EE910: Digital Communication Systems-I

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Lecture #5C: Optimal Detection for Binary Antipodal Signaling

Optimal Detection for Binary Antipodal Signaling

- In binary antipodal signaling scheme $s_1(t) = s(t)$ and $s_2(t) = -s(t)$.
- The probabilities of messages 1 and 2 are p and $1-p$, respectively.
- The vector representations of the two signals are just scalars with $s_1(t) = \sqrt{\mathcal{E}_s}$ and $s_2(t) = -\sqrt{\mathcal{E}_s}$, where $\mathcal{E}_s$ is energy in each signal and is equal to $\mathcal{E}_b$.
- The decision region D_1 is given as,

$$\begin{aligned} D_1 &= \left\{ r : r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln p - \frac{1}{2}\mathcal{E}_b > -r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln(1-p) - \frac{1}{2}\mathcal{E}_b \right\} \\ &= \left\{ r : r > \frac{N_0}{4\sqrt{\mathcal{E}_b}}\ln\frac{1-p}{p} \right\} \\ &= \{ r : r > r_{th} \} \qquad (1) \end{aligned}$$

where the threshold is defined as $r_{th} = \frac{N_0}{4\sqrt{\mathcal{E}_b}}\ln\frac{1-p}{p}$.

Optimal Detection for Binary Antipodal Signaling

- The error probability of this system is derived as

$$\begin{aligned} P_e &= \sum_{m=1}^{2} P_m \sum_{1\le m' \le 2, m' \neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s}_m)d\mathbf{r} \\ &= p\int_{D_2} p(r|s=\sqrt{\mathcal{E}_b})dr + (1-p)\int_{D_1} p(r|s=-\sqrt{\mathcal{E}_b})dr \\ &= p\int_{-\infty}^{r_{th}} p(r|s=\sqrt{\mathcal{E}_b})dr + (1-p)\int_{r_{th}}^{\infty} p(r|s=-\sqrt{\mathcal{E}_b})dr \\ &= p\,P\left[\mathcal{N}(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}) < r_{th}\right] + (1-p)\,P\left[\mathcal{N}(\sqrt{-\mathcal{E}_b}, \frac{N_0}{2}) > r_{th}\right] \\ &= p\,\mathcal{Q}\left(\frac{\sqrt{\mathcal{E}_b} - r_{th}}{\sqrt{\frac{N_0}{2}}}\right) + (1-p)\,\mathcal{Q}\left(\frac{\sqrt{\mathcal{E}_b} + r_{th}}{\sqrt{\frac{N_0}{2}}}\right) \qquad (2) \end{aligned}$$

- In the special case where $p = \frac{1}{2}$, we have $r_{th} = 0$ and the error probability simplifies to $P_e = \mathcal{Q}\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$

Recall

- Q function is closely related to the Gaussian random variable

$$Q(x) = P[\mathcal{N}(0,1) > x] = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \qquad (3)$$

- If X$\sim \mathcal{N}(m, \sigma^2)$, then

$$P[X > \alpha] = Q\left(\frac{\alpha - m}{\sigma}\right) \qquad (4)$$

$$P[X < \alpha] = Q\left(\frac{m - \alpha}{\sigma}\right) \qquad (5)$$

- Some of the important properties of the Q function:

$$\begin{aligned} &Q(0) = 1/2 \qquad && Q(\infty) = 0 \\ &Q(-\infty) = 1 \qquad && Q(-x) = 1 - Q(x) \end{aligned} \qquad (6)$$

Error Probability for Equiprobable Binary Signaling Schemes

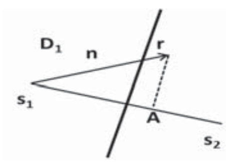


Figure: Decision regions for binary equiprobable signals

- In this case the transmitter transmits one of the two equiprobable signals $s_1(t)$ and $s_2(t)$ over the AWGN channel.
- Since the signals are equiprobable, the two decision regions are separated by the perpendicular bisector of the line connecting s_1 and s_2.

Error Probability for Equiprobable Binary Signaling Schemes

- The error probability is given by

$$P_b = P\left[\frac{\mathbf{n}.(\mathbf{s}_2 - \mathbf{s}_1)}{d_{12}} > \frac{d_{12}}{2}\right] = P\left[\mathbf{n}.(\mathbf{s}_2 - \mathbf{s}_1) > \frac{d_{12}^2}{2}\right] \quad (7)$$

- $\mathbf{n}.(\mathbf{s}_2 - \mathbf{s}_1)$ is a zero-mean Gaussian random variable with variance $\frac{d_{12}^2 N_0}{2}$ and thus we can write

$$P_b = \mathcal{Q}\left(\frac{\frac{d_{12}^2}{2}}{d_{12}\sqrt{\frac{N_0}{2}}}\right) = \mathcal{Q}\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) \quad (8)$$

Error Probability for Equiprobable Binary Signaling Schemes

- Since $\mathcal{Q}$ is a decreasing function, in order to minimize the error probability, the distance between signal points has to be maximized.
- The distance d_{12} is obtained from $d_{12}^2 = \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 dt$
- In the special case that the binary signals are equiprobable and have equal energy, i.e., when $\mathcal{E}_{s1} = \mathcal{E}_{s2} = \mathcal{E}$ and we get $d_{12}^2 = \mathcal{E}_{s1} + \mathcal{E}_{s2} - 2\langle s_1(t), s_2(t)\rangle = 2\mathcal{E}(1-\rho)$

Optimal Detection for Binary Orthogonal Signaling

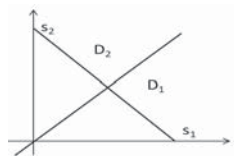


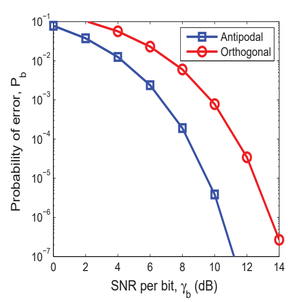
Figure: Signal constellation and decision regions for equiprobable binary orthogonal signaling

- For binary orthogonal signals we have,

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \begin{cases} \mathcal{E}, & i = j \\ 0, & i \neq j \end{cases} \tag{9}$$

  for $1 \leq i, j \leq 2$.
- Since the system is binary, $\mathcal{E}_b = \mathcal{E}$ and $\phi_j(t)$ is choosen as $\phi_j(t) = \frac{s_j(t)}{\sqrt{\mathcal{E}_b}}$ for $j = 1, 2$.

Optimal Detection for Binary Orthogonal Signaling



- The vector representations of the signal set become $\mathbf{s}_1 = (\sqrt{\mathcal{E}_b}, 0)$ and $\mathbf{s}_2 = (0, \sqrt{\mathcal{E}_b})$
- Here $d = \sqrt{2\mathcal{E}_b}$ and $P_b = \mathcal{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right) = \mathcal{Q}\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$