$\boxed{Q \text{ orthogonal}}$ $\iff$ rows & columns

$n \times n$ are orthonormal

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

$$\left.\begin{aligned} \langle q_i, q_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \end{aligned}\right\} \iff Q^T Q = I$$

$$\|q_i\| = 1$$

same for $Q^T \longrightarrow QQ^T = I$

1. $A = A^T$ $\qquad \lambda_i(A)$

(a) $\mathrm{Tr}(A^3)$ $\qquad \lambda_i(A)$

$\underline{A^3}$ $\qquad \underline{\lambda_i^3(A)}$

$\mathrm{Tr}(A^3) = \sum \lambda_i(A^3)$ $\qquad Av = \lambda v$

$= \sum \lambda_i^3(A)$ $\qquad A^2 v = \lambda(Av) = \lambda^2 v$

$\vdots$

(b) $\underline{A^{-2}}$ $\qquad Av = \lambda v$

$A^{-1}Av = \lambda A^{-1} v$

$\lambda_i(A^{-1}) = \dfrac{1}{\lambda_i(A)}$ $\qquad \Rightarrow \dfrac{1}{\lambda} v = A^{-1} v$

$(\lambda_i(A))^{-2}$

(c) $\lambda_i(A-I)$

$$Av = \lambda v$$

$$(A-I)v = Av - v$$

$\lambda_i(A) - 1$

$$= \lambda v - v = \underline{(\lambda-1)}v$$

(d) $\lambda_i(I+2A)$ $\quad 1 + 2\lambda_i(A)$

suppose

$$A = U\Lambda U^T$$

$$\underline{A^2} = U\Lambda \underbrace{U^T U}_{I} \Lambda U^T = U\Lambda^2 U^T$$

$$\Lambda^2 \to \begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \end{bmatrix}$$

$$A^p = U\Lambda^p U^T$$

$$\Lambda^p \to \begin{bmatrix} \lambda_1^p & & \\ & \lambda_2^p & \\ & & \ddots \end{bmatrix}$$

$$I + cA = I + cU\Lambda U^T = U(I + c\Lambda)U^T$$

$I \to UU^T$

$$I + c\Lambda \to \begin{bmatrix} 1+c\lambda_1 & & 0 \\ & 1+c\lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

2. $A \succ 0$ positive definite

$A = A^T$ $\quad \lambda_i(A) > 0$ ✓ $\iff$ $\boxed{x^T A x > 0 \quad \forall x}$ ✓

(a) A^{-1} $\quad A : \lambda_i(A) > 0$

A^{-1} : $\quad (\lambda_i(A))^{-1} > 0$

(b) $\underline{\underline{A_{ii}}}$ $\quad x^T A x > 0 \quad \forall x$

eg. $x = e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$

$e_1^T A e_1 = A_{11} > 0$

$\vdots \qquad A_{22} > 0$

$\vdots$

3. $A^2 = A$ $\quad A = U \Lambda U^T$

$A^2 = U \Lambda U^T U \Lambda U^T = U \Lambda^2 U^T$

$A^2 = A$ $\quad U \Lambda U^T = U \Lambda^2 U^T$

$U^T \times$ $\quad \Lambda U^T = \Lambda^2 U^T$

$\Lambda = \Lambda^2 \qquad \times U$

$$\lambda_i(A) = \lambda_i^2(A) \Rightarrow \lambda_i(A) \in \{0, 1\}$$

$$\underline{A^2 = A \succ 0} \Rightarrow \lambda_i(A) > 0$$

$$\Rightarrow \lambda_i(A) = 1 \Rightarrow \underline{A = I}$$

$m \geq n$

Eg. $B \in \mathbb{R}^{m \times n}$ $\qquad A = \underline{B(B^TB)^{-1}B^T}$

$$A^2 = B(B^TB)^{-1}\underline{B^TB}(\underline{B^TB})^{-1}B^T$$

$$= B(B^TB)^{-1}B^T = A$$

$\underline{B^TB \in \mathbb{R}^{n \times n}}$ inverse exists if B full column rank

4.)

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^T}_{n\times n} = \underbrace{U_1}_{m\times r}\underbrace{\Sigma_1}_{r\times r}\underbrace{V_1^T}_{r\times n}$$

$\text{rank}(A) = r$ $\qquad UU^T = U^TU = I$ $\qquad \underline{U_1^TU_1 = I} \neq U_1U_1^T$

$VV^T = V^TV = I$ $\qquad \underline{V_1^TV_1 = I} \neq V_1V_1^T$

$$\left[\begin{array}{c|c} \sigma_1 \ \ddots \ \sigma_r & 0 \\ \hline 0 & 0 \end{array}\right] \Sigma$$

$\boxed{V_1}$ tall $\boxed{V_1}$

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \quad \text{(rank-1 matrix)}$$

rank-1 decomposition

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

$u_i v_i^T$ rank-one

$$uv^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & \dots & v_m \end{bmatrix}$$

$$= \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \dots & u_1 v_m \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_m \\ \vdots & & & \vdots \\ u_n v_1 & u_n v_2 & \dots & u_n v_m \end{bmatrix}$$

$$\|A\|_F^2 = tr(A^T A)$$

$$\underset{m \times n}{A^T A} = (U \Sigma V^T)^T (U \Sigma V^T)$$

$$= V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$$

$$\underline{\underline{A^T A}} = V \underline{\Sigma^T \Sigma} V^T$$

eigenvalue decomposition

$$\underset{n \times n}{\Sigma^T \Sigma} = \left[\begin{array}{ccc|c} \sigma_1^2 & & 0 & \\ & \sigma_2^2 & & 0 \\ 0 & & \ddots \; \sigma_r^2 & \\ \hline & 0 & & 0 \end{array}\right] \begin{array}{l} \}\, r \\ \\ \}\, n-r \end{array}$$

$n-r$

$$\lambda_i(A^T A) = \begin{cases} \sigma_i^2(A) & i = 1 \dots r \\ 0 & i > r \end{cases}$$

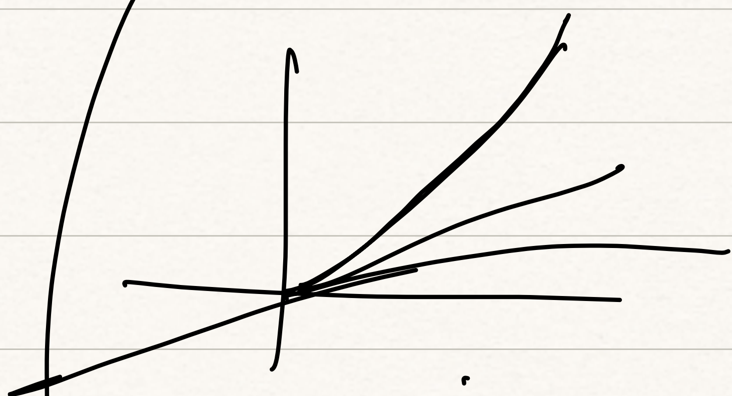
$$\mathrm{Tr}(A^TA) = \sum \lambda_i(A^TA) = \sum \sigma_i^2(A)$$

AA^T

$m \times m$

5. $$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$\boxed{\|A\|_2^{2} = \max_{\|x\|_2=1} \|Ax\|_2^{2}}$$



monotonically increasing

$$P = \min f(x)$$

$$h(P) = \min h(f(x))$$

where h is strictly increasing

$$\|A\|_2^2 = \max_{\|x\|_2^2=1} \|Ax\|_2^2$$

$$\|x\| = 1 \iff \|x\|^2 = 1$$

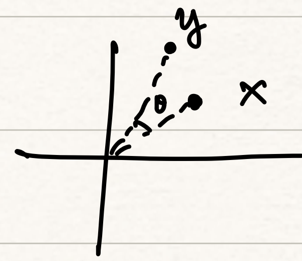
$$\|Ax\|_2^2 = x^T A^T A x$$

$$A^TA = V\Sigma^T\Sigma V^T$$

$$= x^T V \Sigma^T \Sigma \underbrace{V^T x}_{y}$$

V orthogonal $\quad x \quad\quad V^T x \rightarrow$ rotated version of x

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$$



$$y = V^T x$$

$$\underline{\|y\|_2^2} = y^T y = x^T V V^T x = x^T x = \underline{\|x\|_2^2}$$

$$x \longleftrightarrow y \qquad \begin{aligned} x &= Vy \\ y &= V^T x \end{aligned}$$

$$\|Ax\|_2^2 = y^T \Sigma^T \Sigma y$$

$$\|A\|_2^2 = \max_{\underline{\|y\|_2^2 = 1}} y^T \underbrace{\Sigma^T \Sigma}\, y = y^T \Lambda y$$

$$\Lambda = \Sigma^T \Sigma$$

$$= \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

$$y^T \Lambda y = \sum_{i=1}^{r} \sigma_i^2 y_i^2$$

$$= \sigma_1 y_1^2 + \sigma_2^2 y_2^2 \;----.$$

$$y^T \Lambda y = \sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 \cdots + \sigma_r^2 y_r^2$$

$$\sigma_i^2 \le \sigma_{max}^2 \qquad \sigma_{max} = \max_{1 \le i \le r} \sigma_i$$

$$\underbrace{\sum_{i=1}^{r} \sigma_i^2 y_i^2}_{y^T \Lambda y} \le \sum_{i=1}^{r} \sigma_{max}^2 y_i^2 \qquad y_i^2 \ge 0$$

$$\sigma_{max}^2 y_{i_{max}}^2 \qquad \sigma_{max}^2 \sum_{i=1}^{r} y_i^2 = \sigma_{max}^2$$

$$\|y\|_2^2 = 1$$

$$y^T \Lambda y \le \sigma_{max}^2 \quad \text{—} \; (1)$$

equality is attained (what y?)

i_{max} : $\sigma_{i_{max}} = \sigma_{max}$

then take $y_i = \begin{cases} 1 & i = i_{max} \\ 0 & \text{o/w} \end{cases}$

$$\sum y_i^2 = 1 \quad \text{—} \; (2)$$

Summarize : & $\begin{cases} \underline{y^T \Lambda y \le \sigma_{max}^2} \text{ for } \|y\| = 1 \\ \underline{\text{equality for some } y} \end{cases}$

$$\Rightarrow \sigma_{max}^2 = \max_{\|y\|_2 = 1} y^T \Lambda y$$

$$= \max_{\|x\|_2^2 = 1} \|Ax\|_2^2 = \|A\|_2^2$$

$$\|A\|_2 = \sigma_{max}$$

$$\|x\|_p = \sqrt[p]{\sum |x_i|^p} \quad \text{for } x \in \mathbb{R}^n$$