EE908 Assignment-2 Solution

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eMasters in Communication Systems, IITK
EE908: Optimization in SPCOM
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1. Let $\lambda_i(A)$ denote an eigenvalue of a symmetric matrix A. Find the following in terms of $\lambda_i(A)$
   a. $\boldsymbol{Tr(A^3)}$
   Solution:
   According to EVD/spectral theorem, a symmetric real matrix can be decomposed or diagonalized such that:
   $A = Q\Lambda Q^T$
   Where:

   - Λ is a diagonal matrix of eigenvalues of A
   - Q is matrix of corresponding eigenvectors of A which is orthogonal matrix - $QQ^T = Q^TQ = I$
   - From matrix relationships:
     - $Tr(ABC) = Tr(CAB) = Tr(BCA)$

   $\therefore A^3 = ((Q\Lambda Q^T)(Q\Lambda Q^T)(Q\Lambda Q^T) = Q\Lambda(Q^TQ)\Lambda(Q^TQ)\Lambda Q^T = Q\Lambda I\Lambda I\Lambda Q^T = Q\Lambda^3Q^T$
   $\therefore Tr(A^3) = Tr(Q\Lambda^3Q^T) = Tr(QQ^T\Lambda^3) = Tr(I\Lambda^3) = Tr(\Lambda^3)$

   Λ is a diagonal matrix with eigenvalues λ_i on the diagonal. So Λ^3 is also a diagonal matrix with eigenvalues cubed on the diagonal.
   $\therefore \boldsymbol{Tr(A^3) = Tr(\Lambda^3) = \sum_{i=1}^{n}\lambda_i^3}$ **– $\boldsymbol{n}$ is the size of the symmetric matrix**

   b. $\boldsymbol{\lambda_i(A^{-2})}$
   Solution:
   Given:

   - $Av = \lambda_i v \Rightarrow A^{-1}Av = A^{-1}\lambda_i v \Rightarrow v = A^{-1}\lambda_i v \Rightarrow \frac{1}{\lambda_i}v = A^{-1}v$

   $$\Rightarrow A^{-1}v = \frac{1}{\lambda_i}v$$
   Therefore, $\frac{1}{\lambda_i}$ is the eigenvalue(s) of A^{-1} with the same eigenvector(s) v
   $A^{-2} = (A^{-1})^2$
   A is symmetric, hence its inverse is also symmetric and per EVD,
   So, if $\lambda_i(A)$ denotes the eigenvalues of A, then $\frac{1}{\lambda_i(A)}$ are the eigenvalues of A^{-1} as derived above

   Based on the proof from (a) above, if $\frac{1}{\lambda_i(A)}$ are eigenvalues of A^{-1}, then $(A^{-1})^2$ eigenvalues are $\left(\frac{1}{\lambda_i(A)}\right)^2$
   $$\therefore \boldsymbol{\lambda_i(A^{-2}) = \left(\frac{1}{\lambda_i(A)}\right)^2}$$

   c. $\boldsymbol{\lambda_i(A - I)}$

Solution:
Given: $Av = \lambda_i(A)v$
So, let's consider $(A - I)$ matrix and its action on same eigenvector v:
$(A - I)v = Av - Iv = \lambda_i(A)v - Iv = (\lambda_i(A) - 1)v$
$\therefore (A - I)v = (\lambda_i(A) - 1)v$
This satisfies the eigenvalue and eigen vector relationship for A and v.
Therefore, the eigenvalues of matrix $(\boldsymbol{A - I})$ of eigen vector $\boldsymbol{v}$ are $(\boldsymbol{\lambda_i(A) - 1})$
In general, $(A - kI)v = Av - kIv = \lambda_i(A)v - kv = (\lambda_i(A) - k)v$
Therefore, the eigenvalues of matrix $(\boldsymbol{A - kI})$ of eigen vector $\boldsymbol{v}$ are $(\boldsymbol{\lambda_i(A) - k})$

d. $\boldsymbol{\lambda_i(I + 2A)}$
Solution:
Given: $Av = \lambda_i(A)v$
So, let's consider $(I + 2A)$ matrix and its action on same eigenvector v:
$(I + 2A)v = Iv + 2Av = v + 2\lambda_i(A)v = (1 + 2\lambda_i(A))v$
$\therefore (I + 2A)v = (1 + 2\lambda_i(A))v$
This satisfies the eigen value and eigenvector relationship for A and v.
Therefore, the eigenvalues of matrix $(\boldsymbol{I + 2A})$ of eigen vector $\boldsymbol{v}$ are $(\boldsymbol{1 + 2\lambda_i(A)})$
In general, $(I + kA)v = Iv + kAv = v + k\lambda_i(A)v = (1 + k\lambda_i(A))v$
Therefore, the eigenvalues of matrix $(\boldsymbol{I + kA})$ of eigen vector $\boldsymbol{v}$ are $(\boldsymbol{1 + k\lambda_i(A)})$

2. Prove the following results for $A \succ 0$:

a. $\boldsymbol{A^{-1} \succ 0}$
Solution:
Let's prove the inverse of matrix A is also positive definite if A is positive definite leveraging properties of matrix positive definiteness and eigenvalues.

Given $A_{n\times n}$ is positive definite ⇒ All its eigenvalues are positive, $\lambda_i > 0$
As proved in question (1) above, the eigenvalues of A^{-1} will be reciprocals of eigenvalues of A ⇒ $\frac{1}{\lambda_i}$
Since all $\lambda_i > 0 \Rightarrow \frac{1}{\lambda_i} > 0$
$\therefore$ All eigenvalues of $\boldsymbol{A^{-1}}$ are positive which implies that $\boldsymbol{A^{-1}}$ is also positive definite
Hence if $\boldsymbol{A \succ 0}$, then $\boldsymbol{A^{-1} \succ 0}$
QED

b. $\boldsymbol{[A]_{ii} > 0}$ for all $\boldsymbol{i}$, where $\boldsymbol{[A]_{ii}}$ denotes the $\boldsymbol{i}$-th diagonal entry of A
Solution:
If a symmetric $n \times n$ matrix $A = [a_{ij}]$ is positive definite, then for any non-zero column vector v, the quadratic form $v^T Pv$ is strictly positive and its eigenvalues are greater than zero

$$i.e.\, v^T Av > 0, \forall v \neq 0$$

Let's consider a vector (standard basis vector) $e_i = [1,0, \ldots 0, \ldots 0] \in \mathbb{R}^n$ – all elements are 0 except the one at position i which is 1.
For a standard basis vector $e_i \Rightarrow e^T Ae = a_{ii}$
From the positive definiteness of A, $e^T Ae > 0$
$\Rightarrow \boldsymbol{e^T Ae = A_{ii} = [a_{ii}] > 0}$ for all $\boldsymbol{i}$. $\boldsymbol{A_{ii} = [a_{ii}]}$ of this matrix are all diagonal elements.
Hence, if $\boldsymbol{A \succ 0}$, then $\boldsymbol{A_{ii} = [a_{ii}] \succ 0}$
QED

3. A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$
   Solution:
   Given: $A^2 = A$
   EVD for an idempotent matrix:
   $Av = \lambda v$ and $A^2 v = \lambda v$
   $\Rightarrow A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$
   However, from the property of idempotent $\Rightarrow A^2 v = Av = \lambda v$
   $\therefore \lambda^2 v = \lambda v \Rightarrow (\lambda^2 - \lambda)v = 0$
   Since v is an eigenvector and $\neq 0$, hence $(\lambda^2 - \lambda) = 0$
   Therefore, $\boldsymbol{\lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0}$ or $\mathbf{1}$
   QED

4. Given an $m \times n$ matrix A with SVD $A = A = \sum_{i=1}^{r} \sigma_i v_i u_i^T$, show that $\|A\|_F^2 \coloneqq tr(A^T A) = \sum_{i=1}^{r} \sigma_i^2$
   Solution:
   SVD of A: $A = \sum_{i=1}^{r} \sigma_i v_i u_i^T$
   We need to show that $\|A\|_F^2 = Tr(A^T A)$
   Substituting A:
   $A^T A = (\sum_{i=1}^{r} \sigma_i v_i u_i^T)^T (\sum_{i=1}^{r} \sigma_i v_i u_i^T) = (\sum_{i=1}^{r} \sigma_i v_i^T u_i)(\sum_{i=1}^{r} \sigma_i v_i u_i^T) = \sum_{i=1}^{r} \sum_{j=1}^{r} \sigma_i \sigma_j u u_i^T v_j v_j^T$
   $u_i^T u_j$ and $v_i^T v_j$ terms are 1 if $i = j$ else 0 as these singular vectors are orthonormal
   $\therefore Tr(A^T A) = Tr(\sum_{i=1}^{r} \sigma_i^2)$
   QED

5. The ℓ_2 norm of a matrix A is defined as $\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$. Derive an expression for $\|A\|_2$ in terms of $\{\sigma_i\}_{i=1}^{r}$
   Solution:
   $\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$
   $\|x\|_2$ is the Euclidean norm of vector x. To derive an expression for $\|A\|_2$ in terms of the singular values σ_i, we need to find the maximum value of $\|Ax\|_2$ over all unit vectors x.
   $\|Ax\|$ is the Euclidean norm of vector Ax.

   $$\|Ax\|_2 = \left\| \left(\sum_{i=1}^{r} \sigma_i v_i u_i^T \right) x \right\|_2 = \left\| \sum_{i=1}^{r} \sigma_i v_i (u_i^T x) \right\|_2$$

   $u_i^T x$ is a scalar $\Rightarrow \left\| \sum_{i=1}^{r} \sigma_i (u_i^T x) v_i \right\|_2$
   Note above is a linear combination of the singular vectors v_i with coefficients $u_i^T x$. To maximize the norm of this linear combination, choose x such that it aligns with singular vector corresponding to the maximum singular value $\Rightarrow x = u_1$ Assuming σ_1 is maximum singular value.
   $\|Ax\|_2 = \|\sigma_1 v_1\|_2 = \sigma_1 \|v_1\|_2$
   Corresponding to the maximum singular value $\Rightarrow x = u_1$ Assuming σ_1 is maximum singular value.
   Since $\|v_1\|_2 = 1 \Rightarrow \|Ax\|_2 = \sigma 1$
   $\therefore \|A\|_2 = \max\{\sigma_i\}_{i=1}^{r}$
   QED

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