EE901
PROBABILITY AND RANDOM PROCESSES

MODULE -1
INTRODUCTION TO PROBABILITY THEORY

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1

Probability Measure

- Probability measure is a function from $\mathcal{F}$ (set of events) to [0,1] that satisfies

$$\mathbb{P}(\Omega) = 1$$

$$\mathbb{P}(E) \geq 0$$

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$$

for disjoint events $A_1, A_2, A_3, \ldots$

2

Properties of a Probability Measure

3

Property 1: Probability of An Empty Set

$\mathbb{P}(\phi) = 0$

- The probability axioms

$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(E) \geq 0$

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$

for disjoint events $A_1, A_2, A_3, \ldots$

4

Property 1: Probability of An Empty Set

$\mathbb{P}(\phi) = 0$

- The probability axioms

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots)$

for disjoint events $A_1, A_2, A_3, \ldots$

5

Property 1: Probability of An Empty Set

$\mathbb{P}(\phi) = 0$

- The probability axioms

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$$

for disjoint events $A_1, A_2, A_3, \ldots$

6

Property 1: Probability of An Empty Set

$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$ for disjoint events $A_1, A_2, A_3, \ldots$

- Consider the following sequence of sets

$A_1 = E$
$A_2 = \phi$
$A_3 = \phi$
…
$A_i = \phi$
…

$\bigcup_{i=1}^{\infty} A_i$
$= A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_i \cup \ldots$
$= E \cup \phi \cup \phi \cup \ldots \cup \phi \cup \ldots$
$= E \cup \phi$
$= E$

7

Property 1: Probability of An Empty Set

$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$ for disjoint events $A_1, A_2, A_3, \ldots$

$\boxed{\bigcup_{i=1}^{\infty} A_i = E}$

$A_1 = E \longrightarrow \mathbb{P}(E)$
$A_2 = \phi \longrightarrow \mathbb{P}(\phi) = a$
$A_3 = \phi \longrightarrow \mathbb{P}(\phi) = a$
…
$A_i = \phi \longrightarrow \mathbb{P}(\phi) = a$
…

$\sum_{i=1}^{\infty} \mathbb{P}(A_i)$
$= \mathbb{P}(E) + \sum_{i=2}^{\infty} \mathbb{P}(A_i)$
$= \mathbb{P}(E) + \sum_{i=2}^{\infty} a$

8

Property 1: Probability of An Empty Set

$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$ for disjoint events $A_1, A_2, A_3, \ldots$

$\boxed{\bigcup_{i=1}^{\infty} A_i = E}$ $\boxed{\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}(E) + \sum_{i=2}^{\infty} a}$

$\mathbb{P}(E) + \sum_{i=2}^{\infty} a = \mathbb{P}(E)$ $\sum_{i=2}^{\infty} a = 0$ $a = 0$

$\mathbb{P}(\phi) = 0$

9

Property 2: Finite Additivity

- Finite additivity

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots + \mathbb{P}(A_n) = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$$

for disjoint events $A_1, A_2, A_3, \ldots A_n$

- The third probability axiom

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$$ for disjoint events $A_1, A_2, A_3, \ldots$

10

Property 2: Finite Additivity

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$$ for disjoint events $A_1, A_2, A_3, \ldots$

- Consider the following sequence of sets

$$\bigcup_{i=1}^{n} A_i$$

$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

A_1, A_2, A_3, …, A_n, $A_{n+1} = \phi$, $A_{n+2} = \phi$, …

$$\bigcup_{i=1}^{\infty} A_i$$

$= A_1 \cup A_2 \cup A_3 \cup \ldots A_n \cup A_{n+1} \cup A_{n+2} \cup \ldots$

$= A_1 \cup A_2 \cup A_3 \cup \ldots A_n \cup \phi \cup \phi \ \ldots$

$= A_1 \cup A_2 \cup A_3 \cup \ldots A_n \cup \phi$

$= A_1 \cup A_2 \cup A_3 \cup \ldots A_n$

11

Property 2: Finite Additivity

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$$

12

Property 2: Finite Additivity

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \quad \text{for disjoint events } A_1, A_2, A_3, \ldots$$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$$

$A_1 \longrightarrow \mathbb{P}(A_1)$
$A_2 \longrightarrow \mathbb{P}(A_2)$
$A_3 \longrightarrow \mathbb{P}(A_3)$
$\ldots$
$A_n \longrightarrow \mathbb{P}(A_n)$
$A_{n+1} = \phi \longrightarrow 0$
$A_{n+2} = \phi \longrightarrow 0$
$\ldots$

$$\sum_{i=1}^{n} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

13

Property 2: Finite Additivity

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \quad \text{for disjoint events } A_1, A_2, A_3, \ldots$$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$$

$$\sum_{i=1}^{n} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)$$

$$\sum_{i=1}^{n} \mathbb{P}(A_i) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)$$

14

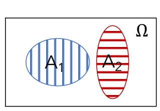
Property 2: Finite Additivity

- Finite additivity

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \ldots + \mathbb{P}(A_n) = \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$$
for disjoint events $A_1, A_2, A_3, \ldots A_n$

For n=2

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$$



15

Property 2

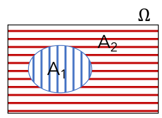
- For any event A

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

Proof:

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$$

$$A_1 = A, \quad A_2 = A^c$$



$$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) \quad = 1$$

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

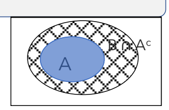
16

Property 3: Monotonicity

- For events A and B

$$\text{If } A \subset B, \text{ then } \mathbb{P}(A) \leq \mathbb{P}(B)$$

Proof:



$$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$$

$$A_1 = A, \quad A_2 = B \cap A^c$$

$$\mathbb{P}(A) + \mathbb{P}(B \cap A^c) = \mathbb{P}(A \cup (B \cap A^c)) = \mathbb{P}(B)$$

$$\mathbb{P}(A) = \mathbb{P}(B) - \underline{\mathbb{P}(B \cap A^c)}_{\geq 0}$$

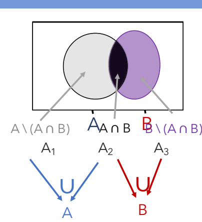
$$\mathbb{P}(A) \leq \mathbb{P}(B)$$

17

Property 4

- For events A and B

$$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B)$$



Proof:

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) = \mathbb{P}(A_1 \cup A_2 \cup A_3)$$

$$= \mathbb{P}(A \cup B)$$

$$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2) \; = \mathbb{P}(A)$$

$$\mathbb{P}(A_1) = \mathbb{P}(A) - \mathbb{P}(A_2)$$

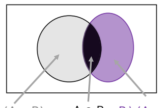
$$\mathbb{P}(A_3) = \mathbb{P}(B) - \mathbb{P}(A_2)$$

18

Property 4

- For events A and B

$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B)$



Proof:

$\mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) = \mathbb{P}(A \cup B)$

$\mathbb{P}(A_1) = \mathbb{P}(A) - \mathbb{P}(A_2)$ $\qquad$ $\mathbb{P}(A_3) = \mathbb{P}(B) - \mathbb{P}(A_2)$

$\mathbb{P}(A) - \cancel{\mathbb{P}(A_2)} + \cancel{\mathbb{P}(A_2)} + \mathbb{P}(B) - \mathbb{P}(A_2) = \mathbb{P}(A \cup B)$

$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B) + \mathbb{P}(A_2)$

19

Property 5: Union Bound

- For events A and B

$\mathbb{P}(A) + \mathbb{P}(B) \geq \mathbb{P}(A \cup B)$

Proof:

$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B)$

≥ 0

$\geq \mathbb{P}(A \cup B)$

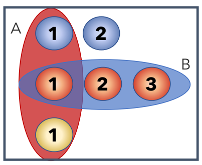
20

Conditional Probability

21

Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



- Each ball is equally likely to be picked
  - 1/6 probability of selecting each ball.
- Event A = A ball with number 1 is chosen

$$\mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$$

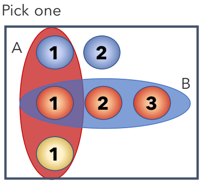
- Event B = A red ball is chosen

$$\mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}$$

22

Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



Event A = choosing a ball with number 1 $\quad \mathbb{P}(A) = \frac{1}{2}$

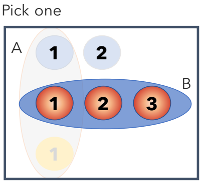
Event B = choosing a red ball $\quad \mathbb{P}(B) = \frac{1}{2}$

- Suppose we saw the color of ball and it is red.
- In other words, B has occurred.
- What is the probability of A?

23

Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



Event A = choosing a ball with number 1 $\quad \mathbb{P}(A) = \frac{1}{2}$

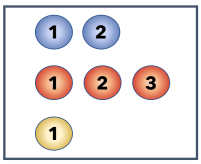
Event B = choosing a red ball $\quad \mathbb{P}(B) = \frac{1}{2}$

- Suppose we saw the color of ball and it is red.
- In other words, B has occurred.
- What is the probability of A?
- Given B, event A is equivalent to A ∩ B

24

Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



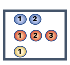
Event A = choosing a ball with number 1 $\quad \mathbb{P}(A) = \frac{1}{2}$

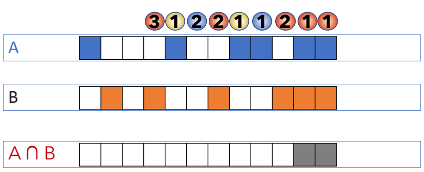
Event B = choosing a red ball $\quad \mathbb{P}(B) = \frac{1}{2}$

- Given B has occurred, what is the probability of A?
- Given B, event A is equivalent to A ∩ B

$$\mathbb{P}(A \cap B) = \frac{1}{6}$$

25





$n(A) = 6$

$n(B) = 6$

$n(A \cap B) = 2$

Event A = choosing a ball with number 1
Event B = choosing a red ball

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{n(A \cap B)/n}{n(B)\ \ /n} = \frac{2}{6}$$

26

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$$\mathbb{P}(A) = \frac{1}{2}$$

$$\mathbb{P}(B) = \frac{1}{2}$$

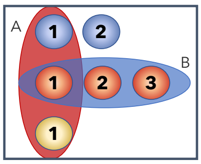
$$\mathbb{P}(A \cap B) = \frac{1}{6}$$

Event A = choosing a ball with number 1
Event B = choosing a red ball

27

Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



Event A = choosing a ball with number 1 $\quad \mathbb{P}(A) = \frac{1}{2}$

Event B = choosing a red ball $\quad \mathbb{P}(B) = \frac{1}{2}$

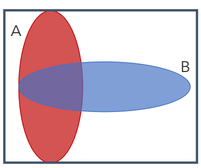
- Occurrence of B has changed the chances of A

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{3}$$

28

Conditional Probability

- Given an event B, the conditional probability of event A is



$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

where $\mathbb{P}(B) > 0$

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\,\mathbb{P}(A \mid B)$$

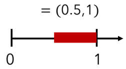
29

Example: Pick a Number

- Pick a number in (0,1)
- Probability measure for an interval (a,b) is

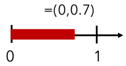
$\mathbb{P}((a,b)) = F(b) - F(a)$

Event A = selected number is bigger than 0.5
= (0.5,1)



$\mathbb{P}(A) = F(1) - F(0.5)$

Event B = selected number is less than 0.7
=(0,0.7)



$\mathbb{P}(B) = F(0.7) - F(0)$

What is probability of A given B?
Given the selected number is less than 0.7, what is the probability it is bigger than 0.5?

30

Example: Pick a Number

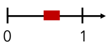
- Pick a number in (0,1)
- Probability measure for an interval (a,b) is

$\mathbb{P}((a,b)) = F(b) - F(a)$

Event A = (0.5,1) $\mathbb{P}(A) = F(1) - F(0.5)$

Event B = (0,0.7) $\mathbb{P}(B) = F(0.7) - F(0)$

A ∩ B = (0.5,0.7) $\mathbb{P}(A \cap B) = F(0.7) - F(0.5)$



$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{F(0.7) - F(0.5)}{F(0.7) - F(0)}$$

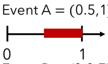
31

Example: Pick a Number

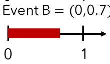
- Pick a number in (0,1)
- Probability measure for an interval (a,b) is

$\mathbb{P}((a,b)) = |b - a|$

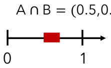
Event A = (0.5,1) $\mathbb{P}(A) = 1 - 0.5 = 0.5$



Event B = (0,0.7) $\mathbb{P}(B) = 0.7$



A ∩ B = (0.5,0.7) $\mathbb{P}(A \cap B) = 0.2$

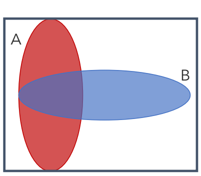


$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.2}{0.7} = \frac{2}{7}$$

32

Conditional Probability

- Given an event B, the conditional probability of event A is



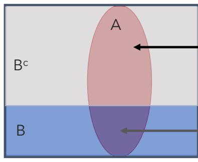
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{where } \mathbb{P}(B) > 0$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\,\mathbb{P}(A \mid B)$$

33

Total Probability Law

$\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}((A \cap B) \cup (A \cap B^c)) = \mathbb{P}(A)$



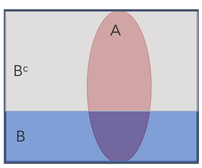
$\mathbb{P}(A \cap B^c) = \mathbb{P}(B^c)\, \mathbb{P}(A \mid B^c)$

$\mathbb{P}(A \cap B) = \mathbb{P}(B)\, \mathbb{P}(A \mid B)$

34

Total Probability Law

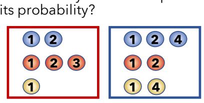
$\mathbb{P}(A) = \mathbb{P}(B)\, \mathbb{P}(A \mid B) + \mathbb{P}(B^c)\, \mathbb{P}(A \mid B^c)$



35

Example: Pick a Bag, then a Ball

- Two bags full of balls
- Each ball has a color and number
- Pick one bag and then pick a ball
- Red bag has ¾ chance to be picked. Given the selected bag, balls are equally likely to be picked.
- Event A = a yellow ball is picked. What is its probability?



- Let B denote the event red bag is picked.
- B^c denotes the event blue bag is picked.

$\mathbb{P}(B) = \frac{3}{4}$ $\qquad \mathbb{P}(B^c) = \frac{1}{4}$

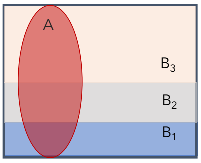
$\mathbb{P}(A|B) = \frac{1}{6}$ $\qquad \mathbb{P}(A|B^c) = \frac{2}{7}$

$$\mathbb{P}(A) = \mathbb{P}(B)\, \mathbb{P}(A \mid B) + \mathbb{P}(B^c)\, \mathbb{P}(A \mid Bc)$$
$$= \frac{3}{4} \times \frac{1}{6} \qquad + \frac{1}{4} \times \frac{2}{7}$$

36

Total Probability Law

$$\mathbb{P}(A) = \mathbb{P}(B_1)\,\mathbb{P}(A \mid B_1) + \mathbb{P}(B_2)\,\mathbb{P}(A \mid B_2) + \mathbb{P}(B_3)\,\mathbb{P}(A \mid B_3)$$



$$\boxed{\mathbb{P}(A) = \sum_i \mathbb{P}(B_i)\,\mathbb{P}(A \mid B_i)}$$

B_i's are mutually exclusive sets,
Form a partition of Ω

37

Conditional Probability as Probability Measure

- Define for every event A

$$\mathbb{P}'(A) = \mathbb{P}(A \mid B)$$

- This is also a valid probability measure for the same sample space and sigma algebra.
- It can be proved by showing that it also satisfies all probability axioms.

38

Independent Events

39

Independent Events

- In an example, we saw that occurrence of an event can affect the probability of other events. So in general, for two events A and B

$$\mathbb{P}(A \mid B) \neq \mathbb{P}(A)$$

- Events A and B are said to be independent if

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

- The condition is equivalent to

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

otherwise the events are said to be dependent.

40

Example: Two Tosses of a Coin

- A fair coin is tossed twice.
- Sample space is

$$\Omega = \{ HH, TH, HT, TT \}$$

- Suppose, each of the 4 outcomes are equally likely.

$$\mathbb{P}(\{HH\}) = \mathbb{P}(\{TH\}) = \mathbb{P}(\{HT\}) = \mathbb{P}(\{TT\}) = \frac{1}{4}$$

41

Example: Two Tosses of a Coin

$$\mathbb{P}(\{HH\}) = \mathbb{P}(\{TH\}) = \mathbb{P}(\{HT\}) = \mathbb{P}(\{TT\}) = \frac{1}{4}$$

- Let A = event that first toss results in Head

$$A = \{ HH, HT \} \qquad \mathbb{P}(A) = \frac{1}{2}$$

- Let B = event that second toss results in Head

$$B = \{ TH, HH \} \qquad \mathbb{P}(B) = \frac{1}{2}$$

- $A \cap B$ = event that both tosses result in Head

$$A \cap B = \{ HH \} \qquad \mathbb{P}(A \cap B) = \frac{1}{4}$$

42

Example: Two Tosses of a Coin

$\mathbb{P}(A) = \frac{1}{2}$ $\quad \mathbb{P}(B) = \frac{1}{2}$ $\quad \mathbb{P}(A \cap B) = \frac{1}{4}$

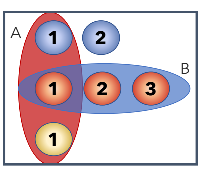
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

- A and B are independent events.

43

Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



Event A = choosing a ball with number 1

Event B = choosing a red ball

$$\mathbb{P}(A) = \frac{1}{2}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{3}$$

- A and B are not independent.

44

Combined Experiments

45

45

Two Random Experiments

- Consider an experiment of coin toss

  Ω_1={ H , T }

  $\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

  $\mathbb{P}_1$: 0 0.5 0.5 1

- Consider an experiment of selecting a ball out of two balls

  Ω_2={ B , R }

  $\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

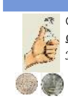
  $\mathbb{P}_2$: 0 0.2 0.8 1

If these two experiments were performed separately with no relation between them, can we call an event in $\mathcal{F}_1$ (say {H}) independent of an event in $\mathcal{F}_2$ (say {B})?

No, the events need to defined in the same probability space.

46

Combined Experiment

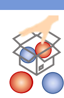


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

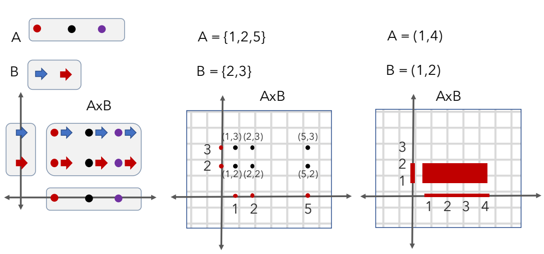
How to combine into one experiment (or one probability space)?

Each outcome of the combined experiment is a pair consisting of outcomes of both experiments.

$$\Omega = \{ (H,R),\ (T,R),\ (H,B),\ (T,B) \}$$

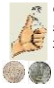
$$\Omega = \Omega_1 \times \Omega_2$$

47



48

Probability Measure for the Combined Experiment

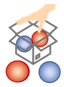


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

How to assign probabilities?

Singleton events are

{(H,R)}, {(H,B)},
{(T,R)}, {(T,B)}

Note that

event {(H,R)} ∪ {(H,B)}= {(H,R), (H,B)} is equivalent to the event {H}

$\mathbb{P}$({(H,R)} ∪ {(H,B)})=$\mathbb{P}_1$({H})

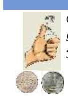
$\mathbb{P}$({(H,R)}) + $\mathbb{P}$({(H,B)})=$\mathbb{P}_1$({H})

Combined experiment

$\Omega = \Omega_1 \times \Omega_2$

49

Probability Measure for the Combined Experiment

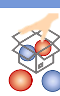


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

How to assign probabilities?

Singleton events are

{(H,R)}, {(H,B)}, {(T,R)}, {(T,B)}

$\mathbb{P}$({(H,R)}) + $\mathbb{P}$({(H,B)})=$\mathbb{P}_1$({H})

$\mathbb{P}$({(T,R)}) + $\mathbb{P}$({(T,B)}) = $\mathbb{P}_1$({T})

$\mathbb{P}$({(H,R)}) + $\mathbb{P}$({(T,R)}) = $\mathbb{P}_2$({R})

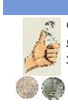
$\mathbb{P}$({(H,B)}) + $\mathbb{P}$({(T,B)}) = $\mathbb{P}_2$({B})

Combined experiment

$\Omega = \Omega_1 \times \Omega_2$

50

Probability Measure for the Combined Experiment

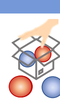


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

How to assign probabilities?

$\boxed{\mathbb{P}(\{(H,R)\})}$ + $\mathbb{P}$({(H,B)})=$\mathbb{P}_1$({H}) = 0.5

$\mathbb{P}$({(T,R)}) + $\mathbb{P}$({(T,B)}) = $\mathbb{P}_1$({T})

$\boxed{\mathbb{P}(\{(H,R)\})}$ + $\mathbb{P}$({(T,R)}) = $\mathbb{P}_2$({R}) = 0.8

$\mathbb{P}$({(H,B)}) + $\mathbb{P}$({(T,B)}) = $\mathbb{P}_2$({B})

Let us assign

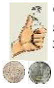
$\mathbb{P}$({(H,R)})=0.4 $\mathbb{P}$({(H,B)})=0.1 $\mathbb{P}$({(T,R)})=0.4 $\mathbb{P}$({(T,B)})=0.1

Combined experiment

$\Omega = \Omega_1 \times \Omega_2$

51

Independence of the Two Random Experiments

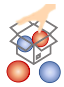


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

What does independence of these two experiments mean?

Consider two events E={H} and F={R}.

In the combined experiment they are equivalently represented by

E = {(H,R), (H,B)} is equivalent to the event {H}

F = {(H,R), (T,R)} is equivalent to the event {R}

E ∩ F = {(H,R)}

$\mathbb{P}$(E ∩ F)=0.4 = $\mathbb{P}$(E)=0.5 × $\mathbb{P}$(F)=0.8

$\mathbb{P}(\{(H,R)\}) = \mathbb{P}_1(\{H\}) \times \mathbb{P}_2(\{R\})$

Combined experiment

$$\Omega = \Omega_1 \times \Omega_2$$

$\mathbb{P}$({(H,R)})=0.4

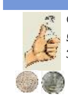
$\mathbb{P}$({(H,B)})=0.1

$\mathbb{P}$({(T,R)})=0.4

$\mathbb{P}$({(T,B)})=0.1

52

Independence of the Two Random Experiments

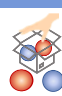


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

$\mathbb{P}(\{(H,R)\}) = \mathbb{P}_1(\{H\}) \times \mathbb{P}_2(\{R\})$

$\mathbb{P}$({(H,R)})=0.4 $\mathbb{P}$({(H,B)})=0.1 $\mathbb{P}$({(T,R)})=0.4 $\mathbb{P}$({(T,B)})=0.1

Combined experiment

$$\Omega = \Omega_1 \times \Omega_2$$

$\mathbb{P}$({(H,R)})=0.4

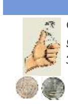
$\mathbb{P}$({(H,B)})=0.1

$\mathbb{P}$({(T,R)})=0.4

$\mathbb{P}$({(T,B)})=0.1

53

Probability Measure for the Combined Experiment

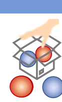


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

A different assignment of probabilities:

$\boxed{\mathbb{P}(\{(H,R)\})}$+ $\mathbb{P}$({(H,B)}) = $\mathbb{P}_1$({H}) = 0.5

$\boxed{\mathbb{P}(\{(H,R)\})}$+ $\mathbb{P}$({(T,R)}) = $\mathbb{P}_2$({R}) = 0.8

Let us assign

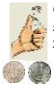
$\mathbb{P}$({(H,R)})=0.3 $\mathbb{P}$({(H,B)})=0.2 $\mathbb{P}$({(T,R)})=0.5 $\mathbb{P}$({(T,B)})=0

Combined experiment

$$\Omega = \Omega_1 \times \Omega_2$$

54

Dependence of the Two Random Experiments

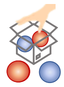


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

Consider two events E={H} and F={R}.

E = {(H,R), (H,B)} is equivalent to the event {H}

F = {(H,R), (T,R)} is equivalent to the event {R}

E ∩ F = {(H,R)}

$\mathbb{P}$(E ∩ F)=0.3 ≠ $\mathbb{P}$(E)=0.5 × $\mathbb{P}$(F)=0.8

Events are not independent.

Combined experiment

$$\Omega = \Omega_1 \times \Omega_2$$

$\mathbb{P}$({(H,R)})=0.3

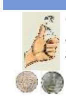
$\mathbb{P}$({(H,B)})=0.2

$\mathbb{P}$({(T,R)})=0.5

$\mathbb{P}$({(T,B)})=0

55

Combining Independent Random Experiments

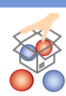


Coin toss

Ω_1={ H , T }

$\mathcal{F}_1$={ {}, {H}, {T}, Ω_1 }

$\mathbb{P}_1$: 0 0.5 0.5 1



Selecting a ball out of two balls

Ω_2={ B , R }

$\mathcal{F}_2$={ {}, {B}, {R}, Ω_2 }

$\mathbb{P}_2$:0 0.2 0.8 1

Combined experiment

$$\Omega = \Omega_1 \times \Omega_2$$

$\mathbb{P}$({(H,R)}) = $\mathbb{P}_1$({H}) × $\mathbb{P}_2$({R})

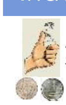
$\mathbb{P}$({(H,B)}) = $\mathbb{P}_1$({H}) × $\mathbb{P}_2$({B})

$\mathbb{P}$({(T,R)}) = $\mathbb{P}_1$({T}) × $\mathbb{P}_2$({R})

$\mathbb{P}$({(T,B)}) = $\mathbb{P}_1$({T}) × $\mathbb{P}_2$({B})

56

Independent Repeated Trials

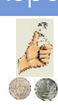


1st Coin toss

Ω ={ H , T }

$\mathcal{F}$={ {}, {H}, {T}, Ω }

$\mathbb{P}$: 0 p 1-p 1



2nd Coin toss

Ω ={ H , T }

$\mathcal{F}$={ {}, {H}, {T}, Ω }

$\mathbb{P}$: 0 p 1-p 1

Combined experiment

$$\Omega' = \Omega \times \Omega = \Omega^2$$

$\mathbb{P}'$({(H,H)}) = $\mathbb{P}$({H}) × $\mathbb{P}$({H}) $= p^2$

$\mathbb{P}'$({(H,T)}) = $\mathbb{P}$({H}) × $\mathbb{P}$({T}) $= p(1-p)$

$\mathbb{P}'$({(T,H)}) = $\mathbb{P}$({T}) × $\mathbb{P}$({H}) $= (1-p)p$

$\mathbb{P}'$({(T,T)}) = $\mathbb{P}$({T}) × $\mathbb{P}$({T}) $= (1-p)^2$

57

n Independent Repeated Trials

Combined experiment

$$\Omega' = \Omega^n$$

$\mathbb{P}'(\{(\text{H,H,H}\ldots,\text{H})\}) = p^n$

$\mathbb{P}'(\{(\text{H,T,H},\ldots,\text{H})\}) = p(1-p)p^{n-2}$

$\mathbb{P}'(\{(\text{T,T,T},\ldots,\text{T})\}) = (1-p)^n$

58