Please submitted by Saturday, 26 Aug. 2022, 11 am, right before the discussion hour.

1. Attempt all problems. There is no penalty for submitting incorrect attempts
2. However, plagiarism will result in serious penalties, such as an F grade.

1. Show that the following two problems are duals of each other.

$$p^\star = \min \ \max_i (\mathbf{P}^T\mathbf{u})_i \tag{1}$$
$$\text{s. t. } \ \mathbf{u} \geq 0 \tag{2}$$
$$\sum_{i=1}^m u_i = 1 \tag{3}$$

and

$$d^\star = \max \ \min_j (\mathbf{P}\mathbf{v})_i \tag{4}$$
$$\text{s. t. } \ \mathbf{v} \geq 0 \tag{5}$$
$$\sum_{j=1}^n v_j = 1 \tag{6}$$

Does it hold that $p^\star = d^\star$? This result is the famous minimax theorem of two-person zero-sum games, first proved in Von Neumann's 1928 paper Zur Theorie der Gesellschaftsspiele.

2. Find the dual of the penalty function approximation

$$\min \sum_{i=1}^m \phi(r_i) \tag{21}$$
$$\text{s. t. } \mathbf{r} = \mathbf{A}\mathbf{x} - \mathbf{b} \tag{22}$$

where ϕ is the deadzone linear penalty function

$$\phi(u) = \begin{cases} 0 & |u| \leq 1 \\ |u| - 1 & |u| > 1 \end{cases} \tag{23}$$

3. Consider the following non-convex problem

$$p^\star = \min \ \mathbf{x}^T\mathbf{A}\mathbf{x} \tag{30}$$
$$\text{s. t. } x_i \in \{-1, 1\} \tag{31}$$

where $\mathbf{A} \in \mathbb{S}^{n\times n}$. Show that

$$n\lambda_{\min}(\mathbf{A}) \leq p^\star \leq \sum_{i,j} A_{ij} \tag{32}$$

(Hint: express the constraint as $x_i^2 = 1$ and use weak duality).

4. Find the dual of the convex piece-wise linear minimization problem:

$$\min \max_{i=1,\ldots,m} (\mathbf{a}_i^T \mathbf{x} + b_i) \tag{38}$$

5. Consider the following convex optimization problem:

$$\min \sum_{i=1}^{m} \exp(x_i - 1) + y \tag{46}$$

$$\text{s. t. } \mathbf{Ax} - \mathbf{b} + y\mathbf{1} \geq 0 \tag{47}$$

Use appropriate change of variables and elimination to show that it can equivalently be written as

$$\min \ \log(\sum_{i=1}^{m} e^{u_i}) \tag{48}$$

$$\text{s. t. } \mathbf{Au} - \mathbf{b} \geq 0 \tag{49}$$

if it holds that $\mathbf{A1} = 1$.