MTH 212: Mid-semester exam

Maximum marks: Euler $\phi(61)$ **Time:** 70 minutes

Instructions. Be sure to show your work and explain your reasoning for full credit. No calculators, phones, notes, etc. are allowed. Unless otherwise stated G always denotes a group.

1. (13+1) Let $G = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ be a group defined by the following table:

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
g_2	g_2	g_5	g_4	g_7	g_6	g_1	g_8	g_3
g_3	g_3	g_8	g_5	g_2	g_7	g_4	g_1	g_6
g_4	g_4	g_3	g_6	g_5	g_8	g_7	g_2	g_1
g_5	g_5	g_6	g_7	g_8	g_1	g_2	g_3	g_4
g_6	g_6	g_1	g_8	g_3	g_2	g_5	g_4	g_7
g_7	g_7	g_4	g_1	g_6	g_3	g_8	g_5	g_2
g_8	g_8	g_7	g_2	g_1	g_4	g_3	g_6	g_5

   (a) Find the identity element, order of each element, inverse of each element, and the centralizer of each element of G.

   (b) What is the center of G?

2. (2+2+2+2) Let $G =< a >$ be a group generated by an element a of order 30.

   (a) Find all elements of G which generate G.

   (b) List all elements in the subgroup $< a^6 >$.

   (c) What are the generators of subgroup $< a^6 >$?

   (d) Find all elements in G of order 3.

3. (2+2+2+1+1+2) Let $\mathbb{R}$ be the additive group of real numbers and let $\mathbb{T}$ be the multiplicative group of complex numbers with absolute value 1. Consider the map $\phi : \mathbb{R} \to \mathbb{T}$ given by $\phi(x) = \cos(x) + i \sin x$.

   (a) Show that ϕ is a homomorphism from $\mathbb{R}$ to $\mathbb{T}$. $e^{i\theta}$

   (b) What is the kernel of ϕ?

   (c) What is the image of ϕ?

   (d) Is ϕ injective?

   (e) Is ϕ surjective?

(f) Is ϕ an isomorphism?

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4. (2+2+1+2) Let G be a group and H be a subgroup of G. Let $a \in G$.

   (a) Define the left coset of H in G containing a.

   (b) Describe the distinct cosets of $\mathbb{Z}$ in $\mathbb{R}$ and hence write $\mathbb{R}$ as disjoint union of cosets of $\mathbb{Z}$ in $\mathbb{R}$.

   (c) Write the index of H in G.

   (d) Let $H = 3\mathbb{Z}$. Which of the following cosets are same.

      i. $11 + H$ and $17 + H$

      ii. $-1 + H$ and $5 + H$

      iii. $7 + H$ and $23 + H$

      iv. $6 + H$ and $2 + H$

5. (2+2+1+0.5) Let $\alpha = (1\ 2\ 4\ 6)(3\ 7\ 5)$ and $\beta = (1\ 4)(2\ 7)(3\ 5)$ be elements in S_7.

   (a) Compute $\alpha\beta$ and $\beta\alpha$.

   (b) Compute α^{-1} and β^{-1}.

   (c) Compute $\alpha\beta\alpha^{-1}$.

   (d) Do α and β commute?

$2 \mid 3, 1, 2, 4$
$2 \mid 3, 1, 1, 2$
$3 \mid 3, 1, 1, 1$

6. (2+1+2+0.5) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 8 & 6 & 7 & 1 & 5 & 9 & 2 \end{pmatrix}$ be an element of S_9.

$\begin{pmatrix} 2 & 3 & 8 & 9 \\ 3 & 8 & 9 & 2 \end{pmatrix}$

   (a) Write α as product of disjoint cycles.

   (b) Find the order of α.

   (c) Write α as product of transpositions.

$(2\ 3\ 8\ 9)$

   (d) Is α an even or odd permutation?

7. (2+2+2+2+2) Fill in the blanks. Give a proof and state any result or theorem you are using to arrive at your answer clearly.

   (a) The minimum possible order of a group consisting of elements of orders 1 through 10, is __________.(Hint: Order of an element divides order of the group).

   (b) Let G be an infinite cyclic group generated by a, and T is an automorphism of G, then $T(a) = a^t$, the possible values of t are ___1___ and __________.

   (c) If $a \in G$ and $o(a) = 18$, then $o(a^2) =$__________.

   (d) Consider the center of a group G, i.e., $Z(G) = \{g \in G | gh = hg \text{ for all } h \in G\}$. If $a \in Z(G)$, then the centralizer of a, $C(a) =$__________.

   (e) Let $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_{25}$ be a group homomorphism. One possible pair of order of $\phi(1)$ is __________.