# eMasters in Communication Systems

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# Elective Module:

# Detection for Wireless Communication

# Chapter 8

# Detection of Random Signals

$\underbrace{\bar{s},\ \bar{s}_0,\ \bar{s}_1}_{\text{Deterministic.}}$

$\bar{s}$ Random signal ?

# Detection of Random Signals

- Consider a **random signal detection** problem
- Under $\mathcal{H}_0$

Signal is absent.

$$\left.\begin{aligned} y(1) &= v(1) \\ y(2) &= v(2) \\ &\vdots \\ y(N) &= v(N) \end{aligned}\right\}$$

iid Gaussian noise samples
mean = 0
var = $\sigma^2$

# Detection of Random Signals

- Consider a **random signal detection** problem
- Under $\mathcal{H}_0$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$\mathcal{H}_0$

$\bar{y} = \bar{v}$

# Detection of Random Signals

- This can be written in the **vector form**.
- Under $\mathcal{H}_0$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\bar{y} = \bar{v}$$

N x 1 noise vector

# Detection of Random Signals

- This can be written in the **vector form**.
- Under $\mathcal{H}_0$

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{y}} = \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{v}}$$

# Detection of Random Signals

Signal Present

- Under $\mathcal{H}_1$

$$y(1) = s(1) + v(1)$$

$$y(2) = s(2) + v(2)$$

$$\vdots$$

$$y(N) = s(N) + v(N)$$

Random

iid Gaussian

mean $= 0$

var $= \sigma_s^2$

$s(i)$ independent of $v(j)$

$E\{s(i)v(j)\} = 0$ for all $i, j$

$$\underbrace{s(1), s(2), \ldots, s(N)}_{\text{iid Gaussian samples.}}$$

# Detection of Random Signals

- Under $\mathcal{H}_1$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\bar{y} = \bar{s} + \bar{v}$$

# Detection of Random Signals

- This can be written in the **vector form**.
- Under $\mathcal{H}_1$

Gaussian signal vector

Gaussian noise vector

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\bar{y} = \bar{s} + \bar{v}$$

# Detection of Random Signals

$E\{\bar{v}\} = 0$

$E\{\bar{v}\bar{v}^T\} = \sigma^2 I$

Noise covariance

- This can be written in the **vector form**.

$E\{\bar{s}\} = 0$ $\quad E\{\bar{s}\bar{s}^T\} = \sigma_s^2 I$

signal covariance matrix

- Under $\mathcal{H}_1$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\bar{y} = \bar{s} + \bar{v}$$

# Detection of Random Signals

- $s(i)$ are i.i.d. **Gaussian** mean $0$ and variance $\sigma_s^2$
- $v(i)$ are i.i.d. **Gaussian** mean $0$ and variance $\sigma^2$

# Detection of Random Signals

likelihood under $\mathcal{H}_0$.
only Gaussian noise present.

- The likelihoods are as follows
- Under $\mathcal{H}_0$ we have $y(i) = v(i)$

$$p(\bar{\mathbf{y}}; \mathcal{H}_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2(1)/2\sigma^2} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2(N)/2\sigma^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\|\bar{y}\|^2}{2\sigma^2}}$$

likelihood under $\mathcal{H}_0$
NULL hypothesis.

# Detection of Random Signals

- The likelihoods are as follows
- Under $\mathcal{H}_0$ we have $y(i) = v(i)$

$$p(\bar{\mathbf{y}}; \mathcal{H}_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2(1)}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2(N)}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\|\bar{\mathbf{y}}\|^2}$$

# Detection of Random Signals

$y(i) = \text{sum of Gaussian} \Rightarrow y(i) = \text{Gaussian}$

- Under $\mathcal{H}_1$ we have $y(i) = s(i) + v(i)$

$E\{y(i)\} = E\{s(i)\} + E\{v(i)\} = 0$

- $y(i)$ are i.i.d. Gaussian mean 0, variance $\sigma^2 + \sigma_s^2$

$y(i) \sim \mathcal{N}(0, \sigma_s^2 + \sigma^2)$.

$$E\{y^2(i)\} = E\{(s(i) + v(i))^2\}$$

$$= E\{s^2(i)\} + E\{v^2(i)\} + \overset{0}{\cancel{2E\{s(i)v(i)\}}}$$

$$= \sigma_s^2 + \sigma^2$$

# Detection of Random Signals

- Under $\mathcal{H}_1$

$$p(\bar{\mathbf{y}};\mathcal{H}_1) = \frac{1}{\sqrt{2\pi(\sigma_s^2+\sigma^2)}} e^{-\frac{y^2(1)}{2(\sigma_s^2+\sigma^2)}} \times \cdots \times \frac{1}{\sqrt{2\pi(\sigma_s^2+\sigma^2)}} e^{-\frac{y^2(N)}{2(\sigma_s^2+\sigma^2)}}$$

$$= \left(\frac{1}{2\pi(\sigma^2+\sigma_s^2)}\right)^{\frac{N}{2}} e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2+\sigma_s^2}} \Bigg\} \begin{array}{l} p(\bar{y};\mathcal{H}_1) \\ \text{likelihood under } \mathcal{H}_1. \end{array}$$

# Detection of Random Signals

- Under $\mathcal{H}_1$

$p(\bar{\mathbf{y}};\mathcal{H}_1)$

$$= \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_s^2)}} e^{-\frac{y^2(1)}{2(\sigma^2+\sigma_s^2)}} \times \cdots \times \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_s^2)}} e^{-\frac{y^2(N)}{2(\sigma^2+\sigma_s^2)}}$$

$$= \left(\frac{1}{2\pi(\sigma^2+\sigma_s^2)}\right)^{\frac{N}{2}} e^{-\frac{1}{2(\sigma^2+\sigma_s^2)}\|\bar{\mathbf{y}}\|^2}$$

# Detection of Random Signals

- Choose $\mathcal{H}_0$ if

Likelihood Ratio Test (LRT) — Threshood.

$$\frac{p(\bar{y};\mathcal{H}_0)}{p(\bar{y};\mathcal{H}_1)} \geq \tilde{\gamma}$$

$$\frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2}\cdot\frac{\|\bar{y}\|^2}{\sigma^2}}}{\left(\frac{1}{2\pi(\sigma^2+\sigma_s^2)}\right)^{\frac{N}{2}} e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2+\sigma_s^2}}} \geq \tilde{\gamma}$$

# Detection of Random Signals

Take natural log on both sides: choose $H_0$ when

$$\frac{N}{2}\ln\left(\frac{\sigma_s^2+\sigma^2}{\sigma^2}\right) + \|\bar{y}\|^2\left(\frac{1}{2}\cdot\frac{1}{\sigma^2+\sigma_s^2} - \frac{1}{2\sigma^2}\right) \geq \ln\tilde{\gamma}$$

$$\frac{N}{2}\ln\left(\frac{\sigma^2+\sigma_s^2}{\sigma^2}\right) - \ln\tilde{\gamma} \geq \|\bar{y}\|^2\left(\frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2+\sigma_s^2)}\right)$$

$$\Rightarrow \quad \|\bar{y}\|^2 \leq \frac{\frac{N}{2}\ln\left(\frac{\sigma_s^2+\sigma^2}{\sigma^2}\right) - \ln\tilde{\gamma}}{\left(\frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2+\sigma_s^2)}\right)} = \gamma$$

# Detection of Random Signals

choose $\mathcal{H}_0$

$$\|\bar{y}\|^2 \leq \gamma$$

# Detection of Random Signals

- Choose $\mathcal{H}_0$ if

$$\frac{p(\bar{\mathbf{y}}; \mathcal{H}_0)}{p(\bar{\mathbf{y}}; \mathcal{H}_1)} \geq \tilde{\gamma}$$

$$\frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\|\bar{\mathbf{y}}\|^2}}{\left(\frac{1}{2\pi(\sigma^2 + \sigma_s^2)}\right)^{\frac{N}{2}} e^{-\frac{1}{2(\sigma^2+\sigma_s^2)}\|\bar{\mathbf{y}}\|^2}} \geq \tilde{\gamma}$$

# Detection of Random Signals

- Choose $\mathcal{H}_0$ if

$$\frac{N}{2}\ln\left(\frac{\sigma^2+\sigma_s^2}{\sigma^2}\right)+\|\bar{\mathbf{y}}\|^2\left(-\frac{1}{2\sigma^2}+\frac{1}{2(\sigma^2+\sigma_s^2)}\right)\geq \ln\tilde{\gamma}$$

$$\|\bar{\mathbf{y}}\|^2 \leq \frac{\frac{N}{2}\ln\left(\frac{\sigma^2+\sigma_s^2}{\sigma^2}\right)-\ln\tilde{\gamma}}{\left(\frac{1}{2\sigma^2}-\frac{1}{2(\sigma^2+\sigma_s^2)}\right)}=\gamma$$

# Detection of Random Signals

energy of signal.

$$\underbrace{\|\bar{y}\|^2 = |y(1)|^2 + \cdots + |y(N)|^2}$$

- Choose $\mathcal{H}_0$ if

$$\|\bar{y}\|^2 < \gamma$$

- Choose $\mathcal{H}_1$ if

$$\|\bar{y}\|^2 > \gamma$$

# Detection of Random Signals

- Choose $\mathcal{H}_0$ if

$$\|\bar{\mathbf{y}}\|^2 \leq \gamma$$

- Choose $\mathcal{H}_1$ if

$$\|\bar{\mathbf{y}}\|^2 > \gamma$$

ENERGY DETECTOR

Very Simple
Low complexity
Efficient

# Detection of Random Signals

- This is termed as the ENERGY. DETECTOR.

Non coherent detector

## Detection of Random Signals

- This is termed as the **Energy Detector.**

# Detection of Random Signals

- The $P_{FA}$ is given as follows under $\mathcal{H}_0$

$$\Pr(||\bar{\mathbf{y}}||^2 \geq \gamma; \mathcal{H}_0)$$

$$\Pr\left(|y(1)|^2 + \cdots + |y(N)|^2 \geqslant \gamma ; \mathcal{H}_0\right)$$

# Detection of Random Signals

under $\mathcal{H}_0$

$$y(i) = v(i) \sim \mathcal{N}(0, \sigma^2)$$

$$\Rightarrow y^2(1) + \cdots + y^2(N) \geq \gamma$$

$$\Rightarrow \left(\frac{y(1)}{\sigma}\right)^2 + \cdots + \left(\frac{y(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$

$$\frac{y(i)}{\sigma} = \frac{v(i)}{\sigma} \sim \mathcal{N}(0, 1)$$

Standard Gaussian RV mean $= 0$, var $= 1$.

# Detection of Random Signals

- The $P_{FA}$ is given as follows

$$\Pr(\|\bar{\mathbf{y}}\|^2 \geq \gamma; \mathcal{H}_0)$$

$$y^2(1) + \cdots + y^2(N) \geq \gamma$$

# Detection of Random Signals

$$\left(\frac{y(1)}{\sigma}\right)^2 + \cdots + \left(\frac{y(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$

$$\underbrace{\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2}_{\text{Sum of Squares of N standard zero mean unit variance Gaussian RVs.}} \geq \frac{\gamma}{\sigma^2}$$

This is $\chi^2_N$ # number of degrees of freedom

# Detection of Random Signals

$$\frac{v(i)}{\sigma} \sim \mathcal{N}(0,1)$$

# Detection of Random Signals

Central Chi Squared RV with N degrees of Freedom

$$\frac{v(i)}{\sigma} \sim \mathcal{N}(0,1)$$

$$\underbrace{\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2}_{\text{Central. } \chi_N^2} \geq \frac{\gamma}{\sigma^2}$$

# Detection of Random Signals

$$\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2$$

- Sum of squares of N iid zero mean Gaussian RVs of unit variance.
- This is $\chi_N^2$ RV.

# Detection of Random Signals

$$\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2$$

- Sum of squares of $N$ iid **zero-mean Gaussian RVs.**
  - This is $\chi_N^2$ RV

$N = $ # degrees of freedom

# Detection of Random Signals

Zero mean.
Each Gaussian has zero mean.

- $\chi_N^2$-Central chi-squared RV with $N$ degrees of freedom

# Detection of Random Signals

- PDF is given as

$\chi^2_N$ : Central ChiSquared RV
N degrees of Freedom

$$f_X(x) = \frac{1}{2^{N/2}\,\Gamma\left(\frac{N}{2}\right)}\, x^{\frac{N}{2}-1}\, e^{-\frac{1}{2}x}, \quad x \geq 0$$

$$\Gamma\left(\frac{N}{2}\right) = \int_0^{\infty} t^{\frac{N}{2}-1} e^{-t}\, dt$$

$$\Gamma(m) = (m-1)!$$

+ve number m.

# Detection of Random Signals

- PDF is given as

$$f_X(x) = \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} x^{\frac{N}{2}-1} e^{-\frac{1}{2}x}, x \geq 0$$

$$\Gamma\left(\frac{N}{2}\right) = \int_0^{\infty} t^{\frac{N}{2}-1} e^{-t} dt$$

# Detection of Random Signals

under $\mathcal{H}_0$, Probability decision $= \mathcal{H}_1$.

- $P_{FA}$ is given as

$$\Pr\left(\underbrace{\left(\frac{v(1)}{\sigma}\right)^2 + \left(\frac{v(2)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2}_{\chi_N^2} > \frac{\gamma}{\sigma^2}\right)$$

$$= Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2}\right)$$

CCDF of chisquared RV with N degrees of freedom.

# Detection of Random Signals

- $P_{FA}$ is given as

$$\Pr\left(\frac{v^2(1)}{\sigma^2}+\frac{v^2(2)}{\sigma^2}+\cdots+\frac{v^2(N)}{\sigma^2}\geq\frac{\gamma}{\sigma^2}\right)$$

$$P_{FA}=Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2}\right)$$

Probability of False Alarm

# Detection of Random Signals

- $Q_{\chi_N^2}(\cdot)$: **CCDF** of the $\chi_N^2$ RV with $N$ degrees of freedom

Complementary cumulative Distribution function

# Detection of Random Signals

- **The CCDF** of the $\chi_N^2$ RV with $N$ degrees of freedom is

$$Q_{\chi_N^2}(x) = \int_x^{\infty} \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} e^{-\frac{1}{2}t}\, t^{\frac{N}{2}-1}\, dt$$

$$t = 2u \Rightarrow dt = 2\,du$$

$$= \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} \int_{\frac{x}{2}}^{\infty} e^{-u} (2u)^{\frac{N}{2}-1}\, 2\,du$$

# Detection of Random Signals

$$Q_{\chi^2_N}(x) = \frac{1}{\Gamma\left(\frac{N}{2}\right)} \underbrace{\int_{\frac{x}{2}}^{\infty} e^{-u}\, u^{\frac{N}{2}-1}\, du}_{\Gamma\left(\frac{N}{2},\frac{x}{2}\right).} \quad \left.\right\} \text{upper incomplete Gamma function}$$

$$Q_{\chi^2_N}(x) = \frac{\Gamma\left(\frac{N}{2},\frac{x}{2}\right)}{\Gamma\left(\frac{N}{2}\right)}$$

# Detection of Random Signals

$$\Gamma(m,x) = \int_x^\infty e^{-t}\, t^{m-1}\, dt \quad \Bigg\} \text{ Upper incomplete Gamma function}$$

# Detection of Random Signals

- **The CCDF** of the $\chi_N^2$ RV with $N$ degrees of freedom is

$$Q_{\chi_N^2}(x) = \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} \int_x^{\infty} t^{\frac{N}{2}-1} e^{-\frac{1}{2}t} dx$$

# Detection of Random Signals

- **The CCDF** of the $\chi_N^2$ RV with $N$ degrees of freedom is

$$Q_{\chi_N^2}(x) = \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} \int_{\frac{x}{2}}^{\infty} (2u)^{\frac{N}{2}-1} e^{-\frac{1}{2}2u} 2du$$

$$= \frac{1}{\Gamma\left(\frac{N}{2}\right)} \underbrace{\int_{\frac{x}{2}}^{\infty} u^{\frac{N}{2}-1} e^{-u} du}_{\text{Upper Incomplete Gamma Function}} = \frac{\Gamma\left(\frac{N}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{N}{2}\right)}$$

# Detection of Random Signals

$$P_{FA} = Q_{\chi^2_N}\left(\frac{\gamma}{\sigma^2}\right)$$

$$P_{FA} = \frac{\Gamma\left(\frac{N}{2}, \frac{\gamma}{2\sigma^2}\right)}{\Gamma\left(\frac{N}{2}\right)}.$$

Probability of <u>False</u> Alarm

# Detection of Random Signals

Probability signal is detected under $\mathcal{H}_1$.

- The $P_D$ is given as follows

$$\Pr(\|\bar{\mathbf{y}}\|^2 \geq \gamma; \mathcal{H}_1)$$

$$= \Pr\left(|y(1)|^2 + \cdots + |y(N)|^2 \geq \gamma ; \mathcal{H}_1\right)$$

# Detection of Random Signals

$$|y(1)|^2 + |y(2)|^2 + \cdots + |y(N)|^2 \geq \gamma \text{ under } \mathcal{H}_1$$

$$y(i) = s(i) + v(i) \sim \mathcal{N}(0, \sigma^2 + \sigma_s^2)$$

$$\underbrace{\left|\frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}}\right|^2 + \cdots + \left|\frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}}\right|^2}_{\text{sum of squares of } N \text{ zero mean iid Gaussian RVs of unit variance}} \geq \frac{\gamma}{\sigma^2+\sigma_s^2}$$

$\mathcal{N}(0,1)$

$\Rightarrow \chi_N^2$ RV

# Detection of Random Signals

- The $P_D$ is given as follows

$$\Pr(\|\bar{\mathbf{y}}\|^2 \geq \gamma; \mathcal{H}_1)$$

$$y^2(1) + \cdots + y^2(N) \geq \gamma$$

# Detection of Random Signals

$$\left(\frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2 + \cdots + \left(\frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2 \geq \frac{\gamma}{\sigma^2+\sigma_s^2}$$

$$\geq \frac{\gamma}{\sigma^2+\sigma_s^2}$$

$\chi_N^2$ Random variable.

# Detection of Random Signals

$$\frac{y(i)}{\sqrt{\sigma^2 + \sigma_s^2}} \sim \mathcal{N}(0, 1)$$

Standard Gaussian RV.

# Detection of Random Signals

$$\frac{y(i)}{\sqrt{\sigma^2+\sigma_s^2}} \sim \mathcal{N}(0,1)$$

Sum of squares of N standard Normal RVs.

$$\underbrace{\left(\frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2+\cdots+\left(\frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2}_{\chi_N^2} \geq \frac{\gamma}{\sigma^2+\sigma_s^2}$$

# Detection of Random Signals

- $P_D$ is given as

$$\Pr\left( \left| \frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}} \right|^2 + \cdots + \left| \frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}} \right|^2 \geq \frac{\gamma}{\sigma^2+\sigma_s^2} \right)$$

$$Q_{\chi^2_N}\left( \frac{\gamma}{\sigma^2+\sigma_s^2} \right)$$

# Detection of Random Signals

- $P_D$ is given as

$$\Pr\left(\left(\frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2+\cdots+\left(\frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2 \geq \frac{\gamma}{\sigma^2+\sigma_s^2}\right)$$

$$P_D = Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2+\sigma_s^2}\right) = \frac{\Gamma\left(\frac{N}{2}, \frac{\gamma}{2(\sigma^2+\sigma_s^2)}\right)}{\Gamma\left(\frac{N}{2}\right)}.$$

Probability of detection

# Detection of Random Signals

$$P_{FA} = Q_{\chi_N^2}\left(\frac{\gamma'}{\sigma^2}\right)$$

$$P_D = Q_{\chi_N^2}\left(\frac{\gamma'}{\sigma^2+\sigma_s^2}\right)$$

ROC = ? $\quad \gamma' = \sigma^2 Q^{-1}_{\chi_N^2}(P_{FA})$

$$P_D = \underbrace{Q_{\chi_N^2}\left(\frac{\sigma^2 Q^{-1}_{\chi_N^2}(P_{FA})}{\sigma^2+\sigma_s^2}\right)}$$

ROC
Receiver operating characteristic

# Detection of Random Signals

- PDF for $N = 2$ is given as

$$f_X(x) = \frac{1}{2^{\frac{N}{2}}\Gamma\frac{N}{2}} \cdot e^{-\frac{1}{2}x} x^{\frac{N}{2}-1} = \frac{1}{2^1 \cdot \Gamma_1} e^{-\frac{1}{2}x} x^0 = \frac{1}{2} e^{-\frac{1}{2}x}$$

$$f_X(x) = \frac{1}{2} e^{-\frac{1}{2}x}$$

$$\Gamma(1) = (1-1)! = 0! = 1$$

$\chi_2^2$ central chi squared RV with 2 degrees of freedom

## Detection of Random Signals

- PDF for $N = 2$ is given as

$$f_X(x) = \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} x^{\frac{N}{2}-1} e^{-\frac{1}{2}x}$$

$$= \frac{1}{2\Gamma(1)} e^{-\frac{1}{2}x} = \frac{1}{2} e^{-\frac{1}{2}x} \quad x \geq 0$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

# Detection of Random Signals

- CCDF for $N = 2$ is given as

$$Q_{\chi_2^2}(x) = \int_x^\infty \frac{1}{2} e^{-\frac{1}{2}t}\, dt$$

$$= -e^{-\frac{1}{2}t}\Big|_x^\infty$$

$$= e^{-\frac{1}{2}x}$$

$$Q_{\chi_2^2}(x)$$

# Detection of Random Signals

- CCDF for $N = 2$ is given as

$$Q_{\chi_2^2}(x) = \int_x^{\infty} \frac{1}{2} e^{-\frac{1}{2}t} = e^{-\frac{1}{2}x}$$

# Detection of Random Signals

- Therefore $P_D, P_{FA}$ are

Receiver Operating characteristic
$N=2$

$$P_D = Q_{\chi^2_2}\left(\frac{\gamma}{\sigma^2+\sigma_s^2}\right) = e^{-\frac{\gamma}{2(\sigma^2+\sigma_s^2)}}$$

$$P_{FA} = Q_{\chi^2_2}\left(\frac{\gamma}{\sigma^2}\right) = e^{-\frac{\gamma}{2\sigma^2}} \Rightarrow \gamma = -2\sigma^2 \ln P_{FA}$$

$$\Rightarrow e^{-\frac{1}{2(\sigma^2+\sigma_s^2)}\cdot(-2\sigma^2 \ln P_{FA})} = e^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}\ln P_{FA}}$$

$$= e^{\ln P_{FA}^{\sigma^2/\sigma^2+\sigma_s^2}} = P_{FA}^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}} = P_D$$

# Detection of Random Signals

Probability of detection $P_D$.

- Therefore $P_D, P_{FA}$ are

$$P_D = Q_{\chi_2^2}\left(\frac{\gamma}{\sigma^2 + \sigma_s^2}\right) = e^{-\frac{1}{2}\frac{\gamma}{\sigma^2+\sigma_s^2}}$$

$$P_{FA} = Q_{\chi_2^2}\left(\frac{\gamma}{\sigma^2}\right) = e^{-\frac{1}{2}\frac{\gamma}{\sigma^2}}$$

Probability of False Alarm. $P_{FA}$.

## Detection of Random Signals

- ROC can be evaluated as

$$\gamma = -2\sigma^2 \ln P_{FA}$$

# Detection of Random Signals

$$P_D = e^{-\frac{1}{2}\frac{\gamma}{\sigma^2+\sigma_s^2}} = e^{-\frac{1}{2}\times\frac{-2\sigma^2 \ln P_{FA}}{\sigma^2+\sigma_s^2}}$$

$$= e^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}\ln P_{FA}} = e^{\ln P_{FA}^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}}}$$

ROC

$$P_D = P_{FA}^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}}$$

# Chi-Squared Approximation

$\chi_N^2$

- **Chi-squared RV** with $N$ degrees of freedom

Central Chi Squared RV

$$u = u_1^2 + u_2^2 + \cdots + u_N^2$$

$$u_i \sim \mathcal{N}(0,1)$$

$u_i$ are iid Standard Normal RVs.

## Chi-Squared Approximation

- **Chi-squared RV** with $N$ degrees of freedom

$$u = u_1^2 + u_2^2 + \cdots + u_N^2$$

$u_i \sim \mathcal{N}(0,1)$.

# Chi-Squared Approximation

$$u = \underline{u_1^2 + u_2^2 + \cdots + u_N^2}$$

- What happens when $N \to \infty$?

$u \to$ Gaussian RV

Central Limit Theorem

$u_i^2$ iid.

Sum of N iid RVs.

# Chi-Squared Approximation

$E\{u\} = ?$
$E\{(u-\bar{u})^2\} = ?$

$$u = u_1^2 + u_2^2 + \cdots + u_N^2 \rightarrow \mathcal{N}(\underline{?}, \underline{?})$$

- What happens when $N \rightarrow \infty$?
  - Becomes **Gaussian** because of ***Central Limit Theorem.***

# Chi-Squared Approximation

- Mean is

$u_i \sim \mathcal{N}(0,1)$

$E\{u_i^2\} = 1$

$$E\{u\} = E\{u_1^2 + u_2^2 + \cdots + u_N^2\}$$

$$= \sum_{i=1}^{N} E\{u_i^2\} = \sum_{i=1}^{N} 1 = N$$

Mean = N

# Chi-Squared Approximation

- Mean is

$$E\{u_i^2\} = 1$$

$$E\{u\} = E\{u_1^2 + u_2^2 + \cdots + u_N^2\}$$
$$= N$$

# Chi-Squared Approximation

- **Variance** can be found as follows

$$\text{Variance} = E\{(u - E\{u\})^2\}$$

$$= E\{u^2\} - \underline{(E\{u\})^2}$$

$$= E\{u^2\} - N^2$$

## Chi-Squared Approximation

- **Variance** can be found as follows

$$\text{Variance} = E\{u^2\} - E^2\{u\}$$

$$= E\{u^2\} - N^2$$

# Chi-Squared Approximation

$$E\{u^2\} = E\left\{\left(\sum_{i=1}^{N} u_i^2\right)^2\right\}$$

$$= E\left\{\sum_{i=1}^{N} u_i^4 + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} u_i^2 u_j^2\right\}$$

$$= \sum_{i=1}^{N} E\{u_i^4\} + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} E\{u_i^2 u_j\}$$

$$E\{u_i^4\} = 3\sigma^4 = 3$$

since $u_i, u_j$ independent

$$E\{u_i^2 u_j^2\} = E\{u_i^2\}E\{u_j^2\}$$

$$= 1 \times 1 = 1$$

# Chi-Squared Approximation

$$E\{u^2\} = \sum_{i=1}^{N} 3 + 2 \sum_{i=1}^{N} \sum_{j=1}^{i-1} 1$$

Total number of combinations $i \neq j = {}^{N}C_2 = \frac{N(N-1)}{2}$

$$= 3N + \cancel{2} \cdot \frac{N(N-1)}{\cancel{2}}$$

$$= 3N + (N^2 - N)$$

$$= N^2 + 2N$$

# Chi-Squared Approximation

$$E\{u^2\} = E\left\{\left(\sum_{i=1}^{N} u_i^2\right)^2\right\}$$

$$= E\left\{\sum_{i=1}^{N} u_i^4 + \sum_{i=1}^{N}\sum_{j=1}^{i-1} 2u_i^2 u_j^2\right\}$$

# Chi-Squared Approximation

$$= \sum_{i=1}^{N} E\{u_i^4\} + \sum_{i=1}^{N} \sum_{j=1}^{i-1} 2E\{u_i^2 u_j^2\}$$

$$= \sum_{i=1}^{N} E\{u_i^4\} + \sum_{i=1}^{N} \sum_{j=1}^{i-1} 2E\{u_i^2 u_j^2\}$$

# Chi-Squared Approximation

$$= \sum_{i=1}^{N} 3 + \sum_{i=1}^{N} \sum_{j=1}^{i-1} 2 = 3N + 2 \times {}^{N}C_2$$

$$= 3N + N(N-1) = N^2 + 2N$$

## Chi-Squared Approximation

- **Variance** can be found as follows

$$\text{Variance} = E\{u^2\} - E\{u\}^2$$

$$= N^2 + 2N - N^2$$

$$= 2N = \underline{\text{Variance of } u.}$$

# Chi-Squared Approximation

- **Variance** can be found as follows

$$\text{Variance} = E\{u^2\} - E^2\{u\}$$

$$= N^2 + 2N - N^2 = 2N$$

# Chi-Squared Approximation

- **Therefore,**

$$\chi_N^2 \approx \sim \mathcal{N}(N, 2N)$$

Normal Distribution

mean

variance

$N \to \infty$

# Chi-Squared Approximation

- Therefore,

$$\chi_N^2 \approx\sim \mathcal{N}(N, 2N)$$

# Chi-Squared Approximation

- The $P_D$ can be found as

Energy detector

$$P_D = Q_{\chi^2_N}\left(\frac{\tilde{\gamma}}{\sigma_s^2+\sigma^2}\right) = \Pr\left(\chi^2_N \geq \frac{\tilde{\gamma}}{\sigma_s^2+\sigma^2}\right)$$

$$= \Pr\left(\mathcal{N}(N, 2N) \geq \frac{\tilde{\gamma}}{\sigma_s^2+\sigma^2}\right)$$

$$= Q\left(\frac{\frac{\tilde{\gamma}}{\sigma_s^2+\sigma^2} - N}{\sqrt{2N}}\right)$$

Q function of standard Gaussian RV

## Chi-Squared Approximation

- The $P_D$ can be found as

$$P_D = Q_{\chi_N^2}\left(\frac{\tilde{\gamma}}{\sigma^2 + \sigma_s^2}\right) = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2 + \sigma_s^2} - N}{\sqrt{2N}}\right)$$

# Chi-Squared Approximation

- The $P_{FA}$ can be found as

$$P_{FA} = Q_{\chi^2_N}\left(\frac{\tilde{\gamma}}{\sigma^2}\right) = \Pr\left(\chi^2_N > \frac{\tilde{\gamma}}{\sigma^2}\right)$$

$$= \Pr\left(\mathcal{N}(N, 2N) > \frac{\tilde{\gamma}}{\sigma^2}\right) \quad \text{For large } N.$$

$$= Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2} - N}{\sqrt{2N}}\right) = P_{FA}$$

## Chi-Squared Approximation

- The $P_{FA}$ can be found as

$$P_{FA} = Q_{\chi_N^2}\left(\frac{\tilde{\gamma}}{\sigma^2}\right) = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2} - N}{\sqrt{2N}}\right)$$

# Chi-Squared Approximation

- ROC found as

$$P_{FA} = Q\left(\frac{\tilde{\gamma}/\sigma^2 - N}{\sqrt{2N}}\right)$$

$$\Rightarrow \tilde{\gamma} = \left(\sqrt{2N}\,Q^{-1}(P_{FA}) + N\right)\sigma^2$$

# Chi-Squared Approximation

- ROC found as

$$P_D = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma_s^2+\sigma^2} - N}{\sqrt{2N}}\right) = Q\left(\frac{\frac{\sigma^2}{\sigma_s^2+\sigma^2}\left(\sqrt{2N}\,Q^{-1}(P_{FA})+N\right)-N}{\sqrt{2N}}\right)$$

$$\underbrace{P_D = Q\left(\frac{\sigma^2}{\sigma_s^2+\sigma^2}Q^{-1}(P_{FA}) - \frac{\sigma_s^2}{\sigma_s^2+\sigma^2}\cdot\sqrt{\frac{N}{2}}\right)}_{\text{Receiver Operating characteristic ROC}}$$

For large N

$$\chi_N^2 \rightarrow \underline{\mathcal{N}(N, 2N)}$$

# Chi-Squared Approximation

- ROC found as

$$P_D = Q\left(\frac{\frac{1}{\sigma^2+\sigma_s^2}\left(\sigma^2\left(\sqrt{2N}Q^{-1}(P_{FA})+N\right)\right)-N}{\sqrt{2N}}\right)$$

$$P_D = Q\left(\frac{\sigma^2}{\sigma^2+\sigma_s^2}Q^{-1}(P_{FA})-\frac{\sigma_s^2}{\sigma^2+\sigma_s^2}\sqrt{\frac{N}{2}}\right)$$

Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.