Solutions of Tutorial-7

Problem set 7.2

1 $A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 0 and 0; $A^{\mathrm{T}}A = \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$ has eigenvalues $\lambda = 16$ and 0. Then $\sigma_1(A) = \sqrt{16} = 4$. The eigenvectors of $A^{\mathrm{T}}A$ and AA^{T} are the columns of $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $U\Sigma V^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = A$.

$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ gives $A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$ with $\lambda_1 = 16$ and $\lambda_2 = 1$. Same U and V.

Then $U\Sigma V^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = A$.

7 This small question is a key to everything. It is based on the associative law $(AA^{\mathrm{T}})A = A(A^{\mathrm{T}}A)$. Here we are applying both sides to an eigenvector v of $A^{\mathrm{T}}A$:

$$(AA^{\mathrm{T}})Av = A(A^{\mathrm{T}}A)v = A\lambda v = \lambda Av.$$

So **Av is an eigenvector of AA^{T}** with the same eigenvalue λ.

14 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

Problem set 9.2

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}Az = \mathbf{0}$ then $(z^{\mathrm{H}}A^{\mathrm{H}})(Az) = 0$. This is $\|Az\|^2 = 0$ so $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $S = S^{\mathrm{H}}$.

11 If $Q^{\mathrm{H}}Q = I$ then $Q^{-1}(Q^{\mathrm{H}})^{-1} = Q^{-1}(Q^{-1})^{\mathrm{H}} = I$ so Q^{-1} is also unitary. Also $(QU)^{\mathrm{H}}(QU) = U^{\mathrm{H}}Q^{\mathrm{H}}QU = U^{\mathrm{H}}U = I$ so QU is unitary.

13 $(z^{\mathrm{H}}A^{\mathrm{H}})(Az) = \|Az\|^2$ is positive unless $Az = \mathbf{0}$. When A has independent columns this means $z = \mathbf{0}$; so $A^{\mathrm{H}}A$ is positive definite.