EE901
PROBABILITY AND RANDOM PROCESSES

MODULE 6
MULTIPLE RANDOM VARIABLES

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Joint Probability Density Function

- If there exists a function $f_{X,Y}(x,y)$ such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

it is known as the joint probability density function of X, Y .

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}[F_{X,Y}(x,y)]$$

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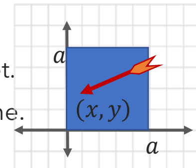
Example: Dart Throw



- $\Omega = B$. Each outcome ω is a 2D coordinate (x, y).
- Assume a uniform probability measure $\mathbb{P}(A) = |A|/a^2$ for any set.
- Let $X(\omega)$ and $Y(\omega)$ denote the x and y coordinate of the outcome.
- Joint CDF $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ OR } y < 0 \\ 1 & \text{if } x > a \text{ AND } y > a \\ \frac{1}{a^2}\min(x,a)\min(y,a) & \text{otherwise} \end{cases}$$

- Joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2} & \text{if } a \geq x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Probability Law in terms of PDF

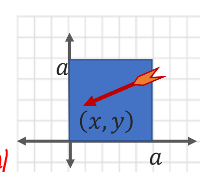
- For any 2D Borel set A

$$\mathbb{P}[(X,Y) \in A] = \int_A f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

- What is probability that the dart hits within 0.2 distance of the center?
- A=Circle of radius 0.2

$A = \{ (x,y) : (x - \frac{a}{2})^2 + (y - \frac{a}{2})^2 = a^2 \}$

$\iint_A \frac{1}{a^2} \, dx\, dy = \frac{1}{a^2} \iint_A dx dy = \frac{1}{a^2} \times |A|$



Marginal Distribution

- The individual distribution of X and Y are called marginal distribution.
- It is possible to compute the marginal distribution from the joint distribution.
- Let us first compute marginal CDFs of X and Y

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$
$$F_{X,Y}(x,\infty) = \mathbb{P}(X \leq x, Y \leq \infty)$$
$$= \mathbb{P}(X \leq x) = F_X(x)$$

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ OR } y < 0 \\ 1 & \text{if } x > a \text{ AND } y > a \\ \frac{1}{a^2}\min(x,a)\min(y,a) & \text{otherwise} \end{cases}$$

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > a \\ x/a & \text{otherwise} \end{cases}$$

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Marginal PMFs

- Let us first compute marginal PMFs of X and Y

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

$$\sum_{y \in \mathcal{R}_Y} p_{X,Y}(x,y) = \sum_{y \in \mathcal{R}_Y} \mathbb{P}(X = x, Y = y) \quad = \mathbb{P}(\cup_{y \in \mathcal{R}_Y}\{X = x, Y = y\})$$
$$= \mathbb{P}(\{X = x, Y \in \mathcal{R}_Y\})$$
$$= \mathbb{P}(\{X = x\}) = p_X(x)$$

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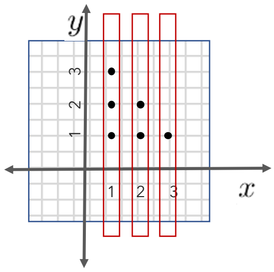
Marginal PMFs

$$p_X(x) = \sum_{y \in \mathcal{R}_Y} p_{X,Y}(x,y)$$



Consider the PMF. Each point is equi-probable.

$$p_X(x) = p_{X,Y}(x,1) + p_{X,Y}(x,2) + p_{X,Y}(x,3)$$

$$p_X(1) = p_{X,Y}(1,1) + p_{X,Y}(1,2) + p_{X,Y}(1,3) = \frac{3}{6} = 0.5$$

$$p_X(2) = p_{X,Y}(2,1) + p_{X,Y}(2,2) + p_{X,Y}(2,3) = \frac{2}{6} = 0.33$$

$$p_X(3) = p_{X,Y}(3,1) + p_{X,Y}(3,2) + p_{X,Y}(3,3) = \frac{1}{6} = 0.17$$

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Marginal PDFs

- Let us first compute marginal PDFs of X and Y

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x$$

$$F_X(x) = F_{X,Y}(x,\infty) = \int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y}\mathrm{d}x$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\mathrm{d}x \quad ✓$$

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y \qquad f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X,Y}(x,\infty)$$

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Marginal PDFs

- Marginal PDFs of X and Y $\quad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y \quad f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X,Y}(x,\infty)$

Example $\quad f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2} & \text{if } a \geq x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > a \\ x/a & \text{otherwise} \end{cases}$$

if $a \geq x \geq 0$

$$f_X(x) = \int_0^a \frac{1}{a^2}\mathrm{d}y \quad = \frac{1}{a}$$

otherwise

$$f_X(x) = \int_0^a 0\mathrm{d}y = 0$$

Independence of RVs

- Random variables X and Y are mutually independent if

$$\mathbb{P}(X \in B_1, Y \in B_2) = \mathbb{P}(X \in B_1)\,\mathbb{P}(Y \in B_2)$$

for any sets B_1 and B_2

- If $B_1 = (-\infty, x]$ and $B_2 = (-\infty, y]$,

$$F_{X,Y}(x,y) = F_X(x)\ F_Y(y)$$
$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\,\mathbb{P}(Y \leq y).$$

Independence of RVs

- Random variables X and Y are mutually independent if

  for any sets B_1 and B_2

- For DRVs, let us take $B_1 = \{x\}, B_2 = \{y\}$

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$$

Independence of RVs

- Random variables X and Y are mutually independent if

$$\mathbb{P}(X \in B_1, Y \in B_2) = \mathbb{P}(X \in B_1)\,\mathbb{P}(Y \in B_2)$$

  for any sets B_1 and B_2

- For CRVs, let us take $B_1 = (x, x + \epsilon)$ and $B_2 = (y, y + \epsilon)$

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

Functions of Multiple Random Variables

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Function of Two RVs

- Let X_1 and X_2 have joint PDF $f_{X_1,X_2}(x_1,x_2)$.
- Let Z be a function of X_1 and X_2

$$Z(\omega) = g(X_1(\omega), X_2(\omega)) \quad \forall \omega \in \Omega$$

$Z = g(X_1, X_2)$

- Z is also a random variable.

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Expectation of A Function of Two RVs

- Let X and Y be two random variables.
- Consider a function $g : \mathcal{R}(X) \times \mathcal{R}(Y) \to \mathbb{R}$.

DRVs

$$\mathrm{E}_{X,Y}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y) \mathrm{d}x\mathrm{d}y \qquad \sum_{x,y} g(x,y)\, p_{XY}(x,y)$$

- Random variables X and Y are mutually independent if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.
- Let $g(x,y) = u(x)v(y)$

$$\begin{aligned}\mathrm{E}_{X,Y}[g(X,Y)] &= \int\int u(x)v(y) f_X(x) f_Y(y) \mathrm{d}x\mathrm{d}y \\ &= \int u(x) f_X(x) \mathrm{d}x \int v(y) f_Y(y) \mathrm{d}y \\ &= \mathrm{E}_X[u(X)]\ \mathrm{E}_Y[v(Y)]\end{aligned}$$

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Expectation of A Function of Two RVs

- X and Y are mutually independent. For general function $g(x,y)$

$$\mathrm{E}_{X,Y}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y) \mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}\mathbb{E}[g(X,Y)] &= \int\int g(x,y) f_{X,Y}(x,y) \mathrm{d}x\ \mathrm{d}y \\ &= \int\int g(x,y) f_X(x) f_Y(y) \mathrm{d}x\ \mathrm{d}y \\ &= \int\int g(x,y) f_X(x) \mathrm{d}x\, f_Y(y) \mathrm{d}y\end{aligned}$$

Given y, think $g(x,y)$ as function of x: $h(x) = g(x,y)$. Then

$$\begin{aligned}\int g(x,y) f_X(x) \mathrm{d}x &= \int h(x) f_X(x) \mathrm{d}x \\ &= \mathbb{E}_X[h(X)] = \mathbb{E}_X[g(X,y)]\end{aligned}$$

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Expectation of A Function of Two RVs

- X and Y are mutually independent. For general function $g(x, y)$

$$\mathrm{E}_{X,Y}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}
\mathbb{E}[g(X,Y)] &= \int\int g(x,y) f_{X,Y}(x,y)\mathrm{d}x\ \mathrm{d}y \\
&= \int\int g(x,y) f_X(x) f_Y(y)\mathrm{d}x\ \mathrm{d}y \\
&= \int\int \underbrace{g(x,y) f_X(x)\mathrm{d}x}\, f_Y(y)\mathrm{d}y \\
&= \int \mathbb{E}_X[g(X,y)] f_Y(y)\mathrm{d}y \\
&= \mathbb{E}_Y[\mathbb{E}_X[g(X,Y)]]
\end{aligned}$$

$\mathbb{E}[g(x,y)] = \int \underbrace{g(x,y)\, f_X(x)\, dx}_{= h(y)}$

Take the expectation with respect to each random variable one by one.

Only for independent RVs

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Distribution of A Function of Two RVs

- Let $Z(\omega) = g(X_1(\omega), X_2(\omega)) \quad \forall \omega \in \Omega$
- For any set B

$$\begin{aligned}
\mathbb{P}[Z \in B] &= \mathbb{P}[g(X_1, X_2) \in B] \\
&= \iint_{g(x_1,x_2)\in B} f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2
\end{aligned}$$

- Example: Let X and Y are two random variables with $f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2} & \text{if } a \geq x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

  Compute the distribution of $Z = X + Y$.

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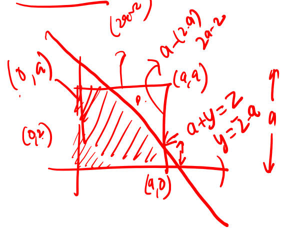
Example: Sum of Two RVs

Example: Let X and Y are two random variables with $f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2} & \text{if } a \geq x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Compute the distribution of $Z = X + Y$.

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) = \int_{x+y \leq z} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$



$z<a$

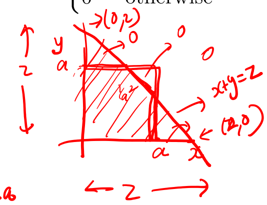
$\frac{1}{a^2} \times \frac{1}{2} z^2$

$\frac{1}{a^2} \times \frac{1}{2} a^2 = \frac{1}{2}$

$z = a$

$z > a$

$1 - \frac{1}{2}\frac{(2a-z)^2}{a^2}$



Example: Sum of Two RVs

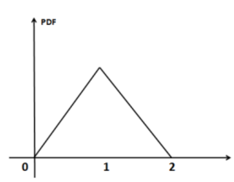
Example: Let X and Y are two random variables with $f_{X,Y}(x,y) = \begin{cases} \frac{1}{a^2} & \text{if } a \geq x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Compute the distribution of $Z = X + Y$.

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) = \int_{x+y \leq z} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

$g(x,y) \leq z$

$$\mathbb{P}[Z \leq z] = \begin{cases} 0 & z < 0 \\ \frac{1}{2}z^2 & 1 > z > 0 \\ 1 - \frac{1}{2}(2-z)^2 & 2 > z > 1 \\ 1 & z \geq 2 \end{cases}$$



Joint MGF

- Joint MGF of RVs are defined as

$$M_{X,Y}(t,s) = \mathbb{E}[e^{tX+sY}] = \int\int e^{tx+sy} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

- If random variables X and Y are mutually independent, then

$$\mathrm{E}_{X,Y}[u(X)v(Y)] = \mathrm{E}_X[u(X)]\ \mathrm{E}_Y[v(Y)]$$

$$M_{X,Y}(t,s) = \mathbb{E}[\underbrace{e^{tX+sY}}_{e^{tX}\ e^{sY}}] = \mathbb{E}[e^{tX}]\mathbb{E}[e^{sY}] = M_X(t)M_Y(s)$$

- Can be helpful in computation of functions of RVs.

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Example: Sum of Two RVs

Example: Let X and Y are two independent exponential random variables with

$$f_X(x) = \lambda e^{-\lambda x}1(x \geq 0) \qquad f_{X,Y}(x,y) = \lambda e^{-\lambda x}\lambda e^{-\lambda y}1(x \geq 0)1(y \geq 0)$$

$$f_Y(y) = \lambda e^{-\lambda y}1(y \geq 0)$$

Compute the distribution of $Z = X + Y$.

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(X+Y \leq z) = \int_{x+y\leq z} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

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Example: Sum of Two RVs

Example: Let X and Y are two independent exponential random variables with

$$f_X(x) = \lambda e^{-\lambda x} 1(x \geq 0) \qquad f_{X,Y}(x,y) = \lambda e^{-\lambda x} \lambda e^{-\lambda y} 1(x \geq 0) 1(y \geq 0)$$

$$f_Y(y) = \lambda e^{-\lambda y} 1(y \geq 0) \qquad \underline{M_X(t) = \frac{1}{1 - t/\lambda}} \qquad \underline{M_Y(t) = \frac{1}{1 - t/\lambda}}$$

Compute the distribution of $Z = X + Y$.

$$M_Z(t) = \mathbb{E}[e^{tZ}] = \mathbb{E}[e^{tX+tY}] = \underline{M_X(t)}\,\underline{M_Y(t)}$$

$$= \left(\frac{1}{1-t/\lambda}\right)\left(\frac{1}{1-t/\lambda}\right)$$

$$= \frac{1}{(1-t/\lambda)^2} \sim$$

$Z \sim \text{Gamma}(2, \lambda)$

$(2, \lambda)$

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Example: Sum of n RVs

Example: Let $\{X_i\}$ are n independent exponential random variables with

$$f_{X_i}(x) = \lambda e^{-\lambda x} 1(x \geq 0) \qquad M_{X_i}(t) = \frac{1}{1 - t/\lambda} \checkmark$$

Compute the distribution of $\underline{Z} = \sum_i X_i$.

$$\underline{M_Z(t)} = \mathbb{E}[e^{tZ}] = \mathbb{E}[e^{tX_1 + tX_2 + \cdots + tX_n}] = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \left(\frac{1}{1-t/\lambda}\right)^n \sim \text{Gamma}(n, \lambda)$$

$Z \sim \text{Gamma}(n, \lambda)$

$\sum X_i \leq Z$

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