$$f(x_1, x_2) = x_1/x_2$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

trace > 0 det < 0

$+$ $-$

neither convex nor concave

A4 1. $x^\theta y^{1-\theta} \le \theta x + (1-\theta) y$

$\log(x)$

$$\log(\theta x + (1-\theta) y) \geq \theta \log(x) + (1-\theta) \log y$$

$$\theta x + (1-\theta) y \geq x^\theta y^{1-\theta}$$

2. $f(\underline{x}) = \dfrac{1}{\sum_{i=1}^{n} \frac{1}{x_i}}$ $w = \sum_{i=1}^{n} 1/x_i$

$f(\underline{x}) = \frac{1}{w}$

$$\left[\nabla^2 f(\underline{x})\right]_{ii} = \frac{2(1 - x_i w)}{x_i^4 w^3}$$

$$\left[\nabla^2 f(\underline{x})\right]_{ij} = \frac{2}{x_i^2 x_j^2 w^3}$$

$$f(\underline{x}) = \frac{1}{w}$$

$$\frac{d}{dx_i}\left(f(x)\right) = \frac{d}{dw}\left(\frac{1}{w}\right)\frac{dw}{dx_i}$$

$$= -\frac{1}{w^2}\frac{1}{x_i^2}$$

$$\frac{d}{dx_j}\left(-\frac{1}{w^2}\frac{1}{x_i^2}\right) = -\frac{d}{dw}\left(\frac{1}{w^2}\right)\frac{1}{x_i^2}\frac{dw}{dx_j}$$

$$= \frac{2}{w^3}\frac{1}{x_i^2}\frac{1}{x_j^2}$$

$$-\frac{d}{dx_i}\left(\frac{1}{w^2}\frac{1}{x_i^2}\right) = -\frac{d}{dw}\left(\frac{1}{w^2}\right)\frac{dw}{dx_i}\frac{1}{x_i^2}$$

$$- \frac{d}{dx_i}\left(\frac{1}{x_i^2}\right)\frac{1}{w^2}$$

$$= -\frac{2}{w^3}\frac{1}{x_i^4} + \frac{2}{x_i^3}\frac{1}{w^2}$$

$$\underline{u}^T \nabla^2 f(\underline{x})\, \underline{u} = \sum u_i^2 \left[\nabla f(\underline{x})\right]_{ii} + \sum_{\substack{i,j \\ i \neq j}} u_i u_j \left[\nabla f\right]_{ij}$$

$$= \sum_{i=1}^{n} \frac{2u_i^2 (1 - x_i w)}{x_i^4 w^3} + \underbrace{\sum_{i \neq j} \frac{2u_i u_j}{x_i^2 x_j^2 w^3}}$$

$$\boxed{\sum_{i,j}} \frac{2u_i u_j}{x_i^2 x_j^2 w^3} - \underline{\sum_{i=1}^{n} \frac{2u_i^2}{x_i^3 w^2}}$$

$$\frac{2u_i^2}{x_i^4 w^3} \qquad \sum_{i=1}^{n} \sum_{j=1}^{n}$$

$$\frac{2}{w^3} \left[\underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{u_i u_j}{x_i^2 x_j^2}} - w \sum_{i=1}^{n} \frac{u_i^2}{x_i^3} \right]$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j = \underline{\left(\sum_{i=1}^{n} a_i \right)^2}$$

$$(a_1 + a_2)^2 = a_1^2 + a_2^2 + a_1 a_2 + a_2 a_1$$

$$\frac{2}{w^3} \left[\left(\sum_{i=1}^{n} \frac{u_i}{x_i^2} \right)^2 - w \sum_{i=1}^{n} \frac{u_i^2}{x_i^3} \right] \leq 0$$

$$\left(\sum a_i b_i\right)^2 \qquad \left(\sum a_i\right) \quad \left(\sum b_i\right)$$

$$(a^T b)^2 \le (a^T a)(b^T b)$$

$$a_i b_i = \frac{u_i}{x_i^2} \qquad a_i^2 = \frac{1}{x_i}$$

$$b_i^2 = \frac{u_i^2}{x_i^3}$$

$$a_i = \frac{1}{\sqrt{x_i}} \qquad b_i = \frac{u_i}{x_i\sqrt{x_i}}$$

$$\Rightarrow \quad \nabla^2 f(\underline{x}) \le 0 \qquad f \text{ concave}$$

3. $f(\lambda_1 x_1 - \lambda_2 x_2 - \dots - \lambda_n x_n)$

$$\ge \lambda_1 f(x_1) - \lambda_2 f(x_2) \dots - \lambda_n f(x_n)$$

$\lambda_i > 0$

$$\lambda_1 - \sum_{i=1}^{n} \lambda_i = 1 \Rightarrow \lambda_1 = 1 + \sum_{i=2}^{n} \lambda_i$$

convexity $\Rightarrow$ $f\left(\sum_{i=1}^{n} \theta_i x_i\right) \le \sum_{i=1}^{n} \theta_i f(x_i)$

$\underline{n=2}$

$$f(\theta_1 y_1 + \theta_2 y_2) \le \theta_1 f(y_1) + \theta_2 f(y_2)$$

$$\longrightarrow f(\lambda_1 x_1 - \lambda_2 x_2) \ge \lambda_1 f(x_1) - \lambda_2 \underline{f}(x_2)$$

$$\lambda_1 f(x_1) \le f(\lambda_1 x_1 - \lambda_2 x_2) + \lambda_2 f(x_2)$$

$$f(\underbrace{x_1}_{\theta_1 y_1+\theta_2 y_2}) \le \frac{1}{\lambda_1} f(\underbrace{\lambda_1 x_1 - \lambda_2 x_2}_{y_1}) + \frac{\lambda_2}{\lambda_1} f(\underbrace{x_2}_{y_2})$$

$$\boxed{\theta_1 = 1/\lambda_1} \qquad \boxed{\theta_2 = \frac{\lambda_2}{\lambda_1}}$$

$$\theta_1 + \theta_2 = \frac{1+\lambda_2}{\lambda_1} = 1$$

$$\boxed{\begin{aligned} y_1 &= \lambda_1 x_1 - \lambda_2 x_2 \\ y_2 &= x_2 \end{aligned}}$$

$$\theta_1 y_1 + \theta_2 y_2 = \frac{1}{\lambda_1}(\lambda_1 x_1 - \lambda_2 x_2) + \frac{\lambda_2}{\lambda_1} x_2$$

$$= x_1$$

<u>General case</u>:

$$\underbrace{\frac{1}{\lambda_1}}_{\theta_1} + \sum_{i=1}^{n} \underbrace{\frac{\lambda_i}{\lambda_1}}_{\theta_i} = 1$$

$$y_1 = \lambda_1 x_1 - \sum_{i=2}^{n} \lambda_i x_i$$

$$y_2 = x_2$$

$$y_3 = x_3$$

$$\vdots$$

$$y_n = x_n$$

4. $f(\underline{x})$ half-space

$$f(\underline{x}) = a^T x + b$$

$$\text{epi } f = \{(\underline{x}, t) \mid f(\underline{x}) \le t\}$$

$$= \{(x,t) \mid a^T x + b \le t\}$$

$$= \{ y \mid c^T y + b \le 0 \}$$

$$y = \begin{bmatrix} \underline{x} \\ t \end{bmatrix} \to \mathbb{R}^{n+1}$$

half space $\subseteq \mathbb{R}^{n+1}$

$$a^T x - t = \begin{bmatrix} a^T & -1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ t \end{bmatrix} = \left\langle \underbrace{\begin{bmatrix} \underline{a} \\ -1 \end{bmatrix}}_{c}, y \right\rangle$$

(b) $\|x\|_2 \le t$

$$f(\underline{x}) = \|x\|_2$$

$$\text{epi } f = \{(\underline{x}, t) \mid \|x\|_2 \le t\}$$

norm cone

(c) $f(\underline{x}) = \max\limits_{1 \le i \le n} a_i^T \underline{x} + b$

$$\text{epi } f = \{(\underline{x}, t) \mid \max_{1 \le i \le n} a_i^T x + b \le t\}$$

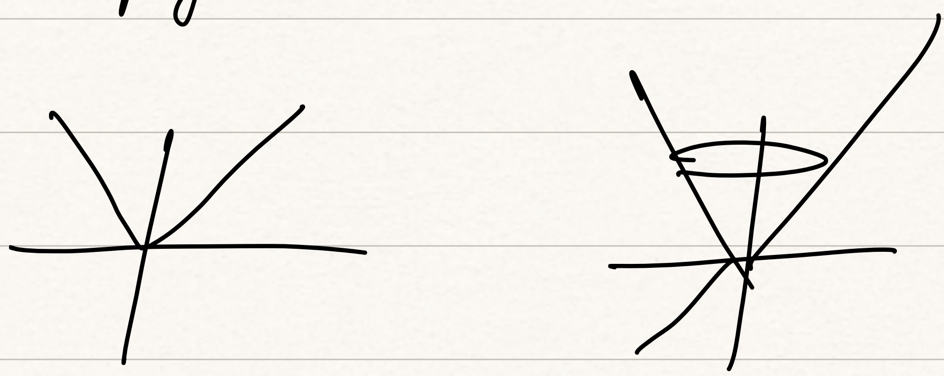
b

$= \{(\underline{x}, t) \mid a_i^T x + b \le t, \ i = 1, 2 \dots n\}$

$= \{(x,t) \mid a_1^T x + b \le t\} \cap \{(\underline{x}, t) \mid a_2^T x + b \le t\}$

$\underbrace{\cap \dots}_{\text{polyhedron}} \cap \{(\underline{x}, t) \mid a_n^T x + b \le t\}$

half spaces



$\left.\begin{array}{l} \|x\|_1 \le t \\ \|x\|_\infty \le t \end{array}\right\}$ polyhedron

5. $D(x,y) = f(\underline{x}) - f(y) - \nabla f(y)^T (\underline{x} - \underline{y})$

Bregman Divergence

f convex $\Rightarrow$ first order condition holds

$D(x,y) \neq D(y,x)$ $\Rightarrow$ $D(x,y) \geq 0$

$f(x) = -\sum \log(x_i)$ convex

$[\nabla f(x)]_i = -1/x_i$

$\nabla f(y)^T (x-y) = -\sum \frac{1}{y_i}(x_i - y_i)$

$= 1 - \sum x_i/y_i$

$\underline{f(\underline{x}) - f(y)} - \nabla f(y)^T (x-y)$

$= -\sum_i \left(\log(x_i/y_i) + 1 - x_i/y_i\right)$

$= D_{IS}(x, y)$

$D_{KL}(\underline{x}, y)$ $\underline{f(\underline{x})} = \underbrace{\sum_{i=1}^{n} x_i \log(x_i)}_{\text{convex}}$

$x_i \log x_i$ $\log x_i + 1$ $\frac{1}{x_i} > 0$

convex

$[\nabla f(y)]_i = 1 + \log y_i$

$\nabla f(y)^T (x-y) = \sum_{i=1}^{n} (x_i - y_i)(1 + \log y_i)$

$f(x) = x$ $x - y - (x-y) = 0$

$f(x) = x^2$ $x^2 - y^2 - 2y(x-y)$

$x^2 - y^2 - 2xy + 2y^2$

$= x^2 + y^2 - 2xy = (x-y)^2$

$f(\underline{x}) = \underline{\underline{\|x\|_2^2}}$

$\|x\|_2^2 - \|y\|^2 - 2y^T(x-y)$

$x^Tx + y^Ty - 2x^Ty$

$= \underline{\underline{\|x-y\|_2^2}} \leftarrow$

$x^Ty = y^Tx$

$\sum x_i y_i = \sum x_i y_i$

$D_f(x,y) \geq 0$

$D_g(x,y) < 0 \leftarrow$ not a valid measure of dissimilarity