EE910: Digital Communication Systems-I

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Lecture #2B: Representation of lowpass and bandpass signals

Commonly Used Signals

- Rectangular signal

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ \frac{1}{2} & t = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Commonly Used Signals

- Sinc signal

$$\mathrm{sinc(t)} = \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases} \quad (2)$$

Commonly Used Signals

- Signum signal

$$sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases} \tag{3}$$

Commonly Used Signals

- Unit step signal

$$u_{-1}(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{4}$$

Commonly Used Signals

- Triangular signal

$$\Lambda(t) = \Pi(t) * \Pi(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ -t+1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (5)$$

Signals and their frequency spectrums

- Fourier transform of a real signal $x(t)$ has *Hermitian* symmetry, i.e. $X(-f) = X^*(f)$ or

$$|X(-f)| = |X(f)| \quad \text{and} \quad \angle X(-f) = -\angle X(f)$$

- The *positive spectrum* and the *negative spectrum*

$$X_+(f) = \begin{cases} X(f) & f > 0 \\ \frac{1}{2}X(0) & f = 0 \\ 0 & f < 0 \end{cases} \qquad X_-(f) = \begin{cases} X(f) & f < 0 \\ \frac{1}{2}X(0) & f = 0 \\ 0 & f > 0 \end{cases}$$

$$\begin{aligned} X_+(f) &= X(f)u_{-1}(f), \ (u_{-1}(.) \text{ is unit-step func.}) \\ X_-(f) &= X(f)u_{-1}(-f) \\ X(f) &= X_+(f) + X_-(f) = X_+(f) + X_+^*(-f) \qquad \text{(for real signal)} \end{aligned}$$

which means knowledge of $X_+(f)$ is sufficient to reconstruct $X(f)$

Bandpass and lowpass signals

- A bandpass signal is real signal whose frequency spectrum is located around some frequency f_0 which is far from zero.
- A lowpass signal is a real signal whose spectrum is located around the zero frequency.

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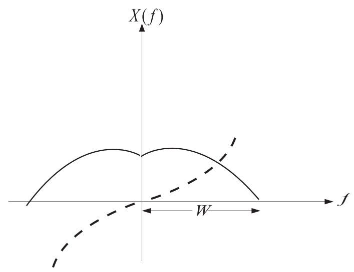

Figure: The spectrum of a real valued lowpass baseband signal

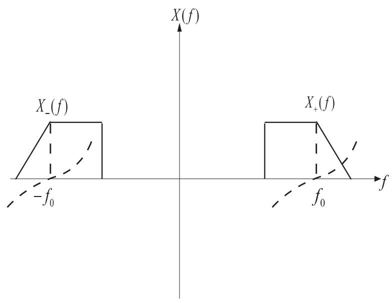

Figure: The spectrum of a real valued bandpass signal

Lowpass equivalent of bandpass signals

- The *analytic signal* or the *pre-envelope*, corresponding to $x(t)$ denoted by the signal $x_+(t)$ whose Fourier transform is $X_+(f)$

$$
\begin{aligned}
x_+(t) &= \mathcal{F}^{-1}\left[X_+(f)\right] \\
&= \mathcal{F}^{-1}\left[X(f)u_{-1}(f)\right] \\
&= x(t) * \left(\frac{1}{2}\delta(t) + j\frac{1}{2\pi t}\right) \\
&= \frac{1}{2}x(t) + \frac{j}{2}\hat{x}(t)
\end{aligned}
$$

where, $\hat{x}(t)$ is *Hilbert transform* of $x(t)$.

Lowpass equivalent of bandpass signals

- *Hilbert transform* of $x(t)$

$$\hat{x}(t) = \frac{1}{\pi t} * x(t)$$

and

$$\mathcal{F}[\hat{x}(t)] = -j\text{sgn}(f)X(f)$$

where

$$\text{sgn}(f) = \begin{cases} -1 & f < 0 \\ 1 & f \geq 0 \end{cases}$$

Lowpass equivalent of bandpass signals

- $x_l(t)$ denotes the *lowpass equivalent* or *complex envelope* of $x(t)$ whose Fourier transform is

$$X_l(f) = \mathcal{F}[x_l(t)] = 2X_+(f+f_0) = 2X(f+f_0)u_{-1}(f+f_0)$$

- Applying the modulation theorem of the Fourier transform

$$\begin{aligned} x_l(t) &= \mathcal{F}^{-1}[X_l(f)] = 2x_+(t)e^{-j2\pi f_0 t} \\ &= (x(t)+j\hat{x}(t))\,e^{-j2\pi f_0 t} \\ &= (x(t)\cos 2\pi f_0 t + \hat{x}(t)\sin 2\pi f_0 t) \\ &\quad +j\,(\hat{x}(t)\cos 2\pi f_0 t - x(t)\sin 2\pi f_0 t) \qquad \textbf{(demodulation)} \end{aligned}$$

- We can write

$$\begin{aligned} x(t) &= \text{Re}\left[x_l(t)e^{j2\pi f_0 t}\right] \qquad \textbf{(modulation)} \\ \Rightarrow X(f) &= \frac{1}{2}[X_l(f-f_0) + X_l^*(-f-f_0)] \end{aligned}$$

Lowpass equivalent of bandpass signals

- *In-phase component* and *quadrature component* of $x_l(t)$

$$x_l(t) = x_i(t) + jx_q(t))$$

where

$$\begin{aligned} x_i(t) &= x(t)\cos 2\pi f_0 t + \hat{x}(t)\sin 2\pi f_0 t \\ x_q(t) &= \hat{x}(t)\cos 2\pi f_0 t - x(t)\sin 2\pi f_0 t \end{aligned}$$

$$\Rightarrow \begin{array}{l} x(t) = x_i(t)\cos 2\pi f_0 t - x_q(t)\sin 2\pi f_0 t \\ \hat{x}(t) = x_q(t)\cos 2\pi f_0 t + x_i(t)\sin 2\pi f_0 t \end{array}$$

Lowpass equivalent of bandpass signals

- $x_l(t)$ in polar coordinates

$$\begin{aligned} \text{envelope of } x(t) \quad r_x(t) &= \sqrt{x_i^2(t) + x_q^2(t)} \\ \text{phase of } x(t) \quad \theta_x(t) &= \arctan \frac{x_q(t)}{x_i(t)} \\ \rightarrow x_l(t) &= r_x(t)e^{j\theta_x(t)} \end{aligned}$$

$$x(t) = \text{Re}\left[r_x(t)e^{(j2\pi f_0 t + \theta_x(t))}\right]$$

$\Rightarrow$ resulting in

$$x(t) = r_x(t)\cos(2\pi f_0 t + \theta_x(t))$$

Energy Considerations

- The *energy* of a signal $x(t)$ is defined is

$$
\begin{aligned}
\mathcal{E}_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \int_{-\infty}^{\infty} |X(f)|^2 df \quad \textbf{(by Rayleigh's relation)} \\
&= \int_{-\infty}^{\infty} |X_+(f) + X_-(f)|^2 df \qquad (6) \\
&= \int_{-\infty}^{\infty} |X_+(f)|^2 df + \int_{-\infty}^{\infty} |X_-(f)|^2 df \quad \left(\mathbf{X_+(f)X_-(f) = 0}\right) \\
&= 2\int_{-\infty}^{\infty} |X_+(f)|^2 df \quad \left(\textbf{for real signals}\right) \\
&= 2\mathcal{E}_{x_+} \qquad (7)
\end{aligned}
$$

Energy Considerations

recall,

$$
\begin{aligned}
X_l(f) &= 2X_+(f+f_0) \\
\Rightarrow \int_{-\infty}^{\infty} \left|\frac{X_l(f)}{2}\right|^2 df &= \int_{-\infty}^{\infty} |X_+(f)|^2 df
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\mathcal{E}_x &= 2\int_{-\infty}^{\infty} |X_+(f)|^2 df \\
&= \int_{-\infty}^{\infty} \left|\frac{X_l(f)}{2}\right|^2 df \\
&= \frac{1}{2}\mathcal{E}_{x_l}
\end{aligned}
$$

Energy Considerations

- Define *inner product of two signals* $x(t)$ and $y(t)$ as

$$\begin{aligned} \langle x(t), y(t)\rangle &\triangleq \int_{-\infty}^{\infty} x(t)y^*(t)dt \\ &= \int_{-\infty}^{\infty} X(f)Y^*(f)df \quad \textbf{(by Parseval's relation)} \\ \Rightarrow \mathcal{E}_x &= \langle x(t), x(t)\rangle \end{aligned}$$

- If $x(t)$ and $y(t)$ are two bandpass signals with lowpass equivalents $x_l(t)$ and $y_l(t)$ with respect to the same center frequency f_0, then

$$\langle x(t), y(t)\rangle = \frac{1}{2}\mathrm{Re}\left[\langle x_l(t), y_l(t)\rangle\right] \tag{8}$$

Bandpass system

- A system whose transfer function is located around a frequency f_0 (far from origin) or, a system whose impulse response $h(t)$ is a bandpass signal. Since $h(t)$ is bandpass,

$$h(t) = \mathrm{Re}\left[h_l(t)e^{j2\pi f_0 t}\right]$$

- If a bandpass signal $x(t)$ is passed through a bandpass system with impulse response $h(t)$ then its output $y(t)$ is also a bandpass signal

$$Y(f) = X(f)H(f)$$

Lowpass equivalent of a bandpass system

- Spectrum of its lowpass equivalent $Y_l(f)$

$$
\begin{aligned}
Y_l(f) &= 2Y(f+f_0)u_{-1}(f+f_0) \\
&= 2H(f+f_0)X(f+f_0)u_{-1}(f+f_0) \\
&= \frac{1}{2}\left[2X(f+f_0)u_{-1}(f+f_0)\right]\left[2H(f+f_0)u_{-1}(f+f_0)\right] \textbf{(true if } f > f_0\textbf{)} \\
&= \frac{1}{2}X_l(f)H_l(f)
\end{aligned}
$$