# Null Spaces and Solution of Linear Equations

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Applied Linear Algebra for Wireless Communications

# Recap and agenda for today's class

- Discussed the following in the last lecture
  - Vector spaces and subspaces
  - Null space of matrix A i.e., $N(A)$ and column space

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  - $N(A)$ and solution of $A\mathbf{x} = \mathbf{b}$

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- Case of a zero nullspace is of the greatest importance
  - It says that All columns have pivots, and they are independent
  - No combination of columns gives zero vector (except zero combination)

# Pivot Columns, Free Columns And Special Solutions

- We will find $N(C)$ where $C = [A\ 2A]$

**Subtract 3 (row 1) from row 2 of $C$**

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{array}{cccc} \begin{bmatrix} \mathbf{1} & 2 & 2 & 4 \\ 0 & \mathbf{2} & 0 & 4 \end{bmatrix} \\ \uparrow\ \uparrow\ \uparrow\ \uparrow \\ \textbf{pivot columns} \quad \textbf{free columns} \end{array}$$

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**Special solutions** $C s = \mathbf{0}$, $U s = \mathbf{0}$

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \textbf{pivot} \\ \leftarrow \quad \textbf{variables} \\ \leftarrow \textbf{free} \\ \leftarrow \quad \textbf{variables} \end{array}$$

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| to be revealed by $R$ | 3 pivots: rank $r = 3$ | $Rs = \mathbf{0}$ means $As = \mathbf{0}$ |

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- Let $A$ has more columns than rows ($n > m$), which is a short wide matrix
- Must have least $n - m$ free variables, since number of pivots cannot exceed $m$
  - $A$ only has $m$ rows, and a row never has two pivots

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$$\boldsymbol{A} = \begin{array}{ccccc} p & p & f & p & f \end{array} \qquad \boldsymbol{R} = \begin{bmatrix} \mathbf{1} & 0 & a & 0 & c \\ 0 & \mathbf{1} & b & 0 & d \\ 0 & 0 & 0 & \mathbf{1} & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{s}_1 = \begin{bmatrix} -a \\ -b \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} \qquad \boldsymbol{s}_2 = \begin{bmatrix} -c \\ -d \\ \mathbf{0} \\ -e \\ \mathbf{1} \end{bmatrix}$$

3 pivot columns $p$
2 free columns $f$
to be revealed by $R$

$I$ in pivot columns
$F$ in free columns
3 pivots: rank $r = 3$

special $R\boldsymbol{s}_1 = \mathbf{0}$ and $R\boldsymbol{s}_2 = \mathbf{0}$
take $-a$ to $-e$ from $R$
$R\boldsymbol{s} = \mathbf{0}$ means $A\boldsymbol{s} = \mathbf{0}$

- Let $A$ has more columns than rows ($n > m$), which is a short wide matrix
- Must have least $n - m$ free variables, since number of pivots cannot exceed $m$
  - $A$ only has $m$ rows, and a row never has two pivots
  - Of course a row might have no pivot which means an extra free variable

# Dimensionality of nullspace

- Counting the pivots leads to an extremely important theorem

$$A = \begin{bmatrix} p & p & f & p & f \\ | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & 0 & a & 0 & c \\ 0 & \mathbf{1} & b & 0 & d \\ 0 & 0 & 0 & \mathbf{1} & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad s_1 = \begin{bmatrix} -a \\ -b \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} \qquad s_2 = \begin{bmatrix} -c \\ -d \\ \mathbf{0} \\ -e \\ \mathbf{1} \end{bmatrix}$$

| | | |
|---|---|---|
| 3 pivot columns $p$ | $I$ in pivot columns | special $Rs_1 = \mathbf{0}$ and $Rs_2 = \mathbf{0}$ |
| 2 free columns $f$ | $F$ in free columns | take $-a$ to $-e$ from $R$ |
| to be revealed by $R$ | 3 pivots: rank $r = 3$ | $Rs = \mathbf{0}$ means $As = \mathbf{0}$ |

- Let $A$ has more columns than rows ($n > m$), which is a short wide matrix
- Must have least $n - m$ free variables, since number of pivots cannot exceed $m$
  - $A$ only has $m$ rows, and a row never has two pivots
  - Of course a row might have no pivot which means an extra free variable
- Nullspace is a subspace. Its "dimension" is the number of free variables