

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 2) ¹

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Problem 2.1

a.

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(a)}{t-a} da$$

Hence :

$$\begin{aligned} -\hat{x}(-t) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(a)}{-t-a} da \\ &= -\frac{1}{\pi} \int_{\infty}^{-\infty} \frac{x(-b)}{-t+b} (-db) \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(b)}{t+b} db \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(b)}{t-b} db = \hat{x}(t) \end{aligned}$$

where we have made the change of variables : $b = -a$ and used the relationship : $x(b) = x(-b)$.

b. In exactly the same way as in part (a) we prove :

$$\hat{x}(t) = \hat{x}(-t)$$

c. $x(t) = \cos \omega_0 t$, so its Fourier transform is : $X(f) = \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)]$, $f_0 = 2\pi\omega_0$. Exploiting the phase-shifting property (2-1-4) of the Hilbert transform :

$$\hat{X}(f) = \frac{1}{2} [-j\delta(f - f_0) + j\delta(f + f_0)] = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] = F^{-1} \{\sin 2\pi f_0 t\}$$

Hence, $\hat{x}(t) = \sin \omega_0 t$.

d. In a similar way to part (c) :

$$\begin{aligned} x(t) = \sin \omega_0 t &\Rightarrow X(f) = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] \Rightarrow \hat{X}(f) = \frac{1}{2} [-\delta(f - f_0) - \delta(f + f_0)] \\ &\Rightarrow \hat{X}(f) = -\frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] = -F^{-1} \{\cos 2\pi f_0 t\} \Rightarrow \hat{x}(t) = -\cos \omega_0 t \end{aligned}$$

e. The positive frequency content of the new signal will be : $(-j)(-j)X(f) = -X(f)$, $f > 0$, while the negative frequency content will be : $j \cdot jX(f) = -X(f)$, $f < 0$. Hence, since $\hat{\hat{X}}(f) = -X(f)$, we have : $\hat{\hat{x}}(t) = -x(t)$.

f. Since the magnitude response of the Hilbert transformer is characterized by : $|H(f)| = 1$, we have that : $|\hat{X}(f)| = |H(f)||X(f)| = |X(f)|$. Hence :

$$\int_{-\infty}^{\infty} |\hat{X}(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and using Parseval's relationship :

$$\int_{-\infty}^{\infty} \hat{x}^2(t) dt = \int_{-\infty}^{\infty} x^2(t) dt$$

g. From parts (a) and (b) above, we note that if $x(t)$ is even, $\hat{x}(t)$ is odd and vice-versa. Therefore, $x(t)\hat{x}(t)$ is always odd and hence : $\int_{-\infty}^{\infty} x(t)\hat{x}(t) dt = 0$.

Problem 2.2

1. Using relations

$$\begin{aligned} X(f) &= \frac{1}{2}X_l(f - f_0) + \frac{1}{2}X_l(-f - f_0) \\ Y(f) &= \frac{1}{2}Y_l(f - f_0) + \frac{1}{2}Y_l(-f - f_0) \end{aligned}$$

and Parseval's relation, we have

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)y(t) dt &= \int_{-\infty}^{\infty} X(f)Y^*(f) df \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2}X_l(f - f_0) + \frac{1}{2}X_l(-f - f_0) \right] \left[\frac{1}{2}Y_l(f - f_0) + \frac{1}{2}Y_l(-f - f_0) \right]^* df \\ &= \frac{1}{4} \int_{-\infty}^{\infty} X_l(f - f_0)Y_l^*(f - f_0) df + \frac{1}{4} \int_{-\infty}^{\infty} X_l(-f - f_0)Y_l^*(-f - f_0) df \\ &= \frac{1}{4} \int_{-\infty}^{\infty} X_l(u)Y_l^*(u) du + \frac{1}{4} \int_{-\infty}^{\infty} X_l^*(v)Y_l(v) dv \\ &= \frac{1}{2} \operatorname{Re} \left[\int_{-\infty}^{\infty} X_l(f)Y_l^*(f) df \right] \\ &= \frac{1}{2} \operatorname{Re} \left[\int_{-\infty}^{\infty} x_l(t)y_l^*(t) dt \right] \end{aligned}$$

where we have used the fact that since $X_l(f - f_0)$ and $Y_l(-f - f_0)$ do not overlap, $X_l(f - f_0)Y_l^*(-f - f_0) = 0$ and similarly $X_l(-f - f_0)Y_l^*(f - f_0) = 0$.

2. Putting $y(t) = x(t)$ we get the desired result from the result of part 1.

Problem 2.3

A well-known result in estimation theory based on the minimum mean-squared-error criterion states that the minimum of \mathcal{E}_e is obtained when the error is orthogonal to each of the functions in the series expansion. Hence :

$$\int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^K s_k f_k(t) \right] f_n^*(t) dt = 0, \quad n = 1, 2, \dots, K \quad (1)$$

since the functions $\{f_n(t)\}$ are orthonormal, only the term with $k = n$ will remain in the sum, so :

$$\int_{-\infty}^{\infty} s(t) f_n^*(t) dt - s_n = 0, \quad n = 1, 2, \dots, K$$

or:

$$s_n = \int_{-\infty}^{\infty} s(t) f_n^*(t) dt \quad n = 1, 2, \dots, K$$

The corresponding residual error \mathcal{E}_e is :

$$\begin{aligned} \mathcal{E}_{\min} &= \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^K s_k f_k(t) \right] \left[s(t) - \sum_{n=1}^K s_n f_n(t) \right]^* dt \\ &= \int_{-\infty}^{\infty} |s(t)|^2 dt - \int_{-\infty}^{\infty} \sum_{k=1}^K s_k f_k(t) s^*(t) dt - \sum_{n=1}^K s_n^* \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^K s_k f_k(t) \right] f_n^*(t) dt \\ &= \int_{-\infty}^{\infty} |s(t)|^2 dt - \int_{-\infty}^{\infty} \sum_{k=1}^K s_k f_k(t) s^*(t) dt \\ &= \mathcal{E}_s - \sum_{k=1}^K |s_k|^2 \end{aligned}$$

where we have exploited relationship (1) to go from the second to the third step in the above calculation.

Note : Relationship (1) can also be obtained by simple differentiation of the residual error with respect to the coefficients $\{s_n\}$. Since s_n is, in general, complex-valued $s_n = a_n + jb_n$ we have to differentiate with respect to both real and imaginary parts :

$$\begin{aligned} \frac{d}{da_n} \mathcal{E}_e &= \frac{d}{da_n} \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^K s_k f_k(t) \right] \left[s(t) - \sum_{n=1}^K s_n f_n(t) \right]^* dt = 0 \\ \Rightarrow - \int_{-\infty}^{\infty} a_n f_n(t) \left[s(t) - \sum_{n=1}^K s_n f_n(t) \right]^* + a_n^* f_n^*(t) \left[s(t) - \sum_{n=1}^K s_n f_n(t) \right] dt &= 0 \\ \Rightarrow -2a_n \int_{-\infty}^{\infty} \text{Re} \left\{ f_n^*(t) \left[s(t) - \sum_{n=1}^K s_n f_n(t) \right] \right\} dt &= 0 \\ \Rightarrow \int_{-\infty}^{\infty} \text{Re} \left\{ f_n^*(t) \left[s(t) - \sum_{n=1}^K s_n f_n(t) \right] \right\} dt &= 0, \quad n = 1, 2, \dots, K \end{aligned}$$

where we have exploited the identity : $(x + x^*) = 2\text{Re}\{x\}$. Differentiation of \mathcal{E}_e with respect to b_n will give the corresponding relationship for the imaginary part; combining the two we get (1).

Problem 2.4

The procedure is very similar to the one for the real-valued signals described in the book (pages 33-37). The only difference is that the projections should conform to the complex-valued vector space :

$$c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1^*(t) dt$$

and, in general for the k -th function :

$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i^*(t) dt, \quad i = 1, 2, \dots, k-1$$

Problem 2.5

The first basis function is :

$$g_4(t) = \frac{s_4(t)}{\sqrt{\mathcal{E}_4}} = \frac{s_4(t)}{\sqrt{3}} = \begin{cases} -1/\sqrt{3}, & 0 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

Then, for the second basis function :

$$c_{43} = \int_{-\infty}^{\infty} s_3(t) g_4(t) dt = -1/\sqrt{3} \Rightarrow g'_3(t) = s_3(t) - c_{43} g_4(t) = \begin{cases} 2/3, & 0 \leq t \leq 2 \\ -4/3, & 2 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

Hence :

$$g_3(t) = \frac{g'_3(t)}{\sqrt{E_3}} = \begin{cases} 1/\sqrt{6}, & 0 \leq t \leq 2 \\ -2/\sqrt{6}, & 2 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

where E_3 denotes the energy of $g'_3(t)$: $E_3 = \int_0^3 (g'_3(t))^2 dt = 8/3$.

For the third basis function :

$$c_{42} = \int_{-\infty}^{\infty} s_2(t) g_4(t) dt = 0 \quad \text{and} \quad c_{32} = \int_{-\infty}^{\infty} s_2(t) g_3(t) dt = 0$$

Hence :

$$g_2'(t) = s_2(t) - c_{42}g_4(t) - c_{32}g_3(t) = s_2(t)$$

and

$$g_2(t) = \frac{g_2'(t)}{\sqrt{\mathcal{E}_2}} = \begin{cases} 1/\sqrt{2}, & 0 \leq t \leq 1 \\ -1/\sqrt{2}, & 1 \leq t \leq 2 \\ 0, & \text{o.w} \end{cases}$$

where : $\mathcal{E}_2 = \int_0^2 (s_2(t))^2 dt = 2$.

Finally for the fourth basis function :

$$c_{41} = \int_{-\infty}^{\infty} s_1(t)g_4(t)dt = -2/\sqrt{3}, \quad c_{31} = \int_{-\infty}^{\infty} s_1(t)g_3(t)dt = 2/\sqrt{6}, \quad c_{21} = 0$$

Hence :

$$g_1'(t) = s_1(t) - c_{41}g_4(t) - c_{31}g_3(t) - c_{21}g_2(t) = 0 \Rightarrow g_1(t) = 0$$

The last result is expected, since the dimensionality of the vector space generated by these signals is 3. Based on the basis functions $(g_2(t), g_3(t), g_4(t))$ the basis representation of the signals is :

$$\begin{aligned} \mathbf{s}_4 &= (0, 0, \sqrt{3}) \Rightarrow \mathcal{E}_4 = 3 \\ \mathbf{s}_3 &= (0, \sqrt{8/3}, -1/\sqrt{3}) \Rightarrow \mathcal{E}_3 = 3 \\ \mathbf{s}_2 &= (\sqrt{2}, 0, 0) \Rightarrow \mathcal{E}_2 = 2 \\ \mathbf{s}_1 &= (2/\sqrt{6}, -2/\sqrt{3}, 0) \Rightarrow \mathcal{E}_1 = 2 \end{aligned}$$

Problem 2.6

Consider the set of signals $\tilde{\phi}_{nl}(t) = j\phi_{nl}(t)$, $1 \leq n \leq N$, then by definition of lowpass equivalent signals and by Equations 2.2-49 and 2.2-54, we see that $\phi_n(t)$'s are $\sqrt{2}$ times the lowpass equivalents of $\phi_{nl}(t)$'s and $\tilde{\phi}_n(t)$'s are $\sqrt{2}$ times the lowpass equivalents of $\tilde{\phi}_{nl}(t)$'s. We also note that since $\phi_n(t)$'s have unit energy, $\langle \phi_{nl}(t), \tilde{\phi}_{nl}(t) \rangle = \langle \phi_{nl}(t), j\phi_{nl}(t) \rangle = -j$ and since the inner product is pure imaginary, we conclude that $\phi_n(t)$ and $\tilde{\phi}_n(t)$ are orthogonal. Using the orthonormality of the set $\phi_{nl}(t)$, we have

$$\langle \phi_{nl}(t), -j\phi_{ml}(t) \rangle = j\delta_{mn}$$

and using the result of problem 2.2 we have

$$\langle \phi_n(t), \tilde{\phi}_m(t) \rangle = 0 \quad \text{for all } n, m$$

We also have

$$\langle \phi_n(t), \phi_m(t) \rangle = 0 \quad \text{for all } n \neq m$$

and

$$\langle \tilde{\phi}_n(t), \tilde{\phi}_m(t) \rangle = 0 \quad \text{for all } n \neq m$$

Using the fact that the energy in lowpass equivalent signal is twice the energy in the bandpass signal we conclude that the energy in $\phi_n(t)$'s and $\tilde{\phi}_n(t)$'s is unity and hence the set of $2N$ signals $\{\phi_n(t), \tilde{\phi}_n(t)\}$ constitute an orthonormal set. The fact that this orthonormal set is sufficient for expansion of bandpass signals follows from Equation 2.2-57.

Problem 2.7

Let $x(t) = m(t) \cos 2\pi f_0 t$ where $m(t)$ is real and lowpass with bandwidth less than f_0 . Then $\mathcal{F}[\hat{x}(t)] = -j \operatorname{sgn}(f) \left[\frac{1}{2} M(f - f_0) + \frac{1}{2} M(f + f_0) \right]$ and hence $\mathcal{F}[\hat{x}(t)] = -\frac{j}{2} M(f - f_0) + \frac{j}{2} M(f + f_0)$ where we have used that fact that $M(f - f_0) = 0$ for $f < 0$ and $M(f + f_0) = 0$ for $f > 0$. This shows that $\hat{x}(t) = m(t) \sin 2\pi f_0 t$. Similarly we can show that Hilbert transform of $m(t) \sin 2\pi f_0 t$ is $-m(t) \cos 2\pi f_0 t$. From above and Equation 2.2-54 we have

$$\mathcal{H}[\phi_n(t)] = \sqrt{2} \phi_{ni}(t) \sin 2\pi f_0 t + \sqrt{2} \phi_{nq}(t) \cos 2\pi f_0 t = -\tilde{\phi}_n(t)$$

Problem 2.8

For real-valued signals the correlation coefficients are given by : $\rho_{km} = \frac{1}{\sqrt{\mathcal{E}_k \mathcal{E}_m}} \int_{-\infty}^{\infty} s_k(t) s_m(t) dt$ and the Euclidean distances by : $d_{km}^{(e)} = \{ \mathcal{E}_k + \mathcal{E}_m - 2\sqrt{\mathcal{E}_k \mathcal{E}_m} \rho_{km} \}^{1/2}$. For the signals in this problem :

$$\mathcal{E}_1 = 2, \mathcal{E}_2 = 2, \mathcal{E}_3 = 3, \mathcal{E}_4 = 3$$

$$\rho_{12} = 0 \quad \rho_{13} = \frac{2}{\sqrt{6}} \quad \rho_{14} = -\frac{2}{\sqrt{6}}$$

$$\rho_{23} = 0 \quad \rho_{24} = 0$$

$$\rho_{34} = -\frac{1}{3}$$

and:

$$\begin{aligned} d_{12}^{(e)} &= 2 & d_{13}^{(e)} &= \sqrt{2 + 3 - 2\sqrt{6}\frac{2}{\sqrt{6}}} = 1 & d_{14}^{(e)} &= \sqrt{2 + 3 + 2\sqrt{6}\frac{2}{\sqrt{6}}} = 3 \\ d_{23}^{(e)} &= \sqrt{2 + 3} = \sqrt{5} & d_{24}^{(e)} &= \sqrt{5} \\ d_{34}^{(e)} &= \sqrt{3 + 3 + 2 * 3\frac{1}{3}} = 2\sqrt{2} \end{aligned}$$

Problem 2.9

We know from Fourier transform properties that if a signal $x(t)$ is real-valued then its Fourier transform satisfies : $X(-f) = X^*(f)$ (Hermitian property). Hence the condition under which $s_l(t)$ is real-valued is : $S_l(-f) = S_l^*(f)$ or going back to the bandpass signal $s(t)$ (using 2-1-5):

$$S_+(f_c - f) = S_+^*(f_c + f)$$

The last condition shows that in order to have a real-valued lowpass signal $s_l(t)$, the positive frequency content of the corresponding bandpass signal must exhibit hermitian symmetry around the center frequency f_c . In general, bandpass signals do not satisfy this property (they have Hermitian symmetry around $f = 0$), hence, the lowpass equivalent is generally complex-valued.

Problem 2.10

a. To show that the waveforms $f_n(t)$, $n = 1, \dots, 3$ are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t)f_n(t)dt = 0, \quad m \neq n$$

Clearly:

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \int_0^4 f_1(t)f_2(t)dt \\ &= \int_0^2 f_1(t)f_2(t)dt + \int_2^4 f_1(t)f_2(t)dt \\ &= \frac{1}{4} \int_0^2 dt - \frac{1}{4} \int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4 - 2) \\ &= 0 \end{aligned}$$

Similarly:

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} f_1(t)f_3(t)dt = \int_0^4 f_1(t)f_3(t)dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt - \frac{1}{4} \int_2^3 dt + \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

and :

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} f_2(t)f_3(t)dt = \int_0^4 f_2(t)f_3(t)dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt + \frac{1}{4} \int_2^3 dt - \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \quad i = 1, 2, 3$$

Hence, they are orthonormal.

b. We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)f_n(t)dt, \quad n = 1, 2, 3$$

$$\begin{aligned} x_1 &= \int_0^4 x(t)f_1(t)dt = -\frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_1^2 dt - \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \\ x_2 &= \int_0^4 x(t)f_2(t)dt = \frac{1}{2} \int_0^4 x(t)dt = 0 \\ x_3 &= \int_0^4 x(t)f_3(t)dt = -\frac{1}{2} \int_0^1 dt - \frac{1}{2} \int_1^2 dt + \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal waveforms $f_n(t)$, $n = 1, 2, 3$ and thus it can not be represented as a linear combination of these functions.

Problem 2.11

a. As an orthonormal set of basis functions we consider the set

$$\begin{aligned} f_1(t) &= \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w} \end{cases} & f_2(t) &= \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w} \end{cases} \\ f_3(t) &= \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w} \end{cases} & f_4(t) &= \begin{cases} 1 & 3 \leq t < 4 \\ 0 & \text{o.w} \end{cases} \end{aligned}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

b. The representation vectors are

$$\begin{aligned}\mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix}\end{aligned}$$

c. The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left| \begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix} \right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned}d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left| \begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix} \right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix} \right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left| \begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix} \right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix} \right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix} \right|^2} = \sqrt{19}\end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 2.12

As a set of orthonormal functions we consider the waveforms

$$f_1(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w} \end{cases} \quad f_2(t) = \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w} \end{cases} \quad f_3(t) = \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}\mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}\end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 2.13

1. $P(E_2) = P(R2, R3, R4) = 3/7$.
2. $P(E_3|E_2) = \frac{P(E_3E_2)}{P(E_2)} = \frac{P(R2)}{3/7} = \frac{1}{3}$.
3. Here $E_4 = \{R2, R4, B2, R1, B1\}$ and $P(E_2|E_4E_3) = \frac{P(E_2E_3E_4)}{P(E_3E_4)} = \frac{P(R2)}{P(R2, B2, R1, B1)} = \frac{1}{4}$.
4. $E_5 = \{R2, R4, B2\}$. We have $P(E_3E_5) = P(R2, B2) = \frac{2}{7}$ and $P(E_3) = P(R1, R2, B1, B2) = \frac{4}{7}$ and $P(E_5) = \frac{3}{7}$. Obviously $P(E_3E_5) \neq P(E_3)P(E_5)$ and the events are not independent.

Problem 2.14

1. $P(R) = P(A)P(R|A) + P(B)P(R|B) + P(C)P(R|C) = 0.2 \times 0.05 + 0.3 \times 0.1 + 0.5 \times 0.15 = 0.01 + 0.03 + 0.075 = 0.115$.
2. $P(A|R) = \frac{P(A)P(R|A)}{P(R)} = \frac{0.01}{0.115} \approx 0.087$.

Problem 2.15

The relationship holds for $n = 2$ (2-1-34) : $p(x_1, x_2) = p(x_2|x_1)p(x_1)$

Suppose it holds for $n = k$, i.e : $p(x_1, x_2, \dots, x_k) = p(x_k|x_{k-1}, \dots, x_1)p(x_{k-1}|x_{k-2}, \dots, x_1) \dots p(x_1)$

Then for $n = k + 1$:

$$\begin{aligned} p(x_1, x_2, \dots, x_k, x_{k+1}) &= p(x_{k+1}|x_k, x_{k-1}, \dots, x_1)p(x_k, x_{k-1}, \dots, x_1) \\ &= p(x_{k+1}|x_k, x_{k-1}, \dots, x_1)p(x_k|x_{k-1}, \dots, x_1)p(x_{k-1}|x_{k-2}, \dots, x_1) \dots p(x_1) \end{aligned}$$

Hence the relationship holds for $n = k + 1$, and by induction it holds for any n .

Problem 2.16

1. Let T and R denote channel input and outputs respectively. Using Bayes rule we have

$$\begin{aligned} p(T = 0|R = A) &= \frac{p(T = 0)p(R = A|T = 0)}{p(T = 0)p(R = A|T = 0) + p(T = 1)p(R = A|T = 1)} \\ &= \frac{0.4 \times \frac{1}{6}}{0.4 \times \frac{1}{6} + 0.6 \times \frac{1}{3}} \\ &= \frac{1}{4} \end{aligned}$$

and therefore $p(T = 1|R = A) = \frac{3}{4}$, obviously if $R = A$ is observed, the best decision would be to declare that a 1 was sent, i.e., $T = 1$, because $T = 1$ is more probable than $T = 0$. Similarly it can be verified that $p(T = 0|R = B) = \frac{4}{7}$ and $p(T = 0|R = C) = \frac{1}{4}$. Therefore, when the output is B, the best decision is 0 and when the output is C, the best decision is $T = 1$. Therefore the decision function d can be defined as

$$d(R) = \begin{cases} 1, & R = A \text{ or } C \\ 0, & R = B \end{cases}$$

This is the optimal decision scheme.

2. Here we know that a 0 is transmitted, therefore we are looking for $p(\text{error}|T = 0)$, this is the probability that the receiver declares a 1 was sent when actually a 0 was transmitted. Since by the decision method described in part 1 the receiver declares that a 1 was sent when $R = A$ or $R = C$, therefore, $p(\text{error}|T = 0) = p(R = A|T = 0) + p(R = C|T = 0) = \frac{1}{3}$.
3. We have $p(\text{error}|T = 0) = \frac{1}{3}$, and $p(\text{error}|T = 1) = p(R = B|T = 1) = \frac{1}{3}$. Therefore, by the total probability theorem

$$\begin{aligned} p(\text{error}) &= p(T = 0)p(\text{error}|T = 0) + p(T = 1)p(\text{error}|T = 1) \\ &= 0.4 \times \frac{1}{3} + 0.6 \times \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

Problem 2.17

Following the same procedure as in example 2-1-1, we prove :

$$p_Y(y) = \frac{1}{|a|} p_X\left(\frac{y-b}{a}\right)$$

Problem 2.18

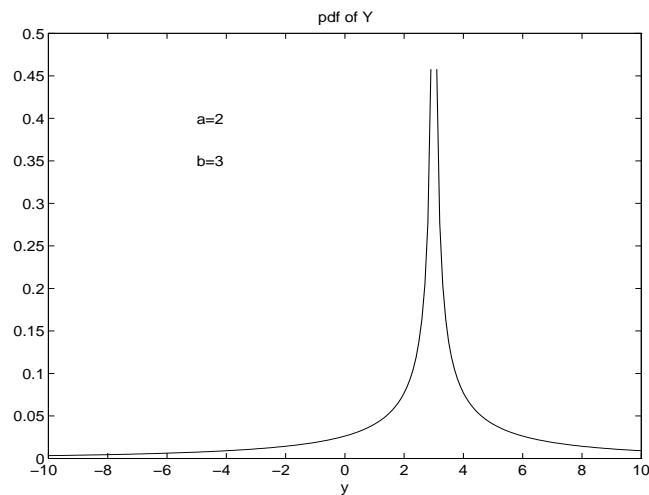
Relationship (2-1-44) gives :

$$p_Y(y) = \frac{1}{3a[(y-b)/a]^{2/3}} p_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

X is a gaussian r.v. with zero mean and unit variance : $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Hence :

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi}[(y-b)/a]^{2/3}} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^{2/3}}$$

**Problem 2.19**

1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $p(X > x) = Q(\frac{x}{\sigma})$ and

$$p(X > 10^{-4}) = Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159$$

$$p(X > 4 \times 10^{-4}) = Q\left(\frac{4 \times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17 \times 10^{-5}$$

$$p(-2 \times 10^{-4} < X \leq 10^{-4}) = 1 - Q(1) - Q(2) = .8182$$

2)

$$p(X > 10^{-4} | X > 0) = \frac{p(X > 10^{-4}, X > 0)}{p(X > 0)} = \frac{p(X > 10^{-4})}{p(X > 0)} = \frac{.159}{.5} = .318$$

Problem 2.20

1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\ &= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}} \end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \leq y \leq b$, then for a fixed y , $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \leq b < y$ for every x ; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned} f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\ &= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} [u_{-1}(y+b) - u_{-1}(y-b)] \end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned} p(Y=b) &= p(X < 0) = F_X(0) = \frac{1}{2} \\ p(Y=a) &= p(X > 0) = 1 - F_X(0) = \frac{1}{2} \end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1 - F_X(0))\delta(y-a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$p(Y = y_n) = p(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^N [F_X(a_{i+1}) - F_X(a_i)] \delta(y - x_i) \\ &= \sum_{i=1}^N \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right) \right] \delta(y - x_i) \end{aligned}$$

Problem 2.21

For n odd, x^n is odd and since the zero-mean Gaussian PDF is even their product is odd. Since the integral of an odd function over the interval $[-\infty, \infty]$ is zero, we obtain $E[X^n] = 0$ for n odd. Let $I_n = \int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2) dx$. Obviously I_n is a constant and its derivative with respect to x is zero, i.e.,

$$\frac{d}{dx} I_n = \int_{-\infty}^{\infty} \left[nx^{n-1} e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{\sigma^2} x^{n+1} e^{-\frac{x^2}{2\sigma^2}} \right] dx = 0$$

which results in the recursion

$$I_{n+1} = n\sigma^2 I_{n-1}$$

This is true for all n . Now let $n = 2k - 1$, we will have $I_{2k} = (2k - 1)\sigma^2 I_{2k-2}$, with the initial condition $I_0 = \sqrt{2\pi\sigma^2}$. Substituting we have

$$\begin{aligned} I_2 &= \sigma^2 \sqrt{2\pi\sigma^2} \\ I_4 &= 3\sigma^2 I_2 = 3\sigma^4 \sqrt{2\pi\sigma^2} \\ I_6 &= 5 \times 3\sigma^2 I_4 = 5 \times 3\sigma^6 \sqrt{2\pi\sigma^2} \\ I_8 &= 7 \times \sigma^2 I_6 = 7 \times 5 \times 3\sigma^8 \sqrt{2\pi\sigma^2} \\ &\vdots \end{aligned}$$

and in general if $I_{2k} = (2k - 1)(2k - 3)(2k - 5) \times \cdots \times 3 \times 1 \sigma^{2k} \sqrt{2\pi\sigma^2}$, then $I_{2k+2} = (2k + 1)\sigma^2 I_{2k} = (2k + 1)(2k - 1)(2k - 3)(2k - 5) \times \cdots \times 3 \times 1 \sigma^{2k+2} \sqrt{2\pi\sigma^2}$. Using the fact that $E[X^{2k}] = I_{2k}/\sqrt{2\pi\sigma^2}$, we obtain

$$I_n = 1 \times 3 \times 5 \times \cdots \times (n - 1) \sigma^n$$

for n even.

Problem 2.22

a. Since (X_r, X_i) are statistically independent :

$$p_{\mathbf{X}}(x_r, x_i) = p_X(x_r)p_X(x_i) = \frac{1}{2\pi\sigma^2}e^{-(x_r^2+x_i^2)/2\sigma^2}$$

Also :

$$\begin{aligned} Y_r + jY_i &= (X_r + X_i)e^{j\phi} \Rightarrow \\ X_r + X_i &= (Y_r + jY_i)e^{-j\phi} = Y_r \cos \phi + Y_i \sin \phi + j(-Y_r \sin \phi + Y_i \cos \phi) \Rightarrow \\ \begin{cases} X_r = Y_r \cos \phi + Y_i \sin \phi \\ X_i = -Y_r \sin \phi + Y_i \cos \phi \end{cases} \end{aligned}$$

The Jacobian of the above transformation is :

$$J = \begin{vmatrix} \frac{\partial X_r}{\partial Y_r} & \frac{\partial X_r}{\partial Y_i} \\ \frac{\partial X_i}{\partial Y_r} & \frac{\partial X_i}{\partial Y_i} \end{vmatrix} = \begin{vmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{vmatrix} = 1$$

Hence, by (2-1-55) :

$$\begin{aligned} p_{\mathbf{Y}}(y_r, y_i) &= p_{\mathbf{X}}((Y_r \cos \phi + Y_i \sin \phi), (-Y_r \sin \phi + Y_i \cos \phi)) \\ &= \frac{1}{2\pi\sigma^2}e^{-(y_r^2+y_i^2)/2\sigma^2} \end{aligned}$$

b. $\mathbf{Y} = \mathbf{A}\mathbf{X}$ and $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$

Now, $p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}}e^{-\mathbf{x}'\mathbf{x}/2\sigma^2}$ (the covariance matrix \mathbf{M} of the random variables x_1, \dots, x_n is $\mathbf{M} = \sigma^2\mathbf{I}$, since they are i.i.d) and $J = 1/|\det(\mathbf{A})|$. Hence :

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \frac{1}{|\det(\mathbf{A})|} e^{-\mathbf{y}'(\mathbf{A}^{-1})'\mathbf{A}^{-1}\mathbf{y}/2\sigma^2}$$

For the pdf's of X and Y to be identical we require that :

$$|\det(\mathbf{A})| = 1 \text{ and } (\mathbf{A}^{-1})'\mathbf{A}^{-1} = \mathbf{I} \implies \mathbf{A}^{-1} = \mathbf{A}'$$

Hence, \mathbf{A} must be a unitary (orthogonal) matrix .

Problem 2.23

Since we are dealing with linear combinations of jointly Gaussian random variables, it is clear that \mathbf{Y} is jointly Gaussian. We clearly have $\mathbf{m}_Y = E[\mathbf{A}\mathbf{X}] = \mathbf{A}\mathbf{m}_X$. This means that $\mathbf{Y} - \mathbf{m}_Y = \mathbf{A}(\mathbf{X} - \mathbf{m}_X)$. Also note that

$$C_Y = E[(\mathbf{Y} - \mathbf{m}_Y)(\mathbf{Y} - \mathbf{m}_Y)'] = E[\mathbf{A}(\mathbf{X} - \mathbf{m}_X)(\mathbf{X} - \mathbf{m}_X)\mathbf{A}']$$

resulting in $C_Y = AC_X A'$.

Problem 2.24

a.

$$\psi_Y(jv) = E[e^{jvY}] = E[e^{jv \sum_{i=1}^n x_i}] = E\left[\prod_{i=1}^n e^{jvx_i}\right] = \prod_{i=1}^n E[e^{jvX}] = (\psi_X(e^{jv}))^n$$

But,

$$p_X(x) = p\delta(x-1) + (1-p)\delta(x) \Rightarrow \psi_X(e^{jv}) = 1 + p + pe^{jv}$$

$$\Rightarrow \psi_Y(jv) = (1 + p + pe^{jv})^n$$

b.

$$E(Y) = -j \frac{d\psi_Y(jv)}{dv} \Big|_{v=0} = -jn(1-p+pe^{jv})^{n-1} jpe^{jv} \Big|_{v=0} = np$$

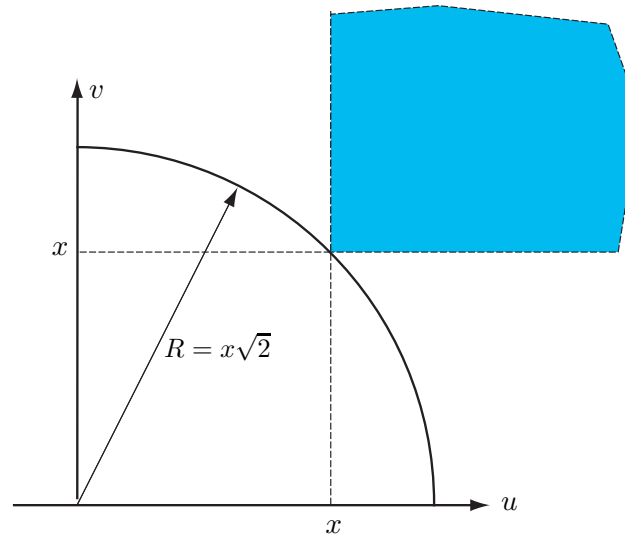
and

$$E(Y^2) = -\frac{d^2\psi_Y(jv)}{d^2v} \Big|_{v=0} = -\frac{d}{dv} [jn(1-p+pe^{jv})^{n-1} jpe^{jv}] \Big|_{v=0} = np + np(n-1)p$$

$$\Rightarrow E(Y^2) = n^2p^2 + np(1-p)$$

Problem 2.25

1. In the figure shown below



let us consider the region $u > x, v > x$ shown as the colored region extending to infinity, call this region \mathcal{R} , and let us integrate $e^{-\frac{u^2+v^2}{2}}$ over this region. We have

$$\begin{aligned} \iint_{\mathcal{R}} e^{-\frac{u^2+v^2}{2}} du dv &= \iint_{\mathcal{R}} e^{-\frac{r^2}{2}} r dr d\theta \\ &\leq \int_{x\sqrt{2}}^{\infty} r e^{-\frac{r^2}{2}} dr \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{\pi}{2} \left[-e^{-\frac{r^2}{2}} \right]_{x\sqrt{2}}^{\infty} \\ &= \frac{\pi}{2} e^{-x^2} \end{aligned}$$

where we have used the fact that region \mathcal{R} is included in the region outside the quarter circle as shown in the figure. On the other hand we have

$$\begin{aligned} \iint_{\mathcal{R}} e^{-\frac{u^2+v^2}{2}} du dv &= \int_x^{\infty} e^{-\frac{u^2}{2}} du \int_x^{\infty} e^{-\frac{v^2}{2}} dv \\ &= \left(\int_x^{\infty} e^{-\frac{u^2}{2}} du \right)^2 \\ &= \left(\sqrt{2\pi} Q(x) \right)^2 \\ &= 2\pi (Q(x))^2 \end{aligned}$$

From the above relations we conclude that

$$2\pi (Q(x))^2 \leq \frac{\pi}{2} e^{-x^2}$$

and therefore, $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$.

2. In $\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2}$ define $u = e^{-\frac{y^2}{2}}$ and $dv = \frac{dy}{y^2}$ and use the integration by parts relation $\int u dv = uv - \int v du$. We have $v = -\frac{1}{y}$ and $du = -ye^{-\frac{y^2}{2}} dy$. Therefore

$$\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2} = \left[-\frac{e^{-\frac{y^2}{2}}}{y} \right]_x^\infty - \int_x^\infty e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x)$$

Now note that $\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2} > 0$ which results in

$$\frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x) > 0 \Rightarrow Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

On the other hand, note that

$$\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2} < \frac{1}{x^2} \int_x^\infty e^{-\frac{y^2}{2}} dy = \frac{\sqrt{2\pi}}{x^2}Q(x)$$

which results in

$$\frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x) < \frac{\sqrt{2\pi}}{x^2}Q(x)$$

or, $\sqrt{2\pi}\frac{1+x^2}{x^2}Q(x) > \frac{e^{-\frac{x^2}{2}}}{x}$ which results in

$$Q(x) > \frac{x}{\sqrt{2\pi}(1+x^2)}e^{-\frac{x^2}{2}}$$

3. From

$$\frac{x}{\sqrt{2\pi}(1+x^2)}e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

we have

$$\frac{1}{\sqrt{2\pi}(\frac{1}{x}+x)}e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

As x becomes large $\frac{1}{x}$ in the denominator of the left hand side becomes small and the two bounds become equal, therefore for large x we have

$$Q(x) \approx \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

Problem 2.26

1. $F_{Y_n}(y) = P[Y_n \leq y] = 1 - P[Y_n > y] = 1 - P[x_1 > y, X_2 > y, \dots, X_n > y] = 1 - (P[X > y])^n$
 where we have used the independence of X_i 's in the last step. But $P[X > y] = \int_y^A \frac{1}{A} dy = \frac{A-y}{A}$.
 Therefore, $F_{Y_n}(y) = 1 - \frac{(A-y)^n}{A^n}$, and $f_{Y_n}(y) = \frac{d}{dy}F_{Y_n}(y) = n\frac{(A-y)^{n-1}}{A^n}$, $0 < y < A$.

2.

$$\begin{aligned} f(y) &= \frac{n}{A} \left(1 - \frac{y}{A}\right)^{n-1} \\ &= \frac{\lambda}{1 - \frac{y}{A}} \left(1 - \frac{ny}{nA}\right)^n \\ &= \frac{\lambda}{1 - \frac{y}{A}} \left(1 - \frac{\lambda y}{n}\right)^n \rightarrow \lambda e^{-\lambda y} \quad y > 0 \end{aligned}$$

Problem 2.27

$$\psi(jv_1, jv_2, jv_3, jv_4) = E \left[e^{j(v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4)} \right]$$

$$E(X_1 X_2 X_3 X_4) = (-j)^4 \frac{\partial^4 \psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4} \Big|_{v_1=v_2=v_3=v_4=0}$$

From (2-1-151) of the text, and the zero-mean property of the given rv's :

$$\psi(j\mathbf{v}) = e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where $\mathbf{v} = [v_1, v_2, v_3, v_4]'$, $\mathbf{M} = [\mu_{ij}]$.

We obtain the desired result by bringing the exponent to a scalar form and then performing quadruple differentiation. We can simplify the procedure by noting that :

$$\frac{\partial \psi(j\mathbf{v})}{\partial v_i} = -\mu'_{i1} \mathbf{v} e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where $\mu'_i = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$. Also note that :

$$\frac{\partial \mu'_{j1} \mathbf{v}}{\partial v_i} = \mu_{ij} = \mu_{ji}$$

Hence :

$$\frac{\partial^4 \psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4} \Big|_{\mathbf{v}=0} = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

Problem 2.28

1) By Chernov bound, for $t > 0$,

$$P[X \geq \alpha] \leq e^{-t\alpha} E[e^{tX}] = e^{-t\alpha} \Theta_X(t)$$

This is true for all $t > 0$, hence

$$\ln P[X \geq \alpha] \leq \min_{t \geq 0} [-t\alpha + \ln \Theta_X(t)] = -\max_{t \geq 0} [t\alpha - \ln \Theta_X(t)]$$

2) Here

$$\ln P[S_n \geq \alpha] = \ln P[Y \geq n\alpha] \leq -\max_{t \geq 0} [tn\alpha - \ln \Theta_Y(t)]$$

where $Y = X_1 + X_2 + \dots + X_n$, and $\Theta_Y(t) = E[e^{X_1+X_2+\dots+X_n}] = [\Theta_X(t)]^n$. Hence,

$$\ln P[S_n \geq \alpha] = -\max_{t \geq 0} n [t\alpha - \ln \Theta_X(t)] = -nI(\alpha) \Rightarrow \frac{1}{n} P[S_n \geq \alpha] \leq e^{-nI(\alpha)}$$

$\Theta_X(t) = \int_0^\infty e^{tx} e^{-x} dx = \frac{1}{1-t}$ as long as $t < 1$. $I(\alpha) = \max_{t \geq 0} (t\alpha + \ln(1-t))$, hence $\frac{d}{dt}(t\alpha + \ln(1-t)) = 0$ and $t^* = \frac{\alpha-1}{\alpha}$. Since $\alpha \geq 0$, $t^* \geq 0$ and also obviously $t^* < 1$. $I(\alpha) = \alpha - 1 + \ln(1 - \frac{\alpha-1}{\alpha}) = \alpha - 1 - \ln \alpha$, using the large deviation theorem

$$\ln P[S_n \geq \alpha] = e^{-n(\alpha-1-\ln \alpha)+o(n)} = \alpha^n e^{-n(\alpha-1)+o(n)}$$

Problem 2.29

For the central chi-square with n degree of freedom :

$$\psi(jv) = \frac{1}{(1 - j2v\sigma^2)^{n/2}}$$

Now :

$$\begin{aligned} \frac{d\psi(jv)}{dv} &= \frac{jn\sigma^2}{(1 - j2v\sigma^2)^{n/2+1}} \Rightarrow E(Y) = -j \frac{d\psi(jv)}{dv} \Big|_{v=0} = n\sigma^2 \\ \frac{d^2\psi(jv)}{dv^2} &= \frac{-2n\sigma^4(n/2+1)}{(1 - j2v\sigma^2)^{n/2+2}} \Rightarrow E(Y^2) = -\frac{d^2\psi(jv)}{dv^2} \Big|_{v=0} = n(n+2)\sigma^2 \end{aligned}$$

The variance is $\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 2n\sigma^4$

For the non-central chi-square with n degrees of freedom :

$$\psi(jv) = \frac{1}{(1 - j2v\sigma^2)^{n/2}} e^{jvs^2/(1-j2v\sigma^2)}$$

where by definition : $s^2 = \sum_{i=1}^n m_i^2$.

$$\frac{d\psi(jv)}{dv} = \left[\frac{jn\sigma^2}{(1-j2v\sigma^2)^{n/2+1}} + \frac{js^2}{(1-j2v\sigma^2)^{n/2+2}} \right] e^{jvs^2/(1-j2v\sigma^2)}$$

Hence, $E(Y) = -j \frac{d\psi(jv)}{dv} \big|_{v=0} = n\sigma^2 + s^2$

$$\frac{d^2\psi(jv)}{dv^2} = \left[\frac{-n\sigma^4(n+2)}{(1-j2v\sigma^2)^{n/2+2}} + \frac{-s^2(n+4)\sigma^2 - ns^2\sigma^2}{(1-j2v\sigma^2)^{n/2+3}} + \frac{-s^4}{(1-j2v\sigma^2)^{n/2+4}} \right] e^{jvs^2/(1-j2v\sigma^2)}$$

Hence,

$$E(Y^2) = -\frac{d^2\psi(jv)}{dv^2} \big|_{v=0} = 2n\sigma^4 + 4s^2\sigma^2 + (n\sigma^2 + s^2)$$

and

$$\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 2n\sigma^4 + 4\sigma^2 s^2$$

Problem 2.30

The Cauchy r.v. has : $p(x) = \frac{a/\pi}{x^2+a^2}$, $-\infty < x < \infty$

a.

$$E(X) = \int_{-\infty}^{\infty} xp(x)dx = 0$$

since $p(x)$ is an even function.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{x^2+a^2} dx$$

Note that for large x , $\frac{x^2}{x^2+a^2} \rightarrow 1$ (i.e non-zero value). Hence,

$$E(X^2) = \infty, \sigma^2 = \infty$$

b.

$$\psi(jv) = E(jvX) = \int_{-\infty}^{\infty} \frac{a/\pi}{x^2+a^2} e^{jvx} dx = \int_{-\infty}^{\infty} \frac{a/\pi}{(x+ja)(x-ja)} e^{jvx} dx$$

This integral can be evaluated by using the residue theorem in complex variable theory. Then, for $v \geq 0$:

$$\psi(jv) = 2\pi j \left(\frac{a/\pi}{x+ja} e^{jvx} \right)_{x=ja} = e^{-av}$$

For $v < 0$:

$$\psi(jv) = -2\pi j \left(\frac{a/\pi}{x-ja} e^{jvx} \right)_{x=-ja} = e^{av}$$

Therefore :

$$\psi(jv) = e^{-a|v|}$$

Note: an alternative way to find the characteristic function is to use the Fourier transform relationship between $p(x), \psi(jv)$ and the Fourier pair :

$$e^{-b|t|} \leftrightarrow \frac{1}{\pi} \frac{c}{c^2 + f^2}, \quad c = b/2\pi, \quad f = 2\pi v$$

Problem 2.31

Since R_0 and R_1 are independent $f_{R_0, R_1}(r_0, r_1) = f_{R_0}(r_0)f_{R_1}(r_1)$ and

$$f_{R_0, R_1}(r_0, r_1) = \begin{cases} \frac{r_0 r_1}{\sigma^4} I_0\left(\frac{\mu r_1}{\sigma^2}\right) e^{-\frac{\mu^2}{2\sigma^2}} e^{-\frac{r_1^2 + r_0^2}{2\sigma^2}}, & r_0, r_1 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} P(R_0 > R_1) &= \iint_{r_0 > r_1} f(r_0, r_1) dr_1 dr_0 \\ &= \int_0^\infty dr_1 \int_{r_1}^\infty f(r_0, r_1) dr_0 \\ &= \int_0^\infty f_{R_1}(r_1) \left(\int_{r_1}^\infty f_{R_0}(r_0) dr_0 \right) dr_1 \\ &= \int_0^\infty f_{R_1}(r_1) \left(\int_{r_1}^\infty \frac{r_0}{\sigma^2} e^{-\frac{r_0^2}{2\sigma^2}} dr_0 \right) dr_1 \\ &= \int_0^\infty f_{R_1}(r_1) \left[-e^{-\frac{r_0^2}{2\sigma^2}} \right]_{r_1}^\infty dr_1 \\ &= \int_0^\infty e^{-\frac{r_1^2}{2\sigma^2}} f_{R_1}(r_1) dr_1 \\ &= \int_0^\infty \frac{r_1}{\sigma^2} I_0\left(\frac{\mu r_1}{\sigma^2}\right) e^{-\frac{\mu^2 + 2r_1^2}{2\sigma^2}} dr_1 \end{aligned}$$

Now using the change of variable $y = \sqrt{2}r_1$ and letting $s = \frac{\mu}{\sqrt{2}}$ we obtain

$$\begin{aligned} P(R_0 > R_1) &= \int_0^\infty \frac{y}{\sqrt{2}\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{2s^2 + y^2}{2\sigma^2}} \frac{dy}{\sqrt{2}} \\ &= \frac{1}{2} e^{-\frac{s^2}{2\sigma^2}} \int_0^\infty \frac{y}{\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{s^2 + y^2}{2\sigma^2}} dy \\ &= \frac{1}{2} e^{-\frac{s^2}{2\sigma^2}} \\ &= \frac{1}{2} e^{-\frac{\mu^2}{4\sigma^2}} \end{aligned}$$

where we have used the fact that $\int_0^\infty \frac{y}{\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{s^2+y^2}{2\sigma^2}} dy = 1$ because it is the integral of a Rician pdf.

Problem 2.32

1. The joint pdf of a, b is :

$$p_{ab}(a, b) = p_{xy}(a - m_r, b - m_i) = p_x(a - m_r)p_y(b - m_i) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(a-m_r)^2+(b-m_i)^2]}$$

2. $u = \sqrt{a^2 + b^2}$, $\phi = \tan^{-1}b/a \Rightarrow a = u \cos \phi$, $b = u \sin \phi$ The Jacobian of the transformation is
 $: J(a, b) = \begin{vmatrix} \partial a / \partial u & \partial a / \partial \phi \\ \partial b / \partial u & \partial b / \partial \phi \end{vmatrix} = u$, hence :

$$\begin{aligned} p_{u\phi}(u, \phi) &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(u \cos \phi - m_r)^2 + (u \sin \phi - m_i)^2]} \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[u^2 + M^2 - 2uM \cos(\phi - \theta)]} \end{aligned}$$

where we have used the transformation :

$$\left\{ \begin{array}{l} M = \sqrt{m_r^2 + m_i^2} \\ \theta = \tan^{-1}m_i/m_r \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} m_r = M \cos \theta \\ m_i = M \sin \theta \end{array} \right\}$$

3.

$$\begin{aligned} p_u(u) &= \int_0^{2\pi} p_{u\phi}(u, \phi) d\phi \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} \int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[-2uM \cos(\phi-\theta)]} d\phi \\ &= \frac{u}{\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{uM \cos(\phi-\theta)/\sigma^2} d\phi \\ &= \frac{u}{\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} I_0(uM/\sigma^2) \end{aligned}$$

Problem 2.33

a. $Y = \frac{1}{n} \sum_{i=1}^n X_i$, $\psi_{X_i}(jv) = e^{-a|v|}$

$$\psi_Y(jv) = E \left[e^{jv \frac{1}{n} \sum_{i=1}^n X_i} \right] = \prod_{i=1}^n E \left[e^{j \frac{v}{n} X_i} \right] = \prod_{i=1}^n \psi_{X_i}(jv/n) = \left[e^{-a|v|/n} \right]^n = e^{-a|v|}$$

b. Since $\psi_Y(jv) = \psi_{X_i}(jv) \Rightarrow p_Y(y) = p_{X_i}(x_i) \Rightarrow p_Y(y) = \frac{a/\pi}{y^2+a^2}$.

c. As $n \rightarrow \infty$, $p_Y(y) = \frac{a/\pi}{y^2+a^2}$, which is not Gaussian ; hence, the central limit theorem does not hold. The reason is that the Cauchy distribution does not have a finite variance.

Problem 2.34

Since \mathbf{Z} and $\mathbf{Z}e^{j\theta}$ have the same pdf, we have $E[\mathbf{Z}] = E[\mathbf{Z}e^{j\theta}] = e^{j\theta}E[\mathbf{Z}]$ for all θ . Putting $\theta = \pi$ gives $E[\mathbf{Z}] = \mathbf{0}$. We also have $E[\mathbf{Z}\mathbf{Z}^t] = E[\mathbf{Z}e^{j\theta}(\mathbf{Z}e^{j\theta})^t]$ or $E[\mathbf{Z}\mathbf{Z}^t] = e^{2j\theta}E[\mathbf{Z}\mathbf{Z}^t]$, for all θ . Putting $\theta = \frac{\pi}{2}$ gives $E[\mathbf{Z}\mathbf{Z}^t] = \mathbf{0}$. Since \mathbf{Z} is zero-mean and $E[\mathbf{Z}\mathbf{Z}^t] = \mathbf{0}$, we conclude that it is proper.

Problem 2.35

Using Equation 2.6-29 we note that for the zero-mean proper case if $\mathbf{W} = e^{j\theta}\mathbf{Z}$, it is sufficient to show that $\det(\mathbf{C}_{\mathbf{W}}) = \det(\mathbf{C}_{\mathbf{Z}})$ and $\mathbf{w}^H \mathbf{C}_{\mathbf{W}}^{-1} \mathbf{w} = \mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1} \mathbf{z}$. But $\mathbf{C}_{\mathbf{W}} = [\mathbf{W}\mathbf{W}^H] = E[e^{j\theta}\mathbf{Z}e^{-j\theta}\mathbf{Z}^H] = E[\mathbf{Z}\mathbf{Z}^H] = \mathbf{C}_{\mathbf{Z}}$, hence $\det(\mathbf{C}_{\mathbf{W}}) = \det(\mathbf{C}_{\mathbf{Z}})$. Similarly, $\mathbf{w}^H \mathbf{C}_{\mathbf{W}}^{-1} \mathbf{w} = e^{-j\theta} \mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1} \mathbf{z} e^{j\theta} = \mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1} \mathbf{z}$. Substituting into Equation 2.6-29, we conclude that $p(\mathbf{w}) = p(\mathbf{z})$.

Problem 2.36

Since \mathbf{Z} is proper, we have $E[(\mathbf{Z} - E(\mathbf{Z}))(\mathbf{Z} - E(\mathbf{Z}))^t] = \mathbf{0}$. Let $\mathbf{W} = \mathbf{A}\mathbf{Z} + \mathbf{b}$, then

$$E[(\mathbf{W} - E(\mathbf{W}))(\mathbf{W} - E(\mathbf{W}))^t] = \mathbf{A}E[(\mathbf{Z} - E(\mathbf{Z}))(\mathbf{Z} - E(\mathbf{Z}))^t]\mathbf{A}^t = \mathbf{0}$$

hence \mathbf{W} is proper.

Problem 2.37

We assume that $x(t), y(t), z(t)$ are real-valued stochastic processes. The treatment of complex-valued processes is similar.

a.

$$R_{zz}(\tau) = E \{ [x(t + \tau) + y(t + \tau)] [x(t) + y(t)] \} = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

b. When $x(t), y(t)$ are uncorrelated :

$$R_{xy}(\tau) = E [x(t + \tau)y(t)] = E [x(t + \tau)] E [y(t)] = m_x m_y$$

Similarly :

$$R_{yx}(\tau) = m_x m_y$$

Hence :

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + 2m_x m_y$$

c. When $x(t), y(t)$ are uncorrelated and have zero means :

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau)$$

Problem 2.38

The power spectral density of the random process $x(t)$ is :

$$\mathcal{S}_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau = N_0/2.$$

The power spectral density at the output of the filter will be :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f) |H(f)|^2 = \frac{N_0}{2} |H(f)|^2$$

Hence, the total power at the output of the filter will be :

$$R_{yy}(\tau = 0) = \int_{-\infty}^{\infty} \mathcal{S}_{yy}(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2} (2B) = N_0 B$$

Problem 2.39

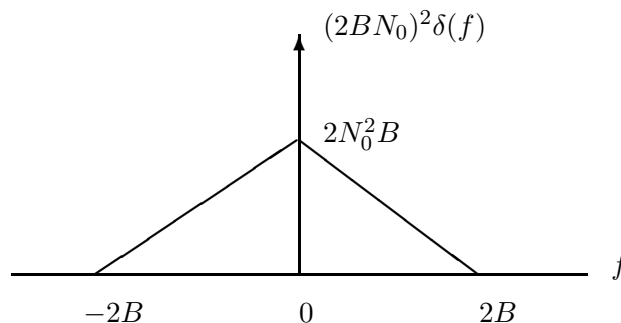
The power spectral density of $X(t)$ corresponds to : $R_{xx}(t) = 2BN_0 \frac{\sin 2\pi Bt}{2\pi Bt}$. From the result of Problem 2.14 :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau) = (2BN_0)^2 + 8B^2 N_0^2 \left(\frac{\sin 2\pi Bt}{2\pi Bt} \right)^2$$

Also :

$$\mathcal{S}_{yy}(f) = R_{xx}^2(0)\delta(f) + 2\mathcal{S}_{xx}(f) * \mathcal{S}_{xx}(f)$$

The following figure shows the power spectral density of $Y(t)$:



Problem 2.40

$$\mathbf{M}_X = E[(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)'], \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \mathbf{m}_x \text{ is the corresponding vector of mean values.}$$

Then :

$$\begin{aligned} \mathbf{M}_Y &= E[(\mathbf{Y} - \mathbf{m}_y)(\mathbf{Y} - \mathbf{m}_y)'] \\ &= E[\mathbf{A}(\mathbf{X} - \mathbf{m}_x)(\mathbf{A}(\mathbf{X} - \mathbf{m}_x))'] \\ &= E[\mathbf{A}(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)' \mathbf{A}'] \\ &= \mathbf{A} E[(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)'] \mathbf{A}' \\ &= \mathbf{A} \mathbf{M}_x \mathbf{A}' \end{aligned}$$

Hence :

$$\mathbf{M}_Y = \begin{bmatrix} \mu_{11} & 0 & \mu_{11} + \mu_{13} \\ 0 & 4\mu_{22} & 0 \\ \mu_{11} + \mu_{31} & 0 & \mu_{11} + \mu_{13} + \mu_{31} + \mu_{33} \end{bmatrix}$$

Problem 2.41

$$Y(t) = X^2(t), \quad R_{xx}(\tau) = E[x(t+\tau)x(t)]$$

$$R_{yy}(\tau) = E[y(t+\tau)y(t)] = E[x^2(t+\tau)x^2(t)]$$

Let $X_1 = X_2 = x(t)$, $X_3 = X_4 = x(t+\tau)$. Then, from problem 2.7 :

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

Hence :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

Problem 2.42

$$p_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-mr^2/\Omega}, \quad X = \frac{1}{\sqrt{\Omega}}R$$

$$\text{We know that : } p_X(x) = \frac{1}{1/\sqrt{\Omega}} p_R\left(\frac{x}{1/\sqrt{\Omega}}\right).$$

Hence :

$$p_X(x) = \frac{1}{1/\sqrt{\Omega}} \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \left(x\sqrt{\Omega}\right)^{2m-1} e^{-m(x\sqrt{\Omega})^2/\Omega} = \frac{2}{\Gamma(m)} m^m x^{2m-1} e^{-mx^2}$$

Problem 2.43

The transfer function of the filter is :

$$H(f) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j2\pi fRC + 1}$$

a.

$$\mathcal{S}_{xx}(f) = \sigma^2 \Rightarrow \mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f) |H(f)|^2 = \frac{\sigma^2}{(2\pi RC)^2 f^2 + 1}$$

b.

$$R_{yy}(\tau) = F^{-1}\{\mathcal{S}_{xx}(f)\} = \frac{\sigma^2}{RC} \int_{-\infty}^{\infty} \frac{\frac{1}{RC}}{(\frac{1}{RC})^2 + (2\pi f)^2} e^{j2\pi f \tau} df$$

Let : $a = RC$, $v = 2\pi f$. Then :

$$R_{yy}(\tau) = \frac{\sigma^2}{2RC} \int_{-\infty}^{\infty} \frac{a/\pi}{a^2 + v^2} e^{jv\tau} dv = \frac{\sigma^2}{2RC} e^{-a|\tau|} = \frac{\sigma^2}{2RC} e^{-|\tau|/RC}$$

where the last integral is evaluated in the same way as in problem P-2.9 . Finally :

$$E[Y^2(t)] = R_{yy}(0) = \frac{\sigma^2}{2RC}$$

Problem 2.44

If $\mathcal{S}_X(f) = 0$ for $|f| > W$, then $\mathcal{S}_X(f)e^{-j2\pi fa}$ is also bandlimited. The corresponding autocorrelation function can be represented as (remember that $\mathcal{S}_X(f)$ is deterministic) :

$$R_X(\tau - a) = \sum_{n=-\infty}^{\infty} R_X\left(\frac{n}{2W} - a\right) \frac{\sin 2\pi W \left(\tau - \frac{n}{2W}\right)}{2\pi W \left(\tau - \frac{n}{2W}\right)} \quad (1)$$

Let us define :

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin 2\pi W \left(t - \frac{n}{2W}\right)}{2\pi W \left(t - \frac{n}{2W}\right)}$$

We must show that :

$$E\left[|X(t) - \hat{X}(t)|^2\right] = 0$$

or

$$E\left[\left(X(t) - \hat{X}(t)\right) \left(X(t) - \sum_{m=-\infty}^{\infty} X\left(\frac{m}{2W}\right) \frac{\sin 2\pi W \left(t - \frac{m}{2W}\right)}{2\pi W \left(t - \frac{m}{2W}\right)}\right)\right] = 0 \quad (2)$$

First we have :

$$E\left[\left(X(t) - \hat{X}(t)\right) X\left(\frac{m}{2W}\right)\right] = R_X\left(t - \frac{m}{2W}\right) - \sum_{n=-\infty}^{\infty} R_X\left(\frac{n-m}{2W}\right) \frac{\sin 2\pi W \left(t - \frac{n}{2W}\right)}{2\pi W \left(t - \frac{n}{2W}\right)}$$

But the right-hand-side of this equation is equal to zero by application of (1) with $a = m/2W$. Since this is true for any m , it follows that $E \left[\left(X(t) - \hat{X}(t) \right) \hat{X}(t) \right] = 0$. Also

$$E \left[\left(X(t) - \hat{X}(t) \right) X(t) \right] = R_X(0) - \sum_{n=-\infty}^{\infty} R_X\left(\frac{n}{2W} - t\right) \frac{\sin 2\pi W \left(t - \frac{n}{2W}\right)}{2\pi W \left(t - \frac{n}{2W}\right)}$$

Again, by applying (1) with $a = t$ and $\tau = t$, we observe that the right-hand-side of the equation is also zero. Hence (2) holds.

Problem 2.45

$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt = P[N \geq x]$, where N is a Gaussian r.v with zero mean and unit variance. From the Chernoff bound :

$$P[N \geq x] \leq e^{-\hat{v}x} E(e^{\hat{v}N}) \quad (1)$$

where \hat{v} is the solution to :

$$E(Ne^{vN}) - xE(e^{vN}) = 0 \quad (2)$$

Now :

$$\begin{aligned} E(e^{vN}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{vt} e^{-t^2/2} dt \\ &= e^{v^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t-v)^2/2} dt \\ &= e^{v^2/2} \end{aligned}$$

and

$$E(Ne^{vN}) = \frac{d}{dv} E(e^{vN}) = ve^{v^2/2}$$

Hence (2) gives :

$$\hat{v} = x$$

and then :

$$(1) \Rightarrow Q(x) \leq e^{-x^2} e^{x^2/2} \Rightarrow Q(x) \leq e^{-x^2/2}$$

Problem 2.46

Since $H(0) = \sum_{n=-\infty}^{\infty} h(n) = 0 \Rightarrow m_y = m_x H(0) = 0$

The autocorrelation of the output sequence is

$$R_{yy}(k) = \sum_i \sum_j h(i)h(j)R_{xx}(k-j+i) = \sigma_x^2 \sum_{i=-\infty}^{\infty} h(i)h(k+i)$$

where the last equality stems from the autocorrelation function of $X(n)$:

$$R_{xx}(k-j+i) = \sigma_x^2 \delta(k-j+i) = \begin{cases} \sigma_x^2, & j = k+i \\ 0, & o.w. \end{cases}$$

Hence, $R_{yy}(0) = 6\sigma_x^2$, $R_{yy}(1) = R_{yy}(-1) = -4\sigma_x^2$, $R_{yy}(2) = R_{yy}(-2) = \sigma_x^2$, $R_{yy}(k) = 0$ otherwise.

Finally, the frequency response of the discrete-time system is :

$$\begin{aligned} H(f) &= \sum_{-\infty}^{\infty} h(n)e^{-j2\pi fn} \\ &= 1 - 2e^{-j2\pi f} + e^{-j4\pi f} \\ &= (1 - e^{-j2\pi f})^2 \\ &= e^{-j2\pi f} (e^{j\pi f} - e^{-j\pi f})^2 \\ &= -4e^{-j\pi f} \sin^2 \pi f \end{aligned}$$

which gives the power density spectrum of the output :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f)|H(f)|^2 = \sigma_x^2 [16 \sin^4 \pi f] = 16\sigma_x^2 \sin^4 \pi f$$

Problem 2.47

$$R(k) = \left(\frac{1}{2}\right)^{|k|}$$

The power density spectrum is

$$\begin{aligned}
 \mathcal{S}(f) &= \sum_{k=-\infty}^{\infty} R(k) e^{-j2\pi f k} \\
 &= \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} e^{-j2\pi f k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{-j2\pi f k} \\
 &= \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{j2\pi f}\right)^k + \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{-j2\pi f}\right)^k - 1 \\
 &= \frac{1}{1 - e^{j2\pi f}/2} + \frac{1}{1 - e^{-j2\pi f}/2} - 1 \\
 &= \frac{2 - \cos 2\pi f}{5/4 - \cos 2\pi f} - 1 \\
 &= \frac{3}{5 - 4 \cos 2\pi f}
 \end{aligned}$$

Problem 2.48

We will denote the discrete-time process by the subscript d and the continuous-time (analog) process by the subscript a . Also, f will denote the analog frequency and f_d the discrete-time frequency.

a.

$$\begin{aligned}
 R_d(k) &= E[X^*(n)X(n+k)] \\
 &= E[X^*(nT)X(nT+kT)] \\
 &= R_a(kT)
 \end{aligned}$$

Hence, the autocorrelation function of the sampled signal is equal to the sampled autocorrelation function of $X(t)$.

b.

$$\begin{aligned}
 R_d(k) &= R_a(kT) = \int_{-\infty}^{\infty} \mathcal{S}_a(F) e^{j2\pi f k T} df \\
 &= \sum_{l=-\infty}^{\infty} \int_{(2l-1)/2T}^{(2l+1)/2T} \mathcal{S}_a(F) e^{j2\pi f k T} df \\
 &= \sum_{l=-\infty}^{\infty} \int_{-1/2T}^{1/2T} \mathcal{S}_a\left(f + \frac{l}{T}\right) e^{j2\pi F k T} df \\
 &= \int_{-1/2T}^{1/2T} \left[\sum_{l=-\infty}^{\infty} \mathcal{S}_a\left(f + \frac{l}{T}\right) \right] e^{j2\pi F k T} df
 \end{aligned}$$

Let $f_d = fT$. Then :

$$R_d(k) = \int_{-1/2}^{1/2} \left[\frac{1}{T} \sum_{l=-\infty}^{\infty} \mathcal{S}_a((f_d + l)/T) \right] e^{j2\pi f_d k} df_d \quad (1)$$

We know that the autocorrelation function of a discrete-time process is the inverse Fourier transform of its power spectral density

$$R_d(k) = \int_{-1/2}^{1/2} \mathcal{S}_d(f_d) e^{j2\pi f_d k} df_d \quad (2)$$

Comparing (1),(2) :

$$\mathcal{S}_d(f_d) = \frac{1}{T} \sum_{l=-\infty}^{\infty} \mathcal{S}_a\left(\frac{f_d + l}{T}\right) \quad (3)$$

c. From (3) we conclude that :

$$\mathcal{S}_d(f_d) = \frac{1}{T} \mathcal{S}_a\left(\frac{f_d}{T}\right)$$

iff :

$$\mathcal{S}_a(f) = 0, \quad \forall f : |f| > 1/2T$$

Otherwise, the sum of the shifted copies of \mathcal{S}_a (in (3)) will overlap and aliasing will occur.

Problem 2.49

$$u(t) = X \cos 2\pi ft - Y \sin 2\pi ft$$

$$E[u(t)] = E(X) \cos 2\pi ft - E(Y) \sin 2\pi ft$$

and :

$$\begin{aligned} R_{uu}(t, t + \tau) &= E \{ [X \cos 2\pi ft - Y \sin 2\pi ft] [X \cos 2\pi f(t + \tau) - Y \sin 2\pi f(t + \tau)] \} \\ &= E(X^2) [\cos 2\pi f(2t + \tau) + \cos 2\pi f\tau] + E(Y^2) [-\cos 2\pi f(2t + \tau) + \cos 2\pi f\tau] \\ &\quad - E(XY) \sin 2\pi f(2t + \tau) \end{aligned}$$

For $u(t)$ to be wide-sense stationary, we must have : $E[u(t)] = \text{constant}$ and $R_{uu}(t, t + \tau) = R_{uu}(\tau)$. We note that if $E(X) = E(Y) = 0$, and $E(XY) = 0$ and $E(X^2) = E(Y^2)$, then the above requirements for WSS hold; hence these conditions are necessary. Conversely, if any of the above

conditions does not hold, then either $E[u(t)] \neq \text{constant}$, or $R_{uu}(t, t + \tau) \neq R_{uu}(\tau)$. Hence, the conditions are also necessary.

Problem 2.50

a.

$$\begin{aligned} R_a(\tau) &= \int_{-\infty}^{\infty} \mathcal{S}_a(f) e^{j2\pi f\tau} df \\ &= \int_{-W}^W e^{j2\pi f\tau} df \\ &= \frac{\sin 2\pi W\tau}{\pi\tau} \end{aligned}$$

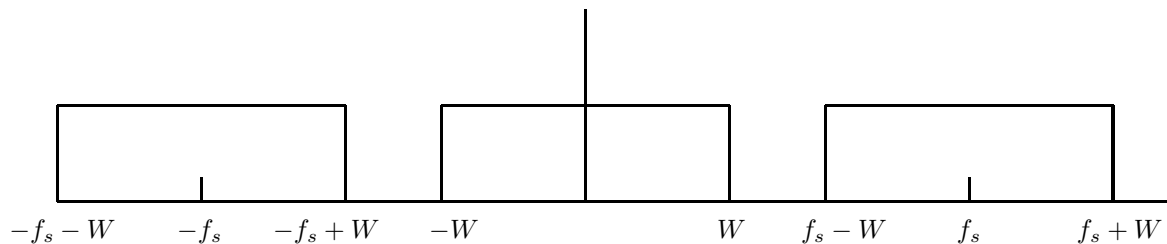
By applying the result in problem 2.21, we have

$$R_d(k) = f_a(kT) = \frac{\sin 2\pi W kT}{\pi kT}$$

b. If $T = \frac{1}{2W}$, then :

$$R_d(k) = \begin{cases} 2W = 1/T, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the sequence $X(n)$ is a white-noise sequence. The fact that this is the minimum value of T can be shown from the following figure of the power spectral density of the sampled process:



We see that the maximum sampling rate f_s that gives a spectrally flat sequence is obtained when :

$$W = f_s - W \Rightarrow f_s = 2W \Rightarrow T = \frac{1}{2W}$$

c. The triangular-shaped spectrum $\mathcal{S}(f) = 1 - \frac{|f|}{W}$, $|f| \leq W$ may be obtained by convolving the rectangular-shaped spectrum $\mathcal{S}_1(f) = 1/\sqrt{W}$, $|f| \leq W/2$. Hence, $R(\tau) = R_1^2(\tau) = \frac{1}{W} \left(\frac{\sin \pi W \tau}{\pi \tau} \right)^2$. Therefore, sampling $X(t)$ at a rate $\frac{1}{T} = W$ samples/sec produces a white sequence with autocorrelation function :

$$R_d(k) = \frac{1}{W} \left(\frac{\sin \pi W k T}{\pi k T} \right)^2 = W \left(\frac{\sin \pi k}{\pi k} \right)^2 = \begin{cases} W, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 2.51

Let's denote : $y(t) = f_k(t)f_j(t)$. Then :

$$\int_{-\infty}^{\infty} f_k(t)f_j(t)dt = \int_{-\infty}^{\infty} y(t)dt = Y(f)|_{f=0}$$

where $Y(f)$ is the Fourier transform of $y(t)$. Since : $y(t) = f_k(t)f_j(t) \longleftrightarrow Y(f) = F_k(f) * F_j(f)$. But :

$$F_k(f) = \int_{-\infty}^{\infty} f_k(t)e^{-j2\pi ft}dt = \frac{1}{2W}e^{-j2\pi fk/2W}$$

Then :

$$Y(f) = F_k(f) * F_j(f) = \int_{-\infty}^{\infty} F_k(a) * F_j(f - a)da$$

and at $f = 0$:

$$\begin{aligned} Y(f)|_{f=0} &= \int_{-\infty}^{\infty} F_k(a) * F_j(-a)da \\ &= \left(\frac{1}{2W} \right)^2 \int_{-\infty}^{\infty} e^{-j2\pi a(k-j)/2W} da \\ &= \begin{cases} 1/2W, & k = j \\ 0, & k \neq j \end{cases} \end{aligned}$$

Problem 2.52

$$B_{eq} = \frac{1}{G} \int_0^{\infty} |H(f)|^2 df$$

For the filter shown in Fig. P2-12 we have $G = 1$ and

$$B_{eq} = \int_0^{\infty} |H(f)|^2 df = B$$

For the lowpass filter shown in Fig. P2-16 we have

$$H(f) = \frac{1}{1 + j2\pi fRC} \Rightarrow |H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}$$

So $G = 1$ and

$$\begin{aligned} B_{eq} &= \int_0^\infty |H(f)|^2 df \\ &= \frac{1}{2} \int_{-\infty}^\infty |H(f)|^2 df \\ &= \frac{1}{4RC} \end{aligned}$$

where the last integral is evaluated in the same way as in problem P-2.9 .

Problem 2.53

a.

$$\begin{aligned} E[z(t)z(t+\tau)] &= E[\{x(t+\tau) + jy(t+\tau)\} \{x(t) + jy(t)\}] \\ &= E[x(t)x(t+\tau)] - E[y(t)y(t+\tau)] + jE[x(t)y(t+\tau)] \\ &\quad + E[y(t)x(t+\tau)] \\ &= R_{xx}(\tau) - R_{yy}(\tau) + j[R_{yx}(\tau) + R_{xy}(\tau)] \end{aligned}$$

But $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{yx}(\tau) = -R_{xy}(\tau)$. Therefore :

$$E[z(t)z(t+\tau)] = 0$$

b.

$$\begin{aligned} V &= \int_0^T z(t) dt \\ E(V^2) &= \int_0^T \int_0^T E[z(a)z(b)] da db = 0 \end{aligned}$$

from the result in (a) above. Also :

$$\begin{aligned} E(VV^*) &= \int_0^T \int_0^T E[z(a)z^*(b)] da db \\ &= \int_0^T \int_0^T N_0 \delta(a-b) da db \\ &= \int_0^T N_0 da = N_0 T \end{aligned}$$

Problem 2.54

$$\begin{aligned}
E[x(t+\tau)x(t)] &= A^2 E[\sin(2\pi f_c(t+\tau) + \theta) \sin(2\pi f_c t + \theta)] \\
&= \frac{A^2}{2} \cos 2\pi f_c \tau - \frac{A^2}{2} E[\cos(2\pi f_c(2t + \tau) + 2\theta)]
\end{aligned}$$

where the last equality follows from the trigonometric identity :

$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$. But :

$$\begin{aligned}
E[\cos(2\pi f_c(2t + \tau) + 2\theta)] &= \int_0^{2\pi} \cos(2\pi f_c(2t + \tau) + 2\theta) p(\theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi f_c(2t + \tau) + 2\theta) d\theta = 0
\end{aligned}$$

Hence :

$$E[x(t+\tau)x(t)] = \frac{A^2}{2} \cos 2\pi f_c \tau$$

Problem 2.55

1) We have $E[Z(t)] = E[X(t)] + jE[Y(t)] = 0 + j0 = 0$ and

$$\begin{aligned}
R_Z(t+\tau, t) &= E[(X(t+\tau) + jY(t+\tau))(X(t) - jY(t))] \\
&= R_X(\tau) + R_Y(\tau) \\
&= 2R_X(\tau)
\end{aligned}$$

because $E[X(t+\tau)Y(t)] = E[Y(t+\tau)X(t)] = E[X(t+\tau)]E[Y(t)] = 0$ (by independence) and therefore $Z(t)$ is obviously stationary. We also note that $R_X(\tau) = R_Y(\tau) = \mathcal{F}^{-1} \left[N_0 \Pi \left(\frac{f}{2W} \right) \right] = 2WN_0 \text{sinc}(2W\tau)$

2) To compute the power spectral density of $Z(t)$, we have $\mathcal{S}_Z(f) = \mathcal{F}[2R_X(\tau)] = 2\mathcal{S}_X(f) = 2N_0 \Pi \left(\frac{f}{2W} \right)$. Note that $\Pi(t)$ is a rectangular pulse defined as

$$\Pi(t) = \begin{cases} 1, & |t| < 1 \\ \frac{1}{2}, & |t| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

3) $E[Z_j] = E \left[\int_{-\infty}^{\infty} Z(t) R_j^*(t) dt \right] = \int_{-\infty}^{\infty} E[Z(t)] R_j^*(t) dt = 0$ since $Z(t)$ is zero-mean. For the

correlation we have

$$\begin{aligned}
 E[Z_j Z_k^*] &= E \left[\int_{-\infty}^{\infty} Z(s) R_j^*(s) ds \int_{-\infty}^{\infty} Z^*(t) R_k(t) dt \right] \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) R_k(t) ds dt \\
 &= \int_{-\infty}^{\infty} R_k(t) \left[\int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) ds \right] dt \quad (**)
 \end{aligned}$$

Using Parseval's Theorem, $\int_{-\infty}^{\infty} x(t)y^*(t) dt = \int_{-\infty}^{\infty} X(f)Y^*(f) df$, we have $(\mathcal{S}_j(f))$ is the Fourier transform of $R_j(t)$.

$$\begin{aligned}
 \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) ds &= \int_{-\infty}^{\infty} e^{-j2\pi ft} 2N_0 \Pi\left(\frac{f}{2W}\right) \mathcal{S}_j^*(f) df \\
 &\stackrel{a}{=} 2 \int_{-W}^W N_0 e^{-j2\pi ft} \mathcal{S}_j^*(f) df \\
 &\stackrel{b}{=} 2 \int_{-\infty}^{\infty} N_0 e^{-j2\pi ft} \mathcal{S}_j^*(f) df
 \end{aligned}$$

where (a) is due to the fact that $\Pi\left(\frac{f}{2W}\right)$ is zero outside the $[-W, W]$ interval and (b) follows from $R_j(t)$ being bandlimited to $[-W, W]$. From above we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) ds &= 2N_0 \left[\int_{-\infty}^{\infty} e^{j2\pi ft} \mathcal{S}_j(f) df \right]^* \\
 &= 2N_0 R_j^*(t)
 \end{aligned}$$

Substituting this result in equation (**) we have

$$\begin{aligned}
 E[Z_j Z_k^*] &= 2 \int_{-\infty}^{\infty} N_0 R_j^*(t) R_k(t) dt \\
 &= \begin{cases} 2N_0, & j = k \\ 0, & j \neq k \end{cases}
 \end{aligned}$$

This shows that Z_j 's are Gaussian random variables (since they are the result of linear operation on a Gaussian process) with mean zero and variance $2N_0$, i.e., $Z_j \sim \mathcal{N}(0, 2N_0)$. Also note that for $j \neq k$, Z_j and Z_k are independent since they are Gaussian and uncorrelated.

4) This is done similar to part 3 (lengthy but straightforward) and the result is that for any k , Z_{kr} and Z_{ki} are zero-mean, independent Gaussian random variables with $E(Z_{kr}^2) = E(Z_{ki}^2) = N_0$ and therefore the random vector $(Z_{1r}, Z_{1i}, Z_{2r}, Z_{2i}, \dots, Z_{nr}, Z_{ni})$ is a $2n$ -dimensional Gaussian vector with independent zero-mean components each having variance N_0 . In standard notation

$$(Z_{1r}, Z_{1i}, Z_{2r}, Z_{2i}, \dots, Z_{nr}, Z_{ni}) \sim \mathcal{N}(\mathbf{0}, N_0 \mathbf{I})$$

where $\mathbf{0}$ is a $2n$ -dimensional zero vector and \mathbf{I} is a $2n \times 2n$ identity matrix.

5) We have

$$\begin{aligned} E[\hat{Z}(t)Z_k^*] &= E[(Z(t) - \sum_{j=1}^N Z_j R_j(t))Z_k^*] \\ &= E[Z(t)Z_k^*] - 2N_0 R_k(t) \end{aligned}$$

where we have used

$$E[Z_j Z_k^*] = \begin{cases} 2N_0, & j = k \\ 0, & j \neq k \end{cases}$$

Now we have

$$\begin{aligned} E[Z(t)Z_k^*] &= E \left[Z(t) \int_{-\infty}^{\infty} Z^*(s) R_k(s) ds \right] \\ &= \int_{-\infty}^{\infty} R_Z(t-s) R_k(s) ds \\ &= \int_{-\infty}^{\infty} R_k(s) R_Z^*(s-t) ds \\ &= 2 \int_{-\infty}^{\infty} \mathcal{S}_k(f) e^{j2\pi f t} N_0 \Pi \left(\frac{f}{2W} \right) df \\ &= \int_{-W}^W 2N_0 \mathcal{S}_k(f) e^{j2\pi f t} dt \\ &\stackrel{a}{=} 2N_0 \int_{-\infty}^{\infty} \mathcal{S}_k(f) e^{j2\pi f t} df \\ &= 2N_0 R_k(t) \end{aligned}$$

(a): because $R_k(t)$ is bandlimited to $[-W, W]$.

From above it follows that $E[\hat{Z}(t)Z_k^*] = 0$ for all $k = 1, 2, \dots, N$. This means that the error term is independent of the projections.

Problem 2.56

1. $\mathcal{S}_{\hat{X}}(f) = |-j \operatorname{sgn}(f)|^2 \mathcal{S}_X(f) = \mathcal{S}_X(f)$, hence $R_{\hat{X}}(\tau) = R_X(\tau)$.
2. $\mathcal{S}_{X\hat{X}}(f) = \mathcal{S}_X(f)(-j \operatorname{sgn}(f))^* = j \operatorname{sgn}(f) \mathcal{S}_X(f)$, therefore, $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$.
3. $R_Z(\tau) = E \left[\left(X(t+\tau) + j\hat{X}(t+\tau) \right) \left(X(t) - j\hat{X}(t) \right) \right]$, expanding we have

$$R_Z(\tau) = R_X(\tau) + R_{\hat{X}}(\tau) - j [R_{X\hat{X}}(\tau) - R_{\hat{X}X}(\tau)]$$

Using $R_{\hat{X}}(\tau) = R_X(\tau)$, and the fact that $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$ is an odd function (since it is the HT of an even signal) we have $R_{\hat{X}X}(\tau) = R_{X\hat{X}}(-\tau) = -R_{X\hat{X}}(\tau)$, we have

$$R_Z(\tau) = 2R_X(\tau) - j2R_{X\hat{X}}(\tau) = 2R_X(\tau) + j2\hat{R}_X(\tau)$$

Taking FT of both sides we have

$$\mathcal{S}_Z(f) = 2\mathcal{S}_X(f) + j2(-j \operatorname{sgn}(f)\mathcal{S}_X(f)) = 2(1 + \operatorname{sgn}(f))\mathcal{S}_X(f) = 4\mathcal{S}_X(f)u_{-1}(f)$$

4. We have

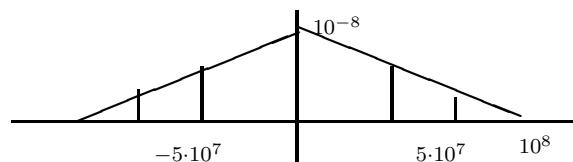
$$\begin{aligned} R_{X_l}(t + \tau, t) &= E \left[Z(t + \tau) e^{-j2\pi f_0(t+\tau)} Z^*(t) e^{j2\pi f_0 t} \right] \\ &= e^{-j2\pi f_0 \tau} R_Z(\tau) \end{aligned}$$

This shows that $X_l(t)$ is WSS (we already know it is zero-mean). Taking FT, we have $\mathcal{S}_{X_l}(f) = \mathcal{S}_Z(f - f_0) = 4\mathcal{S}_X(f - f_0)u_{-1}(f - f_0)$, this shows that $X_l(t)$ is lowpass. Also from above $R_X(\tau) = \frac{1}{2}\operatorname{Re}[R_Z(t)] = \frac{1}{2}\operatorname{Re}[R_{X_l}(\tau)e^{j2\pi f_0 \tau}]$. This shows that $R_{X_l}(\tau)$ is twice the LP equivalent of $R_X(\tau)$.

Problem 2.57

1) The power spectral density $\mathcal{S}_n(f)$ is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$\begin{aligned} P &= \int_{-51 \times 10^6}^{-49 \times 10^6} 10^{-8} \left(1 + \frac{f}{10^8} \right) df + \int_{49 \times 10^6}^{51 \times 10^6} 10^{-8} \left(1 - \frac{f}{10^8} \right) df \\ &= 10^{-8} x \Big|_{-51 \times 10^6}^{-49 \times 10^6} + 10^{-16} \frac{1}{2} x^2 \Big|_{-51 \times 10^6}^{-49 \times 10^6} + 10^{-8} x \Big|_{49 \times 10^6}^{51 \times 10^6} - 10^{-16} \frac{1}{2} x^2 \Big|_{49 \times 10^6}^{51 \times 10^6} \\ &= 2 \times 10^{-2} \end{aligned}$$



2) The output process $N(t)$ can be written as

$$N(t) = N_c(t) \cos(2\pi 50 \times 10^6 t) - N_s(t) \sin(2\pi 50 \times 10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$\begin{aligned} N_c(t) &= N(t) \cos(2\pi 50 \times 10^6 t) + \hat{N}(t) \sin(2\pi 50 \times 10^6 t) \\ N_s(t) &= \hat{N}(t) \cos(2\pi 50 \times 10^6 t) - N(t) \sin(2\pi 50 \times 10^6 t) \end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned} E[|N_c(t)|^2] &= E[|N(t)|^2] \cos^2(2\pi 50 \times 10^6 t) + E[|\hat{N}(t)|^2] \sin^2(2\pi 50 \times 10^6 t) \\ &= E[|N(t)|^2] = 2 \times 10^{-2} \end{aligned}$$

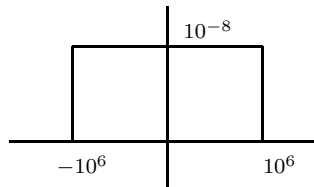
where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2 \times 10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$\mathcal{S}_{N_c}(f) = \mathcal{S}_{N_s}(f) = \begin{cases} \mathcal{S}_N(f - 50 \times 10^6) + \mathcal{S}_N(f + 50 \times 10^6) & |f| \leq 50 \times 10^6 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{S}_{N_c}(f)$ is depicted in the next figure. The power content of $\mathcal{S}_{N_c}(f)$ can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2 \times 10^{-2}$$



4) The power spectral density of the output is given by

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f) |H(f)|^2 = 10^{-6} (|f| - 49 \times 10^6) (10^{-8} - 10^{-16} |f|) \quad \text{for } 49 \times 10^6 \leq |f| \leq 51 \times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= 10^{-6} \left(\int_{-51 \times 10^6}^{-49 \times 10^6} (-f - 49 \times 10^6) (10^{-8} + 10^{-16} f) df \right) \\ &\quad + 10^{-6} \left(\int_{49 \times 10^6}^{51 \times 10^6} (f - 49 \times 10^6) (10^{-8} - 10^{-16} f) df \right) \\ &= 10^{-6} \left(2 \times 10^4 - \frac{4}{3} 10^2 \right) \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} \mathcal{S}_{Y_c}(f) = \mathcal{S}_{Y_s}(f) &= 10^{-6} ((f + 50 \times 10^6) - 49 \times 10^6) (10^{-8} - 10^{-16} (f + 50 \times 10^6)) \\ &\quad + 10^{-6} ((-f - 50 \times 10^6) - 49 \times 10^6) (10^{-8} + 10^{-16} (f - 50 \times 10^6)) \\ &= 10^{-6} (-2 \times 10^{-16} f^2 + 10^{-2}) \end{aligned}$$

for $|f| \leq 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned}
 P_{Y_c} = P_{Y_s} &= 10^{-6} \int_{-10^6}^{10^6} (-2 \times 10^{-16} f^2 + 10^{-2}) df \\
 &= 10^{-6} \left(-2 \times 10^{-16} \frac{1}{3} f^3 \Big|_{-10^6}^{10^6} + 10^{-2} f \Big|_{-10^6}^{10^6} \right) \\
 &= 10^{-6} \left(2 \times 10^4 - \frac{4}{3} 10^2 \right) = P_Y
 \end{aligned}$$

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 3) ¹

Prepared by
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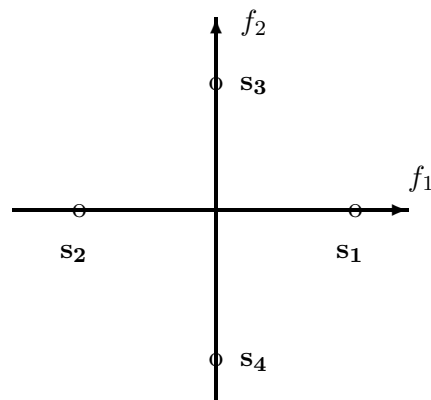
Problem 3.1

Assuming M is even we have

$$\begin{aligned}
 1^2 + 3^2 + 5^2 + \dots + (M-1)^2 &= \sum_{i=1}^M i^2 - \sum_{k=1}^{M/2} (2k)^2 \\
 &= \frac{M(M+1)(2M+1)}{6} - 4 \sum_{k=1}^{M/2} k^2 \\
 &= \frac{M(M+1)(2M+1)}{6} - 4 \frac{M/2(M/2+1)(M+1)}{6} \\
 &= \frac{M(M^2-1)}{6}
 \end{aligned}$$

Problem 3.2

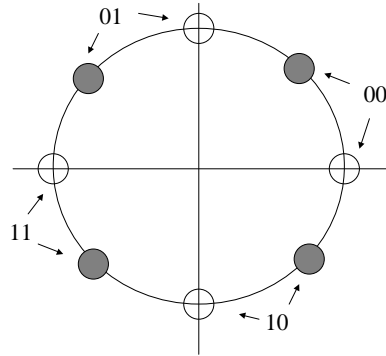
$$\begin{aligned}
 \mathbf{s}_1 &= (\sqrt{\mathcal{E}}, 0) \\
 \mathbf{s}_2 &= (-\sqrt{\mathcal{E}}, 0) \\
 \mathbf{s}_3 &= (0, \sqrt{\mathcal{E}}) \\
 \mathbf{s}_4 &= (0, -\sqrt{\mathcal{E}})
 \end{aligned}$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

Problem 3.3

1.2. The signal space diagram, together with the Gray encoding of each signal point is given in the following figure :



The signal points that may be transmitted at times $t = 2nT$ $n = 0, 1, \dots$ are given with blank circles, while the ones that may be transmitted at times $t = 2nT + 1$, $n = 0, 1, \dots$ are given with filled circles.

Problem 3.4

1. Consider the QAM constellation of Fig. P3-4. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \implies a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between them equal to $\theta = 75^\circ$, we obtain:

$$b^2 = a^2 + A^2 - 2aA \cos 75^\circ \implies b = \frac{1 + \sqrt{3}}{2}A$$

2. If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r \cos 45^\circ \implies r = \frac{A}{\sqrt{2 - \sqrt{2}}}$$

3. The average transmitted power of the PSK constellation is:

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2 - \sqrt{2}}} \right)^2 \Rightarrow P_{\text{PSK}} = \frac{A^2}{2 - \sqrt{2}}$$

whereas the average transmitted power of the QAM constellation:

$$P_{\text{QAM}} = \frac{1}{8} \left(4 \frac{A^2}{2} + 4 \frac{(1 + \sqrt{3})^2}{4} A^2 \right) \Rightarrow P_{\text{QAM}} = \left[\frac{2 + (1 + \sqrt{3})^2}{8} \right] A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2 + (1 + \sqrt{3})^2)(2 - \sqrt{2})} = 1.5927 \text{ dB}$$

Problem 3.5

1. Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, (e.g. going in a clockwise direction : 000, 001, 011, 010, 110, 111, 101, 100). this is not the case for the 8-QAM constellation of Figure P3-4. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \dots, 0\}$ to point A, then point B and C should have the form

$$B = \{0, \dots, 0, 1, 0, \dots, 0\} \quad C = \{0, \dots, 0, 1, 0, \dots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2. Since each symbol conveys 3 bits of information, the resulted symbol rate is :

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

Problem 3.6

The constellation of Fig. P3-6(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is:

$$P_a = \frac{1}{8} \left[4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2 \right] = 6A^2$$

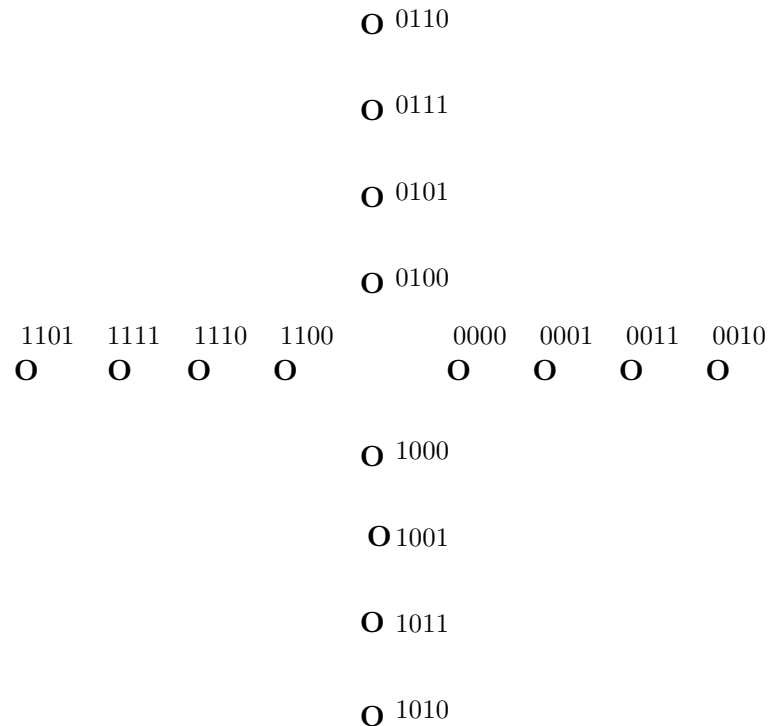
The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A . Thus, the average transmitted power of the second constellation is:

$$P_b = \frac{1}{8} \left[4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2 \right] = \frac{9}{2}A^2$$

Since $P_b < P_a$ the second constellation is more power efficient.

Problem 3.7

One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure.



Problem 3.8

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal

phase of an MSK signal at time instant n is given by

$$\theta(n; \mathbf{a}) = \frac{\pi}{2} \sum_{k=0}^n a_k + \theta_0$$

where θ_0 is the initial phase and a_k is ± 1 depending on the input bit at the time instant k . The following table shows $\theta(n; \mathbf{a})$ for two different values of θ_0 ($0, \pi$), and the four input pairs of data: $\{00, 01, 10, 11\}$.

θ_0	b_0	b_1	a_0	a_1	$\theta(n; \mathbf{a})$
0	0	0	-1	-1	$-\pi$
0	0	1	-1	1	0
0	1	0	1	-1	0
0	1	1	1	1	π
π	0	0	-1	-1	0
π	0	1	-1	1	π
π	1	0	1	-1	π
π	1	1	1	1	2π

Problem 3.9

1.

(i) There are no correlative states in this system, since it is a full response CPM. Based on (3-3-16), we obtain the phase states :

$$\Theta_s = \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\}$$

(ii) Based on (3-3-17), we obtain the phase states :

$$\Theta_s = \left\{ 0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4} \equiv \frac{\pi}{4}, \pi, \frac{15\pi}{4} \equiv \frac{7\pi}{4}, \frac{18\pi}{4} \equiv \frac{\pi}{2}, \frac{21\pi}{4} \equiv \frac{5\pi}{4} \right\}$$

2.

(i) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1/n-2}\}$ take the values ± 1 . Hence there are $3 \times 2 \times 2 = 12$ combined states in all.

(ii) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1/n-2}\}$ take the values ± 1 . Hence there are $8 \times 2 \times 2 = 32$ combined states in all.

Problem 3.10

The bandwidth required for transmission of an M -ary PAM signal is

$$W = \frac{R}{2 \log_2 M} \text{ Hz}$$

Since,

$$R = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 16 \text{ KHz} & M = 4 \\ 10.667 \text{ KHz} & M = 8 \\ 8 \text{ KHz} & M = 16 \end{cases}$$

Problem 3.11

The autocorrelation function for $u_\Delta(t)$ is :

$$\begin{aligned} R_{u_\Delta u_\Delta}(t) &= E[u_\Delta(t + \tau)u_\Delta^*(t)] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E(I_m I_n^*) E[u(t + \tau - mT - \Delta)u^*(t - nT - \Delta)] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{ii}(m - n) E[u(t + \tau - mT - \Delta)u^*(t - nT - \Delta)] \\ &= \sum_{m=-\infty}^{\infty} R_{ii}(m) \sum_{n=-\infty}^{\infty} E[u(t + \tau - mT - nT - \Delta)u^*(t - nT - \Delta)] \\ &= \sum_{m=-\infty}^{\infty} R_{ii}(m) \sum_{n=-\infty}^{\infty} \int_0^T \frac{1}{T} u(t + \tau - mT - nT - \Delta)u^*(t - nT - \Delta) d\Delta \end{aligned}$$

Let $a = \Delta + nT$, $da = d\Delta$, and $a \in (-\infty, \infty)$. Then :

$$\begin{aligned} R_{u_\Delta u_\Delta}(t) &= \sum_{m=-\infty}^{\infty} R_{ii}(m) \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} \frac{1}{T} u(t + \tau - mT - a)u^*(t - a) da \\ &= \sum_{m=-\infty}^{\infty} R_{ii}(m) \frac{1}{T} \int_{-\infty}^{\infty} u(t + \tau - mT - a)u^*(t - a) da \\ &= \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{ii}(m) R_{uu}(\tau - mT) \end{aligned}$$

Thus we have obtained the same autocorrelation function as given by (4.4.11). Consequently the power spectral density of $u_{\Delta}(t)$ is the same as the one given by (4.4.12) :

$$\mathcal{S}_{u_{\Delta}u_{\Delta}}(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_{ii}(f)$$

Problem 3.12

The 16-QAM signal is represented as $s(t) = I_n \cos 2\pi ft + Q_n \sin 2\pi ft$, where $I_n = \{\pm 1, \pm 3\}$, $Q_n = \{\pm 1, \pm 3\}$. A superposition of two 4-QAM (4-PSK) signals is :

$$s(t) = G[A_n \cos 2\pi ft + B_n \sin 2\pi ft] + C_n \cos 2\pi ft + D_n \sin 2\pi ft$$

where $A_n, B_n, C_n, D_n = \{\pm 1\}$. Clearly : $I_n = GA_n + C_n$, $Q_n = GB_n + D_n$. From these equations it is easy to see that $G = 2$ gives the requires equivalence.

Problem 3.13

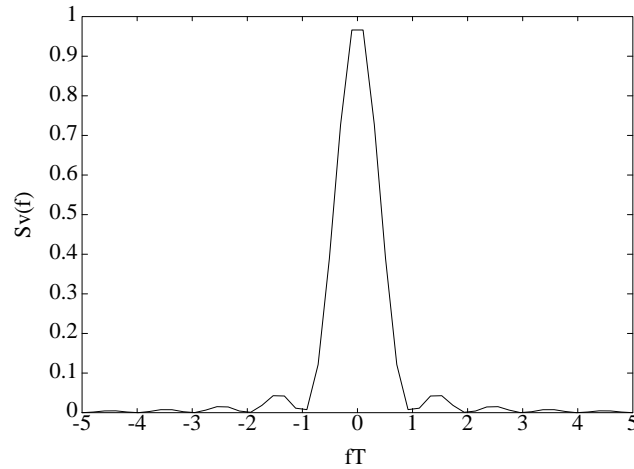
We have that $\mathcal{S}_{uu}(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_{ii}(f)$ But $E(I_n) = 0$, $E(|I_n|^2) = 1$, hence : $R_{ii}(m) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$. Therefore : $\mathcal{S}_{ii}(f) = 1 \Rightarrow \mathcal{S}_{uu}(f) = \frac{1}{T} |G(f)|^2$.

1. For the rectangular pulse :

$$G(f) = AT \frac{\sin \pi f T}{\pi f T} e^{-j2\pi f T/2} \Rightarrow |G(f)|^2 = A^2 T^2 \frac{\sin^2 \pi f T}{(\pi f T)^2}$$

where the factor $e^{-j2\pi f T/2}$ is due to the $T/2$ shift of the rectangular pulse from the center $t = 0$. Hence :

$$\mathcal{S}_{uu}(f) = A^2 T \frac{\sin^2 \pi f T}{(\pi f T)^2}$$

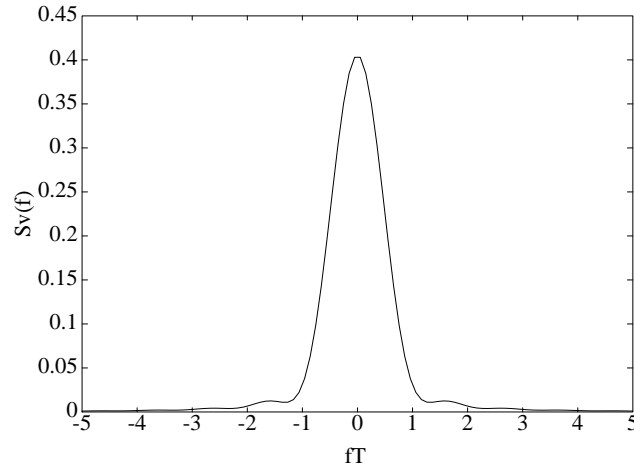


2. For the sinusoidal pulse : $G(f) = \int_0^T \sin \frac{\pi t}{T} \exp(-j2\pi ft) dt$. By using the trigonometric identity $\sin x = \frac{\exp(jx) - \exp(-jx)}{2j}$ it is easily shown that :

$$G(f) = \frac{2AT}{\pi} \frac{\cos \pi T f}{1 - 4T^2 f^2} e^{-j2\pi fT/2} \Rightarrow |G(f)|^2 = \left(\frac{2AT}{\pi} \right)^2 \frac{\cos^2 \pi T f}{(1 - 4T^2 f^2)^2}$$

Hence :

$$\mathcal{S}_{uu}(f) = \left(\frac{2A}{\pi} \right)^2 T \frac{\cos^2 \pi T f}{(1 - 4T^2 f^2)^2}$$



3. The 3-db frequency for (a) is :

$$\frac{\sin^2 \pi f_{3db} T}{(\pi f_{3db} T)^2} = \frac{1}{2} \Rightarrow f_{3db} = \frac{0.44}{T}$$

(where this solution is obtained graphically), while the 3-db frequency for the sinusoidal pulse on (b) is :

$$\frac{\cos^2 \pi T f}{(1 - 4T^2 f^2)^2} = \frac{1}{2} \Rightarrow f_{3db} = \frac{0.59}{T}$$

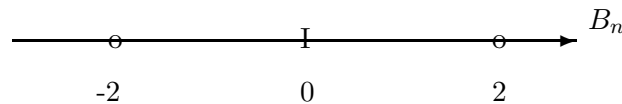
The rectangular pulse spectrum has the first spectral null at $f = 1/T$, whereas the spectrum of the sinusoidal pulse has the first null at $f = 3/2T = 1.5/T$. Clearly the spectrum for the rectangular pulse has a narrower main lobe. However, it has higher sidelobes.

Problem 3.14

1. $B_n = I_n + I_{n-1}$. Hence :

I_n	I_{n-1}	B_n
1	1	2
1	-1	0
-1	1	0
-1	-1	-2

The signal space representation is given in the following figure, with $P(B_n = 2) = P(B_n = -2) = 1/4$, $P(B_n = 0) = 1/2$.



2.

$$\begin{aligned} R_{BB}(m) &= E[B_{n+m}B_n] = E[(I_{n+m} + I_{n+m-1})(I_n + I_{n-1})] \\ &= R_{ii}(m) + R_{ii}(m-1) + R_{ii}(m+1) \end{aligned}$$

Since the sequence $\{I_n\}$ consists of independent symbols :

$$R_{ii}(m) = \begin{cases} E[I_{n+m}]E[I_n] = 0 \cdot 0 = 0, & m \neq 0 \\ E[I_n^2] = 1, & m = 0 \end{cases}$$

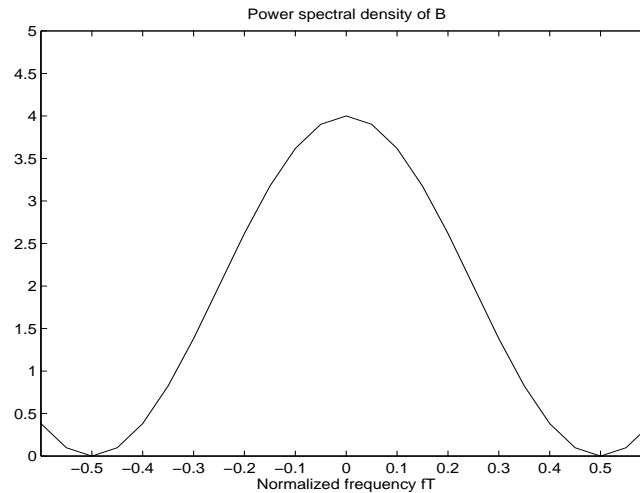
Hence :

$$R_{BB}(m) = \begin{cases} 2, & m = 0 \\ 1, & m = \pm 1 \\ 0, & \text{o.w} \end{cases}$$

and

$$\begin{aligned} \mathcal{S}_{BB}(f) &= \sum_{m=-\infty}^{\infty} R_{BB}(m) \exp(-j2\pi f m T) = 2 + \exp(j2\pi f T) + \exp(-j2\pi f T) \\ &= 2[1 + \cos 2\pi f T] = 4 \cos^2 \pi f T \end{aligned}$$

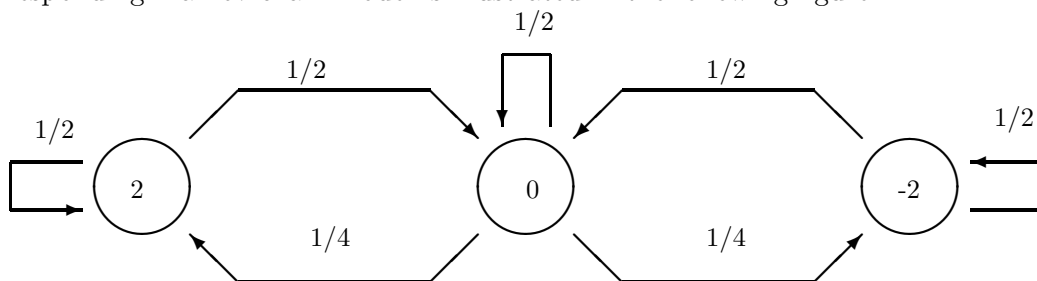
A plot of the power spectral density $\mathcal{S}_B(f)$ is given in the following figure :



3. The transition matrix is :

I_{n-1}	I_n	B_n	I_{n+1}	B_{n+1}
-1	-1	-2	-1	-2
-1	-1	-2	1	0
-1	1	0	-1	0
-1	1	0	1	2
1	-1	0	-1	-2
1	-1	0	1	0
1	1	2	-1	0
1	1	2	1	2

The corresponding Markov chain model is illustrated in the following figure :



Problem 3.15

1. $I_n = a_n - a_{n-2}$, with the sequence $\{a_n\}$ being uncorrelated random variables (i.e $E(a_{n+m}a_n) =$

$\delta(m)$). Hence :

$$\begin{aligned} R_{ii}(m) &= E[I_{n+m}I_n] = E[(a_{n+m} - a_{n+m-2})(a_n - a_{n-2})] \\ &= 2\delta(m) - \delta(m-2) - \delta(m+2) \\ &= \begin{cases} 2, & m = 0 \\ -1, & m = \pm 2 \\ 0, & \text{o.w.} \end{cases} \end{aligned}$$

2. $\mathcal{S}_{uu}(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_{ii}(f)$ where :

$$\begin{aligned} \mathcal{S}_{ii}(f) &= \sum_{m=-\infty}^{\infty} R_{ii}(m) \exp(-j2\pi f m T) = 2 - \exp(j4\pi f T) - \exp(-j4\pi f T) \\ &= 2[1 - \cos 4\pi f T] = 4 \sin^2 2\pi f T \end{aligned}$$

and

$$|G(f)|^2 = (AT)^2 \left(\frac{\sin \pi f T}{\pi f T} \right)^2$$

Therefore :

$$\mathcal{S}_{uu}(f) = 4A^2T \left(\frac{\sin \pi f T}{\pi f T} \right)^2 \sin^2 2\pi f T$$

3. If $\{a_n\}$ takes the values (0,1) with equal probability then $E(a_n) = 1/2$ and $E(a_{n+m}a_n) = \begin{cases} 1/4, & m \neq 0 \\ 1/2, & m = 0 \end{cases} = [1 + \delta(m)]/4$. Then :

$$\begin{aligned} R_{ii}(m) &= E[I_{n+m}I_n] = 2R_{aa}(0) - R_{aa}(2) - R_{aa}(-2) \\ &= \frac{1}{4} [2\delta(m) - \delta(m-2) - \delta(m+2)] \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_{ii}(f) &= \sum_{m=-\infty}^{\infty} R_{ii}(m) \exp(-j2\pi f m T) = \sin^2 2\pi f T \\ \mathcal{S}_{uu}(f) &= A^2T \left(\frac{\sin \pi f T}{\pi f T} \right)^2 \sin^2 2\pi f T \end{aligned}$$

Thus, we obtain the same result as in (b) , but the magnitude of the various quantities is reduced by a factor of 4 .

Problem 3.16

We may use the result in (3.4.27), where we set $K = 2$, $p_1 = p_2 = 1/2$:

$$\mathcal{S}(f) = \frac{1}{T^2} \sum_{l=-\infty}^{\infty} \left| \sum_{i=1}^2 \frac{1}{2} S_i \left(\frac{l}{T} \right) \right|^2 \delta \left(f - \frac{l}{T} \right) + \frac{1}{T} \sum_{i=1}^2 \frac{1}{4} |S_i(f)|^2 - \frac{2}{T} \frac{1}{4} \text{Re} [S_1(f) S_2^*(f)]$$

To simplify the computations we may define the signals over the symmetric interval $-T/2 \leq t \leq T/2$. Then :

$$S_i(f) = \frac{T}{2j} \left[\frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} - \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \right]$$

(the well-known rectangular pulse spectrum, modulated by $\sin 2\pi f_i t$) and :

$$|S_i(f)|^2 = \left(\frac{T}{2}\right)^2 \left[\left(\frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T}\right)^2 + \left(\frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T}\right)^2 \right]$$

where the cross-term involving the product $\frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} \cdot \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T}$ is negligible when $f_i \gg 0$. Also :

$$\begin{aligned} S_1\left(\frac{l}{T}\right) &= \frac{T}{2j} \left[\frac{\sin \pi\left(\frac{l}{T} - \frac{n}{2T}\right)T}{\pi\left(\frac{l}{T} - \frac{n}{2T}\right)T} - \frac{\sin \pi\left(\frac{l}{T} + \frac{n}{2T}\right)T}{\pi\left(\frac{l}{T} + \frac{n}{2T}\right)T} \right] \\ &= \frac{T}{2j} \left[\frac{\sin\left(\pi l - \frac{\pi n}{2}\right)}{\left(\pi l - \frac{\pi n}{2}\right)} - \frac{\sin\left(\pi l + \frac{\pi n}{2}\right)}{\left(\pi l + \frac{\pi n}{2}\right)} \right] \\ &= \frac{T}{2j} 2l(-1)^{l+1} \left(\sin \frac{\pi n}{2}\right) / \pi \left(l^2 - n^2/4\right) \\ &= \frac{lT}{j} (-1)^{l+1} \frac{\sin \frac{\pi n}{2}}{\pi(l^2 - n^2/4)} \end{aligned}$$

and similarly for $S_2(\frac{l}{T})$ (with m instead of n). Note that if $n(m)$ is even then $S_{1(2)}(\frac{l}{T}) = 0$ for all l except at $l = \pm n(m)/2$, where $S_{1(2)}(\frac{n(m)}{2T}) = \pm \frac{T}{2j}$. For this case

$$\frac{1}{T^2} \sum_{l=-\infty}^{\infty} \left| \sum_{i=1}^2 \frac{1}{2} S_i\left(\frac{l}{T}\right) \right|^2 \delta\left(f - \frac{l}{T}\right) = \frac{1}{16} \left[\delta\left(f - \frac{n}{2T}\right) + \delta\left(f + \frac{n}{2T}\right) + \delta\left(f - \frac{m}{2T}\right) + \delta\left(f + \frac{m}{2T}\right) \right]$$

The third term in (3.4.27) involves the product of $S_1(f)$ and $S_2(f)$ which is negligible since they have little spectral overlap. Hence :

$$\mathcal{S}(f) = \frac{1}{16} \left[\delta\left(f - \frac{n}{2T}\right) + \delta\left(f + \frac{n}{2T}\right) + \delta\left(f - \frac{m}{2T}\right) + \delta\left(f + \frac{m}{2T}\right) \right] + \frac{1}{4T} \left[|S_1(f)|^2 + |S_2(f)|^2 \right]$$

In comparison with the spectrum of the MSK signal, we note that this signal has impulses in the spectrum.

Problem 3.17

MFSK signal with waveforms : $s_i(t) = \sin \frac{2\pi i t}{T}$, $i = 1, 2, \dots, M$ $0 \leq t \leq T$

The expression for the power density spectrum is given by (3.4.27) with $K = M$ and $p_i = 1/M$. From Problem 4.23 we have that :

$$S_i(f) = \frac{T}{2j} \left[\frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} - \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \right]$$

for a signal $s_i(t)$ shifted to the left by $T/2$ (which does not affect the power spectrum). We also have that :

$$S_i\left(\frac{n}{T}\right) = \begin{cases} \pm T/2j, & n = \pm i \\ 0, & \text{o.w.} \end{cases}$$

Hence from (3.4.27) we obtain :

$$\begin{aligned} \mathcal{S}(f) &= \frac{1}{T^2} \left(\frac{1}{M}\right)^2 \left(\frac{T^2}{4}\right) \sum_{i=1}^M [\delta(f - f_i) + \delta(f + f_i)] \\ &\quad + \frac{1}{T} \left(\frac{1}{M}\right)^2 \sum_{i=1}^M |S_i(f)|^2 \\ &\quad - \frac{2}{T} \sum_{i=1}^M \sum_{j=i+1}^M \left(\frac{1}{M}\right)^2 \operatorname{Re} [S_i(f) S_j^*(f)] \\ &= \left(\frac{1}{2M}\right)^2 \sum_{i=1}^M [\delta(f - f_i) + \delta(f + f_i)] + \frac{1}{TM^2} \sum_{i=1}^M |S_i(f)|^2 \\ &\quad - \frac{2}{TM^2} \sum_{i=1}^M \sum_{j=i+1}^M \operatorname{Re} [S_i(f) S_j^*(f)] \end{aligned}$$

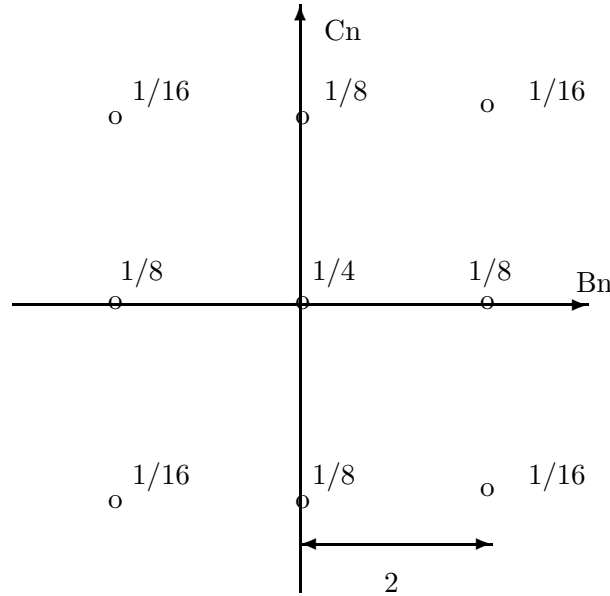
Problem 3.18

QPRS signal $v(t) = \sum_n (B_n + jC_n) u(t - nT)$, $B_n = I_n + I_{n-1}$, $C_n = J_n + J_{n-1}$.

1. Similarly to Problem 3.11, the sequence B_n can take the values : $P(B_n = 2) = P(B_n = -2) = 1/4$, $P(B_n = 0) = 1/2$. The same holds for the sequence C_n ; since these two sequences are independent :

$$P\{B_n = i, C_n = j\} = P\{B_n = i\} P\{C_n = j\}$$

Hence, since they are also in phase quadrature the signal space representation will be as shown in the following figure (next to each symbol is the corresponding probability of occurrence) :



2. If we name $Z_n = B_n + jC_n$:

$$\begin{aligned} R_{ZZ}(m) &= \frac{1}{2} E[(B_{n+m} + jC_{n+m})(B_n - jC_n)] \\ &= \frac{1}{2} \{E[B_{n+m}B_n] + E[C_{n+m}C_n]\} = \frac{1}{2} (R_{BB}(m) + R_{CC}(m)) = R_{BB}(m) = R_{CC}(m) \end{aligned}$$

since the sequences B_n, C_n are independent, and have the same statistics. Now, from Problem 3.11 :

$$R_{BB}(m) = \begin{cases} 2, & m = 0 \\ 1, & m = \pm 1 \\ 0, & \text{o.w} \end{cases} = R_{CC}(m) = R_{ZZ}(m)$$

Hence, from (4-4-11) :

$$R_{vs}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{BB}(m) R_{uu}(\tau - mT) = R_{vc}(\tau) = R_v(\tau)$$

Also :

$$\mathcal{S}_{vs}(f) = \mathcal{S}_{vc}(f) = \mathcal{S}_v(f) = \frac{1}{T} |U(f)|^2 \mathcal{S}_{BB}(f)$$

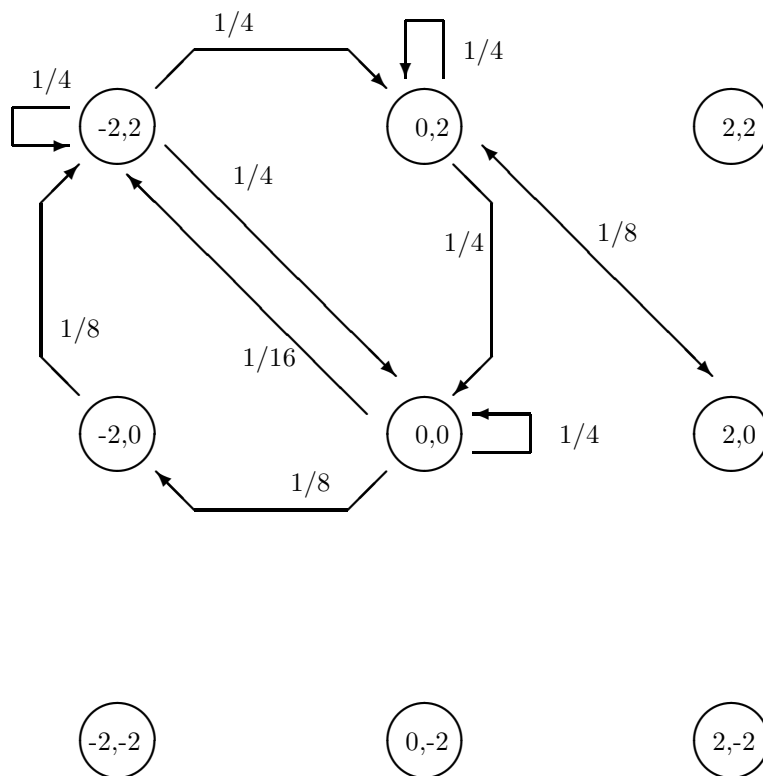
since the corresponding autocorrelations are the same . From Problem 3.11 : $\mathcal{S}_{BB}(f) = 4 \cos^2 \pi f T$, so

$$\mathcal{S}_{vs}(f) = \mathcal{S}_{vc}(f) = \mathcal{S}_v(f) = \frac{4}{T} |U(f)|^2 \cos^2 \pi f T$$

Therefore, the composite QPRS signal has the same power density spectrum as the in-phase and quadrature components.

3. The transition probabilities for the B_n, C_n sequences are independent, so the probability of a transition between one state of the QPRS signal to another state, will be the product of the

probabilities of the respective B-transition and C-transition. Hence, the Markov chain model will be the Cartesian product of the Markov model that was derived in Problem 3.11 for the sequence B_n alone. For example, the transition probability from the state $(B_n, C_n) = (0, 0)$ to the same state will be : $P(B_{n+1} = 0|B_n = 0) \cdot P(C_{n+1} = 0|C_n = 0) = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$ and so on. Below, we give a partial sketch of the Markov chain model; the rest of it can be derived easily, from the symmetries of this model.



Problem 3.19

1. Since : $\mu_a = 0$, $\sigma_a^2 = 1$, we have : $\mathcal{S}_{ss}(f) = \frac{1}{T} |G(f)|^2$. But :

$$\begin{aligned}
 G(f) &= \frac{T}{2} \frac{\sin \pi f T / 2}{\pi f T / 2} e^{-j2\pi f T / 4} - \frac{T}{2} \frac{\sin \pi f T / 2}{\pi f T / 2} e^{-j2\pi f 3T / 4} \\
 &= \frac{T}{2} \frac{\sin \pi f T / 2}{\pi f T / 2} e^{-j\pi f T} (2j \sin \pi f T / 2) \\
 &= jT \frac{\sin^2 \pi f T / 2}{\pi f T / 2} e^{-j\pi f T} \Rightarrow \\
 |G(f)|^2 &= T^2 \left(\frac{\sin^2 \pi f T / 2}{\pi f T / 2} \right)^2 \Rightarrow \\
 \mathcal{S}_{ss}(f) &= T \left(\frac{\sin^2 \pi f T / 2}{\pi f T / 2} \right)^2
 \end{aligned}$$

2. For non-independent information sequence the power spectrum of $s(t)$ is given by : $\mathcal{S}_{ss}(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_{bb}(f)$. But :

$$\begin{aligned}
 R_{bb}(m) &= E [b_{n+m} b_n] \\
 &= E [a_{n+m} a_n] + k E [a_{n+m-1} a_n] + k E [a_{n+m} a_{n-1}] + k^2 E [a_{n+m-1} a_{n-1}] \\
 &= \begin{cases} 1 + k^2, & m = 0 \\ k, & m = \pm 1 \\ 0, & \text{o.w.} \end{cases}
 \end{aligned}$$

Hence :

$$\mathcal{S}_{bb}(f) = \sum_{m=-\infty}^{\infty} R_{bb}(m) e^{-j2\pi f m T} = 1 + k^2 + 2k \cos 2\pi f T$$

We want :

$$\mathcal{S}_{ss}(1/T) = 0 \Rightarrow \mathcal{S}_{bb}(1/T) = 0 \Rightarrow 1 + k^2 + 2k = 0 \Rightarrow k = -1$$

and the resulting power spectrum is :

$$\mathcal{S}_{ss}(f) = 4T \left(\frac{\sin^2 \pi f T / 2}{\pi f T / 2} \right)^2 \sin^2 \pi f T$$

3. The requirement for zeros at $f = l/4T$, $l = \pm 1, \pm 2, \dots$ means : $\mathcal{S}_{bb}(l/4T) = 0 \Rightarrow 1 + k^2 + 2k \cos \pi l / 2 = 0$, which cannot be satisfied for all l . We can avoid that by using precoding in the

form $b_n = a_n + ka_{n-4}$. Then :

$$R_{bb}(m) = \left\{ \begin{array}{ll} 1 + k^2, & m = 0 \\ k, & m = \pm 4 \\ 0, & \text{o.w.} \end{array} \right\} \Rightarrow \mathcal{S}_{bb}(f) = 1 + k^2 + 2k \cos 2\pi f 4T$$

and , similarly to (b), a value of $k = -1$, will zero this spectrum in all multiples of $1/4T$.

Problem 3.20

1. The power spectral density of the FSK signal may be evaluated by using equation (3-4-27) with $K = 2$ (binary) signals and probabilities $p_0 = p_1 = \frac{1}{2}$. Thus, when the condition that the carrier phase θ_0 and θ_1 are fixed, we obtain

$$\mathcal{S}(f) = \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} |S_0(\frac{n}{T}) + S_1(\frac{n}{T})|^2 \delta(f - \frac{n}{T}) + \frac{1}{4T} |S_0(f) - S_1(f)|^2$$

where $S_0(f)$ and $S_1(f)$ are the fourier transforms of $s_0(t)$ and $s_1(t)$. In particular :

$$\begin{aligned} S_0(f) &= \int_0^T s_0(t) e^{-j2\pi ft} dt \\ &= \sqrt{\frac{2\mathcal{E}_b}{T}} \int_0^T \cos(2\pi f_0 t + \theta_0) e^{j2\pi ft} dt, \quad f_0 = f_c - \frac{\Delta f}{2} \\ &= \sqrt{\frac{T\mathcal{E}_b}{2}} \left[\frac{\sin \pi T(f - f_0)}{\pi(f - f_0)} + \frac{\sin \pi T(f + f_0)}{\pi(f + f_0)} \right] e^{-j\pi f T} e^{j\theta_0} \end{aligned}$$

Similarly :

$$\begin{aligned} S_1(f) &= \int_0^T s_1(t) e^{-j2\pi ft} dt \\ &= \sqrt{\frac{T\mathcal{E}_b}{2}} \left[\frac{\sin \pi T(f - f_1)}{\pi(f - f_1)} + \frac{\sin \pi T(f + f_1)}{\pi(f + f_1)} \right] e^{-j\pi f T} e^{j\theta_1} \end{aligned}$$

where $f_1 = f_c + \frac{\Delta f}{2}$. By expressing $\mathcal{S}(f)$ as :

$$\begin{aligned} \mathcal{S}(f) &= \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} \left[|S_0(\frac{n}{T})|^2 + |S_1(\frac{n}{T})|^2 + 2\text{Re}[S_0(\frac{n}{T})S_1^*(\frac{n}{T})] \right] \delta(f - \frac{n}{T}) \\ &\quad + \frac{1}{4T} [|S_0(f)|^2 + |S_1(f)|^2 - 2\text{Re}[S_0(f)S_1^*(f)]] \end{aligned}$$

we note that the carrier phases θ_0 and θ_1 affect only the terms $\text{Re}(S_0 S_1^*)$. If we average over the random phases, these terms drop out. Hence, we have :

$$\begin{aligned}\mathcal{S}(f) &= \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} \left[|S_0(\frac{n}{T})|^2 + |S_1(\frac{n}{T})|^2 \right] \delta(f - \frac{n}{T}) \\ &\quad + \frac{1}{4T} [|S_0(f)|^2 + |S_1(f)|^2]\end{aligned}$$

where :

$$|S_k(f)|^2 = \frac{T\mathcal{E}_b}{2} \left| \frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} + \frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \right|^2, \quad k = 0, 1$$

Note that the first term in $\mathcal{S}(f)$ consists of a sequence of samples and the second term constitutes the continuous spectrum.

2. Note that :

$$|S_k(f)|^2 = \frac{T\mathcal{E}_b}{2} \left[\left(\frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} \right)^2 + \left(\frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \right)^2 \right]$$

because the product

$$\frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} \times \frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \approx 0$$

if f_k is large enough. Hence $|S_k(f)|^2$ decays proportionally to $\frac{1}{(f-f_k)^2} \approx \frac{1}{f^2}$ for $f \gg f_c$. Consequently, $\mathcal{S}(f)$ exhibits the same behaviour.

Problem 3.21

1) The power spectral density of $X(t)$ is given by

$$\mathcal{S}_x(f) = \frac{1}{T} \mathcal{S}_i(f) |U(f)|^2$$

The Fourier transform of $u(t)$ is

$$U(f) = \mathcal{F}[u(t)] = AT \frac{\sin \pi f T}{\pi f T} e^{-j\pi f T}$$

Hence,

$$|U(f)|^2 = (AT)^2 \text{sinc}^2(fT)$$

and therefore,

$$\mathcal{S}_x(f) = A^2 T \mathcal{S}_i(f) \text{sinc}^2(fT) = A^2 T \text{sinc}^2(fT)$$

2) If $u_1(t)$ is used instead of $u(t)$ and the symbol interval is T , then

$$\begin{aligned}\mathcal{S}_x(f) &= \frac{1}{T} \mathcal{S}_a(f) |U_1(f)|^2 \\ &= \frac{1}{T} (A^2 T)^2 \text{sinc}^2(fT) = 4A^2 T \text{sinc}^2(fT)\end{aligned}$$

3) If we precode the input sequence as $b_n = I_n + \alpha I_{n-1}$, then

$$R_b(m) = \begin{cases} 1 + \alpha^2 & m = 0 \\ \alpha & m = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and therefore, the power spectral density $\mathcal{S}_b(f)$ is

$$\mathcal{S}_b(f) = 1 + \alpha^2 + 2\alpha \cos(2\pi fT)$$

To obtain a null at $f = \frac{1}{3T}$, the parameter α should be such that

$$1 + \alpha^2 + 2\alpha \cos(2\pi fT) \big|_{f=\frac{1}{3T}} = 0$$

and α does not have a real-valued solution. Therefore the above precoding cannot result in a PAM system with the desired spectral null.

4) The answer to this question is no. This is because $\mathcal{S}_b(f)$ is an analytic function and unless it is identical to zero it can have at most a countable number of zeros. This property of the analytic functions is also referred as the theorem of isolated zeros.

Problem 3.22

1. Since $X(t) = \text{Re}[\sum_{n=-\infty}^{\infty} (a_n + jb_n)u(t - nT)e^{j2\pi f_c t}]$, the lowpass equivalent signal is $X_l(t) = \sum_{n=-\infty}^{\infty} (a_n + jb_n)u(t - nT)$. The in-phase and quadrature components are thus given by $X_i(t) = \sum_{n=-\infty}^{\infty} a_n \cos 2\pi f_c t$ and $X_q(t) = \sum_{n=-\infty}^{\infty} b_n \sin 2\pi f_c t$.
2. Defining $I_n = a_n + jb_n$, from the values of (a_n, b_n) pair, it is clear that $E[I_n] = 0$ and

$$R_i(m) = E[I_{n+m} I_n^*] = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

and $\mathcal{S}_i(f) = 1$. Also note that $U(f) = 2T \text{sinc}^2(2Tf)$ and from 3.4-16

$$\mathcal{S}_{X_l}(f) = \frac{1}{T} \mathcal{S}_i(f) |U(f)|^2 = 4T \text{sinc}^4(2Tf)$$

Using 2.9-14,

$$\begin{aligned}\mathcal{S}_X(f) &= \frac{1}{4} [4T \text{sinc}^4(2T(f - f_c)) + 4T \text{sinc}^4(2T(f + f_c))] \\ &= T [\text{sinc}^4(2T(f - f_c)) + T \text{sinc}^4(2T(f + f_c))]\end{aligned}$$

3. This is equivalent to a precoding of the form $J_n = I_n + \alpha I_{n-1}$ where $J_n = c_n + jd_n$. Using 3.4-20 we have

$$\begin{aligned}\mathcal{S}_{Y_i}(f) &= \mathcal{S}_{X_i}(f) \left| 1 + \alpha e^{-j2\pi fT} \right|^2 \\ &= 4T \text{sinc}^4(2Tf) \left(1 + |\alpha|^2 + 2 \text{Re}[\alpha e^{-j2\pi fT}] \right)\end{aligned}$$

To have no DC components the PSD must vanish at $f = 0$, or $1 + |\alpha|^2 + 2 \text{Re}[\alpha] = 0$ resulting in $1 + \alpha_r^2 + 2\alpha_r + \alpha_i^2 = 0$. This relation is satisfied for infinitely many α 's, for instance $\alpha = -1$.

Problem 3.23

1. Since $s(t) = \sum_{n=-\infty}^{\infty} \alpha_n g(t - nT) \cos(2\pi f_0 t + \theta_n)$, we have $s_l(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j\theta_n} g(t - nT)$. We know that $\mathcal{S}_{s_l}(f) = \frac{1}{T} \mathcal{S}_a(f) |G(f)|^2$, where $a_n = \alpha_n e^{j\theta_n}$'s are all independent each with probability of $\frac{1}{8}$. Since $g(t) = \Lambda\left(\frac{t-T/2}{T/2}\right)$, therefore $G(f) = e^{-j\pi fT} \frac{T}{2} \text{sinc}^2\left(\frac{Tf}{2}\right)$, and $|G(f)|^2 = \frac{T^2}{4} \text{sinc}^4\left(\frac{Tf}{2}\right)$. We need to find $\mathcal{S}_a(f) = \sum_{m=-\infty}^{\infty} R_a(m) e^{-j2\pi f mT}$, but

$$\begin{aligned}R_a(m) &= E[a_{m+n} a_n^*] \\ &= \begin{cases} E[|a_n|^2], & m = 0 \\ |E(a_n)|^2, & m \neq 0 \end{cases} \\ &= \begin{cases} \frac{1}{8} \sum_{i=1}^7 |\alpha_i|^2, & m = 0 \\ \frac{1}{64} \left| \sum_{i=1}^8 \alpha_i e^{j\theta_i} \right|^2, & m \neq 0 \end{cases} \\ &= \begin{cases} \frac{1}{8} \alpha^2, & m = 0 \\ \frac{1}{64} |\beta|^2, & m \neq 0 \end{cases}\end{aligned}$$

From here we have

$$\begin{aligned}\mathcal{S}_a(f) &= \frac{1}{64} |\beta|^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi f mT} + \left(\frac{1}{8} \alpha^2 - \frac{1}{64} |\beta|^2 \right) \\ &= \frac{1}{8} \left(\alpha^2 - \frac{1}{8} |\beta|^2 \right) + \frac{|\beta|^2}{64T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_{s_l}(f) &= \frac{1}{T} \mathcal{S}_a(f) |G(f)|^2 \\ &= \frac{T}{4} \text{sinc}^4\left(\frac{Tf}{2}\right) \left[\frac{1}{8} \left(\alpha^2 - \frac{1}{8} |\beta|^2 \right) + \frac{|\beta|^2}{64T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \right] \\ &= \frac{T}{32} \left[\left(\alpha^2 - \frac{1}{8} |\beta|^2 \right) \text{sinc}^4\left(\frac{Tf}{2}\right) + \frac{|\beta|^2}{8T} \sum_{m=-\infty}^{\infty} \text{sinc}^4\left(\frac{m}{2}\right) \delta\left(f - \frac{m}{T}\right) \right]\end{aligned}$$

2. In this case $\beta = 0$ and $\alpha^2 = \frac{a^2+b^2}{4}$, therefore, $\mathcal{S}_{s_l}(f) = \frac{T(a^2+b^2)}{128} \text{sinc}^4\left(\frac{Tf}{2}\right)$.
3. For $a = b$ the constellation is on a circle at angles 45° apart, therefore it is 8PSK and $\mathcal{S}_{s_l}(f) = \frac{Ta^2}{64} \text{sinc}^4\left(\frac{Tf}{2}\right)$.
4. b_n 's will still be independent and equiprobable, therefore the previous parts will not change.

Problem 3.24

1. In general

$$\mathcal{S}_v(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_i(f)$$

where

$$\mathcal{S}_i(f) = \sum_{m=-\infty}^{\infty} R_i(m) e^{-j2\pi f m T}$$

and $R_i(m) = E[I_n I_{n+m}]$. Since the sequence I_n is iid we have

$$R_i(m) = \begin{cases} (E[I_n])^2 & m \neq 0 \\ E[I_n^2] & m = 0 \end{cases} = \begin{cases} 0 & m \neq 0 \\ \frac{1}{4}(2^2 + 2^2) = 2 & m = 0 \end{cases} = 2\delta(m)$$

Hence $\mathcal{S}_i(f) = \sum_{m=-\infty}^{\infty} R_i(m) e^{-j2\pi f m T} = 2$ and $\mathcal{S}_v(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_i(f) = \frac{2}{T} |G(f)|^2$, where $G(f) = \mathcal{F}[g(t)] = \frac{T}{2} e^{-j\pi f T/2} \text{sinc}(Tf/2) + T e^{-j\pi f T} \text{sinc}(Tf)$.

2. In this cases the signaling interval is $2T$ and J_n substitutes I_n . We have

$$\begin{aligned}R_j(m) &= E[J_n J_{n+m}] = E[(I_{n-1} + I_n + I_{n+1})(I_{n+m-1} + I_{n+m} + I_{n+m+1})] \\ &= 3R_i(m) + 2R_i(m+1) + 2R_i(m-1) + R_i(m+2) + R_i(m-2) \\ &= 6\delta(m) + 4\delta(m-1) + 4\delta(m+1) + 2\delta(m-2) + 2\delta(m+2) \\ &= \begin{cases} 6 & m = 0 \\ 4 & m = \pm 1 \\ 2 & m = \pm 2 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Hence $\mathcal{S}_j(f) = \sum_{m=-\infty}^{\infty} R_j(m)e^{-j2\pi mf \times 2T} = 6 + 8 \cos(4\pi fT) + 4 \cos(8\pi fT)$ and

$$\mathcal{S}_w(f) = \frac{1}{2T} |G(f)|^2 \mathcal{S}_j(f) = \frac{|G(f)|^2}{T} (3 + 4 \cos 4\pi fT + 2 \cos 8\pi fT)$$

Problem 3.25

1. We have

$$\begin{aligned} R_a(m) &= E[a_{n+m}a_n] \\ &= \begin{cases} E[a_n^2] & m = 0 \\ (E[a_n])^2 & m \neq 0 \end{cases} \\ &= \begin{cases} \frac{5}{4} & m = 0 \\ \frac{1}{16} & m \neq 0 \end{cases} \end{aligned}$$

Using $\mathcal{S}_a(f) = \sum_{m=-\infty}^{\infty} R_a(m)e^{-j2\pi fmT}$, we have

$$\begin{aligned} \mathcal{S}_a(f) &= \frac{19}{16} + \frac{1}{16} \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT} \\ &= \frac{19}{16} + \frac{1}{16T} \delta\left(f - \frac{m}{T}\right) \end{aligned}$$

and since $g(t) = \text{sinc}(t/T)$, we have $G(f) = T\Pi(Tf)$, hence $|G(f)|^2 = T^2\Pi(Tf)$ and

$$\mathcal{S}_v(f) = \frac{1}{T} (T^2\Pi(Tf)) \left[\frac{19}{16} + \frac{1}{16T} \delta\left(f - \frac{m}{T}\right) \right]$$

resulting in

$$\mathcal{S}_v(f) = \frac{19}{16} T\Pi(Tf) + \frac{1}{16} \delta(f)$$

2. The power spectral density is multiplied by $|1 + e^{-j2\pi fT} - e^{-j4\pi fT}|^2 = 3 - 2 \cos(4\pi fT)$.
Therefore

$$\mathcal{S}_u(f) = \frac{19}{16} (3 - 2 \cos 4\pi fT) T\Pi(f) + \frac{1}{16} \delta(f)$$

3. In this case $\mathcal{S}_w(f)$ is multiplied by $|1 + je^{-j2\pi fT}|^2 = (2 + 2 \sin 2\pi fT)$ and

$$\mathcal{S}_w(f) = \frac{19}{8} (1 + \sin 2\pi fT) T\Pi(f) + \frac{1}{8} \delta(f)$$

Problem 3.26

First note that

$$R_a(m) = E[a_n a_{n+m}] = \begin{cases} 1, & n = m \\ 0, & \text{otherwise.} \end{cases}$$

1. For QPSK we have $X_l = \sum_{n=-\infty}^{\infty} (a_{2n} + ja_{2n+1})g_{2T}(t - 2nT) = \sum_{n=-\infty}^{\infty} I_n g_{2T}(t - 2nT)$ where $I_n = a_{2n} + ja_{2n+1}$ and therefore

$$\begin{aligned} R_i(m) &= E[I_{n+m} I_n^*] \\ &= E[(a_{2(n+m)} + ja_{2n+2m+1})(a_{2n} - ja_{2n+1})] \\ &= [2R_a(2m) + jR_a(2m+1) - jR_a(2m-1)] \\ &= \begin{cases} 2, & m = 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $\mathcal{S}_i(f) = \sum_{m=-\infty}^{\infty} R_i(m) e^{-j4\pi f m T} = 2$. Also note that $g_{2T}(t) = \Pi\left(\frac{t-T}{2T}\right)$, and therefore $|G_{2T}(f)|^2 = 4T^2 \text{sinc}^2(2Tf)$. Substituting into $\mathcal{S}_{X_l}(f) = \frac{1}{T} \mathcal{S}_i(f) |G_{2T}(f)|^2$, we have $\mathcal{S}_{X_l}(f) = 8T \text{sinc}^2(2Tf)$.

2. Here $X_l(t) = \sum_{n=-\infty}^{\infty} a_{2n} g_{2T}(t - 2nT) + j \sum_{n=-\infty}^{\infty} a_{2n+1} g_{2T}(t - (2n+1)T)$, and

$$\begin{aligned} R_{X_l}(t + \tau, t) &= E[X_l(t + \tau) X_l^*(t)] \\ &= E \left[\left(\sum_{n=-\infty}^{\infty} a_{2n} g_{2T}(t + \tau - 2nT) + j \sum_{n=-\infty}^{\infty} a_{2n+1} g_{2T}(t + \tau - (2n+1)T) \right) \times \right. \\ &\quad \left. \left(\sum_{m=-\infty}^{\infty} a_{2m} g_{2T}(t - 2mT) - j \sum_{m=-\infty}^{\infty} a_{2m+1} g_{2T}(t - (2m+1)T) \right) \right] \\ &= \sum_{n=-\infty}^{\infty} [g_{2T}(t + \tau - 2nT) g_{2T}(t - 2nT) \\ &\quad + g_{2T}(t + \tau - (2n+1)T) g_{2T}(t - (2n+1)T)] \\ &= \sum_{n=-\infty}^{\infty} g_{2T}(t + \tau - nT) g_{2T}(t - nT) \end{aligned}$$

Same as BPSK with $g(t) = g_{2T}(t)$, therefore we have the same spectrum as in part 1.

3. The only difference here is that instead of the Fourier transform of the rectangular signal we have to use the Fourier transform of the sinusoidal pulse $g_1(t)$, nothing else changes. Noting that $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$ the Fourier transform of the sinusoidal pulse in the problem can be obtained by direct application of the definition of Fourier transform

$$G_1(f) = \int_0^{2T} \sin\left(\frac{\pi t}{2T}\right) e^{-j2\pi f t} dt = \dots = \frac{4T}{\pi} \frac{\cos 2\pi f T}{1 - 16T^2 f^2} e^{-j2\pi f T}$$

and therefore

$$\mathcal{S}(f) = \frac{32T}{\pi^2} \frac{\cos^2 2\pi T f}{(1 - 16T^2 f^2)^2}$$

4. The envelope of X_l , is in general $|X_l(t)|$, so we need to show that $|X_l(t)|$ is independent of t . We have

$$X_l(t) = \sum_{n=-\infty}^{\infty} a_{2n} g_{2T}(t - 2nT) + j \sum_{n=-\infty}^{\infty} a_{2n+1} g_{2T}(t - (2n+1)T)$$

Therefore,

$$\begin{aligned} |X_l(t)|^2 &= \left(\sum_{n=-\infty}^{\infty} a_{2n} g_{2T}(t - 2nT) \right)^2 + \left(\sum_{n=-\infty}^{\infty} a_{2n+1} g_{2T}(t - (2n+1)T) \right)^2 \\ &= \sum_{n=-\infty}^{\infty} g_{2T}^2(t - 2nT) + \sum_{n=-\infty}^{\infty} g_{2T}^2(t - (2n+1)T) \\ &= \left[\sin^2 \left(\frac{\pi t}{2T} \right) + \cos^2 \left(\frac{\pi t}{2T} \right) \right] \quad \text{for all } t \\ &= 1 \end{aligned}$$

where we have used the following facts:

- (a) the duration of $g_{2T}(t)$ is $2T$ and hence $g_{2T}(t - 2nT)$ and $g_{2T}(t - 2mT)$ for $m \neq n$ are non-overlapping and therefore in the expansion of the squares the cross terms vanish (same is true for $g_{2T}(t - (2n+1)T)$ and $g_{2T}(t - (2m+1)T)$.)
- (b) For all n , $a_n^2 = 1$.

Problem 3.27

Since a_n 's are iid

$$R_a(m) = E[a_n a_{n+m}] = \begin{cases} E[a_n^2], & m = 0 \\ E[a_n] E[a_{n+m}], & m \neq 0 \end{cases} = \begin{cases} \frac{1}{2}, & m = 0 \\ \frac{1}{4}, & m \neq 0 \end{cases}$$

1. $b_n = a_{n-1} \oplus a_n$, hence $P(b_n = 0) = P(a_n = a_{n-1} = 0) + P(a_n = a_{n-1} = 1) = \frac{1}{2}$ and $P(b_n = 1) = \frac{1}{2}$. This means that b_n 's individually have the same probabilities as a_n 's but of course unlike a_n 's they are not independent. In fact b_n depends on b_{n-1} and b_{n+1} and since $R_b(m) = E[b_n b_{n+m}] = E[(a_{n-1} \oplus a_n)(a_{n+m-1} \oplus a_{n+m})]$, we conclude that if $m \neq 0, \pm 1$ then $R_b(m) = E[b_n][b_{n+m}] = \frac{1}{4}$ and $R_b(0) = E[b_n^2] = \frac{1}{2}$. On the other hand, for $m = \pm 1$, we have to consider different values that a_{n-1} , a_n , and a_{n+1} can assume. In order for $b_n b_{n+1}$ to be 1, we have to have $b_n = 1$ and $b_{n+1} = 1$. This means that $a_{n-1} = a_{n+1} \neq a_n$ and this can

happen in two cases; $a_{n-1} = a_{n+1} = 1$, $a_n = 0$, and $a_{n-1} = a_{n+1} = 0$, $a_n = 1$, each with probability $1/8$. Therefore, $R_b(\pm 1) = 1/4$ and

$$R_b(m) = \begin{cases} \frac{1}{2}, & m = 0 \\ \frac{1}{4}, & m \neq 0 \end{cases}$$

and $\mathcal{S}(f) = \sum_{m=-\infty}^{\infty} R_b(m) e^{-j2\pi f m T} = \frac{1}{4} + \frac{1}{4} \sum_{m=-\infty}^{\infty} e^{-j2\pi f m T}$. Also, $|G(f)|^2 = T^2 \text{sinc}^2(Tf)$, and

$$\mathcal{S}_v(f) = \frac{1}{T} |G(f)|^2 \mathcal{S}_b(f) = T \text{sinc}^2(Tf) \left[\frac{1}{4} + \frac{1}{4} \sum_{m=-\infty}^{\infty} e^{-j2\pi f m T} \right]$$

We can simplify this using the relation $\sum_{n=-\infty}^{\infty} e^{j2\pi f n T} = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$ to obtain

$$\begin{aligned} \mathcal{S}_v(f) &= T \text{sinc}^2(Tf) \left[\frac{1}{4} + \frac{1}{4T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \right] \\ &= \frac{T}{4} \text{sinc}^2(Tf) + \frac{1}{4} \text{sinc}^2\left(T \frac{m}{T}\right) \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \\ &= \frac{T}{4} \text{sinc}^2(Tf) + \frac{1}{4} \delta(f) \end{aligned}$$

2. Here, $R_b(m) = E[(a_n + a_{n-1})(a_{n+m} + a_{n+m-1})] = 2R_a(m) + R_a(m+1) + R_a(m-1)$, and hence

$$R_b(m) = \begin{cases} \frac{3}{2}, & m = 0 \\ \frac{5}{4}, & m = \pm 1 \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mathcal{S}_b(f) = \sum_{m=-\infty}^{\infty} R_b(m) e^{-j2\pi f m T} = \frac{1}{2} + \frac{1}{4} (e^{j\pi f T} + e^{-j2\pi f T}) + \sum_{m=-\infty}^{\infty} e^{-j2\pi f m T}$$

Therefore,

$$\begin{aligned} \mathcal{S}_v(f) &= T \text{sinc}^2(Tf) \left[\frac{1}{2} + \frac{1}{4} (e^{j2\pi f T} + e^{-j2\pi f T}) + \sum_{m=-\infty}^{\infty} e^{-j2\pi f m T} \right] \\ &= T \text{sinc}^2(Tf) \left[\frac{1}{2} + \frac{1}{2} \cos(2\pi f T) + \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \right] \\ &= \frac{T}{2} \text{sinc}^2(Tf) [1 + \cos(2\pi f T)] + \sum_{m=-\infty}^{\infty} \text{sinc}^2(m) \delta\left(f - \frac{m}{T}\right) \\ &= \frac{T}{2} \text{sinc}^2(Tf) [1 + \cos(2\pi f T)] + \delta(f) \end{aligned}$$

Problem 3.28

Here we have 8 equiprobable symbols given by the eight points in the constellation.

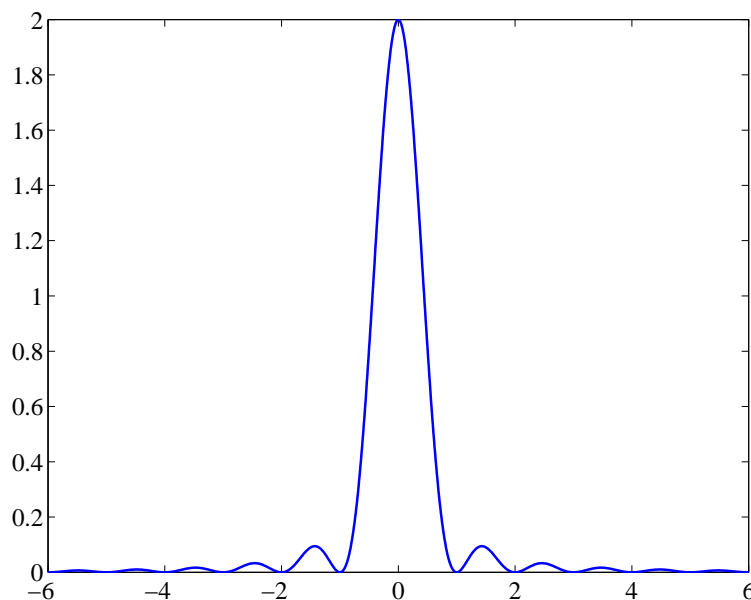
1. Obviously $E(a_n) = 0$ and

$$E[a_n a_m^*] = \begin{cases} E[a_n]E[a_m^*] = 0 & n \neq m \\ E[|a_n|^2] = \frac{r_1^2 + r_2^2}{2} & n = m \end{cases}$$

Hence $\mathcal{S}_a(f) = \frac{r_1^2 + r_2^2}{2}$. Also obviously $|G(f)|^2 = T^2 \text{sinc}^2(Tf)$. Therefore

$$\mathcal{S}_l(f) = T \frac{r_1^2 + r_2^2}{2} \text{sinc}^2(Tf)$$

2. $\mathcal{S}(f) = \frac{1}{4}\mathcal{S}_l(f - f_0) + \frac{1}{4}\mathcal{S}_l(f + f_0)$.
3. In this case $\mathcal{S}_l(f) = Tr^2 \text{sinc}^2(Tf)$. An example of the plot is shown below



Problem 3.29

The MSK and offset QPSK signals have the following form :

$$v(t) = \sum_n [a_n u(t - 2nT) - j b_n u(t - 2nT - T)]$$

where for the QPSK :

$$u(t) = \begin{cases} 1, & 0 \leq t \leq 2T \\ 0, & \text{o.w.} \end{cases}$$

and for MSK :

$$u(t) = \begin{cases} \sin \frac{\pi t}{2T}, & 0 \leq t \leq 2T \\ 0, & \text{o.w.} \end{cases}$$

The derivation is identical to that given in Sec. 3.4.2 with $2T$ substituted for T . Hence, the result is:

$$\begin{aligned} R_{vv}(\tau) &= \frac{1}{2T} \sum_{m=-\infty}^{\infty} R_{ii}(m) R_{uu}(\tau - m2T) \\ &= \frac{1}{2T} \sum_{m=-\infty}^{\infty} (\sigma_a^2 + \sigma_b^2) \delta(m) R_{uu}(\tau - m2T) \\ &= \frac{\sigma_a^2}{T} R_{uu}(\tau) \end{aligned}$$

and :

$$\mathcal{S}_{vv}(f) = \frac{\sigma_a^2}{T} |U(f)|^2$$

For the rectangular pulse of QPSK, we have :

$$R_{uu}(\tau) = 2T \left(1 - \frac{|\tau|}{2T} \right), \quad 0 \leq |\tau| \leq 2T$$

For the MSK pulse :

$$\begin{aligned} R_{uu}(\tau) &= \int_{-\infty}^{\infty} u(t+\tau) u^*(t) dt = \int_0^{2T-\tau} \sin \frac{\pi t}{2T} \sin \frac{\pi(t+\tau)}{2T} dt \\ &= T \left(1 - \frac{|\tau|}{2T} \right) \cos \frac{\pi|\tau|}{2T} + \frac{\tau}{\pi} \sin \frac{\pi|\tau|}{2T} \end{aligned}$$

(check for the 1/2 factor in the defn. Of autocorr. Function!)

Problem 3.30

a. For simplicity we assume binary CPM. Since it is partial response :

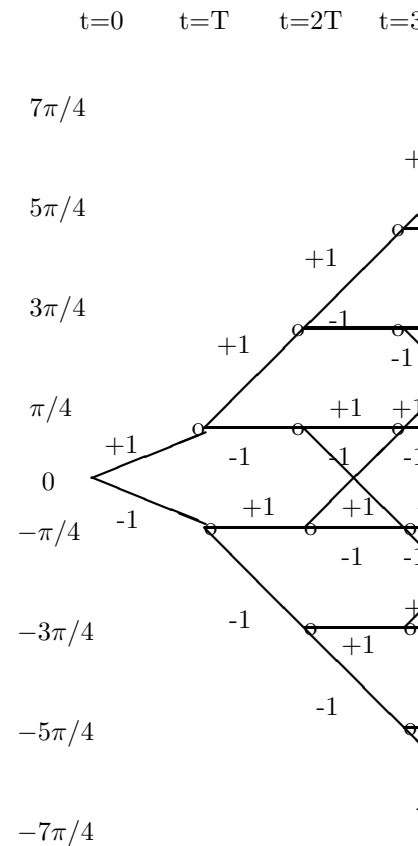
$$\begin{aligned} q(T) &= \int_0^T u(t) dt = 1/4 \\ q(2T) &= \int_0^{2T} u(t) dt = 1/2, \quad q(t) = 1/2, \quad t > 2T \end{aligned}$$

so only the last two symbols will have an effect on the phase :

$$\begin{aligned} R(t; \mathbf{I}) &= 2\pi h \sum_{k=-\infty}^n I_k q(t - kT), \quad nT \leq t \leq nT + T \\ &= \frac{\pi}{2} \sum_{k=-\infty}^{n-2} I_k + \pi (I_{n-1} q(t - (n-1)T) + I_n q(t - nT)), \end{aligned}$$

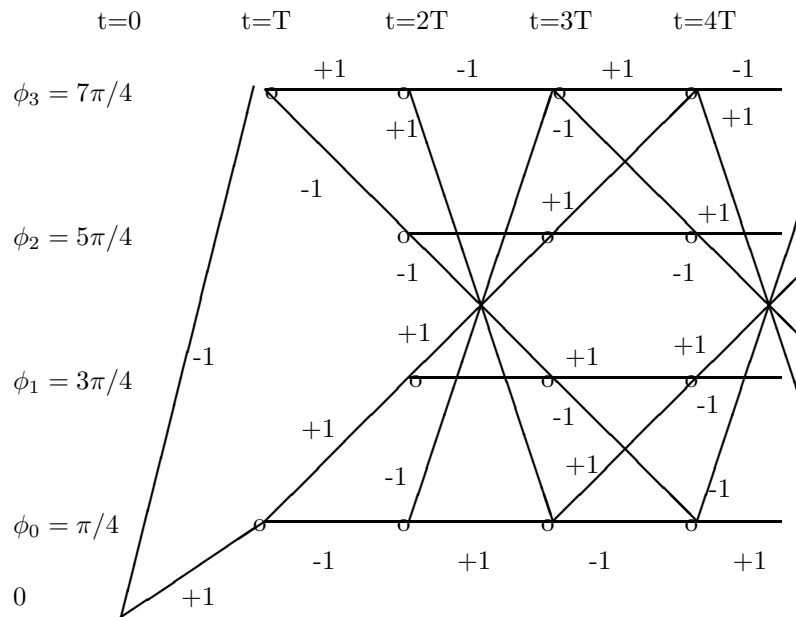
It is easy to see that, after the first symbol, the phase slope is : 0 if I_n, I_{n-1} have different signs, and $\text{sgn}(I_n)\pi/(2T)$ if I_n, I_{n-1} have the same sign. At the terminal point $t = (n+1)T$ the phase is :

$$R((n+1)T; \mathbf{I}) = \frac{\pi}{2} \sum_{k=-\infty}^{n-1} I_k + \frac{\pi}{4} I_n$$

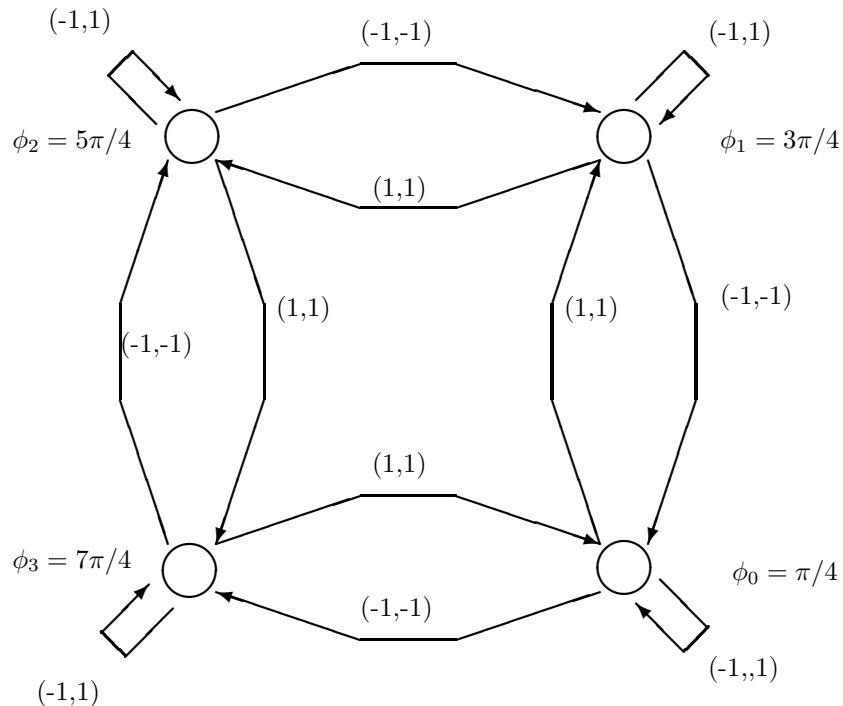


Hence the phase tree is as shown in the following figure :

b. The state trellis is obtained from the phase-tree modulo 2π :



(c) The state diagram is shown in the following figure (with the (I_n, I_{n-1}) or (I_{n-1}, I_n) that cause the respective transitions shown in parentheses)



Problem 3.31

$$R(t; \mathbf{I}) = 2\pi h \sum_{k=-\infty}^n I_k q(t - kT)$$

1. Full response binary CPFSK ($q(T) = 1/2$):

(i) $h = 2/3$. At the end of each bit interval the phase is : $2\pi \frac{2}{3} \frac{1}{2} \sum_{k=-\infty}^n I_k = \frac{2\pi}{3} \sum_{k=-\infty}^n I_k$. Hence the possible terminal phase states are $\{0, 2\pi/3, 4\pi/3\}$.

(ii) $h = 3/4$. At the end of each bit interval the phase is : $2\pi \frac{3}{4} \frac{1}{2} \sum_{k=-\infty}^n I_k = \frac{3\pi}{4} \sum_{k=-\infty}^n I_k$. Hence the possible terminal phase states are $\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}$

2. Partial response $L = 3$, binary CPFSK : $q(T) = 1/6$, $q(2T) = 1/3$, $q(3T) = 1/2$. Hence, at the end of each bit interval the phase is :

$$\pi h \sum_{k=-\infty}^{n-2} I_k + 2\pi h (I_{n-1}/3 + I_n/6) = \pi h \sum_{k=-\infty}^{n-2} I_k + \frac{\pi h}{3} (2I_{n-1} + I_n)$$

The symbol levels in the parenthesis can take the values $\{-3, -1, 1, 3\}$. So :

(i) $h = 2/3$. The possible terminal phase states are :

$$\{0, 2\pi/9, 4\pi/9, 2\pi/3, 8\pi/9, 10\pi/9, 4\pi/3, 14\pi/9, 16\pi/9\}$$

(ii) $h = 3/4$. The possible terminal phase states are : $\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}$

Problem 3.32

We are given by Equation (3.3-33) that the pulses $c_k(t)$ are defined as

$$c_k(t) = s_0(t) \prod_{n=1}^{L-1} s_0[t + (n + La_{k,n})], \quad 0 \leq t \leq T \cdot \min_n [L(2 - a_{k,n} - n)]$$

Hence, the time support of the pulse $c_k(t)$ is

$$0 \leq t \leq T \cdot \min_n [L(2 - a_{k,n} - n)]$$

We need to find the index \hat{n} which minimizes $S = L(2 - a_{k,n}) - n$, or equivalently maximizes $S_1 = La_{k,n} + n$:

$$\hat{n} = \arg \max_n [La_{k,n} + n], \quad n = 1, \dots, L-1, \quad a_{k,n} = 0, 1$$

It is easy to show that

$$\hat{n} = L - 1 \tag{3.0.1}$$

if all $a_{k,n}$, $n = 0, 1, \dots, L-1$ are zero (for a specific k), and

$$\hat{n} = \max \{n : a_{k,n} = 1\} \quad (3.0.2)$$

otherwise.

The first case (1) is shown immediately, since if all $a_{k,n}$, $n = 0, 1, \dots, L-1$ are zero, then $\max_n S_1 = \max_n n$, $n = 0, 1, \dots, L-1$. For the second case (2), assume that there are n_1, n_2 such that : $n_1 < n_2$ and $a_{k,n_1} = 1$, $a_{k,n_2} = 0$. Then $S_1(n_1) = L + n_1 > n_2 (= S_1(n_2))$, since $n_2 - n_1 < L-1$ due to the allowable range of n .

So, finding the binary representation of k , $k = 0, 1, \dots, 2^{L-1} - 1$, we find \hat{n} and the corresponding $S(n)$ which gives the extent of the time support of $c_k(t)$:

$$k = 0 \Rightarrow a_{k,L-1} = 0, \dots, a_{k,2} = 0, \quad a_{k,1} = 0 \Rightarrow \hat{n} = L-1 \Rightarrow S = L+1$$

$$k = 1 \Rightarrow a_{k,L-1} = 0, \dots, a_{k,2} = 0, \quad a_{k,1} = 1 \Rightarrow \hat{n} = 1 \Rightarrow S = L-1$$

$$k = 2/3 \Rightarrow a_{k,L-1} = 0, \dots, a_{k,2} = 1, \quad a_{k,1} = 0/1 \Rightarrow \hat{n} = 2 \Rightarrow S = L-2$$

and so on, using the binary representation of the integers between 1 and $2^{L-1} - 1$.

Problem 3.33

$$s_k(t) = I_k s(t) \Rightarrow S_k(f) = I_k S(f), \quad E(I_k) = \mu_i, \quad \sigma_i^2 = E(I_k^2) - \mu_i^2$$

$$\left| \sum_{k=1}^K p_k S_k(f) \right|^2 = |S(f)|^2 \left| \sum_{k=1}^K p_k I_k \right|^2 = \mu_i^2 |S(f)|^2$$

Therefore, the discrete frequency component becomes :

$$\frac{\mu_i^2}{T^2} \sum_{n=-\infty}^{\infty} \left| S\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right)$$

The continuous frequency component is :

$$\begin{aligned} & \frac{1}{T} \sum_{k=1}^K p_k (1 - p_k) |S_k(f)|^2 - \frac{2}{T} \sum \sum_{i < j} p_i p_j \operatorname{Re} \left[S_i(f) S_j^*(f) \right] \\ &= \frac{1}{T} |S(f)|^2 \left[\sum_{k=1}^K p_k |I_k|^2 - \sum_{k=1}^K p_k^2 |I_k|^2 \right] - \frac{2}{T} \sum \sum_{i < j} p_i p_j |S(f)|^2 \frac{I_i I_j^* + I_i^* I_j}{2} \\ &= \frac{1}{T} |S(f)|^2 \left[\sum_{k=1}^K p_k |I_k|^2 - \sum_{k=1}^K p_k^2 |I_k|^2 - \sum \sum_{i < j} p_i p_j |S(f)| \left(I_i I_j^* + I_i^* I_j \right) \right] \\ &= \frac{1}{T} |S(f)|^2 \left\{ \sum_{k=1}^K p_k |I_k|^2 - \left| \sum_{k=1}^K p_k I_k \right|^2 \right\} \\ &= \frac{\sigma_i^2}{T} |S(f)|^2 \end{aligned}$$

Thus, we have obtained the result in (4.4.18)

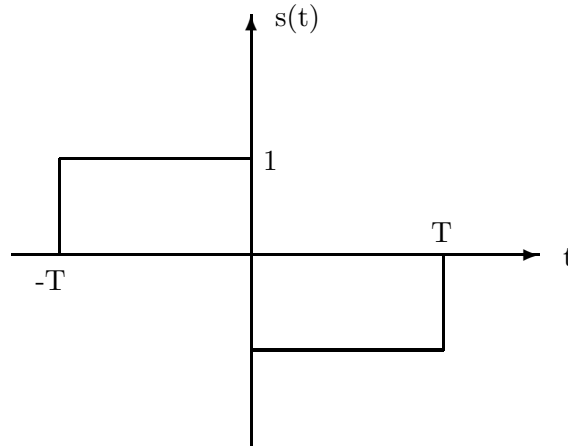
Problem 3.34

The line spectrum in (3.4.27) consists of the term :

$$\frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{k=1}^K p_k S_k \left(\frac{n}{T} \right) \right|^2 \delta \left(f - \frac{n}{T} \right)$$

Now, if $\sum_{k=1}^K p_k s_k(t) = 0$, then $\sum_{k=1}^K p_k S_k(f) = 0$, $\forall f$. Therefore, the condition $\sum_{k=1}^K p_k s_k(t) = 0$ is sufficient for eliminating the line spectrum.

Now, suppose that $\sum_{k=1}^K p_k s_k(t) \neq 0$ for some $t \in [t_0, t_1]$. For example, if $s_k(t) = I_k s(t)$, then $\sum_{k=1}^K p_k s_k(t) = s(t) \sum_{k=1}^K p_k I_k$, where $\sum_{k=1}^K p_k I_k \equiv \mu_i \neq 0$ and $s(t)$ is a signal pulse. Then, the line spectrum vanishes if $S(n/T) = 0$ for all n . A signal pulse that satisfies this condition is shown below :



In this case, $S(f) = T \left(\frac{\sin \pi T f}{\pi T f} \right) \sin \pi T f$, so that $S(n/T) = 0$ for all n . Therefore, the condition $\sum_{k=1}^K p_k s_k(t) = 0$ is not necessary.

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 4) ¹

Prepared by
Kostas Stamatiou

January 11, 2008

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Problem 4.1

$$N_{mr} = \text{Re} \left[\int_0^T z(t) f_m^*(t) dt \right]$$

1. Define $a_m = \int_0^T z(t) f_m^*(t) dt$. Then, $N_{mr} = \text{Re}(a_m) = \frac{1}{2} [a_m + a_m^*]$.

$$E(N_{mr}) = \text{Re} \left[\int_0^T E(z(t)) f_m^*(t) dt \right] = 0$$

since, $E[z(t)] = 0$. Also :

$$E(N_{mr}^2) = E \left[\frac{a_m^2 + (a_m^*)^2 + 2a_m a_m^*}{4} \right]$$

But $E(a_m^2) = E \left[\int_0^T \int_0^T z(a) z(b) f_m^*(a) f_m^*(b) da db \right] = 0$, since $E[z(a)z(b)] = 0$ (Problem 4.3), and the same is true for $E[(a_m^*)^2] = 0$, since $E[z^*(a)z^*(b)] = 0$ Hence :

$$\begin{aligned} E(N_{mr}^2) &= E \left[\frac{a_m a_m^*}{2} \right] = \frac{1}{2} \int_0^T \int_0^T E[z(a)z^*(b)] f_m^*(a) f_m(b) da db \\ &= N_0 \int_0^T |f_m(a)|^2 da = 2\mathcal{E}N_0 \end{aligned}$$

2. For $m \neq k$:

$$\begin{aligned} E[N_{mr}N_{kr}] &= E \left[\frac{a_m + a_m^*}{2} \frac{a_k + a_k^*}{2} \right] \\ &= E \left[\frac{a_m a_k + a_m^* a_k + a_m a_k^* + a_m^* a_k^*}{4} \right] \end{aligned}$$

But, similarly to part (1), $E[a_m a_k] = E[a_m^* a_k^*] = 0$, hence, $E[N_{mr}N_{kr}] = E \left[\frac{a_m^* a_k + a_m a_k^*}{4} \right]$. Now :

$$\begin{aligned} E[a_m a_k^*] &= \int_0^T \int_0^T E[z(a)z^*(b)] f_m^*(a) f_k(b) da db \\ &= 2N_0 \int_0^T f_m^*(a) f_k(a) da = 0 \end{aligned}$$

since, for $m \neq k$, the waveforms are orthogonal.

Similarly : $E[a_m^* a_k] = 0$, hence : $E[N_{mr}N_{kr}] = 0$.

Problem 4.2

Since $\{f_n(t)\}$ constitute an orthonormal basis for the signal space : $r(t) = \sum_{n=1}^N r_n f_n(t)$, $s_m(t) =$

$\sum_{n=1}^N s_{mn} f_n(t)$. Hence, for any m :

$$\begin{aligned}
 C(\mathbf{r}, \mathbf{s}_m) &= 2 \int_0^T r(t) s_m(t) dt - \int_0^T s_m^2(t) dt \\
 &= 2 \int_0^T \sum_{n=1}^N r_n f_n(t) \sum_{l=1}^N s_{ml} f_l(t) dt - \int_0^T \sum_{n=1}^N s_{mn} f_n(t) \sum_{l=1}^N s_{ml} f_l(t) dt \\
 &= 2 \sum_{n=1}^N r_n \sum_{l=1}^N s_{ml} \int_0^T f_n(t) f_l(t) dt - \sum_{n=1}^N s_{mn} \sum_{l=1}^N s_{ml} \int_0^T f_n(t) f_l(t) dt \\
 &= 2 \sum_{n=1}^N r_n s_{mn} - \sum_{n=1}^N s_{mn}^2
 \end{aligned}$$

where we have exploited the orthonormality of $\{f_n(t)\} : \int_0^T f_n(t) f_l(t) dt = \delta_{nl}$. The last form is indeed the original form of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$.

Problem 4.3

$$\mathbf{r} = (r_1, r_2) = (s + n_1, s + n_1 + n_2)$$

The MAP rule for this problem is $\max p(s|\mathbf{r})$. The question is whether decision based on the observation of r_1, r_2 is equivalent to the decision made based on the observation of r_1 alone. This means whether the following conditions are equivalent

$$\begin{aligned}
 p_1 p(r_1, r_2 | s_1) &> p_2 p(r_1, r_2 | s_2) \\
 p_1 p(r_1 | s_1) &> p_2 p(r_2 | s_1)
 \end{aligned}$$

using the chain rule, we check the equivalence of the following

$$\begin{aligned}
 p_1 p(r_1 | s_1) p(r_2 | r_1, s_1) &> p_2 p(r_1 | s_2) p(r_2 | r_1, s_2) \\
 p_1 p(r_1 | s_1) &> p_2 p(r_2 | s_1)
 \end{aligned}$$

or

$$\begin{aligned}
 \frac{p(r_1 | s_1)}{p(r_1 | s_2)} &> \frac{p_2 p(r_2 | r_1, s_2)}{p_1 p(r_2 | r_1, s_1)} \\
 \frac{p(r_1 | s_1)}{p(r_1 | s_2)} &> \frac{p_2}{p_1}
 \end{aligned}$$

For these to be equivalent we have to have $p(r_2 | r_1, s_1) = p(r_2 | r_1, s_2)$ or equivalently if $p(n_2 = r_2 - r_1 | n_1 = r_1 - s_1) = p(n_2 = r_2 - r_1 | n_1 = r_1 - s_2)$ and since $s_1 \neq s_2$ this is equivalent to n_2 being independent of n_1 . Therefore if n_2 and n_1 are independent then r_2 can be ignored, otherwise r_2 has to be used in an optimal decision scheme. A counterexample of dependent noises is the case where $n_2 = -n_1$ and hence $r_2 = s$. Obviously in this case r_2 can not be ignored since it gives the noise-free signal.

Problem 4.4

a. The correlation type demodulator employs a filter :

$$f(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{o.w} \end{cases}$$

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are :

$$r = s_m + n, \quad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance :

$$\sigma_n^2 = \frac{N_0}{2}$$

The probability density function for the sampled output is :

$$\begin{aligned} p(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ p(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\text{PM}(\mathbf{r}, \mathbf{s}_0) = p(r|s_0) > p(r|s_1) = \text{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1 . The decision rule may be expressed as:

$$\frac{\text{PM}(\mathbf{r}, \mathbf{s}_0)}{\text{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \begin{matrix} s_0 \\ > \\ < \\ s_1 \end{matrix} \quad 1$$

or equivalently :

$$\begin{matrix} s_1 \\ r > \frac{1}{2}A\sqrt{T} \\ r < \frac{1}{2}A\sqrt{T} \\ s_0 \end{matrix}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

b. The average probability of error is:

$$\begin{aligned}
 P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
 &= \frac{1}{2} \int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
 &= \frac{1}{2} \int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} dr \\
 &= \frac{1}{2} \int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \frac{1}{2} \int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
 &= Q \left[\frac{1}{2} \sqrt{\frac{2}{N_0}} A\sqrt{T} \right] = Q \left[\sqrt{\text{SNR}} \right]
 \end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

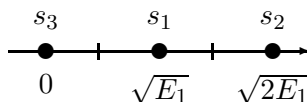
Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 4.5

1. Note that $s_2(t) = 2s_1(t)$ and $s_3(t) = 0s_1(t)$, hence the system is PAM and a singular basis function of the form $\phi_1(t) = \frac{1}{A\sqrt{T}}s_1(t)$ would work

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 < t \leq T/3 \\ -\frac{1}{\sqrt{T}} & T/3 \leq t < T \end{cases}$$

Assuming $E_1 = A^2T$, we have $s_3 = 0$, $s_1 = \sqrt{E_1}$, $s_2 = 2\sqrt{E_1}$. The constellation is shown below.



2. For equiprobable messages the optimal decision rule is the nearest neighbor rule and the perpendicular bisectors are the boundaries of the decision regions as indicated in the figure.
3. This is ternary PAM system with the distance between adjacent points in the constellation being $d = \sqrt{E_1} = A\sqrt{T}$. The average energy is $E_{\text{avg}} = \frac{1}{3}(0 + A^2T + 4A^2T) = \frac{5}{3}A^2T$, and $E_{\text{bavg}} = E_{\text{avg}}/\log_2 3 = \frac{5}{3\log_2 3}A^2T$, from which we obtain

$$d^2 = \frac{3\log_2 3}{5} E_{\text{bavg}} \approx 0.951 E_{\text{bavg}}$$

The error probability of the optimal detector is the average of the error probabilities of the three signals. For the two outer signals error probability is $P(n > d/2) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$ and for the middle point s_1 it is $P(|n| > d/2) = 2Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$. From this,

$$P_e = \frac{4}{3}Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = \frac{4}{3}Q\left(\sqrt{\frac{0.951E_{\text{bavg}}}{2N_0}}\right) = 43Q\left(\sqrt{0.475\frac{E_{\text{bavg}}}{2N_0}}\right)$$

4. $R = R_s \log_2 M = 3000 \times \log_2 3 \approx 4755$ bps.

Problem 4.6

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \implies A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 4.7

1. The PDF of the noise n is :

$$p(n) = \frac{\lambda}{2}e^{-\lambda|n|}$$

where $\lambda = \frac{\sqrt{2}}{\sigma}$ The optimal receiver uses the criterion :

$$\frac{p(r|A)}{p(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \begin{matrix} A \\ > \\ < \\ -A \end{matrix} \begin{matrix} A \\ > \\ < \\ -A \end{matrix} 1 \implies r \begin{matrix} A \\ > \\ < \\ -A \end{matrix} 0$$

The average probability of error is :

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\ &= \frac{1}{2} \int_{-\infty}^0 f(r|A)dr + \frac{1}{2} \int_0^{\infty} f(r|-A)dr \\ &= \frac{1}{2} \int_{-\infty}^0 \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2} \int_0^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\ &= \frac{\lambda}{4} \int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4} \int_A^{\infty} e^{-\lambda|x|}dx \\ &= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{-\frac{\sqrt{2}A}{\sigma}} \end{aligned}$$

2. The variance of the noise is :

$$\begin{aligned} \sigma_n^2 &= \frac{\lambda}{2} \int_{-\infty}^{\infty} e^{-\lambda|x|}x^2dx \\ &= \lambda \int_0^{\infty} e^{-\lambda x}x^2dx = \lambda \frac{2!}{\lambda^3} = \frac{2}{\lambda^2} = \sigma^2 \end{aligned}$$

Hence, the SNR is:

$$\text{SNR} = \frac{A^2}{\sigma^2}$$

and the probability of error is given by:

$$P(e) = \frac{1}{2}e^{-\sqrt{2\text{SNR}}}$$

For $P(e) = 10^{-5}$ we obtain:

$$\ln(2 \times 10^{-5}) = -\sqrt{2\text{SNR}} \implies \text{SNR} = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then the probability of error for antipodal signalling is:

$$P(e) = Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] = Q \left[\sqrt{\text{SNR}} \right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{\text{SNR}} = 4.26$ and therefore $\text{SNR} = 18.1476 = 12.594 \text{ dB}$. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 4.8

1. Since $d_{\min} = 2A$, from the union bound we have $P_e \leq 15Q\left(\sqrt{d_{\min}^2/2N_0}\right) = 15Q\left(\sqrt{2A^2/N_0}\right)$.
2. Three levels of energy are present, $E_1 = A^2 + A^2 = 2A^2$, $E_2 = A^2 + 9A^2 = 10A^2$, and $E_3 = 9A^2 + 9A^2 = 18A^2$. The average energy is $E_{\text{avg}} = \frac{1}{4}E_1 + \frac{1}{2}E_2 + \frac{1}{4}E_3 = 10A^2$. Therefore, $E_{\text{bavg}} = E_{\text{avg}}/\log_2 16 = 2.5A^2$.
3. $P_e \leq 15Q\left(\sqrt{2A^2/N_0}\right) = 15Q\left(\sqrt{4E_{\text{bavg}}/5N_0}\right)$.
4. For a 16-level PAL system

$$P_e \approx 2Q\left(\sqrt{\frac{6 \log_2 M}{M^2 - 1} \frac{E_{\text{bavg}}}{N_0}}\right) = 2Q\left(\sqrt{\frac{24}{255} \frac{E_{\text{bavg}}}{N_0}}\right)$$

The difference is $\frac{4/5}{24/255} = 255/30 \approx 8.5 \sim 9.3$ dB

Problem 4.9

1. $U = \text{Re} \left[\int_0^T r(t)s^*(t)dt \right]$, where $r(t) = \begin{Bmatrix} s(t) + z(t) \\ -s(t) + z(t) \\ z(t) \end{Bmatrix}$ depending on which signal was sent.

If we assume that $s(t)$ was sent :

$$U = \text{Re} \left[\int_0^T s(t)s^*(t)dt \right] + \text{Re} \left[\int_0^T z(t)s^*(t)dt \right] = 2E + N$$

where $E = \frac{1}{2} \int_0^T s(t)s^*(t)dt$, and $N = \text{Re} \left[\int_0^T z(t)s^*(t)dt \right]$ is a Gaussian random variable with zero mean and variance $2EN_0$ (as we have seen in Problem 5.7). Hence, given that $s(t)$ was sent, the probability of error is :

$$P_{e1} = P(2E + N < A) = P(N < -(2E - A)) = Q\left(\frac{2E - A}{\sqrt{2N_0E}}\right)$$

When $-s(t)$ is transmitted : $U = -2E + N$, and the corresponding conditional error probability is :

$$P_{e2} = P(-2E + N > -A) = P(N > (2E - A)) = Q\left(\frac{2E - A}{\sqrt{2N_0E}}\right)$$

and finally, when 0 is transmitted : $U = N$, and the corresponding error probability is :

$$P_{e3} = P(N > A \text{ or } N < -A) = 2P(N > A) = 2Q\left(\frac{A}{\sqrt{2N_0E}}\right)$$

2.

$$P_e = \frac{1}{3}(P_{e1} + P_{e2} + P_{e3}) = \frac{2}{3}\left[Q\left(\frac{2E - A}{\sqrt{2N_0E}}\right) + Q\left(\frac{A}{\sqrt{2N_0E}}\right)\right]$$

3. In order to minimize P_e :

$$\frac{dP_e}{dA} = 0 \Rightarrow A = E$$

where we differentiate $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt$ with respect to x , using the Leibnitz rule : $\frac{d}{dx} \left(\int_{f(x)}^\infty g(a)da \right) = -\frac{df}{dx}g(f(x))$. Using this threshold :

$$P_e = \frac{4}{3}Q\left(\frac{E}{\sqrt{2N_0E}}\right) = \frac{4}{3}Q\left(\sqrt{\frac{E}{2N_0}}\right)$$

Problem 4.10

1. The transmitted energy is :

$$\begin{aligned}\mathcal{E}_1 &= \frac{1}{2} \int_0^T |s_1(t)|^2 dt = A^2T/2 \\ \mathcal{E}_2 &= \frac{1}{2} \int_0^T |s_2(t)|^2 dt = A^2T/2\end{aligned}$$

2. The correlation coefficient for the two signals is :

$$\rho = \frac{1}{2\mathcal{E}} \int_0^T s_1(t)s_2^*(t)dt = 1/2$$

Hence, the bit error probability for coherent detection is :

$$P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}(1 - \rho)}\right) = Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right)$$

3. The bit error probability for non-coherent detection is given by (5-4-53) :

$$P_{2,nc} = Q_1(a, b) - \frac{1}{2}e^{-(a^2+b^2)/2}I_0(ab)$$

where $Q_1(\cdot)$ is the generalized Marcum Q function (given in (2-1-123)) and :

$$a = \sqrt{\frac{\mathcal{E}}{2N_0} \left(1 - \sqrt{1 - |\rho|^2}\right)} = \sqrt{\frac{\mathcal{E}}{2N_0} \left(1 - \frac{\sqrt{3}}{2}\right)}$$

$$b = \sqrt{\frac{\mathcal{E}}{2N_0} \left(1 + \sqrt{1 - |\rho|^2}\right)} = \sqrt{\frac{\mathcal{E}}{2N_0} \left(1 + \frac{\sqrt{3}}{2}\right)}$$

Problem 4.11

1. Taking the inverse Fourier transform of $H(f)$, we obtain :

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \text{sgn}(t) - \text{sgn}(t - T) = 2\Pi\left(\frac{t - \frac{T}{2}}{T}\right) \end{aligned}$$

where $\text{sgn}(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ is a rectangular pulse of unit height and width, centered at $x = 0$.

2. The signal waveform, to which $h(t)$ is matched, is :

$$s(t) = h(T - t) = 2\Pi\left(\frac{T - t - \frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2} - t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t - \frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 4.12

1. The impulse response of the matched filter is :

$$h(t) = s(T - t) = \begin{cases} \frac{A}{T}(T - t) \cos(2\pi f_c(T - t)) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

2. The output of the matched filter at $t = T$ is :

$$\begin{aligned}
 g(T) &= h(t) \star s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \\
 &= \frac{A^2}{T^2} \int_0^T (T-\tau)^2 \cos^2(2\pi f_c(T-\tau))d\tau \\
 &\stackrel{v=T-\tau}{=} \frac{A^2}{T^2} \int_0^T v^2 \cos^2(2\pi f_c v)dv \\
 &= \frac{A^2}{T^2} \left[\frac{v^3}{6} + \left(\frac{v^2}{4 \times 2\pi f_c} - \frac{1}{8 \times (2\pi f_c)^3} \right) \sin(4\pi f_c v) + \frac{v \cos(4\pi f_c v)}{4(2\pi f_c)^2} \right] \Big|_0^T \\
 &= \frac{A^2}{T^2} \left[\frac{T^3}{6} + \left(\frac{T^2}{4 \times 2\pi f_c} - \frac{1}{8 \times (2\pi f_c)^3} \right) \sin(4\pi f_c T) + \frac{T \cos(4\pi f_c T)}{4(2\pi f_c)^2} \right]
 \end{aligned}$$

3. The output of the correlator at $t = T$ is :

$$\begin{aligned}
 q(T) &= \int_0^T s^2(\tau)d\tau \\
 &= \frac{A^2}{T^2} \int_0^T \tau^2 \cos^2(2\pi f_c \tau)d\tau
 \end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

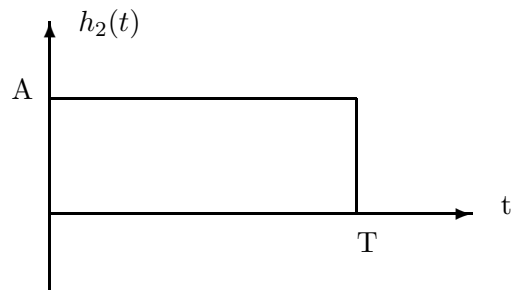
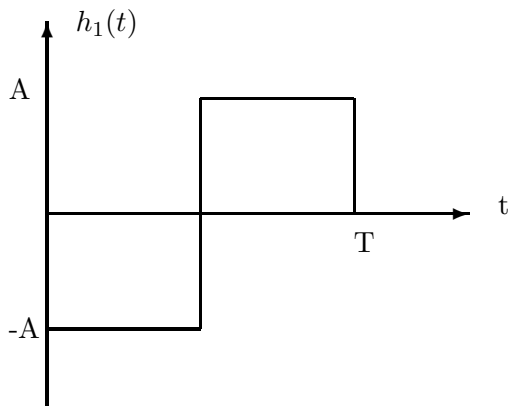
Problem 4.13

1. Since the given waveforms are the equivalent lowpass signals :

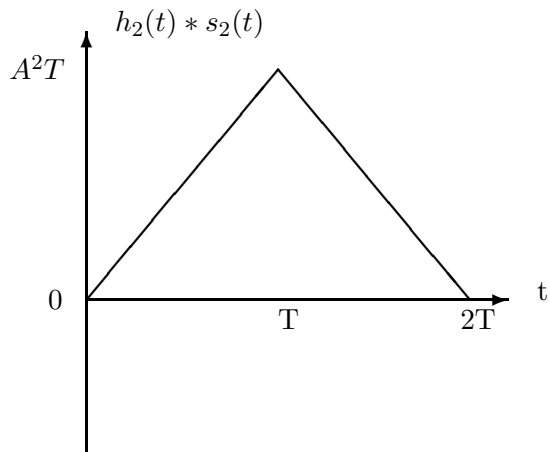
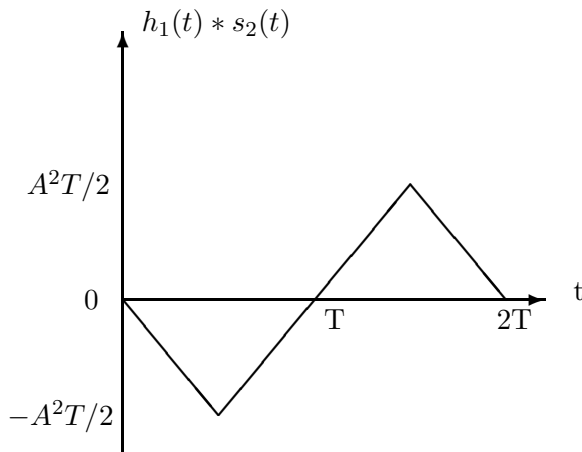
$$\begin{aligned}
 \mathcal{E}_1 &= \frac{1}{2} \int_0^T |s_1(t)|^2 dt = \frac{1}{2} A^2 \int_0^T dt = A^2 T/2 \\
 \mathcal{E}_2 &= \frac{1}{2} \int_0^T |s_2(t)|^2 dt = \frac{1}{2} A^2 \int_0^T dt = A^2 T/2
 \end{aligned}$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also $\rho_{12} = \frac{1}{2\mathcal{E}} \int_0^T s_1(t)s_2^*(t)dt = 0$.

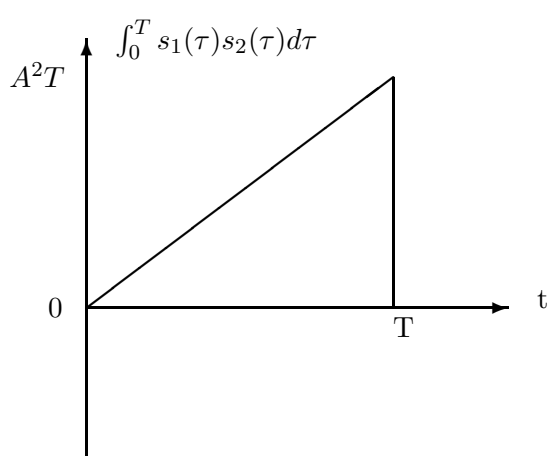
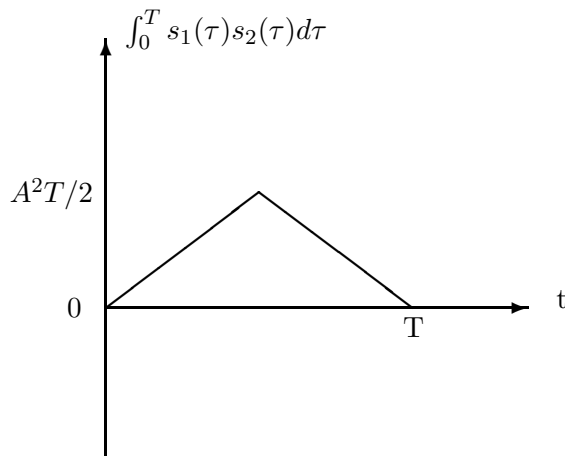
2. Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T-t)$. The following figure shows $h_i(t)$:



3.



4.



5. The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

6. Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.

Problem 4.14

1. This is binary antipodal with equal probabilities, so $r_{th} = 0$ independent of noise level, and $P_e = Q\left(\sqrt{\frac{2E_p}{N_0}}\right)$.
2. The threshold in equiprobable antipodal signaling is zero, and independent of noise level, therefore the receiver of part (1) is also the optimal receiver in this case but since noise is different we have $P_1 = Q\left(\sqrt{\frac{2E_p}{N_1}}\right)$ which is higher than P_e since $N_1 > N_0$ and Q is decreasing.
3. As stated before the optimal receiver in this case is the same as the one designed in part (1). Therefore $P_{e1} = P_1$.
4. When the probabilities are not equal the threshold is $r_{th} = \frac{N_0}{4\sqrt{E_p}} \ln \frac{1-p}{p}$ which now *depends* on the noise level. Therefore the optimal receiver designed in case 1 will not be optimal in case 2. In this case we have

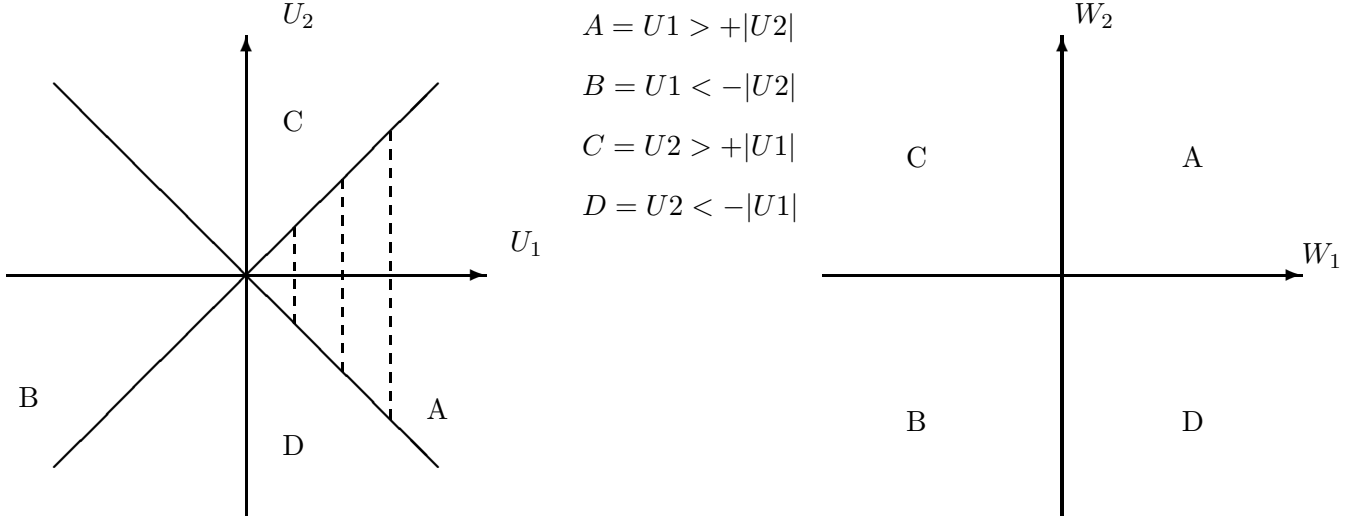
$$P_e = pQ\left(\frac{\sqrt{E_p} - r_{th}}{\sqrt{\frac{N_0}{2}}}\right) + (1-p)Q\left(\frac{\sqrt{E_p} + r_{th}}{\sqrt{\frac{N_0}{2}}}\right)$$

$$P_1 = pQ\left(\frac{\sqrt{E_p} - r_{th}}{\sqrt{\frac{N_1}{2}}}\right) + (1-p)Q\left(\frac{\sqrt{E_p} + r_{th}}{\sqrt{\frac{N_1}{2}}}\right)$$

If we assume that we had designed optimal detector for noise level N_1 , i.e., we had selected $r_{th} = \frac{N_1}{4\sqrt{E_p}} \ln \frac{1-p}{p}$ instead of $r_{th} = \frac{N_0}{4\sqrt{E_p}} \ln \frac{1-p}{p}$, then the probability of error P_{e1} would have been larger than P_e due to higher noise level. But in this case we are using a suboptimal receiver for noise level N_1 ; hence the error probability is even higher than P_{e1} . Therefore we can say $P_1 > P_{e1} > P_e$.

Problem 4.15

The following graph shows the decision regions for the four signals :



As we see, using the transformation $W_1 = U_1 + U_2$, $W_2 = U_1 - U_2$ alters the decision regions to : $(W_1 > 0, W_2 > 0 \rightarrow s_1(t); W_1 > 0, W_2 < 0 \rightarrow s_2(t); \text{ etc.})$. Assuming that $s_1(t)$ was transmitted, the outputs of the matched filters will be :

$$U_1 = 2\mathcal{E} + N_{1r}$$

$$U_2 = N_{2r}$$

where N_{1r}, N_{2r} are uncorrelated (Prob. 5.7) Gaussian-distributed terms with zero mean and variance $2\mathcal{E}N_0$. Then :

$$W_1 = 2\mathcal{E} + (N_{1r} + N_{2r})$$

$$W_2 = 2\mathcal{E} + (N_{1r} - N_{2r})$$

will be Gaussian distributed with means : $E[W_1] = E[W_2] = 2\mathcal{E}$, and variances : $E[W_1^2] = E[W_2^2] = 4\mathcal{E}N_0$. Since U_1, U_2 are independent, it is straightforward to prove that W_1, W_2 are independent, too. Hence, the probability that a correct decision is made, assuming that $s_1(t)$ was transmitted is :

$$\begin{aligned}
 P_{c|s1} &= P[W_1 > 0] P[W_2 > 0] = (P[W_1 > 0])^2 \\
 &= (1 - P[W_1 < 0])^2 = \left(1 - Q\left(\frac{2\mathcal{E}}{\sqrt{4\mathcal{E}N_0}}\right)\right)^2 \\
 &= \left(1 - Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)\right)^2 = \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2
 \end{aligned}$$

where $\mathcal{E}_b = \mathcal{E}/2$ is the transmitted energy per bit. Then :

$$P_{e|s1} = 1 - P_{c|s1} = 1 - \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]$$

This is the exact symbol error probability for the 4-PSK signal, which is expected since the vector space representations of the 4-biorthogonal and 4-PSK signals are identical.

Problem 4.16

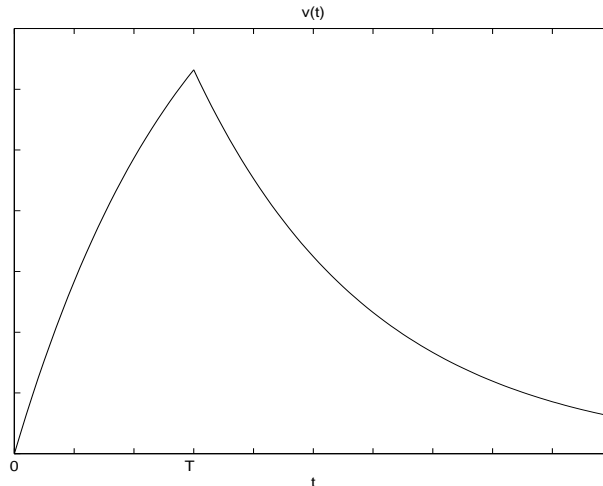
1. The output of the matched filter can be expressed as :

$$y(t) = \text{Re} \left[v(t) e^{j2\pi f_c t} \right]$$

where $v(t)$ is the lowpass equivalent of the output :

$$v(t) = \int_0^t s_0(\tau) h(t - \tau) d\tau = \begin{cases} \int_0^t A e^{-(t-\tau)/T} d\tau = AT (1 - e^{-t/T}), & 0 \leq t \leq T \\ \int_0^T A e^{-(t-\tau)/T} d\tau = AT(e - 1)e^{-t/T}, & T \leq t \end{cases}$$

2. A sketch of $v(t)$ is given in the following figure :



3. $y(t) = v(t) \cos 2\pi f_c t$, where $f_c \gg 1/T$. Hence the maximum value of y corresponds to the maximum value of v , or $y_{\max} = y(T) = v_{\max} = v(T) = AT(1 - e^{-1})$.

4. Working with lowpass equivalent signals, the noise term at the sampling instant will be :

$$v_N(T) = \int_0^T z(\tau) h(T - \tau) d\tau$$

The mean is : $E[v_N(T)] = \int_0^T E[z(\tau)] h(T - \tau) d\tau = 0$, and the second moment :

$$\begin{aligned} E[|v_N(T)|^2] &= E \left[\int_0^T z(\tau) h(T - \tau) d\tau \int_0^T z^*(w) h(T - w) dw \right] \\ &= 2N_0 \int_0^T h^2(T - \tau) d\tau \\ &= N_0 T (1 - e^{-2}) \end{aligned}$$

The variance of the real-valued noise component can be obtained using the relationship $\text{Re}[N] = \frac{1}{2}(N + N^*)$ to obtain : $\sigma_{Nr}^2 = \frac{1}{2} E[|v_N(T)|^2] = \frac{1}{2} N_0 T (1 - e^{-2})$

5. The SNR is defined as :

$$\gamma = \frac{|v_{\max}|^2}{E[|v_N(T)|^2]} = \frac{A^2 T}{N_0} \frac{e - 1}{e + 1}$$

(the same result is obtained if we consider the real bandpass signal, when the energy term has the additional factor 1/2 compared to the lowpass energy term, and the noise term is $\sigma_{N_r}^2 = \frac{1}{2} E[|v_N(T)|^2]$)

6. If we have a filter matched to $s_0(t)$, then the output of the noise-free matched filter will be :

$$v_{\max} = v(T) = \int_0^T s_o^2(t) dt = A^2 T$$

and the noise term will have second moment :

$$\begin{aligned} E[|v_N(T)|^2] &= E\left[\int_0^T z(\tau) s_0(T - \tau) d\tau \int_0^T z^*(w) s_0(T - w) dw\right] \\ &= 2N_0 \int_0^T s_0^2(T - \tau) d\tau \\ &= 2N_0 A^2 T \end{aligned}$$

giving an SNR of :

$$\gamma = \frac{|v_{\max}|^2}{E[|v_N(T)|^2]} = \frac{A^2 T}{2N_0}$$

Compared with the result we obtained in (e), using a sub-optimum filter, the loss in SNR is equal to : $\left(\frac{e-1}{e+1}\right) \left(\frac{1}{2}\right)^{-1} = 0.925$ or approximately 0.35 dB

Problem 4.17

The SNR at the filter output will be :

$$SNR = \frac{|y(T)|^2}{E[|n(T)|^2]}$$

where $y(t)$ is the part of the filter output that is due to the signal $s_l(t)$, and $n(t)$ is the part due to the noise $z(t)$. The denominator is :

$$\begin{aligned} E[|n(T)|^2] &= \int_0^T \int_0^T E[z(a)z^*(b)] h_l(T - a) h_l^*(T - b) da db \\ &= 2N_0 \int_0^T |h_l(T - t)|^2 dt \end{aligned}$$

so we want to maximize :

$$SNR = \frac{\left| \int_0^T s_l(t) h_l(T - t) dt \right|^2}{2N_0 \int_0^T |h_l(T - t)|^2 dt}$$

From Schwartz inequality :

$$\left| \int_0^T s_l(t)h_l(T-t)dt \right|^2 \leq \int_0^T |h_l(T-t)|^2 dt \int_0^T |s_l(t)|^2 dt$$

Hence :

$$SNR \leq \frac{1}{2N_0} \int_0^T |s_l(t)|^2 dt = \frac{\mathcal{E}}{N_0} = SNR_{\max}$$

and the maximum occurs when :

$$s_l(t) = h_l^*(T-t) \Leftrightarrow h_l(t) = s_l^*(T-t)$$

Problem 4.18

The correlation of the two signals in binary FSK is:

$$\rho = \frac{\sin(2\pi\Delta fT)}{2\pi\Delta fT}$$

To find the minimum value of the correlation, we set the derivative of ρ with respect to Δf equal to zero. Thus:

$$\frac{\partial \rho}{\partial \Delta f} = 0 = \frac{\cos(2\pi\Delta fT)2\pi T}{2\pi\Delta fT} - \frac{\sin(2\pi\Delta fT)}{(2\pi\Delta fT)^2}2\pi T$$

and therefore :

$$2\pi\Delta fT = \tan(2\pi\Delta fT)$$

Solving numerically (or graphically) the equation $x = \tan(x)$, we obtain $x = 4.4934$. Thus,

$$2\pi\Delta fT = 4.4934 \implies \Delta f = \frac{0.7151}{T}$$

and the value of ρ is -0.2172 .

We know that the probability of error can be expressed in terms of the distance d_{12} between the signal points, as :

$$P_e = Q \left[\sqrt{\frac{d_{12}^2}{2N_0}} \right]$$

where the distance between the two signal points is :

$$d_{12}^2 = 2\mathcal{E}_b(1 - \rho)$$

and therefore :

$$P_e = Q \left[\sqrt{\frac{2\mathcal{E}_b(1 - \rho)}{2N_0}} \right] = Q \left[\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}} \right]$$

Problem 4.19

1. It is straightforward to see that :

Set I : Four – level PAM

Set II : Orthogonal

Set III : Biorthogonal

2. The transmitted waveforms in the first set have energy : $\frac{1}{2}A^2$ or $\frac{1}{2}9A^2$. Hence for the first set the average energy is :

$$\mathcal{E}_1 = \frac{1}{4} \left(2\frac{1}{2}A^2 + 2\frac{1}{2}9A^2 \right) = 2.5A^2$$

All the waveforms in the second and third sets have the same energy : $\frac{1}{2}A^2$. Hence :

$$\mathcal{E}_2 = \mathcal{E}_3 = A^2/2$$

3. The average probability of a symbol error for M-PAM is (5-2-45) :

$$P_{4,PAM} = \frac{2(M-1)}{M} Q \left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}} \right) = \frac{3}{2} Q \left(\sqrt{\frac{A^2}{N_0}} \right)$$

4. For coherent detection, a union bound can be given by (5-2-25) :

$$P_{4,orth} < (M-1) Q \left(\sqrt{\mathcal{E}_s/N_0} \right) = 3Q \left(\sqrt{\frac{A^2}{2N_0}} \right)$$

while for non-coherent detection :

$$P_{4,orth,nc} \leq (M-1) P_{2,nc} = 3\frac{1}{2}e^{-\mathcal{E}_s/2N_0} = \frac{3}{2}e^{-A^2/4N_0} \quad ??$$

5. It is not possible to use non-coherent detection for a biorthogonal signal set : e.g. without phase knowledge, we cannot distinguish between the signals $u_1(t)$ and $u_3(t)$ (or $u_2(t)/u_4(t)$).

6. The bit rate to bandwidth ratio for M-PAM is given by (5-2-85) :

$$\left(\frac{R}{W} \right)_1 = 2 \log_2 M = 2 \log_2 4 = 4$$

For orthogonal signals we can use the expression given by (5-2-86) or notice that we use a symbol interval 4 times larger than the one used in set I, resulting in a bit rate 4 times smaller :

$$\left(\frac{R}{W} \right)_2 = \frac{2 \log_2 M}{M} = 1$$

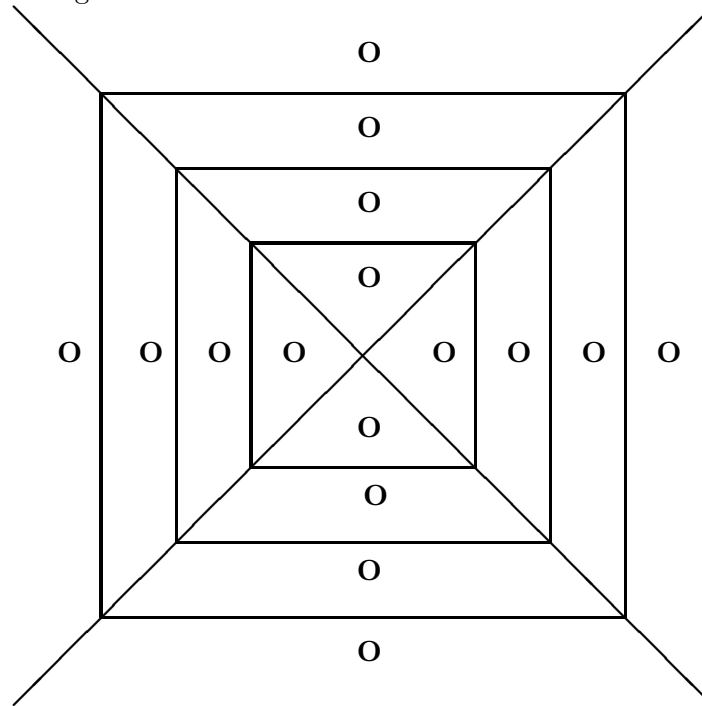
Finally, the biorthogonal set has double the bandwidth efficiency of the orthogonal set :

$$\left(\frac{R}{W}\right)_3 = 2$$

Hence, set I is the most bandwidth efficient (at the expense of larger average power), but set III will also be satisfactory.

Problem 4.20

The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for this QAM constellation are depicted in the next figure:



Problem 4.21

The transmitted signal energy is

$$\mathcal{E}_b = \frac{A^2 T}{2}$$

where T is the bit interval and A is the signal amplitude. Since both carriers are used to transmit information over the same channel, the bit SNR, $\frac{\mathcal{E}_b}{N_0}$, is constant if A^2T is constant. Hence, the desired relation between the carrier amplitudes and the supported transmission rate $R = \frac{1}{T}$ is

$$\frac{A_c}{A_s} = \sqrt{\frac{T_s}{T_c}} = \sqrt{\frac{R_c}{R_s}}$$

With

$$\frac{R_c}{R_s} = \frac{10 \times 10^3}{100 \times 10^3} = 0.1$$

we obtain

$$\frac{A_c}{A_s} = 0.3162$$

Problem 4.22

1. If the power spectral density of the additive noise is $\mathcal{S}_n(f)$, then the PSD of the noise at the output of the prewhitening filter is

$$\mathcal{S}_\nu(f) = \mathcal{S}_n(f)|H_p(f)|^2$$

In order for $\mathcal{S}_\nu(f)$ to be flat (white noise), $H_p(f)$ should be such that

$$H_p(f) = \frac{1}{\sqrt{\mathcal{S}_n(f)}}$$

2. Let $h_p(t)$ be the impulse response of the prewhitening filter $H_p(f)$. That is, $h_p(t) = \mathcal{F}^{-1}[H_p(f)]$. Then, the input to the matched filter is the signal $\tilde{s}(t) = s(t) \star h_p(t)$. The frequency response of the filter matched to $\tilde{s}(t)$ is

$$\tilde{S}_m(f) = \tilde{S}^*(f)e^{-j2\pi ft_0} = S^*(f)H_p^*(f)e^{-j2\pi ft_0}$$

where t_0 is some nominal time-delay at which we sample the filter output.

3. The frequency response of the overall system, prewhitening filter followed by the matched filter, is

$$G(f) = \tilde{S}_m(f)H_p(f) = S^*(f)|H_p(f)|^2e^{-j2\pi ft_0} = \frac{S^*(f)}{\mathcal{S}_n(f)}e^{-j2\pi ft_0}$$

4. The variance of the noise at the output of the generalized matched filter is

$$\sigma^2 = \int_{-\infty}^{\infty} \mathcal{S}_n(f)|G(f)|^2 df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\mathcal{S}_n(f)} df$$

At the sampling instant $t = t_0 = T$, the signal component at the output of the matched filter is

$$\begin{aligned} y(T) &= \int_{-\infty}^{\infty} Y(f) e^{j2\pi fT} df = \int_{-\infty}^{\infty} s(\tau) g(T - \tau) d\tau \\ &= \int_{-\infty}^{\infty} S(f) \frac{S^*(f)}{\mathcal{S}_n(f)} df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\mathcal{S}_n(f)} df \end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{y^2(T)}{\sigma^2} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\mathcal{S}_n(f)} df$$

Problem 4.23

1. The number of bits per symbol is

$$k = \frac{4800}{R} = \frac{4800}{2400} = 2$$

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with $M = 2^k$, is

$$P_M = 1 - \left(1 - 2 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left[\sqrt{\frac{3k\mathcal{E}_b}{(M-1)N_0}} \right] \right)^2$$

With $P_M = 10^{-5}$ and $k = 2$ we obtain

$$Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] = 5 \times 10^{-6} \implies \frac{\mathcal{E}_b}{N_0} = 9.7682$$

2. If the bit rate of transmission is 9600 bps, then

$$k = \frac{9600}{2400} = 4$$

In this case a 16-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2 \left(1 - \frac{1}{4} \right) Q \left[\sqrt{\frac{3 \times 4 \times \mathcal{E}_b}{15 \times N_0}} \right] \right)^2$$

Thus,

$$Q \left[\sqrt{\frac{3 \times \mathcal{E}_b}{15 \times N_0}} \right] = \frac{1}{3} \times 10^{-5} \implies \frac{\mathcal{E}_b}{N_0} = 25.3688$$

3. If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2 \left(1 - \frac{1}{16} \right) Q \left[\sqrt{\frac{3 \times 8 \times \mathcal{E}_b}{255 \times N_0}} \right] \right)^2$$

With $P_M = 10^{-5}$ we obtain

$$\frac{\mathcal{E}_b}{N_0} = 659.8922$$

4. The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

k	2	4	8
SNR (db)	9.89	14.04	28.19

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

Problem 4.24

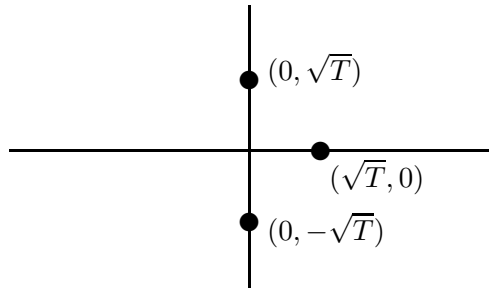
1. Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.
2. As a basis of the signal space we consider the functions:

$$f_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \quad f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is:

$$\begin{aligned} \mathbf{m}_1 &= [\sqrt{T}, 0] \\ \mathbf{m}_2 &= [0, \sqrt{T}] \\ \mathbf{m}_3 &= [0, -\sqrt{T}] \end{aligned}$$

3. The signal constellation is depicted in the next figure :



4. The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T} + n_1, n_2)$, $(n_1, \sqrt{T} + n_2)$ and $(n_1, -\sqrt{T} + n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric (see 5-1-44):

$$C(\mathbf{r}, \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

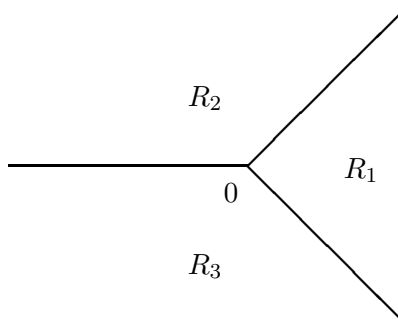
or since $|\mathbf{m}_i|^2$ is the same for all i ,

$$C'(\mathbf{r}, \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for \mathbf{m}_1 is the set of points (r_1, r_2) , such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T}r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T}r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T}r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.



(e) If the signals are equiprobable then:

$$P(e|\mathbf{m}_1) = P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_2|^2 | \mathbf{m}_1) + P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_3|^2 | \mathbf{m}_1)$$

When \mathbf{m}_1 is transmitted then $\mathbf{r} = [\sqrt{T} + n_1, n_2]$ and therefore, $P(e|\mathbf{m}_1)$ is written as:

$$P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, n_1, n_2 are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance N_0 . Hence:

$$\begin{aligned} P(e|\mathbf{m}_1) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{T}}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{T}{N_0}}\right] = 2Q\left[\sqrt{\frac{T}{N_0}}\right] \end{aligned}$$

When \mathbf{m}_2 is transmitted then $\mathbf{r} = [n_1, n_2 + \sqrt{T}]$ and therefore:

$$\begin{aligned} P(e|\mathbf{m}_2) &= P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T}) \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right] \end{aligned}$$

Similarly from the symmetry of the problem, we obtain:

$$P(e|\mathbf{m}_2) = P(e|\mathbf{m}_3) = Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right]$$

Since $Q[\cdot]$ is monotonically decreasing, we obtain:

$$Q\left[\sqrt{\frac{2T}{N_0}}\right] < Q\left[\sqrt{\frac{T}{N_0}}\right]$$

and therefore, the probability of error $P(e|\mathbf{m}_1)$ is larger than $P(e|\mathbf{m}_2)$ and $P(e|\mathbf{m}_3)$. Hence, the message \mathbf{m}_1 is more vulnerable to errors. The reason for that is that it has both threshold lines close to it, while the other two signals have one of the their threshold lines further away.

Problem 4.25

1. In QPSK, $P_e \approx 2Q(\sqrt{2E_b/N_0})$. Here $E_b = \frac{1}{2}E = \frac{1}{4}A^2T$ and $P_e \approx 2Q(\sqrt{A^2T/2N_0})$.
2. For Gray coding, $P_b = Q(\sqrt{2E_b/N_0}) = Q(\sqrt{A^2T/2N_0})$.
3. From 4.6-6, $W = NR_s/2$ where $R_s = 1/T$, hence $W = 1/T$.
4. Here we have orthogonal signaling with $E_b = B^2T_1/2$. The rate is $R = R_s = 1/T_1$. The rate for QPSK is $R = 2R_s = 2/T$. For equality of the rate and bit error probability we must have $2/T = 1/T_1$ and $Q(\sqrt{A^2T/2N_0}) = Q(\sqrt{B^2T_1/2N_0})$ resulting in $T_1 = T/2$ and $B^2 = 2A^2$, or $B = A\sqrt{2}$.

Problem 4.26

1. Signals have equal energy $E = \int_0^T s^2(t) dt = \int_0^{T/2} 4 dt + \int_{T/2}^T 1 dt = 5T/2$. We also have $E_b = \frac{E}{\log_2 M} = E = \frac{5T}{2}$ and $R = \frac{\log_2 M}{T} = \frac{1}{T}$, therefore $T = \frac{1}{R}$. Substituting we obtain, $E_b = \frac{5}{2R}$ and $\frac{E_b}{N_0} = \frac{5}{2RN_0}$.

2. Binary equiprobable, hence $P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$. We have

$$d^2 = \int_0^T (s_1(t) - s_2(t))^2 dt = \int_0^{T/2} (1)^2 dt + \int_{T/2}^T (-1)^2 dt = T$$

and hence $P_e = Q\left(\sqrt{\frac{T}{2N_0}}\right) = Q\left(\sqrt{\frac{1}{2RN_0}}\right)$.

3. From part 1 and 2, we have $P_e = Q\left(\sqrt{\frac{1}{2RN_0}}\right) = Q\left(\sqrt{\frac{1}{5} \frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{1}{10} \frac{2E_b}{N_0}}\right)$. Since for binary antipodal $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$, we see that the current system underperforms the binary antipodal signaling by a factor of 10, or equivalently 10 dB.

4. We have $R_{\text{new}} = \frac{\log_2 4}{T} = \frac{2}{T} = 2R$.

5. We need to find d_{\min}^2 in the new constellation. Obviously, $d_{34}^2 = d_{12}^2 = T$. We also have $d_{13}^2 = d_{42}^2 = \int_0^T 4s_1^2(t) dt = 4E = 10T$. The only distances to be found are $d_{14}^2 = d_{23}^2 = \int_0^T (s_1(t) + s_2(t))^2 dt = \int_0^T 9 dt = 9T$. From these, we notice that $d_{\min}^2 = d_{12}^2 = T = \frac{1}{R}$. The union bound gives

$$P_e \leq (M-1)e^{-\frac{d_{\min}^2}{4N_0}} = 3e^{-\frac{1}{4RN_0}}$$

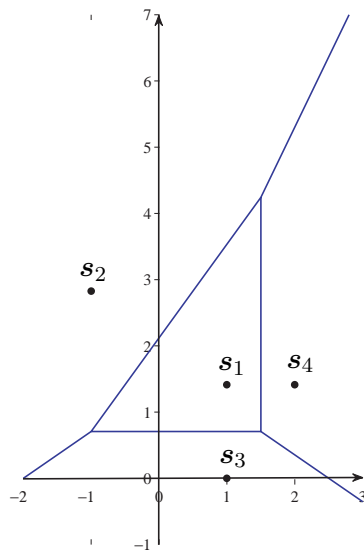
Problem 4.27

1. By inspection a two dimensional basis of the form

$$\phi_1(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \phi_2(t) = \begin{cases} 1/\sqrt{2} & 1 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$

works for this set of signals.

2. We have $\mathbf{s}_1 = (1, \sqrt{2})$, $\mathbf{s}_2 = (-1, 2\sqrt{2})$, $\mathbf{s}_3 = (1, 0)$, $\mathbf{s}_4 = (2, \sqrt{2})$. The constellation is shown below



3. The perpendicular bisectors defining the decision regions are shown on the figure. s_1 has the smallest decision region and hence is most subject to error.
4. D_1 is given by

$$D_1 = \{(r_1, r_2) : r \cdot s_1 - E_1/2 > r \cdot s_2 - E_2/2, r \cdot s_1 - E_1/2 > r \cdot s_3 - E_3/2, r \cdot s_1 - E_1/2 > r \cdot s_4 - E_4/2\}$$

or

$$D_1 = \{(r_1, r_2) : r_1 + \sqrt{2}r_2 - 3/2 > -r_1 + 2\sqrt{2}r_2 - 9/2, r_1 + \sqrt{2}r_2 - 3/2 > r_1 - 1/2, \\ r_1 + \sqrt{2}r_2 - 3/2 > 2r_1 + \sqrt{2}r_2 - 6/2\}$$

resulting in

$$D = \{(r_1, r_2) : 2r_1 - \sqrt{2}r_2 + 3 > 0, \sqrt{2}r_2 - 1 > 0, -r_1 + 3/2 > 0\}$$

resulting in the same region shown in the figure.

Problem 4.28

Using the Pythagorean theorem for the four-phase constellation, we find:

$$r_1^2 + r_1^2 = d^2 \implies r_1 = \frac{d}{\sqrt{2}}$$

The radius of the 8-PSK constellation is found using the cosine rule. Thus:

$$d^2 = r_2^2 + r_2^2 - 2r_2^2 \cos(45^\circ) \implies r_2 = \frac{d}{\sqrt{2 - \sqrt{2}}}$$

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by:

$$P_{4,av} = \frac{d^2}{2}, \quad P_{8,av} = \frac{d^2}{2 - \sqrt{2}}$$

Thus, the additional transmitted power needed by the 8-PSK signal is:

$$P = 10 \log_{10} \frac{2d^2}{(2 - \sqrt{2})d^2} = 5.3329 \text{ dB}$$

We obtain the same results if we use the probability of error given by (see 5-2-61) :

$$P_M = 2Q \left[\sqrt{2\gamma_s} \sin \frac{\pi}{M} \right]$$

where γ_s is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

$$\gamma_{4,s} \sin^2 \frac{\pi}{4} = \gamma_{8,s} \sin^2 \frac{\pi}{8} \implies 10 \log_{10} \frac{\gamma_{8,s}}{\gamma_{4,s}} = 20 \log_{10} \frac{\sin \frac{\pi}{4}}{\sin \frac{\pi}{8}} = 5.3329 \text{ dB}$$

Since we consider that error occur only between adjacent points, the above result is equal to the additional transmitted power we need for the 8-PSK scheme to achieve the same distance d between adjacent points.

Problem 4.29

For 4-phase PSK ($M = 4$) we have the following relationship between the symbol rate $1/T$, the required bandwidth W and the bit rate $R = k \cdot 1/T = \frac{\log_2 M}{T}$ (see 5-2-84):

$$W = \frac{R}{\log_2 M} \rightarrow R = W \log_2 M = 2W = 200 \text{ kbits/sec}$$

For binary FSK ($M = 2$) the required frequency separation is $1/2T$ (assuming coherent receiver) and (see 5-2-86):

$$W = \frac{M}{\log_2 M} R \rightarrow R = \frac{2W \log_2 M}{M} = W = 100 \text{ kbits/sec}$$

Finally, for 4-frequency non-coherent FSK, the required frequency separation is $1/T$, so the symbol rate is half that of binary coherent FSK, but since we have two bits/symbol, the bit rate is the same as in binary FSK :

$$R = W = 100 \text{ kbits/sec}$$

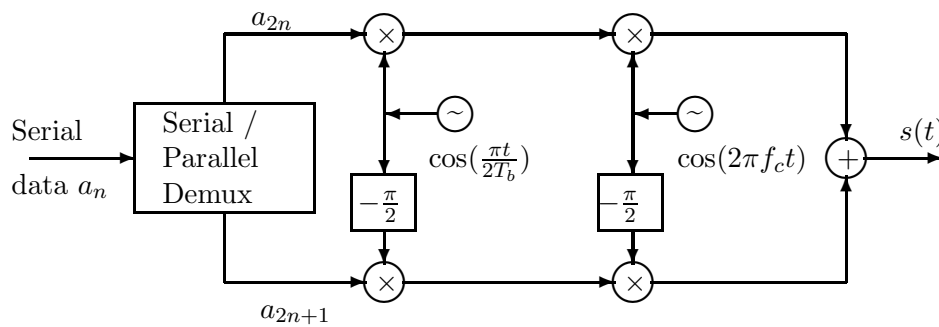
Problem 4.30

1. The envelope of the signal is

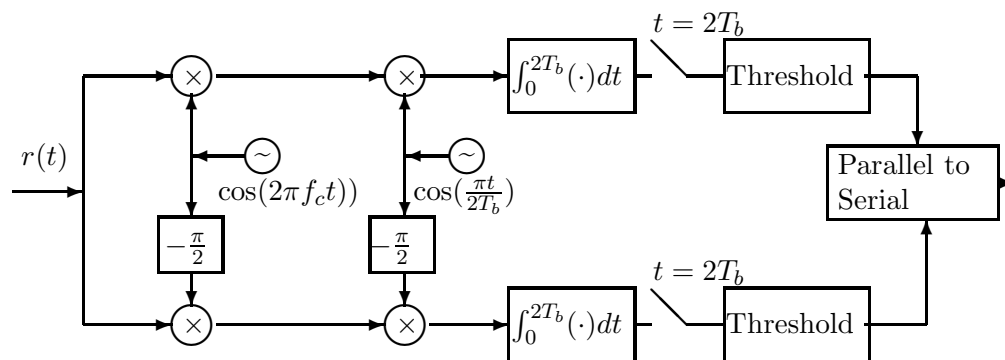
$$\begin{aligned}
 |s(t)| &= \sqrt{|s_c(t)|^2 + |s_s(t)|^2} \\
 &= \sqrt{\frac{2\mathcal{E}_b}{T_b} \cos^2\left(\frac{\pi t}{2T_b}\right) + \frac{2\mathcal{E}_b}{T_b} \sin^2\left(\frac{\pi t}{2T_b}\right)} \\
 &= \sqrt{\frac{2\mathcal{E}_b}{T_b}}
 \end{aligned}$$

Thus, the signal has constant amplitude.

2. The signal $s(t)$ is equivalent to an MSK signal. A block diagram of the modulator for synthesizing the signal is given in the next figure.



3. A sketch of the demodulator is shown in the next figure.



Problem 4.31

A biorthogonal signal set with $M = 8$ signal points has vector space dimensionality 4. Hence, the detector first checks which one of the four correlation metrics is the largest in absolute value, and then decides about the two possible symbols associated with this correlation metric, based on the sign of this metric. Hence, the error probability is the probability of the union of the event E_1 that another correlation metric is greater in absolute value and the event E_2 that the signal correlation metric has the wrong sign. A union bound on the symbol error probability can be given by :

$$P_M \leq P(E_1) + P(E_2)$$

But $P(E_2)$ is simply the probability of error for an antipodal signal set : $P(E_2) = Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)$ and the probability of the event E_1 can be union bounded by :

$$P(E_1) \leq 3[P(|C_2| > |C_1|)] = 3[2P(C_2 > C_1)] = 6P(C_2 > C_1) = 6Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$$

where C_i is the correlation metric corresponding to the i -th vector space dimension; the probability that a correlation metric is greater than the correct one is given by the error probability for orthogonal signals $Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$ (since these correlation metrics correspond to orthogonal signals). Hence :

$$P_M \leq 6Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right) + Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)$$

(sum of the probabilities to choose one of the 6 orthogonal, to the correct one, signal points and the probability to choose the signal point which is antipodal to the correct one).

Problem 4.32

It is convenient to find first the probability of a correct decision. Since all signals are equiprobable,

$$P(C) = \sum_{i=1}^M \frac{1}{M} P(C|\mathbf{s}_i)$$

All the $P(C|\mathbf{s}_i)$, $i = 1, \dots, M$ are identical because of the symmetry of the constellation. By translating the vector \mathbf{s}_i to the origin we can find the probability of a correct decision, given that \mathbf{s}_i was transmitted, as :

$$P(C|\mathbf{s}_i) = \int_{-\frac{d}{2}}^{\infty} f(n_1) dn_1 \int_{-\frac{d}{2}}^{\infty} f(n_2) dn_2 \dots \int_{-\frac{d}{2}}^{\infty} f(n_N) dn_N$$

where the number of the integrals on the right side of the equation is N , d is the minimum distance between the points and :

$$f(n_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n_i^2}{2\sigma^2}} = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_i^2}{N_0}}$$

Hence :

$$\begin{aligned} P(C|\mathbf{s}_i) &= \left(\int_{-\frac{d}{2}}^{\infty} f(n) dn \right)^N = \left(1 - \int_{-\infty}^{-\frac{d}{2}} f(n) dn \right)^N \\ &= \left(1 - Q \left[\frac{d}{\sqrt{2N_0}} \right] \right)^N \end{aligned}$$

and therefore, the probability of error is given by :

$$\begin{aligned} P(e) &= 1 - P(C) = 1 - \sum_{i=1}^M \frac{1}{M} \left(1 - Q \left[\frac{d}{\sqrt{2N_0}} \right] \right)^N \\ &= 1 - \left(1 - Q \left[\frac{d}{\sqrt{2N_0}} \right] \right)^N \end{aligned}$$

Note that since :

$$\mathcal{E}_s = \sum_{i=1}^N s_{m,i}^2 = \sum_{i=1}^N \left(\frac{d}{2} \right)^2 = N \frac{d^2}{4}$$

the probability of error can be written as :

$$P(e) = 1 - \left(1 - Q \left[\sqrt{\frac{2\mathcal{E}_s}{NN_0}} \right] \right)^N$$

Problem 4.33

Consider first the signal :

$$y(t) = \sum_{k=1}^n c_k \delta(t - kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is :

$$\begin{aligned} g(t) &= y(T - t) = y(nT_c - t) = \sum_{k=1}^n c_k \delta(nT_c - kT_c - t) \\ &= \sum_{i=1}^n c_{n-i+1} \delta((i-1)T_c - t) = \sum_{i=1}^n c_{n-i+1} \delta(t - (i-1)T_c) \end{aligned}$$

that is, a sequence of impulses starting at $t = 0$ and weighted by the mirror image sequence of $\{c_i\}$.

Since,

$$s(t) = \sum_{k=1}^n c_k p(t - kT_c) = p(t) \star \sum_{k=1}^n c_k \delta(t - kT_c)$$

the Fourier transform of the signal $s(t)$ is :

$$S(f) = P(f) \sum_{k=1}^n c_k e^{-j2\pi f k T_c}$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is :

$$\begin{aligned} H(f) &= S^*(f) e^{-j2\pi f T} = S^*(f) e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{k=1}^n c_k e^{j2\pi f k T_c} e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{i=1}^n c_{n-i+1} e^{-j2\pi f (i-1) T_c} \\ &= P^*(f) \mathcal{F}[g(t)] \end{aligned}$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter, with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal $y(t) = \sum_{k=1}^n c_k \delta(t - kT_c)$. The output of the matched filter at $t = nT_c$ is (see 5-1-27) :

$$\begin{aligned} \int_{-\infty}^{\infty} |s(t)|^2 dt &= \sum_{k=1}^n c_k^2 \int_{-\infty}^{\infty} p^2(t - kT_c) dt \\ &= T_c \sum_{k=1}^n c_k^2 \end{aligned}$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c .

Problem 4.34

1. The inner product of $s_i(t)$ and $s_j(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} s_i(t) s_j(t) dt &= \int_{-\infty}^{\infty} \sum_{k=1}^n c_{ik} p(t - kT_c) \sum_{l=1}^n c_{jl} p(t - lT_c) dt \\ &= \sum_{k=1}^n \sum_{l=1}^n c_{ik} c_{jl} \int_{-\infty}^{\infty} p(t - kT_c) p(t - lT_c) dt \\ &= \sum_{k=1}^n \sum_{l=1}^n c_{ik} c_{jl} \mathcal{E}_p \delta_{kl} \\ &= \mathcal{E}_p \sum_{k=1}^n c_{ik} c_{jk} \end{aligned}$$

The quantity $\sum_{k=1}^n c_{ik}c_{jk}$ is the inner product of the row vectors \underline{C}_i and \underline{C}_j . Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \mathcal{E}_p \sum_{k=1}^n c_{ik}^2 \delta_{ij} = n\mathcal{E}_p \delta_{ij}$$

Thus, the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

2. Using the results of Problem 5.35, we obtain that the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^n c_{ik}p(t - kT_c)$$

can be realized as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{C}_i = [c_{i1}, \dots, c_{in}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$, we conclude that the n matched filters can be realized by a filter matched to $p(t)$ followed by n discrete-time filters matched to the vectors \underline{C}_i , $i = 1, \dots, n$.

Problem 4.35

1. The optimal ML detector (see 5-1-41) selects the sequence \underline{C}_i that minimizes the quantity:

$$D(\mathbf{r}, \underline{C}_i) = \sum_{k=1}^n (r_k - \sqrt{\mathcal{E}_b} C_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(\mathbf{r}, \underline{C}_1) = \sum_{k=1}^w (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^n (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(\mathbf{r}, \underline{C}_2) = \sum_{k=1}^w (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^n (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n - w$ received elements of \mathbf{r} . That is

$$\sum_{k=w+1}^n (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^n (r_k + \sqrt{\mathcal{E}_b})^2 \begin{matrix} \underline{C}_2 \\ > \\ < \\ \underline{C}_1 \end{matrix} 0$$

or equivalently

$$\sum_{k=w+1}^n r_k \underset{\underline{C}_2}{\overset{\underline{C}_1}{>}} 0$$

2. Since $r_k = \sqrt{\mathcal{E}_b}C_{ik} + n_k$, the probability of error $P(e|\underline{C}_1)$ is

$$\begin{aligned} P(e|\underline{C}_1) &= P\left(\sqrt{\mathcal{E}_b}(n-w) + \sum_{k=w+1}^n n_k < 0\right) \\ &= P\left(\sum_{k=w+1}^n n_k < -(n-w)\sqrt{\mathcal{E}_b}\right) \end{aligned}$$

The random variable $u = \sum_{k=w+1}^n n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$. Hence

$$P(e|\underline{C}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n-w)} \exp\left(-\frac{x^2}{2\pi(n-w)\sigma^2}\right) dx = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

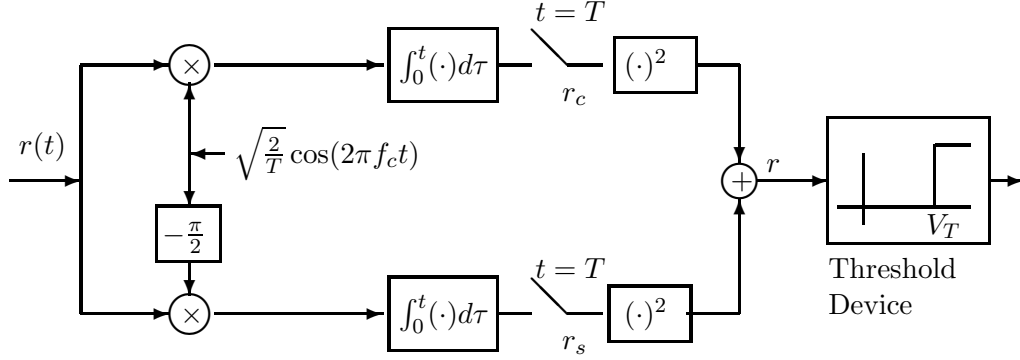
Similarly we find that $P(e|\underline{C}_2) = P(e|\underline{C}_1)$ and since the two sequences are equiprobable

$$P(e) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

3. The probability of error $P(e)$ is minimized when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximized, that is for $w = 0$. This implies that $\underline{C}_1 = -\underline{C}_2$ and thus the distance between the two sequences is the maximum possible.

Problem 4.36

1. The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



2. If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c , r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by :

$$p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is :

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned} r_c &= \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\ &= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t) dt + \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\ &= \frac{2\sqrt{\mathcal{E}_b}}{T_b} \int_0^T \frac{1}{2} (\cos(2\pi 2f_c t + \phi) + \cos(\phi)) dt + n_c \\ &= \sqrt{\mathcal{E}_b} \cos(\phi) + n_c \end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have :

$$r_s = \sqrt{\mathcal{E}_b} \sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ follows the Rician distribution :

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2 + \mathcal{E}_b}{2\sigma^2}} I_0 \left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2} \right) = \frac{2r}{N_0} e^{-\frac{r^2 + \mathcal{E}_b}{N_0}} I_0 \left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0} \right)$$

3. For equiprobable signals the probability of error is given by:

$$P(\text{error}) = \frac{1}{2} \int_{-\infty}^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned} P(\text{error}) &= \frac{1}{2} \int_0^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr \\ &= \frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2 + \mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr + \frac{1}{2} \int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr \end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to: $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation :

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x , that is for high SNR. In this case :

$$\frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2 + \mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr \approx \frac{1}{2} \int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore :

$$\begin{aligned} \frac{1}{2} \int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2} \int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\ &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] \end{aligned}$$

Finally :

$$\begin{aligned} P(\text{error}) &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2} \int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\ &\leq \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2} e^{-\frac{\mathcal{E}_b}{4N_0}} \end{aligned}$$

Problem 4.37

1. In binary DPSK, the information bit 1 is transmitted by shifting the phase of the carrier by π radians relative to the phase in the previous interval, while if the information bit is 0 then the

phase is not shifted. With this in mind :

$$\begin{array}{lcl} \text{Data :} & & 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \\ \text{Phase } \theta : (\pi) & & 0 \quad \pi \quad \pi \quad 0 \quad 0 \quad 0 \quad 0 \quad \pi \quad \pi \quad 0 \quad \pi \quad \pi \end{array}$$

Note : since the phase in the first bit interval is 0, we conclude that the phase before that was π .

2. We know that the power spectrum of the equivalent lowpass signal $u(t)$ is :

$$\Phi_{uu}(f) = \frac{1}{T} |G(f)|^2 \Phi_{ii}(f)$$

where $G(f) = AT \frac{\sin \pi f T}{\pi f T}$, is the spectrum of the rectangular pulse of amplitude A that is used, and $\Phi_{ii}(f)$ is the power spectral density of the information sequence. It is straightforward to see that the information sequence I_n is simply the phase of the lowpass signal, i.e. it is $e^{j\pi}$ or e^{j0} depending on the bit to be transmitted $a_n (= 0, 1)$. We have :

$$I_n = e^{j\theta_n} = e^{j\pi a_n} e^{j\theta_{n-1}} = e^{j\pi \sum_k a_k}$$

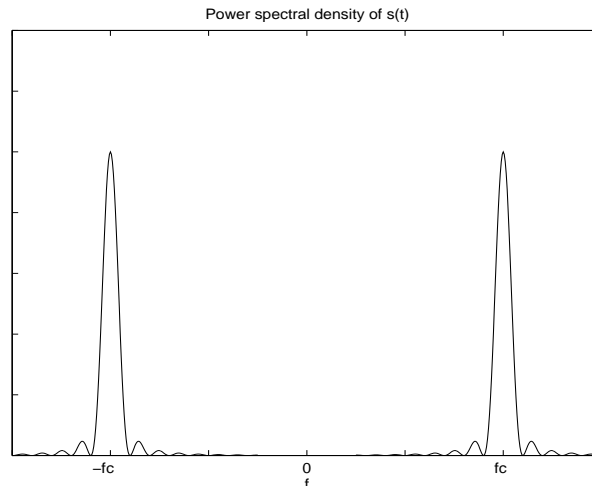
The statistics of I_n are (remember that $\{a_n\}$ are uncorrelated) :

$$\begin{aligned} E[I_n] &= E[e^{j\pi \sum_k a_k}] = \prod_k E[e^{j\pi a_k}] = \prod_k [\tfrac{1}{2}e^{j\pi} - \tfrac{1}{2}e^{j0}] = \prod_k 0 = 0 \\ E[|I_n|^2] &= E[e^{j\pi \sum_k a_k} e^{-j\pi \sum_k a_k}] = 1 \\ E[I_{n+m} I_n^*] &= E[e^{j\pi \sum_{k=0}^{n+m} a_k} e^{-j\pi \sum_{k=0}^n a_k}] = E[e^{j\pi \sum_{k=n+1}^{n+m} a_k}] = \prod_{k=n+1}^m E[e^{j\pi a_k}] = 0 \end{aligned}$$

Hence, I_n is an uncorrelated sequence with zero mean and unit variance, so $\Phi_{ii}(f) = 1$, and

$$\begin{aligned} \Phi_{uu}(f) &= \frac{1}{T} |G(f)|^2 = A^2 T \frac{\sin^2 \pi f T}{\pi^2 f^2 T} \\ \Phi_{ss}(f) &= \frac{1}{2} [\Phi_{uu}(f - f_c) + \Phi_{uu}(-f - f_c)] \end{aligned}$$

A sketch of the signal power spectrum $\Phi_{ss}(f)$ is given in the following figure :



Problem 4.38

1. $D = \text{Re}(V_m V_{m-1}^*)$ where $V_m = X_m + jY_m$. Then :

$$\begin{aligned} D &= \text{Re}((X_m + jY_m)(X_{m-1} - jY_{m-1})) \\ &= X_m X_{m-1} + Y_m Y_{m-1} \\ &= \left(\frac{X_m + X_{m-1}}{2}\right)^2 - \left(\frac{X_m - X_{m-1}}{2}\right)^2 + \left(\frac{Y_m + Y_{m-1}}{2}\right)^2 - \left(\frac{Y_m - Y_{m-1}}{2}\right)^2 \end{aligned}$$

2. $V_k = X_k + jY_k = 2aE \cos(\theta - \phi) + j2aE \sin(\theta - \phi) + N_{k,real} + jN_{k,imag}$. Hence :

$$\begin{aligned} U_1 &= \frac{X_m + X_{m-1}}{2}, & E(U_1) &= 2aE \cos(\theta - \phi) \\ U_2 &= \frac{Y_m + Y_{m-1}}{2}, & E(U_2) &= 2aE \sin(\theta - \phi) \\ U_3 &= \frac{X_m - X_{m-1}}{2}, & E(U_3) &= 0 \\ U_4 &= \frac{Y_m - Y_{m-1}}{2}, & E(U_4) &= 0 \end{aligned}$$

The variance of U_1 is : $E[U_1 - E(U_1)]^2 = E\left[\frac{1}{2}(N_{m,real} + N_{m-1,real})\right]^2 = E[N_{m,real}]^2 = 2\mathcal{E}N_0$, and similarly : $E[U_i - E(U_i)]^2 = 2\mathcal{E}N_0$, $i = 2, 3, 4$. The covariances are (e.g for U_1, U_2): $\text{cov}(U_1, U_2) = E[(U_1 - E(U_1))(U_2 - E(U_2))] = E\left[\frac{1}{4}(N_{m,r} + N_{m-1,r})(N_{m,i} + N_{m-1,i})\right] = 0$, since the noise components are uncorrelated and have zero mean. Similarly for any i, j : $\text{cov}(U_i, U_j) = 0$. The condition $\text{cov}(U_i, U_j) = 0$, implies that these random variables $\{U_i\}$ are uncorrelated, and since they are Gaussian, they are also statistically independent.

Since U_3 and U_4 are zero-mean Gaussian, the random variable $R_2 = \sqrt{U_3^2 + U_4^2}$ has a Rayleigh distribution; on the other hand $R_1 = \sqrt{U_1^2 + U_2^2}$ has a Rice distribution.

3. $W_1 = U_1^2 + U_2^2$, with U_1, U_2 being statistically independent Gaussian variables with means $2aE \cos(\theta - \phi)$, $2aE \sin(\theta - \phi)$ and identical variances $\sigma^2 = 2\mathcal{E}N_0$. Then, W_1 follows a non-central chi-square distribution with pdf given by (2-1-118):

$$p(w_1) = \frac{1}{4\mathcal{E}N_0} e^{-(4a^2 E^2 + w_1)/4\mathcal{E}N_0} I_0\left(\frac{a}{N_0} \sqrt{w_1}\right), \quad w_1 \geq 0$$

Also, $W_2 = U_3^2 + U_4^2$, with U_3, U_4 being zero-mean Gaussian with the same variance. Hence, W_1 follows a central chi-square distribution, with pfd given by (2-1-110) :

$$p(w_2) = \frac{1}{4\mathcal{E}N_0} e^{-w_2/4\mathcal{E}N_0}, \quad w_2 \geq 0$$

4.

$$\begin{aligned}
P_b &= P(D < 0) = P(W_1 - W_2 < 0) \\
&= \int_0^\infty P(w_2 > w_1 | w_1) p(w_1) dw_1 \\
&= \int_0^\infty \left(\int_{w_1}^\infty p(w_2) dw_2 \right) p(w_1) dw_1 \\
&= \int_0^\infty e^{-w_1/4EN_0} p(w_1) dw_1 \\
&= \psi(jv) |_{v=j/4EN_0} \\
&= \frac{1}{(1-2jv\sigma^2)} \exp\left(\frac{jv4a^2E^2}{1-2jv\sigma^2}\right) |_{v=j/4EN_0} \\
&= \frac{1}{2} e^{-a^2E/N_0}
\end{aligned}$$

where we have used the characteristic function of the non-central chi-square distribution given by (2-1-117) in the book.

Problem 4.39

From 4.6-7, $W = RN/2 \log_2 M$, therefor we have

1. BFSK, $M = 2, N = 2, W = R$.
2. 8PSK, $M = 8, N = 2, W = R/3$.
3. QPSK, $M = 4, N = 2, W = R/2$.
4. 64QAM, $M = 64, N = 2, W = R/6$.
5. BPSK, $M = 2, N = 2, W = R$ (here we have excluded the possibility of SSB transmission)
6. 16FSK, $M = 16, N = 16, W = 2R$.

Problem 4.40

1. These are given by Equations 4.2-32 and 4.2-33

$$r_{\text{th}} = \frac{N_0}{4\sqrt{E}} \ln \frac{1-p}{p}$$

$$P_e = pQ\left((\sqrt{E} - r_{\text{th}})/\sqrt{N_0/2}\right) + (1-p)Q\left((\sqrt{E} + r_{\text{th}})/\sqrt{N_0/2}\right)$$

2. When the receiver receives noise only with no trace of signal, it can make any decision and its error probability will be 1/2 regardless of its decision. Therefore without any loss in optimality of the system we can assume that regardless of whether the channel is in service or not, a decision based on r_{th} given above is optimal. Then in half of the times the error probability will be 1/2 and in the remaining half it will be P_e given above, i.e., $P'_e = P_e/2 + 1/4$.

Problem 4.41

1. Binary equiprobable, hence $P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$, where

$$d^2 = \int (s_1(t) - s_2(t))^2 dt = \int_1^2 4^2 dt = 16$$

Hence $P_e = Q\left(\sqrt{\frac{16}{2N_0}}\right) = Q\left(\sqrt{\frac{8}{N_0}}\right)$. Note that here $E_b = \int s_1^2(t) dt = 5$ and therefore $P_e = Q\left(\sqrt{\frac{8}{10} \frac{2E_b}{N_0}}\right)$ which shows a power reduction by a factor of 0.8 or equivalently 0.97 dB compared to binary antipodal signaling. This is obvious from the signaling scheme as well. In the interval from 0 to 1 the signals are equal (complete waste of energy) and do not contribute to the detection at all. In the interval from 1 to 2 the two signals are antipodal. But the energy in the second interval is 80% of the total energy and this is the 0.8 factor.

2. Note that the only relevant information is in the interval from 1 to 2. In the interval from 0 to 1 the two signals are equal and hence the received signals in that interval will be equal regardless of the signal sent. In the interval from 1 to 2 we can define the single basis function

$$\phi(t) = \begin{cases} 1 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

as the only required basis. With this basis $s_1 = 2$ and $s_2 = -2$. Projecting on this basis we have $r_i = s_m + n_i$, for $i = 1, 2$ corresponding to the two paths. Here n_1 and n_2 are independent Gaussian RV's each $\mathcal{N}\left(0, \frac{N_0}{2}\right)$. The decision region D_1 is defined as $D_1 \Leftrightarrow p(r_1, r_2 | s_1) > p(r_1, r_2 | s_2)$, or

$$D_1 \Leftrightarrow Ke^{-\frac{(r_1-2)^2+(r_2-2)^2}{N_0}} > Ke^{-\frac{(r_1+2)^2+(r_2+2)^2}{N_0}}$$

$$\Leftrightarrow (r_1+2)^2 + (r_2+2)^2 - (r_1-2)^2 - (r_2-2)^2 > 0$$

$$\Leftrightarrow r_1 + r_2 > 0$$

and the error probability (using symmetry) is

$$\begin{aligned} P_e &= P(r_1 + r_2 > 0 | s_2 \text{ sent}) \\ &= P(-2 + n_1 - 2 + n_2 > 0) \\ &= P(n_1 + n_2 > 4) \end{aligned}$$

Since n_1 and n_2 are independent $n_1 + n_2$ is zero mean Gaussian with variance N_0 and $P_e = P(n_1 + n_2 > 4) = Q\left(\frac{4}{\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{16}{N_0}}\right)$ which shows a performance improvement of 3 dB compared to single transmission.

3. Here $r_1 = As_m + n_1$ and $r_2 = s_m + n_2$ for $m = 1, 2$, and A is known to the detector. Again here $D_1 \Leftrightarrow p(r_1, r_2 | s_1) > p(r_1, r_2 | s_2)$, or

$$\begin{aligned} D_1 &\Leftrightarrow Ke^{-\frac{(r_1-2A)^2+(r_2-2)^2}{N_0}} > Ke^{-\frac{(r_1+2A)^2+(r_2+2)^2}{N_0}} \\ &\Leftrightarrow (r_1 + 2A)^2 + (r_2 + 2)^2 - (r_1 - 2A)^2 - (r_2 - 2)^2 > 0 \\ &\Leftrightarrow Ar_1 + r_2 > 0 \end{aligned}$$

This is the optimal detection scheme for the receiver who knows A and hence it can compute $Ar_1 + r_2$. The error probability (for someone who does not know A) is

$$P_e = \int P(e|A)f(A)dA = \int_0^1 P(e|A)dA$$

Therefore, we first find $P(e|A)$ as

$$\begin{aligned} P(e|A) &= P(r_1A + r_2 > 0 | s_2 \text{ sent}) \\ &= P(A(-2 + n_1) + (-2 + n_2) > 0) \\ &= P(An_1 + n_2 > 2A + 2) \\ &= P\left(\mathcal{N}\left(0, \frac{N_0}{2}(1 + A^2)\right) > 2 + 2A\right) \\ &= Q\left(\sqrt{\frac{8}{N_0} \frac{(1 + A)^2}{1 + A^2}}\right) \end{aligned}$$

Note that this relation for $A = 0$ and $A = 1$ reduces to the results of parts 1 and 2. Now we have

$$P_e = \int_0^1 Q\left(\sqrt{\frac{8}{N_0} \frac{(1 + A)^2}{1 + A^2}}\right) dA$$

4. Again $D_1 \Leftrightarrow p(r_1, r_2 | s_1) > p(r_1, r_2 | s_2)$ but since the detector does not know A we are in a situation like noncoherent detection, the difference being that here A is random instead of ϕ . We have

$$p(r_1, r_2 | s_m) = \int_0^1 p(r_1, r_2 | s_m, A) dA$$

hence,

$$\begin{aligned}
 D_1 &\Leftrightarrow \int_0^1 e^{-\frac{(r_1-2A)^2+(r_2-2)^2}{N_0}} dA > \int_0^1 e^{-\frac{(r_1+2A)^2+(r_2+2)^2}{N_0}} dA \\
 &\Leftrightarrow e^{\frac{4r_2}{N_0}} \int_0^1 e^{-\frac{4A^2-4r_1A}{N_0}} dA > e^{\frac{-4r_2}{N_0}} \int_0^1 e^{-\frac{4A^2+4r_1A}{N_0}} dA \\
 &\Leftrightarrow \frac{4r_2}{N_0} + \ln \int_0^1 e^{-\frac{4A^2-4r_1A}{N_0}} dA > -\frac{4r_2}{N_0} + \ln \int_0^1 e^{-\frac{4A^2+4r_1A}{N_0}} dA \\
 &\Leftrightarrow r_2 > \frac{N_0}{8} \ln \frac{g(r_1)}{g(-r_1)}
 \end{aligned}$$

where

$$g(r) = \int_0^1 e^{-\frac{4A^2+4rA}{N_0}} dA$$

Problem 4.42

1. This is binary equiprobable antipodal signaling, hence the threshold is zero and $P_e = P(n > A) = Q(A/\sigma)$.
2. Intuitively, it is obvious that in this case no information is transmitted and therefore any decision at the receiver is irrelevant. To make this statement more precise, we make a decision in favor of A is $p(y|A) > p(y|-A)$, or if $1/2p(y|A, \rho = 1) + 1/2p(y|A, \rho = -1) > 1/2p(y|-A, \rho = 1) + 1/2p(y|-A, \rho = -1)$. From $Y = \rho X + N$, the above relation simplifies to

$$\exp[-(y-A)^2/2\sigma^2] + \exp[-(y+A)^2/2\sigma^2] > \exp[-(y+A)^2/2\sigma^2] + \exp[-(y-A)^2/2\sigma^2]$$

Since both sides of inequality are equal, any received Y can be equally detected as A or $-A$, and the error probability will be $1/2$.

3. When $\rho = 0$ no information is transmitted and any decision is irrelevant. When $\rho = 1$, the threshold is zero, thus we can take the threshold to be zero regardless of the value of ρ (which we do not know). Then in half of the cases the error probability will be $1/2$ and in the remaining half it will be $Q(A/\sigma)$, hence $P_e = 1/4 + 1/2Q(A/\sigma)$.
4. This is like part 3 but with decision on favor of A when $1/2p(y|A, \rho = 1) + 1/2p(y|A, \rho = -1) > 1/2p(y|0, \rho = 1) + 1/2p(y|0, \rho = -1)$ resulting in

$$\exp[-(y-A)^2/2\sigma^2] + \exp[-(y+A)^2/2\sigma^2] > 2\exp[-y^2/2\sigma^2]$$

which simplifies to the following decision rule for decision in favor of A

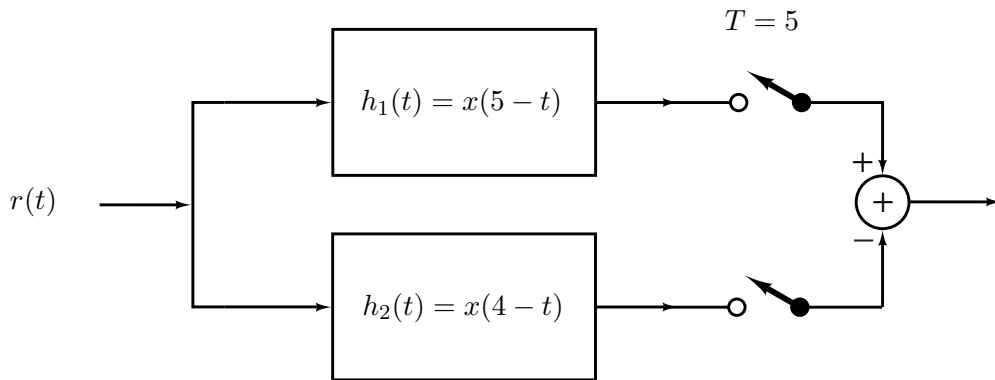
$$\cosh\left(\frac{yA}{\sigma^2}\right) > \exp\left(\frac{A^2}{2\sigma^2}\right)$$

where $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

Problem 4.43

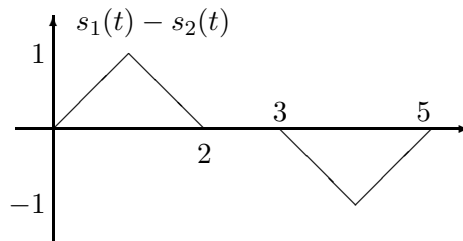
1. We have equiprobable and equal energy signaling, therefore $\eta_1 = \eta_2$ and the decision region D_1 is given by

$$D_1 = \{\mathbf{r} : \mathbf{r} \cdot \mathbf{s}_1 > \mathbf{r} \cdot \mathbf{s}_2\}$$



The sign of the output determines the message.

2. This is binary, equiprobable signaling, hence $p_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$, where $d^2 = \int (s_1(t) - s_2(t))^2 dt$. It is easy to see that $s_1(t) - s_2(t)$ is as shown below



We can easily find this integral as $d^2 = 4 \int_0^1 t^2 dt = \frac{4}{3}$ and $P_e = Q\left(\sqrt{\frac{2}{3N_0}}\right)$.

3. Just use $h(t) = h_1(t) - h_2(t)$.
4. Here we have

$$\begin{aligned} D_1 &\Leftrightarrow p(\mathbf{r}|m_1) > p(\mathbf{r}|m_2) \\ &\sim p(\mathbf{r}|\mathbf{x}_1) > \frac{1}{2}p(\mathbf{r}|\mathbf{x}_2) + \frac{1}{2}p(\mathbf{r}|\mathbf{x}_2) \\ &\sim \frac{1}{2}p(\mathbf{r}|\mathbf{x}_1) > \frac{1}{2}p(\mathbf{r}|\mathbf{x}_2) \end{aligned}$$

where \mathbf{x}_1 and \mathbf{x}_2 are vector representations for $x(t)$ and $x(t-1)$. This means that the decision regions will not change. For the error probability we have two cases, if $s_2(t)$ is $x(t)$, then $d = 0$ and $p_e = Q(0) = \frac{1}{2}$. If $s_2(t) = x(t-1)$, then $p_e = Q\left(\sqrt{\frac{2}{3N_0}}\right)$. Therefore,

$$p_e = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times Q\left(\sqrt{\frac{2}{3N_0}}\right) = \frac{1}{4} + \frac{1}{2}Q\left(\sqrt{\frac{2}{3N_0}}\right)$$

This indicates a major deterioration in performance.

Problem 4.44

1. Using the definition of Q -function $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ we have

$$\begin{aligned} E[Q(\beta X)] &= \int_0^\infty Q(\beta x) \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_0^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{\beta x}^\infty e^{-\frac{t^2}{2}} dt \right] \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \end{aligned}$$

where the region of integration is $0 \leq x < \infty$ and $\beta x \leq t < \infty$. Reordering integrals we have

$$\begin{aligned} E[Q(\beta X)] &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \left[\int_0^{\frac{t}{\beta}} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \right] dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \left(1 - e^{-\frac{t^2}{2\beta^2\sigma^2}} \right) dt \\ &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2} \left(1 + \frac{1}{\beta^2\sigma^2} \right)} dt \\ &= \frac{1}{2} - \sqrt{\frac{\beta^2\sigma^2}{1 + \beta^2\sigma^2}} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{s^2}{2}} ds \\ &= \frac{1}{2} \left(1 - \sqrt{\frac{\beta^2\sigma^2}{1 + \beta^2\sigma^2}} \right) \end{aligned}$$

where we have used the change of variables $s = t \sqrt{\frac{1 + \beta^2\sigma^2}{\beta^2\sigma^2}}$.

2. This is similar to case 1 with $\beta = \sqrt{\frac{2E_b}{N_0}}$, therefore

$$E[P_b] = \frac{1}{2} \left(1 - \sqrt{\frac{\sigma^2 \frac{2E_b}{N_0}}{1 + \sigma^2 \frac{2E_b}{N_0}}} \right)$$

3. Obviously here

$$E[P_b] = \frac{1}{2} \left(1 - \sqrt{\frac{\sigma^2 \frac{E_b}{N_0}}{1 + \sigma^2 \frac{E_b}{N_0}}} \right)$$

4. Note that for $\sigma^2 \frac{E_b}{N_0} \gg 1$, we have

$$\begin{aligned} \sqrt{\frac{\sigma^2 \frac{2E_b}{N_0}}{1 + \sigma^2 \frac{2E_b}{N_0}}} &= \sqrt{1 - \frac{1}{1 + \sigma^2 \frac{2E_b}{N_0}}} \\ &\approx 1 - \frac{1}{2 + 4\sigma^2 \frac{E_b}{N_0}} \end{aligned}$$

where we have used the approximation that for small ϵ , $\sqrt{1 - \epsilon} \approx 1 - \frac{\epsilon}{2}$. From the above we have

$$E[P_b] = \begin{cases} \frac{1}{2} \left(1 - \sqrt{\frac{\overline{\text{SNR}}}{1 + \overline{\text{SNR}}}} \right) \approx \frac{1}{4\overline{\text{SNR}}} & \text{antipodal} \\ \frac{1}{2} \left(1 - \sqrt{\frac{\overline{\text{SNR}}}{2 + \overline{\text{SNR}}}} \right) \approx \frac{1}{2\overline{\text{SNR}}} & \text{orthogonal} \end{cases}$$

5. We have

$$\begin{aligned} E[e^{-\beta X^2}] &= \int_0^\infty e^{-\beta x^2} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_0^\infty \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}(1+2\beta\sigma^2)} dx \\ &= \frac{1}{1 + 2\beta\sigma^2} \int_0^\infty t e^{-\frac{t^2}{2}} dt \\ &= \frac{1}{1 + 2\beta\sigma^2} \\ &\approx \frac{1}{2\beta\sigma^2} \quad \text{for } \beta\sigma^2 \gg 1 \end{aligned}$$

where we have used the change of variables $t = x\sqrt{\frac{1+2\beta\sigma^2}{\sigma^2}}$. We will see later that the error probability for noncoherent detection of BFSK is $P_b = \frac{1}{2}e^{-\frac{E_b}{2N_0}}$ and for binary DPSK is $P_b = \frac{1}{2}e^{-\frac{E_b}{N_0}}$. If we have Rayleigh attenuation as in part 2, we can substitute $\beta = \frac{E_b}{2N_0}$ and $\beta = \frac{E_b}{N_0}$ and obtain

$$E[P_b] = \begin{cases} \frac{1}{2 + \overline{\text{SNR}}} \approx \frac{1}{\overline{\text{SNR}}} & \text{noncoherent BFSK} \\ \frac{1}{2 + 2\overline{\text{SNR}}} \approx \frac{1}{2\overline{\text{SNR}}} & \text{binary DPSK} \end{cases}$$

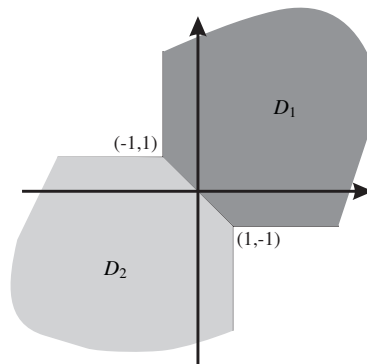
As noticed from parts 4 and 5, all error probabilities are inversely proportional to $\overline{\text{SNR}}$.

Problem 4.45

Here the noise is not Gaussian, therefore none of the results of Gaussian noise can be used. We start from the MAP rule

$$\begin{aligned}
 D_1 &= \{\mathbf{r} : p(\mathbf{s}_1|\mathbf{r}) > p(\mathbf{s}_2|\mathbf{r})\} \\
 &= \{\mathbf{r} : p(\mathbf{s}_1)p(\mathbf{r}|\mathbf{s}_1) > p(\mathbf{s}_2)p(\mathbf{r}|\mathbf{s}_2)\} \\
 &= \{\mathbf{r} : p(\mathbf{r}|\mathbf{s}_1) > p(\mathbf{r}|\mathbf{s}_2)\} \quad \text{since the signals are equiprobable} \\
 &= \{\mathbf{r} : \frac{1}{4}e^{-|r_1-1|-|r_2-1|} > \frac{1}{4}e^{-|r_1+1|-|r_2+1|}\} \quad \text{since noise components are independent} \\
 &= \{\mathbf{r} : |r_1 + 1| + |r_2 + 1| > |r_1 - 1| + |r_2 - 1|\}
 \end{aligned}$$

If $r_1, r_2 > 1$, then $D_1 = \{\mathbf{r} : r_1 + 1 + r_2 + 1 > r_1 - 1 + r_2 - 1\}$ which is always satisfied. Therefore the entire $r_1, r_2 > 1$ region belongs to D_1 . Similarly it can be shown that the entire $r_1, r_2 < -1$ region belongs to D_2 . For $r_1 > 1$ and $r_2 < -1$ we have $D_1 = \{\mathbf{r} : r_1 + 1 - r_2 - 1 > r_1 - 1 - r_2 + 1\}$ or $0 > 0$, i.e., $r_1, r_2 < -1$ can be either in D_1 or D_2 . Similarly we can consider other regions of the plane. The final result is shown in the figure below. Regions D_1 and D_2 are shown in the figure and the rest of the plane can be either D_1 or D_2 .



Problem 4.46

The vector $\mathbf{r} = [r_1, r_2]$ at the output of the integrators is

$$\mathbf{r} = [r_1, r_2] = \left[\int_0^{1.5} r(t)dt, \int_1^2 r(t)dt \right]$$

If $s_1(t)$ is transmitted, then

$$\begin{aligned}\int_0^{1.5} r(t) dt &= \int_0^{1.5} [s_1(t) + n(t)] dt = 1 + \int_0^{1.5} n(t) dt \\ &= 1 + n_1 \\ \int_1^2 r(t) dt &= \int_1^2 [s_1(t) + n(t)] dt = \int_1^2 n(t) dt \\ &= n_2\end{aligned}$$

where n_1 is a zero-mean Gaussian random variable with variance

$$\sigma_{n_1}^2 = E \left[\int_0^{1.5} \int_0^{1.5} n(\tau) n(v) d\tau dv \right] = \frac{N_0}{2} \int_0^{1.5} d\tau = 1.5$$

and n_2 is a zero-mean Gaussian random variable with variance

$$\sigma_{n_2}^2 = E \left[\int_1^2 \int_1^2 n(\tau) n(v) d\tau dv \right] = \frac{N_0}{2} \int_1^2 d\tau = 1$$

The important fact here is that due to the overlap in the integration region n_1 and n_2 are *not* independent. We first find the correlation between the two noise components. We have

$$E[n_1 n_2] = \int_1^{1.5} \frac{N_0}{2} dt = 0.5$$

and the covariance matrix of the noise vector will be

$$\mathbf{C} = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

therefore $\det \mathbf{C} = 1.25$ and

$$\mathbf{C}^{-1} = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 1.2 \end{bmatrix}$$

and the joint density function of the noise components is

$$f(n_1, n_2) = \frac{1}{2\pi \times \sqrt{1.25}} e^{-\frac{1}{2} \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 1.2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}$$

Thus, the vector representation of the received signal (at the output of the integrators) is

$$\mathbf{r} = [1 + n_1, n_2]$$

Similarly we find that if $s_2(t)$ is transmitted, then

$$\mathbf{r} = [0.5 + n_1, 1 + n_2]$$

The optimal decision region compares $f(\mathbf{r}|\mathbf{s}_1)$ and $f(\mathbf{r}|\mathbf{s}_2)$. But we have

$$\begin{aligned} f(\mathbf{r}|\mathbf{s}_1) &= f_{n_1, n_2}(r_1 - 1, r_2) \\ f(\mathbf{r}|\mathbf{s}_2) &= f_{n_1, n_2}(r_1 - 0.5, r_2 - 1) \end{aligned}$$

where f_{n_1, n_2} is given above. Substituting in the expression for $f(n_1, n_2)$ and simplifying we obtain

$$f(\mathbf{r}|\mathbf{s}_1) > f(\mathbf{r}|\mathbf{s}_2) \Leftrightarrow 1.6r_1 - 2.8r_2 + 0.2 > 0$$

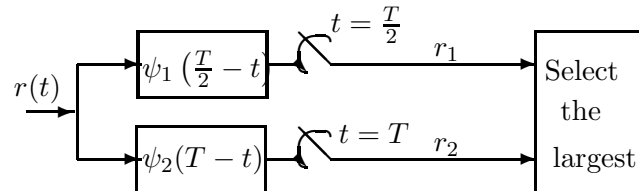
which is the optimal decision rule based on observation of r_1 and r_2 .

Problem 4.47

1) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

$$\psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \quad \psi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

2) The optimal receiver is shown in the next figure



(of course we can use $\psi_1(T - t)$ and sample at T in the upper branch since $\psi_1(T - t)$ is causal. However the implementation shown above is more useful in answering part 4).

3) This is a binary equiprobable system therefore we can use $P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right)$. We have

$$\begin{aligned} d^2 &= \|\mathbf{s}_1 - \mathbf{s}_2\|^2 \\ &= \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 dt \\ &= A^2 T \end{aligned}$$

$$\text{and } P_e = Q\left(\frac{A\sqrt{T}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)$$

4) The signal waveform $\psi_1(\frac{T}{2} - t)$ matched to $\psi_1(t)$ is exactly the same with the signal waveform $\psi_2(T - t)$ matched to $\psi_2(t)$. That is,

$$\psi_1\left(\frac{T}{2} - t\right) = \psi_2(T - t) = \psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the optimal receiver can be implemented by using just one filter with impulse response $\psi_1(t)$ followed by two samplers which sample the output of the matched filter at $t = \frac{T}{2}$ and $t = T$ to produce random variables r_1 and r_2 , respectively.

5) If $s_1(t)$ is transmitted, the received signal $r(t)$ is

$$r(t) = s_1(t) + \frac{1}{2}s_1\left(t - \frac{T}{2}\right) + n(t)$$

and if $s_2(t)$ is transmitted the received signal is

$$r(t) = s_2(t) + \frac{1}{2}s_2\left(t - \frac{T}{2}\right) + n(t)$$

The outputs of the two samplers at $t = \frac{T}{2}$ and $t = T$, which are equal to $\int_{-\infty}^{\infty} r(t)\psi_1(t) dt$ and $\int_{-\infty}^{\infty} r(t)\psi_2(t) dt$, respectively, are given by

$$\begin{aligned} r_1 &= \int_{-\infty}^{\infty} r(t)\psi_1(t) dt = \begin{cases} \alpha + n_1 & s_1(t) \text{ sent} \\ n_1 & s_2(t) \text{ sent} \end{cases} \\ r_2 &= \int_{-\infty}^{\infty} r(t)\psi_2(t) dt = \begin{cases} \frac{\alpha}{2} + n_2 & s_1(t) \text{ sent} \\ \alpha + n_2 & s_2(t) \text{ sent} \end{cases} \end{aligned}$$

where $\alpha = A\sqrt{\frac{T}{2}}$ and n_1 and n_2 are independent Gaussian random variables with mean zero and variance $N_0/2$. Here r_1 and r_2 are all the information observed by the receiver, hence the decision is based on maximizing $f(r_1, r_2|s_m)$. Therefore,

$$D_1 \Leftrightarrow f(r_1, r_2|s_1) > f(r_1, r_2|s_2)$$

or

$$D_1 \Leftrightarrow Ke^{-\frac{(r_1 - \alpha)^2 + (r_2 - \frac{\alpha}{2})^2}{N_0}} > Ke^{-\frac{r_1^2 + (r_2 - \alpha)^2}{N_0}}$$

Simplifying this we obtain the optimal decision rule as

$$D_1 \Leftrightarrow 2r_1 - r_2 > \frac{\alpha}{4} \Leftrightarrow 2r_1 - r_2 > \frac{A}{4}\sqrt{\frac{T}{2}}$$

6) Here again we are comparing $f(r_1, r_2|s_1)$ and $f(r_1, r_2|s_2)$. But since a is random we have to use

$$f(r_1, r_2|s_m) = \int f(a, r_1, r_2|s_m) da = \int f(r_1, r_2|a, s_m)f(a) da = \int_0^1 f(r_1, r_2|a, s_m) da$$

Similar to case 5, we have

$$\begin{aligned} r_1 &= \int_{-\infty}^{\infty} r(t)\psi_1(t) dt = \begin{cases} \alpha + n_1 & s_1(t) \text{ sent} \\ n_1 & s_2(t) \text{ sent} \end{cases} \\ r_2 &= \int_{-\infty}^{\infty} r(t)\psi_2(t) dt = \begin{cases} a\alpha + n_2 & s_1(t) \text{ sent} \\ \alpha + n_2 & s_2(t) \text{ sent} \end{cases} \end{aligned}$$

where $\alpha = A\sqrt{\frac{T}{2}}$. Therefore,

$$f(r_1, r_2|s_1, a) = Ke^{-\frac{(r_1-\alpha)^2+(r_2-a\alpha)^2}{N_0}}$$

$$f(r_1, r_2|s_2, a) = Ke^{-\frac{r_1^2+(r_2-\alpha)^2}{N_0}}$$

From the second equation we have

$$\int_0^1 f(r_1, r_2|s_2, a) da = Ke^{-\frac{r_1^2+(r_2-\alpha)^2}{N_0}}$$

and from the first equation

$$\int_0^1 f(r_1, r_2|s_1, a) da = Ke^{-\frac{(r_1-\alpha)^2}{N_0}} \int_0^1 e^{-\frac{(r_2-a\alpha)^2}{N_0}} da$$

Introducing a change of variable of the form $\beta = \frac{r_2-a\alpha}{\sqrt{N_0}/2}$ and manipulating we obtain

$$D_1 \Leftrightarrow e^{\frac{2\alpha}{N_0}(r_1-r_2)} \left[Q\left(\frac{r_2-\alpha}{\sqrt{N_0}/2}\right) - Q\left(\frac{r_2}{\sqrt{N_0}/2}\right) \right] > \alpha e^{-\frac{r_2^2}{N_0}}$$

where as before $\alpha = A\sqrt{\frac{T}{2}}$. This approach is quite similar to the approach taken when analyzing noncoherent systems and as you see the resulting equations are nonlinear as in that case.

Problem 4.48

1. The receiver does not know the status of the switch: A decision in favor of $s_1(t)$ is made if

$$\frac{1}{2}p(r_1, r_2|s = \sqrt{E}, S = \text{closed}) + \frac{1}{2}p(r_1, r_2|s = \sqrt{E}, S = \text{open}) >$$

$$\frac{1}{2}p(r_1, r_2|s = -\sqrt{E}, S = \text{closed}) + \frac{1}{2}p(r_1, r_2|s = -\sqrt{E}, S = \text{open})$$

resulting in

$$\exp\left(-\frac{(r_1 - \sqrt{E})^2 + (r_2 - \sqrt{E})^2}{N_0}\right) + \exp\left(-\frac{r_1^2 + (r_2 - \sqrt{E})^2}{N_0}\right) >$$

$$\exp\left(-\frac{(r_1 + \sqrt{E})^2 + (r_2 + \sqrt{E})^2}{N_0}\right) + \exp\left(-\frac{r_1^2 + (r_2 + \sqrt{E})^2}{N_0}\right)$$

which can be simplified to the following form after eliminating common terms from both sides

$$\exp\left(\frac{4r_2\sqrt{E}}{N_0}\right) > \frac{1 + \exp(-E/N_0)\exp(-2r_1\sqrt{E}/N_0)}{1 + \exp(-E/N_0)\exp(2r_1\sqrt{E}/N_0)}$$

2. The receiver knows the position of the switch: If the switch is open, then $r_1 = n_1$ is irrelevant that can be ignored by the receiver, thus receiver decides based on r_2 only and the decision threshold is 0. If S is closed then the receiver detects $s_1(t)$ if

$$p(r_1, r_2 | s = \sqrt{E}) > p(r_1, r_2 | s = -\sqrt{E})$$

or

$$\exp\left((r_1 - \sqrt{E})^2/2N_0\right) \exp\left((r_2 - \sqrt{E})^2/2N_0\right) > \exp\left((r_1 + \sqrt{E})^2/2N_0\right) \exp\left((r_2 + \sqrt{E})^2/2N_0\right)$$

resulting in

$$\exp\left((r_1 + r_2)\sqrt{E}/N_0\right) > 1$$

or simply $r_1 + r_2 > 0$. In this case an error occurs if $-\sqrt{E}$ is sent and $r_1 + r_2 > 0$, i.e., if $-\sqrt{E} + n_1 - \sqrt{E} + n_2 > 0$ or if $n_1 + n_2 > 2\sqrt{E}$. Since n_1 and n_2 are iid Gaussian with mean 0 and variance $N_0/2$, $n_1 + n_2$ is a zero-mean Gaussian with variance N_0 and $P(e|S \text{ closed}) = Q\left(2\sqrt{E}/\sqrt{N_0}\right) = Q\left(\sqrt{\frac{4E}{N_0}}\right)$. For the case when the switch is open we have $P(e|S \text{ open}) = Q\left(\sqrt{\frac{2E}{N_0}}\right)$, and

$$P_e = \frac{1}{2}Q\left(\sqrt{\frac{2E}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{4E}{N_0}}\right)$$

3. Transmitter and receiver both know the position of S: If S is open, then the transmitter sends all energy on the second channel, i.e., $\alpha = 0$ and $\beta = \sqrt{2}$. The resulting error probability in this case is $Q(\sqrt{4E/N_0})$. If S is closed the decision rule in favor of $s_1(t)$ becomes

$$\exp\left((r_1 - \alpha\sqrt{E})^2/2N_0\right) \exp\left((r_2 - \beta\sqrt{E})^2/2N_0\right) > \exp\left((r_1 + \alpha\sqrt{E})^2/2N_0\right) \exp\left((r_2 + \beta\sqrt{E})^2/2N_0\right)$$

following the method of part 2 the decision rule simplifies to $\alpha r_1 + \beta r_2 > 0$. Again similar to part 2 an error occurs if $\alpha n_1 + \beta n_2 > (\alpha + \beta)\sqrt{E}$. Note that $\alpha n_1 + \beta n_2$ is Gaussian with mean zero and variance N_0 , hence the error probability when the switch is open is given by $P_e = Q\left((\alpha + \beta)\sqrt{\frac{E}{N_0}}\right)$. The overall error probability in this case is given by

$$P_e = \frac{1}{2}Q\left(\sqrt{\frac{4E}{N_0}}\right) + \frac{1}{2}Q\left((\alpha + \beta)\sqrt{\frac{E}{N_0}}\right)$$

To minimize the error probability we have to maximize $\alpha + \beta$ subject to $\alpha^2 + \beta^2 = 1$. This can be done by the Lagrange multipliers method by differentiating $\alpha + \beta + \lambda(\alpha^2 + \beta^2)$ to obtain $1 + 2\alpha\lambda = 0$ and $1 + 2\beta\lambda = 0$, resulting in $\alpha = \beta = 1$, hence

$$P_e = Q(\sqrt{4E/N_0})$$

Problem 4.49

$r(t) = (r_1(t), r_2(t)) = (s(t) + n_1(t), As(t) + n_2(t))$, where $s(t) = \pm\sqrt{E}\phi(t)$. Projecting on $\phi(t)$ gives $\mathbf{r} = (r_1, r_2) = (s + n_1, As + n_2)$, with $s = \pm\sqrt{E}$.

1. $\hat{s} = \arg \max p(\mathbf{r}|s) = p_{n_1}(r_1 - s) \left(\frac{1}{2}p_{n_2}(r_2 - s) + \frac{1}{2}p_{n_2}(r_2 + s) \right)$. Since $s = \pm\sqrt{E}$, the term in the parenthesis can be ignored and $\hat{s} = \arg \max p(\mathbf{r}|s) = p_{n_1}(r_1 - s)$, resulting in the decision rule $D_1 \Leftrightarrow r_1 > 0$, and $p_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)$.

2. Here $\mathbf{C} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ and $\mathbf{C}^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$. This results in

$$p(n_1, n_2) \propto e^{-\frac{2}{3}(n_1^2 + n_2^2 - n_1 n_2)}$$

and the decision rule becomes $\hat{s} = \arg \max \frac{1}{2}p(r_1 - s, r_2 - s) + \frac{1}{2}p(r_1 - s, r_2 + s)$. This relation cannot be much simplified. Some reduction is possible resulting in

$$D_1 \Leftrightarrow \frac{\sinh\left(\frac{2}{3}\sqrt{E}(r_1 + r_2)\right)}{\sinh\left(2\sqrt{E}(r_2 - r_1)\right)} > e^{-\frac{4}{3}E}$$

3. Here $r(t) = (1 + A)s(t) + n(t)$ where the PSD of $n(t)$ is $\frac{N_1 + N_2}{2}$. The decision rule is $\arg \max \frac{1}{2}p_n(r - 2s) + \frac{1}{2}p_n(r)$, resulting in $\arg \max p_n(r - 2s)$, which reduces to $D_1 \Leftrightarrow r > 0$. Note that although the decision rule in this case is very similar to case 1, but the error probability in this case is much higher.
4. The setting is similar to case 1. The decision rule is $\arg \max \frac{1}{2}p_{n_1}(r_1 - s) (p_{n_2}(r_2 - s) + p_{n_2}(r_2))$. Some reduction is possible,

$$D_1 \Leftrightarrow \frac{\cosh\left(\frac{2r_2\sqrt{E}-E}{2N_2}\right)}{\cosh\left(\frac{2r_2\sqrt{E}+E}{2N_2}\right)} > e^{-\frac{4r_1\sqrt{E}}{N_1} - \frac{4r_2\sqrt{E}}{N_2} + \frac{E}{N_2}}$$

5. In this case $\mathbf{r} = ((1-A)s + n_1, As + n_2)$, and the decision rule becomes $\arg \max \frac{1}{2}p_{n_1}(r_1)p_{n_2}(r_2 - s) + \frac{1}{2}p_{n_1}(r_1 - s)p_{n_2}(r_2)$. This reduces to

$$D_1 \Leftrightarrow \sinh\left(\frac{2r_1\sqrt{E}}{N_1}\right) + \sinh\left(\frac{2r_2\sqrt{E}}{N_2}\right) > 0 \Leftrightarrow r_1 + r_2 > 0$$

Problem 4.50

1. Since a takes only nonnegative values the threshold of the binary antipodal signaling scheme always remains at zero and the decision rule is similar to an AWGN channel.
2. For a given value of a we have $P_b = Q\left(a\sqrt{E_b/N_0}\right)$. We have to average the above value over all a 's to obtain the error probability. This is done similar to the integration in part 1 of Problem 4.44 to give

$$P_b = \frac{1}{2} \left(1 - \sqrt{\frac{E_b/N_0}{1 + E_b/N_0}} \right)$$

where we have used the fact that σ^2 parameter in the Rayleigh distribution given above is $1/2$.

3. For large SNR values

$$\frac{E_b/N_0}{1 + E_b/N_0} = \frac{1}{1 + 1/(E_b/N_0)} \approx 1 - 1/(E_b/N_0) \approx (1 - 1/(2E_b/N_0))^2$$

Hence,

$$P_b \approx \frac{1}{2} (1 - 1 + 1/(2E_b/N_0)) = \frac{1}{4E_b/N_0}$$

4. For an AWGN channel an error probability of 10^{-5} is achieved at E_b/N_0 of 9.6 dB. For a fading channel from $10^{-5} = \frac{1}{4E_b/N_0}$ we have $E_b/N_0 = 10^5/4 = 25000$ or 44 dB, a difference of 34.4 dB.
5. For orthogonal signaling and noncoherent detection $P_b = \frac{1}{2}e^{-a^2 E_b/2N_0}$. We need to average this using the PDF of a to determine the error probability. This is done similar to part 5 of Problem 4.44. The result is $P_b = 1/(2 + E_b/N_0)$.

Problem 4.51

Define $\phi_1(t) = g_1(t)$ and $\alpha = \int g_2(t)g_1(t) dt$, then by Gram-Schmidt $g_2(t) = \alpha\phi_1(t) + \beta\phi_2(t)$ where $\phi_2(t)$ is the normalized version of $g_2(t) - \alpha g_1(t)$, and $\beta = \sqrt{1 - \alpha^2}$. Now from $r(t) = s_1(t) + s_2(t) + n(t)$ we have the following mappings

$$(m_1, m_2) = (1, 1) \Leftrightarrow \mathbf{s}_{11} = (1 + \alpha, \beta) \quad (4.0.1)$$

$$(m_1, m_2) = (1, 2) \Leftrightarrow \mathbf{s}_{12} = (1 - \alpha, -\beta) \quad (4.0.2)$$

$$(m_1, m_2) = (2, 1) \Leftrightarrow \mathbf{s}_{21} = (-1 + \alpha, \beta) \quad (4.0.3)$$

$$(m_1, m_2) = (2, 2) \Leftrightarrow \mathbf{s}_{22} = (-1 - \alpha, -\beta) \quad (4.0.4)$$

and $\mathbf{r} = \mathbf{s} + \mathbf{n}$.

1. For detection of m_1 , we treat m_2 as a RV taking 1 or 2 with equal probability. The detection rule is

$$\arg \max_{s_1 \in \{-1, 1\}} p_{n_1}(r_1 - s_1 - \alpha)p_{n_2}(r_2 - \beta) + p_{n_1}(r_1 - s_1 + \alpha)p_{n_2}(r_2 + \beta)$$

similarly for detection of m_2 we have

$$\arg \max_{(s_{21}, s_{22}) \in \{(\alpha, \beta), (-\alpha, -\beta)\}} p_{n_1}(r_1 - 1 - s_{21})p_{n_2}(r_2 - s_{22}) + p_{n_1}(r_1 + 1 - s_{21})p_{n_2}(r_2 - s_{22})$$

These relations reduce to

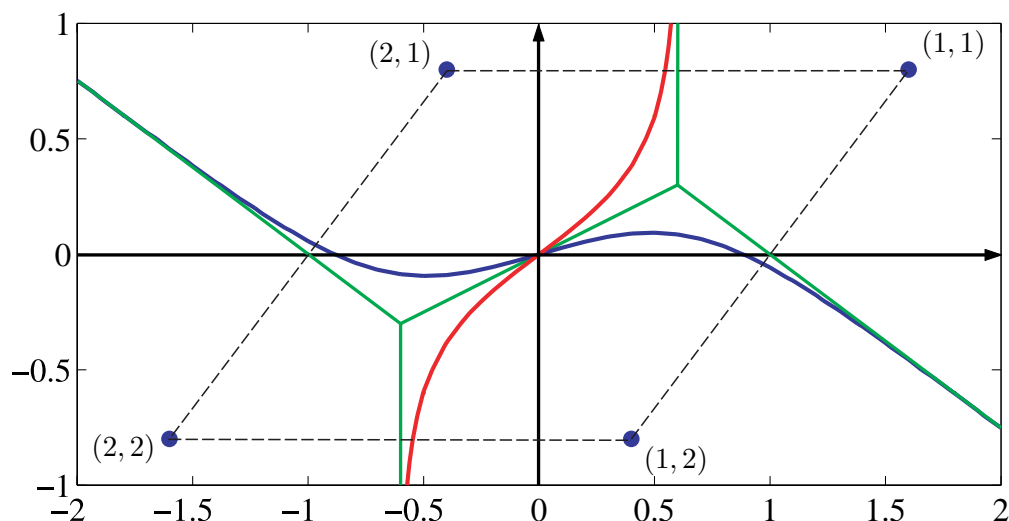
$$D_{+1} \Leftrightarrow e^{\frac{2r_1}{N_0}} \cosh\left(\frac{2\alpha r_1 - 2\alpha + 2\beta r_2}{N_0}\right) > e^{-\frac{2r_1}{N_0}} \cosh\left(\frac{2\alpha r_1 + 2\alpha + 2\beta r_2}{N_0}\right)$$

and

$$D_{(\alpha, \beta)} \Leftrightarrow -e^{\frac{2r_1}{N_0}} \sinh\left(\frac{2\alpha r_1 - 2\alpha + 2\beta r_2}{N_0}\right) > e^{-\frac{2r_1}{N_0}} \sinh\left(\frac{2\alpha r_1 + 2\alpha + 2\beta r_2}{N_0}\right)$$

2. This case easy and straightforward. We have four equiprobable signals whose coordinates are given in Equations 1–4, therefore the optimal decision rule is the nearest neighbor rule for this constellation. The decision region boundaries are line segments.
3. The “real” optimal detector is the one designed in part 1, since it minimizes each error probability individually. However, it has a more complex structure due to nonlinear relations that define the decision boundaries.

The following figure shows the four points in the constellation together with the optimal decision region boundaries for different cases. The red curve shows the decision region boundary for m_1 , the region on the left side of this curve denotes $\hat{m}_1 = 2$ and region on the right denotes $\hat{m}_1 = 1$. The blue curve denotes the boundary of the two decision regions for m_2 . The region above this curve corresponds to $\hat{m}_2 = 1$ and the region below it corresponds to $\hat{m}_2 = 2$. The green lines denote the four decision regions described in case 2 (the plot is given for $\alpha = 0.6, \beta = 0.8$ and $N_0 = 1$).



4. This is an example of a communication situation that even in the binary case one cannot derive a closed form expression for the error probability. In case 2 we cannot find a closed form expression for the error probability either due to the irregular decision regions. Only in the very special case of $\alpha = 0$ and $\beta = 1$ an expression for the error probability can be derived (the problem reduces to QPSK in this case).

Problem 4.52

1. From the law of sines in triangles we obtain $R = \frac{\sqrt{E} \sin \frac{\pi}{M}}{\sin \theta}$.
2. Let the origin be at \sqrt{E} then the error probability is $P[(n_1, n_2) \in \text{shaded area}]$. In polar coordinates $r = \sqrt{n_1^2 + n_2^2}$ and $\phi = \arctan \frac{n_2}{n_1}$ are independent with Rayleigh and uniform on $[0, 2\pi]$ distributions.

$$f(r, \phi) = \begin{cases} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}} & r > 0, 0 \leq \phi < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma^2 = \frac{N_0}{2}$. The error probability is twice the integral over half of the shaded region

$$\begin{aligned} P_e &= 2 \int_{\frac{\pi}{M}}^{\pi} d\phi \int_R^{\infty} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}} dr \\ &= \frac{1}{\pi} \int_{\frac{\pi}{M}}^{\pi} e^{-\frac{E \sin^2 \frac{\pi}{M}}{2\sigma^2 \sin^2(\phi - \pi/M)}} d\phi \end{aligned}$$

Using change of variable $\theta = \phi - \pi/M$, the desired result is obtained.

3. For $M = 2$,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

and from above we have

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{E}{N_0} \frac{1}{\sin^2 \theta}} d\theta$$

, by equating these two results and noting that $E = E_b$, the desired result for $Q(x)$ is obtained.

Problem 4.53

1. We have to look for $\arg \max_m p(\mathbf{r}_1, \mathbf{r}_2 | \mathbf{s}_m) = \arg \max_m p_{\mathbf{n}_1}(\mathbf{r}_1 - \mathbf{s}_m) p_{\mathbf{n}_2}(\mathbf{r}_2 - \mathbf{s}_m)$ by the independence of the noise processes. This reduces to $\arg \max_m \exp \left(-\frac{|\mathbf{r}_1 - \mathbf{s}_m|^2}{N_{01}} - \frac{|\mathbf{r}_2 - \mathbf{s}_m|^2}{N_{02}} \right)$ or $\arg \min_m \frac{|\mathbf{r}_1 - \mathbf{s}_m|^2}{N_{01}} + \frac{|\mathbf{r}_2 - \mathbf{s}_m|^2}{N_{02}}$ which simplifies to

$$\arg \max_m \left(\frac{\mathbf{r}_1 \cdot \mathbf{s}_m}{N_{01}} + \frac{\mathbf{r}_2 \cdot \mathbf{s}_m}{N_{02}} \right) - \frac{E_m}{2} \left(\frac{1}{N_{01}} + \frac{1}{N_{02}} \right)$$

2. In this case the relation reduces to $\arg \max_m (\mathbf{r}_1 + \mathbf{r}_2) \cdot \mathbf{s}_m - E_m$.
3. Obviously only $\mathbf{r}_1 + \mathbf{r}_2$ is needed.
4. Here we are dealing with a one dimensional case and comparing

$$\left(\frac{r_1}{N_{01}} + \frac{r_2}{N_{02}} \right) \sqrt{E_m} - \frac{E_m}{2} \left(\frac{1}{N_{01}} + \frac{1}{N_{02}} \right) > 0$$

which is equivalent to

$$r_1 + \alpha r_2 > r_{th}$$

with $\alpha = \frac{N_{01}}{N_{02}}$ and $r_{th} = \frac{\sqrt{E_m}}{2}(1 + \alpha)$.

5. Of course $\alpha = 1$ and $r_{th} = \sqrt{E_m}$. The detection rule is $r_1 + r_2 > \sqrt{E_m}$. An error occurs when $n_1 + n_2 > \sqrt{E_m}$. But $n_1 + n_2$ is Gaussian with variance N_0 , hence $P_e = Q \left(\sqrt{\frac{E_m}{N_0}} \right)$. For on-off keying with one antenna, $P_e = Q \left(\sqrt{\frac{E_m}{2N_0}} \right)$. Hence having two antennas enhances the performance by 3dB.

Problem 4.54

1. Since the two intervals are non-overlapping we can use two basis for representation of the signal in the two intervals, the vector representation of the received signal will be $(r_1, r_2) = (1 + n_1, A + n_2)$ when $s_1(t)$ is transmitted and $(r_1, r_2) = (-1 + n_1, -A + n_2)$ when $s_2(t)$ is transmitted. The optimal decision rule for $s_1(t)$ is given by

$$\int_0^\infty p(r_1, r_2 | a, s = 1) e^{-a} da > \int_0^\infty p(r_1, r_2 | a, s = -1) e^{-a} da$$

or

$$\int_0^\infty e^{-\frac{(r_1-1)^2 + (r_2-a)^2}{N_0}} e^{-a} da > \int_0^\infty e^{-\frac{(r_1+1)^2 + (r_2+a)^2}{N_0}} e^{-a} da$$

2. Since the amplitude of the multipath link is always positive, it always helps the decision. In fact the system with multipath as described above cannot perform worse than the system without multipath because there is no interference between the direct path and the reflected path and the receiver can always base its decision on the direct path. Therefore the performance of the multipath system is generally superior to the performance of the system without multipath.

Problem 4.55

1. For coherent detection the energy in each signal is E_b and the error probability is $P_b = Q(\sqrt{E_b/N_0})$.
2. In this case the two signals are multiplied by $\sqrt{2}\cos(2\pi f_c t + \theta)$ by the demodulator and then passed through a LPF (the $\sqrt{2}$ factor is introduced to keep energy intact). The output of the LPF, in the absence of noise, will be $\sqrt{E_b}\phi_i(t)\cos\theta$, $i = 1, 2$. This is equivalent to a reduction in signal energy by a factor of $\cos^2\theta$. Note that this will not affect to noise power since by Problem 2.22 noise statistics is invariant to rotations. The error probability is given by

$$P_b = Q\left(\cos\theta\sqrt{\frac{2E_b}{N_0}}\right)$$

and the worst case happens when $\cos\theta = 0$, i.e., for $\theta = \pi/2$.

3. For $\theta = \pi/2$, we have $P_b = \frac{1}{2}$.

Problem 4.56

1. A change of variable of the form $y_i = Rx_i$, $1 \leq i \leq n$ shows that

$$V_n(R) = \int \int \dots \int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq R^2} dx_1 dx_2 \dots dx_n = R^n \int \int \dots \int_{y_1^2 + y_2^2 + \dots + y_n^2 \leq 1} dy_1 dy_2 \dots dy_n = B_n R^n$$

where

$$B_n = \int \int \dots \int_{y_1^2 + y_2^2 + \dots + y_n^2 \leq 1} dy_1 dy_2 \dots dy_n$$

denotes $V(1)$.

2. Note that

$$p(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|\mathbf{y}\|^2}{2}}$$

and

$$V_n(R) - V_n(R - \epsilon) = B_n \left(R^n - R^{n(R-\epsilon)} \right) \approx B_n n R^{n-1} \epsilon$$

where we have used $d(R^n) = nR^{n-1}$.

3. By spherical symmetry of the PDF $p(\mathbf{y})$ is a function of the length of the vector \mathbf{y} and not the direction of it, hence it is a function of $\|\mathbf{y}\|$. The relation $\mathbf{P}[R - \epsilon \leq \|\mathbf{Y}\| \leq R] \approx p_{\|\mathbf{Y}\|}(R)\epsilon$ follows directly from the definition of the PDF.

4. Equating the expressions for $\mathbf{P}[R - \epsilon \leq \|\mathbf{Y}\| \leq R]$ in parts 2 and 3 gives the desired result.

5. This follows directly from part 4 and

$$\int_0^\infty p_{\|\mathbf{Y}\|}(r) dr = 1$$

6. We introduce the change of variable $u = r^2/2$, from which $r dr = du$ and hence $r^{n-1} dr = 2^{\frac{n}{2}-1} u^{\frac{n}{2}-1} du$, hence

$$\begin{aligned} \int_0^\infty r^{n-1} e^{-r^2/2} dr &= 2^{n/2-1} \int_0^\infty u^{n/2-1} e^{-u} du \\ &= 2^{n/2-1} \Gamma(n/2) \end{aligned}$$

Using the result of part 5, we have

$$\frac{nB_n}{(2\pi)^{n/2}} 2^{n/2-1} \Gamma(n/2) = 1$$

from which the desired result is obtained after noting that $\frac{n}{2} \Gamma\left(\frac{n}{2}\right) = \Gamma\left(\frac{n}{2} + 1\right)$.

Problem 4.57

1. Each edge of the n -dimensional cube contains L points of the lattice, the cube contains $L^n = 2^{n\ell}$ pints or $n\ell$ bits. The number of bits per two dimensions is thus $\beta = n\ell/(n/2) = 2\ell$.
2. By definition,

$$\text{CFM}(\mathcal{C}) = \frac{d_{\min}^2(\mathcal{C})}{\mathcal{E}_{\text{avg}/2D}(\mathcal{C})} \quad (*)$$

It is clear that $d_{\min}(\mathcal{C}) = 1$ and by 4.7-25,

$$\begin{aligned}
 \mathcal{E}_{\text{avg}/2D} &\approx \frac{2}{nV(\mathcal{R})} \int_{\mathcal{R}} \|\mathbf{x}\|^2 d\mathbf{x} \\
 &= \frac{2}{nL^n} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \cdots \int_{-L/2}^{L/2} (x_1^2 + x_2^2 + \cdots x_n^2) dx_1 dx_2 \cdots dx_n \\
 &= \frac{2}{nL^n} \sum_{i=1}^n \int_{-L/2}^{L/2} dx_1 \int_{-L/2}^{L/2} dx_2 \cdots \int_{-L/2}^{L/2} x_i^2 dx_i \cdots \int_{-L/2}^{L/2} dx_n \\
 &= \frac{2}{nL^n} nL^{n-1} \frac{1}{3} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] \\
 &= \frac{2L^{n+2}}{12L^n} \\
 &= \frac{L^2}{6} = \frac{2^\beta}{6}
 \end{aligned}$$

Substituting into (*) we have

$$\text{CFM}(\mathcal{C}) \approx \frac{1}{\frac{2^\beta}{6}} = \frac{6}{2^\beta}$$

3. From 4.7-40, we have

$$\begin{aligned}
 \gamma_s(\mathcal{R}) &= \frac{n [V(\mathcal{R})]^{1+\frac{2}{n}}}{12 \int_{\mathcal{R}} \|\mathbf{x}\|^2 d\mathbf{x}} \\
 &\approx \frac{(nL^n)^{1+\frac{2}{n}}}{12 \times \frac{nL^{n+2}}{12}} \\
 &= 1
 \end{aligned}$$

Problem 4.58

$$v(t) = \sum_k [I_k u(t - 2kT_b) + jJ_k u(t - 2kT_b - T_b)] \text{ where } u(t) = \begin{cases} \sin \frac{\pi t}{2T_b}, & 0 \leq t \leq T_b \\ 0, & \text{o.w.} \end{cases}. \text{ Note}$$

that

$$u(t - T_b) = \sin \frac{\pi(t - T_b)}{2T_b} = -\cos \frac{\pi t}{2T_b}, \quad T_b \leq t \leq 3T_b$$

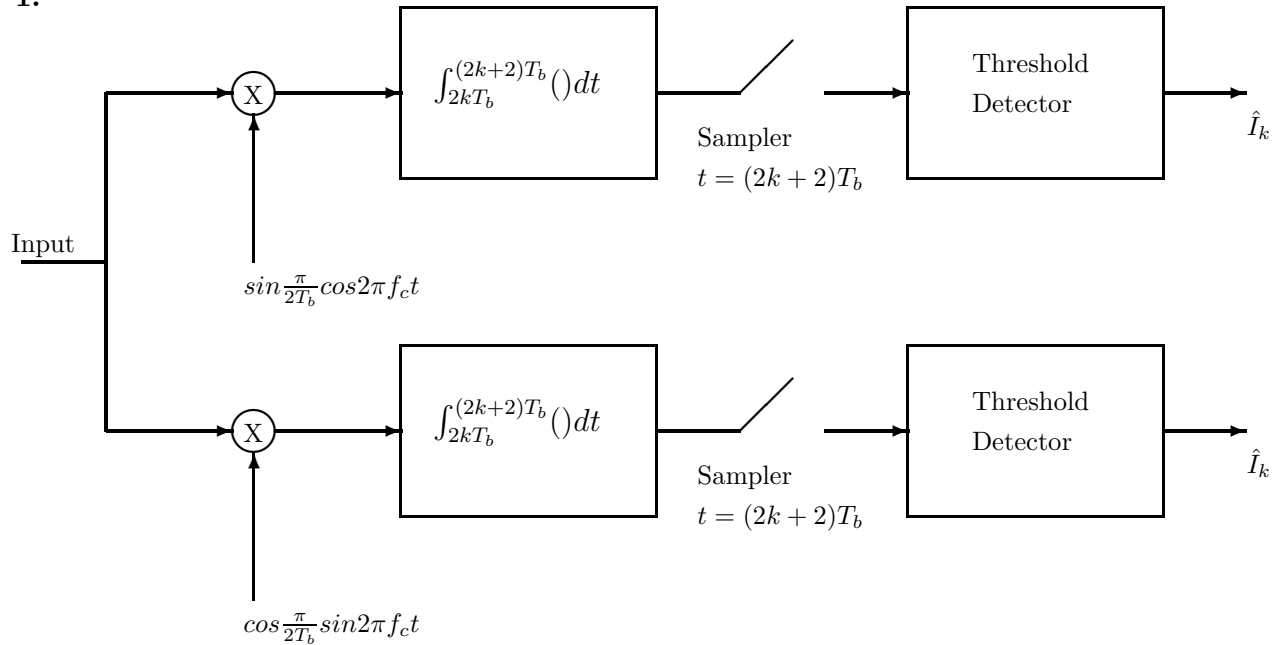
Hence, $v(t)$ may be expressed as :

$$v(t) = \sum_k \left[I_k \sin \frac{\pi(t - 2kT_b)}{2T_b} - jJ_k \cos \frac{\pi(t - 2kT_b)}{2T_b} \right]$$

The transmitted signal is :

$$\text{Re} \left[v(t)e^{j2\pi f_c t} \right] = \sum_k \left[I_k \sin \frac{\pi(t-2kT_b)}{2T_b} \cos 2\pi f_c t + J_k \cos \frac{\pi(t-2kT_b)}{2T_b} \sin 2\pi f_c t \right]$$

1.



2. The offset QPSK signal is equivalent to two independent binary PSK systems. Hence for coherent detection, the error probability is :

$$P_e = Q\left(\sqrt{2\gamma_b}\right), \quad \gamma_b = \frac{\mathcal{E}_b}{N_0}, \quad \mathcal{E}_b = \frac{1}{2} \int_0^T |u(t)|^2 dt$$

3. Viterbi decoding (MLSE) of the MSK signal yields identical performance to that of part (2).

4. MSK is basically binary FSK with frequency separation of $\Delta f = 1/2T$. For this frequency separation the binary signals are orthogonal with coherent detection. Consequently, the error probability for symbol-by-symbol detection of the MSK signal yields an error probability of

$$P_e = Q(\sqrt{\gamma_b})$$

which is 3dB poorer relative to the optimum Viterbi detection scheme.

For non-coherent detection of the MSK signal, the correlation coefficient for $\Delta f = 1/2T$ is :

$$|\rho| = \frac{\sin \pi/2}{\pi/2} = 0.637$$

From the results in Sec. 5-4-4 we observe that the performance of the non coherent detector method is about 4 dB worse than the coherent FSK detector. hence the loss is about 7 dB compared to the optimum demodulator for the MSK signal.

Problem 4.59

1. For n repeaters in cascade, the probability of i out of n repeaters to produce an error is given by the binomial distribution

$$P_i = \binom{n}{i} p^i (1-p)^{n-i}$$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i}$$

Let P_{even} be the probability that an even number of repeaters produces an error. Then

$$P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore,

$$P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^n \binom{n}{i} p^i (1-p)^{n-i} = (p + 1 - p)^n = 1$$

One more relation between P_{even} and P_{odd} can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$\begin{aligned} P_{\text{even}} - P_{\text{odd}} &= \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} \\ &\stackrel{a}{=} \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} \\ &= (1 - p - p)^n = (1 - 2p)^n \end{aligned}$$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for i even and -1 when i is odd. Solving the system

$$\begin{aligned} P_{\text{even}} + P_{\text{odd}} &= 1 \\ P_{\text{even}} - P_{\text{odd}} &= (1 - 2p)^n \end{aligned}$$

we obtain

$$P_n = P_{\text{odd}} = \frac{1}{2}(1 - (1 - 2p)^n)$$

2. Expanding the quantity $(1 - 2p)^n$, we obtain

$$(1 - 2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \dots$$

Since, $p \ll 1$ we can ignore all the powers of p which are greater than one. Hence,

$$P_n \approx \frac{1}{2}(1 - 1 + n2p) = np = 100 \times 10^{-6} = 10^{-4}$$

Problem 4.60

The overall probability of error is approximated by (see 5-5-2)

$$P(e) = KQ \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right]$$

Thus, with $P(e) = 10^{-6}$ and $K = 100$, we obtain the probability of each repeater $P_r = Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] = 10^{-8}$. The argument of the function $Q[\cdot]$ that provides a value of 10^{-8} is found from tables to be

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.61$$

Hence, the required $\frac{\mathcal{E}_b}{N_0}$ is $5.61^2/2 = 15.7$

Problem 4.61

1. The antenna gain for a parabolic antenna of diameter D is :

$$G_R = \eta \left(\frac{\pi D}{\lambda} \right)^2$$

If we assume that the efficiency factor is 0.5, then with :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \quad D = 3 \times 0.3048 \text{ m}$$

we obtain :

$$G_R = G_T = 45.8458 = 16.61 \text{ dB}$$

2. The effective radiated power is :

$$\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}$$

3. The received power is :

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995 \times 10^{-9} = -85.23 \text{ dB} = -55.23 \text{ dBm}$$

Note that :

$$\text{dBm} = 10 \log_{10} \left(\frac{\text{actual power in Watts}}{10^{-3}} \right) = 30 + 10 \log_{10}(\text{power in Watts})$$

Problem 4.62

1. The antenna gain for a parabolic antenna of diameter D is :

$$G_R = \eta \left(\frac{\pi D}{\lambda} \right)^2$$

If we assume that the efficiency factor is 0.5, then with :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \quad \text{and} \quad D = 1 \text{ m}$$

we obtain :

$$G_R = G_T = 54.83 = 17.39 \text{ dB}$$

2. The effective radiated power is :

$$\text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \text{ dB}$$

3. The received power is :

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \text{ dB} = -67.20 \text{ dBm}$$

Problem 4.63

The wavelength of the transmitted signal is:

$$\lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m}$$

The gain of the parabolic antenna is:

$$G_R = \eta \left(\frac{\pi D}{\lambda} \right)^2 = 0.6 \left(\frac{\pi 10}{0.03} \right)^2 = 6.58 \times 10^5 = 58.18 \text{ dB}$$

The received power at the output of the receiver antenna is:

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{3 \times 10^{1.5} \times 6.58 \times 10^5}{(4 \times 3.14159 \times \frac{4 \times 10^7}{0.03})^2} = 2.22 \times 10^{-13} = -126.53 \text{ dB}$$

Problem 4.64

1. Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by :

$$G_R = \eta \left(\frac{\pi D}{\lambda} \right)^2 = 0.5 \left(\frac{3.14159 \times 50}{\frac{3 \times 10^8}{2 \times 10^9}} \right)^2 = 5.483 \times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is :

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^8}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

2. If $\frac{\mathcal{E}_b}{N_0} = 10 \text{ dB} = 10$, then

$$R = \frac{P_R}{N_0} \left(\frac{\mathcal{E}_b}{N_0} \right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

Problem 4.65

The wavelength of the transmission is :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$

If 1 MHz is the passband bandwidth, then the rate of binary transmission is $R_b = W = 10^6$ bps. Hence, with $N_0 = 4.1 \times 10^{-21}$ W/Hz we obtain :

$$\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \implies 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}$$

The transmitted power is related to the received power through the relation (see 5-5-6) :

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} \implies P_T = \frac{P_R}{G_T G_R} \left(4\pi \frac{d}{\lambda} \right)^2$$

Substituting in this expression the values $G_T = 10^{0.6}$, $G_R = 10^5$, $d = 36 \times 10^6$ and $\lambda = 0.75$ we obtain

$$P_T = 0.1185 = -9.26 \text{ dBW}$$

Solutions Manual
for
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(Chapter 5) ¹

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Problem 5.1

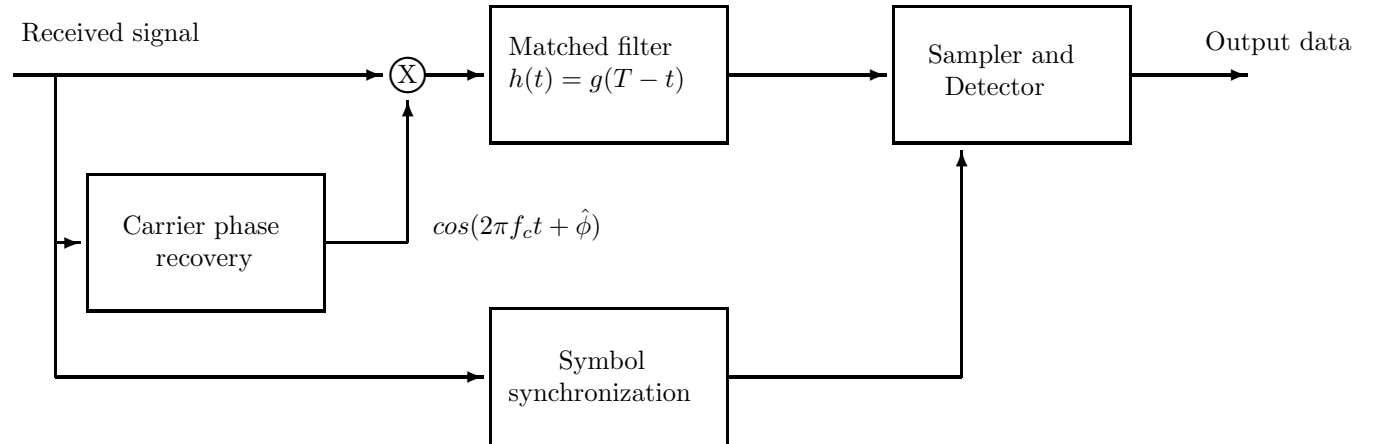
Using the relationship $r(t) = \sum_{n=1}^N r_n f_n(t)$ and $s(t; \psi) = \sum_{n=1}^N s_n(\psi) f_n(t)$ we have :

$$\begin{aligned}
 \frac{1}{N_0} \int [r(t) - s(t; \psi)]^2 dt &= \frac{1}{N_0} \int \left[\sum_{n=1}^N (r_n - s_n(\psi)) f_n(t) \right]^2 dt \\
 &= \frac{1}{N_0} \int \sum_{n=1}^N \sum_{m=1}^N (r_n - s_n(\psi))(r_m - s_m(\psi)) f_n(t) f_m(t) dt \\
 &= \frac{1}{N_0} \sum_{n=1}^N \sum_{m=1}^N (r_n - s_n(\psi))(r_m - s_m(\psi)) \delta_{mn} \\
 &= \frac{1}{N_0} \sum_{n=1}^N (r_n - s_n(\psi))(r_n - s_n(\psi)) \\
 &= \frac{1}{2\sigma^2} \sum_{n=1}^N [r_n - s_n(\psi)]^2
 \end{aligned}$$

where we have exploited the orthonormality of the basis functions $f_n(t) : \int_{T_0} f_n(t) f_m(t) dt = \delta_{mn}$ and $\sigma^2 = \frac{N_0}{2}$.

Problem 5.2

A block diagram of a binary PSK receiver that employs match filtering is given in the following figure :



As we note, the received signal is, first, multiplied with $\cos(2\pi f_c t + \hat{\phi})$ and then fed the matched filter. This allows us to have the filter matched to the baseband pulse $g(t)$ and not to the passband signal.

If we want to have the filter matched to the passband signal, then the carrier phase estimate is fed into the matched filter, which should have an impulse response:

$$\begin{aligned}
 h(t) &= s(T-t) = g(T-t)\cos(2\pi f_c(T-t) + \hat{\phi}) \\
 &= g(T-t)[\cos(2\pi f_c T)\cos(-2\pi f_c t + \hat{\phi}) + \sin(2\pi f_c T)\sin(-2\pi f_c t + \hat{\phi})] \\
 &= g(T-t)\cos(-2\pi f_c t + \hat{\phi}) = g(T-t)\cos(2\pi f_c t - \hat{\phi})
 \end{aligned}$$

where we have assumed that $f_c T$ is an integer so that : $\cos(2\pi f_c T) = 1$, $\sin(2\pi f_c T) = 0$. As we note, in this case the impulse response of the filter should change according to the carrier phase estimate, something that is difficult to implement in practise. Hence, the initial realization (shown in the figure) is preferable.

Problem 5.3

a. The closed loop transfer function is :

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

Since the real part of the roots is negative, the poles lie in the left half plane and therefore, the system is stable.

b. Writing the denominator in the form :

$$D = s^2 + 2\zeta\omega_n s + \omega_n^2$$

we identify the natural frequency of the loop as $\omega_n = 1$ and the damping factor as $\zeta = \frac{1}{\sqrt{2}}$

Problem 5.4

a. The closed loop transfer function is :

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{K}{\tau_1 s^2 + s + K} = \frac{\frac{K}{\tau_1}}{s^2 + \frac{1}{\tau_1}s + \frac{K}{\tau_1}}$$

The gain of the system at $f = 0$ is :

$$|H(0)| = |H(s)|_{s=0} = 1$$

b. The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-1 \pm \sqrt{1 - 4K\tau_1}}{2\tau_1}$$

In order for the system to be stable the real part of the poles must be negative. Since K is greater than zero, the latter implies that τ_1 is positive. If in addition we require that the damping factor $\zeta = \frac{1}{2\sqrt{\tau_1 K}}$ is less than 1, then the gain K should satisfy the condition :

$$K > \frac{1}{4\tau_1}$$

Problem 5.5

The transfer function of the RC circuit is :

$$G(s) = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} = \frac{1 + R_2Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + \tau_2s}{1 + \tau_1s}$$

From the last equality we identify the time constants as :

$$\tau_2 = R_2C, \quad \tau_1 = (R_1 + R_2)C$$

Problem 5.6

Assuming that the input resistance of the operational amplifier is high so that no current flows through it, then the voltage-current equations of the circuit are :

$$\begin{aligned} V_2 &= -AV_1 \\ V_1 - V_2 &= \left(R_1 + \frac{1}{Cs}\right)i \\ V_1 - V_0 &= iR \end{aligned}$$

where, V_1 , V_2 is the input and output voltage of the amplifier respectively, and V_0 is the signal at the input of the filter. Eliminating i and V_1 , we obtain :

$$\frac{V_2}{V_1} = \frac{\frac{R_1 + \frac{1}{Cs}}{R}}{1 + \frac{1}{A} - \frac{R_1 + \frac{1}{Cs}}{AR}}$$

If we let $A \rightarrow \infty$ (ideal amplifier), then :

$$\frac{V_2}{V_1} = \frac{1 + R_1Cs}{RCs} = \frac{1 + \tau_2s}{\tau_1s}$$

Hence, the constants τ_1 , τ_2 of the active filter are given by :

$$\tau_1 = RC, \quad \tau_2 = R_1C$$

Problem 5.7

In the non decision-directed timing recovery method we maximize the function :

$$\Lambda_L(\tau) = \sum_m y_m^2(\tau)$$

with respect to τ . Thus, we obtain the condition :

$$\frac{d\Lambda_L(\tau)}{d\tau} = 2 \sum_m y_m(\tau) \frac{dy_m(\tau)}{d\tau} = 0$$

Suppose now that we approximate the derivative of the log-likelihood $\Lambda_L(\tau)$ by the finite difference :

$$\frac{d\Lambda_L(\tau)}{d\tau} \approx \frac{\Lambda_L(\tau + \delta) - \Lambda_L(\tau - \delta)}{2\delta}$$

Then, if we substitute the expression of $\Lambda_L(\tau)$ in the previous approximation, we obtain :

$$\begin{aligned} \frac{d\Lambda_L(\tau)}{d\tau} &= \frac{\sum_m y_m^2(\tau + \delta) - \sum_m y_m^2(\tau - \delta)}{2\delta} \\ &= \frac{1}{2\delta} \sum_m \left[\left(\int r(t)g(t - mT - \tau - \delta)dt \right)^2 - \left(\int r(t)g(t - mT - \tau + \delta)dt \right)^2 \right] \end{aligned}$$

where $g(-t)$ is the impulse response of the matched filter in the receiver. However, this is the expression of the early-late gate synchronizer, where the lowpass filter has been substituted by the summation operator. Thus, the early-late gate synchronizer is a close approximation to the timing recovery system.

Problem 5.8

An on-off keying signal is represented as :

$$\begin{aligned} s_1(t) &= A \cos(2\pi f_c t + \phi_c), & 0 \leq t \leq T \text{ (binary 1)} \\ s_2(t) &= 0, & 0 \leq t \leq T \text{ (binary 0)} \end{aligned}$$

Let $r(t)$ be the received signal, that is $r(t) = s(t; \phi_c) + n(t)$ where $s(t; \phi_c)$ is either $s_1(t)$ or $s_2(t)$ and $n(t)$ is white Gaussian noise with variance $\frac{N_0}{2}$. The likelihood function, that is to be maximized with respect to ϕ_c over the interval $[0, T]$, is proportional to :

$$\Lambda(\phi_c) = \exp \left[-\frac{2}{N_0} \int_0^T [r(t) - s(t; \phi_c)]^2 dt \right]$$

Maximization of $\Lambda(\phi_c)$ is equivalent to the maximization of the log-likelihood function :

$$\begin{aligned} \Lambda_L(\phi_c) &= -\frac{2}{N_0} \int_0^T [r(t) - s(t; \phi_c)]^2 dt \\ &= -\frac{2}{N_0} \int_0^T r^2(t) dt + \frac{4}{N_0} \int_0^T r(t) s(t; \phi_c) dt - \frac{2}{N_0} \int_0^T s^2(t; \phi_c) dt \end{aligned}$$

Since the first term does not involve the parameter of interest ϕ_c and the last term is simply a constant equal to the signal energy of the signal over $[0, T]$ which is independent of the carrier phase, we can carry the maximization over the function :

$$V(\phi_c) = \int_0^T r(t) s(t; \phi_c) dt$$

Note that $s(t; \phi_c)$ can take two different values, $s_1(t)$ and $s_2(t)$, depending on the transmission of a binary 1 or 0. Thus, a more appropriate function to maximize is the average log-likelihood

$$\bar{V}(\phi_c) = \frac{1}{2} \int_0^T r(t) s_1(t) dt + \frac{1}{2} \int_0^T r(t) s_2(t) dt$$

Since $s_2(t) = 0$, the function $\bar{V}(\phi_c)$ takes the form :

$$\bar{V}(\phi_c) = \frac{1}{2} \int_0^T r(t) A \cos(2\pi f_c t + \phi_c) dt$$

Setting the derivative of $\bar{V}(\phi_c)$ with respect to ϕ_c equal to zero, we obtain :

$$\begin{aligned} \frac{\partial \bar{V}(\phi_c)}{\partial \phi_c} = 0 &= \frac{1}{2} \int_0^T r(t) A \sin(2\pi f_c t + \phi_c) dt \\ &= \cos \phi_c \frac{1}{2} \int_0^T r(t) A \sin(2\pi f_c t) dt + \sin \phi_c \frac{1}{2} \int_0^T r(t) A \cos(2\pi f_c t) dt \end{aligned}$$

Thus, the maximum likelihood estimate of the carrier phase is :

$$\hat{\phi}_{c,ML} = -\arctan \left[\frac{\int_0^T r(t) \sin(2\pi f_c t) dt}{\int_0^T r(t) \cos(2\pi f_c t) dt} \right]$$

Problem 5.9

a. The wavelength λ is :

$$\lambda = \frac{3 \times 10^8}{10^9} \text{ m} = \frac{3}{10} \text{ m}$$

Hence, the Doppler frequency shift is :

$$f_D = \pm \frac{u}{\lambda} = \pm \frac{100 \text{ Km/hr}}{\frac{3}{10} \text{ m}} = \pm \frac{100 \times 10^3 \times 10}{3 \times 3600} \text{ Hz} = \pm 92.5926 \text{ Hz}$$

The plus sign holds when the vehicle travels towards the transmitter whereas the minus sign holds when the vehicle moves away from the transmitter.

b. The maximum difference in the Doppler frequency shift, when the vehicle travels at speed 100 km/hr and $f = 1 \text{ GHz}$, is :

$$\Delta f_{D_{\max}} = 2f_D = 185.1852 \text{ Hz}$$

This should be the bandwidth of the Doppler frequency tracking loop.

c. The maximum Doppler frequency shift is obtained when $f = 1 \text{ GHz} + 1 \text{ MHz}$ and the vehicle moves towards the transmitter. In this case :

$$\lambda_{\min} = \frac{3 \times 10^8}{10^9 + 10^6} \text{ m} = 0.2997 \text{ m}$$

and therefore :

$$f_{D_{\max}} = \frac{100 \times 10^3}{0.2997 \times 3600} = 92.6853 \text{ Hz}$$

Thus, the Doppler frequency spread is $B_d = 2f_{D_{\max}} = 185.3706 \text{ Hz}$.

Problem 5.10

The maximum likelihood phase estimate given by (6-2-38) is :

$$\hat{\phi}_{ML} = -\tan^{-1} \frac{\text{Im} \left[\sum_{n=0}^{K-1} I_n^* y_n \right]}{\text{Re} \left[\sum_{n=0}^{K-1} I_n^* y_n \right]}$$

where $y_n = \int_{nT}^{(n+1)T} r(t)g^*(t-nT)dt$. The $Re(y_n)$, $Im(y_n)$ are statistically independent components of y_n . Since $r(t) = e^{-j\phi} \sum_n I_n g(t-nT) + z(t)$ it follows that $y_n = I_n e^{-j\phi} + z_n$, where the pulse energy is normalized to unity. Then :

$$\sum_{n=0}^{K-1} I_n^* y_n = \sum_{n=0}^{K-1} \left[|I_n|^2 e^{-j\phi} + I_n^* z_n \right]$$

Hence :

$$E \left\{ Im \left[\sum_{n=0}^{K-1} \left[|I_n|^2 e^{-j\phi} + I_n^* z_n \right] \right] \right\} = -K |\bar{I}_n|^2 \sin \phi$$

and

$$E \left\{ Re \left[\sum_{n=0}^{K-1} \left[|I_n|^2 e^{-j\phi} + I_n^* z_n \right] \right] \right\} = -K |\bar{I}_n|^2 \cos \phi$$

Consequently : $E \left[\hat{\phi}_{ML} \right] = -\tan^{-1} \frac{-\sin \phi}{\cos \phi} = \phi$, and hence, $\hat{\phi}_{ML}$ is an unbiased estimate of the true phase ϕ .

Problem 5.11

The procedure that is used in Sec. 5-2-7 to derive the pdf $p(\Theta_r)$ for the phase of a PSK signal may be used to determine the pdf $p(\hat{\phi}_{ML})$. Specifically, we have :

$$\hat{\phi}_{ML} = -\tan^{-1} \frac{Im \left[\sum_{n=0}^{K-1} I_n^* y_n \right]}{Re \left[\sum_{n=0}^{K-1} I_n^* y_n \right]}$$

where $y_n = \int_{nT}^{(n+1)T} r(t)g^*(t-nT)dt$ and $r(t) = e^{-j\phi} \sum_n I_n g(t-nT) + z(t)$. Substitution of $r(t)$ into y_n yields : $y_n = I_n e^{-j\phi} + z_n$. Hence :

$$\sum_{n=0}^{K-1} I_n^* y_n = e^{-j\phi} \sum_{n=0}^{K-1} |I_n|^2 + \sum_{n=0}^{K-1} I_n^* z_n$$

$$U + jV = C e^{-j\phi} + z = C \cos \phi + x + j(y - C \sin \phi)$$

where $C = \sum_{n=0}^{K-1} |I_n|^2$ and $z = \sum_{n=0}^{K-1} I_n^* z_n = x + jy$. The random variables (x,y) are zero-mean, Gaussian random variables with variances σ^2 . Hence :

$$p(U, V) = \frac{1}{2\pi\sigma^2} e^{-[(U-C \cos \phi)^2 - (V-C \sin \phi)^2]}$$

By defining $R = \sqrt{U^2 + V^2}$ and $\hat{\phi}_{ML} = \tan^{-1} \frac{V}{U}$ and making the change in variables, we obtain $p(R, \hat{\phi}_{ML})$ and finally, $p(\hat{\phi}_{ML}) = \int_0^\infty p(r, \hat{\phi}_{ML}) dr$. Upon performing the integration over R , as in Sec. 5-2-7, we obtain :

$$p(\hat{\phi}_{ML}) = \frac{1}{2\pi} e^{-2\gamma \sin^2 \hat{\phi}_{ML}} \int_0^\infty r e^{-(r - \sqrt{4\gamma} \cos \hat{\phi}_{ML})^2 / 2} dr$$

where $\gamma = C^2/2\sigma^2$. The graph of $p(\hat{\phi}_{ML})$ is identical to that given on page 271, Fig. 5-2-9. We observe that $E(\hat{\phi}_{ML}) = \phi$, so that the estimate is unbiased.

Problem 5.12

We begin with the log-likelihood function given in (6-2-35), namely :

$$\Lambda_L(\phi) = \text{Re} \left\{ \left[\frac{1}{N_0} \int_{T_0} r(t) s_l^*(t) dt \right] e^{j\phi} \right\}$$

where $s_l(t)$ is given as : $s_l(t) = \sum_n I_n g(t - nT) + j \sum_n J_n u(t - nT - T/2)$. Again we define $y_n = \int_{nT}^{(n+1)T} r(t) s_l^*(t - nT) dt$. Also, let : $x_n = \int_{(n+1/2)T}^{(n+3/2)T} r(t) s_l^*(t - nT - T/2) dt$. Then :

$$\begin{aligned} \Lambda_L(\phi) &= \text{Re} \left\{ \frac{e^{j\phi}}{N_0} \left[\sum_{n=0}^{K-1} I_n^* y_n - j \sum_{n=0}^{K-1} J_n^* x_n \right] \right\} \\ &= \text{Re} [A \cos \phi + j A \sin \phi] \end{aligned}$$

where $A = \sum_{n=0}^{K-1} I_n^* y_n - j \sum_{n=0}^{K-1} J_n^* x_n$. Thus : $\Lambda_L(\phi) = \text{Re}(A) \cos \phi - \text{Im}(A) \sin \phi$ and :

$$\frac{d\Lambda_L(\phi)}{d\phi} = -\text{Re}(A) \sin \phi - \text{Im}(A) \cos \phi = 0 \Rightarrow$$

$$\hat{\phi}_{ML} = -\tan^{-1} \frac{\text{Im} \left[\sum_{n=0}^{K-1} I_n^* y_n - j \sum_{n=0}^{K-1} J_n^* x_n \right]}{\text{Re} \left[\sum_{n=0}^{K-1} I_n^* y_n - j \sum_{n=0}^{K-1} J_n^* x_n \right]}$$

Problem 5.13

Assume that the signal $u_m(t)$ is the input to the Costas loop. Then $u_m(t)$ is multiplied by $\cos(2\pi f_c t + \hat{\phi})$ and $\sin(2\pi f_c t + \hat{\phi})$, where $\cos(2\pi f_c t + \hat{\phi})$ is the output of the VCO. Hence :

$$\begin{aligned} u_{mc}(t) &= A_m g_T(t) \cos(2\pi f_c t) \cos(2\pi f_c t + \hat{\phi}) - A_m \hat{g}_T(t) \sin(2\pi f_c t) \cos(2\pi f_c t + \hat{\phi}) \\ &= \frac{A_m g_T(t)}{2} [\cos(2\pi 2f_c t + \hat{\phi}) + \cos(\hat{\phi})] - \frac{A_m \hat{g}_T(t)}{2} [\sin(2\pi 2f_c t + \hat{\phi}) - \sin(\hat{\phi})] \\ u_{ms}(t) &= A_m g_T(t) \cos(2\pi f_c t) \sin(2\pi f_c t + \hat{\phi}) - A_m \hat{g}_T(t) \sin(2\pi f_c t) \sin(2\pi f_c t + \hat{\phi}) \\ &= \frac{A_m g_T(t)}{2} [\sin(2\pi 2f_c t + \hat{\phi}) + \sin(\hat{\phi})] - \frac{A_m \hat{g}_T(t)}{2} [\cos(\hat{\phi}) - \cos(2\pi 2f_c t + \hat{\phi})] \end{aligned}$$

The lowpass filters of the Costas loop will reject the double frequency components, so that :

$$\begin{aligned} y_{mc}(t) &= \frac{A_m g_T(t)}{2} \cos(\hat{\phi}) + \frac{A_m \hat{g}_T(t)}{2} \sin(\hat{\phi}) \\ y_{ms}(t) &= \frac{A_m g_T(t)}{2} \sin(\hat{\phi}) - \frac{A_m \hat{g}_T(t)}{2} \cos(\hat{\phi}) \end{aligned}$$

Note that when the carrier phase has been extracted correctly, $\hat{\phi} = 0$ and therefore :

$$y_{mc}(t) = \frac{A_m g_T(t)}{2}, \quad y_{ms}(t) = -\frac{A_m \hat{g}_T(t)}{2}$$

If the second signal, $y_{ms}(t)$ is passed through a Hilbert transformer, then :

$$\hat{y}_{ms}(t) = -\frac{A_m \hat{g}_T(t)}{2} = \frac{A_m g_T(t)}{2}$$

and by adding this signal to $y_{mc}(t)$ we obtain the original unmodulated signal.

Problem 5.14

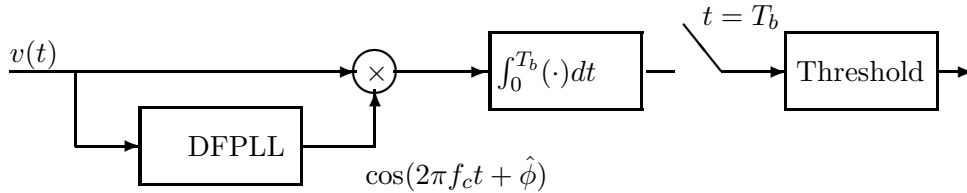
a. The signal $r(t)$ can be written as :

$$\begin{aligned} r(t) &= \pm \sqrt{2P_s} \cos(2\pi f_c t + \phi) + \sqrt{2P_c} \sin(2\pi f_c t + \phi) \\ &= \sqrt{2(P_c + P_s)} \sin \left(2\pi f_c t + \phi + a_n \tan^{-1} \left(\sqrt{\frac{P_s}{P_c}} \right) \right) \\ &= \sqrt{2P_T} \sin \left(2\pi f_c t + \phi + a_n \cos^{-1} \left(\sqrt{\frac{P_c}{P_T}} \right) \right) \end{aligned}$$

where $a_n = \pm 1$ are the information symbols and P_T is the total transmitted power. As it is observed the signal has the form of a PM signal where :

$$\theta_n = a_n \cos^{-1} \left(\sqrt{\frac{P_c}{P_T}} \right)$$

Any method used to extract the carrier phase from the received signal can be employed at the receiver. The following figure shows the structure of a receiver that employs a decision-feedback PLL. The operation of the PLL is described in the next part.



b. At the receiver (DFPLL) the signal is demodulated by crosscorrelating the received signal :

$$r(t) = \sqrt{2P_T} \sin \left(2\pi f_c t + \phi + a_n \cos^{-1} \left(\sqrt{\frac{P_c}{P_T}} \right) \right) + n(t)$$

with $\cos(2\pi f_c t + \hat{\phi})$ and $\sin(2\pi f_c t + \hat{\phi})$. The sampled values at the output of the correlators are :

$$\begin{aligned} r_1 &= \frac{1}{2} \left[\sqrt{2P_T} - n_s(t) \right] \sin(\phi - \hat{\phi} + \theta_n) + \frac{1}{2} n_c(t) \cos(\phi - \hat{\phi} + \theta_n) \\ r_2 &= \frac{1}{2} \left[\sqrt{2P_T} - n_s(t) \right] \cos(\phi - \hat{\phi} + \theta_n) + \frac{1}{2} n_c(t) \sin(\phi - \hat{\phi} + \theta_n) \end{aligned}$$

where $n_c(t)$, $n_s(t)$ are the in-phase and quadrature components of the noise $n(t)$. If the detector has made the correct decision on the transmitted point, then by multiplying r_1 by $\cos(\theta_n)$ and r_2 by $\sin(\theta_n)$ and subtracting the results, we obtain (after ignoring the noise) :

$$\begin{aligned} r_1 \cos(\theta_n) &= \frac{1}{2} \sqrt{2P_T} \left[\sin(\phi - \hat{\phi}) \cos^2(\theta_n) + \cos(\phi - \hat{\phi}) \sin(\theta_n) \cos(\theta_n) \right] \\ r_2 \sin(\theta_n) &= \frac{1}{2} \sqrt{2P_T} \left[\cos(\phi - \hat{\phi}) \cos(\theta_n) \sin(\theta_n) - \sin(\phi - \hat{\phi}) \sin^2(\theta_n) \right] \\ e(t) &= r_1 \cos(\theta_n) - r_2 \sin(\theta_n) = \frac{1}{2} \sqrt{2P_T} \sin(\phi - \hat{\phi}) \end{aligned}$$

The error $e(t)$ is passed to the loop filter of the DFPLL that drives the VCO. As it is seen only the phase θ_n is used to estimate the carrier phase.

c. Having a correct carrier phase estimate, the output of the lowpass filter sampled at $t = T_b$ is :

$$\begin{aligned} r &= \pm \frac{1}{2} \sqrt{2P_T} \sin \cos^{-1} \left(\sqrt{\frac{P_c}{P_T}} \right) + n \\ &= \pm \frac{1}{2} \sqrt{2P_T} \sqrt{1 - \frac{P_c}{P_T}} + n \\ &= \pm \frac{1}{2} \sqrt{2P_T \left(1 - \frac{P_c}{P_T} \right)} + n \end{aligned}$$

where n is a zero-mean Gaussian random variable with variance :

$$\begin{aligned} \sigma_n^2 &= E \left[\int_0^{T_b} \int_0^{T_b} n(t)n(\tau) \cos(2\pi f_c t + \phi) \cos(2\pi f_c \tau + \phi) dt d\tau \right] \\ &= \frac{N_0}{2} \int_0^{T_b} \cos^2(2\pi f_c t + \phi) dt \\ &= \frac{N_0}{4} \end{aligned}$$

Note that T_b has been normalized to 1 since the problem has been stated in terms of the power of the involved signals. The probability of error is given by :

$$P(\text{error}) = Q \left[\sqrt{\frac{2P_T}{N_0} \left(1 - \frac{P_c}{P_T} \right)} \right]$$

The loss due to the allocation of power to the pilot signal is :

$$\text{SNR}_{\text{loss}} = 10 \log_{10} \left(1 - \frac{P_c}{P_T} \right)$$

When $P_c/P_T = 0.1$, then $\text{SNR}_{\text{loss}} = 10 \log_{10}(0.9) = -0.4576$ dB. The negative sign indicates that the SNR is decreased by 0.4576 dB.

Problem 5.15

The received signal-plus-noise vector at the output of the matched filter may be represented as (see (5-2-63) for example) :

$$r_n = \sqrt{\mathcal{E}_s} e^{j(\theta_n - \phi)} + N_n$$

where $\theta_n = 0, \pi/2, \pi, 3\pi/2$ for QPSK, and ϕ is the carrier phase. By raising r_n to the fourth power and neglecting all products of noise terms, we obtain :

$$\begin{aligned} r_n^4 &\approx (\sqrt{\mathcal{E}_s})^4 e^{j4(\theta_n - \phi)} + 4(\sqrt{\mathcal{E}_s})^3 N_n \\ &\approx (\sqrt{\mathcal{E}_s})^3 [\sqrt{\mathcal{E}_s} e^{-j4\phi} + 4N_n] \end{aligned}$$

If the estimate is formed by averaging the received vectors $\{r_n^4\}$ over K signal intervals, we have the resultant vector $U = K\sqrt{\mathcal{E}_s} e^{-j\phi} + 4 \sum_{n=1}^K N_n$. Let $\phi_4 \equiv 4\phi$. Then, the estimate of ϕ_4 is :

$$\hat{\phi}_4 = -\tan^{-1} \frac{\text{Im}(U)}{\text{Re}(U)}$$

N_n is a complex-valued Gaussian noise component with zero mean and variance $\sigma^2 = N_0/2$. Hence, the pdf of $\hat{\phi}_4$ is given by (5-2-55) where :

$$\gamma_s = \frac{(K\sqrt{\mathcal{E}_s})^2}{16(2K\sigma^2)} = \frac{K^2\mathcal{E}_s}{16KN_0} = \frac{K\mathcal{E}_s}{16N_0}$$

To a first approximation, the variance of the estimate is :

$$\sigma_{\hat{\phi}_4}^2 \approx \frac{1}{\gamma_s} = \frac{16}{K\mathcal{E}_s/N_0}$$

Problem 5.16

The PDF of the carrier phase error ϕ_e , is given by :

$$p(\phi_e) = \frac{1}{\sqrt{2\pi}\sigma_\phi} e^{-\frac{\phi_e^2}{2\sigma_\phi^2}}$$

Thus the average probability of error is :

$$\begin{aligned}
 \bar{P}_2 &= \int_{-\infty}^{\infty} P_2(\phi_e) p(\phi_e) d\phi_e \\
 &= \int_{-\infty}^{\infty} Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \cos^2 \phi_e \right] p(\phi_e) d\phi_e \\
 &= \frac{1}{2\pi\sigma_\phi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\frac{2\mathcal{E}_b}{N_0} \cos^2 \phi_e}} \exp \left[-\frac{1}{2} \left(x^2 + \frac{\phi_e^2}{\sigma_\phi^2} \right) \right] dx d\phi_e
 \end{aligned}$$

Problem 5.17

The log-likelihood function of the symbol timing may be expressed in terms of the equivalent low-pass signals as

$$\begin{aligned}
 \Lambda_L(\tau) &= \operatorname{Re} \left[\frac{1}{N_0} \int_{T_0} r(t) s_l^*(t; \tau) dt \right] \\
 &= \operatorname{Re} \left[\frac{1}{N_0} \int_{T_0} r(t) \sum_n I_n^* g^*(t - nT - \tau) dt \right] \\
 &= \operatorname{Re} \left[\frac{1}{N_0} \sum_n I_n^* y_n(\tau) \right]
 \end{aligned}$$

where $y_n(\tau) = \int_{T_0} r(t) g^*(t - nT - \tau) dt$.

A necessary condition for $\hat{\tau}$ to be the ML estimate of τ is

$$\begin{aligned}
 \frac{d\Lambda_L(\tau)}{d\tau} &= 0 \Rightarrow \\
 \frac{d}{d\tau} [\sum_n I_n^* y_n(\tau) + \sum_n I_n y_n^*(\tau)] &= 0 \Rightarrow \\
 \sum_n I_n^* \frac{d}{d\tau} y_n(\tau) + \sum_n I_n \frac{d}{d\tau} y_n^*(\tau) &= 0
 \end{aligned}$$

If we express $y_n(\tau)$ into its real and imaginary parts : $y_n(\tau) = a_n(\tau) + jb_n(\tau)$, the above expression simplifies to the following condition for the ML estimate of the timing $\hat{\tau}$

$$\sum_n \operatorname{Re}[I_n] \frac{d}{d\tau} a_n(\tau) + \sum_n \operatorname{Im}[I_n] \frac{d}{d\tau} b_n(\tau) = 0$$

Problem 5.18

We follow the exact same steps of the derivation found in Sec. 6.4. For a PAM signal $I_n^* = I_n$ and $J_n = 0$. Since the pulse $g(t)$ is real, it follows that $B(\tau)$ in expression (6.4-6) is zero, therefore (6.4-7) can be rewritten as

$$\Lambda_L(\phi, \tau) = A(\tau) \cos \phi$$

where

$$A(\tau) = \frac{1}{N_0} \sum I_n y_n(\tau)$$

Then the necessary conditions for the estimates of ϕ and τ to be the ML estimates (6.4-8) and (6.4-9) give

$$\hat{\phi}_{ML} = 0$$

and

$$\sum_n I_n \frac{d}{d\tau} [y_n(\tau)]_{\tau=\hat{\tau}_{ML}} = 0$$

Problem 5.19

The derivation for the ML estimates of ϕ and τ for an offset QPSK signal follow the derivation found in Sec. 6.4, with the added simplification that, since $w(t) = g(t - T/2)$, we have that $x_n(\tau) = y_n(\tau + T/2)$.

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 6) ¹

Prepared by
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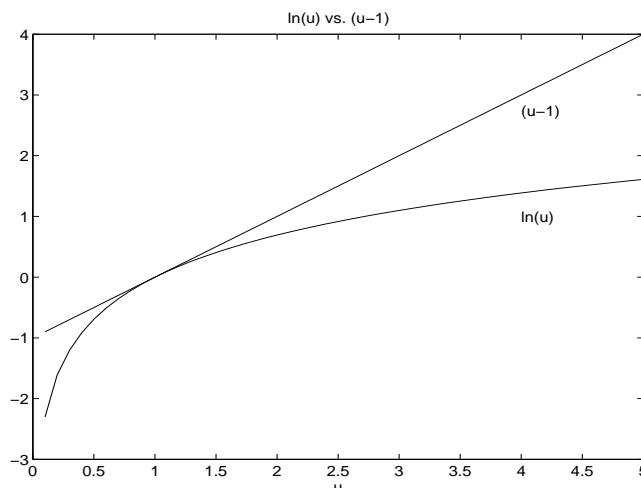
Problem 6.1

Let $f(u) = u - 1 - \ln u$. The first and second derivatives of $f(u)$ are

$$\frac{df}{du} = 1 - \frac{1}{u}$$

and

$$\frac{d^2f}{du^2} = \frac{1}{u^2} > 0, \forall u > 0$$



Hence this function achieves its minimum at $\frac{df}{du} = 0 \Rightarrow u = 1$. The minimum value is $f(u = 1) = 0$ so $\ln u = u - 1$, at $u = 1$. For all other values of $u : 0 < u < \infty, u \neq 1$, we have $f(u) > 0 \Rightarrow u - 1 > \ln u$.

Problem 6.2

We will show that $-I(X; Y) \leq 0$

$$\begin{aligned} -I(X; Y) &= -\sum_i \sum_j P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i)P(y_j)} \\ &= \frac{1}{\ln 2} \sum_i \sum_j P(x_i, y_j) \ln \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \end{aligned}$$

We use the inequality $\ln u \leq u - 1$. We need only consider those terms for which $P(x_i, y_j) > 0$; then, applying the inequality to each term in $I(X; Y)$:

$$\begin{aligned} -I(X; Y) &\leq \frac{1}{\ln 2} \sum_i \sum_j P(x_i, y_j) \left[\frac{P(x_i)P(y_j)}{P(x_i, y_j)} - 1 \right] \\ &= \frac{1}{\ln 2} \sum_i \sum_j [P(x_i)P(y_j) - P(x_i, y_j)] \leq 0 \end{aligned}$$

The first inequality becomes equality if and only if

$$\frac{P(x_i)P(y_j)}{P(x_i, y_j)} = 1 \iff P(x_i)P(y_j) = P(x_i, y_j)$$

when $P(x_i, y_j) > 0$. Also, since the summations

$$\sum_i \sum_j [P(x_i)P(y_j) - P(x_i, y_j)]$$

contain only the terms for which $P(x_i, y_j) > 0$, this term equals zero if and only if $P(X_i)P(Y_j) = 0$, when $P(x_i, y_j) = 0$. Therefore, both inequalities become equalities and hence, $I(X; Y) = 0$ if and only if X and Y are statistically independent.

Problem 6.3

We shall prove that $H(X) - \log n \leq 0$:

$$\begin{aligned} H(X) - \log n &= \sum_{i=1}^n p_i \log \frac{1}{p_i} - \log n \\ &= \sum_{i=1}^n p_i \log \frac{1}{p_i} - \sum_{i=1}^n p_i \log n \\ &= \sum_{i=1}^n p_i \log \frac{1}{np_i} \\ &= \frac{1}{\ln 2} \sum_{i=1}^n p_i \ln \frac{1}{np_i} \\ &\leq \frac{1}{\ln 2} \sum_{i=1}^n p_i \left(\frac{1}{np_i} - 1 \right) \\ &= 0 \end{aligned}$$

Hence, $H(X) \leq \log n$. Also, if $p_i = 1/n \quad \forall i \Rightarrow H(X) = \log n$.

Problem 6.4

1.

$$\begin{aligned}
 H(X) &= -\sum_{k=1}^{\infty} p(1-p)^{k-1} \log_2(p(1-p)^{k-1}) \\
 &= -p \log_2(p) \sum_{k=1}^{\infty} (1-p)^{k-1} - p \log_2(1-p) \sum_{k=1}^{\infty} (k-1)(1-p)^{k-1} \\
 &= -p \log_2(p) \frac{1}{1-(1-p)} - p \log_2(1-p) \frac{1-p}{(1-(1-p))^2} \\
 &= -\log_2(p) - \frac{1-p}{p} \log_2(1-p)
 \end{aligned}$$

2. Clearly $P(X = k|X > K) = 0$ for $k \leq K$. If $k > K$, then

$$P(X = k|X > K) = \frac{P(X = k, X > K)}{P(X > K)} = \frac{p(1-p)^{k-1}}{P(X > K)}$$

But,

$$\begin{aligned}
 P(X > K) &= \sum_{k=K+1}^{\infty} p(1-p)^{k-1} = p \left(\sum_{k=1}^{\infty} (1-p)^{k-1} - \sum_{k=1}^K (1-p)^{k-1} \right) \\
 &= p \left(\frac{1}{1-(1-p)} - \frac{1-(1-p)^K}{1-(1-p)} \right) = (1-p)^K
 \end{aligned}$$

so that

$$P(X = k|X > K) = \frac{p(1-p)^{k-1}}{(1-p)^K}$$

If we let $k = K + l$ with $l = 1, 2, \dots$, then

$$P(X = k|X > K) = \frac{p(1-p)^K(1-p)^{l-1}}{(1-p)^K} = p(1-p)^{l-1}$$

that is $P(X = k|X > K)$ is the geometrically distributed. Hence, using the results of the first part we obtain

$$\begin{aligned}
 H(X|X > K) &= -\sum_{l=1}^{\infty} p(1-p)^{l-1} \log_2(p(1-p)^{l-1}) \\
 &= -\log_2(p) - \frac{1-p}{p} \log_2(1-p)
 \end{aligned}$$

Problem 6.5

The marginal probabilities are given by

$$\begin{aligned}
 P(X = 0) &= \sum_k P(X = 0, Y = k) = P(X = 0, Y = 0) + P(X = 0, Y = 1) = \frac{2}{3} \\
 P(X = 1) &= \sum_k P(X = 1, Y = k) = P(X = 1, Y = 1) = \frac{1}{3} \\
 P(Y = 0) &= \sum_k P(X = k, Y = 0) = P(X = 0, Y = 0) = \frac{1}{3} \\
 P(Y = 1) &= \sum_k P(X = k, Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) = \frac{2}{3}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 H(X) &= -\sum_{i=0}^1 P_i \log_2 P_i = -\left(\frac{1}{3} \log_2 \frac{1}{3} + \frac{1}{3} \log_2 \frac{1}{3}\right) = .9183 \\
 H(X) &= -\sum_{i=0}^1 P_i \log_2 P_i = -\left(\frac{1}{3} \log_2 \frac{1}{3} + \frac{1}{3} \log_2 \frac{1}{3}\right) = .9183 \\
 H(X, Y) &= -\sum_{i=0}^2 \frac{1}{3} \log_2 \frac{1}{3} = 1.5850 \\
 H(X|Y) &= H(X, Y) - H(Y) = 1.5850 - 0.9183 = 0.6667 \\
 H(Y|X) &= H(X, Y) - H(X) = 1.5850 - 0.9183 = 0.6667
 \end{aligned}$$

Problem 6.6

1. The marginal distribution $P(x)$ is given by $P(x) = \sum_y P(x, y)$. Hence,

$$\begin{aligned}
 H(X) &= -\sum_x P(x) \log P(x) = -\sum_x \sum_y P(x, y) \log P(x) \\
 &= -\sum_{x,y} P(x, y) \log P(x)
 \end{aligned}$$

Similarly it is proved that $H(Y) = -\sum_{x,y} P(x, y) \log P(y)$.

2. Using the inequality $\ln w \leq w - 1$ with $w = \frac{P(x)P(y)}{P(x,y)}$, we obtain

$$\ln \frac{P(x)P(y)}{P(x,y)} \leq \frac{P(x)P(y)}{P(x,y)} - 1$$

Multiplying the previous by $P(x, y)$ and adding over x, y , we obtain

$$\sum_{x,y} P(x, y) \ln P(x)P(y) - \sum_{x,y} P(x, y) \ln P(x, y) \leq \sum_{x,y} P(x)P(y) - \sum_{x,y} P(x, y) = 0$$

Hence,

$$\begin{aligned} H(X, Y) &\leq - \sum_{x,y} P(x, y) \ln P(x)P(y) = - \sum_{x,y} P(x, y)(\ln P(x) + \ln P(y)) \\ &= - \sum_{x,y} P(x, y) \ln P(x) - \sum_{x,y} P(x, y) \ln P(y) = H(X) + H(Y) \end{aligned}$$

Equality holds when $\frac{P(x)P(y)}{P(x,y)} = 1$, i.e when X, Y are independent.

3.

$$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

Also, from part 2., $H(X, Y) \leq H(X) + H(Y)$. Combining the two relations, we obtain

$$H(Y) + H(X|Y) \leq H(X) + H(Y) \implies H(X|Y) \leq H(X)$$

Suppose now that the previous relation holds with equality. Then,

$$- \sum_x P(x) \log P(x|y) = - \sum_x P(x) \log P(x) \implies \sum_x P(x) \log \left(\frac{P(x)}{P(x|y)} \right) = 0$$

However, $P(x)$ is always greater or equal to $P(x|y)$, so that $\log(P(x)/P(x|y))$ is non-negative. Since $P(x) > 0$, the above equality holds if and only if $\log(P(x)/P(x|y)) = 0$ or equivalently if and only if $P(x)/P(x|y) = 1$. This implies that $P(x|y) = P(x)$ meaning that X and Y are independent.

Problem 6.7

$$\begin{aligned} H(X, Y) &= H(X, g(X)) = H(X) + H(g(X)|X) \\ &= H(g(X)) + H(X|g(X)) \end{aligned}$$

But, $H(g(X)|X) = 0$, since $g(\cdot)$ is deterministic. Therefore,

$$H(X) = H(g(X)) + H(X|g(X))$$

Since each term in the previous equation is non-negative we obtain

$$H(X) \geq H(g(X))$$

Equality holds when $H(X|g(X)) = 0$. This means that the values $g(X)$ uniquely determine X , or that $g(\cdot)$ is a one to one mapping.

Problem 6.8

$$H(X_1 X_2 \dots X_n) = - \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \dots \sum_{j_n=1}^{m_n} P(x_1, x_2, \dots, x_n) \log P(x_1, x_2, \dots, x_n)$$

Since the $\{x_i\}$ are statistically independent :

$$P(x_1, x_2, \dots, x_n) = P(x_1)P(x_2)\dots P(x_n)$$

and

$$\sum_{j_2=1}^{m_2} \dots \sum_{j_n=1}^{m_n} P(x_1)P(x_2)\dots P(x_n) = P(x_1)$$

(similarly for the other x_i). Then :

$$\begin{aligned} H(X_1 X_2 \dots X_n) &= - \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \dots \sum_{j_n=1}^{m_n} P(x_1)P(x_2)\dots P(x_n) \log P(x_1)P(x_2)\dots P(x_n) \\ &= - \sum_{j_1=1}^{m_1} P(x_1) \log P(x_1) - \sum_{j_2=1}^{m_2} P(x_2) \log P(x_2) \dots - \sum_{j_n=1}^{m_n} P(x_n) \log P(x_n) \\ &= \sum_{i=1}^n H(X_i) \end{aligned}$$

Problem 6.9

The conditional mutual information between x_3 and x_2 given x_1 is defined as :

$$I(x_3; x_2 | x_1) = \log \frac{P(x_3, x_2 | x_1)}{P(x_3 | x_1)P(x_2 | x_1)} = \log \frac{P(x_3 | x_2 x_1)}{P(x_3 | x_1)}$$

Hence :

$$I(x_3; x_2 | x_1) = I(x_3 | x_1) - I(x_3 | x_2 x_1)$$

and

$$\begin{aligned} I(X_3; X_2 | X_1) &= \sum_{x_1} \sum_{x_2} \sum_{x_3} P(x_1, x_2, x_3) \log \frac{P(x_3 | x_2 x_1)}{P(x_3 | x_1)} \\ &= \left\{ \begin{aligned} & - \sum_{x_1} \sum_{x_2} \sum_{x_3} P(x_1, x_2, x_3) \log P(x_3 | x_1) \\ & + \sum_{x_1} \sum_{x_2} \sum_{x_3} P(x_1, x_2, x_3) \log P(x_3 | x_2 x_1) \end{aligned} \right\} \\ &= H(X_3 | X_1) - H(X_3 | X_2 X_1) \end{aligned}$$

Since $I(X_3; X_2|X_1) \geq 0$, it follows that :

$$H(X_3|X_1) \geq H(X_3|X_2X_1)$$

Problem 6.10

Assume that $a > 0$. Then we know that in the linear transformation $Y = aX + b$:

$$p_Y(y) = \frac{1}{a} p_X\left(\frac{y-b}{a}\right)$$

Hence :

$$\begin{aligned} H(Y) &= - \int_{-\infty}^{\infty} p_Y(y) \log p_Y(y) dy \\ &= - \int_{-\infty}^{\infty} \frac{1}{a} p_X\left(\frac{y-b}{a}\right) \log \frac{1}{a} p_X\left(\frac{y-b}{a}\right) dy \end{aligned}$$

Let $u = \frac{y-b}{a}$. Then $dy = a du$, and :

$$\begin{aligned} H(Y) &= - \int_{-\infty}^{\infty} \frac{1}{a} p_X(u) [\log p_X(u) - \log a] a du \\ &= - \int_{-\infty}^{\infty} p_X(u) \log p_X(u) du + \int_{-\infty}^{\infty} p_X(u) \log a du \\ &= H(X) + \log a \end{aligned}$$

In a similar way, we can prove that for $a < 0$:

$$H(Y) = -H(X) - \log a$$

Problem 6.11

The linear transformation produces the symbols :

$$y_i = ax_i + b, \quad i = 1, 2, 3$$

with corresponding probabilities $p_1 = 0.45$, $p_2 = 0.35$, $p_3 = 0.20$. since the $\{y_i\}$ have the same probability distribution as the $\{x_i\}$, it follows that : $H(Y) = H(X)$. Hence, the entropy of a DMS is not affected by the linear transformation.

Problem 6.12

$$\begin{aligned}
H &= \lim_{n \rightarrow \infty} H(X_n | X_1, \dots, X_{n-1}) \\
&= \lim_{n \rightarrow \infty} \left[- \sum_{x_1, \dots, x_n} P(x_1, \dots, x_n) \log_2 P(x_n | x_1, \dots, x_{n-1}) \right] \\
&= \lim_{n \rightarrow \infty} \left[- \sum_{x_1, \dots, x_n} P(x_1, \dots, x_n) \log_2 P(x_n | x_{n-1}) \right] \\
&= \lim_{n \rightarrow \infty} \left[- \sum_{x_n, x_{n-1}} P(x_n, x_{n-1}) \log_2 P(x_n | x_{n-1}) \right] \\
&= \lim_{n \rightarrow \infty} H(X_n | X_{n-1})
\end{aligned}$$

However, for a stationary process $P(x_n, x_{n-1})$ and $P(x_n | x_{n-1})$ are independent of n , so that

$$H = \lim_{n \rightarrow \infty} H(X_n | X_{n-1}) = H(X_n | X_{n-1})$$

Problem 6.13

First, we need the state probabilities $P(x_i)$, $i = 1, 2$. For stationary Markov processes, these can be found, in general, by the solution of the system :

$$P\Pi = P, \quad \sum_i P_i = 1$$

where P is the state probability vector and Π is the transition matrix : $\Pi[ij] = P(x_j | x_i)$. However, in the case of a two-state Markov source, we can find $P(x_i)$ in a simpler way by noting that the probability of a transition from state 1 to state 2 equals the probability of a transition from state 2 to state 1 (so that the probability of each state will remain the same). Hence :

$$P(x_1|x_2)P(x_2) = P(x_2|x_1)P(x_1) \Rightarrow 0.3P(x_2) = 0.2P(x_1) \Rightarrow P(x_1) = 0.6, P(x_2) = 0.4$$

Then :

$$\begin{aligned}
H(X) &= \left\{ \begin{array}{l} P(x_1) [-P(x_1|x_1) \log P(x_1|x_1) - P(x_2|x_1) \log P(x_2|x_1)] + \\ P(x_2) [-P(x_1|x_2) \log P(x_1|x_2) - P(x_2|x_2) \log P(x_2|x_2)] \end{array} \right\} \\
&= 0.6 [-0.8 \log 0.8 - 0.2 \log 0.2] + 0.4 [-0.3 \log 0.3 - 0.7 \log 0.7] \\
&= 0.7857 \text{ bits/letter}
\end{aligned}$$

If the source is a binary DMS with output letter probabilities $P(x_1) = 0.6$, $P(x_2) = 0.4$, its entropy will be :

$$H_{DMS}(X) = -0.6 \log 0.6 - 0.4 \log 0.4 = 0.971 \text{ bits/letter}$$

We see that the entropy of the Markov source is smaller, since the memory inherent in it reduces the information content of each output.

Problem 6.14

Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$, $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$. Since the channel is memoryless : $P(\mathbf{Y}|\mathbf{X}) = \prod_{i=1}^n P(Y_i|X_i)$ and :

$$\begin{aligned} I(\mathbf{X}; \mathbf{Y}) &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X}, \mathbf{Y}) \log \frac{P(\mathbf{Y}|\mathbf{X})}{P(\mathbf{Y})} \\ &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X}, \mathbf{Y}) \log \frac{\prod_i P(Y_i|X_i)}{P(\mathbf{Y})} \end{aligned}$$

For statistically independent input symbols :

$$\begin{aligned} \sum_{i=1}^n I(X_i; Y_i) &= \sum_{i=1}^n \sum_{X_i} \sum_{Y_i} P(X_i, Y_i) \log \frac{P(Y_i|X_i)}{P(Y_i)} \\ &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X}, \mathbf{Y}) \log \frac{\prod_i P(Y_i|X_i)}{\prod_i P(Y_i)} \end{aligned}$$

Then :

$$\begin{aligned} I(\mathbf{X}; \mathbf{Y}) - \sum_{i=1}^n I(X_i; Y_i) &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X}, \mathbf{Y}) \log \frac{\prod_i P(Y_i)}{P(\mathbf{Y})} \\ &= \sum_{\mathbf{Y}} P(\mathbf{Y}) \log \frac{\prod_i P(Y_i)}{P(\mathbf{Y})} = \sum_{\mathbf{Y}} P(\mathbf{Y}) \ln \frac{\prod_i P(Y_i)}{P(\mathbf{Y})} \log e \\ &\leq \sum_{\mathbf{Y}} P(\mathbf{Y}) \left[\frac{\prod_i P(Y_i)}{P(\mathbf{Y})} - 1 \right] \log e \\ &= (\sum_{\mathbf{Y}} \prod_i P(Y_i) - \sum_{\mathbf{Y}} P(\mathbf{Y})) \log e = (1 - 1) \log e = 0 \end{aligned}$$

where we have exploited the fact that : $\ln u \leq u - 1$, with equality iff $u = 1$. Therefore : $I(\mathbf{X}; \mathbf{Y}) \leq \sum_{i=1}^n I(X_i; Y_i)$ with equality iff the set of input symbols is statistically independent.

Problem 6.15

By definition, the differential entropy is

$$H(X) = - \int_{-\infty}^{\infty} p(x) \log p(x) dx$$

For the uniformly distributed random variable :

$$H(X) = - \int_0^a \frac{1}{a} \log \frac{1}{a} dx = \log a$$

1. For $a = 1$, $H(X) = 0$
2. For $a = 4$, $H(X) = \log 4 = 2 \log 2$
3. For $a = 1/4$, $H(X) = \log \frac{1}{4} = -2 \log 2$

Problem 6.16

The source entropy is :

$$H(X) = \sum_{i=1}^5 p_i \log \frac{1}{p_i} = \log 5 = 2.32 \text{ bits/letter}$$

1. When we encode one letter at a time we require $\bar{R} = 3$ bits/letter . Hence, the efficiency is $2.32/3 = 0.77$ (77%).
2. If we encode two letters at a time, we have 25 possible sequences. Hence, we need 5 bits per 2-letter symbol, or $\bar{R} = 2.5$ bits/letter ; the efficiency is $2.32/2.5 = 0.93$.
3. In the case of encoding three letters at a time we have 125 possible sequences. Hence we need 7 bits per 3-letter symbol, so $\bar{R} = 7/3$ bits/letter; the efficiency is $2.32/(7/3) = 0.994$.

Problem 6.17

Given $(n_1, n_2, n_3, n_4) = (1, 2, 2, 3)$ we have :

$$\sum_{k=1}^4 2^{-n_k} = 2^{-1} + 2^{-2} + 2^{-2} + 2^{-3} = \frac{9}{8} > 1$$

Since the Kraft inequality is not satisfied, a binary code with code word lengths (1, 2, 2, 3) that satisfies the prefix condition does not exist.

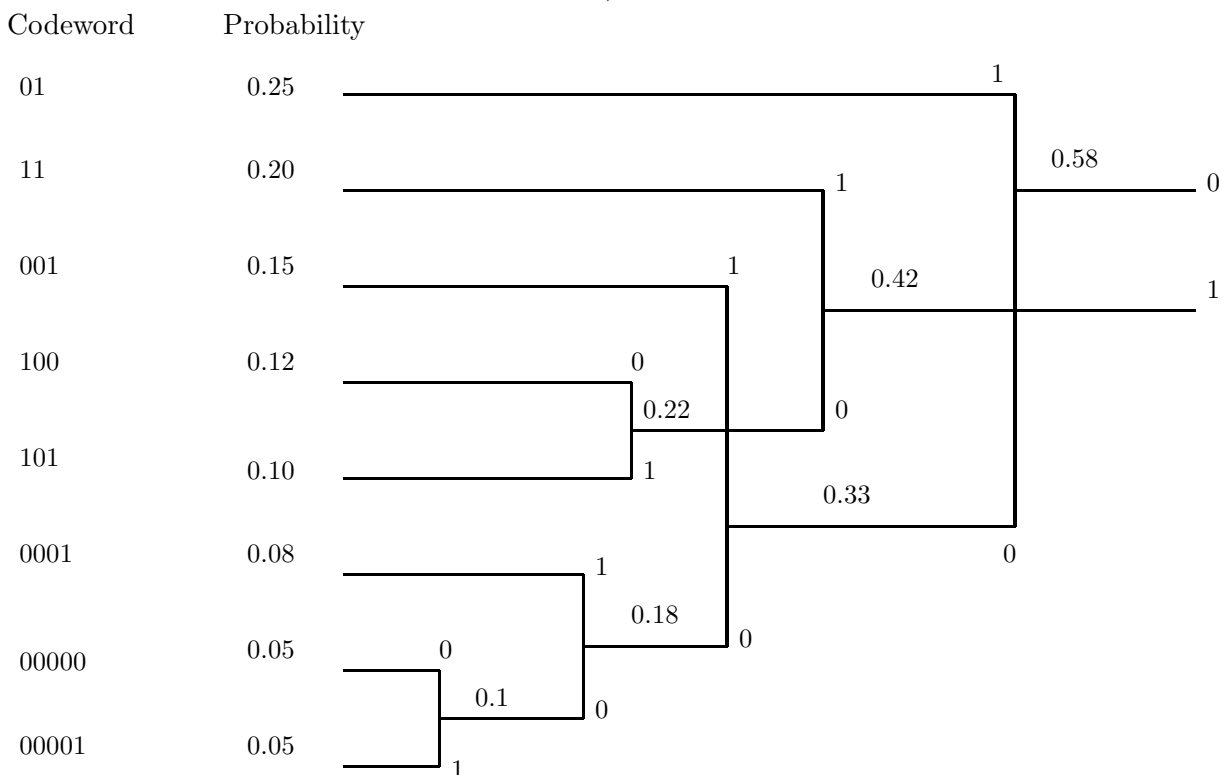
Problem 6.18

$$\sum_{k=1}^{2^n} 2^{-n_k} = \sum_{k=1}^{2^n} 2^{-n} = 2^n 2^{-n} = 1$$

Therefore the Kraft inequality is satisfied.

Problem 6.19

1. The following figure depicts the design of a ternary Huffman code (we follow the convention that the lower-probability branch is assigned a 1) :



2. The average number of binary digits per source letter is :

$$\bar{R} = \sum_i P(x_i)n_i = 2(0.45) + 3(0.37) + 4(0.08) + 5(0.1) = 2.83 \text{ bits/letter}$$

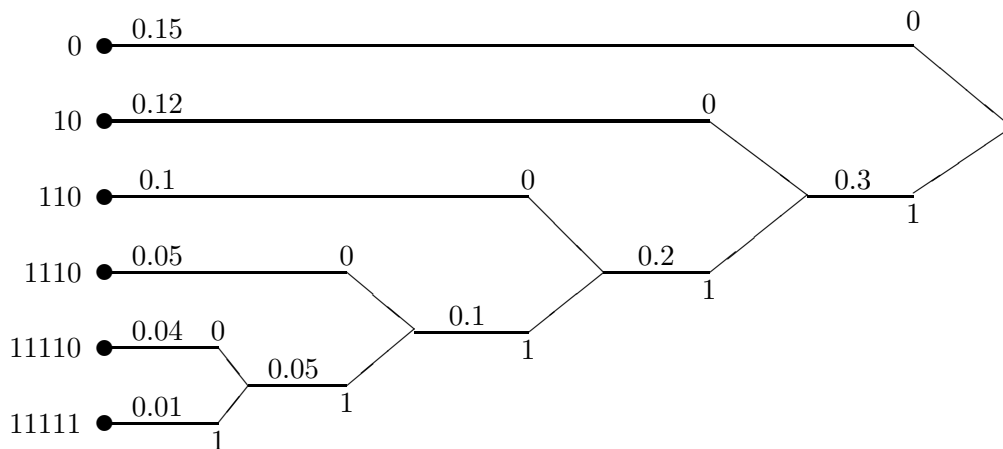
3. The entropy of the source is :

$$H(X) = - \sum_i P(x_i) \log P(x_i) = 2.80 \text{ bits/letter}$$

As it is expected the entropy of the source is less than the average length of each codeword.

Problem 6.20

1. The Huffman tree and the corresponding code is shown below:



The average codeword length is given by

$$\bar{R} = 0.7 \times 1 + 0.1 \times (2 + 3) + 0.05 \times 4 + 5(0.01 + 0.04) = 1.65$$

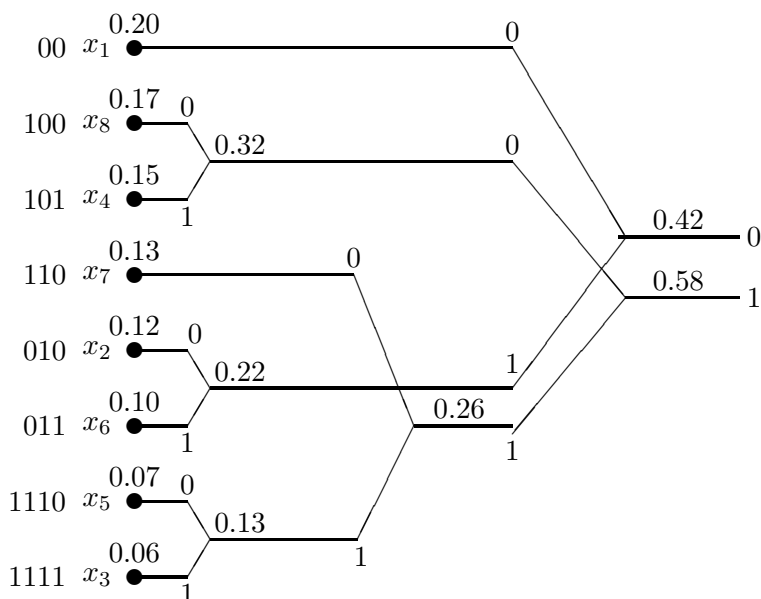
and the minimum possible average codeword length is equal to the entropy of the source computed as

$$H(X) = - \sum p_i \log_2 p_i \approx 1.493$$

2. Yes since 1.5 exceeds the entropy of the source.
 3. Not by using the Huffman code in part 1, since $1.65 > 1.5$.

Problem 6.21

The Huffman tree is shown below



The average codeword length is given by

$$\bar{R} = 2 \times 0.2 + 3(0.17 + 0.15 + 0.13 + 0.12 + 0.1) + 4(0.07 + 0.06) = 2.93$$

and the entropy is given by

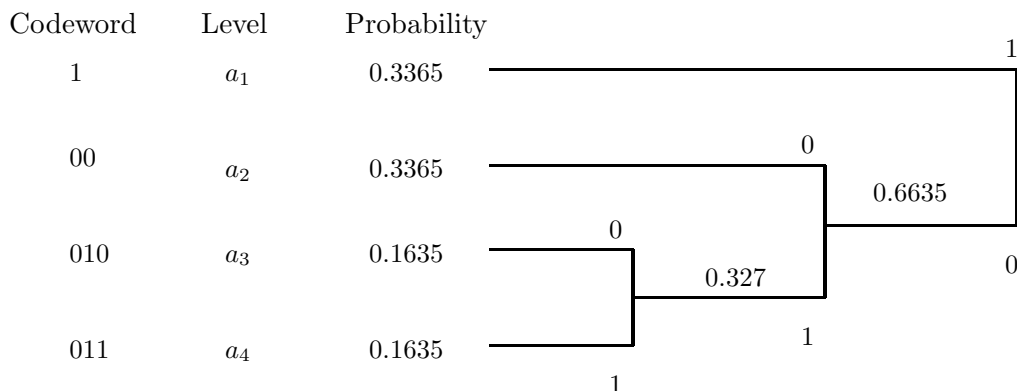
$$H(X) = - \sum_{i=1}^8 p_i \log_2 p_i \approx 2.9035$$

and

$$\eta \approx \frac{2.9035}{2.93} \approx 0.991$$

Problem 6.22

1. The following figure depicts the design of the Huffman code, when encoding a single level at a time :



The average number of binary digits per source level is :

$$\bar{R} = \sum_i P(a_i) n_i = 1.9905 \text{ bits/level}$$

The entropy of the source is :

$$H(X) = - \sum_i P(a_i) \log P(a_i) = 1.9118 \text{ bits/level}$$

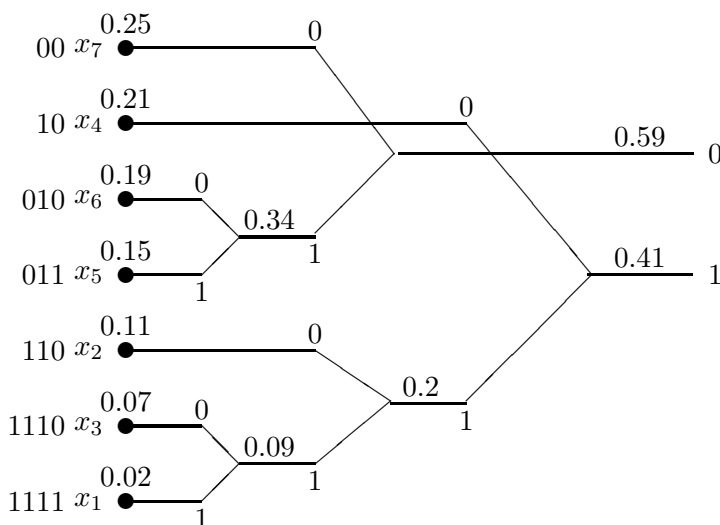
2. Encoding two levels at a time :

Problem 6.23

1. The entropy is

$$H(X) = - \sum_{i=1}^7 p_i \log_2 p_i \approx 2.5703$$

2. The Huffman tree is shown below



and

$$\bar{R} = 2(0.25 + 0.21) + 3(0.19 + 0.15 + 0.11) + 4(0.07 + 0.02) = 2.63$$

3. We have $P(y_1) = 0.02 + 0.11 + 0.15 = 0.28$, $P(y_2) = 0.07 + 0.025 = 0.32$, and $P(y_3) = 0.21 + 0.19 = 0.4$. Therefore, $H(Y) = - \sum_{j=1}^3 P(Y_j) \log_2 P(Y_j) \approx 1.569$.
4. \mathcal{Y} is more predictable because its entropy is lower

Problem 6.24

1. We have

$$P(\hat{X} = 0.5) = \int_0^1 e^{-x} dx = 1 - e^{-1} = 0.6321$$

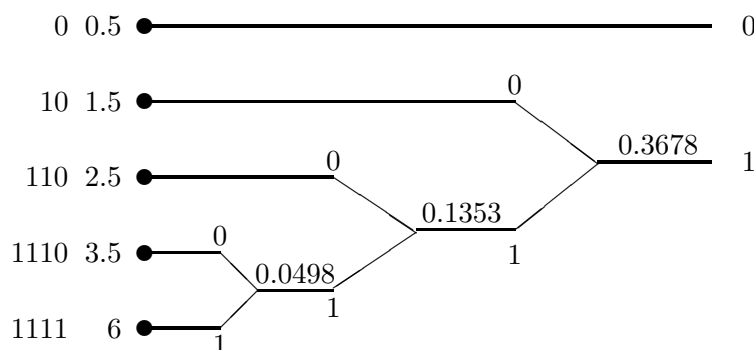
$$P(\hat{X} = 1.5) = \int_1^2 e^{-x} dx = e^{-1} - e^{-2} = 0.2325$$

$$P(\hat{X} = 2.5) = \int_2^3 e^{-x} dx = e^{-2} - e^{-3} = 0.0855$$

$$P(\hat{X} = 3.5) = \int_3^4 e^{-x} dx = e^{-3} - e^{-4} = 0.0315$$

$$P(\hat{X} = 4.5) = \int_4^\infty e^{-x} dx = e^{-4} = 0.0183$$

and the Huffman tree is shown below



2. $H(X) = -\sum_i p_i \log p_i \approx 1.4738$.

3. $\bar{R} = 0.6321 + 2 \times 0.2325 + 3 \times 0.0855 + 4(0.0183 + 0.0314) \approx 1.5528$ and $\eta = 1.4738/1.5528 \approx 0.9491$.

4. Note that \hat{X} is a deterministic function of \tilde{X} since if we map values of 4.5, 5.5, 6.5, 7.5, 8.5, ... of \tilde{X} to 6 we obtain \hat{X} , therefore by the result of problem 6.7, $H(\hat{X}) < H(\tilde{X})$.

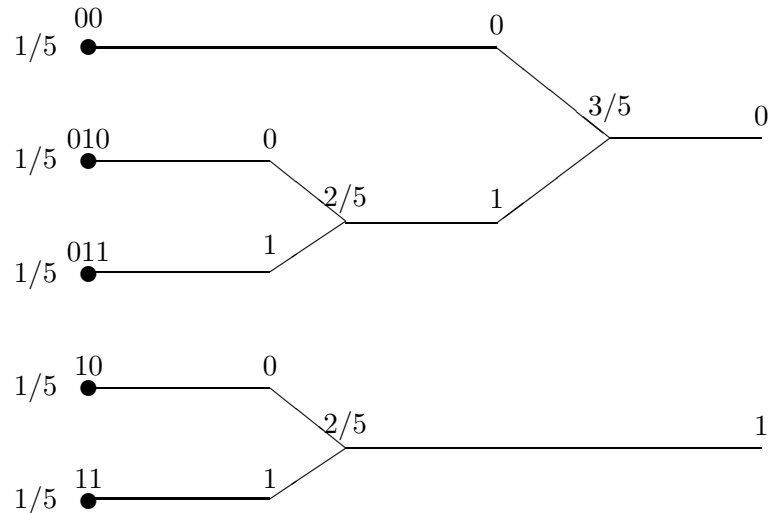
Problem 6.25

1. Since the source is continuous, each source symbol carries infinite number of bits. In order to reconstruct these real numbers with no distortion infinite precision and therefore infinite bits are required and therefore $R = \infty$.

2. For squared error distortion, the best estimate in the absence of any side information is the expected value which is zero in this case and the resulting distortion is equal to variance which is given by

$$\sigma_X^2 = \frac{(4 - (-4))^2}{12} = \frac{16}{3}$$

3. An optimal uniform quantizer generates five equiprobable outputs each with probability $1/5$. The Huffman code is designed in the following diagram



The average codeword length is $\frac{1}{5}(2 + 2 + 2 + 3 + 3) = 12/5 = 2.4$ binary symbols/source symbol, and the resulting transmission rate is $2.4 \times 10000 = 24000$ binary symbols/second. The resulting distortion is

$$D = \frac{(8/5)^2}{12} = \frac{16}{75}$$

4. If Huffman code for *long* sequences is used, the rate will be equal to the entropy (asymptotically) and $R = \log_2 5 = 2.3219$ and the resulting rate is 23219 bits per second, the distortion is as in case 3.

Problem 6.26

1.

$$H(X) = -(.05 \log_2 .05 + .1 \log_2 .1 + .1 \log_2 .1 + .15 \log_2 .15 \\ + .05 \log_2 .05 + .25 \log_2 .25 + .3 \log_2 .3) = 2.5282$$

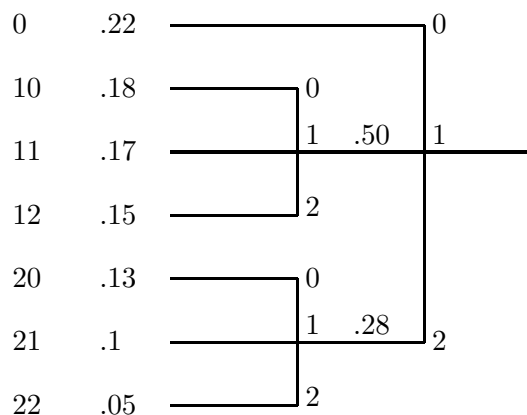
2. After quantization, the new alphabet is $B = \{-4, 0, 4\}$ and the corresponding symbol probabilities are given by

$$\begin{aligned} P(-4) &= P(-5) + P(-3) = .05 + .1 = .15 \\ P(0) &= P(-1) + P(0) + P(1) = .1 + .15 + .05 = .3 \\ P(4) &= P(3) + P(5) = .25 + .3 = .55 \end{aligned}$$

Hence, $H(Q(X)) = 1.4060$. As it is observed quantization decreases the entropy of the source.

Problem 6.27

The following figure depicts the design of a ternary Huffman code.



The average codeword length is

$$\begin{aligned} \bar{R}(X) &= \sum_x P(x)n_x = .22 + 2(.18 + .17 + .15 + .13 + .10 + .05) \\ &= 1.78 \quad (\text{ternary symbols/output}) \end{aligned}$$

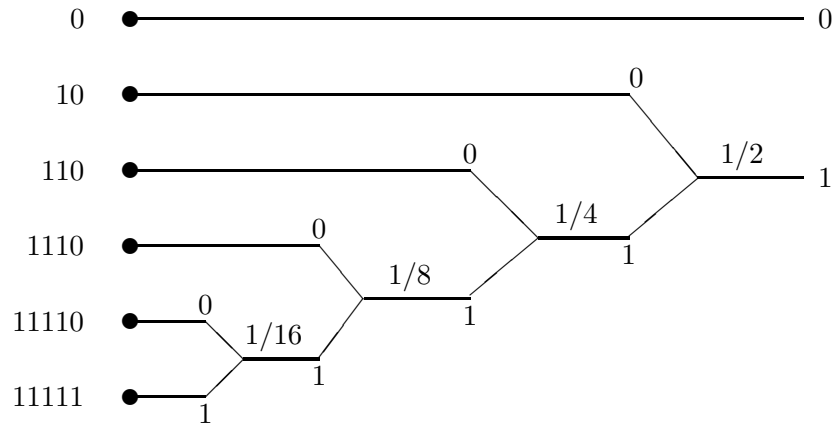
For a fair comparison of the average codeword length with the entropy of the source, we compute the latter with logarithms in base 3. Hence,

$$H(X) = - \sum_x P(x) \log_3 P(x) = 1.7047$$

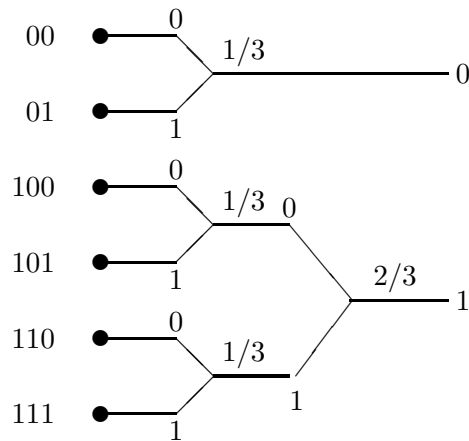
As it is expected $H(X) \leq \bar{R}(X)$.

Problem 6.28

1. We have $H(X) = -\sum_{i=1} 52^{-i} \log_2 2^{-i} - \frac{1}{32} \log_2 \frac{1}{32} = 31/16 = 1.9375$ and $H(Y) = \log_2 6 = 2.58$. Obviously Y is less predictable since its entropy is higher.
2. For X the Huffman tree is



and for Y



For X we have $\bar{R} = 1/2 + 2 \times 1/4 + 3 \times 1/8 + 4 \times 1/16 + 5 \times 2/32 = 31/16$, hence $\eta = 1$. For Y , $\bar{R} = 2 \times 1/3 + 3 \times 2/3 = 2.666$ and $\eta = 2.58/2.666 \approx 0.9675$.

3. The first source has no room for improvement. Therefore we observe an improvement for Y .
4. From elementary probability we know that the PDF of the sum of two independent random variables is the convolution of their PDF's. We can easily obtain the convolution of two sequences

$$(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$$

and

$$(1/2, 1/4, 1/8, 1/16, 1/32, 1/32)$$

to be

$$(1/12, 1/8, 7/48, 5/32, 31/192, 1/6, 1/12, 1/24, 1/48, 1/96, 1/192)$$

The entropy of this PDF is found to be 3.0671. We observe that $H(X) + H(Y) = 4.5225$ and thus $H(X + Y) < H(X) + H(Y)$.

5. We note that $X + Y$ is a function of X and Y and by problem 6.7, $H(X + Y) \leq H(X, Y) \leq H(X) + H(Y)$. For equality we must have 1) (X, Y) be a deterministic function of $X + Y$, i.e., we must be able to uniquely determine both X and Y from their sum. This means that no two pairs of X and Y should result in the same sum, and 2) X and Y have to be independent. Clearly $H(X + Y) > H(X) + H(Y)$ can never happen.

Problem 6.29

1. Simple differentiation proves that $\frac{d^2}{dp^2}H_b(p) = -\frac{\log_2 e}{p(1-p)} < 0$ and therefore the function is concave.
2. We have

$$Q(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-t^2/2} dt$$

resulting in

$$\frac{d^2}{dx^2}Q(x) = \frac{x}{\sqrt{2\pi\sigma^2}}e^{-x^2/2}$$

which is positive for $x > 0$, hence $Q(x)$ is convex.

3. Assume that X is equal to x_1 and x_2 with probabilities λ and $1 - \lambda$ where $0 < \lambda < 1$, then $g(E[X]) = g(\lambda x_1 + (1 - \lambda)x_2)$ and $E[g(X)] = \lambda g(x_1) + (1 - \lambda)g(x_2)$. From the definition of convexity we readily have .
4. The proof is by induction. We assume that $g(E[X]) \leq E[g(X)]$ for all random variables with n possible values and prove that it holds also for RV's with $n + 1$ possible values. Let the possible values be x_i , $1 \leq i \leq n + 1$ and the corresponding probabilities be p_i , $1 \leq i \leq n + 1$.

We have

$$\begin{aligned}
 g(E[X]) &= g\left(\sum_{i=1}^{n+1} p_i x_i\right) \\
 &= g\left((1 - p_{n+1}) \left(\sum_{i=1}^n \frac{p_i}{1 - p_{n+1}} x_i\right) + p_{n+1} x_{n+1}\right) \\
 &\leq (1 - p_{n+1}) g\left(\sum_{i=1}^n \frac{p_i}{1 - p_{n+1}} x_i\right) + p_{n+1} g(x_{n+1}) \\
 &\leq (1 - p_{n+1}) \sum_{i=1}^n \frac{p_i}{1 - p_{n+1}} g(x_i) + p_{n+1} g(x_{n+1}) \\
 &= \sum_{i=1}^{n+1} p_i g(x_i)
 \end{aligned}$$

where in the first inequality we have applied the inequality for a variable with two values and in the second inequality we have applied it to the case with n points. The proof for a general continuous random variable follows from a limiting argument and using the continuity of convex functions on open intervals.

5. This is a direct consequence of Jensen's inequality and the convexity of $Q(x)$ for positive arguments.

Problem 6.30

Parsing the sequence by the rules of the Lempel-Ziv coding scheme we obtain the phrases

0, 00, 1, 001, 000, 0001, 10, 00010, 0000, 0010, 00000, 101, 00001,
000000, 11, 01, 0000000, 110, ...

The number of the phrases is 18. For each phrase we need 5 bits plus an extra bit to represent the new source output.

Dictionary Location	Dictionary Contents	Codeword
1 00001	0	00000 0
2 00010	00	00001 0
3 00011	1	00000 1
4 00100	001	00010 1
5 00101	000	00010 0
6 00110	0001	00101 1
7 00111	10	00011 0
8 01000	00010	00110 0
9 01001	0000	00101 0
10 01010	0010	00100 0
11 01011	00000	01001 0
12 01100	101	00111 1
13 01101	00001	01001 1
14 01110	000000	01011 0
15 01111	11	00011 1
16 10000	01	00001 1
17 10001	0000000	01110 0
18 10010	110	01111 0

Problem 6.31

1. For lossless transmission of a continuous source $R = \infty$.
2. For a discrete source $R = H(\hat{X})$ is the minimum rate. We have to find the probabilities of \hat{X} . We have

$$P(\hat{X} = 0.5) = \int_0^1 \frac{1}{2} e^{-\frac{x}{2}} dx = \left[-e^{-\frac{x}{2}} \right]_0^1 = 1 - e^{-\frac{1}{2}} = 0.3935$$

$$P(\hat{X} = 1.5) = e^{-\frac{1}{2}} - e^{-1} = 0.2387$$

$$P(\hat{X} = 2.5) = e^{-1} - e^{-\frac{3}{2}} = 0.1447$$

$$P(\hat{X} = 6) = e^{-\frac{3}{2}} = 0.2231$$

and

$$\begin{aligned} H(\hat{X}) &= -0.3935 \times \log_2 0.3935 - 0.2387 \times \log_2 0.2387 - 0.1447 \times \log_2 0.1447 \\ &\quad - 0.2231 \times \log_2 0.2231 = 1.9092 \text{ bits/source output} \end{aligned}$$

3. Note that having \tilde{X} we know the i such that $i \leq X < i + 1$, and therefore we know \hat{X} . But having \hat{X} we do not always know \tilde{X} , for instance when $\hat{X} = 6$. This shows that the uncertainty in \tilde{X} is more than the uncertainty in \hat{X} and hence $H(\tilde{X}) > H(\hat{X})$. More rigorously, since knowledge of \tilde{X} determines \hat{X} , we have $H(\hat{X}|\tilde{X}) = 0$ but $H(\tilde{X}|\hat{X}) > 0$. Now expand $H(\hat{X}, \tilde{X})$ in two ways as

$$\begin{aligned} H(\hat{X}, \tilde{X}) &= H(\hat{X}) + H(\tilde{X}) > H(\hat{X}) \\ &= H(\tilde{X}) + H(\hat{X}|\tilde{X}) \\ &= H(\tilde{X}) \end{aligned}$$

Therefore $H(\tilde{X}) > H(\hat{X})$.

4. From part 3, knowledge of \tilde{X} uniquely determines \hat{X} and hence it also uniquely determines Y . Also note that Y uniquely determines i such that $i \leq X < i + 1$, since for different values of i values of Y are different. Since Y uniquely determines i , it also uniquely determines \tilde{X} , hence $Y \Leftrightarrow \tilde{X}$ and $H(Y) = H(\tilde{X})$. Obviously other relations cannot be true since $0 < H(\hat{X}) < H(\tilde{X})$.

Problem 6.32

1.

$$\begin{aligned} H(X) &= - \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \ln\left(\frac{1}{\lambda} e^{-\frac{x}{\lambda}}\right) dx \\ &= - \ln\left(\frac{1}{\lambda}\right) \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx + \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \frac{x}{\lambda} dx \\ &= \ln \lambda + \frac{1}{\lambda} \int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} x dx \\ &= \ln \lambda + \frac{1}{\lambda} \lambda = 1 + \ln \lambda \end{aligned}$$

where we have used the fact $\int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx = 1$ and $E[X] = \int_0^\infty x \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx = \lambda$.

2.

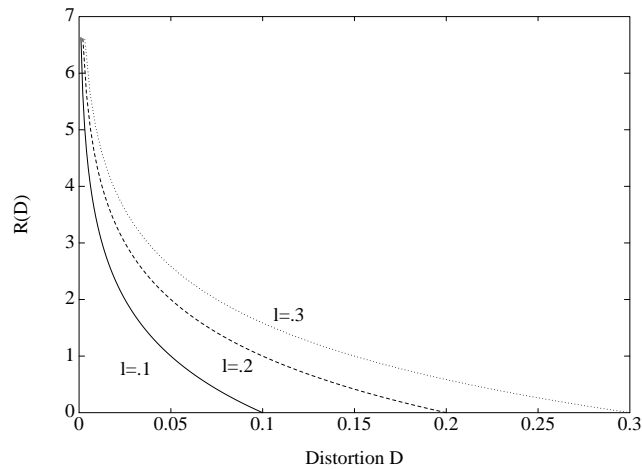
$$\begin{aligned}
 H(X) &= - \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} \ln\left(\frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}\right) dx \\
 &= - \ln\left(\frac{1}{2\lambda}\right) \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} dx + \frac{1}{\lambda} \int_{-\infty}^{\infty} |x| \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} dx \\
 &= \ln(2\lambda) + \frac{1}{\lambda} \left[\int_{-\infty}^0 -x \frac{1}{2\lambda} e^{\frac{x}{\lambda}} dx + \int_0^{\infty} x \frac{1}{2\lambda} e^{-\frac{x}{\lambda}} dx \right] \\
 &= \ln(2\lambda) + \frac{1}{2\lambda} \lambda + \frac{1}{2\lambda} \lambda = 1 + \ln(2\lambda)
 \end{aligned}$$

3.

$$\begin{aligned}
 H(X) &= - \int_{-\lambda}^0 \frac{x+\lambda}{\lambda^2} \ln\left(\frac{x+\lambda}{\lambda^2}\right) dx - \int_0^{\lambda} \frac{-x+\lambda}{\lambda^2} \ln\left(\frac{-x+\lambda}{\lambda^2}\right) dx \\
 &= - \ln\left(\frac{1}{\lambda^2}\right) \left[\int_{-\lambda}^0 \frac{x+\lambda}{\lambda^2} dx + \int_0^{\lambda} \frac{-x+\lambda}{\lambda^2} dx \right] \\
 &\quad - \int_{-\lambda}^0 \frac{x+\lambda}{\lambda^2} \ln(x+\lambda) dx - \int_0^{\lambda} \frac{-x+\lambda}{\lambda^2} \ln(-x+\lambda) dx \\
 &= \ln(\lambda^2) - \frac{2}{\lambda^2} \int_0^{\lambda} z \ln z dz \\
 &= \ln(\lambda^2) - \frac{2}{\lambda^2} \left[\frac{z^2 \ln z}{2} - \frac{z^2}{4} \right]_0^{\lambda} \\
 &= \ln(\lambda^2) - \ln(\lambda) + \frac{1}{2}
 \end{aligned}$$

Problem 6.33

1. Since $R(D) = \log \frac{\lambda}{D}$ and $D = \frac{\lambda}{2}$, we obtain $R(D) = \log\left(\frac{\lambda}{\lambda/2}\right) = \log(2) = 1$ bit/sample.
2. The following figure depicts $R(D)$ for $\lambda = 0.1, .2$ and $.3$. As it is observed from the figure, an increase of the parameter λ increases the required rate for a given distortion.



Problem 6.34

1. Here $R(D) = H_b(p) - H_b(D)$ where D is the Hamming distortion, which is equal to p_e . Hence $R = H_b(.4) - H_b(.1) = 0.502$.
2. For Gaussian sources with squared error distortion, $R(D) = \frac{1}{2} \log_2 \frac{\sigma^2}{D} = \frac{1}{2} \log_2 \frac{4}{1.5} = 0.7075$.
3. $P(Z < 0) = .4$ and $P(Z > 0) = .6$. Since the quantizer is uniform, there are four levels less than zero and four levels higher than zero, for the levels less than zero, $p = .4/4 = .1$ and for positive levels $p = .6/4 = .15$. Since we need error free transmission, $R = H(\hat{X})$ and hence,

$$R = -4 \times 0.1 \log_2 0.1 - 4 \times 0.15 \log_2 0.15 = 2.971$$

Problem 6.35

1. For a Gaussian random variable of zero mean and variance σ^2 the rate-distortion function is given by $R(D) = \frac{1}{2} \log_2 \frac{\sigma^2}{D}$. Hence, the upper bound is satisfied with equality. For the lower bound recall that $H(X) = \frac{1}{2} \log_2(2\pi e \sigma^2)$. Thus,

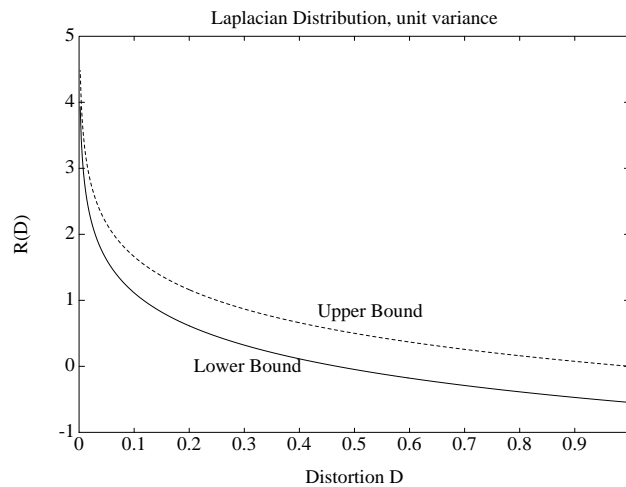
$$\begin{aligned} H(X) - \frac{1}{2} \log_2(2\pi e D) &= \frac{1}{2} \log_2(2\pi e \sigma^2) - \frac{1}{2} \log_2(2\pi e D) \\ &= \frac{1}{2} \log_2 \left(\frac{2\pi e \sigma^2}{2\pi e D} \right) = R(D) \end{aligned}$$

As it is observed the upper and the lower bounds coincide.

2. The differential entropy of a Laplacian source with parameter λ is $H(X) = 1 + \ln(2\lambda)$. The variance of the Laplacian distribution is

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} dx = 2\lambda^2$$

Hence, with $\sigma^2 = 1$, we obtain $\lambda = \sqrt{1/2}$ and $H(X) = 1 + \ln(2\lambda) = 1 + \ln(\sqrt{2}) = 1.3466$ nats/symbol = 1.5 bits/symbol. A plot of the lower and upper bound of $R(D)$ is given in the next figure.

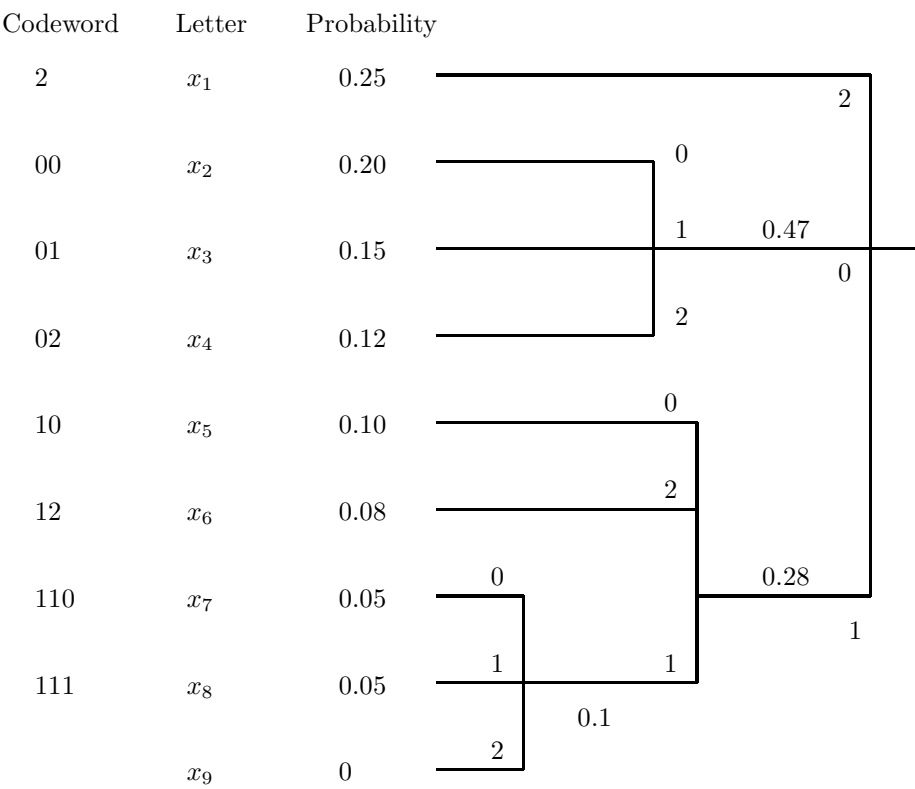
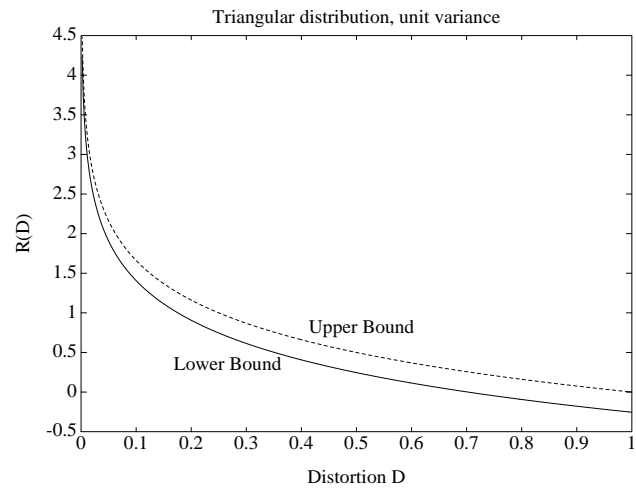


3. The variance of the triangular distribution is given by

$$\begin{aligned} \sigma^2 &= \int_{-\lambda}^0 \left(\frac{x + \lambda}{\lambda^2} \right) x^2 dx + \int_0^{\lambda} \left(\frac{-x + \lambda}{\lambda^2} \right) x^2 dx \\ &= \frac{1}{\lambda^2} \left(\frac{1}{4}x^4 + \frac{\lambda}{3}x^3 \right) \Big|_{-\lambda}^0 + \frac{1}{\lambda^2} \left(-\frac{1}{4}x^4 + \frac{\lambda}{3}x^3 \right) \Big|_0^{\lambda} \\ &= \frac{\lambda^2}{6} \end{aligned}$$

Hence, with $\sigma^2 = 1$, we obtain $\lambda = \sqrt{6}$ and $H(X) = \ln(6) - \ln(\sqrt{6}) + 1/2 = 1.7925$ bits /source output. A plot of the lower and upper bound of $R(D)$ is given in the next figure.

Problem 6.36



$\bar{R} = 1.85$ ternary symbols/letter

Problem 6.37

$$p(\mathbf{X}) = \frac{1}{(2\pi)^{n/2} |M|^{1/2}} e^{-\frac{1}{2} \mathbf{X}' \mathbf{M}^{-1} \mathbf{X}}$$

$$H(\mathbf{X}) = - \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p(\mathbf{X}) \log p(\mathbf{X}) d\mathbf{X}$$

But :

$$\log p(\mathbf{X}) = -\frac{1}{2} \log(2\pi)^n |M| - \left(\frac{1}{2} \log e\right) \mathbf{X}' \mathbf{M}^{-1} \mathbf{X}$$

and

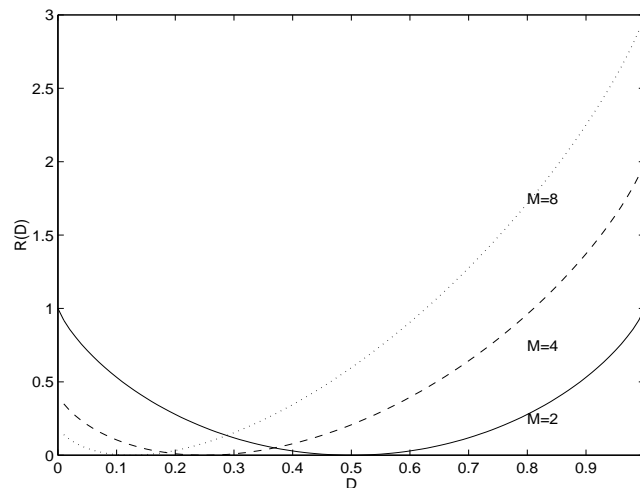
$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\frac{1}{2} \log e\right) \mathbf{X}' \mathbf{M}^{-1} \mathbf{X} p(\mathbf{X}) d\mathbf{X} = \frac{n}{2} \log e$$

Hence :

$$\begin{aligned} H(\mathbf{X}) &= \frac{1}{2} \log(2\pi)^n |M| + \frac{1}{2} \log e^n \\ &= \frac{1}{2} \log(2\pi e)^n |M| \end{aligned}$$

Problem 6.38

$$R(D) = \log M + D \log D + (1 - D) \log \frac{(1 - D)}{M - 1}$$



Problem 6.39

$$d_W(\mathbf{X}, \tilde{\mathbf{X}}) = (\mathbf{X} - \tilde{\mathbf{X}})' \mathbf{W} (\mathbf{X} - \tilde{\mathbf{X}})$$

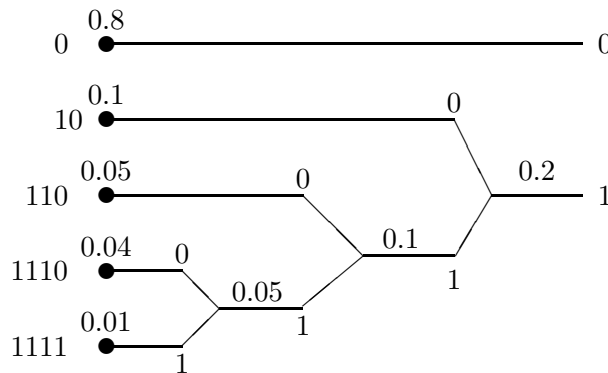
Let $\mathbf{W} = \mathbf{P}'\mathbf{P}$. Then :

$$\begin{aligned} d_W(\mathbf{X}, \tilde{\mathbf{X}}) &= (\mathbf{X} - \tilde{\mathbf{X}})' \mathbf{P}' \mathbf{P} (\mathbf{X} - \tilde{\mathbf{X}}) \\ &= \left(\mathbf{P}(\mathbf{X} - \tilde{\mathbf{X}}) \right)' \mathbf{P} (\mathbf{X} - \tilde{\mathbf{X}}) \\ &= \frac{1}{n} (\mathbf{Y} - \tilde{\mathbf{Y}})' (\mathbf{Y} - \tilde{\mathbf{Y}}) \end{aligned}$$

where by definition : $\mathbf{Y} = \sqrt{n} \mathbf{P} \mathbf{X}$, $\tilde{\mathbf{Y}} = \sqrt{n} \mathbf{P} \tilde{\mathbf{X}}$. Hence : $d_W(\mathbf{X}, \tilde{\mathbf{X}}) = d_2(\mathbf{Y}, \tilde{\mathbf{Y}})$.

Problem 6.40

1. The Huffman tree is shown below



The average codeword length is $\bar{R} = 0.8 \times 1 + 2 \times 0.1 + 3 \times 0.05 + 4(0.01 + 0.04) = 1.35$ and the minimum possible average codeword length given by the entropy is $H(X) = -\sum p_i \log_2 p_i = 1.058$. Obviously $H(X) < \bar{R}$ as expected.

2. The capacity of the channel is $C = 1 - H_b(0.3) = 0.1187$. Since $H(X) > C$ it is not possible to transmit the source reliably.
3. No since $\bar{R} > C$.

Problem 6.41

1. Solving $\frac{1}{2} \log_2 \frac{\sigma^2}{D} = 1 - H_b(\epsilon)$ we obtain

$$D = 2^{2(1-H_b(\epsilon))} \sigma^2$$

2. Here we have to solve $\frac{1}{2} \log_2 \frac{\sigma^2}{D} = \frac{1}{2} \log_2(1 + P/\sigma_n^2)$ resulting in

$$D = \frac{\sigma^2}{1 + P/\sigma_n^2}$$

3. The memory in the source makes it more predictable, thus we expect the distortion to be decreased.

Problem 6.42

1.

$$H(X|G) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, g) \log p(x|g) dx dg$$

But X, G are independent, so : $p(x, g) = p(x)p(g)$, $p(x|g) = p(x)$. Hence :

$$\begin{aligned} H(X|G) &= - \int_{-\infty}^{\infty} p(g) \left[\int_{-\infty}^{\infty} p(x) \log p(x) dx \right] dg \\ &= - \int_{-\infty}^{\infty} p(g) H(X) dg \\ &= H(X) = \frac{1}{2} \log(2\pi e \sigma_x^2) \end{aligned}$$

where the last equality stems from the Gaussian pdf of X .

2.

$$I(X; Y) = H(Y) - H(Y|X)$$

Since Y is the sum of two independent, zero-mean Gaussian r.v's, it is also a zero-mean Gaussian r.v. with variance : $\sigma_y^2 = \sigma_x^2 + \sigma_n^2$. Hence : $H(Y) = \frac{1}{2} \log(2\pi e (\sigma_x^2 + \sigma_n^2))$. Also, since $y = x + g$:

$$p(y|x) = p_g(y - x) = \frac{1}{\sqrt{2\pi\sigma_n}} e^{-\frac{(y-x)^2}{2\sigma_n^2}}$$

Hence :

$$\begin{aligned}
H(Y|X) &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \log p(y|x) dx dy \\
&= - \int_{-\infty}^{\infty} p(x) \log e \int_{-\infty}^{\infty} p(y|x) \ln \left(\frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{(y-x)^2}{2\sigma_n^2}\right) \right) dy dx \\
&= \int_{-\infty}^{\infty} p(x) \log e \left[\int_{-\infty}^{\infty} p_g(y-x) \left(\ln(\sqrt{2\pi}\sigma_n) + \frac{(y-x)^2}{2\sigma_n^2} \right) dy \right] dx \\
&= \int_{-\infty}^{\infty} p(x) \log e \left[\ln(\sqrt{2\pi}\sigma_n) + \frac{1}{2\sigma_n^2} \sigma_n^2 \right] dx \\
&= \left[\log(\sqrt{2\pi}\sigma_n) + \frac{1}{2} \log e \right] \int_{-\infty}^{\infty} p(x) dx \\
&= \frac{1}{2} \log(2\pi e \sigma_n^2) \quad (= H(G))
\end{aligned}$$

where we have used the fact that : $\int_{-\infty}^{\infty} p_g(y-x) dy = 1$, $\int_{-\infty}^{\infty} (y-x)^2 p_g(y-x) dy = E[G^2] = \sigma_n^2$.
From $H(Y)$, $H(Y|X)$:

$$I(X;Y) = H(Y) - H(Y|X) = \frac{1}{2} \log(2\pi e(\sigma_x^2 + \sigma_n^2)) - \frac{1}{2} \log(2\pi e \sigma_n^2) = \frac{1}{2} \log\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right)$$

Problem 6.43

By symmetry of the inputs B and C, their probabilities have to be equal. Therefore we assume $P(A) = 1 - p$, $P(B) = P(C) = p/2$. We then have $P(Y = 1) = 1 - p + p/2 = 1 - p/2$ and $P(Y = 0) = p/2$, thus $H(Y) = H_b(p/2)$. We also note that $H(Y|X = A) = 0$ and $H(Y|X = B) = H(Y|X = C) = 1$, hence $H(Y|X) = (1-p)H(Y|X = A) + p/2H(Y|X = B) + p/2H(Y|X = C) = p$. Therefore,

$$\begin{aligned}
C &= \max_p I(X;Y) \\
&= \max_p H(Y) - H(Y|X) \\
&= \max_p H_b(p/2) - p
\end{aligned}$$

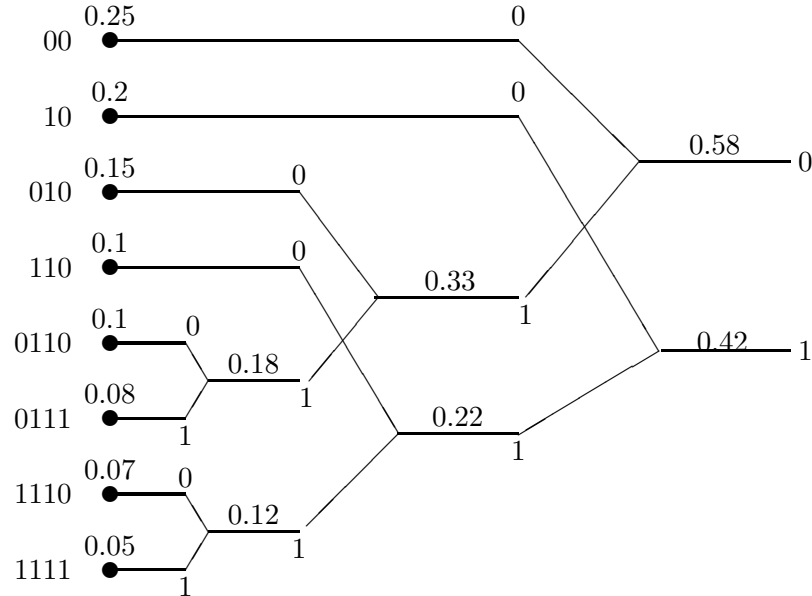
straightforward differentiation results in

$$-\frac{1}{2} \log_2 e \ln \frac{p/2}{1 - p/2} - 1 = 0$$

resulting in $p = 0.4$ and $C = H_b(0.2) - 0.4 = 0.3219$.

Problem 6.44

1. The Huffman tree is shown below



and $\bar{R} = 2(0.25 + 0.2) + 3(0.15 + 0.1) + 4(0.1 + 0.08 + 0.07 + 0.05) = 0.9 + 0.75 + 1.2 = 2.85$.
The minimum average codeword length is $H(X) = -\sum_i p_i \log_2 p_i = 2.8155$.

2. The minimum required channel capacity is equal to the entropy, i.e., $C = 2.8155$. Since the capacity of a BSC is at most 1, this source cannot be transmitted via a BSC.
3. We have to solve $\frac{1}{2} \log_2 \frac{1}{D} = 2.8155$ resulting in $D = 2^{-2 \times 2.8155} = 0.0202$.

Problem 6.45

For channel A, by symmetry of A and B inputs, we need to assume $P(X = A) = P(X = B) = p$ and $P(X = C) = 1 - 2p$, for $0 \leq p \leq 0.5$, and regardless of the value of p we will have $H(Y) = 1$. We note that $H(Y|X = A) = H(Y|X = B) = 0$ and $H(Y|X = C) = 1$, or $H(Y|X) = 1 - 2p$ and $I(X; Y) = H(Y) - H(Y|X) = 1 - (1 - 2p) = 2p$ which is maximized for $p = 1/2$. Therefore we must choose A and B each with probability of $1/2$ and we should not use C. The capacity is 1.

For channel B, the two outputs B' and C' are identical and can be combined resulting a noiseless channel with capacity 1.

For the cascade of two channels the same arguments result in $C = 1$.

Problem 6.46

1. To have a distortion of 1 we have to maintain a rate of at least $R(D) = \frac{1}{2} \log_2 \sigma^2 / D$ when $D = 1$, or $R = \frac{1}{2} \log_2 4/1 = 1$ bits per sample which is equivalent to 8000 bits per second. This means that the channel capacity must be at least 8000 bits per second or

$$8000 = 4000 \log_2(1 + P/N_0W)$$

resulting in $P/N_0W = 3$.

2. For BPSK with hard decision decoding the channel model reduces to a BSC channel with crossover probability of $\epsilon = Q(\sqrt{2E_b/N_0})$. The maximum possible signaling rate over a channel with bandwidth of 4000 Hz from Equation 4.6-6 is $R_s = 2W = 8000$, and we have $8000 = 8000(1 - H_b(\epsilon))$ or $\epsilon = 0$ resulting in $E_b/N_0 = \infty$.

Problem 6.47

1. We have

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) \\ &= \max_{p(x)} H(X) - H(X|Y) \\ &\leq \max_{p(x)} H(X) \\ &= \log_2 3 \end{aligned}$$

2. The input probability that achieves this must be a uniform probability and must satisfy $H(X|Y) = 0$, i.e., X must be a deterministic function of Y is known. For instance the following probability transition matrix would result in $C = \log_2 3$.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3. The minimum achievable distortion is the solution of $\frac{1}{2} \log_2 \frac{1}{D} = \log_2 3$, resulting in $D = 2^{-9}$.

Problem 6.48

1. The entropy of the source is :

$$H(X) = H(0.3) = 0.8813$$

and the capacity of the channel :

$$C = 1 - H(0.1) = 1 - 0.469 = 0.531$$

If the source is directly connected to the channel, then the probability of error at the destination is :

$$\begin{aligned} P(\text{error}) &= P(X=0)P(Y=1|X=0) + P(X=1)P(Y=0|X=1) \\ &= 0.3 \times 0.1 + 0.7 \times 0.1 = 0.1 \end{aligned}$$

2. Since $H(X) > C$, some distortion at the output of the channel is inevitable. To find the minimum distortion, we set $R(D) = C$. For a Bernoulli type of source :

$$R(D) = \begin{cases} H(p) - H(D) & 0 \leq D \leq \min(p, 1-p) \\ 0 & \text{otherwise} \end{cases}$$

and therefore, $R(D) = H(p) - H(D) = H(0.3) - H(D)$. If we let $R(D) = C = 0.531$, we obtain

$$H(D) = 0.3503 \implies D = \min(0.07, 0.93) = 0.07$$

The probability of error is :

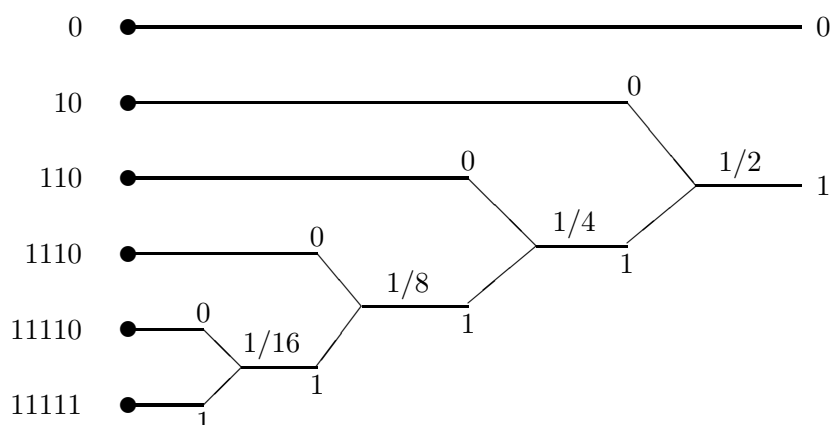
$$P(\text{error}) \leq D = 0.07$$

3. For reliable transmission we must have : $H(X) = C = 1 - H(\epsilon)$. Hence, with $H(X) = 0.8813$ we obtain

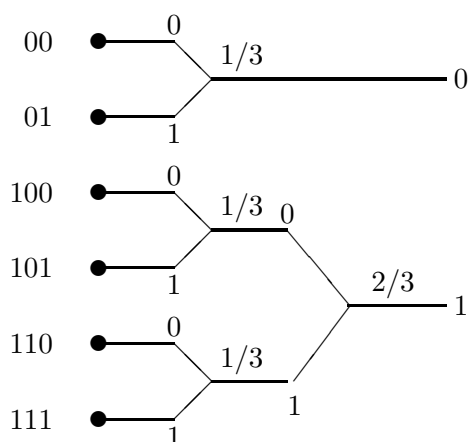
$$0.8813 = 1 - H(\epsilon) \implies \epsilon < 0.016 \text{ or } \epsilon > 0.984$$

Problem 6.49

1. We have $H(S_1) = -\sum_{i=1} 52^{-i} \log_2 2^{-i} - \frac{1}{32} \log_2 \frac{1}{32} = 31/16 = 1.9375$ and $H(S_2) = \log_2 6 = 2.58$. Obviously S_2 is less predictable since its entropy is higher.
2. For X the Huffman tree is



and for Y



For S_1 we have $\bar{R} = 1/2 + 2 \times 1/4 + 3 \times 1/8 + 4 \times 1/16 + 5 \times 2/32 = 31/16$, hence $\eta = 1$. For S_2 , $\bar{R} = 2 \times 1/3 + 3 \times 2/3 = 2.666$ and $\eta = 2.58/2.666 \approx 0.9675$. The Huffman code for the first source is more efficient.

3. The first source has no room for improvement. Therefore we observe an improvement for S_2 .

Problem 6.50

From 6.5-30, it is sufficient to show that

$$\int_{-\infty}^{\infty} p(y|A) \log_2 \frac{p(y|A)}{p(y)} dy = g(A/\sigma)$$

We have

$$\begin{aligned} \int_{-\infty}^{\infty} p(y|A) \log_2 \frac{p(y|A)}{p(y)} dy &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}} \log_2 \frac{2e^{-\frac{(y-A)^2}{2\sigma^2}}}{e^{-\frac{(y-A)^2}{2\sigma^2}} + e^{-\frac{(y+A)^2}{2\sigma^2}}} dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}} \log_2 \frac{2}{1 + e^{-\frac{2yA}{\sigma^2}}} dy \end{aligned}$$

Defining $u = y/\sigma$ and $x = A/\sigma$ gives the desired result.

Problem 6.51

1. By symmetry the capacity is achieved with uniform input distribution. Also note that $H(Y|X = 0) = H(Y|X = 1) = 1$, thus $H(Y|X) = 1$. Uniform input distribution results in an output distribution of $(1/4, 1/2, 1/4)$. Therefore, $C = H(Y) - 1 = 3/2 - 1 = 0.5$.
2. For symmetric binary input channels from Equation 6.8-27, we have $R_0 = 1 - \log_2(1 + \Delta)$, where

$$\Delta = \sum_y \sqrt{p(y|x_1)p(y|x_2)}$$

In this case

$$\Delta = \sqrt{0 \times \frac{1}{2}} + \sqrt{\frac{1}{2} \times \frac{1}{2}} + \sqrt{\frac{1}{2} \times 0} = \frac{1}{2}$$

and $R_0 = 1 - \log_2 \frac{3}{2} = 0.415$.

3. We need to solve $R(D) = 2C = 1$. From $\frac{1}{2} \log_2 \frac{4}{D} = 1$ we have $D = 1$.

Problem 6.52

$$I(x_j; Y) = \sum_{i=0}^{Q-1} P(y_i|x_j) \log \frac{P(y_i|x_j)}{P(y_i)}$$

Since

$$\sum_{j=0}^{q-1} P(x_j) = \sum_{P(x_j) \neq 0} P(x_j) = 1$$

we have :

$$\begin{aligned} I(X;Y) &= \sum_{j=0}^{q-1} P(x_j) I(x_j;Y) = \sum_{P(x_j) \neq 0} C P(x_j) \\ &= C \sum_{P(x_j) \neq 0} P(x_j) = C = \max_{P(x_j)} I(X;Y) \end{aligned}$$

Thus, the given set of $P(x_j)$ maximizes $I(X;Y)$ and the condition is sufficient.

To prove that the condition is necessary, we proceed as follows : Since $P(x_j)$ satisfies the condition $\sum_{j=0}^{q-1} P(x_j) = 1$, we form the cost function :

$$C(X) = I(X;Y) - \lambda \left[\sum_{j=0}^{q-1} P(x_j) - 1 \right]$$

and use the Lagrange multiplier method to maximize $C(X)$. The partial derivative of $C(X)$ with respect to all $P(x_j)$ is :

$$\begin{aligned} \frac{\partial C(X)}{\partial P(x_k)} &= \frac{\partial}{\partial P(x_k)} \left[\sum_{j=0}^{q-1} P(x_j) I(x_j;Y) - \lambda \sum_{j=0}^{q-1} P(x_j) + \lambda \right] \\ &= I(x_k;Y) + \sum_{j=0}^{q-1} P(x_j) \frac{\partial}{\partial P(x_k)} I(x_j;Y) - \lambda = 0 \end{aligned}$$

But :

$$\begin{aligned} \sum_{j=0}^{q-1} P(x_j) \frac{\partial}{\partial P(x_k)} I(x_j;Y) &= -\log e \sum_{j=0}^{q-1} P(x_j) \sum_{i=0}^{Q-1} P(y_i|x_j) \frac{P(y_i)}{P(y_i|x_j) - [P(y_i)]^2} \frac{\partial P(y_i)}{\partial P(x_k)} \\ &= -\log e \sum_{i=0}^{Q-1} \left[\sum_{j=0}^{q-1} \frac{P(x_j) P(y_i|x_j)}{P(y_i)} \right] P(y_i|x_k) \\ &= -\log e \sum_{i=0}^{Q-1} \frac{P(y_i)}{P(y_i)} P(y_i|x_k) = -\log e \end{aligned}$$

Therefore:

$$I(x_k;Y) + \sum_{j=0}^{q-1} P(x_j) \frac{\partial}{\partial P(x_k)} I(x_j;Y) - \lambda = 0 \Rightarrow I(x_k;Y) = \lambda + \log e, \quad \forall x_k$$

Now, we consider two cases :

(i) If there exists a set of $P(x_k)$ such that :

$$\sum_{k=0}^{q-1} P(x_k) = 1, \quad 0 \leq P(x_k) \leq 1, \quad k = 0, 1, \dots, q-1 \quad \text{and} \quad \frac{\partial C(X)}{\partial P(x_k)} = 0$$

then this set will maximize $I(X;Y)$, since $I(X;Y)$ is a convex function of $P(x_k)$. We have :

$$\begin{aligned} C &= \max_{P(x_j)} I(X;Y) = \sum_{j=0}^{q-1} P(x_j) I(x_j;Y) \\ &= \sum_{j=0}^{q-1} P(x_j) [\lambda + \log e] = \lambda + \log e = I(x_j;Y) \end{aligned}$$

This provides a sufficient condition for finding a set of $P(x_k)$ which maximizes $I(X;Y)$, for a symmetric memoryless channel.

(ii) If the set $\{P(x_k)\}$ does not satisfy $0 \leq P(x_k) \leq 1$, $k = 0, 1, \dots, q - 1$, since $I(X; Y)$ is a convex function of $P(x_k)$, necessary and sufficient conditions on $P(x_k)$ for maximizing $I(X; Y)$ are :

$$\frac{\partial I(X; Y)}{\partial P(x_k)} = \mu, \text{ for } P(x_k) > 0, \quad \frac{\partial I(X; Y)}{\partial P(x_k)} \leq \mu, \text{ for } P(x_k) = 0$$

Hence :

$$\begin{aligned} I(x_k; Y) &= \mu + \log e, \quad P(x_k) \neq 0 \\ I(x_k; Y) &\leq \mu + \log e, \quad P(x_k) = 0 \end{aligned}$$

is the necessary and sufficient condition on $P(x_k)$ for maximizing $I(X; Y)$ and $\mu + \log e = C$.

Problem 6.53

a. For a set of equally probable inputs with probability $1/M$, we have :

$$\begin{aligned} I(x_k; Y) &= \sum_{i=0}^{M-1} P(y_i|x_k) \log \frac{P(y_i|x_k)}{P(y_i)} \\ &= P(y_k|x_k) \log \frac{P(y_k|x_k)}{P(y_k)} + \sum_{i \neq k} P(y_i|x_k) \log \frac{P(y_i|x_k)}{P(y_i)} \end{aligned}$$

But $\forall i$:

$$P(y_i) = \sum_{j=0}^{M-1} P(y_i|x_j) = \frac{1}{M} P(y_i|x_i) + \frac{1}{M} \sum_{j \neq i} P(y_i|x_j) = \frac{1}{M} \left(1 - p + (M-1) \frac{p}{M-1} \right) = \frac{1}{M}$$

Hence :

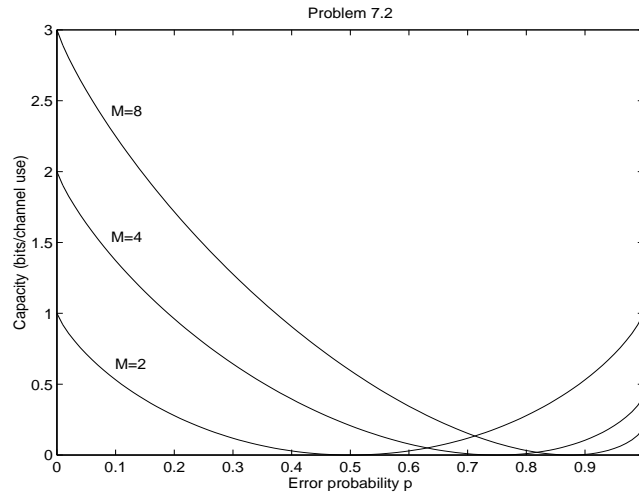
$$\begin{aligned} I(x_k; Y) &= (1-p) \log \frac{(1-p)}{1/M} + (M-1) \frac{p}{M-1} \log \frac{p/M-1}{1/M} \\ &= (1-p) \log (M(1-p)) + p \log \left(\frac{pM}{M-1} \right) \\ &= \log M + (1-p) \log(1-p) + p \log \left(\frac{p}{M-1} \right) \end{aligned}$$

which is the same for all $k = 0, 1, \dots, M-1$. Hence, this set of $\{P(x_k) = 1/M\}$ satisfies the condition of Probl. 7.1.

b. From part (a) :

$$C = \log M + (1-p) \log(1-p) + p \log \left(\frac{p}{M-1} \right)$$

A plot of the capacity is given in the following figure. We note that the capacity of the channel is zero when all transitions are equally likely (i.e. when $1-p = \frac{p}{M-1} \Rightarrow p = \frac{M-1}{M}$ or: $p = 0.5$, $M = 2$; $p = 0.75$, $M = 4$; $p = 0.875$, $M = 8$).



Problem 6.54

In all of these channels, we assume a set of $\{P(x_i)\}$ which, we think, may give the maximum $I(X; Y)$ and see if this set of $\{P(x_i)\}$ satisfies the condition of Probl. 7.1 (or relationship 7-1-21). Since all the channels exhibit symmetries, the set of $\{P(x_i)\}$ that we examine first, is the equiprobable distribution.

a. Suppose $P(x_i) = 1/4, \forall i$. Then : $P(y_1) = P(y_1|x_1)P(x_1) + P(y_1|x_4)P(x_4) = 1/4$. Similarly $P(y_j) = 1/4, \forall j$. Hence :

$$I(x_1; Y) = \sum_{j=1}^4 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = \frac{1}{2} \log \frac{1/2}{1/4} + \frac{1}{2} \log \frac{1/2}{1/4} = \log 2 = 1$$

Similarly : $I(x_i; Y) = 1, i = 2, 3, 4$. Hence this set of input probabilities satisfies the condition of Probl. 7.1 and :

$$C = 1 \text{ bit/symbol sent (bit/channel use)}$$

b. We assume that $P(x_i) = 1/2, i = 1, 2$. Then $P(y_1) = \frac{1}{2} \left(\frac{1}{3} + \frac{1}{6} \right) = \frac{1}{4}$ and similarly $P(y_j) = 1/4, j = 2, 3, 4$. Hence :

$$I(x_1; Y) = \sum_{j=1}^4 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 2 \frac{1}{3} \log \frac{1/3}{1/4} + 2 \frac{1}{6} \log \frac{1/6}{1/4} = 0.0817$$

and the same is true for $I(x_2; Y)$. Thus :

$$C = 0.0817 \text{ bits/symbol sent}$$

c. We assume that $P(x_i) = 1/3$, $i = 1, 2, 3$. Then $P(y_1) = \frac{1}{3} (\frac{1}{2} + \frac{1}{3} + \frac{1}{6}) = \frac{1}{3}$ and similarly $P(y_j) = 1/3$, $j = 2, 3$. Hence :

$$I(x_1; Y) = \sum_{j=1}^3 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = \frac{1}{2} \log \frac{1/2}{1/3} + \frac{1}{3} \log \frac{1/3}{1/3} + \frac{1}{6} \log \frac{1/6}{1/3} = 0.1258$$

and the same is true for $I(x_2; Y), I(x_3; Y)$. Thus :

$$C = 0.1258 \text{ bits/symbol sent}$$

Problem 6.55

We expect the first channel (with exhibits a certain symmetry) to achieve its capacity through equiprobable input symbols; the second one not.

a. We assume that $P(x_i) = 1/2$, $i = 1, 2$. Then $P(y_1) = \frac{1}{2} (0.6 + 0.1) = 0.35$ and similarly $P(y_3) = 0.35$, $P(y_2) = 0.3$. Hence :

$$I(x_1; Y) = \sum_{j=1}^3 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 0.6 \log \frac{0.6}{0.35} + 0.3 \log \frac{0.3}{0.3} + 0.1 \log \frac{0.1}{0.35} = 0.2858$$

and the same is true for $I(x_2; Y)$. Thus, equally likely input symbols maximize the information rate through the channel and :

$$C = 0.2858 \text{ bits/symbol sent}$$

b. We assume that $P(x_i) = 1/2$, $i = 1, 2$. Then $P(y_1) = \frac{1}{2} (0.6 + 0.3) = 0.45$, and similarly $P(y_2) = 0.2, P(y_3) = 0.35$. Hence :

$$I(x_1; Y) = \sum_{j=1}^3 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 0.6 \log \frac{0.6}{0.45} + 0.3 \log \frac{0.3}{0.2} + 0.1 \log \frac{0.1}{0.35} = 0.244$$

But :

$$I(x_2; Y) = \sum_{j=1}^3 P(y_j|x_2) \log \frac{P(y_j|x_2)}{P(y_j)} = 0.3 \log \frac{0.3}{0.45} + 0.1 \log \frac{0.1}{0.2} + 0.6 \log \frac{0.6}{0.35} = 0.191$$

Since $I(x_1; Y) \neq I(x_2; Y)$ the equiprobable input distribution does not maximize the information rate through the channel. To determine $P(x_1), P(x_2)$ that give the channel capacity, we assume that $P(x_1) = a, P(x_2) = 1 - a$; then we find $P(y_i)$, express $I(X; Y)$ as a function of a and set its derivative with respect to a equal to zero.

Problem 6.56

Relationship (7-1-31) gives :

$$C = W \log \left(1 + \frac{P_{av}}{WN_0} \right) = 25.9 \text{ Kbits/sec}$$

Problem 6.57

1. We assume that $P(x_i) = 1/2$, $i = 1, 2$. Then $P(y_1) = \frac{1}{2} \left(\frac{1}{2}(1-p) + \frac{1}{2}p \right) = \frac{1}{4}$ and similarly $P(y_j) = 1/4$, $j = 2, 3, 4$. Hence :

$$\begin{aligned} I(x_1; Y) &= \sum_{j=1}^4 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 2 \frac{1}{2} (1-p) \log \frac{(1-p)/2}{1/4} + 2 \frac{1}{2} p \log \frac{p/2}{1/4} \\ &= 1 + p \log p + (1-p) \log(1-p) \end{aligned}$$

and the same is true for $I(x_2; Y)$. Thus, equiprobable input symbols achieve the channel capacity

$$C = 1 + p \log p + (1-p) \log(1-p) \text{ bits/symbol sent}$$

2. We note that the above capacity is the same to the capacity of the binary symmetric channel. Indeed, if we consider the grouping of the output symbols into $a = \{y_1, y_2\}$ and $b = \{y_3, y_4\}$ we get a binary symmetric channel, with transition probabilities: $P(a|x_1) = P(y_1|x_1) + P(y_2|x_1) = (1-p)$, $P(a|x_2) = p$, etc.

Problem 6.58

We assume that $P(x_i) = 1/3$, $i = 1, 2, 3$. Then $P(y_1) = \frac{1}{3} ((1-p) + p) = \frac{1}{3}$ and similarly $P(y_j) = 1/3$, $j = 2, 3$. Hence :

$$\begin{aligned} I(x_1; Y) &= \sum_{j=1}^3 P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = (1-p) \log \frac{(1-p)}{1/3} + p \log \frac{p}{1/3} \\ &= \log 3 + p \log p + (1-p) \log(1-p) \end{aligned}$$

and the same is true for $I(x_2; Y)$, $I(x_3; Y)$. Thus, equiprobable input symbols achieve the channel capacity :

$$C = \log 3 + p \log p + (1-p) \log(1-p) \text{ bits/symbol sent}$$

Problem 6.59

1. the probability that a codeword transmitted over the BSC is received correctly, is equal to the probability that all R bits are received correctly. Since each bit transmission is independent from the others :

$$P(\text{correct codeword}) = (1 - p)^R$$

2.

$$P(\text{ at least one bit error in the codeword}) = 1 - P(\text{correct codeword}) = 1 - (1 - p)^R$$

3.

$$P(\text{ } n_e \text{ or less errors in R bits}) = \sum_{i=1}^{n_e} \binom{R}{i} p^i (1 - p)^{R-i}$$

4. For $R = 5$, $p = 0.01$, $n_e = 5$:

$$(1 - p)^R = 0.951$$

$$1 - (1 - p)^R = 0.049$$

$$\sum_{i=1}^{n_e} \binom{R}{i} p^i (1 - p)^{R-i} = 1 - (1 - p)^R = 0.049$$

Problem 6.60

$P(X = 0) = a$, $P(X = 1) = 1 - a$. Then : $P(Y = 0) = (1 - p)a$, $P(Y = e) = p(1 - a + a) = p$, $P(Y = 1) = (1 - p)(1 - a)$.

1.

$$\begin{aligned} I(X; Y) &= \sum_{i=1}^2 \sum_{j=1}^3 P(y_j | x_i) P(x_i) \log \frac{P(y_j | x_i)}{P(y_j)} \\ &= a(1 - p) \log \frac{1-p}{a(1-p)} + ap \log \frac{p}{p} + (1 - a)p \log \frac{p}{p} + (1 - a)(1 - p) \log \frac{1-p}{(1-a)(1-p)} \\ &= -(1 - p) [a \log a + (1 - a) \log(1 - a)] \end{aligned}$$

Note that the term $-[a \log a + (1 - a) \log(1 - a)]$ is the entropy of the source.

2. The value of a that maximizes $I(X; Y)$ is found from :

$$\frac{dI(X; Y)}{da} = 0 \Rightarrow \log a + \frac{a}{a} \log e - \log(1 - a) - \frac{1 - a}{1 - a} \log e = 0 \Rightarrow a = 1/2$$

With this value of a , the resulting channel capacity is :

$$C = I(X; Y)|_{a=1/2} = 1 - p \text{ bits/channel use}$$

3. $I(x; y) = \log \frac{P(y|x)}{P(y)}$. Hence :

$$I(0; 0) = \log \frac{1-p}{(1-p)/2} = 1$$

$$I(1; 1) = \log \frac{1-p}{(1-p)/2} = 1$$

$$I(0; e) = \log \frac{p}{p} = 0$$

$$I(1; e) = \log \frac{p}{p} = 0$$

Problem 6.61

1. The capacity of the channel in bits per second is $500 \times \frac{1}{2} \log_2(1 + P/\sigma_2^2) = 250 \log_2(1 + P/\sigma_2^2)$ and the rate of the source, at distortion D is $1000 \times \frac{1}{2} \log_2 \sigma_1^2/D = 500 \log_2 \sigma_1^2/D$. Equating these two and solving for D results in

$$D = \frac{\sigma_1^2}{\sqrt{1 + P/\sigma_2^2}}$$

2. The channel model in a BSC. Since 500 symbols are transmitted per second, the energy in each symbol is $P/500$ and $N_0/2 = \sigma_2^2$. Therefore, the crossover probability of the BSC is

$$\epsilon = Q(\sqrt{2E_b/N_0}) = Q(P/500\sigma_2^2)$$

and the capacity is $C = 500(1 - H_b(Q(P/500\sigma_2^2)))$. The distortion is obtained by equating this with the rate distortion function of the source, i.e., solving

$$500(1 - H_b(Q(P/500\sigma_2^2))) = 500 \log_2 \sigma_1^2/D$$

resulting in

$$D = \sigma_1^2 2^{-(1-H_b(Q(P/500\sigma_2^2)))}$$

3. When the source is not memoryless, it is more predictable from previous samples, i.e., with lower rate we can achieve the same distortion, or with the same rate we can achieve lower distortion, when compared with a memoryless source. Hence we expect the distortion to be lower.

Problem 6.62

1. The required channel capacity is equal to the source entropy, i.e., $H(X) = H_b(1/3) = 0.9183$. The capacity of a BPSK channel with hard decision decoding is $1 - H_b(Q(\sqrt{2RE_b/N_0}))$, where $R = 0.9183$. Therefore we need to have $1 - H_b(Q(\sqrt{2RE_b/N_0})) = 0.9183$ resulting in $Q(\sqrt{2RE_b/N_0}) = 0.0101$ which gives $E_b/N_0 \approx 1.0601$.
2. From Equation 6.8-32, the cutoff rate for a BSC model is given by

$$R_0 = 1 - \log_2(1 + \sqrt{4p(1-p)})$$

where $p = Q(\sqrt{2RE_b/N_0})$. We have $1 - \log_2(1 + \sqrt{4p(1-p)}) = 0.9183$ from which we obtain $p = 8.5072 \times 10^{-4}$. Using $8.5072 \times 10^{-4} = Q(\sqrt{2RE_b/N_0})$ with $R = 0.9183$ gives $E_b/N_0 = 1.7086$.

3. For a Gaussian source $R(D) = \frac{1}{2} \log_2 \frac{1}{D} = \frac{1}{2} \log_2(1/(1/4)) = 1$ and $1 - H_b(Q(\sqrt{2RE_b/N_0})) = 1$ with $R = 1$. This results in $Q(\sqrt{2RE_b/N_0}) = 0$ and $E_b/N_0 = \infty$.

Problem 6.63

1. For this source $R(D) = H_b(0.1) - H_b(D) = 0.469 - H_b(D)$, and $C = 1 - H_b(0.1) = 0.531$, hence $0.531 = 0.469 - H_b(D)$ resulting in $H_b(D) = 0.062$ and $D = P_e \approx 0.007$.
2. Here $R(D) = \frac{1}{2} \log_2 1/D$, hence $\frac{1}{2} \log_2 1/D = 0.531$ and $D = 0.479$.
3. Here $\frac{1}{2} \log_2 1/D = \frac{1}{2} \log_2(1 + P/\sigma^2)$ and $D = \sigma^2/(P + \sigma^2)$.

Problem 6.64

The capacity of the additive white Gaussian channel is :

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N_0 W} \right)$$

For the nonwhite Gaussian noise channel, although the noise power is equal to the noise power in the white Gaussian noise channel, the capacity is higher, The reason is that since noise samples

are correlated, knowledge of the previous noise samples provides partial information on the future noise samples and therefore reduces their effective variance.

Problem 6.65

1. By symmetry the capacity is achieved with a uniform input distribution resulting in the output probability vector $(1/6, 1/6, 1/6, 1/2)$. It is also clear that $H(Y|X = a) = H(Y|X = b) = H(Y|X = c) = 1$, hence $H(Y|X) = 1$. Therefore $C = H(Y) - H(Y|X) = 1.7925 - 1 = 0.7925$.
2. We have

$$P(\hat{X} = 0) = P(X < 2) = \int_0^2 2e^{-2x} dx = 0.9817$$

and $P(\hat{X} = 1) = 1 - 0.9817 = 0.0183$. Therefore, $H(\hat{X}) = H_b(0.0183) = 0.1319$. Since the source entropy is less than the channel capacity, the source can be reliably transmitted over the channel.

Problem 6.66

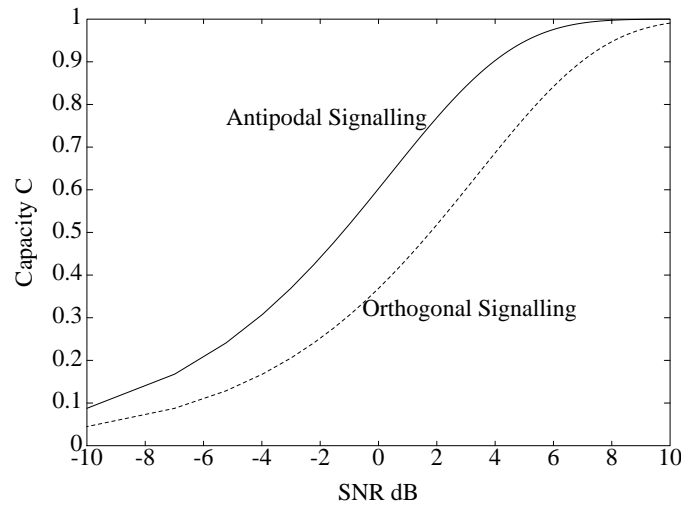
Both channels can be viewed as binary symmetric channels with crossover probability the probability of decoding a bit erroneously. Since :

$$P_b = \begin{cases} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] & \text{antipodal signalling} \\ Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right] & \text{orthogonal signalling} \end{cases}$$

the capacity of the channel is :

$$C = \begin{cases} 1 - H\left(Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]\right) & \text{antipodal signalling} \\ 1 - H\left(Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]\right) & \text{orthogonal signalling} \end{cases}$$

In the next figure we plot the capacity of the channel as a function of $\frac{\mathcal{E}_b}{N_0}$ for the two signalling schemes.



Problem 6.67

1. The capacity of the binary symmetric channel with crossover probability ϵ is :

$$C = 1 - H(\epsilon)$$

where $H(\epsilon)$ is the binary entropy function. The rate distortion function of a zero mean Gaussian source with variance σ^2 per sample is :

$$R(D) = \begin{cases} \frac{1}{2} \log_2 \frac{\sigma^2}{D} & D \leq \sigma^2 \\ 0 & D > \sigma^2 \end{cases}$$

Since $C > 0$, we obtain :

$$\frac{1}{2} \log_2 \frac{\sigma^2}{D} \leq 1 - H(\epsilon) \implies \frac{\sigma^2}{2^{2(1-H(\epsilon))}} \leq D$$

and therefore, the minimum value of the distortion attainable at the output of the channel is :

$$D_{\min} = \frac{\sigma^2}{2^{2(1-H(\epsilon))}}$$

2. The capacity of the additive Gaussian channel is :

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma_n^2} \right)$$

Hence :

$$\frac{1}{2} \log_2 \frac{\sigma^2}{D} \leq C \implies \frac{\sigma^2}{2^{2C}} \leq D \implies \frac{\sigma^2}{1 + \frac{P}{\sigma_n^2}} \leq D$$

The minimum attainable distortion is :

$$D_{\min} = \frac{\sigma^2}{1 + \frac{P}{\sigma_n^2}}$$

3. Here the source samples are dependent and therefore one sample provides information about the other samples. This means that we can achieve better results compared to the memoryless case at a given rate. In other words the distortion at a given rate for a source with memory is less than the distortion for a comparable source with memory. Differential coding methods discussed in Chapter 3 are suitable for such sources.

Problem 6.68

The overall channel is a binary symmetric channel with crossover probability p . To find p note that an error occurs if an odd number of channels produce an error. Thus :

$$p = \sum_{k=\text{odd}} \binom{n}{k} \epsilon^k (1 - \epsilon)^{n-k}$$

Using the results of Problem 5.45, we find that :

$$p = \frac{1}{2} [1 - (1 - 2\epsilon)^n]$$

and therefore :

$$C = 1 - H(p)$$

If $n \rightarrow \infty$, then $(1 - 2\epsilon)^n \rightarrow 0$ and $p \rightarrow \frac{1}{2}$. In this case

$$C = \lim_{n \rightarrow \infty} C(n) = 1 - H\left(\frac{1}{2}\right) = 0$$

Problem 6.69

1. The capacity of the channel is :

$$C_1 = \max_{P(x)} [H(Y) - H(Y|X)]$$

But, $H(Y|X) = 0$ and therefore, $C_1 = \max_{P(x)} H(Y) = 1$ which is achieved for $P(0) = P(1) = \frac{1}{2}$.

2. Let q be the probability of the input symbol 0, and thus $(1 - q)$ the probability of the input symbol 1. Then :

$$\begin{aligned} H(Y|X) &= \sum_x P(x)H(Y|X = x) \\ &= qH(Y|X = 0) + (1 - q)H(Y|X = 1) \\ &= (1 - q)H(Y|X = 1) = (1 - q)H(0.5) = (1 - q) \end{aligned}$$

The probability mass function of the output symbols is :

$$\begin{aligned} P(Y = c) &= qP(Y = c|X = 0) + (1 - q)P(Y = c|X = 1) \\ &= q + (1 - q)0.5 = 0.5 + 0.5q \\ P(Y = d) &= (1 - q)0.5 = 0.5 - 0.5q \end{aligned}$$

Hence :

$$C_2 = \max_q [H(0.5 + 0.5q) - (1 - q)]$$

To find the probability q that achieves the maximum, we set the derivative of C_2 with respect to q equal to 0. Thus,

$$\begin{aligned} \frac{\partial C_2}{\partial q} = 0 &= 1 - [0.5 \log_2(0.5 + 0.5q) + (0.5 + 0.5q) \frac{0.5}{0.5 + 0.5q} \frac{1}{\ln 2}] \\ &\quad - [-0.5 \log_2(0.5 - 0.5q) + (0.5 - 0.5q) \frac{-0.5}{0.5 - 0.5q} \frac{1}{\ln 2}] \\ &= 1 + 0.5 \log_2(0.5 - 0.5q) - 0.5 \log_2(0.5 + 0.5q) \end{aligned}$$

Therefore :

$$\log_2 \frac{0.5 - 0.5q}{0.5 + 0.5q} = -2 \implies q = \frac{3}{5}$$

and the channel capacity is :

$$C_2 = H\left(\frac{1}{5}\right) - \frac{2}{5} = 0.3219$$

3. The transition probability matrix of the third channel can be written as :

$$\mathbf{Q} = \frac{1}{2}\mathbf{Q}_1 + \frac{1}{2}\mathbf{Q}_2$$

where \mathbf{Q}_1 , \mathbf{Q}_2 are the transition probability matrices of channel 1 and channel 2 respectively. We have assumed that the output space of both channels has been augmented by adding two new symbols so that the size of the matrices \mathbf{Q} , \mathbf{Q}_1 and \mathbf{Q}_2 is the same. The transition probabilities to these newly added output symbols is equal to zero. Using the fact that the function $I(\mathbf{p}; \mathbf{Q})$ is a

convex function in \mathbf{Q} we obtain :

$$\begin{aligned}
 C &= \max_{\mathbf{p}} I(X; Y) = \max_{\mathbf{p}} I(\mathbf{p}; \mathbf{Q}) \\
 &= \max_{\mathbf{p}} I(\mathbf{p}; \frac{1}{2}\mathbf{Q}_1 + \frac{1}{2}\mathbf{Q}_2) \\
 &\leq \frac{1}{2} \max_{\mathbf{p}} I(\mathbf{p}; \mathbf{Q}_1) + \frac{1}{2} \max_{\mathbf{p}} I(\mathbf{p}; \mathbf{Q}_2) \\
 &= \frac{1}{2}C_1 + \frac{1}{2}C_2
 \end{aligned}$$

Since \mathbf{Q}_1 and \mathbf{Q}_2 are different, the inequality is strict. Hence :

$$C < \frac{1}{2}C_1 + \frac{1}{2}C_2$$

Problem 6.70

The capacity of a channel is :

$$C = \max_{p(x)} I(X; Y) = \max_{p(x)} [H(Y) - H(Y|X)] = \max_{p(x)} [H(X) - H(X|Y)]$$

Since in general $H(X|Y) \geq 0$ and $H(Y|X) \geq 0$, we obtain :

$$C \leq \min\{\max[H(Y)], \max[H(X)]\}$$

However, the maximum of $H(X)$ is attained when X is uniformly distributed, in which case $\max[H(X)] = \log |\mathcal{X}|$. Similarly : $\max[H(Y)] = \log |\mathcal{Y}|$ and by substituting in the previous inequality, we obtain

$$\begin{aligned}
 C &\leq \min\{\max[H(Y)], \max[H(X)]\} = \min\{\log |\mathcal{Y}|, \log |\mathcal{X}|\} \\
 &= \min\{\log M, \log N\}
 \end{aligned}$$

Problem 6.71

1. Let q be the probability of the input symbol 0, and therefore $(1 - q)$ the probability of the input symbol 1. Then :

$$\begin{aligned}
 H(Y|X) &= \sum_x P(x) H(Y|X = x) \\
 &= qH(Y|X = 0) + (1 - q)H(Y|X = 1) \\
 &= (1 - q)H(Y|X = 1) = (1 - q)H(\epsilon)
 \end{aligned}$$

The probability mass function of the output symbols is :

$$\begin{aligned} P(Y = 0) &= qP(Y = 0|X = 0) + (1 - q)P(Y = 0|X = 1) \\ &= q + (1 - q)(1 - \epsilon) = 1 - \epsilon + q\epsilon \\ P(Y = 1) &= (1 - q)\epsilon = \epsilon - q\epsilon \end{aligned}$$

Hence :

$$C = \max_q [H(\epsilon - q\epsilon) - (1 - q)H(\epsilon)]$$

To find the probability q that achieves the maximum, we set the derivative of C with respect to q equal to 0. Thus :

$$\frac{\partial C}{\partial q} = 0 = H(\epsilon) + \epsilon \log_2(\epsilon - q\epsilon) - \epsilon \log_2(1 - \epsilon + q\epsilon)$$

Therefore :

$$\log_2 \frac{\epsilon - q\epsilon}{1 - \epsilon + q\epsilon} = -\frac{H(\epsilon)}{\epsilon} \implies q = \frac{\epsilon + 2^{-\frac{H(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{H(\epsilon)}{\epsilon}})}$$

and the channel capacity is

$$C = H\left(\frac{2^{-\frac{H(\epsilon)}{\epsilon}}}{1 + 2^{-\frac{H(\epsilon)}{\epsilon}}}\right) - \frac{H(\epsilon)2^{-\frac{H(\epsilon)}{\epsilon}}}{\epsilon(1 + 2^{-\frac{H(\epsilon)}{\epsilon}})}$$

2. If $\epsilon \rightarrow 0$, then using L'Hospital's rule we find that

$$\lim_{\epsilon \rightarrow 0} \frac{H(\epsilon)}{\epsilon} = \infty, \quad \lim_{\epsilon \rightarrow 0} \frac{H(\epsilon)}{\epsilon} 2^{-\frac{H(\epsilon)}{\epsilon}} = 0$$

and therefore

$$\lim_{\epsilon \rightarrow 0} C(\epsilon) = H(0) = 0$$

If $\epsilon = 0.5$, then $H(\epsilon) = 1$ and $C = H(\frac{1}{5}) - \frac{2}{5} = 0.3219$. In this case the probability of the input symbol 0 is

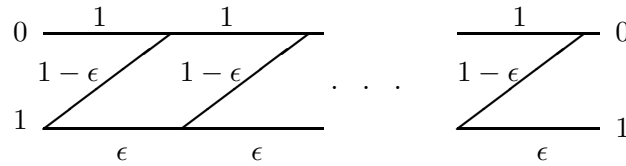
$$q = \frac{\epsilon + 2^{-\frac{H(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{H(\epsilon)}{\epsilon}})} = \frac{0.5 + 0.25 \times (0.5 - 1)}{0.5 \times (1 + 0.25)} = \frac{3}{5}$$

If $\epsilon = 1$, then $C = H(0.5) = 1$. The input distribution that achieves capacity is $P(0) = P(1) = 0.5$.

3. The following figure shows the topology of the cascade channels. If we start at the input labeled 0, then the output will be 0. If however we transmit a 1, then the output will be zero with probability

$$\begin{aligned} P(Y = 0|X = 1) &= (1 - \epsilon) + \epsilon(1 - \epsilon) + \epsilon^2(1 - \epsilon) + \dots \\ &= (1 - \epsilon)(1 + \epsilon + \epsilon^2 + \dots) \\ &= 1 - \epsilon \frac{1 - \epsilon^n}{1 - \epsilon} = 1 - \epsilon^n \end{aligned}$$

Thus, the resulting system is equivalent to a Z channel with $\epsilon_1 = \epsilon^n$.



4. As $n \rightarrow \infty$, $\epsilon^n \rightarrow 0$ and the capacity of the channel goes to 0.

Problem 6.72

The SNR is :

$$\text{SNR} = \frac{2P}{N_0 2W} = \frac{P}{2W} = \frac{10}{10^{-9} \times 10^6} = 10^4$$

Thus the capacity of the channel is :

$$C = W \log_2(1 + \frac{P}{N_0 W}) = 10^6 \log_2(1 + 10000) \approx 13.2879 \times 10^6 \text{ bits/sec}$$

Problem 6.73

1. We have

$$\begin{aligned} D &= E[(X - \hat{X})^2] \\ &= (1 - P_e)E[(X - \hat{X})^2 | \text{no error in channel}] + P_e E[(X - \hat{X})^2 | \text{error in channel}] \end{aligned}$$

But $E[(X - \hat{X})^2 | \text{no error in channel}] = 0.3634$ and

$$\begin{aligned} E[(X - \hat{X})^2 | \text{error}] &= \int_{-\infty}^0 (x - 0.798)^2 f(x) dx + \int_0^{\infty} (x + 0.7998)^2 f(x) dx \\ &= E[X^2] + 0.7982 - 2 \times 0.798 \int_{-\infty}^0 x f(x) dx + 2 \times 0.798 \int_0^{\infty} x f(x) dx \\ &= 1 + 0.6368 - 1.596 \int_{-\infty}^0 -y f(-y) d(-y) + 1.596 \int_0^{\infty} x f(x) dx \\ &= 1.6368 + 3.192 \int_0^{\infty} x f(x) dx = 1.6368 + 3.192 \times 0.39894 = 2.91 \end{aligned}$$

Hence,

$$D = 0.9 \times 0.3634 + 0.1 \times 2.91 = 0.618$$

2. Here we have a binary equiprobable source to be transmitted over a channel with capacity $C = 1 - H_b(0.1) = 0.531$. The minimum error probability of the source is obtained when $R(D) = C$, or when $1 - H_b(P_e) = 0.531$ resulting in $P_e = 0.1$ and thus $D = 0.618$ as in case 1.
3. Since the source \hat{X} is binary and equiprobable, it cannot be compressed by source coding. Hence the results of parts 1 and 2 would not change.
4. In this case we have to solve $\frac{1}{2} \log_2 \frac{1}{D} = 0.531$ resulting in $D = 0.479$.
5. The existence of memory makes the source more predictable and reduces the distortion.

Problem 6.74

1. By symmetry the capacity is achieved using a uniform input probability mass function. This results in an output probability mass function given by $P(a_i) = 1/2n$ for $1 \leq i \leq n$ and $P(E) = 1/2$, resulting in

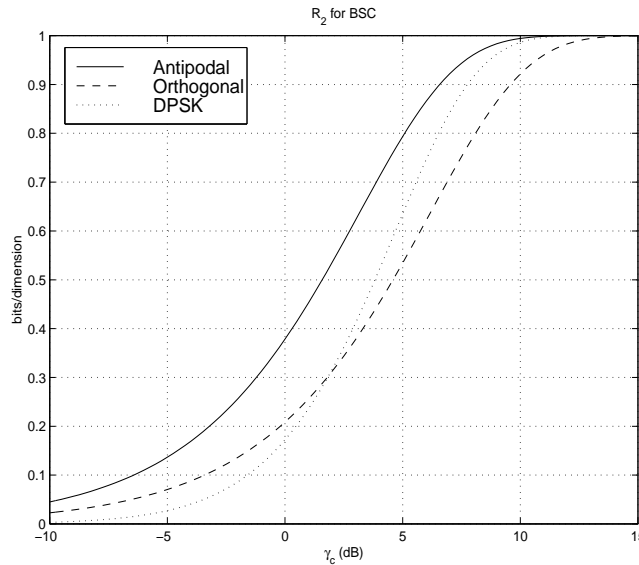
$$H(Y) = -\frac{n}{2n} \log_2 \frac{1}{2n} - \frac{1}{2} \log_2 \frac{1}{2} = 1 + \frac{1}{2} \log_2 n$$

Also we have $H(Y|X = a_i) = 1$ for $1 \leq i \leq n$, hence $H(Y|X) = \sum P(X = a_i)H(Y|X = a_i) = 1$. Therefore, $C = H(Y) - H(Y|X) = \frac{1}{2} \log_2 n$. When $n = 2^m$, the capacity is $C = \frac{m}{2}$.

2. The source entropy is equal to 1, hence if $n \geq 4$, then $C > 1$ and the capacity exceeds source entropy and reliable transmission of the source is possible. If $n < 4$, then the resulting error probability is obtained by solving the equation $R(P_e) = C$ where $R(D)$ is the rate distortion function for a binary equiprobable source with Hamming distortion. This results in $1 - H_b(P_e) = \frac{1}{2} \log_2 n$ and $P_e = H_b^{-1}(1 - \frac{1}{2} \log_2 n)$. For $n = 3$ we have $P_e = H_b^{-1}(1 - \frac{1}{2} \log_2 3) = H_b^{-1}(0.2075) \approx 0.0325$. For $n = 2$, we have $P_e = H_b^{-1}(1 - \frac{1}{2} \log_2 2) = H_b^{-1}(0.5) \approx 0.11$. entropy
3. Here $R(D) = \frac{1}{2} \log_2 \frac{\sigma^2}{D} = \frac{1}{2} \log_2 n$ and $D = \sigma^2/n$.

Problem 6.75

The plot with the cutoff rate R_2 for the BSC, when the three different modulation schemes are employed is given in the following figure:



As we see, orthogonal signaling is 3 dB worse than antipodal signaling. Also, DPSK is the worst scheme in very low SNR's, but approaches antipodal signaling performance as the SNR goes up. Both these conclusions agree with the well-known results on the performance of these schemes, as given by their error probabilities given in Chapter 5.

Problem 6.76

1. The cutoff rate for the binary input, ternary output channel is given by :

$$R_3 = \max_{p_j} \left\{ -\log \sum_{i=0}^2 \left[\sum_{j=0}^1 P_j \sqrt{P(i|j)} \right]^2 \right\}$$

To maximize the term inside the brackets we want to minimize the argument S of the log function : $S = \sum_{i=0}^2 \left[\sum_{j=0}^1 P_j \sqrt{P(i|j)} \right]^2$. Suppose that $P_0 = x, P_1 = 1 - x$. Then :

$$\begin{aligned} S &= (x\sqrt{1-p-a} + (1-x)\sqrt{p})^2 + (x\sqrt{a} + (1-x)\sqrt{a})^2 + (x\sqrt{p} + (1-x)\sqrt{1-p-a})^2 \\ &= 2 \left(1 - a - 2\sqrt{p-p^2-ap} \right) x^2 - 2 \left(1 - a - 2\sqrt{p-p^2-ap} \right) x + 1 \end{aligned}$$

By setting : $\frac{dS}{dx} = 0$, we obtain $x = 1/2$ which corresponds to a minimum for S , since $\frac{d^2S}{dx^2}|_{x=1/2} > 0$. Then :

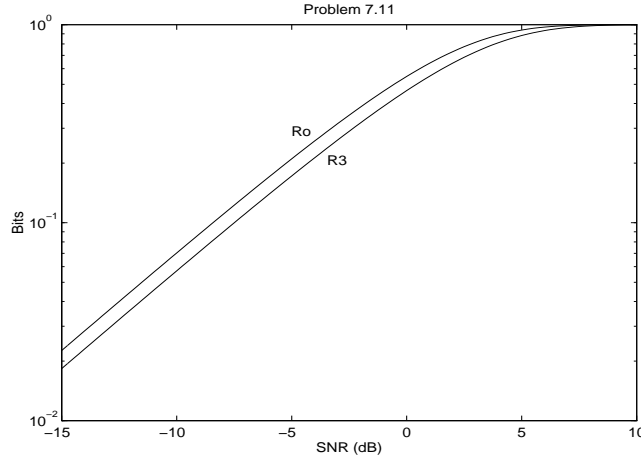
$$R_3 = -\log S = -\log \left\{ \frac{1 + a + 2\sqrt{p-p^2-ap}}{2} \right\} = 1 - \log \left(1 + a + 2\sqrt{p(1-p-a)} \right)$$

2. For $\beta = 0.65\sqrt{N_0/2}$, we have :

$$p = \frac{1}{\sqrt{\pi N_0}} \int_{\beta}^{\infty} \exp(-(x + \sqrt{E_c})^2/N_0) = Q \left[0.65 + \sqrt{2E_c/N_0} \right]$$

$$a = \frac{1}{\sqrt{\pi N_0}} \int_{-\beta}^{\beta} \exp(-(x + \sqrt{E_c})^2/N_0) = Q \left[\sqrt{2E_c/N_0} - 0.65 \right] - Q \left[\sqrt{2E_c/N_0} + 0.65 \right]$$

The plot of R_3 is given in the following figure. In this figure, we have also included the plot of $R_{\infty} = \log \frac{2}{1+\exp(-\sqrt{E_c/N_0})}$. As we see the difference in performance between continuous-output (soft-decision-decoding, R_{∞}) and ternary output (R_3) is approximately 1 dB.



Problem 6.77

From expression (7.2-31) we have that

$$R_0 = -\log_2 \left(\sum_{l=1}^M \sum_{m=1}^M p_l p_m e^{-d_{lm}^2/4N_0} \right)$$

and, since we are given that equiprobable input symbols maximize R_0 , $p_l = p_m = 1/M$ and the above expression becomes

$$R_0 = -\log_2 \left(\frac{1}{M^2} \sum_{l=1}^M \sum_{m=1}^M e^{-d_{lm}^2/4N_0} \right)$$

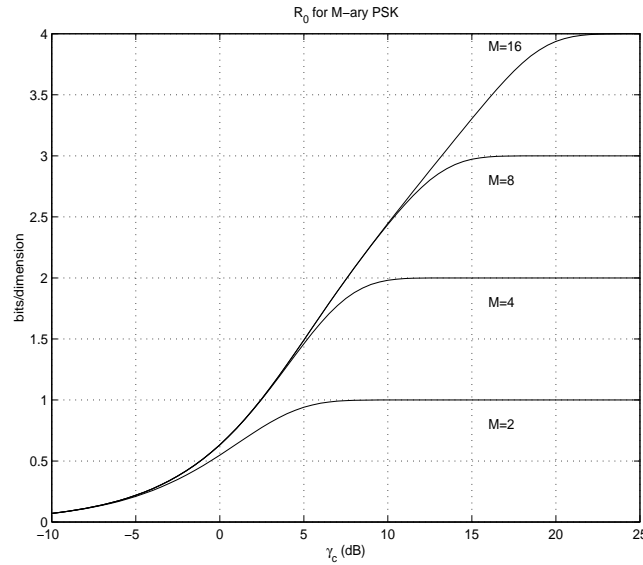
The M-ary PSK constellation points are symmetrically spaced around the unit circle. Hence, the sum of the distances between them is the same, independent of the reference point, or $\sum_{m=1}^M e^{-d_{lm}^2/4N_0}$ is the same for any $l = 0, 1, \dots, M-1$. Hence,

$$\begin{aligned} R_0 &= -\log_2 \left(\frac{M}{M^2} \sum_{l=1}^M e^{-d_{0m}^2/4N_0} \right) \\ &= \log_2 M - \log_2 \sum_{l=1}^M e^{-d_{0m}^2/4N_0} \end{aligned}$$

The distance of equally spaced points around a circle with radius $\sqrt{\mathcal{E}_c}$ is $d_m = 2\sqrt{\mathcal{E}_c} \sin \frac{m\pi}{M}$. So

$$R_0 = \log_2 M - \log_2 \sum_{l=1}^M e^{-(\mathcal{E}_c/N_0) \sin^2 \frac{m\pi}{M}}$$

The plot of R_0 for the various levels of M -ary PSK is given in the following figure:



Problem 6.78

1. For a binary-input symmetric channel from Equation 6.8-27, $R_0 = 1 - \log_2(1 + \Delta)$, where by Equation 6.8-10

$$\Delta = \int_{-\infty}^{\infty} \sqrt{p(y|x_1)p(y|x_2)} dy$$

Hence,

$$\begin{aligned} \Delta &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{4} e^{-|y-1|} e^{-|y+1|}} dy \\ &= \frac{1}{2} \int_{-\infty}^{-1} \sqrt{e^{y-1} e^{y+1}} dy + \frac{1}{2} \int_{-1}^1 \sqrt{e^{y-1} e^{-y-1}} dy + \frac{1}{2} \int_1^{\infty} \sqrt{e^{-y+1} e^{-y-1}} dy \\ &= \frac{1}{2} \int_{-\infty}^{-1} e^y dy + \frac{1}{2} \int_{-1}^1 e^{-1} dy + \frac{1}{2} \int_1^{\infty} e^{-y} dy \\ &= 2e^{-1} \end{aligned}$$

and

$$R_0 = 1 - \log_2(1 + 2e^{-1}) \approx 0.2044$$

2. By symmetry the optimal threshold is zero and the channel crossover probability is

$$p = P(y > 0 | X = -1) = P(n - 1 > 0) = P(n > 1) = \frac{1}{2} \int_1^\infty e^{-n} dn = \frac{1}{2} e^{-1} \approx 0.184$$

3. In this case by Equation 6.8-32,

$$\begin{aligned} R_0 &= \log_2 \frac{2}{1 + \sqrt{4p(1-p)}} \\ &= 0.1723 \end{aligned}$$

Problem 6.79

This is a symmetric channel model, hence we can use Equation 6.8-22

$$R_0 = 2 \log_2 M - \log_2 \left[\int \left(\sum_{\mathbf{x}_i} \sqrt{p(\mathbf{y}|\mathbf{x})} \right)^2 d\mathbf{y} \right]$$

where $\mathbf{x}_i = (0, \dots, 0, \sqrt{E}, 0, \dots, 0)$ is an M dimensional vector with \sqrt{E} at its i th component. We have

$$p(\mathbf{y}|\mathbf{x}_i) = p_n(y_i - \sqrt{E}) \prod_{\substack{j=1 \\ j \neq i}}^M p_n(y_j)$$

and

$$\sum_{\mathbf{x}_i} \sqrt{p(\mathbf{y}|\mathbf{x})} = \sum_{i=1}^M \sqrt{p_n(y_i - \sqrt{E}) \prod_{\substack{j=1 \\ j \neq i}}^M p_n(y_j)}$$

Hence,

$$\begin{aligned}
\left(\sum_{\mathbf{x}_i} \sqrt{p(\mathbf{y}|\mathbf{x})} \right)^2 &= \sum_{i=1}^M p_n(y_i - \sqrt{E}) \prod_{\substack{j=1 \\ j \neq i}}^M p_n(y_j) \\
&\quad + \sum_{i=1}^M \sum_{\substack{k=1 \\ k \neq i}}^M \sqrt{p_n(y_i - \sqrt{E}) p_n(y_k - \sqrt{E})} \sqrt{\left(\prod_{\substack{j=1 \\ j \neq i}}^M p_n(y_j) \right) \left(\prod_{\substack{l=1 \\ l \neq k}}^M p_n(y_l) \right)} \\
&= \sum_{i=1}^M p_n(y_i - \sqrt{E}) \prod_{\substack{j=1 \\ j \neq i}}^M p_n(y_j) \\
&\quad + \sum_{i=1}^M \sum_{\substack{k=1 \\ k \neq i}}^M \sqrt{p_n(y_i - \sqrt{E}) p_n(y_i) p_n(y_k - \sqrt{E}) p_n(y_k)} \prod_{\substack{l=1 \\ l \neq i, l \neq k}}^M p_n(y_l)
\end{aligned}$$

From above we have

$$\begin{aligned}
\int \left(\sum_{\mathbf{x}_i} \sqrt{p(\mathbf{y}|\mathbf{x})} \right)^2 d\mathbf{y} &= M + \sum_{i=1}^M \sum_{\substack{k=1 \\ k \neq i}}^M \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{p_n(y_i - \sqrt{E}) p_n(y_i) p_n(y_k - \sqrt{E}) p_n(y_k)} dy_i dy_k \\
&= M + \sum_{i=1}^M \sum_{\substack{k=1 \\ k \neq i}}^M \left(\int_{-\infty}^{\infty} \sqrt{p_n(y - \sqrt{E}) p_n(y)} dy \right)^2 \\
&= M + M(M-1) \left(\int_{-\infty}^{\infty} \sqrt{p_n(y - \sqrt{E}) p_n(y)} dy \right)^2
\end{aligned}$$

Substituting this result into the relation for R_0 and deleting $\log_2 M$ gives

$$R_0 = \log_2 M - \log_2 \left[1 + (M-1) \left(\int_{-\infty}^{\infty} \sqrt{p_n(y - \sqrt{E}) p_n(y)} dy \right)^2 \right]$$

For a Gaussian channel, $p_n(y) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{y^2}{N_0}}$ and $p_n(y - \sqrt{E}) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y - \sqrt{E})^2}{N_0}}$, therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \sqrt{p_n(y - \sqrt{E}) p_n(y)} dy &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} e^{-\frac{2y^2 - 2\sqrt{E}y + E}{2N_0}} dy \\
&= e^{-\frac{E}{4N_0}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y - \sqrt{E}/2)^2}{N_0}} dy \\
&= e^{-\frac{E}{4N_0}}
\end{aligned}$$

Hence,

$$\left(\int_{-\infty}^{\infty} \sqrt{p_n(y - \sqrt{E}) p_n(y)} dy \right)^2 = e^{-\frac{E}{2N_0}}$$

and

$$R_0 = \log_2 \left[\frac{M}{1 + (M - 1)e^{-\frac{E}{2N_0}}} \right]$$

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 7) ¹

Prepared by
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Problem 7.1

1. Consider the sequence a^i for $i = 1, 2, 3, \dots$. Since the group is finite all these elements cannot be different and for the first time, for some k , we will have $a^k = a^m$ for some $m < k$. This means that $a^j = 1$ for $j = k - m$. Obviously, $a^i \neq 1$ for $i < j$ since otherwise a^k will not be the first element that repeats.
2. If $a^i = a^{i'}$, then $a^{i'-i} = 1$ but since $i' - i < j$, this is impossible.
3. First we note that \mathcal{G}_a is closed under multiplication, in fact we obviously have $a^k \cdot a^m = a^l$ where $l = k + m \pmod j$, and hence $l < j$. Clearly $a^j = 1$ is the identity element and for each a^i , $i < j$, the element a^{j-i} is the inverse.
4. If $\mathcal{G}_a \cap \mathcal{G}_{ba} \neq \emptyset$, then for some $1 \leq k, l \leq j$ we have $b \cdot a^k = a^l$, resulting in $b = a^{l-k}$ if $l > k$ or $b = a^{j+(l-k)}$ if $k > l$. But this is impossible since by assumption $b \notin \mathcal{G}_a$. Also elements of \mathcal{G}_{ba} are distinct since elements of \mathcal{G}_a are distinct and $b \neq 0$.
5. If there exists a $c \notin \mathcal{G}_a \cup \mathcal{G}_{ba}$, then we can generate $\mathcal{G}_{ac} = \{c \cdot a, c \cdot a^2, \dots, c \cdot a^j\}$. The elements of \mathcal{G}_{ac} are distinct and $\mathcal{G}_a \cap \mathcal{G}_{ca} \neq \emptyset$ as shown in part 4. We can also show that $\mathcal{G}_{ba} \cap \mathcal{G}_{ca} \neq \emptyset$ since otherwise $c \cdot a^k = b \cdot a^l$ for some $1 \leq k, l \leq j$, resulting in $c = b \cdot a^{l-k}$ or $c = b \cdot a^{j+(l-k)}$, which is impossible since $c \notin \mathcal{G}_{ba}$. This shows the number of elements in $\{F - \{0\}, \cdot, 1\}$ is at least $3j$. Continuing this argument we get to a point where no new element like c exists that is not in the previously generated cosets, from which we conclude that j , the order of a divides $q - 1$.
6. Let β be a nonzero element and let k denote the order of β , then $\beta^k = 1$. But by part 5, we have $q - 1 = k \cdot l$ for some integer l , therefore, $\beta^{q-1} = (\beta^k)^l = 1$.

Problem 7.2

If $\beta = neq0$, then $\beta^{q-1} = 1$ and hence β satisfies equation $X^q - X = 0$. Obviously $X = 0$ is also a root of this equation. Therefore all elements of the Galois field are the roots of $X^q - X = 0$.

Problem 7.3

The Tables are shown below

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Problem 7.4

There exist nine possible candidates of the form $X^2 + aX + b$ where $a, b \in \{0, 1, 2\}$. From these

candidates X^2 , $X^2 + X$, $X^2 + 2X$ are obviously irreducible. It is easily verified that $X^2 + 2 = (X + 1)(X + 2)$, $X^2 + X + 1 = (X + 2)^2$, and $X^2 + 2X + 1 = (X + 1)^2$. Therefore the only prime polynomials of degree 2 are $X^2 + 1$, $X^2 + X + 2$, and $X^2 + 2X + 2$. Choosing $X^2 + 1$ as the prime polynomial, we obtain the following multiplication table for $\text{GF}(9)$.

\cdot	0	1	2	X	$2X$	$1 + X$	$1 + 2X$	$2 + X$	$2 + 2X$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	X	$2X$	$1 + X$	$1 + 2X$	$2 + X$	$2 + 2X$
2	0	2	1	$2X$	X	$2 + 2X$	$2 + X$	$1 + 2X$	$1 + X$
X	0	X	$2X$	2	1	$X + 2$	$X + 1$	$2X + 2$	$2X + 1$
$2X$	0	$2X$	X	1	2	$2X + 1$	$2X + 2$	$X + 1$	$X + 2$
$1 + X$	0	$1 + X$	$2 + 2X$	$X + 2$	$2X + 1$	$2X$	2	1	X
$1 + 2X$	0	$1 + 2X$	$X + 2$	$X + 1$	$2X + 2$	2	X	$2X$	1
$2 + X$	0	$2 + X$	$1 + 2X$	$2X + 2$	$X + 1$	1	$2X$	X	2
$2 + 2X$	0	$2 + 2X$	$1 + X$	$1 + 2X$	$2 + X$	X	1	2	$2X$

Problem 7.5

In $\text{GF}(8)$ primitive elements are elements of order 7. From Problem 7.1, the order of nonzero elements are factors of $q - 1 = 7$. From this we note that the order of a nonzero element is either 1 or 7. The only element with order 1 is 1. All other nonzero elements have order 7. Therefore primitive elements are 2,3,4,5,6, and 7. Similarly in $\text{GF}(32)$ each nonzero element has order equal to 1 or 31. Since the only element of order one is 1, we conclude that all elements of the field except 0 and 1 are primitive.

Problem 7.6

From Table 7.1-6, we note that $\alpha^{10} + \alpha^5 + 1 = \alpha^2 + \alpha + 1 + \alpha^2 + \alpha = 1$ and $\alpha^{10} \cdot \alpha^5 = \alpha^{15} = 1$, hence is closed under addition and multiplication. It is also clear that α^5 and α^{10} are multiplicative inverses on each other. Therefore, $\{0, 1, \alpha^5, \alpha^{10}\}$ is a subfield of $\text{GF}(16)$, showing that $\text{GF}(4)$ is a subfield of $\text{GF}(16)$.

Problem 7.7

Elements of $\text{GF}(32)$ are the roots of $X^{32} - X = X(X^{31} - 1) = 0$ and elements of $\text{GF}(4)$ are the roots of $X(X^3 - 1) = 0$. Since it can be easily verified that $X^3 - 1$ does not divide $X^{31} - 1$, we conclude that $\text{GF}(4)$ is not a subfield of $\text{GF}(32)$. Also from Problem 7.5 it is seen that all elements of $\text{GF}(32)$, except 0 and 1 are primitive and hence have order equal to 31. This means that they cannot be an element of any subfield.

Problem 7.8

The primitive polynomial of degree five is $X^5 + X^2 + 1$ and the table is given below

Power	Polynomial	Vector
—	0	00000
$\alpha^0 = \alpha^{31}$	1	00001
α^1	α	00010
α^2	α^2	00100
α^3	α^3	01000
α^4	α^4	10000
α^5	$\alpha^2 + 1$	00101
α^6	$\alpha^3 + \alpha$	01010
α^7	$\alpha^4 + \alpha^2$	10100
α^8	$\alpha^3 + \alpha^2 + 1$	01101
α^9	$\alpha^4 + \alpha^3 + \alpha$	11010
α^{10}	$\alpha^4 + 1$	10001
α^{11}	$\alpha^2 + \alpha + 1$	00111
α^{12}	$\alpha^3 + \alpha^2 + \alpha$	01110
α^{13}	$\alpha^4 + \alpha^3 + \alpha^2$	11100
α^{14}	$\alpha^4 + \alpha^3 + \alpha^2 + 1$	11101
α^{15}	$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1$	11111
α^{16}	$\alpha^4 + \alpha^3 + \alpha + 1$	11011
α^{17}	$\alpha^4 + \alpha + 1$	10011
α^{18}	$\alpha + 1$	00011
α^{19}	$\alpha^2 + \alpha$	00110
α^{20}	$\alpha^3 + \alpha^2$	01100
α^{21}	$\alpha^4 + \alpha^3$	11000
α^{22}	$\alpha^4 + \alpha^2 + 1$	10100
α^{23}	$\alpha^3 + \alpha^2 + \alpha + 1$	01111
α^{24}	$\alpha^4 + \alpha^3 + \alpha^2 + \alpha$	11110
α^{25}	$\alpha^4 + \alpha^3 + 1$	11001
α^{26}	$\alpha^4 + \alpha^2 + \alpha + 1$	10111
α^{27}	$\alpha^3 + \alpha + 1$	01011
α^{28}	$\alpha^4 + \alpha^2 + \alpha$	10110
α^{29}	$\alpha^3 + 1$	01001
α^{30}	$\alpha^4 + \alpha$	10010

For $\beta = \alpha^3$ the conjugates are $\beta^2, \beta^4, \beta^8$, and β^{16} and

$$\begin{aligned}\phi_\beta(X) &= (X + \beta)(X + \beta^2)(X + \beta^4)(X + \beta^8)(X + \beta^{16}) \\ &= (X^2 + \alpha X + \alpha^9)(X^2 + \alpha^4 X + \alpha^5)(X + \alpha^{17}) \\ &= (X^4 + \alpha^{30} X^3 + \alpha^9 X^2 + \alpha^{28} X + \alpha^{14})(X + \alpha^{17}) \\ &= X^5 + X^4 + X^3 + X^2 + 1\end{aligned}$$

and for $\gamma = \alpha^3 + \alpha = \alpha^6 = \beta^2$ we have the same set of conjugates and hence the same minimal polynomial.

Problem 7.9

Note that $\sum_{i=1}^p \beta = \beta \sum_{i=1}^n 1$ and since $\beta \neq 0$, then $\sum_{i=1}^p \beta = 0$ if and only if $\sum_{i=1}^n 1 = 0$. Therefore, we need to show that $\sum_{i=1}^n 1 \neq 0$ for $n < p$ and $\sum_{i=1}^p 1 = 0$. Note that if n is the smallest integer such that $\sum_{i=1}^n 1 = 0$ then n must be a prime since if $n = kl$, then

$$\sum_{i=1}^{kl} 1 = \left(\sum_{i=1}^k 1 \right) \left(\sum_{j=1}^l 1 \right) = 0$$

resulting in either $\sum_{i=1}^k 1 = 0$ or $\sum_{j=1}^l 1 = 0$ contradicting that n is the smallest integer satisfying $\sum_{i=1}^n 1 = 0$. Since $\{F, +, 0\}$ is an Abelian group, then by Problem 7.1, the additive order of 1, i.e., the smallest integer such that $\sum_{i=1}^n 1 = 0$, must divide the size of the group, i.e., p^m . Therefore n is a prime that divides p^m , i.e., $n = p$, thus completing the proof.

Problem 7.10

Using the binomial expansion

$$(\alpha + \beta)^p = \alpha^p + \sum_{i=1}^{p-1} \binom{p}{i} \alpha^{p-i} \beta^i + \beta^p$$

Since

$$\binom{p}{i} = \frac{p(p-1)\dots(p-i+1)}{i(i-1)\dots 2 \times 1}$$

is an integer and p is prime, none of the factors $i, i-1, i-2, \dots, 2$ divides p and thus p divides $\binom{p}{i}$. Therefore by the result of Problem 7.9, $\binom{p}{i} \alpha^{p-i} \beta^i = 0$ and hence $(\alpha + \beta)^p = \alpha^p + \beta^p$.

Problem 7.11

A binary LBC is a linear subspace of the n -dimensional binary space. Denoting its dimension by k , we can find a basis of size k for this subspace of the form $\{e_1, \dots, e_k\}$. The codewords can be expressed in the form $c_i = \sum_{j=1}^k a_j e_j$ where $a_j \in \{0, 1\}$. There are a total of 2^k combinations of a_j 's, thus there are 2^k codewords for some $k \leq n$.

Problem 7.12

1. If $d_H(\mathbf{x}, \mathbf{y}) = 0$, then \mathbf{x} and \mathbf{y} agree at all components, hence $\mathbf{x} = \mathbf{y}$, if $\mathbf{x} = \mathbf{y}$ then obviously $d_H(\mathbf{x}, \mathbf{y}) = 0$.

2. $d_H(\mathbf{x}, bmy) = d_H(\mathbf{y}, \mathbf{x})$ is obvious since \mathbf{x} disagrees with \mathbf{y} at the same components that \mathbf{y} disagrees with \mathbf{x} .
3. If for some $1 \leq i \leq n$, $x_i = y_i$ and $y_i = z_i$, then $x_i = z_i$, therefore, the contribution of this component to $d_H(\mathbf{x}, \mathbf{y})$, $d_H(\mathbf{y}, \mathbf{z})$, and $d_H(\mathbf{x}, \mathbf{z})$ is zero. At other components $d_H(x_i, y_i) + d_H(y_i, z_i) \geq 1$, whereas $d_H(x_i, z_i) \leq 1$; i.e., $d_H(x_i, y_i) + d_H(y_i, z_i) \geq d_H(x_i, z_i)$. Adding over all components we conclude $d_H(\mathbf{x}, \mathbf{z}) \leq d_H(\mathbf{x}, \mathbf{y}) + d_H(\mathbf{y}, \mathbf{z})$.

Problem 7.13

- a. Interchanging the first and third rows, we obtain the systematic form :

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

- b.

$$\mathbf{H} = [\mathbf{P}^T | \mathbf{I}_4] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- c. Since we have a (7,3) code, there are $2^3 = 8$ valid codewords, and 2^4 possible syndromes. From these syndromes the all-zero one corresponds to no error, 7 will correspond to single errors and 8

will correspond to double errors (the choice is not unique) :

Error pattern	Syndrome
0 0 0 0 0 0 0	0 0 0 0
0 0 0 0 0 0 1	0 0 0 1
0 0 0 0 0 1 0	0 0 1 0
0 0 0 0 1 0 0	0 1 0 0
0 0 0 1 0 0 0	1 0 0 0
0 0 1 0 0 0 0	1 1 0 1
0 1 0 0 0 0 0	0 1 1 1
1 0 0 0 0 0 0	1 1 1 0
1 0 0 0 0 0 1	1 1 1 1
1 0 0 0 0 1 0	1 1 0 0
1 0 0 0 1 0 0	1 0 1 0
1 0 0 1 0 0 0	0 1 1 0
1 0 1 0 0 0 0	0 0 1 1
1 1 0 0 0 0 0	1 0 0 1
0 1 0 0 0 1 0	0 1 0 1
0 0 0 1 1 0 1	1 0 1 1

d. We note that there are 3 linearly independent columns in \mathbf{H} , hence there is a codeword \mathbf{C}_m with weight $w_m=4$ such that $\mathbf{C}_m\mathbf{H}^T = 0$. Accordingly : $d_{\min} = 4$. This can be also obtained by generating all 8 codewords for this code and checking their minimum weight.

e. 101 generates the codeword : $101 \rightarrow \mathbf{C} = 1010011$. Then : $\mathbf{CH}^T = [0000]$.

Problem 7.14

We have $(n, k) = (n, n - k)$, thus $n - k = k$, hence $n = 2k$ is even and $R = k/n = 1/2$.

Problem 7.15

It is easily verified that each codeword is orthogonal to itself and to all other codewords, hence the code is self dual.

Problem 7.16

$$\mathbf{G}_a = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{G}_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Message \mathbf{X}_m	$\mathbf{C}_{ma} = \mathbf{X}_m \mathbf{G}_a$	$\mathbf{C}_{mb} = \mathbf{X}_m \mathbf{G}_b$
0 0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 1 1	0 0 0 1 0 1 1
0 0 1 0	0 0 1 0 1 1 0	0 0 1 0 1 1 0
0 0 1 1	0 0 1 1 1 0 1	0 0 1 1 1 0 1
0 1 0 0	0 1 0 1 1 0 0	0 1 0 0 1 1 1
0 1 0 1	0 1 0 0 1 1 1	0 1 0 1 1 0 0
0 1 1 0	0 1 1 1 0 1 0	0 1 1 0 0 0 1
0 1 1 1	0 1 1 0 0 0 1	0 1 1 1 0 1 0
1 0 0 0	1 0 1 1 0 0 0	1 0 0 0 1 0 1
1 0 0 1	1 0 1 0 0 1 1	1 0 0 1 1 1 0
1 0 1 0	1 0 0 1 1 1 0	1 0 1 0 0 1 1
1 0 1 1	1 0 0 0 1 0 1	1 0 1 1 0 0 0
1 1 0 0	1 1 1 0 1 0 0	1 1 0 0 0 1 0
1 1 0 1	1 1 1 1 1 1 1	1 1 0 1 0 0 1
1 1 1 0	1 1 0 0 0 1 0	1 1 1 0 1 0 0
1 1 1 1	1 1 0 1 0 0 1	1 1 1 1 1 1 1

As we see, the two generator matrices generate the same set of codewords.

Problem 7.17

The weight distribution of the (7,4) Hamming code is ($n = 7$) :

$$\begin{aligned} A(x) &= \frac{1}{8} [(1+x)^7 + 7(1+x)^3(1-x)^4] \\ &= \frac{1}{8} [8 + 56x^3 + 56x^4 + 8x^7] \\ &= 1 + 7x^3 + 7x^4 + x^7 \end{aligned}$$

Hence, we have 1 codeword of weight zero, 7 codewords of weight 3, 7 codewords of weight 4, and one codeword of weight 7. which agrees with the codewords given in Table 8-1-2.

Problem 7.18

We have $\mathbf{s}_1 = (\sqrt{E_c}, 0)$, $\mathbf{s}_2 = (0, \sqrt{E_c})$ and

$$\Delta = \int \sqrt{p(\mathbf{y}|\mathbf{s}_1)p(\mathbf{y}|\mathbf{s}_2)} d\mathbf{y}$$

Hence

$$\begin{aligned} \Delta &= \int \sqrt{\frac{1}{(\pi N_0)^2} e^{-\frac{1}{N_0}(y_1^2 + (y_1 - \sqrt{E_c})^2 + y_2^2 + (y_2 - \sqrt{E_c})^2)}} dy_1 dy_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi N_0} e^{-\frac{1}{2N_0}(y_1^2 + (y_1 - \sqrt{E_c})^2 + y_2^2 + (y_2 - \sqrt{E_c})^2)} dy_1 dy_2 \\ &= \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{2N_0}(y^2 + (y - \sqrt{E_c})^2)} dy \right)^2 \\ &= \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}((y - \sqrt{E_c}/2)^2 + E_c/4)} dy \right)^2 \\ &= \left(e^{-\frac{E_c}{4N_0}} \right)^2 \\ &= e^{-\frac{E_c}{2N_0}} \end{aligned}$$

Problem 7.19

To find the generator matrix we note that

$$\begin{aligned} \mathbf{G}_0 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ \mathbf{G}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\ \mathbf{G}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_0 \\ \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}$$

Hence,

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To determine the parity check matrix we need to find 5 independent vectors that are orthogonal to the rows of the generator matrix. We can verify that the following vectors satisfy this condition

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and these are exactly the rows of the generator matrix for a Reed-Muller code of the first order and with $n = 16$. Hence the duality of the two codes is shown.

Problem 7.20

The generator matrix for this code is

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$$

and this is equal to \mathbf{G}_0 as given by Equation 7.3-3

Problem 7.21

For M -ary FSK detected coherently, the bandwidth expansion factor is :

$$\left(\frac{W}{R}\right)_{FSK} = \frac{M}{2 \log_2 M}$$

For the Hadamard code : In time T (block transmission time), we want to transmit n bits, so for each bit we have time : $T_b = T/n$. Since for each bit we use binary PSK, the bandwidth requirement is approximately : $W = 1/T_b = n/T$. But $T = k/R$, hence :

$$W = \frac{n}{k} R \Rightarrow \frac{W}{R} = \frac{n}{k}$$

(this is a general result for binary block-encoded signals). For the specific case of a Hadamard code the number of waveforms is $M = 2n$, and also $k = \log_2 M$. Hence :

$$\left(\frac{W}{R}\right)_{Had} = \frac{M}{2 \log_2 M}$$

which is the same as M -ary FSK.

Problem 7.22

From (8-1-47) of the text, the correlation coefficient between the all-zero codeword and the l -th codeword is $\rho_l = 1 - 2w_l/n$, where w_l is the weight of the l -th codeword. For the maximum length shift register codes : $n = 2^m - 1 = M - 1$ (where m is the parameter of the code) and $w_l = 2^{m-1}$ for all codewords except the all-zero codeword. Hence :

$$\rho_l = 1 - \frac{2 \cdot 2^{m-1}}{2^m - 1} = -\frac{1}{2^m - 1} = -\frac{1}{M - 1}$$

for all l . Since the code is linear it follows that $\rho = -1/(M - 1)$ between any pair of codewords.

Note : An alternative way to prove the above is to express each codeword in vector form as

$$s_l = \left[\pm \sqrt{\frac{\mathcal{E}}{n}}, \pm \sqrt{\frac{\mathcal{E}}{n}}, \dots, \pm \sqrt{\frac{\mathcal{E}}{n}} \right] \quad (\text{n elements in all})$$

where $\mathcal{E} = n\mathcal{E}_b$ is the energy per codeword and note that any one codeword differs from each other at exactly 2^{m-1} bits and agrees with the other at $2^{m-1} - 1$ bits. Then the correlation coefficient is :

$$Re[\rho_{mk}] = \frac{s_l \cdot s_k}{|s_l| |s_k|} = \frac{\frac{\mathcal{E}}{n} (2^{m-1} \cdot 1 + (2^{m-1} - 1) \cdot (-1))}{\frac{\mathcal{E}}{n} n} = -\frac{1}{n} = -\frac{1}{M - 1}$$

Problem 7.23

1. The columns of the generator matrix are all sequences of length m , except the all-zero sequence. The codewords are all combinations of these rows. If the all-zero sequence was also included, then when adding a number of rows of the matrix at a certain component i there exists another component j for which the contents are the complements of the contents at i because all possible columns are included in the table. Therefore in any codeword corresponding to a 1, there exist a 0. In other words the number of 1's and 0's are equal. But this is when the all-zero column is added to the matrix, therefore in the original code the number of zeros in any codeword is one less than the number of 1's. This means that each codeword has a weight of 2^{m-1} .
2. From part one we have $2^m - 1$ codewords each with weight 2^{m-1} and one codeword with weight 0. Therefore

$$A(Z) = 1 + (2^m - 1)Z^{2^{m-1}}$$

3. Using MacWilliams identity

$$A(Z) = 2^{-(n-k)}(1+Z)^n A_d\left(\frac{1-Z}{1+Z}\right)$$

with $n = 2^m - 1$, $n - k = m$, and $A_d(Z) = 1 + (2^m - 1)Z^{2^{m-1}}$, we have

$$\begin{aligned} A(Z) &= 2^{-m}(1+Z)^n \left[1 + n(1-Z)^{\frac{n+1}{2}}(1+Z)^{-\frac{n+1}{2}}\right] \\ &= \frac{1}{n+1} \left[(1+Z)^n + n(1-Z)^{\frac{n+1}{2}}(1+Z)^{\frac{n-1}{2}}\right] \end{aligned}$$

Problem 7.24

We know that the (7,4) Huffman code has $d_{\min} = 3$ and weight distribution (Problem 8.3) : $w=0$ (1 codeword), $w=3$ (7 codewords), $w=4$ (7 codewords), $w=7$ (1 codeword).

Hence, for soft-decision decoding (8-1-51) :

$$P_M \leq 7Q\left(\sqrt{\frac{24}{7}}\gamma_b\right) + 7Q\left(\sqrt{\frac{32}{7}}\gamma_b\right) + Q\left(\sqrt{8}\gamma_b\right)$$

or a looser bound (8-1-52) :

$$P_M \leq 15Q\left(\sqrt{\frac{24}{7}}\gamma_b\right)$$

For hard-decision decoding (8-1-82):

$$P_M \leq \sum_{m=2}^7 \binom{7}{m} p^m (1-p)^{7-m} = 1 - \sum_{m=0}^1 \binom{7}{m} p^m (1-p)^{7-m} = 1 - 7p(1-p)^6 - (1-p)^7$$

where $p = Q(\sqrt{2R_c\gamma_b}) = Q\left(\sqrt{\frac{8}{7}\gamma_b}\right)$ or (8-1-90) :

$$P_M \leq 7[4p(1-p)]^{3/2} + 7[4p(1-p)]^2 + [4p(1-p)]^{7/2}$$

or (8-1-91) :

$$P_M \leq 14[4p(1-p)]^{3/2}$$

Problem 7.25

The number of errors is d and the number of components received with no error is $n-d$. Therefore,

$$P(\mathbf{y}|\mathbf{x}) = p^d(1-p)^{n-d} = (1-p)^n \left(\frac{p}{1-p}\right)^d$$

If $p < \frac{1}{2}$, then $p/(1-p) < 1$ and $P(\mathbf{y}|\mathbf{x})$ is a decreasing function of d , hence an ML decoder that maximizes $P(\mathbf{y}|\mathbf{x})$ should minimize d . If $p > \frac{1}{2}$, then an ML decoder should maximize d .

Problem 7.26

Using a symbolic program, the weight distribution polynomial for the (15,11) code is given by

$$1 + 35Z^3 + 105Z^4 + 168Z^5 + 280Z^6 + 435Z^7 + 435Z^8 + 280Z^9 + 168Z^{10} + 105Z^{11} + 35Z^{12} + Z^{15}$$

For decoding error we have

$$P_e = 1 - (1-p)^{15} - 15p(1-p)^{14}$$

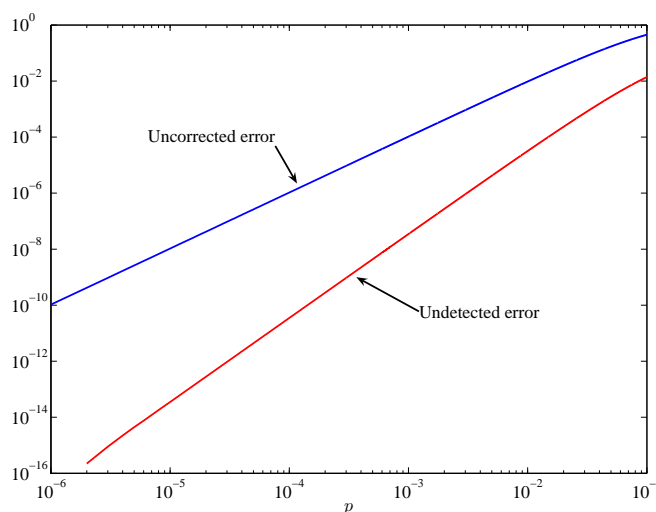
and for undetected error

$$P_u(E) = 2^{-(n-k)}B(1-2p) - (1-p)^n$$

where $B(Z) = 1 + (2^m - 1)Z^{2^{m-1}} = 1 + 15Z^8$. Then

$$P_u(E) = 2^{-4}(1 + 15(1-2p)^8) - (1-p)^{15}$$

Plots of these two error probabilities are given below



Problem 7.27

Using a symbolic program we obtain the wight distribution function

$$\begin{aligned}
& 1 + 651 Z^3 + 9765 Z^4 + 109368 Z^5 + 1057224 Z^6 + 8649279 Z^7 + 60544953 Z^8 + 369776680 Z^9 \\
& + 1996794072 Z^{10} + 9621890019 Z^{11} + 41694856749 Z^{12} + 163568562192 Z^{13} + 584173436400 Z^{14} \\
& + 1908310936455 Z^{15} + 5724932809365 Z^{16} + 15827726179440 Z^{17} + 40448633569680 Z^{18} \\
& + 95799462143175 Z^{19} + 210758816714985 Z^{20} + 431553634502760 Z^{21} + 823875120414360 Z^{22} \\
& + 1468647185710635 Z^{23} + 2447745309517725 Z^{24} + 3818482327223928 Z^{25} + 5580858785942664 Z^{26} \\
& + 7647844002734159 Z^{27} + 9832942289229633 Z^{28} + 11867343566087520 Z^{29} \\
& + 13449656041565856 Z^{30} + 14317376396958243 Z^{31} + 14317376396958243 Z^{32} \\
& + 13449656041565856 Z^{33} + 11867343566087520 Z^{34} + 9832942289229633 Z^{35} + 7647844002734159 Z^{36} \\
& + 5580858785942664 Z^{37} + 3818482327223928 Z^{38} + 2447745309517725 Z^{39} + 1468647185710635 Z^{40} \\
& + 823875120414360 Z^{41} + 431553634502760 Z^{42} + 210758816714985 Z^{43} + 95799462143175 Z^{44} \\
& + 40448633569680 Z^{45} + 15827726179440 Z^{46} + 5724932809365 Z^{47} + 1908310936455 Z^{48} \\
& + 584173436400 Z^{49} + 163568562192 Z^{50} + 41694856749 Z^{51} + 9621890019 Z^{52} + 1996794072 Z^{53} \\
& + 369776680 Z^{54} + 60544953 Z^{55} + 8649279 Z^{56} + 1057224 Z^{57} + 109368 Z^{58} + 9765 Z^{59} + 651 Z^{60} + Z^{63}
\end{aligned}$$

The number of codewords with weight 34 is 11867343566087520.

Problem 7.28

If $\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{c}_j$, then $\mathbf{H}^t(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{H}^t \mathbf{c}_j = \mathbf{0}$, thus $\mathbf{H}^t \mathbf{e}_1 = \mathbf{H}^t \mathbf{e}_2$, i.e., the syndromes are equal.

Problem 7.29

Let the coset leader of the row of the standard array be \mathbf{e} , then the two elements can be denoted as $\mathbf{e} + \mathbf{c}_i$ and $\mathbf{e} + \mathbf{c}_j$ for some i and j . The sum of these two is $\mathbf{c}_i + \mathbf{c}_j$ which is another codeword since the code is linear.

Problem 7.30

1. Assume two elements of the form $\mathbf{e}_i + \mathbf{c}_j$ and $\mathbf{e}_l + \mathbf{c}_m$ are equal. Then

$$\mathbf{e}_i + \mathbf{c}_j = \mathbf{e}_l + \mathbf{c}_m \Rightarrow \mathbf{e}_i = \mathbf{e}_l + \mathbf{c}_m + \mathbf{c}_j = \mathbf{e}_l + \mathbf{c}_p$$

where $\mathbf{c}_p = \mathbf{c}_m + \mathbf{c}_j$. This shows that \mathbf{e}_i and \mathbf{e}_l belong to the same coset and both cannot be coset leaders.

2. Let $\mathbf{e}_i + \mathbf{c}_j$ and $\mathbf{e}_l + \mathbf{c}_m$, where $\mathbf{e}_i \neq \mathbf{e}_l$ have the same syndrome, then

$$\mathbf{H}^t(\mathbf{e}_i + \mathbf{c}_j) = \mathbf{H}^t(\mathbf{e}_l + \mathbf{c}_m) \Rightarrow \mathbf{H}^t(\mathbf{e}_i + \mathbf{e}_l) = \mathbf{H}^t(\mathbf{c}_j + \mathbf{c}_m) = \mathbf{0}$$

Hence $\mathbf{H}^t \mathbf{e}_i = \mathbf{H}^t \mathbf{e}_l$.

Problem 7.31

Since all codewords of the new code have odd parity, the all-zero sequence is not a codeword and hence the resulting code is not linear. Student B's argument is not valid since for nonlinear codes the minimum Hamming distance and the minimum weight of the code are not the same. Although the minimum weight of the code is 1 but the minimum Hamming distance of it is equal to 2.

Problem 7.32

1. The generator matrix is

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

2. $M = 2^k = 16$ codewords and any three columns of \mathbf{H} are linearly independent but we can find four columns that are linearly dependent. Hence $d_{\min} = 3$.
3. The coding gain is $R_c d_{\min} = \frac{1}{2} \times 3 = 1.5$, roughly equal to 1.8 dB.
4. $e_c = \lfloor (d_{\min} - 1)/2 \rfloor = 1$.
5. It is easy to verify that each row of \mathbf{G} is orthogonal to itself and all other rows. Hence any linear combination of the rows is also orthogonal to any linear combination of them.

Problem 7.33

1. This is a nonlinear code since the all-zero sequence is not a codeword.
2. There are $\binom{6}{3} = 20$ codewords, hence $R = \frac{1}{6} \log_2 20 \approx 0.72$.
3. Since all codewords have weight equal to 3, no two codewords can have a distance of 1. On the other hand 111000 and 011100 have a distance of 2, hence $d_{\min} = 2$ and the code can detect one error.
4. The probability of an undetected error is the probability of receiving another codeword. If 111000 is transmitted there exist 9 codewords that are at distance 2 from it and 9 codewords that are at distance 4 from it. There is a single codeword that is at distance 6 from it. The probability of receiving these codewords is the probability of an undetected error, hence

$$P_u = 9p^2(1-p)^4 + 9p^4(1-p)^2 + p^6$$

5. \mathcal{C}_1 must contain all codewords of \mathcal{C} and all linear combinations of them. Since for each codeword in the original code there is another codeword at distance 2 from it, all sequence of weight 2 have to be codewords. With similar reasoning all sequences of weight 4 and the sequence of weight 6 are also codewords. From the inclusion of sequences of weight 2 and 3, we conclude that all sequences of weights 1 and 5 should also be included. Hence \mathcal{C}_1 must be the set of all possible sequences of length 6.

Problem 7.34

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then the standard array is :

000	001	010	011	100	101	110	111
000000	001101	010011	011110	100110	101011	110101	111000
000001	001100	010010	011111	100111	101010	110100	111001
000010	001111	010001	011100	100100	101001	110111	111010
000100	001001	010111	011010	100010	101111	110001	111100
001000	000101	011011	010110	101110	100011	111101	110000
010000	011101	000011	001110	110110	111011	100101	101000
100000	101101	110011	111110	000110	001011	010101	011000
100001	101100	110010	111111	000111	001010	010100	011001

For each column, the first row is the message, the second row is the correct codeword corresponding to this message, and the rest of the rows correspond to the received words which are the sum of the valid codeword plus the corresponding error pattern (coset leader). The error patterns that this code can correct are given in the first column (all-zero codeword), and the corresponding syndromes are :

E_i	$S_i = E_i H^T$
000000	000
000001	001
000010	010
000100	100
001000	101
010000	011
100000	110
100001	111

We note that this code can correct all single errors and one two-bit error pattern.

Problem 7.35

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, the standard array will be :

000	001	010	011	100	101	110	111
0000000	0010111	0101110	0111001	1001011	1011100	1100101	1110010
0000001	0010110	0101111	0111000	1001010	1011101	1100100	1110011
0000010	0010101	0101101	0111011	1001001	1011110	1100111	1110000
0000100	0010011	0101010	0111101	1001111	1011000	1100001	1110110
0001000	0011111	0100110	0110001	1000011	1010100	1101101	1111010
0010000	0000111	0111110	0101001	1011011	1001100	1110101	1100010
0100000	0110111	0001110	0011001	1101011	1111100	1000101	1010010
1000000	1010111	1101110	1111001	0001011	0011100	0100101	0110010
1100000	1110111	1001110	1011001	0101011	0111100	0000101	0010010
1010000	1000111	1111110	1101001	0011011	0001100	0110101	0100010
1001000	1011111	1100110	1110001	0000011	0010100	0101101	0111010
1000100	1010011	1101010	1111101	0001111	0011000	0100001	0110110
1000010	1010101	1101100	1111010	0001001	0011110	0100111	0110000
1000001	1010110	1101111	1111001	0001010	0011101	0100100	0110011
0010001	0000110	0111111	0101001	1011010	1001101	1110100	1100011
0001101	0011010	0100011	0110101	1000110	1010001	1101000	1111111

For each column, the first row is the message, the second row is the correct codeword corresponding to this message, and the rest of the rows correspond to the received words which are the sum of the valid codeword plus the corresponding error pattern (coset leader). The error patterns that this code can correct are given in the first column (all-zero codeword), and the corresponding syndromes

are :

E_i	$S_i = E_i H^T$
0000000	0000
0000001	0001
0000010	0010
0000100	0100
0001000	1000
0010000	0111
0100000	1110
1000000	1011
1100000	0101
1010000	11000
1001000	0011
1000100	1111
1000010	1001
1000001	1010
0010001	0110
0001101	1101

We note that this code can correct all single errors, seven two-bit error patterns, and one three-bit error pattern.

Problem 7.36

1. The parity check equations are $c_1 + c_2 + c_4 = 0$, $c_2 + c_3 + c_5 = 0$, and $c_1 + c_3 + c_6 = 0$. These are obtained from $\mathbf{cH}^t = 0$, where $\mathbf{H} = [\mathbf{P}^t \mathbf{I}]$. From the equations we have

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{G} = [\mathbf{I} \ \mathbf{P}]$,

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

2.

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

3. No two columns of \mathbf{H} are equal, but columns 2, 4, and 5 are linearly dependent. Therefore the maximum number of linearly independent columns which is equal to the minimum distance of the code is 2.

4. $e_c = \lfloor (d_{\min} - 1)/2 \rfloor = 0$.

5. We need to find a codeword at minimum distance from 100000. An obvious choice is 000000.

Problem 7.37

1. Since the rows of \mathbf{G} all have weight four, no linear combination of them can have odd weight. Hence there exist no codeword of weight 1, on the other hand, 000011 is a codeword of weight 2, hence $d_{\min} = 2$. The coding gain is $R_c d_{\min} = \frac{1}{2} \times 2 = 1$ or zero dB.

2. We need to design a LBC with d_{\min} at least three. One example is

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

3. One suggestion is

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Problem 7.38

Since the code is MDS, then $d_{\min} = n - k + 1$ and hence any $n - k$ columns of \mathbf{H} , the parity check matrix of \mathcal{C} , are linearly independent. But \mathbf{H} is the generator matrix of \mathcal{C}^\perp . Let us assume that \mathcal{C}^\perp is not MDS. Since this is an $(n, n - k)$ code, it must have a codeword of weight less than or equal to $n - (n - k) = k$. Let \mathbf{c} be such a codeword of \mathcal{C}^\perp , i.e., let $w(\mathbf{c}) \leq k$. This means that \mathbf{c} at least $n - k$ zeros. But \mathbf{c} is a linear combination of the rows of its generator matrix, i.e., \mathbf{H} . This means that there exists a singular sub-matrix of \mathbf{H} of size $(n - k) \times (n - k)$. This means that the column rank of \mathbf{H} is less than $n - k$ thus contradicting our initial conclusion that $n - k$ columns of \mathbf{H} are linearly independent.

Problem 7.39

1. Since $\binom{n}{i} = \binom{n}{n-i}$, we have

$$\sum_{i=0}^t \binom{n}{i} = \sum_{i=n-t}^n \binom{n}{i}$$

and since $\lambda > 0$ we have $\lambda(n-t) \leq \lambda i$ for all $i \geq n-t$. Therefore

$$\begin{aligned} 2^{\lambda(n-t)} \sum_{i=0}^t \binom{n}{i} &= 2^{\lambda(n-t)} \sum_{i=n-t}^n \binom{n}{i} \\ &\leq \sum_{i=n-t}^n 2^{\lambda i} \binom{n}{i} \\ &\leq \sum_{i=0}^n 2^{\lambda i} \binom{n}{i} = (1 + 2^\lambda)^n \end{aligned}$$

2. From part 1 we have

$$\begin{aligned} \sum_{i=0}^t \binom{n}{i} &\leq (1 + 2^\lambda)^n 2^{-\lambda n(1-t/n)} \\ &= \left(2^{-\lambda(1-p)} + 2^{\lambda p} \right)^n \end{aligned}$$

3. Substitution of $\lambda = \log_2(1-p) - \log_2 p$ in the result of part 2 gives

$$2^{-\lambda(1-p)} + 2^{\lambda p} = p^{1-p}(1-p)^{p-1} + (1-p)^p p^{-p} = p^{-p}(1-p)^{-(1-p)} = 2^{H_b(p)}$$

Hence

$$\sum_{i=0}^t \binom{n}{i} \leq 2^{nH_b(p)}$$

4. First we note that as long as $t < n/2$, the dominating term in the sum is $\binom{n}{t}$. This can be simply shown by showing that for large n we have $\binom{n}{t-1}/\binom{n}{t} \approx t/(t-1)(n-t+1)$ which tends to zero as n tends to infinity. Therefore, for large n ,

$$\sum_{i=0}^t \binom{n}{i} \approx \binom{n}{t}$$

Denoting $q = 1 - p$, we have $t = np$ and $n - t = nq$ and

$$\begin{aligned} \frac{1}{n} \log \binom{n}{t} &\approx \frac{1}{n} \left[\frac{1}{2} \log(2\pi n) + n \log n - n - \frac{1}{2} \log(2\pi np) - np \log(np) + np - \right. \\ &\quad \left. \frac{1}{2} \log(2\pi nq) - nq \log(nq) + nq \right] \\ &\approx -p \log p - q \log q \end{aligned}$$

where we have used the fact that for large n , $\frac{1}{n} \log n$ goes to zero. From above we conclude that

$$\sum_{i=0}^t \binom{n}{t} \approx 2^{nH_b(p)} = 2^{nH_b(t/n)}$$

Problem 7.40

1. Since no all-zero column exists in \mathbf{G} , for each position $1 \leq i \leq n$ there exists at least one codeword that is nonzero at that position, therefore no column of the \mathbf{C} is all-zero. Let $1 \leq i \leq n$, and define

$$C_1 = \{\text{all codewords for which } c_i = 1\}$$

$$C_0 = \{\text{all codewords for which } c_i = 0\}$$

Let $\mathbf{c} \in C_1$, adding it to each $\mathbf{d} \in C_1$ gives $\mathbf{c} + \mathbf{d} \in C_0$, hence $|C_1| \leq |C_0|$. Adding \mathbf{c} to each $\mathbf{d} \in C_0$ gives $\mathbf{c} + \mathbf{d} \in C_1$, hence $|C_0| \leq |C_1|$. From $|C_1| \leq |C_0|$ and $|C_0| \leq |C_1|$, we have $|C_0| = |C_1|$. Since $|C_0| + |C_1| = 2^k$, therefore $|C_0| = |C_1| = 2^{k-1}$.

2. The total weight of the codewords is the weight of \mathbf{C} . Since each column has weight 2^{k-1} , the total weight is $n2^{k-1}$.
3. There are $2^k - 1$ codewords with weight at least d_{\min} and a single codeword of weight zero. Therefore the total weight is at least $(2^k - 1)d_{\min}$. From this we have $(2^k - 1)d_{\min} \leq n2^{k-1}$ or

$$d_{\min} \leq \frac{n2^{k-1}}{2^k - 1}$$

Problem 7.41

We have already generated an extended (8,4) code from the (7,4) Hamming code in Probl. 8.5. Since the generator matrix for the (7,4) Hamming code is not unique, in this problem we will construct the extended code, starting from the generator matrix given in 8-1-7 :

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then :

$$H_e = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We can bring this parity matrix into systematic form by adding rows 1,2,3 into the fourth row :

$$H_{es} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then :

$$G_{e,s} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Problem 7.42

a. The generator polynomial for the (15,11) Hamming code is given as $g(p) = p^4 + p + 1$. We will express the powers p^l as : $p^l = Q_l(p)g(p) + R_l(p)$ $l = 4, 5, \dots, 14$, and the polynomial $R_l(p)$ will give the parity matrix \mathbf{P} , so that \mathbf{G} will be $\mathbf{G} = [\mathbf{I}_{11}|\mathbf{P}]$. We have :

$$\begin{aligned} p^4 &= g(p) + p + 1 \\ p^5 &= pg(p) + p^2 + p \\ p^6 &= p^2g(p) + p^3 + p^2 \\ p^7 &= (p^3 + 1)g(p) + p^3 + p + 1 \\ p^8 &= (p^4 + p + 1)g(p) + p^2 + 1 \\ p^9 &= (p^5 + p^2 + p)g(p) + p^3 + p \\ p^{10} &= (p^6 + p^3 + p^2 + 1)g(p) + p^2 + p + 1 \\ p^{11} &= (p^7 + p^4 + p^3 + p)g(p) + p^3 + p^2 + p \\ p^{12} &= (p^8 + p^5 + p^4 + p^2 + 1)g(p) + p^3 + p^2 + p + 1 \\ p^{13} &= (p^9 + p^6 + p^5 + p^3 + p + 1)g(p) + p^3 + p^2 + 1 \\ p^{14} &= (p^{10} + p^7 + p^6 + p^4 + p^2 + p + 1)g(p) + p^3 + 1 \end{aligned}$$

Using $R_l(p)$ (with $l = 4$ corresponding to the last row of \mathbf{G} ,... $l = 14$ corresponding to the first

row) for the parity matrix \mathbf{P} we obtain :

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

b. In order to obtain the generator polynomial for the dual code, we first factor $p^{15}+1$ into : $p^{15}+1 = g(p)h(p)$ to obtain the parity polynomial $h(p) = (p^{15}+1)/g(p) = p^{11}+p^8+p^7+p^5+p^3+p^2+p+1$. Then, the generator polynomial for the dual code is given by :

$$p^{11}h(p^{-1}) = 1 + p^3 + p^4 + p^6 + p^8 + p^9 + p^{10} + p^{11}$$

Problem 7.43

We can determine \mathbf{G} , in a systematic form, from the generator polynomial $g(p) = p^3 + p^2 + 1$:

$$\begin{aligned} p^6 &= (p^3 + p^2 + p)g(p) + p^2 + p \\ p^5 &= (p^2 + p + 1)g(p) + p + 1 \\ p^4 &= (p + 1)g(p) + p^2 + p + 1 \\ p^3 &= g(p) + p^2 + 1 \end{aligned} \quad \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the parity check matrix for the extended code will be (according to 8-1-15) :

$$\mathbf{H}_e = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and in systematic form (we add rows 1,2,3 to the last one) :

$$\mathbf{H}_{es} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{G}_{es} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Note that \mathbf{G}_{es} can be obtained from the generator matrix \mathbf{G} for the initial code, by adding an overall parity check bit. The code words for the extended systematic code are :

Message \mathbf{X}_m	Codeword \mathbf{C}_m
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 1 0 1 1
0 0 1 0	0 0 1 0 1 1 1 0
0 0 1 1	0 0 1 1 0 1 0 1
0 1 0 0	0 1 0 0 0 1 1 1
0 1 0 1	0 1 0 1 1 1 0 0
0 1 1 0	0 1 1 0 1 0 0 1
0 1 1 1	0 1 1 1 0 0 1 0
1 0 0 0	1 0 0 0 1 1 0 1
1 0 0 1	1 0 0 1 0 1 1 0
1 0 1 0	1 0 1 0 0 0 1 1
1 0 1 1	1 0 1 1 1 0 0 0
1 1 0 0	1 1 0 0 1 0 1 0
1 1 0 1	1 1 0 1 0 0 0 1
1 1 1 0	1 1 1 0 0 1 0 0
1 1 1 1	1 1 1 1 1 1 1 1

An alternative way to obtain the codewords for the extended code is to add an additional check bit to the codewords of the initial (7,4) code which are given in Table 8-1-2. As we see, the minimum weight is 4 and hence : $d_{\min} = 4$.

Problem 7.44

a. We have obtained the generator matrix \mathbf{G} for the (15,11) Hamming code in the solution of Problem 8.4. The shortened code will have a generator matrix \mathbf{G}_s obtained by \mathbf{G} , by dropping its

first 7 rows and the first 7 columns or :

$$\mathbf{G}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Then the possible messages and the codewords corresponding to them will be :

Message \mathbf{X}_m	Codeword \mathbf{C}_m
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 0 1 1
0 0 1 0	0 0 1 0 0 1 1 0
0 0 1 1	0 0 1 1 0 1 0 1
0 1 0 0	0 1 0 0 1 1 0 0
0 1 0 1	0 1 0 1 1 1 1 1
0 1 1 0	0 1 1 0 1 0 1 0
0 1 1 1	0 1 1 1 1 0 0 1
1 0 0 0	1 0 0 0 1 0 1 1
1 0 0 1	1 0 0 1 1 0 0 0
1 0 1 0	1 0 1 0 1 1 0 1
1 0 1 1	1 0 1 1 1 0 1 0
1 1 0 0	1 1 0 0 0 1 1 1
1 1 0 1	1 1 0 1 0 1 0 0
1 1 1 0	1 1 1 0 0 0 0 1
1 1 1 1	1 1 1 1 0 0 1 0

b. As we see the minimum weight and hence the minimum distance is 3 : $d_{\min} = 3$.

Problem 7.45

a.

$$g(p) = (p^4 + p^3 + p^2 + p + 1)(p^4 + p + 1)(p^2 + p + 1) = p^{10} + p^8 + p^5 + p^4 + p^2 + p + 1$$

Factoring p^l , $l = 14, \dots, 10$, into $p^l = g(p)Q_l(p) + R_l(p)$ we obtain the generator matrix in systematic form :

$$\left. \begin{aligned} p^{14} &= (p^4 + p^2 + 1)g(p) + p^9 + p^7 + p^4 + p^3 + p + 1 \\ p^{13} &= (p^3 + p)g(p) + p^9 + p^8 + p^7 + p^6 + p^4 + p^2 + p \\ p^{12} &= (p^2 + 1)g(p) + p^8 + p^7 + p^6 + p^5 + p^3 + p + 1 \\ p^{11} &= pg(p) + p^9 + p^6 + p^5 + p^3 + p^2 + p \\ p^{10} &= g(p) + p^8 + p^5 + p^4 + p^2 + p + 1 \end{aligned} \right\} \Rightarrow$$

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The codewords are obtained from the equation : $\mathbf{C}_m = \mathbf{X}_m \mathbf{G}$, where \mathbf{X}_m is the row vector containing the five message bits.

b.

$$d_{\min} = 7$$

c. The error-correcting capability of the code is :

$$t = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = 3$$

d. The error-detecting capability of the code is : $d_{\min} - 1 = 6$.

e.

$$g(p) = (p^{15} + 1)/(p^2 + p + 1) = p^{13} + p^{12} + p^{10} + p^9 + p^7 + p^6 + p^4 + p^3 + p + 1$$

Then :

$$\begin{aligned} p^{14} &= (p + 1)g(p) + p^{12} + p^{11} + p^9 + p^8 + p^6 + p^5 + p^3 + p^2 + 1 \\ p^{13} &= g(p) + p^{12} + p^{10} + p^9 + p^7 + p^6 + p^4 + p^3 + p + 1 \end{aligned}$$

Hence, the generator matrix is :

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and the valid codewords :

\mathbf{X}_m	Codeword \mathbf{C}_m
0 0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 1	0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
1 0	1 0 1 1 0 1 1 0 1 1 0 1 1 0 1
1 1	1 1 0 1 1 0 1 1 0 1 1 0 1 1 0

The minimum distance is : $d_{\min} = 10$

Problem 7.46

The polynomial $p^7 + 1$ is factors as follows : $p^7 + 1 = (p + 1)(p^3 + p^2 + 1)(p^3 + p + 1)$. The generator polynomials for the matrices $\mathbf{G}_1, \mathbf{G}_2$ are : $g_1(p) = p^3 + p^2 + 1$, $g_2(p) = p^3 + p + 1$. Hence the parity polynomials are : $h_1(p) = (p^7 + 1)/g_1(p) = p^4 + p^3 + p^2 + 1$, $h_2(p) = (p^7 + 1)/g_2(p) = p^4 + p^2 + p + 1$. The generator polynomials for the matrices $\mathbf{H}_1, \mathbf{H}_2$ are : $p^4 h_1(p^{-1}) = 1 + p + p^2 + p^4$, $p^4 h_2(p^{-1}) = 1 + p^2 + p^3 + p^4$. The rows of the matrices $\mathbf{H}_1, \mathbf{H}_2$ are given by : $p^i p^4 h_{1/2}(p^{-1})$, $i = 0, 1, 2$, so :

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Problem 7.47

The generator matrix for the systematic (7,4) cyclic Hamming code is given by (8-1-37) as :

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then, the correctable error patterns E_i with the corresponding syndrome $S_i = E_i H^T$ are :

S_i	E_i
000	0000000
001	0000001
010	0000010
011	0001000
100	0000100
101	1000000
110	0100000
111	0010000

Problem 7.48

a. Let $g(p) = p^8 + p^6 + p^4 + p^2 + 1$ be the generator polynomial of an (n, k) cyclic code. Then, $n - k = 8$ and the rate of the code is

$$R = \frac{k}{n} = 1 - \frac{8}{n}$$

The rate R is minimum when $\frac{8}{n}$ is maximum subject to the constraint that R is positive. Thus, the first choice of n is $n = 9$. However, the generator polynomial $g(p)$ does not divide $p^9 + 1$ and therefore, it can not generate a $(9, 1)$ cyclic code. The next candidate value of n is 10. In this case

$$p^{10} + 1 = g(p)(p^2 + 1)$$

and therefore, $n = 10$ is a valid choice. The rate of the code is $R = \frac{k}{n} = \frac{2}{10} = \frac{1}{5}$.

b. In the next table we list the four codewords of the $(10, 2)$ cyclic code generated by $g(p)$.

Input	$X(p)$	Codeword
00	0	0000000000
01	1	0101010101
10	p	1010101010
11	$p + 1$	1111111111

As it is observed from the table, the minimum weight of the code is 5 and since the code is linear $d_{\min} = w_{\min} = 5$.

c. The coding gain of the $(10, 2)$ cyclic code in part (a) is

$$G_{\text{coding}} = d_{\min} R = 5 \times \frac{2}{10} = 1$$

Problem 7.49

a. For every n

$$p^n + 1 = (p + 1)(p^{n-1} + p^{n-2} + \cdots + p + 1)$$

where additions are modulo 2. Since $p + 1$ divides $p^n + 1$ it can generate a (n, k) cyclic code, where $k = n - 1$.

b. The i^{th} row of the generator matrix has the form

$$\mathbf{g}_i = [0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0 \quad p_{i,1}]$$

where the 1 corresponds to the i -th column (to give a systematic code) and the $p_{i,1}$, $i = 1, \dots, n-1$, can be found by solving the equations

$$p^{n-i} + p_{i,1} = p^{n-i} \bmod p + 1, \quad 1 \leq i \leq n - 1$$

Since $p^{n-i} \bmod p + 1 = 1$ for every i , the generator and the parity check matrix are given by

$$\mathbf{G} = \left(\begin{array}{ccc|c} 1 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \end{array} \right), \quad \mathbf{H} = [1 \quad 1 \quad \cdots \quad 1 \quad | \quad 1]$$

c. A vector $\mathbf{c} = [c_1, c_2, \dots, c_n]$ is a codeword of the $(n, n-1)$ cyclic code if it satisfies the condition $\mathbf{c}\mathbf{H}^t = 0$. But,

$$\mathbf{c}\mathbf{H}^t = 0 = \mathbf{c} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = c_1 + c_2 + \cdots + c_n$$

Thus, the vector \mathbf{c} belongs to the code if it has an even weight. Therefore, the cyclic code generated by the polynomial $p + 1$ is a simple parity check code.

Problem 7.50

a. The generator polynomial of degree $4 = n - k$ should divide the polynomial $p^6 + 1$. Since the polynomial $p^6 + 1$ assumes the factorization

$$p^6 + 1 = (p + 1)^3(p + 1)^3 = (p + 1)(p + 1)(p^2 + p + 1)(p^2 + p + 1)$$

we find that the shortest possible generator polynomial of degree 4 is

$$g(p) = p^4 + p^2 + 1$$

The i^{th} row of the generator matrix \mathbf{G} has the form

$$\mathbf{g}_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p_{i,1} & \cdots & p_{i,4} \end{bmatrix}$$

where the 1 corresponds to the i -th column (to give a systematic code) and the $p_{i,1}, \dots, p_{i,4}$ are obtained from the relation

$$p^{6-i} + p_{i,1}p^3 + p_{i,2}p^2p_{i,3}p + p_{i,4} = p^{6-i} \pmod{p^4 + p^2 + 1}$$

Hence,

$$\begin{aligned} p^5 \pmod{p^4 + p^2 + 1} &= (p^2 + 1)p \pmod{p^4 + p^2 + 1} = p^3 + p \\ p^4 \pmod{p^4 + p^2 + 1} &= p^2 + 1 \pmod{p^4 + p^2 + 1} = p^2 + 1 \end{aligned}$$

and therefore,

$$\mathbf{G} = \left(\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right)$$

The codewords of the code are

$$\begin{aligned} \mathbf{c}_1 &= [0 \ 0 \ 0 \ 0 \ 0 \ 0] \\ \mathbf{c}_2 &= [1 \ 0 \ 1 \ 0 \ 1 \ 0] \\ \mathbf{c}_3 &= [0 \ 1 \ 0 \ 1 \ 0 \ 1] \\ \mathbf{c}_4 &= [1 \ 1 \ 1 \ 1 \ 1 \ 1] \end{aligned}$$

b. The minimum distance of the linear $(6, 2)$ cyclic code is $d_{\min} = w_{\min} = 3$. Therefore, the code can correct

$$e_c = \frac{d_{\min} - 1}{2} = 1 \text{ error}$$

Problem 7.51

1. In general \mathcal{C}_{\max} is not a cyclic code. The codewords of \mathcal{C}_1 are of the form $a(X)g_1(X)$ and the codewords of \mathcal{C}_2 are of the form $b(X)g_2(X)$. If one of $g_1(X)$ and $g_2(X)$ divides the other, it will also divide the set of codewords of \mathcal{C}_{\max} , hence the code will be cyclic. But in general this is not true.
2. \mathcal{C}_{\min} is in general a cyclic code. Let $g(X) = \text{GCD}\{g_1(X), g_2(X)\}$ since both $g_1(X)$ and $g_2(X)$ divide each codeword of \mathcal{C}_{\min} , $g(X)$ also divides each codeword of \mathcal{C}_{\min} . Since $g_1(X)$ and $g_2(X)$ divide $X^n + 1$, $g(X)$ also divides $X^n + 1$. This completes the proof that \mathcal{C}_{\min} is cyclic with the generator polynomial of $g(X)$. Since \mathcal{C}_{\min} contains the codewords that are common to \mathcal{C}_1 and \mathcal{C}_2 , its minimum weight is at least equal to the minimum weight of each code, i.e., $d_{\min} \geq \max\{d_1, d_2\}$.

Problem 7.52

1. No cyclic (5, 3) code can exist since $X^5 + 1 + (X + 1)(X^4 + X^3 + X^2 + X + 1)$ has no factor of degree 2.
2. We have

$$X^{10} + 1 = (X + 1)^2(X^4 + X^3 + X^2 + X + 1)^2$$

There is only one factor of degree eight, which can serve as the generator polynomial. Therefore,

$$g(X) = X^8 + X^6 + X^4 + X^2 + 1$$

3. There are only four codewords in this code. The nonzero codewords are obtained by multiplying $g(X)$ by 1, X , and $X + 1$. Checking these codewords shows that the minimum weight is 5.
4. $e_c = \lfloor (d_{\min} - 1)/2 \rfloor = 2$.
5. We use the bound given by Equation 7.5-17. The resulting code has two codewords of weight 5 and a codeword of weight 10 in addition to the all-zero codeword, therefore $A(Z) = 1 + 2Z^5 + Z^{10}$. We have $\gamma_b = 3 \text{ dB} = 2$, hence $p = Q(\sqrt{2\gamma_b}) = Q(2) = 0.02275$. From Equation 7.5-17 we obtain

$$P_e \leq (A(Z) - 1) \Big|_{Z=\sqrt{4p(1-p)}} \approx 0.0047$$

Problem 7.53

From Table 7.9-1, we see that the factors of $X^{23} + 1$ correspond to 3, 6165, and 5343. The corresponding polynomials are $X + 1$, $X^{11} + X^{10} + X^6 + X^5 + X^4 + X^2 + 1$, and $X^{11} + X^9 + X^7 + X^6 + X^5 + X + 1$. Hence,

$$X^{23} + 1 = (X + 1)(X^{11} + X^{10} + X^6 + X^5 + X^4 + X^2 + 1)(X^{11} + X^9 + X^7 + X^6 + X^5 + X + 1)$$

The possible generator polynomials (including the degenerate ones) are

$$\begin{aligned}
 g_0(X) &= 1 \\
 g_1(X) &= X + 1 \\
 g_2(X) &= X^{11} + X^{10} + X^6 + X^5 + X^4 + X^2 + 1 \\
 g_3(X) &= X^{11} + X^9 + X^7 + X^6 + X^5 + X + 1 \\
 g_4(X) &= (X + 1)(X^{11} + X^{10} + X^6 + X^5 + X^4 + X^2 + 1) \\
 g_5(X) &= (X + 1)(X^{11} + X^9 + X^7 + X^6 + X^5 + X + 1) \\
 g_6(X) &= (X^{11} + X^{10} + X^6 + X^5 + X^4 + X^2 + 1)(X^{11} + X^9 + X^7 + X^6 + X^5 + X + 1) \\
 g_7(X) &= X^{23} + 1
 \end{aligned}$$

For the (23, 12) Golay the generator polynomial is $g_3(X)$. (Compare with Equation 7.9-22)

Problem 7.54

We have $e(X) = a(X)g(X) + s(X)$ and $e^{(1)}(X) = Xe(X) + e_{n-1}(X^n + 1)$. From these relations

$$e^{(1)}(X) = X[a(X)g(X) + s(X)] + e_{n-1}g(X)h(X) = (Xa(X) + e_{n-1}h(X))g(X) + Xs(X)$$

Since the syndrome of $e^{(1)}(X)$ is the remainder of its division by $g(X)$, we conclude that

$$s^{(1)}(X) = Xs(X) \mod g(X)$$

Problem 7.55

The statement is false. Consider the (8, 5) code with $g(X) = X^3 + X^2 + X + 1$. The codeword corresponding to $(X + 1)g(X) = X^4 + 1$, i.e., the minimum weight of this code is 2 which is less than the degree of $g(X)$.

Problem 7.56

From Problem 7.8, $\phi_\alpha(X) = X^5 + X^2 + 1$ and $\phi_{\alpha^3}(X) = X^5 + X^4 + X^3 + X^2 + 1$. Therefore,

$$g(X) = \text{LCM}\{X^5 + X^2 + 1, X^5 + X^4 + X^3 + X^2 + 1\} = X^{10} + X^9 + X^8 + X^6 + X^5 + X^3 + 1$$

From above, $n - k = 31 - k = 10$ and $k = 21$. The rate of the code is $R = 21/31$. The same result can be obtained by using Table 7.10-1 from which 3551 is the octal representation of the generator polynomial.

Problem 7.57

We have $r(X) = 1 + X^3 + X^6 + X^9 + X^{10}$ and

$$\begin{aligned}
 S_1 &= r(\alpha) = 1 + \alpha^3 + \alpha^6 + \alpha^9 + \alpha^{10} = \alpha^3 \\
 S_2 &= r(\alpha^2) = 1 + \alpha^6 + \alpha^{12} + \alpha^{18} + \alpha^{20} = \alpha^6 \\
 S_3 &= r(\alpha^3) = 1 + \alpha^9 + \alpha^{18} + \alpha^{27} + \alpha^{30} = 1 \\
 S_4 &= r(\alpha^4) = 1 + \alpha^{12} + \alpha^{24} + \alpha^5 + \alpha^9 = \alpha^{12}
 \end{aligned}$$

and we have the following table

μ	$\sigma^{(\mu)}(X)$	d_μ	l_μ	$\mu - l_\mu$
-1	1	1	0	-1
0	1	α^3	0	0
1	$1 + \alpha^3 X$	0	1	0
2	$1 + \alpha^3 X$	α^{16}	1	1
3	$1 + \alpha^3 X + \alpha^{13} X^2$	0	2	1
4	$1 + \alpha^3 X + \alpha^{13} X^2$		2	2

Hence, $\sigma(X) = 1 + \alpha^3 X + \alpha^{13} X^2$. This polynomial has two roots, α^{23} and α^{26} which are the reciprocals of error location numbers. The error location numbers are $\beta_1 = \alpha^8$ and $\beta_2 = \alpha^5$, which give the error polynomial of $e(X) = X^5 + X^8$. Therefore, $c(X) = r(X) + e(X) = 1 + X^3 + X^5 + X^6 + X^8 + X^9 + X^{10}$.

Problem 7.58

Here $r(X) = 1 + X^3 + X^5 + X^6 + X^8 + X^9 + X^{10} + X^{28} + X^{29} + X^{30}$. We have

$$S_1 = 1 + \alpha^3 + \alpha^5 + \alpha^6 + \alpha^8 + \alpha^9 + \alpha^{10} + \alpha^{28} + \alpha^{29} + \alpha^{30} = \alpha^8$$

$$S_2 = r(\alpha^2) = S_1^2 = \alpha^{16}$$

$$S_3 = r(\alpha^3) = \alpha^9$$

$$S_4 = r(\alpha^4) = S_2^2 = \alpha$$

and we have the following table

μ	$\sigma^{(\mu)}(X)$	d_μ	l_μ	$\mu - l_\mu$
-1	1	1	0	-1
0	1	α^8	0	0
1	$1 + \alpha^8 X$	0	1	0
2	$1 + \alpha^8 X$	α^2	1	1
3	$1 + \alpha^8 X + \alpha^{25} X^2$	0	2	1
4	$1 + \alpha^8 X + \alpha^{25} X^2$		2	2

resulting in $\sigma(X) = 1 + \alpha^8 X + \alpha^{25} X^2$. This polynomial has no root in $\text{GF}(32)$, therefore the errors in the received sequence cannot be corrected.

Problem 7.59

From Table 8-1-6 we find that the coefficients of the generator polynomial for the (15,7) BCH code are $721 \rightarrow 111010001$ or $g(p) = p^8 + p^7 + p^6 + p^4 + 1$. Then, we can determine the l -th row of the generator matrix \mathbf{G} , using the modulo $R_l(p) : p^{n-l} = Q_l(p)g(p) + R_l(p)$, $l = 1, 2, \dots, 7$. Since

the generator matrix of the shortened code is obtained by removing the first three rows of \mathbf{G} , we perform the above calculations for $l = 4, 5, 6, 7$, only :

$$\begin{aligned} p^{11} &= (p^3 + p^2 + 1)g(p) + p^4 + p^3 + p^2 + 1 \\ p^{10} &= (p^2 + p)g(p) + p^7 + p^6 + p^5 + p^2 + p \\ p^9 &= (p + 1)g(p) + p^6 + p^5 + p^4 + p + 1 \\ p^8 &= (p + 1)g(p) + p^7 + p^6 + p^4 + 1 \end{aligned}$$

Hence :

$$\mathbf{G}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 7.60

Since $N = 2^m - 1 = 7$, we have a code over $\text{GF}(8)$. The generator polynomial is given by

$$g(X) = (X + \alpha)(X + \alpha^2)(X + \alpha^3)(X + \alpha^4) = X^4 + \alpha^3 X^3 + X^2 + \alpha X + \alpha^3$$

where α is a primitive element in $\text{GF}(8)$ and we have used Table 7.1-4 to expand the polynomial. From above we have $N - K = 4$, hence $K = 3$ and $R = 3/7$.

Problem 7.61

With $N = 63 = 2^m - 1$ we have $m = 6$ and the code is over $\text{GF}(2^6)$. The primitive polynomial generating this field is obtained from Table 7.1-5 to be $X^6 + X + 1$, based on which we have to generate the field table. If α is a primitive element in this field we have

$$g(X) = (X + \alpha)(X + \alpha^2)(X + \alpha^3)(X + \alpha^4)(X + \alpha^5)(X + \alpha^6)$$

which simplifies to $g(X) = X^6 + \alpha^{59} X^5 + \alpha^{48} X^4 + \alpha^{43} X^3 + \alpha^{55} X^2 + \alpha^{10} X + \alpha^{21}$. With $N - K = 6$, we have $K = 57$ and there are a total of 64^{57} , equal to

$$8958978968711216842229769122273777112486581988938598139599956403855167484720643781523509973086428463104$$

or roughly 8.96×10^{102} codewords in this code.

Problem 7.62

From Equation 7.11-4 we have

$$A_i = \binom{N}{i} N \sum_{j=0}^{i-D_{\min}} (-1)^j \binom{i-1}{j} (N+1)^{i-j-D_{\min}}, \quad \text{for } D_{\min} \leq i \leq N$$

where in this case $N = 7$ and $D_{\min} = 2t + 1 = 5$. Substituting, we obtain

$$\begin{aligned} A_0 &= 1 \\ A_5 &= 147 \\ A_6 &= 147 \\ A_7 &= 217 \end{aligned}$$

and the weight distribution function is

$$A(Z) = 1 + 147Z^5 + 147Z^6 + 217Z^7$$

Problem 7.63

Let the $n_2 \times n_1$ matrix of Figure 7.13-1 be denoted by \mathbf{C} . Then $c_{ij} = u_{ij}$ for $1 \leq i \leq k_2$ and $1 \leq j \leq k_1$, where u_{ij} denotes the information sequence. Let \mathbf{G}_1 and \mathbf{G}_2 denote the generator matrices of the row and column codes, respectively, then

$$\mathbf{G}_1 = [\mathbf{I}_{k_1} \quad \mathbf{P}^{(1)}] \quad \mathbf{G}_2 = [\mathbf{I}_{k_2} \quad \mathbf{P}^{(2)}]$$

For $1 \leq i \leq k_2$ and $k_1 + 1 \leq j \leq n_1$, using the row code generator matrix we have

$$c_{ij} = \sum_{l=1}^{k_1} c_{il} P_{lj}^{(1)}$$

and for $k_2 + 1 \leq i \leq n_2$ and $1 \leq j \leq k_1$ using the column code we have

$$c_{ij} = \sum_{m=1}^{k_2} c_{mj} P_{mi}^{(2)}$$

For $k_2 + 1 \leq i \leq n_2$ and $k_1 \leq j \leq n_1$, if we use the row code we have

$$c_{ij} = \sum_{l=1}^{k_1} c_{il} P_{lj}^{(1)} = \sum_{l=1}^{k_1} P_{lj}^{(1)} \sum_{m=1}^{k_2} c_{ml} P_{mi}^{(2)}$$

and if we use the column code we obtain

$$c_{ij} = \sum_{m=1}^{k_2} c_{mj} P_{mi}^{(2)} = \sum_{m=1}^{k_2} P_{mi}^{(2)} \sum_{l=1}^{k_1} c_{ml} P_{lj}^{(1)}$$

It is clear from the above expressions that the two c_{ij} 's are equal.

Problem 7.64

We first prove that there exists no nonzero code word with weight less than $d_1 d_2$ and then show that there exists at least one codeword with weight $d_1 d_2$. Let \mathbf{c} be a nonzero codeword, and let $c_{ij} = 1$. Since the minimum distance of the row code is d_1 , there must be at least d_1 ones in the i th row of the matrix. In each column j such that $c_{ij} = 1$, there must be at least d_2 ones, hence the weight of the codeword is at least $d_1 d_2$, i.e., $d \geq d_1 d_2$. Now we prove that a codeword exists

with weight d_1d_2 . Let the $bmc^{(1)}$ be a codeword of length n_1 and weight d_1 for the row code and $\mathbf{c}^{(2)}$ be a codeword of length n_2 and weight d_2 for the column code. Assume we pick d_2 copies of $\mathbf{c}^{(1)}$ and locate them at the d_2 row locations for which $c_i^{(2)} = 1$, $1 \leq i \leq n_2$ and fill the rest of the code matrix by zeros. The weight of the resulting matrix will be d_1d_2 and since each row of it is either an all zero sequence or $bmc^{(1)}$ and each column of it is either the all-zero sequence or $\mathbf{c}^{(2)}$, the matrix satisfies the row and column code restrictions and thus is a valid codeword of the product code. This proves that $d = d_1d_2$.

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 8) ¹

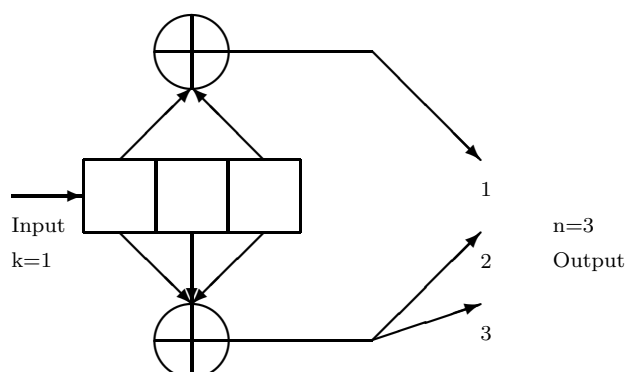
Prepared by
Kostas Stamatiou

January 30, 2008

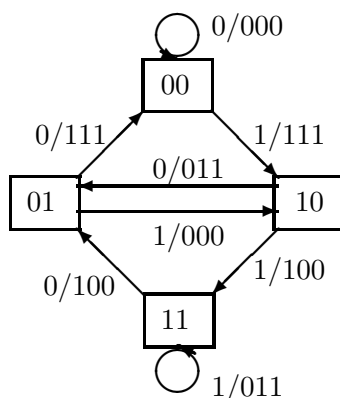
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Problem 8.1

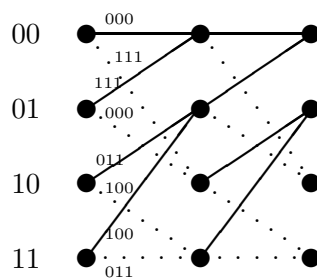
(a) The encoder for the $(3,1)$ convolutional code is depicted in the next figure.



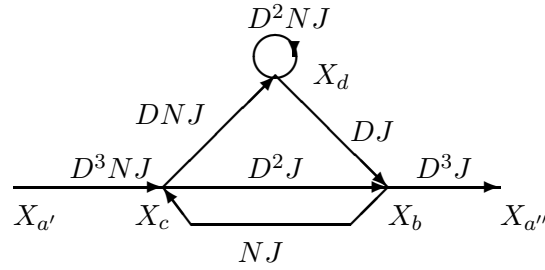
(b) The state transition diagram for this code is depicted in the next figure.



(c) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



(d) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned}
 X_c &= D^3NJX_{a'} + NJX_b \\
 X_b &= D^2JX_c + DJX_d \\
 X_d &= DNJX_c + D^2NJX_d \\
 X_{a''} &= D^3JX_b
 \end{aligned}$$

Eliminating X_b , X_c and X_d results in

$$T(D, N, J) = \frac{X_{a''}}{X_{a'}} = \frac{D^8NJ^3(1 + NJ - D^2NJ)}{1 - D^2NJ(1 + NJ^2 + J - D^2J^2)}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

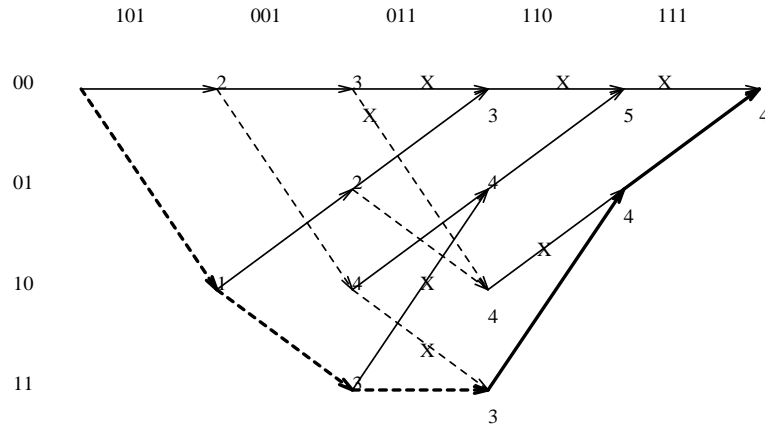
$$T_1(D) = T(D, N, J)|_{N=J=1} = \frac{D^8(1 - 2D^2)}{1 - D^2(3 - D^2)} = D^8 + 2D^{10} + \dots$$

Hence, $d_{\text{free}} = 8$

(e) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 8.2

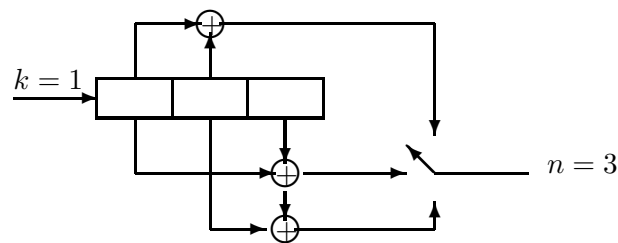
The code of Problem 8-1 is a $(3,1)$ convolutional code with $K = 3$. The length of the received sequence \mathbf{y} is 15. This means that 5 symbols have been transmitted, and since we assume that the information sequence has been padded by two 0's, the actual length of the information sequence is 3. The following figure depicts 5 frames of the trellis used by the Viterbi decoder. The numbers on the nodes denote the metric (Hamming distance) of the survivor paths (the non-survivor paths are shown with an X). In the case of a tie of two merging paths at a node, we have purged the upper path.



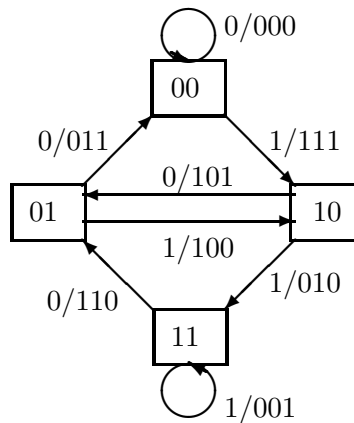
The decoded sequence is $\{111, 100, 011, 100, 111\}$ (i.e the path with the minimum final metric - heavy line) and corresponds to the information sequence $\{1, 1, 1\}$ followed by two zeros.

Problem 8.3

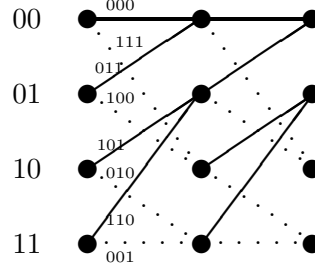
(a) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



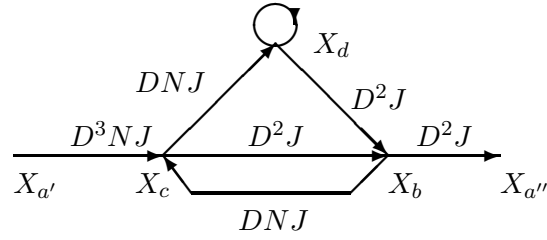
(b) The state transition diagram for this code is shown below



(c) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



(d) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= D^2JX_c + D^2JX_d \\ X_d &= DNJX_c + DNJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b , X_c and X_d results in

$$T(D, N, J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

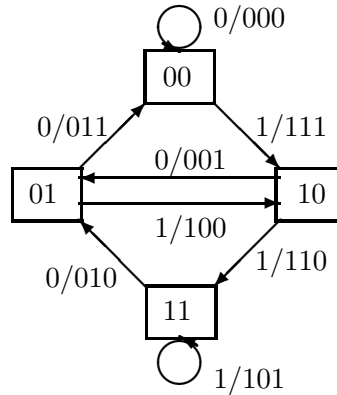
$$T_1(D) = T(D, N, J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \dots$$

Hence, $d_{\text{free}} = 7$

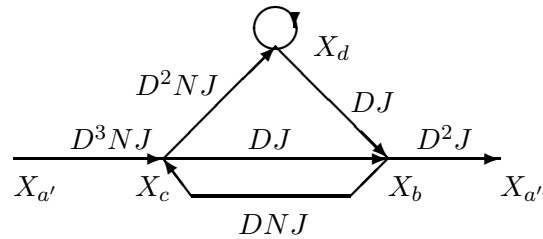
(e) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 8.4

(a) The state transition diagram for this code is depicted in the next figure.



(b) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= DJX_c + DJX_d \\ X_d &= D^2NJX_c + D^2NJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b , X_c and X_d results in

$$T(D, N, J) = \frac{X_{a''}}{X_{a'}} = \frac{D^6NJ^3}{1 - D^2NJ - D^2NJ^2}$$

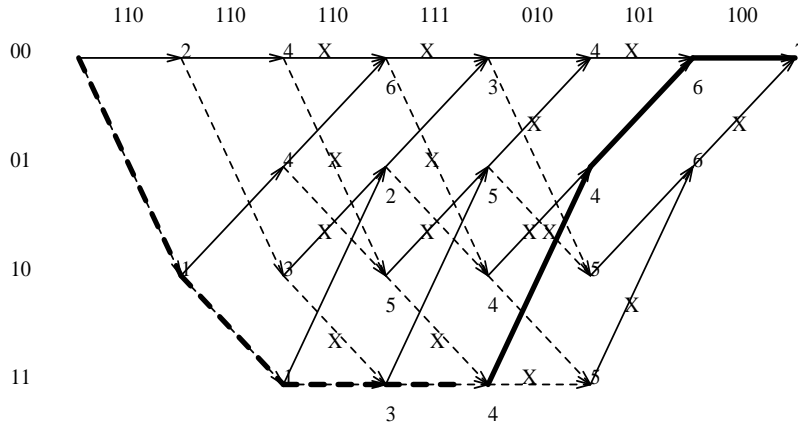
(c) To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D, N, J)|_{N=J=1} = \frac{D^6}{1 - 2D^2} = D^6 + 2D^8 + 4D^{10} + \dots$$

Hence, $d_{\text{free}} = 6$

(d) The following figure shows 7 frames of the trellis diagram used by the Viterbi decoder. It is assumed that the input sequence is padded by two zeros, so that the actual length of the information

sequence is 5. The numbers on the nodes indicate the Hamming distance of the survivor paths. The deleted branches have been marked with an X. In the case of a tie we deleted the upper branch. The survivor path at the end of the decoding is denoted by a thick line.



The information sequence is 11110 and the corresponding codeword 111 110 101 101 010 011 000...

(e) An upper to the bit error probability of the code is given by

$$P_b \leq \left. \frac{dT(D, N, J=1)}{dN} \right|_{N=1, D=\sqrt{4p(1-p)}}$$

But

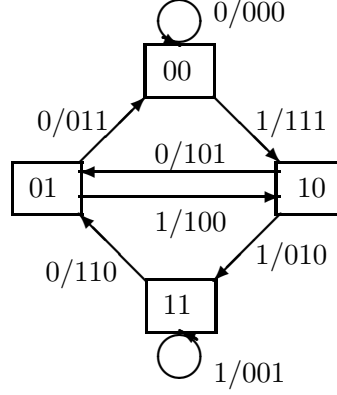
$$\frac{dT(D, N, 1)}{dN} = \frac{d}{dN} \left[\frac{D^6 N}{1 - 2D^2 N} \right] = \frac{D^6 - 2D^8(1 - N)}{(1 - 2D^2 N)^2}$$

and since $p = 10^{-5}$, we obtain

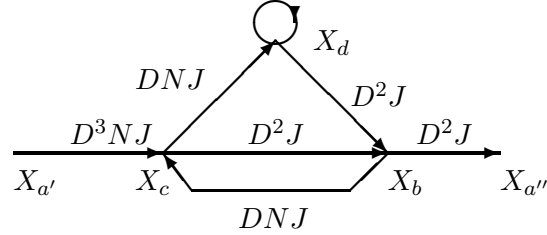
$$P_b \leq \left. \frac{D^6}{(1 - 2D^2)^2} \right|_{D=\sqrt{4p(1-p)}} \approx 6.14 \cdot 10^{-14}$$

Problem 8.5

(a) The state transition diagram for this code is shown below



(b) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= D^2JX_c + D^2JX_d \\ X_d &= DNJX_c + DNJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b , X_c and X_d results in

$$T(D, N, J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

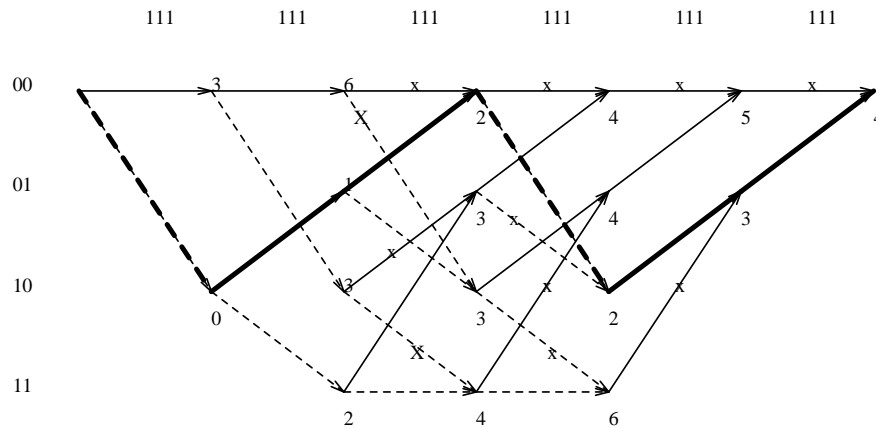
(c) To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D, N, J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \dots$$

Hence, $d_{\text{free}} = 7$. The path, which is at a distance d_{free} from the all zero path, is the path $X_a \rightarrow X_c \rightarrow X_b \rightarrow X_a$.

(d) The following figure shows 6 frames of the trellis diagram used by the Viterbi algorithm to decode the sequence $\{111, 111, 111, 111, 111, 111\}$. The numbers on the nodes indicate the Hamming distance of the survivor paths from the received sequence. The branches that are dropped by the

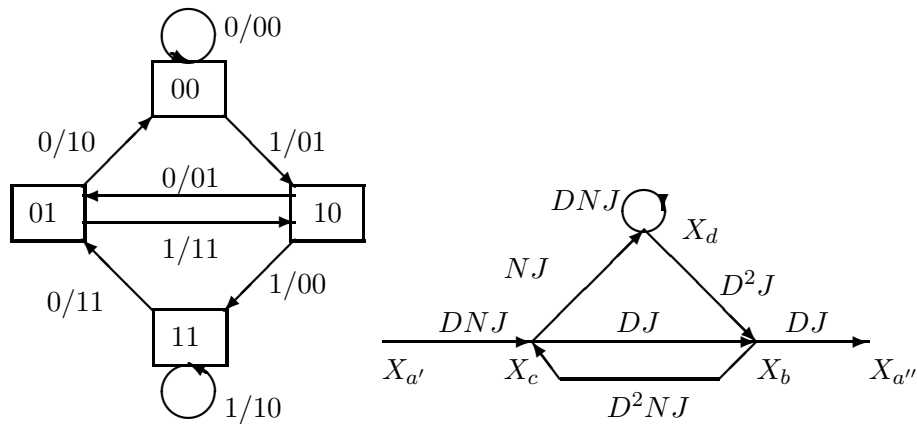
Viterbi algorithm have been marked with an X. In the case of a tie of two merging paths, we delete the upper path.



The decoded sequence is $\{111, 101, 011, 111, 101, 011\}$ which corresponds to the information sequence $\{x_1, x_2, x_3, x_4\} = \{1, 0, 0, 1\}$ followed by two zeros.

Problem 8.6

(a) The state transition diagram and the flow diagram used to find the transfer function for this code are depicted in the next figure.



Thus,

$$\begin{aligned} X_c &= DNJX_{a'} + D^2NJX_b \\ X_b &= DJX_c + D^2JX_d \\ X_d &= NJX_c + DNJX_d \\ X_{a''} &= DJX_b \end{aligned}$$

and by eliminating X_b , X_c and X_d , we obtain

$$T(D, N, J) = \frac{X_{a''}}{X_{a'}} = \frac{D^3 N J^3}{1 - DNJ - D^3 N J^2}$$

To find the transfer function of the code in the form $T(D, N)$, we set $J = 1$ in $T(D, N, J)$. Hence,

$$T(D, N) = \frac{D^3 N}{1 - DN - D^3 N}$$

(b) To find the free distance of the code we set $N = 1$ in the transfer function $T(D, N)$, so that

$$T_1(D) = T(D, N)|_{N=1} = \frac{D^3}{1 - D - D^3} = D^3 + D^4 + D^5 + 2D^6 + \dots$$

Hence, $d_{\text{free}} = 3$

(c) An upper bound on the bit error probability, when hard decision decoding is used, is given by (see (8-2-34))

$$P_b \leq \frac{1}{k} \left. \frac{dT(D, N)}{dN} \right|_{N=1, D=\sqrt{4p(1-p)}}$$

Since

$$\left. \frac{dT(D, N)}{dN} \right|_{N=1} = \frac{d}{dN} \frac{D^3 N}{1 - (D + D^3)N} \Big|_{N=1} = \frac{D^3}{(1 - (D + D^3))^2}$$

with $k = 1$, $p = 10^{-6}$ we obtain

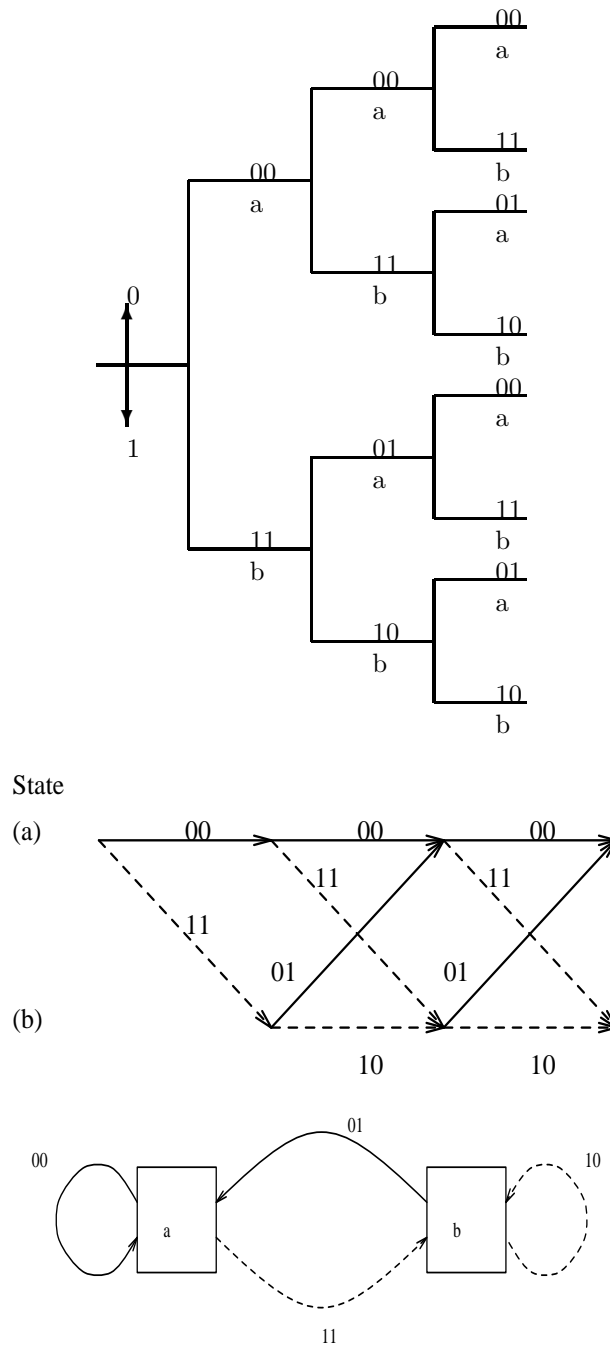
$$P_b \leq \left. \frac{D^3}{(1 - (D + D^3))^2} \right|_{D=\sqrt{4p(1-p)}} = 8.0321 \times 10^{-9}$$

Problem 8.7

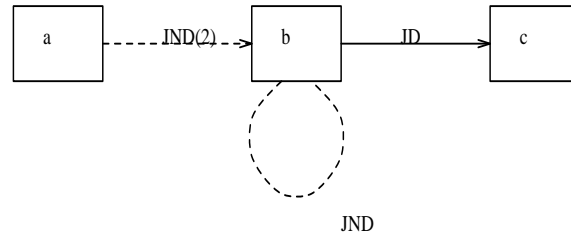
(a)

$$g_1 = [10], \quad g_2 = [11], \quad \text{states : (a) = [0], (b) = [1]}$$

The tree diagram, trellis diagram and state diagram are given in the following figures :



(b) Redrawing the state diagram :



$$X_b = JND^2 X_a + JND X_b \Rightarrow X_b = \frac{JND^2}{1 - JND} X_a$$

$$X_c = JD X_b \Rightarrow \frac{X_c}{X_a} = T(D, N, J) = \frac{J^2 ND^3}{1 - JND} = J^2 ND^3 + J^3 N^2 D^4 + \dots$$

Hence :

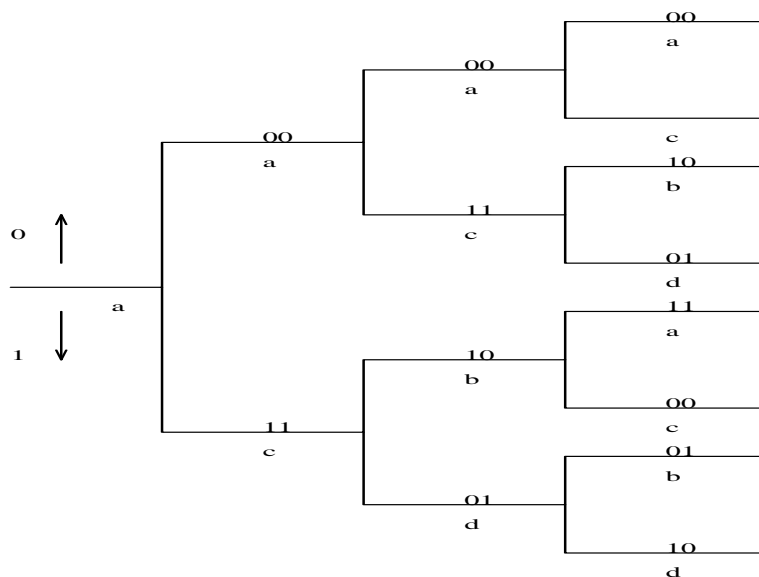
$$d_{\min} = 3$$

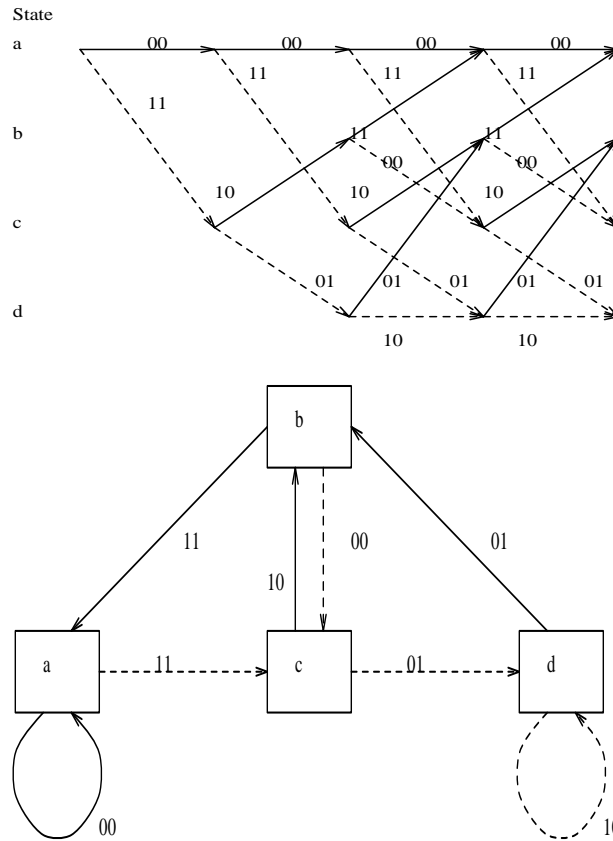
Problem 8.8

(a)

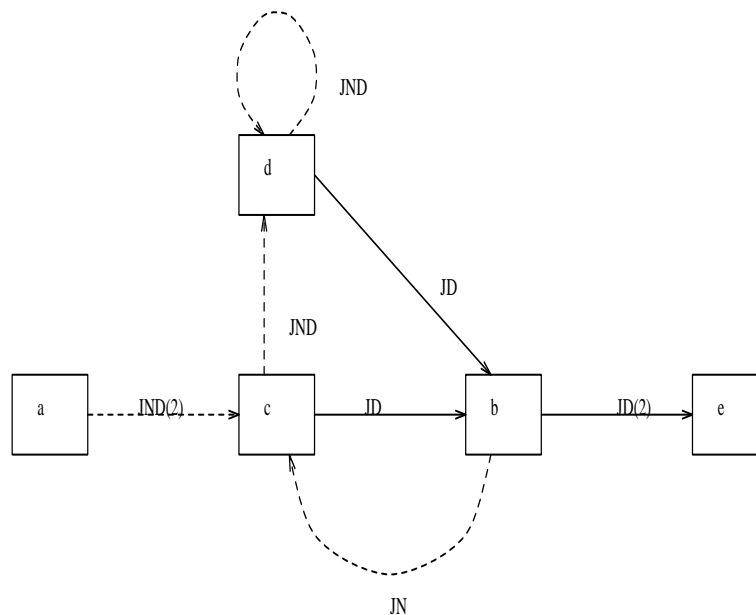
$$g_1 = [111], g_2 = [101], \text{ states : } (a) = [00], (b) = [01], (c) = [10], (d) = [11]$$

The tree diagram, trellis diagram and state diagram are given in the following figures :





(b) Redrawing the state diagram :



$$\left\{ \begin{array}{l} X_c = JND^2X_a + JNX_b \\ X_b = JDX_c + JDX_d \\ X_d = JNDX_d + JNDX_c = NX_b \end{array} \right\} \Rightarrow X_b = \frac{J^2ND^3}{1 - JND(1 + J)}X_a$$

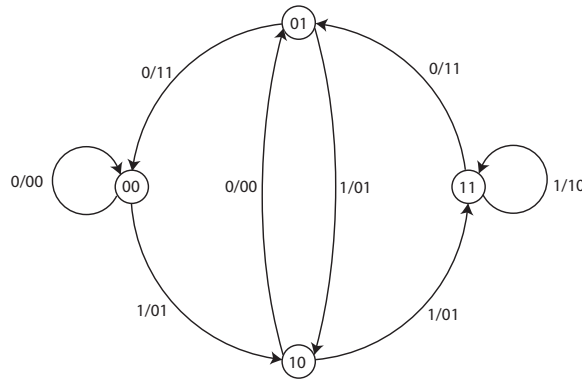
$$X_e = JD^2X_b \Rightarrow \frac{X_e}{X_a} = T(D, N, J) = \frac{J^3ND^5}{1 - JND(1 + J)} = J^3ND^5 + J^4N^2D^6(1 + J) + \dots$$

Hence :

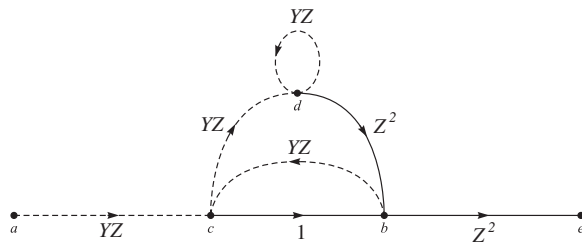
$$d_{\min} = 5$$

Problem 8.9

1. The state transition diagram is shown below



2. The transition diagram is shown below



and the Equations are

$$X_c = YZX_a + YZX_b$$

$$X_b = X_c + Z^2X_d$$

$$X_d = YZX_c + YZX_d$$

$$X_e = Z^2X_b$$

Eliminating X_b , X_c , and X_d , we obtain

$$T(Y, Z) = \frac{X_e}{X_a} = \frac{YZ^3 - Y^2Z^4 + YZ^6}{1 - 2YZ + Y^2Z^2 - Y^2Z^4}$$

3. The code is not catastrophic since in the state transition diagram there are no loops of wight zero corresponding to inputs of nonzero weight.
4. From expansion of $T(Y, Z)$ we observe that the lowest power of Z is 3, hence $d_{\text{free}} = 3$. This is also observed from the state transition diagram.
5. We use Equations 8.2-16–8.2-19 to find a bound on the bit error probability. First we note that $T(Y, Z) = YZ^3 + Y^2Z^4 + Y^3Z^5 + \dots$, hence $a_3 = a_4 = a_5 = 1$ and

$$P_2(3) = \sum_{k=2}^3 \binom{3}{k} p^k (1-p)^{3-k} \approx 3p^2$$

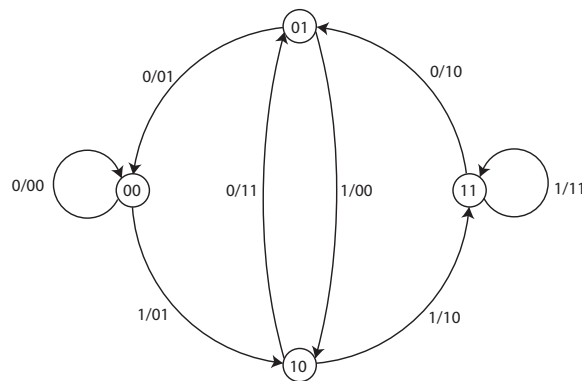
$$P_2(4) = \sum_{k=3}^4 \binom{4}{k} p^k (1-p)^{4-k} + \frac{1}{2} \binom{4}{2} p^2 (1-p)^2 \approx 3p^2$$

Hence

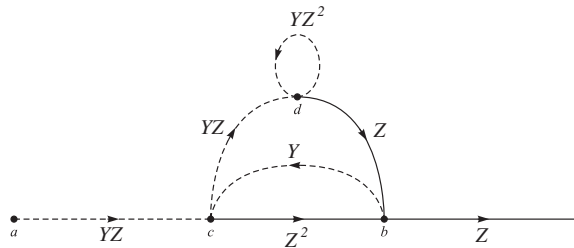
$$P_e \lesssim 6p^2 = 6 \times 10^{-6}$$

Problem 8.10

1. The state transition diagram is shown below



2. Non catastrophic since in the state transition diagram there are no loops of wight zero corresponding to inputs of nonzero weight.
3. The transition diagram is shown below



and the Equations are

$$\begin{aligned} X_c &= YZX_a + YX_b \\ X_b &= Z^2X_c + ZX_d \\ X_d &= YZX_c + YZ^2X_d \\ X_e &= ZX_b \end{aligned}$$

Eliminating X_b , X_c , and X_d , we obtain

$$T(Y, Z) = \frac{X_e}{X_a} = \frac{YZ^4 + Y^2Z^4 - Y^2Z^6}{1 - 2YZ^2 - Y^2Z^2 + Y^2Z^4}$$

4. Expanding $T(Y, Z)$, we have

$$T(Y, Z) = YZ^4 + Y^2Z^4 + 3Y^3Z^6 + Y^4Z^6 + \dots$$

Hence, $d_{\text{free}} = 4$.

5. We use Equations 8.2-16–8.2-19 to find a bound on the bit error probability. First we note that $T(Z) = 2Z^4 + 4Z^6 + \dots$, hence $a_4 = 2$, $a_6 = 3$ and

$$P_2(4) = \sum_{k=3}^4 \binom{4}{k} p^k (1-p)^{4-k} + \frac{1}{2} \binom{4}{2} p^2 (1-p)^2 \approx 3p^2$$

Hence

$$P_e \lesssim a_4 \times P_2(4) = 6 \times 10^{-6}$$

Problem 8.11

1. We have

$$p(\mathbf{r}|\mathbf{s}) = p(\mathbf{n} = \mathbf{r} - \mathbf{s}) \propto e^{-\sum_j |r_j - c_j|}$$

where \propto denotes proportionality. This shows that $p(\mathbf{r}|\mathbf{s})$ is maximized when $\sum_j |r_j - c_j|$ is minimized, thus giving the optimal decoding rule.

2. Since soft decision decoding is employed, we use the bound given by Equation 8.2.15 with Δ given by Equation 8.2.10. From 8.2.10 we have

$$\begin{aligned}
 \Delta &= \int_{-\infty}^{\infty} \sqrt{p(r|c = \sqrt{E_c})p(r|c = -\sqrt{E_c})} dr \\
 &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{4}e^{-|r-\sqrt{E_c}|}e^{-|r+\sqrt{E_c}|}} dr \\
 &= \int_{-\infty}^{-\sqrt{E_c}} \sqrt{\frac{1}{4}e^{r-\sqrt{E_c}}e^{r+\sqrt{E_c}}} dr \\
 &\quad + \int_{-\sqrt{E_c}}^{\sqrt{E_c}} \sqrt{\frac{1}{4}e^{r-\sqrt{E_c}}e^{-r-\sqrt{E_c}}} dr \\
 &\quad + \int_{\sqrt{E_c}}^{\infty} \sqrt{\frac{1}{4}e^{-r+\sqrt{E_c}}e^{-r-\sqrt{E_c}}} dr \\
 &= e^{-\sqrt{E_c}} (1 + \sqrt{E_c})
 \end{aligned}$$

The transfer function for this convolutional code was derived in the solution to Problem 8.10 and is given by

$$T(Y, Z) = T(Y, Z) = YZ^4 + Y^2Z^4 + 3Y^3Z^6 + Y^4Z^6 + \dots$$

Therefore,

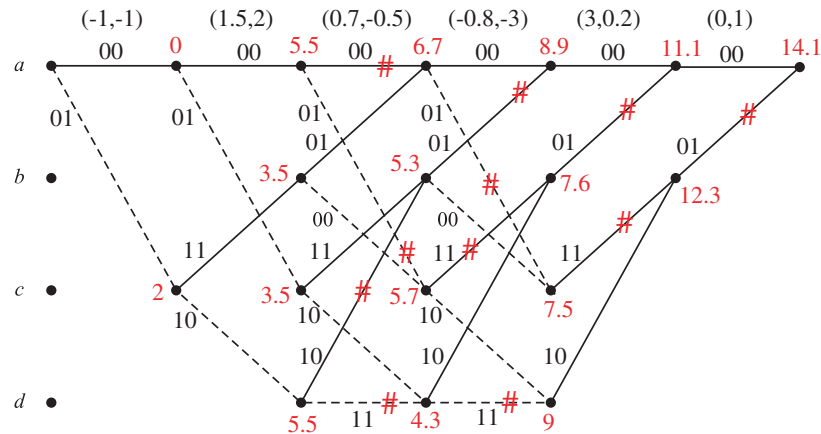
$$\left. \frac{1}{k} \frac{\partial}{\partial Y} T(Y, Z) \right|_{Y=1} = 3Z^4 + 13Z^6 + \dots$$

Substituting $Z = \Delta$, we obtain

$$\begin{aligned}
 P_b &\leq 3\Delta^4 + 13\Delta^6 + \dots \\
 &\approx 2.9415
 \end{aligned}$$

when we put $E_c = 1$. This bound is obviously useless since at such a low SNR, the Bhat-acharyya union bound is very loose.

3. Since the length of the received sequence is 12 and $n = 2$, we need a trellis of depth 6 which is shown below

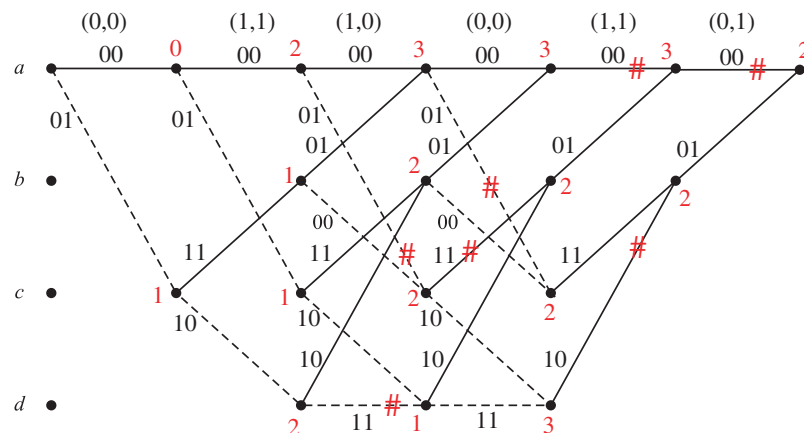


Note that on the trellis each 0 corresponds to $-\sqrt{E_c} = -1$ and each 1 corresponds to $\sqrt{E_c} = 1$. The accumulated metrics are shown in red on each node. The # denotes a path that is not a survivor. From the trellis it is clear that the optimal path corresponds to the information sequence 100000. Since the last two zeros indicate the two zeros padded to the original information sequence we conclude that the transmitted sequence is 1000.

4. After hard decision decoding we have

$$\mathbf{y} = (0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1)$$

The trellis and the metrics for hard decision decoding are shown below



and the decoded sequence is either 110100 or 010100, thus the information sequence can be either 1101 or 0101.

5. For hard decision decoding we have

$$\begin{aligned}
 p &= P(r > 0 | c = -1) \\
 &= \int_0^{\infty} \frac{1}{2} e^{-|r+1|} \\
 &= \frac{1}{2} e^{-1} \approx 0.184
 \end{aligned}$$

and from Equation 8.2-14 we have $\Delta = \sqrt{4p(1-p)} = 0.775$. If we use this new Δ in the error bound expression of part 3 we obtain

$$P_b \lesssim 3\Delta^4 + 13\Delta^6 \approx 3.9$$

This bound is also useless similar to the bound in part 3.

Problem 8.12

1. Here $k = 2$, $n = 3$, and $K = 2$, hence the number of states is $2^{k(K-1)} = 2^2 = 4$.

2. This is similar to Example 8.1-5 and the resulting state diagram is shown below

4. The code is not catastrophic since there exists no loop of nonzero weight corresponding to an input sequence of weight zero.
5. For $p = 10^{-4}$ we have $\Delta = \sqrt{4p(1-p)} \approx 0.02$, and

$$\left. \frac{1}{k} \frac{\partial}{\partial Y} T(Y, Z) \right|_{Y=1} = \frac{1}{2} Z^2 + 4Z^3 + 7Z^4 + \dots$$

resulting in

$$P_b \lesssim 0.000233$$

Problem 8.13

1. For this code we have

$$\mathbf{g}_1 = [1 \ 0 \ 0 \ 0] \quad \mathbf{g}_2 = [0 \ 1 \ 0 \ 0] \quad \mathbf{g}_3 = [1 \ 1 \ 1 \ 0]$$

Therefore,

$$\begin{aligned} \mathbf{g}_1^{(1)} &= [1 \ 0] & \mathbf{g}_2^{(1)} &= [0 \ 0] \\ \mathbf{g}_1^{(2)} &= [0 \ 0] & \mathbf{g}_2^{(2)} &= [1 \ 0] \\ \mathbf{g}_1^{(3)} &= [1 \ 1] & \mathbf{g}_2^{(3)} &= [1 \ 0] \end{aligned}$$

resulting in

$$\mathbf{G}(D) = \begin{bmatrix} 1 & 0 & 1+D \\ 0 & 1 & 1 \end{bmatrix}$$

2. From \mathbf{u} we have $\mathbf{u}^{(1)} = (10110)$ and $\mathbf{u}^{(2)} = (01101)$, hence

$$\begin{aligned} u^{(1)}(D) &= 1 + D^2 + D^3 \\ u^{(2)}(D) &= D + D^2 + D^4 \end{aligned}$$

and

$$\mathbf{u}(D) = \begin{bmatrix} 1 + D^2 + D^3 & D + D^2 + D^4 \end{bmatrix}$$

and

$$\mathbf{c}(D) = \mathbf{u}(D)\mathbf{G}(D) = \begin{bmatrix} 1 + D^2 + D^3 & D + D^2 + D^4 & 1 \end{bmatrix}$$

Hence,

$$c(D) = c^{(1)}(D^3) + Dc^{(2)}(D^3) + D^2c^{(3)}(D^3) = 1 + D^2 + D^4 + D^6 + D^7 + D^9 + D^{13}$$

and $\mathbf{c} = (101010110100010)$.

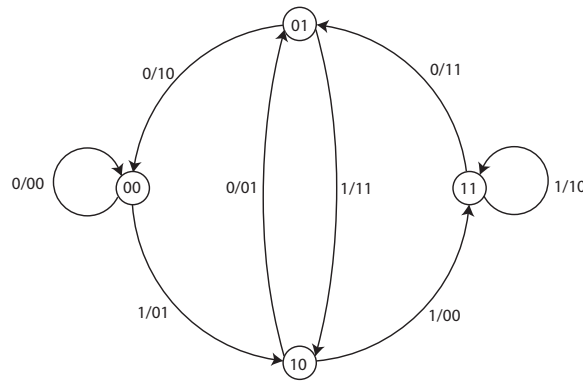
3. Working directly on the finite-state machine describing the code, we obtain the same sequence as in the previous part.
4. We need to use Equation 8.1-38, to obtain

$$\text{GCD}\{1, 1, 1 + D\} = 1 = D^0$$

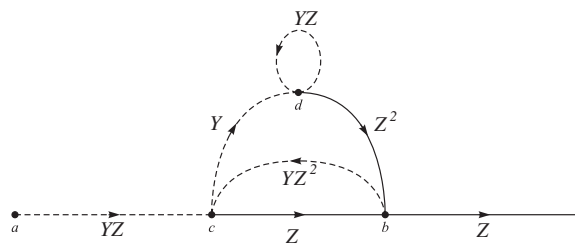
Thus the code is not catastrophic.

Problem 8.14

1. The state transition diagram is shown below



and the state diagram is



Solving the following equations

$$\begin{aligned} X_b &= ZX_c + Z^2X_d \\ X_c &= YZX_a + YZ^2X_b \\ X_d &= YX_c + YZX_d \\ X_e &= ZX_b \end{aligned}$$

results in

$$T(Y, Z) = \frac{X_e}{X_a} = \frac{YZ^3}{1 - YZ - YZ^3}$$

or $T(Y, Z) = YZ^3 + Y^2Z^4 + Y^3Z^5 + (Y^2 + Y^4)Z^6 + (2Y^3 + Y^5)Z^7 + (3Y^4 + Y^6)Z^8 + \dots$

2. The code is not catastrophic since there are no loops of output weight equal to zero corresponding to no-zero input weights.
3. From the expression for $T(Y, Z)$ we see that $d_{\text{free}} = 3$
4. The crossover probability of the BSC is given by $p = Q(\sqrt{2R_c\gamma_b})$ where $R_c = \frac{1}{2}$ and $\gamma_b = E_b/N_0 = 10^{12.6/10} \approx 18.2$. Hence $p = Q(\sqrt{18.2}) \approx Q(4.27) \approx 10^{-5}$. From Equations 8.2-14 and 8.2-15 we obtain $\Delta = \sqrt{4 \times 10^{-5}(1 - 10^{-5})} \approx 0.00632$ and

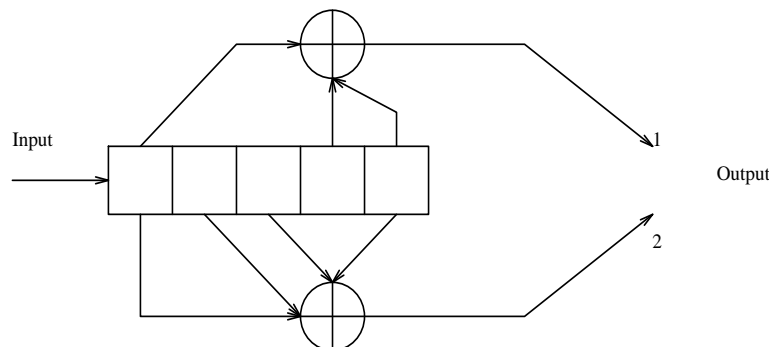
$$\frac{\partial}{\partial Y} T(Y, Z) = \frac{YZ^3(Z + Z^3)}{(1 - YZ - YZ^3)^2} + \frac{Z^3}{1 - YZ - YZ^3}$$

resulting in $P_b \leq 2.55657 \times 10^{-7}$.

Problem 8.15

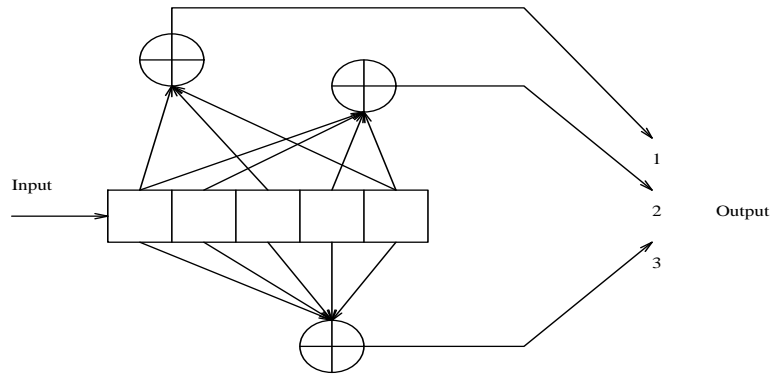
(a)

$$g_1 = [23] = [10011], \quad g_2 = [35] = [11101]$$



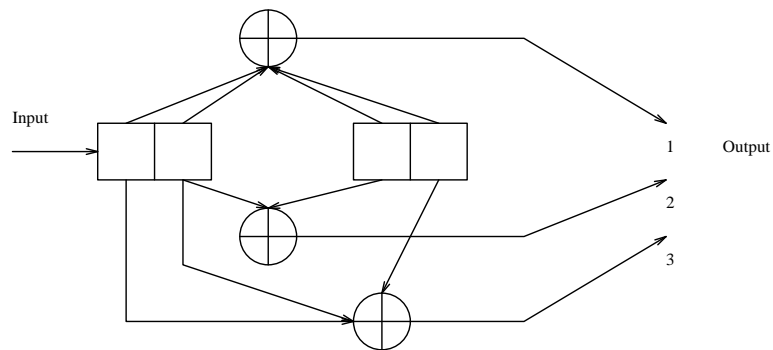
(b)

$$g_1 = [25] = [10101], \quad g_2 = [33] = [11011], \quad g_3 = [37] = [11111]$$



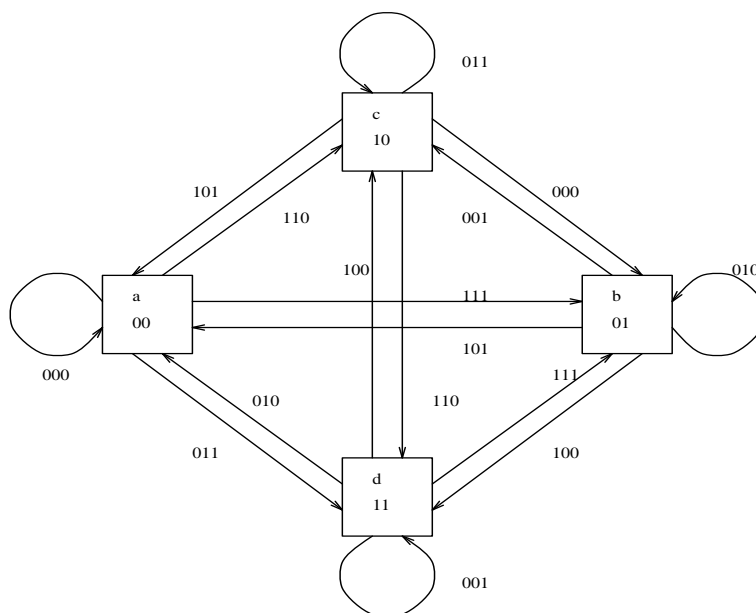
(c)

$$g_1 = [17] = [1111], g_2 = [06] = [0110], g_3 = [15] = [1101]$$



Problem 8.16

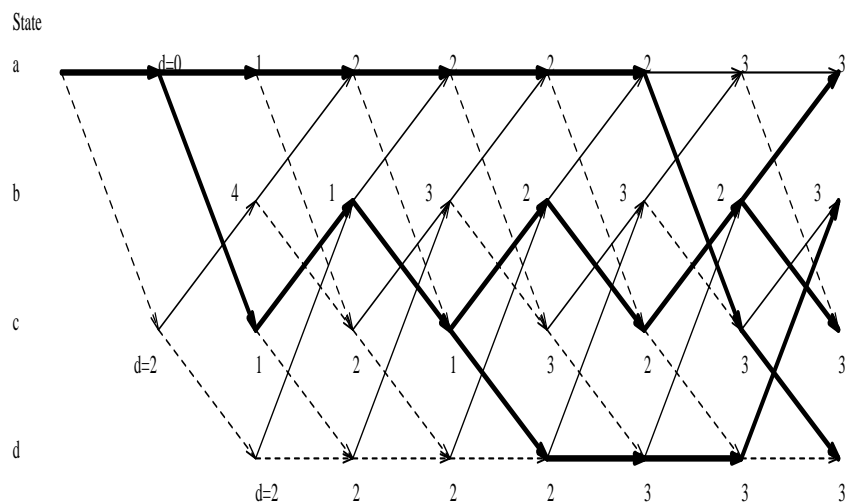
For the encoder of Probl. 8.15(c), the state diagram is as follows :



The 2-bit input that forces the transition from one state to another is the 2-bits that characterize the terminal state.

Problem 8.17

The encoder is shown in Probl. 8.8. The channel is binary symmetric and the metric for Viterbi decoding is the Hamming distance. The trellis and the surviving paths are illustrated in the following figure :



Problem 8.18

In Probl. 8.8 we found :

$$T(D, N, J) = \frac{J^3 ND^5}{1 - JND(1 + J)}$$

Setting $J = 1$:

$$T(D, N) = \frac{ND^5}{1 - 2ND} \Rightarrow \frac{dT(D, N)}{dN} = \frac{D^5}{(1 - 2ND)^2}$$

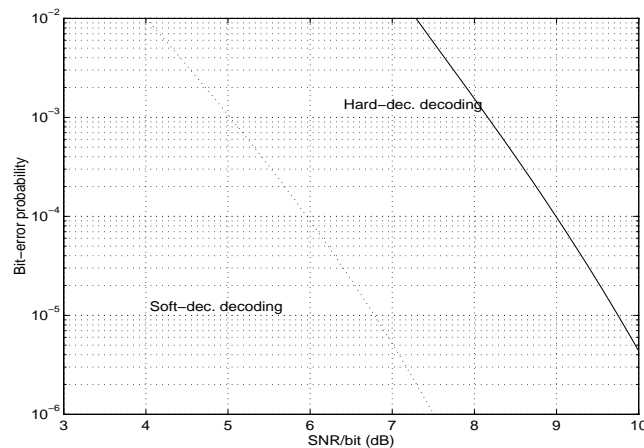
For soft-decision decoding the bit-error probability can be upper-bounded by :

$$P_{bs} \leq \frac{1}{2} \frac{dT(D, N)}{dN} \Big|_{N=1, D=\exp(-\gamma_b R_c)} = \frac{1}{2} \frac{D^5}{(1 - 2ND)^2} \Big|_{N=1, D=\exp(-\gamma_b/2)} = \frac{1}{2} \frac{\exp(-5\gamma_b/2)}{(1 - \exp(-\gamma_b/2))^2}$$

For hard-decision decoding, the Chernoff bound is :

$$P_{bh} \leq \frac{dT(D, N)}{dN} \Big|_{N=1, D=\sqrt{4p(1-p)}} = \frac{\left[\sqrt{4p(1-p)} \right]^{5/2}}{\left[1 - 2\sqrt{4p(1-p)} \right]^2}$$

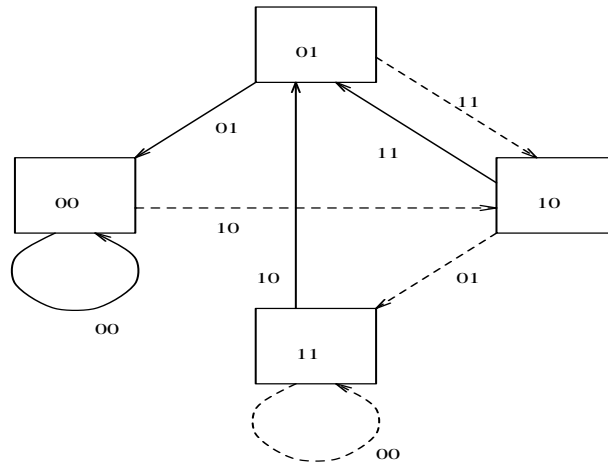
where $p = Q(\sqrt{\gamma_b R_c}) = Q(\sqrt{\gamma_b/2})$ (assuming binary PSK). A comparative plot of the bit-error probabilities is given in the following figure :



Problem 8.19

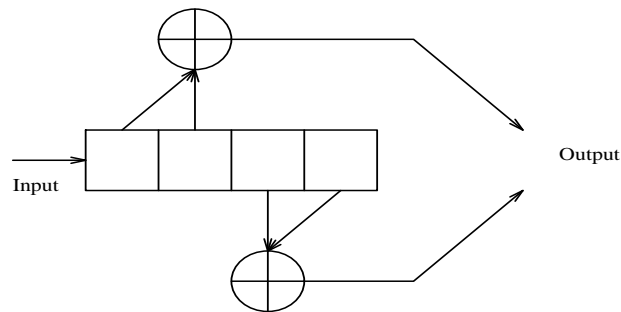
$$g_1 = [110], g_2 = [011], \quad \text{states : (a) = [00], (b) = [01], (c) = [10], (d) = [11]}$$

The state diagram is given in the following figure :

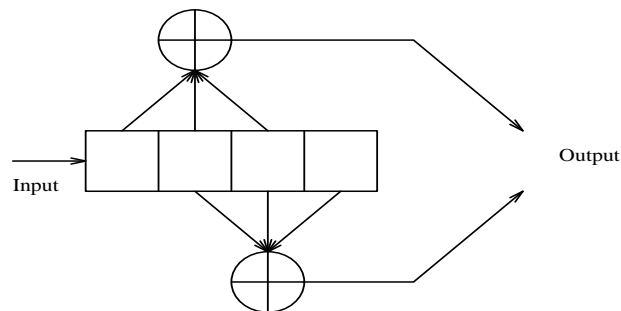


We note that this is a catastrophic code, since there is a zero-distance path from a non-zero state back to itself, and this path corresponds to input 1.

A simple example of an $K = 4$, rate $1/2$ encoder that exhibits error propagation is the following :



The state diagram for this code has a self-loop in the state 111 with input 1, and output 00. A more subtle example of a catastrophic code is the following :



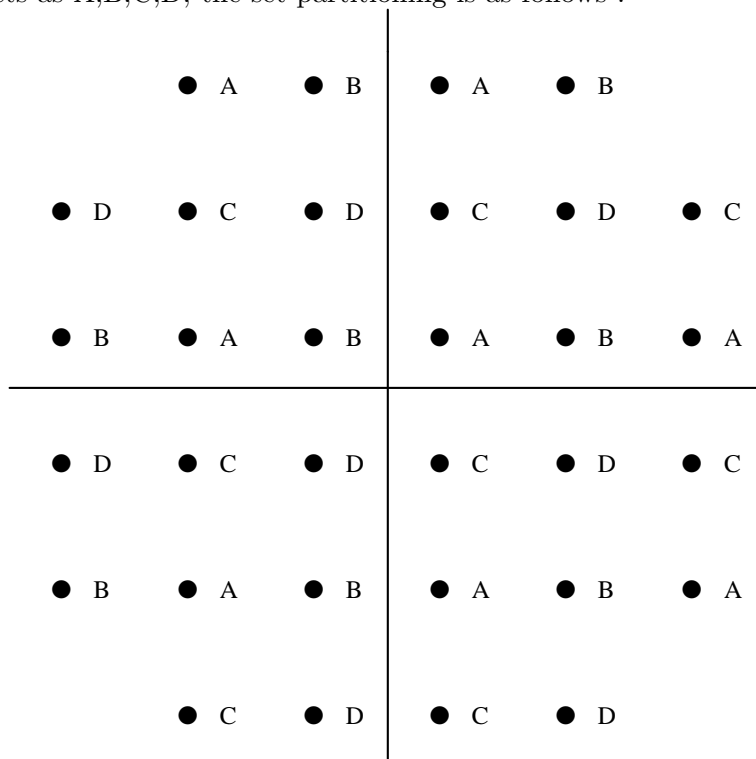
In this case there is a zero-distance path generated by the sequence 0110110110..., which encompasses the states 011, 101, and 110. that is, if the encoder is in state 011 and the input is 1,

the output is 00 and the new state is 101. If the next bit is 1, the output is again 00 and the new state is 110. Then if the next bit is a zero, the output is again 00 and the new state is 011, which is the same state that we started with. Hence, we have a closed path in the state diagram which yields an output that is identical to the output of the all-zero path, but which results from the input sequence 110110110...

For an alternative method for identifying rate $1/n$ catastrophic codes based on observation of the code generators, please refer to the paper by Massey and Sain (1968).

Problem 8.20

There are 4 subsets corresponding to the four possible outputs from the rate $1/2$ convolutional encoder. Each subset has eight signal points, one for each of the 3-tuples from the uncoded bits. If we denote the sets as A,B,C,D, the set partitioning is as follows :



The minimum distance between adjacent points in the same subset is doubled.

Problem 8.21

Over P frames, the number of information bits that are being encoded is

$$k_P = P \sum_{j=1}^J N_j$$

The number of bits that are being transmitted is determined as follows: For a particular group of bits j , $j = 1, \dots, J$, we may delete, with the corresponding puncturing matrix, x_j out of nP bits, on the average, where x may take the values $x = 0, 1, \dots, (n-1)P - 1$. Remembering that each frame contains N_j bits of the particular group, we arrive at the total average number of bits for each group

$$n(j) = N_j(nP - x_j) \Rightarrow n(j) = N_j(P + M_j), \quad M_j = 1, 2, \dots, (n-1)P$$

In the last group $j = J$ we should also add the $K - 1$ overhead information bits, that will add up another $(K - 1)(P + M_J)$ transmitted bits to the total average number of bits for the J^{th} group. Hence, the total number of bits transmitted over P frames be

$$n_P = (K - 1)(P + M_J) + \sum_{j=1}^J JN_j(P + M_j)$$

and the average effective rate of this scheme will be

$$R_{av} = \frac{k_P}{n_P} = \frac{\sum_{j=1}^J N_j P}{\sum_{j=1}^J JN_j(P + M_j) + (K - 1)(P + M_J)}$$

Problem 8.22

By definition, from Equation 8.8-26, we have

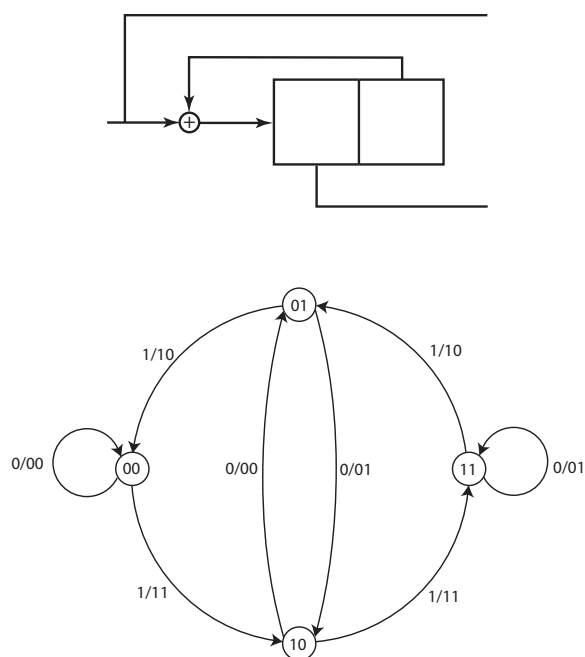
$$\begin{aligned} \max^* \{x, y\} &= \ln(e^x + e^y) \\ &= \begin{cases} \ln[e^x(1 + e^{y-x})], & x > y \\ \ln[e^y(1 + e^{x-y})], & y > x \end{cases} \\ &= \begin{cases} x + \ln(1 + e^{y-x}), & x > y \\ y + \ln(1 + e^{x-y}), & y > x \end{cases} \\ &= \max\{x, y\} + \ln(1 + e^{|x-y|}) \end{aligned}$$

To prove the second relation we note that $\max^* \{x, y\} = \ln(x^x + e^y)$, or $e^{\max^* \{x, y\}} = e^x + e^y$. Therefore

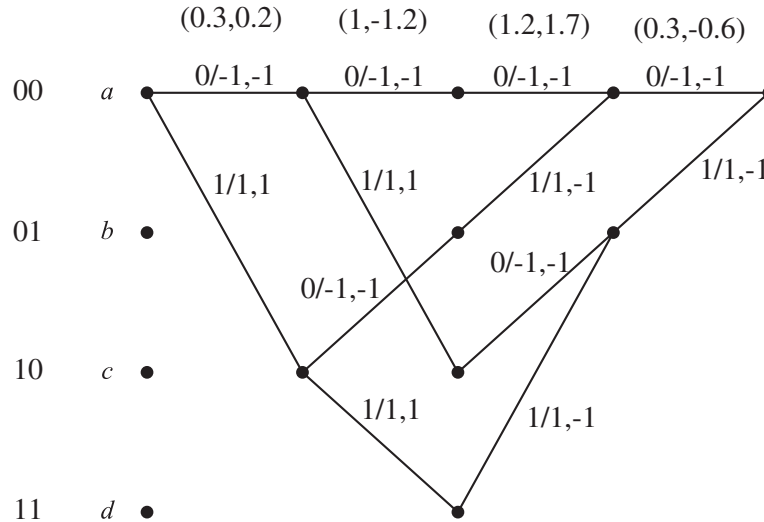
$$\begin{aligned} \max^* \{x, y, z\} &= \ln(e^x + e^y + e^z) \\ &= \ln(e^{\max^* \{x, y\}} + e^z) \\ &= \max^* \{\max^* \{x, y\}, z\} \end{aligned}$$

Problem 8.23

The encoder and the state transition diagram are shown below



The trellis diagram for the received sequence (i.e., a terminated trellis of depth 4) is shown below



where $0/-1, -1$ on a trellis branch indicates that this branch corresponds to an information bit $u_i = 0$ and encoded bits $0, 0$ that are modulated to $-\sqrt{E_c}, -\sqrt{E_c} = -1, -1$.

1. To use the BCJR algorithm we first use Equation 8.8-20 to compute γ 's, then use the recursive relations 8.8-11 and 8.8-14 to compute α 's and β 's and finally use Equation 8.8-17 to find the likelihood values. In using these equations we assume equal probability for input sequence, i.e., $P(u_i = 0) = P(u_i = 1) = 1/2$, and put $N_0 = 4$ and $\mathcal{E}_c = 1$. Using these relations we obtain the following values (note that states a, b, c, and d are represented by numbers 1 to 4)

$\gamma_1(1, 1) = 0.0181939$	$\alpha_1(1) = 0.0181939$	$\beta_3(1) = 0.0250553$
$\gamma_1(1, 3) = 0.0299966$	$\alpha_1(3) = 0.0299966$	$\beta_3(2) = 0.0338211$
$\gamma_2(1, 1) = 0.0144918$	$\alpha_2(1) = 0.000263662$	$\beta_2(1) = 0.0000480464$
$\gamma_2(1, 3) = 0.0118649$	$\alpha_2(2) = 0.000434705$	$\beta_2(2) = 0.00015952$
$\gamma_2(3, 2) = 0.0144918$	$\alpha_2(3) = 0.000215869$	$\beta_2(3) = 0.0000648559$
$\gamma_2(3, 4) = 0.0118649$	$\alpha_2(4) = 0.000355907$	$\beta_2(4) = 0.000215329$
$\gamma_3(1, 1) = 0.00191762$	$\alpha_3(1) = 3.28818 \times 10^{-6}$	$\beta_1(1) = 1.46579 \times 10^{-6}$
$\gamma_3(2, 1) = 0.00636672$	$\alpha_3(2) = 2.6799 \times 10^{-6}$	$\beta_1(3) = 4.86659 \times 10^{-6}$
$\gamma_3(3, 2) = 0.00191762$	$\alpha_4(1) = 1.73024 \times 10^{-7}$	$\beta_0(1) = 1.72649 \times 10^{-7}$
$\gamma_3(4, 2) = 0.00636672$		
$\gamma_4(1, 1) = 0.0250553$		
$\gamma_4(2, 1) = 0.0338211$		

Using these values in Equation 8.8-17, we obtain

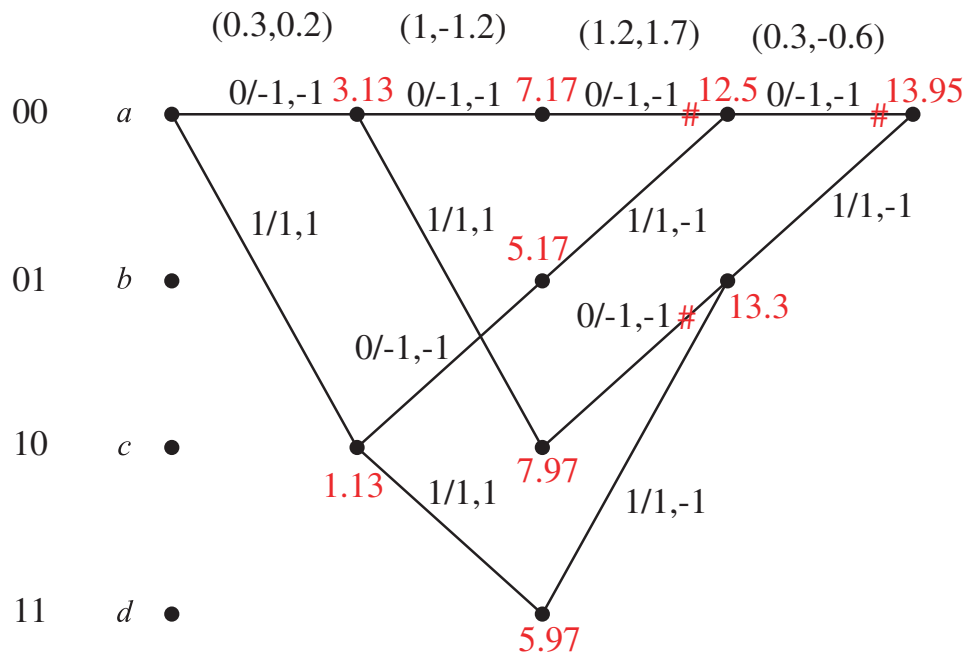
$$L(u_1) \approx 1.7 > 0 \Rightarrow \hat{u}_1 = 1$$

$$L(u_2) \approx 0.1 > 0 \Rightarrow \hat{u}_2 = 1$$

$$L(u_3) \approx 1.7 > 0 \Rightarrow \hat{u}_3 = 1$$

$$L(u_4) \approx 0.095 > 0 \Rightarrow \hat{u}_4 = 1$$

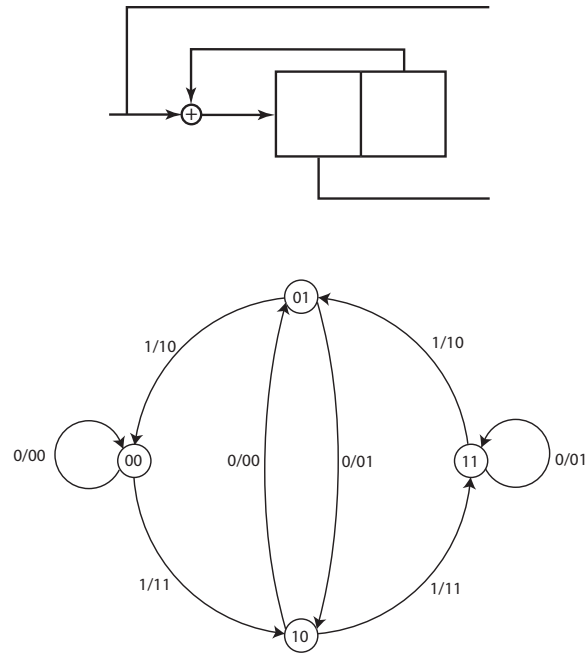
2. For Viterbi algorithm we have to minimize the Euclidean distance, the trellis with accumulated metrics (in red) is shown below



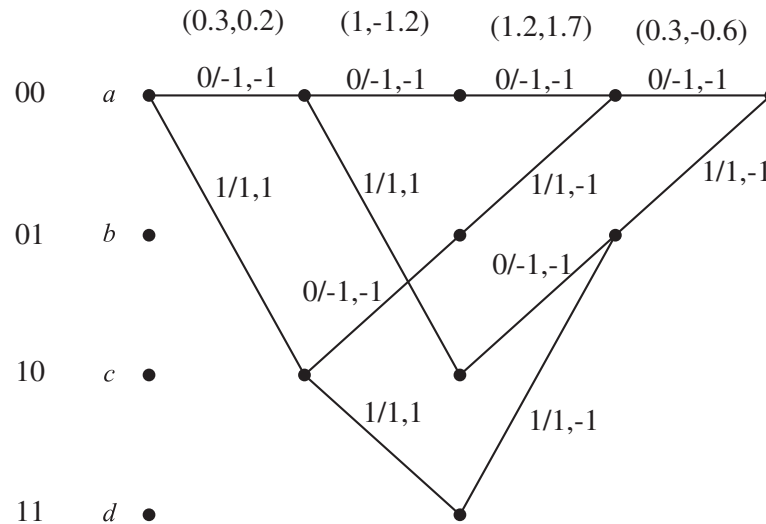
and hence the Viterbi algorithm detects $\hat{\mathbf{u}} = (1, 1, 1, 1)$.

Problem 8.24

The encoder and the state transition diagram are shown below



The trellis diagram for the received sequence (i.e., a terminated trellis of depth 4) is shown below



where 0/ -1, -1 on a trellis branch indicates that this branch corresponds to an information bit $u_i = 0$ and encoded bits 0,0 that are modulated to $-\sqrt{E_c}$, $-\sqrt{E_c} = -1, -1$.

To use the Max-Log-APP algorithm we first use Equation 8.8-19 to compute $\tilde{\gamma}$'s, in computing

these values we drop the term corresponding to the constant coefficient and define

$$\tilde{\gamma}_i = -\frac{\|\mathbf{y}_i - \mathbf{c}_i\|^2}{N_0}$$

We then use the recursive relations in 8.8-27 to compute $\tilde{\alpha}$'s and $\tilde{\beta}$'s and finally use Equation 8.8-28 to find the likelihood values. In using these equations we assume equal probability for input sequence, i.e., $P(u_i = 0) = P(u_i = 1) = 1/2$, and put $N_0 = 4$ and $\mathcal{E}_c = 1$. Using these relations we obtain the following values (note that states a, b, c, and d are represented by numbers 1 to 4)

$\tilde{\gamma}_1(1, 1) = -0.7825$	$\tilde{\alpha}_1(1) = -0.7825$	$\tilde{\beta}_3(1) = -0.4625$
$\tilde{\gamma}_1(1, 3) = -0.2825$	$\tilde{\alpha}_1(3) = -0.2825$	$\tilde{\beta}_3(2) = -0.1625$
$\tilde{\gamma}_2(1, 1) = -1.01$	$\tilde{\alpha}_2(1) = -1.7925$	$\tilde{\beta}_2(1) = -3.495$
$\tilde{\gamma}_2(1, 3) = -1.21$	$\tilde{\alpha}_2(2) = -1.4925$	$\tilde{\beta}_2(2) = -2.295$
$\tilde{\gamma}_2(3, 2) = -1.21$	$\tilde{\alpha}_2(3) = -1.9925$	$\tilde{\beta}_2(3) = -3.195$
$\tilde{\gamma}_2(3, 4) = -1.01$	$\tilde{\alpha}_2(4) = -1.2925$	$\tilde{\beta}_2(4) = -1.995$
$\tilde{\gamma}_3(1, 1) = -3.0325$	$\tilde{\alpha}_3(1) = -3.325 \times 10^{-6}$	$\tilde{\beta}_1(1) = -3.405$
$\tilde{\gamma}_3(2, 1) = -1.8325$	$\tilde{\alpha}_3(2) = -3.125 \times 10^{-6}$	$\tilde{\beta}_1(3) = -3.005$
$\tilde{\gamma}_3(3, 2) = -3.0325$	$\tilde{\alpha}_4(1) = -3.2875 \times 10^{-7}$	$\tilde{\beta}_0(1) = -3.283$
$\tilde{\gamma}_3(4, 2) = -1.8325$		
$\tilde{\gamma}_4(1, 1) = -0.4625$		
$\tilde{\gamma}_4(2, 1) = -0.1625$		

Using these values in Equation 8.8-17, we obtain

$$L(u_1) \approx 0.9 > 0 \Rightarrow \hat{u}_1 = 1$$

$$L(u_2) \approx 0 \Rightarrow \hat{u}_2 = 0 \text{ or } 1$$

$$L(u_3) \approx 1.9 > 0 \Rightarrow \hat{u}_3 = 1$$

$$L(u_4) \approx 0.5 > 0 \Rightarrow \hat{u}_4 = 1$$

Problem 8.25

1. Let us define new random variables $W_i = 1 - 2X_i$, obviously W_i 's are independent and each W_i takes values of +1 and -1 with probabilities $p(0)$ and $p(1)$, respectively. This means that $E[W_i] = p_i(0) - p_i(1)$. We also define

$$T = \prod_{i=1}^n W_i$$

T is equal to $+1$ or -1 if an even or odd number of W_i 's are equal to -1 , or equivalently if an even or odd number of X_i 's are equal to 1. Similarly Y is equal to 0 or 1 if an even or odd number of X_i 's are equal to 1. This shows that $T = 1$ if and only if $Y = 0$ and $T = -1$ if and only if $Y = 1$; i.e., $P[T = 1] = P[Y = 0] = p(0)$ and $P[T = -1] = P[Y = 1] = p(1)$. Therefore, $E[T] = P(T = 1) - P(T = -1) = p(0) - p(1)$ and by independence of W_i 's we have

$$E[T] = \prod_{i=1}^n E[W_i] = \prod_{i=1}^n (p_i(0) - p_i(1)) = p(0) - p(1)$$

2. From part 1 and $p(0) + p(1) = 1$ we obtain

$$p(0) = \frac{1}{2} + \frac{1}{2} \prod_{i=1}^n (p_i(0) - p_i(1))$$

$$p(1) = \frac{1}{2} - \frac{1}{2} \prod_{i=1}^n (p_i(0) - p_i(1))$$

3. Note that

$$p_j(0) - p_j(1) = 1 - 2p_j(1) \quad (*)$$

at a check node the messages are binary and the check node function is true if the binary sum of messages is equal to zero. Therefore from part 2 and relation $(*)$ we obtain Equation 8.10-27.

Problem 8.26

For an equality constraint node (cloning node) that generates replicas of x_i we have

$$g(x_1, x_2, \dots, x_n) = \prod_{j \neq i} \delta[x_j = x_i]$$

Using Equation 8.10-22 we have

$$\begin{aligned} \mu_{g \rightarrow x_i}(x_i) &= \sum_{\sim x_i} \prod_{j \neq i} \delta[x_j = x_i] \prod_{j \neq i} \mu_{x_j \rightarrow g}(x_j) \\ &= \prod_{j \neq i} \mu_{x_j \rightarrow g}(x_i) \end{aligned}$$

where we have used the relation

$$\sum_{x_j} \delta[x_j = x_i] \mu_{x_j \rightarrow g}(x_j) = \mu_{x_i \rightarrow g}(x_i)$$

Problem 8.27

From \mathbf{H} we have the following parity check equations

$$c_3 + c_6 + c_7 + c_8 = 0$$

$$c_1 + c_2 + c_5 + c_{12} = 0$$

$$c_4 + c_9 + c_{10} + c_{11} = 0$$

$$c_2 + c_6 + c_7 + c_{10} = 0$$

$$c_1 + c_3 + c_8 + c_{11} = 0$$

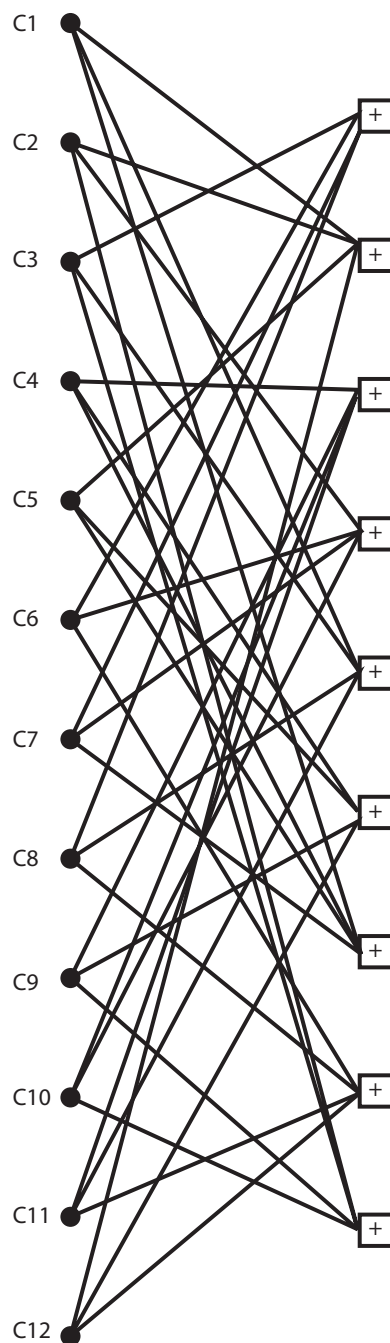
$$c_4 + c_5 + c_9 + c_{12} = 0$$

$$c_1 + c_4 + c_5 + c_7 = 0$$

$$c_6 + c_8 + c_{11} + c_{12} = 0$$

$$c_2 + c_3 + c_9 + c_{10} = 0$$

resulting in the Tanner graph shown below



Problem 8.28

For a repetition code $\mathbf{G} = [1 \ 1 \ 1 \ \dots \ 1]$ and hence

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which clearly corresponds to an irregular LDPC matrix.

Problem 8.29

We have

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the parity check equations are

$$c_1 + c_2 = 0$$

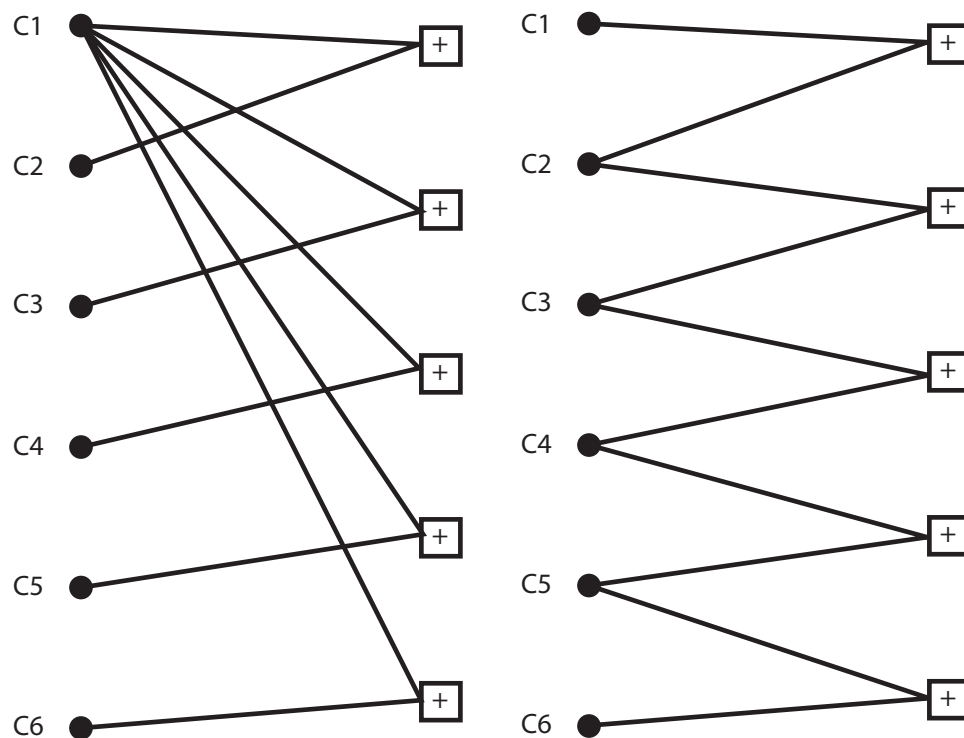
$$c_1 + c_3 = 0$$

$$c_1 + c_4 = 0$$

$$c_1 + c_5 = 0$$

$$c_1 + c_6 = 0$$

Two Tanner graphs for this code are shown below



Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 9) ¹

Prepared by
Kostas Stamatiou

January 15, 2008

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Problem 9.1

We want $y(t) = Kx(t - t_0)$. Then :

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ Y(f) &= \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt = K \exp(-j2\pi ft_0) X(f) \end{aligned}$$

Therefore :

$$A(f) e^{-j\theta(f)} = K e^{-j2\pi ft_0} \Rightarrow \left\{ \begin{array}{ll} A(f) = K, & \text{for all } f \\ \theta(f) = 2\pi ft_0 \pm n\pi, & n = 0, 1, 2, \dots \end{array} \right\}$$

Note that $n\pi$, n odd, results in a sign inversion of the signal.

Problem 9.2

(a) Since $\cos(a + \pi/2) = -\sin(a)$, we can write :

$$X(f) = \left\{ \begin{array}{ll} T, & 0 \leq |f| \leq \frac{1-\beta}{2T} \\ \frac{T}{2} \left[1 - \sin \frac{\pi T}{\beta} \left(f - \frac{1}{2T} \right) \right], & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \end{array} \right\}$$

Then, taking the first two derivatives with respect to f :

$$X'(f) = \left\{ \begin{array}{ll} -\frac{T^2\pi}{2\beta} \cos \frac{\pi T}{\beta} \left(f - \frac{1}{2T} \right), & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{array} \right\}$$

and :

$$X''(f) = \left\{ \begin{array}{ll} \frac{T^3\pi^2}{2\beta^2} \sin \frac{\pi T}{\beta} \left(f - \frac{1}{2T} \right), & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{array} \right\}$$

Therefore the second derivative can be expressed as :

$$X''(f) = -\frac{\pi^2 T^2}{\beta^2} \left[X(f) - \frac{T}{2} \text{rect} \left(\frac{1-\beta}{2T} f \right) - \frac{T}{2} \text{rect} \left(\frac{1+\beta}{2T} f \right) \right]$$

where :

$$\text{rect}(af) = \left\{ \begin{array}{ll} 1, & |f| \leq a \\ 0, & \text{o.w} \end{array} \right\}$$

Since the Fourier transform of dx/dt is $j2\pi f X(f)$, we exploit the duality between (f, t) , take the inverse Fourier transform of $X''(f)$ and obtain :

$$-4\pi^2 t^2 x(t) = -\frac{\pi^2 T^2}{\beta^2} \left[x(t) - \frac{T}{2} \frac{1}{\pi t} \sin \frac{1-\beta}{2T} 2\pi t - \frac{T}{2} \frac{1}{\pi t} \sin \frac{1+\beta}{2T} 2\pi t \right]$$

Solving for $x(t)$ we obtain :

$$\begin{aligned} x(t) &= \frac{1}{1-4\beta^2 t^2/T} \left[\frac{1}{2\pi t/T} \left(\sin \frac{1-\beta}{2T} 2\pi t + \sin \frac{1+\beta}{2T} 2\pi t \right) \right] \\ &= \frac{1}{1-4\beta^2 t^2/T} \left[\frac{1}{\pi t/T} \left(\sin \frac{\pi t}{T} \cos \frac{\pi \beta t}{T} \right) \right] \end{aligned}$$

(b) When $\beta = 1$, $X(f)$ is non-zero in $|f| \leq 1/T$, and :

$$X(f) = \frac{T}{2} (1 + \cos \pi T f)$$

The Hilbert transform is :

$$\hat{X}(f) = \begin{cases} -j\frac{T}{2} (1 + \cos \pi T f), & 0 \leq f \leq 1/T \\ j\frac{T}{2} (1 + \cos \pi T f), & -1/T \leq f \leq 0 \end{cases}$$

Then :

$$\begin{aligned} \hat{x}(t) &= \int_{-\infty}^{\infty} \hat{X}(f) \exp(j2\pi f t) dt \\ &= \int_{-1/T}^0 \hat{X}(f) \exp(j2\pi f t) dt + \int_0^{1/T} \hat{X}(f) \exp(j2\pi f t) dt \end{aligned}$$

Direct substitution for $\hat{X}(f)$ yields the result :

$$\hat{x}(t) = \frac{T}{\pi t} \left[\frac{\sin^2 \pi t/T - 4t^2/T^2}{1 - 4t^2/T^2} \right]$$

Note that $\hat{x}(t)$ is an odd function of t .

(c) No, since $\hat{x}(0) = 0$ and $\hat{x}(nT) \neq 0$, for $n \neq 0$. Also $\sum_{n=-\infty}^{\infty} \hat{X}(f + n/2T) \neq \text{constant}$ for $|f| \leq 1/2T$.

(d) The single-sideband signal is :

$$x(t) \cos 2\pi f_c t \pm \hat{x}(t) \sin 2\pi f_c t = \text{Re} \left[(x(t) \pm j\hat{x}(t)) e^{j2\pi f_c t} \right]$$

The envelope is $a(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$. For $\beta = 1$:

$$a(t) = \frac{1}{\pi t/T} \frac{1}{1 - 4t^2/T^2} \sqrt{(1 - 8t^2/T^2) \sin^2(\pi t/T) + 16t^4/T^4}$$

Problem 9.3

(a) $\sum_k h(t - kT) = u(t)$ is a periodic signal with period T . Hence, $u(t)$ can be expanded in the Fourier series :

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi nt/T}$$

where :

$$\begin{aligned} u_n &= \frac{1}{T} \int_{-T/2}^{T/2} u(t) \exp(-j2\pi nt/T) dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} h(t - kT) \exp(-j2\pi nt/T) dt \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} h(t - kT) \exp(-j2\pi nt/T) dt \\ &= \frac{1}{T} \int_{-\infty}^{\infty} h(t) \exp(-j2\pi nt/T) dt = \frac{1}{T} H\left(\frac{n}{T}\right) \end{aligned}$$

Then : $u(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right) e^{j2\pi nt/T} \Rightarrow U(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right) \delta\left(f - \frac{n}{T}\right)$. Since $x(t) = u(t)g(t)$, it follows that $X(f) = U(f) * G(f)$. Hence :

$$X(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right) G\left(f - \frac{n}{T}\right)$$

(b)

(i)

$$\sum_{k=-\infty}^{\infty} h(kT) = u(0) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right)$$

(ii)

$$\sum_{k=-\infty}^{\infty} h(t - kT) = u(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right) e^{j2\pi nt/T}$$

(iii) Let

$$v(t) = h(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} h(kT) \delta(t - kT)$$

Hence :

$$V(f) = \sum_{k=-\infty}^{\infty} h(kT) e^{-j2\pi f kT}$$

But

$$\begin{aligned} V(f) &= H(f) * \text{Fourier transform of } \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ &= H(f) * \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H\left(f - \frac{n}{T}\right) \end{aligned}$$

(c) The criterion for no intersymbol interference is $\{h(kT) = 0, k \neq 0 \text{ and } h(0) = 1\}$. If the above condition holds, then from (iii) above we have :

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} H(f - \frac{n}{T}) = \sum_{k=-\infty}^{\infty} h(kT) e^{-j2\pi f k T} = 1$$

Conversely, if $\frac{1}{T} \sum_{n=-\infty}^{\infty} H(f - \frac{n}{T}) = 1, \forall f \Rightarrow \sum_{k=-\infty}^{\infty} h(kT) e^{-j2\pi f k T} = 1, \forall f$. This is possible only if the left-hand side has no dependence on f , which means $h(kT) = 0$, for $k \neq 0$. Then $\sum_{k=-\infty}^{\infty} h(kT) e^{-j2\pi f k T} = h(0) = 1$.

Problem 9.4

$$x(t) = e^{-\pi a^2 t^2} \Rightarrow X(f) = \frac{1}{a} e^{-\pi f^2 / a^2}$$

Hence :

$$X(0) = \frac{1}{a}, X(W) = \frac{1}{a} e^{-\pi W^2 / a^2}$$

We have :

$$\frac{X(W)}{X(0)} = 0.01 \Rightarrow e^{-\pi W^2 / a^2} = 0.01 \Rightarrow W^2 = -\frac{a^2}{\pi} \ln(0.01)$$

But due to the condition for the reduced ISI :

$$x(T) = e^{-\pi a^2 T^2} = 0.01 \Rightarrow T^2 = -\frac{1}{\pi a^2} \ln(0.01)$$

Hence $WT = \frac{-1}{\pi} \ln(0.01) = 1.466$ or :

$$W = \frac{1.466}{T}$$

For the raised cosine spectral characteristic (with roll-off factor 1) $W = 1/T$. Hence, the Gaussian shaped pulse requires more bandwidth than the pulse having the raised cosine spectrum.

Problem 9.5

The impulse response of a square-root raised cosine filter is given by

$$x_{ST}(t) = \int_{-\frac{1+\beta}{2T}}^{\frac{1+\beta}{2T}} \sqrt{X_{rc}(f)} e^{j2\pi ft} df$$

where $X_{rc}(f)$ is given by (9.2-26). Splitting the integral in three parts we obtain

$$x_{ST}(t) = \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \sqrt{T/2} \sqrt{1 + \cos\left(\frac{\pi T}{\beta}\left(-f - \frac{1-\beta}{2T}\right)\right)} e^{j2\pi ft} df \quad (9.0.1)$$

$$+ \int_{-\frac{1-\beta}{2T}}^{\frac{1-\beta}{2T}} \sqrt{T} e^{j2\pi ft} df \quad (9.0.2)$$

$$+ \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \sqrt{T/2} \sqrt{1 + \cos\left(\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right)} e^{j2\pi ft} df \quad (9.0.3)$$

The second term (2) gives immediately

$$(2) = \frac{\sqrt{T}}{\pi t} \sin(\pi(1-\beta)t/T)$$

The third term can be solved with the transformation $\lambda = f - \frac{1-\beta}{2T}$. Then

$$(3) = \int_0^{\frac{\beta}{T}} \sqrt{T/2} \sqrt{1 + \cos\left(\frac{\pi T \lambda}{\beta}\right)} e^{j2\pi t(\lambda + \frac{1+\beta}{2T})} d\lambda$$

Using the relationship $1 + \cos 2A = 2 \cos^2 A \Rightarrow \sqrt{1 + \cos 2A} = \sqrt{2} |\cos A| = \sqrt{2} \cos A$, we can rewrite the above expression as

$$(3) = \int_0^{\frac{\beta}{T}} \sqrt{T} \cos\left(\frac{\pi T \lambda}{2\beta}\right) e^{j2\pi t(\lambda + \frac{1+\beta}{2T})} d\lambda$$

Since $\cos A = \frac{e^{jA} + e^{-jA}}{2}$, the above integral simplifies to the sum of two simple exponential argument integrals.

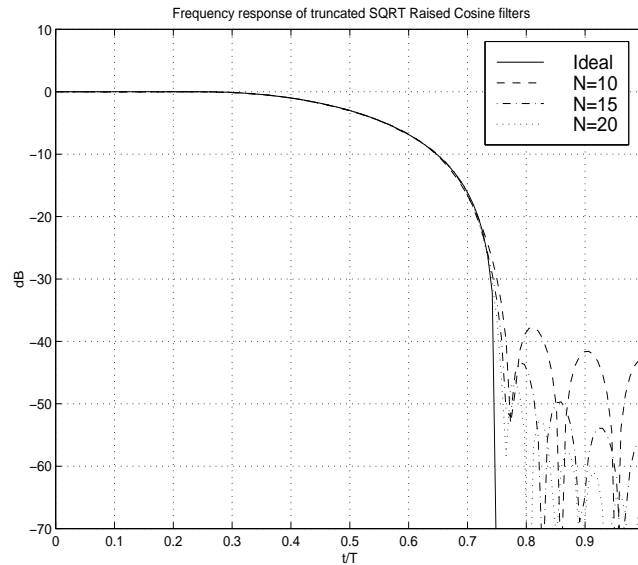
Similarly to (3), the first term (1) can be solved with the transformation $\lambda = f + \frac{1-\beta}{2T}$ (notice that $\cos(\frac{\pi T}{\beta}(-f - \frac{1-\beta}{2T})) = \cos(\frac{\pi T}{\beta}(f + \frac{1-\beta}{2T}))$). Then again, the integral simplifies to the sum of two simple exponential argument integrals. Proceeding with adding (1),(2),(3) we arrive at the desired result.

Problem 9.6

(a)(b) In order to calculate the frequency response based on the impulse response, we need the values of the impulse response at $t = 0, \pm T/2$, which are not given directly by the expression of Problem 9.5. Using L'Hospital's rule it is straightforward to show that:

$$x(0) = \frac{1}{2} + \frac{2}{\pi}, \quad x(\pm T/2) = \frac{\sqrt{2}}{2} \frac{(2 + \pi)}{2\pi}$$

Then, the frequency response of the filters with $N = 10, 15, 20$ compared to the frequency response of the ideal square-root raised cosine filter are depicted in the following figure.

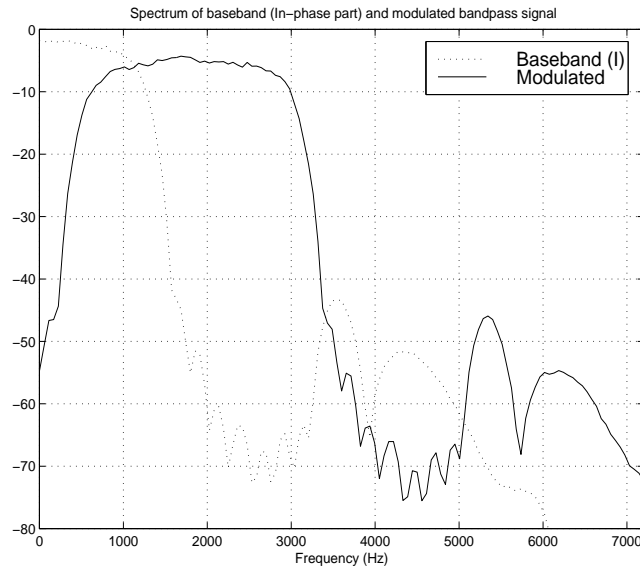


As we see, there is no significant difference in the passband area of the filters, but the realizable, truncated filters do have spectral sidelobes outside their $(1 + \beta)/T$ nominal bandwidth. Still, depending on how much residual ISI an application can tolerate, even the $N = 10$ filter appears an acceptable approximation of the ideal (non-realizable) square-root raised cosine filter.

Problem 9.7

(a),(b) Given a mathematical package like MATLAB, the implementation in software of the digital modulator of Fig P9.7 is relatively straightforward. One comment is that the interpolating filters should have a nominal passband of $[-\pi/3, \pi/3]$, since the interpolation factor applied to the samples at the output of the shaping filter is 3. We chose our interpolation filters (designed with the MATLAB `fir1` function) to have a cutoff frequency (-3 dB frequency) of $\pi/5$. This corresponds to the highest frequency with significant signal content, since with the spectrum of the baseband signal should be (approximately, due to truncation effects) limited to $(1 + 0.25)/2T$, so sampled at $6T$ it should be limited to a discrete frequency of $(2 * \pi * (1 + 0.25)/2T)/6 \approx 0.21 * \pi$.

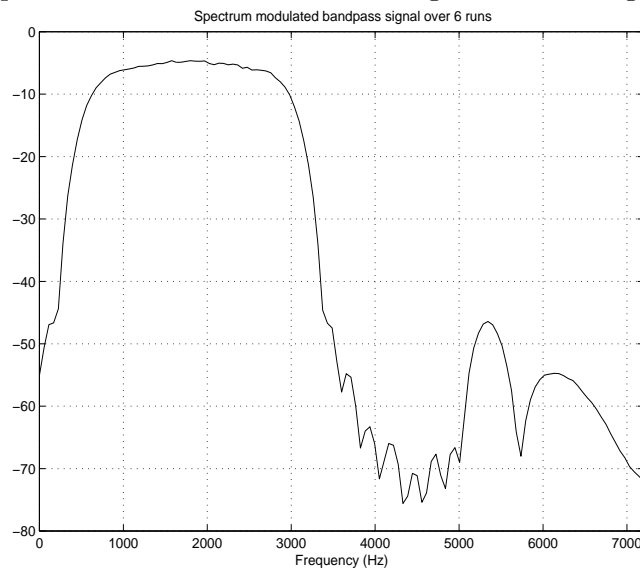
The plot with the power spectrum of the digital signal sequence is given in the following figure. We have also plotted the power spectrum of the baseband in-phase (I component) of the signal.



We notice the rather significant sidelobe that is due to the non-completely eliminated image of the spectrum that was generated by the interpolating process. We could mitigate it by choosing an interpolation filter with lower cut-off frequency, but then, we would lose a larger portion of the useful signal as well. The best solution would be to use a longer interpolation filter.

(c)

By repeating the experiment for a total of 6 runs we get the following figure



We notice the smoother shape of the PSD, and we can verify that indeed the spectrum is centered around 1800 Hz.

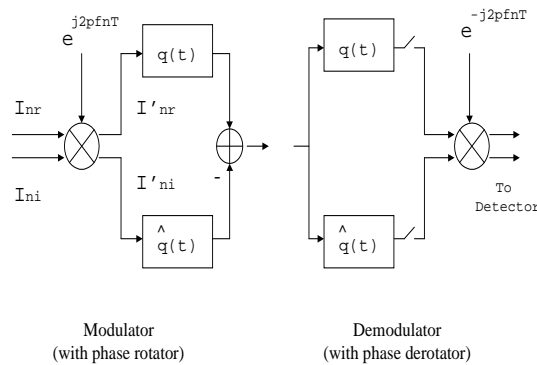
Problem 9.8

(a) The alternative expression for $s(t)$ can be rewritten as

$$\begin{aligned}
 s(t) &= \operatorname{Re} \left\{ \sum_n I'_n Q(t - nT) \right\} \\
 &= \operatorname{Re} \left\{ \sum_n I_n e^{j2\pi f_c nT} g(t - nT) [\cos 2\pi f_c(t - nT) + j \sin(2\pi f_c(t - nT))] \right\} \\
 &= \operatorname{Re} \left\{ \sum_n I_n g(t - nT) [\cos 2\pi f_c nT + j \sin 2\pi f_c nT] [\cos 2\pi f_c(t - nT) + j \sin(2\pi f_c(t - nT))] \right\} \\
 &= \operatorname{Re} \left\{ \sum_n I_n g(t - nT) [\cos 2\pi f_c nT \cos 2\pi f_c(t - nT) - \sin 2\pi f_c nT \sin 2\pi f_c(t - nT) \right. \\
 &\quad \left. + j \sin 2\pi f_c nT \cos 2\pi f_c(t - nT) + j \cos 2\pi f_c nT \sin 2\pi f_c(t - nT)] \right\} \\
 &= \operatorname{Re} \left\{ \sum_n I_n g(t - nT) [\cos 2\pi f_c t + j \sin 2\pi f_c t] \right\} \\
 &= \operatorname{Re} \left\{ \sum_n I_n g(t - nT) e^{j2\pi f_c t} \right\} = s(t)
 \end{aligned}$$

So, indeed the alternative expression for $s(t)$ is a valid one.

(b)



Problem 9.9

(a) From the impulse response of the pulse having a square-root raised cosine characteristic, which is given in problem 9.5, we can see immediately that $x_{SQ}(t) = x_{SQ}(-t)$, i.e. the pulse $g(t)$ is an even function. We know that the product of an even function times an even function has even symmetry, while the product of even times odd has odd symmetry. Hence $q(t)$ is even, while $\hat{q}(t)$ is odd. Hence, the product $q(t)\hat{q}(t)$ has odd symmetry. We know that the (symmetric around 0) integral of an odd function is zero, or

$$\int_{-\infty}^{\infty} q(t)\hat{q}(t)dt = \int_{-(1+\beta)/2T}^{(1+\beta)/2T} q(t)\hat{q}(t)dt = 0$$

(b) We notice that when $f_c = k/T$, where k is an integer, then the rotator/derotator of a carrierless QAM system (described in Problem 9.8) gives a trivial rotation of an integer number of full circles ($2\pi kn$), and the carrierless QAM/PSK is equivalent to CAP.

Problem 9.10

(a)

(i) $x_0 = 2$, $x_1 = 1$, $x_2 = -1$, otherwise $x_n = 0$. Then :

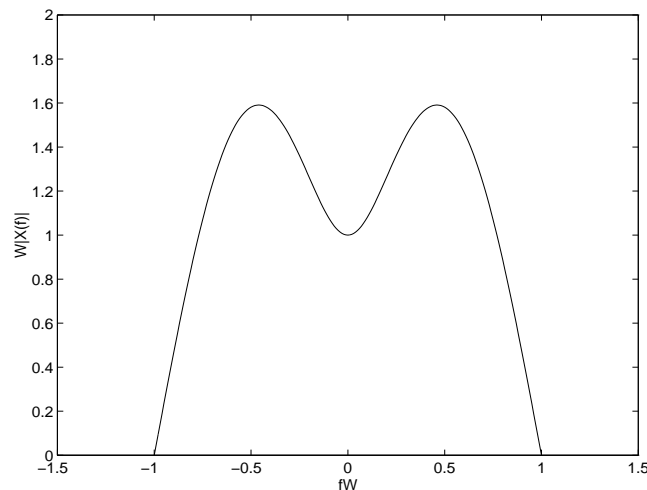
$$x(t) = 2\frac{\sin(2\pi Wt)}{2\pi Wt} + \frac{\sin(2\pi W(t - 1/2W))}{2\pi W(t - 1/2W)} - \frac{\sin(2\pi W(t - 1/W))}{2\pi W(t - 1/W)}$$

and :

$$X(f) = \frac{1}{2W} [2 + e^{-j\pi f/W} - e^{-j2\pi f/W}], \quad |f| \leq W \Rightarrow$$

$$|X(f)| = \frac{1}{2W} \left[6 + 2 \cos \frac{\pi f}{W} - 4 \cos \frac{2\pi f}{W} \right]^{1/2}, \quad |f| \leq W$$

The plot of $|X(f)|$ is given in the following figure :



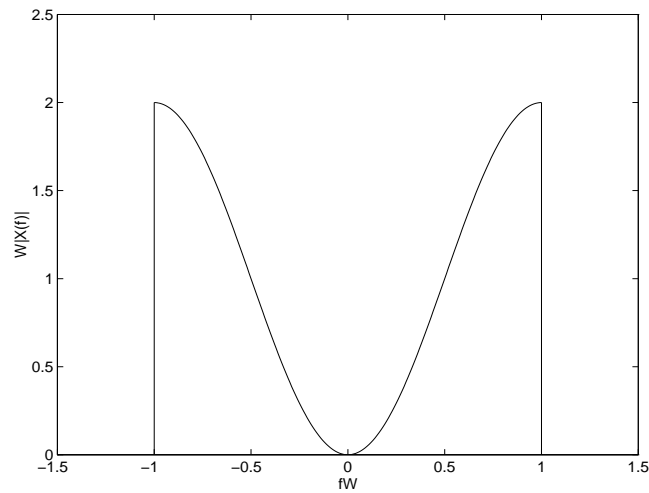
(ii) $x_{-1} = -1$, $x_0 = 2$, $x_1 = -1$, otherwise $x_n = 0$. Then :

$$x(t) = 2\frac{\sin(2\pi Wt)}{2\pi Wt} - \frac{\sin(2\pi W(t + 1/2W))}{2\pi W(t + 1/2W)} - \frac{\sin(2\pi W(t - 1/2W))}{2\pi W(t - 1/2W)}$$

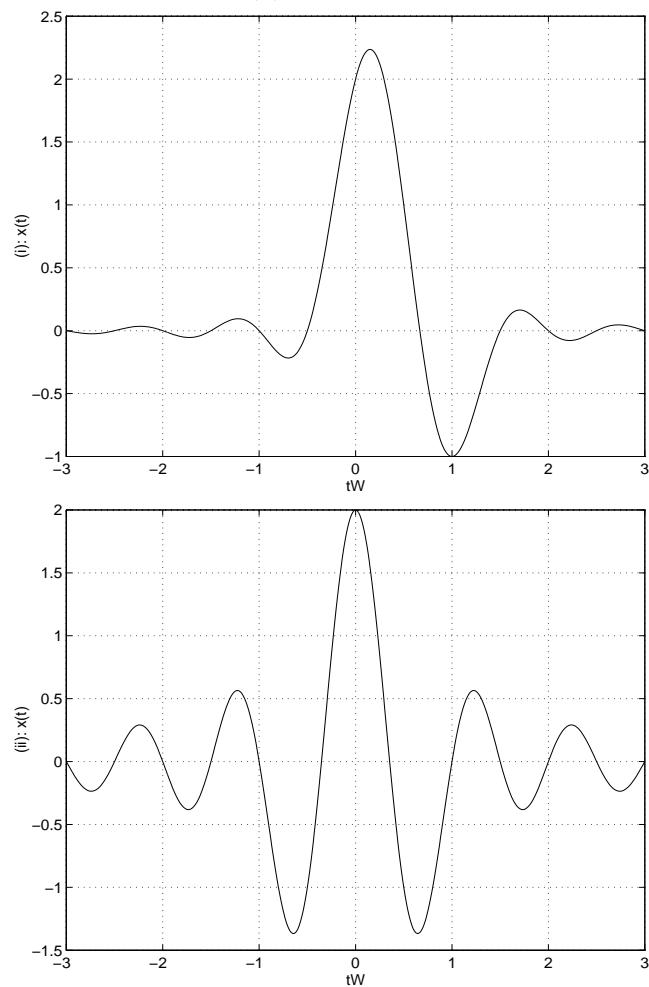
and :

$$X(f) = \frac{1}{2W} [2 - e^{-j\pi f/W} - e^{+j\pi f/W}] = \frac{1}{2W} \left[2 - 2 \cos \frac{\pi f}{W} \right] = \frac{1}{W} \left[1 - \cos \frac{\pi f}{W} \right], \quad |f| \leq W$$

The plot of $|X(f)|$ is given in the following figure :



(b) Based on the results obtained in part (a) :



(c) The possible received levels at the receiver are given by :

(i)

$$B_n = 2I_n + I_{n-1} - I_{n-2}$$

where $I_m = \pm 1$. Hence :

$$P(B_n = 0) = 1/4$$

$$P(B_n = -2) = 1/4$$

$$P(B_n = 2) = 1/4$$

$$P(B_n = -4) = 1/8$$

$$P(B_n = 4) = 1/8$$

(ii)

$$B_n = 2I_n - I_{n-1} - I_{n+1}$$

where $I_m = \pm 1$. Hence :

$$P(B_n = 0) = 1/4$$

$$P(B_n = -2) = 1/4$$

$$P(B_n = 2) = 1/4$$

$$P(B_n = -4) = 1/8$$

$$P(B_n = 4) = 1/8$$

Problem 9.11

The bandwidth of the bandpass channel is $W = 4$ KHz. Hence, the rate of transmission should be less or equal to 4000 symbols/sec. If a 8-QAM constellation is employed, then the required symbol rate is $R = 9600/3 = 3200$. If a signal pulse with raised cosine spectrum is used for shaping, the maximum allowable roll-off factor is determined by :

$$1600(1 + \beta) = 2000$$

which yields $\beta = 0.25$. Since β is less than 50%, we consider a larger constellation. With a 16-QAM constellation we obtain :

$$R = \frac{9600}{4} = 2400$$

and :

$$1200(1 + \beta) = 2000$$

or $\beta = 2/3$, which satisfies the required conditions. The probability of error for an M -QAM constellation is given by :

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where :

$$P_{\sqrt{M}} = 2 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}} \right]$$

With $P_M = 10^{-6}$ we obtain $P_{\sqrt{M}} = 5 \times 10^{-7}$ and therefore using the last equation and the table of values for the $Q(\cdot)$ function, we find that the average transmitted energy is :

$$\mathcal{E}_{av} = 24.70 \times 10^{-9}$$

Note that if the desired spectral characteristic $X_{rc}(f)$ is split evenly between the transmitting and receiving filter, then the energy of the transmitting pulse is :

$$\int_{-\infty}^{\infty} g_T^2(t) dt = \int_{-\infty}^{\infty} |G_T(f)|^2 df = \int_{-\infty}^{\infty} X_{rc}(f) df = 1$$

Hence, the energy $\mathcal{E}_{av} = P_{av}T$ depends only on the amplitude of the transmitted points and the symbol interval T . Since $T = \frac{1}{2400}$, the average transmitted power is :

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = 24.70 \times 10^{-9} \times 2400 = 592.8 \times 10^{-7}$$

If the points of the 16-QAM constellation are evenly spaced with minimum distance between them equal to d , then there are four points with coordinates $(\pm \frac{d}{2}, \pm \frac{d}{2})$, four points with coordinates $(\pm \frac{3d}{2}, \pm \frac{3d}{2})$, and eight points with coordinates $(\pm \frac{3d}{2}, \pm \frac{d}{2})$, or $(\pm \frac{d}{2}, \pm \frac{3d}{2})$. Thus, the average transmitted power is :

$$P_{av} = \frac{1}{2 \times 16} \sum_{i=1}^{16} (A_{mc}^2 + A_{ms}^2) = \frac{1}{32} \left[4 \times \frac{d^2}{2} + 4 \times \frac{9d^2}{2} + 8 \times \frac{10d^2}{4} \right] = \frac{5}{4} d^2$$

Since $P_{av} = 592.8 \times 10^{-7}$, we obtain

$$d = \sqrt{4 \frac{P_{av}}{5}} = 0.0069$$

Problem 9.12

The channel (bandpass) bandwidth is $W = 4000$ Hz. Hence, the lowpass equivalent bandwidth will extend from -2 to 2 KHz.

(a) Binary PAM with a pulse shape that has $\beta = \frac{1}{2}$. Hence :

$$\frac{1}{2T}(1 + \beta) = 2000$$

so $\frac{1}{T} = 2667$, and since $k = 1$ bit/symbols is transmitted, the bit rate is 2667 bps.

(b) Four-phase PSK with a pulse shape that has $\beta = \frac{1}{2}$. From (a) the symbol rate is $\frac{1}{T} = 2667$ and

the bit rate is 5334 bps.

(c) $M = 8$ QAM with a pulse shape that has $\beta = \frac{1}{2}$. From (a), the symbol rate is $\frac{1}{T} = 2667$ and hence the bit rate $\frac{3}{T} = 8001$ bps.

(d) Binary FSK with noncoherent detection. Assuming that the frequency separation between the two frequencies is $\Delta f = \frac{1}{T}$, where $\frac{1}{T}$ is the bit rate, the two frequencies are $f_c + \frac{1}{2T}$ and $f_c - \frac{1}{2T}$. Since $W = 4000$ Hz, we may select $\frac{1}{2T} = 1000$, or, equivalently, $\frac{1}{T} = 2000$. Hence, the bit rate is 2000 bps, and the two FSK signals are orthogonal.

(e) Four FSK with noncoherent detection. In this case we need four frequencies with separation of $\frac{1}{T}$ between adjacent frequencies. We select $f_1 = f_c - \frac{1.5}{T}$, $f_2 = f_c - \frac{1}{2T}$, $f_3 = f_c + \frac{1}{2T}$, and $f_4 = f_c + \frac{1.5}{T}$, where $\frac{1}{2T} = 500$ Hz. Hence, the symbol rate is $\frac{1}{T} = 1000$ symbols per second and since each symbol carries two bits of information, the bit rate is 2000 bps.

(f) $M = 8$ FSK with noncoherent detection. In this case we require eight frequencies with frequency separation of $\frac{1}{T} = 500$ Hz for orthogonality. Since each symbol carries 3 bits of information, the bit rate is 1500 bps.

Problem 9.13

(a) The bandwidth of the bandpass channel is :

$$W = 3000 - 600 = 2400 \text{ Hz}$$

Since each symbol of the QPSK constellation conveys 2 bits of information, the symbol rate of transmission is :

$$R = \frac{1}{T} = \frac{2400}{2} = 1200 \text{ symbols/sec}$$

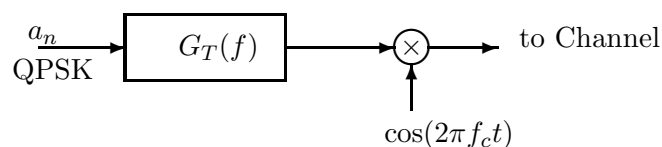
Thus, for spectral shaping we can use a signal pulse with a raised cosine spectrum and roll-off factor $\beta = 1$, since the spectral requirements will be $\frac{1}{2T}(1 + \beta) = \frac{1}{T} = 1200\text{Hz}$. Hence :

$$X_{rc}(f) = \frac{T}{2}[1 + \cos(\pi T|f|)] = \frac{1}{1200} \cos^2\left(\frac{\pi|f|}{2400}\right)$$

If the desired spectral characteristic is split evenly between the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$, then

$$G_T(f) = G_R(f) = \sqrt{\frac{1}{1200}} \cos\left(\frac{\pi|f|}{2400}\right), \quad |f| < \frac{1}{T} = 1200$$

A block diagram of the transmitter is shown in the next figure.



(b) If the bit rate is 4800 bps, then the symbol rate is

$$R = \frac{4800}{2} = 2400 \text{ symbols/sec}$$

In order to satisfy the Nyquist criterion, the the signal pulse used for spectral shaping, should have roll-off factor $\beta = 0$ with corresponding spectrum :

$$X(f) = T, \quad |f| < 1200$$

Thus, the frequency response of the transmitting filter is $G_T(f) = \sqrt{T}$, $|f| < 1200$.

Problem 9.14

The bandwidth of the bandpass channel is :

$$W = 3300 - 300 = 3000 \text{ Hz}$$

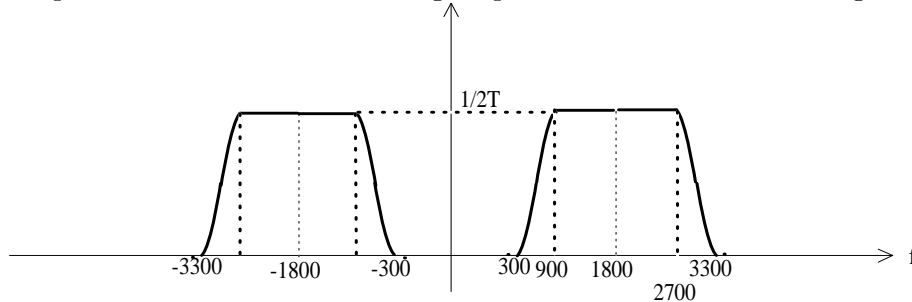
In order to transmit 9600 bps with a symbol rate $R = \frac{1}{T} = 2400$ symbols per second, the number of information bits per symbol should be :

$$k = \frac{9600}{2400} = 4$$

Hence, a $2^4 = 16$ QAM signal constellation is needed. The carrier frequency f_c is set to 1800 Hz, which is the mid-frequency of the frequency band that the bandpass channel occupies. If a pulse with raised cosine spectrum and roll-off factor β is used for spectral shaping, then for the bandpass signal with bandwidth W :

$$\frac{1}{2T}(1 + \beta) = \frac{W}{2} = 1500 \Rightarrow \beta = 0.25$$

A sketch of the spectrum of the transmitted signal pulse is shown in the next figure.



Problem 9.15

The SNR at the detector is :

$$\frac{\mathcal{E}_b}{N_0} = \frac{P_b T}{N_0} = \frac{P_b(1 + \beta)}{N_0 W} = 30 \text{ dB}$$

Since it is desired to expand the bandwidth by a factor of $\frac{10}{3}$ while maintaining the same SNR, the received power P_b should increase by the same factor. Thus the additional power needed is

$$P_a = 10 \log_{10} \frac{10}{3} = 5.2288 \text{ dB}$$

Hence, the required transmitted power is :

$$P_S = -3 + 5.2288 = 2.2288 \text{ dBW}$$

Problem 9.16

The pulse $x(t)$ having the raised cosine spectrum given by (9-2-26/27) is :

$$x(t) = \text{sinc}(t/T) \frac{\cos(\pi\beta t/T)}{1 - 4\beta^2 t^2/T^2}$$

The function $\text{sinc}(t/T)$ is 1 when $t = 0$ and 0 when $t = nT$. Therefore, the Nyquist criterion will be satisfied as long as the function $g(t)$ is :

$$g(t) = \frac{\cos(\pi\beta t/T)}{1 - 4\beta^2 t^2/T^2} = \begin{cases} 1 & t = 0 \\ \text{bounded} & t \neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $4\beta^2 t^2/T^2 = 1$ or $\beta t = \frac{T}{2}$. However :

$$\lim_{\beta t \rightarrow \frac{T}{2}} \frac{\cos(\pi\beta t/T)}{1 - 4\beta^2 t^2/T^2} = \lim_{x \rightarrow 1} \frac{\cos(\frac{\pi}{2}x)}{1 - x}$$

and by using L'Hospital's rule :

$$\lim_{x \rightarrow 1} \frac{\cos(\frac{\pi}{2}x)}{1 - x} = \lim_{x \rightarrow 1} \frac{\pi}{2} \sin(\frac{\pi}{2}x) = \frac{\pi}{2} < \infty$$

Hence :

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

Problem 9.17

Substituting the expression of $X_{rc}(f)$ given by (8.2.22) in the desired integral, we obtain :

$$\begin{aligned}
 \int_{-\infty}^{\infty} X_{rc}(f) df &= \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \frac{T}{2} \left[1 + \cos \frac{\pi T}{\beta} \left(-f - \frac{1-\beta}{2T} \right) \right] df + \int_{-\frac{1-\beta}{2T}}^{\frac{1-\beta}{2T}} T df \\
 &\quad + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \frac{T}{2} \left[1 + \cos \frac{\pi T}{\beta} \left(f - \frac{1-\beta}{2T} \right) \right] df \\
 &= \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \frac{T}{2} df + T \left(\frac{1-\beta}{T} \right) + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \frac{T}{2} df \\
 &\quad + \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \cos \frac{\pi T}{\beta} \left(f + \frac{1-\beta}{2T} \right) df + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \cos \frac{\pi T}{\beta} \left(f - \frac{1-\beta}{2T} \right) df \\
 &= 1 + \int_{-\frac{\beta}{T}}^0 \cos \frac{\pi T}{\beta} x dx + \int_0^{\frac{\beta}{T}} \cos \frac{\pi T}{\beta} x dx \\
 &= 1 + \int_{-\frac{\beta}{T}}^{\frac{\beta}{T}} \cos \frac{\pi T}{\beta} x dx = 1 + 0 = 1
 \end{aligned}$$

Problem 9.18

Let $X(f)$ be such that

$$\operatorname{Re}[X(f)] = \begin{cases} T\Pi(fT) + U(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases} \quad \operatorname{Im}[X(f)] = \begin{cases} V(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases}$$

with $U(f)$ even with respect to 0 and odd with respect to $f = \frac{1}{2T}$. Since $x(t)$ is real, $V(f)$ is odd with respect to 0 and by assumption it is even with respect to $f = \frac{1}{2T}$. Then,

$$\begin{aligned}
 x(t) &= \mathcal{F}^{-1}[X(f)] \\
 &= \int_{-\frac{1}{T}}^{\frac{1}{2T}} X(f) e^{j2\pi ft} df + \int_{-\frac{1}{2T}}^{\frac{1}{T}} X(f) e^{j2\pi ft} df + \int_{\frac{1}{2T}}^{\frac{1}{T}} X(f) e^{j2\pi ft} df \\
 &= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} T e^{j2\pi ft} df + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)] e^{j2\pi ft} df \\
 &= \operatorname{sinc}(t/T) + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)] e^{j2\pi ft} df
 \end{aligned}$$

Consider first the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f) e^{j2\pi ft} df$. Clearly,

$$\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f) e^{j2\pi ft} df = \int_{-\frac{1}{T}}^0 U(f) e^{j2\pi ft} df + \int_0^{\frac{1}{T}} U(f) e^{j2\pi ft} df$$

and by using the change of variables $f' = f + \frac{1}{2T}$ and $f' = f - \frac{1}{2T}$ for the two integrals on the right hand side respectively, we obtain

$$\begin{aligned}
 & \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f) e^{j2\pi ft} df \\
 &= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' - \frac{1}{2T}) e^{j2\pi f't} df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f't} df' \\
 &\stackrel{a}{=} \left(e^{j\frac{\pi}{T}t} - e^{-j\frac{\pi}{T}t} \right) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f't} df' \\
 &= 2j \sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f't} df'
 \end{aligned}$$

where for step (a) we used the odd symmetry of $U(f')$ with respect to $f' = \frac{1}{2T}$, that is

$$U(f' - \frac{1}{2T}) = -U(f' + \frac{1}{2T})$$

For the integral $\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f) e^{j2\pi ft} df$ we have

$$\begin{aligned}
 & \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f) e^{j2\pi ft} df \\
 &= \int_{-\frac{1}{2T}}^0 V(f) e^{j2\pi ft} df + \int_0^{\frac{1}{2T}} V(f) e^{j2\pi ft} df \\
 &= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T}) e^{j2\pi f't} df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T}) e^{j2\pi f't} df'
 \end{aligned}$$

However, $V(f)$ is odd with respect to 0 and since $V(f' + \frac{1}{2T})$ and $V(f' - \frac{1}{2T})$ are even, the translated spectra satisfy

$$\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T}) e^{j2\pi f't} df' = - \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T}) e^{j2\pi f't} df'$$

Hence,

$$\begin{aligned}
 x(t) &= \text{sinc}(t/T) + 2j \sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f't} df' \\
 &\quad - 2 \sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f't} df'
 \end{aligned}$$

and therefore,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Thus, the signal $x(t)$ satisfies the Nyquist criterion.

Problem 9.19

The bandwidth of the channel is :

$$W = 3000 - 300 = 2700 \text{ Hz}$$

Since the minimum transmission bandwidth required for bandpass signaling is R , where R is the rate of transmission, we conclude that the maximum value of the symbol rate for the given channel is $R_{\max} = 2700$. If an M -ary PAM modulation is used for transmission, then in order to achieve a bit-rate of 9600 bps, with maximum rate of R_{\max} , the minimum size of the constellation is $M = 2^k = 16$. In this case, the symbol rate is :

$$R = \frac{9600}{k} = 2400 \text{ symbols/sec}$$

and the symbol interval $T = \frac{1}{R} = \frac{1}{2400}$ sec. The roll-off factor β of the raised cosine pulse used for transmission is determined by noting that $1200(1 + \beta) = 1350$, and hence, $\beta = 0.125$. Therefore, the squared root raised cosine pulse can have a roll-off of $\beta = 0.125$.

Problem 9.20

Since the one-sided bandwidth of the ideal lowpass channel is $W = 2400$ Hz, the rate of transmission is :

$$R = 2 \times 2400 = 4800 \text{ symbols/sec}$$

(remember that PAM can be transmitted single-sideband; hence, if the lowpass channel has bandwidth from $-W$ to W , the passband channel will have bandwidth equal to W ; on the other hand, a PSK or QAM system will have passband bandwidth equal to $2W$). The number of bits per symbol is

$$k = \frac{14400}{4800} = 3$$

Hence, the number of transmitted symbols is $2^3 = 8$. If a duobinary pulse is used for transmission, then the number of possible transmitted symbols is $2M - 1 = 15$. These symbols have the form

$$b_n = 0, \pm 2d, \pm 4d, \dots, \pm 12d$$

where $2d$ is the minimum distance between the points of the 8-PAM constellation. The probability mass function of the received symbols is

$$P(b = 2md) = \frac{8 - |m|}{64}, \quad m = 0, \pm 1, \dots, \pm 7$$

An upper bound of the probability of error is given by (see (9-3-18))

$$P_M < 2 \left(1 - \frac{1}{M^2}\right) Q \left[\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6}{M^2 - 1} \frac{k\mathcal{E}_{b,av}}{N_0}} \right]$$

With $P_M = 10^{-6}$ and $M = 8$ we obtain

$$\frac{k\mathcal{E}_{b,av}}{N_0} = 1.3193 \times 10^3 \implies \mathcal{E}_{b,av} = 0.088$$

Problem 9.21

(a) The spectrum of the baseband signal is (see (4-4-12))

$$\Phi_V(f) = \frac{1}{T} \Phi_i(f) |X_{rc}(f)|^2 = \frac{1}{T} |X_{rc}(f)|^2$$

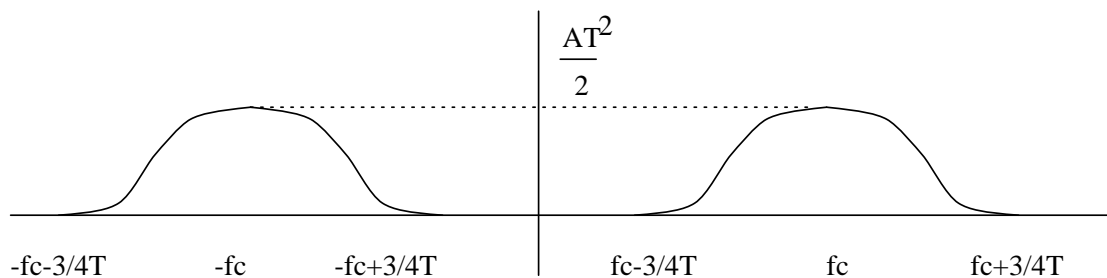
where $T = \frac{1}{2400}$ and

$$X_{rc}(f) = \begin{cases} T & 0 \leq |f| \leq \frac{1}{4T} \\ \frac{T}{2} (1 + \cos(2\pi T(|f| - \frac{1}{4T}))) & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0 & \text{otherwise} \end{cases}$$

If the carrier signal has the form $c(t) = A \cos(2\pi f_c t)$, then the spectrum of the DSB-SC modulated signal, $\Phi_U(f)$, is

$$\Phi_U(f) = \frac{A}{2} [\Phi_V(f - f_c) + \Phi_V(f + f_c)]$$

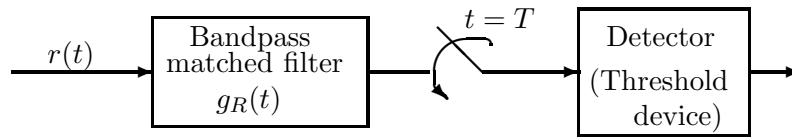
A sketch of $\Phi_U(f)$ is shown in the next figure.



(b) Assuming bandpass coherent demodulation using a matched filter, the received signal $r(t)$ is first passed through a linear filter with impulse response

$$g_R(t) = Ax_{rc}(T - t) \cos(2\pi f_c(T - t))$$

The output of the matched filter is sampled at $t = T$ and the samples are passed to the detector. The detector is a simple threshold device that decides if a binary 1 or 0 was transmitted depending on the sign of the input samples. The following figure shows a block diagram of the optimum bandpass coherent demodulator.



Problem 9.22

(a) The power spectral density of $X(t)$ is given by (see (4-4-12))

$$\Phi_x(f) = \frac{1}{T} \Phi_a(f) |G_T(f)|^2$$

The Fourier transform of $g(t)$ is

$$G_T(f) = \mathcal{F}[g(t)] = AT \frac{\sin \pi f T}{\pi f T} e^{-j\pi f T}$$

Hence,

$$|G_T(f)|^2 = (AT)^2 \text{sinc}^2(fT)$$

and therefore,

$$\Phi_x(f) = A^2 T \Phi_a(f) \text{sinc}^2(fT) = A^2 T \text{sinc}^2(fT)$$

(b) If $g_1(t)$ is used instead of $g(t)$ and the symbol interval is T , then

$$\begin{aligned} \Phi_x(f) &= \frac{1}{T} \Phi_a(f) |G_{2T}(f)|^2 \\ &= \frac{1}{T} (A2T)^2 \text{sinc}^2(f2T) = 4A^2 T \text{sinc}^2(f2T) \end{aligned}$$

(c) If we precode the input sequence as $b_n = a_n + \alpha a_{n-3}$, then

$$\phi_b(m) = \begin{cases} 1 + \alpha^2 & m = 0 \\ \alpha & m = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

and therefore, the power spectral density $\Phi_b(f)$ is

$$\Phi_b(f) = 1 + \alpha^2 + 2\alpha \cos(2\pi f 3T)$$

To obtain a null at $f = \frac{1}{3T}$, the parameter α should be such that

$$1 + \alpha^2 + 2\alpha \cos(2\pi f 3T)|_{f=\frac{1}{3}} = 0 \implies \alpha = -1$$

(c) The answer to this question is no. This is because $\Phi_b(f)$ is an analytic function and unless it is identical to zero it can have at most a countable number of zeros. This property of the analytic functions is also referred as the theorem of isolated zeros.

Problem 9.23

The roll-off factor β is related to the bandwidth by the expression $\frac{1+\beta}{T} = 2W$, or equivalently $R(1 + \beta) = 2W$. The following table shows the symbol rate for the various values of the excess bandwidth and for $W = 1500$ Hz.

β	.25	.33	.50	.67	.75	1.00
R	2400	2256	2000	1796	1714	1500

The above results were obtained with the assumption that double-sideband PAM is employed, so the available lowpass bandwidth will be from $-W = \frac{3000}{2}$ to W Hz. If single-sideband transmission is used, then the spectral efficiency is doubled, and the above symbol rates R are doubled.

Problem 9.24

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a duobinary transmitting filter.

Data seq. D_n :	1	0	0	1	0	1	1	0	0	1	0
Precoded seq. P_n :	0	1	1	1	0	0	1	0	0	0	1
Transmitted seq. I_n :	-1	1	1	1	-1	-1	1	-1	-1	-1	1
Received seq. B_n :	0	2	2	0	-2	0	0	-2	-2	0	2
Decoded seq. D_n :	1	0	0	1	0	1	1	0	0	1	0

Problem 9.25

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a modified duobinary transmitting filter.

Data seq. D_n :	1	0	0	1	0	1	1	0	0	1	0
Precoded seq. P_n :	0	0	1	0	1	1	1	0	0	0	1
Transmitted seq. I_n :	-1	-1	1	-1	1	1	1	-1	-1	-1	-1
Received seq. B_n :	2	0	0	2	0	-2	-2	0	0	2	0
Decoded seq. D_n :	1	0	0	1	0	1	1	0	0	1	0

Problem 9.26

Let $X(z)$ denote the \mathcal{Z} -transform of the sequence x_n , that is

$$X(z) = \sum_n x_n z^{-n}$$

Then the precoding operation can be described as

$$P(z) = \frac{D(z)}{X(z)} \mod -M$$

where $D(z)$ and $P(z)$ are the \mathcal{Z} -transforms of the data and precoded sequences respectively. For example, if $M = 2$ and $X(z) = 1 + z^{-1}$ (duobinary signaling), then

$$P(z) = \frac{D(z)}{1 + z^{-1}} \implies P(z) = D(z) - z^{-1}P(z)$$

which in the time domain is written as

$$p_n = d_n - p_{n-1}$$

and the subtraction is mod-2.

However, the inverse filter $\frac{1}{X(z)}$ exists only if x_0 , the first coefficient of $X(z)$ is relatively prime with M . If this is not the case, then the precoded symbols p_n cannot be determined uniquely from the data sequence d_n .

In the example given in the book, where $x_0 = 2$ we note that whatever the value of d_n (0 or 1), the value of $(2d_n \mod 2)$ will be zero, hence this precoding scheme cannot work.

Problem 9.27

(a) The frequency response of the RC filter is

$$C(f) = \frac{\frac{1}{j2\pi RCf}}{R + \frac{1}{j2\pi RCf}} = \frac{1}{1 + j2\pi RCf}$$

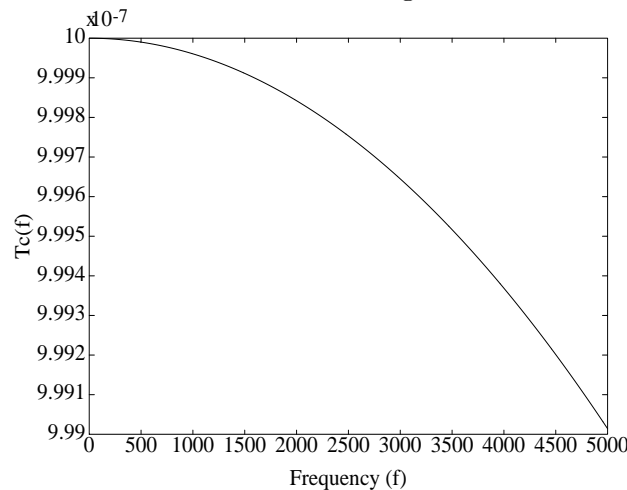
The amplitude and the phase spectrum of the filter are :

$$|C(f)| = \left(\frac{1}{1 + 4\pi^2(RC)^2 f^2} \right)^{\frac{1}{2}}, \quad \theta_c(f) = \arctan(-2\pi RCf)$$

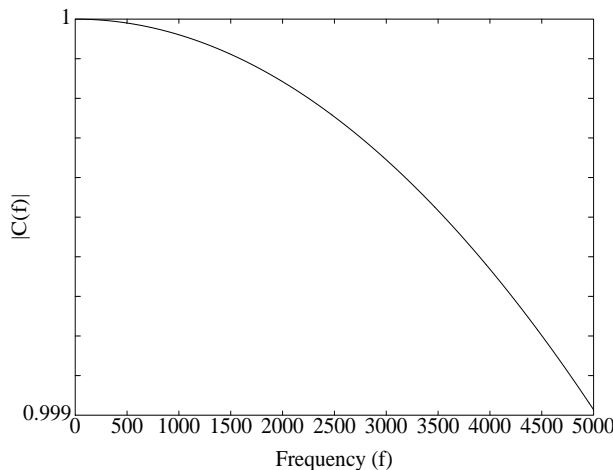
The envelope delay is

$$\tau_c(f) = -\frac{1}{2\pi} \frac{d\theta_c(f)}{df} = -\frac{1}{2\pi} \frac{-2\pi RC}{1 + 4\pi^2(RC)^2 f^2} = \frac{RC}{1 + 4\pi^2(RC)^2 f^2}$$

A plot of $\tau(f)$ with $RC = 10^{-6}$ is shown in the next figure :



(b) The following figure is a plot of the amplitude characteristics of the RC filter, $|C(f)|$. The values of the vertical axis indicate that $|C(f)|$ can be considered constant for frequencies up to 2000 Hz. Since the same is true for the envelope delay, we conclude that a lowpass signal of bandwidth $\Delta f = 1$ KHz will not be distorted if it passes the RC filter.



Problem 9.28

Let $G_T(f)$ and $G_R(f)$ be the frequency response of the transmitting and receiving filter. Then, the condition for zero ISI implies

$$G_T(f)C(f)G_R(f) = X_{rc}(f) = \begin{cases} T, & 0 \leq |f| \leq \frac{1}{4T} \\ \frac{T}{2}[1 + \cos(2\pi T(|f| - \frac{1}{T}))], & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0, & |f| > \frac{3}{4T} \end{cases}$$

Since the additive noise is white, the optimum transmitting and receiving filter characteristics are given by (see (9-2-81))

$$|G_T(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}, \quad |G_R(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}$$

Thus,

$$|G_T(f)| = |G_R(f)| = \begin{cases} \left[\frac{T}{1+0.3 \cos 2\pi f T} \right]^{\frac{1}{2}}, & 0 \leq |f| \leq \frac{1}{4T} \\ \left[\frac{T(1+\cos(2\pi T(|f| - \frac{1}{T}))}{2(1+0.3 \cos 2\pi f T)} \right]^{\frac{1}{2}}, & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0, & \text{otherwise} \end{cases}$$

Problem 9.29

A 4-PAM modulation can accomodate $k = 2$ bits per transmitted symbol. Thus, the symbol interval duration is :

$$T = \frac{k}{9600} = \frac{1}{4800} \text{ sec}$$

Since, the channel's bandwidth is $W = 2400 = \frac{1}{2T}$, in order to achieve the maximum rate of transmission, $R_{\max} = \frac{1}{2T}$, the spectrum of the signal pulse should be :

$$X(f) = T \Pi \left(\frac{f}{2W} \right)$$

Then, the magnitude frequency response of the optimum transmitting and receiving filter is (see (9-2-81))

$$|G_T(f)| = |G_R(f)| = \left[1 + \left(\frac{f}{2400} \right)^2 \right]^{\frac{1}{4}} \Pi \left(\frac{f}{2W} \right) = \begin{cases} \left[1 + \left(\frac{f}{2400} \right)^2 \right]^{\frac{1}{4}}, & |f| < 2400 \\ 0, & \text{otherwise} \end{cases}$$

Problem 9.30

We already know that

$$\begin{aligned}\sigma_v^2 &= \int_{-\infty}^{\infty} \Phi_{nn}(f) |G_R(f)|^2 df \\ P_{av} &= \frac{d^2}{T} \int_{-W}^W |G_T(f)|^2 df \\ |G_T(f)| &= \frac{|X_{rc}(f)|}{|G_R(f)||C(f)|}, |f| \leq W\end{aligned}$$

From these

$$\frac{\sigma_v^2}{d^2} = \frac{1}{P_{av}T} \int_{-W}^W \Phi_{nn}(f) |G_R(f)|^2 df \int_{-W}^W \frac{|X_{rc}(f)|^2}{|G_R(f)|^2 |C(f)|^2} df \quad (9.0.4)$$

The optimum $|G_R(f)|$ can be found by applying the Cauchy-Schwartz inequality

$$\int_{-\infty}^{\infty} |U_1(f)|^2 df \int_{-\infty}^{\infty} |U_2(f)|^2 df \geq \left[\int_{-\infty}^{\infty} |U_1(f)||U_2(f)| df \right]^2$$

where $|U_1(f)|, |U_2(f)|$ are defined as

$$\begin{aligned}|U_1(f)| &= |\sqrt{\Phi_{nn}(f)}| |G_R(f)| \\ |U_2(f)| &= \frac{|X_{rc}(f)|}{|G_R(f)||C(f)|}\end{aligned}$$

The minimum value of (1) is obtained when $|U_1(f)|$ is proportional to $|U_2(f)|$, i.e. $|U_1(f)| = K|U_2(f)|$ or, equivalently, when

$$|G_R(f)| = \sqrt{K} \frac{|X_{rc}(f)|^{1/2}}{[N_0/2]^{1/4} |C(f)|^{1/2}}, \quad |f| \leq W$$

where K is an arbitrary constant. By setting it appropriately,

$$|G_R(f)| = \frac{|X_{rc}(f)|^{1/2}}{|C(f)|^{1/2}}, \quad |f| \leq W$$

The corresponding modulation filter has a magnitude characteristic of

$$|G_T(f)| = \frac{|X_{rc}(f)|}{|G_R(f)||C(f)|} = \frac{|X_{rc}(f)|^{1/2}}{|C(f)|^{1/2}}, \quad |f| \leq W$$

Problem 9.31

In the case where the channel distortion is fully precompensated at the transmitter, the loss of SNR is given by

$$10 \log L_1, \text{ with } L_1 = \int_{-W}^W \frac{X_{rc}(f)}{|C(f)|^2}$$

whereas in the case of the equally split filters, the loss of SNR is given by

$$10 \log [L_2]^2, \text{ with } L_2 = \int_{-W}^W \frac{X_{rc}(f)}{|C(f)|}$$

Assuming that $1/T = W$, so that we have a raised cosine characteristic with $\beta = 0$, we have

$$X_{rc}(f) = \frac{1}{2W} \left[1 + \cos \frac{\pi|f|}{W} \right]$$

Then

$$\begin{aligned} L_1 &= 2 \int_0^W \frac{1}{2W} \frac{[1 + \cos \frac{\pi f}{W}]}{|C(f)|^2} \\ &= 2 \left[\int_0^{W/2} \frac{1}{2W} \frac{[1 + \cos \frac{\pi f}{W}]}{1} + \int_{W/2}^W \frac{1}{2W} \frac{[1 + \cos \frac{\pi f}{W}]}{1/4} \right] \\ &= \frac{5\pi - 6}{2\pi} \end{aligned}$$

Hence, the loss for the first type of filters is $10 \log L_1 = 1.89$ dB.

In a similar way,

$$\begin{aligned} L_2 &= 2 \int_0^W \frac{1}{2W} \frac{[1 + \cos \frac{\pi f}{W}]}{|C(f)|} \\ &= 2 \left[\int_0^{W/2} \frac{1}{2W} \frac{[1 + \cos \frac{\pi f}{W}]}{1} + \int_{W/2}^W \frac{1}{2W} \frac{[1 + \cos \frac{\pi f}{W}]}{1/2} \right] \\ &= \frac{3\pi - 2}{2\pi} \end{aligned}$$

Hence, the loss for the second type of filters is $10 \log [L_2]^2 = 1.45$ dB. As expected, the second type of filters which split the channel characteristics between the transmitter and the receiver exhibit a smaller SNR loss.

Problem 9.32

Suppose that $a_m = +1$ is the transmitted signal. Then the probability of error will be :

$$\begin{aligned} P_{e|1} &= P(y_m < 0 | a_m = +1) \\ &= P(1 + n_m + i_m < 0) \\ &= \frac{1}{4} P(1/2 + n_m < 0) + \frac{1}{4} P(3/2 + n_m < 0) + \frac{1}{2} P(1 + n_m < 0) \\ &= \frac{1}{4} Q \left[\frac{1}{2\sigma_n} \right] + \frac{1}{4} Q \left[\frac{3}{2\sigma_n} \right] + \frac{1}{2} Q \left[\frac{1}{\sigma_n} \right] \end{aligned}$$

Due to the symmetry of the intersymbol interference, the probability of error, when $a_m = -1$ is transmitted, is the same. Thus, the above result is the average probability of error.

Problem 9.33

(a) If the transmitted signal is :

$$r(t) = \sum_{n=-\infty}^{\infty} I_n h(t - nT) + n(t)$$

then the output of the receiving filter is :

$$y(t) = \sum_{n=-\infty}^{\infty} I_n x(t - nT) + \nu(t)$$

where $x(t) = h(t) \star h(t)$ and $\nu(t) = n(t) \star h(t)$. If the sampling time is off by 10%, then the samples at the output of the correlator are taken at $t = (m \pm \frac{1}{10})T$. Assuming that $t = (m - \frac{1}{10})T$ without loss of generality, then the sampled sequence is :

$$y_m = \sum_{n=-\infty}^{\infty} I_n x((m - \frac{1}{10})T - nT) + \nu((m - \frac{1}{10})T)$$

If the signal pulse is rectangular with amplitude A and duration T , then $\sum_{n=-\infty}^{\infty} I_n x((m - \frac{1}{10})T - nT)$ is nonzero only for $n = m$ and $n = m - 1$ and therefore, the sampled sequence is given by :

$$\begin{aligned} y_m &= I_m x(-\frac{1}{10}T) + I_{m-1} x(T - \frac{1}{10}T) + \nu((m - \frac{1}{10})T) \\ &= \frac{9}{10} I_m A^2 T + I_{m-1} \frac{1}{10} A^2 T + \nu((m - \frac{1}{10})T) \end{aligned}$$

The variance of the noise is :

$$\sigma_\nu^2 = \frac{N_0}{2} A^2 T$$

and therefore, the SNR is :

$$\text{SNR} = \left(\frac{9}{10} \right)^2 \frac{2(A^2 T)^2}{N_0 A^2 T} = \frac{81}{100} \frac{2A^2 T}{N_0}$$

As it is observed, there is a loss of $10 \log_{10} \frac{81}{100} = -0.9151$ dB due to the mistiming.

(b) Recall from part (a) that the sampled sequence is

$$y_m = \frac{9}{10} I_m A^2 T + I_{m-1} \frac{1}{10} A^2 T + \nu_m$$

The term $I_{m-1} \frac{A^2 T}{10}$ expresses the ISI introduced to the system. If $I_m = 1$ is transmitted, then the probability of error is

$$\begin{aligned}
 P(e|I_m = 1) &= \frac{1}{2}P(e|I_m = 1, I_{m-1} = 1) + \frac{1}{2}P(e|I_m = 1, I_{m-1} = -1) \\
 &= \frac{1}{2\sqrt{\pi N_0 A^2 T}} \int_{-\infty}^{-A^2 T} e^{-\frac{\nu^2}{N_0 A^2 T}} d\nu + \frac{1}{2\sqrt{\pi N_0 A^2 T}} \int_{-\infty}^{-\frac{8}{10} A^2 T} e^{-\frac{\nu^2}{N_0 A^2 T}} d\nu \\
 &= \frac{1}{2}Q \left[\sqrt{\frac{2A^2 T}{N_0}} \right] + \frac{1}{2}Q \left[\sqrt{\left(\frac{8}{10}\right)^2 \frac{2A^2 T}{N_0}} \right]
 \end{aligned}$$

Since the symbols of the binary PAM system are equiprobable the previous derived expression is the probability of error when a symbol by symbol detector is employed. Comparing this with the probability of error of a system with no ISI, we observe that there is an increase of the probability of error by

$$P_{\text{diff}}(e) = \frac{1}{2}Q \left[\sqrt{\left(\frac{8}{10}\right)^2 \frac{2A^2 T}{N_0}} \right] - \frac{1}{2}Q \left[\sqrt{\frac{2A^2 T}{N_0}} \right]$$

Problem 9.34

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$h(t) = \mathcal{F}^{-1}[H(f)] = \delta(t) + \frac{\alpha}{2}\delta(t - t_0) + \frac{\alpha}{2}\delta(t + t_0)$$

Hence,

$$y(t) = s(t) \star h(t) = s(t) + \frac{\alpha}{2}s(t - t_0) + \frac{\alpha}{2}s(t + t_0)$$

(b) If the signal $s(t)$ is used to modulate the sequence $\{I_n\}$, then the transmitted signal is :

$$u(t) = \sum_{n=-\infty}^{\infty} I_n s(t - nT)$$

The received signal is the convolution of $u(t)$ with $h(t)$. Hence,

$$\begin{aligned}
 y(t) &= u(t) \star h(t) = \left(\sum_{n=-\infty}^{\infty} I_n s(t - nT) \right) \star \left(\delta(t) + \frac{\alpha}{2}\delta(t - t_0) + \frac{\alpha}{2}\delta(t + t_0) \right) \\
 &= \sum_{n=-\infty}^{\infty} I_n s(t - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n s(t - t_0 - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n s(t + t_0 - nT)
 \end{aligned}$$

Thus, the output of the matched filter $s(-t)$ at the time instant t_1 is :

$$\begin{aligned} w(t_1) &= \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} s(\tau - nT) s(\tau - t_1) d\tau \\ &\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} s(\tau - t_0 - nT) s(\tau - t_1) d\tau \\ &\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} s(\tau + t_0 - nT) s(\tau - t_1) d\tau \end{aligned}$$

If we denote the signal $s(t) \star s(t)$ by $x(t)$, then the output of the matched filter at $t_1 = kT$ is :

$$\begin{aligned} w(kT) &= \sum_{n=-\infty}^{\infty} I_n x(kT - nT) \\ &\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n x(kT - t_0 - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n x(kT + t_0 - nT) \end{aligned}$$

(c) With $t_0 = T$ and $k = n$ in the previous equation, we obtain :

$$\begin{aligned} w_k &= I_k x_0 + \sum_{n \neq k} I_n x_{k-n} \\ &\quad + \frac{\alpha}{2} I_k x_{-1} + \frac{\alpha}{2} \sum_{n \neq k} I_n x_{k-n-1} + \frac{\alpha}{2} I_k x_1 + \frac{\alpha}{2} \sum_{n \neq k} I_n x_{k-n+1} \\ &= I_k \left(x_0 + \frac{\alpha}{2} x_{-1} + \frac{\alpha}{2} x_1 \right) + \sum_{n \neq k} I_n \left[x_{k-n} + \frac{\alpha}{2} x_{k-n-1} + \frac{\alpha}{2} x_{k-n+1} \right] \end{aligned}$$

The terms under the summation is the ISI introduced by the channel. If the signal $s(t)$ is designed so as to satisfy the Nyquist criterion, then :

$$x_k = 0, \quad k \neq 0$$

and the above expression simplifies to :

$$w_k = I_k + \frac{\alpha}{2} (I_{k+1} + I_{k-1})$$

Problem 9.35

(a) Each segment of the wire-line can be considered as a bandpass filter with bandwidth $W = 1200$ Hz. Thus, the highest bit rate that can be transmitted without ISI by means of binary PAM is :

$$R = 2W = 2400 \text{ bps}$$

(b) The probability of error for binary PAM transmission is :

$$P_2 = Q \left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right]$$

Hence, using mathematical tables for the function $Q[\cdot]$, we find that $P_2 = 10^{-7}$ is obtained for :

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.2 \implies \frac{\mathcal{E}_b}{N_0} = 13.52 = 11.30 \text{ dB}$$

(c) The received power P_R is related to the desired SNR per bit through the relation :

$$\frac{P_R}{N_0} = \frac{1}{T} \frac{\mathcal{E}_b}{N_0} = R \frac{\mathcal{E}_b}{N_0}$$

Hence, with $N_0 = 4.1 \times 10^{-21}$ we obtain :

$$P_R = 4.1 \times 10^{-21} \times 1200 \times 13.52 = 6.6518 \times 10^{-17} = -161.77 \text{ dBW}$$

Since the power loss of each segment is :

$$L_s = 50 \text{ Km} \times 1 \text{ dB/Km} = 50 \text{ dB}$$

the transmitted power at each repeater should be :

$$P_T = P_R + L_s = -161.77 + 50 = -111.77 \text{ dBW}$$

Problem 9.36

$$x_n = \int_{-\infty}^{\infty} h(t + nT) h^*(t) dt$$

$$v_k = \int_{-\infty}^{\infty} z(t) h^*(t - kT) dt$$

Then :

$$\begin{aligned} \frac{1}{2} E [v_j v_k^*] &= \frac{1}{2} E \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z(a) h^*(a - jT) z^*(b) h(b - kT) da db \right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} E [z(a) z^*(b)] h^*(a - jT) h(b - kT) da db \\ &= N_0 \int_{-\infty}^{\infty} h^*(a - jT) h(a - kT) da = N_0 x_{j-k} \end{aligned}$$

Problem 9.37

In the case of duobinary signaling, the output of the matched filter is :

$$x(t) = \text{sinc}(2Wt) + \text{sinc}(2Wt - 1)$$

and the samples x_{n-m} are given by :

$$x_{n-m} = x(nT - mT) = \begin{cases} 1 & n - m = 0 \\ 1 & n - m = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the metric $CM(\mathbf{I})$ in the Viterbi algorithm becomes

$$\begin{aligned} CM(\mathbf{I}) &= 2 \sum_n I_n r_n - \sum_n \sum_m I_n I_m x_{n-m} \\ &= 2 \sum_n I_n r_n - \sum_n I_n^2 - \sum_n I_n I_{n-1} \\ &= \sum_n I_n (2r_n - I_n - I_{n-1}) \end{aligned}$$

Problem 9.38

(a) The output of the matched filter demodulator is :

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} I_k \int_{-\infty}^{\infty} g_T(\tau - kT_b) g_R(t - \tau) d\tau + \nu(t) \\ &= \sum_{k=-\infty}^{\infty} I_k x(t - kT_b) + \nu(t) \end{aligned}$$

where,

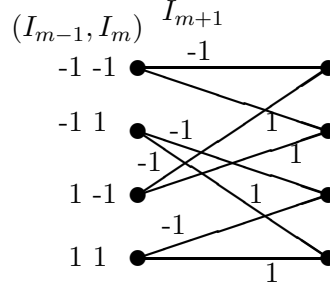
$$x(t) = g_T(t) \star g_R(t) = \frac{\sin \frac{\pi t}{T}}{\frac{\pi t}{T}} \frac{\cos \frac{\pi t}{T}}{1 - 4 \frac{t^2}{T^2}}$$

Hence,

$$\begin{aligned} y(mT_b) &= \sum_{k=-\infty}^{\infty} I_k x(mT_b - kT_b) + \nu(mT_b) \\ &= I_m + \frac{1}{\pi} I_{m-1} + \frac{1}{\pi} I_{m+1} + \nu(mT_b) \end{aligned}$$

The term $\frac{1}{\pi} I_{m-1} + \frac{1}{\pi} I_{m+1}$ represents the ISI introduced by doubling the symbol rate of transmission.

(b) In the next figure we show one trellis stage for the ML sequence detector. Since there is postcursor ISI, we delay the received signal, used by the ML decoder to form the metrics, by one sample. Thus, the states of the trellis correspond to the sequence (I_{m-1}, I_m) , and the transition labels correspond to the symbol I_{m+1} . Two branches originate from each state. The upper branch is associated with the transmission of -1 , whereas the lower branch is associated with the transmission of 1 .



Problem 9.39

(a) The output of the matched filter at the time instant mT is :

$$y_m = \sum_k I_m x_{k-m} + \nu_m = I_m + \frac{1}{4}I_{m-1} + \nu_m$$

The autocorrelation function of the noise samples ν_m is

$$E[\nu_k \nu_j] = \frac{N_0}{2} x_{k-j}$$

Thus, the variance of the noise is

$$\sigma_\nu^2 = \frac{N_0}{2} x_0 = \frac{N_0}{2}$$

If a symbol by symbol detector is employed and we assume that the symbols $I_m = I_{m-1} = \sqrt{\mathcal{E}_b}$ have been transmitted, then the probability of error $P(e|I_m = I_{m-1} = \sqrt{\mathcal{E}_b})$ is :

$$\begin{aligned} P(e|I_m = I_{m-1} = \sqrt{\mathcal{E}_b}) &= P(y_m < 0 | I_m = I_{m-1} = \sqrt{\mathcal{E}_b}) \\ &= P(\nu_m < -\frac{5}{4}\sqrt{\mathcal{E}_b}) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\mathcal{E}_b}} e^{-\frac{\nu_m^2}{N_0}} d\nu_m \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}} e^{-\frac{\nu^2}{2}} d\nu = Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] \end{aligned}$$

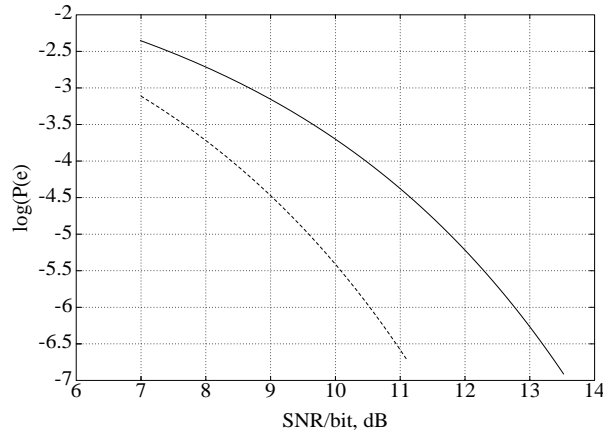
If however $I_{m-1} = -\sqrt{\mathcal{E}_b}$, then :

$$P(e|I_m = \sqrt{\mathcal{E}_b}, I_{m-1} = -\sqrt{\mathcal{E}_b}) = P(\frac{3}{4}\sqrt{\mathcal{E}_b} + \nu_m < 0) = Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Since the two symbols $\sqrt{\mathcal{E}_b}$, $-\sqrt{\mathcal{E}_b}$ are used with equal probability, we conclude that :

$$\begin{aligned} P(e) &= P(e|I_m = \sqrt{\mathcal{E}_b}) = P(e|I_m = -\sqrt{\mathcal{E}_b}) \\ &= \frac{1}{2}Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{2}Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] \end{aligned}$$

(b) In the next figure we plot the error probability obtained in part (a) ($\log_{10}(P(e))$) vs. the SNR per bit and the error probability for the case of no ISI. As it observed from the figure, the relative difference in SNR of the error probability of 10^{-6} is 2 dB.



Problem 9.40

For the DFE we have that :

$$\hat{I}_k = \sum_{j=-K_1}^0 c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}$$

We want to minimize $J = E \left| I_k - \hat{I}_k \right|^2$. Taking the derivative of J , with respect to the real and imaginary parts of $c_l = a_l + jb_l$, $1 \leq l \leq K_2$, we obtain :

$$\begin{aligned} \frac{\partial J}{\partial a_l} = 0 &\Rightarrow E \left[-I_{k-l} \left(I_k^* - \hat{I}_k^* \right) - I_{k-l}^* \left(I_k - \hat{I}_k \right) \right] = 0 \Rightarrow \\ &E \left[\text{Re} \left\{ I_{k-l}^* \left(I_k - \hat{I}_k \right) \right\} \right] = 0 \end{aligned}$$

and similarly :

$$\frac{\partial J}{\partial b_l} = 0 \Rightarrow E \left[\text{Im} \left\{ I_{k-l}^* \left(I_k - \hat{I}_k \right) \right\} \right] = 0$$

Hence,

$$E \left[I_{k-l}^* \left(I_k - \hat{I}_k \right) \right] = 0, \quad 1 \leq l \leq K_2 \quad (1)$$

Since the information symbols are uncorrelated : $E[I_k I_l^*] = \delta_{kl}$. We also have :

$$\begin{aligned} E[I_k u_l^*] &= E\left[I_k \left(\sum_{m=0}^L f_m^* I_{l-m}^* + n_l^*\right)\right] \\ &= f_{l-k}^* \end{aligned}$$

Hence, equation (1) gives :

$$\begin{aligned} E[I_k I_{k-l}^*] &= E[\hat{I}_k I_{k-l}^*], \quad 1 \leq l \leq K_2 \Rightarrow \\ 0 &= E\left[\left(\sum_{j=-K_1}^0 c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}\right) I_{k-l}^*\right] \Rightarrow \\ 0 &= \left(\sum_{j=-K_1}^0 c_j f_{l-j}\right) + c_l \Rightarrow \\ c_l &= -\sum_{j=-K_1}^0 c_j f_{l-j}, \quad 1 \leq l \leq K_2 \end{aligned}$$

which is the desired equation for the feedback taps.

Problem 9.41

(a) The equivalent discrete-time impulse response of the channel is :

$$h(t) = \sum_{n=-1}^1 h_n \delta(t - nT) = 0.3\delta(t + T) + 0.9\delta(t) + 0.3\delta(t - T)$$

If by $\{c_n\}$ we denote the coefficients of the FIR equalizer, then the equalized signal is :

$$q_m = \sum_{n=-1}^1 c_n h_{m-n}$$

which in matrix notation is written as :

$$\begin{pmatrix} 0.9 & 0.3 & 0. \\ 0.3 & 0.9 & 0.3 \\ 0. & 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The coefficients of the zero-force equalizer can be found by solving the previous matrix equation. Thus,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{pmatrix}$$

(b) The values of q_m for $m = \pm 2, \pm 3$ are given by

$$\begin{aligned} q_2 &= \sum_{n=-1}^1 c_n h_{2-n} = c_1 h_1 = -0.1429 \\ q_{-2} &= \sum_{n=-1}^1 c_n h_{-2-n} = c_{-1} h_{-1} = -0.1429 \\ q_3 &= \sum_{n=-1}^1 c_n h_{3-n} = 0 \\ q_{-3} &= \sum_{n=-1}^1 c_n h_{-3-n} = 0 \end{aligned}$$

Problem 9.42

(a) The output of the zero-force equalizer is :

$$q_m = \sum_{n=-1}^1 c_n x_{m_n}$$

With $q_0 = 1$ and $q_m = 0$ for $m \neq 0$, we obtain the system :

$$\begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solving the previous system in terms of the equalizer's coefficients, we obtain :

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0.000 \\ 0.980 \\ 0.196 \end{pmatrix}$$

(b) The output of the equalizer is :

$$q_m = \begin{cases} 0 & m \leq -4 \\ c_{-1}x_{-2} = 0 & m = -3 \\ c_{-1}x_{-1} + c_0x_{-2} = -0.49 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_0x_2 + x_1c_1 = 0.0098 & m = 2 \\ c_1x_2 = 0.0098 & m = 3 \\ 0 & m \geq 4 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\dots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0, \dots\}$$

and its span is 6 symbols.

Problem 9.43

(a) If $\{c_n\}$ denote the coefficients of the zero-force equalizer and $\{q_m\}$ is the sequence of the equalizer's output samples, then :

$$q_m = \sum_{n=-1}^1 c_n x_{m-n}$$

where $\{x_k\}$ is the noise free response of the matched filter demodulator sampled at $t = kT$. With $q_{-1} = 0$, $q_0 = q_1 = \mathcal{E}_b$, we obtain the system :

$$\begin{pmatrix} \mathcal{E}_b & 0.9\mathcal{E}_b & 0.1\mathcal{E}_b \\ 0.9\mathcal{E}_b & \mathcal{E}_b & 0.9\mathcal{E}_b \\ 0.1\mathcal{E}_b & 0.9\mathcal{E}_b & \mathcal{E}_b \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{E}_b \\ \mathcal{E}_b \end{pmatrix}$$

The solution to the system is :

$$\begin{pmatrix} c_{-1} & c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0.2137 & -0.3846 & 1.3248 \end{pmatrix}$$

(b) The set of noise variables $\{\nu_k\}$ at the output of the sampler is a gaussian distributed sequence with zero-mean and autocorrelation function :

$$R_\nu(k) = \begin{cases} \frac{N_0}{2}x_k & |k| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the autocorrelation function of the noise at the output of the equalizer is :

$$R_n(k) = R_\nu(k) \star c(k) \star c(-k)$$

where $c(k)$ denotes the discrete time impulse response of the equalizer. Therefore, the autocorrelation sequence of the noise at the output of the equalizer is :

$$R_n(k) = \frac{N_0 \mathcal{E}_b}{2} \begin{cases} 0.9402 & k = 0 \\ 1.3577 & k = \pm 1 \\ -0.0546 & k = \pm 2 \\ 0.1956 & k = \pm 3 \\ 0.0283 & k = \pm 4 \\ 0 & \text{otherwise} \end{cases}$$

To find an estimate of the error probability for the sequence detector, we ignore the residual interference due to the finite length of the equalizer, and we only consider paths of length two. Thus, if we start at state $I_0 = 1$ and the transmitted symbols are $(I_1, I_2) = (1, 1)$ an error is made by the sequence detector if the path $(-1, 1)$ is more probable, given the received values of r_1 and r_2 . The metric for the path $(I_1, I_2) = (1, 1)$ is :

$$\mu_2(1, 1) = [r_1 - 2\mathcal{E}_b \quad r_2 - 2\mathcal{E}_b] \mathbf{C}^{-1} \begin{bmatrix} r_1 - 2\mathcal{E}_b \\ r_2 - 2\mathcal{E}_b \end{bmatrix}$$

where :

$$\mathbf{C} = \frac{N_0 \mathcal{E}_b}{2} \begin{pmatrix} 0.9402 & 1.3577 \\ 1.3577 & 0.9402 \end{pmatrix}$$

Similarly, the metric of the path $(I_1, I_2) = (-1, 1)$ is

$$\mu_2(-1, 1) = [r_1 \quad r_2] \mathbf{C}^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Hence, the probability of error is :

$$P_2 = P(\mu_2(-1, 1) < \mu_2(1, 1))$$

and upon substitution of $r_1 = 2\mathcal{E}_b + n_1$, $r_2 = 2\mathcal{E}_b + n_2$, we obtain :

$$P_2 = P(n_1 + n_2 < -2\mathcal{E}_b)$$

Since n_1 and n_2 are zero-mean Gaussian variables, their sum is also zero-mean Gaussian with variance :

$$\sigma_2 = (2 \times 0.9402 + 2 \times 1.3577) \frac{N_0 \mathcal{E}_b}{2} = 4.5958 \frac{N_0 \mathcal{E}_b}{2}$$

and therefore :

$$P_2 = Q \left[\sqrt{\frac{8\mathcal{E}_b}{4.5958N_0}} \right]$$

The bit error probability is $\frac{P_2}{2}$.

Problem 9.44

The optimum tap coefficients of the zero-force equalizer can be found by solving the system:

$$\begin{pmatrix} 1.0 & 0.3 & 0.0 \\ 0.2 & 1.0 & 0.3 \\ 0.0 & 0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.3409 \\ 1.1364 \\ -0.2273 \end{pmatrix}$$

The output of the equalizer is :

$$q_m = \begin{cases} 0 & m \leq -3 \\ c_{-1}x_{-1} = -0.1023 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_1x_1 = -0.0455 & m = 2 \\ 0 & m \geq 3 \end{cases}$$

Hence, the residual ISI sequence is :

$$\text{residual ISI} = \{\dots, 0, -0.1023, 0, 0, 0, -0.0455, 0, \dots\}$$

Problem 9.45

(a) If we assume that the signal pulse has duration T , then the output of the matched filter at the time instant $t = T$ is :

$$\begin{aligned}
 y(T) &= \int_0^T r(\tau)s(\tau)d\tau \\
 &= \int_0^T (s(\tau) + \alpha s(\tau - T) + n(\tau))s(\tau)d\tau \\
 &= \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\
 &= \mathcal{E}_s + n
 \end{aligned}$$

where \mathcal{E}_s is the energy of the signal pulse and n is a zero-mean Gaussian random variable with variance $\sigma_n^2 = \frac{N_0\mathcal{E}_s}{2}$. Similarly, the output of the matched filter at $t = 2T$ is :

$$\begin{aligned}
 y(2T) &= \alpha \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\
 &= \alpha\mathcal{E}_s + n
 \end{aligned}$$

(b) If the transmitted sequence is :

$$x(t) = \sum_{n=-\infty}^{\infty} I_n s(t - nT)$$

with I_n taking the values 1, -1 with equal probability, then the output of the demodulator at the time instant $t = kT$ is

$$y_k = I_k \mathcal{E}_s + \alpha I_{k-1} \mathcal{E}_s + n_k$$

The term $\alpha I_{k-1} \mathcal{E}_s$ expresses the ISI due to the signal reflection. If a symbol by symbol detector is employed and the ISI is ignored, then the probability of error is :

$$\begin{aligned}
 P(e) &= \frac{1}{2}P(\text{error}|I_n = 1, I_{n-1} = 1) + \frac{1}{2}P(\text{error}|I_n = 1, I_{n-1} = -1) \\
 &= \frac{1}{2}P((1 + \alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2}P((1 - \alpha)\mathcal{E}_s + n_k < 0) \\
 &= \frac{1}{2}Q\left[\sqrt{\frac{2(1 + \alpha)^2\mathcal{E}_s}{N_0}}\right] + \frac{1}{2}Q\left[\sqrt{\frac{2(1 - \alpha)^2\mathcal{E}_s}{N_0}}\right]
 \end{aligned}$$

(c) To find the error rate performance of the DFE, we assume that the estimation of the parameter α is correct and that the probability of error at each time instant is the same. Since the transmitted

symbols are equiprobable, we obtain :

$$\begin{aligned}
 P(e) &= P(\text{error at } k | I_k = 1) \\
 &= P(\text{error at } k-1)P(\text{error at } k | I_k = 1, \text{error at } k-1) \\
 &\quad + P(\text{no error at } k-1)P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\
 &= P(e)P(\text{error at } k | I_k = 1, \text{error at } k-1) \\
 &\quad + (1 - P(e))P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\
 &= P(e)p + (1 - P(e))q
 \end{aligned}$$

where :

$$\begin{aligned}
 p &= P(\text{error at } k | I_k = 1, \text{error at } k-1) \\
 &= \frac{1}{2}P(\text{error at } k | I_k = 1, I_{k-1} = 1, \text{error at } k-1) \\
 &\quad + \frac{1}{2}P(\text{error at } k | I_k = 1, I_{k-1} = -1, \text{error at } k-1) \\
 &= \frac{1}{2}P((1 + 2\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2}P((1 - 2\alpha)\mathcal{E}_s + n_k < 0) \\
 &= \frac{1}{2}Q \left[\sqrt{\frac{2(1 + 2\alpha)^2 \mathcal{E}_s}{N_0}} \right] + \frac{1}{2}Q \left[\sqrt{\frac{2(1 - 2\alpha)^2 \mathcal{E}_s}{N_0}} \right]
 \end{aligned}$$

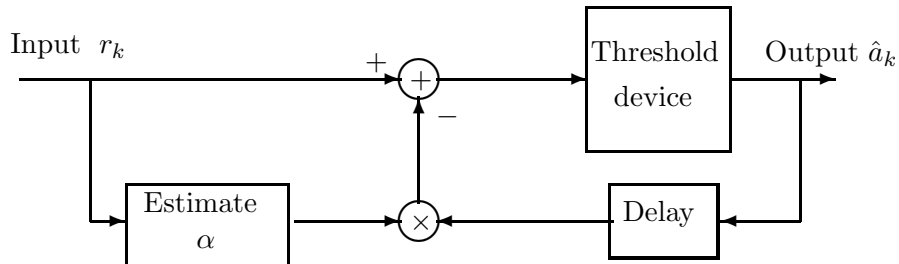
and

$$\begin{aligned}
 q &= P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\
 &= P(\mathcal{E}_s + n_k < 0) = Q \left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} \right]
 \end{aligned}$$

Solving for $P(e)$, we obtain :

$$P(e) = \frac{q}{1 - p + q} = \frac{Q \left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} \right]}{1 - \frac{1}{2}Q \left[\sqrt{\frac{2(1+2\alpha)^2 \mathcal{E}_s}{N_0}} \right] - \frac{1}{2}Q \left[\sqrt{\frac{2(1-2\alpha)^2 \mathcal{E}_s}{N_0}} \right] + Q \left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} \right]}$$

A sketch of the detector structure is shown in the next figure.



Problem 9.46

A discrete time transversal filter equivalent to the cascade of the transmitting filter $g_T(t)$, the channel $c(t)$, the matched filter at the receiver $g_R(t)$ and the sampler, has tap gain coefficients $\{x_m\}$, where

$$x_m = \begin{cases} 0.9 & m = 0 \\ 0.3 & m = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The noise ν_k , at the output of the sampler, is a zero-mean Gaussian sequence with autocorrelation function :

$$E[\nu_k \nu_l] = \sigma^2 x_{k-l}, \quad |k-l| \leq 1$$

If the \mathcal{Z} -transform of the sequence $\{x_m\}$, $X(z)$, assumes the factorization :

$$X(z) = F(z)F^*(z^{-1})$$

then the filter $1/F^*(z^{-1})$ can follow the sampler to white the noise sequence ν_k . In this case the output of the whitening filter, and input to the MSE equalizer, is the sequence :

$$u_n = \sum_k I_k f_{n-k} + n_k$$

where n_k is zero mean Gaussian with variance σ^2 . The optimum coefficients of the MSE equalizer, c_k , satisfy :

$$\sum_{n=-1}^1 c_n \Gamma_{kn} = \xi_k, \quad k = 0, \pm 1$$

where :

$$\Gamma(n-k) = \begin{cases} x_{n-k} + \sigma^2 \delta_{n,k}, & |n-k| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\xi(k) = \begin{cases} f_{-k}, & -1 \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

With

$$X(z) = 0.3z + 0.9 + 0.3z^{-1} = (f_0 + f_1 z^{-1})(f_0 + f_1 z)$$

we obtain the parameters f_0 and f_1 as :

$$f_0 = \begin{cases} \pm\sqrt{0.7854} \\ \pm\sqrt{0.1146} \end{cases}, \quad f_1 = \begin{cases} \pm\sqrt{0.1146} \\ \pm\sqrt{0.7854} \end{cases}$$

The parameters f_0 and f_1 should have the same sign since $f_0 f_1 = 0.3$. However, the sign itself does not play any role if the data are differentially encoded. To have a stable inverse system $1/F^*(z^{-1})$,

we select f_0 and f_1 in such a way that the zero of the system $F^*(z^{-1}) = f_0 + f_1 z$ is inside the unit circle. Thus, we choose $f_0 = \sqrt{0.1146}$ and $f_1 = \sqrt{0.7854}$ and therefore, the desired system for the equalizer's coefficients is

$$\begin{pmatrix} 0.9 + 0.1 & 0.3 & 0.0 \\ 0.3 & 0.9 + 0.1 & 0.3 \\ 0.0 & 0.3 & 0.9 + 0.1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.7854} \\ \sqrt{0.1146} \\ 0 \end{pmatrix}$$

Solving this system, we obtain

$$c_{-1} = 0.8596, \quad c_0 = 0.0886, \quad c_1 = -0.0266$$

Problem 9.47

(a) The spectrum of the band limited equalized pulse is

$$\begin{aligned} X(f) &= \begin{cases} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\frac{\pi n f}{W}} & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2W} \left[2 + 2 \cos \frac{\pi f}{W} \right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{W} \left[1 + \cos \frac{\pi f}{W} \right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $W = \frac{1}{2T_b}$

(b) The following table lists the possible transmitted sequences of length 3 and the corresponding output of the detector.

-1	-1	-1	-4
-1	-1	1	-2
-1	1	-1	0
-1	1	1	2
1	-1	-1	-2
1	-1	1	0
1	1	-1	2
1	1	1	4

As it is observed there are 5 possible output levels b_m , with probability $P(b_m = 0) = \frac{1}{4}$, $P(b_m = \pm 2) = \frac{1}{4}$ and $P(b_m = \pm 4) = \frac{1}{8}$.

(c) The transmitting filter $G_T(f)$, the receiving filter $G_R(f)$ and the equalizer $G_E(f)$ satisfy the condition

$$G_T(f)G_R(f)G_E(f) = X(f)$$

The power spectral density of the noise at the output of the equalizer is :

$$\mathcal{S}_\nu(f) = S_n(f)|G_R(f)G_E(f)|^2 = \sigma^2|G_R(f)G_E(f)|^2$$

With

$$G_T(f) = G_R(f) = P(f) = \frac{\pi T_{50}}{2} e^{-\pi T_{50}|f|}$$

the variance of the output noise is :

$$\begin{aligned} \sigma_\nu^2 &= \sigma^2 \int_{-\infty}^{\infty} |G_R(f)G_E(f)|^2 df = \sigma^2 \int_{-\infty}^{\infty} \left| \frac{X(f)}{G_T(f)} \right|^2 df \\ &= \sigma^2 \int_{-W}^W \frac{4}{\pi^2 T_{50}^2 W^2} \frac{|1 + \cos \frac{\pi f}{W}|^2}{e^{-2\pi T_{50}|f|}} df \\ &= \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \int_0^W \left(1 + \cos \frac{\pi f}{W} \right)^2 e^{2\pi T_{50}f} df \end{aligned}$$

The value of the previous integral can be found using the formula :

$$\begin{aligned} &\int e^{ax} \cos^n bx dx \\ &= \frac{1}{a^2 + n^2 b^2} \left[(a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + n(n-1)b^2 \int e^{ax} \cos^{n-2} bx dx \right] \end{aligned}$$

Thus, we obtain :

$$\begin{aligned} \sigma_\nu^2 &= \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \times \left[(e^{2\pi T_{50}W} - 1) \left(\frac{1}{2\pi T_{50}} + \frac{2\pi T_{50} + \pi \frac{1}{W^2 T_{50}}}{4\pi^2 T_{50}^2 + 4\frac{\pi^2}{W^2}} \right) \right. \\ &\quad \left. - \frac{4\pi T_{50}}{4\pi^2 T_{50}^2 + \frac{\pi^2}{W^2}} (e^{2\pi T_{50}W} + 1) \right] \end{aligned}$$

To find the probability of error using a symbol by symbol detector, we follow the same procedure as in Section 9.2.3. The results are the same with that obtained from a 3-point PAM constellation $(0, \pm 2)$ used with a duobinary signal with output levels having the probability mass function given in part (b). An upper bound of the symbol probability of error is :

$$\begin{aligned} P(e) &< P(|y_m| > 1 | b_m = 0)P(b_m = 0) + 2P(|y_m - 2| > 1 | b_m = 2)P(b_m = 2) \\ &\quad + 2P(|y_m + 4| > 1 | b_m = -4)P(b_m = -4) \\ &= P(|y_m| > 1 | b_m = 0) [P(b_m = 0) + 2P(b_m = 2) + P(b_m = -4)] \\ &= \frac{7}{8} P(|y_m| > 1 | b_m = 0) \end{aligned}$$

But

$$P(|y_m| > 1 | b_m = 0) = \frac{2}{\sqrt{2\pi}\sigma_\nu} \int_1^\infty e^{-x^2/2\sigma_\nu^2} dx$$

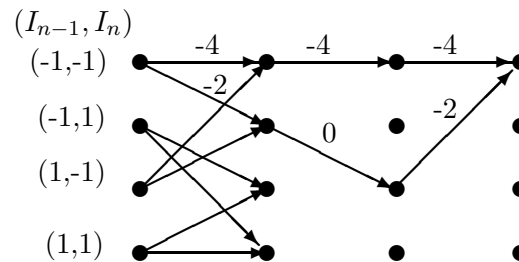
Therefore,

$$P(e) < \frac{14}{8} Q \left[\frac{1}{\sigma_\nu} \right]$$

Problem 9.48

Since the partial response signal has memory length equal to 2, the corresponding trellis has 4 states which we label as (I_{n-1}, I_n) . The following figure shows three frames of the trellis. The labels of the branches indicate the output of the partial response system. As it is observed the free distance between merging paths is 3, whereas the Euclidean distance is equal to

$$d_E = 2^2 + 4^2 + 2^2 = 24$$



Problem 9.49

(a) $X(z) = F(z)F^*(z^{-1}) = \frac{1}{2}z + 1 + \frac{1}{2}z^{-1}$. Then, the covariance matrix $\mathbf{\Gamma}$ is :

$$\mathbf{\Gamma} = \begin{bmatrix} 1 + N_0 & 1/2 & 0 \\ 1/2 & 1 + N_0 & 1/2 \\ 0 & 1/2 & 1 + N_0 \end{bmatrix} \text{ and } \xi = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

The optimum equalizer coefficients are given by :

$$\begin{aligned}
 \mathbf{C}_{opt} &= \mathbf{\Gamma}^{-1}\xi \\
 &= \frac{1}{\det(\mathbf{\Gamma})} \begin{bmatrix} (1+N_0)^2 - 1/4 & -\frac{1}{2}(1+N_0) & 1/4 \\ -\frac{1}{2}(1+N_0) & (1+N_0)^2 & -\frac{1}{2}(1+N_0) \\ 1/4 & -\frac{1}{2}(1+N_0) & (1+N_0)^2 - 1/4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \\
 &= \frac{1}{\sqrt{2}\det(\mathbf{\Gamma})} \begin{bmatrix} N_0^2 + \frac{3}{2}N_0 + \frac{1}{4} \\ N_0^2 + \frac{3}{2}N_0 + \frac{1}{2} \\ -\frac{N_0}{2} - \frac{1}{4} \end{bmatrix}
 \end{aligned}$$

where $\det(\mathbf{\Gamma}) = (1+N_0) \left[(1+N_0)^2 - \frac{1}{2} \right]$

(b)

$$\begin{aligned}
 \det(\mathbf{\Gamma} - \lambda \mathbf{I}) &= (1+N_0 - \lambda) \left[(1+N_0 - \lambda)^2 - \frac{1}{2} \right] \Rightarrow \\
 \lambda_1 &= 1+N_0, \lambda_2 = \frac{1}{\sqrt{2}} + 1+N_0, \lambda_3 = 1 - \frac{1}{\sqrt{2}} + N_0
 \end{aligned}$$

and the corresponding eigenvectors are :

$$\mathbf{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

(c)

$$J_{\min}(K)|_{K=1} = J_{\min}(1) = 1 - \xi' \mathbf{\Gamma}^{-1} \xi = \frac{2N_0^3 + 4N_0^2 + 2N_0 + 3/4}{2N_0^3 + 4N_0^2 + 5N_0 + 1}$$

(d)

$$\gamma = \frac{1 - J_{\min}(1)}{J_{\min}(1)} = \frac{2N_0^2 + 3N_0 + 3/4}{2N_0^3 + 4N_0^2 + 1/4}$$

Note that as $N_0 \rightarrow 0$, $\gamma \rightarrow 3$. For $N_0 = 0.1$, $\gamma = 2.18$ for the 3-tap equalizer and $\gamma = \sqrt{1 + \frac{2}{N_0}} - 1 = 3.58$, for the infinite-tap equalizer (as in example 10-2-1). Also, note that $\gamma = \frac{1}{N_0} = 10$ for the case of no intersymbol interference.

Problem 9.50

For the DFE we have that :

$$\hat{I}_k = \sum_{j=-K_1}^0 c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}, \text{ and } \epsilon_k = I_k - \hat{I}_k$$

The orthogonality principle is simply :

$$\left\{ \begin{array}{l} E [\epsilon_k u_{k-l}^*] = 0, \text{ for } -K_1 \leq l \leq 0 \\ E [\epsilon_k I_{k-l}^*] = 0, \text{ for } 1 \leq l \leq K_2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} E [I_k u_{k-l}^*] = E [\hat{I}_k u_{k-l}^*], -K_1 \leq l \leq 0 \\ E [I_k I_{k-l}^*] = E [\hat{I}_k I_{k-l}^*], 1 \leq l \leq K_2 \end{array} \right\}$$

Since the information symbols are uncorrelated : $E [I_k I_l^*] = a \delta_{kl}$, where $a = E [|I_k|^2]$ is a constant whose value is not needed since it will be present in all terms and thus, cancelled out. In order to solve the above system, we also need $E [u_k u_l^*]$, $E [I_k u_l^*]$. We have :

$$\begin{aligned} E [u_k u_l^*] &= E \left[\left(\sum_{n=0}^L f_n I_{k-n} + n_k \right) \left(\sum_{m=0}^L f_m^* I_{l-m}^* + n_l^* \right) \right] \\ &= a \sum_{m=0}^L f_m^* f_{m+k-l} + N_0 \delta_{kl} \end{aligned}$$

and

$$\begin{aligned} E [I_k u_l^*] &= E \left[I_k \left(\sum_{m=0}^L f_m^* I_{l-m}^* + n_l^* \right) \right] \\ &= a f_{l-k}^* \end{aligned}$$

Hence, the second equation of the orthogonality principle gives :

$$\begin{aligned} E [I_k I_{k-l}^*] &= E [\hat{I}_k I_{k-l}^*], 1 \leq l \leq K_2 \Rightarrow \\ 0 &= E \left[\left(\sum_{j=-K_1}^0 c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j} \right) I_{k-l}^* \right] \Rightarrow \\ 0 &= a \left(\sum_{j=-K_1}^0 c_j f_{l-j} \right) + a c_l \Rightarrow \\ c_l &= - \sum_{j=-K_1}^0 c_j f_{l-j}, 1 \leq l \leq K_2 \end{aligned}$$

which is the desired equation for the feedback taps. The first equation of the orthogonality principle will give :

$$E [I_k u_{k-l}^*] = E [\hat{I}_k u_{k-l}^*], -K_1 \leq l \leq 0 \Rightarrow$$

$$a f_{-l}^* = E \left[\left(\sum_{j=-K_1}^0 c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j} \right) u_{k-l}^* \right] \Rightarrow$$

$$a f_{-l}^* = \sum_{j=-K_1}^0 c_j \left(a \sum_{m=0}^L f_m^* f_{m+l-j} + N_0 \delta_{kl} \right) + a \sum_{j=1}^{K_2} c_j f_{j-l}^*, -K_1 \leq l \leq 0$$

Substituting the expression for c_j , $1 \leq j \leq K_2$, that we found above :

$$f_{-l}^* = \sum_{j=-K_1}^0 c_j \left(\sum_{m=0}^L f_m^* f_{m+l-j} + N_0 \delta_{kl} \right) - \sum_{j=1}^{K_2} \sum_{m=-K_1}^0 c_m f_{j-m} f_{j-l}^*, -K_1 \leq l \leq 0 \Rightarrow$$

$$f_{-l}^* = \sum_{j=-K_1}^0 c_j \left(\sum_{m=0}^L f_m^* f_{m+l-j} + N_0 \delta_{kl} \right) - \sum_{j=-K_1}^0 c_j \sum_{m=1}^{K_2} f_{m-j} f_{m-l}^*, -K_1 \leq l \leq 0 \Rightarrow$$

$$\sum_{j=-K_1}^0 c_j \psi_{lj} = f_{-l}^*, -K_1 \leq l \leq 0$$

where $\psi_{lj} = \sum_{m=0}^{-l} f_m^* f_{m+l-j} + N_0 \delta_{lj}$, which is the desired expression for the feedforward taps of the equalizer.

Problem 9.51

The tap coefficients for the feedback section of the DFE are given by the equation :

$$\begin{aligned} c_k &= -\sum_{j=-K_1}^0 c_j f_{k-j}, \quad k = 1, 2, \dots, K_2 \\ &= -(c_0 f_k + c_{-1} f_{k+1} + \dots + c_{-K_1} f_{k+K_1}) \end{aligned}$$

But $f_k = 0$ for $k < 0$ and $k > L$. Therefore :

$$c_L = -c_0 f_L, \quad c_{L+1} = 0, \quad c_{L+2} = 0, \quad \text{etc}$$

Problem 9.52

(a) The tap coefficients for the feedback section of the DFE are given by the equation : $c_k = -\sum_{j=-K_1}^0 c_j f_{k-j}$, $1 \leq k \leq K_2$, and for the feedforward section as the solution to the equations : $\sum_{j=-K_1}^0 c_j \psi_{lj} = -f_{-l}^*$, $K_1 \leq l \leq 0$. In this case, $K_1 = 1$, and hence : $\sum_{j=-K_1}^0 c_j \psi_{lj} = -f_{-l}^*$, $l = -1, 0$ or :

$$\begin{aligned} \psi_{0,0} c_0 + \psi_{0,-1} c_{-1} &= f_0^* \\ \psi_{-1,0} c_0 + \psi_{-1,-1} c_{-1} &= f_1^* \end{aligned}$$

But $\psi_{lj} = \sum_{m=0}^{-l} f_m^* f_{m+l-j} + N_0 \delta_{lj}$, so the above system can be written :

$$\begin{bmatrix} \frac{1}{2} + N_0 & \frac{1}{2} \\ \frac{1}{2} & 1 + N_0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_{-1} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

so :

$$\begin{bmatrix} c_0 \\ c_{-1} \end{bmatrix} = \frac{1}{\sqrt{2} (N_0^2 + \frac{3}{2}N_0 + \frac{1}{4})} \begin{bmatrix} \frac{1}{2} + N_0 \\ N_0 \end{bmatrix} \approx \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2}N_0 \end{bmatrix}, \quad \text{for } N_0 \ll 1$$

The coefficient for the feedback section is :

$$c_1 = -c_0 f_1 = -\frac{1}{\sqrt{2}} c_0 \approx -1, \quad \text{for } N_0 \ll 1$$

(b)

$$J_{\min}(1) = 1 - \sum_{j=-K_1}^0 c_j f_{-j} = \frac{2N_0^2 + N_0}{2(N_0^2 + \frac{3}{2}N_0 + \frac{1}{4})} \approx 2N_0, \quad \text{for } N_0 \ll 1$$

(c)

$$\gamma = \frac{1 - J_{\min}(1)}{J_{\min}(1)} = \frac{1 + 4N_0}{2N_0(1 + 2N_0)} \approx \frac{1}{2N_0}, \quad N_0 \ll 1$$

(d) For the infinite tap DFE, we have from example 9-5-1 :

$$J_{\min} = \frac{2N_0}{1 + N_0 + \sqrt{(1 + N_0)^2 - 1}} \approx 2N_0, \quad N_0 \ll 1$$

$$\gamma_{\infty} = \frac{1 - J_{\min}}{J_{\min}} = \frac{1 - N_0 \sqrt{(1 + N_0)^2 - 1}}{2N_0}$$

(e) For $N_0 = 0.1$, we have :

$$J_{\min}(1) = 0.146, \quad \gamma = 5.83 \text{ (7.66 dB)}$$

$$J_{\min} = 0.128, \quad \gamma_{\infty} = 6.8 \text{ (8.32 dB)}$$

For $N_0 = 0.01$, we have :

$$J_{\min}(1) = 0.0193, \quad \gamma = 51 \text{ (17.1 dB)}$$

$$J_{\min} = 0.0174, \quad \gamma_{\infty} = 56.6 \text{ (17.5 dB)}$$

The three-tap equalizer performs very well compared to the infinite-tap equalizer. The difference in performance is 0.6 dB for $N_0 = 0.1$ and 0.4 dB for $N_0 = 0.01$.

Problem 9.53

(a) We have that :

$$\begin{aligned} \frac{1}{2T} &= 900, \quad \frac{1+\beta}{2T} = 1200 \Rightarrow \\ 1 + \beta &= 1200/900 = 4/3 \Rightarrow \beta = 1/3 \end{aligned}$$

(b) Since $1/2T = 900$, the pulse rate $1/T$ is 1800 pulses/sec.

(c) The largest interference is caused by the sequence : $\{1, -1, s, 1, -1, 1\}$ or its opposite in sign. This interference is constructive or destructive depending on the sign of the information symbol s . The peak distortion is $\sum_{k=-2, k \neq 0}^3 f_k = 1.6$

(d) The probability of the worst-case interference given above is $(\frac{1}{2})^5 = 1/32$, and the same is the probability of the sequence that causes the opposite-sign interference.

Problem 9.54

(a)

$$F(z) = 0.8 - 0.6z^{-1} \Rightarrow$$

$$X(z) \equiv F(z)F^*(z^{-1}) = (0.8 - 0.6z^{-1})(0.8 - 0.6z) = 1 - 0.48z^{-1} - 0.48z$$

Thus, $x_0 = 1$, $x_{-1} = x_1 = -0.48$.

(b)

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} \left| H \left(\omega + \frac{2\pi n}{T} \right) \right|^2 = X(e^{j\omega T}) = 1 - 0.48e^{-j\omega T} - 0.48e^{j\omega T} = 1 - 0.96 \cos \omega T$$

(c) For the linear equalizer base on the mean-square-error criterion we have :

$$\begin{aligned} J_{\min} &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{1+N_0-0.96 \cos \omega T} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{N_0}{1+N_0-0.96 \cos \theta} d\theta \\ &= \frac{1}{2\pi} \left(\frac{N_0}{1+N_0} \right) \int_{-\pi}^{\pi} \frac{1}{1-a \cos \theta} d\theta, \quad a = \frac{0.96}{1+N_0} \end{aligned}$$

But :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1-a \cos \theta} d\theta = \frac{1}{\sqrt{1-a^2}}, \quad a^2 < 1$$

Therefore :

$$J_{\min} = \frac{N_0}{1+N_0} \frac{1}{\sqrt{1-\left(\frac{0.96}{1+N_0}\right)^2}} = \frac{N_0}{\sqrt{(1+N_0)^2 - (0.96)^2}}$$

(d) For the decision-feedback equalizer :

$$J_{\min} = \frac{2N_0}{1+N_0 + \sqrt{(1+N_0)^2 - (0.96)^2}}$$

which follows from the result in example 10.3.1. Note that for $N_0 \ll 1$,

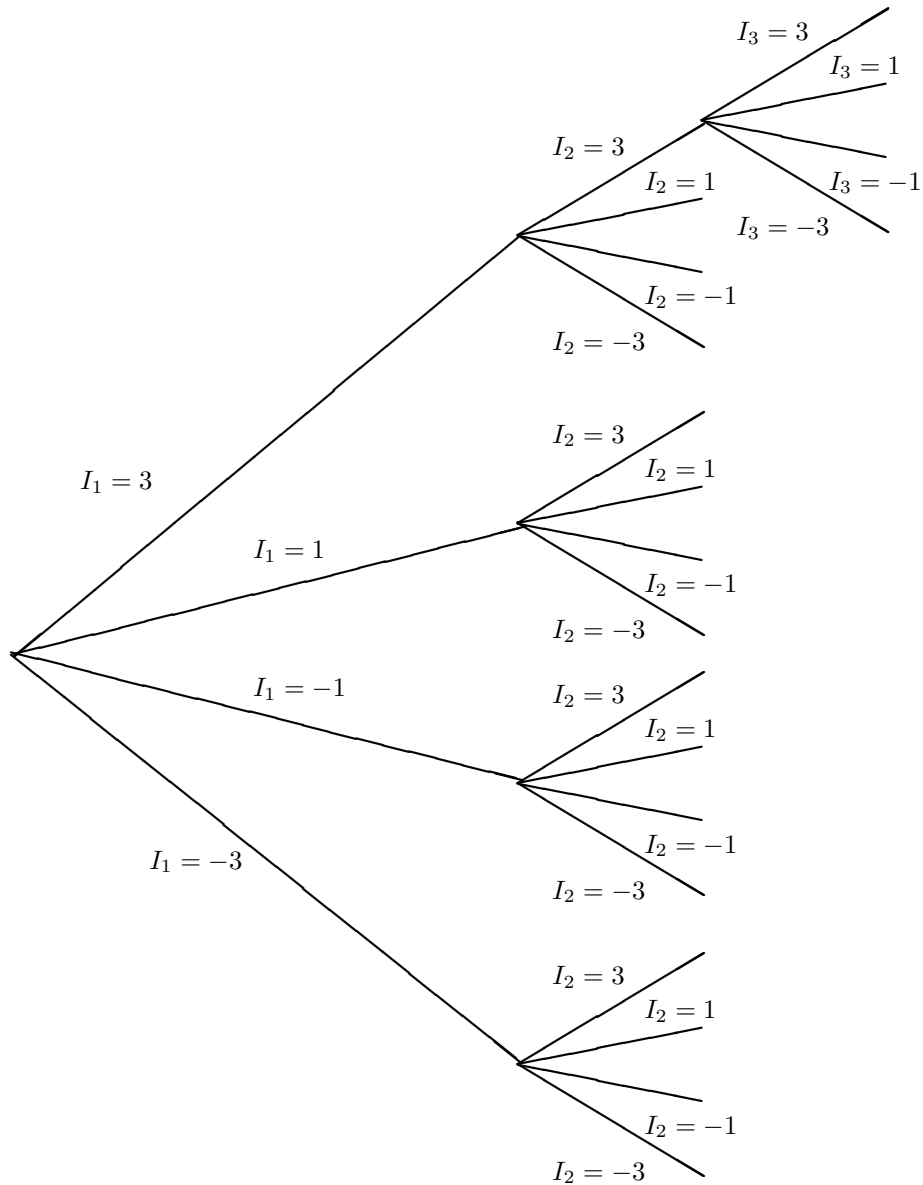
$$J_{\min} \approx \frac{2N_0}{1 + \sqrt{1 - (0.96)^2}} \approx 1.56N_0$$

In contrast, for the linear equalizer we have :

$$J_{\min} \approx \frac{N_0}{\sqrt{1 - (0.96)^2}} \approx 3.57N_0$$

Problem 9.55

(a) Part of the tree structure is shown in the following figure :



(b) There are four states in the trellis (corresponding to the four possible values of the symbol I_{k-1}), and for each one there are four paths starting from it (corresponding to the four possible values of the symbol I_k). Hence, 16 probabilities must be computed at each stage of the Viterbi algorithm.

(c) Since, there are four states, the number of surviving sequences is also four.

(d) The metrics are

$$(y_1 - 0.8I_1)^2, \quad i = 1 \quad \text{and} \quad \sum_i (y_i - 0.8I_i + 0.6I_{i-1})^2, \quad i \geq 2$$

$$\mu_1(I_1 = 3) = [0.5 - 3 * 0.8]^2 = 3.61$$

$$\mu_1(I_1 = 1) = [0.5 - 1 * 0.8]^2 = 0.09$$

$$\mu_1(I_1 = -1) = [0.5 + 1 * 0.8]^2 = 1.69$$

$$\mu_1(I_1 = -3) = [0.5 + 3 * 0.8]^2 = 8.41$$

$$\mu_2(I_2 = 3, I_1 = 3) = \mu_1(3) + [2 - 2.4 + 3 * 0.6]^2 = 5.57$$

$$\mu_2(3, 1) = \mu_1(1) + [2 - 2.4 + 1 * 0.6]^2 = 0.13$$

$$\mu_2(3, -1) = \mu_1(-1) + [2 - 2.4 - 1 * 0.6]^2 = 6.53$$

$$\mu_2(3, -3) = \mu_1(-3) + [2 - 2.4 - 3 * 0.6]^2 = 13.25$$

$$\mu_2(1, 3) = \mu_1(3) + [2 - 0.8 + 3 * 0.6]^2 = 12.61$$

$$\mu_2(1, 1) = \mu_1(1) + [2 - 0.8 + 1 * 0.6]^2 = 3.33$$

$$\mu_2(1, -1) = \mu_1(-1) + [2 - 0.8 - 1 * 0.6]^2 = 2.05$$

$$\mu_2(1, -3) = \mu_1(-3) + [2 - 0.8 - 3 * 0.6]^2 = 8.77$$

$$\mu_2(-1, 3) = \mu_1(3) + [2 + 0.8 + 3 * 0.6]^2 = 24.77$$

$$\mu_2(-1, 1) = \mu_1(1) + [2 + 0.8 + 1 * 0.6]^2 = 11.65$$

$$\mu_2(-1, -1) = \mu_1(-1) + [2 + 0.8 - 1 * 0.6]^2 = 6.53$$

$$\mu_2(-1, -3) = \mu_1(-3) + [2 + 0.8 - 3 * 0.6]^2 = 9.41$$

$$\mu_2(-3, 3) = \mu_1(3) + [2 + 2.4 + 3 * 0.6]^2 = 42.05$$

$$\mu_2(-3, 1) = \mu_1(1) + [2 + 2.4 + 1 * 0.6]^2 = 25.09$$

$$\mu_2(-3, -1) = \mu_1(-1) + [2 + 2.4 - 1 * 0.6]^2 = 16.13$$

$$\mu_2(-3, -3) = \mu_1(-3) + [2 + 2.4 - 3 * 0.6]^2 = 15.17$$

The four surviving paths at this stage are $\min_{I_1} [\mu_2(x, I_1)]$, $x = 3, 1, -1, -3$ or :

$$I_2 = 3, I_1 = 1 \text{ with metric } \mu_2(3, 1) = 0.13$$

$$I_2 = 1, I_1 = -1 \text{ with metric } \mu_2(1, -1) = 2.05$$

$$I_2 = -1, I_1 = -1 \text{ with metric } \mu_2(-1, -1) = 6.53$$

$$I_2 = -3, I_1 = -3 \text{ with metric } \mu_2(-3, -3) = 15.17$$

Now we compute the metrics for the next stage :

$$\mu_3(I_3 = 3, I_2 = 3, I_1 = 1) = \mu_2(3, 1) + [-1 - 2.4 + 1.8]^2 = 2.69$$

$$\mu_3(3, 1, -1) = \mu_2(1, -1) + [-1 - 2.4 + 0.6]^2 = 9.89$$

$$\mu_3(3, -1, -1) = \mu_2(-1, -1) + [-1 - 2.4 - 0.6]^2 = 22.53$$

$$\mu_3(3, -3, -3) = \mu_2(-3, -3) + [-1 - 2.4 - 1.8]^2 = 42.21$$

$$\begin{aligned}\mu_3(1, 3, 1) &= \mu_2(3, 1) + [-1 - 0.8 + 1.8]^2 = 0.13 \\ \mu_3(1, 1, -1) &= \mu_2(1, -1) + [-1 - 0.8 + 0.6]^2 = 7.81 \\ \mu_3(1, -1, -1) &= \mu_2(-1, -1) + [-1 - 0.8 - 0.6]^2 = 12.29 \\ \mu_3(1, -3, -3) &= \mu_2(-3, -3) + [-1 - 0.8 - 1.8]^2 = 28.13\end{aligned}$$

$$\begin{aligned}\mu_3(-1, 3, 1) &= \mu_2(3, 1) + [-1 + 0.8 + 1.8]^2 = 2.69 \\ \mu_3(-1, 1, -1) &= \mu_2(1, -1) + [-1 + 0.8 + 0.6]^2 = 2.69 \\ \mu_3(-1, -1, -1) &= \mu_2(-1, -1) + [-1 + 0.8 - 0.6]^2 = 7.17 \\ \mu_3(-1, -3, -3) &= \mu_2(-3, -3) + [-1 + 0.8 - 1.8]^2 = 19.17\end{aligned}$$

$$\begin{aligned}\mu_3(-3, 3, 1) &= \mu_2(3, 1) + [-1 + 2.4 + 1.8]^2 = 10.37 \\ \mu_3(-3, 1, -1) &= \mu_2(1, -1) + [-1 + 2.4 + 0.6]^2 = 2.69 \\ \mu_3(-3, -1, -1) &= \mu_2(-1, -1) + [-1 + 2.4 - 0.6]^2 = 7.17 \\ \mu_3(-3, -3, -3) &= \mu_2(-3, -3) + [-1 + 2.4 - 1.8]^2 = 15.33\end{aligned}$$

The four surviving sequences at this stage are $\min_{I_2, I_1} [\mu_3(x, I_2, I_1)]$, $x = 3, 1, -1, -3$ or :

$$\begin{aligned}I_3 = 3, I_2 = 3, I_1 = 1 &\text{ with metric } \mu_3(3, 3, 1) = 2.69 \\ I_3 = 1, I_2 = 3, I_1 = 1 &\text{ with metric } \mu_3(1, 3, 1) = 0.13 \\ I_3 = -1, I_2 = 3, I_1 = 1 &\text{ with metric } \mu_3(-1, 3, 1) = 2.69 \\ I_3 = -3, I_2 = 1, I_1 = -1 &\text{ with metric } \mu_3(-3, 1, -1) = 2.69\end{aligned}$$

(e) For the channel, $\delta_{\min}^2 = 1$ and hence :

$$P_4 = 8Q \left(\sqrt{\frac{6}{15} \gamma_{av}} \right)$$

Problem 9.56

(a)

$$\begin{aligned}b_k &= \frac{1}{K} \sum_{n=0}^{K-1} E_n e^{j2\pi nk/K} \\ &= \frac{1}{K} \sum_{n=0}^{K-1} \sum_{l=0}^{K-1} c_l e^{-j2\pi nl/K} e^{j2\pi nk/K} \\ &= \frac{1}{K} \sum_{l=0}^{K-1} c_l \sum_{n=0}^{K-1} e^{j2\pi n(k-l)/K}\end{aligned}$$

But

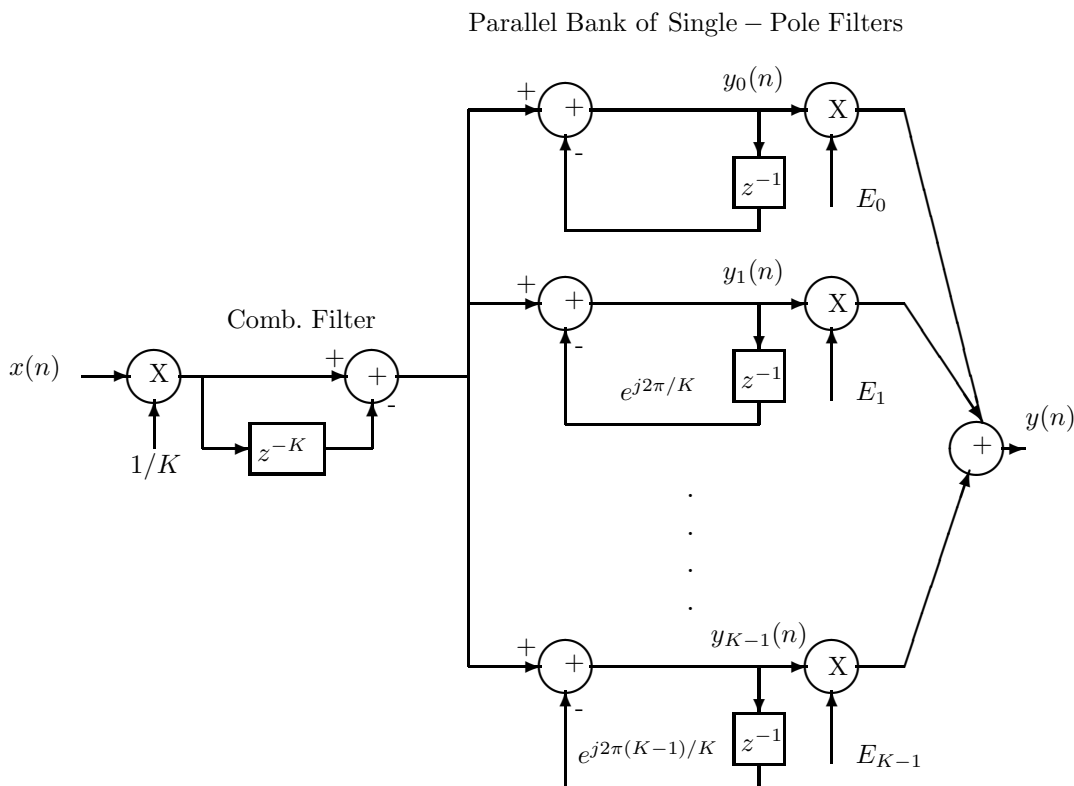
$$\sum_{n=0}^{K-1} e^{j2\pi n(k-l)/K} = \begin{cases} 0, & k \neq l \\ K, & k = l \end{cases}$$

Hence, $b_k = c_k$.

(b)

$$\begin{aligned} E(z) &= \sum_{k=0}^{K-1} c_k z^{-k} \\ &= \sum_{k=0}^{K-1} \left[\frac{1}{K} \sum_{n=0}^{K-1} E_n e^{j2\pi nk/K} \right] z^{-k} \\ &= \frac{1}{K} \sum_{n=0}^{K-1} E_n \left[\sum_{k=0}^{K-1} \left(e^{j2\pi n/K} z^{-1} \right)^k \right] \\ &= \frac{1}{K} \sum_{n=0}^{K-1} E_n \frac{1 - z^{-K}}{1 - \exp(j2\pi n/K) z^{-1}} \\ &= \frac{1 - z^{-K}}{K} \sum_{n=0}^{K-1} \frac{E_n}{1 - \exp(j2\pi n/K) z^{-1}} \end{aligned}$$

(c) The block diagram is as shown in the following figure :



(d) The adjustable parameters in this structure are $\{E_0, E_1, \dots, E_{K-1}\}$, i.e. the DFT coefficients of the equalizer taps. For more details on this equalizer structure, see the paper by Proakis (IEEE Trans. on Audio and Electroacc., pp 195-200, June 1970).

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 10) ¹

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January 15, 2008

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Problem 10.1

(a)

$$F(z) = \frac{4}{5} + \frac{3}{5}z^{-1} \Rightarrow X(z) = F(z)F^*(z^{-1}) = 1 + \frac{12}{25}(z + z^{-1})$$

Hence :

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & \frac{12}{25} & 0 \\ \frac{12}{25} & 1 & \frac{12}{25} \\ 0 & \frac{12}{25} & 1 \end{bmatrix} \quad \xi = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$$

and :

$$\mathbf{C}_{opt} = \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \mathbf{\Gamma}^{-1}\xi = \frac{1}{\beta} \begin{bmatrix} 1 - a^2 & -a & a^2 \\ -a & 1 & -a \\ a^2 & -a & 1 - a^2 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$$

where $a = 0.48$ and $\beta = 1 - 2a^2 = 0.539$. Hence :

$$\mathbf{C}_{opt} = \begin{bmatrix} 0.145 \\ 0.95 \\ -0.456 \end{bmatrix}$$

(b) The eigenvalues of the matrix $\mathbf{\Gamma}$ are given by :

$$|\mathbf{\Gamma} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 1 - \lambda & 0.48 & 0 \\ 0.48 & 1 - \lambda & 0.48 \\ 0 & 0.48 & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1, 0.3232, 1.6768$$

The step size Δ should range between :

$$0 \leq \Delta \leq 2/\lambda_{\max} = 1.19$$

(c) Following equations (10-3-3)-(10-3-4) we have :

$$\psi = \begin{bmatrix} 1 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}, \quad \psi \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.25 \end{bmatrix}$$

and the feedback tap is :

$$c_1 = -c_0 f_1 = -0.75$$

Problem 10.2

(a)

$$\Delta_{\max} = \frac{2}{\lambda_{\max}} = \frac{2}{1 + \frac{1}{\sqrt{2}} + N_0} = \frac{2}{1.707 + N_0}$$

(b) From (10-1-31) :

$$J_{\Delta} = \Delta^2 J_{\min} \sum_{k=1}^3 \frac{\lambda_k^2}{1 - (1 - \Delta \lambda_k)^2} \approx \frac{1}{2} \Delta J_{\min} \sum_{k=1}^3 \lambda_k$$

Since $\frac{J_{\Delta}}{J_{\min}} = 0.01$:

$$\Delta \approx \frac{0.07}{1 + N_0} \approx 0.06$$

(c) Let $\mathbf{C}' = \mathbf{V}^t \mathbf{C}$, $\xi' = \mathbf{V}^t \xi$, where \mathbf{V} is the matrix whose columns form the eigenvectors of the covariance matrix $\mathbf{\Gamma}$ (note that $\mathbf{V}^t = \mathbf{V}^{-1}$). Then :

$$\begin{aligned} \mathbf{C}_{(n+1)} &= (\mathbf{I} - \Delta \mathbf{\Gamma}) \mathbf{C}_{(n)} + \Delta \xi \Rightarrow \\ \mathbf{C}_{(n+1)} &= (\mathbf{I} - \Delta \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}) \mathbf{C}_{(n)} + \Delta \xi \Rightarrow \\ \mathbf{V}^{-1} \mathbf{C}_{(n+1)} &= \mathbf{V}^{-1} (\mathbf{I} - \Delta \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}) \mathbf{C}_{(n)} + \Delta \mathbf{V}^{-1} \xi \Rightarrow \\ \mathbf{C}'_{(n+1)} &= (\mathbf{I} - \Delta \mathbf{\Lambda}) \mathbf{C}'_{(n)} + \Delta \xi' \end{aligned}$$

which is a set of three de-coupled difference equations (de-coupled because $\mathbf{\Lambda}$ is a diagonal matrix). Hence, we can write :

$$c'_{k,(n+1)} = (1 - \Delta \lambda_k) c'_{k,(n)} + \Delta \xi'_k, \quad k = -1, 0, 1$$

The steady-state solution is obtained when $c'_{k,(n+1)} = c'_k$, which gives :

$$c'_k = \frac{\xi'_k}{\lambda_k}, \quad k = -1, 0, 1$$

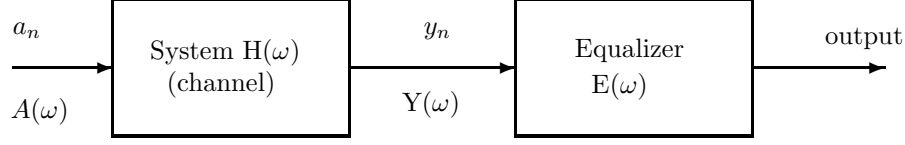
or going back to matrix form :

$$\begin{aligned} \mathbf{C}' &= \mathbf{\Lambda}^{-1} \xi' \Rightarrow \\ \mathbf{C} &= \mathbf{V} \mathbf{C}' = \mathbf{V} \mathbf{\Lambda}^{-1} \mathbf{V}^{-1} \xi \Rightarrow \\ \mathbf{C} &= (\mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1})^{-1} \xi = \mathbf{\Gamma}^{-1} \xi \end{aligned}$$

which agrees with the result in Probl. 9.49(a).

Problem 10.3

Suppose that we have a discrete-time system with frequency response $H(\omega)$; this may be equalized by use of the DFT as shown below :



$$A(\omega) = \sum_{n=0}^{N-1} a_n e^{-j\omega n} \quad Y(\omega) = \sum_{n=0}^{N-1} c_n e^{-j\omega n} = A(\omega)H(\omega)$$

Let :

$$E(\omega) = \frac{A(\omega)Y^*(\omega)}{|Y(\omega)|^2}$$

Then by direct substitution of $Y(\omega)$ we obtain :

$$E(\omega) = \frac{A(\omega)A^*(\omega)H^*(\omega)}{|A(\omega)|^2 |H(\omega)|^2} = \frac{1}{H(\omega)}$$

If the sequence $\{a_n\}$ is sufficiently padded with zeros, the N -point DFT simply represents the values of $E(gw)$ and $H(\omega)$ at $\omega = \frac{2\pi}{N}k = \omega_k$, for $k = 0, 1, \dots, N-1$ without frequency aliasing. Therefore the use of the DFT as specified in this problem yields $E(\omega_k) = \frac{1}{H(\omega_k)}$, independent of the properties of the sequence $\{a_n\}$. Since $H(\omega)$ is the spectrum of the discrete-time system, we know that this is equivalent to the folded spectrum of the continuous-time system (i.e the system which was sampled). For further details for the use of a pseudo-random periodic sequence to perform equalization we refer to the paper by Qureshi (1985).

Problem 10.4

The MSE performance index at the time instant k is

$$J(\mathbf{c}_k) = E \left[\left| \sum_{n=-N}^N c_{k,n} v_{k-n} - I_k \right|^2 \right]$$

If we define the gradient vector \mathbf{G}_k as

$$\mathbf{G}_k = \frac{\partial J(\mathbf{c}_k)}{2\partial \mathbf{c}_k}$$

then its l - th element is

$$\begin{aligned} G_{k,l} = \frac{\partial J(\mathbf{c}_k)}{2\partial c_{k,l}} &= \frac{1}{2}E \left[2 \left(\sum_{n=-N}^N c_{k,n} v_{k-n} - I_k \right) v_{k-l}^* \right] \\ &= E \left[-\epsilon_k v_{k-l}^* \right] = -E \left[\epsilon_k v_{k-l}^* \right] \end{aligned}$$

Thus, the vector \mathbf{G}_k is

$$\mathbf{G}_k = \begin{pmatrix} -E[\epsilon_k v_{k+N}^*] \\ \vdots \\ -E[\epsilon_k v_{k-N}^*] \end{pmatrix} = -E[\epsilon_k \mathbf{V}_k^*]$$

where \mathbf{V}_k is the vector $\mathbf{V}_k = [v_{k+N} \cdots v_{k-N}]^T$. Since $\hat{\mathbf{G}}_k = -\epsilon_k \mathbf{V}_k^*$, its expected value is

$$E[\hat{\mathbf{G}}_k] = E[-\epsilon_k \mathbf{V}_k^*] = -E[\epsilon_k \mathbf{V}_k^*] = \mathbf{G}_k$$

Problem 10.5

The tap-leakage LMS algorithm is :

$$\mathbf{C}(n+1) = w\mathbf{C}(n) + \Delta\epsilon(n)\mathbf{V}^*(n) = w\mathbf{C}(n) + \Delta(\mathbf{\Gamma}\mathbf{C}(n) - \xi) = (w\mathbf{I} - \Delta\mathbf{\Gamma})\mathbf{C}(n) - \Delta\xi$$

Following the same diagonalization procedure as in Problem 10.2 or Section (10-1-3) of the book, we obtain :

$$\mathbf{C}'(n+1) = (w\mathbf{I} - \Delta\mathbf{\Lambda})\mathbf{C}'(n) - \Delta\xi'$$

where $\mathbf{\Lambda}$ is the diagonal matrix containing the eigenvalues of the correlation matrix $\mathbf{\Gamma}$. The algorithm converges if the roots of the homogeneous equation lie inside the unit circle :

$$|w - \Delta\lambda_k| < 1, \quad k = -N, \dots, -1, 0, 1, \dots, N$$

and since $\Delta > 0$, the convergence criterion is :

$$\Delta < \frac{1+w}{\lambda_{\max}}$$

Problem 10.6

The estimate of g can be written as : $\hat{g} = h_0 x_0 + \dots + h_{M-1} x_{M-1} = \mathbf{x}^T \mathbf{h}$, where \mathbf{x}, \mathbf{h} are column vectors containing the respective coefficients. Then using the orthogonality principle we obtain the optimum linear estimator \mathbf{h} :

$$E[\mathbf{x}\epsilon] = 0 \Rightarrow E[\mathbf{x}(g - \mathbf{x}^T \mathbf{h})] = 0 \Rightarrow E[\mathbf{x}g] = E[\mathbf{x}\mathbf{x}^T] \mathbf{h}$$

or :

$$\mathbf{h}_{opt} = \mathbf{R}_{xx}^{-1} \mathbf{c}$$

where the $M \times M$ correlation matrix \mathbf{R}_{xx} has elements :

$$R(m, n) = E[x(m)x(n)] = E[g^2] u(m)u(n) + \sigma_w^2 \delta_{nm} = Gu(m)u(n) + \sigma_w^2 \delta_{nm}$$

where we have used the fact that g and w are independent, and that $E[g] = 0$. Also, the column vector $\mathbf{c} = E[\mathbf{x}g]$ has elements :

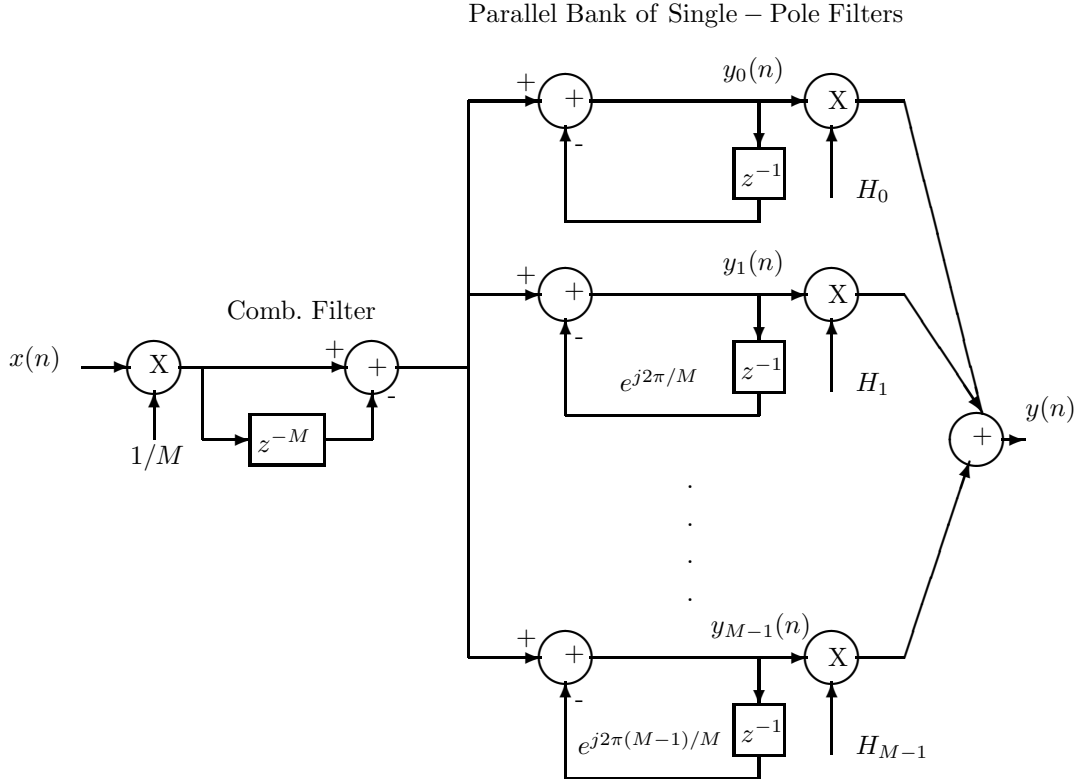
$$c(n) = E[x(n)g] = Gu(n)$$

Problem 10.7

(a) The time-update equation for the parameters $\{H_k\}$ is :

$$H_k^{(n+1)} = H_k^{(n)} + \Delta \epsilon^{(n)} y_k^{(n)}$$

where n is the time-index, k is the filter index, and $y_k^{(n)}$ is the output of the k -th filter with transfer function : $(1 - z^{-M}) / (1 - e^{j2\pi k/M} z^{-1})$ as shown in the figure below :



The error $\epsilon(n)$ is calculated as : $\epsilon(n) = I_n - y(n)$, and then it is fed back in the adaptive part of the equalizer, together with the quantities $y_k^{(n)}$, to update the equalizer parameters H_k .

(b) It is straightforward to prove that the transfer function of the k -th filter in the parallel bank has a resonant frequency at $f_k = 2\pi \frac{k}{M}$, and is zero at the resonant frequencies of the other filters $f_m = 2\pi \frac{m}{M}$, $m \neq k$. Hence, if we choose as a test signal sinusoids whose frequencies coincide with the resonant frequencies of the tuned circuits, this allows the coefficient H_k for each filter to be adjusted independently without any interaction from the other filters.

Problem 10.8

(a) The gradient of the performance index J with respect to h is : $\frac{dJ}{dh} = 2h + 40$. Hence, the time update equation becomes :

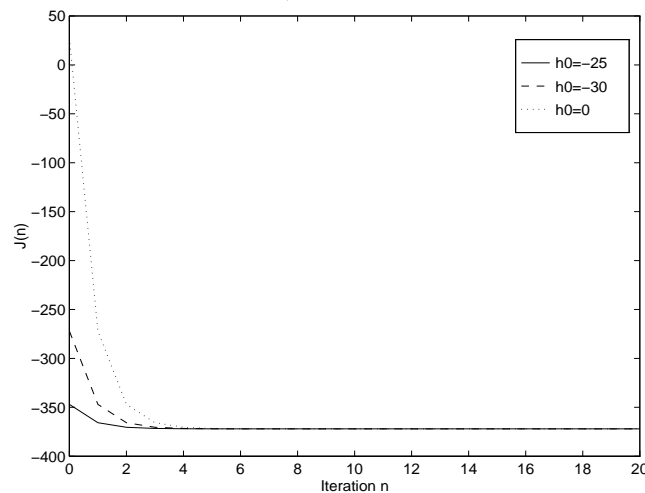
$$h_{n+1} = h_n - \frac{1}{2}\Delta(2h_n + 40) = h_n(1 - \Delta) - 20\Delta$$

This system will converge if the homogeneous part will vanish away as $n \rightarrow \infty$, or equivalently if : $|1 - \Delta| < 1 \Leftrightarrow 0 < \Delta < 2$.

(b) We note that J has a minimum at $h = -20$, with corresponding value : $J_{\min} = -372$. To illustrate the convergence of the algorithm let's choose : $\Delta = 1/2$. Then : $h_{n+1} = h_n/2 - 10$, and, using induction, we can prove that :

$$h_{n+1} = \left(\frac{1}{2}\right)^n h_0 - 10 \left[\sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \right]$$

where h_0 is the initial value for h . Then, as $n \rightarrow \infty$, the dependence on the initial condition h_0 vanishes and $h_n \rightarrow -10 \frac{1}{1-1/2} = -20$, which is the desired value. The following plot shows the expression for J as a function of n , for $\Delta = 1/2$ and for various initial values h_0 .



Problem 10.9

The linear estimator for x can be written as : $\hat{x}(n) = a_1x(n-1) + a_2x(n-2) = [x(n-1) \ x(n-2)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

Using the orthogonality principle we obtain :

$$E \left\{ \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} \epsilon \right\} = 0 \Rightarrow E \left\{ \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} \left(x(n) - [x(n-1) \ x(n-2)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) \right\} = 0$$

or :

$$\begin{bmatrix} \gamma_{xx}(-1) \\ \gamma_{xx}(-2) \end{bmatrix} = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ b^2 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

This is a well-known fact from Statistical Signal Processing theory : a first-order AR process (which has autocorrelation function $\gamma(m) = a^{|m|}$) has a first-order optimum (MSE) linear estimator : $\hat{x}_n = ax_{n-1}$.

Problem 10.10

In Probl. 10.9 we found that the optimum (MSE) linear predictor for $x(n)$, is $\hat{x}(n) = bx(n-1)$. Since it is a first order predictor, the corresponding lattice implementation will comprise of one stage, too, with reflection coefficient a_{11} . This coefficient can be found using (10-4-28) :

$$a_{11} = \frac{\gamma_{xx}(1)}{\gamma_{xx}(0)} = b$$

Then, we verify that the residue $f_1(n)$ is indeed the first-order prediction error : $f_1(n) = x(n) - bx(n-1) = x(n) - \hat{x}(n) = e(n)$

Problem 10.11

The system $C(z) = \frac{1}{1-0.9z^{-1}}$ has an impulse response : $c(n) = (0.9)^n$, $n \geq 0$. Then, we write the input $y(n)$ to the adaptive FIR filter :

$$y(n) = \sum_{k=0}^{\infty} c(k)x(n-k) + w(n)$$

Since the sequence $\{x(n)\}$ corresponds to the information sequence that is transmitted through a channel, we will assume that is uncorrelated with zero mean and unit variance. Then the optimum (according to the MSE criterion) estimator of $x(n)$ will be : $\hat{x}(n) = [y(n) \ y(n-1)] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$. Using the orthogonality criterion we obtain the optimum coefficients $\{b_i\}$:

$$\begin{aligned} E \left\{ \begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} \epsilon \right\} &= 0 \Rightarrow E \left\{ \begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} \left(x(n) - [y(n) \ y(n-1)] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \right) \right\} = 0 \\ \Rightarrow \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} &= \left\{ E \begin{bmatrix} y(n)y(n) & y(n)y(n-1) \\ y(n-1)y(n) & y(n-1)y(n-1) \end{bmatrix} \right\}^{-1} \left\{ E \begin{bmatrix} y(n)x(n) \\ y(n-1)x(n) \end{bmatrix} \right\} \end{aligned}$$

The various correlations are as follows :

$$E[y(n)x(n)] = E \left[\sum_{k=0}^{\infty} c(k)x(n-k)x(n) + w(n)x(n) \right] = c(0) = 1$$

where we have used the fact that : $E[x(n-k)x(n)] = \delta_k$, and that $\{w(n)\} \{x(n)\}$ are independent. Similarly :

$$E[y(n-1)x(n)] = E \left[\sum_{k=0}^{\infty} c(k)x(n-k-1)x(n) + w(n)x(n) \right] = 0$$

$$\begin{aligned} E[y(n)y(n)] &= E \left[\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c(k)c(j)x(n-k)x(n-j) \right] + \sigma_w^2 \\ &= \sum_{j=0}^{\infty} c(j)c(j) + \sigma_w^2 = \sum_{j=0}^{\infty} (0.9)^{2j} + \sigma_w^2 = \\ &= \frac{1}{1-0.81} + \sigma_w^2 = \frac{1}{0.19} + \sigma_w^2 \end{aligned}$$

and :

$$\begin{aligned} E[y(n)y(n-1)] &= E \left[\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c(k)c(j)x(n-k)x(n-1-j) \right] \\ &= \sum_{j=0}^{\infty} c(j)c(j+1) = \sum_{j=0}^{\infty} (0.9)^{2j+1} \\ &= 0.9 \frac{1}{1-0.81} = 0.9 \frac{1}{0.19} \end{aligned}$$

Hence :

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{0.19} + 0.1 & 0.9\frac{1}{0.19} \\ 0.9\frac{1}{0.19} & \frac{1}{0.19} + 0.1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.85 \\ -0.75 \end{bmatrix}$$

It is interesting to note that in the absence of noise (i.e when the term $\sigma_w^2 = 0.1$ is missing from the diagonal of the correlation matrix), the optimum coefficients are : $B(z) = b_0 + b_1 z^{-1} = 1 - 0.9z^{-1}$, i.e. the equalizer function is the inverse of the channel function (in this case the MSE criterion coincides with the zero-forcing criterion). However, we see that, in the presence of noise, the MSE criterion gives a slightly different result from the inverse channel function, in order to prevent excessive noise enhancement.

Problem 10.12

(a) If we denote by \mathbf{V} the matrix whose columns are the eigenvectors $\{\mathbf{v}_i\}$:

$$\mathbf{V} = [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_N]$$

then its conjugate transpose matrix is :

$$\mathbf{V}^{*t} = \begin{bmatrix} \mathbf{v}_1^{*t} \\ \mathbf{v}_2^{*t} \\ \dots \\ \mathbf{v}_N^{*t} \end{bmatrix}$$

and $\mathbf{\Gamma}$ can be written as :

$$\mathbf{\Gamma} = \sum_{i=1}^N \lambda_i \mathbf{v}_i \mathbf{v}_i^{*t} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{*t}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{\Gamma}$. Then, if we name $\mathbf{X} = \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{V}^{*t}$, we see that :

$$\mathbf{X} \mathbf{X} = \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{V}^{*t} \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{V}^{*t} = \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{\Lambda}^{1/2} \mathbf{V}^{*t} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{*t} = \mathbf{\Gamma}$$

where we have used the fact that the matrix \mathbf{V} is unitary : $\mathbf{V} \mathbf{V}^{*t} = \mathbf{I}$. Hence, since $\mathbf{X} \mathbf{X} = \mathbf{\Gamma}$, this shows that the matrix $\mathbf{X} = \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{V}^{*t} = \sum_{i=1}^N \lambda_i^{1/2} \mathbf{v}_i \mathbf{v}_i^{*t}$ is indeed the square root of $\mathbf{\Gamma}$.

(b) To compute $\mathbf{\Gamma}^{1/2}$, we first determine $\mathbf{V}, \mathbf{\Lambda}$ (i.e the eigenvalues and eigenvectors of the correlation matrix). Then :

$$\mathbf{\Gamma}^{1/2} = \sum_{i=1}^N \lambda_i^{1/2} \mathbf{v}_i \mathbf{v}_i^{*t} = \mathbf{V} \mathbf{\Lambda}^{1/2} \mathbf{V}^{*t}$$

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 11) ¹

Prepared by
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Problem 11.1

(a)

$$\begin{aligned}
 U &= \sum_{n=1}^N X_n \\
 E[U] &= \sum_{n=1}^N E[X_n] = Nm \\
 \sigma_u^2 &= E[U^2] - E^2[U] = E\left[\sum_n \sum_m X_n X_m\right] - N^2 m^2 \\
 &= N(\sigma^2 + m^2) + N(N-1)\sigma^2 - N^2 m^2 = N\sigma^2
 \end{aligned}$$

Hence :

$$(SNR)_u = \frac{N^2 m^2}{2N\sigma^2} = \frac{N}{2} \frac{m^2}{\sigma^2}$$

(b)

$$\begin{aligned}
 V &= \sum_{n=1}^N X_n^2 \\
 E[V] &= \sum_{n=1}^N E[X_n^2] = N(\sigma^2 + m^2)
 \end{aligned}$$

For the variance of V we have :

$$\sigma_V^2 = E[V^2] - E^2[V] = E[V^2] - N^2(\sigma^2 + m^2)$$

But :

$$E[V^2] = \sum_n \sum_m E(X_n^2 X_m^2) = \sum_{n=1}^N E(X_n^4) + \sum_n \sum_{m, n \neq m} E(X_n^2) E(X_m^2) = NE(X^4) + N(N-1)E^2(X^2)$$

To compute $E(X^4)$ we can use the fact that a zero-mean Gaussian RV Y has moments :

$$E[Y^k] = \begin{cases} 0, & k : \text{odd} \\ 1 \cdot 3 \cdot \dots (k-1)\sigma^k & k : \text{even} \end{cases}$$

Hence :

$$\left\{ \begin{array}{l} E[(X-m)^3] = 0 \\ E[(X-m)^4] = 3\sigma^4 \end{array} \right\} \Rightarrow E[X^4] = m^4 + 6\sigma^2 m^2 + 3\sigma^4$$

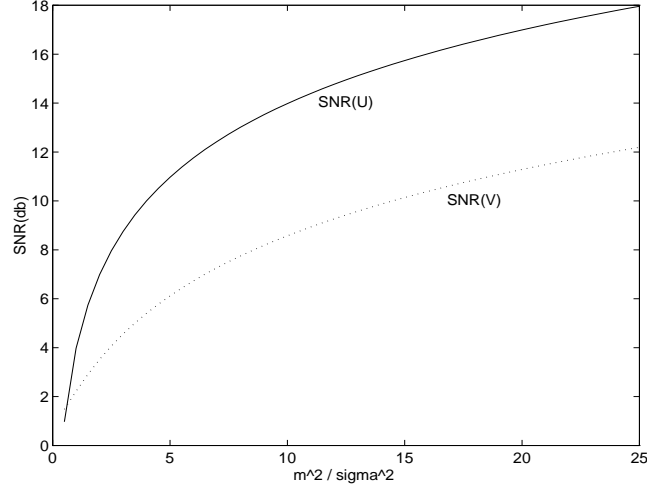
Then :

$$\begin{aligned}
 E[V^2] &= N(m^4 + 6\sigma^2 m^2 + 3\sigma^4) + N(N-1)(\sigma^2 + m^2) \Rightarrow \\
 \sigma_V^2 &= E[V^2] - N^2(\sigma^2 + m^2) = 2N\sigma^2(\sigma^2 + 2m^2)
 \end{aligned}$$

Note : the above result could be obtained by noting that V is a non-central chi-square RV, with N degrees of freedom and non-centrality parameter equal to Nm^2 ; then we could apply directly expression (2-3-40). Having obtained σ_V^2 , we have :

$$(SNR)_V = \frac{N^2 (m^2 + \sigma^2)^2}{2N\sigma^2 (\sigma^2 + 2m^2)} = \frac{N ((m^2/\sigma^2) + 1)^2}{4 (2(m^2/\sigma^2) + 1)}$$

(c) The plot is given in the following figure for $N=5$:



(d) In multichannel operation with coherent detection the decision variable is U as given in (a). With square-law detection, the decision variable is of the form $\sum_{n=1}^N |X_n + jY_n|^2$ where X_n and Y_n are Gaussian. We note that V is not the exact model for square-law detection, but, nevertheless, the effect of the non coherent combining loss is evident in the $(SNR)_V$.

Problem 11.2

(a) r is a Gaussian random variable. If $\sqrt{\mathcal{E}_b}$ is the transmitted signal point, then :

$$E(r) = E(r_1) + E(r_2) = (1 + k)\sqrt{\mathcal{E}_b} \equiv m_r$$

and the variance is :

$$\sigma_r^2 = \sigma_1^2 + k^2 \sigma_2^2$$

The probability density function of r is

$$p(r) = \frac{1}{\sqrt{2\pi}\sigma_r} e^{-\frac{(r-m_r)^2}{2\sigma_r^2}}$$

and the probability of error is :

$$\begin{aligned} P_2 &= \int_{-\infty}^0 p(r) dr \\ &= Q\left(\sqrt{\frac{m_r^2}{\sigma_r^2}}\right) \end{aligned}$$

where

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+k)^2 \mathcal{E}_b}{\sigma_1^2 + k^2 \sigma_2^2}$$

The value of k that maximizes this ratio is obtained by differentiating this expression and solving for the value of k that forces the derivative to zero. Thus, we obtain

$$k = \frac{\sigma_1^2}{\sigma_2^2}$$

Note that if $\sigma_1 > \sigma_2$, then $k > 1$ and r_2 is given greater weight than r_1 . On the other hand, if $\sigma_2 > \sigma_1$, then $k < 1$ and r_1 is given greater weight than r_2 . When $\sigma_1 = \sigma_2$, $k = 1$ (equal weight).

(b) When $\sigma_2^2 = 3\sigma_1^2$, $k = \frac{1}{3}$, and

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1 + \frac{1}{3})^2 \mathcal{E}_b}{\sigma_1^2 + \frac{1}{9}(3\sigma_1^2)} = \frac{4}{3} \left(\frac{\mathcal{E}_b}{\sigma_1^2} \right)$$

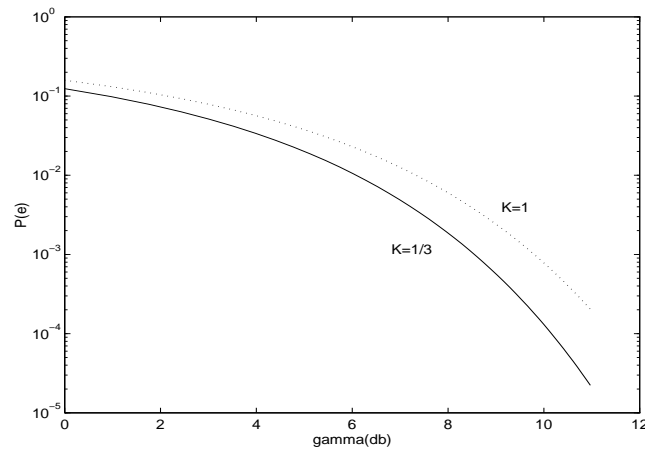
On the other hand, if k is set to unity we have

$$\frac{m_r^2}{\sigma_r^2} = \frac{4\mathcal{E}_b}{\sigma_1^2 + 3\sigma_1^2} = \frac{\mathcal{E}_b}{\sigma_1^2}$$

Therefore, the optimum weighting provides a gain of :

$$10 \log \frac{4}{3} = 1.25 \text{ dB}$$

This is illustrated in the following figure, where $\gamma = \frac{\mathcal{E}_b}{\sigma_1^2}$.



Problem 11.3

(a) If the sample rate $\frac{1}{T_s} = \tilde{N} \cdot \Delta f = W$, does not alter with the insertion of the cyclic prefix (which indeed is the case in most multicarrier systems), then the bandwidth requirements for the system remain the same. However, keeping the same sample rate means that the block length is increased by a factor of $\frac{v}{N}$, and the effective throughput is reduced to $\frac{1}{1+\frac{v}{N}} = \frac{N}{N+v}$ of the previous one. This is usually compensated by the elimination of ISI, which allows the use of higher order alphabets in each one of the subcarriers.

If the sample rate is increased by a factor of $\left(\frac{N}{N+v}\right)^{-1}$, so that the block length after the insertion of the cyclic prefix will be the same as before, then the bandwidth requirements for the system are increased by the same factor : $W' = W \frac{N+v}{N}$. However, this second case is rarely used in practice.

(b) If the real and imaginary parts of the information sequence $\{X_k\}$ have the same average energy : $E[Re(X_k)]^2 = E[Im(X_k)]^2$, then it is straightforward to prove that the time-domain samples $\{x_n\}$, that are the output of the IDFT, have the same average energy:

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (Re(X_k) + j \cdot Im(X_k)) \exp(j2\pi nk/N), \quad n = 0, 1, \dots, N-1$$

and :

$$E[x_n^2] = \epsilon$$

for all $n = 0, 1, \dots, N-1$. Hence, the energy of the cyclic-prefixed block, will be increased from $N\epsilon$ to $(N+v)\epsilon$. However, the power requirements will remain the same, since the duration of the prefixed block is also increased from NT_s to $(N+v)T_s$.

For an analysis of the case where the real and imaginary parts of the information sequence do not have the same average energy, we refer the interested reader to the paper by Chow et al. (1991).

Problem 11.4

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad k = 0, \dots, N-1$$

and for the padded sequence :

$$X'(k) = \sum_{n=0}^{N+L-1} x'(n)e^{-j2\pi nk/(N+L)} = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/(N+L)}, \quad k = 0, \dots, N+L-1$$

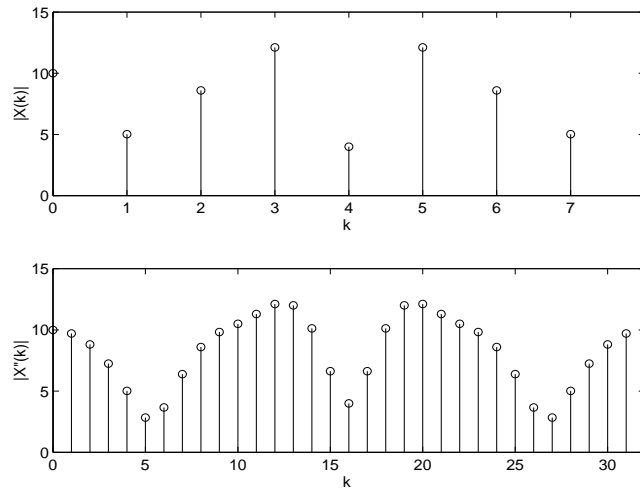
where we have used the fact that : $x'(n) = 0$, $n = N, N+1, \dots, N+L-1$. We have also chosen to use the traditional definition of the DFT (without a scaling factor in front of the sum). Then:

$$X(0) = \sum_{n=0}^{N-1} x(n) = X'(0)$$

If we plot $|X(k)|$ and $|X'(k)|$ in the same graph, with the x-axis being the normalized frequency $f = \frac{k}{N}$ or $f = \frac{k}{N+L}$, respectively, then we notice that the second graph is just an interpolated version of the first. This can be seen if $N+L$ is an integer multiple of N : $N+L = mN$. Then :

$$X'(mk) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nmk/mN} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = X(k), \quad k = 0, 1, \dots, N-1$$

This is illustrated in the following plot, for a random sequence $x(n)$, of length $N = 8$, which is padded with $L = 24$ zeros.



Problem 11.5

The analog signal is :

$$x(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kt/T}, \quad 0 \leq t < T$$

The subcarrier frequencies are : $F_k = k/T$, $k = 0, 1, \dots, \tilde{N}$, and, hence, the maximum frequency in the analog signal is : \tilde{N}/T . If we sample at the Nyquist rate : $2\tilde{N}/T = N/T$, we obtain the discrete-time sequence :

$$x(n) = x(t = nT/N) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi k(nT/N)/T} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1$$

which is simply the IDFT of the information sequence $\{X_k\}$.

Problem 11.6

The resetting of the filter state every N samples, is equivalent to a filter with system function :

$$H_n(z) = \frac{1 - z^{-N}}{1 - \exp(j2\pi n/N)z^{-1}}$$

We will make use of the relationship that gives the sum of finite geometric series : $\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a}$. Using this we can re-write each system function $H_n(z)$ as :

$$H_n(z) = \frac{1 - z^{-N}}{1 - \exp(j2\pi n/N)z^{-1}} = \frac{1 - [\exp(j2\pi n/N)z^{-1}]^N}{1 - \exp(j2\pi n/N)z^{-1}} = \sum_{k=0}^{N-1} (\exp(j2\pi n/N)z^{-1})^k$$

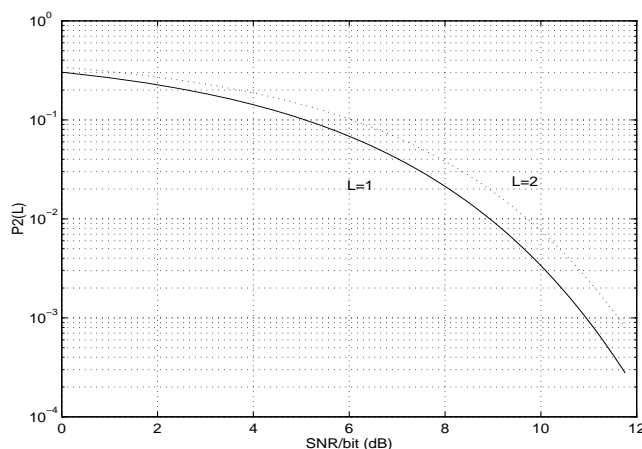
or :

$$H_n(z) = \sum_{k=0}^{N-1} \exp(j2\pi nk/N)z^{-k}, \quad n = 0, 1, \dots, N-1$$

which is exactly the transfer function of the transversal filter which calculates the n -th IDFT point of a sequence.

Problem 11.7

We assume binary ($M = 2$) orthogonal signalling with square-law detection (DPSK signals will have the same combining loss). Using (11-1-24) and (11-1-14) we obtain the following graph for $P_2(L)$, where $\text{SNR/bit} = 10 \log_{10} \gamma_b$:



From this graph we note that the combining loss for $\gamma_b = 10$ is approximately 0.8 dB.

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 12) ¹

Prepared by
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January 29, 2008

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Problem 12.1

$$g(t) = \sqrt{\frac{16\mathcal{E}_c}{3T_c}} \cos^2 \frac{\pi}{T_c} \left(t - \frac{T_c}{2} \right), \quad 0 \leq t \leq T_c$$

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \\ &= \sqrt{\frac{16\mathcal{E}_c}{3T_c}} \int_0^{T_c} \cos^2 \frac{\pi}{T_c} \left(t - \frac{T_c}{2} \right) e^{-j2\pi ft} dt \end{aligned}$$

But $\cos^2 \frac{\pi}{T_c} \left(t - \frac{T_c}{2} \right) = \frac{1}{2} \left[1 + \cos \frac{2\pi}{T_c} \left(t - \frac{T_c}{2} \right) \right]$. Then

$$G(0) = \frac{1}{2} \sqrt{\frac{16\mathcal{E}_c}{3T_c}} T_c = \sqrt{\frac{4\mathcal{E}_c T_c}{3}} \Rightarrow |G(0)|^2 = \frac{4\mathcal{E}_c T_c}{3}$$

and

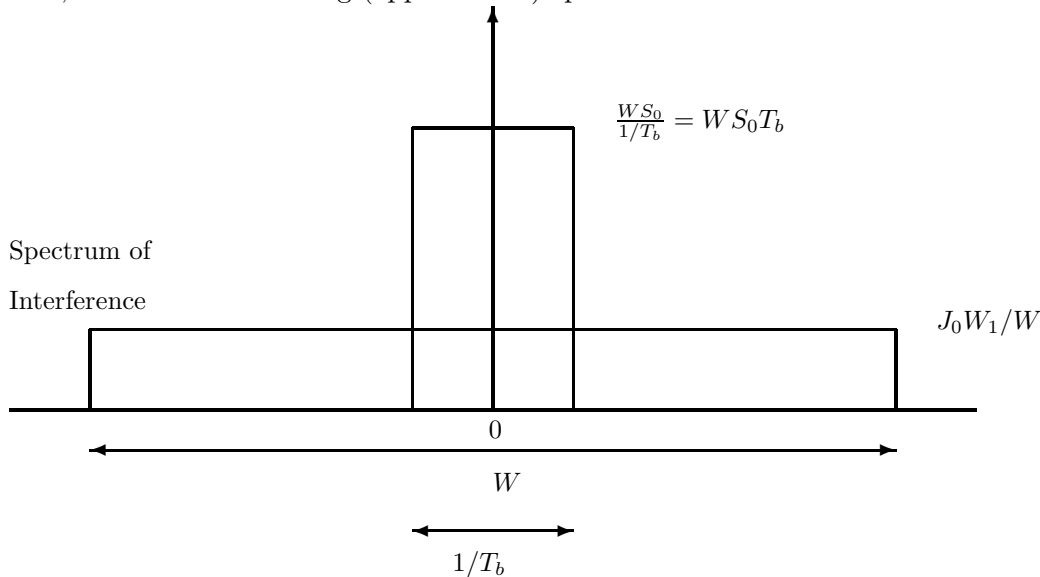
$$\sigma_m^2 = 2w_m J_{av} |G(0)|^2 = \frac{8}{3} \mathcal{E}_c T_c w_m J_{av}, \quad \mathcal{E}_c = R_c \mathcal{E}_b$$

Hence,

$$P_M \leq \sum_{m=2}^M Q \left(\sqrt{6 \frac{R_c \mathcal{E}_b}{J_{av} T_c} w_m} \right)$$

Problem 12.2

The PN spread spectrum signal has a bandwidth W and the interference has a bandwidth W_1 , where $W \gg W_1$. Upon multiplication of the received signal $r(t)$ with the PN reference at the receiver, we have the following (approximate) spectral characteristics



After multiplication with the PN reference, the interference power in the bandwidth $1/T_b$ occupied by the signal is

$$\left(\frac{J_0 W_1}{W}\right) \left(\frac{1}{T_b}\right) = \frac{J_0 W_1}{W T_b}$$

Prior to multiplication, the noise power is $J_0 W$. Therefore, in the bandwidth of the information-bearing signal, there is a reduction in the interference power by a factor $W T_b = \frac{T_b}{T_c} = L_c$, which is just the processing gain of the PN spread spectrum signal.

Problem 12.3

The concatenation of a Reed-Solomon (31,3) code with the Hadamard (16,5) code results in an equivalent binary code of block length $n = n_1 n_2 = 31 \times 16 = 496$ bits. There are 15 information bits conveyed by each code word, i.e. $k = k_1 k_2 = 15$. Hence, the overall code rate is $R_c = 15/496$, which is the product of the two code rates. The minimum distances are

$$\text{Reed - Solomon code : } D_{\min} = 31 - 3 + 1 = 29$$

$$\text{Hadamard code : } d_{\min} = \frac{n_2}{2} = 8$$

Hence, the minimum distance of the overall code is $d_{\min} = 28 \times 8 = 232$. A union bound on the probability of error based on the minimum distance of the code is

$$P_M \leq (M - 1) Q \left(\sqrt{4 \frac{\mathcal{E}_b}{J_{av} T_c} R_c d_{\min}} \right)$$

where $M = 2^{15} = 32768$. Also, $\mathcal{E}_b = P_{av} T_b$. Thus,

$$P_M \leq 2^{15} Q \left(\sqrt{4 \frac{P_{av} T_b}{J_{av} T_c} \frac{k}{n} d_{\min}} \right)$$

But $k T_b = n T_c$ and $d_{\min} = 232$. Hence,

$$P_M \leq 2^{15} Q \left(\sqrt{\frac{528}{J_{av}/P_{av}}} \right)$$

Due to the large number of codewords, this union bound is very loose. A much tighter bound is

$$P_M \leq \sum_{m=2}^M Q \left(\sqrt{\frac{4 w_m}{J_{av}/P_{av}}} \right)$$

but the evaluation of the bound requires the use of the weight distribution of the concatenated code.

Problem 12.4

For hard-decision decoding we have

$$P_M \leq (M-1) [4p(1-p)]^{d_{\min}/2} = 2^{m+1} [4p(1-p)]^{2^{m-2}}$$

where $p = Q\left(\sqrt{\frac{4W/R}{J_{av}/P_{av}}} R_c\right) = Q\left(\sqrt{4\frac{P_{av}}{J_{av}}}\right)$. Note that in the presence of a strong jammer, the probability p is large, i.e close to 1/2. For soft-decision decoding, the error probability bound is

$$P_M \leq (M-1)Q\left(\sqrt{\frac{4W/R}{J_{av}/P_{av}}} R_c d_{\min}\right)$$

We select $\frac{W}{R} = \frac{n}{k} = \frac{1}{R_c}$ and hence:

$$\begin{aligned} P_M &\leq 2^{m+1}Q\left(\sqrt{\frac{2^{m+1}}{J_{av}/P_{av}}}\right) < 2^m \exp\left(-\frac{2^m}{J_{av}/P_{av}}\right) \\ &< \exp\left(-\frac{P_{av}}{J_{av}} \left[2^m - \frac{J_{av}}{P_{av}} m \ln 2\right]\right) \end{aligned}$$

For a give jamming margin, we can find an m which is sufficiently large to achieve the desired level of performance.

Problem 12.5

(a) The coding gain is

$$R_c d_{\min} = \frac{1}{2} \times 10 = 5 \text{ (7dB)}$$

(b) The processing gain is W/R , where $W = 10^7 \text{ Hz}$ and $R = 1000 \text{ bps}$. Hence,

$$\frac{W}{R} = \frac{2 \times 10^7}{10^3} = 2 \times 10^4 \text{ (43dB)}$$

(c) The jamming margin is

$$\begin{aligned} \left(\frac{J_{av}}{P_{av}}\right) &= \frac{\left(\frac{2W}{R}\right) \cdot (R_c d_{\min})}{\left(\frac{\varepsilon_b}{J_0}\right)} \Rightarrow \\ \left(\frac{J_{av}}{P_{av}}\right)_{dB} &= \left(\frac{2W}{R}\right)_{dB} + (CG)_{dB} - \left(\frac{\varepsilon_b}{J_0}\right)_{dB} \\ &= 43 + 7 - 10 = 40 \text{ dB} \end{aligned}$$

Problem 12.6

We assume that the interference is characterized as a zero-mean AWGN process with power spectral density J_0 . To achieve an error probability of 10^{-5} , the required $\mathcal{E}_b/J_0 = 10$. Then, by using the relation in (12-2-46) and (12-2-37), we have

$$\begin{aligned}\frac{2W/R}{J_{av}/P_{av}} &= \frac{2W/R}{N_u-1} = \frac{\mathcal{E}_b}{J_0} \\ 2W/R &= \left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1) \\ W &= R\left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1)/2\end{aligned}$$

where $R = 10^4$ bps, $N_u = 30$ and $\mathcal{E}_b/J_0 = 10$. Therefore,

$$W = 1.45 \times 10^6 \text{ Hz}$$

The minimum chip rate is $1/T_c = W = 1.45 \times 10^6$ chips/sec.

Problem 12.7

To achieve an error probability of 10^{-6} , we require

$$\left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 10.5dB$$

Then, the number of users of the CDMA system is

$$\begin{aligned}N_u &= \frac{2W/R}{\mathcal{E}_b/J_0} + 1 \\ &= \frac{2000}{11.3} + 1 = 177 \text{ users}\end{aligned}$$

If the processing gain is reduced to $W/R = 500$, then

$$N_u = \frac{1000}{11.3} + 1 = 89 \text{ users}$$

Problem 12.8

(a) We are given a system where $(J_{av}/P_{av})_{dB} = 20 \text{ dB}$, $R = 1000 \text{ bps}$ and $(\mathcal{E}_b/J_0)_{dB} = 10 \text{ dB}$. Hence, using the relation in (12-2-37) we obtain

$$\left(\frac{2W}{R}\right)_{dB} = \left(\frac{J_{av}}{P_{av}}\right)_{dB} + \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 30 \text{ dB}$$

$$\frac{2W}{R} = 1000$$

$$W = 1000R/2 = 5 \times 10^5 \text{ Hz}$$

(b) The duty cycle of a pulse jammer for worst-case jamming is

$$\alpha^* = \frac{0.71}{\mathcal{E}_b/J_0} = \frac{0.7}{10} = 0.07$$

The corresponding probability of error for this worst-case jamming is

$$P_2 = \frac{0.083}{\mathcal{E}_b/J_0} = \frac{0.083}{10} = 8.3 \times 10^{-3}$$

Problem 12.9

(a) We have $N_u = 15$ users transmitting at a rate of 10,000 *bps* each, in a bandwidth of $W = 1 \text{ MHz}$. The \mathcal{E}_b/J_0 is

$$\begin{aligned} \frac{\mathcal{E}_b}{J_0} &= \frac{2W/R}{N_u-1} = \frac{2 \times 10^6 / 10^4}{14} = \frac{200}{14} \\ &= 14.28 \text{ (11.54 dB)} \end{aligned}$$

(b) The processing gain is 100.

(c) With $N_u = 30$ and $\mathcal{E}_b/J_0 = 14.28$, the processing gain should be increased to

$$W/R = (14.28)(29) = 414$$

Hence, the bandwidth must be increased to $W = 4.14 \text{ MHz}$.

Problem 12.10

The processing gain is given as

$$\frac{W}{R} = 500 \text{ (27 dB)}$$

The (\mathcal{E}_b/J_0) required to obtain an error probability of 10^{-5} for binary PSK is 9.5 dB. Hence, the jamming margin is

$$\begin{aligned} \left(\frac{J_{av}}{P_{av}}\right)_{dB} &= \left(\frac{2W}{R}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 30 - 9.5 \\ &= 20.5 \text{ dB} \end{aligned}$$

Problem 12.11

If the jammer is a pulse jammer with a duty cycle $\alpha = 0.01$, the probability of error for binary PSK is given as

$$P_2 = \alpha Q \left(\sqrt{\frac{4W/R}{J_{av}/P_{av}}} \right)$$

For $P_2 = 10^{-5}$, and $\alpha = 0.01$, we have

$$Q \left(\sqrt{\frac{4W/R}{J_{av}/P_{av}}} \right) = 10^{-3}$$

Then,

$$\frac{2W/R_b}{J_{av}/P_{av}} = \frac{1000}{J_{av}/P_{av}} = 5$$

and

$$\frac{J_{av}}{P_{av}} = 200 \text{ (23 dB)}$$

Problem 12.12

$$c(t) = \sum_{n=-\infty}^{\infty} c_n p(t - nT_c)$$

The power spectral density of $c(t)$ is given by (3.4.15) as

$$S_c(f) = \frac{1}{T_c} S_c(f) |P(f)|^2$$

where

$$|P(f)|^2 = (AT_c)^2 \sin^2(fT_c), \quad T_c = 1\mu \text{ sec}$$

and $S_c(f)$ is the power spectral density of the sequence $\{c_n\}$. Since the autocorrelation of the sequence $\{c_n\}$ is periodic with period N and is given as

$$R_c(m) = \begin{cases} N, & m = 0, \pm N, \pm 2N, \dots \\ -1, & \text{otherwise} \end{cases}$$

then, $R_c(m)$ can be represented in a discrete Fourier series as

$$\phi_c(m) = \frac{1}{N} \sum_{k=0}^{N-1} r_c(k) e^{j2\pi mk/N}, \quad m = 0, 1, \dots, N-1$$

where $\{r_c(k)\}$ are the Fourier series coefficients, which are given as

$$r_c(k) = \sum_{m=0}^{N-1} R_c(m) e^{-j2\pi km/N}, \quad k = 0, 1, \dots, N-1$$

and $r_c(k + nN) = r_c(k)$ for $n = 0, \pm 1, \pm 2, \dots$. The latter can be evaluated to yield

$$\begin{aligned} r_c(k) &= N + 1 - \sum_{m=0}^{N-1} e^{-j2\pi km/N} \\ &= \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ N + 1, & \text{otherwise} \end{cases} \end{aligned}$$

The power spectral density of the sequence $\{c_n\}$ may be expressed in terms of $\{r_c(k)\}$. These coefficients represent the power in the spectral components at the frequencies $f = k/N$. Therefore, we have

$$S_c(f) = \frac{1}{N} \sum_{k=-\infty}^{\infty} r_c(k) \delta\left(f - \frac{k}{NT_c}\right)$$

Finally, we have

$$S_c(f) = \frac{1}{NT_c} \sum_{k=-\infty}^{\infty} r_c(k) \left| P\left(\frac{k}{NT_c}\right) \right|^2 \delta\left(f - \frac{k}{NT_c}\right)$$

Problem 12.13

Without loss of generality, let us assume that $N_1 < N_2$. Then, the period of the sequence obtained by forming the modulo-2 sum of the two periodic sequences is

$$N_3 = kN_2$$

where k is the smallest integer multiple of N_2 such that kN_2/N_1 is an integer. For example, suppose that $N_1 = 15$ and $N_2 = 63$. Then, we find the smallest multiple of 63 which is divisible by $N_1 = 15$, without a remainder. Clearly, if we take $k = 5$ periods of N_2 , which yields a sequence of $N_3 = 315$, and divide N_3 by N_1 , the result is 21. Hence, if we take $21N_1$ and $5N_2$, and modulo-2 add the resulting sequences, we obtain a single period of length $N_3 = 21N_1 = 5N_2$ of the new sequence.

Problem 12.14

(a) The period of the maximum length shift register sequence is

$$N = 2^{10} - 1 = 1023$$

Since $T_b = NT_c$, then the processing gain is

$$N \frac{T_b}{T_c} = 1023 (30dB)$$

(b) According to (12-2-37), the jamming margin is

$$\begin{aligned} \left(\frac{J_{av}}{P_{av}} \right)_{dB} &= \left(\frac{2W}{R_b} \right)_{dB} - \left(\frac{\varepsilon_b}{J_0} \right)_{dB} \\ &= 33 - 10 \\ &= 23dB \end{aligned}$$

where $J_{av} = 2J_0W \approx 2J_0/T_c = 2J_0 \times 10^6$

Problem 12.15

(a) The length of the shift-register sequence is

$$\begin{aligned} L &= 2^m - 1 = 2^{15} - 1 \\ &= 32767 \text{ bits} \end{aligned}$$

For binary FSK modulation, the minimum frequency separation is $2/T$, where $1/T$ is the symbol (bit) rate. The hop rate is 100 hops/sec . Since the shift register has $L = 32767$ states and each state utilizes a bandwidth of $2/T = 200 \text{ Hz}$, then the total bandwidth for the FH signal is 6.5534 MHz .

(b) The processing gain is W/R . We have,

$$\frac{W}{R} = \frac{6.5534 \times 10^6}{100} = 6.5534 \times 10^4 \text{ bps}$$

(c) If the noise is AWG with power spectral density N_0 , the probability of error expression is

$$P_2 = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{2W/R}{P_N/P_{av}}}\right)$$

Problem 12.16

(a) If the hopping rate is 2 hops/bit and the bit rate is 100 bits/sec, then, the hop rate is 200 hops/sec. The minimum frequency separation for orthogonality $2/T = 400 \text{ Hz}$. Since there are $N = 32767$ states of the shift register and for each state we select one of two frequencies separated by 400 Hz , the hopping bandwidth is 13.1068 MHz .

(b) The processing gain is W/R , where $W = 13.1068 \text{ MHz}$ and $R = 100 \text{ bps}$. Hence

$$\frac{W}{R} = 0.131068 \text{ MHz}$$

(c) The probability of error in the presence of AWGN is given by (12-3-2) with $L = 2$ chips per hop.

Problem 12.17

(a) The total SNR for three hops is $20 \sim 13 \text{ dB}$. Therefore the SNR per hop is $20/3$. The probability of a chip error with noncoherent detection is

$$p = \frac{1}{2} e^{-\frac{\mathcal{E}_c}{2N_0}}$$

where $\mathcal{E}_c/N_0 = 20/3$. The probability of a bit error is

$$\begin{aligned}
 P_b &= 1 - (1 - p)^2 \\
 &= 1 - (1 - 2p + p^2) \\
 &= 2p - p^2 \\
 &= e^{-\frac{\mathcal{E}_c}{2N_0}} - \frac{1}{2}e^{-\frac{\mathcal{E}_c}{N_0}} \\
 &= 0.0013
 \end{aligned}$$

b) In the case of one hop per bit, the SNR per bit is 20, Hence,

$$\begin{aligned}
 P_b &= \frac{1}{2}e^{-\frac{\mathcal{E}_c}{2N_0}} \\
 &= \frac{1}{2}e^{-10} \\
 &= 2.27 \times 10^{-5}
 \end{aligned}$$

Therefore there is a loss in performance of a factor 57 AWGN due to splitting the total signal energy into three chips and, then, using hard decision decoding.

Problem 12.18

(a) We are given a hopping bandwidth of 2 GHz and a bit rate of 10 kbs. Hence,

$$\frac{W}{R} = \frac{2 \times 10^9}{10^4} = 2 \times 10^5 \text{ (53dB)}$$

(b) The bandwidth of the worst partial-band jammer is α^*W , where

$$\alpha^* = 2/(\mathcal{E}_b/J_0) = 0.2$$

Hence

$$\alpha^*W = 0.4\text{GHz}$$

(c) The probability of error with worst-case partial-band jamming is

$$\begin{aligned}
 P_2 &= \frac{e^{-1}}{(\mathcal{E}_b/J_0)} = \frac{e^{-1}}{10} \\
 &= 3.68 \times 10^{-2}
 \end{aligned}$$

Problem 12.19

The error probability for the binary convolutional code is upper-bounded as

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(d)$$

where $P_2(d)$ is the probability of error in a pairwise comparison of two paths that are separated in Hamming distance by d . For square-law detected and combined binary FSK in AWGN, $P_2(d)$ is

$$P_2(d) = \frac{1}{2^{2d-1}} \exp(-\gamma_b R_c d/2) \sum_{n=0}^{d-1} \left[\frac{1}{n!} \left(\frac{\gamma_b R_c d}{2} \right)^n \sum_{r=0}^{d-1-n} \binom{2d-1}{r} \right]$$

Problem 12.20

For hard-decision Viterbi decoding of the convolutional code, the error probability is

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(d)$$

where $P_2(d)$ is given by (8.2.16) when d is odd and by (8.2.17) when d is even. Alternatively, we may use the Chernoff bound for $P_2(d)$, which is : $P_2(d) \leq [4p(1-p)]^{d/2}$. In both cases, $p = \frac{1}{2} \exp(-\gamma_b R_c/2)$.

Problem 12.21

For fast frequency hopping at a rate of L hops/bit and for soft-decision decoding, the performance of the binary convolutional code is upper bounded as

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(Ld)$$

where

$$P_2(Ld) = \frac{1}{2^{2Ld-1}} \exp(-\gamma_b R_c d/2) \sum_{n=0}^{Ld-1} \left[\frac{1}{n!} \left(\frac{\gamma_b R_c d}{2} \right)^n \sum_{r=0}^{Ld-1-n} \binom{2Ld-1}{r} \right]$$

Note that the use of L hops/coded bit represents a repetition of each coded bit by a factor of L . Hence, the convolutional code is in cascade with the repetition code. The overall code rate of R_c/L and the distance properties of the convolutional code are multiplied by the factor L , so that the binary event error probabilities are evaluated at distances of Ld , where $d_{free} \leq d \leq \infty$.

Problem 12.22

For fast frequency hopping at a rate of L hops/bit and for hard-decision Viterbi decoding, the performance of the binary convolutional code is upper bounded as

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(Ld)$$

where

$$P_2(d) \leq [4p(1-p)]^{d/2}$$

and

$$p = \frac{1}{2^{2L-1}} e^{-\gamma_b R_c/2} \sum_{n=0}^{L-1} \left[\left(\frac{\gamma_b R_c}{2} \right)^n \frac{1}{n!} \sum_{r=0}^{L-1-n} \binom{2L-1}{r} \right]$$

On the other hand, if each of the L chips is detected independently, then :

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(Ld)$$

where $P_2(Ld) \leq [4p(1-p)]^{Ld/2}$ and $p = \frac{1}{2} \exp(\gamma_b R_c/2)$.

Problem 12.23

There are 64 decision variables corresponding to the 64 possible codewords in the (7,2) Reed-Solomon code. With $d_{\min} = 6$, we know that the performance of the code is no worse than the performance of an $M = 64$ orthogonal waveform set, where the SNR per bit is degraded by the factor 6/7. Thus, an upper bound on the code word error probability is

$$P_{64} \leq (M-1)P_2 = 63P_2$$

where

$$P_2 \leq \frac{1}{2^{2L-1}} \exp\left(-\gamma_b \frac{6}{7} k/2\right) \sum_{n=0}^{L-1} c_n \left(\frac{3}{7} k \gamma_b\right)^n$$

With $L = 6$ and $k = 6$, P_2 becomes

$$P_2 \leq \frac{1}{2^{11}} \exp(-18\gamma_b/7) \sum_{n=0}^5 c_n \left(\frac{18}{7} \gamma_b\right)^n$$

where

$$c_n = \frac{1}{n!} \sum_{r=0}^{5-n} \binom{11}{r}$$

Problem 12.24

In the worst-case partial-band interference channel, the (7,2) Reed-Solomon code provides an effective order of diversity $L = 6$. Hence

$$\begin{aligned} P_{64} &\leq 63P_2(6) = 63 \left(\frac{1.47}{\frac{6}{7}\gamma_b} \right)^6 \\ &\leq \frac{1.6 \times 10^3}{\gamma_b^6}, \text{ for } \frac{\gamma_b}{L} = \frac{\gamma_b}{6} \geq 3 \end{aligned}$$

Problem 12.25

$$P_2(a) = aQ \left(\sqrt{\frac{2a\mathcal{E}_b}{J_0}} \right) = a \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2a\mathcal{E}_b}{J_0}}}^{+\infty} e^{-t^2/2} dt$$

Hence, the maximum occurs when

$$\begin{aligned} \frac{dP_2(a)}{da} &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2a\mathcal{E}_b}{J_0}}}^{+\infty} e^{-t^2/2} dt - \frac{1}{2} \sqrt{\frac{a\mathcal{E}_b}{\pi J_0}} e^{-\frac{a\mathcal{E}_b}{J_0}} = 0 \Rightarrow \\ \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2a\mathcal{E}_b}{J_0}}}^{+\infty} e^{-t^2/2} dt &= \frac{1}{2} \sqrt{\frac{a\mathcal{E}_b}{\pi J_0}} e^{-\frac{a\mathcal{E}_b}{J_0}} \end{aligned}$$

The value of a that satisfies the above equation can be found numerically (or graphically) and is equal to $a = \frac{0.71}{\mathcal{E}_b/J_0}$. Substitution of this result into $P_2(a)$ yields (for $\mathcal{E}_b/J_0 \geq 0.71$)

$$P_2 = \frac{0.71}{\mathcal{E}_b/J_0} Q \left(\sqrt{2 \cdot 0.71} \right) = \frac{0.083}{\mathcal{E}_b/J_0}$$

Problem 12.26

The problem is to determine

$$E \left[\exp \left(-v \sum_{k=1}^L \beta_k |2\mathcal{E}_c + N_{1k}|^2 - v \sum_{k=1}^L \beta_k |N_{2k}|^2 \right) \right]$$

where the $\{\beta_k\}$ are fixed and the $\{N_{1k}\}, \{N_{2k}\}$ are complex-valued Gaussian random variables with zero-mean and variances equal to σ_k^2 . We note that $\beta_k = 1/\sigma_k^2$ and, hence,

$$\frac{1}{2} E |N_{1k}|^2 = \frac{1}{2} E |N_{2k}|^2 = \sigma_k^2$$

$$\frac{1}{2} \beta_k E |N_{1k}|^2 = \frac{1}{2} \beta_k E |N_{2k}|^2 = 1$$

Since the $\{N_{1k}\}$, and $\{N_{2k}\}$ are all statistically independent

$$E \left[\exp \left(-v \sum_{k=1}^L \beta_k |2\mathcal{E}_c + N_{1k}|^2 \right) \right] = \prod_{k=1}^L E \left[\exp \left(-v \beta_k |2\mathcal{E}_c + N_{1k}|^2 \right) \right]$$

and similarly for $E \left[\exp \left(-v \sum_{k=1}^L \beta_k |N_{2k}|^2 \right) \right]$. Now, we use the characteristic function for the sum of the squares of two Gaussian random variables (the two Gaussian random variables are the real and imaginary parts of $2\mathcal{E}_c + N_{1k}$). That is

$$\psi_Y(ju) = E(e^{juY}) = \frac{1}{1 - ju\sigma^2} \exp \left(\frac{ju \sum_{i=1}^m m_i^2}{1 - ju\sigma^2} \right)$$

where $Y = X_1^2 + X_2^2$, m_i is the mean of X_i and σ^2 is the variance of X_i . Hence,

$$E \left[\exp \left(-v \left| \sqrt{\beta_k} 2\mathcal{E}_c + \sqrt{\beta_k} N_{1k} \right|^2 \right) \right] = \frac{1}{1 + 2v} \exp \left(\frac{-4\beta_k v \mathcal{E}_c^2}{1 + 2v} \right)$$

and

$$E \left[\exp \left(-v \left| \sqrt{\beta_k} N_{2k} \right|^2 \right) \right] = \frac{1}{1 - 2v}$$

Consequently

$$P_2(\beta) = \prod_{k=1}^L \frac{1}{1 - 4v^2} \exp \left(\frac{-4\beta_k v \mathcal{E}_c^2}{1 + 2v} \right)$$

Problem 12.27

The function

$$f(v, a) = \left[\frac{a}{1 - 4v^2} \exp \left(\frac{-2av \mathcal{E}_c^2}{J_0(1 + 2v)} \right) \right]^L$$

is to be minimized with respect to v and maximized with respect to a . Thus,

$$\frac{\partial}{\partial v} f(v, a) = 0 \Rightarrow 8v(1 + 2v) - \frac{2a\mathcal{E}_c}{J_0}(1 - 2v) = 0$$

and

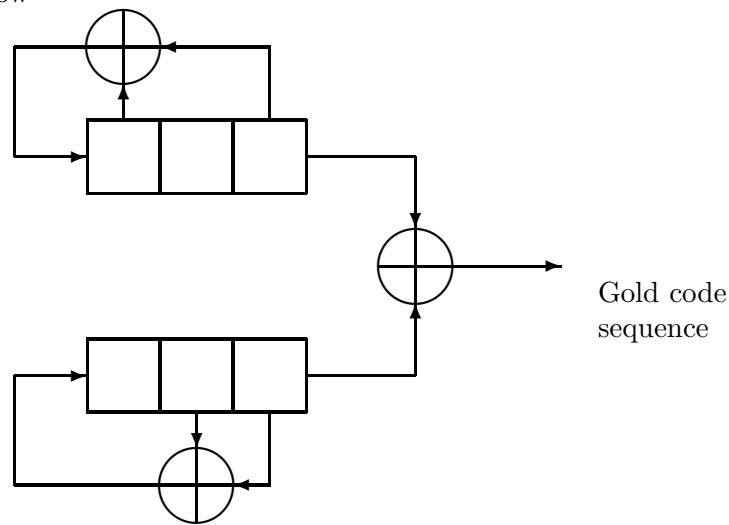
$$\frac{\partial}{\partial a} f(v, a) = 0 \Rightarrow 1 - \frac{2av\mathcal{E}_c}{J_0(1 + 2v)} = 0$$

The simultaneous solution of these two equations yields the result $v = 1/4$ and $a = \frac{3J_0}{\mathcal{E}_c} \leq 1$. For these values, the function $f(v, a)$ becomes

$$f\left(\frac{1}{4}, \frac{3J_0}{\mathcal{E}_c}\right) = \left(\frac{4}{e\gamma_c}\right)^L = \left(\frac{1.47}{\gamma_c}\right)^L \quad \text{for } \gamma_c = \frac{\mathcal{E}_c}{J_0} \geq 3$$

Problem 12.28

The Gold code sequences of length $n = 7$ may be generated by the use of two feedback shift registers of length 4 as shown below



These sequences have the following cross-correlation values

Values of correlation	Frequency of this value
-5	1
3	3
-1	3

Problem 12.29

The method of Omura and Levitt (1982) based on the cut-off rate R_0 , which is described in Section 12.2.3, can be used to evaluate the error rate of the coded system with interleaving in pulse interference. This computational method yields results that are reasonably close to the results given by Martin and McAdam (1980) and which are illustrated in Fig. 12.2.12 for the rate 1/2 convolutional coder with soft-decision decoding.

Problem 12.30

(a) For the coded and interleaved DS binary PSK modulation with pulse jamming and soft-decision decoding, the cutoff rate is

$$R_0 = 1 - \log_2 \left[1 + ae^{-a\mathcal{E}_c/N_0} \right]$$

Hence,

$$\begin{aligned} \log_2 \left[1 + ae^{-a\mathcal{E}_c/N_0} \right] &= 1 - R_0 \Rightarrow \\ \left[1 + ae^{-a\mathcal{E}_c/N_0} \right] &= 2^{1-R_0} \Rightarrow \\ \frac{\mathcal{E}_c}{N_0} &= \frac{1}{a} \ln \frac{a}{2^{1-R_0}-1} \end{aligned}$$

and since $\mathcal{E}_c = \mathcal{E}_b R$

$$\frac{\mathcal{E}_b}{N_0} = \frac{1}{aR} \ln \frac{a}{2^{1-R_0}-1}$$

(b)

$$\frac{d}{da} \left(\frac{\mathcal{E}_b}{N_0} \right) = -\frac{1}{a^2 R} \ln \frac{a}{2^{1-R_0}-1} + \frac{1}{a^2 R} = 0$$

Hence, the worst-case a is

$$a^* = (2^{1-R_0} - 1) e$$

provided that $(2^{1-R_0} - 1) e < 1$, or equivalently : $R_0 > 1 - \log_2(1 + e^{-1}) = 0.548$.

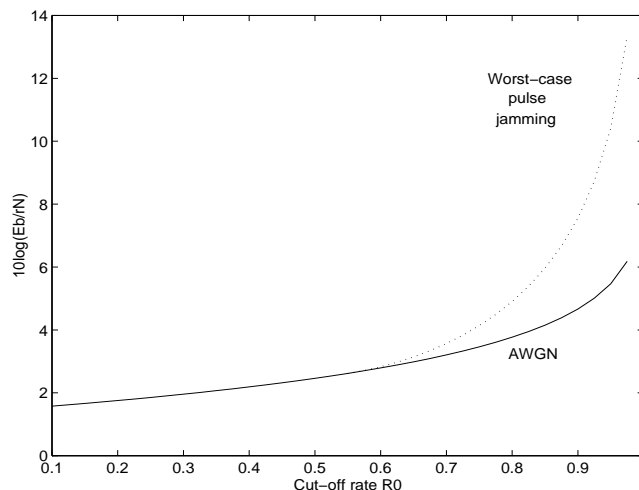
If $R_0 \leq 0.548$, then $a = 1$ maximizes $\frac{\mathcal{E}_b}{N_0}$ and, hence :

$$\frac{\mathcal{E}_b}{N_0} = \frac{1}{R} \ln \frac{1}{2^{1-R_0}-1}, \quad a = 1$$

which is the result for the AWGN channel. For $R_0 > 0.548$ and $a = a^*$ we have

$$\frac{\mathcal{E}_b}{N_0} = \frac{e^{-1}}{R(2^{1-R_0}-1)}, \quad R_0 > 0.548$$

(c) The plot of $10 \log \left(\frac{\mathcal{E}_b}{rN_0} \right)$ versus R_0 , is given in the following figure:



Clearly, there is no penalty in SNR due to worst case pulse jamming for rates below 0.548. Even for $R_0 = 0.8$ the SNR loss is relatively small. As $R_0 \rightarrow 1$, the relative difference between pulse jamming and AWGN becomes large.

Problem 12.31

(a)

$$R_0 = \log_2 q - \log_2 [1 + (q-1)a \exp(-a\mathcal{E}_c/2N_0)] \Rightarrow$$

$$1 + (q-1)a \exp(-a\mathcal{E}_c/2N_0) = q2^{-R_0}$$

and since $\mathcal{E}_c = R\mathcal{E}_b$

$$\frac{\mathcal{E}_b}{N_0} = \frac{2}{aR} \ln \frac{(q-1)a}{q2^{-R_0} - 1}$$

(b)

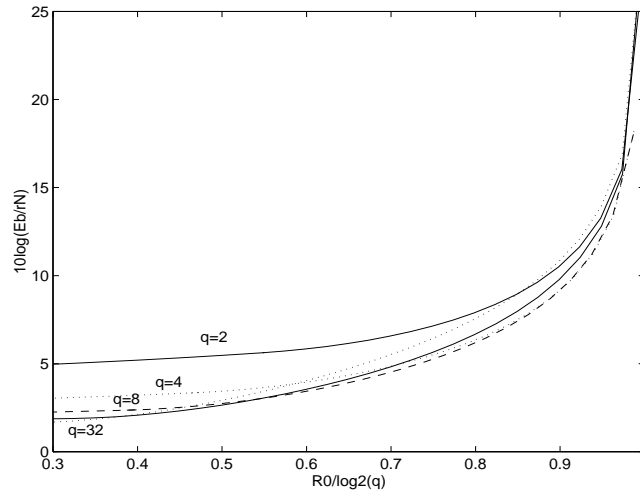
$$\frac{d}{da} \left(\frac{\mathcal{E}_b}{N_0} \right) = -\frac{2}{a^2 R} \ln \frac{(q-1)a}{q2^{-R_0} - 1} + \frac{2}{a^2 R} = 0 \Rightarrow$$

$$a^* = \frac{(q2^{-R_0} - 1) e}{q - 1}$$

provided that $\frac{(q2^{-R_0} - 1)e}{q - 1} < 1$ or, equivalently, $R_0 > \log_2 \frac{q}{(q-1)/e + 1}$. For $a = a^*$, the SNR/bit becomes

$$\frac{\mathcal{E}_b}{N_0} = \frac{2(q-1)}{R[q2^{-R_0} - 1]e}, \quad \text{for } R_0 > \log_2 \frac{q}{(q-1)/e + 1}$$

(c) The plots are given in the following figure



For low rates, the loss due to partial band jamming is negligible if coding is used. Increasing q reduces the SNR/bit at low rates. At very high rates, a large q implies a large SNR loss. For $q = 2$, there is a 3dB loss relative to binary PSK. As $q \rightarrow \infty$, the orthogonal FSK approaches -1.6dB as $R_0 \rightarrow 0$.

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 13) ¹

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Problem 13.1

Based on the info about the scattering function we know that the multipath spread is $T_m = 1 \text{ ms}$, and the Doppler spread is $B_d = 0.2 \text{ Hz}$.

- (a) (i) $T_m = 10^{-3} \text{ sec}$
(ii) $B_d = 0.2 \text{ Hz}$
(iii) $(\Delta t)_c \approx \frac{1}{B_d} = 5 \text{ sec}$
(iv) $(\Delta f)_c \approx \frac{1}{T_m} = 1000 \text{ Hz}$
(v) $T_m B_d = 2 \cdot 10^{-4}$

(b) (i) Frequency non-selective channel : This means that the signal transmitted over the channel has a bandwidth less than 1000 Hz.

(ii) Slowly fading channel : the signaling interval T is $T \ll (\Delta t)_c$.

(iii) The channel is frequency selective : the signal transmitted over the channel has a bandwidth greater than 1000 Hz.

(c) The signal design problem does not have a unique solution. We should use orthogonal M=4 FSK with a symbol rate of 50 symbols/sec. Hence $T = 1/50 \text{ sec}$. For signal orthogonality, we select the frequencies with relative separation $\Delta f = 1/T = 50 \text{ Hz}$. With this separation we obtain $10000/50=200$ frequencies. Since four frequencies are required to transmit 2 bits, we have up to 50^{th} -order diversity available. We may use simple repetition-type diversity or a more efficient block or convolutional code of rate $\geq 1/50$. The demodulator may use square-law combining.

Problem 13.2

(a)

$$P_{2h} = p^3 + 3p^2(1 - p)$$

where $p = \frac{1}{2+\bar{\gamma}_c}$, and $\bar{\gamma}_c$ is the received SNR/cell.

- (b) For $\bar{\gamma}_c = 100$, $P_{2h} \approx 10^{-6} + 3 \cdot 10^{-4} \approx 3 \cdot 10^{-4}$
For $\bar{\gamma}_c = 1000$, $P_{2h} \approx 10^{-9} + 3 \cdot 10^{-6} \approx 3 \cdot 10^{-6}$

(c) Since $\bar{\gamma}_c \gg 1$, we may use the approximation : $P_{2s} \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c} \right)^L$, where L is the order of diversity. For $L=3$, we have :

$$P_{2s} \approx \frac{10}{\bar{\gamma}_c^3} \Rightarrow \left\{ \begin{array}{ll} P_{2s} \approx 10^{-5}, & \bar{\gamma}_c = 100 \\ P_{2s} \approx 10^{-8}, & \bar{\gamma}_c = 1000 \end{array} \right\}$$

(d) For hard-decision decoding :

$$P_{2h} = \sum_{k=\frac{L+1}{2}}^L \binom{L}{k} p^k (1-p)^{L-k} \leq [4p(1-p)]^{L/2}$$

where the latter is the Chernoff bound, L is odd, and $p = \frac{1}{2+\bar{\gamma}_c}$. For soft-decision decoding :

$$P_{2s} \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c} \right)^L$$

Problem 13.3

(a) For a fixed channel, the probability of error is : $P_e(a) = Q\left(\sqrt{\frac{a^2 2\mathcal{E}}{N_0}}\right)$. We now average this conditional error probability over the possible values of α , which are $a=0$, with probability 0.1, and $a=2$ with probability 0.9. Thus :

$$P_e = 0.1Q(0) + 0.9Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) = 0.05 + 0.9Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right)$$

(b) As $\frac{\mathcal{E}}{N_0} \rightarrow \infty$, $P_e \rightarrow 0.05$

(c) When the channel gains a_1, a_2 are fixed, the probability of error is :

$$P_e(a_1, a_2) = Q\left(\sqrt{\frac{(a_1^2 + a_2^2) 2\mathcal{E}}{N_0}}\right)$$

Averaging over the probability density function $p(a_1, a_2) = p(a_1) \cdot p(a_2)$, we obtain the average probability of error :

$$\begin{aligned} P_e &= (0.1)^2 Q(0) + 2 \cdot 0.9 \cdot 0.1 \cdot Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + (0.9)^2 Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right) \\ &= 0.005 + 0.18Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + 0.81Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right) \end{aligned}$$

(d) As $\frac{\mathcal{E}}{N_0} \rightarrow \infty$, $P_e \rightarrow 0.005$

Problem 13.4

(a)

$$T_m = 1 \text{ sec} \Rightarrow (\Delta f)_c \approx \frac{1}{T_m} = 1 \text{ Hz}$$

$$B_d = 0.01 \text{ Hz} \Rightarrow (\Delta t)_c \approx \frac{1}{B_d} = 100 \text{ sec}$$

(b) Since $W = 5 \text{ Hz}$ and $(\Delta f)_c \approx 1 \text{ Hz}$, the channel is frequency selective.

(c) Since $T = 10 \text{ sec} < (\Delta t)_c$, the channel is slowly fading.

(d) The desired data rate is not specified in this problem, and must be assumed. Note that with a pulse duration of $T = 10 \text{ sec}$, the binary PSK signals can be spaced at $1/T = 0.1 \text{ Hz}$ apart. With a bandwidth of $W = 5 \text{ Hz}$, we can form 50 subchannels or carrier frequencies. On the other hand, the amount of diversity available in the channel is $W/(\Delta f)_c = 5$. Suppose the desired data rate is 1 bit/sec. Then, ten adjacent carriers can be used to transmit the data in parallel and the information is repeated five times using the total number of 50 subcarriers to achieve 5-th order diversity. A subcarrier separation of 1 Hz is maintained to achieve independent fading of subcarriers carrying the same information.

(e) We use the approximation :

$$P_2 \approx \binom{2L-1}{L} \left(\frac{1}{4\bar{\gamma}_c} \right)^L$$

where $L=5$. For $P_2 = 10^{-6}$, the SNR required is :

$$(126) \left(\frac{1}{4\bar{\gamma}_c} \right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 10.4 \text{ (10.1 dB)}$$

(f) The tap spacing between adjacent taps is $1/5 = 0.2 \text{ seconds}$. the total multipath spread is $T_m = 1 \text{ sec}$. Hence, we employ a RAKE receiver with at least 5 taps.

(g) Since the fading is slow relative to the pulse duration, in principle we can employ a coherent receiver with pre-detection combining.

(h) For an error rate of 10^{-6} , we have :

$$P_2 \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c} \right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 41.6 \text{ (16.1 dB)}$$

Problem 13.5**(a)**

$$p(n_1, n_2) = \frac{1}{2\pi\sigma^2} e^{-(n_1^2 + n_2^2)/2\sigma^2}$$

$$U_1 = 2\mathcal{E} + N_1, \quad U_2 = N_1 + N_2 \Rightarrow N_1 = U_1 - 2\mathcal{E}, \quad N_2 = U_2 - U_1 + 2\mathcal{E}$$

where we assume that $s(t)$ was transmitted. Then, the Jacobian of the transformation is :

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

and :

$$\begin{aligned} p(u_1, u_2) &= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} [(U_1 - 2\mathcal{E})^2 + (U_2 - (U_1 - 2\mathcal{E}))^2]} \\ &= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} [(U_1 - 2\mathcal{E})^2 + U_2^2 + (U_1 - 2\mathcal{E})^2 - 2U_2(U_1 - 2\mathcal{E})]} \\ &= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{\sigma^2} [(U_1 - 2\mathcal{E})^2 + \frac{1}{2}U_2^2 - U_2(U_1 - 2\mathcal{E})]} \end{aligned}$$

The derivation is exactly the same for the case when $-s(t)$ is transmitted, with the sole difference that $U_1 = -2\mathcal{E} + N_1$.

(b) The likelihood ratio is :

$$\Lambda = \frac{p(u_1, u_2 | +s(t))}{p(u_1, u_2 | -s(t))} = \exp \left[-\frac{1}{\sigma^2} (-8\mathcal{E}U_1 + 4\mathcal{E}U_2) \right] >^{+s(t)} 1$$

or :

$$\ln \Lambda = \frac{8\mathcal{E}}{\sigma^2} \left(U_1 - \frac{1}{2}U_2 \right) >^{+s(t)} 0 \Rightarrow U_1 - \frac{1}{2}U_2 >^{+s(t)} 0$$

Hence $\beta = -1/2$.

(c)

$$U = U_1 - \frac{1}{2}U_2 = 2\mathcal{E} + \frac{1}{2}(N_1 - N_2)$$

$$E[U] = 2\mathcal{E}, \quad \sigma_U^2 = \frac{1}{4}(\sigma_{n1}^2 + \sigma_{n2}^2) = \mathcal{E}N_0$$

Hence:

$$p(u) = \frac{1}{\sqrt{2\pi\mathcal{E}N_0}} e^{-(u-2\mathcal{E})^2/2\mathcal{E}N_0}$$

(d)

$$\begin{aligned} P_e &= P(U < 0) \\ &= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi\mathcal{E}N_0}} e^{-(u-2\mathcal{E})^2/2\mathcal{E}N_0} du \\ &= Q\left(\frac{2\mathcal{E}}{\sqrt{\mathcal{E}N_0}}\right) = Q\left(\sqrt{\frac{4\mathcal{E}}{N_0}}\right) \end{aligned}$$

(e) If only U_1 is used in reaching a decision, then we have the usual binary PSK probability of error : $P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)$, hence a loss of 3 dB, relative to the optimum combiner.

Problem 13.6

(a)

$$U = \operatorname{Re} \left[\sum_{k=1}^L \beta_k U_k \right] > 0$$

where $U_k = 2Ea_k e^{-j\phi_k} + v_k$ and where v_k is zero-mean Gaussian with variance $2EN_{0k}$. Hence, U is Gaussian with :

$$\begin{aligned} E[U] &= \operatorname{Re} \left[\sum_{k=1}^L \beta_k E(U_k) \right] \\ &= 2\mathcal{E} \cdot \operatorname{Re} \left[\sum_{k=1}^L a_k \beta_k e^{-j\phi_k} \right] \\ &= 2\mathcal{E} \sum_{k=1}^L a_k |\beta_k| \cos(\theta_k - \phi_k) \equiv m_u \end{aligned}$$

where $\beta_k = |\beta_k| e^{j\theta_k}$. Also :

$$\sigma_u^2 = 2\mathcal{E} \sum_{k=1}^L |\beta_k|^2 N_{0k}$$

Hence :

$$p(u) = \frac{1}{\sqrt{2\pi}\sigma_u} e^{-(u-m_u)^2/2\sigma_u^2}$$

(b) The probability of error is :

$$P_2 = \int_{-\infty}^0 p(u) du = Q\left(\sqrt{2\gamma}\right)$$

where :

$$\gamma = \frac{\mathcal{E} \left[\sum_{k=1}^L a_k |\beta_k| \cos(\theta_k - \phi_k) \right]^2}{\sum_{k=1}^L |\beta_k|^2 N_{0k}}$$

(c) To maximize P_2 , we maximize γ . It is clear that γ is maximized with respect to $\{\theta_k\}$ by selecting $\theta_k = \phi_k$ for $k = 1, 2, \dots, L$. Then we have :

$$\gamma = \frac{\mathcal{E} \left[\sum_{k=1}^L a_k |\beta_k| \right]^2}{\sum_{k=1}^L |\beta_k|^2 N_{0k}}$$

Now :

$$\frac{d\gamma}{d|\beta_l|} = 0 \Rightarrow \left(\sum_{k=1}^L |\beta_k|^2 N_{0k} \right) a_l - \left(\sum_{k=1}^L a_k |\beta_k| \right) |\beta_l| N_{0l} = 0$$

Consequently :

$$|\beta_l| = \frac{a_l}{N_{0l}}$$

and :

$$\gamma = \frac{\mathcal{E} \left[\sum_{k=1}^L \frac{a_k^2}{N_{ok}} \right]^2}{\sum_{k=1}^L \frac{a_k^2}{N_{ok}} N_{0k}} = \mathcal{E} \sum_{k=1}^L \frac{a_k^2}{N_{ok}}$$

The above represents maximal ratio combining.

Problem 13.7

(a)

$$p(u_1) = \frac{1}{(2\sigma_1^2)^L (L-1)!} u_1^{L-1} e^{-u_1/2\sigma_1^2}, \quad \sigma_1^2 = 2\mathcal{E}N_0(1 + \bar{\gamma}_c)$$

$$p(u_2) = \frac{1}{(2\sigma_2^2)^L (L-1)!} u_2^{L-1} e^{-u_2/2\sigma_2^2}, \quad \sigma_2^2 = 2\mathcal{E}N_0$$

$$P_2 = P(U_2 > U_1) = \int_0^\infty P(U_2 > U_1 | U_1) p(U_1) dU_1$$

But :

$$\begin{aligned} P(U_2 > U_1 | U_1) &= \int_{u_1}^\infty p(u_2) du_2 = \int_{u_1}^\infty \frac{1}{(2\sigma_2^2)^L (L-1)!} u_2^{L-1} e^{-u_2/2\sigma_2^2} du_2 \\ &= \left[\frac{1}{(2\sigma_2^2)^L (L-1)!} u_2^{L-1} e^{-u_2/2\sigma_2^2} (-2\sigma_2^2) \right]_{u_1}^\infty - \int_{u_1}^\infty \frac{(-2\sigma_2^2)(L-1)}{(2\sigma_2^2)^L (L-1)!} u_2^{L-2} e^{-u_2/2\sigma_2^2} du_2 \\ &= \frac{1}{(2\sigma_2^2)^{L-1} (L-1)!} u_1^{L-1} e^{-u_1/2\sigma_2^2} + \int_{u_1}^\infty \frac{1}{(2\sigma_2^2)^{L-1} (L-2)!} u_2^{L-2} e^{-u_2/2\sigma_2^2} du_2 \end{aligned}$$

Continuing, in the same way, the integration by parts, we obtain :

$$P(U_2 > U_1 | U_1) = e^{-u_1/2\sigma_2^2} \sum_{k=0}^{L-1} \frac{(u_1/2\sigma_2^2)^k}{k!}$$

Then :

$$\begin{aligned} P_2 &= \int_0^\infty \left[e^{-u_1/2\sigma_2^2} \sum_{k=0}^{L-1} \frac{(u_1/2\sigma_2^2)^k}{k!} \right] \frac{1}{(2\sigma_1^2)^L (L-1)!} u_1^{L-1} e^{-u_1/2\sigma_1^2} du_1 \\ &= \sum_{k=0}^{L-1} \frac{1}{k! (2\sigma_2^2)^k (2\sigma_1^2)^L (L-1)!} \int_0^\infty u_1^{L-1+k} e^{-u_1(1/\sigma_1^2 + 1/\sigma_2^2)/2} du_1 \end{aligned}$$

The integral that exists inside the summation is equal to :

$$\begin{aligned} \int_0^\infty u^{L-1+k} e^{-ua} du &= \\ \left[\frac{u^{L-1+k} e^{-ua}}{(-a)} \right]_0^\infty - \frac{L-1+k}{(-a)} \int_0^\infty u^{L-2+k} e^{-ua} du &= \\ \frac{L-1+k}{a} \int_0^\infty u^{L-2+k} e^{-ua} du \end{aligned}$$

where $a = (1/\sigma_1^2 + 1/\sigma_2^2)/2 = \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2\sigma_2^2}$. Continuing the integration by parts, we obtain :

$$\int_0^\infty u^{L-1+k} e^{-ua} du = \frac{1}{a^{L+k}} (L-1+k)! = \left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^{L+k} (L-1+k)!$$

Hence :

$$\begin{aligned} P_2 &= \sum_{k=0}^{L-1} \frac{1}{k!(2\sigma_2^2)^k (2\sigma_1^2)^L (L-1)!} \int_0^\infty u_1^{L-1+k} e^{-u_1(1/\sigma_1^2 + 1/\sigma_2^2)/2} du_1 \\ &= \sum_{k=0}^{L-1} \frac{1}{k!(2\sigma_2^2)^k (2\sigma_1^2)^L (L-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^{L+k} (L-1+k)! \\ &= \sum_{k=0}^{L-1} \left(\frac{L-1+k}{k} \right) \frac{\sigma_1^{2k} \sigma_2^{2L}}{(\sigma_1^2 + \sigma_2^2)^{L+k}} = \sum_{k=0}^{L-1} \left(\frac{L-1+k}{k} \right) \frac{(2EN_0(1+\bar{\gamma}_c))^k (2EN_0)^L}{(2EN_0(1+\bar{\gamma}_c) + 2EN_0)^{L+k}} \\ &= \sum_{k=0}^{L-1} \left(\frac{L-1+k}{k} \right) \frac{(2EN_0(1+\bar{\gamma}_c))^k (2EN_0)^L}{(2EN_0(2+\bar{\gamma}_c))^{L+k}} = \left(\frac{1}{2+\bar{\gamma}_c} \right)^L \sum_{k=0}^{L-1} \left(\frac{L-1+k}{k} \right) \left(\frac{1+\bar{\gamma}_c}{2+\bar{\gamma}_c} \right)^k \end{aligned}$$

which is the desired expression (14-4-15) with $\mu = \frac{\bar{\gamma}_c}{2+\bar{\gamma}_c}$.

Problem 13.8

$$U = \sum_{k=1}^L U_k$$

(a) $U_k = 2Ea_k + v_k$, where v_k is Gaussian with $E[v_k] = 0$ and $\sigma_v^2 = 2EN_0$. Hence, for fixed $\{a_k\}$, U is also Gaussian with : $E[U] = \sum_{k=1}^L E[U_k] = 2E \sum_{k=1}^L a_k$ and $\sigma_u^2 = L\sigma_v^2 = 2LEN_0$. Since U is Gaussian, the probability of error, conditioned on a fixed number of gains $\{a_k\}$ is

$$P_b(a_1, a_2, \dots, a_L) = Q\left(\frac{2E \sum_{k=1}^L a_k}{\sqrt{2LEN_0}}\right) = Q\left(\sqrt{\frac{2E \left(\sum_{k=1}^L a_k\right)^2}{LN_0}}\right)$$

(b) The average probability of error for the fading channel is the conditional error probability averaged over the $\{a_k\}$. Hence :

$$P_d = \int_0^\infty da_1 \int_0^\infty da_2 \dots \int_0^\infty da_L P_b(a_1, a_2, \dots, a_L) p(a_1) p(a_2) \dots p(a_L)$$

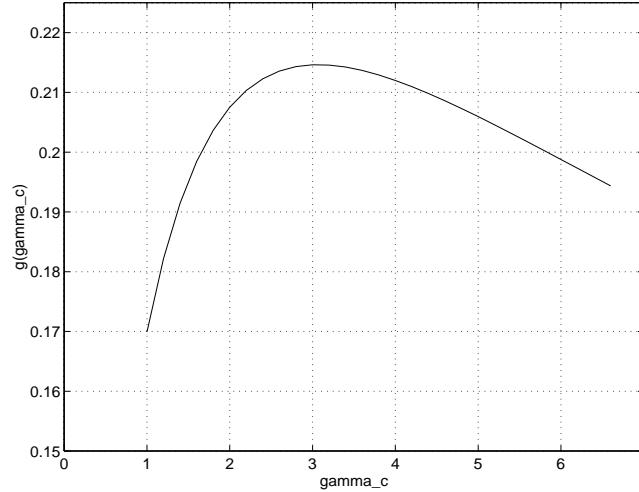
where $p(a_k) = \frac{a_k}{\sigma^2} \exp(-a_k^2/2\sigma^2)$, where σ^2 is the variance of the Gaussian RV's associated with the Rayleigh distribution of the $\{a_k\}$ (not to be confused with the variance of the noise terms). Since $P_b(a_1, a_2, \dots, a_L)$ depends on the $\{a_k\}$ through their sum, we may let : $X = \sum_{k=1}^L a_k$ and, thus, we have the conditional error probability $P_b(X) = Q\left(\sqrt{2EX/(LN_0)}\right)$. The average error probability is :

$$P_b = \int_0^\infty P_b(X) p(X) dX$$

The problem is to determine $p(X)$. Unfortunately, there is no closed form expression for the pdf of a sum of Rayleigh distributed RV's. Therefore, we cannot proceed any further.

Problem 13.9

(a) The plot of $g(\bar{\gamma}_c)$ as a function of $\bar{\gamma}_c$ is given below :



The maximum value of $g(\bar{\gamma}_c)$ is approximately 0.215 and occurs when $\bar{\gamma}_c \approx 3$.

(b) $\bar{\gamma}_c = \bar{\gamma}_b/L$. Hence, for a given $\bar{\gamma}_b$ the optimum diversity is $L = \bar{\gamma}_b/\bar{\gamma}_c = \bar{\gamma}_b/3$.

(c) For the optimum diversity we have :

$$P_2(L_{opt}) < 2^{-0.215\bar{\gamma}_b} = e^{-\ln 2 \cdot 0.215\bar{\gamma}_b} = e^{-0.15\bar{\gamma}_b} = \frac{1}{2} e^{-0.15\bar{\gamma}_b + \ln 2}$$

For the non-fading channel : $P_2 = \frac{1}{2}e^{-0.5\gamma_b}$. Hence, for large SNR ($\bar{\gamma}_b \gg 1$), the penalty in SNR is:

$$10 \log_{10} \frac{0.5}{0.15} = 5.3 \text{ dB}$$

Problem 13.10

The radio signal propagates at the speed of light, $c = 3 \times 10^8 \text{ m/sec}$. The difference in propagation delay for a distance of 300 meters is

$$T_d = \frac{300}{3 \times 10^8} = 1 \mu \text{ sec}$$

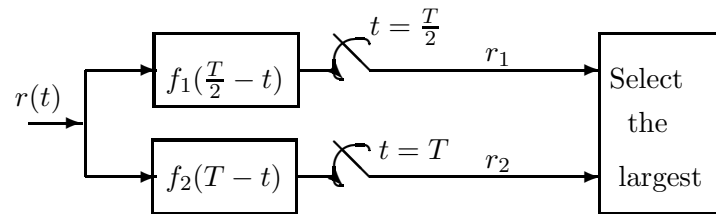
The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is $W = 1 \text{ MHz}$. Hence, the minimum chip rate is 10^6 chips per second.

Problem 13.11

(a) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

$$f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \quad f_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

(b) The optimal receiver is shown in the next figure



(c) Assuming that the signal $s_1(t)$ is transmitted, the received vector at the output of the samplers is

$$\mathbf{r} = \left[\sqrt{\frac{A^2 T}{2}} + n_1, n_2 \right]$$

where n_1, n_2 are zero mean Gaussian random variables with variance $\frac{N_0}{2}$. The probability of error $P(e|s_1)$ is

$$\begin{aligned} P(e|s_1) &= P(n_2 - n_1 > \sqrt{\frac{A^2 T}{2}}) \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_{\frac{A^2 T}{2}}^{\infty} e^{-\frac{x^2}{2N_0}} dx = Q \left[\sqrt{\frac{A^2 T}{2N_0}} \right] \end{aligned}$$

where we have used the fact the $n = n_2 - n_1$ is a zero-mean Gaussian random variable with variance N_0 . Similarly we find that $P(e|s_1) = Q \left[\sqrt{\frac{A^2 T}{2N_0}} \right]$, so that

$$P(e) = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q \left[\sqrt{\frac{A^2 T}{2N_0}} \right]$$

(d) The signal waveform $f_1(\frac{T}{2} - t)$ matched to $f_1(t)$ is exactly the same with the signal waveform $f_2(T - t)$ matched to $f_2(t)$. That is,

$$f_1(\frac{T}{2} - t) = f_2(T - t) = f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the optimal receiver can be implemented by using just one filter followed by a sampler which samples the output of the matched filter at $t = \frac{T}{2}$ and $t = T$ to produce the random variables r_1 and r_2 respectively.

(e) If the signal $s_1(t)$ is transmitted, then the received signal $r(t)$ is

$$r(t) = s_1(t) + \frac{1}{2}s_1(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= A\sqrt{\frac{2}{T}}\frac{T}{4} + \frac{3A}{2}\sqrt{\frac{2}{T}}\frac{T}{4} + n_1 = \frac{5}{2}\sqrt{\frac{A^2 T}{8}} + n_1 \\ r_2 &= \frac{A}{2}\sqrt{\frac{2}{T}}\frac{T}{4} + n_2 = \frac{1}{2}\sqrt{\frac{A^2 T}{8}} + n_2 \end{aligned}$$

If the optimal receiver uses a threshold V to base its decisions, that is

$$\begin{array}{c} s_1 \\ r_1 - r_2 > V \\ < \\ s_2 \end{array}$$

then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V) = Q \left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}} \right]$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2\left(t - \frac{T}{2}\right) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= n_1 \\ r_2 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 \\ &= \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_2 \end{aligned}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \frac{5}{2}\sqrt{\frac{A^2T}{8}} + V) = Q \left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}} \right]$$

Thus, the average probability of error is given by

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\ &= \frac{1}{2}Q \left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}} \right] + \frac{1}{2}Q \left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}} \right] \end{aligned}$$

The optimal value of V can be found by setting $\frac{\partial P(e)}{\partial V}$ equal to zero. Using Leibnitz rule to differentiate definite integrals, we obtain

$$\frac{\partial P(e)}{\partial V} = 0 = \left(2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}} \right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}} \right)^2$$

or by solving in terms of V

$$V = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

(f) Let a be fixed to some value between 0 and 1. Then, if we argue as in part (e) we obtain

$$\begin{aligned} P(e|s_1, a) &= P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V(a)) \\ P(e|s_2, a) &= P(n_1 - n_2 > (a+2)\sqrt{\frac{A^2T}{8}} + V(a)) \end{aligned}$$

and the probability of error is

$$P(e|a) = \frac{1}{2}P(e|s_1, a) + \frac{1}{2}P(e|s_2, a)$$

For a given a , the optimal value of $V(a)$ is found by setting $\frac{\partial P(e|a)}{\partial V(a)}$ equal to zero. By doing so we find that

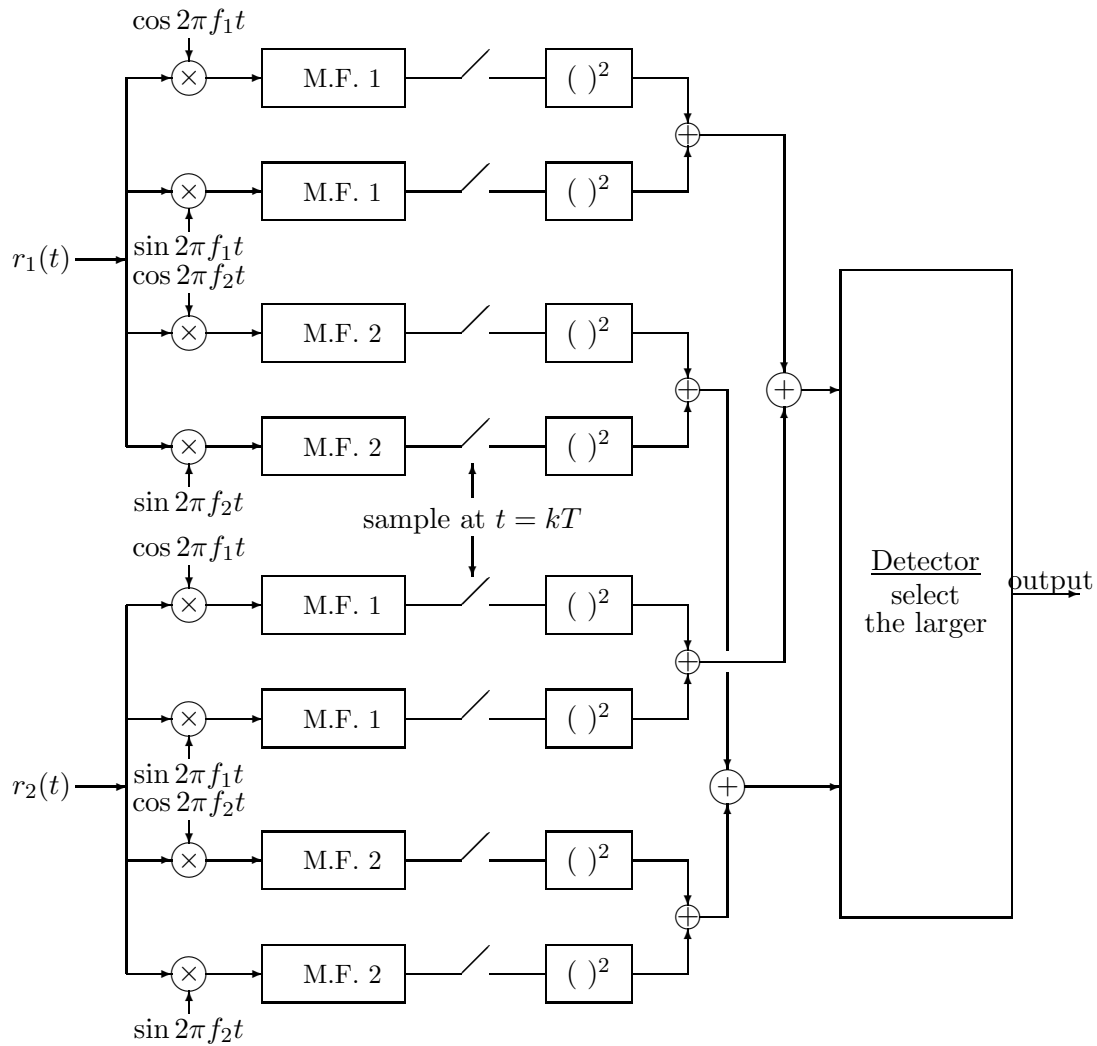
$$V(a) = -\frac{a}{4}\sqrt{\frac{A^2T}{2}}$$

The mean square estimation of $V(a)$ is

$$V = \int_0^1 V(a)f(a)da = -\frac{1}{4}\sqrt{\frac{A^2T}{2}} \int_0^1 ada = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

Problem 13.12

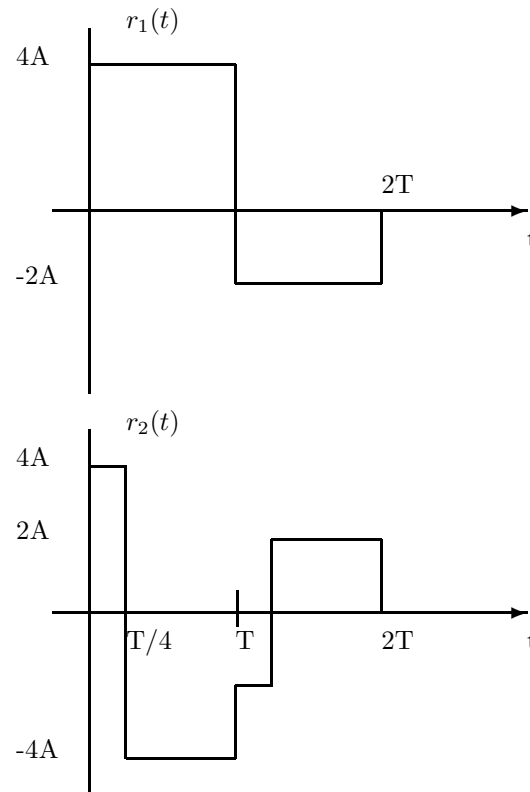
(a)



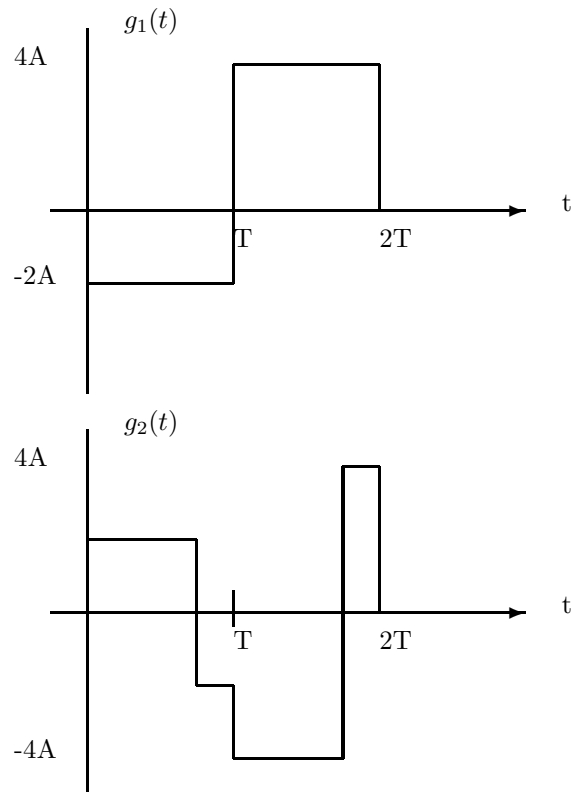
(b) The probability of error for binary FSK with square-law combining for $L = 2$ is given in Figure 14-4-7. The probability of error for $L = 1$ is also given in Figure 14-4-7. Note that an increase in SNR by a factor of 10 reduces the error probability by a factor of 10 when $L = 1$ and by a factor of 100 when $D = 2$.

Problem 13.13

(a) The noise-free received waveforms $\{r_i(t)\}$ are given by : $r_i(t) = h(t) * s_i(t)$, $i = 1, 2$, and they are shown in the following figure :



(b) The optimum receiver employs two matched filters $g_i(t) = r_i(2T - t)$, and after each matched filter there is a sampler working at a rate of $1/2T$. The equivalent lowpass responses $g_i(t)$ of the two matched filters are given in the following figure :



Problem 13.14

Since a follows the Nakagami-m distribution :

$$p_a(a) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega} \right)^m a^{2m-1} \exp(-ma^2/\Omega), \quad a \geq 0$$

where : $\Omega = E(a^2)$. The pdf of the random variable $\gamma = a^2 \mathcal{E}_b / N_0$ is specified using the usual method for a function of a random variable :

$$a = \sqrt{\gamma \frac{N_0}{\mathcal{E}_b}}, \quad \frac{d\gamma}{da} = 2a \mathcal{E}_b / N_0 = 2\sqrt{\gamma \mathcal{E}_b / N_0}$$

Hence :

$$\begin{aligned}
 p_\gamma(\gamma) &= \left(\frac{d\gamma}{da}\right)^{-1} p_a\left(\sqrt{\gamma \frac{N_0}{\mathcal{E}_b}}\right) \\
 &= \frac{1}{2\sqrt{\gamma \mathcal{E}_b/N_0}} \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \left(\sqrt{\gamma \frac{N_0}{\mathcal{E}_b}}\right)^{2m-1} \exp\left(-m\gamma \frac{N_0}{\mathcal{E}_b} / \Omega\right) \\
 &= \frac{m^m}{\Gamma(m)} \frac{\gamma^{m-1}}{\Omega^m (\mathcal{E}_b/N_0)^m} \exp(-m\gamma / (\mathcal{E}_b \Omega / N_0)) \\
 &= \frac{m^m}{\Gamma(m)} \frac{\gamma^{m-1}}{\bar{\gamma}^m} \exp(-m\gamma / \bar{\gamma})
 \end{aligned}$$

where $\bar{\gamma} = E(a^2) \mathcal{E}_b / N_0$.

Problem 13.15

(a) By taking the conjugate of $r_2 = h_1 s_2^* - h_2 s_1^* + n_2$

$$\begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

Hence, the soft-decision estimates of the transmitted symbols (s_1, s_2) will be

$$\begin{aligned}
 \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} &= \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} \\
 &= \frac{1}{h_1^2 + h_2^2} \begin{bmatrix} h_1^* r_1 - h_2 r_2^* \\ h_2^* r_1 + h_1 r_2^* \end{bmatrix}
 \end{aligned}$$

which corresponds to dual-diversity reception for s_i .

(b) The bit error probability for dual diversity reception of binary PSK is given by Equation (14.4-15), with $L = 2$ and $\mu = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}}$ (where the average SNR per channel is $\bar{\gamma}_c = \frac{\mathcal{E}}{N_0} E[h^2] = \frac{\mathcal{E}}{N_0}$). Then (14.4-15) becomes

$$\begin{aligned}
 P_2 &= \left[\frac{1}{2}(1-\mu)\right]^2 \left\{ 1 \binom{1}{0} + \left[\frac{1}{2}(1+\mu)\right] \binom{2}{1} \right\} \\
 &= \left[\frac{1}{2}(1-\mu)\right]^2 [2+\mu]
 \end{aligned}$$

When $\bar{\gamma}_c \gg 1$, then $\frac{1}{2}(1-\mu) \approx 1/4\bar{\gamma}_c$ and $\mu \approx 1$. Hence, for large SNR the bit error probability for binary PSK can be approximated as

$$P_2 \approx 3 \left(\frac{1}{4\bar{\gamma}_c} \right)^2$$

(c) The bit error probability for dual diversity reception of binary PSK is given by Equation (14.4-41), with $L = 2$ and μ as above. Replacing we get

$$P_2 = \frac{1}{2} \left[1 - \frac{\mu}{\sqrt{2 - \mu^2}} \left(1 + \frac{1 - \mu^2}{2 - \mu^2} \right) \right]$$

Problem 13.16

The DFT of $\mathbf{r}(m)$ is

$$\mathbf{R}(m) = \mathbf{W}^H \mathbf{H}(m) \mathbf{W} \mathbf{s}(m) + \mathbf{W}^H \mathbf{n}(m) = \mathbf{G}(m) \mathbf{s}(m) + \mathbf{W}^H \mathbf{n}(m)$$

where $\mathbf{G}(m)$ is defined in equation 13.6-26 as $\mathbf{G}(m) = \mathbf{W}^H \mathbf{H}(m) \mathbf{W}$. We select $\mathbf{b}_k(m)$ to minimize

$$\begin{aligned} \mathbb{E} [|s_k(m) - \mathbf{b}_k^H \mathbf{R}(m)|^2] &= \mathbb{E} [|s_k|^2] - \mathbb{E} [s_k(m) \mathbf{R}^H(m)] \mathbf{b}_k(m) \\ &\quad - \mathbf{b}_k^H(m) \mathbb{E} [\mathbf{R}(m) s_k^*(m)] + \mathbf{b}_k^H(m) \mathbb{E} [\mathbf{R}(m) \mathbf{R}^H(m)] \mathbf{b}_k(m) \end{aligned}$$

which yields the result

$$\mathbf{b}_k(m) = \mathbb{E} [\mathbf{R}(m) \mathbf{R}^H(m)]^{-1} \mathbb{E} [\mathbf{R}(m) s_k^*(m)]$$

But

$$\begin{aligned} \mathbb{E} [\mathbf{R}(m) \mathbf{R}^H(m)] &= \mathbb{E} [(\mathbf{G}(m) \mathbf{s}(m) + \mathbf{W}^H \mathbf{n}(m))(\mathbf{s}^H(m) \mathbf{G}^H(m) + \mathbf{n}^H(m) \mathbf{W})] \\ &= \mathbf{G}(m) \mathbb{E} [\mathbf{s}(m) \mathbf{s}^H(m)] \mathbf{G}^H(m) + \mathbf{W}^H \mathbb{E} [\mathbf{n}(m) \mathbf{n}^H(m)] \mathbf{W} \\ &= \mathbf{G}(m) \mathbf{G}^H(m) + N_0 \mathbf{I}_N \end{aligned}$$

since $\mathbb{E} [\mathbf{s}(m) \mathbf{s}^H(m)] = \mathbf{I}_N$ and $\mathbf{W}^H \mathbb{E} [\mathbf{n}(m) \mathbf{n}^H(m)] \mathbf{W} = N_0 \mathbf{W}^H \mathbf{W} = N_0 \mathbf{I}_N$. Also, $\mathbb{E} [\mathbf{R}(m) s_k^*(m)] = \mathbf{g}_k(m)$ by definition. Therefore, equation 13.6-24 is proved.

The minimum MSE is

$$\begin{aligned} \mathbb{E} [|s_k(m) - \hat{\mathbf{b}}_k^H(m) \mathbf{R}(m)|^2] &= \mathbb{E} [|s_k(m)|^2] - \mathbf{b}_k^H(m) \mathbb{E} [\mathbf{R}(m) s_k^*(m)] \\ &= 1 - \mathbf{g}_k^H(m) (\mathbf{G}(m) \mathbf{G}^H(m) + N_0 \mathbf{I}_N)^{-1} \mathbf{g}_k(m) \end{aligned}$$

Problem 13.17

We have

$$\begin{aligned}
\mathbb{E} [\alpha'(t + \tau)(\alpha'(t))^*] &= \mathbb{E} \left[\lim_{h_1 \rightarrow 0} \frac{\alpha(t + \tau + h_1) - \alpha(t + \tau)}{h_1} \lim_{h_2 \rightarrow 0} \frac{\alpha^*(t + h_2) - \alpha^*(t)}{h_2} \right] \\
&= \lim_{h_1, h_2 \rightarrow 0} \left(\frac{\mathbb{E} [\alpha(t + \tau + h_1)\alpha^*(t + h_2)] - \mathbb{E} [\alpha(t + \tau + h_1)\alpha^*(t)]}{h_1 h_2} \right. \\
&\quad \left. - \frac{\mathbb{E} [\alpha(t + \tau)\alpha^*(t + h_2)] - \mathbb{E} [\alpha(t + \tau)\alpha^*(t)]}{h_1 h_2} \right) \\
&= \lim_{h_1, h_2 \rightarrow 0} \frac{R_\alpha(\tau + h_1 - h_2) - R_\alpha(\tau + h_1) - R_\alpha(\tau - h_2) + R_\alpha(\tau)}{h_1 h_2} \\
&= -R''_\alpha(\tau)
\end{aligned}$$

where the second equality is due to the fact that both of the limits exist and the last equality is derived by the definition of the second derivative.

Problem 13.18

The zero-mean complex-valued Gaussian process $\alpha_k(t)$ is passed through a differentiator whose transfer function is $H(f) = j2\pi f$. Therefore, the power spectral density of the signal at the output of the differentiator is

$$S_y(f) = |H(f)|^2 S(f)$$

where $S(f)$ is the power spectral density of $\alpha_k(t)$. Consequently,

$$\mathbb{E} [|\alpha'_k(t)|^2] = \int_{-\infty}^{+\infty} S_y(f) df = \int_{-f_m}^{f_m} (2\pi f)^2 S(f) df$$

where $S(f)$ is given by equation 13.6-5. Thus,

$$\int_{-f_m}^{f_m} \frac{(2\pi f)^2}{\pi f_m \sqrt{1 - (f/f_m)^2}} df = 2\pi^2 f_m^2$$

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 14) ¹

Prepared by
Kostas Stamatiou

March 9, 2008

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Problem 14.1

1. We use the water filling argument similar to what is given by Equations 14.1-10–14.1-13. To determine C_1 we need to find K such that

$$\int_{-\infty}^{\infty} (K - N_1(f))^+ df = 10^{-2}$$

or

$$\int_0^{2000} (K - 0.5 \times 10^{-9} f) df + \int_{2000}^{3000} (K - 10^{-6}) df = \frac{1}{2} \times 10^{-2}$$

Solving for K gives $K = \frac{7}{3} \times 10^{-6}$. The capacity is given by

$$\begin{aligned} C_1 &= \frac{1}{2} \int_{-\infty}^{\infty} \log_2 \left(1 + \frac{(K - N_1(f))^+}{N_1(f)} \right) df \\ &= \int_0^{2000} \log_2 \left(1 + \frac{7/3 \times 10^{-6} - 1/2 \times 10^{-9} f}{1/2 \times 10^{-9} f} \right) df + \int_{2000}^{3000} \log_2 \left(1 + \frac{7/3 \times 10^{-6} - 10^{-6}}{10^{-6}} \right) df \\ &= 6552.57 \text{ bps} \end{aligned}$$

For the second channel $C_2 = \infty$ since there exists in interval in the bandwidth in which $N_2(f) = 0$.

2. For this source $R(D) = R(P_e) = H_b(p) - H_b(P_e) = H_b(0.4) - H_b(P_e)$. The resulting P_e is the solution of $7500R(P_e) = C$. For the first channel we have $7500(H_b(0.4) - H_b(P_e)) = 6552.57$, resulting in $H_b(0.4) - H_b(P_e) = 0.874$ or $H_b(P_e) = H_b(0.4) - 0.874 = 0.971 - 0.874 = 0.097$ resulting in $P_e = 0.0125$. For the second channel since the capacity is infinite, we have $P_e = 0$.
3. In the first case where the two channels are directly concatenated, by independence of the two noise processes the PSD of the noise is the sum of the two noise PSD's, $N(f) = N_1(f) + N_2(f)$. The equivalent channel is AWGN with noise PSD (for positive frequencies)

$$N(f) = \begin{cases} \frac{10^{-9}}{2} f & 0 \leq f < 1\text{kHz} \\ \frac{3}{2} \times 10^{-9} f - 10^{-6} & 1 \leq f < 2\text{kHz} \\ 2 \times 10^{-6} & 2 \leq f < 3\text{kHz} \\ 2 \times 10^{-5} & f \geq 3\text{kHz} \end{cases}$$

The capacity of this channel is computed using waterfilling, i.e., by finding a K such that

$$\int_0^{1000} \left(K - \frac{10^{-9}}{2} f \right) df + \int_{1000}^{2000} \left[K - \left(\frac{3}{2} \times 10^{-9} f - 10^{-6} \right) \right] df + \int_{2000}^{3000} (K - 2 \times 10^{-6}) df = \frac{1}{2} \times 10^{-2}$$

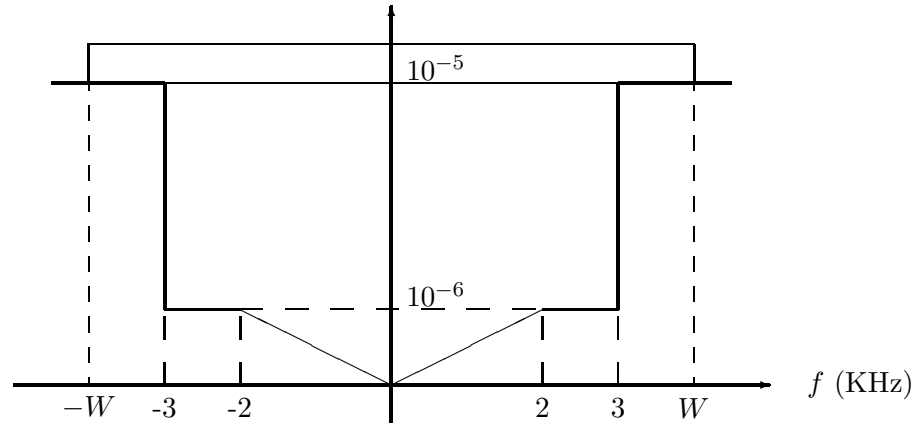
Solving for K gives $K = \frac{17}{6} \times 10^{-6}$, and Using the capacity relation we obtain $C = 5033.3$ bps. Now we have to solve $7500(H_b(0.4) - H_b(P_e)) = 5033.3$, which results in $P_e = 0.0532$.

In the case where an arbitrary processor can be used between the two channels, since the capacity of the second channel is infinite, we can transmit at any rate lower than the capacity of the first channel, therefore $C = \min\{C_1, C_2\} = 6552.57$ and hence we have the same results as in part 1, namely $P_e = 0.0125$.

4. In this case the waterfilling argument can still be used but the input power will be spread over an infinite bandwidth of the channel since K exceeds the 10^{-5} level. Here if $K = 10^{-5}$, then

$$P = 10^{-6} \times 2000 + 6000 * (10^{-5} - 10^{-6}) = 2 \times 10^{-3} + 6 \times 10^{-2} - 6 \times 10^{-3} = 56 \times 10^{-3}$$

and the remaining power, i.e., $100 \times 10^{-3} - 56 \times 10^{-3} = 44 \times 10^{-3}$ is spread over an infinite bandwidth with an effectively zero height. We can look at this case as shown in the figure below in which W goes to ∞ and the height of the region between $-W$ and W is $\frac{44 \times 10^{-3}}{2W}$.



The capacity is given by

$$\begin{aligned} C &= \int_0^{2000} \log_2 \left(1 + \frac{10^{-5} - \frac{1}{2} \times 10^{-9} f}{\frac{1}{2} \times 10^{-9} f} \right) df + \int_{2000}^{3000} \log_2 \left(1 + \frac{10^{-5} - 10^{-6}}{10^{-6}} \right) df \\ &\quad + \lim_{W \rightarrow \infty} \int_{3000}^W \log_2 \left(1 + \frac{\frac{44 \times 10^{-3}}{2W}}{10^{-5}} \right) df \\ &= 15053 \text{ bps} \end{aligned}$$

Problem 14.2

1. Since the capacities of both channels is zero, the ergodic capacity is zero.
2. In this case, although the capacity of each channel is zero but since the transmitter can control the state and the receiver observes the state, the transmitter can send information by selection of the channel state. In fact in this case the channel state transfers information. Since there are two states, the capacity of the channel is one bit per transmission.

Problem 14.3

Since the channel state is available only at the transmitter, random variable V is degenerate and

$$C = \max_{p(t)} I(T; Y)$$

Problem 14.4

This is the case where $V = U$. We have

$$\begin{aligned} \max_{p(t)} I(T; Y|V) &= \max_{p(t)} I(T; Y|U) \\ &= \max_{p(t)} \sum_u p(u) I(T; Y|U = u) \\ &\leq \sum_u p(u) \max_{p(t)} I(T; Y|U = u) \\ &= \sum_u p(u) \max_{p(x|u)} I(X; Y|U = u) \end{aligned}$$

where the last step follows from the fact that since $x = t(u)$, for a given u , any PDF on t induces a PDF on X . On the other hand we have

$$\max_{p(t)} I(T; Y|V) \geq \sum_u p(u) \max_{p(x|u)} I(X; Y|U = u)$$

since the right hand side corresponds to a particular probability distribution on T in which the probability distribution of the u th component of T is selected to be $p(x|u)$ independent of other components. These two relations show that in this case

$$C = \sum_u p(u) \max_{p(x|u)} I(X; Y|U = u)$$

Problem 14.5

1. In this case the channel is a BSC with crossover probability

$$\begin{aligned} P(Y = 1|X = 0) &= \frac{p}{2}P(Y = 1|X = 0, S = 0) + \frac{p}{2}P(Y = 1|X = 0, S = 1) + (1 - p)P(Y = 1|X = 0, S = 2) \\ &= 0 + \frac{p}{2} + 0 = \frac{p}{2} \end{aligned}$$

Therefor, the capacity is $C = 1 - H_b(p/2)$.

2. In this case we can use Equation 14.1-25 with $U = S$. Hence,

$$C = \sum_u P(s) \max_{P(x|s)} I(X; Y|S = s)$$

which means that the capacity is the weighted average of the three capacities, i.e.,

$$C = \frac{p}{2}C_0 + \frac{p}{2}C_1 + (1 - p)C_2 = 0 + 0 + 1 - p = 1 - p$$

Problem 14.6

It can be easily shown that if $Z = 0$, we have a binary input, binary output channel with $P(Y = 0|X = 0) = 1$ and $P(Y = 0|X = 1) = 1 - P(Y = 1|X = 1) = p$. Similarly if $Z = 1$ we have a channel with $P(Y = 0|X = 0) = 1 - P(Y = 1|X = 0) = p$ and $P(Y = 1|X = 1) = 1$. From this it is straightforward to show that

$$\begin{aligned} C &= \max_{p(x|z)} I(X; Y|Z) \\ &= \max_{\epsilon} \left[\frac{1}{2}h(\epsilon(1 - p)) + \frac{1}{2}H_b((1 - \epsilon)(1 - p)) - \frac{1}{2}H_b(p) \right] \end{aligned} \quad (14.0.1)$$

where $H_b(\cdot)$ denotes the binary entropy function. The above function is maximized for $\epsilon = 1/2$ and we get $C = H_b(\frac{1-p}{2}) - \frac{1}{2}H_b(p)$.

Problem 14.7

1. We have $P(Y = 0|X = 0) = 1$ and $P(Y = 0|X = 1) = \frac{1}{4}$, therefore the overall channel is equivalent to a Z -channel. For this Z -channel

$$I(X; Y) = H(Y) - H(Y|X) = H_b(3p/4) - pH_b(1/4)$$

Differentiating the mutual information and finding the root of the derivative gives

$$p = \frac{4\sqrt[3]{\frac{1}{4}}}{4 + 3\sqrt[3]{\frac{1}{4}}} \approx 0.4278$$

and the capacity is given by $C \approx 0.5582$ bits per transmission.

2. When channel state information is available at both sides, the capacity is a weighted sum of the two capacities. For the first channel $C_1 = 1$ and for the second channel $C_2 = 0.3219$ (see problem 6.69). Therefore $C = 0.5 + 0.5 \times 0.3219 = 0.6609$ bits per transmission.
3. If only the receiver knows the state of the channel the capacity is given by

$$C = \max_{p(x)} I(X; Y|S) = \max_{p(x)} \left[\frac{1}{2} I(X; Y|S = 1) + \frac{1}{2} I(X; Y|S = 2) \right]$$

Assuming $p = P(X = 1) = 1 - P(X = 0)$, we have $I(X; Y|S = 1) = H_b(p)$ and for the Z-channel the output probabilities will be $P(Y = 1) = (1 + p)/2$ and $P(Y = 0) = (1 - p)/2$. Therefore,

$$\begin{aligned} I(X; Y|S = 2) &= H(Y|S = 2) - H(Y|X, S = 2) \\ &= H(Y|S = 2) - P(X = 0)H(Y|X = 0, S = 2) - P(X = 1)H(Y|X = 1, S = 2) \\ &= H_b\left(\frac{1+p}{2}\right) - (1-p)H_b(0.5) - p \times 0 \\ &= H_b\left(\frac{1+p}{2}\right) - (1-p) \end{aligned}$$

Resulting in

$$C = \frac{1}{2} \left[H_b(p) + H_b\left(\frac{1+p}{2}\right) - (1-p) \right]$$

To find the optimal p we differentiate with respect to p and find the root to obtain $p = 0.5268$ resulting in $C = 0.6570$.

Problem 14.8

We need to show that

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty p(y|x, r, \theta) p(r) dr d\theta = \frac{1}{\pi(N_0 + |x|^2)} e^{-\frac{|y|^2}{N_0 + |x|^2}}$$

where

$$p(y|x, r, \theta) = \frac{1}{\pi N_0} e^{-\frac{|y - r e^{j\theta} x|^2}{N_0}}$$

Here x and y are complex constants, and $p(r) = 2re^{-r^2}$ for $r > 0$. We can change to the Cartesian coordinates and define

$$\begin{aligned} re^{j\theta} &= \alpha + j\beta \\ \frac{y}{x} &= \delta = \delta_r + j\delta_i \\ \frac{N_0}{|x|^2} &= \gamma \end{aligned}$$

where α and β are independent zero-mean Gaussian RV's each with variance $1/2$, and δ_r , δ_i , and γ are real constants. With these changes we have

$$\begin{aligned} \frac{1}{\pi N_0} e^{-\frac{|y-re^{j\theta}x|^2}{N_0}} &= \frac{1}{\pi N_0} e^{-\frac{|x|^2}{N_0} \left(\frac{y}{x} - re^{j\theta}\right)^2} \\ &= \frac{1}{\pi N_0} e^{-\frac{1}{\gamma}[(\alpha-\delta_r)^2 + (\beta-\delta_i)^2]} \end{aligned}$$

Also note that by changing the integration to Cartesian coordinates $rdrd\theta = d\alpha d\beta$, hence

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty p(y|x, r, \theta) p(r) dr d\theta &= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\pi N_0} e^{-\frac{1}{\gamma}[(\alpha-\delta_r)^2 + (\beta-\delta_i)^2]} 2e^{-(\alpha^2 + \beta^2)} d\alpha d\beta \\ &= \frac{1}{\pi N_0} \int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{(\alpha-\delta_r)^2 + \gamma\alpha^2}{\gamma}} d\alpha \int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{(\beta-\delta_i)^2 + \gamma\beta^2}{\gamma}} d\beta \\ &= \frac{1}{\pi N_0} g(\alpha, \delta_r) g(\beta, \delta_i) \end{aligned}$$

where

$$g(\phi, \omega) = \int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{(\phi-\omega)^2 + \gamma\phi^2}{\gamma}} d\phi$$

But

$$\begin{aligned} g(\phi, \omega) &= \int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{(\phi-\omega)^2 + \gamma\phi^2}{\gamma}} d\phi \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{1}{\gamma} \left[\left(\phi \sqrt{1+\gamma} - \frac{\omega}{\sqrt{1+\gamma}} \right)^2 + \frac{\gamma\omega^2}{1+\gamma} \right]} d\phi \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{\omega^2}{1+\gamma}} e^{-u^2} \sqrt{\frac{\gamma}{1+\gamma}} du \\ &= \sqrt{\frac{\gamma}{1+\gamma}} e^{-\frac{\omega^2}{1+\gamma}} \\ &= \sqrt{\frac{N_0}{N_0 + |x|^2}} e^{-\frac{\omega^2}{1+\gamma}} \end{aligned}$$

where we have introduced the change of variable

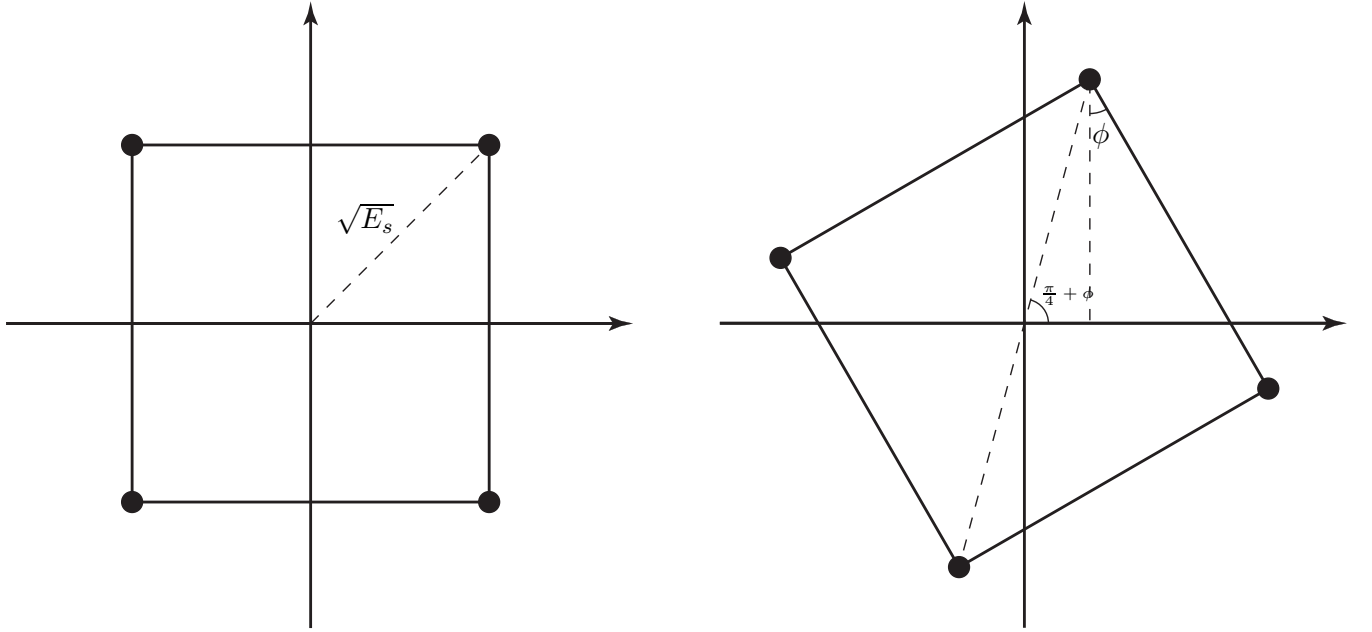
$$u = \frac{\phi \sqrt{1+\gamma} - \frac{\omega}{\sqrt{1+\gamma}}}{\sqrt{\gamma}}$$

Using this result, we obtain

$$\begin{aligned}
 \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty p(y|x, r, \theta) p(r) dr d\theta &= \frac{1}{\pi N_0} g(\alpha, \delta_r) g(\beta, \delta_i) \\
 &= \frac{1}{\pi N_0} \sqrt{\frac{N_0}{N_0 + |x|^2}} e^{-\frac{\delta_r^2}{1+\gamma}} \sqrt{\frac{N_0}{N_0 + |x|^2}} e^{-\frac{\delta_i^2}{1+\gamma}} \\
 &= \frac{1}{\pi (N_0 + |x|^2)} e^{-\frac{|y|^2}{N_0 + |x|^2}}
 \end{aligned}$$

which concludes the proof.

Problem 14.9



Consider a rotation of ϕ as shown in the figure above. With this rotation we have $\Gamma_{\min} = \min\{\Gamma_e, \Gamma_d\}$ where Γ_e corresponds to two points in the constellation that share an edge and Γ_d corresponds to two diagonal points in the constellation. We have

$$\Gamma_{mm'} = (\delta^2(\mathbf{x}_m, \mathbf{x}_{m'}))^{-\frac{1}{d_H(\mathbf{x}_m, \mathbf{x}_{m'})}}$$

and

$$\delta^2(\mathbf{x}_m, \mathbf{x}_{m'}) = \frac{1}{(\overline{E_s})^{d_H(\mathbf{x}_m, \mathbf{x}_{m'})}} \prod_{j \in \mathcal{J}_{mm'}} d_{mm'j}^2$$

where

$$d_H(\mathbf{x}_m, \mathbf{x}_{m'}) = |\mathcal{J}_{mm'}| = |\{1 \leq j \leq n : x_{mj} \neq x_{m'j}\}|$$

and

$$\overline{E_s} = \frac{1}{M} \sum_{m=1}^M \|\mathbf{x}_m\|^2 = E_s$$

We consider two cases separately:

1. Γ_e : In this case for $0 < \phi < \frac{\pi}{4}$, $d_H(\mathbf{x}_m, \mathbf{x}_{m'}) = 2$ and $d_{mm'1} = \sqrt{2E_s} \cos \phi$, $d_{mm'2} = \sqrt{2E_s} \sin \phi$. Therefore,

$$\delta^2(\mathbf{x}_m, \mathbf{x}_{m'}) = \frac{1}{E_s^2} 4E_s^2 \sin^2 \phi \cos^2 \phi = \sin^2 2\phi$$

Hence,

$$\Gamma_e = (\delta^2(\mathbf{x}_m, \mathbf{x}_{m'}))^{\frac{1}{d_H(\mathbf{x}_m, \mathbf{x}_{m'})}} = \sin(2\phi)$$

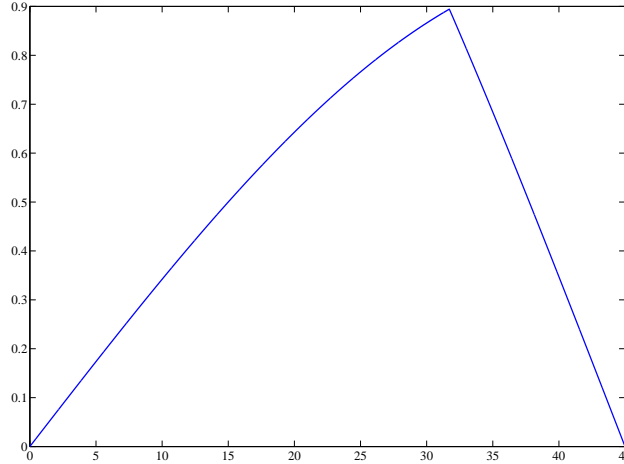
2. Γ_d : Similar to the other case we have $d_{mm'1} = 2\sqrt{E_s} \cos(\phi + \pi/4)$, $d_{mm'2} = 2\sqrt{E_s} \sin(\phi + \pi/4)$. Hence,

$$\delta^2(\mathbf{x}_m, \mathbf{x}_{m'}) = \frac{1}{E_s^2} 16E_s^2 \sin^2(\phi + \pi/4) \cos^2(\phi + \pi/4) = 4 \sin^2(2\phi + \pi/2) = 4 \cos^2(2\phi)$$

Therefore,

$$\Gamma_e = (\delta^2(\mathbf{x}_m, \mathbf{x}_{m'}))^{\frac{1}{d_H(\mathbf{x}_m, \mathbf{x}_{m'})}} = 2 \cos(2\phi)$$

From above, $\Gamma_{\min} = \min\{\sin(2\phi), 2 \cos(2\phi)\}$. A plot of Γ_{\min} versus ϕ (in degrees) is given below.



The maximum of this function is achieved for $\sin(\phi) = 2 \cos(2\phi)$ or $\phi = \frac{1}{2} \arctan 2 = 31.72^\circ$.

Problem 14.10

If channel state information is not available at the receiver, we have

$$\begin{aligned}
 \hat{m} &= \arg \max_m P_m p(\mathbf{y} | \mathbf{x}_m) \\
 &= \arg \max_m P_m \int_0^\infty p(r, \mathbf{y} | \mathbf{x}_m) dr \\
 &= \arg \max_m P_m \int_0^\infty p(r) \frac{1}{(\sqrt{\pi N_0})^N} e^{-\frac{1}{N_0} \|\mathbf{y} - r\mathbf{x}\|^2} \\
 &= \arg \max_m P_m \int_0^\infty p(r) e^{-\frac{1}{N_0} \|\mathbf{y} - r\mathbf{x}\|^2}
 \end{aligned}$$

where $p(r)$ denotes the Ricean PDF.

If channel state information is available at the receiver, we can interpret it as one component of the received signal, i.e., (\mathbf{y}, R) is the received signals. In this case we have

$$\begin{aligned}
 \hat{m} &= \arg \max_m P_m p(\mathbf{y}, r | \mathbf{x}_m) \\
 &= \arg \max_m P_m p(r) p(\mathbf{y} | \mathbf{x}_m, r) \\
 &= \arg \max_m P_m p(\mathbf{y} | \mathbf{x}_m, r) \\
 &= \arg \max_m P_m e^{-\frac{1}{N_0} \|\mathbf{y} - r\mathbf{x}\|^2}
 \end{aligned}$$

which is similar to the decision rule for a Gaussian channel in which the signal \mathbf{x}_m has been replaced by $r\mathbf{x}_m$.

Problem 14.11

Using Equation 13.4-13 with $\bar{\gamma}_c = 100 \sim 20$ dB, we have

$$p(\gamma_b) = \frac{1}{(N_r - 1)! 100^{N_r}} \gamma_b^{N_r - 1} e^{-\frac{\gamma_b}{100}}$$

and we need to determine

$$P[\gamma_b < 10] = \int_0^{10} p(\gamma_b) d\gamma_b$$

1. $N_r = 1$, here

$$\int_0^{10} p(\gamma_b) d\gamma_b = \int_0^{10} \frac{1}{100} e^{-\frac{\gamma_b}{100}} d\gamma_b = 0.0952$$

2. $N_r = 2$, in this case

$$\int_0^{10} \frac{1}{10000} \gamma_b e^{-\frac{\gamma_b}{100}} d\gamma_b = 0.0047$$

where we have used integration by parts.

3. For $N_r = 4$ we have

$$\int_0^{10} \frac{1}{6 \times 10^8} \gamma_b^3 e^{-\frac{\gamma_b}{100}} d\gamma_b = 3.85 \times 10^{-6}$$

Problem 14.12

1. Let $\beta = \sqrt{1 - \alpha^2}$, where we assume α is real and $|\alpha| < 1$. We have the recursive relation

$$h(m) = \beta h(m-1) + \alpha w(m), \quad m = 1, 2, 3, \dots$$

which defines a difference equation with initial condition $h(0)$. Solution of this equation gives

$$h(n) = \beta^n h(0) + \sum_{i=0}^n \beta^{n-i} w(i)$$

The autocorrelation function is given by

$$\begin{aligned} R_h(n, m) &= \mathbb{E}[h(n)h^*(m)] \\ &= \mathbb{E}\left[\left(\beta^n h(0) + \sum_{i=0}^n \beta^{n-i} w(i)\right) \left(\beta^m h^*(0) + \sum_{j=0}^m \beta^{m-j} w^*(j)\right)\right] \\ &= \beta^{n+m} + \alpha^2 \sum_{i=0}^m \sum_{j=0}^n \beta^{n+m-i-j} R_w(i-j) \\ &= \beta^{n+m} + \alpha^2 \sum_{i=0}^{\min\{m, n\}} \beta^{n+m-2i} \end{aligned}$$

which results in

$$R_h(n, m) = \alpha^2 \beta^{m+n} + \beta^{|m-n|}$$

which goes to $R_h(k) = \beta^k$ in the steady-state, i.e., when m and n are large. In the above expression $k = m - n$.

2. The coherence time T_c is obtained when $k = T_c/T_s$ is such that $\beta^k = 1/2$. From this relation we obtain

$$2^{-\frac{T_s}{T_c}} = \beta \Rightarrow \alpha = \sqrt{1 - 2^{-\frac{2T_s}{T_c}}}$$

3. This is a MMSE estimation problem, therefore we can invoke the orthogonality principle

$$\begin{aligned} \mathbb{E}[(h(m) - b_1 h(m-n) - b_2 h(m-n-1))h^*(m-n)] &= 0 \\ \mathbb{E}[(h(m) - b_1 h(m-n) - b_2 h(m-n-1))h^*(m-n-1)] &= 0 \end{aligned}$$

resulting in

$$\begin{aligned} R_h(n) &= b_1 R_h(0) + b_2 R_h(1) \\ R_h(n) &= b_1 R_h(1) - b_2 R_h(0) \end{aligned}$$

From these equations we obtain $b_2 = 0$ and $b_1 = \beta^n$.

Problem 14.13

For soft decision decoding, from Equation 8.1-21, We obtain

$$\frac{\partial}{\partial Y} T(Y, Z) \Big|_{Y=1} = Z^6 + 4Z^8 + 12Z^{10} + \dots$$

hence, $d_{\text{free}} = 6$, and $\beta_6 = 1$, $\beta_8 = 4$, $\beta_{10} = 12$, \dots . From Equations 14.7-13, 14.7-12, and 14.7-6, we have $p = \frac{1}{2+R_c\bar{\gamma}_b} = \frac{3}{6+\bar{\gamma}_b}$,

$$P_b < \frac{1}{3} [P_2(6) + 4P_2(8) + 12P_2(10) + \dots]$$

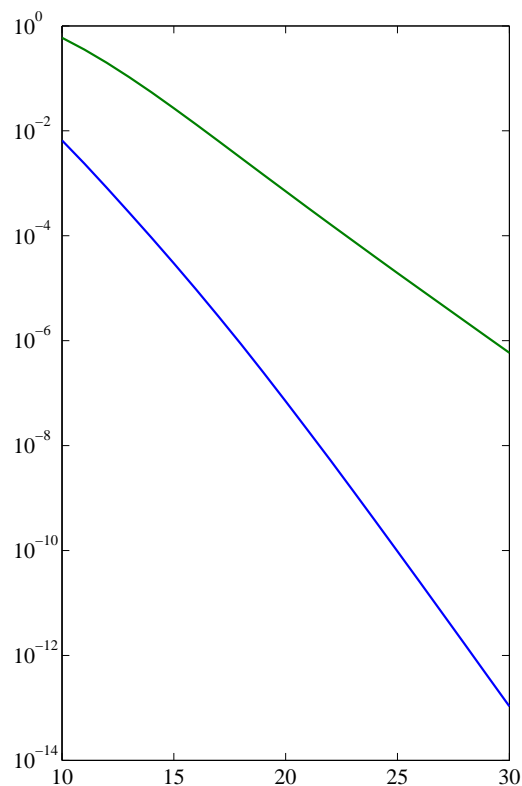
and

$$P_2(d) = p^d \sum_{k=0}^{d-1} \binom{d-1+k}{k} (1-p)^k$$

For hard decision decoding, we essentially have the same expressions, except that

$$P_b < \frac{1}{3} [Z^6 + 4Z^8 + 12Z^{10} + \dots] \Big|_{Z=\sqrt{4p(1-p)}}$$

Plots are shown below. The horizontal axis denotes $\bar{\gamma}_b$ in dB, the upper curve corresponds to hard decision and the lower one corresponds to soft decision decoding.



Problem 14.14

The channel model is $y_i = R_i x_i + n_i$ and the PEP can be written as

$$\begin{aligned} P_{m \rightarrow m'} &= \int_0^\infty \mathbf{P}[m \rightarrow m' | R = r] p(r) dr \\ &= \mathbf{E}[m \rightarrow m' | R] \end{aligned}$$

But

$$\begin{aligned} \mathbf{E}[m \rightarrow m' | R] &= \mathbf{E} \left[Q \left(\sqrt{\frac{d_{mm'}^2}{2N_0}} \right) \right] \\ &\leq \mathbf{E} \left[e^{-\frac{d_{mm'}^2}{4N_0}} \right] \\ &= \prod_{i=1}^n \mathbf{E} \left[e^{-\frac{R_i^2 |x_{mi} - x_{m'i}|^2}{4N_0}} \right] \end{aligned}$$

Note that R_i^2 is a χ^2 RV with two degrees of freedom and $\mathbb{E} \left[e^{-\frac{R_i^2 |x_{mi} - x_{m'i}|^2}{4N_0}} \right] = \mathbb{E} \left[e^{tR_i^2} \right]$ is the moment generating of this RV computed at $t = -\frac{|x_{mi} - x_{m'i}|^2}{4N_0}$. From Table 2.3.3, we have $\Theta(t) = \Phi(-jt) = 1/(1 - 2t\sigma^2)$, i.e.,

$$\mathbb{E} \left[e^{-\frac{R_i^2 |x_{mi} - x_{m'i}|^2}{4N_0}} \right] = \frac{1}{1 + \frac{\sigma^2 |x_{mi} - x_{m'i}|^2}{2N_0}}$$

which is the desired result.

Problem 14.15

Since there are no parallel branches in the trellis the diversity order is at least 2. The diversity order is actually equal to 2 if we consider the path corresponding to E followed by B . The two paths (A,A,A) and (C,A,B) that start at state 0 and end at the same state have a distance of $d_{AC}^2 + d_{AB}^2 = 2 + (2 - \sqrt{2}) = 4 - \sqrt{2} \approx 2.586$. Inspection shows that no closer paths exist. Therefore the free Euclidean distance for this code is 2.586. The product distance for this code is $2(2 - \sqrt{2}) = 1.1716$.

Problem 14.16

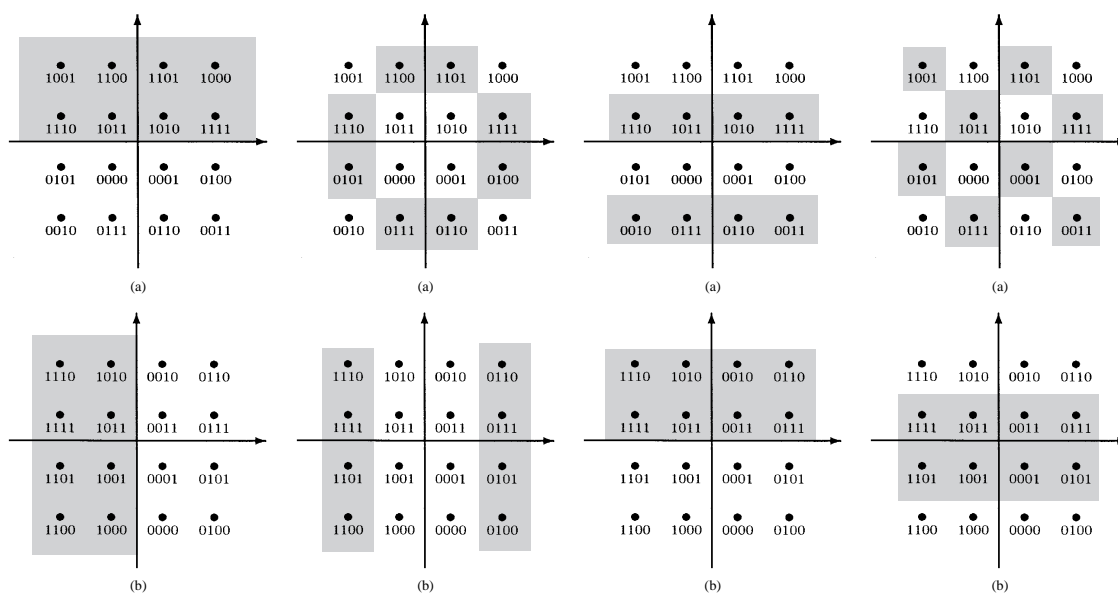
In this case we consider that paths corresponding to (E,B,G) and (C,A,F) which start and finish at the zero state. The Euclidean distance between these paths is $d_{EC}^2 + d_{AB}^2 + d_{FG}^2 = 2 + (2 - \sqrt{2}) + (2 - \sqrt{2}) \approx 3.1716$. Computing the Euclidean distance between other paths shows that this is the free Euclidean distance for this code. To find the product distance we consider the two paths of length 2, (E,B) and (A,C) which provide a product distance of $4(2 - \sqrt{2}) = 2.3432$. Inspection of all paths of length 2 shows that this is the minimum value of the product of the squared distances over all paths of length 2.

Problem 14.17

In Figure 14.5-4 consider the two paths starting from zero state and ending in the zero state corresponding to (AA,AA) and (BB,BH). This Euclidean distance between these paths is $4(2 - \sqrt{2}) \approx 2.343$. Inspection shows that this is the free Euclidean distance for this code. For Figure 14.5-5 the free Euclidean distance corresponds to the paths (AA,AA) and (AC,HB) resulting in $2(2 - \sqrt{2}) + 2 \approx 3.1716$. The gain on AWGN channel is $10 \log_{10} \frac{3.1716}{2.343} \approx 1.315$.

Problem 14.18

The figure shown below gives the answer to the problem χ_1^i is shown in gray. The four columns from left to right correspond to $i = 1, 2, 3, 4$, respectively.



Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 15) ¹

Prepared by
Kostas Stamatiou

March 12, 2008

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Problem 15.1

From equation (15.4-25), the estimate for symbol s_1 is

$$\hat{s}_1 = y_1 h_{11}^* + y_2^* h_{12}$$

where y_1 and y_2 are given in equation (15.4-17). Substituting for y_1 and y_2 in the equation for \hat{s}_1 , we obtain

$$\hat{s}_1 = [|h_{11}|^2 + |h_{12}|^2]s_1 + h_{11}^* n_1 + h_{12} n_2^*$$

Since the modulation is binary PSK, $s_1 = \pm\sqrt{\mathcal{E}_b}$. Conditioned on h_{11} and h_{12} , the additive noise is a complex-valued Gaussian random variable with zero mean and total variance

$$\sigma^2 = |h_{11}|^2 E[|n_1|^2] + |h_{12}|^2 E[|n_2|^2] = N_o [|h_{11}|^2 + |h_{12}|^2]$$

Hence, the conditional error probability is, from equation (13.4-8)

$$P_b(\gamma_b) = Q(\sqrt{2\gamma_b})$$

where

$$\gamma_b = \frac{\mathcal{E}_b}{N_o} [|h_{11}|^2 + |h_{12}|^2]$$

By averaging $P_b(\gamma_b)$ over the Rayleigh fading channel statistics, we obtain the probability of error, which is given by equation (13.4-15) for $L = 2$ as

$$P_b = \frac{1}{4}(1 - \mu)^2(2 + \mu)$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}}$$

and $\bar{\gamma}_c = \mathcal{E}_b E[|h_{11}|^2]/N_o = \mathcal{E}_b E[|h_{12}|^2]/N_o = \mathcal{E}_b/N_o$.

Problem 15.2

The capacity of the SIMO channel with selection diversity is

$$C = \log_2 \left(1 + \frac{\mathcal{E}}{N_o} |h|_{\max}^2 \right)$$

where $|h|_{\max}^2 = \max\{|h_1|^2, \dots, |h_{N_R}|^2\}$. We define

$$\gamma_{sc} = \frac{\mathcal{E}}{N_o} |h|_{\max}^2$$

Then, $C = \log_2(1 + \gamma_{sc})$.

To determine the ergodic capacity $\bar{C} = E[C]$, we need the pdf of γ_{sc} . We assume that the channel coefficients of the SIMO channel are i.i.d., complex-valued Gaussian, with zero mean and identical variance. Hence, we define

$$\bar{\gamma} = \frac{\mathcal{E}}{N_o} E[|h_i|^2], \text{ for } 1 \leq i \leq N_R$$

The pdf of γ_i is

$$p(\gamma_i) = \frac{1}{\bar{\gamma}} e^{-\gamma_i/\bar{\gamma}}, \quad \gamma_i > 0$$

The probability that all γ_i are less than γ is

$$P(\gamma_1 < \gamma, \gamma_2 < \gamma, \dots, \gamma_{N_R} < \gamma) = \left[1 - e^{-\gamma/\bar{\gamma}}\right]^{N_R}$$

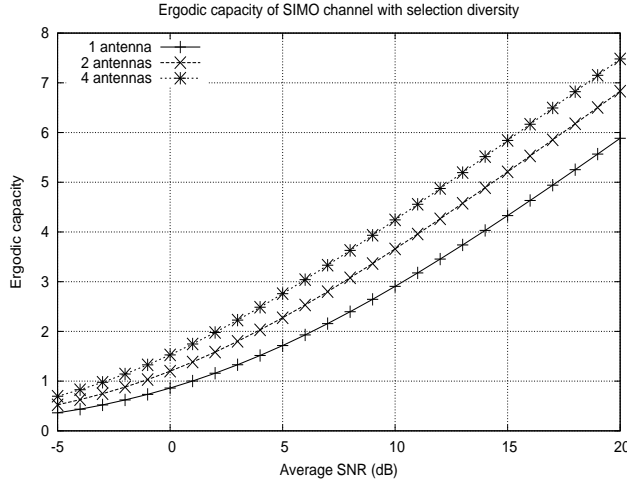
This is the cdf of γ_{sc} . Hence, the pdf of γ_{sc} is

$$p(\gamma_{sc}) = \frac{N_R}{\bar{\gamma}} e^{-\gamma_{sc}/\bar{\gamma}} \left[1 - e^{-\gamma_{sc}/\bar{\gamma}}\right]^{N_R-1}$$

The ergodic capacity \bar{C} is obtained by evaluating the integral

$$\bar{C} = \int_0^{+\infty} \log_2(1 + \gamma_{sc}) p(\gamma_{sc}) d\gamma_{sc}$$

A plot of \bar{C} vs. $\bar{\gamma}$ is shown below.



Problem 15.3

By using the SVD equation for \mathbf{H} , we have

$$\begin{aligned}
 \mathbf{H}\mathbf{H}^H &= (\mathbf{U}\Sigma\mathbf{V}^H)(\mathbf{U}\Sigma\mathbf{V}^H)^H \\
 &= \mathbf{U}\Sigma\mathbf{V}^H\mathbf{V}\Sigma\mathbf{U}^H \\
 &= \mathbf{U}\Sigma\Sigma\mathbf{U}^H \\
 &= \sum_{i=1}^{N_R} \sigma_i^2 \mathbf{u}_i \mathbf{u}_i^H \\
 &= \sum_{i=1}^r \sigma_i^2 \mathbf{u}_i \mathbf{u}_i^H
 \end{aligned}$$

From the decomposition of $\mathbf{H}\mathbf{H}^H$ in equation (15.2-3), we have

$$\mathbf{H}\mathbf{H}^H = \sum_{i=1}^{N_R} \lambda_i \mathbf{q}_i \mathbf{q}_i^H$$

Hence, $\lambda_i = \sigma_i^2$ for $i = 1, 2, \dots, r$ and $\lambda_i = 0$ for $i = r + 1, \dots, N_R$.

Problem 15.4

$$\bar{C} = E \left[\sum_{i=1}^N \log_2 \left(1 + \frac{\mathcal{E}_s}{NN_o} \lambda_i \right) \right]$$

For large N ,

$$\log_2 \left(1 + \frac{\gamma \lambda_i}{N} \right) = \frac{1}{\ln 2} \ln \left(1 + \frac{\gamma \lambda_i}{N} \right) \simeq \frac{\gamma \lambda_i}{N \ln 2}$$

Hence,

$$\begin{aligned}\bar{C} &\simeq \frac{\gamma}{N \ln 2} \sum_{i=1}^N E[\lambda_i] \\ &= \frac{\gamma \lambda_{\text{av}}}{\ln 2} \\ &= \frac{\mathcal{E}_s}{N_o \ln 2} \lambda_{\text{av}}\end{aligned}$$

where

$$\lambda_{\text{av}} = \frac{1}{N} \sum_{i=1}^N E[\lambda_i]$$

Problem 15.5

(a) The capacity of a deterministic SIMO channel with AWGN is given by equation (15.2-15) as

$$C_{\text{SIMO}} = \log_2 \left(1 + \frac{\mathcal{E}_s}{N_o} \sum_{i=1}^{N_R} |h_i|^2 \right) \text{ bps/Hz}$$

With the condition that $|h_i|^2 = 1$ for $i = 1, 2, \dots, N_R$, we have

$$C_{\text{SIMO}} = \log_2 \left(1 + \frac{N_R \mathcal{E}_s}{N_o} \right) \text{ bps/Hz}$$

(b) Since $|h_i|^2 = 1$ for all i , knowledge of the channel at the transmitter does not result in any increase in channel capacity.

Problem 15.6

(a) The capacity of the deterministic MISO channel with AWGN is given by equation (15.2-17) as

$$C_{\text{MISO}} = \log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_o} \sum_{i=1}^{N_T} |h_i|^2 \right) \text{ bps/Hz}$$

With $|h_i|^2 = 1$ for $i = 1, \dots, N_T$, we have

$$C_{\text{MISO}} = \log_2 \left(1 + \frac{\mathcal{E}_s}{N_o} \right) \text{ bps/Hz}$$

(b) The capacity of a SISO channel is

$$\begin{aligned} C_{\text{SISO}} &= \log_2 \left(1 + \frac{\mathcal{E}_s}{N_o} |h|^2 \right) \\ &= \log_2 \left(1 + \frac{\mathcal{E}_s}{N_o} \right) \end{aligned}$$

and the capacity of a SIMO channel is

$$\begin{aligned} C_{\text{SIMO}} &= \log_2 \left(1 + \frac{\mathcal{E}_s}{N_o} \sum_{i=1}^{N_R} |h_i|^2 \right) \\ &= \log_2 \left(1 + \frac{N_R \mathcal{E}_s}{N_o} \right) \end{aligned}$$

Hence $C_{\text{MISO}} = C_{\text{SISO}}$ and $C_{\text{SIMO}} > C_{\text{MISO}}$.

Problem 15.7

(a) By using the Lagrange multiplier $-1/\alpha$, we have

$$J(\lambda_1, \lambda_2, \dots, \lambda_N) = \sum_{i=1}^N \log_2 \left(1 + \frac{\mathcal{E}_s}{N N_o} \lambda_i \right) - \frac{1}{\alpha} \left(\sum_{i=1}^N \lambda_i - \beta \right)$$

We differentiate $J(\lambda_1, \lambda_2, \dots, \lambda_N)$ with respect to $\lambda_1, \lambda_2, \dots, \lambda_N$ and find that $\lambda_i = \beta/N$ for all i . Hence,

$$C = \sum_{i=1}^N \log_2 \left(1 + \frac{\beta \mathcal{E}_s}{N^2 N_o} \right) = N \log_2 \left(1 + \frac{\beta \mathcal{E}_s}{N^2 N_o} \right)$$

(b) $\mathbf{H}\mathbf{H}^H = \mathbf{Q}\mathbf{A}\mathbf{Q}^H = \frac{\beta}{N}\mathbf{Q}\mathbf{I}_N\mathbf{Q}^H = \frac{\beta}{N}\mathbf{I}_N$, since $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}_N$.

(c)

$$\begin{aligned}
\|\mathbf{H}\|_F^2 &= \sum_{i=1}^N \sum_{j=1}^N |h_{ij}|^2 = N^2 \\
&= \sum_{i=1}^N \lambda_i = \frac{\beta}{N} N = \beta
\end{aligned}$$

Hence, $\beta = N^2$ and $C = N \log_2(1 + \mathcal{E}_s/N_o)$.

Problem 15.8

(a)

$$\begin{aligned}
\mathbf{y} &= \mathbf{H}\mathbf{s} + \mathbf{n} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{s} \\
\mathbf{U}^H\mathbf{y} &= \mathbf{y}' = \mathbf{\Sigma}\mathbf{V}^H\mathbf{s} + \mathbf{U}^H\mathbf{n} \\
&= \mathbf{\Sigma}\mathbf{s}' + \mathbf{U}^H\mathbf{n}
\end{aligned}$$

where $\mathbf{s}' = \mathbf{V}^H\mathbf{s}$.

(b)

$$\begin{aligned}
\mathbf{n}' &= \mathbf{U}^H\mathbf{n} \\
\mathbf{R}_{n'} &= E[\mathbf{n}'\mathbf{n}'^H] = \mathbf{U}^H E[\mathbf{n}\mathbf{n}^H] \mathbf{U} = \mathbf{I}
\end{aligned}$$

(c) As shown in equation (15.2-12)

$$C = \sum_{i=1}^r \log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_o} \lambda_i \right)$$

where λ_i is the i^{th} eigenvalue of $\mathbf{H}\mathbf{H}^H$. But $\lambda_i = \sigma_i^2$, where σ_i is the i^{th} singular value of \mathbf{H} . Since the channel is known at the transmitter, it can allocate a variable amount of power to each of the N_T transmitted symbols. Thus, if \mathcal{E}_s is the total energy transmitted across the N_T antennas, then we may allocate $\alpha_i \mathcal{E}_s$ to the i^{th} antenna, so that $\sum_{i=1}^{N_T} \alpha_i = N_T$. Then, the weights $\{\alpha_i\}$ are selected to maximize C . Thus

$$C = \max_{\{\alpha_i\}} \sum_{i=1}^r \log_2 \left(1 + \frac{\mathcal{E}_s \alpha_i \sigma_i^2}{N_T N_o} \right)$$

subject to the constraint that $\sum_{i=1}^{N_T} \alpha_i = N_T$. Using the Lagrange multiplier $-1/\beta$, we solve for the optimum power (energy) allocation by differentiating

$$J = \sum_{i=1}^r \log_2 \left(1 + \frac{\mathcal{E}_s \alpha_i \sigma_i^2}{N_T N_o} \right) - \frac{1}{\beta} \left(\sum_{i=1}^{N_T} \alpha_i - N_T \right)$$

Thus

$$\frac{\gamma_i}{1 + \alpha \gamma_i} - \frac{1}{\beta} = 0$$

and, hence

$$\alpha_i = \beta - \frac{1}{\gamma_i}$$

where $\gamma_i = \frac{\mathcal{E}_s \sigma_i^2}{N_T N_o}$.

Problem 15.9

(a) The pdf of

$$X = \sum_{i=1}^{N_T} |h_i|^2$$

where $\{h_i\}$ are i.i.d. complex-valued Gaussian with mean 0 and variance 1, is chi-squared with $2N_T$ degrees of freedom. Hence

$$p_X(x) = \frac{1}{(N_T - 1)!} x^{N_T - 1} e^{-x}, \quad x > 0$$

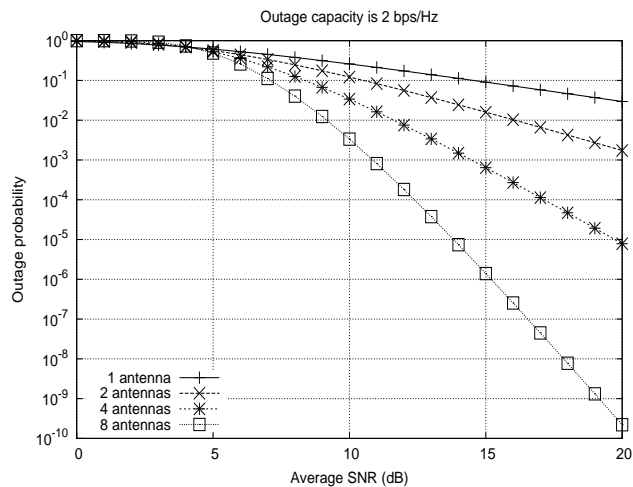
(b)

$$\begin{aligned} C &= \log_2 \left(1 + \frac{\gamma}{N_T} \sum_{i=1}^{N_T} |h_i|^2 \right), \quad \gamma = \mathcal{E}_s / N_o \\ &= \log_2 \left(1 + \frac{\gamma X}{N_T} \right) \end{aligned}$$

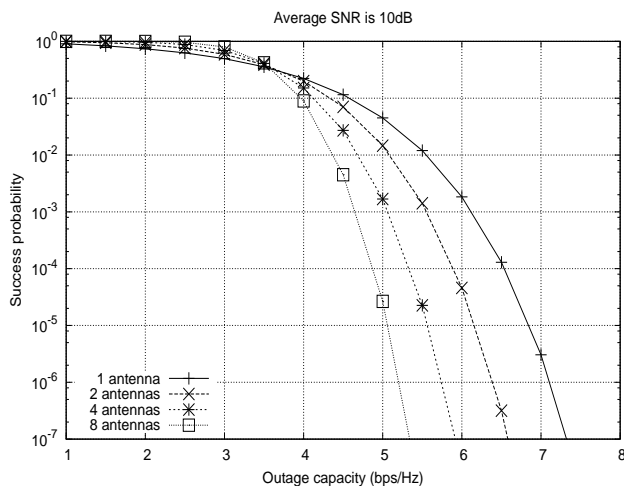
Hence, $X = \frac{2^C - 1}{\gamma} N_T$ and

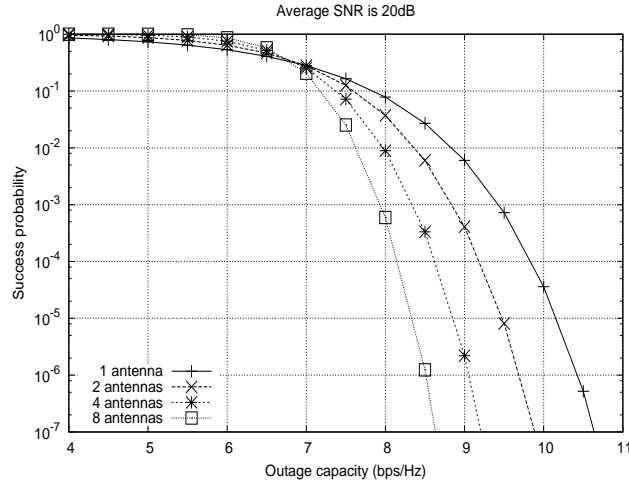
$$\begin{aligned} P_{\text{out}} &= P(C \leq C_p) \\ &= P \left(\log_2 \left(1 + \frac{\gamma X}{N_T} \right) \leq C_p \right) \\ &= P \left(X \leq \frac{2^{C_p} - 1}{\gamma} N_T \right) \end{aligned}$$

(c) The following is a plot of P_{out} vs. γ for $C_p = 2\text{bps/Hz}$ and $N_T = 1, 2, 4, 8$.

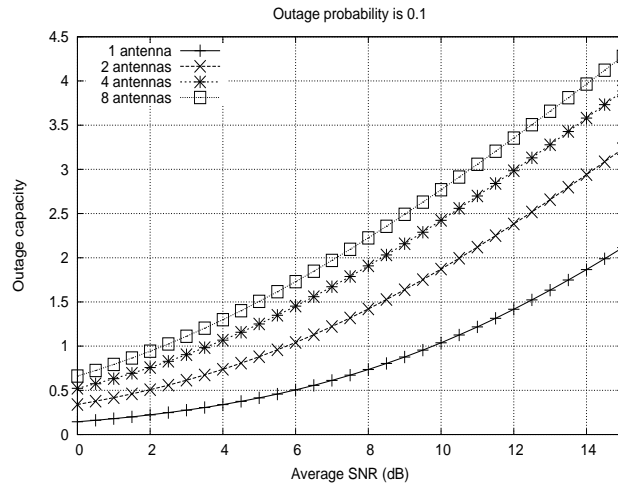


(d) The following plots are of the “success probability” $1 - P_{\text{out}}$ vs. C for $\gamma = 10, 20\text{dB}$, respectively, and $N_T = 1, 2, 4, 8$.





(e) Setting $P_{\text{out}} = 0.1$, we plot C vs. γ for $N_T = 1, 2, 4, 8$.



Problem 15.10

(a) $E[y] = \mathbf{h}^H \mathbf{s}$ and $\sigma_y^2 = E[|y|^2]$. The SNR is defined as

$$\gamma = \frac{|\mathbf{h}^H \mathbf{s}|^2}{2\sigma_y^2}$$

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The maximum γ is obtained when \mathbf{s} is colinear with \mathbf{h} , i.e., \mathbf{s} is proportional to \mathbf{h}^* . With the normalization factor $\|\mathbf{h}\|$ that is included in the denominator, we have

$$\mathbf{s} = \frac{\mathbf{h}^H}{\|\mathbf{h}\|} s'$$

and $\gamma_{\max} = (s')^2 / (2\sigma_y^2)$.

(b) The capacity of the MISO $(N_T, 1)$ channel with AWGN is given as

$$C = \max \sum_{i=1}^r \log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_o} \alpha_i \lambda_i \right)$$

with the constraint that $\sum_{i=1}^r \alpha_i = N_T$. It follows from the above that

$$C_{\text{MISO}} = \log_2 \left(1 + \frac{\mathcal{E}_s}{N_o} \|\mathbf{h}\|_F^2 \right)$$

when the channel is known at the transmitter.

(c) $C_{\text{MISO}} = C_{\text{SIMO}}$ in this case.

Problem 15.11

We have

$$P_{\text{out}} = P(C \leq C_{\text{out}}) = P(C \leq 2)$$

and

$$C_{\text{MISO}} = \log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_o} \sum_{j=1}^{N_T} |h_j|^2 \right)$$

Let $X = \sum_{j=1}^{N_T} |h_j|^2$. Then

$$C_{\text{MISO}} = \log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_o} X \right)$$

and

$$\begin{aligned} P_{\text{out}} &= P \left(\log_2 \left(1 + \frac{\mathcal{E}_s}{N_T N_o} X \right) \leq 2 \right), \text{ where } N_T = 4 \\ &= P \left(X \leq \frac{3 \times 4}{\gamma} \right) \\ &= P \left(X \leq \frac{12}{100} \right) \end{aligned}$$

where the pdf of X is

$$p_X(x) = \frac{1}{6}x^3e^{-x}, \quad x > 0$$

Therefore,

$$P_{\text{out}} = \int_0^{\frac{12}{100}} \frac{1}{6}x^3e^{-x}dx$$

Problem 15.12

$$C_{\text{SIMO}} = \log_2 \left(1 + \gamma \sum_{i=1}^{N_R} |h_i|^2 \right), \quad \gamma = \mathcal{E}_s/N_o$$

where $E[h_i] = 0$ and $E[|h_i|^2] = 1$.

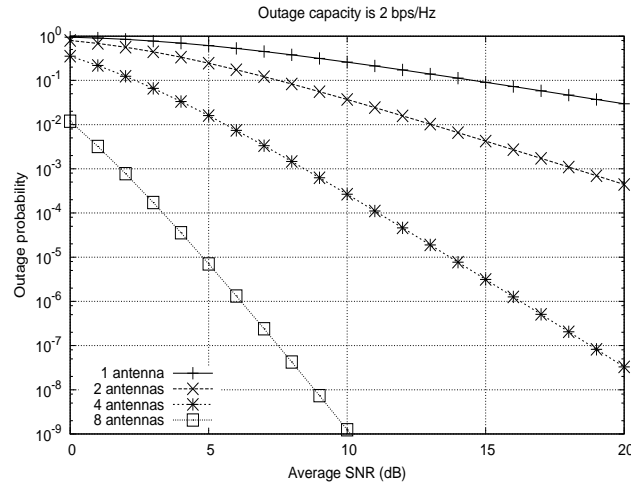
(a) Let $X = \sum_{i=1}^{N_R} |h_i|^2$. Then

$$p_X(x) = \frac{1}{(N_R - 1)!} x^{N_R - 1} e^{-x}, \quad x > 0$$

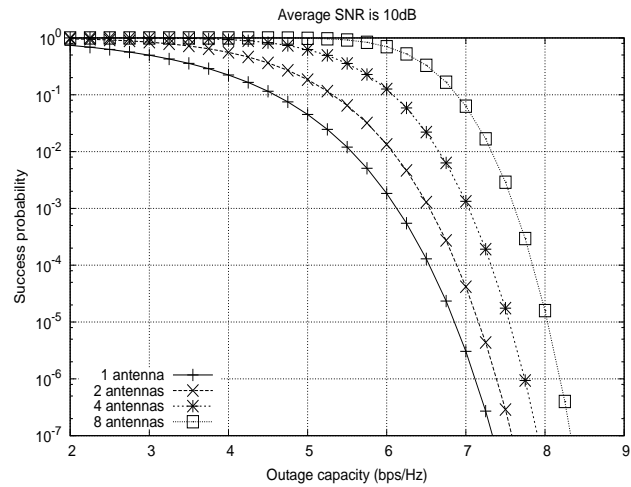
(b)

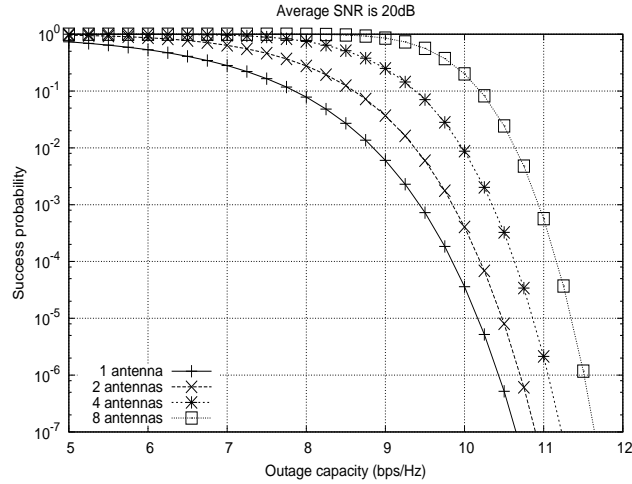
$$\begin{aligned} P_{\text{out}} &= P(C \leq C_{\text{out}}) \\ &= P(\log_2(1 + \gamma X) \leq C_{\text{out}}) \\ &= P\left(X \leq \frac{2^{C_{\text{out}}} - 1}{\gamma}\right) \end{aligned}$$

(c) The following is a plot of P_{out} vs. γ for $C_p = 2\text{bps/Hz}$ and $N_T = 1, 2, 4, 8$.

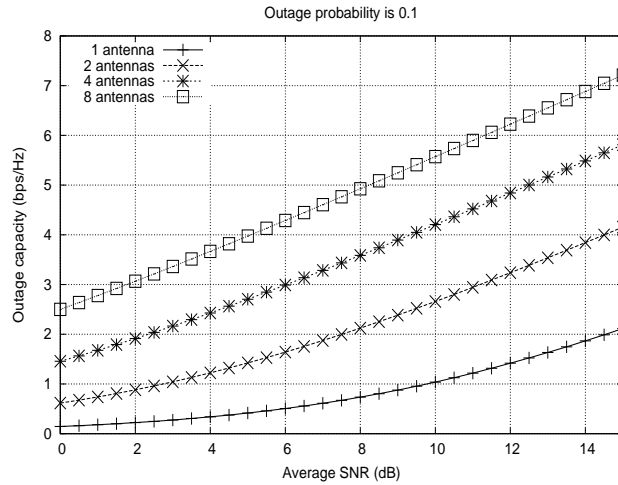


(d) The following plots are of the “success probability” $1 - P_{\text{out}}$ vs. C for $\gamma = 10, 20\text{dB}$, respectively, and $N_T = 1, 2, 4, 8$.





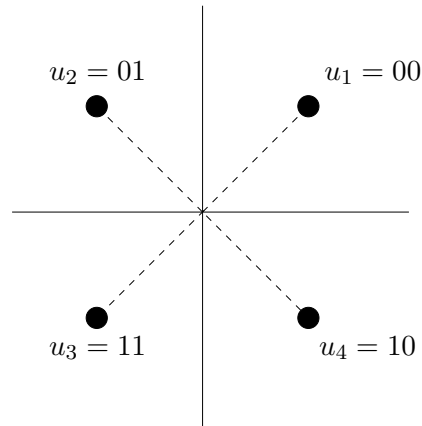
(e) Setting $P_{\text{out}} = 0.1$, we plot C vs. γ for $N_T = 1, 2, 4, 8$.



Problem 15.13

An $(N_T, N_R) = (2, N_R)$ MIMO system employs the Alamouti code with QPSK modulation. The input bit stream is

01	10	10	01	11	00
s_1	s_2	s_3	s_4	s_5	s_6



Problem 15.14

We wish to show that the decision metrics in equations (15.4-25) and (15.4-22) are equivalent. The received signals are

$$\begin{aligned} y_1 &= h_{11}s_1 + h_{12}s_2 + n_1 \\ y_2 &= -h_{11}s_2^* + h_{12}s_1^* + n_2 \end{aligned}$$

and

$$\begin{aligned} \hat{s}_1 &= y_1 h_{11}^* + y_2^* h_{12} \\ &= [h_{11}s_1 + h_{12}s_2 + n_1]h_{11}^* + [-h_{11}^*s_2 + h_{12}^*s_1 + n_2^*]h_{12} \\ &= (|h_{11}|^2 + |h_{12}|^2)s_1 + h_{11}^*n_1 + h_{12}n_2^* \end{aligned}$$

Signalling interval	Antenna 1	Antenna 2
1	$s_1 = v_2$	$s_2 = v_4$
2	$-s_2^* = v_3$	$s_1^* = v_3$
3	$s_3 = v_4$	$s_4 = v_2$
4	$-s_4^* = v_1$	$s_3^* = v_1$
5	$s_5 = v_3$	$s_6 = v_1$
6	$-s_6^* = v_2$	$s_5^* = v_2$

The decision metric $\mu(s_1)$ may be expressed as

$$\begin{aligned}
\mu(s_1) &= 2\text{Re}\{y_1 h_{11}^* s_1^* + y_2^* h_{12} s_1^*\} - |s_1|^2(|h_{11}|^2 + |h_{12}|^2) \\
&= 2\text{Re}\{s_1^*(y_1 h_{11}^* + y_2^* h_{12}) - \frac{1}{2}s_1 s_1^*(|h_{11}|^2 + |h_{12}|^2)\} \\
&= 2\text{Re}\{s_1^* \hat{s}_1 - \frac{1}{2}s_1(|h_{11}|^2 + |h_{12}|^2)\}
\end{aligned}$$

We substitute for $\hat{s}_1 = (|h_{11}|^2 + |h_{12}|^2)s_1 + h_{11}^* n_1 + h_{12} n_2^*$ into $\mu(s_1)$ and obtain

$$\begin{aligned}
\mu(s_1) &= 2\text{Re}\{s_1^*(|h_{11}|^2 + |h_{12}|^2)s_1/2 + (h_{11}^* n_1 + h_{12} n_2^*)\} \\
&= \text{Re}\{s_1^*(|h_{11}|^2 + |h_{12}|^2)s_1 + 2(h_{11}^* n_1 + h_{12} n_2^*)\} \\
&= \text{Re}\{s_1^* \hat{s}_1\}
\end{aligned}$$

We note that $\mu(s_1)$ is simply the correlation of \hat{s}_1 with s_1^* . The noise components in \hat{s}_1 and $\mu(s_1)$ have identical statistics.

Problem 15.15

Since

$$\mathbf{G}^H \mathbf{G} = (|s_1|^2 + |s_2|^2 + |s_3|^2) \mathbf{I}_3$$

and the received signals are

$$\begin{aligned} y_1 &= h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + n_1 \\ y_2 &= -h_{11}s_2^* + h_{12}s_1^* + n_2 \\ y_3 &= h_{11}s_3^* - h_{13}s_1^* + n_3 \\ y_4 &= h_{12}s_3^* - h_{13}s_2^* + n_4 \end{aligned}$$

the maximum likelihood detector bases its decisions on the estimates $\hat{s}_1, \hat{s}_2, \hat{s}_3$, where

$$\begin{aligned} \hat{s}_1 &= h_{11}^*y_1 + h_{12}y_2^* - h_{13}y_3^* \\ \hat{s}_2 &= h_{12}^*y_1 - h_{11}y_2^* - h_{13}y_4^* \\ \hat{s}_3 &= h_{13}^*y_1 + h_{11}y_3^* - h_{13}y_4^* \end{aligned}$$

For example, if we substitute for y_1, y_2 and y_3 in the estimate \hat{s}_1 , we obtain

$$\hat{s}_1 = (|h_{11}|^2 + |h_{12}|^2 + |h_{13}|^2)s_1 + h_{11}^*n_1 + h_{12}^*n_2^* - h_{13}n_3^*$$

Similar expressions are obtained for \hat{s}_2 and \hat{s}_3 . The noise components, conditioned on h_{11}, h_{12} and h_{13} are zero mean, complex-valued Gaussian.

Problem 15.16

\mathbf{G} is given by equation (15.4-42). The received signal vector is

$$\mathbf{y} = \mathbf{G}\mathbf{h} + \mathbf{n} = \mathbf{G}[h_{11} \ h_{12} \ h_{13} \ h_{14}]^T + [n_1 \ n_2 \ \dots \ n_8]^T$$

The received signal components are:

$$\begin{aligned} y_1 &= h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + h_{14}s_4 + n_1 \\ y_2 &= -h_{11}s_2 + h_{12}s_1 - h_{13}s_4 + h_{14}s_3 + n_2 \\ y_3 &= -h_{11}s_3 + h_{12}s_4 + h_{13}s_1 - h_{14}s_2 + n_3 \\ y_4 &= -h_{11}s_4 - h_{12}s_3 + h_{13}s_2 + h_{14}s_1 + n_4 \\ y_5 &= h_{11}s_1^* + h_{12}s_2^* + h_{13}s_3^* + h_{14}s_4^* + n_5 \\ y_6 &= -h_{11}s_2^* + h_{12}s_1^* - h_{13}s_4^* + h_{14}s_3^* + n_6 \\ y_7 &= -h_{11}s_3^* + h_{12}s_4^* + h_{13}s_1^* - h_{14}s_2^* + n_7 \\ y_8 &= -h_{11}s_4 - h_{12}s_3^* + h_{13}s_2^* + h_{14}s_1^* + n_8 \end{aligned}$$

Since $\mathbf{G}^H \mathbf{G} = \sum_{i=1}^4 |s_i|^2 \mathbf{I}_4$, the maximum-likelihood detector bases its decisions on the estimates:

$$\begin{aligned}\hat{s}_1 &= h_{11}^* y_1 + h_{12}^* y_2 + h_{13}^* y_3 + h_{14}^* y_4 + h_{11} y_5^* + h_{12} y_6^* + h_{13} y_7^* + h_{14} y_8^* \\ \hat{s}_2 &= h_{12}^* y_1 - h_{11}^* y_2 - h_{14}^* y_3 + h_{13}^* y_4 + h_{12} y_5^* - h_{11} y_6^* - h_{14} y_7^* + h_{13} y_8^* \\ \hat{s}_3 &= h_{13}^* y_1 + h_{14}^* y_2 - h_{11}^* y_3 - h_{12}^* y_4 + h_{13} y_5^* + h_{14} y_6^* + h_{13} y_7^* - h_{12} y_8^* \\ \hat{s}_4 &= h_{14}^* y_1 - h_{13}^* y_2 + h_{12}^* y_3 - h_{11}^* y_4 + h_{14} y_5^* - h_{13} y_6^* + h_{12} y_7^* - h_{11} y_8^*\end{aligned}$$

For example, if we substitute for y_i , $i = 1, 2, \dots, 8$ in the expression for \hat{s}_1 , we obtain

$$\hat{s}_1 = (|h_{11}|^2 + |h_{12}|^2 + |h_{13}|^2 + |h_{14}|^2) s_1 + \text{noise terms}$$

Similar expressions are obtained for \hat{s}_2 , \hat{s}_3 and \hat{s}_4 .

Problem 15.17

From equation (15.3-2), the output of the MRC is

$$\mu_j = \mathbf{h}_j^H \mathbf{y}_j = \sqrt{\frac{\mathcal{E}_b}{N_T}} s_j \|\mathbf{h}_j\|_F^2 + \mathbf{h}_j^H \mathbf{n}_j, \quad j = 1, 2, \dots, N_T$$

For BPSK modulation, assume $s_j = \pm 1$ and consider the case of the signal transmitted from antenna 1. Then, with $s_1 = 1$,

$$\mu_1 = \sqrt{\frac{\mathcal{E}_b}{N_T}} s_1 \|\mathbf{h}_1\|_F^2 + \mathbf{h}_1^H \mathbf{n}_1$$

When $E[\mathbf{h}_1] = \mathbf{0}$ and $E[|h_{i1}|^2] = E[\alpha_i^2]$, the random variable $\|\mathbf{h}_1\|_F^2$ is chi-squared distributed with $2N_R$ degrees of freedom. Conditioned on \mathbf{h}_1 , the error probability is

$$P_b(\gamma_b) = Q\left(\sqrt{2\gamma_b/N_T}\right), \quad \gamma_b = \frac{\mathcal{E}_b}{N_o} \sum_{i=1}^{N_R} \alpha_i^2$$

The pdf of γ_b is

$$p(\gamma_b) = \frac{1}{\bar{\gamma}_c^{N_R} (N_R - 1)!} \gamma_b^{N_R - 1} e^{-\gamma_b/\bar{\gamma}_c}, \quad \gamma_b > 0$$

where $\bar{\gamma}_c = \frac{\mathcal{E}_b}{N_o} E[\alpha_i^2]$, for all i . By averaging $P_b(\gamma_b)$ over the pdf $p(\gamma_b)$, we obtain (see equation 13.4-15)

$$P_b = \frac{1}{2}(1 - \mu)^{N_R} \sum_{k=0}^{N_R-1} \binom{N_R - 1 + k}{k} \left[\frac{1}{2}(1 + \mu) \right]^k$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}_c/N_T}{1 + \bar{\gamma}_c/N_T}}$$

Problem 15.18

For the SIMO (1,2) system, with transmitted energy $\sqrt{\mathcal{E}_b}$ and AWGN, we simply have the performance of a conventional dual diversity system with MRC. Hence, from equation (13.4-15), with $L = 2$, we have

$$P_{\text{SIMO}} = \frac{1}{4}(1 - \mu)^2(2 + \mu)$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}}$$

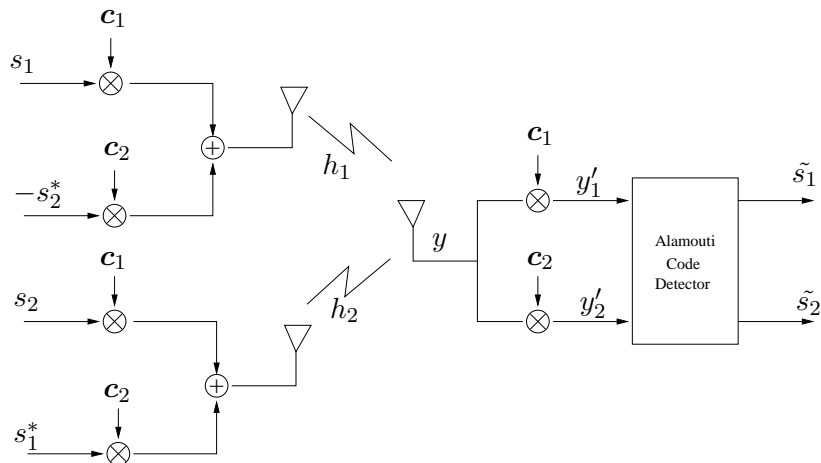
and $\bar{\gamma}_c = \frac{\mathcal{E}_b}{N_o} E[\alpha_1^2] = \frac{\mathcal{E}_b}{N_o} E[\alpha_2^2]$.

For the MISO (2,1) system that employs the Alamouti code, the energy per symbol (bit) is \mathcal{E}_b over the two signal intervals and, hence, for the decision variable

$$\hat{s}_1 = y_1 h_{11}^* + y_2^* h_{12} = (|h_{11}|^2 + |h_{12}|^2)s_1 + h_{11}^* n_1 + h_{12} n_2^*$$

the error probability is the same as that for the SIMO (1,2) system.

Problem 15.19



(b)

$$\begin{aligned}
 \mathbf{y} &= (\mathbf{c}_1 s_1 - \mathbf{c}_2 s_2^*) h_1 + (\mathbf{c}_1 s_2 + \mathbf{c}_2 s_1^*) h_2 + \text{noise} \\
 \mathbf{c}_1^H \mathbf{y} &= y_1' = s_1 h_1 + s_2 h_2 \\
 \mathbf{c}_2^H \mathbf{y} &= y_2' = -s_2^* h_1 + s_1^* h_2
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \hat{s}_1 &= h_1^* y_1' + h_2 y_2'^* \\
 \hat{s}_2 &= h_2^* y_1' - h_1 y_2'^*
 \end{aligned}$$

(c) One advantage is that the two symbols are transmitted and detected in the same time interval, so the channel is not required to be constant over two symbol intervals. A second advantage is that the data rate is doubled by the use of the two orthogonal spreading codes. The one disadvantage is that the transmitted signal bandwidth is expanded by the use of spread spectrum signals.

Problem 15.20

(a)

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}$$

The ICD computes

$$\mathbf{H}^{-1}\mathbf{y} = \mathbf{H}^{-1}\mathbf{H}\mathbf{s} + \mathbf{H}^{-1}\mathbf{n} = \mathbf{s} + \mathbf{H}^{-1}\mathbf{n}$$

Let $\mathbf{v} = \mathbf{H}^{-1}\mathbf{n}$. Then

$$\begin{aligned}\mathbf{R}_v &= E[\mathbf{v}\mathbf{v}^H] = E[\mathbf{H}^{-1}\mathbf{n}\mathbf{n}^H(\mathbf{H}^{-1})^H] \\ &= N_o E[\mathbf{H}^{-1}(\mathbf{H}^{-1})^H]\end{aligned}$$

(b) No. The optimum detector is a maximum-likelihood detector.

(c) The probability of the ICD may be expressed as

$$P_{bk} = Q(\sqrt{2\gamma_k}), \quad k = 1, 2, \dots, N_T$$

where γ_k is the SNR for the k^{th} received symbol. The SNR is variable because the noise variance may differ among the N_T symbols.

(d) If $\hat{\mathbf{s}} = \mathbf{W}^H\mathbf{y}$, then the noise component is $\mathbf{W}^H\mathbf{n}$. Hence

$$\mathbf{R}_w = E[\mathbf{W}^H\mathbf{n}\mathbf{n}^H\mathbf{W}] = N_o\mathbf{W}^H\mathbf{W}$$

But

$$\begin{aligned}\mathbf{W}^H\mathbf{W} &= (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H[(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H]^H \\ &= (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{H}[(\mathbf{H}^H\mathbf{H})^{-1}]^H \\ &= [(\mathbf{H}^H\mathbf{H})^{-1}]^H\end{aligned}$$

As in the case of the ICD, this linear detector is not optimum.

Problem 15.21

$$\mathbf{H} = \begin{bmatrix} 0.4 & 0.5 \\ 0.7 & 0.3 \end{bmatrix} \quad \mathbf{H}\mathbf{H}^H = \begin{bmatrix} 0.41 & 0.43 \\ 0.43 & 0.58 \end{bmatrix}$$

(a) The eigenvalues of $\mathbf{H}\mathbf{H}^H$ are found as follows:

$$\begin{vmatrix} 0.41 - \lambda & 0.43 \\ 0.43 & 0.58 - \lambda \end{vmatrix} = \lambda^2 - 0.99\lambda + 0.0529 = 0$$

Hence, $\lambda_1 = 0.933$ and $\lambda_2 = 0.0567$.

The singular values of \mathbf{H} are $\sigma_1 = \sqrt{\lambda_1} = 0.9661$ and $\sigma_2 = \sqrt{\lambda_2} = 0.2381$. The SVD of \mathbf{H} is $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$ and

$$\mathbf{H}\mathbf{H}^H = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H)^H = \mathbf{U}\mathbf{\Sigma}^2\mathbf{V}^H = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2)$. Therefore, \mathbf{U} may be determined by computing the eigenvectors of $\mathbf{H}\mathbf{H}^H$. Thus

$$\begin{bmatrix} 0.41 - \lambda & 0.43 \\ 0.43 & 0.58 - \lambda \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With $\lambda = \lambda_1$, we find that the normalized eigenvectors are

$$\mathbf{u}_1 = \begin{bmatrix} 0.63485 \\ 0.77263 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.77263 \\ -0.63485 \end{bmatrix}$$

We should note that the eigenvectors are not unique, i.e., $-\mathbf{u}_1$ and $-\mathbf{u}_2$ are also possible eigenvectors.

To determine \mathbf{V} , we may compute

$$\mathbf{H}^H\mathbf{H} = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H)^H(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H) = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^H$$

Hence,

$$\mathbf{H}^H\mathbf{H} = \begin{bmatrix} 0.65 & 0.41 \\ 0.41 & 0.34 \end{bmatrix}$$

Clearly, the eigenvalues of $\mathbf{H}^H\mathbf{H}$ are the same as for $\mathbf{H}\mathbf{H}^H$. Now, we solve for the eigenvectors of $\mathbf{H}^H\mathbf{H}$. Hence,

$$\begin{bmatrix} 0.65 - \lambda & 0.41 \\ 0.41 & 0.34 - \lambda \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This yields

$$\mathbf{v}_1 = \begin{bmatrix} 0.82269 \\ 0.56850 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.56850 \\ 0.82269 \end{bmatrix}$$

(b) When the channel is known at the transmitter and the receiver, the MIMO channel is equivalent to two parallel SISO channels whose outputs are given by equation (15.2-27). The capacity of the MIMO channel is given by equation (15.2-29) as

$$C = \max_{\sigma_{ks}^2} \sum_{k=1}^2 \log_2 \left(1 + \frac{\gamma}{2} \lambda_k \sigma_{ks}^2 \right), \quad \gamma = \frac{\mathcal{E}_s}{N_o}$$

with the constraint that $\sigma_{1s}^2 + \sigma_{2s}^2 = N_T = 2$. The optimum power allocation can be determined by maximizing

$$J(\sigma_{1s}^2, \sigma_{2s}^2) = \sum_{k=1}^2 \log_2 \left(1 + \frac{\gamma}{2} \lambda_k \sigma_{ks}^2 \right) - \frac{1}{\mu} \left(\sum_{k=1}^2 \sigma_{ks}^2 - 2 \right)$$

where $-1/\mu$ is the Lagrange multiplier. By differentiating $J(\sigma_{1s}^2, \sigma_{2s}^2)$ with respect to σ_{1s}^2 and σ_{2s}^2 , we find that

$$\sigma_{ks}^2 = \mu - \frac{2}{\gamma \lambda_k}, \quad k = 1, 2$$

(c) When \mathbf{H} is known at the receiver only, the capacity is given as

$$\begin{aligned} C &= \sum_{i=1}^2 \log_2 \left(1 + \frac{\mathcal{E}_s}{2N_o} \lambda_i \right) \\ &= \log_2 \left(1 + \frac{\gamma \lambda_1}{2} \right) \left(1 + \frac{\gamma \lambda_2}{2} \right), \quad \gamma = \frac{\mathcal{E}_s}{N_o} \end{aligned}$$

Problem 15.22

The capacity of the MISO (2,1) channel which is known at the receiver only, with AWGN, is given as

$$\begin{aligned} C_{\text{MISO}} &= \log_2 \left(1 + \frac{\mathcal{E}_s}{2N_o} \sum_{i=1}^2 |h_i|^2 \right) \\ &= \log_2 \left(1 + \frac{\gamma}{2} X \right) \end{aligned}$$

where $\gamma = \mathcal{E}_s/N_o$ and $X = |h_1|^2 + |h_2|^2$. The outage probability is

$$P_{\text{out}} = P\left(X \leq \frac{2^C - 1}{\gamma/2}\right)$$

When the MISO (2,1) channel is known at the transmitter, the capacity of the channel is (see problem 15.10b),

$$C_{\text{MISO}} = \log_2(1 + \gamma X)$$

The outage probability is

$$P_{\text{out}} = P\left(X \leq \frac{2^C - 1}{\gamma}\right)$$

We observe that the outage probability for the MISO (2,1) channel that is known at the transmitter is lower than that for the MISO(2,1) channel known at the receiver only. The SNR advantage of the former MISO system over the latter is 3dB.

Problem 15.23

(a)

$$\begin{aligned} \mathbf{G}^H \mathbf{G} &= \begin{bmatrix} s_1^* & -s_2 & -s_3 & s_4^* \\ s_2^* & s_1 & -s_4 & -s_3^* \\ s_3^* & -s_4 & s_1 & -s_2^* \\ s_4^* & s_3 & s_2 & s_1^* \end{bmatrix} \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^4 |s_i|^2 & 0 & 0 & 2\text{Re}\{s_1 s_4^* - s_2 s_3^*\} \\ 0 & \sum_{i=1}^4 |s_i|^2 & 2\text{Re}\{s_2 s_3^* - s_1 s_4^*\} & 0 \\ 0 & 2\text{Re}\{s_2 s_3^* - s_1 s_4^*\} & \sum_{i=1}^4 |s_i|^2 & 0 \\ 2\text{Re}\{s_1 s_4^* - s_2 s_3^*\} & 0 & 0 & \sum_{i=1}^4 |s_i|^2 \end{bmatrix} \end{aligned}$$

Since $\mathbf{G}^H \mathbf{G}$ is not a diagonal matrix, the code is not orthogonal.

(b) The received signals in the four signal intervals are:

$$\begin{aligned} y_1 &= h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + h_{14}s_4 + n_1 \\ y_2 &= -h_{11}s_2^* + h_{12}s_1^* - h_{13}s_4^* + h_{14}s_3^* + n_2 \\ y_3 &= -h_{11}s_3^* - h_{12}s_4^* + h_{13}s_1^* + h_{14}s_2^* + n_3 \\ y_4 &= h_{11}s_4 - h_{12}s_3 - h_{13}s_2 + h_{14}s_1 + n_4 \end{aligned}$$

where n_i , $i = 1, \dots, 4$ are statistically independent, zero mean Gaussian random variables. The conditional pdf of y_i , $i = 1, \dots, 4$ is proportional to the following quadratic terms:

$$\begin{aligned} p(y_1, y_2, y_3, y_4 | \mathbf{H}, \mathbf{s}) &\sim |y_1 - (h_{11}s_1 + h_{12}s_2 + h_{13}s_3 + h_{14}s_4)|^2 \\ &+ |y_2 - (-h_{11}s_2^* + h_{12}s_1^* - h_{13}s_4^* + h_{14}s_3^*)|^2 \\ &+ |y_3 - (-h_{11}s_3^* - h_{12}s_4^* + h_{13}s_1^* + h_{14}s_2^*)|^2 \\ &+ |y_4 - (h_{11}s_4 - h_{12}s_3 - h_{13}s_2 + h_{14}s_1)|^2 \end{aligned}$$

By expanding the quadratic terms, it is easily shown that $p(y_1, y_2, y_3, y_4 | \mathbf{H}, \mathbf{s}) \sim \mu(s_2, s_3) + \mu(s_1, s_4)$, where each of the metrics depends on only two signal components. Therefore, the ML detector can perform pairwise ML detection. This pairwise detection can also be observed from the form of $\mathbf{G}^H \mathbf{G}$.

(c) The order of diversity achieved by the code is 4.

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 16) ¹

Prepared by
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March 12, 2008

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Problem 16.1

$$g_k(t) = e^{j\theta_k} \sum_{n=0}^{L-1} a_k(n)p(t - nT_c)$$

The unit energy constraint is :

$$\int_0^T g_k(t)g_k^*(t)dt = 1$$

We also define as cross-correlation :

$$\rho_{ij}(\tau) = \int_0^T g_i(t)g_j^*(t - \tau)dt$$

(a) For synchronous transmission, the received lowpass-equivalent signal $r(t)$ is again given by (16-3-9), while the log-likelihood ratio is :

$$\begin{aligned} \Lambda(\mathbf{b}) &= \int_0^T \left| r(t) - \sum_{k=1}^K \sqrt{E_k} b_k g_k(t) \right|^2 dt \\ &= \int_0^T |r(t)|^2 dt + \sum_k \sum_j \sqrt{E_k} \sqrt{E_j} b_j b_k^* \int_0^T g_j(t) g_k^*(t) dt \\ &\quad - 2 \operatorname{Re} \left[\sum_{k=1}^K \sqrt{E_k} b_k \int_0^T r(t) g_k^*(t) dt \right] \\ &= \int_0^T |r(t)|^2 dt + \sum_k \sum_j \sqrt{E_k} \sqrt{E_j} b_j b_k \rho_{jk}(0) \\ &\quad - 2 \operatorname{Re} \left[\sum_{k=1}^K \sqrt{E_k} b_k r_k \right] \end{aligned}$$

where $r_k = \int_0^T r(t) g_k^*(t) dt$, and we assume that the information sequence $\{b_k\}$ is real. Hence, the correlation metrics can be expressed in a similar form to (16-3-15) :

$$C(\mathbf{r}_k, \mathbf{b}_k) = 2\mathbf{b}_K^t \operatorname{Re}(\mathbf{r}_K) - \mathbf{b}_K^t \mathbf{R}_s \mathbf{b}_K$$

The only difference from the real-valued case of the text is that the correlation matrix \mathbf{R}_s uses the complex-valued cross-correlations given above :

$$\mathbf{R}_s[ij] = \begin{cases} \rho_{ij}^*(0), & i \leq j \\ \rho_{ij}(0), & i > j \end{cases}$$

and that the matched filters producing $\{r_k\}$ employ the complex-conjugate of the signature waveforms $\{g_k(t)\}$.

(b) Following the same procedure as in the text, we see that the correlator outputs are :

$$r_k(i) = \int_{iT+\tau_k}^{(i+1)T+\tau_k} r(t) g_k^*(t - iT - \tau_k) dt$$

and that these can be expressed in matrix form as :

$$\mathbf{r} = \mathbf{R}_N \mathbf{b} + \mathbf{n}$$

where $\mathbf{r}, \mathbf{b}, \mathbf{n}$ are given by (16-3-20)-(16-3-22) and :

$$\mathbf{R}_N = \begin{bmatrix} \mathbf{R}_a(0) & \mathbf{R}_a(-1) & \mathbf{0} & \cdots & \cdots \\ \mathbf{R}_a(1) & \mathbf{R}_a(0) & \mathbf{R}_a(-1) & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R}_a(1) & \mathbf{R}_a(0) \end{bmatrix} \Rightarrow$$

$$\mathbf{R}_N = \begin{bmatrix} \mathbf{R}_a(0) & \mathbf{R}_a^H(1) & \mathbf{0} & \cdots & \cdots \\ \mathbf{R}_a(1) & \mathbf{R}_a(0) & \mathbf{R}_a^H(1) & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R}_a(1) & \mathbf{R}_a(0) \end{bmatrix}$$

where $\mathbf{R}_a(m)$ is a $K \times K$ matrix with elements :

$$R_{kl}(m) = \int_{-\infty}^{\infty} g_k^*(t - \tau_k) g_l(t + mT - \tau_l) dt$$

and we have exploited the fact (which holds in the real-valued case, too) that :

$$\mathbf{R}_a(m) = \mathbf{R}_a^{*t}(-m) = \mathbf{R}_a^H(-m)$$

Finally, we note that $\mathbf{R}_a(0) = \mathbf{R}_s$, the correlation matrix of the real-valued case.

Problem 16.2

The capacity per user C_K is :

$$C_K = \frac{1}{K} W \log_2 \left(1 + \frac{P}{W N_0} \right), \quad \lim_{K \rightarrow \infty} C_K = 0$$

and the total capacity :

$$K C_K = W \log_2 \left(1 + \frac{P}{W N_0} \right)$$

which is independent of K . By using the fact that : $P = C_K \mathcal{E}_b$ we can rewrite the above equations as :

$$C_K = \frac{1}{K} W \log_2 \left(1 + \frac{C_K \mathcal{E}_b}{W N_0} \right) \Rightarrow$$

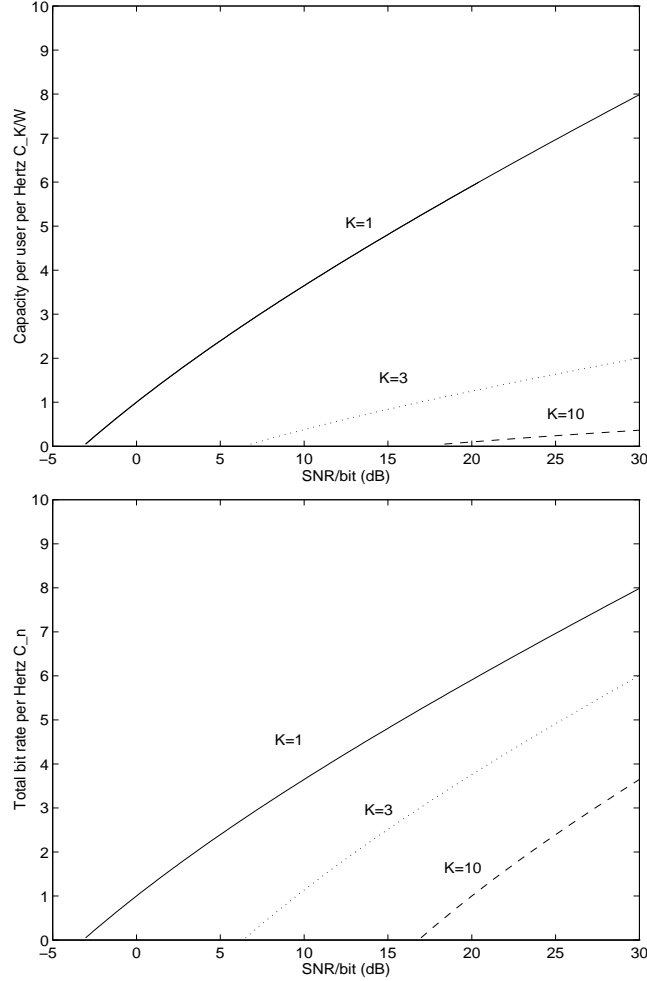
$$K \frac{C_K}{W} = \log_2 \left(1 + \frac{C_K \mathcal{E}_b}{W N_0} \right) \Rightarrow$$

$$\frac{\mathcal{E}_b}{N_0} = \frac{(2^K)^{\frac{C_K}{W}} - 1}{\frac{C_K}{W}}$$

which is the relationship between the SNR and the normalized capacity per user. The relationship between the normalized total capacity $C_n = K \frac{C_K}{W}$ and the SNR is :

$$\frac{\mathcal{E}_b}{N_0} = K \frac{2^{C_n} - 1}{C_n}$$

The corresponding plots for these last two relationships are given in the following figures :



As we observe the normalized capacity per user C_K/W decreases to 0 as the number of user increases. On the other hand, we saw that the total normalized capacity C_n is constant, independent of the number of users K . The second graph is explained by the fact that as the number of users increases, the capacity per user C_K , decreases and hence, the $\text{SNR/bit} = P/C_K$ increases, for the same user power P . That's why the curves are shifted to the right, as $K \rightarrow \infty$.

Problem 16.3

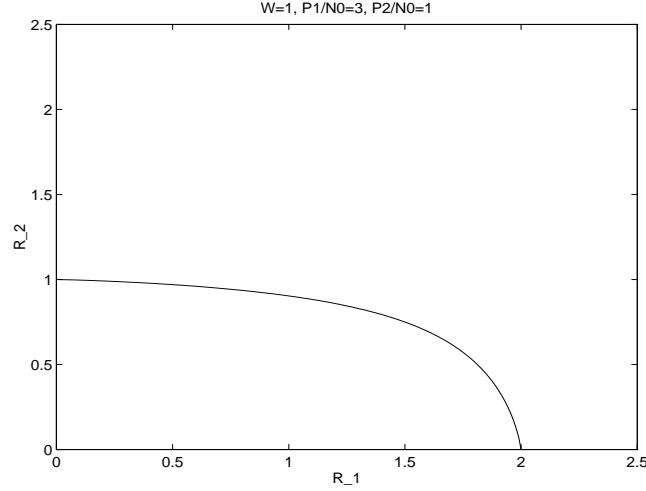
(a)

$$C_1 = aW \log_2 \left(1 + \frac{P_1}{aWN_0} \right)$$

$$C_2 = (1-a)W \log_2 \left(1 + \frac{P_2}{(1-a)WN_0} \right)$$

$$C = C_1 + C_2 = W \left[a \log_2 \left(1 + \frac{P_1}{aWN_0} \right) + (1-a) \log_2 \left(1 + \frac{P_2}{(1-a)WN_0} \right) \right]$$

As a varies between 0 and 1, the graph of the points (C_1, C_2) is given in the following figure:



(b) Substituting $P_1/a = P_2/(1-a) = P_1 + P_2$, in the expression for $C = C_1 + C_2$, we obtain :

$$\begin{aligned} C = C_1 + C_2 &= W \left[a \log_2 \left(1 + \frac{P_1+P_2}{WN_0} \right) + (1-a) \log_2 \left(1 + \frac{P_1+P_2}{WN_0} \right) \right] \\ &= W \log_2 \left(1 + \frac{P_1+P_2}{WN_0} \right) \end{aligned}$$

which is the maximum rate that can be satisfied, based on the inequalities that the rates R_1, R_2 must satisfy. Hence, the distribution of the bandwidth according to the SNR of each user, produces the maximum achievable rate.

Problem 16.4

(a) Since the transmitters are peak-power-limited, the constraint on the available power holds for the allocated time frame when each user transmits. This is more restrictive than an average-power limited TDMA system, where the power is averaged over all the time frames, so each user can transmit in his allocated frame with power P_i/a_i , where a_i is the fraction of the time that the user transmits.

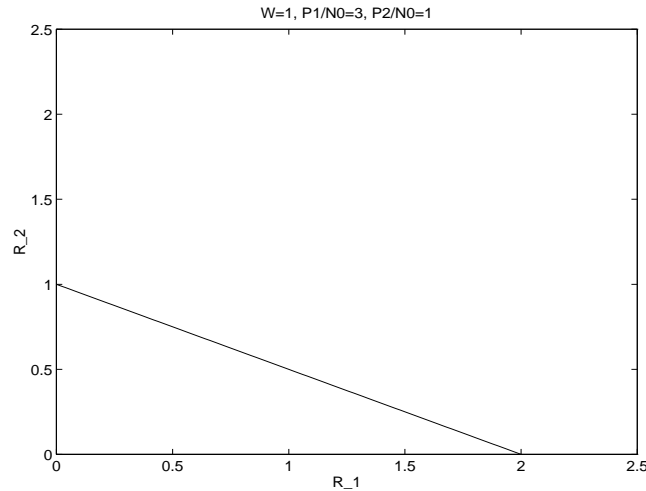
Hence, in the peak-power limited system :

$$C_1 = aW \log_2 \left(1 + \frac{P_1}{WN_0} \right)$$

$$C_2 = (1 - a)W \log_2 \left(1 + \frac{P_2}{WN_0} \right)$$

$$C = C_1 + C_2 = W \left[a \log_2 \left(1 + \frac{P_1}{WN_0} \right) + (1 - a) \log_2 \left(1 + \frac{P_2}{WN_0} \right) \right]$$

(b) As a varies between 0 and 1, the graph of the points (C_1, C_2) is given in the following figure



We note that the peak-power-limited TDMA system has a more restricted achievable region (R_1, R_2) , compared to the FDMA system of problem 16.3.

Problem 16.5

(a) Since the system is average-power limited, the i -th user can transmit in his allocated time-frame with peak-power P_i/a_i , where a_i is the fraction of the time that the user transmits.

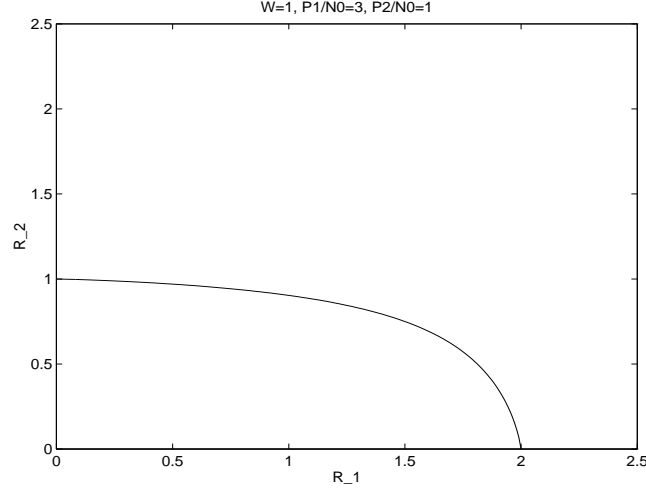
Hence, in the average-power limited system :

$$C_1 = aW \log_2 \left(1 + \frac{P_1/a}{WN_0} \right)$$

$$C_2 = (1 - a)W \log_2 \left(1 + \frac{P_2/(1 - a)}{WN_0} \right)$$

$$C = C_1 + C_2 = W \left[a \log_2 \left(1 + \frac{P_1}{aWN_0} \right) + (1 - a) \log_2 \left(1 + \frac{P_2}{(1 - a)WN_0} \right) \right]$$

(b) As a varies between 0 and 1, the graph of the points (C_1, C_2) is given in the following figure



(c) We note that the expression for the total capacity is the same as that of the FDMA in Problem 16.2. Hence, if the time that each user transmits is proportional to the transmitter's power : $P_1/a = P_2/(1-a) = P_1 + P_2$, we have :

$$\begin{aligned} C &= C_1 + C_2 = W \left[a \log_2 \left(1 + \frac{P_1+P_2}{WN_0} \right) + (1-a) \log_2 \left(1 + \frac{P_1+P_2}{WN_0} \right) \right] \\ &= W \log_2 \left(1 + \frac{P_1+P_2}{WN_0} \right) \end{aligned}$$

which is the maximum rate that can be satisfied, based on the inequalities that the rates R_1, R_2 must satisfy. Hence, the distribution of the time that each user transmits according to the respective SNR produces the maximum achievable rate.

Problem 16.6

(a) We have

$$r_1 = \int_0^T r(t)g_1(t)dt$$

Since $\int_0^T g_1(t)g_1(t)dt = 1$, and $\int_0^T g_1(t)g_2(t)dt = \rho$

$$r_1 = \sqrt{\mathcal{E}_1}b_1 + \sqrt{\mathcal{E}_2}b_2\rho + n_1$$

where $n_1 = \int_0^T n(t)g_1(t)dt$. Similarly

$$r_2 = \sqrt{\mathcal{E}_1}b_1\rho + \sqrt{\mathcal{E}_2}b_2 + n_2$$

where $n_2 = \int_0^T n(t)g_2(t)dt$

(b) We have $E[n_1] (= m_1) = E[n_2] (= m_2) = 0$. Hence

$$\begin{aligned}\sigma_1^2 = E[n_1^2] &= E \left[\int_0^T \int_0^T g_1(a)g_1(b)n(a)n(b)dadb \right] \\ &= \frac{N_0}{2} \int_0^T g_1(a)g_1(a)da \\ &= \frac{N_0}{2}\end{aligned}$$

In the same way, $\sigma_1^2 = E[n_1^2] = \frac{N_0}{2}$. The covariance is equal to

$$\begin{aligned}\mu_{12} = E[n_1n_2] - E[n_1]E[n_2] &= E[n_1n_2] \\ &= E \left[\int_0^T \int_0^T g_1(a)g_2(b)n(a)n(b)dadb \right] \\ &= \frac{N_0}{2} \int_0^T g_1(a)g_2(a)da \\ &= \frac{N_0}{2}\rho\end{aligned}$$

(c) Given b_1 and b_2 , then (r_1, r_2) follow the pdf of (n_1, n_2) which are jointly Gaussian with a pdf given by (2-1-150) or (2-1-156). Using the results from (b)

$$\begin{aligned}p(r_1, r_2 | b_1, b_2) &= p(n_1, n_2) \\ &= \frac{1}{2\pi \frac{N_0}{2} \sqrt{1-\rho^2}} \exp \left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)} \right]\end{aligned}$$

where $x_1 = r_1 - \sqrt{\mathcal{E}_1}b_1 - \sqrt{\mathcal{E}_2}b_2\rho$ and $x_2 = r_2 - \sqrt{\mathcal{E}_2}b_2 - \sqrt{\mathcal{E}_1}b_1\rho$

Problem 16.7

We use the result for r_1, r_2 from Problem 5.6 (a) (or the equivalent expression (16.3-40)). Then, assuming $b_1 = 1$ was transmitted, the probability of error for b_1 is

$$\begin{aligned}P_1 &= P(\text{error}_1 | b_2 = 1)P(b_2 = 1) + P(\text{error}_1 | b_2 = -1)P(b_2 = -1) \\ &= Q \left(\sqrt{2 \frac{(\sqrt{\mathcal{E}_1} + \rho\sqrt{\mathcal{E}_2})^2}{N_0}} \right) \frac{1}{2} + Q \left(\sqrt{2 \frac{(\sqrt{\mathcal{E}_1} - \rho\sqrt{\mathcal{E}_2})^2}{N_0}} \right) \frac{1}{2}\end{aligned}$$

The same expression is obtained when $b_1 = -1$ is transmitted. Hence

$$P_1 = \frac{1}{2}Q \left(\sqrt{2 \frac{(\sqrt{\mathcal{E}_1} + \rho\sqrt{\mathcal{E}_2})^2}{N_0}} \right) + \frac{1}{2}Q \left(\sqrt{2 \frac{(\sqrt{\mathcal{E}_1} - \rho\sqrt{\mathcal{E}_2})^2}{N_0}} \right)$$

Similarly

$$P_2 = \frac{1}{2}Q \left(\sqrt{2 \frac{(\sqrt{\mathcal{E}_2} + \rho\sqrt{\mathcal{E}_1})^2}{N_0}} \right) + \frac{1}{2}Q \left(\sqrt{2 \frac{(\sqrt{\mathcal{E}_2} - \rho\sqrt{\mathcal{E}_1})^2}{N_0}} \right)$$

Problem 16.8

(a)

$$P(b_1, b_2 | r(t), 0 \leq t \leq T) = \frac{p(r(t), 0 \leq t \leq T | b_1, b_2) P(b_1, b_2)}{p(r(t), 0 \leq t \leq T)}$$

But $P(b_1, b_2) = P(b_1)P(b_2) = 1/4$ for any pair of (b_1, b_2) and $p(r(t), 0 \leq t \leq T)$ is independent of (b_1, b_2) . Hence

$$\arg \max_{b_1, b_2} P(b_1, b_2 | r(t), 0 \leq t \leq T) = \arg \max_{b_1, b_2} p(r(t), 0 \leq t \leq T | b_1, b_2)$$

which shows the equivalence between the MAP and ML criteria, when b_1, b_2 are equiprobable.

(b) Sufficient statistics for $r(t), 0 \leq t \leq T$ are the correlator outputs r_1, r_2 at $t = T$. From Problem 16.6 the joint pdf of r_1, r_2 given b_1, b_2 is

$$p(r_1, r_2 | b_1, b_2) = \frac{1}{2\pi \frac{N_0}{2} \sqrt{1 - \rho^2}} \exp \left\{ -\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1 - \rho^2)} \right\}$$

where $x_1 = r_1 - \sqrt{\mathcal{E}_1} b_1 - \sqrt{\mathcal{E}_2} b_2 \rho$ and $x_2 = r_2 - \sqrt{\mathcal{E}_2} b_2 - \sqrt{\mathcal{E}_1} b_1 \rho$

The ML detector searches for the arguments b_1, b_2 that maximize $p(r_1, r_2 | b_1, b_2)$. We see that the term outside the exponent and the denominator of the exponent do not depend on b_1, b_2 . Hence :

$$\begin{aligned} (b_1, b_2) &= \arg \max \exp [-(x_1^2 - 2\rho x_1 x_2 + x_2^2)] \\ &= \arg \max [-(x_1^2 - 2\rho x_1 x_2 + x_2^2)] \end{aligned}$$

Expanding x_1, x_2 and remembering that additive terms which are constant independent of b_1, b_2 (e.g. r_i^2 , or $b_i^2 (= 1)$) do not affect the argument of the maximum, we arrive at

$$\begin{aligned} (b_1, b_2) &= \arg \max (2(1 - \rho^2)\sqrt{\mathcal{E}_1} b_1 r_1 + 2(1 - \rho^2)\sqrt{\mathcal{E}_2} b_2 r_2 - 2(1 - \rho^2)\sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 b_2 \rho) \\ &= \arg \max (\sqrt{\mathcal{E}_1} b_1 r_1 + \sqrt{\mathcal{E}_2} b_2 r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 b_2 \rho) \end{aligned}$$

Problem 16.9

(a)

$$\begin{aligned} P(b_1 | r(t), 0 \leq t \leq T) &= P(b_1 | r_1, r_2) \\ &= P(b_1, b_2 = 1 | r_1, r_2) + P(b_1, b_2 = -1 | r_1, r_2) \end{aligned}$$

But

$$P(b_1, b_2 = x | r_1, r_2) = \frac{p(r_1, r_2 | b_1, b_2 = x)}{p(r_1, r_2)} P(b_1, b_2 = x)$$

and $p(r_1, r_2)$ and $P(b_1, b_2 = x)$ do not depend on the value of b_1 . Hence

$$\arg \max_{b_1} P(b_1 | r(t), 0 \leq t \leq T) = \arg \max_{b_1} (p(r_1, r_2 | b_1, b_2 = 1) + p(r_1, r_2 | b_1, b_2 = -1))$$

From Problem 15.6 the joint pdf of r_1, r_2 given b_1, b_2 is

$$p(r_1, r_2 | b_1, b_2) = \frac{1}{2\pi \frac{N_0}{2} \sqrt{1 - \rho^2}} \exp \left\{ -\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1 - \rho^2)} \right\}$$

where $x_1 = r_1 - \sqrt{\mathcal{E}_1} b_1 - \sqrt{\mathcal{E}_2} b_2 \rho$ and $x_2 = r_2 - \sqrt{\mathcal{E}_2} b_2 - \sqrt{\mathcal{E}_1} b_1 \rho$. Expanding x_1, x_2 and remembering that additive terms which are constant independent of b_1, b_2 (e.g. r_i^2 , or $b_i^2 (= 1)$) do not affect the argument of the maximum, we arrive at

$$\begin{aligned} \arg \max_{b_1} P(b_1 | r(t), 0 \leq t \leq T) &= \arg \max \left[\exp \left(\frac{\sqrt{\mathcal{E}_1} b_1 r_1 + \sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 \rho}{N_0} \right) \right. \\ &\quad \left. + \exp \left(\frac{\sqrt{\mathcal{E}_1} b_1 r_1 - \sqrt{\mathcal{E}_2} r_2 + \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 \rho}{N_0} \right) \right] \\ &= \arg \max \left[\exp \left(\frac{\sqrt{\mathcal{E}_1} b_1 r_1}{N_0} \right) \right. \\ &\quad \left. \times \left(\exp \left(\frac{\sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 \rho}{N_0} \right) + \exp \left(\frac{-\sqrt{\mathcal{E}_2} r_2 + \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 \rho}{N_0} \right) \right) \right] \\ &= \arg \max \left[\exp \left(\frac{\sqrt{\mathcal{E}_1} b_1 r_1}{N_0} \right) \cdot 2 \cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 \rho}{N_0} \right) \right] \\ &= \arg \max \left[\frac{\sqrt{\mathcal{E}_1} b_1 r_1}{N_0} + \ln \cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 \rho}{N_0} \right) \right] \end{aligned}$$

(b) From part(a)

$$\begin{aligned} b_1 = 1 &\Leftrightarrow \frac{\sqrt{\mathcal{E}_1} r_1}{N_0} + \ln \cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0} \right) > \\ &\quad \frac{-\sqrt{\mathcal{E}_1} r_1}{N_0} + \ln \cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 + \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0} \right) \\ &\Leftrightarrow 2 \frac{\sqrt{\mathcal{E}_1} r_1}{N_0} + \ln \left(\frac{\cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0} \right)}{\cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 + \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0} \right)} \right) > 0 \end{aligned}$$

Hence

$$b_1 = \operatorname{sgn} \left[r_1 - \frac{N_0}{2\sqrt{\mathcal{E}_1}} \ln \left(\frac{\cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 + \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0} \right)}{\cosh \left(\frac{\sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0} \right)} \right) \right]$$

Problem 16.10

As $N_0 \rightarrow 0$, the probability in expression (16.3-62) will be dominated by the term which has the smallest argument in the Q function. Hence

$$\text{effective SNR} = \min_{b_j} \frac{\left[\sqrt{\mathcal{E}_k} + \sum_{j \neq k} \sqrt{\mathcal{E}_j} b_j \rho_{jk} \right]^2}{N_0}$$

The minimum over b_j is achieved when all terms add destructively to the $\sqrt{\mathcal{E}_k}$ term (or, it is 0, if the term inside the square is negative). Therefore

$$\eta_k = \left[\max \left\{ 0, 1 - \sum_{j \neq k} \sqrt{\frac{\mathcal{E}_j}{\mathcal{E}_k}} |\rho_{jk}| \right\} \right]^2$$

Problem 16.11

The probability that the ML detector makes an error for the first user is :

$$\begin{aligned}
 P_1 &= \sum_{b_1, b_2} P(\hat{b}_1 \neq b_1 | b_1, b_2) (P(b_1, b_2)) \\
 &= \frac{1}{4} (P[++ \rightarrow -+] + P[++ \rightarrow --]) \\
 &\quad + \frac{1}{4} (P[-+ \rightarrow ++] + P[-+ \rightarrow +-]) \\
 &\quad + \frac{1}{4} (P[+- \rightarrow --] + P[+- \rightarrow -+]) \\
 &\quad + \frac{1}{4} (P[-- \rightarrow +-] + P[-- \rightarrow ++])
 \end{aligned}$$

where $P[b_1 b_2 \rightarrow \hat{b}_1 \hat{b}_2]$ denotes the probability that the detector chooses $(\hat{b}_1 \hat{b}_2)$ conditioned on (b_1, b_2) having being transmitted. Due to the symmetry of the decision statistic, the above relationship simplifies to

$$\begin{aligned}
 P_1 &= \frac{1}{2} (P[-- \rightarrow +-] + P[-- \rightarrow ++]) \\
 &\quad + \frac{1}{2} (P[-+ \rightarrow ++] + P[-+ \rightarrow +-])
 \end{aligned} \tag{16.0.1}$$

From Problem 16.8 we know that the decision of this detector is based on

$$(\hat{b}_1, \hat{b}_2) = \arg \max \left(S(b_1, b_2) = \sqrt{\mathcal{E}_1} b_1 r_1 + \sqrt{\mathcal{E}_2} b_2 r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} b_1 b_2 \rho \right)$$

Hence, $P[-- \rightarrow +-]$ can be upper bounded as

$$P[-- \rightarrow +-] \leq P[S(-->+-) < S(+--)](-->transmitted]$$

This is a bound and not an equality since the if $S(-->+-) < S(+--)$ then $(-->)$ is not chosen, but not necessarily in favor of $(+-)$; it may have been in favor of $(++)$ or $(-+)$.

The last bound is easy to calculate :

$$\begin{aligned}
 &P[S(-->+-) < S(+--)](-->transmitted] \\
 &= P \left[-\sqrt{\mathcal{E}_1} r_1 - \sqrt{\mathcal{E}_2} r_2 - \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho < \sqrt{\mathcal{E}_1} r_1 - \sqrt{\mathcal{E}_2} r_2 + \sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho \right. \\
 &\quad \left. | r_1 = -\sqrt{\mathcal{E}_1} - \sqrt{\mathcal{E}_2} \rho + n_1; r_2 = -\sqrt{\mathcal{E}_1} - \sqrt{\mathcal{E}_2} \rho + n_1 \right] \\
 &= P \left[n_1 > \sqrt{\mathcal{E}_1} \right] = Q \left(\sqrt{\frac{2\mathcal{E}_1}{N_0}} \right)
 \end{aligned}$$

Similarly, for the other three terms of (1) we obtain :

$$\begin{aligned}
 P[-- \rightarrow ++] &\leq P[S(-->++) < S(++>)](-->transmitted] \\
 &= P[\sqrt{\mathcal{E}_1} n_1 + \sqrt{\mathcal{E}_2} n_2 > \mathcal{E}_1 + \mathcal{E}_2 + 2\sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho] \\
 &= Q \left(\sqrt{\frac{2\mathcal{E}_1 + \mathcal{E}_2 + 2\sqrt{\mathcal{E}_1 \mathcal{E}_2} \rho}{N_0}} \right)
 \end{aligned}$$

$$\begin{aligned}
P[-+ \rightarrow +-] &\leq P[S(-+) < S(+ -)|(-+) \text{ transmitted}] \\
&= P[\sqrt{\mathcal{E}_1}n_1 - \sqrt{\mathcal{E}_2}n_2 > \mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho] \\
&= Q\left(\sqrt{2\frac{\mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right)
\end{aligned}$$

$$\begin{aligned}
P[-+ \rightarrow ++] &\leq P[S(-+) < S(++)|(-+) \text{ transmitted}] \\
&= P[n_1 > \sqrt{\mathcal{E}_1}] \\
&= Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right)
\end{aligned}$$

By adding the four terms we obtain

$$\begin{aligned}
P_1 &\leq Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right) \\
&\quad + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1 + \mathcal{E}_2 + 2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right)
\end{aligned}$$

But we note that if $\rho \geq 0$, the last term is negligible, while if $\rho \leq 0$, then the second term is negligible. Hence, the bound can be written as

$$P_1 \leq Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_2}|\rho|}{N_0}}\right)$$

Problem 16.12

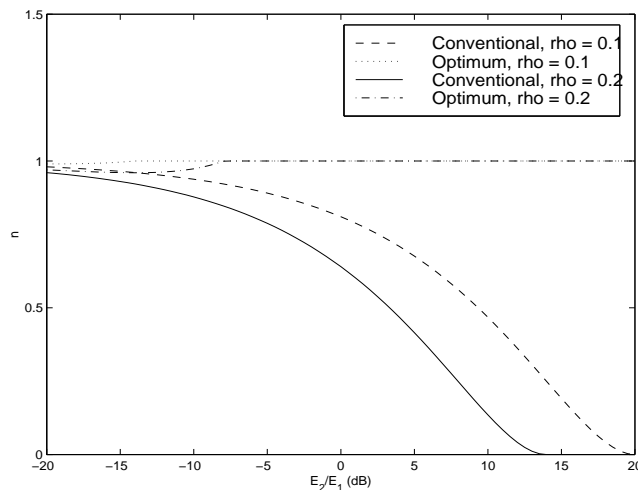
(a) We have seen in Prob. 16.11 that the probability of error for user 1 can be upper bounded by

$$P_1 \leq Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_2}|\rho|}{N_0}}\right)$$

As $N_0 \rightarrow 0$ the probability of error will be dominated by the Q function with the smallest argument. Hence

$$\begin{aligned}
\eta_1 &= \min \left\{ \left(\frac{2\mathcal{E}_1}{N_0}\right) / \left(\frac{2\mathcal{E}_1}{N_0}\right), 2\frac{\mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_2}|\rho|}{N_0} / \left(\frac{2\mathcal{E}_1}{N_0}\right) \right\} \\
&= \min \left\{ 1, 1 + \frac{\mathcal{E}_2}{\mathcal{E}_1} - 2\sqrt{\frac{\mathcal{E}_2}{\mathcal{E}_1}}|\rho| \right\}
\end{aligned}$$

(b) The plot of the asymptotic efficiencies is given in the following figure



We notice the much better performance of the optimal detector especially when the interferer (user 2) is much stronger than the signal. We also notice that the performance of the conventional detector decreases as $|\rho|$ (i.e interference) increases, which agrees with the first observation.

Problem 16.13

The decision rule for the decorrelating detector is $\hat{\mathbf{b}}_2 = \text{sgn}(\mathbf{b}_2^0)$, where \mathbf{b}_2^0 is the output of the decorrelating operation as given by equation (16.3-41). The signal component for the first term in the equation is $\sqrt{\mathcal{E}_1}$. The noise component is

$$n = \frac{n_1 - \rho n_2}{1 - \rho^2}$$

with variance

$$\begin{aligned} \sigma_1^2 = E[n^2] &= \frac{E[n_1 - \rho n_2]^2}{(1 - \rho^2)^2} \\ &= \frac{E[n_1^2] + \rho^2 E[n_2^2] - 2\rho E[n_1 n_2]}{(1 - \rho^2)^2} \\ &= \frac{N_0}{2} \frac{1 + \rho^2}{(1 - \rho^2)^2} \\ &= \frac{N_0}{2} \frac{1}{(1 - \rho^2)} \end{aligned}$$

Hence

$$P_1 = Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}(1 - \rho^2)}\right)$$

Similarly, for the second user

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_2}{N_0}(1 - \rho^2)}\right)$$

Problem 16.14

(a) The matrix \mathbf{R}_s is

$$\mathbf{R}_s = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Hence the linear transformation \mathbf{A}_0 for the two users will be

$$\mathbf{A}_0 = \left(\mathbf{R}_s + \frac{N_0}{2} \mathbf{I} \right)^{-1} = \begin{bmatrix} 1 + \frac{N_0}{2} & \rho \\ \rho & 1 + \frac{N_0}{2} \end{bmatrix}^{-1} = \frac{1}{\left(1 + \frac{N_0}{2}\right)^2 - \rho^2} \begin{bmatrix} 1 + \frac{N_0}{2} & -\rho \\ -\rho & 1 + \frac{N_0}{2} \end{bmatrix}$$

(b) The limiting form of A_0 , as $N_0 \rightarrow 0$ is obviously

$$A_0 \rightarrow \frac{1}{1 - \rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

which is the same as the transformation for the decorrelating detector, as given by expression (16.3-37).

(c) The limiting form of A_0 , as $N_0 \rightarrow \infty$ is

$$A_0 \approx \frac{1}{\left(\frac{N_0}{2}\right)^2} \begin{bmatrix} \frac{N_0}{2} & -\rho \\ -\rho & 1 + \frac{N_0}{2} \end{bmatrix} \rightarrow \frac{1}{\left(\frac{N_0}{2}\right)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is simply a (scaled) form of the conventional single-user detector, since the decision for each user is based solely on the output of the particular user's correlator.

Problem 16.15

(a) The performance of the receivers, when no post-processing is used, is the performance of the conventional multiuser detection.

(b) Since : $y_1(l) = b_1(l)w_1 + b_2(l)\rho_{12}^{(1)} + b_2(l-1)\rho_{21}^{(1)} + n$, the decision variable $z_1(l)$, for the first user after post-processing, is equal to :

$$z_1(l) = b_1(l)w_1 + n + \rho_{21}^{(1)}e_2(l-1) + \rho_{12}^{(1)}e_2(l)$$

where n is Gaussian with zero mean and variance $\sigma^2 w_1$ and, by definition : $e_2(l) \equiv b_2(l) - \text{sgn}[y_2(l)]$. We note that $e_2(l)$ is not orthogonal to $e_2(l-1)$, in general; however these two quantities are

orthogonal when conditioned on $b_1(l)$. The distribution of $e_2(l-1)$, conditioned on $b_1(l)$ is :

$$P[e_2(l-1) = +2|b_1(l)] = \frac{1}{4}Q\left[\frac{w_2 + \rho_{12}^{(2)} + \rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right] + \frac{1}{4}Q\left[\frac{w_2 - \rho_{12}^{(2)} + \rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right]$$

$$P[e_2(l-1) = -2|b_1(l)] = \frac{1}{4}Q\left[\frac{w_2 - \rho_{12}^{(2)} - \rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right] + \frac{1}{4}Q\left[\frac{w_2 + \rho_{12}^{(2)} - \rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right]$$

$$P[e_2(l-1) = 0|b_1(l)] = 1 - P[e_2(l-1) = 2|b_1(l)] - P[e_2(l-1) = -2|b_1(l)]$$

The distribution of $e_2(l)$, given $b_1(l)$, is similar, just exchange $\rho_{12}^{(2)}$ with $\rho_{21}^{(2)}$. Then, the probability of error for user 1 is :

$$P[\hat{b}_1(l) \neq b_1(l)] = \sum_{\substack{a \in \{-2, 0, 2\} \\ b \in \{-1, +1\} \\ c \in \{-2, 0, 2\}}} \frac{1}{2} P[e_2(l-1) = a|b_1(l) = b] P[e_2(l) = c|b_1(l) = b] \times \\ \times Q\left[\frac{w_1 + (\rho_{12}^{(1)}c + \rho_{21}^{(1)}a)b_1(l)}{\sigma\sqrt{w_1}}\right]$$

The distribution of $e_2(l-1)$, conditioned on $b_1(l)$, when $\sigma \rightarrow 0$ is :

$$P[e_2(l-1) = a|b_1(l)] \approx \frac{1}{4}Q\left[\frac{w_2 - |\rho_{12}^{(2)}| + a\rho_{21}^{(2)}b_1(l)/2}{\sigma\sqrt{w_2}}\right], \quad a = \pm 2$$

$$P[e_2(l-1) = 0|b_1(l)] = 1 - P[e_2(l-1) = 2|b_1(l)] - P[e_2(l-1) = -2|b_1(l)]$$

This distribution may be concisely written as :

$$P[e_2(l-1) = a|b_1(l)] \approx \frac{1}{4}Q\left[\frac{|a|}{2} \frac{w_2 - |\rho_{12}^{(2)}| + \frac{a}{2}\rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right]$$

which is exponentially tight. The limiting form of the probability of error is (dropping constants)

$$P[\hat{b}_1(l) \neq b_1(l)] \approx \sum_{\substack{a \in \{-2, 0, 2\} \\ b \in \{-1, +1\} \\ c \in \{-2, 0, 2\}}} Q\left[\frac{|a|}{2} \frac{w_2 - |\rho_{12}^{(2)}| + \frac{a}{2}\rho_{21}^{(2)}b}{\sigma\sqrt{w_2}}\right] \times \\ \times Q\left[\frac{|c|}{2} \frac{w_2 - |\rho_{21}^{(2)}| + \frac{c}{2}\rho_{12}^{(2)}b}{\sigma\sqrt{w_2}}\right] \times Q\left[\frac{w_1 + (\rho_{12}^{(1)}c + \rho_{21}^{(1)}a)b}{\sigma\sqrt{w_1}}\right]$$

(c) Consider the special case :

$$\begin{aligned} \operatorname{sgn}\left(\rho_{12}^{(1)}\right) &= \operatorname{sgn}\left(\rho_{12}^{(2)}\right) \\ \operatorname{sgn}\left(\rho_{21}^{(1)}\right) &= \operatorname{sgn}\left(\rho_{21}^{(2)}\right) \end{aligned}$$

as would occur for far-field transmission (this case is the most prevalent in practice ; other cases follow similarly). Then, the slowest decaying term corresponds to either :

$$\text{sgn} \left(b\rho_{21}^{(1)} a \right) = \text{sgn} \left(b\rho_{12}^{(1)} c \right) = -1$$

for which the resulting term is :

$$Q \left[\sqrt{\frac{w_1}{\sigma^2}} \sqrt{2} \left\{ \sqrt{\frac{w_2}{w_1}} - \frac{|\rho_{12}^{(2)}| + |\rho_{21}^{(2)}|}{\sqrt{w_1} \sqrt{w_2}} \right\} \right] \cdot Q \left[\sqrt{\frac{w_1}{\sigma^2}} \left\{ 1 - 2 \frac{|\rho_{12}^{(1)}| + |\rho_{21}^{(1)}|}{w_1} \right\} \right]$$

or the case when either ma or $c = 0$. In this case the term is :

$$Q \left[\sqrt{\frac{w_1}{\sigma^2}} \left\{ \sqrt{\frac{w_2}{w_1}} - \frac{|\rho_{12}^{(2)}| + |\rho_{21}^{(2)}|}{\sqrt{w_1} \sqrt{w_2}} \right\} \right] \cdot Q \left[\sqrt{\frac{w_1}{\sigma^2}} \left\{ 1 - 2 \frac{\max(|\rho_{12}^{(1)}|, |\rho_{21}^{(1)}|)}{w_1} \right\} \right]$$

or the case when $a = c = 0$ for which the term is :

$$Q \left[\sqrt{\frac{w_1}{\sigma^2}} \right]$$

Therefore, the asymptotic efficiency of this detector is :

$$\eta_1 = \min \left[\begin{array}{l} 1, \max^2 \left\{ 0, \sqrt{\frac{w_2}{w_1}} - \frac{|\rho_{12}^{(2)}| + |\rho_{21}^{(2)}|}{\sqrt{w_1} \sqrt{w_2}} \right\} + \max^2 \left\{ 0, 1 - 2 \frac{\max(|\rho_{12}^{(1)}|, |\rho_{21}^{(1)}|)}{w_1} \right\}, \\ 2 \max^2 \left\{ 0, \sqrt{\frac{w_2}{w_1}} - \frac{|\rho_{12}^{(2)}| + |\rho_{21}^{(2)}|}{\sqrt{w_1} \sqrt{w_2}} \right\} + \max^2 \left\{ 0, 1 - 2 \frac{\max(|\rho_{12}^{(1)}|, |\rho_{21}^{(1)}|)}{w_1} \right\} \end{array} \right]$$

Problem 16.16

(a) The normalized offered traffic per user is : $G_{user} = \lambda \cdot T_p = \left(\frac{1}{60} \text{ pack/sec} \right) \cdot \left(\frac{100}{2400} \text{ sec} \right) = 1/1440$. The maximum channel throughput S_{\max} occurs when $G_{\max} = 1/2$; hence, the number of users that will produce the maximum throughput for the system is : $G_{\max}/G_{user} = 720$.

(b) For slotted Aloha, the maximum channel throughput occurs when $G_{\max} = 1$; hence, the number of users that will produce the maximum throughput for the system is : $G_{\max}/G_{user} = 1440$.

Problem 16.17

A , the average normalized rate for retransmissions, is the total rate of transmissions (G) times the probability that a packet will overlap. This last probability is equal to the probability that another

packet will begin from T_p seconds before until T_p seconds after the start time of the original packet. Since the start times are Poisson-distributed, the probability that the two packets will overlap is $1 - \exp(-2\lambda T_p)$. Hence,

$$A = G(1 - e^{-2G}) \Rightarrow G = S + G(1 - e^{-2G}) \Rightarrow S = Ge^{-2G}$$

Problem 16.18

(a) Since the number of arrivals in the interval T , follows a Poisson distribution with parameter λT , the average number of arrivals in the interval T , is $E[k] = \lambda T$.

(b) Again, from the well-known properties of the Poisson distribution : $\sigma^2 = (\lambda T)^2$.

(c)

$$P(k \geq 1) = 1 - P(k = 0) = 1 - e^{-\lambda T}$$

(d)

$$P(k = 1) = \lambda T e^{-\lambda T}$$

Problem 16.19

(a) Since the average number of arrivals in 1 sec is $E[k] = \lambda T = 10$, the average time between arrivals is 1/10 sec.

(b)

$$P(\text{at least one arrival within 1 sec}) = 1 - e^{-10} \approx 1$$

$$P(\text{at least one arrival within 0.1 sec}) = 1 - e^{-1} = 0.63$$

Problem 16.20

(a) The throughput S and the normalized offered traffic G are related as $S = Ge^{-G} = 0.1$. Solving numerically for G , we find $G = 0.112$.

(b) The average number of attempted transmissions to send a packet, is : $G/S = 1.12$.

Problem 16.21

(a)

$$\tau_d = (2 \text{ km}) \cdot \left(5 \frac{\mu\text{s}}{\text{km}}\right) = 10 \mu\text{s}$$

(b)

$$T_p = \frac{1000 \text{ bits}}{10^7 \text{ bits/sec}} = 10^{-4} \text{ s}$$

(c)

$$a = \frac{\tau_d}{T_p} = \frac{1}{10}$$

Hence, a carrier-sensing protocol yields a satisfactory performance.

(d) For non-persistent CDMA :

$$S = \frac{Ge^{-aG}}{G(1+2a) + e^{-aG}}$$

The maximum bus utilization occurs when :

$$\frac{dS}{dG} = 0$$

Differentiating the above expression with respect to G , we obtain :

$$e^{-aG} - aG^2(1+2a) = 0$$

which, when solved numerically, gives : $G_{\max} = 2.54$. Then , the maximum throughput will be :

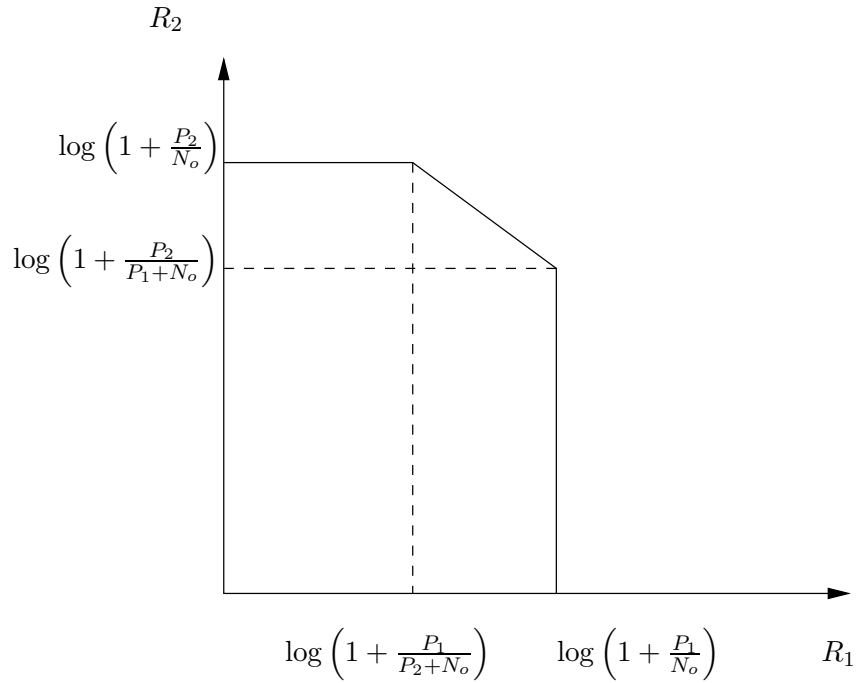
$$S_{\max} = \frac{Ge^{-aG}}{G(1+2a) + e^{-aG}} = 0.515$$

and the maximum bit rate :

$$S_{\max} \cdot 10^7 \text{ bits/sec} = 5.15 \text{ Mbits/sec}$$

Problem 16.22

The capacity region for $K = 2$ users is shown below:



(a)

$$\begin{aligned}
 R_1 &= \log \left(1 + \frac{P_1}{N_o} \right) \\
 R_2 &= \log \left(1 + \frac{P_1 + P_2}{N_o} \right) - \log \left(1 + \frac{P_1}{N_o} \right) = \log \left(1 + \frac{P_2}{P_1 + N_o} \right)
 \end{aligned}$$

(b) $P_1 = 10P_2$. Then,

$$\begin{aligned}
 R_1 &= \log \left(1 + \frac{P_1}{N_o} \right) \\
 R_2 &= \log \left(1 + \frac{P_2}{10P_2 + N_o} \right) \\
 R_1 + R_2 &= \log \left(1 + \frac{10P_2}{N_o} \right) + \log \left(1 + \frac{P_2}{10P_2 + N_o} \right) \\
 &= \log(1 + 10\gamma_2) \left(1 + \frac{\gamma_2}{1 + 10\gamma_2} \right), \quad \gamma_2 = P_2/N_o \\
 &= \log(1 + 11\gamma_2)
 \end{aligned}$$

(c)

$$\begin{aligned}
R_1 &= \log \left(1 + \frac{P_1}{P_2 + N_o} \right) \\
R_2 &= \log \left(1 + \frac{P_2}{N_o} \right) \\
R_1 + R_2 &= \log \left(1 + \frac{10P_2}{P_2 + N_o} \right) + \log \left(1 + \frac{P_2}{N_o} \right) \\
&= \log \left(\frac{1 + 11\gamma_2}{1 + \gamma_2} \right) (1 + \gamma_2) = \log(1 + 11\gamma_2)
\end{aligned}$$

Therefore, the sum rate is the same as in (b).