EE901 PROBABILITY AND RANDOM PROCESSES

MODULE 11 DISTRIBUTION OF RANDOM PROCESSES

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Mean of a Random Process

- At a fixed t, $X(t)$ is a random variable.
- The mean of a random process at time t is given as

$$m_X(t) = \mathbb{E}[X(t)]$$

- For a different t, we may get a different mean value.
- Hence, $m_X(t)$ is a function of t.

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Example: Mean of a Random Process

$$X(\omega, t) = A\cos(2\pi f t \; + \Phi(\omega))$$

- The random process at time t_1 is given as

$$Z = X(t_1) = A\cos(2\pi f t_1 \; + \Phi)$$

- Ensemble mean at time t_1

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}[A\cos(2\pi f t_1 \; + \Phi)] \\ &= \int_{-\infty}^{\infty} A\cos\left(2\pi f t_1 + \phi\right) \mathbb{1}(0 < \phi < 2\pi) \frac{1}{2\pi} \mathrm{d}\phi \\ &= \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f t_1 + \phi) \mathrm{d}\phi = 0.\end{aligned}$$

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Mean

- At a fixed t, $X(t)$ is a random variable.

- Its distribution can be found as

$$F_{X(t_1)}(x_1) \; = \mathbb{P}(X(t_1) \leq \; x_1)$$

- The mean of a random process at time t can also be computed as

$$m_X(t) = \mathbb{E}[X(t)] = \int x \, f_{X(t)}(x)\mathrm{d}x$$

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Example: Mean of a Bernoulli Process

Consider A Bernoulli process $X(\omega, t)$ where each of $X(\omega, k)$ is a Bernoulli RV with probability p.

$Z = X(\omega, t_1)$ is a Bernoulli random variable

$$m(t) = \mathbb{E}[X(\omega, t)] \quad = p$$

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Variance

Variance of a random process at time t can be written as

$$\text{Var}(X(t)) = \mathbb{E}\left[X^2(t)\right] - \left(\mathbb{E}\left[X(t)\right]\right)^2$$

Note that at a fixed t, $X(t)$ is a random variable.

The variance is also a function of time t.

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Auto-correlation of a random process

- The auto-correlation of a random process describes the correlation between values of the process at different points in time, as a function of these two time instants

$$R_X(t_1, t_2) = \mathbb{E}[(\underset{z_1}{X(t_1)})\underset{z_2}{X(t_2)}]$$

$\mathbb{E}[z_1 z_2]$

- Note that $Z_1 = X(\omega, t_1)$ and $Z_2 = X(\omega, t_2)$ are RVs. If their distributions are known, $\mathbb{E}[Z_1 Z_2]$ can also be computed from their distributions.

$$E[Z_1 Z_2] = \int Z_1 Z_2 f_{Z_1 Z_2}(z_1, z_2)\, dz_1\, dz_2$$

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Example

Example: Let $X(\omega, t) \;=\; A\cos(2\pi f\, t \;+\; \Phi(\omega))$,

where $\Phi(\omega)$ is uniformly distributed over $(-\pi, +\pi)$

Now, ACF of $X(\omega, t)$ is given by

$$\begin{aligned} R(t_1, t_2) &= \mathbb{E}[\overset{x(t_1)}{A\cos(2\pi f t_1 + \Phi)}\overset{x(t_2)}{A\cos(2\pi f t_2 + \Phi)}] \\ &= \mathbb{E}[A^2\cos(2\pi f t_1 + \Phi)\cos(2\pi f t_2 + \Phi)] \\ &= \mathbb{E}\left[\frac{A^2}{2}\cos(2\pi f(t_1 - t_2))\right] \\ &= \frac{A^2}{2}\cos(2\pi f(t_1 - t_2)) \end{aligned}$$

- ACF here is dependent on the difference only.

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Example

- Consider a Bernoulli process

$$R_X(t_1, t_2) = \mathbb{E}[(X(t_1))X(t_2)]$$

$\mathbb{E}[Z_1 Z_2] = \mathbb{E}[Z_1]\,\mathbb{E}[Z_2]$

- $X(\omega, t_1) = Z_1$ is a Bernoulli random variable $\sim p$
- $X(\omega, t_2) = Z_2$ is another Bernoulli random variable $\sim p$

If $t_1 = t_2$, $Z_1 = Z_2 = Z$, then

$$R_X(t_1, t_1) = \mathbb{E}[Z^2] = p$$

If $t_1 \neq t_2$, then

$$\mathbb{E}[Z_1 Z_2] = \mathbb{E}[Z_1]E[Z_2] = pp = p^2$$

(as they are independent.)

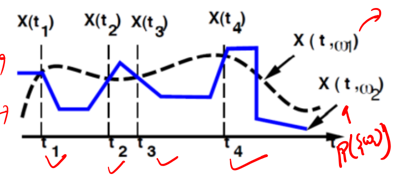
Auto-covariance of Random Processes

- Auto-covariance is defined as

$$C_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mathbb{E}[X(t_1)])(X(t_2) - \mathbb{E}[X(t_2)])]$$

Finite Dimensional Distribution

- Consider a number n. Let $(t_1, t_2, t_3 \cdots t_n)$ be n time instants.
- The nth order FDD is defined as the joint distribution of $X(\omega, t_1), X(\omega, t_2), X(\omega, t_3), \cdots, X(\omega, t_n)$
- Let's denote these RVs as $X_1, X_2, X_3, \cdots, X_n$.



$$F_{X_1, X_2, \cdots, X_n}(x_1, x_2, \dots, x_n) = \mathbb{P}(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n)$$

$$= \mathbb{P}(X(t_1) \le x_1, X(t_2) \le x_2, \cdots, X(t_n) \le x_n)$$

$X(t_1)$ $X(\omega, t_1)$

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Wide Sense Stationary Processes

A random process is said to be a Wide Sense Stationary (WSS) Process if it satisfies the following properties

1. Its mean is a constant with time

$$E[X(t)] = \text{constant}$$

2. Its ACF is a function of time difference τ only

$$\mathbb{E}[X(t)X(t+\tau)] = f(\tau)$$

$R_X(t, t+\tau)$ t_1 t_2 $f(t_2 - t_1)$ $f(t_1 - t_2)$

$$R_X(t_1, t_2) = f(t_2 - t_1)$$

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Power Spectral Density

- Power Spectral Density (PSD) of a WSS $X(t)$ is the Fourier Transform of ACF of $X(t)$

$\mathbb{E}[X(t)X(t+0)]$

$\mathbb{E}[X(t)^2]$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} \mathrm{d}\tau$$

1^0

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} \mathrm{d}f$$

At $t = 0$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f) \mathrm{d}f$$

$-\infty$ PSD ∞

$\mathbb{E}[|X(t)|^2]$

Example

- Let a RP X has autocorrelation given as

$$R_X(\tau) = e^{-2\alpha|\tau|}$$

$$S_X(\omega) = \mathcal{F}\{R_X(\tau)\} = \frac{4\alpha}{4\alpha^2 + \omega^2}$$

Example:

Let $X(t) = a\cos(\omega_0 t + \Theta)$, $\Theta \sim \text{Uniform}[0, 2\pi]$. Find $R_X(\tau)$ and $S_X(\omega)$.

Solution:

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PSD Interpretation

$$R_X(0) = \mathbb{E}[|X(t)|^2] = \int_{-\infty}^{\infty} S_X(f)\mathrm{d}f$$

f_1 f_2

- Integrating $S_X(f)$ over a band gives the expected power of the signal content between these frequencies.

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Cross Correlation

- The cross-correlation function of $X(t)$ and $Y\,(t)$ is

$$R_{X,Y}(t_1, t_2) = \mathbb{E}[X(t_1)Y\,(t_2)]$$

- The cross-covariance function of $X(t)$ and $Y\,(t)$ is

$$C_{X,Y}\,(t_1, t_2) \;= \mathbb{E}\left[\left(X(t_1) - m_X(t_1)\right)\left(Y\,(t_2) - m_Y(t_2)\right)\right]$$

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EE901
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Module 12
Random Processes Through Linear Systems

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