Perspective Function & Minimization

Perspective function transform

$$g\left(\begin{bmatrix} x \\ t \end{bmatrix}\right) = t f(x/t), \qquad t > 0$$

f convex $\Rightarrow$ g convex

Eg $f(x) = x^T x$ $\qquad \nabla^2 f(x) = 2I \succeq 0$

$$g(x) = t\left(\left(\frac{x}{t}\right)^T\left(\frac{x}{t}\right)\right) = \frac{x^T x}{t} \qquad t > 0 \quad \text{convex}$$

Eg $f(x) = -\log(x)$ convex

$$g(x,t) = -t\log(x/t)$$

Minimization :

$$f : \mathbb{R}^{m} \times \mathbb{R}^{n} \to \mathbb{R}$$

$x \quad y$

$f(x,y)$ convex jointly in (x,y)

C convex set $\qquad g(x) = \min_{y \in C} f(x,y)$

Contrast with pointwise max. result

Eg : $f(\underline{x},\underline{y}) = \|\underline{x}-\underline{y}\| = \left\| \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \right\|$ convex

$g(\underline{x}) = \min_{y \in S} \|x-y\| = \text{dist}(x,S)$