eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 4

Vector Parameter Estimation

Least Squares Solution

How to derive this?

- The Least Squares solution can be **derived** as follows

$$\hat{h} = \left(X^T X\right)^{-1} X^T \bar{y}$$

Least Squares Solution

Least Squares Solution

- The LS cost function is

$$\min \|\bar{y} - X\bar{h}\|^2$$

minimize this

$N \times 1$ $N \times M$ $M \times 1$

Least Squares Solution

- The LS cost function is

$$\left\| \bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}} \right\|^2$$

Least Squares Solution

- The LS cost function can be simplified as

$$\left\|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\right\|^2$$

$$= (\bar{y} - X\bar{h})^T(\bar{y} - X\bar{h}) = (\bar{y}^T - \bar{h}^T X^T)(\bar{y} - X\bar{h})$$

$$= \bar{y}^T\bar{y} - \bar{h}^T X^T \bar{y} - \bar{y}^T X \bar{h} + \bar{h}^T X^T X \bar{h}$$

EQUAL!

Least Squares Solution

$$\left(\bar{h}^T X^T \bar{y}\right)^T = \bar{y}^T X \bar{h}$$

$$[5]^T = [5]$$

Scalar & Transpose
$\Rightarrow$ EQUAL!

$$\|\bar{y} - X\bar{h}\|^2 = \underbrace{\bar{y}^T\bar{y} - 2\bar{h}^T X^T \bar{y} + \bar{h}^T X^T X \bar{h}}_{\text{Expansion}}$$

Least Squares Solution

Find minimum wrt $\bar{h}$

- The **LS cost function** can be simplified as

$$\left\|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\right\|^2$$

$$= \left(\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\right)^T \left(\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\right)$$

$$= \bar{\mathbf{y}}^T \bar{\mathbf{y}} - 2\bar{\mathbf{h}}^T \mathbf{X}^T \bar{\mathbf{y}} + \bar{\mathbf{h}}^T \mathbf{X}^T \mathbf{X}\bar{\mathbf{h}}$$

Least Squares Solution

- To minimize, we now calculate the **gradient** and set to zero

$$\nabla f(\bar{\mathbf{h}}) = \begin{bmatrix} \partial f / \partial h_1 \\ \partial f / \partial h_2 \\ \vdots \\ \partial f / \partial h_M \end{bmatrix}$$

Partial derivatives w.r.to each component of $\bar{h}$

Least Squares Solution

- We now calculate the gradient

$$\nabla f(\bar{\mathbf{h}}) = \begin{bmatrix} \frac{\partial f(\bar{\mathbf{h}})}{\partial h_1} \\ \frac{\partial f(\bar{\mathbf{h}})}{\partial h_2} \\ \vdots \\ \frac{\partial f(\bar{\mathbf{h}})}{\partial h_M} \end{bmatrix}$$

$\underline{M \times 1}$

Least Squares Solution

$\bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}$

- We use the following principles

$$\bar{\mathbf{c}}^T\bar{\mathbf{h}} = c_1h_1 + \cdots + c_Mh_M = \bar{\mathbf{h}}^T\bar{\mathbf{c}}$$

$$\nabla\bar{\mathbf{c}}^T\bar{\mathbf{h}} = \nabla\bar{\mathbf{h}}^T\bar{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} = \bar{c}$$

Principle #1

$$\nabla \bar{h}^T\bar{c} = \nabla \bar{c}^T\bar{h} = \bar{c}$$

Least Squares Solution

- We use the following principles

$$\bar{\mathbf{c}}^T \bar{\mathbf{h}} = c_1 h_1 + \cdots + c_M h_M$$

$$\nabla \bar{\mathbf{c}}^T \bar{\mathbf{h}} = \nabla \bar{\mathbf{h}}^T \bar{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}$$

Least Squares Solution

- For a symmetric matrix $\mathbf{P} = \mathbf{P}^T$

$$\nabla\left(\bar{\mathbf{h}}^T \mathbf{P} \bar{\mathbf{h}}\right) = 2P\bar{h}$$

Principle #2

Least Squares Solution

- For a symmetric matrix $\mathbf{P} = \mathbf{P}^T$

$$\nabla \bar{\mathbf{h}}^T \mathbf{P} \bar{\mathbf{h}} = 2\mathbf{P}\bar{\mathbf{h}}$$

Least Squares Solution

- Therefore, it follows that

$$\nabla \left\| \bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}} \right\|^2$$

$$= \cancel{\nabla\, \bar{y}^T \bar{y}} - \nabla 2\, \bar{h}^T \underbrace{X^T \bar{y}}_{\bar{c}} + \nabla \bar{h}^T X^T X \bar{h}$$

$$= 0 - 2X^T \bar{y} + 2\left(X^T X\right)\bar{h}$$

Now set this to 0.

$= 0$ since $\bar{y}$ is a given fixed vector!

Principle 1 with $\bar{c} = X^T \bar{y}$

$P = X^T X$

$P = P^T$

Principle 2 $X^T X = P$

Least Squares Solution

- Therefore, it follows that

To minimize
set this to zero.

$$\nabla \left\| \bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}} \right\|^2 =$$

$$\nabla \left(\bar{\mathbf{y}}^T \bar{\mathbf{y}} - 2\bar{\mathbf{h}}^T \mathbf{X}^T \bar{\mathbf{y}} + \bar{\mathbf{h}}^T \mathbf{X}^T \mathbf{X}\bar{\mathbf{h}} \right)$$

$$= -2\mathbf{X}^T \bar{\mathbf{y}} + 2\mathbf{X}^T \mathbf{X}\bar{\mathbf{h}}$$

Least Squares Solution

- Setting gradient equal to zero yields

$$-2X^T\bar{y} + 2X^TX\bar{h} = 0$$

$$\Rightarrow \cancel{2}X^TX\bar{h} = \cancel{2}X^T\bar{y}$$

ML Estimate.

$$\Rightarrow \hat{h} = \left(X^TX\right)^{-1}X^T\bar{y}$$

LEAST SQUARES SOLUTION.

Least Squares Solution

- Setting gradient equal to zero yields

$$-2\mathbf{X}^T\bar{\mathbf{y}} + 2\mathbf{X}^T\mathbf{X}\bar{\mathbf{h}} = \mathbf{0}$$

$$\Rightarrow \mathbf{X}^T\mathbf{X}\bar{\mathbf{h}} = \mathbf{X}^T\bar{\mathbf{y}}$$

$$\Rightarrow \hat{\mathbf{h}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

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