Solutions of Tutorial-3

Problem set 3.1

1 $x + y \neq y + x$ and $x + (y + z) \neq (x + y) + z$ and $(c_1 + c_2)x \neq c_1 x + c_2 x$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times x equals x. Rules (1)-(4) for addition $x + y$ still hold since addition is not changed.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

17 (a) The invertible matrices do not include the zero matrix, so they are not a subspace

(b) The sum of singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular: not a subspace.

Problem set 3.2

1 (a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3 (b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free x_3 Pivot x_1, x_2

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$

(b) Free variable x_3: solution $(1, -1, 1)$. Special solution for each free variable.

5 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

8 If column 4 of a 3 by 5 matrix is all zero then x_4 is a *free* variable. Its special solution is $x = (0, 0, 0, 1, 0)$, because 1 will multiply that zero column to give $Ax = \mathbf{0}$.

9 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

10 If a matrix has n columns and r pivots, there are $n - r$ special solutions. The nullspace contains only $x = \mathbf{0}$ when $r = n$. The column space is all of $\mathbf{R}^m$ when $r = m$. All those statements are important!

16 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through the special solution $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

Problem set 3.3

5 $$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} \text{ solvable if } b_3 - 2b_1 - b_2 = 0.$$

Back-substitution gives the particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{x} = \boldsymbol{0}$: $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

11 A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and also $\boldsymbol{x} = \boldsymbol{0}$ solve $A\boldsymbol{x} = \boldsymbol{0}$

(b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \boldsymbol{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$