EE908 Assignment-1 Solution

----

eMasters in Communication Systems, IITK
EE908: Optimization in SPCOM
Instructor: Prof. Ketan Rajawat
Student Name: Venkateswar Reddy Melachervu
Roll No: 23156022

1.Inner product and norms

(a) Question: $\|u+v\|_2^2 = \|u\|_2^2 + \|v\|_2^2$ **if and only if** $u^T v = 0$

Solution:

Expanding left side:
$\|u+v\|_2^2 = (u+v)^T(u+v)$
$= u^T u + u^T v + v^T u + v^T v = \|u\|_2^2 + u^T v + v^T u + \|v\|_2^2$
Note that for vectors u, v, the dot product $u^T v = v^T u$
$\therefore \|u+v\|_2^2 = \|u\|_2^2 + u^T v + v^T u + \|v\|_2^2 = \|u\|_2^2 + 2u^T v + \|v\|_2^2 - (1)$
Suppose that $\|u+v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 - (2)$
Substituting (2) in LHS of (1)
$\Rightarrow \|u\|_2^2 + \|v\|_2^2 = \|u\|_2^2 + 2u^T v + \|v\|_2^2$
$\Rightarrow \|u\|_2^2 + \|v\|_2^2 - \|u\|_2^2 - \|v\|_2^2 = 2u^T v$
$\Rightarrow 0 = 2u^T v$
$\Rightarrow u^T v = 0$
Hence it is proved that $\|\boldsymbol{u}+\boldsymbol{v}\|_2^2 = \|\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}\|_2^2$ **if and only if** $\boldsymbol{u}^T\boldsymbol{v} = \boldsymbol{0}$
QED

(b) Question: $2\langle a, b\rangle + 2\langle x, y\rangle = \langle a+x, b+y\rangle + \langle a-x, b-y\rangle$

Solution:

Let's prove this by showing both sides are equal.
Rewriting both sides using transpose notation of inner product:
$2[a^T b + x^T y] = (a+x)^T(b+y) + (a-x)^T(b-y)$
Expanding the RHS
$\Rightarrow a^T b + \cancel{a^T y} + \cancel{x^T b} + x^T y + a^T b \cancel{- a^T y} \cancel{- x^T b} + x^T y = 2[a^T b + x^T y]$
$\therefore \boldsymbol{LHS} = \boldsymbol{RHS}$ **- Hence proved.**
QED

(c) Question: $\|x\|_1 \le \sqrt{n}\|x\|_2$

Solution:

Ask: Prove that level 1 norm of vector x equals $\sqrt{n}$ times the level 2 norm of x
Say $x \in \mathbb{R}^n$
$\|x\|_1$ – level 1 norm $\boldsymbol{\ell_1}$ of x aka ***Manhattan norm*** **(Taxicab distance)** – sum of the absolute values of the components of x
$\|x\|_2$ – level 2 norm $\boldsymbol{\ell_2}$ of x aka **Euclidean norm or magnitude/distance** – square root of the squares of the components of x

$$\|x\|_1 = |x_1| + |x_2| + \cdots |x_n = \sum_{i=1}^{n} |x_i| \Rightarrow \|x\|_1^2 = (|x_1| + |x_2| + \cdots |x_n|)^2$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \Rightarrow \|x\|_2^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

$$\|x\|_1^2 = \left|\sum_{i=1}^{n} 1.|x_i|\right|^2$$

Per Cauchy-Schwartz inequality,

$$\left|\sum_{i=1}^{n} 1.|x_i|\right|^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |x_i|^2\right) \le n\|x\|_2^2 \Rightarrow \|x\|_1 \le \sqrt{n\|x\|_2^2}$$

$\Rightarrow \|\boldsymbol{x}\|_1 \le \sqrt{\boldsymbol{n}}\|\boldsymbol{x}\|_2$ **– hence proved**

QED

(d) Question: Let's prove this by showing both sides are equal.

(e) Solution:

$\|x\|_1 \ge \|x\|_2 \ge \|x\|_\infty$

$\|x\|_\infty$ – **maximum norm or Chebyshev norm or Infinity norm $\boldsymbol{\ell}_\infty$** of x – the maximum value from the absolute values of each element in vector x or a matrix $= \|x\|_\infty = \max(|x_1|, |x_2|, \ldots |x_n|)$

From the proof in **c) solution** using Cauchy-Schwartz inequality,

$\|x\|_1 \le \sqrt{n}\|x\|_2$, here x is a vector with n components where $n \ge 1$

$\therefore \|x\|_1 \ge \|x\|_2$

Now let's prove $\|x\|_2 \ge \|x\|_\infty$.

For any value i such that $|x_i|$ is maximum in x, then we have:

$|x_i|^2 \le \sum_{i=1}^{n} |x_i|^2$

Taking square root both sides:

$|x_i| \le \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_2 \Rightarrow \|x\|_\infty = \max(|x_i|) \le \|x\|_2$

Therefore, $\|x\|_2 \ge \|x\|_\infty$

$\Rightarrow \|x\|_1 \ge \|x\|_2$ and $\|x\|_2 \ge \|x\|_\infty$

Hence, $\|\boldsymbol{x}\|_1 \ge \|\boldsymbol{x}\|_2 \ge \|\boldsymbol{x}\|_\infty$ is proved

QED

2. Question: Prove the triangle inequality for matrices – $\|A + B\|_F \le \|A\|_F + \|B\|_F$

Solution:

If A is a matrix, then $\|A\|_F$ is **Frobenius norm** is akin to Euclidean norm for vectors and is the square root of sum of the squares of all the elements of A.

If A is an $m \times n$ matrix,

$\Rightarrow \|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$, where a_{ij} are elements of matrix A.

$\Rightarrow \|B\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |b_{ij}|^2}$, where b_{ij} are elements of matrix B

$$\Rightarrow \|A + B\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |(a_{ij} + b_{ij})|^2}$$

Using Cauchy-Shwartz inequality for the inner product of two vectors,

$|(a_{ij}+b_{ij})^2| \leq (|a_{ij}|+|b_{ij}|)^2$
Expanding right side,
$(|a_{ij}|+|b_{ij}|)^2 \leq |a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2$
Summing over $i\ and\ j$, we've:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}+b_{ij}|^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\left(|a_{ij}|^2 + 2|a_{ij}||b_{ij}| + |b_{ij}|^2\right)$$

$$\therefore \|A+B\|_F \leq \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}2|a_{ij}||b_{ij}| + \sum_{i=1}^{m}\sum_{j=1}^{n}|b_{ij}|^2$$

Notice that, from Cauchy-Shwartz inequality applied to the inner product of the matrices A and B,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}||b_{ij}| \leq \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}\ \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|b_{ij}|^2}$$

Therefore, we've:
$\|A+B\|_F^2 \leq \|A\|_F^2 + 2\|A\|_F.\|B\|_F + \|B\|_F^2$
Taking square root both sides,
$\boldsymbol{\|A+B\|_F \leq \|A\|_F + \|B\|_F}$
Hence, it is proved.
QED

3.Prove the following inequalities
(a). **Problem:** $2\langle x,y\rangle \leq \|\mathrm{x}\|^2 + \|y\|^2$
Solution:

$$2\langle x,y\rangle = x^Ty = 2\sum_{i=1}^{n}x_iy_i$$

$$\|x\|^2 = \|x\|_2^2 = \sum_{i=1}^{n}|x_i|^2$$

$$\|y\|^2 = \|y\|_2^2 = \sum_{i=1}^{n}|y_i|^2$$

From the given statement,

$$2\langle x,y\rangle \leq \|\mathrm{x}\|^2 + \|y\|^2 \Rightarrow 2\sum_{i=1}^{n}x_iy_i \leq \sum_{i=1}^{n}|x_i|^2\sum_{i=1}^{n}|y_i|^2$$

For any two real numbers x and y, we know that $2ab \leq a^2+b^2$
∴ Appling this inequality to $\boldsymbol{\|x\|, \|y\|}$
$\boldsymbol{2\langle x,y\rangle, = \|x\|\|y\| \leq \|x\|^2 + \|y\|^2}$
Hence proved.
QED
(b). **Problem:** $2\langle x,y\rangle \leq \epsilon\|\mathrm{x}\|^2 + \frac{1}{\epsilon}\|y\|^2$ for any $\epsilon > 0$
Solution:

(c). **Problem:** $\|\mathrm{x}+\mathrm{y}\|^2 \leq (1+\epsilon)\|x\|^2 + \left(1+\frac{1}{\epsilon}\right)\|y\|^2$ for any $\epsilon > 0$
Solution:
Using Cauchy-Schwart inequality,
$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\sqrt{\|x\|^2\|y\|^2} + \|y\|^2$

C-501, Salarpuria Serenity, 5th Main, Sector 7, HSR Layout, Bengaluru, Karnataka, PIN 560102, India
Mobile: +91 97012 22130, Email: vmelachervu@gmail.com, vmela23@iitk.ac.in, Website: www.linkedin.com/in/vmelachervu

Applying AM-GM inequality (Arithmetic Mean-Geometric Mean),

$$\frac{\|x\|+\|y\|}{2} \geq \sqrt{\|x\|\|y\|} \Rightarrow \|x\|+\|y\| = 2\sqrt{\|x\|\|y\|} \Rightarrow \sqrt{\|x\|\|y\|} = \frac{1}{2}(\|x\|+\|y\|)$$

$$\Rightarrow \|x\|\|y\| = \frac{1}{4}(\|x\|+\|y\|)^2$$

Substituting in above equation,

$$\|x+y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \leq \|x\|^2 + 2 * \frac{1}{4}(\|x\|+\|y\|)^2 + \|y\|^2$$

$$\Rightarrow \|x+y\|^2 \leq \|x\|^2 + \frac{1}{2} * (\|x\|^2 + 2\|x\|\|y\| + \|y\|^2) + \|y\|^2$$

$$\Rightarrow \|x+y\|^2 \leq \left(1+\frac{1}{2}\right)\|x\|^2 + \|x\|\|y\| + \left(\frac{1}{2}+1\right)\|y\|^2 \leq (1+\epsilon)\|x\|^2 + \left(1+\frac{1}{\epsilon}\right)\|y\|^2$$

QED

(d). **Problem:** $\|\mathrm{x}_1 + x_2 + \cdots + x_n\|^2 \leq n\|x_1\|^2 + n\|x_2\|^2 + \cdots + n\|x_n\|^2$

Solution:

Expanding left side,

$$\|\mathrm{x}_1 + x_2 + \cdots + x_n\|^2 = \langle x_1 + x_2 + \cdots + x_n, x_1 + x_2 + \cdots + x_n\rangle$$
$$= \|x_1\|^2 + 2\langle x_1, x_2\rangle + \cdots + 2\langle x_1, x_n\rangle + 2\langle x_2, x_3\rangle + \cdots + \|x_n\|^2$$

Using Cauchy-Schwart inequality,

$$\|x_1\|^2 + 2\langle x_1, x_2\rangle + \cdots + 2\langle x_1, x_n\rangle + 2\langle x_2, x_3\rangle + \cdots + \|x_n\|^2$$
$$\leq \|x_1\|^2 + 2\|x_1\|.\|x_2\| + \cdots 2\|x_1\|.\|x_n\| + 2\|x_2\|.\|x_3\| + \cdots + \|x_n\|^2$$

$$\therefore \|\mathrm{x}_1 + x_2 + \cdots + x_n\|^2 \leq n\|x_1\|^2 + n\|x_2\| + \cdots + n\|x_n\|^2$$

QED

-- End of the Document --

C-501, Salarpuria Serenity, 5th Main, Sector 7, HSR Layout, Bengaluru, Karnataka, PIN 560102, India
Mobile: +91 97012 22130, Email: vmelachervu@gmail.com, vmela23@iitk.ac.in, Website: www.linkedin.com/in/vmelachervu