eMasters in Communication Systems

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Elective Module:

Detection for Wireless Communication

Chapter 5

Bayesian Detection

min probability of Error Decision rule.

Min P_e Detector

Bayesian Detection

- Consider a **Bayesian Binary Hypothesis Testing:**

$0 \leq \pi_0, \pi_1 \leq 1$

$$P(\mathcal{H}_0) = \pi_0$$
$$P(\mathcal{H}_1) = \pi_1$$

$\pi_0 + \pi_1 = 1$

Prior probabilities of hypotheses.

Bayesian Detection

- Consider a **Bayesian Binary Hypothesis Testing**:

$$P(\mathcal{H}_0) = \pi_0$$
$$P(\mathcal{H}_1) = \pi_1$$

Bayesian Detection

- ***Decision region*** for $\mathcal{H}_1$ is $R_1 \subset \mathbb{R}^N$

if $\bar{y} \in R_1$ then choose $\mathcal{H}_1$,
$\bar{y} \in R_0$ then choose $\mathcal{H}_0$.

- ***Decision region*** for $\mathcal{H}_0$ is $R_0 = \mathbb{R}^N - R_1$

$$R_1 \cup R_0 = \mathbb{R}^N.$$

$$R_1 \cap R_0 = \emptyset.$$

Bayesian Detection

- ***Decision region*** for $\mathcal{H}_1$ be $\bar{\mathbf{y}} \in R_1$

- ***Decision region*** for $\mathcal{H}_0$ is $\bar{\mathbf{y}} \in R_0 = \mathcal{R}^N - R_1$

Bayesian Detection

- The probability of error P_e is given as

Probability of False Alarm

Probability of miss detection

$$P_e = \Pr(\mathcal{H}_0)\overbrace{\Pr(\mathcal{H}_1|\mathcal{H}_0)}^{P_{FA}} + \Pr(\mathcal{H}_1)\overbrace{\Pr(\mathcal{H}_0|\mathcal{H}_1)}^{P_{MD}}$$

$$= \pi_0 \cdot \underset{\bar{y}\in R_1;\,\mathcal{H}_0}{\Pr(\mathcal{H}_1|\mathcal{H}_0)} + \pi_1 \cdot \underset{\bar{y}\in R_0;\,\mathcal{H}_1}{\Pr(\mathcal{H}_0|\mathcal{H}_1)}$$

Bayesian Detection

- The probability of error P_e is given as

$$
\begin{aligned}
P_e &= P(\mathcal{H}_0|\mathcal{H}_1)P(\mathcal{H}_1) + P(\mathcal{H}_1|\mathcal{H}_0)\Pr(\mathcal{H}_0) \\
&= P(\mathcal{H}_0|\mathcal{H}_1)\pi_1 + P(\mathcal{H}_1|\mathcal{H}_0)\pi_0
\end{aligned}
$$

Bayesian Detection

Pr $\bar{y} \in R_0$ given $\mathcal{H}_1$. Pr $\bar{y} \in R_1$ given $\mathcal{H}_0$.

- P_e can be expressed as

$$P_e = \pi_1 \underbrace{\int_{R_0} p(\bar{y}|\mathcal{H}_1)\,d\bar{y}} + \pi_0 \underbrace{\int_{R_1} p(\bar{y}|\mathcal{H}_0)\,d\bar{y}}$$

$$= \pi_1 \int_{R_0} p(\bar{y}|\mathcal{H}_1)\,d\bar{y} + \pi_0 \cdot \left(1 - \int_{R_0} p(\bar{y};\mathcal{H}_0)\,d\bar{y}\right)$$

Bayesian Detection

$$\int_{R_1} p(\bar{y}|\mathcal{H}_0)\,d\bar{y} + \int_{R_0} p(\bar{y}|\mathcal{H}_0)\,d\bar{y} = 1$$

$$\overbrace{P_e}^{\text{minimize.}} = \underbrace{\int_{R_0} \left(\pi_1 p(\bar{y}|\mathcal{H}_1) - \pi_0 p(\bar{y}|\mathcal{H}_0)\right) d\bar{y}}_{\text{minimize.}} + \overset{\text{constant}}{\pi_0}.$$

Bayesian Detection

To minimize $\int_{R_0} \left(\pi_1 p(\bar{y}|\mathcal{H}_1) - \pi_0 p(\bar{y}|\mathcal{H}_0) \right) d\bar{y}$

include all $\bar{y}$ in R_0 such that

$$\pi_1 p(\bar{y}|\mathcal{H}_1) - \pi_0 p(\bar{y}|\mathcal{H}_0) < 0$$

Bayesian Detection

- P_e can be expressed as

$$\pi_1 \int_{R_0} p(\bar{\mathbf{y}}|\mathcal{H}_1)d\bar{\mathbf{y}} + \pi_0 \int_{R_1} p(\bar{\mathbf{y}}|\mathcal{H}_0)d\bar{\mathbf{y}}$$

Bayesian Detection

$$\pi_1 \int_{R_0} p(\bar{\mathbf{y}}|\mathcal{H}_1)d\bar{\mathbf{y}} + \pi_0 \left(1 - \int_{R_0} p(\bar{\mathbf{y}}|\mathcal{H}_0)d\bar{\mathbf{y}} \right)$$

$$= \int_{R_0} \big(\pi_1 p(\bar{\mathbf{y}}|\mathcal{H}_1) - \pi_0 p(\bar{\mathbf{y}}|\mathcal{H}_0)\big)d\bar{\mathbf{y}} + \pi_0$$

Bayesian Detection

- To minimize, choose R_0 as

$\bar{y}$ such that

$$\pi_1 \, p(\bar{y}|\mathcal{H}_1) - \pi_0 \, p(\bar{y}|\mathcal{H}_0) \leq 0$$

$$\Rightarrow \frac{p(\bar{y}|\mathcal{H}_0)}{p(\bar{y}|\mathcal{H}_1)} \geq \frac{\pi_1}{\pi_0} \qquad \tilde{\gamma}$$

Reduces to Likelihood Ratio Test (LRT)

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0}.$$

Bayesian Detection

- To minimize, choose R_0 as

$$\pi_1 p(\bar{\mathbf{y}}|\mathcal{H}_1) - \pi_0 p(\bar{\mathbf{y}}|\mathcal{H}_0) \leq 0$$

$$\Rightarrow \pi_0 p(\bar{\mathbf{y}}|\mathcal{H}_0) \geq \pi_1 p(\bar{\mathbf{y}}|\mathcal{H}_1)$$

$$\Rightarrow \frac{p(\bar{\mathbf{y}}|\mathcal{H}_0)}{p(\bar{\mathbf{y}}|\mathcal{H}_1)} \geq \frac{\pi_1}{\pi_0}$$

Bayesian Detection

- It reduces to LRT with

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0} = \frac{Pr(\mathcal{H}_1)}{Pr(\mathcal{H}_0)}$$

Bayesian Detection

- It reduces to LRT with

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0} = \frac{\Pr(\mathcal{H}_1)}{\Pr(\mathcal{H}_0)}$$

MAP Decision Rule

Bayes Rule:

- Choose $\mathcal{H}_0$ if

$$\underbrace{\Pr(\mathcal{H}_0|\bar{y}) \qquad \Pr(\mathcal{H}_1|\bar{y})}_{\text{A posteriori probabilities.}}$$

$$\pi_0\, p(\bar{y}|\mathcal{H}_0) \geq \pi_1\, p(\bar{y}|\mathcal{H}_1).$$

$$\Rightarrow \frac{\pi_0\, p(\bar{y}|\mathcal{H}_0)}{\pi_0\, p(\bar{y}|\mathcal{H}_0) + \pi_1\, p(\bar{y}|\mathcal{H}_1)} \geq \frac{\pi_1\, p(\bar{y}|\mathcal{H}_1)}{\pi_0\, p(\bar{y}|\mathcal{H}_0) + \pi_1\, p(\bar{y}|\mathcal{H}_1)}.$$

$$\Rightarrow \frac{\Pr(\mathcal{H}_0)\, p(\bar{y}|\mathcal{H}_0)}{\Pr(\mathcal{H}_0)\, p(\bar{y}|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\, p(\bar{y}|\mathcal{H}_1)} \geq \frac{\Pr(\mathcal{H}_1)\, p(\bar{y}|\mathcal{H}_1)}{\Pr(\mathcal{H}_0)\, p(\bar{y}|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\, p(\bar{y}|\mathcal{H}_1)}$$

MAP Decision Rule

- Chose $\mathcal{H}_0$ if

$$\Rightarrow \frac{p(\bar{\mathbf{y}}|\mathcal{H}_0)}{p(\bar{\mathbf{y}}|\mathcal{H}_1)} \geq \frac{\pi_1}{\pi_0}$$

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0}$$

$$\pi_1 = \Pr(\mathcal{H}_1)$$

$$\pi_0 = \Pr(\mathcal{H}_0)$$

$$\Rightarrow \Pr(\mathcal{H}_0)\, p(\bar{\mathbf{y}}|\mathcal{H}_0) \geq \Pr(\mathcal{H}_1)\, p(\bar{\mathbf{y}}|\mathcal{H}_1)$$

$$\Rightarrow \frac{\Pr(\mathcal{H}_0)\, p(\bar{\mathbf{y}}|\mathcal{H}_0)}{\Pr(\mathcal{H}_0)\, p(\bar{\mathbf{y}}|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\, p(\bar{\mathbf{y}}|\mathcal{H}_1)}$$

$$\geq \frac{\Pr(\mathcal{H}_1)\, p(\bar{\mathbf{y}}|\mathcal{H}_1)}{\Pr(\mathcal{H}_0)\, p(\bar{\mathbf{y}}|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\, p(\bar{\mathbf{y}}|\mathcal{H}_1)}$$

MAP Decision Rule

- Choose $\mathcal{H}_0$ if

$$\Pr(\mathcal{H}_0 | \bar{y}) \geq \Pr(\mathcal{H}_1 | \bar{y})$$

Choose hypothesis with maximum aposteriori probability (MAP).

MAP decision rule!

MAP Decision Rule

Aposteriori Probabilities .
$\Pr(\mathcal{H}_0 | \bar{y}) \geq \Pr(\mathcal{H}_1 | \bar{y})$

- Choose $\mathcal{H}_0$ if

$$\Rightarrow \Pr(\mathcal{H}_0|\bar{\mathbf{y}}) \geq \Pr(\mathcal{H}_1|\bar{\mathbf{y}})$$

choose $\mathcal{H}_1$ if $\Pr(\mathcal{H}_0 | \bar{y}) < \Pr(\mathcal{H}_1 | \bar{y})$

- i.e. choose hypothesis with the ***Maximum Aposteriori Probability*** (MAP)

MAP Decision Rule

- This is termed as the **Maximum Aposteriori Probability (MAP) rule**
- Therefore, MAP rule **minimizes the** Probability of error Pe.

MAP Decision Rule

- This is termed as the **Maximum Aposteriori Probability (MAP) rule**

- Therefore, MAP rule **minimizes the probability of error**

MAP Decision Rule

- Finally, when $\pi_0 = \pi_1 = \frac{1}{2}$

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0} = \frac{1/2}{1/2} = 1 \Rightarrow ML$$

- MAP reduces to ML Decision rule!

MAP Decision Rule

- Finally, when $\pi_0 = \pi_1 = \frac{1}{2}$

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0} = 1$$

- MAP reduces to ML!

MAP Decision Rule

- When prior probabilities are equal, MAP becomes the Maximum Likelihood (ML) rule!

MAP Decision Rule

- When prior probabilities are equal, MAP becomes the **ML decision rule**

Bayesian Detection

$E\{\bar{v}\} = 0$
$E\{\bar{v}\bar{v}^T\} = \sigma^2 I$

- Consider now the signal detection problem

noise

$\mathcal{H}_0$: $\bar{y} = \bar{v}$

$\mathcal{H}_1$: $\bar{y} = \bar{s} + \bar{v}$

Bayesian Detection

- Consider now the signal detection problem

$$\mathcal{H}_0 : \bar{\mathbf{y}} = \bar{\mathbf{v}}$$

$$\mathcal{H}_1 : \bar{\mathbf{y}} = \bar{\mathbf{s}} + \bar{\mathbf{v}}$$

LRT

$\bar{s}^T\bar{y} = \sum_{i=1}^{N-1} y(i)s(i)$

$\tilde{\gamma} = \frac{\pi_1}{\pi_0}$

- Choose $\mathcal{H}_0$ if

$$\bar{s}^T\bar{y} = \sum_{i=1}^{N} y(i)s(i) \le \frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \gamma$$

$$\gamma = \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \frac{\pi_1}{\pi_0}}{2}$$

MAP rule

min P_e.

choose $\mathcal{H}_0$ if $\bar{s}^T\bar{y} \le \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \frac{\pi_1}{\pi_0}}{2}$

LRT

- Choose $\mathcal{H}_0$ if

$$\sum_{i=1}^{N} y(i)s(i) \leq \frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \gamma$$

γ

$$\gamma = \frac{\|\bar{\mathbf{s}}\|^2}{2} - \sigma^2 \ln \frac{\pi_1}{\pi_0}$$

$\tilde{\gamma}$

LRT

- Choose $\mathcal{H}_0$ if

$$\sum_{i=1}^{N} y(i)s(i) \leq \frac{\|\bar{\mathbf{s}}\|^2}{2} - \sigma^2 \ln \frac{\pi_1}{\pi_0}$$

LRT

- Choose $\mathcal{H}_1$ if

$$\bar{s}^T \bar{y} = \sum_i y(i) s(i) > \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2}$$

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0}$$

$$\bar{s}^T \bar{y} > \frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right).$$

LRT

- Choose $\mathcal{H}_1$ if

$$\bar{\mathbf{s}}^T \bar{\mathbf{y}} > \frac{\|\bar{\mathbf{s}}\|^2}{2} - \sigma^2 \ln \frac{\pi_1}{\pi_0}$$

Performance ML

What is min P_e?

- The min P_e can now be determined as follows

Performance ML

Probability of False Alarm

- Recall, P_{FA} is

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{\sigma\|\bar{s}\|}\right)$$

$$= Q\left(\frac{\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right)$$

Performance ML

- Recall, P_{FA} is

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln\frac{\pi_1}{\pi_0}}{2\sigma\|\bar{\mathbf{s}}\|}\right)$$

Performance ML

Probability of Detection.

- Recall, P_D is

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right) - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$$= Q\left(\frac{-\frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{\sigma\|\bar{s}\|}\right)$$

$$= Q\left(\frac{-\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{\sigma\|\bar{s}\|}\right)$$

Performance ML

- Recall, P_D is

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{-\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln\frac{\pi_1}{\pi_0}}{2\sigma\|\bar{\mathbf{s}}\|}\right)$$

Performance ML

$$1 - Q(-x) = Q(x)$$

$$P_{MD} = 1 - P_D = 1 - Q\left(-\frac{\|\bar{s}\|^2 + 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right)$$

$$= Q\left(\frac{\|\bar{s}\|^2 + 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right)$$

Probability of misdetection

Performance ML

$$P_{MD} = 1 - P_D = 1 - Q\left(\frac{-||\bar{\mathbf{s}}||^2 - 2\sigma^2 \ln\frac{\pi_1}{\pi_0}}{2\sigma||\bar{\mathbf{s}}||}\right)$$

$$= Q\left(\frac{||\bar{\mathbf{s}}||^2 + 2\sigma^2 \ln\frac{\pi_1}{\pi_0}}{2\sigma||\bar{\mathbf{s}}||}\right)$$

Performance ML

- Therefore, P_e is

$$\pi_0 P_{FA} + \pi_1 P_{MD}$$

$$= \pi_0 Q\left(\frac{\|s\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|s\|}\right) + \pi_1 Q\left(\frac{\|s\|^2 + 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|s\|}\right)$$

Performance ML

Probability of Error

P_{FA}

P_{MD}:

- Therefore, P_e is

$Pr(\mathcal{H}_0)\, Pr(\mathcal{H}_1 | \mathcal{H}_0)$

$+ Pr(\mathcal{H}_1) \cdot Pr(\mathcal{H}_0 | \mathcal{H}_0)$

$$\pi_0 P_{FA} + \pi_1 P_{MD}$$

$$= \pi_0 Q\left(\frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \frac{\pi_1}{\pi_0}}{2\sigma\|\bar{\mathbf{s}}\|}\right) + \pi_1 Q\left(\frac{\|\bar{\mathbf{s}}\|^2 + 2\sigma^2 \ln \frac{\pi_1}{\pi_0}}{2\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$= \pi_0 Q\left(\frac{\|\bar{s}\|}{2\sigma} - \frac{\sigma}{\|\bar{s}\|} \ln\left(\frac{\pi_1}{\pi_0}\right)\right) + \pi_1 Q\left(\frac{\|\bar{s}\|}{2\sigma} + \frac{\sigma}{\|\bar{s}\|} \ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

Performance ML

$$\frac{\|s\|}{\sigma} = \sqrt{\frac{\|s\|^2}{\sigma^2}} = \sqrt{SNR}.$$

- This can be simplified as

$$SNR = \frac{\|s\|^2}{\sigma^2}$$

$$P_e = \pi_0 \, Q\left(\frac{1}{2}\sqrt{SNR} - \frac{1}{\sqrt{SNR}} \ln\left(\frac{\pi_1}{\pi_0}\right)\right) + \pi_1 \, Q\left(\frac{1}{2}\sqrt{SNR} + \frac{1}{\sqrt{SNR}} \ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

Performance ML

- This can be simplified as

Minimum Prob of Error

$$\pi_0 Q\left(\frac{1}{2}\sqrt{SNR} - \frac{1}{\sqrt{SNR}}\ln\frac{\pi_1}{\pi_0}\right)$$
$$+ \pi_1 Q\left(\frac{1}{2}\sqrt{SNR} + \frac{1}{\sqrt{SNR}}\ln\frac{\pi_1}{\pi_0}\right)$$

Simple example

$= 10$

$\pi_0 = 1 - \pi_1 = 0.2$

- $SNR = 10\ dB.\ \pi_1 = 0.8$
- **Minimum probability of error?**

Simple example

- We have

$$SNR = 10\ \text{dB} = 10$$

Simple example

- We have

$$SNR = \frac{\|\bar{\mathbf{s}}\|^2}{\sigma^2} = 10$$

Simple example

- Min P_e is

$$\frac{\pi_1}{\pi_0} = \frac{0.8}{0.2} = 4$$

$$P_e = \pi_1 \times Q\left(\frac{1}{\sqrt{SNR}}\ln\left(\frac{\pi_1}{\pi_0}\right) + \frac{1}{2}\sqrt{SNR}\right) + \pi_0 \times Q\left(\frac{1}{2}\sqrt{SNR} - \frac{1}{\sqrt{SNR}}\ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

$$= 0.8\,Q\left(\frac{1}{2}\sqrt{10} + \frac{1}{\sqrt{10}}\ln 0.4\right) + 0.2\,Q\left(\frac{1}{2}\sqrt{10} - \frac{1}{\sqrt{10}}\ln 0.4\right)$$

$$= 0.8\,Q(2.01) + 0.2\,Q(1.14)$$

$$= 0.0408$$

Min P_e.

Simple example

- Min P_e is

$$\pi_1 \times Q\left(\frac{1}{\sqrt{SNR}}\ln\left(\frac{\pi_1}{\pi_0}\right)+\frac{1}{2}\sqrt{SNR}\right)+\pi_0 \times Q\left(\frac{1}{2}\sqrt{SNR}-\frac{1}{\sqrt{SNR}}\ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

$$= 0.8 \times Q\left(\frac{1}{\sqrt{10}}\ln(4)+\frac{1}{2}\sqrt{10}\right)+0.2 \times Q\left(\frac{1}{2}\sqrt{10}-\frac{1}{\sqrt{10}}\ln(4)\right)$$

$$= 0.8 \times Q(2.01) + 0.2 \times Q(1.14) = 0.0408$$

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

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