EE901
Probability and Random Processes

Module 4
Expectation and Moments

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Expectation

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Expectation

- Random variables take different values each time the experiment is performed.
- On the average, what value can we expect?
- We call it mean/average/expected value. How to calculate it?
- Suppose random variable takes only one value, then the mean would be this value only.
- Suppose random variables takes two value a and b, with equal probability. The mean would be $(a+b)/2$.
- What if these values don't have equal chances?

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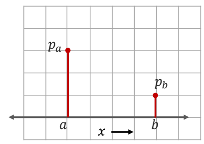
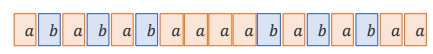
Expectation

- Consider a RV X which takes two values a and b with probability p_a and p_b
- If the experiment is repeated N times,
- Approximately
  - $N_a = N\,p_a$ times, the outcome is a
  - $N_b = N\,p_b$ times, the outcome is b.
- Average is

$$\frac{a + a + \ldots.\ N_a \text{ times } \ + b + b + \ldots\ N_b \text{ times}}{N}$$

$$= \frac{aN_a + bN_b}{N} = \frac{Np_a a + Np_b b}{N} = ap_a + bp_b$$





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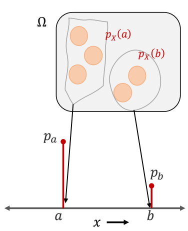
Expectation



p_a = chance of occurrence of an outcome that will result in X being a

p_b = chance of occurrence of an outcome that will result in X being b

$$\mu = ap_a + bp_b$$

$$\mu = \mathbb{P}(\{X = a\}) \times a + \mathbb{P}(\{X = b\}) \times b$$

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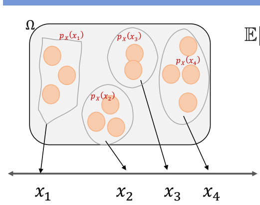
Expectation of Discrete Random Variable



$$\mathbb{E}[X] = \mu = \sum_i \mathbb{P}(\{X = x_i\}) \times x_i$$

$$= \sum_i p_X(x_i) x_i$$

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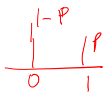
Examples

- Bernoulli RV with parameter p



$$\mathbb{E}[X] \quad 0 \times (1-p) + 1 \times p$$
$$= p$$

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Examples

- Dice roll

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$
$$= \frac{1}{6}[1+2+3+4+5+6]$$
$$= 3.5$$

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Examples

- Poisson RV with parameter λ

$$E[X] = \sum_{x=0}^{\infty} P_X(x) \cdot x$$
$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} x = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda \cdot \lambda^{x-1}}{(x-1)!}$$
$$= \sum_{k=0}^{\infty} (e^{-\lambda} \cdot \lambda) \frac{\lambda^k}{(k)!} = e^{-\lambda} \lambda \cdot e^{\lambda}$$
$$= \lambda$$

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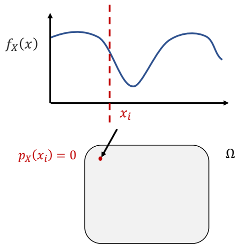
Expectation for Continuous Random Variable

$$\mathbb{E}[X] = \sum_i \mathbb{P}(\{X = x_i\}) x_i$$

For continuous RV, $\mathbb{P}(\{X = x_i\}) = 0$



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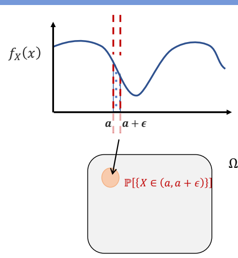
Expectation for Continuous Random Variable

$$\mathbb{P}[\{X \in (a, a+\epsilon)\}] = f_X(a)\,\epsilon$$



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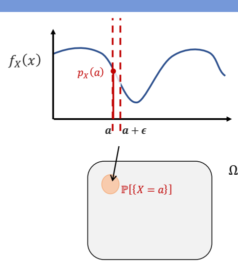
Expectation for Continuous Random Variable

$p_X(a) =$

$$\mathbb{P}[\{X \in (a, a+\epsilon)\}] = f_X(a)\,\epsilon$$



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Expectation for Continuous Random Variable

$p_X(a_i) =$
$\mathbb{P}[\{X \in (a_i, a_i + \epsilon)\}] = f_X(a_i)\,\epsilon$

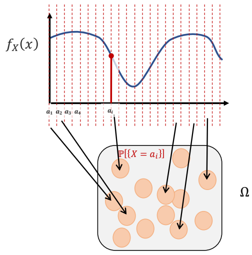
Divide the complete range into ϵ length intervals

$a_1 = a$

$a_2 = a_1 + \epsilon$

$a_3 = a_2 + \epsilon = a_1 + 2\epsilon$

$a_i = a_{i-1} + \epsilon = a_1 + (i-1)\epsilon$



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Expectation for Continuous Random Variable

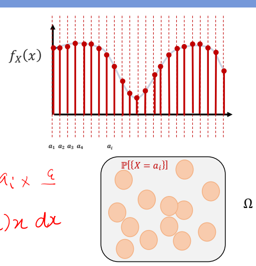
$p_X(a_i) =$
$\mathbb{P}[\{X \in (a_i, a_i + \epsilon)\}] = f_X(a_i)\,\epsilon$

$$\mathbb{E}[X] = \sum_i \mathbb{P}(\{X = x_i\})x_i$$

$$= \sum_i f_X(a_i)\,\epsilon\, a_i \quad \sum f_X(a_i) a_i \times \epsilon$$

As $\epsilon \to 0$,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} f_X(a)\, a\, \mathrm{d}a = \int_{-\infty}^{\infty} f_X(a)\, x\, dx$$



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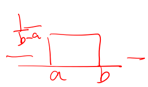
Examples

- Uniform random variable (a, b)

$$\int_a^b x\cdot f_X(x)\,dx$$



$$= \int_a^b x\cdot\frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\cdot\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{a+b}{2}$$

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Examples

- Exponential random variable (λ)

$$\int_0^\infty x\, f_X(x)\, dx$$

$$\lambda e^{-\lambda x}$$

$$\Rightarrow \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx \qquad \lambda x = t$$

$$= \int_0^\infty \frac{t}{\lambda} \cdot \lambda e^{-t} \frac{dt}{\lambda}$$

$$= \frac{1}{\lambda} \int_0^\infty e^{-t}\, t\, dt = \frac{1}{\lambda}$$

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Expectation of Function of RV

$$\mathbb{E}[X] = \sum_i x_i p_X(x_i) \qquad \mathbb{E}[X] = \int x\, f_X(x)\, \mathrm{d}x$$

Let g be some function

$$\mathbb{E}[g(X)] = \sum_i g(x_i)\, p_X(x_i) \qquad Y \qquad \mathbb{E}[g(X)] = \int g(x) f_X(x) \mathrm{d}x$$

x_1 x_2 $g(x_1)$ $g(x_2)$ $p_X(x_1)$ $p_X(x_2)$

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Linearity

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b \qquad g(x) = ax + b$$

$$\mathbb{E}[g(x)]$$

$$= \sum_{x_i} g(x_i) P_X(x_i)$$

$$= \sum_{x_i} (a x_i + b) P_X(x_i)$$

$$= a \underbrace{\sum x_i P_X(x_i)} + b \underbrace{\sum P_X(x_i)}_{=1}$$

$$= a\mathbb{E}[X] + b$$

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Expectation and CCDF

Let X be a RV taking positive values only, then $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(\{X > x\})\mathrm{d}x$

$$\int_0^\infty \mathbb{P}[\{X>x\}]\,dx$$

$$\int_0^\infty \int_x^\infty f_X(t)\,dt\,dx$$

$$= \int_0^\infty f_X(t)\; t \; dt = \mathbb{E}(X)$$

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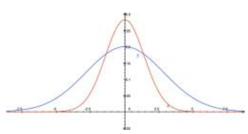
Moments

- The nth moment of a RV X is defined as
  $$\mu_n = \mathbb{E}[X^n].$$
- In particular,
  - $\mathbb{E}[X]$ is the first moment.
  - $\mathbb{E}[X^2]$ is the second moment.
- For some random variables, first moment is enough to fix the distribution.
  - Example: Exponential distribution
- However, for others, first moment is not enough.



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Central Moments

- The nth central moment of a RV X is defined as

$$v_n = \mathbb{E}[(X - \mathbb{E}[X])^n]$$

- The variance for a random variable X

$$\mathrm{Var}(X) = v_2 = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

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Variance

- Variance represents the spread of a random variable
- It shows the mean value of square deviation of a RV from its mean.
- For a constant random variable, variance is 0.
- Consider the following two RVs

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = -1 \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x = 1 \end{cases} \qquad p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = -2 \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x = 2 \end{cases}$$

- Both has mean 0. First has ½ variance and second has 2 variance.

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Properties of Variance

$$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Proof:

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}(X))^2\right] \\ &= \mathbb{E}\left[X^2 + (\mathbb{E}(X))^2 - 2X\mathbb{E}[X]\right] \\ &= \mathbb{E}[X^2] + (\mathbb{E}[X])^2 - 2\mathbb{E}[X]\mathbb{E}[X] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}$$

$\mathbb{E}[\underbrace{\mathbb{E}(X)^2}]$

$(\mathbb{E}[X])^2$

$\mathbb{E}[X\,\mathbb{E}[X]]$

$\mathbb{E}[X] \cdot \mathbb{E}[X]$

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Properties of Variance

- Let $\text{Var}(X) = \sigma^2$. What will the variance of aX?
- Let

$$\begin{aligned} Y &= a\,X, \\ \mathbb{E}[Y] &= a\,\mathbb{E}[X] \\ \text{Var}(Y) &= \mathbb{E}[(Y - \mathbb{E}[Y])^2] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\ &= \mathbb{E}[a^2X^2] - \mathbb{E}[aX]^2 \\ &= a^2\mathbb{E}[X^2] - a^2\mathbb{E}[X]^2 \\ &= a^2\sigma^2 \end{aligned}$$

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Properties of Variance

- Let $\mathrm{Var}(X) = \sigma^2$. What will the variance of $X + b$?
- Let

$$Y = X + b,$$
$$\mathbb{E}[Y] = \mathbb{E}[X] + b$$
$$\mathrm{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}[Y])^2]$$

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Properties of Variance

- Let $\mathrm{Var}(X) = \sigma^2$.

$$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$$

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Moment Generating Function

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Moment Generating Function

- The moment generating function (MGF) of a random variable is an alternative characterization in the place of its probability distribution.
- Instead of directly working with PDF/PMF and CDF, many analytical results can be computed easily with MGF.
- If MGFs of two random variables are the same, they have the same distribution.

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MGF

- The MGF of a random variable is given as:

$$M_X(t) = \mathbb{E}[e^{t\cancel{x}}] = \mathbb{E}[e^{tX}]$$

- For DRV

$$M_X(t) = \mathbb{E}[e^{t\cancel{x}}] = \sum_x e^{tx} p_X(x)$$

- For CRV

$$M_X(t) = \mathbb{E}[e^{t\cancel{x}}] = \int_{\mathbb{R}} e^{tx} f_X(x) \mathrm{d}x$$

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MGF of Exponential RV

- Example: Let X is an exponential random variable with parameter λ. The density function of X is given as

$$\boxed{f_X(x) = \lambda e^{-\lambda x} 1(x \geq 0)}$$

$$M_X(t) = \int_{\mathbb{R}} e^{tx} f_X(x) \mathrm{d}x$$
$$= \int_0^{\infty} e^{tx} \lambda\, e^{-\lambda x} dx = \frac{1}{1 - \frac{t}{\lambda}}$$

- If we know that X is a random variable whose $M_X(t) = \frac{1}{1-\frac{t}{2}}$, then we can say that X is an exponential random variable with parameter 2.

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MGF of Few Distributions

Bernoulli (p): $M_X(t) = 1 - p + pe^t$

Uniform (a, b):
$M_X(t) = \frac{\exp(tb) - \exp(ta)}{t(b-a)}$

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Moments and MGF

- As its name implies, the moment generating function can be used to compute a distribution's moments.
- The nth moment of a RV X is defined as

$$\mu_n = \mathbb{E}[X^n].$$

- The nth moment is equal to the nth derivative of the moment-generating function, evaluated at 0:

$$\mu_n = \underline{M_X^{(n)}}(0). \quad f^{(1)} \quad f^{(2)}$$

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Moments and MGF

$$M_X(t) = \mathbb{E}[e^{tx}] = \int_{\mathbb{R}} e^{tx} f_X(x)\mathrm{d}x$$

- Differentiating with respect to t,

$$M_X^{(1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[e^{tx}] = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} e^{tx} f_X(x)\mathrm{d}x$$

$$= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t} e^{tx} f_X(x)\mathrm{d}x$$

$$= \int_{\mathbb{R}} xe^{tx} f_X(x)\mathrm{d}x$$

- Put $t = 0$

$$M_X^{(1)}(0) = \int_{\mathbb{R}} x f_X(x)\mathrm{d}x = \mathbb{E}[X]$$

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Moments and MGF

$$\boxed{M_X^{(1)}(t) = \int_{\mathbb{R}} x e^{tx} f_X(x) \mathrm{d}x}$$

- Differentiating with respect to t,

$$M_X^{(2)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} x e^{tx} f_X(x) \mathrm{d}x$$

$$= \int_{\mathbb{R}} x \frac{\mathrm{d}}{\mathrm{d}t} e^{tx} f_X(x) \mathrm{d}x$$

$$= \int_{\mathbb{R}} x^2 e^{tx} f_X(x) \mathrm{d}x$$

- Put $t = 0$

$$M_X^{(2)}(0) = \int_{\mathbb{R}} x^2 f_X(x) \mathrm{d}x = \mathbb{E}[X^2]$$

- Similarly, other moments can be derived as $\boxed{\mathbb{E}[X^n] = M_X^{(n)}(0)}$

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Moments and MGF

Bernoulli (p): $M_X(t) = 1 - p + pe^t$

$\frac{d}{dt} M_X(t) = pe^t \to p \quad \mathbb{E}[X]$

$\frac{d}{dt} pe^t = pe^t \to p \quad \mathbb{E}[X^2]$

$Var = p - p^2 = p(1-p)$

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Moments and MGF

- MGF can be written in terms of moments.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx$$

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Moments and MGF

- MGF can be written in terms of moments.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \qquad f_X(x)\,dx$$

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Moments and MGF

- MGF can be written in terms of moments.

$$M_X(t) = \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} + \cdots\right) f_X(x)\,dx$$

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Moments and MGF

- MGF can be written in terms of moments.

$$M_X(t) = \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} + \cdots\right) f_X(x)\,dx$$

$$= 1 + tm_1 + \frac{t^2m_2}{2!} + \cdots + \frac{t^nm_n}{n!} + \cdots,$$

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