EE910: Digital Communication Systems-I

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

May 9, 2022

Lecture #5B: Waveform and vector AWGN channels

Equivalence of Waveform and Vector AWGN channels

- The waveform AWGN channel is described by the input-output relation

$$r(t) = s_m(t) + n(t) \quad (1)$$

  where $s_m(t)$ is one of the possible M signals and n(t) is a zero-mean white Gaussian process with power spectral density $N_0/2$.
- Using the Gram-Schmidt procedure, we can derive an orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$ for representation of the signals.

Equivalence of Waveform and Vector AWGN channels

- The noise process cannot be completely expanded in terms of the basis $\{\phi_j(t)\}_{j=1}^N$.
- One component, denoted by $n_1(t)$ is part of the noise process that can be expanded in terms of $\{\phi_j(t)\}_{j=1}^N$ and the the other part, denoted by $n_2(t)$, is the part that cannot be expressed in terms of this basis function.
- Thus, we can write $n_1(t) = \sum_{j=1}^N n_j\phi_j(t)$, where $n_j = \langle n(t), \phi_j(t)\rangle$ and $n_2(t) = n(t) - n_1(t)$.

Equivalence of Waveform and Vector AWGN channels

- $s_m(t)$ can be written as $s_m(t) = \sum_{j=1}^{N} s_{mj}\phi_j(t)$, where $s_{mj} = \langle s_m(t), \phi_j(t)\rangle$.
- Thus (1) can be written as $r(t) = \sum_{j=1}^{N}(s_{mj} + n_j)\phi_j(t) + n_2(t)$.
- We define $r_j = s_{mj} + n_j$ where, $r_j = \langle s_m(t), \phi_j(t)\rangle + \langle n(t), \phi_j(t)\rangle = \langle r(t), \phi_j(t)\rangle$.
- Thus we have $r(t) = \sum_{j=1}^{N} r_j\phi_j(t) + n_2(t)$.

Equivalence of Waveform and Vector AWGN channels

- n_j is defined by $n_j = \int_{-\infty}^{\infty} n(t)\phi_j(t)dt$.
- The mean of n_j is given as,

$$E[n_j] = E\left[\int_{-\infty}^{\infty} n(t)\phi_j(t)dt\right] = \int_{-\infty}^{\infty} E[n(t)]\phi_j(t)dt = 0, \qquad (2)$$

where the last equality holds since $n(t)$ is zero-mean.

Equivalence of Waveform and Vector AWGN channels

- The covariance of n_i and n_j is,

$$
\begin{aligned}
COV[n_i n_j] &= E[n_i n_j] - E[n_i]E[n_j] \\
&= E\left[\int_{-\infty}^{\infty} n(t)\phi_i(t)dt \int_{-\infty}^{\infty} n(s)\phi_j(s)ds\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[n(t)n(s)]\phi_i(t)\phi_j(s)dtds \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\delta(t-s)\phi_i(t)dt\right]\phi_j(s)ds \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}\phi_i(s)\phi_j(s)ds \\
&= \begin{cases} \frac{N_0}{2}, & i=j \\ 0, & i \neq j \end{cases} \qquad (3)
\end{aligned}
$$

Equivalence of Waveform and Vector AWGN channels

- $n_2(t) = n(t) - n_1(t)$, which is a linear combination of two jointly Gaussian processes, is itself a Gaussian process.
- The covariance at any given t is,

$$
\begin{aligned}
COV[n_j n_2(t)] &= E[n_j n_2(t)] \\
&= E[n_j n(t)] - E[n_j n_1(t)] \\
&= E\left[n(t)\int_{-\infty}^{\infty} n(s)\phi_j(s)ds\right] - E\left[n_j \sum_{i=1}^{N} n_i\phi_i(t)\right] \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}\delta(t-s)\phi_j(s)ds - \frac{N_0}{2}\phi_j(t) \\
&= \frac{N_0}{2}\phi_j(t) - \frac{N_0}{2}\phi_j(t) \\
&= 0 \qquad (4)
\end{aligned}
$$

- The AWGN waveform channel of the form $r(t) = s_m(t) + n(t)$, $1 \leq m \leq M$ is equivalent to the N- dimensional vector channel $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$, $1 \leq m \leq M$.

Optimal Detection for the Vector AWGN Channel

- The MAP detector for the AWGN vector channel is given by

$$\begin{aligned}
\hat{m} &= \arg\max_{1\leq m\leq M} \left[P_m p(\mathbf{r}|\mathbf{s}_m)\right] \\
&= \arg\max_{1\leq m\leq M} P_m[p_n(\mathbf{r}-s_m)] \\
&= \arg\max_{1\leq m\leq M} \left[P_m\left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}}\right] \\
&= \arg\max_{1\leq m\leq M} \left[P_m e^{-\frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}}\right] \\
&= \arg\max_{1\leq m\leq M} \left[\ln P_m - \frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}\right] \\
&= \arg\max_{1\leq m\leq M} \left[\frac{N_0}{2}\ln P_m - \frac{1}{2}\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\
&= \arg\max_{1\leq m\leq M} \left[\frac{N_0}{2}\ln P_m - \frac{1}{2}(\|\mathbf{r}\|^2 + \|\mathbf{s}_m\|^2 - 2\mathbf{r}.\mathbf{s}_m)\right] \qquad (5)
\end{aligned}$$

Optimal Detection for the Vector AWGN Channel

-

$$\begin{aligned}
&\overset{(a)}{=} \arg\max_{1\leq m\leq M} \left[\frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m + \mathbf{r}.\mathbf{s}_m\right] \\
&\overset{(b)}{=} \arg\max_{1\leq m\leq M} \left[\eta_m + \mathbf{r}.\mathbf{s}_m\right] \qquad (6)
\end{aligned}$$

where we have used the following steps in simplifying the expression: (a): $\|\mathbf{s}_m\|^2 = \mathcal{E}_m$ and (b): $\eta_m = \frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m$ as the bias term.

Optimal Detection for the Vector AWGN Channel

- In the special case where the signals are equiprobable, i.e., $P_m = 1/M$ for all m,

$$\begin{aligned}\hat{m} &= \arg\max_{1\leq m\leq M}\left[\frac{N_0}{2}\ln P_m - \frac{1}{2}\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\ &= \arg\max_{1\leq m\leq M}\left[-\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\ &= \arg\min_{1\leq m\leq M}\|\mathbf{r}-\mathbf{s}_m\| \qquad (7)\end{aligned}$$

- When the signals are both equiprobable and have equal energy, the bias terms defined as $\eta_m = \frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m$ are independent of m and can be dropped. The optimal detection rule in this case reduces to

$$\hat{m} = \arg\max_{1\leq m\leq M}\mathbf{r}\,.\,\mathbf{s}_m \qquad (8)$$

Optimal Detection for the Vector AWGN Channel

- The decision region D_m is given as

$$\mathbf{D}_m = \{\mathbf{r}\in\mathcal{R}^N : \mathbf{r}.\mathbf{s}_m + \eta_m > \mathbf{r}.\mathbf{s}_{m'} + \eta_{m'}, \text{for all } 1\leq m'\leq M \text{ and } m'\neq m\}$$

- The boundaries of the decision regions in general are hyperplanes and are of the form $\mathbf{r}.(\mathbf{s}_m - \mathbf{s}_{m'}) \geq \eta_{m'} - \eta_m$.
- The optimal MAP detection rule in an AWGN channel can be written in the form

$$\hat{m} = \arg\max_{1\leq m\leq M}\left[\frac{N_0}{2}\ln P_m + \int_{-\infty}^{\infty} r(t)s_m(t)dt - \frac{1}{2}\int_{-\infty}^{\infty} s_m^2(t)dt\right] \qquad (9)$$

where we have used the relation $\mathbf{r}.\mathbf{s}_m = \int_{-\infty}^{\infty} r(t)s_m(t)dt$ and $\mathcal{E} = \|\mathbf{s}\|^2 = \int_{-\infty}^{\infty} s_m^2(t)dt$.

Optimal Detection for the Vector AWGN Channel

- The ML detector has the following form

$$\hat{m} = \underset{1 \le m \le M}{\arg\max} \left[\int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} s_m^2(t) dt \right] \tag{10}$$

- The distance metric is defined as

$$D(\mathbf{r}, \mathbf{s_m}) = \|\mathbf{r} - \mathbf{s}_m\|^2 = \int_{-\infty}^{\infty} (r(t) - s_m(t))^2 dt \tag{11}$$

- Modified distance metric is defined as $D'(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r}.\mathbf{s}_m + \|s_m\|^2$

Optimal Detection for the Vector AWGN Channel

- The correlation metric is defined as the negative of the modified distance metric and is denoted by $C(\mathbf{r}, \mathbf{s}_m)$.
- With these definitions the optimal detection rule (MAP rule) in general can be written as

$$\hat{m} = \underset{1 \le m \le M}{\arg\max} \left[N_0 \ln P_m - D(\mathbf{r}, \mathbf{s}_m) \right] = \underset{1 \le m \le M}{\arg\max} \left[N_0 \ln P_m + C(\mathbf{r}, \mathbf{s}_m) \right]$$

- The ML detection rule becomes $\hat{m} = \underset{1 \le m \le M}{\arg\max}\, C(\mathbf{r}, \mathbf{s}_m)$