

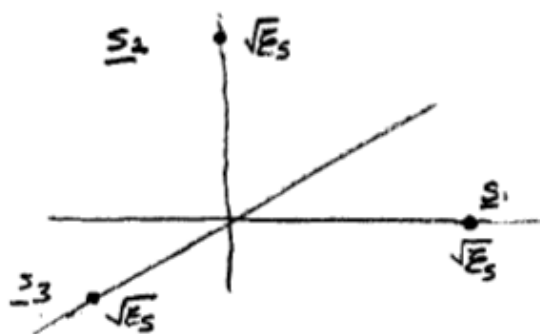
4.2.4 Orthogonal, Biorthogonal and Simplex Signals

- In PAM, QAM and PSK, we had only one basis function. For orthogonal, biorthogonal and simplex signals, however, we use more than one orthogonal basis function, so N -dimensional. Examples: the Fourier basis; time-translated pulses; the Walsh-Hadamard basis.
 - We've become used to SER getting worse quickly as we add bits to the symbol – but with these orthogonal signals it actually gets *better*.
 - The drawback is bandwidth occupancy; the number of dimensions in bandwidth W and symbol time T_s is

$$N \approx 2WT_s$$
 - So we use these sets when the power budget is tight, but there's plenty of bandwidth.

Orthogonal Signals

- With orthogonal signals, we select only one of the orthogonal basis functions for transmission:



$$\underline{s}_1 = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} \quad \underline{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_s} \\ 0 \end{bmatrix} \quad \underline{s}_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{E_s} \end{bmatrix}$$

The number of signals M equals the number of dimensions N .

- Examples of orthogonal signals are frequency-shift keying (FSK), pulse position modulation (PPM), and choice of Walsh-Hadamard functions (note that with Fourier basis, it's FSK, *not* OFDM).

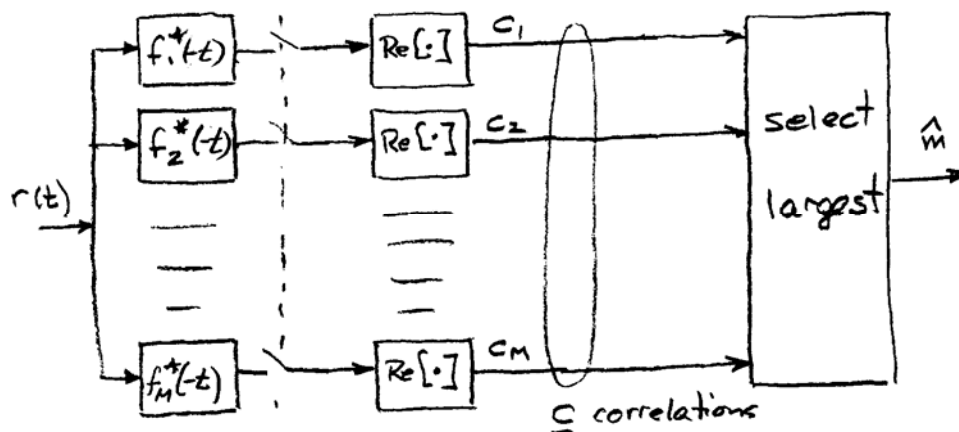
- Energy and distance:

- The signals are equidistant, as can be seen from the sketch or from

$$\mathbf{s}_i - \mathbf{s}_j = \begin{bmatrix} \vdots \\ \sqrt{E_s} \\ \vdots \\ -\sqrt{E_s} \\ \vdots \end{bmatrix} \begin{matrix} \leftarrow \text{location } i \\ \leftarrow \text{location } j \end{matrix}$$

$$\text{so } \|\mathbf{s}_i - \mathbf{s}_j\| = \sqrt{2E_s} = \sqrt{2\log_2(M)E_b} = \sqrt{2kE_b} = d_{\min}, \quad \forall i, j$$

- Effect of adding bits:
 - For a fixed energy per bit, adding more bits *increases* the minimum distance – strong contrast to PAM, QAM, PSK.
 - But adding each bit doubles the number of signals M , which equals the number of dimensions N – and that doubles the bandwidth!
- Error analysis for orthogonal signals. Equal energy, equiprobable signals, so receiver is



- Error probability is same for all signals. If s_1 was sent, then the correlation vector is

$$\mathbf{c} = \begin{bmatrix} \sqrt{E_s} + n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix} \quad \text{with real components.}$$

- Assume s_1 was sent. The probability of a *correct* symbol decision, conditioned on the value of the received c_1 , is

$$P_{cs}(c_1) = P[(n_2 < c_1) \wedge (n_3 < c_1) \wedge \dots \wedge (n_M < c_1)]$$

$$= \left(1 - Q\left(\frac{c_1}{\sqrt{N_0/2}}\right)\right)^{M-1}$$

- Then the *unconditional* probability of correct symbol detection is

$$P_{cs} = \int_{-\infty}^{\infty} \left(1 - Q\left(\frac{c_1}{\sqrt{N_0/2}}\right)\right)^{M-1} p_{c_1}(c_1) dc_1$$

$$= \int_{-\infty}^{\infty} \left(1 - Q\left(\frac{c_1}{\sqrt{N_0/2}}\right)\right)^{M-1} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{N_0/2}} \exp\left(-\frac{1}{2} \frac{2}{N_0} (c_1 - \sqrt{E_s})^2\right) dc_1$$

$$= \int_{-\infty}^{\infty} (1 - Q(u))^{M-1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} (u - \sqrt{2\gamma_s})^2\right) du$$

Needs a numerical evaluation.

- The unconditional probability of symbol error (SER) is

$$P_{es} = 1 - P_{cs}$$

- Now for the *bit* error rate.

○ All errors equally likely, at $\frac{P_{es}}{M-1} = \frac{P_{es}}{2^k-1}$. No point in Gray coding.

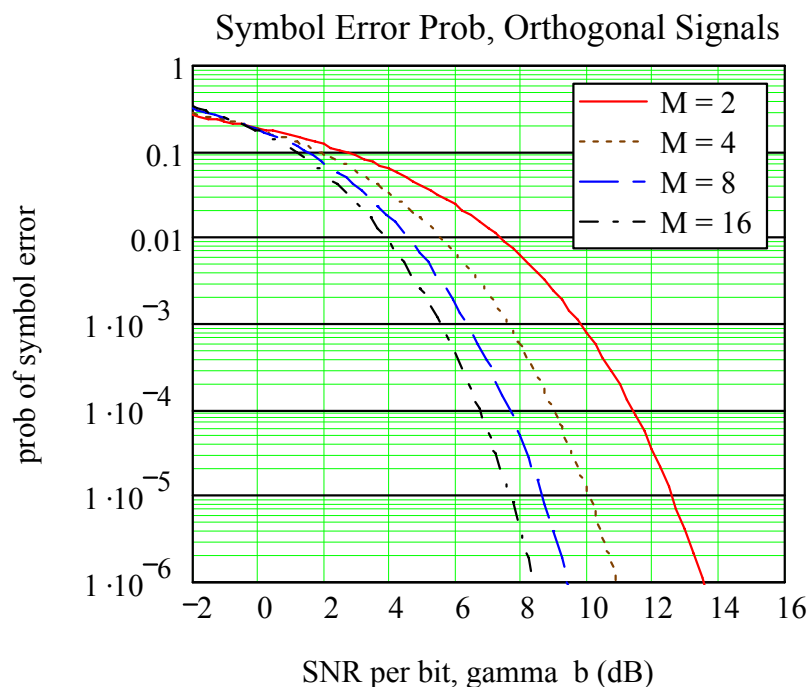
○ Can get i bit errors in $\binom{k}{i}$ equiprobable ways, so

$$P_{eb} = \frac{P_s}{2^k-1} \sum_{i=1}^k i \binom{k}{i} = \frac{P_s}{2^k-1} \sum_{i=1}^k k \binom{k-1}{i-1}$$

$$= k \frac{2^{k-1}}{2^k-1} P_s \approx \frac{k}{2} P_s$$

About half the bits are in error in the average symbol error.

- The SER for orthogonal signals is striking:



The SER *improves* as we add bits!

But the bandwidth doubles with each additional bit.

- Why does SER drop as M increases?
 - All points are separated by the same $d = \sqrt{2\log_2(M)E_b}$. Since d^2 increases linearly with $k = \log_2(M)$, the *pairwise* error probability (between two specific points) *decreases* exponentially with k , the number of bits per symbol.
 - Offsetting this is the number $M-1$ of neighbours of any point: $M-1$ ways of getting it wrong, and M *increases* exponentially with k .
 - Which effect wins? Can't get much analytical traction from the integral expression two pages back. So fall back to a union bound.
- Union bound analysis:
 - Without loss of generality, assume \mathbf{s}_1 was transmitted.
 - Define E_m , $m = 2, \dots, M$ as the event that \mathbf{r} is closer to \mathbf{s}_m than to \mathbf{s}_1 .

- Note that E_m is a *pairwise* error. It does not imply that the receiver's decision is m . In fact, we have events E_2, \dots, E_M and they are not mutually exclusive, since \mathbf{r} could lie closer to two or more points than to \mathbf{s}_1 :



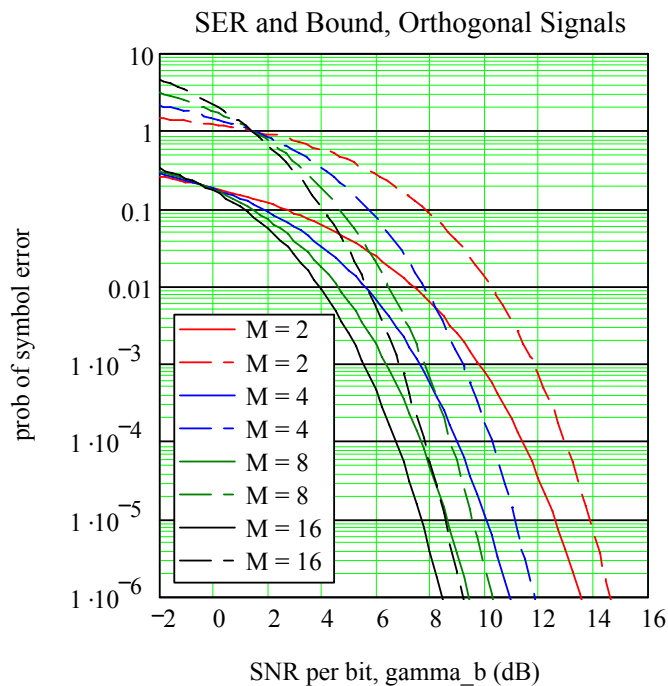
- The overall error event $E = E_2 \cup E_3 \cup \dots \cup E_M$, so the SER is bounded by the sum of the pairwise event probabilities¹

$$\begin{aligned}
 P_{es} &= P[E] = P[E_2 \cup E_3 \cup \dots \cup E_M] \leq \sum_{m=2}^M P[E_m] \\
 &= (M-1)Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) = (M-1)Q\left(\sqrt{\log_2(M)}\gamma_b\right) \\
 &\leq 2^k \frac{1}{2} e^{-k\gamma_b/2} < e^{-k\left(\frac{\gamma_b}{2} - \ln(2)\right)}
 \end{aligned}$$

¹ A general expansion is

$$\begin{aligned}
 P[A_1 \cup A_2 \cup \dots \cup A_M] &= \sum_{i=1}^M P[A_i] - \sum_{i=1}^M \sum_{\substack{j=1 \\ j \neq i}}^M P[A_i \cap A_j] \\
 &+ \sum_{i=1}^M \sum_{\substack{j=1 \\ j \neq i}}^M \sum_{\substack{k=1 \\ k \neq i, k \neq j}}^M P[A_i \cap A_j \cap A_k] - \dots \text{ etc.}
 \end{aligned}$$

- From the bound, if $\gamma_b > 2\ln(2) = 1.386$, then loading more bits onto a symbol causes the upper bound on SER to drop exponentially to zero.
- Conversely, if $\gamma_b < 2\ln(2)$, increasing k causes the upper bound on SER to rise exponentially (SER itself saturates at 1).
- Too bad the dimensionality, the number of correlators and the bandwidth also increase exponentially with k .
- This scheme is called “block orthogonal coding.” Thresholds are characteristic of coded systems.
- Remember that this analysis is based on bounds...



True SERs are solid lines,
bounds are dashed.

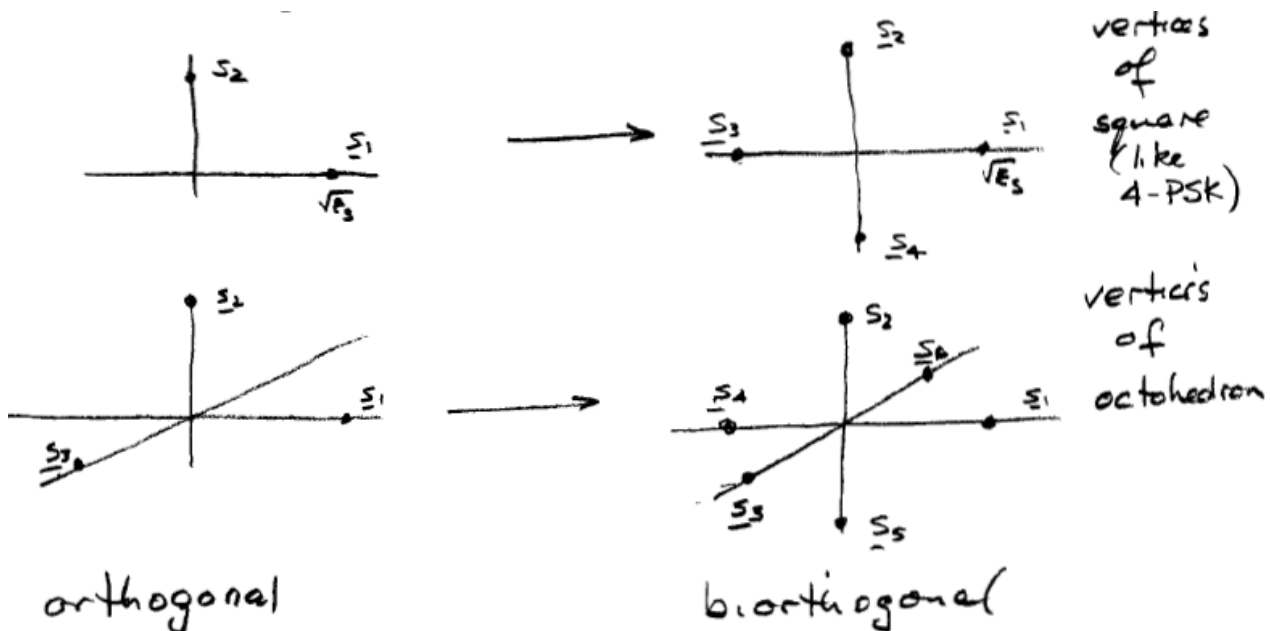
Upper bound is useful to
show decreasing SER, but is
too loose for a good
approximation.

Biorthogonal Signals

- If you have coherent detection, why waste the other side of the axis?

Double the number of signal points (add one bit) with $\pm\sqrt{E_s}$. Now

$M = 2N$, with no increase in bandwidth. Or keep the same M and cut the bandwidth in half.



- Energy and distance:

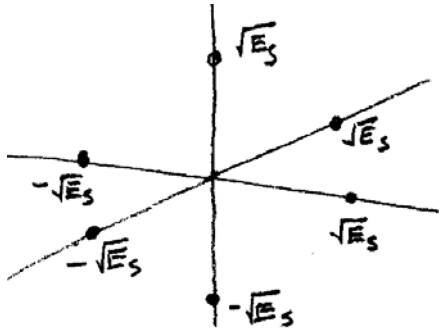
All signal points are equidistant from s_i

$$\|s_i - s_k\| = \sqrt{2E_s} = d_{\min} \quad (\text{the same as orthogonal signals})$$

except one – the reflection through the origin – which is farther away

$$\|s_i - s_i\| = 2\sqrt{E_s}$$

- Symbol error rate:



The probability of error is messy, but the union bound is easy. A signal is equidistant from all other signals but its own complement.

- So the union bound on SER is

$$\begin{aligned}
 P_s &\leq (M-2)Q(\sqrt{\gamma_s}) + Q(\sqrt{2\gamma_s}) \\
 &\leq (M-2)Q(\sqrt{\gamma_s}) \quad \text{second term redundant (Sec'n 4.5.4)} \\
 &= (M-2)Q(\sqrt{\log_2(M) \gamma_b})
 \end{aligned}$$

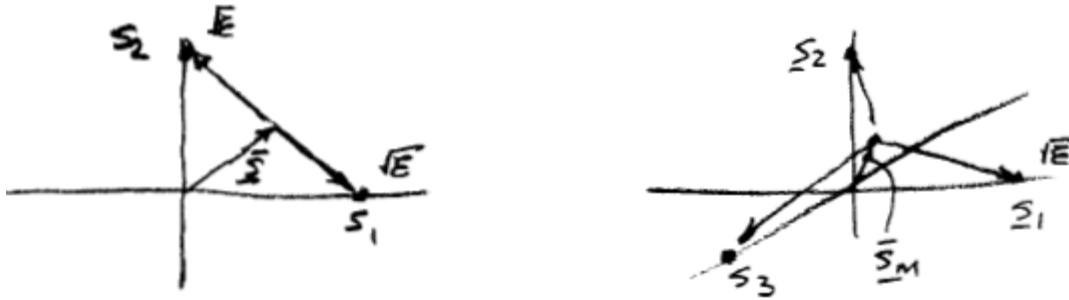
which is slightly less than orthogonal signaling, for the same M and same γ_b . The major benefit is that biorthogonal needs only half the bandwidth of orthogonal, since it has half the number of dimensions.

- And the bit error probability is

$$P_b \approx \frac{\log_2(M)}{2} P_s$$

Simplex Signals

- The orthogonal signals can be seen as a mean values, shared by all, and signal-dependent increments:



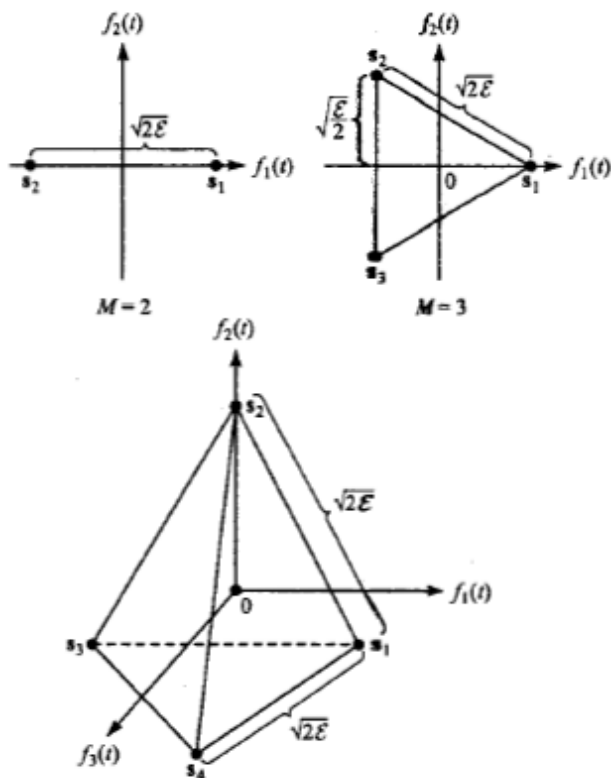
The mean signal (the centroid) is $\bar{\mathbf{s}} = \frac{1}{M} \sum_{m=1}^M \mathbf{s}_m$.

- The mean does not contribute to distinguishability of signals – so why not subtract it?

$$\mathbf{s}'_m = \mathbf{s}_m - \bar{\mathbf{s}} \quad \mathbf{s}'_m = \sqrt{E_s} \begin{bmatrix} -1/M \\ -1/M \\ \vdots \\ 1 - 1/M \\ \vdots \\ -1/M \end{bmatrix} \leftarrow \text{location } m$$

where $\sqrt{E_s}$ is the original signal energy.

- Removing the mean lowers the dimensionality by one. After rotation into tidier coordinates:

**FIGURE 4.3-10**

Signal space diagrams for M -ary simplex signals.

Vertices of
 - line segment
 - equilateral triangle
 - tetrahedron.

- They still have equal energy (though no longer orthogonal). That energy is

$$E'_s = \|\mathbf{s}'_m\|^2 = E_s \left(M \frac{1}{M^2} + 1 - \frac{2}{M} \right) = E \left(1 - \frac{1}{M} \right)$$

The energy saving is small for larger M .

- The correlation among signals is:

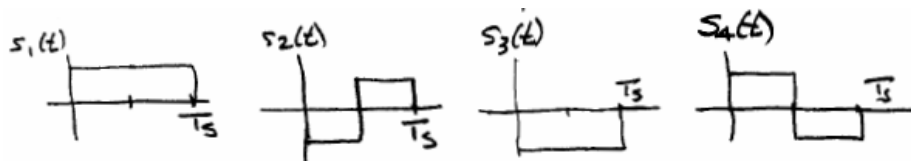
$$\text{Re}[\rho_{mn}] = \frac{(\mathbf{s}'_m, \mathbf{s}'_n)}{\|\mathbf{s}'_m\| \cdot \|\mathbf{s}'_n\|} = \frac{-\frac{1}{M}}{1 - \frac{1}{M}} = -\frac{1}{M-1} \quad \text{for } m \neq n$$

A uniform negative correlation.

- The SER is easy. Translation doesn't change the error rate, so the SER is that of orthogonal signals with a $M/(M-1)$ SNR boost.

4.2.5 Vertices of a Hypercube and Generalizations

- More signals defined on a multidimensional space. Unlike orthogonal, we will now allow more than one dimension to be used in a symbol.
- Vertices of a hypercube is straightforward: two-level PAM (binary antipodal) on each of the N dimensions:
 - With the Fourier basis, it is BPSK on each frequency at once – a simple OFDM
 - With the time translate basis, it is a classical NRZ transmission:



- Signal vectors:

$$\mathbf{s}_m = \begin{bmatrix} s_{m1} \\ s_{m2} \\ \vdots \\ s_{mN} \end{bmatrix} \quad \text{with} \quad s_{mn} = \pm \sqrt{E_s/N}$$

for $m = 1, \dots, M$ and $M = 2^N$.

- In space, it looks like this

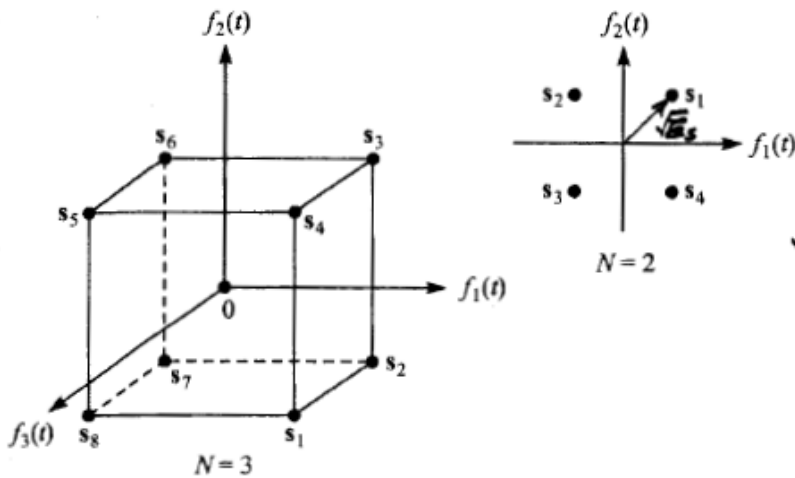


FIGURE 4.3-11

Signal space diagrams for signals generated from binary codes.

"vertices of a hypercube"

- Every point has the same distance from the origin, hence the same energy

$$\|\mathbf{s}_m\|^2 = E_s = \log_2(M) E_b = N E_b$$

- Minimum distance occurs for differences in a single coordinate:

$$\mathbf{s}_m = \sqrt{\frac{E_s}{N}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{s}_n = \sqrt{\frac{E_s}{N}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

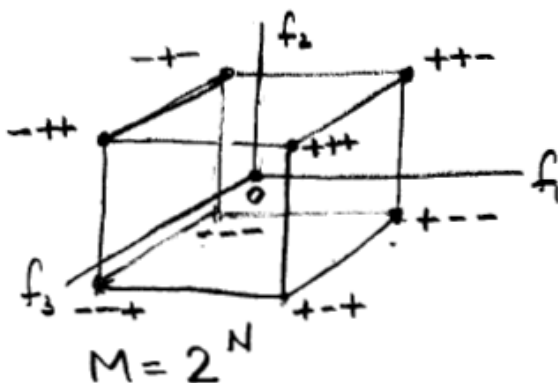
$$d_{\min} = \|\mathbf{s}_m - \mathbf{s}_n\| = 2\sqrt{\frac{E_s}{N}} = 2\sqrt{E_b}$$

More generally, for d_H disagreements (Hamming distance),

$$\|\mathbf{s}_m - \mathbf{s}_n\| = 2\sqrt{\frac{d_H E_s}{N}} = 2\sqrt{d_H E_b}$$

- What are the effects on d_{\min} and bandwidth if we increase the number of bits, keeping E_b fixed?
- It's easy to generalize from binary to PAM, PSK, QAM, etc. on each of the dimensions:
 - With the Fourier basis, it is OFDM
 - With time translates, it is the usual serial transmission.

- Error analysis:



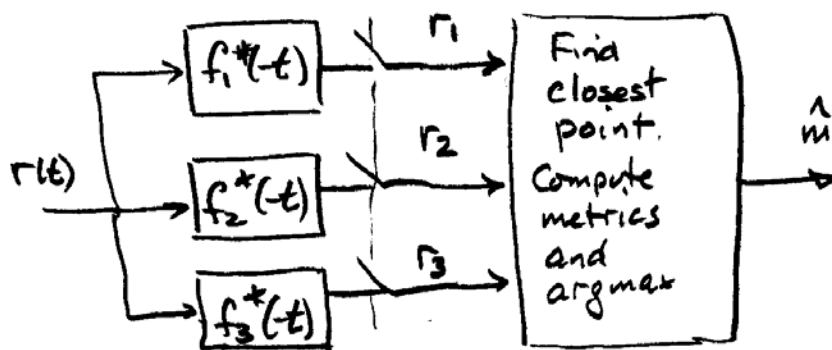
All points have same energy

$$E_s = N E_b$$

Euclidean distance for Hamming

distance h is $d = 2\sqrt{h E_b}$

- If labeling is done independently on different dimensions (see sketch), then the independence of the noise causes it to decompose to N independent detectors. So the following structure

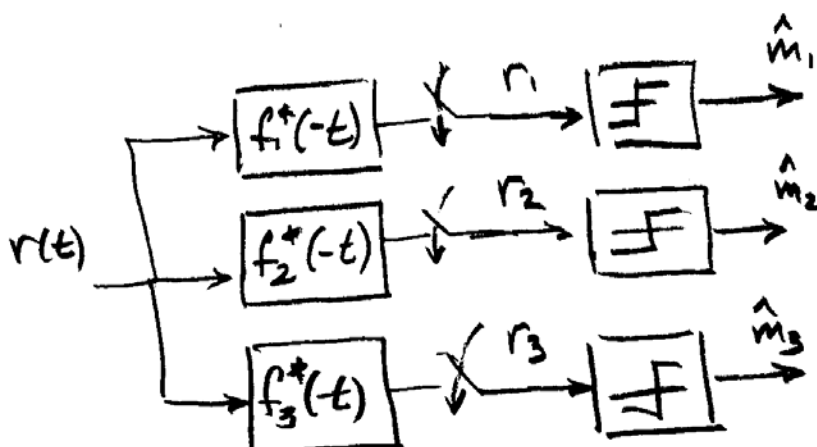


$$P_s \leq N Q(\sqrt{2\gamma_b})$$

$$= \log_2(M) Q(\sqrt{2\gamma_b})$$

P_b depends on labels

becomes



P_s is meaningless

$P_b = Q(\sqrt{2\gamma_b})$, just
binary antipodal

- Generalization: use multilevel signals (PAM) in each dimension. Again, if labeling is independent by dimension, then it's just independent and parallel use, like QAM. Not too exciting. Yet. These signals form a finite lattice.
- Generalization: use a subset of the cube vertices, with points selected for greater minimum distance. This is a binary block code.
 - Advantage: greater Euclidean distance
 - Disadvantage: lower data rate

