Q: $\|v\|_P^2 = v^T P v \quad P \in S^n$ symmetric

when $\|v\|_P$ valid

whenever P is positive definite

$v^T P v > 0 \quad \forall\, v \neq 0$

EVD?

$$P = Q \Lambda Q^T \qquad v^T P v = \underbrace{v^T Q}_{z^T} \Lambda \underbrace{Q^T v}_{z} = z^T \Lambda z \qquad z = Q^T v$$

$$= \sum z_i^2 \lambda_i$$

$\sqrt{v^T P v}$ valid norm when $\lambda_i > 0 \quad \forall\, i = 1, 2 \ldots n$

valid norm wrt z.

but also $z = Q^T v$ & $v = Qz$

one-to-one mapping z & v

Recap (1) $v^T P v > 0 \quad \forall\, v \neq 0$ equivalent

(2) $\lambda_i(P) > 0 \quad \forall\, i$ P is p.d.

$P \succ 0$

notation

$P > 0 \not\Rightarrow$ entries of P are positive

$\Rightarrow$ e.v. of P are positive

(3) $P = LL^T$ where L lower triangular & full rank

Cholesky decomposition

note $v^T P v = v^T L L^T v = (L^T v)^T (L^T v) = \|L^T v\|_2^2$

so $\|L^T v\|_2^2 > 0 \quad \forall\, v \neq 0$

$L^T v \neq 0$ when $v \neq 0$ → holds when L is full rank

faster way to check if $P \succ 0$

Positive Semidefinite matrix $\quad P \in S^n$

(a) $P \succeq 0$ (not entriwise)

(b) $\lambda_i(P) \geq 0 \quad i = 1, 2 \ldots n$

(c) $v^T P v \geq 0 \quad \forall\, v \in \mathbb{R}^n$

(d) $P = AA^T$ for any $A \in \mathbb{R}^{n \times m} \quad m \leq n$

Suppose $P = AA^T$

$$v^T P v = v^T A A^T v = \|A^T v\|_2^2 \geq 0 \quad \forall\, v$$

(d) ⇒ (c) ⇒ $P \succeq 0$

Note: only for symmetric matrices

(non-symmetric case: $\lambda_i(P)$ may be complex)

notion of PSD would be different

Note : $\|v\|_P$ for $P \succeq 0$ not a norm

(not definite)