EE901 PROBABILITY AND RANDOM PROCESSES

MODULE 8
CONDITIONAL DISTRIBUTION

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Independence of RVs

- The random variables X and Y are mutually independent if

$$\mathbb{P}(X \in B_1, Y \in B_2) = \mathbb{P}(X \in B_1)\mathbb{P}(Y \in B_2)$$

for any sets B_1 and B_2

- This implies

$$F_{X,Y}(x,y) = F_X(x)\ F_Y(y)$$
$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$
$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

- What about RVs that are not independent?

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Conditional Distribution Given an Event

- Conditional distribution (CDF) of a random variable X given an event A

$$F_{X|A}(x) = \mathbb{P}_{X|A}(\{\omega : X(\omega) \le x\}|A)$$
$$= \frac{\mathbb{P}(\{\omega \colon X(\omega) \le x \;\&\; \omega \in A\})}{\mathbb{P}(A)}$$

Example: Pick a number between 0 and 1

$\mathbb{P}((a,b)) = (b-a)$

X be the random variable $X(\omega) = \omega$.

Let event $A = [0, 0.5] = \{\omega\colon 0 \le \omega \le 0.5\}$.

So that, $\mathbb{P}(A) = (0.5 - 0) = 0.5$.

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Conditional Distribution Given an Event

Example: Pick a number between 0 and 1. $X(\omega) = \omega$

$F_{X|A}(x) = \mathbb{P}(X \le x | A)$

$A = \{\omega\colon 0 \le \omega \le 0.5\}$. $\mathbb{P}(A) = 0.5$.

$$\mathbb{P}(\{\omega : X(\omega) \le x\} \cap A) = \mathbb{P}(\{\omega \colon X(\omega) \le x \;\&\; \omega \in A\})$$
$$= \mathbb{P}(\{\omega \colon \omega \le x \;\&\; 0 \le \omega \le 0.5\})$$

x	$X^{-1}(B)$	$\mathbb{P}(\{\omega : X(\omega) \le x\} \cap A)$	$\mathbf{F}_{X\|A}(x)$
$x < 0$	ϕ	0	$0/0.5 = 0$
$x = 0$	$\{0\}$	0	$0/0.5 = 0$
$0 < x < 0.5$	$\{\omega\colon \omega \le x\} = [0, x]$	x	$x/0.5 = 2x$
$x \ge 0.5$	$[0, 0.5]$	0.5	$0.5/0.5 = 1$

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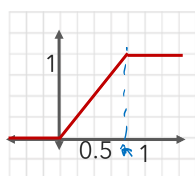
Conditional Distribution Given an Event

Example: Pick a number between 0 and 1. $X(\omega) = \omega$

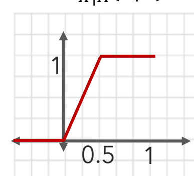
$A = \{\omega\colon 0 \le \omega \le 0.5\}$. $\mathbb{P}(A) = 0.5$.

Example: Pick a number between 0 and 1.

$F_X(x)$



$F_{X|A}(x|A)$



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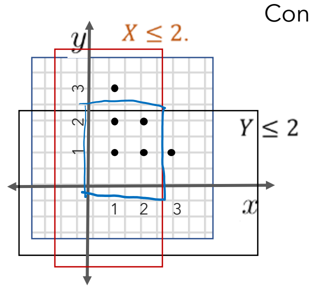
Conditional Distribution of Y given X

Consider the random variable X and Y with the following joint PMF. Each point is equiprobable.



$p_{X,Y}(x,y)$

Conditional on $Y \le 2$, probability that $X \le 2$.

$P(A|B)$ $\frac{P(A \cap B)}{P(B)}$

$$\mathbb{P}(\underset{A}{X \le 2}|\underset{B}{Y \le 2}) = \frac{\mathbb{P}(\{X \le 2\} \cap \{Y \le 2\})}{\mathbb{P}(\{Y \le 2\})}$$

$$= \frac{\mathbb{P}(\{X \le 2 \;\&\; Y \le 2\})}{\mathbb{P}(\{Y \le 2\})}$$

$$= \frac{\sum_{x \le 2, y \le 2} p_{X,Y}(x,y)}{\sum_{y \le 2} p_Y(y)}$$

$$p_Y(y) = \sum_{x=1,2,3} p_{X,Y}(x,y) = \begin{cases} 3/6 & \text{if } y = 1 \\ 2/6 & \text{if } y = 2 \\ 1/6 & \text{if } y = 3 \end{cases}$$

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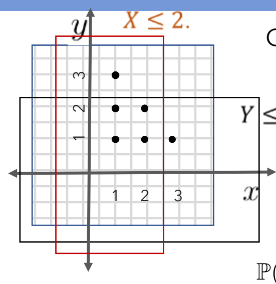
Conditional Distribution of Y given X



$p_{X,Y}(x,y)$

Conditional on Y ≤ 2, probability that $X \leq 2$.

$$\mathbb{P}(X \leq 2|Y \leq 2) = \frac{\sum_{x\leq 2, y\leq 2} p_{X,Y}(x,y)}{\sum_{y \leq 2} p_Y(y)}$$

$$p_Y(y) = \begin{cases} 3/6 & \text{if } y = 1 \\ 2/6 & \text{if } y = 2 \\ 1/6 & \text{if } y = 3 \end{cases}$$

$$\mathbb{P}(X \leq 2|Y \leq 2) = \frac{1/6 + 1/6 + 1/6 + 1/6}{3/6 + 2/6} = \frac{4}{5}$$

This can be seen a conditional CDF of X given Y. $F_{X|Y}(x|y) = \mathbb{P}\left(X \leq x | Y \leq y\right)$

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Dependent Variables

- Note that

$$F_{X|Y}(x|Y \leq y) = \mathbb{P}(X \leq x|Y \leq y) = \frac{\mathbb{P}(\{X \leq x \ \& \ Y \leq y\})}{\mathbb{P}(\{Y \leq y\})} = \frac{F_{X,Y}(x,y)}{F_Y(y)}$$

- For independent variables

$$F_{X,Y}(x,y) = F_X(x) \ F_Y(y)$$

$$F_{X|Y}(x|Y \leq y) = \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{F_X(x) F_Y(y)}{F_Y(y)} = F_X(x)$$

- For a general case,

$$F_{X|Y}(x|Y \leq y) \neq F_X(x)$$

- These RVs are called dependent variables.

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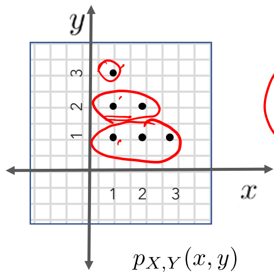
Conditional PMFs

The conditional PMF of X given Y

$$p_{X|Y}(x|y) = \mathbb{P}[X = x|Y = y]$$
$$= \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$p_Y(y) = \begin{cases} 3/6 & \text{if } y = 1 \\ 2/6 & \text{if } y = 2 \\ 1/6 & \text{if } y = 3 \end{cases}$$

Example:



$p_{X,Y}(x,y)$

$y=1$

$$P_{X|Y}(x|1) = \begin{cases} \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{1/6}{3/6} = \frac{1}{3} & x=1 \\ \frac{1}{3} & x=2, x=3 \end{cases}$$

$$y=2 \quad P_{X|Y}(x|2) = \begin{cases} \frac{1/6}{2/6} = \frac{1}{2} & x=1,2 \end{cases}$$

$$y=3 \quad P_{X|Y}(x|3) = \left\{ \frac{1/6}{1/6} = 1 \right. \quad x=1$$

Conditional PDFs

The conditional PDF of X given Y

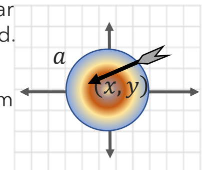
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Conditional CDF of X given Y can be written as

$$F_{X|Y}(x|Y=y) = \int_{-\infty}^{x} f_{X|Y}(x|y)\mathrm{d}x = \int_{-\infty}^{x} \frac{f_{X,Y}(x,y)}{f_Y(y)}\mathrm{d}x$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$$

Example: Dart Throw



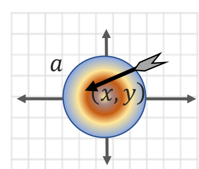
- Consider a random experiment where a dart is thrown on a circular board B. The outcome is the location where the dart hits the board. Board radius is a.
- $\Omega = B$. Each outcome ω is a 2D coordinate (x, y). Assume a uniform probability measure which means
  - $\mathbb{P}(A) = \frac{|A|}{\pi a^2}$ for any set A on the board.
  - Let $X(\omega)$ and $Y(\omega)$ denote the x and y coordinate of the outcome.
- Joint PDF of X and Y can be computed as

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi a^2} & \text{if } x^2+y^2 \le a^2 \\ 0 & \text{otherwise} \end{cases} \quad \rightarrow (x,y) \in B$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example: Dart Throw

Joint PDF of X and Y $\quad f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi a^2} & \text{if } x^2+y^2 \le a^2 \\ 0 & \text{otherwise} \end{cases}$



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\pi a^2} \underbrace{1(x^2+y^2 \le a^2)\mathrm{d}x}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi a^2} 1(x^2 \le a^2 - y^2)\mathrm{d}x = \frac{1}{\pi a^2} 2\sqrt{a^2-y^2}$$

$1(-\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2})$

$-\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2}$

Functions of Multiple Random Variables

Expectation of A Function of Two RVs

- Let X and Y be two random variables.
- Consider a function $g : \mathcal{R}(X) \times \mathcal{R}(Y) \rightarrow \mathbb{R}$.

$$
\begin{aligned}
\mathrm{E}_{X,Y}[g(X,Y)] &= \int\int g(x,y) f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int\int g(x,y) f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int\int g(x,y) f_{X|Y}(x|y) f_Y(y)\mathrm{d}x\mathrm{d}y \\
&= \int\left(\int g(x,y) f_{X|Y}(x|y)\mathrm{d}x\right) f_Y(y)\mathrm{d}y = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}\left[g(X,Y)\right]\right]
\end{aligned}
$$

$\int$ $f_Y(y)dy$

$$\mathbb{E}_{X,Y}\left[g(X,Y)\right] = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}\left[g(X,Y)\right]\right]$$

Conditional Expectation

The expression

$$\mathbb{E}_{X|Y}\left[g(X,Y)|Y=y\right] = \int g(x,y) f_{X|Y}(x|y)\mathrm{d}x$$

is known as conditional expectation of X given Y

- It is a function of Y and thus, itself a random variable.

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Joint Moments

- Consider two random variables X, Y.
- Covariance

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}_{XY}[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\end{aligned}$$

- Correlation

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

$-1 \leq$ ≤ 1

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Joint Moments for Independent RVs

- If X, Y are independent.

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\mathrm{d}x\mathrm{d}y$$

$$= \underbrace{\int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x}_{E(X)} \underbrace{\int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y}_{E(Y)} = \mathbb{E}[X]\mathbb{E}[Y]$$

$$\text{Cov}(X,Y) = \mathbb{E}_{XY}[(X - \mathbb{E}(X)(Y - \mathbb{E}(Y))]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$\text{Cov}(X,Y) = 0$$
$$\text{Corr}(X,Y) = 0$$

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Uncorrelated and Orthogonal RVs

- Two random variables are called to uncorrelated if $\text{Cov}(X,Y) = 0$
- Independent variables are uncorrelated
- However, uncorrelated variables may not be independent.
- Two random variables are called to orthogonal if

$$\mathbb{E}[XY] = 0$$

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Variance of the Sum of two RVs

- Consider a random variable Z such that

$$Z = X + Y$$

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathbb{E}[(Z - \mathbb{E}(Z))^2] \\
&= \mathbb{E}[(X + Y - \mathbb{E}[X] - \mathbb{E}[Y])^2] \\
&= \mathbb{E}[(X - \mathbb{E}(X))^2] + \mathbb{E}[(Y - \mathbb{E}(Y))^2] + 2\mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)
\end{aligned}$$

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