Weighted norm: $\|v\|_{\underline{a}} = \sum_{i=1}^{n} a_i v_i^2$

$v \in \mathbb{R}^n$ $\quad\quad$ $\underline{a} \in \mathbb{R}^n$ $\quad\quad$ weights

Valid? $\quad$ $a_i > 0$

counter-example: suppose that $a_1 = 0$, $a_i > 0 \ \forall i \geq 2$

we can find v s.t. $\|\underline{v}\|_{\underline{a}} = 0$ while $\underline{v} \neq 0$

(Definiteness violated)

Eg: $v_i = \begin{cases} 1 & i = 1 \\ 0 & i \geq 0 \end{cases}$

$\Rightarrow$ $a_i > 0 \ \forall i$ for valid norm

Alternatively: $\|v\|_A^2 = v^T A v = [v_1\ v_2 \ldots v_n] \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$= \sum a_i v_i^2$

valid? $\quad$ $[A]_{ii} > 0$

Generalize: $A \in S^n$ $\quad$ symmetric $n \times n$ matrix

define $\|v\|_A$

Aside

Symmetric Eigenvalue Decomposition

$$A = Q\Lambda Q^T \qquad Q \in \mathbb{R}^{n\times n}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

eigenvalues $\qquad$ $q_i \in \mathbb{R}^n$ eigenvectors

$$A = Q\Lambda Q^T$$
$$= \sum_{i=1}^{n} \lambda_i q_i q_i^T \qquad \text{(alternative way)}$$

note $\quad q_i q_i^T \quad n\times n$ matrix

$\text{rank}(q_i q_i^T) = 1 \qquad$ sum of rank-1 matrices

Also $\quad Q$ orthogonal $\qquad QQ^T = Q^TQ = I$

and $\quad Q^{-1} = Q^T$

also $\quad \langle q_i, q_j \rangle = 0 \quad i \neq j$

$\qquad\qquad = 1 \quad i = j$

$$Aq_i = \left(\sum_{j=1}^{n} \lambda_j q_j q_j^T\right) q_i = \lambda_i q_i$$

Recall: $tr(AB) = tr(BA)$

$tr(ABC) = tr(CAB) = tr(BCA)$

$$tr(A) = tr(Q\Lambda Q^T) = tr(Q^TQ\Lambda) = tr(\Lambda) = \sum \lambda_i$$

$$det(A) = \prod_{i=1}^{n} \lambda_i$$

$det(A) > 0$ only when $\lambda_i > 0 \ \forall \ i = 1, \dots n$

$$\det(A-\lambda I) = (\lambda-\lambda_1)(\lambda-\lambda_2)\ \cdots\ (\lambda-\lambda_n)$$

characteristic equation

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad A-\lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (3-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda = 2, 4 \quad \text{roots}$$

eigen values

$$Aq_i = \lambda_i q_i = 2q_i$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = \begin{bmatrix} 2q_{11} \\ 2q_{12} \end{bmatrix}$$

$$3q_{11} + q_{12} = 2q_{11} \Rightarrow q_{11} + q_{12} = 0$$

$$\Rightarrow q_1 = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \text{ for some } \alpha$$

$$\|q_1\| = 1 \quad \text{so} \quad \alpha^2 + \alpha^2 = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$$

$$\text{so} \quad q_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \qquad q_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

so that $\langle q_1, q_2 \rangle = 0$