# Lagrange Duality

- beyond just "solving" problems
- bounding techniques
- closed-form solution
- good algorithms

$$x^* = \arg\min f_0(x)$$

$\lambda_1, \lambda_2 \dots \lambda_m$ $\quad f_i(x) \le 0 \quad i = 1 \dots m$

$\nu_1, \nu_2 \dots \nu_p$ $\quad h_j(x) = 0 \quad j = 1 \dots p$

dual variables $\quad x \in \mathcal{D}$ (domain)

Lagrange multipliers

$$p = f_0(x^*) = \begin{cases} \infty & \text{infeasible} \\ -\infty & \text{unbounded below} \\ \text{finite} & \text{o/w} \end{cases}$$

primal

$$L(\underline{x}, \underline{\lambda}, \underline{\nu}) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(\underline{x}) + \sum_{j=1}^{p} \nu_j h_j(x)$$

$\mathbb{R}^n \quad \mathbb{R}^m \quad \mathbb{R}^p$

- affine in $\underline{\lambda}, \underline{\nu}$
- when problem is convex, $\lambda_i \ge 0$
  - $f_0$ convex
  - $f_i$ convex
  - $h_j$ affine

  $\Big]$ $L(\underline{x}, \underline{\lambda}, \underline{\nu})$ convex in $\underline{x}$

Dual function: $\quad g(\underline{\lambda}, \underline{\nu}) = \min_{x \in \mathcal{D}} L(\underline{x}, \underline{\lambda}, \nu)$ $\qquad$ pointwise min of affine functions

concave in $\underline{\lambda}, \underline{\nu}$

(by definition, even when original problem is not convex)

Consider: $\tilde{x}$ feasible $\Rightarrow$ $\tilde{x} \in \mathcal{D}$

$f_i(\tilde{x}) \leq 0 \quad i = 1 \ldots m$

$h_j(\tilde{x}) = 0 \quad j = 1 \ldots p$

$$f_0(\tilde{x}) \geq f_0(\tilde{x}) + \overbrace{\sum \lambda_i f_i(\tilde{x})}^{-ve} + \sum \nu_j \underset{}{\overset{0}{\overset{\parallel}{h_j(\tilde{x})}}}$$

$$= L(\tilde{x}, \underline{\lambda}, \underline{\nu})$$

$$\geq \min_{x \in \mathcal{D}} L(x, \underline{\lambda}, \underline{\nu}) \qquad \text{(by defintion)}$$

$$= g(\underline{\lambda}, \underline{\nu}) \qquad \text{dual function}$$

$$\underset{\text{primal objective}}{f_0(\tilde{x})} \geq \underset{\text{dual function}}{g(\underline{\lambda}, \underline{\nu})} \qquad \text{for } \lambda \geq 0$$