y_2(x_0)) = 0$ for some $x_0 \in I$ then y_1 and y_2 are linearly dependent.

Then by the first part, $W \equiv 0$ on I

- Thus if $W(x_1) \neq 0$ at some point $x_1 \in J$ then y_1 & y_2 are linearly independent.

Note:

$$W(y_1, y_2) = \left(\frac{y_2}{y_1}\right)' y_1^2 \quad (\text{if } y_1 \neq 0)$$

or,

$$W(y_1, y_2) = \left(\frac{y_1}{y_2}\right)' y_2^2 \quad (\text{if } y_2 \neq 0)$$

- $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (Wronski determinant or Wronskian)

Existence of a General Solution:

Theorem: If $p(x)$ and $q(x)$ are continuous functions on an open interval I, then the ODE

$$y'' + p(x)y' + q(x)y = 0 \quad \text{---- (1)}$$

has a general solution on I.

Proof: Using the first theorem, the ODE ① has a solution $y_1(x)$

satisfying $y_1(x_0)=1$, $y_1'(x_0)=0$ for some $x_0 \in I$

and a solution $y_2(x)$ satisfying

$$y_2(x_0)=0, \quad y_2'(x_0)=1$$

Thus for $x_0 \in I$ we are considering two IVP:

① $y''+p(x)y'+q(x)y=0$

with initial conditions $y'(x_0)=1$, $y'(x_0)=0$

and

② $y''+p(x)y'+q(x)y=0$

with initial conditions $y'(x_0)=0$, $y'(x_0)=1$

Now the Wronskian of these two solutions at $x=x_0$

$$W(y_1(x_0), y_2(x_0)) = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$$
$$= 1\times 1 - 0\times 0 = 1 \neq 0.$$

So, $y_1(x)$ and $y_2(x)$ are linearly independent on I. They form a basis of solutions of ①

Thus $y(x) = c_1 y_1(x) + c_2 y_2(x)$ with arbitrary c_1 & c_2 is a general solution of ①.

Theorem (A general solution includes all solutions):

Consider the ODE : $y'' + p(x)y' + q(x) = 0$ ....① where $p(x)$ and $q(x)$ are continuous functions on some open interval I.

Then every solution $Y(x)$ of ① on I, is of the form $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ where y_1 and y_2 is any basis of solutions of ① on I and c_1, c_2 are suitable constants.

Note: Hence ① doesn't have singular solutions (i.e. solutions not obtainable from a general solution)

Proof: Let $Y(x)$ be any solution of ① on I.

By the previous theorem, the ODE ① has a general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$ on the interval I.

Choose $x_0 \in I$ and consider the system of equations:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = Y(x_0)$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = Y'(x_0)$$

$$\Rightarrow \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} Y(x_0) \\ Y'(x_0) \end{pmatrix}$$

Since y_1, y_2 is a basis, their Wronskian W at x_0 is not zero.

Hence there is a unique solution $c_1 = \tilde{c}_1$ and $c_2 = \tilde{c}_2$

With $\tilde{c}_1$ and $\tilde{c}_2$,

Then $y^*(x) = \tilde{c}_1 y_1(x) + \tilde{c}_2 y_2(x)$ is a particular solution of ①

with the condition,

$$y^*(x_0) = \tilde{c}_1 y_1(x_0) + \tilde{c}_2 y_2(x_0) = Y(x_0)$$
$$y^{*\prime}(x_0) = \tilde{c}_1 y_1'(x_0) + \tilde{c}_2 y_2'(x_0) = Y'(x_0)$$

By the uniqueness of solution of IVP,

$$y^* \equiv Y \text{ on } I$$
$$\left(y^*(x) = Y(x) \ \forall x \in I \right)$$

So, $Y(x) = \tilde{c}_1 y_1(x) + \tilde{c}_2 y_2(x)$. (QED)

Ex: Find Wronskian. Show linear independence by using quotient and confirm it by the second theorem.

Given $y_1 = e^{-x}\cos\omega x$, $y_2 = e^{-x}\sin\omega x$.

- $\dfrac{y_2}{y_1} = \tan\omega x$ (not a Constant)

  So, y_1 and y_2 are linearly independent

- Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{-x}\cos\omega x & e^{-x}\sin\omega x \\ -e^{-x}\cos\omega x - \omega e^{-x}\sin\omega x & -e^{-x}\sin\omega x + \omega e^{-x}\cos\omega x \end{vmatrix}$$

$$= \cancel{-e^{-2x}\sin\omega x\cos\omega x} + \omega e^{-2x}\cos^2\omega x + \cancel{e^{-2x}\sin\omega x\cos\omega x} + \omega e^{-2x}\sin^2\omega x$$

$$= \omega e^{-2x}$$

The solutions are linearly independent iff $\omega \neq 0$

$\left(\dfrac{y_2}{y_1} = \tan\omega x = 0 \text{ if } \omega = 0 \Rightarrow y_2 = 0 \text{ and then } y_1 \text{ and } y_2 \text{ are linearly dependent}\right)$

Note: A second order linear homogeneous ODE for which $e^{-x}\cos\omega x$ and $e^{-x}\sin\omega x$ are solutions is $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + (1+\omega^2)\,y = 0$

(Because from the two functions we can write the characteristic equation as: $(\lambda+1-i\omega)(\lambda+1+i\omega)=0 \Rightarrow \lambda^2+2\lambda+(1+\omega^2)=0$

<u>Ex:</u> $y''-2y'+y=0$

characteristic Equation $\lambda^2-2\lambda+1=0$

$\Rightarrow (\lambda-1)^2=0 \Rightarrow \lambda=1,1$

So, the general solution: $y(x)=c_1e^x+c_2xe^x$

Wronskian of e^x and xe^x:

$$W(e^x, xe^x)=\begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix} = e^{2x}+\cancel{xe^{2x}}-\cancel{xe^{2x}}$$

$$=e^{2x}\neq 0$$

So, the two solutions e^x and xe^x are linearly independent.

<u>Ex:</u> (a) Find a second order homogeneous linear ODE for which the given functions are solutions.

(b) Show linear independence by the Wronskian.

(c) Solve the initial value problem:

$e^{-kx}\cos\pi x$, $e^{-kx}\sin\pi x$

$y(0)=1$, $y'(0)=-k-\pi$

(a) The characteristic equation is:

$$[\lambda - (-k + i\pi)]\,[\lambda - (-k - i\pi)] = 0$$

$$\Rightarrow [(\lambda + k) + i\pi]\,[(\lambda + k) - i\pi] = 0$$

$$\Rightarrow (\lambda + k)^2 - i^2\pi^2 = 0 \Rightarrow (\lambda + k)^2 + \pi^2 = 0$$

$$\Rightarrow \lambda^2 + 2k\lambda + (k^2 + \pi^2) = 0$$

Hence the ODE is:

$$\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + (k^2 + \pi^2)y = 0$$

(b) The Wronskian of $e^{-kx}\cos\pi x$ and $e^{-kx}\sin\pi x$ is:

$$W(e^{-kx}\cos\pi x, e^{-kx}\sin\pi x) = \begin{vmatrix} e^{-kx}\cos\pi x & e^{-kx}\sin\pi x \\ -ke^{-kx}\cos\pi x - \pi e^{-kx}\sin\pi x & -ke^{-kx}\sin\pi x + \pi e^{-kx}\cos\pi x \end{vmatrix}$$

$$= \cancel{-ke^{-2kx}\cos\pi x\sin\pi x} + \pi e^{-2kx}\cos^2\pi x + \cancel{ke^{-2kx}\sin\pi x\cos\pi x} + \pi e^{-2kx}\sin^2\pi x$$

$$= \pi e^{-2kx}(\cos^2\pi x + \sin^2\pi x) = \pi e^{-2kx} \neq 0$$

Hence $e^{-kx}\cos\pi x$ and $e^{-kx}\sin\pi x$ are linearly independent.

(c) The general solution of the ODE will be

$$y(x) = c_1 e^{-kx}\cos\pi x + c_2 e^{-kx}\sin\pi x$$

$$y'(x) = c_1\left(-ke^{-kx}\cos\pi x - \pi e^{-kx}\sin\pi x\right) + c_2\left(-ke^{-kx}\sin\pi x + \pi e^{-kx}\cos\pi x\right)$$

Now $y(0) = 1 \Rightarrow c_1 + c_2 \times 0 = 1 \Rightarrow \boxed{c_1 = 1}$

$$y'(0) = -k-\pi \Rightarrow c_1(-k) + c_2\pi = -k-\pi$$

$$\Rightarrow -\cancel{k} + c_2\pi = -\cancel{k} - \pi \quad (\text{Since } c_1 = 1)$$

$$\Rightarrow c_2\pi = -\pi \Rightarrow \boxed{c_2 = -1}$$

So, the solution of the initial value problem is:

$$y = e^{-kx}\cos\pi x - e^{-kx}\sin\pi x$$

$$\Rightarrow \boxed{y = e^{-kx}(\cos\pi x - \sin\pi x)}$$

Ex: (a) Find a second order homogeneous ODE whose solutions are x^2 and $x^2 \ln x$

(b) Show linear independence by the wronskian.

(c) Solve the initial value problem:

$y(1) = 4$, $y'(1) = 6$

(a) $m=2$ is a real double root of the Auxiliary Equation (corresponding to Euler-Cauchy equation)

The auxiliary equation is $(m-2)^2 = 0$

$$\Rightarrow \quad m^2 - 4m + 4 = 0$$

$$\Rightarrow \quad m(m-1) - 3m + 4 = 0$$

Hence the ODE is $x^2 y'' - 3x y' + 4y = 0$

(b) Now Wronskian of x^2 and $x^2 \ln x$ is:

$$W(x^2, x^2 \ln x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + \frac{x^{\cancel{2}}}{\cancel{x}} \end{vmatrix}$$

$$= \cancel{2x^3 \ln x} + x^3 - \cancel{2x^3 \ln x} = x^3 \neq 0 \text{ for } x \neq 0$$

Hence the two solutions x^2 and $x^2 \ln x$ are linearly independent and the general solution of the ODE is:

$$y(x) = (c_1 + c_2 \ln x) x^2 = c_1 x^2 + c_2 x^2 \ln x$$

(c) Now $y(x) = c_1 x^2 + c_2 x^2 \ln x$

$$\Rightarrow \quad y'(x) = 2c_1 x + 2c_2 x \ln x + c_2 x^{\cancel{2}} \frac{1}{\cancel{x}}$$

$$\Rightarrow \quad y'(x) = 2c_1 x + 2c_2 x \ln x + c_2 x$$

Now $y(1)=4 \Rightarrow c_1 + 0 = 4 \Rightarrow c_1 = \boxed{4}$

$y'(1) = 6 \Rightarrow 2c_1 + c_2 = 6$

$\Rightarrow 8 + c_2 = 6 \Rightarrow c_2 = \boxed{-2}$

Therefore the solution to the initial value problem is :

$$\boxed{y(x) = (4 - 2\ln x)x^2}$$

MTH 204: Lecture 10

Differential Operator:

- Here an operator is a transformation that transforms a function into another function.

(Recall: Transformation from a vector space to another vector space)

- The Differential operator D transforms a differentiable function into its derivative.

e.g. consider $D: C^1 \longrightarrow C$

where C^1 is the space of all differentiable functions with continuous derivative

and C is the space of all continuous functions.

Notation: $D = \frac{d}{dx}$

So, $Dy = y' = \frac{dy}{dx}$

For higher derivatives $D^2 y = D(Dy) = y''$

For a homogeneous linear ODE with constant Coefficients $y'' + ay' + by = 0$ can be written as $Ly = P(D)y = 0$

where $L = P(D) = D^2 + aD + bI$

(second order differential operator)

Here I is the identity operator: $Iy = y$

Note that L is defined only on C^2.

- **<u>L is a Linear operator:</u>**

If y and w are two twice differentiable functions, then

$L(cy + dw) = c\,L(y) + d\,L(w)$ where c and d are constants.

$$L(e^{\lambda x}) = P(D)(e^{\lambda x}) = (D^2 + aD + bI)(e^{\lambda x})$$

$$= (\lambda^2 + a\lambda + b)(e^{\lambda x}) = P(\lambda)(e^{\lambda x}) = 0$$

Thus $e^{\lambda x}$ is a solution of the ODE iff λ is a solution of the characteristic equation $P(\lambda) = 0$

- $P(\lambda)$ is a polynomial in the usual sense of algebra. Replacing λ by the operator, we get the operator polynomial

$P(D)$: Operational Calculus.

- $P(D)$ can be treated like an algebraic quantity and we can factor it.

Ex: Factor and solve:

$$\left(D^2 - 4.00D + 3.84I\right)y = 0$$

$$\Rightarrow \quad (D - 2.4I)(D - 1.6I)y = 0$$

So, $y_1 = e^{2.4x}$ and $y_2 = e^{1.6x}$ is a basis of the solution and the general solution is:

$$y = c_1 e^{2.4x} + c_2 e^{1.6x}$$

Note: If $L(D)z = 0$ then z is an eigen function of L corresponding to the eigenvalue 0.

Note:

Let $L(D)y = y$ i.e. y is an eigenfunction of L corresponding to eigen value 1.

Then $\left(D^2 + aD + bI\right)y = y$.

$$\Rightarrow \left[D^2 + aD + (b-1)I\right]y = 0$$

$\Rightarrow$ y is eigenfunction of $L_1 = D^2 + aD + (b-1)I$ corresponding to the eigen value 0.

Thus solution of a Homogeneous Linear Differential Equation can be thought of eigen function of the corresponding Differential Operator.

Ex: Find all eigen functions corresponding to zero eigen value of the differential operator $D^2 - 2D + 3I$.

- $(D^2 - 2D + 3I)\, y = 0$

$$\Rightarrow y'' - 2y' + 3y = 0$$

Characteristic Equation: $\lambda^2 - 2\lambda + 3 = 0$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \sqrt{2}\, i$$

So, $y(x) = e^x \left[c_1 \cos(\sqrt{2}\, x) + c_2 \sin(\sqrt{2}\, x) \right]$

Ex: If we want the eigen function of $(D^2 - 2D + 3I)$ corresponding to eigenvalue 1, then $(D^2 - 2D + 3I)\, y(x) = y(x)$ and so we need to find the solution of

$$y'' - 2y' + 2y = 0$$

The characteristic Equation is $\lambda^2 - 2\lambda + 2 = 0$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

So, $y(x) = e^x \left[c_1 \cos(x) + c_2 \sin x \right]$

is an eigen function of $D^2 - 2D + 3I$ corresponding to to eigen value 1.

Note: Let $y'' + ay' + by = 0$

$$\Leftrightarrow (D^2 + aD + bI)y = 0$$

$$\Leftrightarrow L(D)y = 0 \text{ where } L(D) = D^2 + aD + bI$$

$$\Leftrightarrow y \in \ker L(D)$$

Thus, solving the ODE, $y'' + ay' + by = 0$ is same as finding kernel of $L = D^2 + aD + bI$

Ex: Find the kernel of $D^2 - 3D - 40I$

$y \in \ker(D^2 - 3D - 40I)$

$$\Leftrightarrow (D^2 - 3D - 40I)y = 0$$

Characteristic Equation: $\lambda^2 - 3\lambda - 40 = 0$

$$\Rightarrow (\lambda - 8)(\lambda + 5) = 0$$

$$\Rightarrow \lambda = 8, -5$$

Thus $\ker(D^2 - 3D - 40I)$
$$= \{c_1 e^{8x} + c_2 e^{-5x} : c_1, c_2 \in \mathbb{R}\}$$

Higher Order Linear ODEs

- A nth order ODE is linear if it can be written as: $y^{(n)}(x) + p_{n-1}(x)\, y^{(n-1)}(x) + \cdots + p_1(x) y'(x) + p_0(x) y(x) = r(x)$

Otherwise, it is nonlinear.
It is homogeneous if $r(x) = 0$

Theorem (Principle of Superposition):
The linear combination of any numbers of solutions of a homogeneous linear ODE is also a solution.

General Solution:
The general solution of the ODE on an open interval I is of the form
$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$
where $\{y_1, y_2, \ldots, y_n\}$ is a set of linearly

independent (in I) solutions.

- A set $\{y_1, y_2, \dots, y_n\}$ is linearly indepent in I if

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$
$$\Rightarrow c_1 = c_2 = \dots = c_n = 0$$

Ex: $\{x^2, 5x, 2x\}$ is linearly dependent

Since $0(x^2) + 1(5x) + \left(-\frac{5}{2}\right)(2x) = 0$

Ex: $\{1, x, x^2\}$ is linearly independent on any open interval

Since $c_1 \times 1 + c_2 x + c_3 x^2 = 0 = 0 \times 1 + 0 \times x + 0 \times x^2$

$$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$$

- We will learn method of testing independence with the Wronskian.

Basis of a general solution:

Ex: $y^{(4)} - 5y'' + 4y = 0$

Let us try a solution of the form $e^{\lambda x}$

Then $\lambda^4 e^{\lambda x} - 5\lambda^2 e^{\lambda x} + 4e^{\lambda x} = 0$

$\Rightarrow \quad e^{\lambda x}(\lambda^4 - 5\lambda^2 + 4) = 0$

$\Rightarrow \quad \lambda^4 - 5\lambda^2 + 4 = 0$

$\Rightarrow \quad (\lambda^2 - 4)(\lambda^2 - 1) = 0 \Rightarrow \lambda^2 = 4$ or $\lambda^2 = 1$

$\Rightarrow \quad \lambda = 2, -2, 1, -1$

So, the general solution is:

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

Initial Value Problem:

An initial value problem consists of n initial conditions

$$y(x_0) = K_0, \quad y'(x_0) = K_1, \ldots, y^{(n-1)}(x_0) = K_{n-1}$$

These n conditions are used to determine the constants of the general solution:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

Ex: $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$, $y(1) = 2$, $y'(1) = 1$, $y''(1) = -4$

This is a third order Euler-Cauchy problem.

Let us try a solution $y = x^m$

Then $x^3\left[m(m-1)(m-2)x^{m-3}\right]-3x^2\left[m(m-1)x^{m-2}\right]$
$+6x\left(mx^{m-1}\right)-6x^m=0$

$\Rightarrow \left[m(m-1)(m-2)-3m(m-1)+6m-6\right]x^m=0$

$\Rightarrow (m-1)\left[m^2-2m-3m+6\right]x^m=0$

$\Rightarrow (m-1)(m-2)(m-3)\,x^m=0$

$\Rightarrow m=1,2,3$ and the general solution is : $\boxed{y=c_1x+c_2x^2+c_3x^3}$

For the particular solution, $y(1)=2$, $y'(1)=1$, $y''(1)=-4$

Hence, $c_1+c_2+c_3=2$ .... (1)

$y'(x)=c_1+2c_2x+3c_3x^2 \Rightarrow c_1+2c_2+3c_3=1$ ..... (2)

$y''(x)=2c_2+6c_3x \Rightarrow 2c_2+6c_3=-4$ ...... (3)

Now (2) − (1) $\Rightarrow c_2+2c_3=-1$

and (3) $\Rightarrow c_2+3c_3=-2$

Subtracting $-c_3=1 \Rightarrow \boxed{c_3=-1}$

Then $c_2+2\times(-1)=-1 \Rightarrow \boxed{c_2=1}$

and (1) $\Rightarrow \boxed{c_1=2}$

So, the particular solution is :

$$\boxed{y_p=2x+x^2-x^3}$$

Linear Independence of Solutions : Wronskian :

Consider the homogeneous linear ODE :

$$y^{(n)}(x) + p_{n-1}(x)\, y^{(n-1)}(x) + \cdots\cdots + p_1(x)\, y'(x) + p_0(x)\, y(x) = 0$$

Assume that the functions $p_{n-1}(x), \ldots, p_1(x), p_0(x)$ are continuous on an open interval I.

A set of solution $\{y_1, y_2, \ldots, y_n\}$ on I is linearly dependent iff their wronskian,

$$W(x) = W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-2)} & \cdots\cdots & y_n^{(n-1)} \end{vmatrix} = 0$$

for some $x_0 \in I$.

Furthermore, if $W(x) = 0$ for some $x = x_0 \in I$, then $W(x) = 0 \;\forall\; x \in I$

Hence if there is an $x_1 \in I$ at which W is not zero, then $y_1, y_2, \ldots, y_n$ are linearly independent on I, so that they form a basis of solutions of (2) on I.

<u>Ex:</u> $1, x, x^2$ are solutions of the ODE :

$$y''' = 0$$

The coefficient functions are continuous everywhere (i.e. on $\mathbb{R}$)

Now $W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 1(2) = 2 \neq 0$

Y

Hence $1, x, x^2$ are linearly independent on $\mathbb{R}$ and forms a basis of solutions of the ODE.

<u>Ex:</u> $x^2, 5x, 2x$ are solutions of the ODE

$$y''' = 0$$

However,

$$W(x) = \begin{vmatrix} x^2 & 5x & 2x \\ 2x & 5 & 2 \\ 2 & 0 & 0 \end{vmatrix} = 2(10x - 10x) = 0$$

Hence $x^2, 5x, 2x$ are linearly dependent.

Existence and Uniqueness of Solutions:

Existence and Uniqueness Theorem for IVP:

Consider the Initial value problem:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)}(x) + \cdots + p_1(x)\, y'(x) + p_0(x)\, y(x) = 0$$

with n initial conditions:

$$y(x_0) = K_0,\ y'(x_0) = K_0,\ \ldots,\ y^{(n-1)}(x_0) = K_{n-1}.$$

If the coefficients $p_{n-1}(x), \ldots, p_1(x)$ are continuous on some open interval I and $x_0 \in I$, then the IVP has a unique solution $y(x)$ on I.

Existence of a General solution:

Consider the homogeneous linear ODE:

$$y^{(n)}(x) + p_{n-1}(x)\, y^{(n-1)}(x) + \cdots + p_1(x)\, y'(x) + p_0(x)\, y(x) = 0$$

If the coefficient functions $p_{n-1}(x), \ldots, p_1(x), p_0(x)$ are continuous on some open interval I, then there exists a general solution of the ODE on I.

General solution Includes All Solutions:

Consider the homogeneous linear ODE

$$y^{(n)}(x) + p_{n-1}(x)\, y^{(n-1)}(x) + \cdots + p_1(x)\, y'(x) + p_0(x)\, y(x) = 0$$

If the coefficient functions $p_{n-1}(x), \ldots, p_1(x), p_0(x)$ are continuous on some open interval I, then every solution $y = Y(x)$ on I is of the form: $Y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ where $\{y_1, y_2, \ldots, y_n\}$ is a basis of solutions of ODE on I and $c_1, c_2, \ldots, c_n$ are suitable constants.

Hence the ODE has no singular solution (that is, solutions that cannot be obtained from the general solution)

Homogeneous Linear ODEs with constant Coefficients :

Let us consider an nth order homogeneous linear ODE with constant coefficients

$$y^{(n)} + p_{n-1}\, y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

If we substitute $y = e^{\lambda x}$ as a possible solution, we get the characteristic equation as:

$$\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0$$

where roots may be real or complex, and have any degree of multiplicity.

- **Distinct Real Roots:**

If all the n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and different, then the corresponding general solution is $y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}$ (The n solutions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$ are linearly independent).

- **Simple complex roots:**

The (part of the) general solution associated to a couple of conjugated (simple) complex roots $\lambda_1, \lambda_2 = \alpha \pm i\omega$ is

$$e^{\alpha x}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right) = e^{\alpha x}\cos(\omega x - \delta)$$

Ex: $y''' - y'' + 100y' - 100y = 0$

The characteristic equation is:

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

$$\Rightarrow \lambda^2(\lambda-1)+100(\lambda-1)=0$$
$$\Rightarrow (\lambda-1)(\lambda^2+100)=0 \Rightarrow \lambda=1, \pm 10i$$

So, the general solution is:
$$y = c_1 e^x + c_2 \cos(10x) + c_3 \sin(10x)$$

Multiple Real roots:

The (part of the) general solution associated to a real root λ of order m is
$$\left(c_0 + c_1 x + \cdots + c_{m-1} x^{m-1}\right) e^{\lambda x}$$

Ex: $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$

The characteristic equation is:
$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$
$$\Rightarrow \lambda^2(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = 0$$
$$\Rightarrow (\lambda-0)^2(\lambda-1)^3 = 0$$
$$\Rightarrow \lambda = 0 \text{ (multiplicity 2)}$$
and $\lambda = 1$ (multiplicity 3)

Hence the general solution is:
$$y = (c_1 + c_2 x + c_3 x^2)e^x + (c_4 + c_5 x)$$

Multiple Real roots : Proof :

Consider the Linear ODE with constant coefficients

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

and define the differential operator :

$$L = D^n + p_{n-1} D^{n-1} + \cdots + p_1 D + p_0 I$$

The ODE can be written as $Ly = 0$

For $y = e^{\lambda x}$ is a solution of the ODE then,

$$L(e^{\lambda x}) = e^{\lambda x} P(\lambda) = 0$$

where $P(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1 \lambda + p_0$

Now if λ_1 is a root of $P(\lambda)$ of order m, then

$$L(e^{\lambda x}) = e^{\lambda x}(\lambda - \lambda_1)^m P_1(\lambda) \text{ where } P_1(\lambda)$$

$$\frac{\partial L(e^{\lambda x})}{\partial \lambda} = m(\lambda - \lambda_1)^{m-1}\left[e^{\lambda x} P_1(\lambda)\right] + (\lambda - \lambda_1)^m \frac{\partial\left[e^{\lambda x} P_1(\lambda)\right]}{\partial \lambda} \quad \cdots\cdots (1)$$

Since all derivatives are continuous,

$$\frac{\partial L(e^{\lambda x})}{\partial \lambda} = L\left(\frac{\partial (e^{\lambda x})}{\partial \lambda}\right) = L(x e^{\lambda x})$$

$$\Rightarrow \quad \left.\frac{\partial L(e^{\lambda x})}{\partial \lambda}\right|_{\lambda = \lambda_1} = L(x e^{\lambda_1 x}) \quad \cdots\cdots (2)$$

From ① and ②

$$L(xe^{\lambda_1 x}) = \frac{\partial L(e^{\lambda x})}{\partial \lambda}\Big|_{\lambda=\lambda_1} = m(\lambda_1-\lambda_1)^{m-1}\left[e^{\lambda_1 x}P_1(\lambda_1)\right] + (\lambda_1-\lambda_1)^{m}\frac{\partial\left[e^{\lambda x}P_1(\lambda)\right]}{\partial\lambda}\Big|_{\lambda=\lambda_1}$$

$$= 0$$

$\Rightarrow \quad L(xe^{\lambda_1 x}) = 0$

$\Rightarrow \quad xe^{\lambda_1 x}$ is a solution of the ODE.

We can repeat this step and produce $x^2e^{\lambda_1 x}, \ldots\ldots, x^{m-1}e^{\lambda_1 x}$ by another $(m-2)$ such differentiations with respect to λ. Going one step further will no longer give zero on the right because the lowest power of $\lambda-\lambda_1$ would then be $(\lambda-\lambda_1)^0$ and this will be multiplied by $m!\,P_1(\lambda)$ where $P_1(\lambda_1) \neq 0$ ($P_1(\lambda)$ has no factor $(\lambda-\lambda_1)$).

Thus $e^{\lambda_1 x}, \; xe^{\lambda_1 x}, \ldots, x^{m-1}e^{\lambda_1 x}$ are precisely the solutions associated to λ.

By calculating the Wronskian of the above solutions, we can see that it is $e^{\lambda m x}$ times a determinant which is the wronskian of $1, x, x^2, \ldots x^{m-1}$. The later is $(1!\,2!\ldots(m-1)!) \neq 0$

$(1, x, x^2, \ldots, x^{m-1}$ are solutions of $y^{(m)} = 0)$

Therefore $e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots\ldots, x^{m-1}e^{\lambda_1 x}$ are linearly independent solutions of the ODE.

Multiple Complex Conjugate roots:

The part of the general solution associated to a couple of complex conjugate roots

$$\lambda_1, \lambda_2 = \alpha \pm i\omega$$

of order m is:

$$e^{\alpha x}\Big(\left(c_0 + c_1 x + \cdots + c_{m-1}x^{m-1}\right)\cos(\omega x) + \left(d_0 + d_1 x + \cdots\cdots + d_{m-1}x^{m-1}\right)\sin(\omega x)\Big)$$

or can be written as:

$$e^{\alpha x}\Big(A_0\cos(\omega x - \delta_0) + A_1 x\cos(\omega x - \delta_1) + \cdots\cdots + A_{m-1}x^{m-1}\cos(\omega x - \delta_{m-1})\Big)$$

Nonhomogeneous Linear ODEs:

A nonhomogeneous linear ODEs of nth order is given by $y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots\cdots + p_1(x)\,y = r(x)$
and $r(x) \not\equiv 0$

A general solution on an open interval I is of the form $y(x) = y_h(x) + y_p(x)$

where $y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is a general solution of the corresponding homogeneous ODE

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0 \quad \text{on } I.$$

y_p is any solution of the nonhomogeneous ODE on I containing no arbitrary constants.

If $p_{n-1}(x), \ldots, p_1(x)$ and $r(x)$ are continuous functions on I, then a general solution of the ODE exists and includes all solutions. Thus the ODE has no singular solution.

Method of Undetermined Coefficients:

For a linear nonhomogeneous ODE with constant coefficients

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = r(x)$$

we can use the method of Undetermined coefficients in the same way as in second order ODEs.

Ex: $y''' + 3y'' + 3y' + y = 30e^{-x}$; $y(0) = 3$, $y'(0) = -3$, $y''(0) = -47$

The characteristic equation of the associated homogeneous equation: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$$\Rightarrow (\lambda+1)^3 = 0 \Rightarrow \lambda = -1 \text{ (Multiple root of order 3)}$$

Hence $y_h = (c_1 + c_2 x + c_3 x^2)e^{-x}$

Now note that the R.H.S. $r(x) = 30e^{-x}$ also satisfies the homogeneous ODE and it corresponds to a triple root of the characteristic equation.

Let us try $y_p = Ax^3 e^{-x}$

$$y_p' = 3Ax^2 e^{-x} - Ax^3 e^{-x} = A(3x^2 - x^3)e^{-x}$$

$$y_p'' = A(6x - 3x^2)e^{-x} - A(3x^2 - x^3)e^{-x}$$

$$\Rightarrow y_p'' = A(6x - 6x^2 + x^3)e^{-x}$$

$$\Rightarrow y_p''' = A(6 - 12x + 3x^2)e^{-x} - A(6x - 6x^2 + x^3)e^{-x}$$

$$\Rightarrow y_p''' = A(6 - 18x + 9x^2 - x^3)e^{-x}$$

Substituting in the ODE we get

$$A(6 - 18x + 9x^2 - x^3)e^{-x} + 3A(6x - 6x^2 + x^3)e^{-x} + 3A(3x^2 - x^3)e^{-x} + Ax^3 e^{-x} = 30e^{-x}$$

$$\Rightarrow \left.\begin{aligned} &A(-x^3 + 3x^3 - 3x^3 + x^3) = 0 \\ &A(9x^2 - 18x^2 + 9x^2) = 0 \end{aligned}\right\}$$

$A(-18x + 18x) = 0$ }

$6A = 30 \quad \Rightarrow A = 5$

So, $y_p(x) = 5x^3 e^{-x}$

Therefore the general solution of the given ODE is:

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + 5x^3 e^{-x}$$

$$\Rightarrow \boxed{y = (c_1 + c_2 x + c_3 x^2 + 5x^3) e^{-x}}$$

Now $y(0) = 3 \Rightarrow c_1 = \boxed{3}$

$$y'(x) = (c_2 + 2c_3 x + 15x^2) e^{-x} - (c_1 + c_2 x + c_3 x^2 + 5x^3) e^{-x}$$

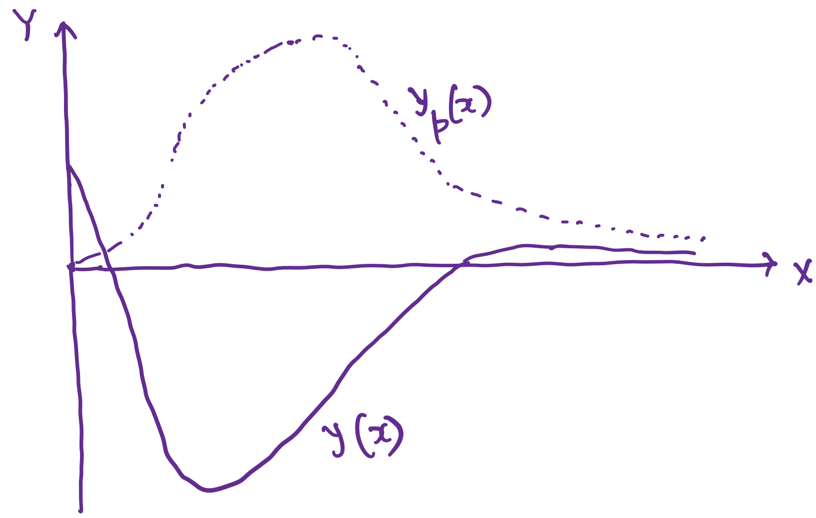
$y'(0) = -3 \Rightarrow c_2 - c_1 = -3 \Rightarrow c_2 = c_1 - 3 = 3 - 3 = \boxed{0}$

Then $y'(x) = (2c_3 x + 15x^2) e^{-x} - (3 + c_3 x^2 + 5x^3) e^{-x}$

$$\Rightarrow y''(x) = (2c_3 + 30x) e^{-x} - (2c_3 x + 15x^2) e^{-x}$$
$$- (2c_3 x + 15x^2) e^{-x} + (3 + c_3 x^2 + 5x^3) e^{-x}$$

$y''(0) = -47 \Rightarrow 2c_3 + 3 = -47 \Rightarrow 2c_3 = -50 \Rightarrow c_3 = \boxed{-25}$

Therefore the particular solution of the given ODE (IVP) is: $\boxed{y = (3 - 25x^2 + 5x^3) e^{-x}}$



Method of Variation of Parameters:

The method of variation of parameters also extends to arbitrary order n.

Let $\{y_1, y_2, \ldots, y_n\}$ be a basis of solutions of the homogeneous equation $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0$

Then a particular solution y_p of the nonhomogeneous equation: $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x) = r(x)$ is given by the formula:

$$y_p(x) = y_1(x) \int \frac{W_1(x)}{W(x)}\, r(x)\, dx + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)}\, r(x)\, dx$$

$$= \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)}\, r(x)\, dx$$

on an open interval I on which $p_{n-1}(x), \ldots, p_1(x), r(x)$ are continuous. Here W is the wronskian of $\{y_1, y_2, \ldots, y_n\}$ and W_k is obtained from W by replacing kth column of W by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Ex: $\quad x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x \quad (x > 0)$

This is an Euler-Cauchy equation of order 3.
For the general solution of the homogeneous equation we try $y = x^m$.

Then $x^3 m(m-1)(m-2)x^{m-3} - 3x^2 m(m-1)x^{m-2} + 6xmx^{m-1}$
$- 6x^m = 0$

$$\Rightarrow \quad [m(m-1)(m-2) - 3m(m-1) + 6m - 6]x^m = 0$$

$$\Rightarrow \quad (m-1)[m(m-2) - 3m + 6]x^m = 0$$

$$\Rightarrow (m-1)(m^2 - 5m + 6) = 0 \Rightarrow (m-1)(m-2)(m-3) = 0$$

$$\Rightarrow \quad m = 1, 2, 3$$

$\Rightarrow$ general solution of the homogeneous equation

$$y_h = c_1 x + c_2 x^2 + c_3 x^3$$

Thus $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$, $r(x) = \dfrac{x^4 \ln x}{x^3} = x \ln x$

Now $W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x\left(12x^2 - 6x^2\right) - 1\left(6x^3 - 2x^3\right)$

$= 6x^3 - 4x^3 = \boxed{2x^3}$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = 1\left(3x^4 - 2x^4\right) = \boxed{x^4}$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = (-1)\left(3x^3 - x^3\right) = \boxed{-2x^3}$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = (1)\left(2x^2 - x^2\right) = \boxed{x^2}$$

Then $y_p(x) = x\int \frac{x^4}{2x^3}\, x\ln x\, dx + x^2\int \frac{(-2x^3)}{2x^3}\, x\ln x\, dx$

$+ x^3\int \frac{x^2}{2x^3}\, x\ln x\, dx$

$$= \frac{1}{2}x\int x^2\ln x\, dx - x^2\int x\ln x\, dx + \frac{x^3}{2}\int \ln x\, dx$$

$$= \frac{1}{2}x\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] - x^2\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right] + \frac{x^3}{2}\left[x\ln x - x\right]$$

$$= \frac{1}{6}x^4\ln x - \frac{1}{2}x^4\ln x + \frac{1}{2}x^4\ln x - \frac{x^4}{18} + \frac{x^4}{4} - \frac{x^4}{2}$$

$$= \frac{1}{6}x^4\ln x - \frac{2x^4 - 9x^4 + 18x^4}{36} = \frac{1}{6}x^4\ln x - \frac{11}{36}x^4$$

$$= \boxed{\frac{1}{6}x^4\left(\ln x - \frac{11}{6}\right)}$$

Thus the general solution of the given ODE is:

$$y(x) = y_h(x) + y_p(x) = \boxed{c_1 x + c_2 x^2 + c_3 x^3 + \frac{x^4}{6}\left(\ln x - \frac{11}{6}\right)}$$

<u>An Example:</u>

$$y'' - 3y' + 2y = 10\sin x$$

For the solution of the homogeneous equation, the characteristic equation is: $\lambda^2 - 3\lambda + 2 = 0$

$$\Rightarrow (\lambda - 2)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 2$$

So, the solution of the associated homogeneous equation is: $y_h = c_1 e^x + c_2 e^{2x}$

- For a particular solution of the nonhomogeneous equation, first we use Method of Undetermined coefficients.

Let $y_p = A\sin x + B\cos x$

Then $y_p' = A\cos x - B\sin x$ and $y_p'' = -A\sin x - B\cos x$

Then $[-A\sin x - B\cos x] - 3(A\cos x - B\sin x) + 2[A\sin x + B\cos x]$

$= 10\sin x$

$\Rightarrow \sin x[-A + 3B + 2A] + \cos x[-B - 3A + 2B] = 10\sin x$

$$\Rightarrow \left.\begin{array}{r} A + 3B = 10 \\ -3A + B = 0 \end{array}\right\} \Rightarrow 10A = 10 \Rightarrow A = \boxed{1}$$

and $B = \boxed{3}$

So, $\boxed{y_p(x) = \sin x + 3\cos x}$

We can also use method of Variation of Parameters to obtain $y_p(x)$:

$y_1 = e^x, \quad y_2 = e^{2x}$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

Then $u(x) = -\int \frac{e^{2x}}{e^{3x}} 10\sin x\,dx = -10\int e^{-x}\sin x\,dx$

$= -10\left[\sin x(-e^{-x}) + \int \cos x\, e^{-x}\,dx\right]$

$$= 10e^{-x}\sin x - 10\left[\left(-e^{-x}\cos x\right) - \int \sin x\, e^{-x}\,dx\right]$$

$$= 10e^{-x}\sin x + 10e^{-x}\cos x - u(x)$$

$$\Rightarrow 2u(x) = 10e^{-x}\left[\sin x + \cos x\right]$$

$$\Rightarrow u(x) = 5e^{-x}\left[\sin x + \cos x\right]$$

Now $v(x) = \displaystyle\int \frac{e^{x}}{e^{3x}} 10\sin x\,dx = 10\int e^{-2x}\sin x\,dx$

$$= 10\left[\frac{e^{-2x}}{(-2)}\sin x - \int \frac{e^{-2x}}{(-2)}\cos x\,dx\right]$$

$$= 10\left[-\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\left\{\frac{e^{-2x}}{(-2)}\cos x - \int \frac{e^{-2x}}{(-2)}(-\sin x)\,dx\right\}\right]$$

$$= -5e^{-2x}\sin x - \frac{5}{2}e^{-2x}\cos x - \frac{5}{2}\int e^{-2x}\sin x\,dx$$

$$= -5e^{-2x}\left[\sin x + \frac{1}{2}\cos x\right] - \frac{1}{4}v(x)$$

$$\Rightarrow \frac{5}{4}v(x) = -5e^{-2x}\left[\sin x + \frac{1}{2}\cos x\right]$$

$$\Rightarrow v(x) = -4e^{-2x}\sin x - 2e^{-2x}\cos x$$

So, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$

$$= e^{x}\left[5e^{-x}(\sin x + \cos x)\right] + e^{2x}\left[-4e^{-2x}\sin x - 2e^{-2x}\cos x\right]$$

$$= 5\sin x + 5\cos x - 4\sin x - 2\cos x$$

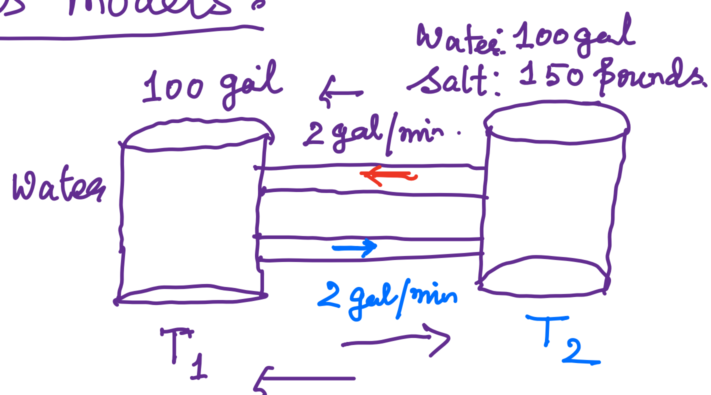
$$\Rightarrow \boxed{y_p(x) = \sin x + 3\cos x}$$

MTH 204: Lecture 11

System of ODEs

- **System of ODEs as models:**

(1) Mixing Tanks:



(System of Tanks)

Problem: To determine the amount of salt in T_1 and T_2 at time t.

Let the amount of salt in T_1 at time t be $y_1(t)$ and the amount of salt in T_2 at time t be $y_2(t)$

$$y_1' = \text{inflow} - \text{outflow} = \left(\frac{y_2}{100} \times 2 - \frac{y_1}{100} \times 2\right) \frac{\text{lb}}{\cancel{\text{gal}}} \frac{\text{gal}}{\text{min}} = \frac{\text{lb}}{\text{min}}$$

and

$$y_2' = \text{inflow} - \text{outflow} = \left(\frac{y_1}{100} \times 2 - \frac{y_2}{100} \times 2\right) \frac{\text{lb}}{\cancel{\text{gal}}} \frac{\cancel{\text{gal}}}{\text{min}} = \frac{\text{lb}}{\text{min}}$$

$$\Rightarrow \left.\begin{array}{l} y_1' = -.02y_1 + .02y_2 \\ y_2' = .02y_1 - .02y_2 \end{array}\right\}$$

$$\Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -.02 & .02 \\ .02 & -.02 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

$Y = Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

$\Rightarrow \boxed{Y' = AY}$ where $A = \begin{pmatrix} -.02 & .02 \\ .02 & -.02 \end{pmatrix}$

• We try a solution of the form:

$$Y = x e^{\lambda t}$$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} e^{\lambda t}$

$\Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \lambda x_1 e^{\lambda t} \\ \lambda x_2 e^{\lambda t} \end{pmatrix}$

$\Rightarrow Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \lambda e^{\lambda t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Then $Y' = \lambda X e^{\lambda t} \Rightarrow \lambda x e^{\lambda t} = A x e^{\lambda t}$.

$$\Rightarrow AX = \lambda X$$

Thus $Y = xe^{\lambda t}$ is a solution of the system of the ODE if X is an eigenvector of the matrix A corresponding to the eigen value λ of A.

Now $A = \begin{pmatrix} -.02 & .02 \\ .02 & -.02 \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (\lambda + .02)^2 - (.02)^2 = 0$

$$\Rightarrow \lambda^2 + .04\lambda = 0 \Rightarrow \lambda(\lambda + .04) = 0$$

$$\Rightarrow \lambda = 0, -.04$$

For $\lambda = 0$, $AX = 0.X \Rightarrow \begin{pmatrix} -.02 & .02 \\ .02 & -.02 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{array}{l} \Rightarrow -.02x_1 + .02x_2 = 0 \\ \Rightarrow x_1 = x_2 \end{array}\right\} \Rightarrow X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \left(\text{or } (1 \;\; 1)^T \right)$$

For $\lambda = -.04$, $AX = -.04X \Rightarrow \begin{pmatrix} -.02 & .02 \\ .02 & -.02 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -.04 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left.\Rightarrow .02x_1 + .02x_2 = 0 \Rightarrow x_1 = -x_2 \right\} \Rightarrow X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left(\text{or } (1 \;\; -1)^T \right)$$

Hence, the general solution is:

$$Y = c_1 X_1 e^{\lambda_1 t} + c_2 X_2 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-.04t}$$

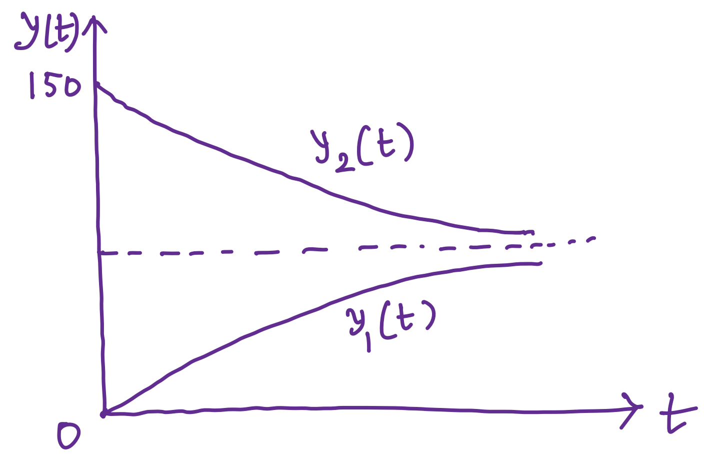
Now given $y_1(0) = 0$, $y_2(0) = 150$ $\left(\text{so, } Y(0) = \begin{pmatrix} 0 \\ 150 \end{pmatrix} \right)$

Then $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 150 \end{pmatrix} \Rightarrow \left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 - c_2 = 150 \end{array}\right\}$

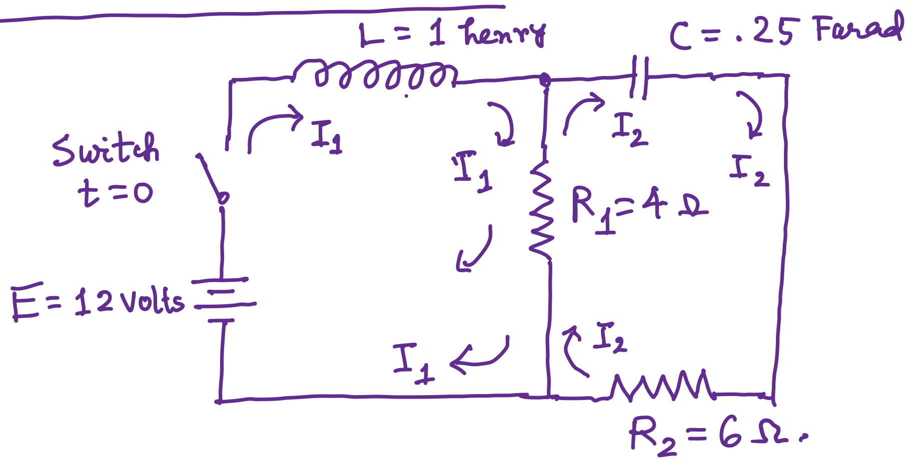
$$\Rightarrow c_1 = 75, \quad c_2 = -75$$

Therefore $Y = 75 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 75 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-.04t}$

$$\Rightarrow \left.\begin{array}{l} y_1(t) = 75 - 75e^{-.04t} \\ y_2(t) = 75 + 75e^{-.04t} \end{array}\right\}$$



Ex: Electrical Network :



$$L I_1' + R_1(I_1 - I_2) = E \quad \Rightarrow \quad I_1' = -R_1 I_1 + R_1 I_2 + E \quad \cdots\cdots (1)$$

(given $L = 1$)

and

$$R_1(I_2 - I_1) + \frac{1}{C}\int I_2\,dt + R_2 I_2 = 0$$

$$\Rightarrow \quad R_1(I_2' - I_1') + \frac{1}{C} I_2 + R_2 I_2' = 0$$

$$\Rightarrow \quad (R_1 + R_2) I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$\Rightarrow (R_1+R_2)I_2' - R_1(-R_1 I_1 + R_1 I_2 + E) + \frac{1}{C} I_2 = 0 \quad \ldots\ (2)$$

Now $R_1 = 4$, $R_2 = 6$, $E = 12$, $C = 0.25$

Then (1) $\Rightarrow$ $\boxed{I_1' = -4I_1 + 4I_2 + 12}$

and (2) $\Rightarrow$ $10 I_2' - 4(-4I_1 + 4I_2 + 12) + \frac{I_2}{.25} = 0$

$$\Rightarrow I_2' = .4(-4I_1 + 4I_2 + 12) - \frac{I_2}{10 \times .25} = 0$$

$$\Rightarrow I_2' = -1.6 I_1 + 1.6 I_2 + 4.8 - .4 I_2$$

$$\Rightarrow \boxed{I_2' = -1.6 I_1 + 1.2 I_2 + 4.8}$$

Thus the system of ODE is

$$\left.\begin{aligned} I_1' &= -4I_1 + 4I_2 + 12 \\ I_2' &= -1.6 I_1 + 1.2 I_2 + 4.8 \end{aligned}\right\}$$

or $$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

This is a nonhomogeneous system of ODE

First we find the general solution of the homogeneous system of ODE

$$I' = AI \quad \text{where } I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

and $A = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$

First we find the eigen values of A.

$\det(A - \lambda I) = 0 \Rightarrow (-4-\lambda)(1.2-\lambda) - 4(-1.6) = 0$

$$\Rightarrow -4.8 + 4\lambda - 1.2\lambda + \lambda^2 + 6.4 = 0$$

$$\Rightarrow \lambda^2 + 2.8\lambda + 1.6 = 0$$

$$\Rightarrow (\lambda + 2)(\lambda + .8) = 0$$

So, the eigen values are $\lambda = -2, -.8$

For eigen vector corresponding to the eigenvalue $\lambda = -2$

we solve: $\begin{pmatrix} -2 & 4 \\ -1.6 & 3.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}$$

$$\Rightarrow 2x_1 = 4x_2 \Rightarrow x_1 = 2x_2$$

So, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = -2$

For eigen vector corresponding to the eigen value $\lambda = -.8$

we solve: $\begin{pmatrix} -3.2 & 4 \\ -1.6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} -3.2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 2x_2 &= 0 \end{aligned}\right\}$$

$$\Rightarrow 1.6x_1 = 2x_2 \Rightarrow x_2 = \frac{1.6}{2}x_1$$

$x_1 = 1$, $x_2 = \frac{1.6}{2}x_1 = \frac{1.6}{2} = .8$

$\begin{pmatrix} 1 \\ .8 \end{pmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = -.8$

So, the general solution of the homogeneous system of ODE is:

$$I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ .8 \end{pmatrix} e^{-.8t}$$

Now for a particular solution of the nonhomogeneous system of ODE $I' = AI + C$ $\left[\text{Here } C = \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}\right]$

we try a constant vector $I = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Then $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

$$-4a_1 + 4a_2 + 12 = 0$$
$$-1.6a_1 + 1.2a_2 + 4.8 = 0$$

$$\left.\begin{array}{l} 4a_1 - 4a_2 = 12 \Rightarrow a_1 - a_2 = 3 \\ -1.6a_1 + 1.2a_2 = 4.8 \Rightarrow 1.6a_1 - 1.2a_2 = 4.8 \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} 1.2a_1 - \cancel{1.2a_2} = 3.6 \\ -\,1.6a_1 \underset{+}{-} \cancel{1.2a_2} \underset{-}{=} 4.8 \end{array}\right\}$$
$$\overline{-.4a_1 = -1.2}$$

$$\Rightarrow \boxed{a_1 = 3}$$

Now from $a_1 - a_2 = 3$, we get $3 - a_2 = 3$

$$\Rightarrow \boxed{a_2 = 0}$$

So, the general solution of the nonhomogeneous ODE is:

$$I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ .8 \end{pmatrix} e^{-.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

where c_1 and c_2 are arbitrary constants.

To determine c_1, c_2 we use the initial condition $I(0) = 0$

Then $\quad 2c_1 + c_2 + 3 = 0 \Rightarrow \quad 2c_1 + c_2 = -3$

$\quad c_1 + .8c_2 = 0 \Rightarrow \underline{-2c_1 + 1.6c_2 = 0}$

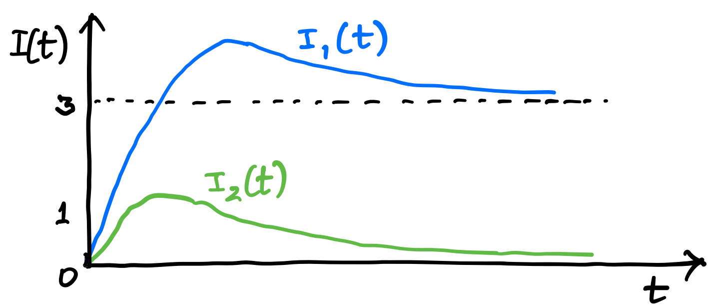
$$-.6c_2 = -3$$

$$\Rightarrow c_2 = \frac{-3}{-.6} = \boxed{5}$$

Then $\quad 2c_1 + 5 = -3 \Rightarrow 2c_1 = -8 \Rightarrow \boxed{c_1 = -4}$

Thus the solution of the nonhomogeneous system of ODE is: $I = -4\begin{pmatrix}2\\1\end{pmatrix}e^{-2t} + 5\begin{pmatrix}1\\.8\end{pmatrix}e^{-.8t} + \begin{pmatrix}3\\0\end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} I_1(t) &= -8e^{-2t} + 5e^{-.8t} + 3 \\ I_2(t) &= 4e^{-2t} + 4e^{-.8t} \end{aligned}\right\}$$



Conversion of an n-th order ODE to a system of ODE:

An n-th order ODE $\quad y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$ can be converted to a system of n first order ODEs by setting $\quad y = y_1 \quad$ and

$$y_1' = y_2$$
$$y_2' = y_3$$
$$\vdots$$
$$y_{n-1}' = y_n$$
$$y_n' = F(t, y_1, y_2, \dots, y_n)$$

Note that $y_1 = y$, $y_2 = y_1' = y'$, $y_3 = y_2' = y''$, ... $y_{n-1} = y_{n-2}' = y^{(n-2)}$,
$y_n = y_{n-1}' = y^{(n-1)}$

Ex: Consider the second order ODE :

$$my'' + cy' + ky = 0$$

Then $\quad y'' = -\frac{c}{m} y' - \frac{k}{m} y$

Let $y = y_1$, $y_1' = y_2$ So, $y' = y_2$ and $y'' = y_2'$

Therefore the second order ODE can be written as a system of ODE :

$$\left. \begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{k}{m} y_1 - \frac{c}{m} y_2 \end{aligned} \right\}$$

$$\Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The characteristic equation is :

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda\left(-\frac{c}{m} - \lambda\right) - \left(-\frac{k}{m}\right) = 0$$

$$\Rightarrow \quad \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Let us now assume that in this problem,

$m = 1$, $c = 2$ and $k = 0.75$

Then the characteristic equation becomes:

$$\lambda^2 + 2\lambda + .75 = 0$$

$$\Rightarrow \lambda^2 + 1.5\lambda + .5\lambda + .75 = 0$$

$$\Rightarrow \lambda(\lambda + 1.5) + .5(\lambda + 1.5) = 0$$

$$\Rightarrow (\lambda + 1.5)(\lambda + .5) = 0$$

So, the two eigen values are $\lambda = -.5$, $\lambda = -1.5$

For eigen vector corresponding to the eigenvalue $\lambda = -.5$ we solve:

$$\begin{pmatrix} .5 & 1 \\ -.75 & -1.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} .5x_1 + x_2 = 0 \\ -.75x_1 - 1.5x_2 = 0 \end{array}\right\} \Rightarrow .5x_1 + x_2 = 0 \Rightarrow x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = -2x_2$$

Thus $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = -.5$

For eigen vector corresponding to the eigenvalue $\lambda = -1.5$, we solve:

$$\begin{pmatrix} 1.5 & 1 \\ -.75 & -.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 1.5x_1 + x_2 = 0 \\ -.75x_1 - .5x_2 = 0 \end{array}\right\} \Rightarrow 1.5x_1 + x_2 = 0$$

$$\Rightarrow x_2 = -1.5x_1$$

Hence $\begin{pmatrix} 1 \\ -1.5 \end{pmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = -1.5$

Thus the general solution of the system of ODE is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-.5t} + c_2 \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} e^{-1.5t}$$

Note that the first component is

$$y = y_1 = 2c_1 e^{-.5t} + c_2 e^{-1.5t}$$

(which is the expected solution of the second order ODE) and the second component is y_2 which is the derivative of the first component y_1.

MTH 204 : Lecture 12

Basic Theory :

In general an ODE system is of the form:

$$\left.\begin{aligned} y_1' &= f_1(t, y_1, \dots, y_n) \\ y_2' &= f_2(t, y_1, \dots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, \dots, y_n) \end{aligned}\right\}$$

We can write the system as a vector equation by $Y' = f(t, Y)$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ and $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$

An initial value problem needs n initial conditions at t_0 i.e. $y_1(t_0) = K_1$, $y_2(t_0) = K_2$, ..., $y_n(t_0) = K_n$

This can also be written as

$$Y(t_0) = K \quad \text{where} \quad K = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix}$$

$$Y(t_0) = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix}$$

Existence and Uniqueness :

Let $f_1, f_2, \dots, f_n$ be continuous functions with continuous partial derivatives $\frac{\partial f_1}{\partial y_1}, \frac{\partial f_1}{\partial y_2}, \dots\dots, \frac{\partial f_n}{\partial y_n}$ in some domain R of the $t\, y_1\, y_2 \dots y_n$ space containing the point $(t_0, K_1, \dots, K_n)$

Then the ODE system has a solution on some interval $t_0 - \alpha < t < t_0 + \alpha$ satisfying the initial conditions, and this solution is unique.

Linear Systems:

$$\left.\begin{aligned}
y_1' &= a_{11}(t)\, y_1 + a_{12}(t)\, y_2 + \dots + a_{1n}(t)\, y_n + g_1(t) \\
y_2' &= a_{21}(t)\, y_1 + a_{22}(t)\, y_2 + \dots + a_{2n}(t)\, y_n + g_2(t) \\
&\ \ \vdots \\
y_n' &= a_{n1}(t)\, y_1 + a_{n2}(t)\, y_2 + \dots + a_{nn}(t)\, y_n + g_n(t)
\end{aligned}\right\}$$

The above can be written as $\boxed{Y' = AY + g}$

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ and $g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}$

and $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

If $g = 0$, the system is homogeneous.
(i.e. $Y' = AY$)

Existence and Uniqueness:

Let the a_{ij}'s and g_i's functions be continuous functions of t in an open interval I containing t_0, then there exists a solution, satisfying the initial conditions and this solution is unique.

Superposition Principle:

The linear combination of any two solutions Y_1 and Y_2 of the Homogeneous Equation is also a solution of the Homogeneous Equation.

Proof:

Let $Y = c_1 Y_1 + c_2 Y_2$

Now, $Y' = (c_1 Y_1 + c_2 Y_2)' = c_1 Y_1' + c_2 Y_2'$

$$= c_1 A Y_1 + c_2 A Y_2 = A(c_1 Y_1 + c_2 Y_2) = AY$$

General Solution : Wronskian :

If the a_{ij}'s functions are continuous, then the general solution of the Homogeneous equation can be written as :

$$Y = c_1 Y_1 + c_2 Y_2 + \cdots\cdots + c_n Y_n$$

where $Y_1, Y_2, \ldots, Y_n$ constitute a basis or fundamental system of solutions and there is no singular solution.

We can write the n basis functions as columns of a matrix $\tilde{Y}$

$$\tilde{Y} = (Y_1, Y_2, \ldots, Y_n) = \begin{pmatrix} Y_1^{(1)} & Y_2^{(1)} & \cdots & Y_n^{(1)} \\ Y_1^{(2)} & Y_2^{(2)} & & Y_n^{(2)} \\ \vdots & \vdots & & \vdots \\ Y_1^{(n)} & Y_2^{(n)} & & Y_n^{(n)} \end{pmatrix}$$

and write the general solution as ;

$$Y = \tilde{Y} C \quad \text{where} \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$\left(Y = c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n = [Y_1 \; Y_2 \cdots Y_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right)$$

The Wronskian W is the determinant of $\tilde{Y}$

$$\text{i.e.} \quad W = |\tilde{Y}|$$

<u>Note:</u> The solutions form a basis on the interval iff $W \neq 0$ at some point on the interval. W is either identically zero or nowhere zero on the interval.

Constant Coefficient System :

Consider $Y' = AY$ where A is a $(n \times n)$ matrix of constants.

We try a function of the form $Y = Xe^{\lambda t}$

$$\Rightarrow Y' = \lambda X e^{\lambda t}$$

$$\Rightarrow \quad \lambda X e^{\lambda t} = A X e^{\lambda t}$$

$$\Rightarrow \quad AX = \lambda X$$

Note that if $Y = \begin{pmatrix} e^{\lambda t} x_1 \\ \vdots \\ e^{\lambda t} x_n \end{pmatrix}$ then $Y' = \begin{pmatrix} \lambda e^{\lambda t} x_1 \\ \vdots \\ \lambda e^{\lambda t} x_n \end{pmatrix}$

Thus Y is a solution of the system of ODE if X is an eigenvector of A.

If A has n distinct eigen values, then the general solution is

$$Y = c_1 X_1 e^{\lambda_1 t} + \cdots + c_n X_n e^{\lambda_n t}$$

The Wronskian of the basis of solutions is

$$W = \left| X_1 e^{\lambda_1 t} \cdots X_n e^{\lambda_n t} \right|_{n \times n}$$

$$= e^{\lambda_1 t + \cdots + \lambda_n t} \left| X_1 \cdots X_n \right|_{n \times n}$$

- The exponential term cannot be zero and the determinant of the matrix of eigen vectors cannot be zero because they are linearly independent vectors since they correspond to distinct eigenvalues. This proves that there is no singular solution if all eigenvalues are distinct.

(It is true whenever the n-solutions are linearly independent)

Phase Plane Method : (Qualitative Method):

We will look at systems with constant coefficients consisting of two ODEs

$$Y' = AY \quad \Rightarrow \quad \left.\begin{array}{l} y_1' = a_{11} y_1 + a_{12} y_2 \\ y_2' = a_{21} y_1 + a_{22} y_2 \end{array}\right\} \cdots (1)$$

We can graph solutions of (1):

$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ as two curves over the t-axis, one for each component of $Y(t)$

But we can also graph it as a single curve in the $y_1 y_2$ plane (Parametric representation with parameter t)

Such a curve is called a Trajectory (or orbit or path) of (1).

The $y_1 y_2$-plane is called the Phase plane.

If we fill the phase plane with trajectories of (1), we obtain Phase portrait of (1).

A phase portrait gives a good general qualitative impression of the entire family of solutions.

Ex:

$$Y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} Y$$

$\Rightarrow A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \Rightarrow$ characteristic polynomial of A: $(\lambda+3)^2 - 1 = 0$

$$\Rightarrow \lambda^2 + 6\lambda + 8 = 0$$

$$\Rightarrow (\lambda+4)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -2, -4$$

For $\lambda = -2$, $AX = \lambda X \Rightarrow \left.\begin{array}{l} -3x_1 + x_2 = -2x_1 \\ x_1 - 3x_2 = -2x_2 \end{array}\right\}$

$$\Rightarrow -x_1 + x_2 = 0 \Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = -4$, $AX = \lambda X \Rightarrow \left.\begin{array}{l} -3x_1 + x_2 = -4x_1 \\ x_1 - 3x_2 = -4x_2 \end{array}\right\}$

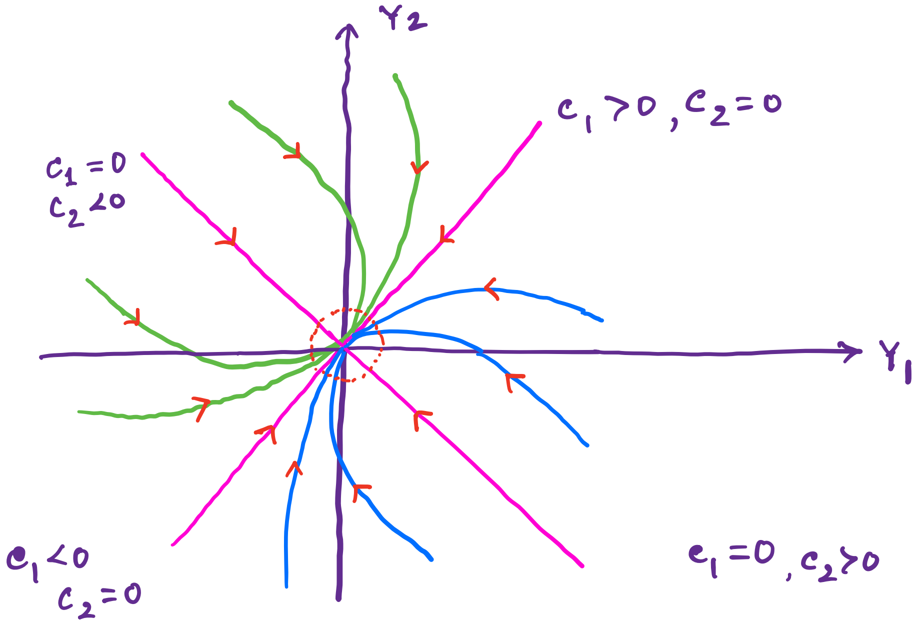
$$\Rightarrow x_1 + x_2 = 0 \Rightarrow X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then $Y = c_1 X_1 e^{\lambda_1 t} + c_2 X_2 e^{\lambda_2 t}$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

If $c_1 = 0$, $y_1 = c_2 e^{-4t}$, $y_2 = -c_2 e^{-4t} \Rightarrow y_1 = -y_2$

If $c_2 = 0$, $y_1 = c_1 e^{-2t}$, $y_2 = c_1 e^{-2t} \Rightarrow y_1 = y_2$



Critical points:

A critical point is a point at which $Y' = 0$.
They are also called equilibrium solutions.

From the equation of the system

$$Y' = AY,$$

slope of the trajectories in the phase plane at a given point (y_1, y_2):

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

(At every point, there is a unique tangent direction $\frac{dy_2}{dy_1}$ of the trajectory passing through that point.)

At critical points, this ratio becomes undefined $\left(\frac{0}{0}\right)$.

There are five types of critical points:
improper nodes, proper nodes, saddle points,

$$= a_1 x + a_0 \left(1 - \sum_{m=1}^{\infty} \frac{x^{2m}}{(2m-1)} \right)$$

Table of Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$		$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\ n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\ n = 1, 2, 3, \ldots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at) + at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at) - at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at) + at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b) + a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b) - a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\ n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t) = u(t-c)$	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\ n = 1, 2, 3, \ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T) = f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s) - f(0)$	36.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$			

centers and spiral points.

Improper node: An improper node is a critical point at which all trajectories, except two of them have the same limiting direction of the tangent. The two exceptional directions also have a limiting direction of the tangent which however is different.

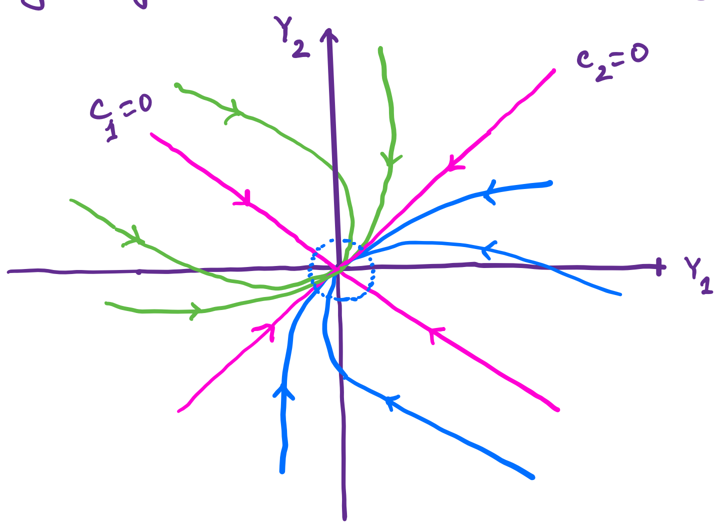
Ex ①: $Y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} Y$

We have calculated the general solution of the above system as: $Y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

The common limiting direction at 0 is the direction of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ because e^{-4t} goes to zero faster than e^{-2t} as t increases to ∞.

The two exceptional limiting tangent directions are those of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(0,0) is an improper node.



Ex: Proper node:

Consider the system $Y' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y$

The characteristic equation is $(1-\lambda)^2 = 0 \Rightarrow \lambda = 1$

From $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \left.\begin{array}{l} x_1 = x_1 \\ x_2 = x_2 \end{array}\right\} \Rightarrow$ Two linearly independent eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

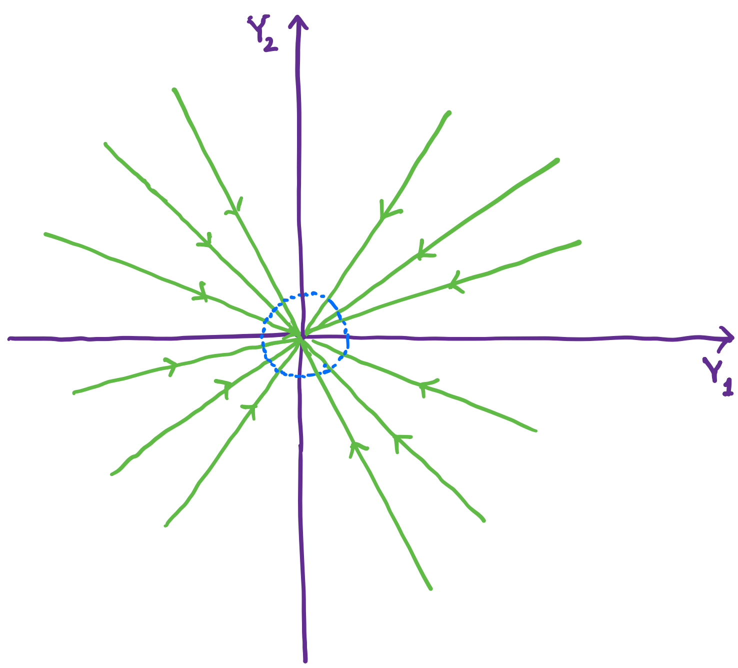
So, the general solution is: $Y = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$

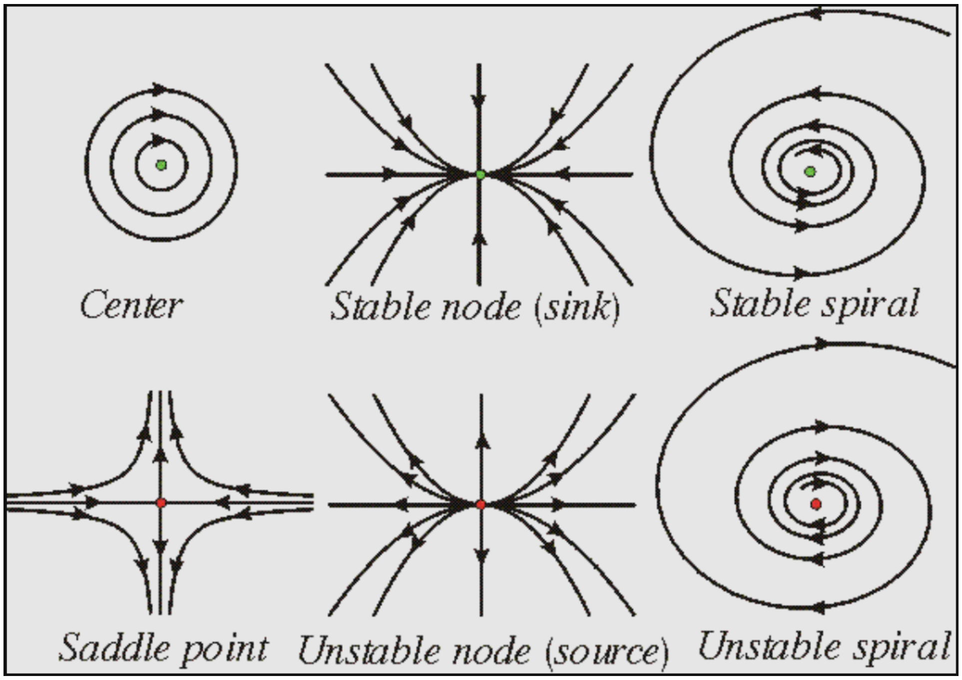
A proper node is a critical point at which every trajectory has a definite limiting direction and for any given direction d, there is a trajectory having d as its limiting direction.

$Y_1 = c_1 e^t, \; Y_2 = c_2 e^t$

$\Rightarrow \quad c_2 Y_2 = c_1 Y_1$

(0,0) is a proper node.




Center
Stable node (sink)
Stable spiral
Saddle point
Unstable node (source)
Unstable spiral

Ex: Saddle point

Let $Y' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y$: The characteristic equation is $(1-\lambda)(-1-\lambda) = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = 1, -1$, From $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_1 = x_1, -x_2 = x_2 \}$

From $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \left.\begin{matrix} x_1 = -x_1 \\ -x_2 = -x_2 \end{matrix}\right\} \Rightarrow$ Two linearly independent eigen vectors

are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So, the general solution is $Y = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$

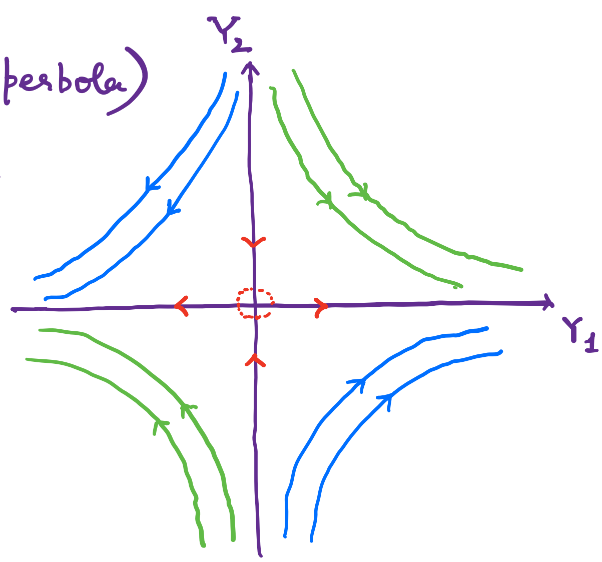
A saddle point is a critical point at which there are <u>two incoming trajectories</u>, <u>two outgoing trajectories</u> and all other trajectories in a neighborhood of the critical point bypass it.

$Y_1 = c_1 e^{t}$, $Y_2 = c_2 e^{-t}$

$\Rightarrow Y_1 Y_2 = c_1 c_2$ (rectangular hyperbola)

If $c_2 = 0$, $Y_2 = 0$, $Y_1 = c_1 e^{t} \left.\begin{matrix} \to \infty \text{ if } c_1 > 0 \\ \to -\infty \text{ if } c_1 < 0 \end{matrix}\right\}$

If $c_1 = 0$, $Y_1 = 0$, $Y_2 = c_2 e^{-t} \left.\begin{matrix} \to 0 \text{ if } c_2 > 0 \\ \to 0 \text{ if } c_2 < 0 \end{matrix}\right\}$



Ex: Center: $Y' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} Y$. The characteristic equation is:

$$\left.\begin{aligned} &(-\lambda)(-\lambda)+4=0 \Rightarrow \lambda^2+4=0 \\ \Rightarrow \lambda = \pm 2i : \text{From } &\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_2 = 2i x_1, -4x_1 = 2i x_2 \end{aligned}\right\}$$

From $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_2 = -2i x_1, \; -4x_1 = -2i x_2 \}$

Two linearly eigenvectors are $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$

So, the general solution is $Y = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it}$

- A center is a critical point that is enclosed by infinitely many closed trajectories.

$$\left.\begin{aligned} Y_1 &= c_1 e^{2it} + c_2 e^{-2it} \\ Y_2 &= 2i c_1 e^{2it} - 2i c_2 e^{-2it} \end{aligned}\right\}$$

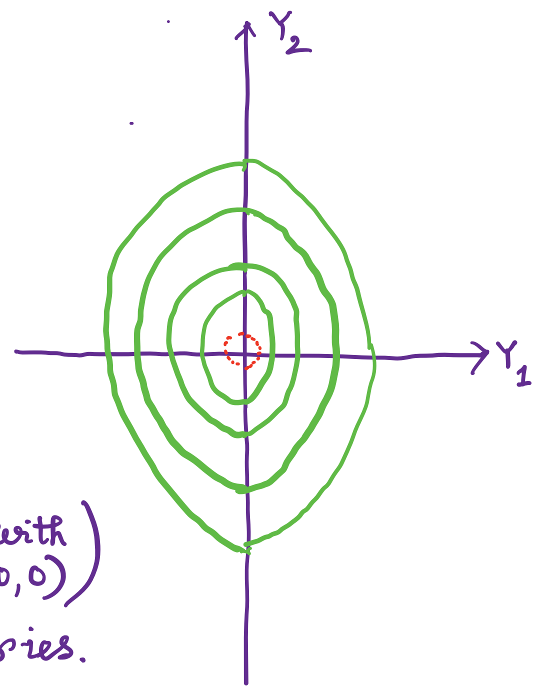
$\Rightarrow Y_1' = Y_2$ and $Y_2' = -4Y_1$

$\Rightarrow 4Y_1 Y_1' = -Y_2 Y_2'$

$\Rightarrow \int 4Y_1 \, dY_1 = -\int Y_2 \, dY_2$

$\Rightarrow 2Y_1^2 + \frac{Y_2^2}{2} = C$

$\Rightarrow Y_1^2 + \frac{Y_2^2}{4} = C$ (family of ellipse with center $(0,0)$)



$(0,0)$ is the center of the trajectories.

(Eigen values are purely imaginary)

Spiral point:

A Spiral point is a critical point P_0 about which the trajectories spiral approaching P_0 as $t \to \infty$ (or tracing these spirals in the opposite sense, away from P_0)

Ex: $Y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} Y$. The characteristic equation is

$$(-1-\lambda)(-1-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i$$

From $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-1+i)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \left.\begin{matrix} -x_1 + x_2 = (-1+i)x_1 \\ -x_1 - x_2 = (-1+i)x_2 \end{matrix}\right\} \Rightarrow \begin{matrix} x_2 = ix_1 \\ -x_1 = ix_2 \end{matrix}$

From $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-1-i)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \left.\begin{matrix} -x_1 + x_2 = (-1-i)x_1 \\ -x_1 - x_2 = (-1-i)x_2 \end{matrix}\right\} \Rightarrow \begin{matrix} x_2 = -ix_1 \\ -x_1 = -ix_2 \end{matrix}$

Two linearly independent eigen vectors are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

So, the general solution is:

$$Y = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

$$\Rightarrow \left.\begin{matrix} Y_1 = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} \\ Y_2 = ic_1 e^{(-1+i)t} - ic_2 e^{(-1-i)t} \end{matrix}\right\}$$

$$\Rightarrow Y_1' = c_1(-1+i)e^{(-1+i)t} + c_2(-1-i)e^{(-1-i)t}$$

$$\Rightarrow Y_1' = (-c_1 e^{(-1+i)t} - c_2 e^{(-1-i)t}) + ic_1 e^{(-1+i)t} - ic_2 e^{(-1-i)t}$$

$$\Rightarrow Y_1' = -Y_1 + Y_2 \quad \ldots\ldots\ldots (1)$$

and $Y_2' = i(-1+i)c_1 e^{(-1+i)t} - i(-1-i)c_2 e^{(-1-i)t}$

$$= \left(-c_1 e^{(-1+i)t} - c_2 e^{(-1-i)t}\right) - \left(i c_1 e^{(-1+i)t} - i c_2 e^{(-1-i)t}\right)$$

$$Y_2' = -Y_1 - Y_2 \quad \ldots\ldots (2)$$

Now $(1) \times Y_1 + (2) \times Y_2 \Rightarrow Y_1'Y_1 + Y_2'Y_2 = -Y_1^2 + Y_1Y_2 - Y_1Y_2 - Y_2^2$

$$\Rightarrow Y_1'Y_1 + Y_2'Y_2 = -Y_1^2 - Y_2^2 \Rightarrow Y_1'Y + Y_2'Y_2 = -(Y_1^2 + Y_2^2)$$

We now introduce polar coordinates r and t such that $r^2 = Y_1^2 + Y_2^2$ (ie $Y_1 = r\cos t$, $Y_2 = r\sin t$)

$$\Rightarrow \quad 2r\frac{dr}{dt} = 2Y_1\frac{dY_1}{dt} + 2Y_2\frac{dY_2}{dt} = 2Y_1Y_1' + 2Y_2Y_2'$$

$$\Rightarrow 2r\frac{dr}{dt} = -2r^2 \Rightarrow \frac{dr}{dt} = -r$$

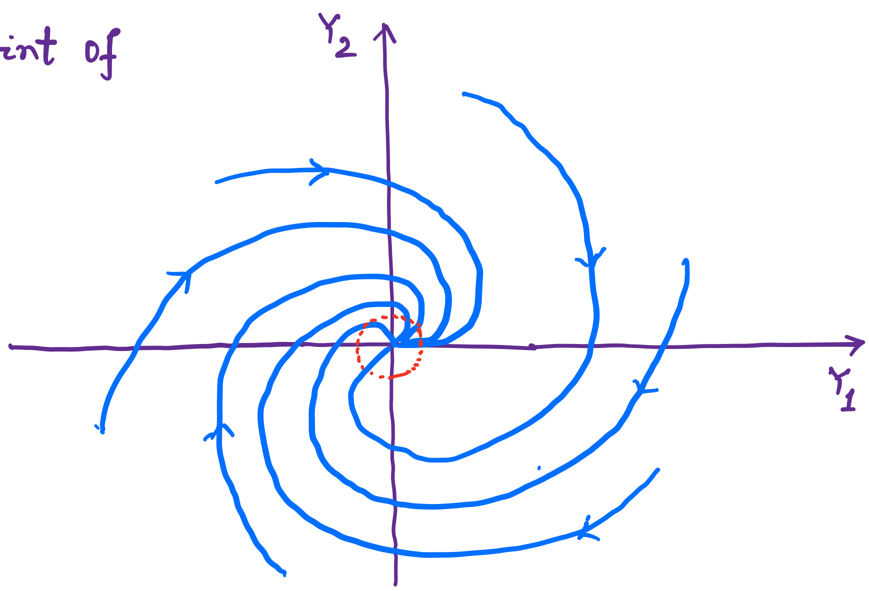
$$\Rightarrow \frac{dr}{r} = -dt \Rightarrow \ln|r| = -t + c_1$$

$$\Rightarrow \boxed{r = ce^{-t}} \quad (\text{where } c = e^{c_1})$$

For each real c, this is a spiral

(0,0) is a spiral point of the trajectories.

(Eigen values are complex but not purely imaginary).



Degenerate node:

Ex: $Y' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} Y$ Let $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equation is $(4-\lambda)(2-\lambda)+1=0$

$\Rightarrow 8-6\lambda+\lambda^2+1=0 \Rightarrow \lambda^2-6\lambda+9=0 \Rightarrow (\lambda-3)^2=0$

Thus $\lambda_1 = 3$ is an eigen value of the coefficient matrix with multiplicity 2

From the equation $\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \left.\begin{array}{l} 4x_1 + x_2 = 3x_1 \\ -x_1 + 2x_2 = 3x_2 \end{array}\right\}$

$\Rightarrow \left.\begin{array}{l} x_1 + x_2 = 0 \\ -x_1 - x_2 = 0 \end{array}\right\}$ So, we get only one independent eigen vector $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So, one of the solutions is of the form

$$Y_1 = x_1 e^{\lambda_1 t} = x_1 e^{3t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$$

For the second solution we try

$Y_2 = t x_1 e^{\lambda_1 t} + u e^{\lambda_1 t}$ where u is a constant vector.

Then $Y_2' = x_1 e^{\lambda_1 t} + t\lambda_1 x_1 e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t}$

Since $Y_2' = AY_2$, we have

$$x_1 e^{\lambda_1 t} + t\lambda_1 x_1 e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t} = te^{\lambda_1 t} Ax_1 + e^{\lambda_1 t} Au$$

$$\Rightarrow x_1 e^{\lambda_1 t} + \cancel{t\lambda_1 x_1 e^{\lambda_1 t}} + \lambda_1 u e^{\lambda_1 t} = \cancel{t\lambda_1 x_1 e^{\lambda_1 t}} + e^{\lambda_1 t} Au$$

$$\Rightarrow x_1 e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t} = Au e^{\lambda_1 t}$$

$$\Rightarrow x_1 + \lambda_1 u = Au$$

$$\Rightarrow (A - \lambda_1 I)u = x_1$$

$$\Rightarrow \begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

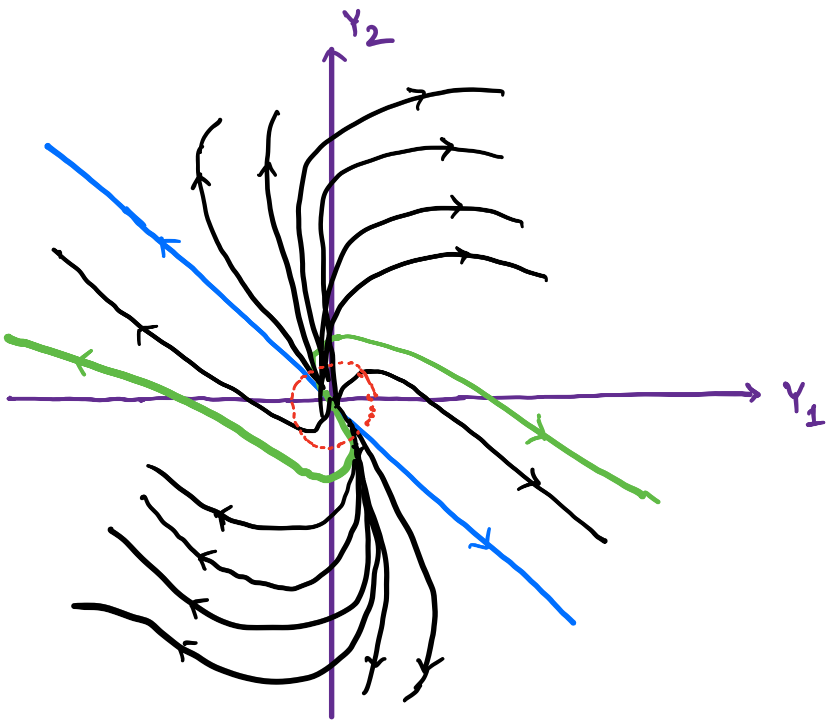
$$\Rightarrow \left.\begin{matrix} u_1 + u_2 = 1 \\ -u_1 - u_2 = -1 \end{matrix}\right\} \Rightarrow u_1 + u_2 = 1$$

So, a solution, linearly independent of $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, the general solution is:

$$Y = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \right)$$

The critical point at the origin is called a degenerate node.



<u>Note:</u> When the matrix A is not diagonalizable and there is one linearly independent eigen vector X_1 corresponding to an eigen value λ_1, we may complete the fundamental system with solutions of the form:

$$Y_2 = (t X_1 + v_1) e^{\lambda_1 t}$$

$$Y_3 = \left(\frac{1}{2} t^2 X_1 + t v_1 + v_2\right) e^{\lambda_1 t}$$

$$Y_4 = \left(\frac{1}{3} t^3 X_1 + \frac{1}{2} t^2 v_1 + t v_2 + v_3\right) e^{\lambda_1 t}$$

$$\vdots$$

Addendum to Lecture 12

Classification of Critical points:

(1) Eigen values are real, distinct

(a) Both positive : improper node, unstable

(b) Both negative : improper node, stable, asymptotically stable

(c) One positive, another negative : Saddle point unstable

(2) Eigen values are real, equal :

(a) Positive, eigen space 1-dimensional : degenerate node unstable

(b) Positive, eigen space 2-dimensional : proper node (non degenerate) (Star) unstable

(c) Negative, eigen space 1-dimensional : degenerate node, stable and asymptotically stable

(d) Negative, eigen space 2-dimensional : proper node (Star) (non degenerate) ; stable and asymptotically stable

(3) Eigen values are Purely imaginary (real part zero): center, stable (not asymptotically stable)

(4) Eigen values are complex (but not purely imaginary):

(a) Real part positive: Spiral point, unstable

(b) Real part negative: Spiral point, stable and asymptotically stable

MTH 204: Lecture 13

Critical point classification

Consider the linear system: $Y' = AY = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} Y$

The characteristic polynomial of A:

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$= \lambda^2 - Tr(A)\lambda + \det(A).$$

Let us define $\Delta = Tr(A)^2 - 4\{\det(A)\}$.

The eigen values of A are the roots of the equation $\det(A - \lambda I) = 0$

Hence the eigen values are $\lambda_1, \lambda_2 = \dfrac{Tr(A) \pm \sqrt{\Delta}}{2}$

Note that $\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)$

Therefore $\lambda_1 + \lambda_2 = Tr(A) = p$ (say)

and $\lambda_1 \lambda_2 = \det(A) = q$ (say)

and $\Delta = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2$

Based on the sign of $Tr(A)$, $det(A)$ and Δ we can classify the critical points.

Criteria for Critical points:

Type	$p = Tr(A)$ $= \lambda_1 + \lambda_2$	$q = det(A)$ $= \lambda_1 \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Comments
Node		> 0	≥ 0	Real, same sign
Saddle point		< 0		Real, opposite sign
Ceter	$= 0$		< 0	Pure imaginary
Spiral point	$\neq 0$		< 0	Complex, not pure imaginary

Stability:

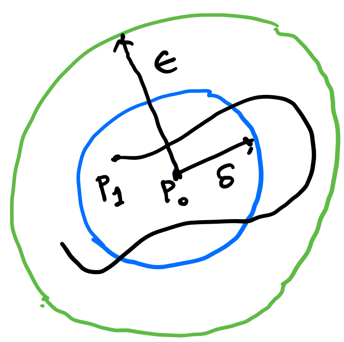
Stability means, roughly speaking, that a small change (a small disturbance) of a physical system at some instant changes the behavior of the system only slightly all all future times t.

Stable Critical point:

A critical point P_0 is stable if all trajectories

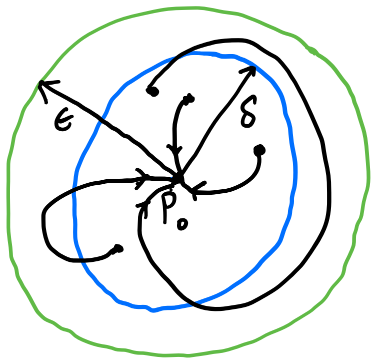
of the ODE that at some instant are close to P_0 remain close to P_0 at all future times;
More precisely: If every disk D_ϵ of radius ϵ with center P_0 there is a disk D_δ of radius δ with center P_0 such that every trajectory of the ODE that has a point P_1 (corresponding to $t = t_1$, say) in D_δ has all its points corresponding to $t \geq t_1$ in D_ϵ.

If a critical point is not stable, it is unstable.



Asymptotically stable critical point:

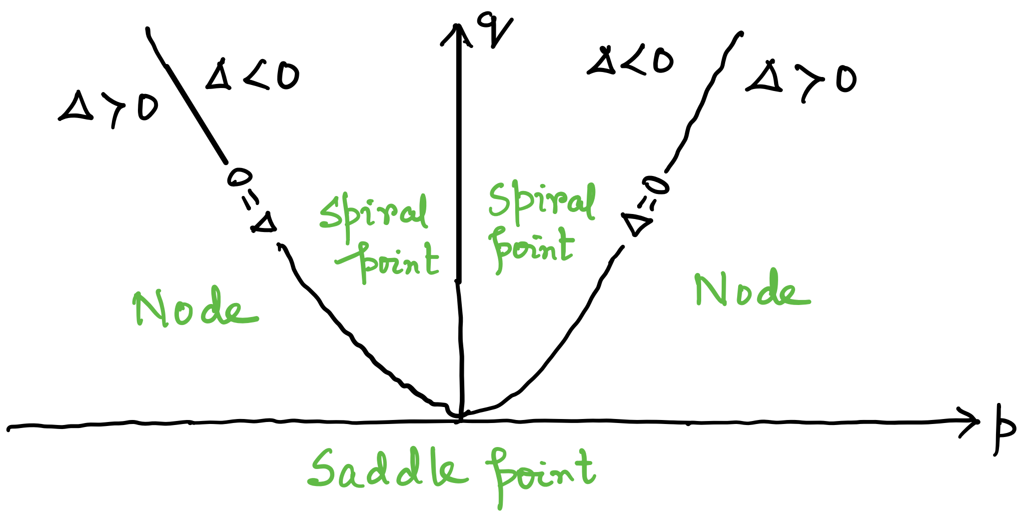
A critical point P_0 is asymptotically stable (stable and attractive) if P_0 is stable and every trajectory that has a point D_δ approaches P_0 as $t \longrightarrow \infty$.



(stable and attractive critical point P_0)

Criteria for Critical points:

Type	$p = \mathrm{Tr}(A) = \lambda_1 + \lambda_2$	$q = \det A = \lambda_1 \lambda_2$
Asymptotically stable	< 0	0
Stable	< 0	> 0
Unstable	> 0 or	< 0



Stability Chart of the system

Stable and attractive : The second quadrant without the q-axis.

Stability also on the positive q-axis (which corresponds to centers)

Unstable : The first quadrant excluding the q-axis and the region below the p-axis

Example : Free motions of a Mass on a Spring :

$$my'' + cy' + ky = 0 \qquad c > 0$$

$$\Rightarrow \quad y'' = -\frac{k}{m}y - \frac{c}{m}y'$$

Let $\quad y_1 = y \; , \; y_1' = y_2$

Then

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{k}{m}y_1 - \frac{c}{m}y_2 \end{aligned}\right\}$$

$$\Rightarrow \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \quad Y' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} Y = AY \text{ (say)}$$

Now $\det(A-\lambda I) = (-\lambda)\left(-\lambda-\frac{c}{m}\right) + \frac{k}{m}$

$$= \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

If λ_1 and λ_2 are two eigen values, then

$p = \lambda_1 + \lambda_2 = tr(A) = -\frac{c}{m}$

$q = \lambda_1 \lambda_2 = \det(A) = \frac{k}{m}$.

and $\Delta = p^2 - 4q = \left(\frac{c}{m}\right)^2 - 4\frac{k}{m} = \frac{c^2-4mk}{m}$

Thus we have the following situations:

No damping: $c=0$, $p=0$, $q>0$: a center

Underdamping: $c^2 < 4mk$, $p<0$, $q>0$, $\Delta<0$

a stable and attractive spiral point

Critical damping: $c^2 = 4mk$, $p<0$, $q>0$, $\Delta=0$

a stable and attractive node.

Overdamping: $c^2 > 4mk$, $p<0$, $q>0$, $\Delta>0$

a stable and attractive node.

Qualitative Methods for Nonlinear Systems

- Qualitative methods are methods of obtaining qualitative information on solutions without actually solving a system.

- Useful for systems whose solution by analytic method is difficult or impossible.

For nonlinear system $Y' = f(Y)$ ie
$$\begin{cases} y_1' = f_1(y_1, \dots, y_n) \\ y_2' = f_2(y_1, \dots, y_n) \\ \dots\dots\dots\dots \\ y_n' = f_n(y_1, \dots, y_n) \end{cases}$$

we may have several critical points

ie. points $(y_1, y_2, \dots, y_n)$ at which $f_i(y_1, y_2, \dots, y_n) = 0$ for $i = 1, 2, \dots, n$

- If Y_0 is a critical point, we may shift the origin so that Y_0 is centered

$$\tilde{Y} = Y - Y_0$$
$$\tilde{Y}' = f(\tilde{Y} + Y_0) \Rightarrow \tilde{Y}' = \tilde{f}(\tilde{Y})$$

and study the local behavior of the system of ODE around 0.

We will also assume that the critical points are isolated.

First we may need to linearize the ODE system.
We will assume that the system is autonomous ie the independent variable t doesnot occur explicitly.

Linearization of Autonomous Nonlinear System:

The system $\tilde{Y}' = \tilde{f}(\tilde{Y})$ can be linearized as:

$$\tilde{Y}' = \tilde{f}(\tilde{Y}) \approx A\tilde{Y}.$$

where A is the Jacobian of the function $\tilde{f}$ evaluated at the origin

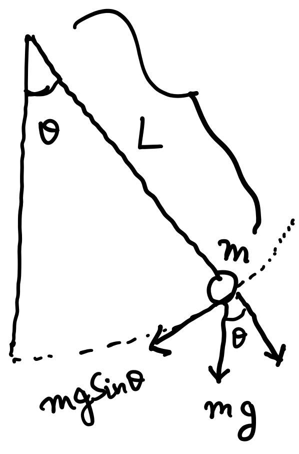
$$A = \left.\begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} & \cdots & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_n} \\ \vdots & & & \\ \frac{\partial \tilde{f}_n}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_n}{\partial \tilde{y}_2} & \cdots & \frac{\partial \tilde{f}_n}{\partial \tilde{y}_n} \end{pmatrix}\right|_{\tilde{y} = 0}$$

Theorem:

If $\tilde{f}$ has continuous components and continuous partial derivatives in a neighborhood of the critical point 0 and $\det(A) \neq 0$, then the kind and stability of the critical point of the nonlinear system of ODE is the same as those of the linearized system. Exceptions occur if A has equal or pure imaginary eigenvalues, then the nonlinear system may have the same kind of critical point as the linearized system or a spiral point.

Ex: Free Undamped Pendulum:

Ex: Free Undamped Pendulum:



Gravity compensates the acceleration of the bob.

$$mL\theta'' + mg\sin\theta = 0$$

$$\Rightarrow \theta'' + k\sin\theta = 0 \text{ where } k = \frac{g}{L}.$$

To find the critical points, we convert the equation into a system of ODEs:

$$\left.\begin{aligned} y_1 &= \theta, \quad y_2 = y_1' \\ \text{Then} \quad y_2' &+ k\sin y_1 = 0. \end{aligned}\right\}$$

Thus $Y' = \begin{pmatrix} y_2 \\ -k\sin y_1 \end{pmatrix}$

The critical points are $\begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ for any integer n.

Let us look at the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The Jacobian of f at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \left.\begin{pmatrix} 0 & 1 \\ -k\cos y_1 & 0 \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$$

$Tr(A) = 0, \quad \det A = K > 0$

So, $Y = 0$ is a center (always stable)

(Same is true at all points $(0, 2\pi n)$ since the sin function is periodic with period 2π)

Let us now look at the point $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$.

We center this critical point by using the transformation $\tilde{Y} = Y - \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ $\left(\text{ie } \begin{matrix} \tilde{y}_1 = y_1 - \pi \\ \tilde{y}_2 = y_2 \end{matrix} \right)$

The system of ODE now becomes:

$$\tilde{Y}' = \begin{pmatrix} \tilde{y}_2 \\ -K \sin(\tilde{y}_1 + \pi) \end{pmatrix}$$

Let us calculate the jacobian of $\tilde{f}$ at $(0,0)$

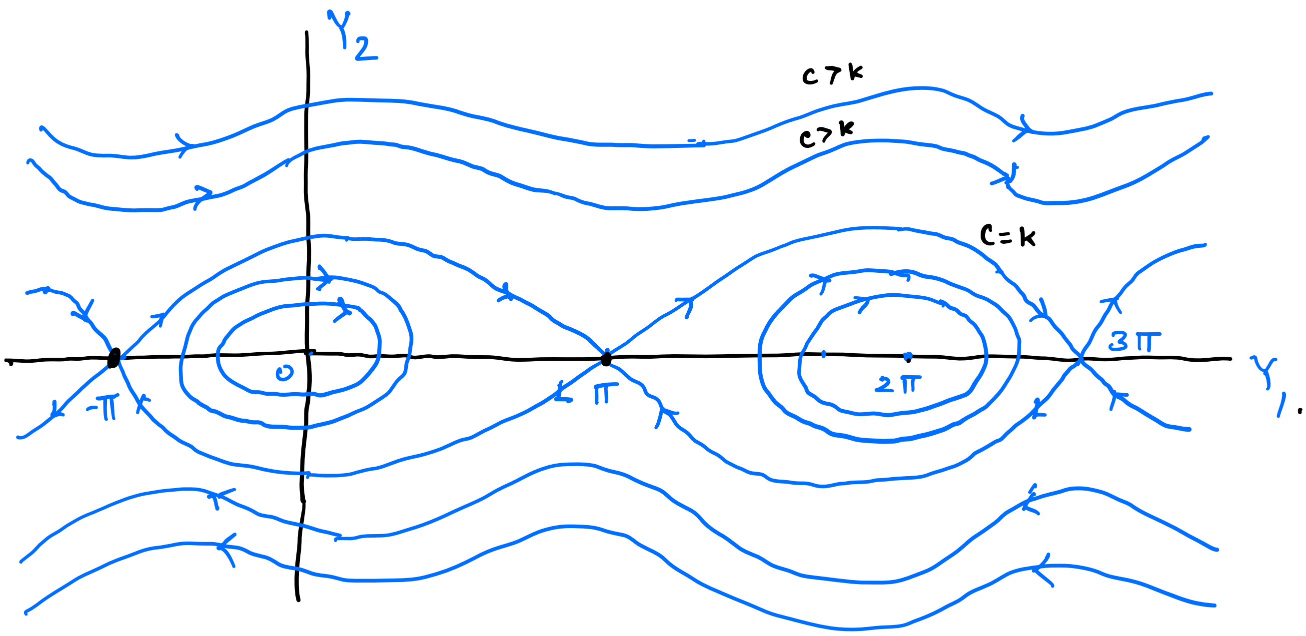
$$A = \left. \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} \end{pmatrix} \right|_{\tilde{y}=0} = \left. \begin{pmatrix} 0 & 1 \\ -K \cos(\tilde{y}_1 + \pi) & 0 \end{pmatrix} \right|_{\tilde{Y}=0}$$

$$= \begin{pmatrix} 0 & 1 \\ K & 0 \end{pmatrix}$$

Now $Tr(A) = 0$, $\det(A) = -k < 0$

So, $Y = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ is a saddle point (unstable).

The same is true for all points $\begin{pmatrix} 0 \\ \pi + 2n\pi \end{pmatrix}$ since Sin function is periodic with period 2π



Example: Damped Pendulum:

$$\theta'' + c\theta' + k\sin\theta = 0$$

Let $y_1 = \theta$, $y_2 = y_1'$

Then $y_2' + c\,y_2 + k\sin(y_1) = 0$

$$\Rightarrow Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -k\sin y_1 - c y_2 \end{pmatrix}$$

Critical points are $\begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ for any integer n

(as in the case of free undamped pendulum)

First consider the critical point $(0,0)$:

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\right|_{Y=0} = \left.\begin{pmatrix} 0 & 1 \\ -k\cos(y_1) & -c \end{pmatrix}\right|_{Y=0}$$

$$= \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}$$

$$\operatorname{Tr}(A) = -c < 0, \quad \det(A) = k > 0, \quad \Delta = -c + 4k^2$$

- If $\Delta < 0$, we have a stable and attractive spiral point.
- If $\Delta > 0$, then it is a stable and attractive node.

Let us now look at the critical point $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$

We center it by the transformation:

$$\tilde{Y} = Y - \begin{pmatrix} \pi \\ 0 \end{pmatrix} \quad \left(\text{i.e. } \tilde{y}_1 = y_1 - \pi, \ \tilde{y}_2 = y_2 \right.$$

Then the ODE becomes

$$\tilde{Y}' = \begin{pmatrix} \tilde{y}_2 & 0 \\ -k \sin(\tilde{y}_1 + \pi) & -c\tilde{y}_2 \end{pmatrix}$$

Now

$$A = \left. \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} \end{pmatrix} \right|_{\tilde{Y}=0} = \left. \begin{pmatrix} 0 & 1 \\ -k\cos(\tilde{y}_1 + \pi) & -c \end{pmatrix} \right|_{\tilde{Y}=0}$$

$$= \begin{pmatrix} 0 & 1 \\ k & -c \end{pmatrix}$$

Note that $Tr(A) = -c < 0$, $\det A = -k < 0$

So, $Y = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ is a saddle point (unstable)

The same happens to all points $(0, \pi + 2\pi n)$ since sin function is periodic with period 2π.

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$

$$(1-x^2)y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots - 2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots$$

$$-2xy' = -2a_1 x - 4a_2 x^2 - 6a_3 x^3 - 8a_4 x^4 - \cdots$$

$$2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots$$

So, $2a_2 + 2a_0 = 0 \Rightarrow \boxed{a_2 = -a_0}$

$6a_3 = 0 \Rightarrow \boxed{a_3 = 0}$

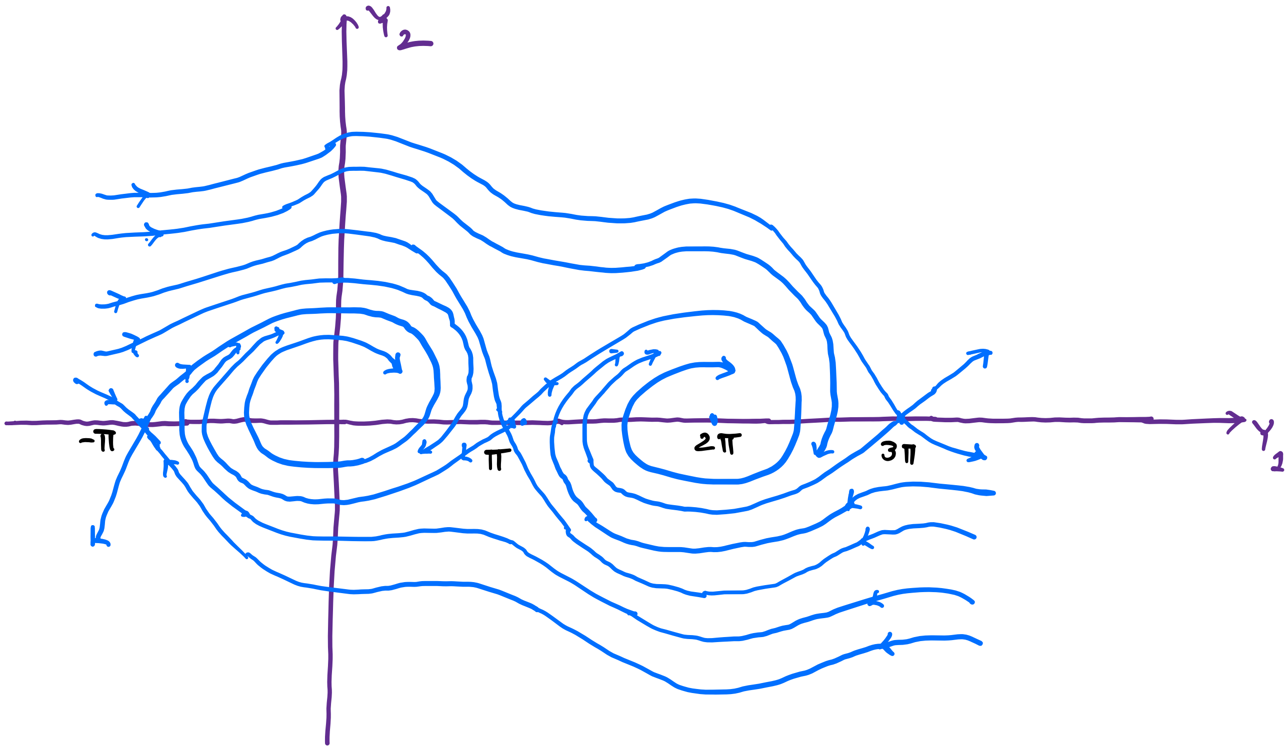
$12a_4 - 4a_2 = 0 \Rightarrow a_4 = \frac{1}{3}a_2 = \boxed{-\frac{1}{3}a_0}$

$\vdots$

The general solution of the equation is:

$$y = a_1 x + a_0 - a_0 x^2 - \frac{1}{3}a_0 x^4 - \cdots$$

$$y = a_1 x + a_0\left(1 - \frac{x^2}{1} - \frac{1}{3}x^4 - \cdots\right)$$


y_2
y_1
$-\pi$
π
2π
3π

Ex: Lotka-Volterra population model:

- Rabbits have unlimited food supply. Hence, if there were no foxes, there number $y_1(t)$ would grow exponentially, $y_1' = a y_1$

- Actually y_1 is decreased because of the kill by foxes, say at a rate proportional to $y_1 y_2$ where $y_2(t)$ is the number of foxes.
  Hence $y_1' = a y_1 - b y_1 y_2$ where $a > 0$, $b > 0$

- If there were no rabbits, then $y_2(t)$ would exponentially decrease to zero: $y_2' = -l y_2$
  However y_2 is increased by a rate proportional to the number of encounters between predator and prey; together we have $y_2' = -l y_2 + k y_1 y_2$ where $k > 0$ and $l > 0$

Thus
$$\left.\begin{aligned} y_1' &= a y_1 - b y_1 y_2 \\ y_2' &= k y_1 y_2 - l y_2 \end{aligned}\right\}$$

Critical points are solutions of

$$0 = y_1' = y_1 (a - b y_2)$$
$$0 = y_2' = (k y_1 - l) y_2$$

That is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{l}{k} \\ \frac{a}{b} \end{pmatrix}$

First let us look at $(0,0)$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\right|_{Y=0} = \left.\begin{pmatrix} a - b y_2 & -b y_1 \\ k y_2 & k y_1 - l \end{pmatrix}\right|_{Y=0}$$

$$= \begin{pmatrix} a & 0 \\ 0 & -l \end{pmatrix}$$

Eigen values are $\lambda_1 = a$, $\lambda_2 = -l$

They have different signs & so we have a saddle point.

For the critical point $\left(\frac{l}{k}, \frac{a}{b}\right)$ we use the transformation:

$$\tilde{Y} = Y - \begin{pmatrix} \frac{l}{k} \\ \frac{a}{b} \end{pmatrix}$$

The system of ODE becomes:

$$\tilde{Y}' = \begin{pmatrix} \left(\tilde{y}_1 + \frac{l}{k}\right) \times \left[a - b\left(\tilde{y}_2 + \frac{a}{b}\right)\right] \\ \left[k\left(\tilde{y}_1 + \frac{l}{k}\right) - l\right]\left(\tilde{y}_2 + \frac{a}{b}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\tilde{y}_1 + \frac{l}{k}\right) \times \left(-b\tilde{y}_2\right) \\ \left(k\tilde{y}_1\right) \times \left(\tilde{y}_2 + \frac{a}{b}\right) \end{pmatrix}$$

The Jacobian of $\tilde{f}$ at $(0,0)$:

$$A = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} \end{pmatrix} = \left.\begin{pmatrix} -b\tilde{y}_2 & -b\left(\tilde{y}_1 + \frac{l}{k}\right) \\ k\left(\tilde{y}_2 + \frac{a}{b}\right) & k\tilde{y}_1 \end{pmatrix}\right|_{\tilde{Y}=0}$$

$$= \begin{pmatrix} 0 & -\frac{l}{k}b \\ k\frac{a}{b} & 0 \end{pmatrix}$$

$$\mathrm{Tr}(A) = 0 \qquad \det A = al > 0$$

So, the critical point is a stable center.

Let us solve the equation around the

critical point:

$$\left.\begin{aligned} \tilde{y}_1' &= -\frac{l}{k} b \tilde{y}_2 \\ \tilde{y}_2' &= k \frac{a}{b} \tilde{y}_1 \end{aligned}\right\}$$

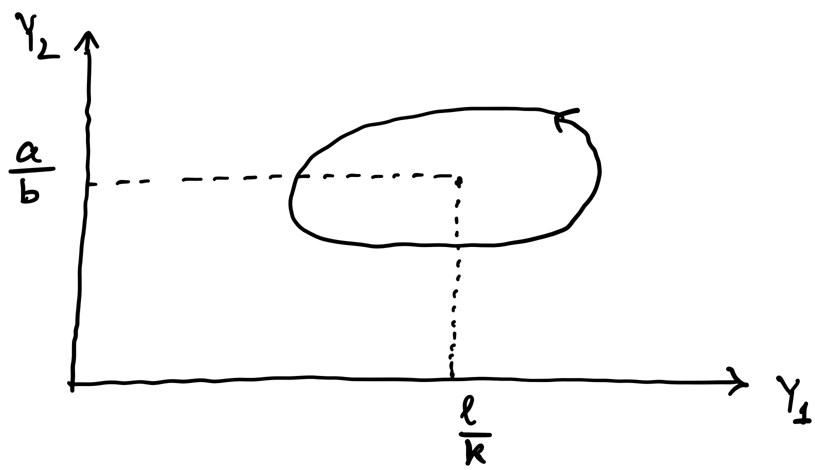
$$\Rightarrow \quad k \frac{a}{b} \tilde{y}_1 \tilde{y}_1' = -\frac{l}{k} b \tilde{y}_2 \tilde{y}_2'$$

$$\Rightarrow \quad k \frac{a}{b} \tilde{y}_1 \tilde{y}_1{}' = -\frac{l}{k} b \tilde{y}_2 \tilde{y}_2{}'$$

$$\Rightarrow \quad k \frac{a}{2b} \tilde{y}_1^{\,2} = -\frac{l}{2k} b \tilde{y}_2^{\,2} + c$$

$$\Rightarrow \quad \frac{ak}{b} \tilde{y}_1^{\,2} + \frac{bl}{k} \tilde{y}_2^{\,2} = c$$

$$\Rightarrow \quad \boxed{\frac{ak}{b}\left(y_1 - \frac{l}{k}\right)^2 + \frac{bl}{k}\left(y_2 - \frac{a}{b}\right)^2 = c}$$



Transformation to a first order equation in the phase plane:

Consider a second order autonomous ODE

$$F(y, y', y'') = 0$$

We make a change of variables

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y_1' \end{aligned}\right\}$$

and $\quad y'' = y_2' = \dfrac{dy_2}{dt} = \dfrac{dy_2}{dy_1}\dfrac{dy_1}{dt} = \dfrac{dy_2}{dy_1} y_2$

The ODE becomes $\boxed{F\left(y_1, y_2, \dfrac{dy_2}{dy_1} y_2\right) = 0}$

<u>Ex:</u> <u>Free Undamped Pendulum:</u>

$$\theta'' + k(\sin\theta) = 0$$

Making the substitution:

$$\frac{dy_2}{dy_1} y_2 + k\sin(y_1) = 0$$

$$y_2\, dy_2 = -k\sin(y_1)\, dy_1$$

$$\Rightarrow \boxed{\frac{y_2^2}{2} = k\cos(y_1) + C}$$

MTH 204 : Lecture 14

Nonhomogeneous linear system of ODEs:

A nonhomogeneous linear system of ODEs can be given by:

$$Y' = A(t)Y + g(t) \quad \text{......} (1)$$

where the vector $g(t)$ is not identically zero.

If the entries of the matrix A and g vector are continuous on some interval J of the t-axis, then the general solution of (1) can be expressed as: $Y = Y_h + Y_p$

where Y_h is the general solution of the associated homogeneous system of ODEs and Y_p is a particular solution of (1) on J

Method of Undetermined coefficients:

- This method is applied when A is a constant matrix and g is a sum of constant, powers, exponentials, Sine/Cosine functions.

Ex:

$$Y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} Y + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$$

- First let us find the general solution of the homogeneous system.

For $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$,

$$\det(A - \lambda I) = (-3-\lambda)^2 - 1 = \lambda^2 + 6\lambda + 9 - 1$$
$$= \lambda^2 + 6\lambda + 8 = (\lambda+4)(\lambda+2)$$

So, the eigen values are $-2, -4$

For eigen vector corresponding to the eigen value $\lambda = -2$,

$$\begin{bmatrix} -3+2 & 1 \\ 1 & -3+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{r} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{array}\right\} \Rightarrow x_1 = x_2$$

So, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = -2$

Now for eigen vector corresponding to the eigen value $\lambda = -4$

$$\begin{bmatrix} -3+4 & 1 \\ 1 & -3+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{aligned}\right\} \Rightarrow x_1 = -x_2$$

So, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigen vector of A

corresponding to the eigen value -4

So, $Y_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

Note that e^{-2t} on the RHS also appears in the general solution of the associated homogeneous equation

Therefore we try a particular solution of the form :

$$Y_p = (tu + v)\, e^{-2t} \quad (u, v \text{ are vectors})$$

Then $Y_p' = -2(tu+v)e^{-2t} + u e^{-2t}$

$= (-2tu - 2v + u)e^{-2t}$

Substituting in the ODE,

$$(-2tu - 2v + u)e^{-2t} = A(tu+v)e^{-2t} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$$

$$\Rightarrow \quad -2tu - 2v + u = tAu + Av + \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

Equating the coefficients of t, we get

$$Au = -2u$$

$\Rightarrow$ u is an eigen vector of A corresponding to the eigen value $\lambda = -2$

Thus $u = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for some scalar a

Now equating the coefficients without t,

$$u - 2v = Av + \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad Av + 2v = u - \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad (A + 2I)v = u - \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

$$\Rightarrow \left[\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a+6 \\ a-2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -v_1 + v_2 &= a+6 \\ v_1 - v_2 &= a-2 \end{aligned}\right\}$$

For this system to be compatible we need

$$a+6 = -(a-2) \Rightarrow 2a = -4$$
$$\Rightarrow a = -2$$

Then $v_2 = v_1 + 4$

We take $v_1 = 0$ Then $v_2 = 4$

Thus $u = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Therefore $Y_p = \left[t\begin{pmatrix} -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}\right]e^{-2t}$

Hence the general solution is

$$Y = Y_h + Y_p$$

$$= c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-4t} + \left[t\begin{pmatrix} -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}\right]e^{-2t}$$

$$\Rightarrow \quad Y = \begin{pmatrix} c_1 - 2t \\ c_1 - 2t + 4 \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t}$$

Method of Variation of Parameters

This method can be applied to nonhomogeneous linear system $Y' = A(t)Y + g(t)$ for nonconstant matrix $A(t)$ and general g. It yields a particular solution Y_p on some open interval J on the t-axis if the general solution of the associated homogeneous system $Y' = A(t)Y$ is known.

If the general solution of the associated homogeneous system is of the form

$$Y_h = (Y_1 \; Y_2 \cdots Y_n)\,\mathbf{c} = Y(t)\,\mathbf{c}$$

then we will look for a solution of the form

$$Y_p = Y(t)\,u(t)$$

Then $Y_p' = A(t)\,Y(t)\,u(t) + g(t)$

$$\Rightarrow \quad Y'(t)\,u(t) + Y(t)\,u'(t) = A(t)\,Y(t)\,u(t) + g(t)$$

$$\Rightarrow \quad Y'u + Yu' = AYu + g$$

Since the columns of Y are solutions of the associated homogeneous system,

we have $Y' = AY$

Then $Yu' = g$

$$\Rightarrow \boxed{u' = Y^{-1}g}$$

Ex:

$$Y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} Y + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$$

In the previous example, we have calculated the general solution of the associated homogeneous system:

$$Y_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

$$= \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = YC$$

So, $Y_1 = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$ $\quad Y_2 = \begin{pmatrix} e^{-4t} \\ -e^{-4t} \end{pmatrix}$

$$Y^{-1} = \frac{1}{-2e^{-6t}} \begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix}$$

So, $u' = Y^{-1} g = \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix} \begin{pmatrix} -6e^{-2t} \\ 2e^{-2t} \end{pmatrix}$

$$= \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix}$$

Then $u = \int \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix} dt = \begin{pmatrix} -2t \\ -\frac{4e^{2t}}{2} \end{pmatrix} = \begin{pmatrix} -2t \\ -2e^{2t} \end{pmatrix}$

Then $Y_p = Y u = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} -2t \\ -2e^{2t} \end{pmatrix}$

$$\Rightarrow \quad Y_p = \begin{pmatrix} -2te^{-2t} - 2e^{-2t} \\ -2te^{-2t} + 2e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2t-2 \\ -2t+2 \end{pmatrix} e^{-2t}$$

Therefore

$$Y = Y_h + Y_p$$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

$$+ \begin{pmatrix} -2t-2 \\ -2t+2 \end{pmatrix} e^{-2t}$$

$$= \begin{pmatrix} c_1 - 2t - 2 \\ c_1 - 2t + 2 \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t}$$

- compare with the previous solution.

SOME IMPORTANT MACLAURIN SERIES

MACLAURIN SERIES	INTERVAL OF CONVERGENCE
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$	$-1 < x < 1$
$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots$	$-1 < x < 1$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	$-\infty < x < +\infty$
$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$-\infty < x < +\infty$
$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$-\infty < x < +\infty$
$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \le 1$
$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 \le x \le 1$
$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$	$-\infty < x < +\infty$
$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$	$-\infty < x < +\infty$
$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!} x^k$	$-1 < x < 1^*$ $(m \ne 0, 1, 2, \ldots)$

Series solutions of ODEs:

- A power series is an infinite series of the form:

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\cdots$$

Ex: Taylor Series: $f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$

Ex: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$ (geometric series) $(|x|<1)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

Ex: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\cdots$

(centered around zero)

$$e^x = e + e(x-1) + e\frac{1}{2!}(x-1)^2 + e\frac{1}{3!}(x-1)^3 + \cdots$$

(centered around 1)

Series Expansions You Should Know				
e^x	=	$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	=	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x$	=	$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots$	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x$	=	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\cosh x$	=	$1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$	=	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
$\sinh x$	=	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$	=	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x}$	=	$1 + x + x^2 + x^3 + \ldots$	=	$\sum_{n=0}^{\infty} x^n$
$\frac{1}{1+x}$	=	$1 - x + x^2 - x^3 + \ldots$	=	$\sum_{n=0}^{\infty} (-x)^n$
$\tan^{-1} x$	=	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$\ln(1+x)$	=	$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$	=	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Power Series Method:

$$y' - y = 0$$

We look for a solution of the form:

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

Substituting in the equation: $y' - y = 0$

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = 0$$

$$\Rightarrow (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

$$\Rightarrow \quad a_1 - a_0 = 0 \quad \Rightarrow a_1 = a_0$$

$$2a_2 - a_1 = 0 \quad \Rightarrow 2a_2 = a_1 = a_0 \Rightarrow a_2 = \frac{1}{2}a_0$$

$$3a_3 - a_2 = 0 \Rightarrow 3a_3 = a_2 \Rightarrow a_3 = \frac{1}{3.2}a_0$$

$$\Rightarrow a_3 = \frac{1}{3!}a_0$$

$$\vdots$$

In general $a_k = \frac{1}{k!}a_0$

So, $$Y = a_0 + a_0 x + \frac{1}{2}a_0 x^2 + \frac{1}{3!}a_0 x^3 + \cdots$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right)$$

$$= a_0 e^x$$

- The power series method can be applied to linear ODEs with variable coefficients such as:

$$y'' + p(x)y' + q(x)y = 0$$

because the coefficients p and q can also be substituted by a power series.

Ex: $$(1-x^2)y'' - 2xy' + 2y = 0$$

Table of Integrals

1 $\int u\,dv = uv - \int v\,du$

2 $\int u^n du = \frac{1}{n+1}u^{n+i} + C$

3 $\int \frac{du}{u} = \ln|u| + C$

4 $\int e^u du = e^u + C$

5 $\int a^u du = \frac{1}{\mathrm{In}(a)}a^u + C$

6 $\int \mathrm{sen}(u)\,du = -\cos(u) + C$

7 $\int \cos(u)\,du = \mathrm{sen}(u) + C$

8 $\int \sec^2(u)\,du = \mathrm{tg}(u) + C$

9 $\int \mathrm{cossec}^2(u)\,du = -\mathrm{cotg}(u) + C$

10 $\int \sec(u)\ \mathrm{tg}(u)\,du = \sec(u) + C$

11 $\int \frac{\mathrm{cotg}(u)}{\mathrm{sen}(u)}\,du = -\frac{1}{\mathrm{sen}(u)} + C$

12 $\int \mathrm{tg}(u)\,du = \ln|\sec(u)| + C$

13 $\int \mathrm{cotg}(u)\,du = \ln|\mathrm{sen}(u)| + C$

14 $\int \sec(u)\,du = \ln|\sec(u) + \mathrm{tg}(u)| + C$

15 $\int \frac{du}{\mathrm{sen}(u)} = \ln\left|\frac{1}{\mathrm{sen}(u)} - \frac{\cos(u)}{\mathrm{sen}(u)}\right| + C$

16 $\int \frac{du}{\sqrt{a^2-u^2}} = \mathrm{arc\ sen}(\frac{u}{a}) + C$

17 $\int \frac{du}{a^2+u^2} = \frac{1}{a}\mathrm{arc\ tg}(\frac{u}{a}) + C$

18 $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\mathrm{arc\ sec}(\frac{u}{a}) + C$

19 $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20 $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

21 $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

22 $\int u^2\sqrt{a^2+u^2}\,du = \frac{(a^2u+2u^3)\sqrt{a^2+u^2}}{8} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

23 $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

24 $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

25 $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

26 $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$

27 $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

28 $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

29 $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

30 $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\mathrm{arc\ sen}(\frac{u}{a}) + C$

31 $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\mathrm{arc\ sen}(\frac{u}{a}) + C$

32 $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

33 $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \mathrm{arc\ sen}(\frac{u}{a}) + C$

34 $\int \frac{u^2du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\mathrm{arc\ sen}(\frac{u}{a}) + C$

35 $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2-u^2}+a}{u}\right| + C$

36 $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

37 $\int (a^2+u^2)^{3/2}\,du = -\frac{(2u^3-5a^2u)\sqrt{a^2-u^2}}{8} + \frac{3a^4}{8}\mathrm{arc\ sen}(\frac{u}{a}) + C$

38 $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

39 $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

40 $\int u^2\sqrt{u^2-a^2}\,du = -\frac{(2u^3-a^2u)\sqrt{u^2-a^2}}{8} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

41 $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\ \mathrm{arc\ cos}(\frac{a}{|u|}) + C$

42 $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

43 $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

44 $\int \frac{u^2du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

45 $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

46 $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

47 $\int \frac{udu}{a+bu} = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right) + C$

48 $\int \frac{u^2du}{a+bu} = \frac{\left[(a+bu)^2-4a(a+bu)+2a^2\ln|a+bu|\right]}{2b^3} + C$

49 $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

50 $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

51 $\int \frac{udu}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

52 $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

53 $\int \frac{u^2du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right) + C$

54 $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

55 $\int \frac{udu}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

56 $\int \frac{u^2du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

57 $\int \frac{du}{u\sqrt{a+bu}}\,du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + c,\ \text{se } a>0$

58 $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

59 $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

60 $\int u^n\sqrt{a+bu}\,du = \frac{2\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]}{b(2n+3)}$

61 $\int \frac{u^n du}{\sqrt{a+du}} = \frac{2u^n\sqrt{a+bu}}{b(2n-1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}du}{\sqrt{a+bu}}$

62 $\int \frac{u^{-n}du}{\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{u^{-n+1}du}{\sqrt{a+bu}}$

63 $\int \mathrm{sen}^2(u)\,du = \frac{1}{2}u - \frac{1}{4}\mathrm{sen}(2u) + C$

64 $\int \cos^2(u)\,du = \frac{1}{2}u + \frac{1}{4}\mathrm{sen}(2u) + C$

65 $\int \mathrm{tg}^2(u)\,du = \mathrm{tg}(u) - u + C$

66 $\int \mathrm{cotg}^2(u)\,du = -\mathrm{cotg}(u) - u + C$

67 $\int \mathrm{sen}^3(u)\,du = -\frac{\left[2+\mathrm{sen}^2(u)\right]\cos(u)}{3} + C$

68 $\int \cos^3 u\,du = \frac{\left[2+\cos^2(u)\right]\mathrm{sen}(u)}{3} + C$

69 $\int \mathrm{tg}^3(u)\,du = \frac{\mathrm{tg}^2(u)}{2} + \ln|\cos(u)| + C$

70 $\int \mathrm{cotg}^3(u)\,du = -\frac{\mathrm{cotg}^2(u)}{2} - \ln|\mathrm{sen}(u)| + C$

71 $\int \sec^3(u)\,du = -\frac{\sec(u)\mathrm{tg}(u)}{2} - \frac{\ln|\mathrm{sen}(u)+\mathrm{tg}(u)|}{2} + C$

72 $\int \frac{du}{\mathrm{sen}^3(u)} = -\frac{\mathrm{cotg}(u)}{2\mathrm{sen}(u)} + \frac{\ln|\mathrm{cossec}(u)-\mathrm{cotg}(u)|}{2} + C$

73 $\int \mathrm{sen}^n(u)\,du = -\frac{\mathrm{sen}^{n-1}(u)\cos(u)}{n} + \frac{n-1}{n}\int \mathrm{sen}^{n-2}(u)\,du$

74 $\int \cos^n(u)\,du = \frac{\cos^{n-1}(u)\mathrm{sen}(u)}{n} + \frac{n-1}{n}\int \cos^{n-2}(u)\,du$

75 $\int \mathrm{tg}^n(u)\,du = \frac{\mathrm{tg}^{n-1}(u)}{n-1} - \int \mathrm{tg}^{n-2}(u)\,du$

76 $\int \mathrm{cotg}^n(u)\,du = -\frac{\mathrm{cotg}^{n-1}(u)}{n-1} - \int \mathrm{cotg}^{n-2}(u)\,du$

77 $\int \sec^n(u)\,du = \frac{\mathrm{tg}(u)\sec^{n-2}(u)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(u)\,du$

78 $\int \frac{du}{\mathrm{sen}^n(u)} = -\frac{\mathrm{cotg}(u)}{(n-1)\mathrm{sen}^{n-2}(u)} + \frac{n-2}{n-1}\int \frac{du}{\mathrm{sen}^{n-2}(u)}$

79 $\int \mathrm{sen}(au)\,\mathrm{sen}(bu)\,du = \frac{\mathrm{sen}(a-b)u}{2(a-b)} - \frac{\mathrm{sen}(a+b)u}{2(a+b)} + C$

80 $\int \cos(au)\cos(bu)\,du = \frac{\mathrm{sen}(a-b)u}{2(a-b)} + \frac{\mathrm{sen}(a+b)u}{2(a+b)} + C$

MTH 204: Lecture 15

- <u>Convergent Sequence:</u> A sequence of real numbers $\{a_n\}$ is said to converge to a real number l if for any given $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that $|a_n - l| < \epsilon \quad \forall\, n \geq N$

- <u>Convergent Series:</u> A series of real numbers $\sum_{n=1}^{\infty} a_n$ is said to be convergent

  if the sequence of its partial sum

  $$S_1 = a_1, \quad S_2 = a_1 + a_2, \quad S_3 = a_1 + a_2 + a_3, \ldots\ldots$$

  $$S_n = \sum_{i=1}^{n} a_i, \ldots$$ is convergent.

  The limit of this sequence is called the sum of the series $\sum_{n=1}^{\infty} a_n$.

- <u>Sequence of functions:</u> A sequence of functions $\{f_n\}$ is said to converge to a function f on an interval I if for each $x \in I$, the sequence $\{f_n(x)\}$ converges to $f(x)$.

(i.e. for any given $\epsilon > 0$ and for $x \in I$, there exists a positive integer $N = N(\epsilon, x)$ such that $|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq N(\epsilon, x)$)

- **Series of functions:** A series of functions $\sum_{n=1}^{\infty} f_n$ is said to be convergent on an interval I if the sequence of its partial sums $S_1 = f_1$, $S_2 = f_1 + f_2$, $\ldots$

$$S_n = \sum_{i=1}^{n} f_i$$ is convergent on I.

The limit function of this sequence is called the sum of the series (of functions)

- **Power series:** A power series is a series of functions of the form $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ ...... (1)

$f_n = (x - x_0)^n$

where $a_0, a_1, a_2, \dots$ are constants (coefficients) and x_0 is called the center of the series.

In particular if $x_0 = 0$, we get a power series of the form $\sum_{m=1}^{\infty} a_m x^m$

- Now if $x = x_0$, then the power series $\sum_{m=0}^{\infty} a_m (x-x_0)^m$ reduces to a_0 (a constant)

and so the power series ① converges at $x = x_0$

In some cases, this may be the only value of x for which ① converges.

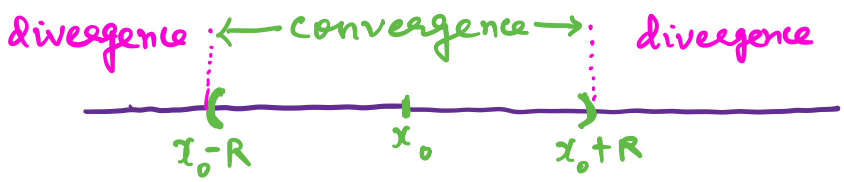
If there are other values of x for which the series converges, these values form an interval called "the interval of convergence".

- The interval may be finite with midpoint x_0. Then the power series ① converges for all x in $(x_0 - R, x_0 + R)$ (ie in $\{x \in \mathbb{R} : |x - x_0| < R\}$)

and diverges for $|x - x_0| > R$

- The interval may also be infinite ie.

the series may Converge for all $x \in \mathbb{R}$



The quantity R is called the Radius of convergence.

- If the power series converges for all x, $R = \infty$ (i.e. $\frac{1}{R} = 0$)

- The radius of convergence can be determined from the coefficients of the series by the formulas:

(a) $$R = \frac{1}{\lim\limits_{m \to \infty} \left(|a_m|\right)^{1/m}}$$

(b) $$R = \frac{1}{\lim\limits_{m \to \infty} \left|\frac{a_{m+1}}{a_m}\right|}$$

provided the limits exist.

(If $R = \infty$, then the power series (1) converges only at the center x_0.)

Existence of Power series solution:

If p, q and r in $y'' + p(x)y' + q(x)y = r(x)$ have power series representation at $x = x_0$, then every solution of the differential equation is differentiable in a neighborhood of x_0 and can be represented as power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Ex: 1 $\quad e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m$

$$\frac{a_{m+1}}{a_m} = \frac{\frac{1}{(m+1)!}}{\frac{1}{m!}} = \frac{m!}{(m+1)!} = \frac{1}{m+1}$$

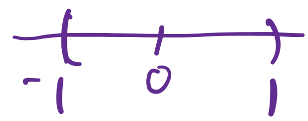
$$\lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \frac{1}{m+1} = 0$$

So, $\quad R = \dfrac{1}{\lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right|} = \boxed{\infty}$

(2) $$\frac{1}{1-x} = 1 + x + x^2 + \cdots\cdots$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1$$

So, $$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{1} = \boxed{1}$$



(3) $$\sum_{n=0}^{\infty} n!\, x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty}(n+1) = \infty$$

So, $$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\infty} = \boxed{0}$$

Operations on Power series:

(1) Termwise addition: Two power series may be added term by term. More precisely: if the series $\sum_{m=0}^{\infty} a_m (x-x_0)^m$ and $\sum_{m=0}^{\infty} b_m (x-x_0)^m$ have positive radius of convergence and their sums are $f(x)$ and $g(x)$, then the series $\sum_{m=0}^{\infty} (a_m+b_m)(x-x_0)^m$ converges and represents $f(x)+g(x)$ for each x that lies in the interior of the convergence interval common to each of the two series

(2) Termwise Multiplication: Two power series may be multiplied term by term. More precisely: suppose the two series in (1) have positive radius of convergence and let $f(x)$ and $g(x)$ be their sums. Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $(x-x_0)^m$, that is,

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-x_0)^2 + \cdots\cdots$$

$$= \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots\cdots + a_m b_0)(x-x_0)^m$$ converges and represents $f(x)g(x)$ for each x in the interior of the convergence interval of each of the two given series.

(4) Vanishing of all coefficients (Identity Theorem for Power Series)

If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.

(4) <u>Termwise Differentiation:</u> A power series may be differentiated term by term. More precisely:

If $y(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m$ converges for $|x-x_0| < R$ where $R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x:

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x-x_0)^{m-1} \quad (|x-x_0| < R)$$

Similarly for the second and further derivatives.

Legendre's Equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{---- (1)}$$

We can transform this equation into

$$y'' - \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0$$

The coefficient functions are differentiable at $x=0$ but not at $x = \pm 1$

Thus we can use a power series

around 0 as a solution.

Let $y = \sum_{m=0}^{\infty} a_m x^m$, $y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Let $k = n(n+1)$

Substituting in ①, we get

$$(1-x^2)\left(\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}\right) - 2x\left(\sum_{m=1}^{\infty} m a_m x^{m-1}\right) + k\left(\sum_{m=0}^{\infty} a_m x^m\right) = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^{m} - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

Let us substitute $m-2 = s$ in the first summation
$m = s+2$

and $m = s$ in the other summations.

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s$$

$$- \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0$$

<u>Coefficient of x^0 :</u>

$$2 \times 1 \times a_2 + k a_0 = 0 \Rightarrow 2a_2 + ka_0 = 0$$

<u>Coefficient of x^1 :</u>

$$3 \times 2 \times a_3 - 2a_1 + k a_1 = 0$$

$$\Rightarrow 6a_3 + (k-2)a_1 = 0$$

<u>Coefficient of x^s :</u>

$$(s+2)(s+1)a_{s+2} + (-s^2 + s - 2s + k)a_s = 0$$

$$\Rightarrow (s+2)(s+1)a_{s+2} - (s^2 + s - k)a_s = 0$$

$$a_2 = -\frac{k}{2} a_0 = -\frac{n(n+1)}{2} a_0$$

$$a_3 = -\frac{(k-2)}{6} a_1 = -\frac{\{n(n+1)-2\}}{6} a_1$$

$$\Rightarrow a_3 = -\frac{(n^2+n-2)}{6} a_1$$

$$\Rightarrow a_3 = -\frac{(n+2)(n-1)}{6} a_1$$

In general,

$$a_{s+2} = \frac{s^2+s-k}{(s+2)(s+1)} a_s$$

$$= \frac{s^2+s-n(n+1)}{(s+2)(s+1)} a_s$$

$$= \frac{s^2+(n+1)s-ns-n(n+1)}{(s+2)(s+1)} a_s$$

$$a_{s+2} = \frac{(s+n+1)(s-n)}{(s+2)(s+1)} a_s$$

$$\Rightarrow \boxed{a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s}$$

for $s = 0, 1, 2, \ldots$

(Recursion formula)

Let $s=2$:

$$a_4 = -\frac{(n-2)(n+3)}{4.3} a_2$$

$$= \left\{-\frac{(n-2)(n+3)}{4.3}\right\}\left\{-\frac{n(n+1)}{2!}\right\} a_0$$

$$= \frac{(n-2)n(n+1)(n+3)}{4!} a_0$$

Let $s=3$

$$a_5 = -\frac{(n-3)(n+4)}{5.4} a_3$$

$$= \left\{-\frac{(n-3)(n+4)}{5.4}\right\}\left\{-\frac{(n+2)(n-1)}{3!} a_1\right\}$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

Thus inserting these expressions for the coefficients in $y = \sum_{m=0}^{\infty} a_m x^m$, we obtain

that there are actually two independent solutions and the general solution in $(-1, 1)$ is a linear combination of the two:

$$y = a_0 y_1 + a_1 y_2$$

where $y_1 = \left\{ 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \cdots\cdots \right\}$

and $y_2 = \left\{ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots\cdots \right\}$

(Both the series converges in $(-1, 1)$)

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Since $\frac{y_1}{y_2}$ is not a constant,

y_1 and y_2 are linearly independent.

So, $y = a_0 y_1 + a_1 y_2$ is a general solution of the ODE on the interval $(-1, 1)$

Polynomial solution:

Consider the recursion relation:

$$\boxed{a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s}$$

If n is a positive integer then

For $s = n$: $a_{s+2} = 0$

$$\Rightarrow \quad 0 = a_{n+2} = a_{n+4} = a_{n+6} = \cdots\cdots$$

If n is even, y_1 is a polynomial of degree n.

If n is odd, y_2 is a polynomial of degree n.

These polynomials will be referred to as $P_n(x)$ (Legendre Polynomial)

We will choose the highest coefficient (of degree n) to be

$$a_n = \begin{cases} \dfrac{(2n)!}{2^n (n!)^2} & \text{if } n \text{ is a positive integer} \\ = 1 & \text{if } n = 0 \end{cases}$$

From the recursion relation,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \quad (\text{for } s \le n-2)$$

we get

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

$$= -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2}$$

$$= - \frac{\cancel{n}(\cancel{n-1})}{2(2n-1)} \frac{(2n)!}{\underset{(n-2)!}{\cancel{n!}}\, 2^n\, n!}$$

$$= - \frac{(2n)!}{2^n (n-2)!\,(n-1)!\, n \times 2\, (2n-1)}$$

$$= - \frac{(2n-2)!}{2^n (n-2)!\,(n-1)!}$$

Then similarly,

$$a_{n-4} = - \frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(n-2)(n-3)}{4(2n-3)} \frac{(2n-2)!}{2^n (n-2)!\,(n-1)!}$$

$$= \frac{(2n-4)!}{2^n\, 2!\, (n-2)!\, (n-4)!}$$

In general, when

$n - 2m \geqslant 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!}$$

The resulting solution of Legendre's differential equation is called the Legendre Polynomial of degree n and is denoted by $P_n(x)$. Hence,

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!} x^{n-2m}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n\, 1!\,(n-1)!\,(n-2)!} x^{n-2} + \cdots\cdots$$

where
$$M = \frac{n}{2} \quad (\text{if } n \text{ is even})$$
$$= \frac{n-1}{2} \quad (\text{if } n \text{ is odd})$$

Then,

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \qquad P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

Note: The Legendre Polynomials $P_n(x)$ are **orthogonal** on the interval $[-1, 1]$

In fact, $\int_{-1}^{1} P_i(x) P_j(x)\,dx = \frac{2}{2i+1}\delta_{ij}$

So, $\int_{-1}^{1} P_i(x) P_j(x)\,dx = 0$ for $i \neq j$

Note: $P_0(x) = 1$, $P_1(x) = x$

and **Bonnet's recursion formula**

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

for $n = 1, 2, \ldots$