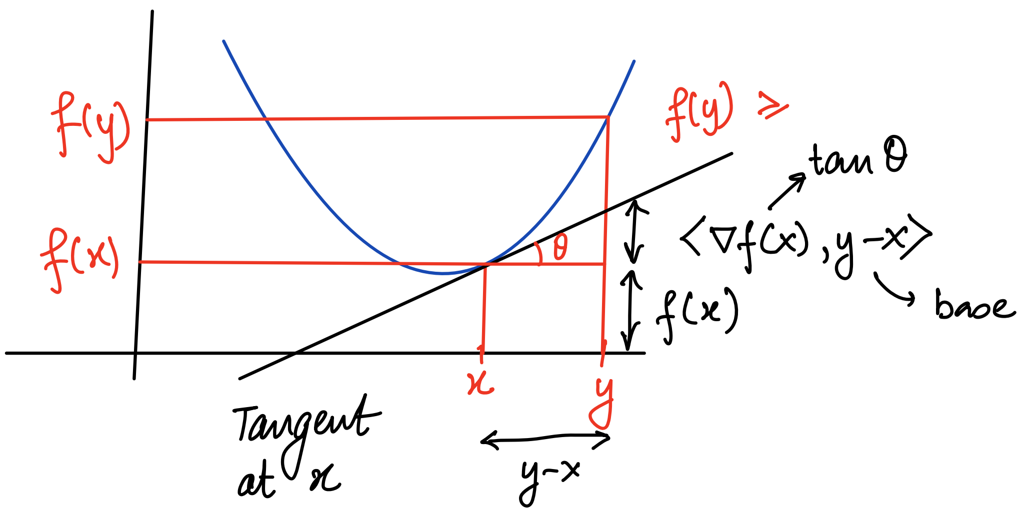
First and Second Order Conditions

First order condition (gradient-based)

$$f(y) \geq f(\underline{x}) + \langle \nabla f(\underline{x}), y - \underline{x} \rangle$$

note $\quad [\nabla f(\underline{x})]_i = \dfrac{\partial f}{\partial x_i} \qquad x \in \mathbb{R}^n$

Eg $\quad n=1 \qquad \nabla f(x) = \dfrac{df}{dx} \qquad$ (slope)



Note : bound holds $\forall y \in \mathrm{dom} f$ (global behavior)
but depends only on $\nabla f(x)$ (local property)

Eg $\quad$ Suppose $\exists x_0$ s.t. $\nabla f(x_0) = 0$

Ist order: $$f(y) - f(x_0) \geq \langle \underbrace{\nabla f(x_0)}_{=0}, y - x_0 \rangle$$

$$\Rightarrow f(y) \geq f(x_0) \quad \forall \; y \in \operatorname{dom} f$$

$$\Rightarrow x_0 = \arg\min_x f(x)$$

$$\text{or } f(x_0) = \min_x f(x)$$

Second order condition

Hessian: $$[\nabla^2 f]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad \begin{array}{l} i = 1 \ldots n \\ j = 1 \ldots n \end{array}$$

Eg: $f(x) = a^T x + b \qquad \nabla^2 f = 0$

IInd order condition: $$\underbrace{\nabla^2 f(x) \succeq 0}_{\text{P.S.D.}} \quad \forall \; x \in \operatorname{dom} f$$