eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 6

Channel Equalization

Equalization

intersymbol interference

- Equalization is used to remove the effect of **Intersymbol Interference (ISI)**.

Arises due to Delay Spread.

Multipath Propagation

Equalization

- Consider the ISI channel modeled as

output at time k

input symbol at time k

$$y(k) = h(0)x(k) + h(1)x(k-1) + \cdots$$
$$+ h(L-1)x(k-L+1) + v(k)$$

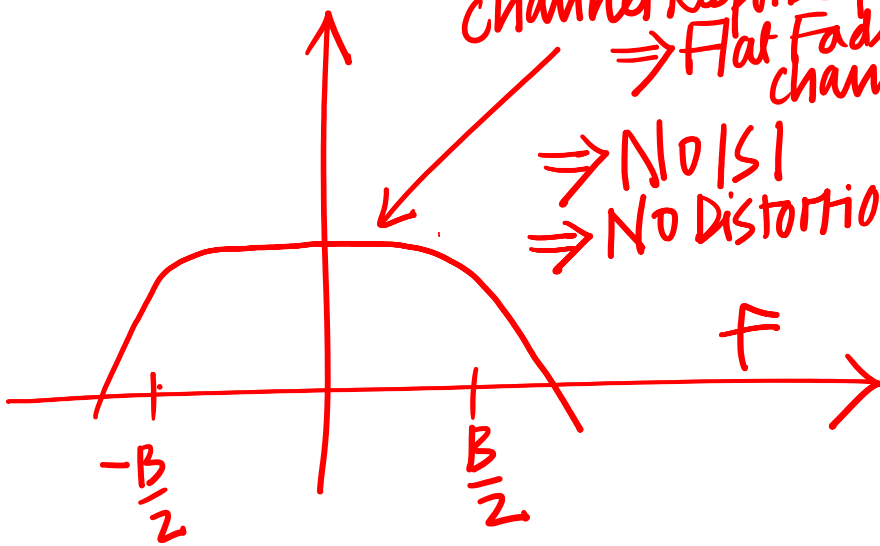
$$\underbrace{h(0), h(1), \ldots, h(L-1)}_{\text{channel Taps.}}$$

$$\underbrace{x(k-1), x(k-2), \ldots, x(k-L+1)}_{\text{interference}}$$

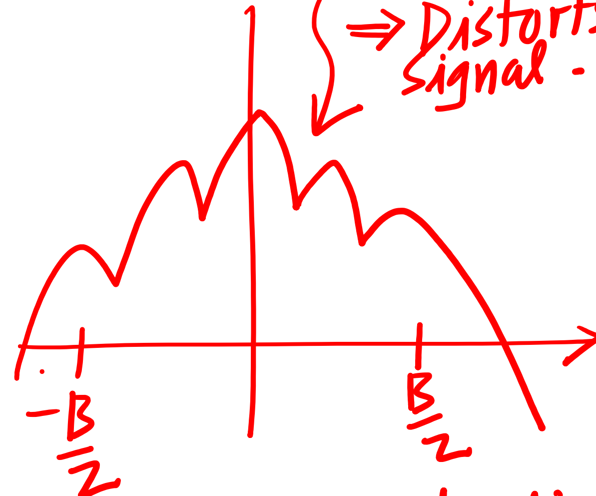
Past symbols.

Equalization

Channel Response Flat
⇒ Flat Fading channel
⇒ No ISI
⇒ No Distortion



Channel Response NOT Flat
⇒ Distorts Signal.



⇒ Frequency selective channel.
⇒ Distortion
⇒ intersymbol interference.

Equalization

channel taps -

- Consider the ISI channel modeled as

$$\underbrace{y(k) = h(0)x(k) + h(1)x(k-1) + \cdots + h(L-1)x(k-L+1) + v(k)}$$

$$y = \underbrace{h * x} + v$$

Linear convolution

convolutive channel.

Equalization

- Consider $L = 2$ tap channel.

interference from 1 past symbol

$$y(k) = h(0)x(k) + h(1)x(k-1) + v(k)$$

Equalization

- Consider $L = 2$

How to remove ISI TO extract $x(k)$?

$$y(k) = h(0)x(k) + h(1)x(k-1) + v(k)$$

This is EQUALIZATION.

Equalization

- Perform equalization using $y(k+1)$, $y(k)$
- We have

$$y(k+1) = h(0)\,x(k+1) + h(1)\,x(k) + v(k+1)$$

$$y(k) = h(0)\,x(k) + h(1)\,x(k-1) + v(k)$$

Equalization

- Perform equalization using $y(k+1)$, $y(k)$

$$\begin{aligned} y(k+1) &= h(0)x(k+1) + h(1)x(k) + v(k+1) \\ y(k) &= h(0)x(k) + h(1)x(k-1) + v(k) \end{aligned}$$

Equalization

- This can be written in the vector form

Filter matrix

$$\underbrace{\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}}_{\bar{y}} = \underbrace{\begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} v(k+1) \\ v(k) \end{bmatrix}}_{\bar{v}}$$

vector notation

$$\underset{2\times 1}{\bar{y}} = \underset{2\times 3}{H}\,\underset{3\times 1}{\bar{x}} + \underset{2\times 1}{\bar{v}}$$

Equalization

- This can be written in the vector form

$$\overbrace{\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}}^{\bar{y}(k)} = \overbrace{\begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix}}^{H} \overbrace{\begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}}^{\bar{x}(k)} + \underbrace{\begin{bmatrix} v(k+1) \\ v(k) \end{bmatrix}}_{\bar{v}(k)}$$

Equalization

- The compact vector form is

$$\underbrace{\overset{2\times 1}{\bar{\mathbf{y}}(k)} = \overset{2\times 3}{\mathbf{H}}\,\overset{3\times 1}{\bar{\mathbf{x}}(k)} + \overset{2\times 1}{\bar{\mathbf{v}}(k)}}$$

Equalization

Linear Equalizer

- Let the equalizer weights be c_0, c_1
- Equalizer is

$$\overbrace{c_0 y(k+1) + c_1 y(k)}^{\text{EQUALIZER}}$$

$$= \underbrace{\begin{bmatrix} c_0 & c_1 \end{bmatrix}}_{\bar{c}^T} \underbrace{\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}}_{\bar{y}(k)} = \bar{c}^T \bar{y}(k)$$

Equalization

- Let the equalizer weights be c_0, c_1
- Equalizer is

$$c_0 y(k+1) + c_1 y(k)$$

$$= \begin{bmatrix} c_0 & c_1 \end{bmatrix} \begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix} = \underline{c}^T \underline{y}(k)$$

Equalization

$$\bar{c}^T \bar{y}(k) = \bar{c}^T \left(H \bar{x}(k) + \bar{v}(k) \right)$$

$$= \underbrace{\bar{c}^T H \bar{x}(k) + \bar{c}^T \bar{v}(k)}_{\text{EQUALIZER OUTPUT}}.$$

Equalization

- Substituting the model, this is

$$\bar{\mathbf{c}}^T\left(\mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{v}}(k)\right)$$

$$= \bar{\mathbf{c}}^T\mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{c}}^T\bar{\mathbf{v}}(k)$$

$$\bar{c}^T H \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}$$

Equalization

$$\underline{\underline{\bar{c}^T H}} \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}$$

should suppress $x(k+1)$, $x(k-1)$
Recover $x(k)$

$\Rightarrow$ ideally it should be

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix} = x(k)$$

Equalization

- Let us now examine the term $\bar{\mathbf{c}}^T \mathbf{H} \bar{\mathbf{x}}(k)$

$$\bar{\mathbf{c}}^T \mathbf{H} \bar{\mathbf{x}}(k) = \bar{\mathbf{c}}^T \mathbf{H} \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}$$

ideally

$$\bar{c}^T H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Equalization

- In order to recover $x(k)$

$$\bar{\mathbf{c}}^T \mathbf{H} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

column vector

$$\Rightarrow \mathbf{H}^T \bar{\mathbf{c}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Equalization

- In order to recover $x(k)$

$$\bar{\mathbf{c}}^T \mathbf{H} = [0\ 1\ 0]$$

$$\Rightarrow \mathbf{H}^T \bar{\mathbf{c}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Equalization

- Therefore, we solve

$$\min \underbrace{\left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - H^T \bar{c} \right\|^2}_{\text{Least squares Problem.}}$$

$H^T \bar{c}$ should approximate $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as closely as possible!!

Equalization

- Therefore, we solve

$$\min \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \mathbf{H}^T \bar{\mathbf{c}} \right\|^2$$

Equalization

- The equalizer vector is given as

$$\bar{c} = \left(\left(H^T\right)^T H^T \right)^{-1} \left(H^T\right)^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underbrace{\bar{c} = \left(H H^T \right)^{-1} H \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\substack{\text{Equalizer Vector} \\ \text{Equalizer Filter.}}}$$

Equalization

- The equalizer vector is given as

Equalizer Filter

channel matrix

$$\bar{\mathbf{c}} = \left((\mathbf{H}^T)^T \mathbf{H}^T \right)^{-1} (\mathbf{H}^T)^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= (\mathbf{H}\mathbf{H}^T)^{-1} \mathbf{H} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Equalization Example

inter symbol interference

- Consider the ISI channel

$h(0) = 1$

$$y(k) = x(k) + \frac{1}{3}x(k-1) + v(k)$$

$h(1) = \frac{1}{3}$

interference

$y(k) = h(0)x(k) + h(1)x(k-1) + v(k).$

- Design a 2 tap equalizer for this channel

$\bar{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$

Equalization Example

- The matrix $\mathbf{H}$ is

2x3

$$\mathbf{H} = \begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix}$$

h(0)

h(1)

$$= \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

Equalization Example

- The matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}}^{\text{channel matrix}}$$

Equalization Example

- The equalizer vector is

$$\underline{\mathbf{H} = (\mathbf{H}\mathbf{H}^{T})^{-\mathbf{1}}\mathbf{H}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Equalization Example

- Let us evaluate $\mathbf{HH}^T$

$$\mathbf{HH}^T = \underbrace{\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \\ 0 & \frac{1}{3} \end{bmatrix}}_{H^T}$$

$$= \begin{bmatrix} \frac{10}{9} & \frac{1}{3} \\ \frac{1}{3} & \frac{10}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix} = HH^T$$

2x2

Equalization Example

- Let us evaluate $\mathbf{H}\mathbf{H}^T$

$$\mathbf{H}\mathbf{H}^T = \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} & \frac{1}{3} \\ \frac{1}{3} & \frac{10}{9} \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$$

Equalization Example

- Therefore

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(\mathbf{HH}^T)^{-1} = \left(\frac{1}{9}\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}\right)^{-1}$$

$$= \frac{9}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}$$

Equalization Example

- Therefore

$$(\mathbf{HH}^T)^{-1} = \frac{9}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}$$

Equalization Example

$c_0 = \frac{3}{91}$

$c_1 = \frac{81}{91}$.

$\bar{c} = \begin{bmatrix} \frac{3}{91} \\ \frac{81}{91} \end{bmatrix}$

- The equalizer vector is

$$\mathbf{c} = (\mathbf{H}\mathbf{H}^T)^{-1}\mathbf{H}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{9}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\bar{c}$ equalizer vector

$$= \frac{9}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = \frac{9}{91}\begin{bmatrix} \frac{1}{3} \\ 9 \end{bmatrix} = \frac{3}{91}\begin{bmatrix} 1 \\ 27 \end{bmatrix}$$

Equalization Example

- The equalizer vector is

2 Tap Equalizer

$$\mathbf{c} = (\mathbf{H}\mathbf{H}^T)^{-1}\mathbf{H}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{3}{91}\\ \frac{81}{91}\end{bmatrix}$$

$$= \frac{9}{91}\begin{bmatrix}10 & -3\\ -3 & 10\end{bmatrix}\begin{bmatrix}1 & \frac{1}{3} & 0\\ 0 & 1 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{3}{91}\begin{bmatrix}1\\27\end{bmatrix}$$

Equalizer output:

$$c_0\, y(k+1) + c_1\, y(k)$$

$$= \underline{\frac{3}{91} y(k+1) + \frac{81}{91} y(k)}.$$

Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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