EE901
Probability and Random Processes

Module 5
Functions of Random Variables

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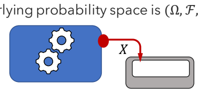
Function of a Random Variable

- Random variables take different values each time the experiment is performed.
- Consider a physical system (for example, a machine) which gives a random output X. Suppose it is an electric current.
- Its random output results from a random experiment going inside the machine.
- Every value of X corresponds to one outcome.
- So X is a random variable. Underlying probability space is $(\Omega, \mathcal{F}, \mathbb{P})$.



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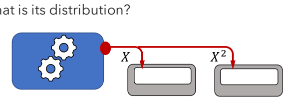
Function of a Random Variable

- We may be interested in the power of the output signal $Y = X^2$ or any other function of X.
- What is Y?
- Y also takes many values. One corresponding to each value of X and hence, each outcome.
- Is Y a random variable? If yes, what is its distribution?



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Transformation of Discrete Random Variables

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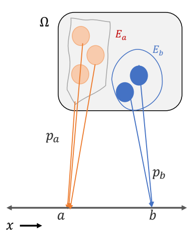
Example: DRV with Two Values



Consider a RV X which takes two values a and b with probability p_a and p_b

$$X: \Omega \to \mathbb{R}$$

$$X(\omega) = \begin{cases} a & \text{if } \omega \in E_a \\ b & \text{if } \omega \in E_b \end{cases}$$

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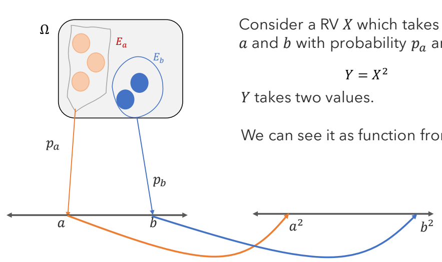
Example: DRV with Two Values



Consider a RV X which takes two values a and b with probability p_a and p_b

$$Y = X^2$$

Y takes two values.

We can see it as function from Ω to $\mathbb{R}$

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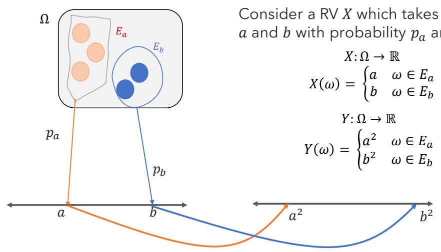
Example: DRV with Two Values



Consider a RV X which takes two values a and b with probability p_a and p_b

$$X:\Omega \to \mathbb{R}$$

$$X(\omega) = \begin{cases} a & \omega \in E_a \\ b & \omega \in E_b \end{cases}$$

$$Y:\Omega \to \mathbb{R}$$

$$Y(\omega) = \begin{cases} a^2 & \omega \in E_a \\ b^2 & \omega \in E_b \end{cases}$$

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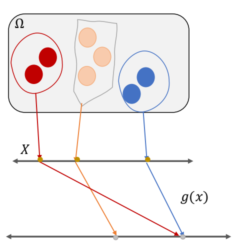
Transformation of a RV



Consider a RV X
Let a function g be defined on the range of X

$$g : \mathcal{R}(X) \to \mathbb{R}$$

What is $g(X)$?

Whenever we perform the experiment, we look at the output $g(X)$.

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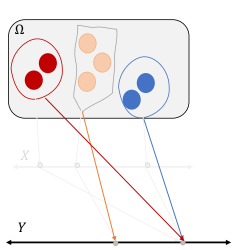
Transformation of a RV



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$$g : \mathcal{R}(X) \to \mathbb{R}$$

What is $g(X)$?

Whenever we perform the experiment, we look at the output $g(X)$.

Let $Y = g(X)$
Then, for each ω

$$Y(\omega) = g\big(X(\omega)\big)$$

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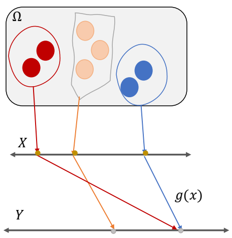
Transformation of a RV



Consider a RV X
Let a function g be defined on the range of X

$$g : \mathcal{R}(X) \to \mathbb{R}$$

Let $Y = g(X)$
Then for each ω

$$Y(\omega) = g(X(\omega))$$

$g(X)$ is a random variable.

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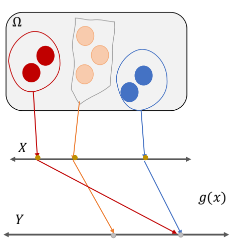
Distribution of a Function of RV



$Y = g(X)$ is a random variable.

What will be its distribution?

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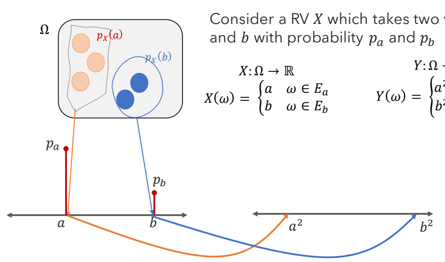
Example: DRV with Two Values



Consider a RV X which takes two values a and b with probability p_a and p_b

$$X: \Omega \to \mathbb{R} \qquad X(\omega) = \begin{cases} a & \omega \in E_a \\ b & \omega \in E_b \end{cases}$$

$$Y: \Omega \to \mathbb{R} \qquad Y(\omega) = \begin{cases} a^2 & \omega \in E_a \\ b^2 & \omega \in E_b \end{cases}$$

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Distribution

Consider a RV X which takes two values a and b with probability p_a and p_b

$$X:\Omega \to \mathbb{R} \qquad X(\omega) = \begin{cases} a & \omega \in E_a \\ b & \omega \in E_b \end{cases}$$

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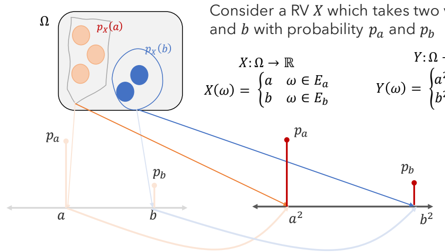

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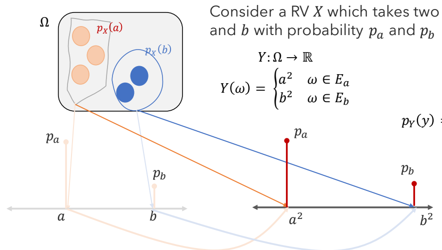
Distribution

Consider a RV X which takes two values a and b with probability p_a and p_b

$$Y:\Omega \to \mathbb{R} \qquad Y(\omega) = \begin{cases} a^2 & \omega \in E_a \\ b^2 & \omega \in E_b \end{cases}$$

$$p_Y(y) = \begin{cases} a^2 & p_a \\ b^2 & p_b \end{cases}$$



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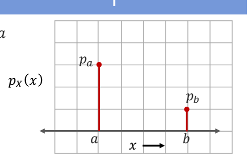
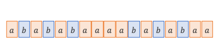
Frequency Interpretation and Expectation

- Consider a RV X which takes two values a and b with probability p_a and p_b
- If the experiment is repeated N times,
- Approximately
  - $N_a = N\,p_a$ times, the outcome is a
  - $N_b = N\,p_b$ times, the outcome is b.
- Average is

$$\frac{a + a + \dots.\ N_a \text{ times } + b + b + \dots\ N_b \text{ times}}{N}$$
$$= \frac{aN_a + bN_b}{N} = \frac{Np_a a + Np_b b}{N} = ap_a + bp_b$$





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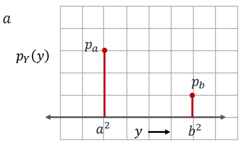
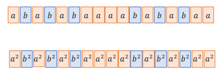
Frequency Interpretation and Expectation

- Consider a RV X which takes two values a and b with probability p_a and p_b
- If the experiment is repeated N times,
- Approximately
  - $N_a = N\,p_a$ times, the outcome is a^2
  - $N_b = N\,p_b$ times, the outcome is b^2.
- Average is

$$\frac{a^2+a^2+\ldots.\ N_a \text{ times } + b^2+b^2+\ldots\ N_b \text{ times}}{N}$$

$$= \frac{a^2N_a+b^2N_b}{N} = \frac{Np_a a^2 + Np_b b^2}{N} = a^2p_a + b^2p_b$$





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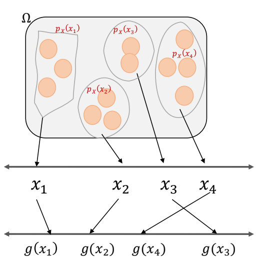
Expectation



For a DRV X

$$\mathbb{E}[X] = \sum_i p_X(x_i)x_i$$

For a DRV X and a function g

$$\mathbb{E}[g(X)] = \sum_i p_X(x_i)g(x_i)$$

Let $Y = g(X)$

What if we consider PMF of Y directly to compute $\mathbb{E}[Y]$

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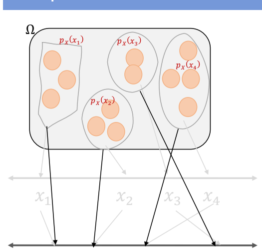
Expectation



For a DRV Y and a function g

$$\mathbb{E}[Y] = \sum_i p_Y(y_i)\,y_i$$

Y takes the same number of values

Corresponding to each x_i, there is a point where probability of Y, $p_Y(y_i)$, is concentrated

$$y_i = g(x_i)$$

$$p_Y(y) = \begin{cases} g(x_1) & \text{with probability } p_1 \\ g(x_2) & \text{with probability } p_2 \\ \ldots & \ldots \end{cases}$$

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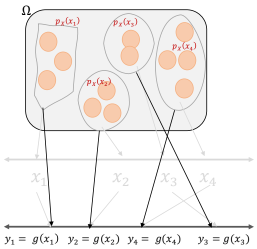
Expectation of Function of DRV



For a DRV Y and a function g

$$p_Y(y) = \begin{cases} g(x_1) & \text{with probability } p_1 \\ g(x_2) & \text{with probability } p_2 \\ \dots & \dots \end{cases}$$

y_i $p_X(x_i)$

$$\mathbb{E}[Y] = \sum_i p_Y(y_i)\, y_i$$
$$= \sum_i p_X(x_i)\, g(x_i)$$
$$= \mathbb{E}[g(X)]$$

What happens when function g does not map to unique values?

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Example

- Consider a discrete random variable X with PMF

$$p_X(x) = \begin{cases} 1/3 & \text{for } x = -1 \\ 1/3 & \text{for } x = 0 \\ 1/3 & \text{for } x = 1 \end{cases}$$

- Let $Y = X^2$

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Example

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Example

$$\mathbb{P}[Y=0] = \mathbb{P}[X^2=0] = \mathbb{P}[X=0] = \frac{1}{3}$$

$$\begin{aligned}\mathbb{P}[Y=1] &= \mathbb{P}[X^2=1] = \mathbb{P}[X=+1,\ \text{or}\ X=-1] \\ &= \mathbb{P}\big[\{\omega : X(\omega)=1\} \cup \{\omega : X(\omega)=-1\}\big] = \mathbb{P}[E_1 \cup E_2]\end{aligned}$$

Events E_1 and E_2 are disjoint,

$$= \mathbb{P}\big[\{\omega : X(\omega)=1\}\big] + \mathbb{P}\big[\{\omega : X(\omega)=-1\}\big]$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

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Example

$\mathbb{P}[Y=0] = \frac{1}{3}$ $\mathbb{P}[Y=1] = \frac{2}{3}.$

CDF

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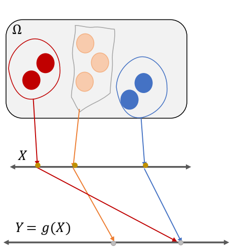
Distribution of a Function of DRV



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Distribution of a Function of DRV

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Expectation of a Function of DRV

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