Examples

Dual is formulation dependent

$$\min_x \; -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

$y_i = b_i - a_i^T x$

or $y = \underline{b} - Ax$

$$= \min_{(x,y)} \; -\sum_{i=1}^{m} \log(y_i)$$

$$y = b - Ax \quad \text{----} \; \nu$$

$$L(x,y,\nu) = -\sum_{i=1}^{m} \log(y_i) + \sum \nu_i (y_i - b_i + a_i^T x)$$

given ν,

$$= \underbrace{\sum_{i=1}^{m} \left(-\log y_i + \nu_i y_i\right)}_{\text{depends on } y_i} + \underbrace{\sum_{i=1}^{m} \nu_i (a_i^T x)}_{\text{depends on } x} - \underbrace{\sum_{i=1}^{m} \nu_i b_i}_{\text{constant}}$$

$$g(\nu) = \min_{x,y} L(x,y,\nu)$$

$$= \min_{y>0} \sum_{i=1}^{m} (\nu_i y_i - \log y_i) + \min_x \sum \nu_i (a_i^T x) - \nu^T b$$

$$= \sum_{i=1}^{m} \underbrace{\min_{y_i > 0} (\nu_i y_i - \log y_i)}_{} + \min_x \left(\sum_{i=1}^{m} \underline{a_i} \nu_i\right)^T x - \nu^T b$$

① $\frac{d}{dy_i}(\;) = 0$

$$\nu_i - \frac{1}{y_i} = 0$$

or $y_i = 1/\nu_i$ only valid when $\nu_i > 0$

note : when $v_i \leq 0$ then $v_i y_i - \log(y_i) \rightarrow -\infty$

(unbounded below)

so:

if $v_i > 0$ then objective $= 1 + \log(v_i)$

if $v_i < 0$ then objective $\rightarrow -\infty$

so $\min_{y_i \geq 0} v_i y_i - \log(y_i) = \begin{cases} 1 + \log(v_i) & v_i > 0 \\ -\infty & v_i \leq 0 \end{cases}$

② $\min_x \underbrace{\left(\sum a_i v_i\right)}_{c}^T x = \min_x c^T x = \begin{cases} 0 & c = 0 \\ -\infty & \text{o/w} \end{cases}$

$c = 0$ when $\sum a_i v_i = 0$

or $A^T v = 0$

Summary :

$$g(v) = \begin{cases} \sum_{i=1}^{m} (1 + \log(v_i)) + 0 - b^T v & v \geq 0 \quad A^T v = 0 \\ -\infty & \text{o/w} \end{cases}$$

or

$$-g(v) = \begin{cases} -\sum_{i=1}^{m} (1 + \log(v_i)) + b^T v & A^T v = 0, v \geq 0 \\ \infty & \text{o/w} \end{cases}$$

Recall: extended function value definition
(convex function $= \infty$ outside dom)

$$\Rightarrow \quad \operatorname{dom} g = \{ \nu \mid -g(\nu) < \infty \}$$
$$= \{ \nu \mid A^T \nu = 0 \,, \nu \geq 0 \}$$

Dual Problem:

$$\max_{\nu} \quad m + \sum_{i=1}^{m} \log(\nu_i) - b^T \nu$$

$$\text{s.t} \quad \nu \geq 0$$
$$A^T \nu = 0$$

$\nu \in \operatorname{dom} g$ (implicit)

<u>Note</u>

- → dual is formulation-dependent
- — pay attention to domains