Matrices – Inverse and Transpose

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed the following in the last lecture
  - matrices, their addition and multiplication rules
  - Solution of equations using eliminatin matrices
- Discuss the following today
  - Matrix transpose, inverse and their properties

Inverse Matrices (1)

- Square matrix A is invertible if there exists a matrix A^{-1} that "inverts" A:

$$A^{-1}A = \mathbf{I} \text{ and } AA^{-1} = \mathbf{I}$$

- Not all matrices have inverses. Is A invertible?
- Note 1: Inverse exists if and only if elimination produces n pivots
  - (row exchanges are allowed)
- Note 2: A cannot have two different inverses
- Let $BA = I$ and $AC = I$ then $B = C$
- Proof

$$B(A\mathbf{C}) = (BA)\mathbf{C} \text{ gives } B\mathbf{I} = \mathbf{IC} \text{ or } B = \mathbf{C}$$

- Left-inverse B and a right-inverse C must be the same matrix

Inverse Matrices (2)

- Note 3: If A is invertible, the one and only solution to $A\mathbf{x} = \mathbf{b}$ $\mathbf{x} = A^{-1}\mathbf{b}$:
  - Proof: $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$
- Let there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ then A is not invertible
  - No matrix can bring $\mathbf{0}$ back to $\mathbf{x}$
- If A is invertible, then $A\mathbf{x} = \mathbf{0}$ can only have the zero solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$
- A is 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Number $ad - bc$ is the determinant of A
  - A matrix is invertible if its determinant is not zero

Inverse Matrices (3)

- A diagonal matrix has an inverse provided no diagonal entries are zero:

$$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$$

- If A and B are invertible then so is AB i.e., $(AB)^{-1} = B^{-1}A^{-1}$
- Proof:

$$(AB)(B^{-1}A^{-1}) = (ABB^{-1})A^{-1}) = AIA^{-1} = AA^{-1} = I$$

- Inverses come in reverse order

$$(AB\mathbf{C})^{-1} = \mathbf{C}^{-1}B^{-1}A^{-1}$$

Transpose of a matrix (1)

- "Transpose" of A, which is denoted by A^T

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- Columns of A^T are the rows of A - exchange rows and columns $(A^T)_{ij} = A_{ij}$
- Transpose of $A + B = (A + B)^T = A^T + B^T$
- Transpose of AB is $(AB)^T = B^T A^T$. Proof:
  - Start by considering B just a vector $\mathbf{x} \Rightarrow (A\mathbf{x})^T = \mathbf{x}^T A^T$
  - $A\mathbf{x}$ combines the columns of A while $\mathbf{x}^T A^T$ combines the rows of A^T
  - It is the same combination of the same vectors
  - Transpose of column Ax is the row $\mathbf{x}^T A^T$
  - Fits our formula $(A\mathbf{x})^T = \mathbf{x}^T A^T$

Transpose of a matrix (2)

- Now we can prove the formula $(AB)^T = B^T A^T$, when B has several columns
- If $B = [\mathbf{x}_1\ \mathbf{x}_2]$ has two columns, apply the same idea to each column
- Columns of AB are $A[\mathbf{x}_1\ \mathbf{x}_2]$ Their transposes appear in rows of $B^T A^T$

$$\text{Transposing } AB = \begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 & \dots \end{bmatrix} \text{ gives } \begin{bmatrix} \mathbf{x}_1^T A^T \\ \mathbf{x}_2^T A^T \\ \vdots \end{bmatrix} \text{ which is } B^T A^T$$

- $(A^{-1})^T = (A^T)^{-1}$. Proof

$$A^{-1}A = \mathbf{I} \text{ is transposed to } A^T (A^{-1})^T = \mathbf{I} \Rightarrow (A^T)^{-1} = (A^{-1})^T$$

Inner Products, symmetric matrices

- Dot product (inner product) of $\mathbf{x}$ and $\mathbf{y}$ is the sum of numbers $x_i y_i$
  - Now we have a better way to write $\mathbf{x} \cdot \mathbf{y}$ i.e., $\mathbf{x}^T \mathbf{y}$
- A symmetric matrix has $S^T = S$. This means that $S_{ji} = S_{ij}$
- $(A^T A)^T$ is $A^T (A^T)^T$ which is $A^T A$ again

Review of key ideas

- Inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$
- A is invertible if and only if it has n pivots (row exchanges allowed)
- Important: If $A\mathbf{x} = 0$ for a nonzero vector $\mathbf{x}$, then A has no inverse
- Inverse of AB is the reverse product $B^{-1}A^{-1}$
- $(AB)^T = B^T A^T$ and $(A^{-I})^T = (A^T)^{-1}$
- Dot product is $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$. Then $(A\mathbf{x})^T\mathbf{y}$ equals the dot product $\mathbf{x}^T(A^T\mathbf{y})$