Solving Linear equations

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed the following in the last lecture
  - matrices and linear equations,
  - independence and dependence of vectors
- Discuss the following today
  - solution of "linear" equations'
- Reference for today's class - Chap 2.1 of the book

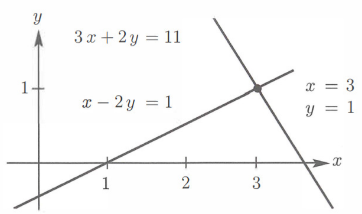
Vectors and Linear Equations – row picture (1)

- Central problem of linear algebra is to solve a system of equations:

$$x - 2y = 1 \tag{1}$$
$$3x + 2y = 11 \tag{2}$$

- Linear equations: unknowns are only multiplied by numbers
  - we never see x times y
- We begin with the row picture and look at a row at a time
- Eq. (1) produces a straight line in the xy plane
  - Point $x = 1, y = 0$ is on the line because it solves that equation
  - Point $x = 3, y = 1$ is also on the line because 3 - 2 $= 1$

Vectors and Linear Equations – row picture (2)



- First and second lines in "row picture" are Eq. (1) and Eq. (2), respectively
- Note the point $x = 3, y = 1$ where the two lines meet
  - Point $(3, 1)$ lies on both lines and solves both equations

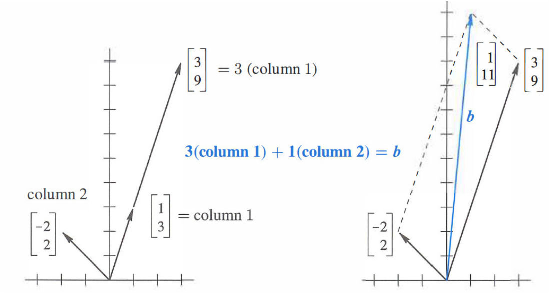
Vectors and Linear Equations – column picture (1)

- Column picture – recognize same linear system as a "vector equation"
  - Instead of numbers we need to see vectors
- We separate original system into its columns and get a vector equation

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \mathbf{b}$$

- Need to find the combination of those vectors that equals to $\mathbf{b}$
- With $x = 3$ and $y = 1$ (the same numbers as before), we get $\mathbf{b}$
- Columns picture combines column vectors on LHS to produce vector $\mathbf{b}$

Vectors and Linear Equations – column picture (2)



- Figure is the "column picture" of two equations in two unknowns
- Solution is same in both pictures

Vectors and Linear Equations -summary

- Coefficient matrix on the left side of the equations is the 2 by 2 matrix A:

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

- Its rows give the row picture and columns give the columns picture
    - Same number, different pictures, same equations
- We combine those equations into a matrix problem $A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

3 Equations in 3 Unknowns (1)

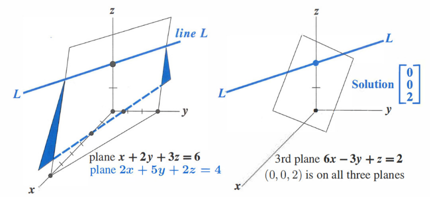
- Three unknowns are x, y, z – we have three linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

- We look for numbers x, y, z that solve all three equations at once
    - Those desired numbers might or might not exist
    - For this system, they do exist
- When the number of unknowns matches the number of equations,
    - There is usually one solution
- Before solving the problem, we visualize it both ways

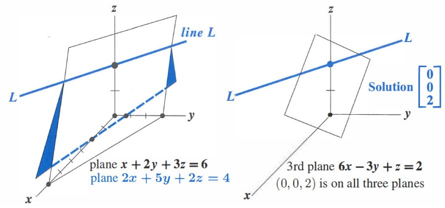
3 Equations in 3 Unknowns – row picture (1)

- Row picture shows three planes meeting at a single point



- In the row picture, each equation produces a plane in three-dimensional space
- First plane comes from the first equation $x + 2y + 3z = 6$
  - It crosses x, y and z axes at: $(6, 0, 0)$ and $(0, 3, 0)$ and $(0, 0, 2)$
- Vector $(x, y, z) = (0, 0, 0)$ does not solve $x + 2y + 3z = 6$
  - Plane does not contain origin

3 Equations in 3 Unknowns – row picture (2)



- 2nd plane is from second equation, intersects 1st plane in line L
- Usual result of two equations in three unknowns is a line L of solutions
- Third equation gives a third plane, it cuts the line L at a single point
  - That point lies on all three planes and it solves all three equations
- Three planes meet at the solution (which we haven't found yet)

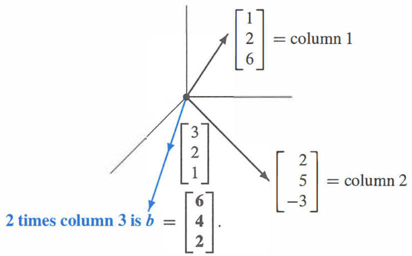
3 Equations in 3 Unknowns – column picture (1)

- Column form will now show immediately why $z = 2$
- Column picture combines three columns to produce $\mathbf{b} = (6, 4, 2)$
- Column picture starts with the vector form of the equations $A\mathbf{x} = \mathbf{b}$:

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{b}$$

- Linear combinations of those columns can produce any vector $\mathbf{b}$
- Need to multiply three column vectors by numbers x, y, z to produce $\mathbf{b}$
  - Coefficients we need are $x = 0, y = 0$, and $z = 2$
  - Combination that produces $b = (6, 4, 2)$ is just 2 times the third column

3 Equations in 3 Unknowns – column picture (2)



- Figure above shows this column picture
- Three planes in row picture meet at that same solution point $(0, 0, 2)$

Matrix Form of the Equations (1)

- We have three rows in row picture and three columns in the column picture
- Three rows and three columns contain nine numbers
- These nine numbers fill a 3 by 3 matrix "coefficient matrix" A

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$$

- Matrix equation $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Matrix Form of the Equations (2)

- What does it mean to "multiply A times $\mathbf{x}$"?
- We can multiply by rows or by columns

$$\mathbf{Ax} = \begin{bmatrix} (row1).\mathbf{x} \\ (row2).\mathbf{x} \\ (row3).\mathbf{x} \end{bmatrix}$$

- Multiplication by columns: $A\mathbf{x}$ is a combination of column vectors

$$\mathbf{Ax} = x\,(\mathbf{column\ 1}) + y\,(\mathbf{column\ 2}) + z\,(\mathbf{column\ 3})$$

- We see $A\mathbf{x}$ as a combination of the columns of A

Matrix notation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- First index gives the row number, so that a_{ij} is an entry in row i
- Second index j gives the column number

Review of important ideas

- Matrix-vector multiplication $A\mathbf{x}$ can be computed by dot products
- But $A\mathbf{x}$ must be understood as a combination of the columns of A
- Column picture: $A\mathbf{x} = \mathbf{b}$ asks for a combination of columns to produce $\mathbf{b}$
- Row picture: each equation in $A\mathbf{x} = \mathbf{b}$ gives a line $(n = 2)$ or a plane $(n = 3)$ or a "hyperplane" $(n > 3)$
- They intersect at the solution or solutions, if any