eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 9

LMMSE

Linear MMSE

Linear minimum Mean square error

LMMSE

- **LMMSE = Linear Minimum Mean Square Error.**

LMMSE Estimator.

LMMSE

- Recall MMSE is given as

$$\underbrace{\min \underbrace{E\left\{\left\|\underbrace{\hat{\mathbf{h}}-\bar{\mathbf{h}}}_{\text{Square Error}}\right\|^2\right\}}_{\text{Mean Square error}}}_{\text{Minimum Mean Square Error}}$$

LMMSE

- Recall MMSE is given as

$$\underbrace{\min \underbrace{E\left\{\underbrace{\left\|\hat{\mathbf{h}}-\bar{\mathbf{h}}\right\|^2}_{\text{Square Error}}\right\}}_{\text{Mean Square Error}}}_{\text{Minimum Mean Square Error}}$$

LMMSE

$f_{\bar{h}|\bar{y}}(\bar{h}|\bar{y})$

- Furthermore

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\}$$

Conditional Expectation

LMMSE

- However, frequency this is **extremely challenging** to determine.

Determine Conditional PDF + Expected Value $\Rightarrow$ DIFFICULT!

Esp when $\bar{h}, \bar{y}$ NOT Gaussian
$\bar{h}, \bar{y}$ arbitrarily distributed.

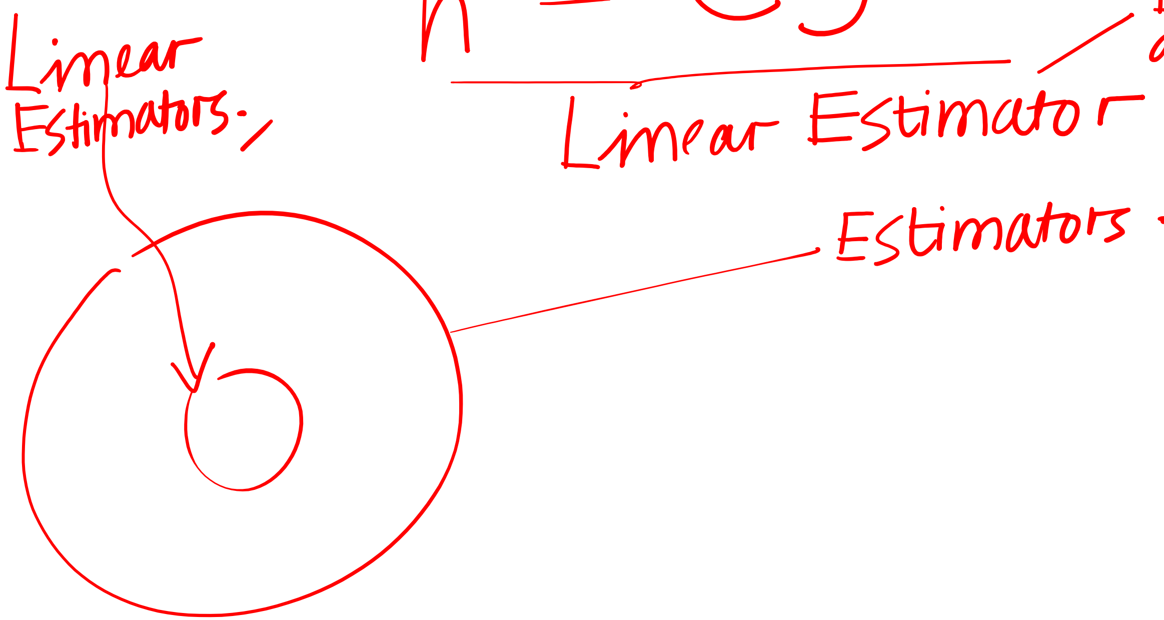
LMMSE

- Hence, we settle for the **best linear estimator** given as

$$\underbrace{\hat{h}}_{M \times 1} = \underbrace{C}_{M \times N}\,\underbrace{\bar{y}}_{N \times 1}$$

Linear Estimator

Easier to determine but suboptimal.



LMMSE

- Hence, we settle for the **best linear estimator.**

$C = ?$

$M \times N$.

$$\hat{\mathbf{h}} = \mathbf{C}\bar{\mathbf{y}}$$

Linear Estimator.

LMMSE = Best Linear Estimator that has minimum mean square error !

LMMSE

- This cost function to optimize is

$$\min E\left\{ \left\| \overset{C\bar{y}}{\cancel{\hat{h}}} - \bar{h} \right\|^2 \right\}$$

$$\text{min. } E\left\{ \left\| C\bar{y} - \bar{h} \right\|^2 \right\}$$

Linear <u>Estimators.</u>

LMMSE

- This is given as

$$\underbrace{\min E\left\{\underbrace{\left\|\underbrace{\mathbf{C}\bar{\mathbf{y}}}_{\text{Linear}}-\bar{\mathbf{h}}\right\|^2}_{\text{Square Error}}\right\}}_{\text{Mean Square Error}}$$

Linear Minimum Mean Square Error

Estimator is constrained to be Linear.

LMMSE

$$\|\bar{x}\|^2 = \bar{x}^T\bar{x} = \mathrm{Tr}\{\bar{x}^T\bar{x}\}$$
$$= \mathrm{Tr}\{\bar{x}\bar{x}^T\}$$

- This can be simplified as follows

$$\|\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\|^2 = (C\bar{y}-\bar{h})^T(C\bar{y}-\bar{h})$$
$$= \mathrm{Tr}\{(C\bar{y}-\bar{h})^T(C\bar{y}-\bar{h})\}$$
$$= \mathrm{Tr}\{(C\bar{y}-\bar{h})(C\bar{y}-\bar{h})^T\}$$

LMMSE

$\|\bar{x}\|^2 = \mathrm{Tr}\{\bar{x}\bar{x}^T\} = \mathrm{Tr}\left\{\begin{bmatrix} x_1^2 & & & \\ & x_2^2 & & \\ & & \ddots & \\ & & & x_n^2 \end{bmatrix}\right\}$

- This can be simplified as follows $= x_1^2 + x_2^2 + \cdots + x_n^2$

$$\left\|\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right\|^2 = \left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)^T\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)$$
$$= \mathrm{Tr}\left\{\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)^T\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)\right\}$$
$$= \mathrm{Tr}\left\{\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)^T\right\}$$

LMMSE

- This can be simplified as follows

$$\mathrm{Tr}\left\{\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)\left(\mathbf{C}\bar{\mathbf{y}} - \bar{\mathbf{h}}\right)^T\right\}$$

$$= \mathrm{Tr}\left\{ C\bar{y}\bar{y}^T C^T - \bar{h}\bar{y}^T C^T - C\bar{y}\bar{h}^T + \bar{h}\bar{h}^T \right\}$$

LMMSE

- This can be simplified as follows

$$\mathrm{Tr}\left\{\left(\mathbf{C}\bar{\mathbf{y}}-\bar{\mathbf{h}}\right)\left(\mathbf{C}\bar{\mathbf{y}}-\bar{\mathbf{h}}\right)^{H}\right\}$$
$$=\mathrm{Tr}\{\mathbf{C}\bar{\mathbf{y}}\bar{\mathbf{y}}^{T}\mathbf{C}^{T}-\bar{\mathbf{h}}\bar{\mathbf{y}}^{T}\mathbf{C}^{T}-\mathbf{C}\bar{\mathbf{y}}\bar{\mathbf{h}}^{T}+\bar{\mathbf{h}}\bar{\mathbf{h}}^{T}\}$$

LMMSE

interchange Tr, E → Because both are Linear!

- This can be simplified as follows

$$E\left\{\left\|\mathbf{C}\bar{\mathbf{y}}-\bar{\mathbf{h}}\right\|^2\right\}$$

$$= E\left\{ Tr\left\{ C\bar{y}\bar{y}^T C^T - \bar{h}\bar{y}^T C^T - C\bar{y}\bar{h}^T + \bar{h}\bar{h}^T \right\}\right\}$$

$$= Tr\left\{ E\left\{ C\bar{y}\bar{y}^T C^T - \bar{h}\bar{y}^T C^T - C\bar{y}\bar{h}^T + \bar{h}\bar{h}^T \right\}\right\}$$

$$= Tr\left\{ C R_{yy} C^T - R_{hy} C^T - C R_{yh} + R_{hh} \right\}$$

LMMSE

Find C that minimizes this -

- This can be simplified as follows

$$E\left\{\left\|\mathbf{C}\bar{\mathbf{y}}-\bar{\mathbf{h}}\right\|^2\right\}$$

$$= \underbrace{\mathrm{Tr}\left\{ C R_{yy} C^T - R_{hy} C^T - C R_{yh} + R_{hh} \right\}}_{\text{mean square error of Linear Estimator } \hat{h} = C\bar{y}}$$

LMMSE

- This can be simplified as follows

$$E\left\{\left\|\mathbf{C}\bar{\mathbf{y}}-\bar{\mathbf{h}}\right\|^{2}\right\}$$

$$=E\left\{\mathrm{Tr}\{\mathbf{C}\bar{\mathbf{y}}\bar{\mathbf{y}}^{T}\mathbf{C}^{T}-\bar{\mathbf{h}}\bar{\mathbf{y}}^{T}\mathbf{C}^{T}-\mathbf{C}\bar{\mathbf{y}}\bar{\mathbf{h}}^{T}+\bar{\mathbf{h}}\bar{\mathbf{h}}^{T}\}\right\}$$

$$=\mathrm{Tr}\left\{E\{\mathbf{C}\bar{\mathbf{y}}\bar{\mathbf{y}}^{T}\mathbf{C}^{T}-\bar{\mathbf{h}}\bar{\mathbf{y}}^{T}\mathbf{C}^{T}-\mathbf{C}\bar{\mathbf{y}}\bar{\mathbf{h}}^{T}+\bar{\mathbf{h}}\bar{\mathbf{h}}^{T}\}\right\}$$

LMMSE

- This can be simplified as follows

$$MSE = Tr\left\{ C R_{yy} C^T - R_{hy} C^T - C R_{yh} + R_{hh} \right\}$$

$$= Tr\left\{ \underbrace{(C R_{yy} - R_{hy}) R_{yy}^{-1} (C R_{yy} - R_{hy})^T}_{\text{Function of } C} + \underbrace{R_{hh} - R_{hy} R_{yy}^{-1} R_{yh}}_{\text{independent of } C}. \right\}$$

Function of C

independent of C .

LMMSE

- This can be simplified as follows

R_{yy} is PSD

$\Rightarrow R_{yy}^{-1}$ is positive semidefinite

$\bar{x}^T R_{yy}^{-1} \bar{x} \geq 0$.

$$= \mathrm{Tr}\{\mathbf{C}\mathbf{R}_{yy}\mathbf{C}^T - \mathbf{R}_{hy}\mathbf{C}^T - \mathbf{C}\mathbf{R}_{yh} + \mathbf{R}_{hh}\}$$

$$= \mathrm{Tr}\Big\{\underbrace{(\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy})\mathbf{R}_{yy}^{-1}(\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy})^T}_{\text{Function of } C.\ \geq 0}$$
$$+ \mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}\Big\}$$

First term always ≥ 0

minimum is 0.

LMMSE

- Minimization reduces to

only first term depends on C.

$$\min.\ \mathrm{Tr}\left\{(CR_{yy}-R_{hy})R_{yy}^{-1}(CR_{yy}-R_{hy})^T + R_{hh} - R_{hy}R_{yy}^{-1}R_{yh}\right\}$$

$$= \min\ \underbrace{\mathrm{Tr}\left\{(CR_{yy}-R_{hy})R_{yy}^{-1}(CR_{yy}-R_{hy})^T\right\}}_{\geq 0} + \mathrm{Tr}\left\{R_{hh} - R_{hy}R_{yy}^{-1}R_{yh}\right\}$$

Does NOT depend on C.

First Term always ≥ 0

$\Rightarrow$ minimum value $= 0$.

LMMSE

- Minimization reduces to

$$\min \operatorname{Tr}\left\{\left(\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy}\right)\mathbf{R}_{yy}^{-1}\left(\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy}\right)^{T} + \mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}\right\}$$

$$= \min \underbrace{\operatorname{Tr}\left\{\left(\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy}\right)\mathbf{R}_{yy}^{-1}\left(\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy}\right)^{T}\right\}}_{\text{minimum value is zero}} + \operatorname{Tr}\left\{\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}\right\}$$

$\bar{x}^T R_{yy}^{-1} \bar{x} = 0$ only if $\bar{x} = 0$ — Positive definite

minimum value $= 0$.

To achieve minimum MSE set first term to zero.

LMMSE

- Minimum occurs for

C for which minimum MSE is achieved

$$C R_{yy} - R_{ny} = 0$$

$$\Rightarrow \quad C R_{yy} = R_{ny}$$

$$\Rightarrow \quad C = R_{ny} R_{yy}^{-1}$$

LMMSE

- Minimum occurs for

$$\mathbf{C}\mathbf{R}_{yy} - \mathbf{R}_{hy} = 0$$

$$\Rightarrow \mathbf{C} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}$$

LMMSE

Linear minimum Mean square error Estimator.

- Therefore, the LMMSE estimate is

$$\hat{h} = C\bar{y}$$

$$\hat{h} = R_{hy} R_{yy}^{\dagger} \bar{y}$$

LMMSE Estimator

LMMSE

$C = R_{hy} R_{yy}^{-1}$

- Therefore, the LMMSE estimate is

$M \times N$

$$\underbrace{\hat{\mathbf{h}} = \mathbf{C}\bar{\mathbf{y}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}}_{\text{LMMSE Estimate}}$$

LMMSE

Mean square error

- The corresponding MSE is

$$E\left\{ \|\hat{h} - \bar{h}\|^2 \right\} = \mathrm{Tr}\left\{ R_{hh} - R_{hy} R_{yy}^{-1} R_{yh} \right\}.$$

LMMSE

- The corresponding MSE is

$$\mathrm{Tr}\left\{\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}\right\}$$

LMMSE

This is also MMSE when $\bar{h}, \bar{y}$ are Gaussian.

- Seems **exactly similar** to MMSE!!!

$$\underbrace{\hat{\mathbf{h}} = \mathbf{C}\bar{\mathbf{y}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}}_{\text{LMMSE Estimate}}$$

— $\bar{h}, \bar{y}$ are arbitrarily distributed.

- Then what is the **DIFFERENCE**?

LMMSE

$\hat{h} = R_{hy} R_{yy}^{-1} \bar{y}$

- Recall that this is MMSE only when $\bar{h}, \bar{y}$ are **jointly Gaussian**

$$\underbrace{\hat{\mathbf{h}} = \mathbf{C}\bar{\mathbf{y}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}}_{\text{LMMSE Estimate}}$$

when $\bar{h}, \bar{y}$ are arbitrarily distributed.

- Else it is only **LMMSE!!!**

LMMSE

- To summarize…

$\hat{\mathrm{h}}(\bar{\mathrm{y}}) = \mathrm{R}_{hy}\mathrm{R}_{yy}^{-1}\bar{\mathrm{y}}$	MMSE	LMMSE
$\bar{\mathrm{y}}, \bar{\mathrm{h}}$ **Jointly Gaussian**	YES	YES
$\bar{\mathrm{y}}, \bar{\mathrm{h}}$ **Arbitrary PDF**	NO	YES

LMMSE

- To summarize…

$\hat{\mathrm{h}}(\bar{\mathrm{y}}) = \mathrm{R}_{hy}\mathrm{R}_{yy}^{-1}\bar{\mathrm{y}}$	MMSE	LMMSE
$\bar{\mathrm{y}}, \bar{\mathrm{h}}$ **Jointly Gaussian**	Yes	Yes
$\bar{\mathrm{y}}, \bar{\mathrm{h}}$ **Arbitrary PDF**	NO	Yes

LMMSE Estimation

M Transmit antennas.
1 Receive antenna.

- Consider now the MISO channel estimation model

$N \times 1$ $\quad$ $M \times 1$ $\quad$ $N \times M$

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

$N \times 1$
Noise vector.

LMMSE Estimation

- Therefore, the LMMSE estimate is given as

$$\underbrace{\hat{\mathbf{h}} = \left(X^T X + \frac{1}{SNR} I\right)^{-1} X^T \bar{y}}_{\text{LMMSE when } \bar{y}, \bar{h} \text{ are NOT necessarily Gaussian.}}$$

LMMSE Estimation

- Therefore, the LMMSE estimate is given as

$$\hat{\mathbf{h}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$= \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}$$

LMMSE Estimation

$E\{\bar{h}\}=0$

$E\{\bar{y}\}=0$.

- Note that this is valid even when $\bar{\mathbf{h}}, \bar{\mathbf{y}}$ are **NOT jointly Gaussian**-only **zero-mean**

Zero Mean.
NOT necessarily Gaussian.

$$\hat{\mathbf{h}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$= \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}$$

LMMSE Estimation

- The **error covariance** of the LMMSE is given as

$$\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

$$= \sigma^2\left(X^T X + \frac{1}{SNR} I\right)^{-1}!$$

LMMSE Estimation

- The **error covariance** of the LMMSE is given as

$$MSE = \sigma^2 \, Tr\left\{ \left(X^T X + \frac{1}{SNR} I \right)^{-1} \right\}.$$

$$\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

$$= \sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$

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