Elimination Using Matrices

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed the following in the last lecture
  - systematic way to solve "linear" equations'
- Discuss the following today
  - matrices, their addition and multiplication rules (Chap 2.4 of the book)
  - Solution of equations using eliminatin matrices (Chap 2.3 of the book)

Matrix and rules for its addition and multiplication

- A matrix is a rectangular array of numbers or "entries".
- When A has m rows and n columns, it is a m by n matrix
- Matrices can be added if they have same shapes
- Fundamental law of matrix multiplication: $(AB)C = A(BC)$
- Let A be $m \times n$ and B is $n \times p$ – product AB is m by p

$$(\boldsymbol{m} \times n)(n \times \boldsymbol{p}) = (\boldsymbol{m} \times \boldsymbol{p}) \qquad \begin{bmatrix} m \text{ rows} \\ n \text{ columns} \end{bmatrix} \begin{bmatrix} n \text{ rows} \\ p \text{ columns} \end{bmatrix} = \begin{bmatrix} m \text{ rows} \\ p \text{ columns} \end{bmatrix}$$

- To multiply AB: If A has n columns, B must have n rows

Matrix multiplication - first two ways

- 1st way:

$$\begin{bmatrix} & * & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ & * & & \\ & * & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

A is 4 by 5 $\quad$ B is 5 by 6 $\quad$ AB is $(4 \times 5)(5 \times 6) = 4$ by 6

- Entry in row i and column j of AB = (row i of A) $\cdot$ (column j of B)
- 2nd way:

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$$

- Matrix A times every column of B

Matrix multiplication - third and fourth ways

- 3rd way:
$$[\text{row } i \text{ of A}]\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} = [\text{row } i \text{ of AB}]$$
- Every row of A times matrix B
- 4th way:
$$\begin{bmatrix}\text{col 1} & \text{col 2} & \text{col 3}\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}\text{row 1} & \cdot & \cdot & \cdot\\ \text{row 2} & \cdot & \cdot & \cdot\\ \text{row 3} & \cdot & \cdot & \cdot\end{bmatrix} = (\text{col 1})\,(\text{row 1}) + (\text{col 2})\,(\text{row 2}) + (\text{col 3})\,(\text{row 3}).$$
- Multiply columns 1 to n of A times rows 1 to n of B and add the matrices

$$AB = \begin{bmatrix}\boldsymbol{a} & b\\ \boldsymbol{c} & d\end{bmatrix}\begin{bmatrix}\boldsymbol{E} & \boldsymbol{F}\\ G & H\end{bmatrix} = \begin{bmatrix}\boldsymbol{aE} + bG & \boldsymbol{aF} + bH\\ \boldsymbol{cE} + dG & \boldsymbol{cF} + dH\end{bmatrix}$$

Add columns of A times rows of B $\quad AB = \begin{bmatrix}a\\ c\end{bmatrix}\begin{bmatrix}E & F\end{bmatrix} + \begin{bmatrix}b\\ d\end{bmatrix}\begin{bmatrix}G & H\end{bmatrix}$

Laws for Matrix Operations

- $A + B = B + A$ (Commutative law)
- $c(A + B) = cA + cB$ (Distributive law)
- $A + (B + C) = (A + B) + C$ (Associative law)
- $A(B + C) = AB + BC$ (Distributive law from left)
- $(A + B)C = AC + BC$ (Distributive law from right)
- $A(BC) = (AB)C$ (Associative law of ABC)
- $AB \neq BA$

Block Matrices and Block Multiplication

- If blocks of A can multiply blocks of B, then block multiplication of AB is allowed
- Cuts between columns of A match cuts between rows of B

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} \end{bmatrix}$$

- Let the blocks of A be its n columns, and the blocks of B be its n rows
  - Then block multiplication AB adds up columns times rows:

$$\mathbf{AB} = \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \mathbf{b}_1 & - \\ & \vdots & \\ - & \mathbf{b}_n & - \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 + \cdots + \mathbf{a}_n\mathbf{b}_n \end{bmatrix}$$

Elimination by blocks

- Here is a numerical example:

$$\begin{bmatrix}1 & 4\\1 & 5\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}3 & 2\end{bmatrix} + \begin{bmatrix}4\\5\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}$$
$$= \begin{bmatrix}3 & 2\\3 & 2\end{bmatrix} + \begin{bmatrix}4 & 0\\5 & 0\end{bmatrix}$$

- Using inverse matrices, a block matrix E can elimintate a whole block column
- Suppose a matrix has four blocks A, B, C, D

$$\mathbf{EA} = \left[\begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & \mathbf{I} \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array}\right]$$

- This is called the Schur complement

Matrix Form of One Elimination Step (1)

- The 3 by 3 example in the previous section has the short form $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \qquad \text{is the same as} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}.$$

- $Ax = \mathbf{b}$ is a convenient form for original equation – what about elimination ?
- In this example, 2 times Eq. (1) is subtracted from Eq. (2)
- On right side, 2 times the first component of $\mathbf{b}$ is subtracted from its second

$$\textbf{First step} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \quad \text{changes to} \quad \boldsymbol{b}_{\text{new}} \quad \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

Matrix Form of One Elimination Step (2)

- We want to subtract using a matrix! The same result $\mathbf{b}_{new} = E\mathbf{b}$ is achieved

$$E\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

- Multiplying by an "elimination matrix" E times $\mathbf{b}$, it subtracts $2b_1$ from b_2
- Rows 1 and 3 stay same. First and third rows of E are from identity matrix I

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- They don't change the first and third numbers (2 and 10)
- New second component "4" that appeared after elimination step

Definition of Elimination matrix

- Elimination matrix E_{ij} has an extra nonzero entry $-l$ in the i, j position
- Matrix E_{21} has $-l$ in the $2, 1$ position

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$$

- E_{ij} subtracts a multiple l of row j from row i

Complete elimination process using matrices

- What about the left side of $A\mathbf{x} = \mathbf{b}$?

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

- Purpose of E_{21} is to produce a zero in the $(2, 1)$ position of the matrix
  - Both sides will be multiplied by this E_{21}
- Purpose of E_{31} is to produce a zero in the $(3, 1)$ position of the matrix
- Elimination procedure: start with A, apply E's to produce zeros below pivots and end with a triangular U

Elimination using augumented matrix

- We earlier separately applied elimination matrices to both sides of $A\mathbf{x} = \mathbf{b}$
- We now apply it only once using an "augmented matrix"

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

- Elimination acts on whole rows of this matrix
- With $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ they happen together:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

Matrix P_{ij} for a row Exchange

- To exchange or "permute" rows we use permutation matrix P_{ij}
  - Recall that a row exchange is needed when zero is in the pivot position
- Lower down, that pivot column may contain a non-zero
- By exchanging the two rows, we have a pivot and elimination goes forward
- What matrix P_{23} exchanges row 2 with row 3

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Review of key ideas

$$\begin{bmatrix} & * & & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ & * & & & \\ & * & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

A is 4 by 5 $\qquad$ B is 5 by 6 $\qquad$ AB is $(4 \times 5)(5 \times 6) = 4$ by 6

- $A(BC) = (AB)C$ (surprisingly important)
- Block multiplication is allowed when the block shapes match correctly