EE901
Probability and Random Processes

Module -1
Introduction to Probability Theory

Abhishek Gupta
Electrical Engineering
IIT Kanpur

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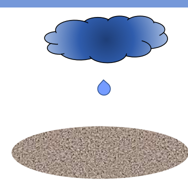
Random Experiments

Choose a number



Finite number of possibilities	Countable number of possibilities	Uncountable number of possibilities

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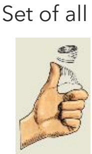
Sample Space

- Set of all possible outcomes



$$\Omega = \{H, T\}$$

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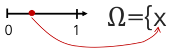
Examples

- Coin toss  $\Omega = \{ H , T \}$
- Dice roll  $\Omega = \{ 1,2,3,4,5,6 \}$
- Picking a number

$\Omega = \{ 1,2,3,4,... \} = \mathbb{N}$
- Pick a real number between 0 & 1 

$\Omega = \{ x : 0 < x < 1 \} = (0,1)$

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Sample Space

- A sample space is the set of all possible outcomes
- Elements must be
  - Mutually exclusive
  - Collectively exhaustive
  - Finest grain (How to chose granularity?)

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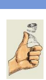
Sample Space of a Coin Toss



Set of all possible outcomes $\Omega = \{ H , T \}$

- Are there other outcomes possible?
  - No, but we can reorganize the outcomes!
- If we are interested in the number of turns the coin takes before hitting the ground,
  - Possible outcomes are "coin takes 0 turn", "coin takes 1 turn" and so on.
  - Sample space is $\Omega = \{ 0,1,2,3,4,... \}$

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Sample Space of a Coin Toss

Ω={ H , T }

Ω={ 0,1,2,3,4, . . .}

- If we are interested in both results,
  - An outcomes is in the form
    - The coin takes *n* turns and show side *p*
    - Denoted by *(n,p)*
    - *Sample space is*

Ω={ (0,H),(0,T), (1,H), (1,T), . . .}

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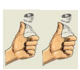
Sample Space of Two Coin Tosses



Ω = { H H , H T, T H, T T }

- Other possible sample spaces:

Number of coins showing Head Ω={ 0,1,2}

If the two coins show the same side Ω={ YES, NO}

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Dice as a Coin

- Suppose we need to decide between two options, but we have a dice instead of a coin.

Ω = { 1, 2, 3, 4, 5, 6 }

- So we roll the dice, and
  If an even number appears -> Option A
  If an odd number appears -> Option B

Ω = { E, O}

E denotes even number outcome
O denotes odd number outcome

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Events

- An event E is a set of outcomes.

Dice Roll. Ω={ 1,2,3,4,5,6}
A={1,2,3} = A number less than 4 appears.
B={2,4,6} = An even number appears.
C={6} = Number 6 appears.
{6} vs 6

- $E \subset \Omega$ (a subset of the sample space).
- We say that event E occurs if any outcome in event E ($\omega \in E$) has occurred.

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Operations on Events

- Events are sets, so we can define union, intersection and complements on them

Dice Roll. Ω={ 1,2,3,4,5,6}
A={1,2} = A number less than 3 appears.
B={3} = Number 3 appears.
C={2,4,6} = An even number appears.

$A \cup B$ ={1,2,3} = A number less than 4 appears.

$A \cup B$ will occur if A or B or both occur.

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Operations on Events

- Events are sets, so we can define union, intersection and complements on them

Dice Roll. Ω={ 1,2,3,4,5,6}
A={1,2} = A number less than 3 appears.
B={3} = Number 3 appears.
C={2,4,6} = An even number appears.

$A \cap C$ ={2} = Number 2 occurs

$A \cap C$ will occur if A and C both occur.

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Operations on Events

- Events are sets, so we can define union, intersection and complements on them

Dice Roll. Ω={ 1,2,3,4,5,6}

A={1,2} = A number less than 3 appears.

B={3} = Number 3 appears.

C={2,4,6} = An even number appears.

C^c={1,3,5} = An odd number appears

C^c will occur if C does not occur.

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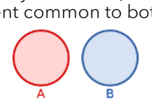
Mutually Exclusive Events

- A and B are mutually exclusive (also called disjoint) events, if there is no element common to both



$A \cap B = \{\} = \phi$

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Probability

- Each event E is assigned a probability: $\mathbb{P}(E)$
  - Denotes the collective chance of occurrence of any of the outcomes in it.
- It is a function from set of events $\mathcal{F}$ to set of real numbers $\mathbb{R}$.
- What properties should this function have?

Dice Roll. Ω={1,2,3,4,5,6}

A={1}, B={2}, $A \cup B$ ={1,2}

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Properties of Probability

- Each event E is assigned a probability: $\mathbb{P}(\mathrm{E})$
- What properties do we want in this function?

Dice Roll. Ω={1,2,3,4,5,6}

A={1}, B={2}, A ∪ B ={1,2}

What is $\mathbb{P}(\Omega)$?

Ω will always occur. Its probability should be 1. $\mathbb{P}(\Omega) = 1$

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Properties of Probability

- Each event E is assigned a probability: $\mathbb{P}(\mathrm{E})$
- What properties do we want in this function?

$\mathbb{P}(\Omega) = 1$

Dice Roll. Ω={1,2,3,4,5,6}

A={1}, B={2}, A ∪ B ={1,2}

Lower limit on $\mathbb{P}$(E)?

Every event has a non-negative chance. $\mathbb{P}(\mathrm{E}) \geq 0$

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Properties of Probability

- Each event E is assigned a probability: $\mathbb{P}(\mathrm{E})$
- What properties do we want in this function?

$\mathbb{P}(\Omega) = 1$ $\mathbb{P}(\mathrm{E}) \geq 0$

Dice Roll. Ω={1,2,3,4,5,6}

A={1}, B={2}, A ∪ B ={1,2}

A and B are different singleton events.

If $\mathbb{P}$(A) and $\mathbb{P}$(B) are known, what will be $\mathbb{P}$(A ∪ B) ?

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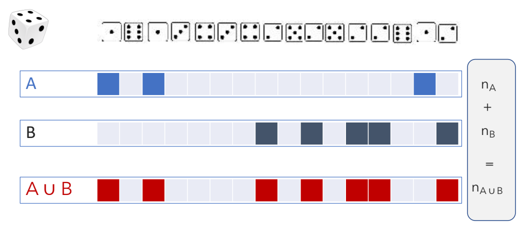

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$$n_A \quad + \quad n_B \quad = \quad n_{A \cup B}$$

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$$\frac{n_A}{n} + \frac{n_B}{n} = \frac{n_{A \cup B}}{n}$$

$$\boxed{\mathbb{P}(A) \quad + \quad \mathbb{P}(B) = \quad \mathbb{P}(A \cup B\,)}$$

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Properties of Probability

- Each event E is assigned a probability: $\mathbb{P}(\mathrm{E})$
- What properties do we want in this function?

$\boxed{\mathbb{P}(\Omega) = 1}$ $\boxed{\mathbb{P}(\mathrm{E}) \geq 0}$ $\boxed{\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B)}$

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Probability Axioms

- Each event E is assigned a probability: $\mathbb{P}(\mathrm{E})$

  which satisfies

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(\mathrm{E}) \geq 0$

$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B)$

for any disjoint events A and B

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Probability Axioms

- Each event E is assigned a probability: $\mathbb{P}(\mathrm{E})$

  which satisfies

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(\mathrm{E}) \geq 0$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$

for disjoint events $A_1, A_2, A_3, \ldots$

- Probability measure is a function from $\mathcal{F}$(set of events) to $[0,1]$.
- Is the definition complete?

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Probability Measure

- Probability measure is a function from $\mathcal{F}$ (set of events) to [0,1] that satisfies

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(E) \geq 0$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$
for disjoint events $A_1, A_2, A_3, \ldots$

- How to assign probabilities?
- How to choose $\mathcal{F}$?

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Class of Events $\mathcal{F}$

- Set of events (subsets of Ω)
- Should include Ω.
- Should include all those "interesting events" for which we want to know probabilities
- Usually,

If we are interested in the probability of A, we are also interested in the probability of A not occurring. A^c

If we are interested in the probability of A and B, we are also interested in the probability of A and B both occurring simultaneously. $A \cap B$

If we are interested in the probability of A and B, we are also interested in the probability of at least A or B occurring. $A \cup B$

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Class of Events $\mathcal{F}$

- Set of events (subsets of Ω)

$\Omega \in \mathcal{F}$.

If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

If $A \in \mathcal{F}$, and $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

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Class of Events $\mathcal{F}$

- Set/Collection of events (subsets of sample space Ω)

  $\Omega \in \mathcal{F}$.

  If $\mathrm{A} \in \mathcal{F}$, then $\mathrm{A}^c \in \mathcal{F}$.

  If $\mathrm{A_i} \in \mathcal{F}\ \forall\, i = 1, 2, 3, \ldots$, then $\mathrm{A_1} \cup \mathrm{A_2} \cup \mathrm{A_3} \cup \ \ldots\ \in \mathcal{F}$.

- Such a structure is known as $\sigma -$ algebra.
- $\mathcal{F}$ is a collection of subsets of Ω satisfying above properties.
- There can be many possibilities, we can choose as per our interest.

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Example of $\mathcal{F}$

Collection of subsets of sample space Ω
$\Omega \in \mathcal{F}$.
If $\mathrm{A} \in \mathcal{F}$, then $\mathrm{A}^c \in \mathcal{F}$.
If $\mathrm{A_i} \in \mathcal{F}\ \forall\, i = 1, 2, 3, \ldots$, then $\mathrm{A_1} \cup \mathrm{A_2} \cup \mathrm{A_3} \cup \ \ldots\ \in \mathcal{F}$.

Tetrahedral Dice Roll. Ω={1,2,3,4}.
We are interested in the exact outcome. What should be $\mathcal{F}$?

$$\mathcal{F} = \left\{ \begin{array}{l} \{1\}, \{2\}, \{3\}, \{4\},\ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \\ \{1,2,3,4\}, \{\} \end{array} \right\}$$

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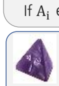
Example of $\mathcal{F}$

Collection of subsets of sample space Ω
$\Omega \in \mathcal{F}$.
If $\mathrm{A} \in \mathcal{F}$, then $\mathrm{A}^c \in \mathcal{F}$.
If $\mathrm{A_i} \in \mathcal{F}\ \forall\, i = 1, 2, 3, \ldots$, then $\mathrm{A_1} \cup \mathrm{A_2} \cup \mathrm{A_3} \cup \ \ldots\ \in \mathcal{F}$.



Tetrahedral Dice Roll. Ω={1,2,3,4}.
We are interested in the knowing whether outcome is even or odd.
What should be $\mathcal{F}$?

$$\mathcal{F} = \left\{ \begin{array}{l} \{1,3\}, \{2,4\}, \\ \{1,2,3,4\},\ \{\} \end{array} \right\}$$

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Example of $\mathcal{F}$

Collection of subsets of sample space Ω
$\Omega \in \mathcal{F}$.
If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
If $A_i \in \mathcal{F}\ \forall\, i = 1, 2, 3, \ldots$, then $A_1 \cup A_2 \cup A_3 \cup \ldots \in \mathcal{F}$.

Tetrahedral Dice Roll. Ω={1,2,3,4}.
What is the smallest $\mathcal{F}$ possible?

$$\mathcal{F} = \left\{ \begin{array}{l} \{1,2,3,4\}, \\ \{\} \end{array} \right\} = \{\, \Omega, \phi \,\}$$

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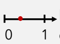
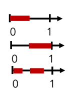
Example of $\mathcal{F}$



Pick a number between 0 and 1, $\Omega = (0,1)$. We are interested in the exact outcome. What is $\mathcal{F}$?

$$\mathcal{F} = \left\{ \begin{array}{l} \text{(0,x) Ex. (0,.3) , (0,.4), ...} \\ \text{[x,1) Ex. [.3,1) , [.4,1), ...} \\ \text{(0,x) U (y,z)} \\ \text{.....} \qquad \{\}, \Omega \end{array} \right\}$$



Set of all possible open intervals, and their countable unions and intersections. It will contain closed intervals also.
Termed Borel sigma algebra.

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Probability Space

For any random experiment, we have

- Ω Sample space: set of all possible outcomes
- $\mathcal{F}$ Sigma algebra: set of all events we like to consider
- $\mathbb{P}$ Probability measure: a function to assign probability to each event

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Probability Space

For any random experiment, we have

Sample space: set of all possible outcomes

Sigma algebra: set of all events we like to consider

Probability measure: a function to assign probability to each event

$$\mathcal{P} = \{\Omega, \mathcal{F}, \mathbb{P}\}$$

Probability Space

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Probability Measure

- Probability measure is a function from $\mathcal{F}$ (set of events) to [0,1] that satisfies

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(E) \geq 0$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$
for disjoint events $A_1, A_2, A_3, \ldots$

- How to assign probabilities?
- How to choose $\mathcal{F}$?

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Probability Measure

- Probability measure is a function from $\mathcal{F}$ (set of events) to [0,1] that satisfies

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for disjoint events $A_1, A_2, A_3, \ldots$

- How to assign probabilities?
- How to choose $\mathcal{F}$?

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Probability Measure

Assignment of the probability function depends on whether there are finite, countable or uncountable number of outcomes.

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }.
$\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

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Probability Measure for a Coin Toss

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }. $\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

$\mathbb{P}(\ \{\}\ )$

$\mathbb{P}(\ \{H\}\ )$

$\mathbb{P}(\ \{T\}\ )$

$\mathbb{P}(\{H,T\}) \quad = 1$

Probability Axioms

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(E) \geq 0$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$

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Probability Measure for a Coin Toss

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }. $\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

$\mathbb{P}(\ \{\}\ )$

$\mathbb{P}(\ \{H\}\ )$

$\mathbb{P}(\ \{T\}\ )$

$\mathbb{P}(\{H,T\}) \quad = 1$

Probability Axioms

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(E) \geq 0$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$

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Probability Measure for a Coin Toss

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }. $\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

$\boxed{\mathbb{P}(\{\}) = 0}$

$\mathbb{P}(\{H\})$

$\mathbb{P}(\{T\})$

$\mathbb{P}(\{H,T\}) = 1$

$\boxed{\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)}$

$A_1 = \Omega$

$A_2 = \{\}$.

$A_1 \cup A_2 = \Omega$

$\mathbb{P}(\Omega) + \mathbb{P}(\{\}) = \mathbb{P}(\Omega)$

$1 \quad + \mathbb{P}(\{\}) = 1$

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Probability Measure for a Coin Toss

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }. $\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

$\mathbb{P}(\{\}) = 0$

$\boxed{\mathbb{P}(\{H\}) = a \geq 0}$

$\boxed{\mathbb{P}(\{T\})}$

$\mathbb{P}(\{H,T\}) = 1$

Probability Axioms

$\mathbb{P}(\Omega) = 1$

$\boxed{\mathbb{P}(E) \geq 0}$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$

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Probability Measure for a Coin Toss

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }. $\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

$\mathbb{P}(\{\}) = 0$

$\mathbb{P}(\{H\}) = a$

$\boxed{\mathbb{P}(\{T\}) = 1 - a}$

$\mathbb{P}(\{H,T\}) = 1$

$\boxed{\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)}$

$A_1 = \{H\}, \qquad A_2 = \{T\}$

$A_1 \cup A_2 = \Omega$

$\mathbb{P}(\{H\}) + \mathbb{P}(\{T\}) = \mathbb{P}(\Omega)$

$a \quad + \mathbb{P}(\{T\}) = 1$

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Probability Measure for a Coin Toss

Let us take finite case first *i.e.* Ω has finite elements.

Coin toss. Ω={ H , T }. $\mathcal{F}$ = { {}, {H}, {T}, {H,T} }.
Goal: Assign probability to each element in $\mathcal{F}$ so that it satisfies probability axioms.

$\mathbb{P}(\{\}) = 0$

$\mathbb{P}(\{H\}) = a$

$\mathbb{P}(\{T\}) = 1 - a$

$\mathbb{P}(\{H,T\}) = 1$

What should be the value of a?

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Probability Measure for a Dice Roll

Tetrahedral dice roll. Ω={ 1,2,3,4 }.
$\mathcal{F}$ = {{},{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,3},{1,2,4},{1,3,4},{2,3,4},{1,2,3,4}}

$\mathbb{P}(\{\}) = 0$

$\mathbb{P}(\{1,2,3,4\}) = 1$

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Probability Measure for a Dice Roll

Tetrahedral dice roll. Ω={ 1,2,3,4 }.
$\mathcal{F}$ = {{},{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,3},{1,2,4},{1,3,4},{2,3,4},{1,2,3,4}}

$\mathbb{P}(\{1\}) = a_1$

$\mathbb{P}(\{2\}) = a_2$

$\mathbb{P}(\{3\}) = a_3$

$\mathbb{P}(\{4\}) = 1 - a_1 - a_2 - a_3$

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Probability Measure for a Dice Roll

Tetrahedral dice roll. Ω={ 1,2,3,4 }.
$\mathcal{F}$ = {{},{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,3},{1,2,4},{1,3,4},{2,3,4},{1,2,3,4}}

$\mathbb{P}(\ \{1\}\ ) = a_1$

$\mathbb{P}(\ \{2\}\ ) = a_2$

$\mathbb{P}(\ \{3\}\ ) = a_3$

$\mathbb{P}(\ \{4\}\ ) = 1 - a_1 - a_2 - a_3$

$\mathbb{P}(\{1,2\})$

$= \mathbb{P}(\{1\}) + \mathbb{P}(\{2\})$

$= a_1 + a_2$

$\boxed{\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)}$

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Probability Assignment for Finite Sample Space

- Start with singleton events
- Assign positive numbers to each event ensuring the sum is equal to 1.
- For rest of the events
  - List all outcomes in each event .
  - List corresponding singleton events of each of these outcomes.
  - Add the probabilities of these singleton events.
- E = { $\omega_1, \omega_2, \omega_3$}

  Outcomes: ω_1, ω_2, ω_3

  $\{\omega_1\} = E_1$ $\{\omega_2\} = E_2$ $\{\omega_3\} = E_3$

  $\mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3)$

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Probability Measure for Countable Ω

- The same procedure works for a countable sample space

Choose a number. Ω={ 1,2,3,4, …. }.

$\mathbb{P}(\ \{1\}\ ) = a_1$

$\mathbb{P}(\ \{2\}\ ) = a_2$

…

$\mathbb{P}(\ \{i\}\ ) = a_i$

…

$$\sum_i a_i = 1$$

Example

$$a_i = \frac{1}{2^i}$$

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Probability Measure for Uncountable Ω

0 1 Pick a number between 0 and 1, $\Omega = (0,1)$. $\mathcal{F}$ is Borel sigma algebra.

Elements of sigma algebra are open intervals, their complements and their countable union/intersections

Let's start with the events of the following simple interval form

A=(0,x) 0 1

Assign the length of this interval as the probability for this event $\mathbb{P}((0,x))$ =x.

For singleton events of form {x}, the length is zero. Hence $\mathbb{P}(\{x\}) = 0$.

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Probability Measure for Uncountable Ω

0 1 Pick a number between 0 and 1, $\Omega = (0,1)$. $\mathcal{F}$ is Borel sigma algebra.

$\mathbb{P}((0,x))$ =x. $\mathbb{P}(\{x\}) = 0$.

For events of the form (a, b), the probability can be computed from the probability axiom

$A_1 = (0, a)$ $A_2 = \{a\}$ $A_3 = (a, b)$ $A_1 \cup A_2 \cup A_3 = (0, b)$

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) = \mathbb{P}(A_1 \cup A_2 \cup A_3)$$
$$a \quad + \quad 0 + \mathbb{P}(A_3) = b$$
$$\mathbb{P}(A_3) = b - a$$

which is length of the interval (a, b) and is consistent with the definition.

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Probability Measure for Uncountable Ω

0 1 Pick a number between 0 and 1, $\Omega = (0,1)$. $\mathcal{F}$ is Borel sigma algebra.

$\mathbb{P}((0,x))$ =x. $\mathbb{P}(\{x\}) = 0$. $\mathbb{P}((a,b)) = b - a$.

Now, any set in the sigma algebra can be written as countable union of above sets. The third axiom can be used to compute the probability.

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$$

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Probability Measure for Uncountable Ω

Pick a number between 0 and 1, $\Omega = (0,1)$. $\mathcal{F}$ is Borel sigma algebra.

Elements of sigma algebra are open intervals, their complements and their countable union/intersections
Let's start with the events of the following simple interval form

A=(0,x)

Assign the length of this interval as the probability for this event $\mathbb{P}(\,(0,x)\,)=x$

For singleton events of form {x}, the length is zero. Hence $\mathbb{P}(\{x\}) = 0$.

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Elements of sigma algebra are open intervals, their complements and their countable union/intersections
Let's start with the events of the following simple interval form

A=(0,x)

Assign the any increasing function F(x) of x as the probability for this event
$\mathbb{P}(\,(0,x)\,)=F(x)$
such that F(1)=1, F(0)=0.
For singleton events of form {x}, take $\mathbb{P}(\{x\}) = 0$.

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For singleton events of form {x}, take $\mathbb{P}(\{x\}) = 0$.

For events (a,b), the probability can be computed from the probability axiom as
$\mathbb{P}\big((a,b)\big) = F(b) - F(a)$.

Now, any set in the sigma algebra can be written as countable union of above sets.
The third axiom can be used to compute the probability.

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