Vector Spaces and Subspaces

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

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  - Matrix transpose, inverse and their properties

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  - Discuss vector spaces and subspaces

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- One-dimensional space $\mathbf{R}^1$ is a line (like the x axis)

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- A real vector space is a set of "vectors" together with eight rules for vector addition and for multiplication by real numbers

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  - Choose $c = 0$ and rule requires $0\mathbf{v}$ to be in the subspace

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- A subspace containing $\mathbf{v}$ and $\mathbf{w}$ must contain all linear combinations $c\mathbf{v} + d\mathbf{w}$

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- System $A\mathbf{x} = \mathbf{b}$ is solvable if and only if **b** is in the column space of A

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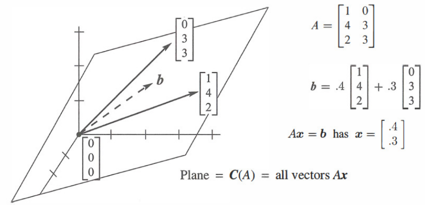

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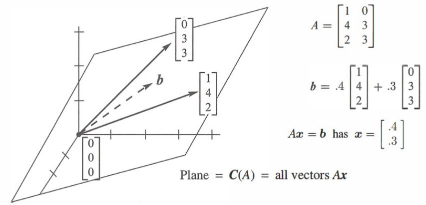

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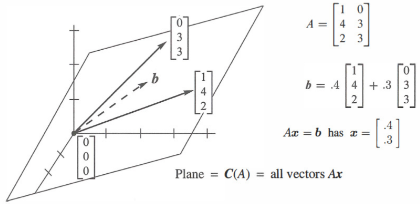

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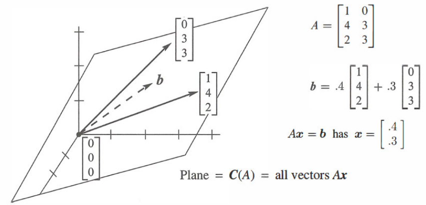

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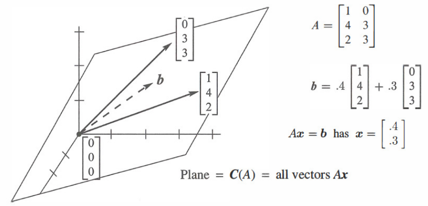

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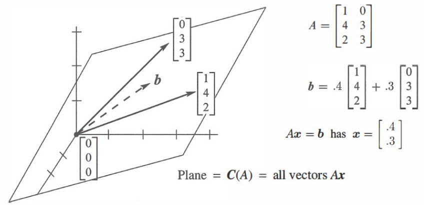

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- There is only one equation – Second Eq. is the first Eq. multiplied by 3

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- Special solution is $s = (-2, 1)$