Fundamental Spaces $\quad A \in \mathbb{R}^{m \times n}$

range space $\mathcal{R}(A) = \{ \underline{v} \in \mathbb{R}^m \mid \underline{v} = A\underline{u} \}$ $\quad u \in \mathbb{R}^n$

(column space)

$$v = u_1 \underline{a}_1 + u_2 \underline{a}_2 \dots + u_n \underline{a}_n$$

$\mathcal{R}(A)$ is a subspace

$v_1, v_2 \in \mathcal{R}(A) \Rightarrow u_1, u_2$ s.t.

$v_1 = Au_1$

$v_2 = Au_2$

$$\underbrace{\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2}_{\in \mathcal{R}(A)} = A(\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2)$$

$\dim(\mathcal{R}(A))$ = # linearly indep. columns of A

= column rank of A

Null Space $\quad \mathcal{N}(A) = \{ u \in \mathbb{R}^n \mid Au = 0 \}$

$\mathcal{R}(A^T) = \{ u \in \mathbb{R}^n \mid u = A^T v ;\ v \in \mathbb{R}^m \}$

$\mathcal{N}(A^T) = \{ v \in \mathbb{R}^m \mid A^T v = 0 \}$

<u>Recap</u>

$R(A) \rightarrow \mathbb{R}^m$ $\qquad$ $R(A^T) \rightarrow \mathbb{R}^n$

A

$N(A) \rightarrow \mathbb{R}^n$ $\qquad$ $N(A^T) \rightarrow \mathbb{R}^m$

$$w \in R(A)^{\perp} \iff w^T(\underbrace{Au}_{v \in R(A)}) = 0 \quad \forall u$$

$$\Rightarrow \quad u^T A^T w = 0 \quad \forall u$$

or $u \perp A^T w \quad \forall u$

not possible unless $A^T w = 0$

$w \in N(A^T)$

so $R(A)^{\perp} = N(A^T)$

$N(A)^{\perp} = R(A^T)$ $\leftarrow$ prove this

$$\text{rank}(A) = \dim(R(A)) = \dim(R(A^T)) = r \leq m, n$$

SVD

$v_1, v_2 \ldots v_r \in R(A)$ $\qquad$ $v_{r+1}, \ldots, v_m \in N(A^T) = R(A)^{\perp}$

$u_1, u_2 \ldots u_r \in R(A^T)$ $\qquad$ $u_{r+1} \ldots u_n \in R(A^T)^{\perp} = N(A)$

Singular Value Decomposition

$$A = \underset{m\times m}{\begin{bmatrix} \underbrace{v_1 \cdots v_r}_{R(A)} & \underbrace{v_{r+1} \cdots v_m}_{N(A^T)} \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & 0 & \\ & \ddots \sigma_r & & \\ 0 & & & 0 \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} u_1^T \\ \vdots \\ u_r^T \\ \vdots \\ u_n^T \end{bmatrix}} \begin{matrix} \} R(A^T) \\ \\ \} N(A) \end{matrix}$$

$\sigma_i > 0$

$$= \sum_{i=1}^{r} \sigma_i v_i u_i^T \quad \text{sum of rank-1 matrices}$$

$$= V \Sigma U^T$$

does not depend on v_{r+1} or u_{r+1}

SVD vs EVD

$$A = V \Sigma U^T \qquad V^T V = V V^T = I$$
$$U^T U = U U^T = I$$

$$A^T A = (V \Sigma U^T)^T (V \Sigma U^T)$$
$$= U \Sigma^T \underbrace{V^T V}_{I} \Sigma U^T$$
$$= U \underbrace{\Sigma^T \Sigma}_{\text{eigenvalues}} U$$

$$\Sigma^T \Sigma = \left[\begin{array}{cc|c} \sigma_1^2 & 0 & \\ & \ddots\ \sigma_r^2 & 0 \\ \hline 0 & & 0 \end{array}\right]$$

$$(\sigma_i(A))^2 = \lambda_i(A^T A) \qquad i = 1 \ldots r$$
$$= \lambda_i(AA^T) \qquad \text{similarly}$$