EE910: Digital Communication Systems-I

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Lecture #2D: Some useful random variables

Bernoulli Random Variable

- The Bernoulli random variable is a discrete binary-valued random variable taking values 1 and 0 with probabilities p and $1-p$, respectively.
- Therefore the probability mass function (PMF) for this random variable is given by

$$\mathrm{P}[X=1]=p \quad \mathrm{P}[X=0]=1-p \tag{1}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= p \\ \mathrm{VAR}[X] &= p(1-p) \end{aligned} \tag{2}$$

The Binomial Random Variable

- The binomial random variable models the sum of n independent Bernoulli random variables with common parameter p.
- The PMF of this random variable is given by

$$\mathrm{P}[X=k]=\binom{n}{k} p^k(1-p)^{n-k}, \quad k=0,1,\ldots,n \tag{3}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= np \\ \mathrm{VAR}[X] &= np(1-p) \end{aligned} \tag{4}$$

The Uniform Random Variable

- The uniform random variable is a continuous random variable with PDF

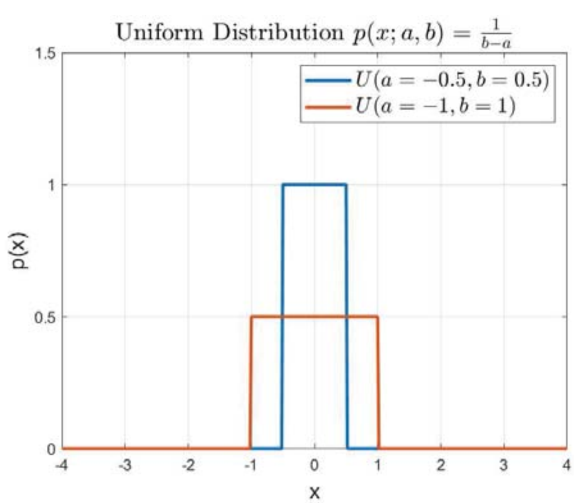
$$p(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $b > a$ and the interval $[a, b]$ is the range of the random variable.

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= \frac{b-a}{2} \\ \mathrm{VAR}[X] &= \frac{(b-a)^2}{12} \end{aligned} \quad (6)$$

The Uniform Random Variable



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The Gaussian (Normal) Random Variable

- The Gaussian random variable is described in terms of two parameters $m \in \mathbb{R}$ and $\sigma > 0$ by the PDF

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{7}$$

- We usually use the shorthand form $\mathcal{N}\left(m, \sigma^2\right)$ to denote the PDF of Gaussian random variables and write $X \sim \mathcal{N}\left(m, \sigma^2\right)$.
- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= m \\ \mathrm{VAR}[X] &= \sigma^2 \end{aligned} \tag{8}$$

The Gaussian (Normal) Random Variable

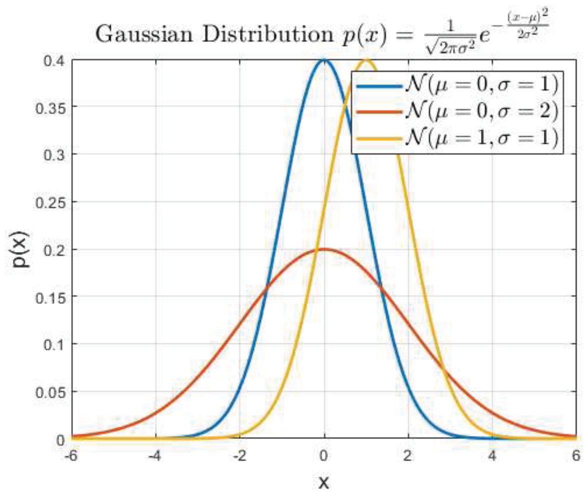
- A Gaussian random variable with $m = 0$ and $\sigma = 1$ is called a standard normal.
- A function closely related to the Gaussian random variable is the Q function defined as

$$Q(x) = \mathrm{P}[\mathcal{N}(0,1) > x] = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$$

- The CDF of a Gaussian random variable is given by

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}} dt \\ &= 1 - \int_x^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}} dt \\ &= 1 - \int_{\frac{x-m}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ &= 1 - Q\left(\frac{x-m}{\sigma}\right) \end{aligned} \tag{9}$$

The Gaussian (Normal) Random Variable



The Rayleigh Random Variable

- If X_1 and X_2 are two iid Gaussian random variables each distributed according to $\mathcal{N}(0, \sigma^2)$, then

$$X = \sqrt{X_1^2 + X_2^2} \tag{10}$$

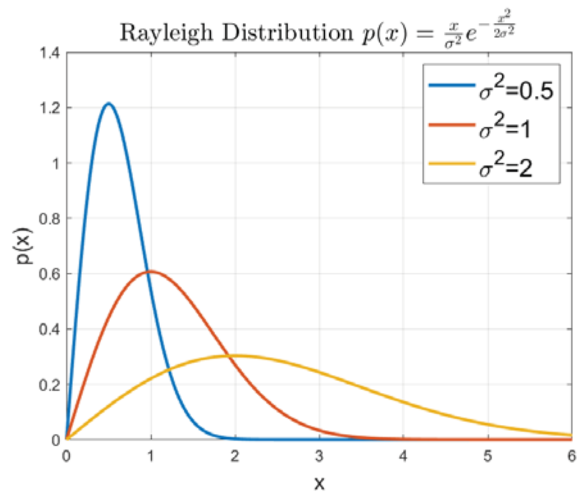
  is a Rayleigh random variable.
- The PDF of a Rayleigh random variable is given by

$$p(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= \sigma\sqrt{\frac{\pi}{2}} \\ \mathrm{VAR}[X] &= \left(2 - \frac{\pi}{2}\right)\sigma^2 \end{aligned} \tag{12}$$

The Rayleigh Random Variable



The Ricean Random Variable

- If X_1 and X_2 are two independent Gaussian random variables distributed according to $\mathcal{N}\left(m_1, \sigma^2\right)$ and $\mathcal{N}\left(m_2, \sigma^2\right)$ (i.e., the variances are equal and the means may be different), then

$$X = \sqrt{X_1^2 + X_2^2} \tag{13}$$

is a Ricean random variable with PDF given by

$$p(x) = \begin{cases} \frac{x}{\sigma^2} I_0\left(\frac{sx}{\sigma^2}\right) e^{-\frac{x^2+s^2}{2\sigma^2}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $s = \sqrt{m_1^2 + m_2^2}$ and $I_0(x)$ is the modified Bessel function of the first kind and order zero.

The Ricean Random Variable

- For $s = 0$, the Ricean random variable reduces to a Rayleigh random variable.
- For large s the Ricean random variable can be well approximated by a Gaussian random variable.
- The CDF of a Ricean random variable can be expressed as

$$F(x) = \begin{cases} 1 - Q_1\left(\frac{s}{\sigma}, \frac{x}{\sigma}\right) & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where $Q_1(a, b) = \int_b^\infty x e^{-\frac{a^2+x^2}{2}} I_0(ax)dx$, known as Marcum Q function

The Nakagami Random Variable

- Nakagami-m distribution is frequently used to characterize the statistics of signals transmitted through multipath fading channels.
- The PDF for this distribution is given by Nakagami (1960) as

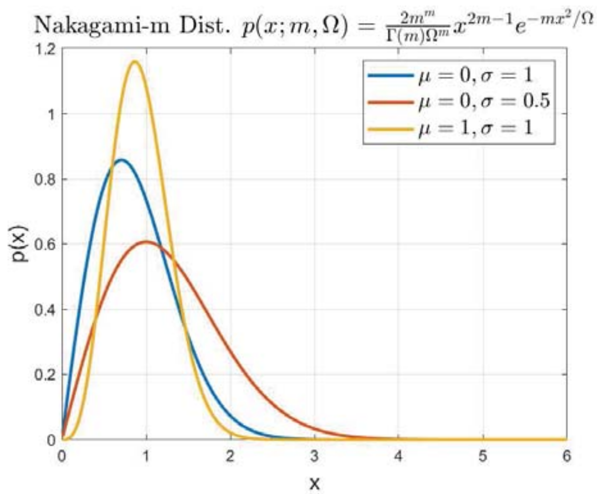
$$p(x) = \begin{cases} \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m x^{2m-1} e^{-mx^2/\Omega} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where Ω is defined as

$$\Omega = \mathrm{E}\left[X^2\right] \tag{17}$$

and the parameter m is defined as the ratio of moments $\left(\frac{\Omega^2}{E[(X^2-\Omega)^2]}\right)$, called the fading figure,

The Nakagami Random Variable



The Lognormal Random Variable

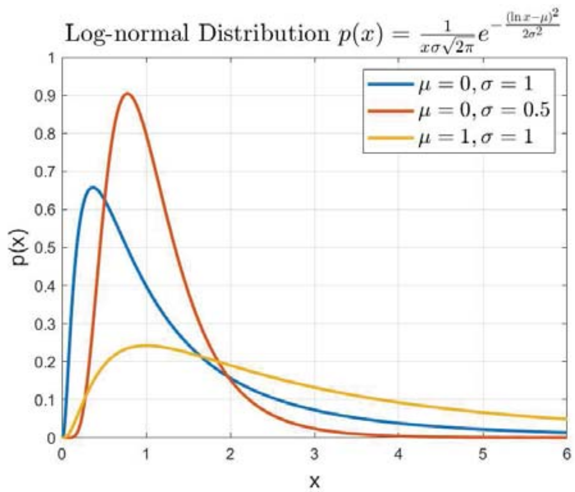
- Suppose that a random variable Y is normally distributed with mean m and variance σ^2. Let us define a new random variable X that is related to Y through the transformation $Y = \ln X$ (or $X = e^Y$). Then the PDF of X is

$$p(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}x} e^{-(\ln x - m)^2/2\sigma^2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= e^{m+\frac{\sigma^2}{2}} \\ \mathrm{VAR}[X] &= e^{2m+\sigma^2}\left(e^{\sigma^2} - 1\right) \end{aligned} \tag{19}$$

The Lognormal Random Variable



Jointly Gaussian Random Variables

- An $n \times 1$ column random vector $\boldsymbol{X}$ with components $\{X_i, 1 \leq i \leq n\}$ is called a Gaussian vector, and its components are called jointly Gaussian random variables or, multivariate Gaussian random variables if the joint PDF of X_i 's can be written as

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}(\det C)^{1/2}} e^{-\frac{1}{2}(x-m)^t C^{-1}(x-m)} \tag{20}$$

where $\boldsymbol{m}$ and $\boldsymbol{C}$ are the mean vector and covariance matrix, respectively, of $\boldsymbol{X}$ and are given by

$$\begin{aligned} m &= \mathrm{E}[\boldsymbol{X}] \\ \boldsymbol{C} &= \mathrm{E}\left[(\boldsymbol{X}-\boldsymbol{m})(\boldsymbol{X}-\boldsymbol{m})^t\right] \end{aligned} \tag{21}$$

Jointly Gaussian Random Variables

- From this definition it is clear that

$$C_{ij} = \text{COV}[X_i, X_j] \tag{22}$$

and therefore C is a symmetric matrix.
- We know that $\boldsymbol{C}$ is nonnegative definite.
- In the special case of $n = 2$, we have

$$\begin{aligned} \boldsymbol{m} &= \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \\ \boldsymbol{C} &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \end{aligned} \tag{23}$$

where

$$\rho = \frac{\text{COV}[X_1, X_2]}{\sigma_1\sigma_2} \tag{24}$$

is the correlation coefficient of the two random variables.

Jointly Gaussian Random Variables

- In this case the PDF reduces to

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1-m_1}{\sigma_1}\right)^2 + \left(\frac{x_2-m_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-m_1}{\sigma_1}\right)\left(\frac{x_2-m_2}{\sigma_2}\right)}{2(1-\rho^2)}} \tag{25}$$

where m_1, m_2, σ_1^2 and, σ_2^2 are means and variances of the two random variables and ρ is their correlation coefficient.
- For the special case when $\rho = 0$ (i.e., when the two random variables are uncorrelated), we have

$$p(x_1, x_2) = \mathcal{N}\left(m_1, \sigma_1^2\right) \times \mathcal{N}\left(m_2, \sigma_2^2\right) \tag{26}$$

- This means that the two random variables are independent, and therefore for this case independence and uncorrelatedness are equivalent.
- This property is true for general jointly Gaussian random variables.

Jointly Gaussian Random Variables

- Linear combinations of jointly Gaussian random variables are also jointly Gaussian.
- In other words, if $\boldsymbol{X}$ is a Gaussian vector, the random vector $\boldsymbol{Y} = \boldsymbol{AX}$, where the invertible matrix $\boldsymbol{A}$ represents a linear transformation, is also a Gaussian vector whose mean and covariance matrix are given by

$$\begin{aligned} m_Y &= Am_X \\ C_Y &= AC_X A^t \end{aligned} \tag{27}$$

Jointly Gaussian Random Variables

- In summary, jointly Gaussian random variables have the following important properties:
  - For jointly Gaussian random variables, uncorrelated is equivalent to independent.
  - Linear combinations of jointly Gaussian random variables are themselves jointly Gaussian.
  - The random variables in any subset of jointly Gaussian random variables are jointly Gaussian, and any subset of random variables conditioned on random variables in any other subset is also jointly Gaussian (all joint subsets and all conditional subsets are Gaussian).
- Any set of independent Gaussian random variables is jointly Gaussian, but this is not necessarily true for a set of dependent Gaussian random variables.