Optimality Conditions

$$\min f(x)$$
$$x \in \mathcal{X} : \{x \mid g_i(x) \le 0, Ax = b\} \quad \text{(convex)}$$

what properties does x^* satisfy?

first order: $f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle \quad x, y \in \mathcal{X}$

suppose for some y: $\langle \nabla f(y), x - y \rangle \geq 0 \quad \forall x \in \mathcal{X}$

then: $f(x) \geq f(y) \quad \forall x \in \mathcal{X}$

or $y = \underset{x \in \mathcal{X}}{\arg\min} f(x) = x^*$

$$\boxed{\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \mathcal{X}}$$

Eg: $\mathcal{X} = \mathbb{R}^n$

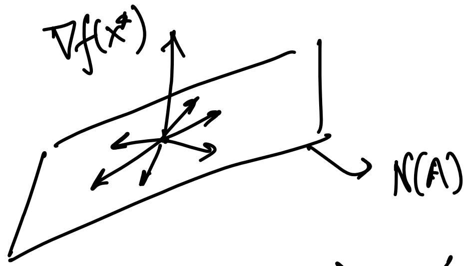
$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow \nabla f(x^*) = 0$

Eg $\min\limits_{x \in \mathcal{X} = \{x \mid Ax = b\}} f(x)$

$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \mathcal{X}$

note: $x \in \mathcal{X} \Rightarrow Ax = b$
$x^* \in \mathcal{X} \Rightarrow Ax^* = b$] $A(x - x^*) = 0$
or $x - x^* \in N(A)$
$\forall \; x \in \mathcal{X}$



$\Rightarrow \langle \nabla f(x^*), x - x^* \rangle = 0$

$\Rightarrow \nabla f(x^*) \in N(A)^{\perp} = \mathcal{R}(A^T)$
or $\exists v: \quad \nabla f(x^*) = A^T v$

Eg $\min \; f_1(x_1) + f_2(x_2)$
$x_1 + x_2 = 1$

$A = [1 \;\; 1]$
$A^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\nabla f(x) = \begin{bmatrix} f_1'(x_1) \\ f_2^2(x_2) \end{bmatrix}$

$$\begin{bmatrix} f_1'(x_1^*) \\ f_2'(x_2^*) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v \qquad v \in \mathbb{R}$$

or (1) $f_1'(x_1^*) = f_2'(x_2^*) = v^{\in \mathbb{R}}$ first order condition
(2) $x_1^* + x_2^* = 1$ feasibility