Assignment 2

Ques1: What is the minimum distance b/w two parallel half spaces $\{x \in \mathbb{R}^n \mid a^T x \le b_1\}$ and $\{x \in \mathbb{R}^n \mid a^T x \ge b_2\}$?

Soln: The minimum distance b/w $\{x \in \mathbb{R}^n \mid a^T x \le b_1\}$ and $\{x \in \mathbb{R}^n \mid a^T x \ge b_2\}$ is absolute distance b/w b_1 and b_2, divided by norm of a

$$d = \frac{|b_1 - b_2|}{\|a\|} \qquad a \ne 0 \text{ vector}$$

In this, the half spaces must be non empty & intersecting each other otherwise for disjoint half spaces, $d_{min} = 0$.

Ques2: Is the following set affine: $\{x \in \mathbb{R}^n \mid \|x - x_1\|_1 \le \|x - x_2\|_1\}$?

Soln: For affine set, if x and y are two pts in the set, then any pt on the line segment connecting x and y is also in the set.

Let x, y be two pts in the set, i.e. $\|x - x_1\|_1 \le \|x - x_2\|_1$ — (1)

$\|y - x_1\|_1 \le \|y - x_2\|_1$ — (2)

Consider a pt z on line segment x & y. Then, z can be written as

$$z = \lambda x + (1-\lambda) y \qquad 0 \le \lambda \le 1$$

λ is scalar

$$\|z - x_1\|_1 = \|(\lambda x + (1-\lambda) y) - x_1\|_1$$

Using triangle inequality

$$\|z - x_1\|_1 \le \|\lambda x - x_1\|_1 + \|(1-\lambda) y - x_1\|_1$$

$$\|z - x_1\|_1 \le \lambda \|x - x_2\|_1 + (1-\lambda)\|y - x_2\|_1 \quad \text{(from (1) \& (2))}$$

$$\|z - x_1\|_1 \le \lambda \|x - x_2\|_1 + \lambda \|y - x_2\|_1 + \lambda \|y - x_2\| - \lambda \|y - x_2\|$$

$$\|z - x_1\|_1 \le \lambda \|x - x_2\|_1 + (1-\lambda) \|y - x_2\|_1$$

$$\|z - x_1\|_1 \leq \|x - x_2\|$$

Therefore z also satisfies which shows

$\{x \in R^n \mid \|x - x_1\|_1 \leq \|x - x_2\|_1\}$ is an affine set.

Que 3:- Show that the set $\{x \in R^n \mid \|x\|_\infty \leq 1\}$ is a polyhedra and express it an intersection of finite number of half spaces (and hyperplanes, if required).

Soln: The max (∞) norm for vector $x \in R^n$ is

$$\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$$

The given set can be expressed as an intersection of half spaces, each half space out of n will represent one plane.

for ith dimension

→ if $x_i \leq 1$, then $x \leq 1$ in ith dimension for half space: $x_i \leq 1$

→ if $x_i \geq 1$, then $-x \leq 1$ in the ith dimension for half space $-x_i \leq 1$

with all n dimensions, total $2n$ half spaces will be there which forms intersection of half spaces for polyhedron.

Therefore, $\{x \in R^n \mid \|x\|_{max} \leq 1\}$ is a polyhedron defined by half spaces having inequalities

$$x_i \leq \ ; \ 1 \leq i \leq n$$

$$-x_i \leq 1 \ ; \ 1 \leq i \leq n$$

Que 4:- Given θ, consider the set $S = \{x \in \mathbb{R}^n \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$; $a \neq b$

Soln:- (a) S is a half space for $\theta = 1$

$S = \{x \in \mathbb{R}^n \mid \|x-a\|_2 \leq \|x-b\|\}$

This represents a set of all pts that are closer to pt 'a' than 'b'. That is, this is the half space defined by the perpendicular bisector of line segment connecting a & b.

$\therefore$ set S is a half space when $\theta = 1$

(b) for $\theta < 1$

$S = \{x \in \mathbb{R}^n \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$

for $\theta < 1$, the set S is convex since it forms an ellipse $\|x-a\|_2 = \|x-b\|_2$. Any 2 pts in the set can be connected by a straight line that lies within the set S.

(c) Eg to prove S can be non-convex for $\theta > 1$

Let $n=2$ and consider pts $a = (0,0)$ & $b = (1,1)$.

for $\theta > 1$, S contains all pts closer to $(0,0)$ than to $(1,1)$, except within ellipse, given by $\|x-(0.5, 0.5)\| = \theta \|(x) - (1,1)\|$ which will be a hole in S, making it non convex.

PTO

Q5:- Show that the intersection of two convex cones is a convex cone.

Soln:- Let C_1 & C_2 be two convex cones in vector space V.

$$C = C_1 \cap C_2$$

C will be a convex cone if it follows convexity & cone property.

1) convexity property

Since x & y are in C,

$x \in C_1$ and $x \in C_2$

$y \in C_1$ and $y \in C_2$

As C_1 & C_2 are convex cones

vectors $u, v \in V$, scalar $\mu, \alpha \geq 0$

$u \in C_1$ and $v \in C_1 \Rightarrow \mu u + \alpha v \in C_1$

lly $u \in C_2$ and $v \in C_2 \Rightarrow \mu u + \alpha v \in C_2$

for convex combination

$$\lambda x + (1-\lambda) y$$

$$\lambda x + (1-\lambda) y = \lambda(x+y) + (1-\lambda) y$$

As C_1 & C_2 are cones

$\lambda(x+y) + (1-\lambda) y \in C_1$

$\lambda(x+y) + (1-\lambda) y \in C_2$

Holds for C also $(C_1 \cap C_2)$

2) Cone property

As C_1 & C_2 are cones, their intersection should also be cone.

C $(C_1 \cap C_2)$.