

eMasters in Communication Systems

**Prof. Aditya
Jagannatham**



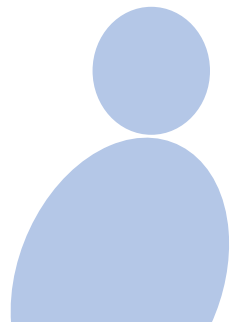
Elective Module:

**Estimation for Wireless
Communication**



Chapter 9

LMMSE Interpretation

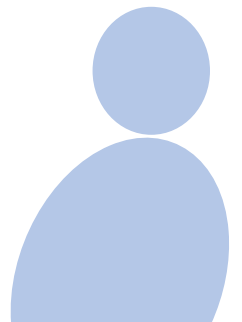


LMMSE Interpretation

- Let us now take a **deeper look** at LMMSE.

$$\min E\{\|c\bar{y} - \bar{h}\|^2\}.$$

Linear minimum
Mean square error Estimator.

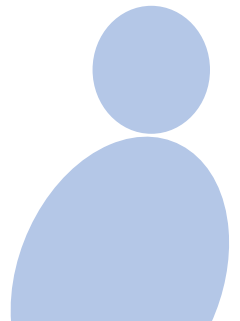


LMMSE

- Recall the LMMSE estimate is

$$\hat{h} = C\bar{y} = R_{hy} R_{yy}^{-1} \bar{y}$$

LMMSE
 h, \bar{y} arbitrarily distributed.
Zero mean.
 $E\{\bar{h}\} = 0$ $E\{\bar{y}\} = 0$.



LMMSE

- Recall the LMMSE estimate is

$$\underbrace{\hat{\mathbf{h}} = \mathbf{C}\bar{\mathbf{y}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}}_{\text{LMMSE Estimate}}$$



LMMSE Non-zero Mean

- We first derive the LMMSE estimate for non-zero mean parameter/observation

$$\left. \begin{aligned} E\{\mathbf{h}\} &= \bar{\mu}_h \\ E\{\bar{y}\} &= \bar{\mu}_y \end{aligned} \right\} \text{Non zero mean.} \rightarrow$$



LMMSE Non-zero Mean

- Let

$$\begin{aligned} E\{\bar{\mathbf{h}}\} &= \bar{\boldsymbol{\mu}}_h \quad M \times 1 \\ E\{\bar{\mathbf{y}}\} &= \bar{\boldsymbol{\mu}}_y \quad N \times 1 \end{aligned}$$

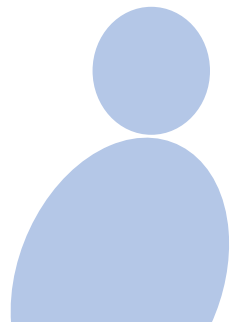


LMMSE Non-zero Mean

- Make zero-mean

$$\left\{ \begin{array}{l} \bar{h} - \bar{\mu}_h : E\{\bar{h} - \bar{\mu}_h\} = 0 \\ \bar{y} - \bar{\mu}_y : E\{\bar{y} - \bar{\mu}_y\} = 0 \end{array} \right.$$

Zero mean Quantities -
 \Rightarrow use previous result.

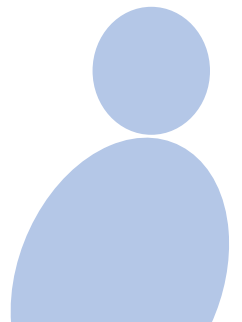


LMMSE Non-zero Mean

- Make zero-mean

$$E\{\bar{\mathbf{h}} - \bar{\boldsymbol{\mu}}_h\} = 0$$

$$E\{\bar{y} - \bar{\mu}_y\} = 0$$



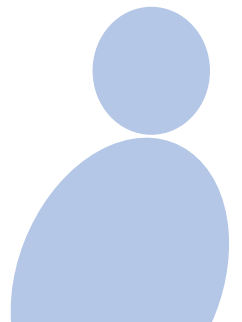
LMMSE Non-zero Mean

- The LMMSE estimate for non-zero mean is given as follows

$$\hat{h} - \bar{\mu}_h = R_{hy} R_{yy}^{-1} (\bar{y} - \bar{\mu}_y)$$

$$\Rightarrow \hat{h} = R_{hy} R_{yy}^{-1} (\bar{y} - \bar{\mu}_y) + \bar{\mu}_h.$$

LMMSE for Arbitrarily distributed h, y
Non-zero mean.



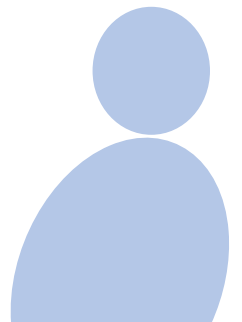
LMMSE Non-zero Mean

- The LMMSE estimate for non-zero mean is given as follows

$$\hat{\mathbf{h}} - \bar{\boldsymbol{\mu}}_h = \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y)$$

$$\hat{\mathbf{h}} = \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y) + \bar{\boldsymbol{\mu}}_h$$

Prior information



LMMSE Non-zero Mean

- Note

$$\mathbf{R}_{yy} = \overbrace{E \{ (\bar{y} - \bar{\mu}_y)(\bar{y} - \bar{\mu}_y)^T \}}^{\text{covariance matrix}}$$

$$\mathbf{R}_{hy} = \underbrace{E \{ (\bar{h} - \bar{\mu}_h)(\bar{y} - \bar{\mu}_y)^T \}}_{\text{cross covariance matrix}}$$



LMMSE Non-zero Mean

- Note

$$\mathbf{R}_{yy} = E \left\{ (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y)(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y)^T \right\}$$
$$\mathbf{R}_{hy} = E \left\{ (\bar{\mathbf{h}} - \bar{\boldsymbol{\mu}}_h)(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y)^T \right\}$$



LMMSE Linear Model $\bar{y} = X\bar{h} + \bar{v}$

- Therefore, the LMMSE for the linear **MISO** channel estimation **model** is

$$\hat{h} = \left(X^T X + \frac{1}{\text{SNR}} I \right) X^T (\bar{y} - \bar{\mu}_y) + \bar{\mu}_h.$$

LMMSE for MISO with non zero means.
Linear Parameter estimation model.



LMMSE Linear Model

- Therefore, the LMMSE for the linear **MISO channel estimation** model is

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{X}^T (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y) + \bar{\boldsymbol{\mu}}_h$$



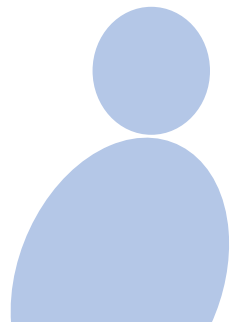
LMMSE Linear Model

- Further, $\bar{\mu}_y$ is derived as follows

$$\underline{\bar{y}} = \underline{X\bar{h}} + \underline{\bar{v}}$$

- Since noise is zero-mean, we have ^{$E\{\bar{v}\}=0$}

$$\begin{aligned}\bar{\mu}_y &= E\{\bar{y}\} = E\{X\bar{h} + \bar{v}\} \\ &= E\{X\bar{h}\} + \cancel{E\{\bar{v}\}} \\ \bar{\mu}_y &= X\bar{\mu}_h\end{aligned}$$



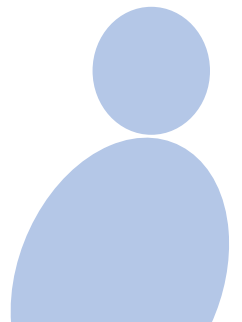
LMMSE Linear Model

- Further, $\bar{\mu}_y$ is derived as follows

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

- Since noise is zero-mean, we have

$$\bar{\mu}_y = E\{\bar{\mathbf{y}}\} = E\{\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}\} = \mathbf{X}\bar{\mu}_h$$



LMMSE Linear Model

- Therefore,

$$\begin{aligned}\hat{\mathbf{h}} &= \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{X}^T (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y) + \bar{\boldsymbol{\mu}}_h \\ &= \left(X^T X + \frac{1}{SNR} I \right)^{-1} X^T (\bar{y} - X \bar{\mu}_h) + \bar{\mu}_h\end{aligned}$$



LMMSE Linear Model

- Therefore,

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{X}^T (\bar{\mathbf{y}} - \mathbf{X} \bar{\boldsymbol{\mu}}_h) + \bar{\boldsymbol{\mu}}_h$$



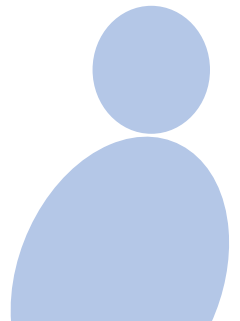
LMMSE SISO Model

Single input
Single output.

- For simplicity consider the SISO Model

$$\begin{array}{l} N \text{ outputs - } \left\{ \begin{array}{l} y(1) = hx(1) + v(1) \\ y(2) = hx(2) + v(2) \\ \vdots \\ y(N) = hx(N) + v(N) \end{array} \right. \end{array}$$

↑
channel coefficient -



LMMSE SISO Model

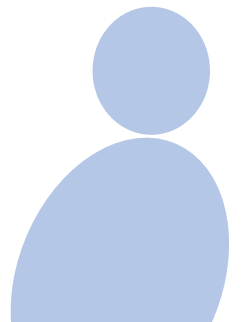
- For simplicity consider the SISO Model

$$y(1) = hx(1) + v(1)$$

$$y(2) = hx(2) + v(2)$$

$$\vdots$$

$$y(N) = hx(N) + v(N)$$



Wireless System Model

- The different quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$



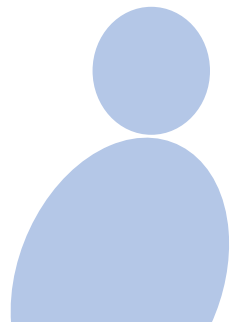
Wireless System Model

- The different quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Pilot vector

Output vector.



Wireless System Model

- Therefore,

$$\bar{y} = \bar{x}h + \bar{v}$$



Wireless System Model

- The mean satisfies

$$E\{\bar{v}\} = 0.$$

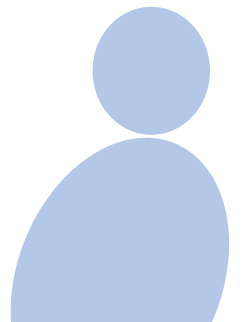
$$E\{h\} = \mu_h.$$

$$\bar{\mu}_y = E\{\bar{y}\} = \bar{x}\mu_h.$$

$$\bar{y} = \bar{x}h + \bar{v}$$

Mean of \bar{y}
Mean of h .

$$\Rightarrow \bar{\mu}_y = \bar{x}\mu_h$$



LMMSE Linear Model

- Therefore, the LMMSE for the linear **SISO** **channel estimation** model is

$$\hat{h} = \left(\bar{x}^T \bar{x} + \frac{1}{\text{SNR}} \mathbf{1} \right)^{-1} \bar{x}^T (\bar{y} - \bar{x} \mu_h) + \mu_h$$
$$= \frac{\bar{x}^T (\bar{y} - \bar{x} \mu_h)}{\|\bar{x}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h \quad \left\{ \begin{array}{l} \text{LMMSE Estimate} \\ \text{of SISO channel.} \end{array} \right.$$

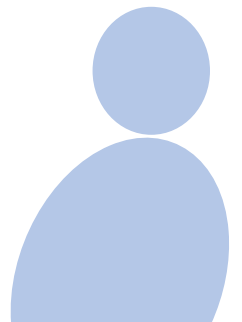


LMMSE Linear Model

- Therefore, the LMMSE for the linear **SISO channel estimation** model is

$$\hat{h} = \left(\bar{\mathbf{x}}^T \bar{\mathbf{x}} + \frac{1}{SNR} 1 \right)^{-1} \bar{\mathbf{x}}^T (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y) + \mu_h$$

$$\hat{h} = \frac{\bar{\mathbf{x}}^T (\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y)}{\|\bar{\mathbf{x}}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h$$



LMMSE Linear Model

- This can be simplified as

$$\begin{aligned}\hat{h} &= \frac{\bar{x}^T (\bar{y} - \bar{x} \mu_u)}{\|\bar{x}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_u. \\ &= \frac{\frac{\bar{x}^T}{\sigma^2} (\bar{y} - \bar{x} \mu_u)}{\frac{\|\bar{x}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_u.\end{aligned}$$



LMMSE Linear Model

- This can be simplified as

$$\begin{aligned}\hat{h} &= \frac{\frac{\bar{x}^T \bar{y}}{\sigma^2} - \frac{\|\bar{x}\|^2}{\sigma^2} \mu_h}{\frac{\|\bar{x}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h. \\ &= \frac{\frac{\bar{x}^T \bar{y}}{\sigma^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{\|\bar{x}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} = \frac{\frac{\bar{x}^T \bar{y} / \|\bar{x}\|^2}{\sigma^2 / \|\bar{x}\|^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2 / \|\bar{x}\|^2} + \frac{1}{\sigma_h^2}}\end{aligned}$$



LMMSE Linear Model

- This can be simplified as

$$\hat{h} = \frac{\bar{\mathbf{x}}^T (\bar{y} - \bar{\mu}_y)}{\|\bar{\mathbf{x}}\|^2 + \frac{\sigma^2}{\sigma_h^2}} + \mu_h$$

$$= \frac{\frac{\bar{\mathbf{x}}^T}{\sigma^2} (\bar{y} - \bar{\mathbf{x}}\mu_h)}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h$$

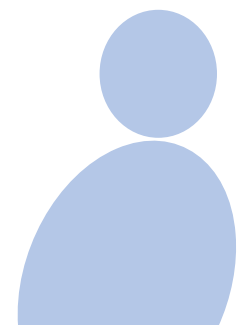


LMMSE Linear Model

- This can be simplified as

$$\hat{h} = \frac{\frac{\bar{\mathbf{x}}^T}{\sigma^2} (\bar{\mathbf{y}} - \bar{\mathbf{x}}\mu_h)}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h$$

$$= \frac{\frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\sigma^2} - \frac{\mu_h \|\bar{\mathbf{x}}\|^2}{\sigma^2}}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}} + \mu_h = \frac{\frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\sigma^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2} + \frac{1}{\sigma_h^2}}$$



LMMSE Linear Model

- This can intuitively explained as follows

$$\hat{h} = \frac{\frac{\bar{x}^T y / \|\bar{x}\|^2}{\sigma^2 / \|\bar{x}\|^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2 / \|\bar{x}\|^2} + \frac{1}{\sigma_h^2}} = \frac{W_1(\text{ML}) + W_2(\text{Prior})}{W_1 + W_2}$$

ML Estimate

MSE of ML Estimate

Prior

Variance of Prior

Weighted Linear Combination

LMMSE Estimate!

$W_1 = \frac{1}{\sigma^2 / \|\bar{x}\|^2} \propto \text{Reliability!}$

$W_2 = \frac{1}{\sigma_h^2} \propto \text{Reliability!}$

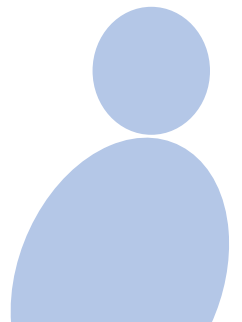
LMMSE Linear Model

- This can intuitively explained as follows

$$\hat{h} = \frac{\frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}} / \|\bar{\mathbf{x}}\|^2}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \frac{1}{\sigma_h^2}} = \underbrace{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}_{\text{Linear combination of ML/Prior -}}$$

Weights $\propto \frac{1}{\text{MSE/variance}}$

higher MSE/var \Rightarrow Lower weight



LMMSE Linear Model

- Therefore, LMMSE is performing *weighted Linear combination of ML, Prior*
Weights $\propto \frac{1}{\text{MSE/Variance}}$

$$\hat{h} = \frac{\frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}} / \|\bar{\mathbf{x}}\|^2}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \frac{1}{\sigma_h^2}} = \frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{ML MSE}} + \frac{1}{\text{Prior var.}}}$$



LMMSE Linear Model

- Therefore, LMMSE is performing **a linear combination** of **ML** and **prior**
 - With weights given by **inverse of var/MSE.**

$$\hat{h} = \frac{\frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}} / \|\bar{\mathbf{x}}\|^2}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma^2 / \|\bar{\mathbf{x}}\|^2} + \frac{1}{\sigma_h^2}} = \frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{ML MSE}} + \frac{1}{\text{Prior var.}}}$$



LMMSE Linear Model

Very noisy observations.
 $\sigma^2 \rightarrow \infty$

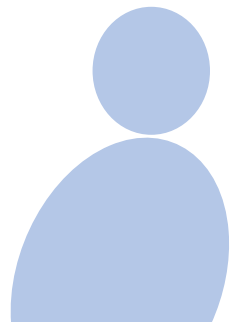
- When ML MSE $\rightarrow \infty$

$$\hat{h} = \frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}$$

Annotations: A red circle around "ML Est" has an arrow pointing to 0. A red circle around "ML MSE" has an arrow pointing to 0.

$$\rightarrow \frac{\frac{\mu_h}{\sigma_h^2}}{\frac{1}{\sigma_h^2}} \rightarrow \mu_h$$

Annotation: "Prior information." with an arrow pointing to μ_h .



LMMSE Linear Model

- When ML MSE $\rightarrow \infty$

$$\hat{h} = \frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{ML MSE}} + \frac{1}{\text{Prior var.}}} \rightarrow \frac{\frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{Prior var.}}} = \mu_h$$



LMMSE Linear Model

- When Prior var $\rightarrow \infty$

$\sigma_h^2 \rightarrow \infty$
Prior info is unreliable
 \Rightarrow non-informative prior.

$$\hat{h} = \frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{1}{\text{ML MSE}} + \frac{1}{\text{Prior var.}}} \rightarrow \frac{\frac{\text{ML Est}}{\cancel{\text{ML MSE}}} + \overset{\rightarrow 0}{\frac{\text{Prior}}{\cancel{1}}}}{\frac{1}{\cancel{\text{ML MSE}}} + \overset{\rightarrow 0}{\frac{1}{\cancel{\text{Prior var.}}}}} \rightarrow \frac{\text{ML Estimate} \cdot \bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$



LMMSE Linear Model

- When Prior var $\rightarrow \infty$

$$\hat{h} = \frac{\frac{\text{ML Est}}{\text{ML MSE}} + \frac{\text{Prior}}{\text{Prior var.}}}{\frac{\text{ML MSE}}{1} + \frac{1}{\text{Prior var.}}} \rightarrow \frac{\frac{\text{ML Est}}{\text{ML MSE}}}{\frac{\text{ML MSE}}{1}} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\|\bar{\mathbf{x}}\|^2}$$



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Three options provided below for the font size.

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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