

Schur's Complement

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$



$$A = A^T \quad C = C^T$$

$$X = X^T \in S^n$$

$$S_1 = C - B^T A^{-1} B$$

A invertible

$$S_2 = A - B C^{-1} B^T$$

C invertible

Key result

$$(a) \quad X > 0 \quad \Leftrightarrow \quad \begin{matrix} S_1 > 0 \\ A > 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} S_2 > 0 \\ C > 0 \end{matrix}$$

$$(b) \quad \det(X) = \det(A) \det(S_1) = \det(C) \det(S_2)$$

Eg

$$D = \begin{bmatrix} tI & \underline{x} \\ \underline{x}^T & t \end{bmatrix}$$

A diagram of the matrix D from the previous block. The top-left block is labeled 'tI' and has an arrow pointing to it from the text 'tI' below. The top-right block is labeled 'x' and has an arrow pointing to it from the text 'x' below. The bottom-left block is labeled 'x^T' and has an arrow pointing to it from the text 'x^T' below. The bottom-right block is labeled 't' and has an arrow pointing to it from the text 't' below.

$$A = tI \quad B = \underline{x} \quad C = t \quad t > 0$$

$$S_1 = t - \underline{x}^T t^{-1} \underline{x} = t - \frac{\underline{x}^T \underline{x}}{t}$$

$$\text{so } D \geq 0 \quad \Leftrightarrow \quad \begin{matrix} \underline{x}^T \underline{x} \leq t^2 \\ t > 0 \end{matrix} \quad \text{or} \quad \begin{matrix} \|\underline{x}\| \leq t \\ (\text{SDC}) \end{matrix}$$