eMasters in Communication Systems

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<u>Elective Module:</u>

Estimation for Wireless Communication

Chapter 5

Multiple Input-
Multiple Output.

MIMO Channel Estimation

MIMO

- MIMO denotes

Key Technology in 4G/5G.

Multiple input
Multiple output

Multiple Transmit
Antennas.

Multiple Rx
antennas.

⇒ Spatial Multiplexing
⇒ Parallel Transmission
of Multiple streams.
over same Time & Frequency

⇒ Very high data rates!!

MIMO

- MIMO denotes **Multiple-Input Multiple-Output**

MIMO

- MIMO is a **key technology** in 4G/ 5G

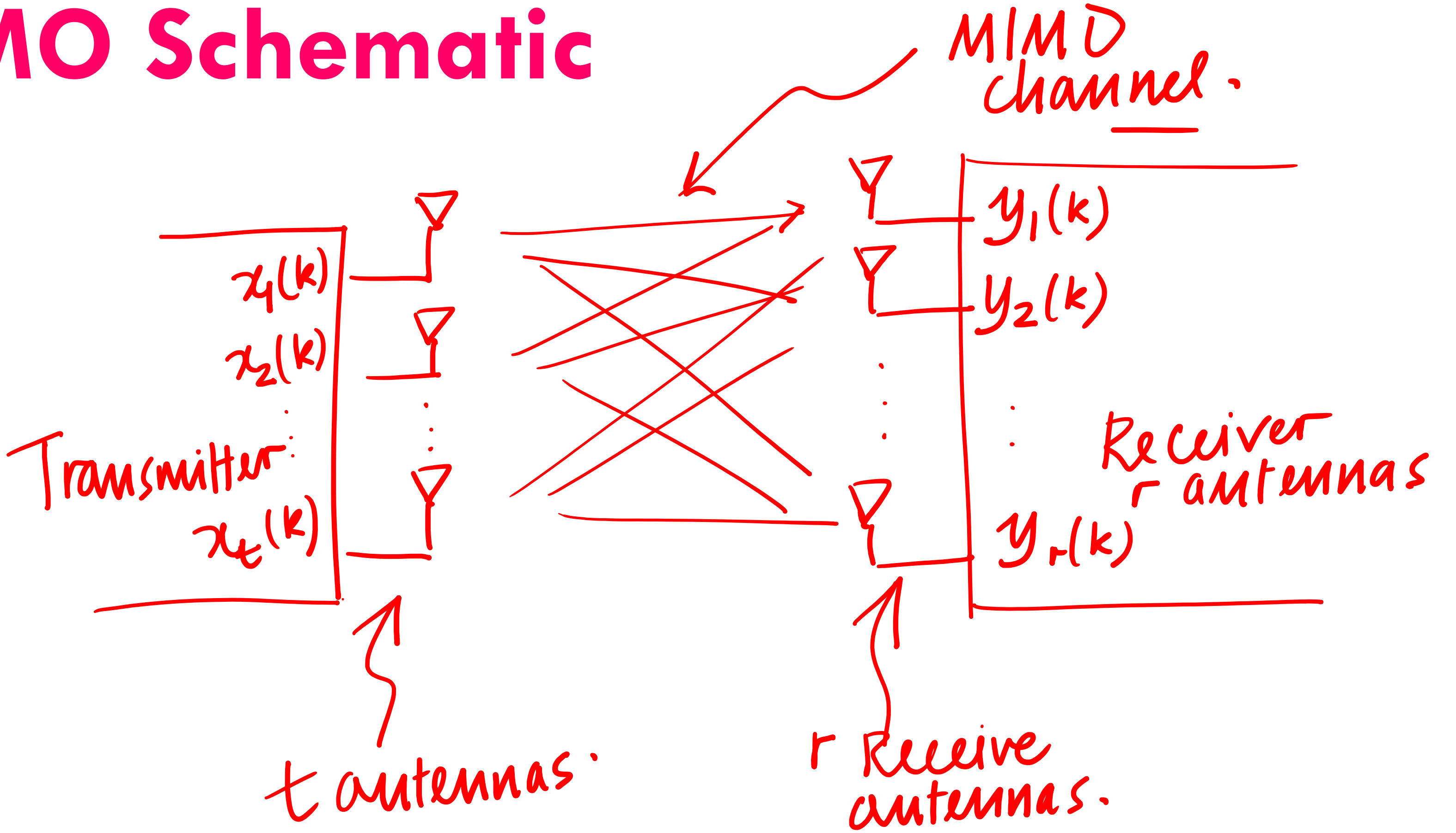
802.11 n/ac/ax
WiFi

LTE

NR

Significantly high rates!!

MIMO Schematic



MIMO Schematic

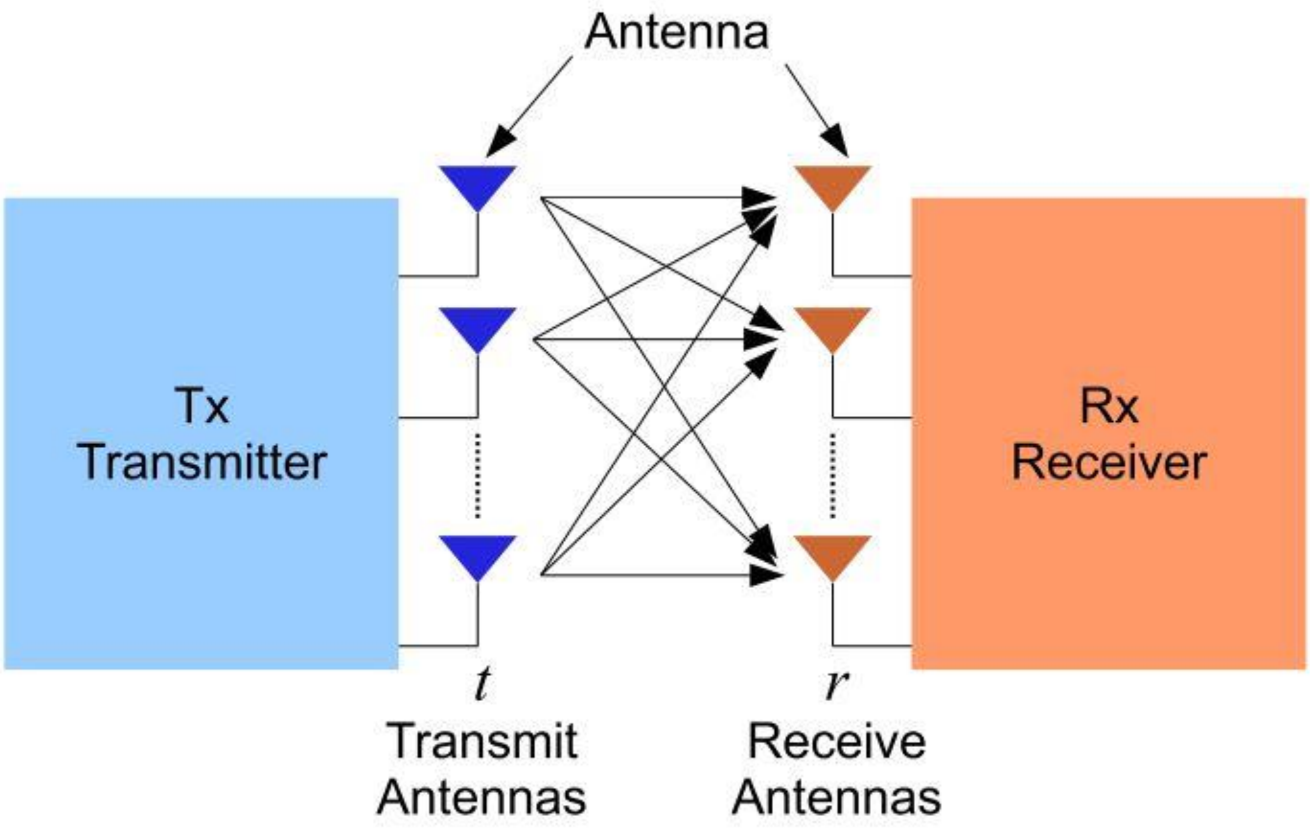


Figure: MIMO System

MIMO

- Number of receive antennas $= r$
- Number of transmit antennas $= t$

MIMO

- MIMO system model is given as

r output symbols Time k.

t input symbols Time k

r noise samples Time k.

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}_{\bar{y}(k)} = \underbrace{\overbrace{\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}}^{r \times t \text{ matrix}}}_{H} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\bar{x}(k)} + \underbrace{\begin{bmatrix} v_1(k) \\ v_2(k) \\ \vdots \\ v_r(k) \end{bmatrix}}_{\bar{v}(k)}$$

$$\bar{y}(k) = H\bar{x}(k) + \bar{v}(k).$$

MIMO

- MIMO system model is given as

output vector $r \times 1$

$r \times t$ MIMO channel

Transmit vector $t \times 1$

$r \times 1$ noise vector

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}_{\bar{\mathbf{y}}(k)} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\bar{\mathbf{x}}(k)} + \underbrace{\begin{bmatrix} v_1(k) \\ v_2(k) \\ \vdots \\ v_r(k) \end{bmatrix}}_{\bar{\mathbf{v}}(k)}$$

4×2 MIMO

$\Rightarrow$ 4 Rx antennas

2 Tx antennas.

MIMO

- h_{ij} denotes

ith row jth col.
→ channel between
ith Rx antenna
jth Tx antenna.

r×t MIMO channel matrix

$$\underbrace{\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}}_{\mathbf{H}}$$

2nd Receive antenna

1st Transmit antenna.

MIMO

- MIMO system model can be represented in the **compact fashion**

r×t MIMO channel.

$$\bar{y}(k) = H\bar{x}(k) + \bar{v}(k)$$

r×1 output vector

t×1 transmit vector

r×1 noise vector.

MIMO

- MIMO system model can be represented in the compact fashion

k = Time instant.

$$\bar{\mathbf{y}}(k) = \mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{v}}(k)$$

MIMO

N output vectors.

- Consider the transmission of N pilot vectors

N pilot vectors.

$$\begin{aligned}
\bar{\mathbf{y}}(1) &= \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1) \\
\bar{y}(2) &= H\bar{x}(2) + \bar{v}(2) \\
&\vdots \\
\bar{y}(N) &= H\bar{x}(N) + \bar{v}(N)
\end{aligned}$$

MIMO

For MIMO channel Estimation.

- Consider the transmission of N pilot vectors

$$
\begin{aligned}
\bar{\mathbf{y}}(1) &= \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1) \\
\bar{\mathbf{y}}(2) &= \mathbf{H}\bar{\mathbf{x}}(2) + \bar{\mathbf{v}}(2) \\
&\vdots \\
\bar{\mathbf{y}}(N) &= \mathbf{H}\bar{\mathbf{x}}(N) + \bar{\mathbf{v}}(N)
\end{aligned}
$$

MIMO

- We can concatenate them as

Placing side-by-side.

$$\underbrace{\overset{r \times N \text{ matrix}}{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{y}(1) & \bar{y}(2) & \cdots & \bar{y}(N) \\ \vdots & \vdots & & \vdots \end{bmatrix}}}_{Y} = H \underbrace{\overbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{x}(1) & \bar{x}(2) & \cdots & \bar{x}(N) \\ \vdots & \vdots & & \vdots \end{bmatrix}}^{t \times N \text{ pilot matrix}}}_{X} + \underbrace{\overset{r \times N \text{ noise matrix}}{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{v}(1) & \bar{v}(2) & \cdots & \bar{v}(N) \\ \vdots & \vdots & & \vdots \end{bmatrix}}}_{V}$$

MIMO

- We can concatenate them as

r x N

$$\underbrace{[\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \ldots \quad \bar{\mathbf{y}}(N)]}_{\mathbf{Y}}$$

t x N

$$= \mathbf{H}\underbrace{[\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \ldots \quad \bar{\mathbf{x}}(N)]}_{\mathbf{X}}$$

r x N

$$+ \underbrace{[\bar{\mathbf{v}}(1) \quad \bar{\mathbf{v}}(2) \quad \ldots \quad \bar{\mathbf{v}}(N)]}_{\mathbf{V}}$$

MIMO Estimation Model

- This can be represented in the compact fashion

Pilot matrix

$$Y = HX + V$$

Output matrix

MIMO channel Matrix

Noise matrix

MIMO Estimation Model

- This can be represented in the compact fashion

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{V}$$

MIMO Channel Estimation

- Note that in this case $\mathbf{X}$ is a wide matrix

$t \times N$

$$\underbrace{\left[\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \ldots \quad \bar{\mathbf{x}}(N)\right]}_{\bar{\mathbf{X}}}$$

t = # rows = # Transmit antennas.

N = # columns = # pilot vectors.

$N \geq t \Rightarrow$ # columns $\geq$ # rows

$\Rightarrow X$ is a wide matrix

MIMO Estimation Model

- The pseudo-inverse of $\mathbf{X}$ is given as

$$\underbrace{X^T(XX^T)^{-1}}_{\text{Pseudo inverse of wide matrix}}$$

acts as a right inverse

$$X \cdot X^T(XX^T)^{-1} = I$$

MIMO Estimation Model

- The pseudo-inverse of $\mathbf{X}$ is given as

$$\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-\mathbf{1}}$$

MIMO Estimation Model

- The MIMO channel estimate is

$$\underbrace{\hat{H} = YX^T\left(XX^T\right)^{-1}}_{\text{MIMO channel Estimate}}$$

MIMO Estimate

- The MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1}$$

$X \sim$ wide matrix

$t \times N$

$N \geq t$

$\Rightarrow$ #columns $\geq$ #rows.

MIMO Estimation Example

$N = 4$ pilot vectors.

- Consider the MIMO channel estimation problem with pilot vectors

$$\bar{\mathbf{x}}(1) = [3 \quad -2]^T, \bar{\mathbf{x}}(2) = [-2 \quad 3]^T$$
$$\bar{\mathbf{x}}(3) = [4 \quad 2]^T, \bar{\mathbf{x}}(4) = [2 \quad 2]^T$$

$$\bar{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \bar{x}(2) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \bar{x}(3) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \bar{x}(4) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

- What is the pilot matrix?

MIMO Estimation Example

- The pilot matrix is

t = # Transmit antennas
= 2
N = # pilot vectors
= 4.

$$\mathbf{X} = [\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \bar{\mathbf{x}}(3) \quad \bar{\mathbf{x}}(4)]$$

$$= \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}$$

2×4
$t \times N$
$t = 2$
$N = 4$

MIMO Estimation Example

- The pilot matrix is

$$\mathbf{X} = [\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \bar{\mathbf{x}}(3) \quad \bar{\mathbf{x}}(4)] = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}$$

MIMO Estimation Example

- The output vectors are

$$
\begin{aligned}
\bar{\mathbf{y}}(1) &= [-2, 1, -3]^T, \\
\bar{\mathbf{y}}(2) &= [-1, 3, 3]^T, \\
\bar{\mathbf{y}}(3) &= [-1, -2, 2]^T, \\
\bar{\mathbf{y}}(4) &= [-3, -1, 1]^T
\end{aligned}
$$

- What is the **output matrix**?

MIMO Estimation Example

- The output matrix is

r = # Receive antennas
= 3
$\Rightarrow$ 3 x 2 MIMO system.

$$\mathbf{Y} = [\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \bar{\mathbf{y}}(3) \quad \bar{\mathbf{y}}(4)]$$

$$= \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}$$

3 x 4 matrix
r x N
r = 3

MIMO Estimation Example

- The output matrix is

$$\mathbf{Y} = [\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \bar{\mathbf{y}}(3) \quad \bar{\mathbf{y}}(4)] = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}$$

MIMO Estimation Example

- The channel estimate is given as follows

$$\hat{H} = YX^T(XX^T)^{-1}$$

MIMO Estimation Example

- The channel estimate is given as follows

Diagonal matrix

$\Rightarrow$ rows orthogonal.

$\Rightarrow$ orthogonal Pilot

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\underline{(\mathbf{X}\mathbf{X}^T)^{-1}}$$

$$XX^T = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}$$

MIMO Estimation Example

$$XX^T = \overbrace{\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}}^{\text{diagonal matrix}}$$

Inversion is easy!!

$\Rightarrow$ inner product of rows of X is zero

$\Rightarrow$ rows are orthogonal.

$\Rightarrow$ Orthogonal pilot matrix

Typically preferred.

MIMO Estimation Example

- Let us first evaluate

$$\mathbf{X}\mathbf{X}^T$$

$$= \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}$$

MIMO Estimation Example

$$\mathbf{X}\mathbf{X}^T = \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}$$

$$\left(\mathbf{X}\mathbf{X}^T\right)^{-1} = \begin{bmatrix} \frac{1}{33} & 0 \\ 0 & \frac{1}{21} \end{bmatrix}$$

MIMO Estimation Example

- Let us now evaluate

$$\mathbf{YX}^T$$

$$= \overbrace{\begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}}^{Y} \overbrace{\begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}}^{X^T}$$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} = YX^T$$

3X2

MIMO Estimation Example

- Therefore,

$$\mathbf{YX}^T = \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix}$$

MIMO Estimation Example

Estimate of MIMO channel.

- Finally, the MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1}$$

$\hat{H} \sim 3\times 2$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} \frac{1}{33} & 0 \\ 0 & \frac{1}{21} \end{bmatrix} = \begin{bmatrix} -\frac{14}{33} & -\frac{7}{21} \\ -\frac{13}{33} & \frac{1}{21} \\ -\frac{5}{33} & \frac{21}{21} \end{bmatrix}$$

MIMO Estimation Example

- Finally, the MIMO channel estimate is

$$\widehat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T\left(\mathbf{X}\mathbf{X}^T\right)^{-\mathbf{1}}$$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} \frac{1}{33} & 0 \\ 0 & \frac{1}{21} \end{bmatrix} = \begin{bmatrix} -\frac{14}{33} & -\frac{7}{21} \\ -\frac{13}{33} & \frac{1}{21} \\ -\frac{5}{33} & \frac{21}{21} \end{bmatrix}$$

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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