Positive Definite Matrices

Rohit Budhiraja, IITK

Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed eigenvalues and eigenvectors for a symmetric matrix A
- Discuss the concept of positive definite matrices
  - Chapter 6.5 of the book

Positive Definite Matrices

- Positive definite matrices - symmetric matrices with positive eigenvalues λ
- How to recognize positive definite matrices
  - You may say, just find eigenvalues and test whether they are > 0
  - This is what we want to avoid - calculating eigenvalues is a work
- When eigenvalues are needed, we can calculate them
- But if we just want to know whether they are $+$ve, there are faster ways
- Here are two goals of today's class
  - To find quick tests on a symmetric matrix that guarantee positive eigenvalues
  - To explain important applications of positive definiteness
- Every eigenvalue is real because the matrix is symmetric

Positive Definite Matrices – Test-1

- When is 2×2 matrix $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ have $\lambda_1 > 0$ and $\lambda_2 > 0$
- Test1: Eigenvalues of S are positive if and only if

$$a > 0 \text{ and } ac - b^2 > 0$$

- Proof: Their product $\lambda_1 \lambda_2$ is equal to the determinant i.e., $ac - b^2$
  - If $\lambda_1 \lambda_2 > 0$ then $ac - b^2 > 0$
- Their sum $\lambda_1 + \lambda_2$ is equal to the trace i.e., $a + c$
  - If $\lambda_1 + \lambda_2 > 0$ then $a + c > 0$
  - Then a and c are both positive, (if a or c is not positive, $ac - b^2 > 0$ will fail)
  - If $a > 0$ and $ac - b^2 > 0$ then it automatically implies $c > 0$
- This test uses the 1 by 1 determinant a and the 2 by 2 determinant $ac - b^2$
- When S is 3 by 3, we need to also check det $S > 0$

Positive Definite Matrices – Test-2

- When is 2×2 matrix $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ have $\lambda_1 > 0$ and $\lambda_2 > 0$
- Test2: Eigenvalues of S are positive if and only if the pivots are positive

$$a > 0 \text{ and } \frac{ac - b^2}{a} > 0$$

- $a > 0$ is required in both tests
  - So $ac > b^2$ is also required, for the determinant test and now the pivot test
- Point is to recognize that ratio as the second pivot of S

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \xrightarrow[\text{The multiplier is } b/a]{\text{The first pivot is } a} \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a}b \end{bmatrix} \quad \textit{The second pivot is } c - \frac{b^2}{a} = \frac{ac - b^2}{a}$$

Positive Definite Matrices – Test-2

- This connects two big parts of linear algebra
  - Positive eigenvalues mean positive pivots and vice versa
- Each pivot is a ratio of upper left determinants
  - Pivots give a quick test for $A > 0$
  - Pivots are a lot faster to compute than the eigenvalues
- Satisfying to see pivots and determinants and eigenvalues come together

Positive Definite Matrices – Test 3

- We have $S\mathbf{x} = \lambda\mathbf{x}$. We now perform inner product with $\mathbf{x}$

$$\mathbf{x}^T S\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} \Rightarrow \mathbf{x}^T S\mathbf{x} = \lambda||\mathbf{x}||^2$$

- Since $\lambda > 0$ for a positive definite matrix. This implies that $\mathbf{x}^T S\mathbf{x} > 0$
- In many applications this number is the energy of the system
- Idea is that $\mathbf{x}^T S\mathbf{x}$ is positive for all nonzero vectors $\mathbf{x}$, not just eigenvectors
- If S and T are positive definite, so is $S + T$
    - Reason: $\mathbf{x}^T(S + T)\mathbf{x} = \mathbf{x}^T S\mathbf{x} + \mathbf{x}^T T\mathbf{x}$
    - Since, both $\mathbf{x}^T S\mathbf{x} > 0$ and $\mathbf{x}^T T\mathbf{x} > 0$, their sum is > 0
- $\mathbf{x}^T S\mathbf{x}$ also connects with our final way to recognize a positive definite matrix

Positive Definite Matrices – Test 4

- Consider matrix A, possibly rectangular
  - We know that $S = A^T A$ is square and symmetric
- More than that, S will be positive definite when A has independent columns:
- Test: If the columns of A are independent, then $S = A^T A$ is positive definite

$$\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T A\mathbf{x} = ||A\mathbf{x}||^2$$

- If A has independent columns, vector $A\mathbf{x}$ is not zero when $\mathbf{x} \neq 0$
- Then $\mathbf{x}^T S\mathbf{x}$ is the positive number $||A\mathbf{x}||^2$ and the matrix S is positive definite

Positive Definite Matrices – Summary

- When a symmetric matrix S has one of these five properties, it has them all:
  - All n eigenvalues of S are positive
  - All n upper left determinants are positive
  - All n pivots of S are positive
  - $\mathbf{x}^T S \mathbf{x}$ is positive except at $\mathbf{x} = 0$. This is the energy-based definition
  - S equals $A^T A$ for a matrix A with independent columns
- Cholesky factorization of positive definite matrix

$$S = Q\Lambda Q^T = Q\sqrt{\Lambda}\sqrt{\Lambda}^T Q^T = Q\sqrt{\Lambda}(Q\sqrt{\Lambda})^T = A_1 A_1^T$$

- Square-root of a positive definite matrix

$$\begin{aligned} \sqrt{S} &= Q\sqrt{\Lambda}Q^T \\ \sqrt{S}\sqrt{S} &= Q\sqrt{\Lambda}Q^T Q\sqrt{\Lambda}Q^T = Q\Lambda Q^T = S \end{aligned}$$

Positive Semidefinite Matrices

- All eigenvalues of S are ≥ 0
- $\mathbf{x}^T S \mathbf{x} \geq 0$