1.

$$\boxed{\begin{array}{c}\min\ c^T x\\ Ax \le b\end{array}}$$

A full rank

A^{-1}

$\underline{x \le A^{-1} b}$ X

$-3x \le 1 \not\Leftrightarrow x \le -1/3$

$A = [-3]$

new variable $\underline{y := Ax}$

(a) $x = A^{-1} b$

$$\begin{array}{c}\min\ c^T x\\ Ax \le b\end{array}$$

$x \in \mathbb{R}^n$

$\underline{Ax = b \le b}$ ✓

$\underline{A^{-1} b}$ feasible

cannot be infeasible

(b) $\underline{y := Ax}$ $\quad \underline{x = A^{-1} y}$

$$\boxed{\begin{array}{c}\min_y\ c^T (A^{-1} y)\\ y \le b\end{array}} \to \tilde{c}^T y$$

where

$\tilde{c}^T = c^T A^{-1}$

or $\tilde{c} = (A^{-1})^T c$

$$\boxed{\begin{array}{c}\min_y\ \tilde{c}^T y\\ y \le b\end{array}} \to \text{optimization } \cancel{y} \text{ wrt } y$$

$$\boxed{\begin{aligned} \min\ & \tilde{c}_1 y_1 + \tilde{c}_2 y_2 + \dots \tilde{c}_n y_n \\ & y_1 \le b_1 \\ & y_2 \le b_2 \\ & \vdots \\ & y_n \le b_n \end{aligned}}$$

e.g. $\tilde{c}_1 > 0$ $\quad$ $\tilde{c}_1 y_1 < 0$ for $y_1 < 0$

$y_1 \to -\infty$ $\quad$ $\tilde{c}_1 y_1 \to -\infty$

$\tilde{c}^T y \to -\infty$ if $y_i \to -\infty$

corresp. to $\tilde{c}_i > 0$

summary: if $\left[(A^{-1})^T c\right]_i > 0$

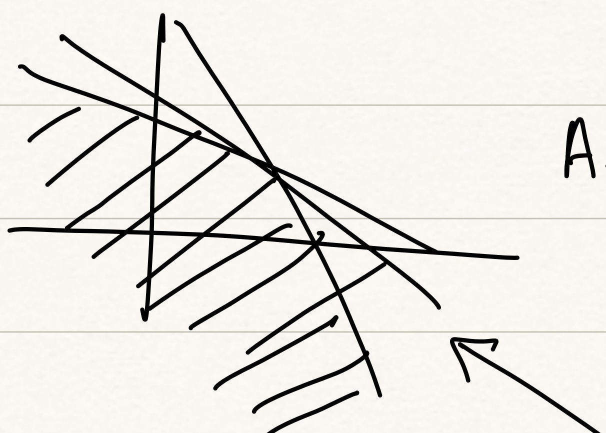
then problem unbounded below

(c) $(A^{-1})^T c \le 0$ $\quad$ $y_i = b_i \ \forall i$

$y = \underline{b}$

$\tilde{c}^T y = c^T x = c^T A^{-1} b$ (finite solution)

when $(A^{-1})^T c \le 0$

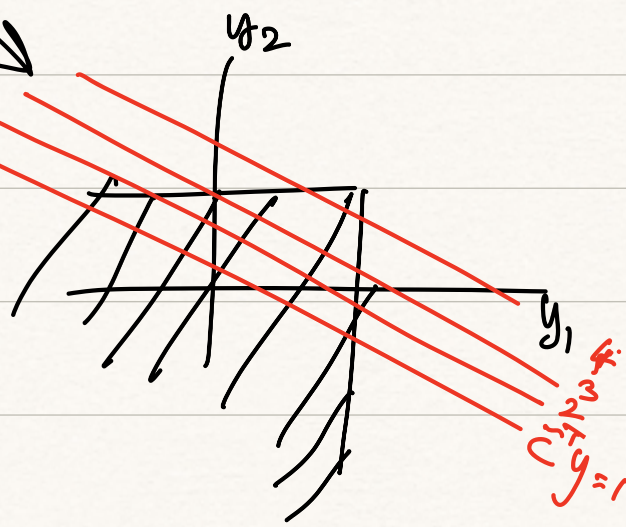


$A\cancel{x} \le b$

$y \le b$

$x = A^{-1} y \in \mathbb{R}^n$

$y \in \mathbb{R}^n$



2.

$\min\ c^T x$ $Ax = b$ $x_i \geqslant 0$	$\min\ a^T y$ $Gy \le h$ $Fy = u$
Simplex form $\quad P_2$	P_1 $\quad y$

$$Gy \le h \iff Gy + z = h$$

$$z \geqslant 0$$

$$\begin{bmatrix} G & I \\ F & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} h \\ u \end{bmatrix}$$

$$z \geqslant 0$$

$$y = y_+ - y_-$$

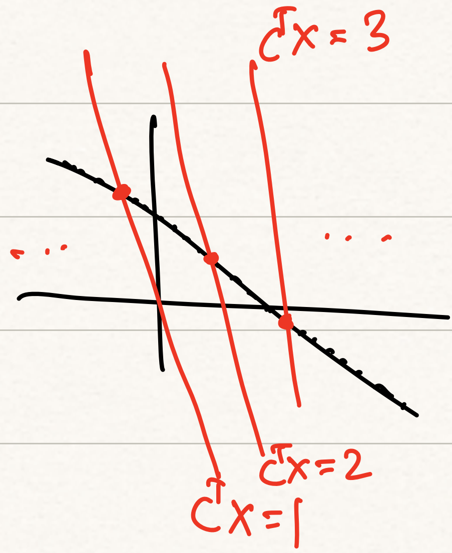
$$y = v - w \qquad v, w \geqslant 0$$

$$\underbrace{\begin{bmatrix} G & -G & I \\ F & -F & 0 \end{bmatrix}}_{A} \begin{bmatrix} v \\ w \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} h \\ u \end{bmatrix}}_{b} \qquad c = \begin{bmatrix} a \\ -a \end{bmatrix}$$

$$x = \begin{bmatrix} v \\ w \\ z \end{bmatrix} \succeq 0$$

3 (a)

$$\boxed{\begin{array}{l} \min \quad c^T x \\ \quad Ax = b \end{array}}$$



(a) $Ax = b$ has no solution

$b \notin R(A)$

(b) $b \in R(A)$

$$\{x \mid Ax = b\} = \{u \mid \underline{x = Cu + x_0}\}$$
$$\underline{u \in \mathbb{R}}$$

columns of C are $N(A)$

$x = Cu + x_0$ → particular solution

$$Ax = ACu + \underline{\underline{Ax_0}}$$

$AC = 0$ $\qquad$ b

$Ax = b$ remove constraint

$$\boxed{\min_{u} \quad c^T(Cu + x_0)}$$

$C^T c = 0$ $\qquad$ any u solves & any solution to $Ax=b$

$C^T \underline{c} \neq 0$

solves this problem

$[C^T \underline{c}]_i \neq 0 \quad u_i \to \pm\infty$

$c^T C u \longrightarrow -\infty$

(unbounded below)

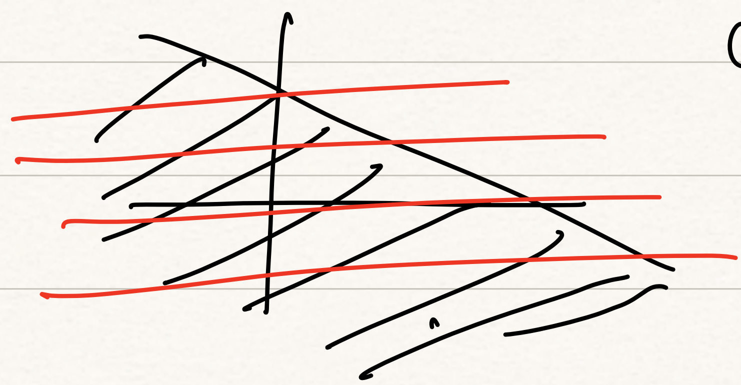
$C^T \underline{c} = 0$ or $\underline{c} \in N(C^T)$

$\underline{c} \in N(A)$

$\underline{c} \in N(A)^{\perp} = R(A^T)$

4. $\min c^T x$

$a^T x \leq b \qquad a \neq 0$



(a) never infeasible

(b) $y \perp a$

so $y^T a = 0$

$x = z + y$

$a^T x = a^T z + a^T y$

$= a^T z \leq b$

$\min_{z,y} \; c^T(z+y)$

$a^T z \leq b$

$$\min_{z} \quad c^T z + c^T y$$
$$a^T z \leq b$$

$c^T y \neq 0$

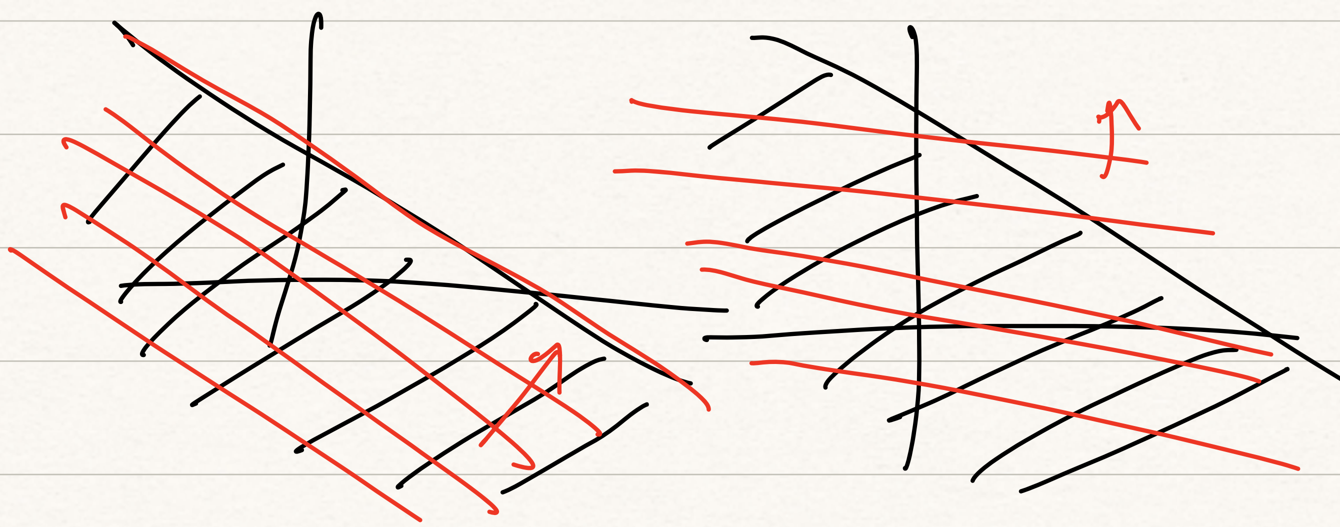
$c^T y \longrightarrow -\infty$

$c = -\underline{a}\beta$

$c^T y = -(a^T y)\beta = 0$

$c = -\beta \underline{a}$ finite solution

$c \neq -\beta a$ unbounded below



5. $\min \; x^T A x \qquad A \succ 0$

$\|x\|_2^2 = 1 \qquad A = U \Sigma U^T$ e.v.d.

$y = U^T x \iff x = Uy$

$\|x\|_2^2 = \|Uy\|_2^2 = \|y\|_2^2$

$$x^T A x = y^T U^T \underbrace{U \Sigma U^T}_{A} U y = y^T \Sigma y$$

$$\min_{\|y\|_2^2 = 1} y^T \Sigma y$$

$$\Sigma = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$y^T \Sigma y = \sum \lambda_i y_i^2$$

$\sum y_i = 1$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2$$

$\lambda_{min} \le \lambda_i \le \lambda_{max}$

$$\underset{\overset{\vee}{1}}{\lambda_{min}\left(\sum y_i^2\right)} \le \lambda_1 y_1^2 + \lambda_2 y_2^2 \cdots \lambda_n y_n^2 \le \underset{\overset{\shortparallel}{1}}{\lambda_{max}\left(\sum y_i^2\right)}$$

$$\lambda_{min} \le y^T \Sigma y \le \lambda_{max}$$

equality when

$$\left.\begin{array}{ll} y_i = 1 & \lambda_i = \lambda_{min} \\ y_1 \ldots y_{i-1}, y_{i+1} \cdots y_n = 0 & \end{array}\right] \|y\| = 1$$

$$\min_{y^T y = 1} y^T \Sigma y = \lambda_{min}(A)$$