6. $\min\limits_{Ax=b} \|x\|_2 = \min\limits_{Ax=b} \frac{1}{2}\|x\|_2^2$, $f(x) = \frac{1}{2}\|x\|_2^2$

$\underline{\nabla f(x^*) \in R(A^T)}$

$$\boxed{\begin{aligned} &\min \tfrac{1}{2}\|x\|_2^2 \\ &Ax=b \end{aligned}}$$

$x^* \in R(A^T) \iff \exists v \quad A^T v = x^*$

$AA^T v = Ax^* = b$

$v = (AA^T)^{-1} b$

$x^* = A^T v = A^T (AA^T)^{-1} b$

$\underline{\langle \nabla f(x^*), x - x^* \rangle \geq 0} \quad \forall\ x \in \mathcal{X}$

$$\boxed{\begin{aligned} &\min f(x) \\ &x \in \mathcal{X} \end{aligned}}$$

1. $\min \max (P^T u)_i \iff \min t$

$u \geq 0$ $\qquad \max\limits_i (P^T u)_i \leq t$

$\sum u_i = 1$ $\qquad u \geq 0$

$(P^T u)_i \leq t \qquad \sum u_i = 1$

$\forall i$

dual variables

$$\boxed{\begin{aligned} &\min_{u,t}\ t \\ &P^T u \leq t\mathbb{1} \\ &u \geq 0 \\ &\mathbb{1}^T u = 1 \end{aligned}}$$

v $\qquad P^T u \leq t\mathbb{1}$

λ $\qquad -u \leq 0$

ν

$$1 - \mathbb{1}^T u$$

$$L(u, t, v, \lambda, \nu) = t + v^T(P^T u - t\mathbb{1}) - \lambda^T u + \nu(\mathbb{1}^T u - 1)$$

$$= t(1 - \mathbb{1}^T v) + u^T(Pv - \lambda + \nu\mathbb{1}) - \nu$$

$$0 \qquad\qquad 0$$

$$\min_t \; t(1 - \mathbb{1}^T v) = \begin{cases} 0 & 1 = \mathbb{1}^T v \\ -\infty & \text{o/w} \end{cases}$$

$$\min_u \; u^T(Pv - \lambda + \nu\mathbb{1}) = \begin{cases} 0 & Pv - \lambda + \nu\mathbb{1} = 0 \\ -\infty & \text{o/w} \end{cases}$$

Dual Problem

$$\max_{\lambda, v, \nu} \; -\nu$$

$$Pv + \nu\mathbb{1} \geq 0$$

$$\mathbb{1}^T v = 1$$

$$\boxed{\begin{matrix} Pv + \nu\mathbb{1} = \lambda \\ \lambda \geq 0 \end{matrix}} \leftarrow Pv - \lambda + \nu\mathbb{1} = 0 \rightarrow Pv + \nu\mathbb{1} \geq 0$$

$$v \geq 0, \; \lambda \geq 0$$

$$\nu = -w$$

$$d^* = \max_{v \geq 0} \min_j (Pv)_j$$

$$\max \; w$$

$$\sum v_i = 1 \searrow$$

$$Pv \geq w\mathbb{1}$$

$$v \geq 0$$

$$\mathbb{1}^T v = 1$$

$$\max \; w$$

$$\min (Pv)_j \geq w$$

Eg $\qquad \underline{f(x)} = \underline{y \geq 0} \qquad \longleftrightarrow \qquad \underline{f(x)} \geq 0$

2. $\min_{r, x} \sum_{i=1}^{m} \phi_i(r_i)$ $\qquad \phi(u) = \begin{cases} 0 & |u| \leq 1 \\ |u| - 1 & |u| > 1 \end{cases}$

$r = Ax - b \; \ldots\ldots \; \underline{\nu}$

$$L = \sum \phi_i(r_i) + \nu^T(r - Ax + b)$$
$$= \sum \phi_i(r_i) + \sum_i \nu_i (r_i - a_i^T x + b_i)$$
$$= \sum_{i=1}^{m} \left(\phi_i(r_i) + \nu_i r_i\right) - \nu^T A x + \underline{\nu^T b}$$

$\min_{r, x} L$ $\qquad \min_{x} -x^T(A^T \nu) = \begin{cases} 0 & A^T\nu = 0 \\ -\infty & o/w \end{cases}$

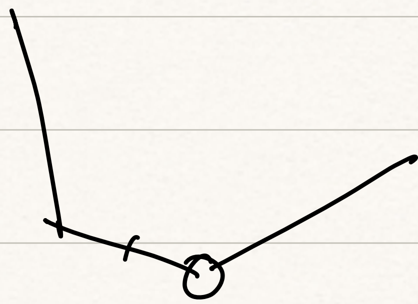
$$\min_{\underline{r}} \sum_{i=1}^{m} \phi(r_i) + \nu_i r_i$$

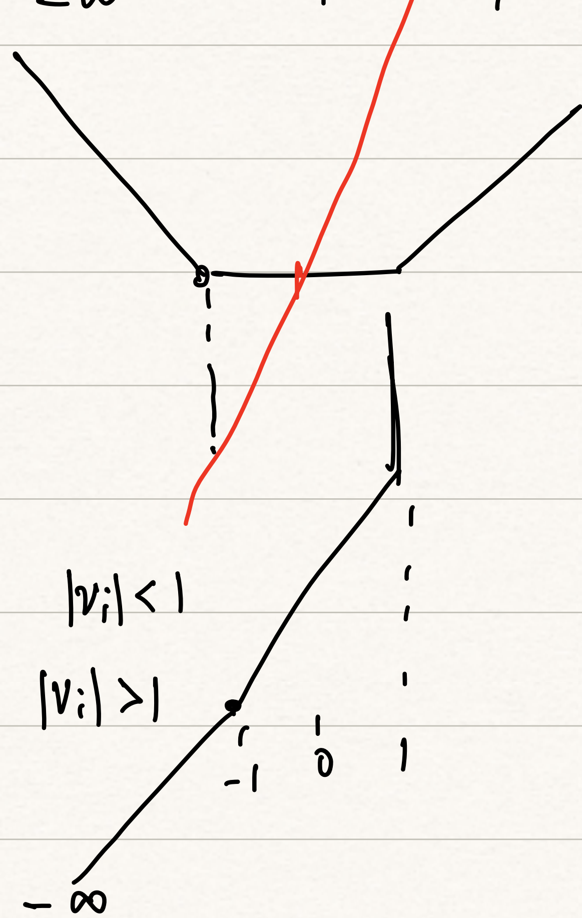
$$= \sum_{i=1}^{m} \underbrace{\min_{r_i} \phi(r_i) + \nu_i r_i}$$

$\phi(r_i) + \underline{v_i r_i}$

$v_i > 0$

$-v_i$	when $v_i > 0$, small	
v_i	when $v_i < 0$, small	
$-\infty$	$v_i > 0$, $v_i > 1$	
$-\infty$	$v_i < 0$	$v_i < -1$





$$\min \phi(r_i) + v_i r_i = \begin{cases} -|v_i| & |v_i| < 1 \\ -\infty & |v_i| > 1 \end{cases}$$

$$\max \; b^T v - \sum_{i=1}^{m} |v_i| \quad \longleftrightarrow \quad b^T v - \|v\|_1$$

$$|v_i| \le 1 \quad \forall i \quad \longleftrightarrow \quad \|v\|_\infty \le 1$$

$$A^T v = 0$$

3. $p^* = \min x^T A x$ $\quad\quad$ $A = A^T$

$\nearrow$ $x_i \in \{-1, 1\} \longleftrightarrow x_i^2 = 1$

$x_i = 1$ then
$x^T A x = \sum_{i,j} A_{ij}$

$\min x^T A x$

$x_i^2 = 1 \quad \ldots\ldots \nu_i$

$L(x, \underline{\nu}) = x^T A x + \underline{\sum \nu_i x_i^2} - \mathbf{1}^T \underline{\nu}$

$$[x_1 \ \ldots \ x_n] \begin{bmatrix} \nu_1 & & 0 \\ & \nu_2 & \\ 0 & & \ddots \ \nu_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{x}^T \mathrm{Diag}(\underline{\nu}) \underline{x}$$

$L(x, \nu) = x^T (A + \mathrm{Diag}(\nu)) x - \mathbf{1}^T \nu$

$$\min_x x^T (A + \mathrm{Diag}(\nu)) x = \begin{cases} 0 & A + \mathrm{Diag}(\nu) \succeq 0 \\ -\infty & A + \mathrm{Diag}(\nu) \not\succeq 0 \end{cases}$$

<u>Aside</u> $\quad x^T Q x \geq 0$ when $Q \succeq 0$

$Q \not\succeq 0 \quad \min x^T Q x = -\infty$

$Q v = \lambda v \quad \underline{\underline{\lambda < 0}}$

$x = \alpha v \quad\quad x^T Q x = (\alpha v)^T Q (\alpha v)$
$= \alpha^2 \lambda v^T v \longrightarrow -\infty$

$\alpha \to \infty$

Dual Problem $\quad \max_{\nu} \ -\mathbf{1}^T \nu$

s.t. $A + \mathrm{Diag}(\nu) \succeq 0$

$\nu_i = -\lambda_{min}(A)$

$\mathrm{Diag}(\underline{\nu}) = -\lambda_{min}(A) I$

e.v. of $\underline{A + \mathrm{Diag}(\nu)}$ are $\quad \lambda_i - \lambda_{min}(A) \geq 0$

feasible solution $\quad$ since $\lambda_i \geq \lambda_{min}(A)$

$$-\mathbf{1}^T \nu = n \lambda_{min}(A)$$

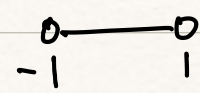
$$\underline{\sum A_{ij}} \geq p^* \geq n \lambda_{min}(A)$$

$$x^T A x \geq \lambda_{min}(A) \|x\|^2$$

$\hookrightarrow n$ when $x_i = 1$

$$x^T A x \geq n \lambda_{min}(A) \quad \forall x$$

$$p^* = \min_x x^T A x \geq n \lambda_{min}(A)$$



$$\inf(-1, 1) = -1$$

$$\min(-1, 1) \neq -1$$

4. $\min_x \max_i (a_i^T x + b_i)$

$$\min_x t$$

$\max_i a_i^T x + b_i \leq t \iff a_i^T x + b_i \leq t \;\; \forall i$

$Ax + b \leq t\mathbb{1} \;\; \forall i$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ \vdots \end{bmatrix}$$

$$\min_{x,t} t$$

$Ax+b \leq t\mathbb{1} \; \ldots \; \underline{\lambda}$

$$L(x,t,\lambda) = t + \lambda^T(Ax+b-t\mathbb{1})$$

$$t(1-\mathbb{1}^T\lambda) + x^T(A^T\lambda) + \lambda^T b$$

$$\rightarrow \min_t (1-\mathbb{1}^T\lambda)t = \begin{cases} 0 & 1 = \lambda^T 1 \\ -\infty & o/w \end{cases}$$

$$\min_x x^T(A^T\lambda) = \begin{cases} 0 & A^T\lambda = 0 \\ -\infty & o/w \end{cases}$$

$$\max \lambda^T b$$

$\lambda \geq 0$

$\lambda^T 1 = 1$

$A^T\lambda = 0$

5. $\min \sum_{i=1}^{m} e^{x_i - 1} + y$ $\qquad\qquad$ $\min \log\left(\sum e^{u_i}\right)$

$\underline{A}\underline{x} - b + y\underline{1} \geq 0 \quad \longleftrightarrow \quad Au - b \geq 0$

$u = \underline{x} + y\underline{1}$ $\qquad$ $Au = Ax + y\underbrace{A\underline{1}}_{=\underline{1}} = Ax + y\underline{1}$

$$\min_{\underline{u}, y} \sum \exp(u_i - y - 1) + y$$

$$A\underline{u} - b \geq 0$$

$$\min_{\substack{\underline{u} \\ Au - b \geq 0}} \min_{y} \underbrace{\underset{+1}{\underline{\sum \exp(u_i - y. - 1)}} + y}$$

$$-\underbrace{\sum_i \exp(u_i - y - 1)} + 1 = 0$$

$-e^{-(y+1)} \sum e^{u_i} + 1 = 0 \Rightarrow 1 = e^{-(y+1)} \sum e^{u_i}$

$e^{y+1} = \sum e^{u_i}$

$y = \underline{\log\left(\sum e^{u_i}\right)} - 1$

$+1 + y = \underline{\log\left(\sum e^{u_i}\right)}$