

eMasters in Communication Systems

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Jagannatham**



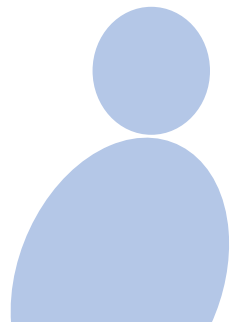
Elective Module:

**Estimation for Wireless
Communication**



Chapter 4

Vector Parameter Estimation



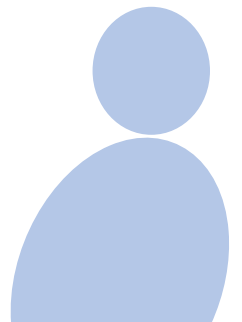
Least Squares Solution

- The Least Squares solution can be **derived** as follows

How to derive this?

$$\hat{h} = (X^T X)^{-1} X^T \bar{y}$$

Least Squares Solution

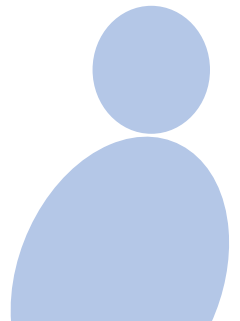


Least Squares Solution

- The LS cost function is

$$\min \left\| \underbrace{\bar{y}}_{N \times 1} - \underbrace{X}_{N \times M} \underbrace{h}_{M \times 1} \right\|^2$$

minimize
this



Least Squares Solution

- The LS cost function is

$$\|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\|^2$$



Least Squares Solution

- The LS cost function can be simplified as

$$\begin{aligned} & \|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\|^2 \\ = & (\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}})^T (\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}) = (\bar{\mathbf{y}}^T - \bar{\mathbf{h}}^T \mathbf{X}^T) (\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}) \\ = & \bar{\mathbf{y}}^T \bar{\mathbf{y}} - \bar{\mathbf{h}}^T \mathbf{X}^T \bar{\mathbf{y}} - \bar{\mathbf{y}}^T \mathbf{X} \bar{\mathbf{h}} + \bar{\mathbf{h}}^T \mathbf{X}^T \mathbf{X} \bar{\mathbf{h}} \end{aligned}$$

EQUAL!



Least Squares Solution

$$\left(\bar{h}^T X^T \bar{y} \right)^T = \bar{y}^T X \bar{h}$$

Scalar & Transpose
 \Rightarrow EQUAL!

$$[5]^T = [5]$$

$$\| \bar{y} - X \bar{h} \|^2 = \bar{y}^T \bar{y} - 2 \bar{h}^T X^T \bar{y} + \bar{h}^T X^T X \bar{h}$$

Expansion

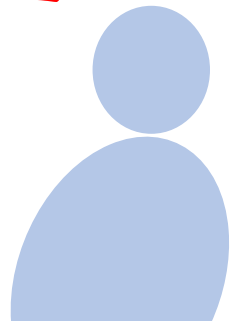


Least Squares Solution

- The **LS cost function** can be simplified as

*Find minimum
wrt $\bar{\mathbf{h}}$*

$$\begin{aligned} & \|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\|^2 \\ &= (\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}})^T (\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}) \\ &= \bar{\mathbf{y}}^T \bar{\mathbf{y}} - 2\bar{\mathbf{h}}^T \mathbf{X}^T \bar{\mathbf{y}} + \bar{\mathbf{h}}^T \mathbf{X}^T \mathbf{X} \bar{\mathbf{h}} \end{aligned}$$

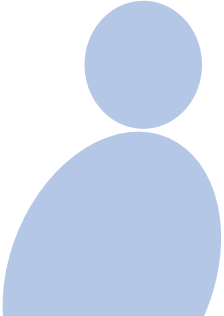


Least Squares Solution

- To minimize, we now calculate the **gradient** and set to zero

$$\nabla f(\bar{\mathbf{h}}) = \begin{bmatrix} \frac{\partial f}{\partial h_1} \\ \frac{\partial f}{\partial h_2} \\ \vdots \\ \frac{\partial f}{\partial h_m} \end{bmatrix}$$

Partial derivatives
wrt to each component
of $\bar{\mathbf{h}}$



Least Squares Solution

- We now calculate the gradient

$$\nabla f(\bar{\mathbf{h}}) = \begin{bmatrix} \frac{\partial f(\bar{\mathbf{h}})}{\partial h_1} \\ \frac{\partial f(\bar{\mathbf{h}})}{\partial h_2} \\ \vdots \\ \frac{\partial f(\bar{\mathbf{h}})}{\partial h_M} \end{bmatrix} \quad \text{--- } \underline{M \times 1}$$



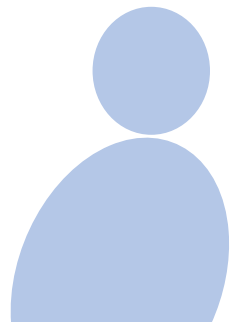
Least Squares Solution $\bar{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}$

- We use the following principles

$$\bar{\mathbf{c}}^T \bar{\mathbf{h}} = c_1 h_1 + \cdots + c_M h_M = \bar{\mathbf{h}}^T \bar{\mathbf{c}}$$

$$\nabla \bar{\mathbf{c}}^T \bar{\mathbf{h}} = \nabla \bar{\mathbf{h}}^T \bar{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} = \bar{\mathbf{c}} \quad \text{Principle \#1}$$

$$\nabla \bar{\mathbf{h}}^T \bar{\mathbf{c}} = \nabla \bar{\mathbf{c}}^T \bar{\mathbf{h}} = \bar{\mathbf{c}}$$



Least Squares Solution

- We use the following principles

$$\bar{\mathbf{c}}^T \bar{\mathbf{h}} = c_1 h_1 + \cdots + c_M h_M$$

$$\nabla \bar{\mathbf{c}}^T \bar{\mathbf{h}} = \nabla \bar{\mathbf{h}}^T \bar{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}$$



Least Squares Solution

- For a symmetric matrix $\mathbf{P} = \mathbf{P}^T$

$$\nabla (\bar{\mathbf{h}}^T \mathbf{P} \bar{\mathbf{h}}) = 2 \mathbf{P} \bar{\mathbf{h}}$$

Principle #2



Least Squares Solution

- For a symmetric matrix $\mathbf{P} = \mathbf{P}^T$

$$\nabla \bar{\mathbf{h}}^T \mathbf{P} \bar{\mathbf{h}} = 2\mathbf{P} \bar{\mathbf{h}}$$



Least Squares Solution

- Therefore, it follows that

$$\nabla \|\bar{y} - X\bar{h}\|^2$$

$= \nabla \left(\bar{y}^T \bar{y} - 2 \bar{h}^T \underbrace{X^T \bar{y}}_{\bar{c}} + \bar{h}^T X^T X \bar{h} \right)$

$= 0 - 2 X^T \bar{y} + 2 (X^T X) \bar{h}$

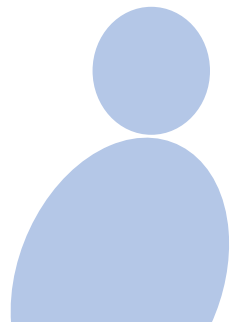
Since \bar{y} is a given fixed vector!
 Principle 1 with $\bar{c} = X^T \bar{y}$
 $P = X^T X$
 $P = P^T$
 Principle 2 $X^T X = P$
 Now set this to 0.

Least Squares Solution

- Therefore, it follows that

To minimize
set this to zero.

$$\begin{aligned} \nabla \|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\|^2 &= \\ \nabla (\bar{\mathbf{y}}^T \bar{\mathbf{y}} - 2\bar{\mathbf{h}}^T \mathbf{X}^T \bar{\mathbf{y}} + \bar{\mathbf{h}}^T \mathbf{X}^T \mathbf{X} \bar{\mathbf{h}}) &= \\ \underline{-2\mathbf{X}^T \bar{\mathbf{y}} + 2\mathbf{X}^T \mathbf{X} \bar{\mathbf{h}}} \end{aligned}$$



Least Squares Solution

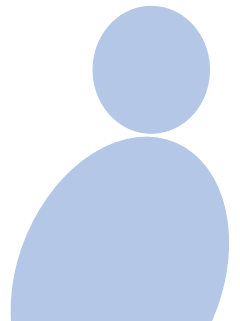
- Setting gradient equal to zero yields

$$-2X^T\bar{y} + 2X^TX\bar{h} = 0$$

$$\Rightarrow \cancel{2}X^TX\bar{h} = \cancel{2}X^T\bar{y} \quad \text{ML Estimate.}$$

$$\Rightarrow \hat{h} = (X^TX)^{-1}X^T\bar{y}$$

LEAST SQUARES SOLUTION.



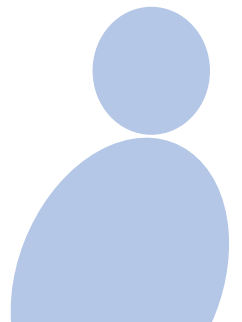
Least Squares Solution

- Setting gradient equal to zero yields

$$-2\mathbf{X}^T \bar{\mathbf{y}} + 2\mathbf{X}^T \mathbf{X} \bar{\mathbf{h}} = \mathbf{0}$$

$$\Rightarrow \mathbf{X}^T \mathbf{X} \bar{\mathbf{h}} = \mathbf{X}^T \bar{\mathbf{y}}$$

$$\Rightarrow \underline{\hat{\mathbf{h}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \bar{\mathbf{y}}}$$



Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

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