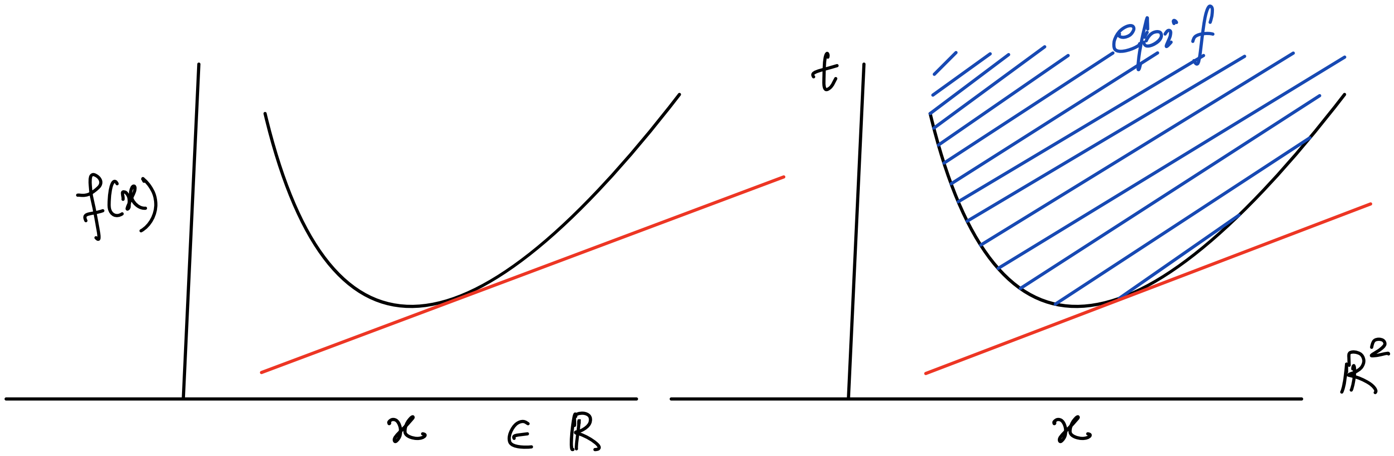
Convex Sets and Functions

$f: \mathbb{R}^n \to \mathbb{R}$

$$\text{epi}\, f = \left\{ \begin{bmatrix} \underline{x} \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \,\middle|\, f(\underline{x}) \le t \quad \forall\, x \in \text{dom}\, f \right\}$$

Eg $f(\underline{x}) = \|\underline{x}\|$ then $\text{epi}\, f: \left\{ \begin{bmatrix} \underline{x} \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \|\underline{x}\| \le t \right\}$

Eg



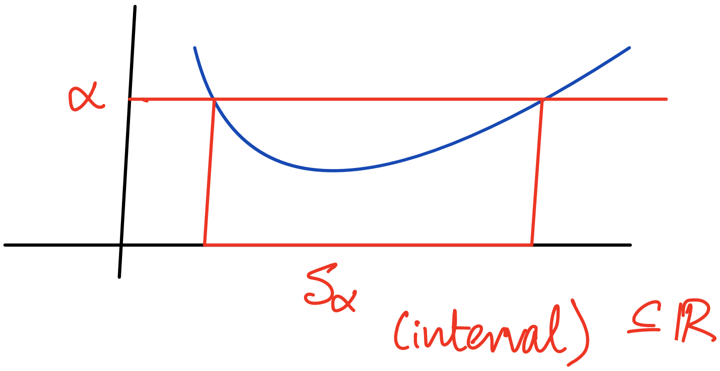
Result : f convex function $\Leftrightarrow$ epi f convex set

Compare : 1st order condition vs. supporting hyperplane

Quasi-convex functions

Given α : define $S_\alpha = \{\underline{x} \mid f(\underline{x}) \le \alpha\} \subseteq \mathbb{R}^n$

(contrast with epi f) Sublevel set



Result : $f(\underline{x})$ convex $\Rightarrow$ S_α convex (when non-empty)

Proof: S_α : $x, y \in S_\alpha$

$x \in S_\alpha \Rightarrow f(x) \le \alpha$

$y \in S_\alpha \Rightarrow f(y) \le \alpha$

$$
\begin{aligned}
f(\theta x + (1-\theta) y) &\le \theta f(x) + (1-\theta) f(y) \quad \text{Zeroth-order} \\
&\le \theta\alpha + (1-\theta)\alpha \\
&= \alpha
\end{aligned}
$$

$\Rightarrow \theta x + (1-\theta) y \in S_\alpha \Rightarrow S_\alpha$ convex

Converse not true

– possible that f non-convex but S_α still convex set

Eg: $f(x) = \log(x)$ $\quad$ f concave not convex

$$\frac{d^2 f}{dx^2} = -\frac{1}{x^2} < 0$$

$$S_\alpha = \{x \mid \log(x) \leq \alpha\}$$

or $\{0 < x \leq e^\alpha\}$ $\quad$ interval, convex

Quasi-convex functions : f s.t. S_α (sub-level set) convex set

$\hookrightarrow$ includes convex functions

$$S^\alpha \text{ (superlevel set)} = \{x \mid f(\underline{x}) \geq \alpha\}$$

Quasi-concave : f s.t. S^α convex set

– Generalize convexity idea

monotonic functions are both quasi-convex &
quasi-concave