Please submitted by Saturday, 19 Aug. 2023, 11 am, right before the discussion hour.

1. Attempt all 5 problems. There is no penalty for submitting incorrect attempts
2. However, plagiarism will result in serious penalties, such as an F grade.

1. Express the following problem as an SOCP

$$\min\ \mathbf{c}^T\mathbf{x} \tag{1}$$
$$\text{s. t. } \mathbf{x}^T\mathbf{x} \le yz \tag{2}$$
$$y^2 + z^2 \le 1 \tag{3}$$
$$y \ge 0, z \ge 0 \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^n$, and y, $z \in \mathbb{R}$.

2. Formulate the following problems as SOCP:

(a)

$$\max \left(\sum_{i=1}^{m} 1/(\mathbf{a}_i^T\mathbf{x} - b_i)\right)^{-1} \tag{7}$$
$$\text{s. t. } \mathbf{a}_i^T\mathbf{x} - b_i \ge 0 \tag{8}$$

(b)

$$\min t \tag{13}$$
$$\text{s. t. } 1/t \le \mathbf{a}_i^T\mathbf{x}/b_i \le t \tag{14}$$

over $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

3. Solve the least-norm problem

$$\min\ \|\mathbf{x}\|_2 \tag{21}$$
$$\text{s. t. } \mathbf{Ax} = \mathbf{b} \tag{22}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m < n$ and $\mathbf{b} \in \mathcal{R}(\mathbf{A})$.

4. Solve the following regularized least-squares problem

$$\min\ \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \lambda\,\|\mathbf{x}\|_2^2 \tag{24}$$

where the regularization parameter $\lambda > 0$. Express the solution such that no assumptions are needed on the rank of matrix $\mathbf{A}$. Comment on the solution for the cases $0 < \lambda \ll 1$ and $\lambda \gg 1$.

5. Consider the following robust optimization problem

$$\min\ \mathbf{c}^T\mathbf{x} \tag{25}$$
$$\text{s. t. } \mathbf{Ax} \le \mathbf{b} \qquad \forall \mathbf{A} \in \mathcal{A} \tag{26}$$

where $\mathcal{A} = \{\mathbf{A} \in \mathbb{R}^{m\times n} | \bar{A}_{ij} - V_{ij} \le A_{ij} \le \bar{A}_{ij} + V_{ij}\ \forall\, i, j\}$. This problem can be interpreted as an LP with infinite number of constraints, one for each value that A_{ij} can take. In other words, the solution $\mathbf{x}$ must satisfy the constraints for all possible values of A_{ij}.

(a) Show that in the constraint

$$\sum_j A_{ij} x_j \leq b_i \qquad\qquad \forall\ \bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij} \tag{27}$$

can equivalently be written as

$$\sum_j \bar{A}_{ij} x_j + \sum V_{ij} |x_j| \leq b_i \tag{28}$$

(b) Express the robust problem as an LP.