EE910: Digital Communication Systems-I

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Lecture #7A: Noncoherent detection of carrier modulated signals

Noncoherent Detection

- In the detection schemes we have studied so far, we made the implicit assumption that the signals $\{s_m(t), 1 \leq m \leq M\}$ are available at the receiver.
- This assumption was in the form of either the availability of the signals themselves or the availability of an orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$.
- There are many cases where cannot make such an assumption
  - One of the cases in which such an assumption is invalid occurs when transmission over the channel introduces random changes to the signal as either a random attenuation or a random phase shift.
  - Another situation that results in imperfect knowledge of the signals at the receiver arises when the transmitter and the receiver are not perfectly synchronized.

Noncoherent Detection

- In this case, although the receiver knows the general shape of $\{s_m(t)\}$, due to imperfect synchronization with the transmitter, it can use only signals in the form of $\{s_m(t - t_d)\}$, where t_d represents the time slip between the transmitter and the receiver clocks.
- This time slip can be modelled as a random variable.
- To study the effect of random parameters of this type on the optimal receiver design and performance, we consider the transmission of a set of signals over the AWGN channel with some random parameter denoted by the random vector θ.
- We assume that signals $\{s_m(t), 1 \leq m \leq M\}$ are transmitted, and the received signal r(t) can be written as

$$\mathbf{r} = s_m(t;\theta) + n(t) \tag{1}$$

where θ is in general a vector-valued random variable.

Noncoherent Detection

- We can find an orthonormal basis for expansion of the random process $s_m(t;\theta)$ and the same orthonormal basis can be used for expansion of the white Gaussian noise process n(t).
- By using this basis, the waveform channel given in Equation (1).5-1 becomes equivalent to the vector channel

$$\mathbf{r} = s_{m,\theta} + n \tag{2}$$

for which the optimal detection rule is given by

$$\begin{aligned}\hat{m} &= \underset{1\leq m\leq M}{argmax} P_m p(\mathbf{r}|m) \\ &= \underset{1\leq m\leq M}{argmax} P_m \int p(\mathbf{r}|m,\theta)p(\theta)d\theta \\ &= \underset{1\leq m\leq M}{argmax} P_m \int p_n(\mathbf{r} - s_{m,\theta})p(\theta)d\theta\end{aligned} \tag{3}$$

Noncoherent Detection

- Equation (3) represents the optimal decision rule and the resulting decision regions.
- The minimum error probability, when the optimal detection rule of Equation (3) is employed, is given by

$$\begin{aligned}P_e &= \sum_{m=1}^{M} P_m \int_{D_m^c} \left(\int p(\mathbf{r}|m,\theta)p(\theta)d\theta \right) d\mathbf{r} \\ &= \sum_{m=1}^{M} P_m \sum_{m'=1,m'\neq m}^{M} \int_{D_m^c} \left(\int p(\mathbf{r}|m,\theta)p(\theta)d\theta \right) d\mathbf{r}\end{aligned} \tag{4}$$

- Equations (3) and (4) are quite general and can be used for all types of uncertainties in channel parameters.

Noncoherent Detection

- Consider a binary antipodal signalling system where equiprobable signals $s_1(t) = s(t)$ and $s_2(t) = -s(t)$ are used on an AWGN channel with noise power spectral density of $\frac{N_0}{2}$.
- Consider a channel that introduces a random gain of A which can take only nonnegative values.
- This channel can be modelled as

$$r(t) = As_m(t) + n(t) \tag{5}$$

where A is a random gain with PDF p(A) and $p(A) = 0$ for $A < 0$.

Noncoherent Detection

- Using Equation (3), and noting that $p(\mathbf{r}|m, A) = p_n(r - As_m)$,$D_1$, the optimal decision region for $s_1(t)$ is given by

$$D_1 = r : \int_0^{\infty} e^{-\frac{r - A\sqrt{\mathcal{E}_b})^2}{N_0}} p(A)dA > \int_0^{\infty} e^{-\frac{(r + A\sqrt{\mathcal{E}_b})^2}{N_0}} p(A)dA \tag{6}$$

- Equation (6) simplifies to

$$D_1 = r : \int_0^{\infty} e^{-\frac{A^2\mathcal{E}_b}{N_0}} \left(e^{\frac{2rA\sqrt{\mathcal{E}_b}}{N_0}} - e^{\frac{2rA\sqrt{\mathcal{E}_b}}{N_0}} \right) p(A)dA > 0 \tag{7}$$

- Since A takes only positive values, the expression inside the paranthesis is positive if and only if $r > 0$. Therefore,

$$D_1 = \{r : r > 0\} \tag{8}$$

Noncoherent Detection

- To compute the error probability we have

$$P_b = \int_0^\infty \left(\int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r+A\sqrt{\mathcal{E}_b})^2}{N_0}} dr \right) p(A) dA$$
$$= \int_0^\infty Q\left(A\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) p(A) dA \tag{9}$$
$$= E\left(Q\left(A\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) \right)$$

where the expectation is taken with respect to A.

Noncoherent Detection

- For instance, if A takes values $\frac{1}{2}$ and 1 with equal probability, then

$$P_b = \frac{1}{2} Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) + \frac{1}{2} Q\left(\sqrt{\frac{\mathcal{E}_b}{2N_0}} \right) \tag{10}$$

- It is important to note that in this case the average received energy per bit is $\mathcal{E}_{bavg} = \frac{1}{2}\mathcal{E}_b + \frac{1}{2}(\frac{1}{4}\mathcal{E}_b) = \frac{5}{8}\mathcal{E}_b$

Noncoherent detection of carrier modulated signals

- For carrier modulated signals, $\{s_{ml}(t), 1 \leq m \leq M\}$ are bandpass signals with lowpass equivalents $\{s_{ml}(t), 1 \leq m \leq M\}$ where

$$s_m(t) = Re[s_{ml}(t)e^{j2\pi f_c t}] \tag{11}$$

- Received signal in the AWGN channel is given by

$$r(t) = s_m(t - t_d) + n(t) \tag{12}$$

where t_d indicates the random time asynchronism between the clocks of the transmitter and receiver.

Noncoherent detection of carrier modulated signals

- We can see that the received random process r(t) is a function of three random phenomena,
  - The message m, which is selected with probability P_m,
  - The random variable t_d, and
  - The random process n(t).
- From equations (11) and (12) we have

$$\begin{aligned} r(t) &= Re[s_{ml}(t - t_d)e^{j2\pi f_c(t-t_d)}] + n(t) \\ &= Re[s_{ml}(t - t_d)e^{-j2\pi f_c t_d}e^{j2\pi f_c t}] + n(t) \end{aligned} \tag{13}$$

- Therefore, the lowpass equivalent of $s_m(t - t_d)$ is equal to $s_{ml}(t - t_d)e^{-j2\pi f_c t_d}$

Noncoherent detection of carrier modulated signals

- In practice $t_d \ll T_s$, where T_s is the symbol duration.
- Thus the effect of a time shift of size t_d on $s_{ml}(t)$ is negligible.
- However the term $e^{-j2\pi f_c t_d}$ can introduce a large phase shift $\phi = -2\pi f_c t_d$.
- Since t_d is random and even small values of t_d can cause large phase shifts that are folded modulo 2π.
- We can model ϕ as a random variable uniformly distributed between 0 and 2π.
- This model of the channel and detection of signals under this assumption is called noncoherent detection

Noncoherent detection of carrier modulated signals

- In the noncoherent case

$$Re[r_l(t)e^{j2\pi f_c t}] = Re[(e^{j\phi}s_{ml}(t) + n_l(t))e^{j2\pi f_c t}] \tag{14}$$

or, in the baseband, we have

$$r_l(t) = e^{j\phi}s_{ml}(t) + n_l(t) \tag{15}$$

- Since the lowpass noise process $n_l(t)$ is circular and its statistics are independent of any rotation; hence we can ignore the effect of phase rotation on the noise component.

Noncoherent detection of carrier modulated signals

- For the phase coherent case where the receiver knows ϕ, it can compensate for it, and the lowpass equivalent channel will have the familiar form of
$$r_l(t) = s_{ml}(t) + n_l(t) \tag{16}$$
- In the noncoherent case,the vector equivalent of equation (16) is given by
$$\mathbf{r}_l = e^{j\phi}\mathbf{s}_{ml} + \mathbf{n}_l \tag{17}$$

Noncoherent detection of carrier modulated signals

- The optimal detector for the baseband vector channel of equation (17) is given by
$$\hat{m} = \underset{1\leq m\leq M}{\arg\max} \frac{P_m}{2\pi}\int_0^{2\pi} p_{n_l}(\mathbf{r}_l - e^{j\phi}\mathbf{s}_{ml})d\phi \tag{18}$$
- Note that $n_l(t)$ is a complex baseband random process with power spectral density of $2N_0$ in the [-W,W] frequency band.
- The projections of this process on an orthonormal basis will have complex i.i.d. zero-mean Gaussian components with variance $2N_0$ (variance N_0 per real and imaginary components).
- Therefore we can write
$$\hat{m} = \underset{1\leq m\leq M}{\arg\max} \frac{P_m}{2\pi}\frac{1}{(4\pi N_0)^N}\int_0^{2\pi} e^{-\frac{||\mathbf{r}_l - e^{j\phi}\mathbf{s}_{ml}||^2}{4N_0}}d\phi \tag{19}$$

Noncoherent detection of carrier modulated signals

- Expanding the exponent, and dropping terms that do not depend on m, and noting that $||\mathbf{s}_{ml}||^2 = 2\mathcal{E}_m$, we obtain

$$\begin{aligned}
\hat{m} &= \arg\max_{1\le m\le M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[r_l . e^{j\phi} s_{ml}]} d\phi \\
&= \arg\max_{1\le m\le M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[(r_l . s_{ml}) e^{-j\phi}]} d\phi \\
&= \arg\max_{1\le m\le M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[|r_l . s_{ml}| e^{-j(\phi-\theta)}]} d\phi \\
&= \arg\max_{1\le m\le M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} |r_l . s_{ml}| cos(\phi-\theta)} d\phi
\end{aligned} \tag{20}$$

where θ denotes the phase of $\mathbf{r}_l . s_{ml}$

Noncoherent detection of carrier modulated signals

- Note that the integrand in equation (20) is a periodic function of ϕ with period 2π, and we are integrating over a complete period; therefore θ has no effect on the result.
- Using the relation

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x cos\phi} d\phi \tag{21}$$

where $I_0(x)$ is the modified Bessel function of the first kind and order zero, we obtain

$$\hat{m} = \arg\max_{1\le m\le M} P_m e^{-\frac{\mathcal{E}_m}{2N_0}} I_0\left(\frac{|\mathbf{r}_l . \mathbf{s}_{ml}|}{2N_0}\right) \tag{22}$$

Noncoherent detection of carrier modulated signals

- In general, the decision rule is given in equation (22) cannot be made simpler.
- However, in the case of equiprobable and equal-energy signals, the terms P_m and $\mathcal{E}_m$ can be ignored, and the optimal detection rule becomes

$$\hat{m} = \arg\max_{1 \leq m \leq M} I_0\left(\frac{|\mathbf{r}_l.\mathbf{s}_{ml}|}{2N_0}\right) \tag{23}$$

- Since for $x > 0$, $I_0(x)$ is an increasing function of x, the decision rule in this case reduces to

$$\hat{m} = \arg\max_{1 \leq m \leq M} |\mathbf{r}_l.\mathbf{s}_{ml}| \tag{24}$$

Noncoherent detection of carrier modulated signals

- From equation (24) it is clear that an optimal noncoherent detector first demodulates the received signal, using its nonsynchronized local oscillator, to obtain $r_l(t)$, the lowpass equivalent of the received signal.
- It then correlates $r_l(t)$ with all $s_{ml}(t)$'s and chooses the one that has the maximum absolute value, or envelope.
- This detector is called envelope detector.
- Note that equation (24) can also be written as

$$\hat{m} = \arg\max_{1 \leq m \leq M} \left| \int_{-\infty}^{\infty} r_l(t) s_{ml}^*(t) dt \right| \tag{25}$$

Envelope Detector