EE910: Digital Communication Systems-I

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Lecture #8B: Optimum Receivers for CPM Signals

Optimum Receiver for CPM Signals

- The transmitted CPM signal may be expressed as

$$s(t) = \sqrt{\frac{2\mathcal{E}}{T}} cos[2\pi f_c t + \phi(t; I)] \tag{1}$$

- The filtered received signal for an additive Gaussian noise channel is

$$r(t) = s(t) + n(t) \tag{2}$$

where

$$n(t) = n_i(t) cos(2\pi f_c t) - n_q(t) \sin(2\pi f_c t) \tag{3}$$

Optimum Demodulation and Detection of CPM

- The optimum receiver for CPM signal consists of a correlator followed by a maximum likelihood (ML) sequence detector.
- The ML sequence detector searches all the paths through the state trellis for minimum Euclidean distance path.
- Viterbi algorithm is used for performing this search.

Optimum Demodulation and Detection of CPM

- The carrier phase for a CPM signal with a fixed modulation index h may be expressed as

$$\begin{aligned}\phi(t;\mathbf{I}) &= 2\pi h \sum_{k=-\infty}^{n} I_k q(t-KT) \\ &= \pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^{n} I_k q(t-KT) \\ &= \theta_n + \theta(t;I), \qquad nT \le t \le (n+1)T\end{aligned} \tag{4}$$

where we have assumed that q(t) = 0 for $t < 0$, q(t) = $\frac{1}{2}$ for $t \ge LT$, and

$$q(t) = \int_0^t g(\tau)d\tau \tag{5}$$

Optimum Demodulation and Detection of CPM

- Now, when h is rational, i.e., h = m/p where m and p are relatively prime positive integers, the CPM scheme can be represented by a trellis. In this case, there are p phase states

$$\Theta_s = \left\{ \ 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \ldots\ldots, \frac{(p-1)\pi m}{p} \ \right\} \tag{6}$$

when m is even, and 2p phase states

$$\Theta_s = \left\{ \ 0, \frac{\pi m}{p}, \ldots\ldots, \frac{(2p-1)\pi m}{p} \ \right\} \tag{7}$$

- On the other hand, if L$>$ 1, we have an additional number of states due to the partial response character of the signal pulse g(t).

$$\theta(t;\mathbf{I}) = 2\pi h \sum_{k=n-L+1}^{n-1} I_k q(t-kT) + 2\pi h I_n q(t-KT) \tag{8}$$

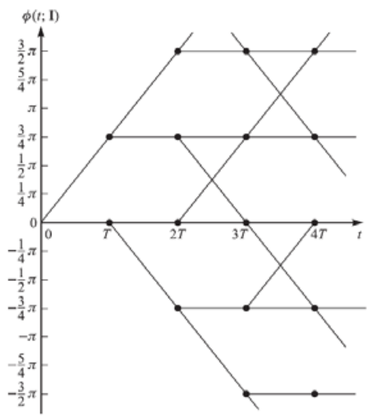
Optimum Demodulation and Detection of CPM

- The state of the CPM signal (or the modulator) at time t = nT may be expressed as the combined phase state and correlative state, denoted as
$$S_n = \{\theta_n, I_{n-1}, I_{n-2}, ..., I_{n-L+1}\} \tag{9}$$
- For a partial response signal pulse of length LT, where L > 1. In this case, the number of states is
$$N_s = \begin{cases} pM^{L-1} & (even \quad m) \\ 2pM^{L-1} & (odd \quad m) \end{cases} \tag{10}$$
when $h = m/p$.

Optimum Demodulation and Detection of CPM

- Now, suppose the state of the modulator at t = nT is S_n. The effect of the new symbol in the time interval $nT \leq t \leq (n+1)T$ is to change the state from S_n to S_{n+1}. Hence, at t = (n + 1)T , the state becomes
$$S_{n+1} = \{\theta_{n+1}, I_n, I_{n-1}, ..., I_{n-L+2}\} \tag{11}$$
where
$$\theta_{n+1} = \theta_n + \pi h I_{n-L+1} \tag{12}$$

Optimum Demodulation and Detection of CPM



Phase tree for L=2 partial response CPM with h=$\frac{3}{4}$.

Optimum Demodulation and Detection of CPM

- For CPM signals, the logarithm of the probability of the observed signal r(t) conditioned on a particular sequence of transmitted symbols $\mathbf{I}$ is proportional to the cross-correlation metric

$$\begin{aligned} CM_n(\mathbf{I}) &= \int_{-\infty}^{(n+1)T} r(t)\cos[w_c t + \phi(t;\mathbf{I})]dt \qquad (13) \\ &= CM_{n-1}(\mathbf{I}) + \int_{nT}^{(n+1)T} r(t)\cos[w_c t + \phi(t;\mathbf{I})]dt \end{aligned}$$

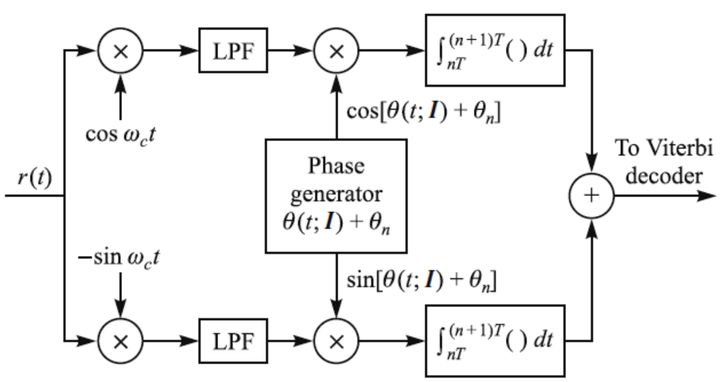
Optimum Demodulation and Detection of CPM

- The term $CM_{n-1}(\mathbf{I})$ represents the metrics for the surviving sequences up to time nT, and the term

$$\nu_n(\mathbf{I};\theta_n) = \int_{nT}^{(n+1)T} r(t)\cos[w_c t + \theta(t;\mathbf{I}) + \theta_n]dt \qquad (14)$$

represents the additional increments to the metrics contributed by the signal in the time interval $nT \leq t \leq (n+1)T$.

Performance of CPM Signals



Performance of CPM Signals

- There are pM^L(or $2pM^L$) different values of $\nu_n(\mathbf{I}, \theta_n)$ computed in each signal interval.
- Each value is used to increment the metrics corresponding to pM^{L-1}(or $2pM^{L-1}$) surviving sequences from the previous signaling interval.
- The number of surviving sequences at each state of the Viterbi decoding process is pM^{L-1} (or $2pM^{L-1}$).
- For each surviving sequence, we have M new increments of $\nu_n(\mathbf{I}, \theta_n)$ that are added to the existing metrics to yield pM^L(or $2pM^L$) sequences with pM^L(or $2pM^L$) metrics.

Performance of CPM Signals

- This number is then reduced back to pM^{L-1}(or $2pM^{L-1}$) survivors with corresponding metrics by selecting the most probable sequence of the M sequences merging at each node of the trellis and discarding the other $M-1$ sequences.
- In evaluating the performance of CPM signals achieved with ML sequence detection, we must determine the minimum Euclidean distance of paths through the trellis that separate at the node at $t = 0$ and remerge at a later time at the same node.

Performance of CPM Signals

- The Euclidean distance between the two signals over an interval of length NT, where 1/T is the symbol rate, is defined as,

$$
\begin{aligned}
d_{ij}{}^2 &= \int_0^{NT} [s_i(t) - s_j(t)]^2 dt \\
&= \int_0^{NT} s_i(t)^2 dt + \int_0^{NT} s_j(t)^2 dt - 2\int_0^{NT} s_i(t)s_j(t)dt \\
&= 2N\mathcal{E} - 2\frac{2\mathcal{E}}{T}\int_0^{NT} cos[\omega_c t + \phi(t; I_i)]cos[\omega_c t + \phi(t; I_j)]dt \qquad (15) \\
&= 2N\mathcal{E} - \frac{2\mathcal{E}}{T}\int_0^{NT} cos[\phi(t; I_i) - \phi(t; I_j)]dt \\
&= \frac{2\mathcal{E}}{T}\int_0^{NT} \{1 - cos[\phi(t; I_i) - \phi(t; I_j)]\}dt
\end{aligned}
$$

Performance of CPM Signals

- It is desirable to express the distance $\delta_{ij}{}^2$ in terms of the bit energy. Since $\mathcal{E} = \mathcal{E}_b log_2 M$, Equation (15) becomes

$$d_{ij}{}^2 = 2\mathcal{E}_b \delta_{ij}{}^2 \qquad (16)$$

where $\delta_{ij}{}^2$ is defined as

$$\delta_{ij}{}^2 = \frac{log_2 M}{T}\int_0^{NT} \{1 - cos[\phi(t; I_i) - \phi(t; I_j)]\}dt \qquad (17)$$

- Furthermore, we observe that

$$\phi(t; I_i) - \phi(t; I_j) = \phi(t; I_i - I_j), \quad with\ \xi = I_i - I_j \qquad (18)$$

- Thus we have

$$\delta_{ij}{}^2 = \frac{log_2 M}{T}\int_0^{NT} \{1 - cos[\phi(t; \xi)]\}dt \qquad (19)$$

Performance of CPM Signals

- The error rate performances for CPM is dominated by the term corresponding to the minimum Euclidean distance, and it may be expressed as

$$P_m = K_{\delta_{\min}} Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}\delta^2_{\min}}\right) \tag{20}$$

where $K_{\delta_{\min}}$ is the number of paths having the minimum distance

- We have

$$\begin{aligned}\delta^2_{\min} &= \lim_{N\to\infty} \min_{i,j} {\delta_{ij}}^2 \\ &= \lim_{N\to\infty} \min_{i,j} {\delta_{ij}}^2 \quad \left\{\frac{log_2 M}{T}\int_0^{NT}\{1 - cos[\phi(t;\xi)]\}dt\right\}\end{aligned} \tag{21}$$

- Note that for conventional binary PSK with no memory, N = 1 and $\delta^2_{\min} = {\delta_{12}}^2 = 2$

Performance of CPM Signals

- Since $\delta^2_{\min}$ characterizes the performance of CPM, we investigate the effect on $\delta^2_{\min}$ resulting from varying the alphabet size M, the modulation index h, and the length of the transmitted pulse in partial response CPM.
- First, we consider full response (L = 1) CPM. If we take M = 2 we note that the sequences

$$\begin{aligned}\mathbf{I}_i &= +1, -1, I_2, I_3 \\ \mathbf{I}_j &= -1, +1, I_2, I_3\end{aligned} \tag{22}$$

which differ for k = 0, 1 and agree for k $\geq$ 2, result in two phase trajectories that merge after the second symbol.

- This corresponds to the difference sequence

$$\xi = \{2, -2, 0, 0, 0.....\} \tag{23}$$

Performance of CPM Signals

- The Euclidean distance for this sequence is easily calculated from Equation (19), and provides an upper bound on $\delta^2_{\min}$.
- This upper bound for CPFSK with M = 2 is

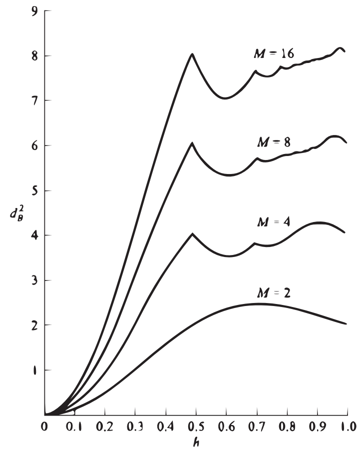
$$d_B{}^2(h) = 2\Big(1 - \frac{\sin 2\pi h}{2\pi h}\Big), \qquad M = 2 \tag{24}$$

- For $M \geq 2$ and full response CPM, the phase trajectories merge at t = 2T.

Performance of CPM Signals

- Hence, an upper bound on $\delta^2_{\min}$ can be obtained by considering the phase difference sequence
$\xi = \{\alpha, -\alpha, 0, 0, \cdots\}$ where $\alpha = \pm 2, \pm 4, \cdots, \pm 2(M-1)$
- This sequence yields the upper bound for M-ary CPFSK as

$$d_B{}^2(h) = \min_{1 \leq k \leq M-1} 2 log_2\, M\Big(1 - \frac{\sin 2k\pi h}{2k\pi h}\Big), \tag{25}$$



Upper bound $d_B{}^2(h)$ versus h for M = 2, 4, 8, 16 for full response CPM with rectangular pulses

- Large performance gains can also be achieved with maximum-likelihood sequence detection of CPM by using partial response signals.
- For example, for partial response, raised cosine pulses given by

$$g(t) = \begin{cases} \frac{1}{2LT}\left(1 - cos\frac{2\pi t}{2LT}\right) & 0 \leq t \leq LT \\ 0 & otherwise \end{cases} \quad (26)$$

  - As L increases, the distance bound $d_B{}^2(h)$ also achieves higher values.
  - The performance of CPM improves as the correlative memory L increases, but h must also be increased in order to achieve the larger values of $d_B{}^2(h)$.

Performance of CPM Signals

- Upper bound ${d_B}^2(h)$ on the minimum distance for partial response (raised cosine pulse) binary CPM.