eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 11

MMSE/LMMSE

MIMO Estimation and Receivers

MIMO

r = # Receive antennas.
t = # Transmit antennas.

- MIMO system model is given as

$$\underbrace{\overbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}^{r \times 1}}_{\bar{\mathbf{y}}(k)} = \underbrace{\overbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix}}^{r \times t}}_{\mathbf{H}} \underbrace{\overbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}^{t \times 1}}_{\bar{\mathbf{x}}(k)} + \underbrace{\overbrace{\begin{bmatrix} v_1(k) \\ v_2(k) \\ \vdots \\ v_r(k) \end{bmatrix}}^{r \times 1}}_{\bar{\mathbf{v}}(k)}$$

output vector | channel matrix | input vector | noise vector

MIMO

MIMO input output model.

- MIMO system model can be represented in the **compact fashion**

$$\bar{y}(k) = H\bar{x}(k) + \bar{v}(k).$$

MIMO

- MIMO system model can be represented in the compact fashion

$$\bar{\mathbf{y}}(k) = \mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{v}}(k)$$

MIMO

- Consider the transmission of N pilot vectors

$$\bar{\mathbf{y}}(1) = \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1)$$

$$\bar{y}(2) = H\bar{x}(2) + \bar{v}(2)$$

$$\vdots$$

$$\bar{y}(N) = H\bar{x}(N) + \bar{v}(N).$$

MIMO

- Consider the transmission of N pilot vectors

$$\bar{\mathbf{y}}(1) = \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1)$$
$$\bar{\mathbf{y}}(2) = \mathbf{H}\bar{\mathbf{x}}(2) + \bar{\mathbf{v}}(2)$$
$$\vdots$$
$$\bar{\mathbf{y}}(N) = \mathbf{H}\bar{\mathbf{x}}(N) + \bar{\mathbf{v}}(N)$$

MIMO

- We can concatenate them as

$Y \sim$ output matrix

$X \sim$ Pilot matrix.

$$\underbrace{\left[\bar{y}(1)\ \bar{y}(2)\ \cdots\ \bar{y}(N)\right]}_{Y \sim r \times N.}$$

$$= H \underbrace{\left[\bar{x}(1)\ \bar{x}(2)\ \cdots\ \bar{x}(N)\right]}_{X \sim t \times N} + \overbrace{\left[\bar{v}(1)\ \bar{v}(2)\ \cdots\ \bar{v}(N)\right]}^{V \sim r \times N.}$$

MIMO

$Y \sim r \times N$ $X \sim t \times N$

$V \sim r \times N$.

- We can concatenate them as

$$\underbrace{[\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \ldots \quad \bar{\mathbf{y}}(N)]}_{\mathbf{Y}}$$

$$= \mathbf{H}\,\underbrace{[\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \ldots \quad \bar{\mathbf{x}}(N)]}_{\mathbf{X}}$$

$$+ \underbrace{[\bar{\mathbf{v}}(1) \quad \bar{\mathbf{v}}(2) \quad \ldots \quad \bar{\mathbf{v}}(N)]}_{\mathbf{V}}$$

MIMO Estimation Model

- This can be represented in the compact fashion

$$Y = HX + N.$$

MIMO Estimation Model

- This can be represented in the compact fashion

Widematrix.

$t \times N$

$$\underbrace{\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{V}}_{\text{MIMO channel Estimation model.}}$$

$$\text{Pinv of } X = X^T (X X^T)^{-1}$$

$$X^H (X X^H)^{-1}$$

ML MIMO Estimate

- The ML MIMO channel estimate is

$$\underbrace{\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-1}}_{\text{ML channel estimate}}$$

$$\min. \ \|Y - HX\|_F^2$$

Frobenius norm.

LMMSE MIMO Estimate

Linear minimum mean square error estimate -

- The LMMSE MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

$$SNR = \frac{\sigma_h^2}{\sigma^2}$$

$SNR \rightarrow \infty$

$LMMSE \rightarrow ML$

MIMO Estimation Example

- Consider the MIMO channel estimation problem with pilot vectors

$$\bar{\mathbf{x}}(1) = [3 \quad -2]^T, \bar{\mathbf{x}}(2) = [-2 \quad 3]^T$$
$$\bar{\mathbf{x}}(3) = [4 \quad 2]^T, \bar{\mathbf{x}}(4) = [2 \quad 2]^T$$

$N = 4$

$N = 4 \quad t = 2$

- What is the pilot matrix?

2×4

$$X = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}$$

MIMO Estimation Example

- The pilot matrix is

$$\mathbf{X} = [\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \bar{\mathbf{x}}(3) \quad \bar{\mathbf{x}}(4)]$$
$$= \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}$$

MIMO Estimation Example

- The output vectors are

$$
\begin{aligned}
\bar{\mathbf{y}}(1) &= [-2, 1, -3]^T, \\
\bar{\mathbf{y}}(2) &= [-1, 3, 3]^T, \\
\bar{\mathbf{y}}(3) &= [-1, -2, 2]^T, \\
\bar{\mathbf{y}}(4) &= [-3, -1, 1]^T
\end{aligned}
$$

$r = 3$

$r \times t = 3 \times 2$ MIMO system. 3×4

- What is the **output matrix**?

$$
Y = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}
$$

MIMO Estimation Example

- The output matrix is

$$\mathbf{Y} = [\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \bar{\mathbf{y}}(3) \quad \bar{\mathbf{y}}(4)] = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}$$

MIMO Estimation Example

- The LMMSE channel estimate is given as follows

$$\widehat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

MIMO Estimation Example

- Let us first evaluate

$$\mathbf{X}\mathbf{X}^T$$

$$= \underbrace{\begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}}_{X^T} = \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}$$

MIMO Estimation Example

Let $SNR = -6\ dB = \frac{1}{4}$

$$\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR}\mathbf{I}\right)^{-1} = \left(\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4\mathbf{I}\right)^{-1}$$

$$= \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 37 & 0 \\ 0 & 25 \end{bmatrix}$$

MIMO Estimation Example

Let $SNR = -6\ dB$

$$\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR}\mathbf{I}\right)^{-1} = \left(\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4\mathbf{I}\right)^{-1}$$

$$= \begin{bmatrix} \frac{1}{37} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$$

MIMO Estimation Example

- Let us now evaluate $\mathbf{YX}^T$

$$= \underbrace{\begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}}_{X^T} = \overbrace{\begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix}}^{YX^T}$$

MIMO Estimation Example

- Finally, the MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} \frac{1}{37} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} -\frac{14}{37} & -\frac{7}{25} \\ -\frac{13}{37} & \frac{1}{25} \\ -\frac{5}{37} & \frac{21}{25} \end{bmatrix}$$

MIMO Estimation Example

- Finally, the MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} \frac{1}{37} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} -\frac{14}{37} & -\frac{7}{25} \\ -\frac{13}{37} & \frac{1}{25} \\ -\frac{5}{37} & \frac{21}{25} \end{bmatrix}$$

$\hat{H}$

LMMSE MIMO
channel Estimate.

MIMO Receivers

MIMO Receivers

$\bar{y}\cancel{(k)} = H\bar{x}\cancel{(k)} + \bar{v}\cancel{(k)}$

Dropped index k.

- Consider the MIMO model

$$\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{v}}$$

- Time index k dropped for simplicity

MIMO Receivers

$$\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{v}}$$

We know H.

Given $\bar{y}$, $\hat{x} = ?$

- How to determine $\bar{\mathbf{x}}$ given $\bar{\mathbf{y}}$?

$\hat{x} = ?$

What is estimate of transmit vector?

MIMO ZF Receiver

Least Squares Receiver

$$\min \|\bar{\mathbf{y}} - \mathbf{H}\bar{\mathbf{x}}\|^2$$

- This is termed the **least-squares problem.**

$$\hat{x} = (H^H H)^{-1} H^H \bar{y}$$

Zero Forcing (ZF) Receiver

MIMO Receiver

$$\hat{\mathbf{x}} = \underline{(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H}\bar{\mathbf{y}}$$

- This is termed as the **zero-forcing (ZF) Receiver.**

$(H^H H)^{-1} H^H$. is pseudoinverse of H.

H is tall matrix

$r \geq t$.

MIMO ZF Receiver Example

Example:

Consider

4×2

$r = 4$ Receive antennas-

$t = 2$ transmit antennas-

$$\bar{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

What is $\hat{\mathbf{x}}$?

MIMO ZF Receiver Example

The **ZF estimate** can be calculated as follows

$$\mathbf{H}^T\mathbf{H} = \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}}^{H^T} \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}^{H}$$

$$= \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

MIMO ZF Receiver Example

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(\mathbf{H}^T\mathbf{H})^{-1} = \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

MIMO ZF Receiver Example

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(\mathbf{H}^T\mathbf{H})^{-\mathbf{1}} = \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

MIMO ZF Receiver Example

$$(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T = \overbrace{\frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}}^{(H^TH)^{-1}}\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}}^{H^T}$$

$$= \underbrace{\begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{10} & -\frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix}}_{(H^TH)^{-1}H^T}$$

MIMO ZF Receiver Example

$$\hat{\mathbf{x}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\bar{\mathbf{y}}$$

ZF Receiver
Linear Receiver

$$= \begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{10} & -\frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}}_{\bar{y}}$$

$$= \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} = \hat{x} = \text{Zero Forcing Estimate}$$

MIMO ZF Receiver Example

- Therefore, the **ZF estimate** is

output of ZF Receiver.

$$\hat{\mathbf{x}} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

LMMSE Receiver

- Another popular MIMO receiver is the **LMMSE Receiver**

ML : Deterministic

MMSE/LMMSE : Random -

LMMSE

$$\min E\{\|\underbrace{\mathbf{C}^H}\bar{\mathbf{y}} - \bar{\mathbf{x}}\|^2\}$$

$$\underbrace{\hat{x} = C^H \bar{y}}_{\text{Estimate is Linear Transformation of output.}}$$

LMMSE Receiver

$E\{x(i)\} = 0$
$E\{|x(i)|^2\} = P.$

- Consider the symbols to be i.i.d. mean 0, power P.

P = power of symbols.

$$E\{\bar{\mathbf{x}}\bar{\mathbf{x}}^H\} = P\mathbf{I}$$

LMMSE Receiver

$SNR = \frac{P}{N_0}.$

- The LMMSE Receiver is

$$\hat{\mathbf{x}} = \left(\mathbf{H}^H \mathbf{H} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{H}^H \bar{\mathbf{y}}$$

$$SNR = \frac{P}{N_0}$$

MIMO LMMSE Receiver Example

Example:

Consider

4x2

$$\bar{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

What is $\hat{\mathbf{x}}$ when $SNR = -3dB = \frac{1}{2}$

MIMO LMMSE Receiver Example

The **LMMSE estimate** can be calculated as follows

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$$

MIMO LMMSE Receiver Example

- $SNR = -3\ dB \approx \frac{1}{2}$

$$\mathbf{H}^T\mathbf{H} + \frac{1}{SNR}\mathbf{I} = 4I + 2I = 6I$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

MIMO LMMSE Receiver Example

- $SNR = -3\ dB \approx \frac{1}{2}$

$$\mathbf{H}^T\mathbf{H} + \frac{1}{SNR}\mathbf{I} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

MIMO LMMSE Receiver Example

$$\mathbf{H}^T\mathbf{H} + \frac{1}{SNR}\mathbf{I} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\left(\mathbf{H}^T\mathbf{H} + \frac{1}{SNR}\mathbf{I}\right)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

MIMO LMMSE Receiver Example

$$\mathbf{H}^T\mathbf{H} + \frac{1}{SNR}\mathbf{I} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\left(\mathbf{H}^T\mathbf{H} + \frac{1}{SNR}\mathbf{I}\right)^{-\mathbf{1}} = \frac{1}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

MIMO LMMSE Receiver Example

$$\left(\mathbf{H}^T\mathbf{H} + \frac{1}{\mathrm{SNR}}\mathbf{I}\right)^{-1}\mathbf{H}^T = \frac{1}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{6}\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{H^T}$$

MIMO LMMSE Receiver Example

$$\hat{\mathbf{x}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\bar{\mathbf{y}}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\underbrace{\begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}}_{\bar{y}}$$

$$= \frac{1}{6}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \hat{x}_{LMMSE}$$

LMMSE Receiver output.

MIMO LMMSE Receiver Example

- Therefore, the **LMMSE estimate** is

$$\hat{\mathbf{x}} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

LMMSE Receiver.

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