Solutions of Tutorial-6

Problem set 6.1

4 $\det(A-\lambda I) = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$. Then A has $\lambda_1=-3$ and $\lambda_2=2$ (check trace $=-1$ and determinant $=-6$) with $\boldsymbol{x}_1=(3,-2)$ and $\boldsymbol{x}_2=(1,1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2=9$ and $\lambda_2^2=4$.

6 A and B have $\lambda_1=1$ and $\lambda_2=1$. AB and BA have $\lambda^2-4\lambda+1$ and the quadratic formula gives $\lambda=2\pm\sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved at the end of Section 6.2).

9 (a) Multiply by A: $A(A\boldsymbol{x})=A(\lambda\boldsymbol{x})=\lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x}=\boldsymbol{\lambda}^2\boldsymbol{x}$

(b) Multiply by A^{-1}: $\boldsymbol{x}=A^{-1}A\boldsymbol{x}=A^{-1}\lambda\boldsymbol{x}=\lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x}=\frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$

(c) Add $I\boldsymbol{x}=\boldsymbol{x}$: $(A+I)\boldsymbol{x}=(\boldsymbol{\lambda}+\mathbf{1})\boldsymbol{x}$.

13 (a) $P\boldsymbol{u}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u})=\boldsymbol{u}$ so $\lambda=1$ (b) $P\boldsymbol{v}=(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v}=\boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})=\mathbf{0}$

(c) $\boldsymbol{x}_1=(-1,1,0,0)$, $\boldsymbol{x}_2=(-3,0,1,0)$, $\boldsymbol{x}_3=(-5,0,0,1)$ all have $P\boldsymbol{x}=0\boldsymbol{x}=\mathbf{0}$.

19 (a) rank $=2$ (b) $\det(B^{\mathrm{T}}B)=0$ (d) eigenvalues of $(B^2+I)^{-1}$ are $1,\frac{1}{2},\frac{1}{5}$.

32 (a) $\boldsymbol{u}$ is a basis for the nullspace (we know $A\boldsymbol{u}=0\boldsymbol{u}$); $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space (we know $A\boldsymbol{v}$ and $A\boldsymbol{w}$ are in the column space).

(b) $A(\boldsymbol{v}/3+\boldsymbol{w}/5)=3\boldsymbol{v}/3+5\boldsymbol{w}/5=\boldsymbol{v}+\boldsymbol{w}$. So $\boldsymbol{x}=\boldsymbol{v}/3+\boldsymbol{w}/5$ is a particular solution to $A\boldsymbol{x}=\boldsymbol{v}+\boldsymbol{w}$. Add any $c\boldsymbol{u}$ from the nullspace

(c) If $A\boldsymbol{x}=\boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

Problem set 6.2

11 (a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda=2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 (a) False: don't know if $\lambda=0$ or not.

(b) True: an eigenvector is missing, which can only happen for a repeated eigenvalue.

(c) True: We know there is only one line of eigenvectors.

23 If $A=X\Lambda X^{-1}$ then $B=\begin{bmatrix}A & 0\\ 0 & 2A\end{bmatrix}=\begin{bmatrix}X & 0\\ 0 & X\end{bmatrix}\begin{bmatrix}\Lambda & 0\\ 0 & 2\Lambda\end{bmatrix}\begin{bmatrix}X^{-1} & 0\\ 0 & X^{-1}\end{bmatrix}$. So B has the original λ's from A and the additional eigenvalues $2\lambda_1,\ldots,2\lambda_n$ from $2A$.

27 $R = X\sqrt{\Lambda}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$.

$\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace (their sum) is not real so $\sqrt{B}$ cannot be real. Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has *two* imaginary eigenvalues $\sqrt{-1} = i$ and $-i$, real trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Problem set 6.4

11 If λ is complex then $\overline{\lambda}$ is also an eigenvalue $(A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}})$. Always $\lambda + \overline{\lambda}$ is real. The trace is real so the third eigenvalue of a 3 by 3 real matrix must be real.

12 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* always real. Can't assume real eigenvectors!

23 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$ (c) True from $S^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

Problem set 6.5

2 Only $S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues since $101 > 10^2$.

$\boldsymbol{x}^{\mathrm{T}}S_1\boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms; S_2 has trace c_0; S_3 has det $= 0$.

4 $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$; $x^2 + 6xy + 9y^2 = (x + 3y)^2$.

14 The eigenvalues of S^{-1} are positive because they are $1/\lambda(S)$. Also the entries of S^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}}S^{-1}\boldsymbol{x} = (S^{-1}\boldsymbol{x})^{\mathrm{T}}S(S^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$.

15 Since $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$ and $\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0$ we have $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$. Then $S + T$ is a positive definite matrix. The second proof uses the test $S = A^{\mathrm{T}}A$ (independent columns in A): If $S = A^{\mathrm{T}}A$ and $T = B^{\mathrm{T}}B$ pass this test, then $S + T = \begin{bmatrix} A & B \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A \\ B \end{bmatrix}$ also passes, and must be positive definite.

16 $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1, -10, 0)$ because the second pivot is *negative*.

18 If $S\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. If S is positive definite this leads to $\lambda = \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} > 0$ (ratio of positive numbers). So positive energy $\Rightarrow$ positive eigenvalues.

28 $\det S = (1)(10)(1) = 10$; $\lambda = 2$ and 5; $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$, $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; the λ's are positive. So S is positive definite.