Optimization Problems

objective

General Form : $x^* = \arg\min_{x} f_0(\underline{x})$

inequality constraints

optimum

s.t. $f_i(\underline{x}) \leq 0 \quad i = 1 \ldots m$

optimization variable

$h_j(\underline{x}) = 0 \quad j = 1 \ldots p$

equality constraints

Note : $x \in \text{dom } f_0$, $\text{dom } f_i$, $\text{dom } h_i$

implicit constraints

$$\boxed{x \in \mathcal{D}} := \text{dom } f_0 \bigcap \left(\bigcap_{i=1}^{m} \text{dom } f_i \right) \bigcap \left(\bigcap_{j=1}^{p} \text{dom } h_j \right)$$

usually not written

Eg : $$\min_{\underline{x} \in \mathbb{R}^n} \; c^T x - \sum_{i=1}^{m} \log(a_i^T x - b_i)$$

- no explicit constraints
- but problem should be well defined

$$\Rightarrow x \in \text{dom } f_0 = \{ \underline{x} \mid a_i^T x - b_i > 0, \; i = 1, 2 \ldots m \}$$

Feasible solution: $\tilde{\underline{x}}$ feasible if $\tilde{\underline{x}} \in \mathcal{D}$

$$f_i(\tilde{\underline{x}}) \leq 0 \qquad i = 1, 2 \ldots m$$
$$h_j(\tilde{\underline{x}}) = 0 \qquad j = 1, 2 \ldots p$$

also: $f_0(x^*) = \min_{\underline{x}} f_0(x) = \begin{cases} \text{finite solvable} \\ -\infty \quad \text{unbounded below} \\ \infty \quad \text{infeasible} \end{cases}$

$$\text{s.t. } f_i(\underline{x}) \leq 0 \quad i = 1 \ldots m$$
$$h_j(\underline{x}) = 0 \quad j = 1, \ldots p$$

Eg

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} a_i x_i = \begin{cases} 0 & a_i > 0 \ \forall i \\ (x^* = 0) & \\ -\infty & a_1 < 0 \\ & a_2 \ldots a_n \geq 0 \end{cases}$$

$$x_i \geq 0 \qquad i = 1, 2 \ldots n$$

take $x_1 \to \infty$

Note: $\underbrace{a_1 x_1}_{\text{unbounded below}} + \underbrace{a_2 x_2 + \ldots + a_n x_n}_{\text{minimized for } x_2 = x_3 = \ldots = 0}$

unbounded below

$x_1 \to \infty$

Feasibility problem : find x

$$f_i(x) \leq 0$$
$$h_j(x) = 0$$

or

$$\begin{aligned} &\min\ 0 \\ &f_i(x) \leq 0 \\ &h_j(x) = 0 \end{aligned} \quad = \begin{cases} 0 & \text{if feasible solution exists} \\ \infty & \text{if infeasible} \end{cases}$$

* Any optimization problem can be written in standard form.

Eg:
$$\begin{aligned} &\min f_0(\underline{x}) \\ &l_i \leq x_i \leq u_i \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\min f_0(\underline{x}) \\ &x_i - u_i \leq 0 \quad i=1,\dots,n \\ &l_i - x_i \leq 0 \quad i=1,\dots,n \end{aligned}$$