Second Order Examples

Eg $f(x) = \frac{1}{2} x^T P x + q^T x + r$ where $P \in S^n$

$$= \frac{1}{2} \sum_{i,j} P_{ij} x_i x_j + \sum_i q_i x_i + r$$

$$\frac{\partial f}{\partial x_i \partial x_j} = \frac{P_{ij}}{2} + \frac{P_{ji}}{2} = P_{ij}$$

$$\Rightarrow \nabla^2 f(\underline{x}) = P \qquad \text{(constant)}$$

$$f \text{ convex} \iff P \succeq 0$$

Eg 1: $f(x) = x \log x$, $x > 0$

$$\frac{df}{dx} = \log x + 1 \qquad \frac{d^2 f}{dx^2} = \frac{1}{x} > 0 \quad \forall\, x > 0$$

$$\Rightarrow f \text{ convex}$$

Eg 2: $H(x) = -x \log x - (1-x) \log(1-x)$ $\quad x \in (0,1)$

$$\frac{d^2 H}{dx^2} = -\frac{1}{x} - \frac{1}{1-x} < 0 \Rightarrow H \text{ concave}$$

Eg 3: $f(\underline{x}) = \log\left(\sum_{i=1}^{n} e^{x_i}\right)$

log-sum-exp.
(Geometric Programs)

$z_i = e^{x_i} \qquad f(x) = \log\left(\sum z_i\right)$

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial z_i} \underbrace{\frac{\partial z_i}{\partial x_i}}_{e^{x_i} = z_i} = \frac{z_i}{\sum_i z_i}$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial z_i}\left(\frac{\partial f}{\partial x_i}\right)\frac{\partial z_i}{\partial x_i}$$

$$= -\frac{z_i^2}{\left(\sum_k z_k\right)^2} + \frac{z_i}{\left(\sum z_k\right)}$$

Likewise $i \neq j$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial z_j}\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial z_j}{\partial x_j}\right) = \frac{-z_i z_j}{\left(\sum z_k\right)^2} \qquad i \neq j$$

How to verify $\nabla^2 f(x) \succeq 0 \quad \forall x$?

– cannot calculate eigenvalues

Recall: A P.S.D. $\iff$ $u^T A u \geq 0 \; \forall\, u \in \mathbb{R}^n$

$$u^T \nabla^2 f(x) u = \sum_{i,j} [\nabla^2 f]_{ij} u_i u_j$$

$$= \underbrace{\sum_{i \neq j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) u_i u_j}_{\text{off-diagonal}} + \underbrace{\sum_{i=1}^{n} u_i^2 \frac{\partial^2 f}{\partial x_i^2}}_{\text{diagonal}}$$

$$= -\sum_{i \neq j} \frac{u_i u_j z_i z_j}{\left(\sum_{k=1}^n z_k\right)^2} + \sum_{i=1}^{n} \frac{u_i^2 z_i}{\left(\sum_{k=1}^n z_k\right)} - \sum_{i=1}^{n} \frac{u_i^2 z_i^2}{\left(\sum_{k=1}^n z_k\right)^2}$$

$$-\frac{1}{\left(\sum_{k=1}^n z_k\right)^2} \left[\sum_{i,j} u_i u_j z_i z_j - \left(\sum_{k=1}^n z_k\right)\left(\sum_{i=1}^n u_i^2 z_i\right) \right]$$

Now use Cauchy-Schwarz Inequality

$$(\underline{a}^T \underline{b})^2 \leq (\underline{a}^T \underline{a})(\underline{b}^T \underline{b})$$

what should be a & b?

$$(a^T b)^2 = \left(\sum_i a_i b_i\right)^2 = \sum_{i,j} a_i a_j b_i b_j = \left(\sum u_i z_i\right)^2$$

$a_i b_i = u_i z_i$ $\quad a_i^2 = z_i$ $\quad b_i^2 = u_i^2 z_i$

$\quad\quad\quad\quad a_i = \sqrt{z_i}$ $\quad b_i = u_i \sqrt{z_i}$

$\Rightarrow \quad \nabla^2 f(x) \succeq 0 \quad$ P.S.D $\quad \Rightarrow \quad f$ convex

Key points : (a) calculate $u^T \nabla^2 f(x) u$

(b) use Cauchy-Schwarz Inequality

H.W. show that $\quad g(\underline{x}) = \left(\prod_{i=1}^{n} x_i \right)^{1/n}$ concave

Geometric Mean