Jacobi's Principle

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We start with principle of least action, viz.

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i) dt = 0 \tag{1}$$

for the path that the particle takes. In our case, we are going to introduce an arbitrary parameter τ and treat the time as one of the position coördinates. Primes denoting differentiation[1] with respect to τ, we have

$$\delta \int_{\tau_1}^{\tau_2} \left(L \left(q_i, \frac{q_i'}{t'} \right) t' \right) d\tau = 0 \tag{2}$$

Now we can apply the Euler-Lagrange equations for t to this integral to get

$$\frac{\partial (Lt')}{\partial t} - \frac{d}{d\tau} \left(\frac{\partial (Lt')}{\partial t'} \right) = 0. \tag{3}$$

Since t does not appear in the integrand, $\dfrac{\partial (Lt')}{\partial t} = 0$, and thus $\dfrac{\partial (Lt')}{\partial t'}$ is constant with respect to τ.

The generalized momentum p_i for a variable q_i will be defined as $\dfrac{\partial L}{\partial \dot{q}_i}$; in our case we also consider the momentum associated with time, viz. $p_t = \dfrac{\partial (Lt')}{\partial t'}$.

Now,

$$\begin{aligned}
p_t &= \frac{\partial (Lt')}{\partial t'} \\
&= L - \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{q_i'}{t'^2} \right) t' \\
&= \left(L - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \\
&= \left(L - \sum_{i=1}^{n} p_i \dot{q}_i \right) \\
&= -E.
\end{aligned}$$

[1] Actually, the q_i' are independent of the q_i but we add the auxiliary conditions $\frac{dq_i}{d\tau} = q_i'$.

where E is the energy of the system.

Forming $\bar{L} = Lt' - p_t t'$, we will show that

$$\delta \int_{\tau_1}^{\tau_2} \bar{L} d\tau = 0 \tag{4}$$

Now,

$$\delta \int_{\tau_1}^{\tau_2} p_t t' d\tau. \tag{5}$$

will be true if and only if the Euler-Lagrange equations hold for all the variables. This is clearly true, since

$$\frac{\partial(p_t t')}{\partial q_i} - \frac{d}{d\tau}\left(\frac{\partial(p_t t')}{\partial q_i'}\right) = 0 \tag{6}$$

p_t being constant and the q_i, q_i' being independent of t'. Also,

$$\frac{\partial(p_t t')}{\partial t} - \frac{d}{d\tau}\left(\frac{\partial(p_t t')}{\partial t'}\right) = 0 \tag{7}$$

since $\frac{\partial(p_t t')}{\partial t'} = p_t$, which is constant.

We already know $\delta \int_{\tau_1}^{\tau_2} Lt' d\tau = 0$, so, adding the two, we get $\delta \int_{\tau_1}^{\tau_2} \bar{L} d\tau = 0$. From above,

$$\bar{L} = Lt' - p_t t' = (L - p_t)t' = \left(\sum_{i=1}^{n} p_i \dot{q}_i\right) t', \tag{8}$$

so

$$\int_{\tau_1}^{\tau_2} \bar{L} d\tau = 2 \int_{\tau_1}^{\tau_2} Tt' d\tau. \tag{9}$$

It was pointed out by Jacobi that this cannot be simplified to $\int_{\tau_1}^{\tau_2} \bar{L} d\tau = 2 \int_{t_1}^{t_2} T dt$, because t cannot be treated as an independent variable in the variational problem. Instead, we take advantage of the fact that

$$T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 \tag{10}$$

or

$$T = \frac{1}{2}\frac{\left(\frac{ds}{d\tau}\right)^2}{t'^2} \tag{11}$$

Since $T = E - V$,

$$t' = \frac{1}{\sqrt{2(E-V)}}\frac{ds}{d\tau}, \tag{12}$$

giving finally

$$2\int_{\tau_1}^{\tau_2} Tt' d\tau = \int_{\tau_1}^{\tau_2} \sqrt{2(E-V)}\frac{ds}{d\tau} d\tau = \int_{\tau_1}^{\tau_2} \sqrt{2(E-V)} ds, \tag{13}$$

as desired. Recalling that $\delta \displaystyle\int_{\tau_1}^{\tau_2} \bar{L} d\tau = 0$, we see that

$$\delta \int_{\tau_1}^{\tau_2} \sqrt{2(E-V)} ds = 0 \tag{14}$$

as well. This condition determines the particle's path, and is known as Jacobi's principle. Note that this determines the path a particle takes in space, but says nothing about time.

Fermat's principle of least time states that light takes the path that takes the least time, that is,

$$\delta \int dt = 0 \tag{15}$$

Rewriting dt as nds, n being the index of refraction, we get

$$\delta \int nds = 0. \tag{16}$$

As can be seen, this bears a striking resemblance to Jacobi's principle. If a material has an index of refraction $n(x, y, z)$, then light will travel through it the same way a particle would be affected by a potential field $V(x, y, x)$ provided that $n = \sqrt{2(E-V)}$. This is not the entirety of the optico-mechanical analogy, but it is part of it. The optico-mechanical analogy was used in the development of the old quantum theory, including de Broglie's Nobel Prize-winning dissertation on matter waves.