Live Interaction #2:

8th October 2023

E-masters Communication Systems

Estimation for Wireless

- Channel estimation:

$$y(k) = hx(k) + v(k)$$

- Problem:

$$\begin{aligned} y(1) &= hx(1) + v(1) \\ y(2) &= hx(2) + v(2) \\ &\vdots \\ y(N) &= hx(N) + v(N) \end{aligned}$$

- Problem can be formulated as

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{\mathbf{y}}} = h \underbrace{\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}}_{\bar{\mathbf{x}}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{\mathbf{v}}}$$

$$\bar{\mathbf{y}} = \bar{\mathbf{x}}h + \bar{\mathbf{v}}$$

- **ML (Maximum Likelihood)**:

$$\hat{h} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{||\bar{\mathbf{x}}||^2}$$

- For complex quantities, we have to use

$$\hat{h} = \frac{\bar{\mathbf{x}}^H \bar{\mathbf{y}}}{||\bar{\mathbf{x}}||^2}$$

- Properties of $\hat{h}$

$$E\{\hat{h}\} = E\left\{\frac{\bar{\mathbf{x}}^T\bar{\mathbf{y}}}{\|\bar{\mathbf{x}}\|^2}\right\} = E\left\{\frac{\bar{\mathbf{x}}^T(\bar{\mathbf{x}}h + \bar{\mathbf{v}})}{\|\bar{\mathbf{x}}\|^2}\right\}$$

$$= E\left\{\frac{\|\bar{\mathbf{x}}\|^2 h + \bar{\mathbf{x}}^T\bar{\mathbf{v}}}{\|\bar{\mathbf{x}}\|^2}\right\}$$

$$= E\left\{h + \frac{\bar{\mathbf{x}}^T\bar{\mathbf{v}}}{\|\bar{\mathbf{x}}\|^2}\right\}$$

$$= h + \frac{\bar{\mathbf{x}}^T}{\|\bar{\mathbf{x}}\|^2}\underbrace{E\{\bar{\mathbf{v}}\}}_{0} = h$$

$$\underbrace{E\{\hat{h}\} = h}_{\text{Unbiased Estimator}}$$

- Mean square error (MSE)

$$E\left\{\left(\hat{h} - h\right)^2\right\} = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

- **Cramer-Rao Bound (CRB)**:
- **Cramer-Rao Lower Bound (CRLB)**:
- Fundamental lower bound on the **MSE of an unbiased estimator**.
- Legend:

- Calyampudi Radhakrishna Rao
- **Indian American Stastician**
- He is our **legend**.

- **Harald Cramér**:
- CRB is applicable only for **unbiased estimator**.

$$E\{\hat{h}\} = h$$

$$E\left\{\left(\hat{h} - h\right)^2\right\} \geq \frac{1}{I(h)}$$

- $I(h)$: **Fisher information**.

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}; h)\right)^2\right\}$$

- Example:

$$p(\bar{\mathbf{y}}; h) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2}$$

$$\ln p(\bar{\mathbf{y}}; h)$$

$$= -\frac{N}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k) - hx(k)\right)^2$$

$$\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}; h) = 0 - \frac{1}{2\sigma^2}\sum_{k=1}^{N} 2\left(-x(k)\right)\left(\left(y(k) - hx(k)\right)\right)$$

$$\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}};h) = \frac{1}{\sigma^2}\sum_{k=1}^{N} x(k)v(k)$$

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}};h)\right)^2\right\} = \frac{1}{\sigma^4}E\left\{\left(\sum_{k=1}^{N} x(k)v(k)\right)^2\right\}$$

$$= \frac{1}{\sigma^4}\sum_{k=1}^{N}\sigma^2 x^2(k) = \frac{1}{\sigma^2}\sum_{k=1}^{N} x^2(k)$$

$$= \frac{1}{\sigma^2}\|\bar{\mathbf{x}}\|^2$$

$$E\left\{\left(\hat{h}-h\right)^2\right\} \geq \frac{1}{I(h)} = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

- Such an estimator which achieves the CRB is termed an **efficient estimator**.
- **Assignment #2 deadline: Saturday 14th October 11:59 AM**.
- **Assignment discussion: Saturday 14th October 1:00 PM**.
- **Quiz #1: Sunday 15th October 12:30 to 1:10 PM**.
- **Live interaction Monday 16th October 9:00 – 10:00 PM.**