Homework 4 solution

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Problem 1 (20 points)

In an additive white Gaussian noise channel with noise power-spectral density of $\frac{N_0}{2}$, two equiprobable messages are transmitted by

$$s_1(t) = \begin{cases} \frac{At}{T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = \begin{cases} A(1 - \frac{t}{T}) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the structure of the optimal receiver.

(b) Determine the probability of error.

Solution 1

(a) The two equiprobable signals have the same energy and therefore the optimal receiver bases its decisions on the rule

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt \gtrless_{s_2}^{s_1} \int_{-\infty}^{\infty} r(t)s_2(t)dt$$

(b) If the message signal $s_1(t)$ is transmitted, then $r(t) = s_1(t) + n(t)$ and the decision rule becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty}(s_1(t)+n(t))(s_1(t)-s_2(t))dt \\
&= \int_{-\infty}^{\infty} s_1(t)(s_1(t)-s_2(t))dt + \int_{-\infty}^{\infty} n(t)(s_1(t)-s_2(t))dt \\
&= \int_{-\infty}^{\infty} s_1(t)(s_1(t)-s_2(t))dt + n \gtrless_{s_2}^{s_1} 0
\end{aligned}
$$

where n is a zero mean Gaussian random variable with variance

$$
\begin{aligned}
\sigma_n^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_1(\tau)-s_2(\tau))(s_1(v)-s_2(v))E[n(\tau)n(v)]d\tau dv \\
&= \int_0^T\int_0^T (s_1(\tau)-s_2(\tau))(s_1(v)-s_2(v))\frac{N_0}{2}\delta(\tau-v)d\tau dv \\
&= \frac{N_0}{2}\int_0^T (s_1(\tau)-s_2(\tau))^2 d\tau \\
&= \frac{N_0}{2}\int_0^T (\frac{2A\tau}{T}-A)^2 d\tau \\
&= \frac{N_0}{2}\frac{A^2T}{3}
\end{aligned}
$$

Since

$$
\int_{-\infty}^{\infty} s_1(t)(s_1(t)-s_2(t))dt = \int_0^T \frac{At}{T}(\frac{2At}{T}-A)dt = \frac{A^2T}{6}
$$

the probability of error $P(e|s_1)$ is given by

$$
\begin{aligned}
P(e|s_1) &= P(\frac{A^2T}{6}+n<0) \\
&= \frac{1}{\sqrt{2\pi\frac{A^2TN_0}{6}}}\int_{-\infty}^{-\frac{A^2T}{6}} \exp(-\frac{x^2}{2\frac{A^2TN_0}{6}})dx \\
&= Q[\sqrt{\frac{A^2T}{6N_0}}]
\end{aligned}
$$

Similarly we find that

$$
P(e|s_2) = Q[\sqrt{\frac{A^2T}{6N_0}}]
$$

and since the two signals are equiprobable, the average probability of error is given by

$$
\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\
&= Q[\sqrt{\frac{A^2T}{6N_0}}] \\
&= Q[\sqrt{\frac{\varepsilon_s}{2N_0}}]
\end{aligned}
$$

Problem 2 (20 points)

Consider a biorthogonal signal set with $M = 8$ signal points. Determine a union bound for the probability of a symbol error as a function of ε_b/N_0. The signal points are equally likely a priori. Where the biorthogonal signal set has the form

$s_1 = [\sqrt{\varepsilon_s}, 0, 0, 0]$ $s_2 = [0, \sqrt{\varepsilon_s}, 0, 0]$ $s_3 = [0, 0, \sqrt{\varepsilon_s}, 0]$ $s_4 = [0, 0, 0, \sqrt{\varepsilon_s}]$
$s_5 = [-\sqrt{\varepsilon_s}, 0, 0, 0]$ $s_6 = [0, -\sqrt{\varepsilon_s}, 0, 0]$ $s_7 = [0, 0, -\sqrt{\varepsilon_s}, 0]$ $s_8 = [0, 0, 0, -\sqrt{\varepsilon_s}]$

Solution 2

For each point s_i, there are $M - 2 = 6$ points at a distance

$$d_{i,k} = |s_i - s_k| = \sqrt{2\varepsilon_s}$$

and one vector $(-s_i)$ at a distance $d_{i,m} = 2\sqrt{\varepsilon_s}$. Hence, the union bound on the probability of error $P(e|s_i)$ is given by

$$P_{UB}(e|s_i) = \sum_{k=1,k\neq i}^{M} Q[\frac{d_{i,k}}{\sqrt{2N_0}}] = 6Q[\sqrt{\frac{\varepsilon_s}{N_0}}] + Q[\sqrt{\frac{2\varepsilon_s}{N_0}}]$$

Since all the signals are equiprobable, we find that

$$P_{UB}(e) = 6Q[\sqrt{\frac{\varepsilon_s}{N_0}}] + Q[\sqrt{\frac{2\varepsilon_s}{N_0}}]$$

With $M = 8 = 2^3$, $\varepsilon_s = 3\varepsilon_b$ and therefore,

$$P_{UB}(e) = 6Q[\sqrt{\frac{3\varepsilon_b}{N_0}}] + Q[\sqrt{\frac{6\varepsilon_b}{N_0}}]$$

Problem 3 (15 points)

The 16-QAM signal constellation shown in Figure 1 is an international standard for telephone-line modems (called V.29). Determine the optimum decision boundaries for the detector, assuming that the SNR is sufficiently high so that errors only occur between adjacent points.

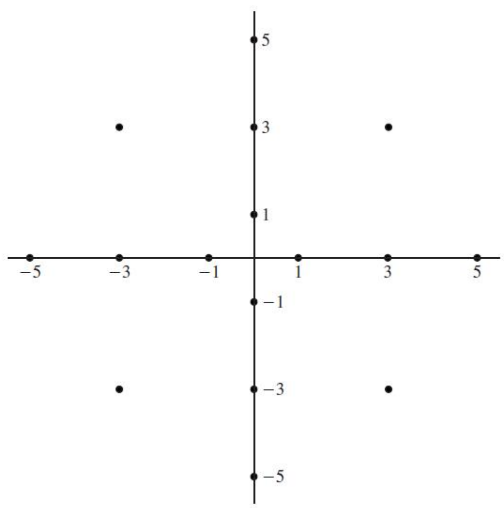
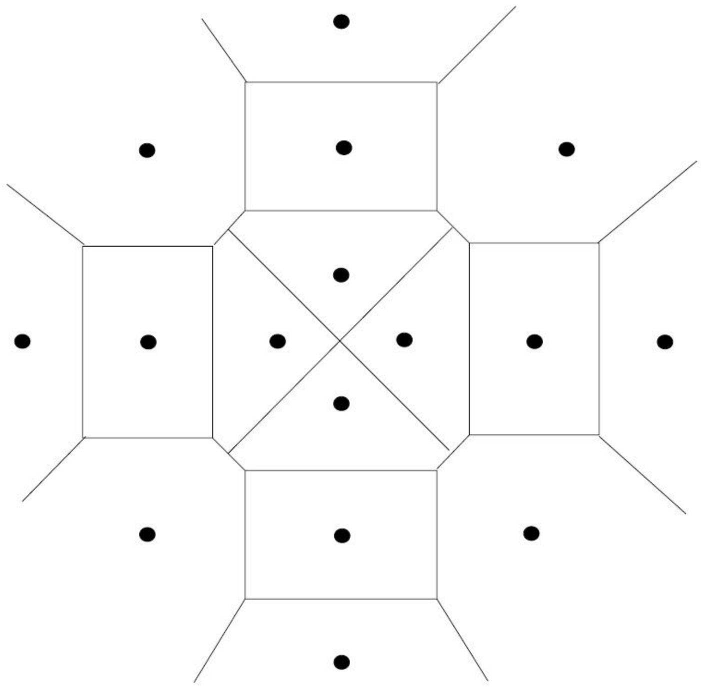


Figure 1

Solution 3

The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for the V.29 constellation are depicted in the next figure.



Problem 4 (30 points)

Consider the octal signal-point constellations in Figure 2.

(a) The nearest neighbor signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles.

(b) The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle.

(c) Determine the average transmitter powers for the two signal constellations and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that all signal points are equally probable).

(d) Compare the SNR required for the 8-point QAM modulation with that of an 8-point PSK modulation having the same error probability.

Solution 4

(a) Using the Pythagorean theorem we can find the radius of the inner circle as

$$a^2 + a^2 = A^2 \Rightarrow a = \frac{1}{\sqrt{2}}A$$

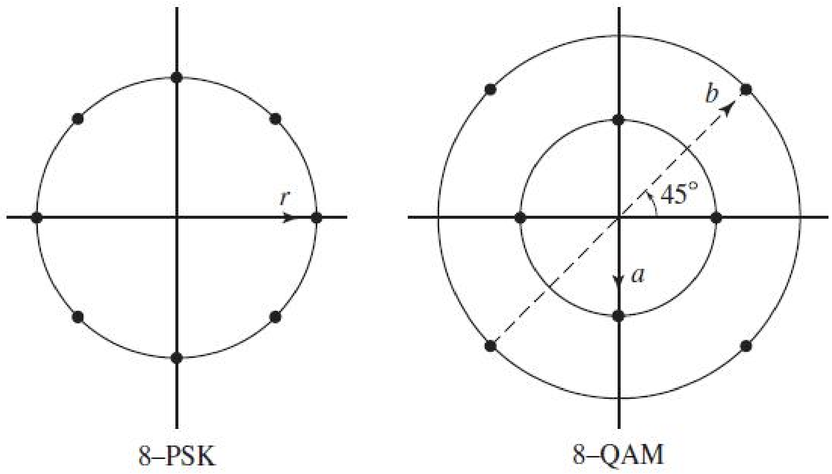


Figure 2

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 105°$, we obtain

$$b^2 = a^2 + A^2 - 2aA\cos 105° \Rightarrow b = \frac{1+\sqrt{3}}{2}A$$

(b) If we denote by r the radius of the circle, then using the cosine theorem we obtain

$$A^2 = r^2 + r^2 - 2r^2 \cos 45° \Rightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

(c) The average transmitted power of the PSK constellation is

$$P_{PSK} = 8 \times \frac{1}{8} \times (\frac{A}{\sqrt{2-\sqrt{2}}})^2 \Rightarrow P_{PSK} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation

$$P_{QAM} = \frac{1}{8}(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2) \Rightarrow P_{QAM} = [\frac{2+(1+\sqrt{3})^2}{8}]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is

$$\text{gain} = \frac{P_{PSK}}{P_{QAM}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927dB$$

(d) The probability of error for an M-ary PSK signal is

$$P_M = 2Q[\sqrt{\frac{2\varepsilon_s}{N_0}} \sin \frac{\pi}{M}]$$

whereas the probability of error for an M-ary QAM signal is upper bounded by

$$P_M = 4Q[\sqrt{\frac{3\varepsilon_{av}}{(M-1)N_0}}]$$

Since, the probability of error is dominated by the argument of the Q function, the two signals will achieve the same probability of error if

$$\sqrt{2SNR_{PSK}} \sin \frac{\pi}{M} = \sqrt{\frac{3SNR_{QAM}}{M-1}}$$

With $M = 8$ we obtain

$$\sqrt{2SNR_{PSK}} \sin \frac{\pi}{8} = \sqrt{\frac{3SNR_{QAM}}{7}} \Rightarrow \frac{SNR_{PSK}}{SNR_{QAM}} = \frac{3}{7 \times 2 \times 0.3827^2} = 1.4627$$

Problem 5 (15 points)

Suppose that the loop filter for a PLL has the transfer function

$$G(s) = \tfrac{1}{s+\sqrt{2}}$$

(a) Determine the closed-loop transfer function $H(s)$. [Hint:J. Proakis, M. Salehi, Digital Communications, 5th Edition, McGraw-Hill, 2007]
(b) Determine the damping factor and the natural frequency of the loop.

Solution 5

(a) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

(b) Writing the denominator in the form

$$D = s^2 + 2\varsigma w_n s + \omega_n^2$$

we identify the natural frequency of the loop as $\omega_n = 1$ and the damping factor as $\varsigma = \frac{1}{\sqrt{2}}$.