1. Show that the set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^T\mathbf{y} \leq 1 \text{ for all } \mathbf{y} \text{ with } \mathbf{y} \in \mathcal{C}\}$ is always convex, even if $\mathcal{C}$ is not convex. Show that $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \geq 0, \mathbf{x}^T\mathbf{y} \leq 1 \text{ for all } \mathbf{y} \text{ with } \|\mathbf{y}\|_2 = 1\}$ is convex.

Solution: The set can be written as

$$\bigcap_{\mathbf{y}\in\mathcal{C}} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T\mathbf{y} \leq 1\} \tag{1}$$

which is an intersection of half-spaces and hence convex. The case where $\mathcal{C} = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y}\| = 1\}$ is a special case.

2. Consider the two-dimensional positive semidefinite cone $\mathbb{S}^2_+$ defined as

$$\{\mathbf{X} = \left[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right] | x, y, z \in \mathbb{R}, \mathbf{X} \succeq 0\}$$

Show that it can equivalently be expressed as $\{x, y, z \in \mathbb{R} | x \geq 0, z \geq 0, xz \geq y^2\}$.

Solution: The matrix $\mathbf{X}$ is PSD if and only if $\mathbf{u}^T\mathbf{X}\mathbf{u} \geq 0$ for all $\mathbf{u}$, which means that

$$u_1^2 x + u_2^2 z + 2u_1u_2y \geq 0 \tag{2}$$

for all u_1, u_2. Consider the following cases:

1. Case $\mathbf{u} = 0$: This case is trivial.
2. Case $u_1 = 0, u_2 \neq 0$: In this case, the inequality can be written as $z \geq 0$.
3. Case $u_1 \neq 0, u_2 = 0$: In this case, the inequality can be written as $x \geq 0$.
4. Case $u_1 \neq 0, u_2 \neq 0$: Dividing by $u_2^2 > 0$ and letting $u = u_1/u_2$, we have the quadratic equation:

$$u^2x + 2uy + z \geq 0 \tag{3}$$

   The minimum of this quadratic equation is at $u = -y/x$. There are two cases to consider. Suppose that $y = 0$, in which case, $\mathbf{X}$ is a diagonal matrix and is PSD if $x \geq 0$ and $z \geq 0$. In the case that $-y/x \neq 0$, it is possible to have $\frac{u_1}{u_2} = -\frac{y}{x} \neq 0$. So the quadratic equation is greater than or equal to zero if and only if its minimum value is non-negative, i.e., $xz - y^2 \geq 0$.

Combining all the different cases, we have that $\mathbf{X} \succeq 0$ if an only if $x, z \geq 0, xz \geq y^2$. So the set $\mathbf{X} \succeq 0$ can be expressed as $\{x, y, z \in \mathbb{R} | x \geq 0, z \geq 0, xz \geq y^2\}$.

3. For $m < n$, let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$ be full row rank. Show that any affine set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}\}$ can be expressed in the form $\{\mathbf{C}\mathbf{u} + \mathbf{v} | \mathbf{u} \in \mathbb{R}^{n-m}\}$. For example, the set $\{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 1\}$ can be expressed as $\{[u\ 1-u]^T | u \in \mathbb{R}\}$.

Solution: From the rank nullity theorem, we know $rank(A) + nullity(A) = n$, where $nullity(A)$ is the dimension of null space of the matrix A. Let $\mathcal{N}(A)$ represents null space of A. Let $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_{n-m}$ be the basis of $\mathcal{N}(A)$, where each $\mathbf{c}_i \in \mathbb{R}^n$. Let $\mathbf{v}$ be one of the solutions to $\mathbf{Av} = \mathbf{b}$. Let $\mathbf{C} = [\mathbf{c}_1 \; ... \; \mathbf{c}_{n-m}] \in \mathbb{R}^{n \times n-m}$. Then for any $\mathbf{u} \in \mathbb{R}^{n-m}$, $\mathbf{Cu} \in \mathcal{N}(A)$. Hence the affine set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} = \mathbf{b}\}$ can be expressed in the form $\{\mathbf{Cu} + \mathbf{v} | \mathbf{u} \in \mathbb{R}^{n-m}\}$.

4. Show that the set of all doubly stochastic matrices is convex polyhedral in $\mathbb{R}^{n \times n}$. A doubly stochastic matrix is a square matrix with nonnegative entries with the property that the sum of entries in every row and column is exactly 1.

Solution: Let all the columns of matrix are stalked into one big vector $\mathbf{x} \in \mathbb{R}^{n^2}$. Then the set of all doubly stochastic matrices can be written as,

$$\{\mathbf{x} \in \mathbb{R}^{n^2} | \sum_{i=1}^{n} x_{nj+i} = 1 \; \forall 0 \leq j \leq n-1, \sum_{i=0}^{n-1} x_{ni+j} = 1 \; \forall 1 \leq j \leq n\}.$$

5. Which of the following sets are convex (provide proof or counterexample)
   (a) $\{\mathbf{x} \in \mathbb{R}^n | \min_i x_i = 1\}$
   (b) $\{\mathbf{x} \in \mathbb{R}^n | \left\|\mathbf{x}\right\|_1 \leq \sqrt{n} \left\|\mathbf{x}\right\|_2\}$

Solution: This is not a convex set. For instance consider the case of $n = 2$, where we are $\min\{x_1, x_2\} = 1$, which is basically two lines that are perpendicular to each other, and hence not convex. The counter example is, vectors $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$ belong to the set but not $\begin{bmatrix}1.5\\1.5\end{bmatrix}$.

The second condition is always true, so the set that satisfies them is $\mathbb{R}^n$ which is clearly convex. We can express $\left\|\mathbf{x}\right\|_1$ as $\mathbf{a}^T\mathbf{x}$ where $a_i = sign(x_i)$. From Cauchy-Schwarz inequality, for any $\mathbf{x} \in \mathbb{R}^n$, we can write $\left\|\mathbf{x}\right\|_1 = \mathbf{a}^T\mathbf{x} \leq \left\|\mathbf{a}\right\|_2 \left\|\mathbf{x}\right\|_2 = \sqrt{n} \left\|\mathbf{x}\right\|_2$.