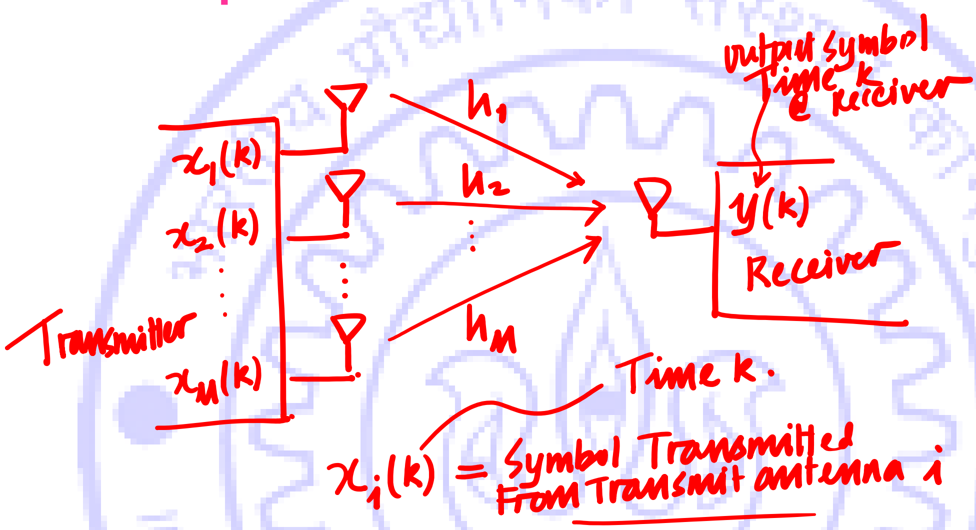
Live Interaction #1:

16th October 2023

E-masters Communication Systems

Estimation for Wireless

- **Vector parameter estimation:**



$$y(k) = x_1(k)h_1 + x_2(k)h_2 + \cdots + x_M(k)h_M + v(k)$$

$$y(k) = \bar{\mathbf{x}}^T(k)\bar{\mathbf{h}} + v(k)$$

- How many pilot vectors? N

$$y(1) = \bar{\mathbf{x}}^T(1)\bar{\mathbf{h}} + v(1)$$
$$y(2) = \bar{\mathbf{x}}^T(2)\bar{\mathbf{h}} + v(2)$$
$$\vdots$$
$$y(N) = \bar{\mathbf{x}}^T(N)\bar{\mathbf{h}} + v(N)$$

- Net model can be expressed as:

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\substack{\bar{\mathbf{y}} \\ N\times 1}} = \underbrace{\begin{bmatrix} \bar{\mathbf{x}}^T(1) \\ \bar{\mathbf{x}}^T(2) \\ \vdots \\ \bar{\mathbf{x}}^T(N) \end{bmatrix}}_{\substack{\mathbf{X} \\ N\times M}} \underbrace{\bar{\mathbf{h}}}_{M\times 1} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\substack{\bar{\mathbf{v}} \\ N\times 1}}$$

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

- $N \geq M \Rightarrow \mathbf{X}$ is a **tall-matrix**.
- Likelihood:

$$y(k) = \bar{\mathbf{x}}^T(k)\bar{\mathbf{h}} + v(k)$$

$$f_{Y(k)}\big(y(k)\big) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y(k)-\bar{\mathbf{x}}^T(k)\bar{\mathbf{h}}\right)^2}{2\sigma^2}}$$

$$p\big(\bar{\mathbf{y}};\bar{\mathbf{h}}\big) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-\bar{\mathbf{x}}^T(k)\bar{\mathbf{h}}\right)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\left\|\bar{\mathbf{y}}-\mathbf{X}\bar{\mathbf{h}}\right\|^2}$$

- Maximum likelihood reduces to

$$\underbrace{\min\left\|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\right\|^2}_{\textbf{Least squares}}$$

$$\hat{\mathbf{h}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

- Properties of estimator:

$$E\{\hat{\mathbf{h}}\} = \bar{\mathbf{h}}$$

$$E\left\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\right\} = \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$$

$$\text{MSE} = \sum_{i=1}^{M} E\left\{\left(\hat{h}_i - h_i\right)^2\right\}$$

$$= \text{Tr}\left\{E\left\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\right\}\right\}$$

$$= \text{Tr}\left\{E\left\{\begin{bmatrix}\hat{h}_1 - h_1 \\ \hat{h}_2 - h_2 \\ \vdots \\ \hat{h}_M - h_M\end{bmatrix}\begin{bmatrix}\hat{h}_1 - h_1 & \hat{h}_2 - h_2 & \dots & \hat{h}_M - h_M\end{bmatrix}\right\}\right\}$$

$$E\left\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\right\} = \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$$

$$\text{MSE} = E\left\{\text{Tr}\left\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\right\}\right\}$$

$$= \text{Tr}\left\{E\left\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\right\}\right\}$$

$$= \text{Tr}\{\sigma^2(\mathbf{X}^T\mathbf{X})^{-1}\} = \sigma^2\text{Tr}\{(\mathbf{X}^T\mathbf{X})^{-1}\}$$

- Example:

$$\mathbf{X} = \begin{bmatrix}1 & 1 \\ -1 & -1 \\ -1 & 1 \\ 1 & -1\end{bmatrix}, \bar{\mathbf{y}} = \begin{bmatrix}-2 \\ 1 \\ 3 \\ -2\end{bmatrix}, \sigma^2 = \frac{1}{2}$$

- Find ML estimate, error covariance, MSE.

$$\hat{\mathbf{h}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix}1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1 \\ -1 & -1 \\ -1 & 1 \\ 1 & -1\end{bmatrix}$$

$$= 4\mathbf{I}$$

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T = \frac{1}{4}\mathbf{I}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\hat{\mathbf{h}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\hat{\mathbf{h}} = \frac{1}{4}\begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1/2 \end{bmatrix}$$

- Error covariance

$$\sigma^2(\mathbf{X}^T\mathbf{X})^{-1} = \frac{1}{2} \times \frac{1}{4}\mathbf{I} = \frac{1}{8}\mathbf{I} = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{bmatrix}$$

- MSE

$$\mathrm{MSE} = \mathrm{Tr}\left\{\left(\begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{bmatrix}\right)\right\} = \frac{1}{4}$$

- **Assignment #3 deadline: 21st October Saturday 11:59PM.**
- **Live interaction 26th October 9-10 PM.**
- **Assignment #4 deadline: 28th October Saturday 11:59 AM.**

- **Assignment #3, #4 discussion: 28th October Saturday 3-4 PM**.
- **Quiz #2 28th October Saturday 6-6:45 PM**.