EE910: Digital Communication Systems-I

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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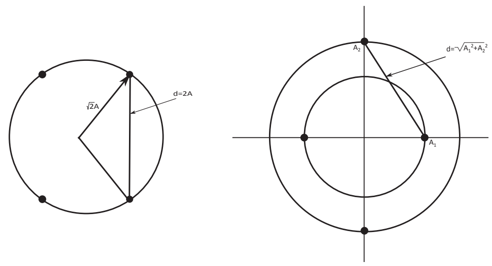
Lecture #6B: Optimal Detection and Error Probability for QAM Signalling

Optimal Detection and Error Probability for QAM Signalling

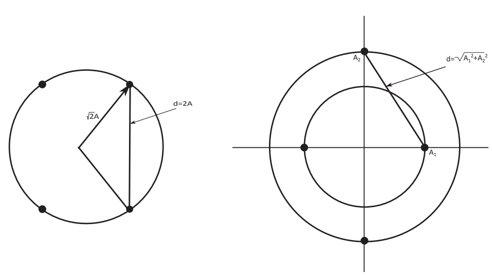
- To determine the probability of error for QAM, we must specify the signal point constellation.
- Consider QAM signal sets that have $M = 4$ points as shown in figure.



(a) (b)

Optimal Detection and Error Probability for QAM Signalling

- Consider QAM signal sets that have $M = 4$ points as shown in figure.



(a) (b)

- The first is a four-phase modulated signal, and the second is a QAM signal with two amplitude levels, labelled A_1 and A_2, and four phases.

Optimal Detection and Error Probability for QAM Signalling

- Impose the condition that $d_{min} = 2A$ for both signal constellations.
- Let us evaluate the average transmitted power, based on the premise that all signal points are equally probable.
- For the four-phase signal,we have

$$\mathcal{E}_{avg} = 2A^2 \tag{1}$$

Optimal Detection and Error Probability for QAM Signalling

- For the two-amplitude, four-phase QAM, we place the points on circles of radii A and $\sqrt{3}A$. Thus, $d_{min} = 2A$, and

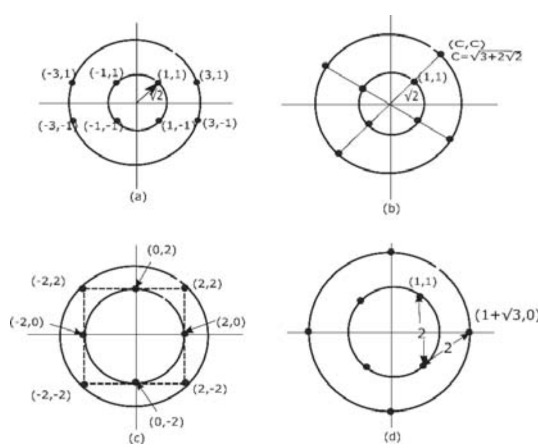
$$\mathcal{E}_{avg} = \frac{1}{4}\left[2(3A^2) + 2A^2\right] = 2A^2 \tag{2}$$

which is the average power as the $M = 4$ phase signal constellation.
- Hence, the error rate performance of the two signal sets is the same.
- There is no advantage of the two-amplitude QAM signal set over $M = 4$ phase modulation.

Optimal Detection and Error Probability for QAM Signalling

- Let us consider $M = 8QAM$. In this case, there are many possible signal constellations.
- We shall consider the four signal constellations shown in figure (next page) all of which consist of two amplitudes and have a minimum distance between signal points of 2A.
- The coordinates (A_{mc}, A_{ms}) for each signal point, normalized by A, are shown in figure.

Optimal Detection and Error Probability for QAM Signalling



Optimal Detection and Error Probability for QAM Signalling

- Assuming that the signal points are equally probable, the average transmitted signal energy is

$$\begin{aligned}\mathcal{E}_{avg} &= \frac{1}{M}\sum_{m=1}^{M}\left(A_{mc}^2 + A_{ms}^2\right) \\ &= \frac{A^2}{M}\sum_{m=1}^{M}\left(a_{mc}^2 + a_{ms}^2\right)\end{aligned} \tag{3}$$

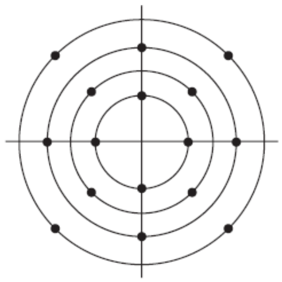
where (a_{mc}, a_{ms}) are the coordinates of the signal points, normalized by A.

Optimal Detection and Error Probability for QAM Signalling

- The two signal sets (a) and (c) in the figure contain signal points that fall on a rectangular grid and have $\mathcal{E}_{avg} = 6A^2$.The signal set (b) requires an average transmitted energy $\mathcal{E}_{avg} = 6.83A^2$, and (d) requires $\mathcal{E}_{avg} = 4.73A^2$
- The fourth signal set requires approximately 1 dB less energy than the first two and 1.6dB less energy than the third, to achieve the same probability of error.
- This signal constellation is known to be the best eight-point QAM constellation because it requires the least power for a given minimum distance between signal points.

Optimal Detection and Error Probability for QAM Signalling

- For $M \geq 16$, there are many more possibilities for selecting the QAM signal points in two-dimensional space.
- For example, we may choose a circular multi amplitude constellation for $M = 16$, as shown in figure



$M = 16$

Optimal Detection and Error Probability for QAM Signalling

- In this case, the signal points at a given amplitude level are phase-rotated by $\frac{1}{4}\pi$ relative to the signal points at adjacent amplitude levels.
- The circular 16-QAM constellation is not the best 16-point QAM signal constellation for the AWGN channel.

Optimal Detection and Error Probability for QAM Signalling

- Rectangular QAM signal constellations have the distinct advantage of being easily generated as two PAM signals impressed on the in-phase and quadrature carriers.
- Although they are not the best M-ary QAM signal constellations for $M \geq 16$, the average transmitted power required to achieve a given minimum distance is only slightly greater than the average required power for the best M-ary QAM signal constellation.
- Thus, rectangular M-ary QAM signals are most frequently used in practice.

Optimal Detection and Error Probability for ASK or PAM Signalling

- In the special case where k is even and the constellation is square, it is possible to derive an exact expression for the error probability.
- The minimum distance of this constellation is given by

$$d_{min} = \sqrt{\frac{6\log_2 M}{M-1}\mathcal{E}_{bavg}} \tag{4}$$

- This constellation can be considered as two $\sqrt{M}$-ary PAM constellations in the in-phase and quadrature directions.

Optimal Detection and Error Probability for ASK or PAM Signalling

- An error occurs if either n_1 or n_2 is large enough to cause an error in one of the two PAM signals.
- The probability of a correct detection for this QAM constellation is therefore the product of correct decision probabilities for constituent PAM systems, i.e.,

$$P_{c,M-QAM} = P^2_{c,\sqrt{M}-PAM} = \left(1 - P_{e,\sqrt{M}-PAM}\right)^2 \tag{5}$$

Optimal Detection and Error Probability for QAM Signalling

- This results in

$$\begin{aligned} P_{e,M-QAM} &= 1 - \left(1 - P_{e,\sqrt{M}-PAM}\right)^2 \\ &= 2P_{e,\sqrt{M}-PAM}\left(1 - \frac{1}{2}P_{e,\sqrt{M}-PAM}\right) \end{aligned} \tag{6}$$

- From equation

$$\begin{aligned} P_e &= \frac{1}{M}\sum_{m=1}^{M} P[error|m\ sent] \\ &= \frac{1}{M}\left[2(M-2)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) + 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)\right] \\ &= \frac{2(M-1)}{M}Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \end{aligned} \tag{7}$$

we have

$$P_{e,\sqrt{M}-PAM} = 2\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \tag{8}$$

Optimal Detection and Error Probability for QAM Signalling

- Substituting the value for d_{min} from equation (4), we get

$$P_{e,\sqrt{M}-PAM} = 2\Big(1 - \frac{1}{\sqrt{M}}\Big) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \tag{9}$$

Optimal Detection and Error Probability for QAM Signalling

- Substituting Equation (9) into Equation (6) yields

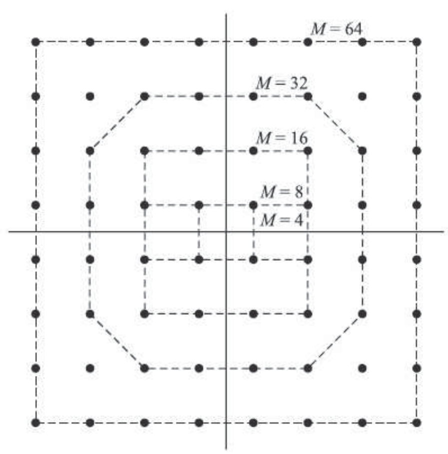
$$\begin{aligned} P_{e,M-QAM} &= 4\big(1 - \frac{1}{\sqrt{M}}\big) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \\ &\quad x\left(1 - \Big(1 - \frac{1}{\sqrt{M}}\Big) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\right) \\ &\leq 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \end{aligned} \tag{10}$$

Optimal Detection and Error Probability for QAM Signalling

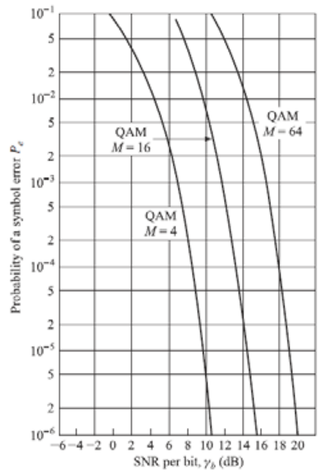
- For large M and moderate to high SNR per bit, the upper bound given by above equation is quite tight.
- Although above equation is obtained for square constellations, for large M it gives a good approximation for general QAM constellations with $M = 2^k$ points which are either in the shape of a square (when k is even) or in the shape of a cross (when k is odd).

Optimal Detection and Error Probability for QAM Signalling

- These types of constellations are illustrated in the below figure



Error probability of M-ary QAM as a function of SNR per bit



Optimal Detection and Error Probability for QAM Signalling

- Comparing the error performance of M-ary QAM with M-ary ASK and MPSK, we observe that unlike PAM and PSK Signalling in which in the penalty for increasing the rate was 6 dB/bit, in QAM this penalty is 3 dB/bit.
- This shows that QAM is more power efficient compared with PAM and PSK.
- The advantage of PSK is, however, its constant-envelope properties.

Optimal Detection and Error Probability for QAM Signalling

- QPSK can be considered as $4QAM$ with a square constellation. Using Equation (10) with $M = 4$, we obtain

$$
\begin{aligned}
P_4 &= 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right] \\
&\leq 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)
\end{aligned}
\tag{11}
$$

- For 16-QAM with a rectangular constellation we obtain

$$
\begin{aligned}
P_{16} &= 3Q\left(\sqrt{\frac{4}{5}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\left[1 - \frac{3}{4}Q\left(\sqrt{\frac{4}{5}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\right] \\
&\leq 3Q\left(\sqrt{\frac{4}{5}\frac{\mathcal{E}_{bavg}}{N_0}}\right)
\end{aligned}
\tag{12}
$$

Optimal Detection and Error Probability for QAM Signalling

- For nonrectangular QAM signal constellations, we may upper-bound the error probability by use of the union bound as

$$
P_M \leq (M-1)Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) \tag{13}
$$

where d_{min} is the minimum Euclidean distance of the constellation

- This bound may be loose when M is large. In such a case, we may approximate P_M by replacing $M - 1$ by N_{min}, where N_{min} is the largest number of neighbouring points that are at distance d_{min} from any constellation point.

Optimal Detection and Error Probability for QAM Signalling

- It is interesting to compare the performance of QAM with that of PSK for any given signal size M, since both types of signals are two-dimensional.
- For $M-ary$ PSK, the probability of a symbol error is approximated as

$$P_M \approx 2Q\left(\sqrt{(2\log_2 M)sin^2(\frac{\pi}{M})\frac{\mathcal{E}_b}{N_0}}\right) \tag{14}$$

Optimal Detection and Error Probability for QAM Signalling

- For $M-ary$ QAM, we may use the expression (10).

$$\begin{aligned} P_{e,M-QAM} &= 4(1-\frac{1}{\sqrt{M}})Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \\ &\quad x\left(1-\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\right) \\ &\le 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \end{aligned} \tag{15}$$

Optimal Detection and Error Probability for QAM Signalling

- Since the error probability is dominated by the argument of the Q function, we may simply compare the arguments of Q for the two signal formats.
- Thus, the ratio of these two arguments is

$$R_M = \frac{\frac{3}{M-1}}{2sin^2(\frac{\pi}{M})} \tag{16}$$

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Optimal Detection and Error Probability for QAM Signalling

- When M $= 4$, we have $R_M = 1$. Hence, $4 - PSK$ and $4 - QAM$ yield comparable performance for the same SNR per symbol.
- When $M > 4$, we find that $R_M > 1$, so that $M - ary$ QAM yields better performance than $M - ary$ PSK.
- The following table illustrates the SNR advantage of QAM over PSK for several values of M.

M	$10 \log \mathcal{R}_M$
8	1.65
16	4.20
32	7.02
64	9.95

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