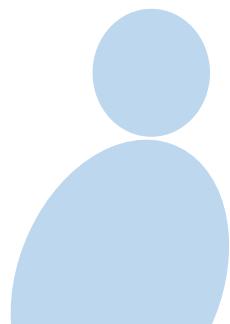


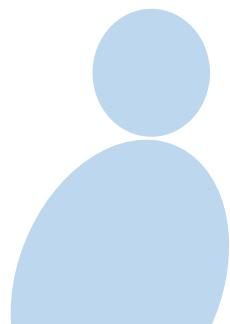
eMasters in Communication Systems

**Prof. Aditya
Jagannatham**



Elective Module:

**Advanced ML
Techniques**

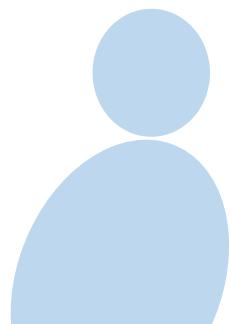


Chapter 5

Naïve Bayes

Bayesian
Technique :

- Prior
- Posterior



Naïve Bayes

- **Naïve Bayes** is best suited for ML applications...

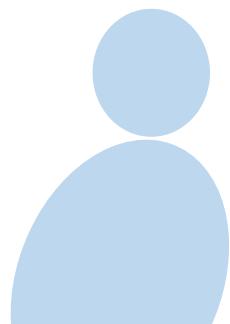
Binary Response

$$y \in \{0, 1\}$$

- wherein the feature vectors \bar{x} are

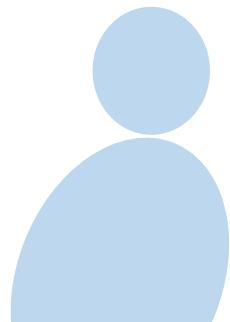
Discrete

	y	\bar{x}
Linear regression	Continuous	Continuous
Logistic Regression	Discrete	Continuous
Naïve Bayes	Discrete	Discrete



Naïve Bayes

- **Naïve Bayes** is best suited for ML applications...
 - wherein the feature vectors \bar{x} are **discrete**

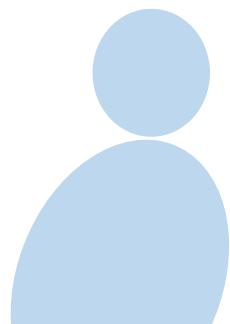


Naïve Bayes

$y = 1$ Spam
 $y = 0$ Genuine

- Example: ML-based e-mail spam filter
- Like/ dislike an item based on other preferences

movie



Naïve Bayes

Email classification
Spam or Genuine

- Consider a **feature vector** \bar{x} of size \underline{N}
- where N is the number of words in the

English Dictionary

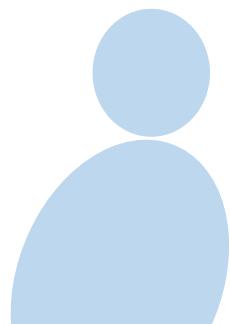
$$\bar{x} = [\quad]$$

$\sim 100,000$ words · each entry corresponds to a word.

$\approx 100,000$

Naïve Bayes

- Consider a **feature vector** \bar{x} of size N
 - where N is the number of words in the **English language dictionary**

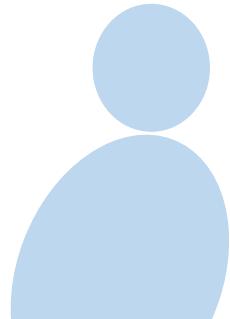


Naïve Bayes

- The labels $y = 1, 0$ indicate SPAM,
Genuine e-mails, respectively

Naïve Bayes

- The labels $y = 1, 0$ indicate **spam, genuine** e-mails, respectively

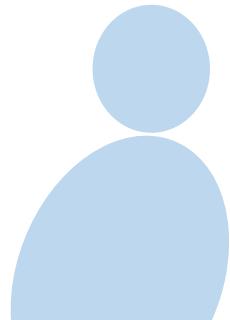


Naïve Bayes

- The entry $x_j = 1$, if the email contains the j -th word of the **dictionary**,...

- else $x_j = 0$

Each j corresponds to
a word in dictionary



Naïve Bayes

$$N = \# \text{Features}$$

Feature
vector

$$\bar{\mathbf{x}} =$$

$$\begin{bmatrix} \vdots \\ 1 \\ 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

4562
able
above
abroad
access

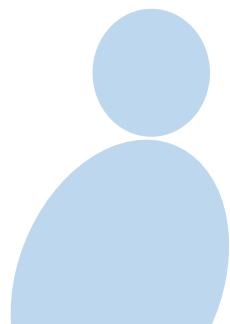
4563
4564
4565
:
input vector

$$x_{4562} = 1$$

$$x_{4563} = 0$$

$$x_{4564} = 0$$

$$x_{4565} = 1$$

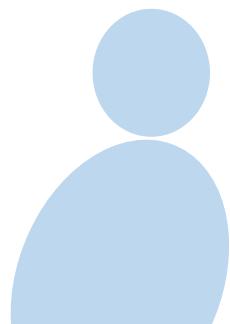


Naïve Bayes

Simple, NOT
complicated.

- Naïve Bayes assumption:
- The different words are conditionally independent. given the label y

Derogatory

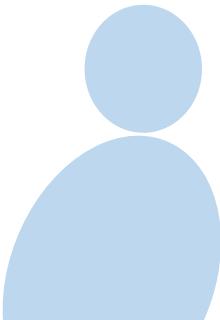


Naïve Bayes

- Naïve Bayes assumption:
- The different words are **conditionally independent** given the label y

Simplistic
assumption

Reasonably
True!



Naïve Bayes

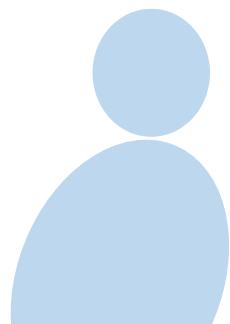
$$p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = u)$$

$u \in \{0, 1\}$
 $v_i \in \{0, 1\}$

$$= p(x_1 = v_1, x_2 = v_2, \dots, x_N = v_N | y = u)$$

$$= p(x_1 = v_1 | y = u) \times \dots \times p(x_N = v_N | y = u)$$

$$= \prod_{j=1}^N p(x_j = v_j | y = u)$$



Naïve Bayes Assumption

$$p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = u)$$

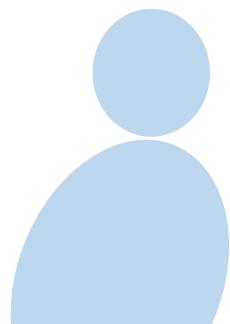
$$= p(x_1 = v_1, \dots, x_N = v_N | y = u)$$

$$= p(x_1 = v_1 | y = u) \times \dots \times p(x_N = v_N | y = u)$$

$$= \prod_{j=1}^N p(x_j = v_j | y = u)$$

Conditional
independence

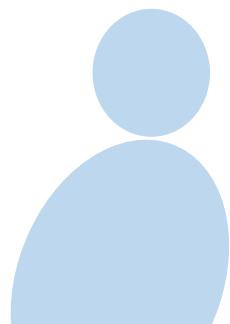
Conditionally independent
given response.



Naïve Bayes

- The quantities $p(x_j = v_j | y = u)$ are the prior probabilities.
 - How to calculate these?

Prior
Posterior



Naïve Bayes

- The quantities $p(x_j = v_j | y = u)$ are the **prior probabilities**
 - How to calculate these?

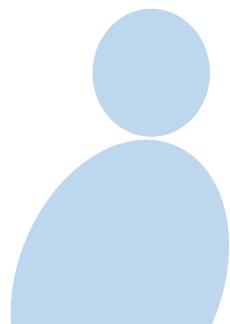
$$P(x_{4562} = 1 | y = 1)$$

$$P(x_{4562} = 1 | y = 0)$$

Probabilities that j^{th} word occurs in genuine/spam emails.

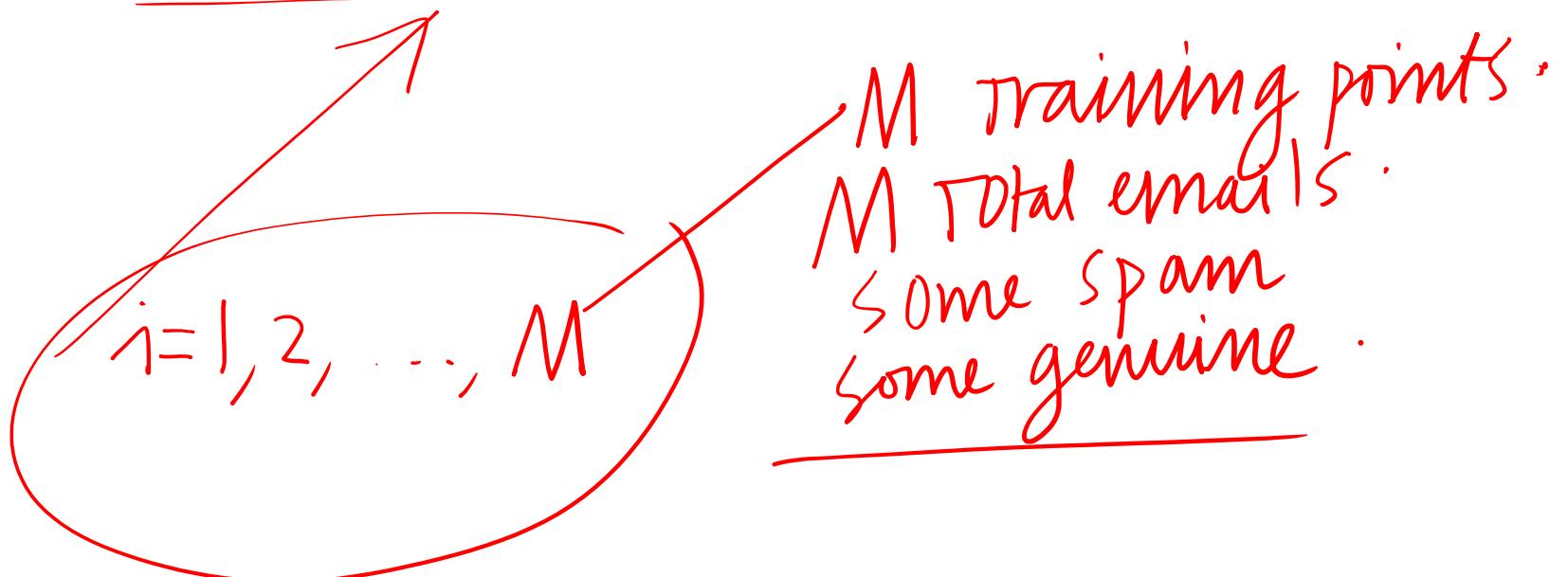
What is probability word able occurs in a spam email.

What is the probability word able occurs in a genuine email.



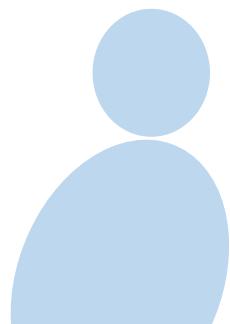
Naïve Bayes

- Consider the availability of M **training pairs** $(\bar{x}(i), y(i))$



Naïve Bayes

- The various **prior probabilities** can now be calculated as follows



Naïve Bayes

$1(x) = \begin{cases} 1 & \text{if } x \text{ is true} \\ 0 & \text{else.} \end{cases}$

$$p(x_j = 1 | y = 1) = \frac{\text{number of spam emails with } j\text{th word}}{\text{Total Number of spam emails}}$$

Prior probability.
 j th word occurs in
spam email.

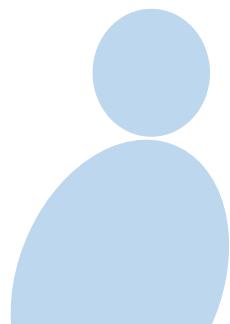
$$= \frac{\sum_{i=1}^M 1(x_j(i) = 1, y(i) = 1)}{\sum_{i=1}^M 1(y(i) = 1)}$$

TOTAL number
OF SPAM
Emails with
jth word.
TOTAL number of
spam emails.

Naïve Bayes

$$p(x_j = 1 | y = 0) = \frac{\text{Number of genuine emails with } j\text{th word.}}{\text{Total number of genuine emails.}}$$
$$= \frac{\sum_{i=1}^M 1(x_j(i) = 1, y(i) = 0)}{\sum_{i=1}^M 1(y(i) = 0)}$$

Prior probability
jth word occurs in
a genuine email.

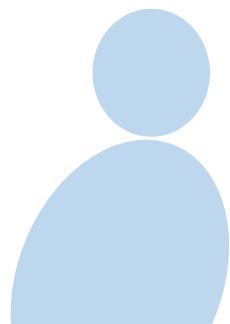


Naïve Bayes

$$p(x_j = 1 | y = 1) = \frac{\sum_{i=1}^M \mathbf{1}(x_j(i) = 1, y(i) = 1)}{\sum_{i=1}^M \mathbf{1}(y(i) = 1)}$$

$$p(x_j = 1 | y = 0) = \frac{\sum_{i=1}^M \mathbf{1}(x_j(i) = 1, y(i) = 0)}{\sum_{i=1}^M \mathbf{1}(y(i) = 0)}$$

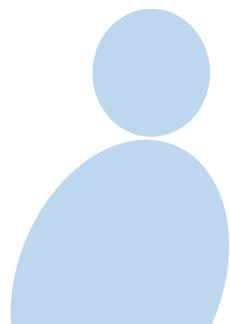
Prior
probabilities -



Naïve Bayes

$$p(y = 1) = \frac{\text{Number of Spam emails}}{\text{Total number of emails}}$$
$$= \frac{\sum_{i=1}^M 1(y(i) = 1)}{M}$$

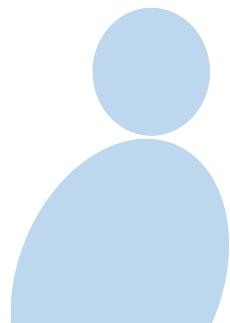
Prior probability
of spam email



Naïve Bayes

$$p(y = 1) = \frac{\sum_{i=1}^M \mathbf{1}(y(i) = 1)}{M}$$

Prior probability
IF getting a spam
email.



Naïve Bayes

$$p(x_j = 0 | y = 1) = 1 - P(x_j = 1 | y = 1).$$

$$p(x_j = 0 | y = 0) = 1 - P(x_j = 1 | y = 0)$$

Prior prob. j^{th} word
does NOT occurs in genuine
 $= 1 - \text{prob. } j^{\text{th}} \text{ word occurs}$
in genuine.

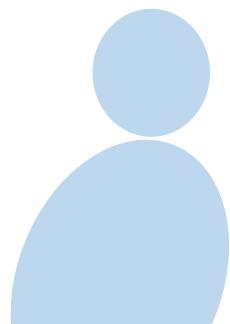
Prior prob. j^{th} word
does NOT occurs in spam
 $= 1 - \text{prob. } j^{\text{th}} \text{ word occurs}$
in spam.

Naïve Bayes

$$p(x_j = 0|y = 1) = 1 - p(x_j = 1|y = 1)$$

$$p(x_j = 0|y = 0) = 1 - p(x_j = 1|y = 0)$$

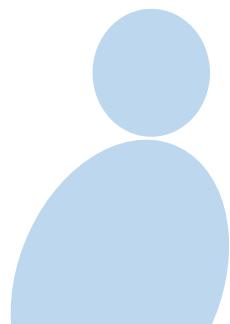
Prior probabilities
of Non occurrence



Naïve Bayes

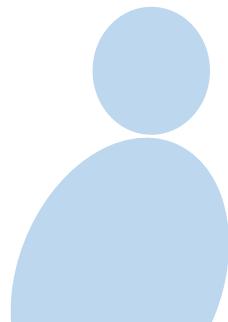
$$p(y=0) = 1 - p(y=1)$$

Prob. of genuine email
 $= 1 - \text{prob. Spam-email}$



Naïve Bayes

$$p(y = 0) = 1 - p(y = 1)$$



Naïve Bayes

- Finally, the posterior probabilities are calculated as follows

Naïve Bayes

$$p(y = 1 | \bar{x} = \bar{v}) = \frac{P(\bar{x} = \bar{v} | y = 1) P(y = 1)}{P(\bar{x} = \bar{v})}$$

$$p(y = 0 | \bar{x} = \bar{v}) = \frac{P(\bar{x} = \bar{v} | y = 0) P(y = 0)}{P(\bar{x} = \bar{v})}$$

$x_j = 1$ if j^{th} word
occurs in email

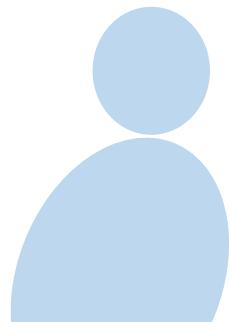
$x_j = 0$ else

Follow from
Bayes rule .

Naïve Bayes

$$p(y = 1 | \bar{\mathbf{x}} = \bar{\mathbf{v}}) = \frac{p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = 1) \times p(y = 1)}{p(\bar{\mathbf{x}} = \bar{\mathbf{v}})}$$

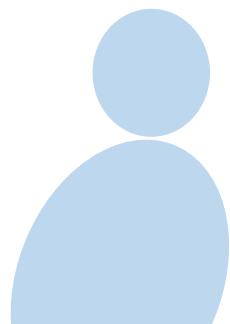
$$p(y = 0 | \bar{\mathbf{x}} = \bar{\mathbf{v}}) = \frac{p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = 0) \times p(y = 0)}{p(\bar{\mathbf{x}} = \bar{\mathbf{v}})}$$



Naïve Bayes

- E-mail is classified as **spam** if

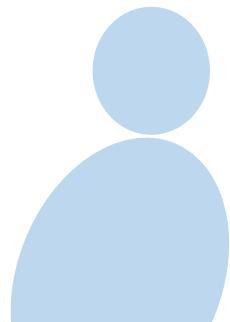
$$P(y=1 | \bar{x} = \bar{v}) > \underbrace{P(y=0 | \bar{x} = \bar{v})}_{\text{Probability genuine given } \bar{x}}.$$



Naïve Bayes

- E-mail is classified as **spam** if

$$p(y = 1 | \bar{x} = \bar{v}) > p(y = 0 | \bar{x} = \bar{v})$$



Naïve Bayes

$$p(y = 1 | \bar{x} = \bar{v}) > p(y = 0 | \bar{x} = \bar{v})$$

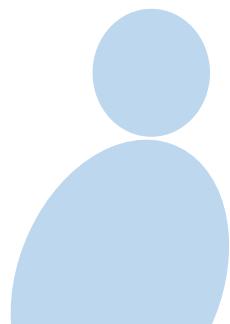
$$\Rightarrow \frac{p(\bar{x} = \bar{v} | y=1) p(y=1)}{p(\bar{x} = \bar{v})} > \frac{p(\bar{x} = \bar{v} | y=0) p(y=0)}{p(\bar{x} = \bar{v})}$$

condition for spam

$$\Rightarrow p(\bar{x} = \bar{v} | y=1) p(y=1) > p(\bar{x} = \bar{v} | y=0) p(y=0)$$

$$p(\bar{x} = \bar{v} | y=1) p(y=1) < p(\bar{x} = \bar{v} | y=0) p(y=0)$$

condition for genuine .



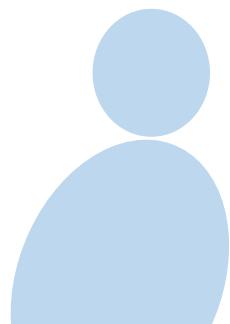
Naïve Bayes

$$p(y = 1 | \bar{\mathbf{x}} = \bar{\mathbf{v}}) > p(y = 0 | \bar{\mathbf{x}} = \bar{\mathbf{v}})$$

$$\Rightarrow \frac{p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = 1) \times p(y = 1)}{p(\bar{\mathbf{x}} = \bar{\mathbf{v}})} > \frac{p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = 0) \times p(y = 0)}{p(\bar{\mathbf{x}} = \bar{\mathbf{v}})}$$

$$\Rightarrow p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = 1) \times p(y = 1) \\ > p(\bar{\mathbf{x}} = \bar{\mathbf{v}} | y = 0) \times p(y = 0)$$

condition for spam .



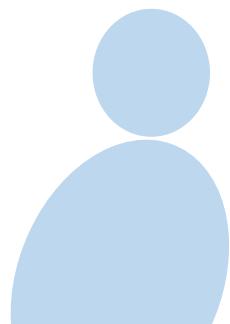
Naïve Bayes

$$P(y=1) > \left(\prod_{j=1}^N P(x_j = v_j | y=1) \right) P(y=0)$$
$$\quad \quad \quad \left(\prod_{j=1}^N P(x_j = v_j | y=0) \right) P(y=0)$$

Naïve Bayes

- This implies

$$\underbrace{\prod_{j=1}^N p(x_j = v_j | y = 1) \times p(y = 1)}_{Q_1} > \underbrace{\prod_{j=1}^N p(x_j = v_j | y = 0) \times p(y = 0)}_{Q_0}$$



Naïve Bayes

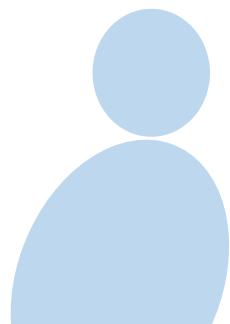
$$p(y=1 \mid \bar{x} = \bar{v}) \cdot p(y=0 \mid \bar{x} = \bar{v}).$$

- Taking the logarithm yields

classified as spam if

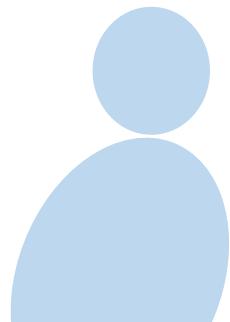
$$\left(\sum_{j=1}^N \ln p(x_j=v_j | y=1) \right) + \ln p(y=1) \\ > \left(\sum_{j=1}^N \ln p(x_j=v_j | y=0) \right) + \ln p(y=0)$$

log posterior probability



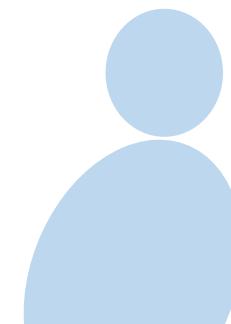
Naïve Bayes

$$\Rightarrow \sum_{j=1}^N \ln(p(x_j = v_j | y = 1)) + \ln p(y = 1)$$
$$> \sum_{j=1}^N \ln(p(x_j = v_j | y = 0)) + \ln p(y = 0)$$



Laplace Smoothing

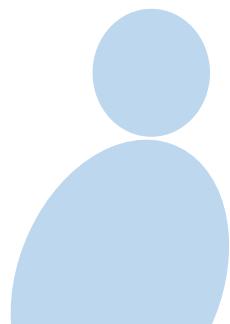
- Naïve Bayes has a problem



Laplace Smoothing

- Lets say a new word “IITK” appears in your e-mail
 - which is NOT present in any training e-mails

NOT present in
any genuine
or spam email.



Laplace Smoothing

- Lets say **index** of “IITK” in the **dictionary** is j .

25,634

Laplace Smoothing

Prior prob. of IITK in Spam = 0

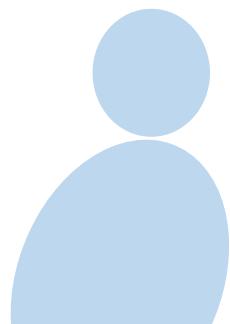
- The prior probabilities will be

$$p(x_j = 1 | y = 1) = \frac{\text{Number of spam emails containing IITK}}{\text{Total number of spam emails}} = 0$$

$$p(x_j = 1 | y = 0) = \frac{\sum_{i=1}^M 1(x_j(i) = 1, y(i) = 0)}{\sum_{i=1}^M 1(y(i) = 0)} = 0$$

Prior prob of IITK in genuine emails = $\frac{\text{Number of genuine emails with IITK}}{\text{Total number of genuine emails}} = 0$

\Rightarrow Posterior prob = 0



Laplace Smoothing

- The prior probabilities will be

$$p(x_j = 1|y = 1) = \frac{\sum_{i=1}^M 1(x_j(i) = 1, y(i) = 1)}{\sum_{i=1}^M 1(y(i) = 1)} = 0$$

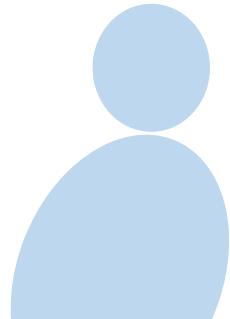
$$p(x_j = 1|y = 0) = \frac{\sum_{i=1}^M 1(x_j(i) = 1, y(i) = 0)}{\sum_{i=1}^M 1(y(i) = 0)} = 0$$

Prior prob = 0
→ Posterior prob = 0 → This causes problems in classification



Laplace Smoothing

- These cause problems in computation of the **posterior probabilities**.



Laplace Smoothing

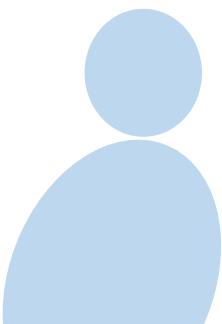
- Therefore, we use the following **prior probabilities** instead

$$p(x_j = 1 | y = 1) = \frac{1 + \sum_{i=1}^M 1(x_j(i) = 1, y(i) = 1)}{2 + \sum_{i=1}^M 1(y(i) = 1)}$$

Laplace Smoothing

$$p(x_j = 1 | y = 0) = \frac{1 + \sum_{i=1}^M 1(x_j(i) = 1, y(i) = 0)}{2 + \sum_{i=1}^M 1(y(i) = 0)}$$

Because of addition of 1,
Prior probabilities are
never 0.



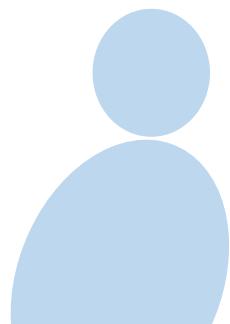
Laplace Smoothing

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$$p(x_j = 1|y = 0) = \frac{1 + \sum_{i=1}^M 1(x_j(i) = 1, y(i) = 0)}{2 + \sum_{i=1}^M 1(y(i) = 0)}$$

LAPLACE SMOOTHING!



Laplace Smoothing

- This is termed **Laplace smoothing**

Avoids prior probabilities = 0

Equiv to adding 10 training set

{

2 spam emails

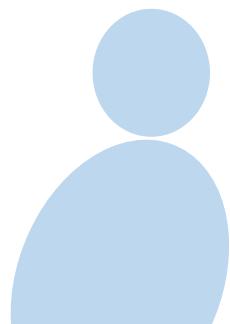
one containing all
words in dictionary

one blank

2 genuine emails.

one containing
all words

one blank



Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.

