Second Order Cone Program

SOCP

$$\min_x \; f^T x$$

$$\|A_i x - b_i\| \leq c_i^T x + d_i \qquad i = 1, \dots m$$

$$Fx = g$$

$$c_i^T x + d_i \geq 0$$

$$\underbrace{\underbrace{\|A_i x - b_i\|}_{\text{norm}} - \underbrace{c_i^T x - d_i}_{\text{affine}}}_{\text{convex}} \leq 0$$

Note: form is important

Eg $\quad \|x\| \leq t \qquad$ convex − affine

but $\iff \|x\|^2 \leq t^2$

or $\underbrace{\|x\|^2 - t^2}_{\text{not convex}} \leq 0$

Eg: $n = 1 \qquad \nabla^2 (x^2 - t^2) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

Algorithms

$$LP \subseteq QP \subseteq QCQP \subseteq SOCP$$

$$\min \tfrac{1}{2} x^T P x + q^T x$$

$P=0$

$$\tfrac{1}{2} x^T P_i x + q_i^T x + r_i \le 0$$

$$q_i^T x + r_i \le 0 \xleftarrow{P_i = 0} \tfrac{1}{2} x^T P_i x + q_i^T x + r_i \le 0$$

$\longleftarrow$ faster algorithms

Claim: $SOCP \supseteq QCQP$

$$\left.\begin{array}{l} \min \tfrac{1}{2} x^T P_0 x + q_0^T x \\ \tfrac{1}{2} x^T P_i x + q_i^T x + r_i \le 0 \end{array}\right] \quad P_0, P_i \succeq 0 \text{ P.S.D.}$$

$$\underbrace{\tfrac{1}{2} x^T P_i x + q_i^T x + r_i}_{\|\sqrt{P_i}\, x + \sqrt{P_i}^{-1} q_i\|^2 - q_i^T P_i^{-1} q_i + r_i} \le 0$$

or $$\|\sqrt{P_i}\, x + \sqrt{P_i}^{-1} q_i\|_2 \le \sqrt{q_i^T P_i q_i - r_i}$$

≥ 0

(problem infeasible otherwise)

$$\min \ \|\sqrt{P_0}x + \sqrt{P_0}^{-1}q_0\|_2$$

$$\|\sqrt{P_i}x + \sqrt{P_i}^{-1}q_i\|_2 \le \sqrt{q_i^T P_i^{-1} q_i - r_i}$$

(epigraph trick)

$$\min \ t$$

$$\|\sqrt{P_0}x + \sqrt{P_0}^{-1}q_0\|_2 \le t$$ SOCP contraints

$$\|\sqrt{P_i}x + \sqrt{P_i}^{-1}q_i\|_2 \le \sqrt{q_i^T P_i^{-1} q_i - r_i}$$

norm ball constraints (also SOCP)