EE908 Assignment-6 Solution

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eMasters in Communication Systems, IITK
EE908: Optimization in SPCOM
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Q1. Express the following problem as an SOCP

$\min c^T x$
$s.t. x^T x \le yz$
$y^2 + z^2 \le 1$
$y \ge 0, z \ge 0$
where $x \in \mathbb{R}^n$, and $y, z \in \mathbb{R}$

Solution:
SOCP standard form:
$\min f^T x$
$s.t. \|A_i x - b_i\| \le c_i^T x + d_i, i = 1,2, \dots m$
$Fx = g$

First constraint:
$x^T x \le yz \Rightarrow \|x\|_2^2 \le yz \Rightarrow \|x\|_2 \le \sqrt{yz}$
Say $t = \sqrt{yz}$
$\|x\|_2 \le t$

Second constraint:
$y^2 + z^2 \le 1$ is already resembling like a norm constraint

∴ SOCP form is:
$\min c^T x$
$s.t. \|x\|_2 \le t$
$\|y, z\|_2 \le 1 , y \ge 0, z \ge 0$

Q2. Formulate the following problem as SOCP:

(a) $\max \left(\sum_{i=1}^{m} \frac{1}{a_i^T x - b_i}\right)^{-1} \quad s.t. (a_i^T x - b_i) \ge 0$

Solution:

Say $t = \left(\sum_{i=1}^{m} \frac{1}{a_i^T x - b_i}\right)^{-1} \Rightarrow \sum_{i=1}^{m} \frac{1}{a_i^T x - b_i} = \frac{1}{t}$

Let's define $z_i = \frac{1}{a_i^T x - b_i} \Rightarrow z_i(a_i^T - b_i) = 1$

This condition can be written as a SOC constraint:

$$\left\| \begin{bmatrix} z_i \\ a_i^T x - b_i \end{bmatrix} \right\| \le z_i + a_i^T x - b_i$$

Then the problem becomes:
$\max t$
$s.t. \left\| \begin{bmatrix} z_i \\ a_i^T x - b_i \end{bmatrix} \right\| \le z_i + a_i^T x - b_i, i = 1,2, \dots m$

"... dare to dream; care to win ..."

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(b) $\min t \; s.t. \frac{1}{t} \le \frac{a_i^T x}{b_i} \le t$ over $x \in \mathbb{R}$ *and* $t \in \mathbb{R}$

Solution:

Let's define $z_i = \frac{a_i^T x}{b_i} \Rightarrow b_i z_i = a_i^T x$

Then the constraint becomes:

$\left\| \begin{bmatrix} z_i \\ b_i \end{bmatrix} \right\| \le t$

Then the problems becomes:

$\min t$

$s.t. \left\| \begin{bmatrix} z_i \\ b_i \end{bmatrix} \right\| \le t$

This is an SOCP form

Q3. Solve the least-norm problem

$\min \|x\|_2 \;\; s.t. Ax = b$ where $A \in \mathbb{R}^{m \times n}$ with $m < n$ and $b \in \mathcal{R}(A)$

Solution:

Since b is in the range/column space of A and m<n (wide matrix), the system of equations are over-determined – more equations than unknowns and the A will have a pseudo-inverse.

$$\therefore x = (A^T A)^{-1} A^T b$$

Therefore, the min $\|x\|_2 = \|(A^T A)^{-1} A^T b\|_2$

Q4. Solve the following regularized least-square problem

$$\min \|Ax - b\|_2^2 + \lambda \|x\|_2^2$$

Where the regularized parameter $\lambda > 0$. Express the solution such that no assumptions are needed on the rank of matrix A. Comment on the solution for the cases $0 < \lambda \ll 1$ and $\lambda \gg 1$

Solution:

Let's differentiate the objective function w.r.t x and set it to zero to solve for x.

$\nabla \|Ax - b\|_2^2 + \lambda \|x\|_2^2 = 2A^T(Ax - b) + 2\lambda x = 0$

$A^T Ax - A^T b + \lambda x = 0 \Rightarrow A^T Ax + \lambda x = A^T b \Rightarrow (A^T A + \lambda I) = A^T b$

$\Rightarrow x = (A^T A + \lambda I)^{-1} A^T b$

Case $0 < \lambda \ll 1$:

Regularization term has little effect on the solution. Solution is similar to ordinary LS problem and solution vector x may have large components

Case $\lambda \gg 1$

Regularization term dominates the objective function. Solution vector will be closer to zero.

Q5. Consider the following robust optimization problem

$$\min c^T x \;\; s.t. Ax \le b \;\; \forall A \in \mathcal{A}$$

Where $\mathcal{A} = \{A \in \mathbb{R}^{m \times n} \mid \bar{A}_{ij} - V_{ij} \le A_{ij} \le \bar{A}_{ij} + V_{ij} \; \forall \; i, j\}$. This problem can be interpreted as an LP with infinite number of constraints, one for each value that A_{ij} can take. In other words, the solution x must satisfy the constraints for all possible values of A_{ij}.

(a) Show that in the constraint

$\sum_j A_{ij} x_j \le b_i \;\; \forall \bar{A}_{ij} - V_{ij} \le A_{ij} \le \bar{A}_{ij} + V_{ij}$ can equivalently be written as

$$\sum_j \bar{A}_{ij} x_j + \sum V_{ij} |x_j| \le b_i$$

Solution:

Lower bound of the constraint $\Rightarrow \sum_j (\bar{A}_{ij} - V_{ij}) x_j \le b_i$

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$$\sum_j \bar{A}_{ij}x_j - \sum_j V_{ij}x_j \leq b_i$$

Upper bound

$$\sum_j (\bar{A}_{ij} + V_{ij})x_j \leq b_i \Rightarrow \sum_j \bar{A}_{ij}x_j + \sum_j V_{ij}x_j \leq b_i$$

Combining both

$$\sum_j \bar{A}_{ij}x_j + \sum_j V_{ij}|x_j| \leq b_i$$

QED

(b) Express the robust problem as an LP

Solution:

Say $y_{ij} = A_{ij} - \bar{A}_{ij}$

Then the constraint becomes:

$$\sum_j (\bar{A}_{ij} + y_{ij})x_j \leq b_j \text{ }, \forall y_{ij} \in [-V_{ij}, V_{ij}] \ \forall i$$

Rewriting the objective function and constraints in terms of x and y

$\min c^T x$

$$s.t. \sum_j (\bar{A}_{ij} + y_{ij})x_j < b_i \ \ \forall y_{ij} \in [-V_{ij}, V_{ij}] \ \forall i \ and \ y_{ij} \in [-V_{ij}, V_{ij}] \ \forall i, j$$

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