EE910: Digital Communication Systems-I

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Lecture #2E: Complex random variables

Complex Random Variable

- A complex random Z= X + jY can be considered as a pair of real random variables.
- A complex random variable can be treated as a two-dimensional random vector with components X and Y.
- The PDF of a complex random variable is defined to be the joint PDF of its real and complex parts.
- If X and Y are jointly Gaussian random variables, then Z is a complex Gaussian random variable.
- The PDF of a zero-mean complex Gaussian random variable Z with i.i.d. real and imaginary parts is given by

$$
\begin{aligned}
p(z) &= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \\
&= \frac{1}{2\pi\sigma^2} e^{-\frac{|z|^2}{2\sigma^2}}
\end{aligned}
$$

Complex Random Vectors

- For a complex random variable Z, the mean and variance are defined by

$$
\begin{aligned}
E[Z] &= E[X] + jE[Y] \qquad (1) \\
Var[Z] &= E[|Z|^2] - |E[Z]|^2 = Var[X] + Var[Y]
\end{aligned}
$$

- A complex random vector is defined as $Z = X + jY$, where X and Y are real-valued random vectors of size n. Real-valued matrices for a complex random vector Z are defines as

$$
\begin{aligned}
C_X &= E[(X - E(X))(X - E[X])^t] \qquad (2) \\
C_Y &= E[(Y - E(Y))(Y - E[Y])^t] \\
C_{XY} &= E[(X - E(X))(Y - E[Y])^t] \\
C_{YX} &= E[(Y - E(Y))(X - E[X])^t]
\end{aligned}
$$

Matrices C_X and C_Y are the covariance matrices of real random vectors X and Y.

- $C_{YX} = {C_{XY}}^t$

Complex Random Vectors

- Let 2n-dimensional real vector is defines as

$$\tilde{Z} = \begin{pmatrix} X \\ Y \end{pmatrix} \quad (3)$$

then the PDF of the complex vector Z is the PDF of the real vector $\tilde{Z}$.

- The covariance matrix of $\tilde{Z}$, can be written as

$$C_{\tilde{Z}} = \begin{pmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{pmatrix}$$

-

$$C_Z = E[(Z - E[Z])(Z - E[Z])^H] \quad (4)$$
$$\tilde{C_Z} = E[(Z - E[Z])(Z - E[Z])^t]$$

C_Z and $\tilde{C_Z}$ are called the covariance and the pseudocovariance of the complex random vector Z, respectively.

Complex Random Vectors

- From definition, we have

$$C_Z = C_X + C_Y + j(C_{YX} - C_{XY}) \quad (5)$$
$$\tilde{C_Z} = C_X - C_Y + j(C_{XY} + C_{YX})$$
$$C_X = \frac{1}{2} Re[C_Z + \tilde{C_Z}]$$
$$C_Y = \frac{1}{2} Re[C_Z - \tilde{C_Z}]$$
$$C_{YX} = \frac{1}{2} Im[C_Z + \tilde{C_Z}]$$
$$C_{XY} = \frac{1}{2} Im[\tilde{C_Z} - C_Z]$$

Proper and Circularly Symmetric Random Vectors

- A complex random vector Z is called proper if its pseudocovariance is zero, i.e., if $\tilde{C}_Z = 0$.
- For a proper random vector

$$\begin{aligned} C_X &= C_Y \\ C_{XY} &= -C_{YX} \end{aligned} \quad (6)$$

- Also,

$$\begin{aligned} C_Z &= 2C_X + 2jC_{YX} \\ C_X = C_Y &= \frac{1}{2}Re[C_Z] \\ C_{YX} = -C_{XY} &= \frac{1}{2}Im[C_Z] \\ C_{\tilde{Z}} &= \begin{pmatrix} C_X & C_{XY} \\ -C_{XY} & C_X \end{pmatrix} \end{aligned} \quad (7)$$

Proper and Circularly Symmetric Random Vectors

- For n= 1

$$\begin{aligned} Var[X] &= Var[Y] \\ Cov[X,Y] &= -Cov[Y,X] \end{aligned} \quad (8)$$

which means that Z is proper if X and Y have equal variances and are uncorrelated.

- If the complex random vector $Z = X + jY$ is Gaussian, meaning that X and Y are jointly Gaussian, then we have

$$p(z) = p(\tilde{z}) = \frac{1}{(2\pi)^n (detC_{\tilde{z}})^{\frac{1}{2}}} e^{-\frac{1}{2}(\tilde{z}-\tilde{m})^t C_{\tilde{z}}{}^{-1}(\tilde{z}-\tilde{m})} \quad (9)$$

where

$$\tilde{m} = E[\tilde{Z}]$$

Proper and Circularly Symmetric Random Vectors

- If Z is a proper n-dimensional complex Gaussian random vector, with mean $m = E[Z]$ and nonsingular covariance matrix C_Z, its PDF can be written as

$$p(z) = \frac{1}{\pi^n det C_Z} e^{-\frac{1}{2}(z-m)^\dagger {C_Z}^{-1}(z-m)} \tag{10}$$

- A complex random vector Z is called circularly symmetric or circular if rotating the vector by any angle does not change its PDF.
- For complex Gaussian random vectors being zero-mean and proper is equivalent to being circular.

Proper and Circularly Symmetric Random Vectors

- If $\boldsymbol{Z}$ is circular, then it is zero-mean and proper.
- Since $\boldsymbol{Z}$ and $\boldsymbol{Z}e^{j\theta}$ have the same pdf, we have $E[\boldsymbol{Z}] = E\left[\boldsymbol{Z}e^{j\theta}\right] = e^{j\theta}E[\boldsymbol{Z}]$ for all θ.
- Putting $\theta = \pi$ gives $E[\boldsymbol{Z}] = \mathbf{0}$.
- We also have $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = E\left[\boldsymbol{Z}e^{j\theta}\left(\boldsymbol{Z}e^{j\theta}\right)^t\right]$ or $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = e^{2j\theta}E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right]$, for all θ.
- Putting $\theta = \frac{\pi}{2}$ gives $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = \mathbf{0}$.
- Since $\boldsymbol{Z}$ is zero-mean and $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = \mathbf{0}$, we conclude that it is proper.

Proper and Circularly Symmetric Random Vectors

- If Z is a zero-mean proper Gaussian complex vector, then Z is circular.
- If Z is a proper n-dimensional complex Gaussian random vector, with mean $m = E[Z]$ and nonsingular covariance matrix C_Z, its PDF can be written as

$$p(z) = \frac{1}{\pi^n det C_Z} e^{-\frac{1}{2}(z-m)^\dagger {C_Z}^{-1}(z-m)}$$

- We note that for the zero-mean proper case if $\boldsymbol{W} = e^{j\theta}\boldsymbol{Z}$, it is sufficient to show that $\det(\boldsymbol{C_W}) = \det(\boldsymbol{C_Z})$ and $\boldsymbol{w}^H \boldsymbol{C}_{\boldsymbol{W}}^{-1} \boldsymbol{w} = \boldsymbol{z}^H \boldsymbol{C}_{\boldsymbol{Z}}^{-1} \boldsymbol{z}$.
- But $\boldsymbol{C_W} = \left[\boldsymbol{W}\boldsymbol{W}^H\right] = E\left[e^{j\theta}\boldsymbol{Z}e^{-j\theta}\boldsymbol{Z}^H\right] = E\left[\boldsymbol{Z}\boldsymbol{Z}^H\right] = \boldsymbol{C_Z}$, hence $\det(\boldsymbol{C_W}) = \det(\boldsymbol{C_Z})$. Similarly, $\boldsymbol{w}^H \boldsymbol{C}_{\boldsymbol{W}}^{-1} \boldsymbol{w} = e^{-j\theta}\boldsymbol{z}^H \boldsymbol{C}_{\boldsymbol{Z}}^{-1} \boldsymbol{z} e^{j\theta} = \boldsymbol{z}^H \boldsymbol{C}_{\boldsymbol{Z}}^{-1} \boldsymbol{z}$.
- Substituting this, we conclude that $p(\boldsymbol{w}) = p(\boldsymbol{z})$.

Proper and Circularly Symmetric Random Vectors

- If Z is a proper complex vector, then any transform of the form $\mathbf{W} = \mathbf{AZ} + \mathbf{b}$ is also a proper complex vector.
- Since $\boldsymbol{Z}$ is proper, we have $E\left[(\boldsymbol{Z} - E(\boldsymbol{Z}))(\boldsymbol{Z} - E(\boldsymbol{Z}))^t\right] = \mathbf{0}$.
- Let $\boldsymbol{W} = \boldsymbol{AZ} + \boldsymbol{b}$, then

$$E\left[(\boldsymbol{W} - E(\boldsymbol{W}))(\boldsymbol{W} - E(\boldsymbol{W}))^t\right] = \boldsymbol{A}E\left[(\boldsymbol{Z} - E(\boldsymbol{Z}))(\boldsymbol{Z} - E(\boldsymbol{Z}))^t\right]\boldsymbol{A}^t = \mathbf{0} \tag{11}$$

- Hence $\boldsymbol{W}$ is proper.

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