Semidifinite Programs: LMI form

$$\min \ c^T x$$

$$F(\underline{x}) := G + F_1 x_1 + F_2 x_2 + \cdots + F_n x_n \preceq 0$$

$$Ax = b$$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$G, F_i \in S^m$$

Linear matrix inequality (LMI)

Eg : $[G]_{jj} = g_j \qquad [F_i]_{jj} = f_{ij}$

G, F_i diagonal

$$F(\underline{x}) = \begin{bmatrix} g_1 + \sum_{i=1}^n f_{i1} x_i & 0 & 0 & \cdots & 0 \\ & g_2 + \sum_{i=1}^n f_{i2} x_i & & & \\ & & \ddots & & \\ & & & & g_m + \sum_{i=1}^n f_{im} x_i \end{bmatrix}$$

$$F(\underline{x}) \preceq 0 \iff g_j + \sum_{i=1}^m f_{ij} x_i \leq 0 \qquad \text{(LP)}$$

Is SDP convex?

$$F(x) \preceq 0 \iff \lambda_i(F(x)) \leq 0$$

or $\lambda_{max}(F(x)) \leq 0$

$\hookrightarrow$ max. eigenvalue

Claim: $\lambda_{max}(F(\underline{x}))$ convex in x for $G, F_i \in S^m$

note: $$\lambda_{max}(A) = \max_{\|y\| \leq 1} y^T A y$$

$$y^T F(x) y = y^T G y + \sum_{i=1}^{n} (y^T F_i y) x_i$$

affine function of x

$$\lambda_{max}(F(\underline{x})) = \max_{\|y\|=1} y^T F(\underline{x}) y$$

(pointwise max. of affine)

$\Rightarrow$ $\lambda_{max}(F(\underline{x}))$ convex in $\underline{x}$

$\Rightarrow$ SDP are convex.

<u>Note</u>: $\lambda_{min}(F(x))$ is concave in x $\Rightarrow$ $F(x) \succeq 0$ also valid

note $F(x) = G + \sum F_i x_i \preceq 0$

$-F(x) = -G - \sum F_i x_i = H(x) \succeq 0$

both valid