eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 9

LMMSE Interpretation

LMMSE Interpretation

- Let us now take a **deeper look** at LMMSE.

$$\min E\left\{\|C\bar{y}-\bar{h}\|^2\right\}.$$

Linear minimum
Mean square error Estimator.

LMMSE

- Recall the LMMSE estimate is

$$\hat{h} = C\bar{y} = R_{hy} R_{yy}^{-1} \bar{y}.$$

LMMSE

$\bar{h}, \bar{y}$ arbitrarily distributed.

Zero mean.

$E\{\bar{h}\} = 0.$ $E\{\bar{y}\} = 0.$

LMMSE

- Recall the LMMSE estimate is

$$\underbrace{\hat{\mathbf{h}} = \mathbf{C}\bar{\mathbf{y}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}}_{\text{LMMSE Estimate}}$$

LMMSE Non-zero Mean

- We first derive the LMMSE estimate for **non-zero mean** parameter/observation

$$\left.\begin{aligned} E\{\bar{h}\} &= \bar{\mu}_h. \\ E\{\bar{y}\} &= \bar{\mu}_y \end{aligned}\right\} \text{Non-zero mean.}$$

LMMSE Non-zero Mean

- Let

$$E\{\bar{\mathbf{h}}\} = \bar{\boldsymbol{\mu}}_h \quad M \times 1$$

$$E\{\bar{\mathbf{y}}\} = \bar{\boldsymbol{\mu}}_y \quad N \times 1$$

LMMSE Non-zero Mean

- Make zero-mean

$$\left\{\begin{array}{ll} \bar{h} - \bar{\mu}_h : & E\{\bar{h} - \bar{\mu}_h\} = 0 \\ \bar{y} - \bar{\mu}_y : & E\{\bar{y} - \bar{\mu}_y\} = 0 \end{array}\right.$$

Zeromean Quantities.

$\Rightarrow$ use previous result.

LMMSE Non-zero Mean

- Make zero-mean

$$E\left\{\bar{\mathbf{h}} - \bar{\boldsymbol{\mu}}_h\right\} = 0$$

$$E\left\{\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right\} = 0$$

LMMSE Non-zero Mean

- The LMMSE estimate for non-zero mean is given as follows

$$\hat{h} - \bar{\mu}_h = R_{hy} R_{yy}^{-1} (\bar{y} - \bar{\mu}_y)$$

$$\Rightarrow \hat{h} = R_{hy} R_{yy}^{-1} (\bar{y} - \bar{\mu}_y) + \bar{\mu}_h.$$

LMMSE for Arbitrarily distributed $\bar{h}, \bar{y}$
Non-zero mean.

LMMSE Non-zero Mean

- The LMMSE estimate for non-zero mean is given as follows

$$\hat{\mathbf{h}} - \bar{\boldsymbol{\mu}}_h = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y)$$

Prior information

$$\hat{\mathbf{h}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y) + \bar{\boldsymbol{\mu}}_h$$

LMMSE Non-zero Mean

- Note

$$\mathbf{R}_{yy} = \overbrace{E\left\{(\bar{y}-\bar{\mu}_y)(\bar{y}-\bar{\mu}_y)^T\right\}}^{\text{covariance matrix}}$$

$$\mathbf{R}_{hy} = \underbrace{E\left\{(\bar{h}-\bar{\mu}_h)(\bar{y}-\bar{\mu}_y)^T\right\}}_{\text{cross covariance matrix}}$$

LMMSE Non-zero Mean

- Note

$$\mathbf{R}_{yy} = E\left\{\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right)\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right)^T\right\}$$

$$\mathbf{R}_{hy} = E\left\{\left(\bar{\mathbf{h}} - \bar{\boldsymbol{\mu}}_h\right)\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right)^T\right\}$$

LMMSE Linear Model

$$\underbrace{\bar{y} = X\bar{h} + \bar{v}}_{\text{Linear model.}}$$

- Therefore, the LMMSE for the linear **MISO channel estimation** model is

With non zero mean.

$$\underbrace{\hat{h} = \left(X^T X + \frac{1}{SNR} I\right) X^T \left(\bar{y} - \bar{\mu}_y\right) + \bar{\mu}_h .}_{\text{LMMSE For MISO with non zero means.}}$$

Linear Parameter estimation model.

LMMSE Linear Model

- Therefore, the LMMSE for the linear **MISO channel estimation** model is

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}\mathbf{X}^T\left(\bar{\mathbf{y}} - \overline{\boldsymbol{\mu}}_y\right) + \overline{\boldsymbol{\mu}}_h$$

LMMSE Linear Model

- Further, $\bar{\boldsymbol{\mu}}_y$ is derived as follows

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

- Since noise is zero-mean, we have $E\{\bar{v}\} = 0$

$$\bar{\boldsymbol{\mu}}_y = E\{\bar{\mathbf{y}}\} = E\{X\bar{h} + \bar{v}\}$$

$$= E\{X\bar{h}\} + \cancel{E\{\bar{v}\}}^{0}$$

$$\bar{\mu}_y = X\bar{\mu}_h$$

LMMSE Linear Model

- Further, $\bar{\boldsymbol{\mu}}_y$ is derived as follows

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

- Since noise is zero-mean, we have

$$\bar{\boldsymbol{\mu}}_y = E\{\bar{\mathbf{y}}\} = E\left\{\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}\right\} = \mathbf{X}\bar{\boldsymbol{\mu}}_h$$

LMMSE Linear Model

- Therefore,

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}\mathbf{X}^T(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y) + \bar{\boldsymbol{\mu}}_h$$

$$= \left(X^T X + \frac{1}{SNR} I\right)^{-1} X^T(\bar{y} - X\bar{\mu}_h) + \bar{\mu}_h$$

LMMSE Linear Model

- Therefore,

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}\mathbf{X}^T(\bar{\mathbf{y}} - \mathbf{X}\overline{\boldsymbol{\mu}}_h) + \overline{\boldsymbol{\mu}}_h$$

LMMSE SISO Model

Single input single output.

- For simplicity consider the SISO Model

$$\text{N outputs.}\left\{\begin{aligned} y(1) &= hx(1) + v(1) \\ y(2) &= hx(2) + v(2) \\ &\vdots \\ y(N) &= hx(N) + v(N) \end{aligned}\right.$$

channel coefficient.

LMMSE SISO Model

- For simplicity consider the SISO Model

$$y(1) = hx(1) + v(1)$$
$$y(2) = hx(2) + v(2)$$
$$\vdots$$
$$y(N) = hx(N) + v(N)$$

Wireless System Model

- The different quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \qquad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Wireless System Model

- The different quantities are

Pilot vector

output vector.

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Wireless System Model

- Therefore,

$$\bar{\mathbf{y}} = \bar{\mathbf{x}}h + \bar{\mathbf{v}}$$

Wireless System Model

- The mean satisfies

$E\{\bar{v}\} = 0.$

$E\{h\} = \mu_h.$

$\bar{\mu}_y = E\{\bar{y}\} = \bar{x}\mu_h.$

$$\bar{\mathbf{y}} = \bar{\mathbf{x}}h + \bar{\mathbf{v}}$$

Mean of $\bar{y}$

Mean of h.

$$\Rightarrow \bar{\boldsymbol{\mu}}_y = \bar{\mathbf{x}}\mu_h$$

LMMSE Linear Model

- Therefore, the LMMSE for the linear **SISO channel estimation** model is

$$\hat{h} = \left(\bar{x}^T\bar{x} + \frac{1}{SNR}1\right)^{-1}\bar{x}^T(\bar{y} - \bar{x}\mu_h) + \mu_h$$

$$= \frac{\bar{x}^T(\bar{y} - \bar{x}\mu_h)}{\|\bar{x}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h.$$

LMMSE Estimate of <u>SISO channel</u>.

LMMSE Linear Model

- Therefore, the LMMSE for the linear **SISO channel estimation** model is

$$\hat{h} = \left(\bar{\mathbf{x}}^T\bar{\mathbf{x}} + \frac{1}{SNR}1\right)^{-1} \bar{\mathbf{x}}^T\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right) + \mu_h$$

$$\hat{h} = \frac{\bar{\mathbf{x}}^T\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right)}{\|\bar{\mathbf{x}}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h$$

LMMSE Linear Model

- This can be simplified as

$$\hat{h} = \frac{\bar{x}^T(\bar{y} - \bar{x}\mu_h)}{\|\bar{x}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h.$$

$$= \frac{\frac{\bar{x}^T}{\sigma^2}(\bar{y} - \bar{x}\mu_h)}{\frac{\|\bar{x}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h.$$

LMMSE Linear Model

- This can be simplified as

$$\hat{h} = \frac{\frac{\bar{x}^T\bar{y}}{\sigma^2} - \frac{\|\bar{x}\|^2}{\sigma^2}\mu_h}{\frac{\|\bar{x}\|^2}{\sigma^2}+\frac{1}{\sigma_h^2}} + \mu_h$$

$$= \frac{\frac{\bar{x}^T\bar{y}}{\sigma^2}+\frac{\mu_h}{\sigma_h^2}}{\frac{\|\bar{x}\|^2}{\sigma^2}+\frac{1}{\sigma_h^2}} = \frac{\frac{\bar{x}^T\bar{y}/\|\bar{x}\|^2}{\sigma^2/\|\bar{x}\|^2}+\frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/\|\bar{x}\|^2}+\frac{1}{\sigma_h^2}}$$

LMMSE Linear Model

- This can be simplified as

$$\hat{h} = \frac{\bar{\mathbf{x}}^T\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right)}{\|\bar{\mathbf{x}}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h$$

$$= \frac{\frac{\bar{\mathbf{x}}^T}{\sigma^2}(\bar{\mathbf{y}} - \bar{\mathbf{x}}\mu_h)}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h$$

LMMSE Linear Model

- This can be simplified as

$$\hat{h} = \frac{\frac{\bar{\mathbf{x}}^T}{\sigma^2}(\bar{\mathbf{y}} - \bar{\mathbf{x}}\mu_h)}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h$$

$$= \frac{\frac{\bar{\mathbf{x}}^T\bar{\mathbf{y}}}{\sigma^2} - \frac{\mu_h\|\bar{\mathbf{x}}\|^2}{\sigma^2}}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h = \frac{\frac{\bar{\mathbf{x}}^T\bar{\mathbf{y}}}{\sigma^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}}$$

LMMSE Linear Model

- This can intuitively explained as follows

ML Estimate

MSE of ML Estimate

Prior

variance of Prior

$$\hat{h} = \frac{\dfrac{\bar{x}^T\bar{y}/\|\bar{x}\|^2}{\sigma^2/\|\bar{x}\|^2} + \dfrac{\mu_h}{\sigma_h^2}}{\dfrac{1}{\sigma^2/\|\bar{x}\|^2} + \dfrac{1}{\sigma_h^2}} = \frac{w_1(ML) + w_2(Prior)}{w_1 + w_2}$$

Weighted Linear combination

LMMSE Estimate!

$$w_1 = \frac{1}{\sigma^2/\|\bar{x}\|^2} \propto \text{Reliability!}$$

$$w_2 = \frac{1}{\sigma_h^2} \propto \text{Reliability!}$$

LMMSE Linear Model

- This can intuitively explained as follows

$$\hat{h} = \frac{\frac{\bar{\mathbf{x}}^T\bar{\mathbf{y}}/\|\bar{\mathbf{x}}\|^2}{\sigma^2/\|\bar{\mathbf{x}}\|^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2/\|\bar{\mathbf{x}}\|^2} + \frac{1}{\sigma_h^2}} = \overbrace{\frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{ML MSE}} + \frac{1}{\text{Prior var.}}}}^{\text{Linear combination of ML/Prior.}}$$

weights $\propto \frac{1}{\text{MSE/variance}}$.

higher MSE/var $\Rightarrow$ Lower weight.

LMMSE Linear Model

- Therefore, LMMSE is performing weighted Linear combination of ML, Prior

Weights $\cdot \propto \cdot \dfrac{1}{\text{MSE/variance}}$.

$$\hat{h} = \frac{\dfrac{\bar{\mathbf{x}}^T \bar{\mathbf{y}} / \|\bar{\mathbf{x}}\|^2}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \dfrac{\mu_h}{\sigma_h^2}}{\dfrac{1}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \dfrac{1}{\sigma_h^2}} = \frac{\dfrac{\text{ML Est}}{\text{ML MSE}} + \dfrac{\text{Prior}}{\text{Prior var.}}}{\dfrac{1}{\text{ML MSE}} + \dfrac{1}{\text{Prior var.}}}$$

LMMSE Linear Model

- Therefore, LMMSE is performing **a linear combination** of ML and prior
  - With weights given by **inverse of var/MSE.**

$$\hat{h} = \frac{\dfrac{\bar{\mathbf{x}}^T\bar{\mathbf{y}}/\|\bar{\mathbf{x}}\|^2}{\sigma^2/\|\bar{\mathbf{x}}\|^2} + \dfrac{\mu_h}{\sigma_h^2}}{\dfrac{1}{\sigma^2/\|\bar{\mathbf{x}}\|^2} + \dfrac{1}{\sigma_h^2}} = \frac{\dfrac{\text{ML Est}}{\text{ML MSE}} + \dfrac{\text{Prior}}{\text{Prior var.}}}{\dfrac{1}{\text{ML MSE}} + \dfrac{1}{\text{Prior var.}}}$$

LMMSE Linear Model

Very noisy observations.
$\sigma^2 \to \infty$.

- When $\text{ML MSE} \to \infty$

0

0

$$\hat{h} = \frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{ML MSE}} + \frac{1}{\text{Prior var.}}} \rightarrow \frac{\frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma_h^2}} \rightarrow \mu_h$$

Prior information.

LMMSE Linear Model

- When $\text{ML MSE} \to \infty$

$$\hat{h} = \frac{\dfrac{\text{ML Est}}{\text{ML MSE}} + \dfrac{\text{Prior}}{\text{Prior var.}}}{\dfrac{1}{\text{ML MSE}} + \dfrac{1}{\text{Prior var.}}} \to \frac{\dfrac{\text{Prior}}{\text{Prior var.}}}{\dfrac{1}{\text{Prior var.}}} = \mu_h$$

LMMSE Linear Model

- When $\text{Prior var} \to \infty$

$\sigma_h^2 \to \infty$

Prior info is unreliable

$\Rightarrow$ non-informative prior.

$$\hat{h} = \frac{\dfrac{\text{ML Est}}{\text{ML MSE}} + \underbrace{\dfrac{\text{Prior}}{\text{Prior var.}}}_{\to 0}}{\dfrac{1}{\text{ML MSE}} + \underbrace{\dfrac{1}{\text{Prior var.}}}_{\to 0}} \to \frac{\dfrac{\text{ML Est}}{\cancel{\text{ML MSE}}}}{\dfrac{1}{\cancel{\text{ML MSE}}}} \to \underbrace{\frac{x^T y}{\|x\|^2}}_{\text{ML Estimate.}}$$

LMMSE Linear Model

- When $\text{Prior var} \rightarrow \infty$

$$\hat{h} = \frac{\dfrac{\text{ML Est}}{\text{ML MSE}} + \dfrac{\text{Prior}}{\text{Prior var.}}}{\dfrac{1}{\text{ML MSE}} + \dfrac{1}{\text{Prior var.}}} \rightarrow \frac{\dfrac{\text{ML Est}}{\text{ML MSE}}}{\dfrac{1}{\text{ML MSE}}} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\|\bar{\mathbf{x}}\|^2}$$

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Three options provided below for the font size.

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

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