# EE910: Digital Communication Systems-I

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April 11, 2022

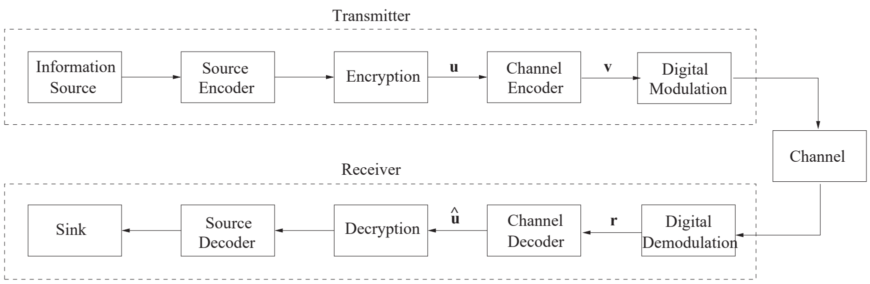
**Lecture #1A: An introduction to digital communications**

## Books

- John G. Proakis and Masoud Salehi, "Digital Communications", 5th edition, McGraw Hill, 2008.
- Bernard Sklar and Pabitra Kumar Ray, "Digital Communications: Fundamentals and Applications", 2nd Edition, Prentice Hall
- John R. Barry, Edward A. Lee and David G. Messerschmitt, "Digital Communication", 3rd Edition, Springer.
- Michael P. Fitz, "Fundamentals of Communication Systems", 1st Edition, McGraw Hill,



## Block Diagram of a Digital Communication System



## Source Coding

- *Function*: To minimize the number of bits per unit time required to represent the source output.
- This process is known as *source coding* or *data compression*
- *Examples*: Huffman coding, Lempel-Ziv algorithm.
- The output of the source encoder is referred to as the *information sequence*.



## Source Coding

- Use the statistical structure of a source to represent its output efficiently.
- Example: A bag contains 50% black balls, 25% red balls, 12.5% blue balls, 12.5% green balls. You are randomly picking a ball from the bag and want to convey the information about the color of the ball.
- Simple encoding (Dumb way!), black=00, red=01, blue=10, green=11. An average of 2.0 bits/color
- Smart way? black=0, red=10, blue=110, green=111. An average of 1.75 bits/color
- Can you figure out the color of the balls from the sequence 0110100111?
- Black, blue, red, black, green.
- Main principle of data compression: "Only infrmatn esentil to understnd mst b tranmitd."

# Encryption

- *Function*: To make source bits transmission secure.
- This process of converting source bits (message text) into a source stream that looks like meaningless random bits of data (cipher text) is known as *encryption*.
- *Examples*: Data Encryption Standard (DES), RSA system.



# Encryption

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I | J | K | L | M |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

- Example Message: SEE ME IN MALL
  Take keyword as INFOSEC
  Vigenere cipher works as follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| S | E | E | M | E | I | N | M | A | L | L |
| I | N | F | O | S | E | C | I | N | F | O |
| A | R | J | A | W | M | P | U | N | Q | Z |

# Channel Coding

- *Function*: To correct transmission errors introduced by the channel.
- The process of introducing some redundant bits to a sequence of information bits in a controlled manner to correct transmission errors is known as *channel coding* or *error control coding*.
- *Example*: Repetition code, Reed-Solomon codes, CRC codes.
- The encoded sequence that is the output of the channel encoder is referred to as *codeword*.



# Channel Coding

- Example: Repetition codes
- Rate R=1/2 code

$$0 \rightarrow 00 \qquad 1 \rightarrow 11$$

- Rate R=1/3 code

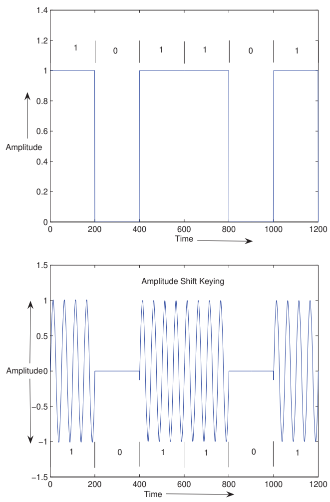
$$0 \rightarrow 000 \qquad 1 \rightarrow 111$$

# Modulation

- *Function*: To map the codewords into waveforms which are then transmitted over the physical medium known as the channel.
- *Examples*: Phase shift keying (PSK), quadrature amplitude modulation (QAM).



# Modulation



## Channel

- The physical transmission medium; it can be wireless or wireline.
- Corrupts transmitted waveforms due to various effects such as noise, interference, fading, and multipath transmission.
- *Examples*: Binary erasure channel (BEC), Additive white Gaussian noise (AWGN) channel.



## Demodulation

- *Function*: To convert received noisy waveform to a sequence of bits, which is an estimate of the transmitted data bits. This is known as *hard demodulation*.
- If the demodulator outputs are unquantized (or has more than two quantization levels), this is known as *soft demodulation*.
- Soft demodulation has significant improvement over hard demodulation.

## Channel Decoding

- *Function*: To estimate the information bits $\hat{\mathbf{u}}$, and correct the transmission errors.
- If $\hat{\mathbf{u}} \neq \mathbf{u}$, decoding errors have occurred.
- The performance of the channel decoder is usually measured by the *bit error rate* (BER) or the *frame error rate* (FER) of the decoded information sequence.
- The BER is defined as the expected number of information bit decoding errors per decoded information bit.
- The coded sequences can be broken up into blocks of data *frames*. A frame error occurs if any information bit in that data frame is in error. The decoded FER is the percentage of frames in error.



## Decryption

- *Function*: To recover the plain text from the cipher text with the help of key.
- It is in the key that the security of a modern cipher lies, not in the details of the cipher.

## Source Decoding

- *Function*: To reconstruct the original source bits from the decoded information sequence.
- Due to channel errors, the final reconstructed signal may be distorted.

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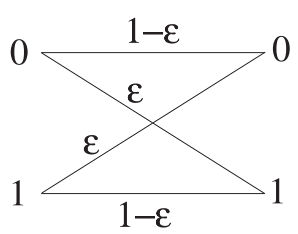
**Lecture #1B: Communication channels and models**

# Communication Channels

- Channel is the transmission medium over which we transmit information bits.
- Examples
  - Wireline channel
  - Wireless channel
  - Underwater acoustic channel
  - Storage channel



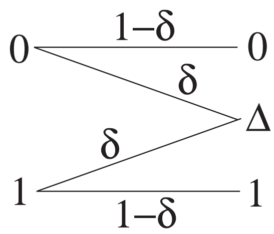
# Binary Symmetric Channel




**Model for Binary Symmetric Channel**

- It is a binary input, binary output symmetric channel.
- $\epsilon$ denotes the crossover probability.

## Binary Erasure Channel

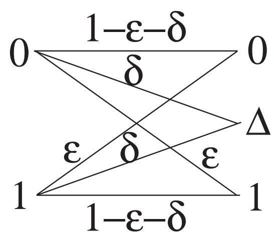



**Model for Binary Erasure Channel**

- It is a binary input, ternary output symmetric channel.
- $\delta$ denotes the erasure probability.



## Binary Symmetric Erasure Channel




**Model for Binary Symmetric Erasure Channel**

- It is a binary input, ternary output symmetric channel.
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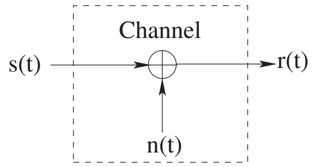
## Additive Noise Channel

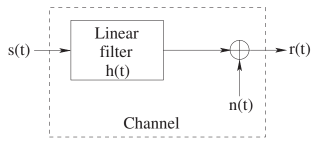



**Model for Additive Noise Channel**

- In this model, transmitted signal $s(t)$ is corrupted by an additive random noise process $n(t)$.
- Typically, noise is characterized statistically as a Gaussian noise process.
- Noise that has constant power spectral density is known as white noise.
- Additive white Gaussian noise (AWGN) channel is used to model a broad class of physical communication channels.



## Linear Filter Channel



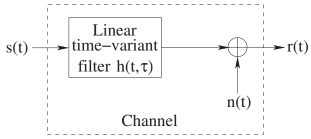

**Model for Linear Filter Channel**

- If channel input is $s(t)$, output $r(t)$ is given by

$$r(t) = \int_{-\infty}^{\infty} h(\tau)s(t-\tau)d\tau + n(t) \quad (1)$$

where $h(t)$ is the impulse response of the linear filter, $n(t)$ is the additive random noise process.

# Linear Time-Variant Filter Channel




**Model for Linear Time-Variant Filter Channel**

- In this case, the linear filter is characterized by time-variant channel impulse response $h(t,\tau)$.
- A typical time-variant channel impulse response $h(t,\tau)$ is of the form

$$h(t,\tau) = \sum_{i=1}^{L} a_k(t)\delta(\tau - \tau_k) \tag{2}$$

where the $\{a_k(t)\}$ represents the time-variant attenuation factors for the L multipath propagation paths and $\{\tau_k\}$ are the corresponding time delays.

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**Lecture #2A: Review of signals**

## Signals: Introduction

- A signal, $x(t)$, is defined to be a function of time ($t \in \mathcal{R}$).
- Signals in engineering systems are typically described with five different mathematical classifications:
  - Deterministic or random
  - Energy or power
  - Periodic or aperiodic
  - Complex or real
  - Continuous time or discrete time



## Deterministic vs. random signal

- A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time.
- Deterministic signals can be defined exactly by a mathematical formula.
- In contrast, there is uncertainty with respect to the value of a random signal at some instant of time.
- Random signals are modeled in probabilistic terms.

## Energy signal

- The energy, $E_x$, of a signal $x(t)$ is given by

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{1}$$

- $x(t)$ is called an energy signal when $E_x < \infty$.
- Energy signals are normally associated with finite duration waveforms.



## Energy signal

- Example:

$$x(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \qquad E_x = 1 \tag{2}$$

## Power signal

- A signal is called a power signal if it does not have finite energy.
- The signal power, $P_x$, is given by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{3}$$

Note that if $E_x < \infty$, then $P_x = 0$ and if $P_x > 0$, then $E_x = \infty$.



## Power signal

- Example:

$$\begin{aligned} x(t) &= \cos(2\pi f_c t) \\ P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos^2(2\pi f_c t)\, dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_c t)\right) dt = \frac{1}{2} \end{aligned} \tag{4}$$

## Periodic vs. Aperiodic signal

- A periodic signal is one that repeats itself in time.
- $x(t)$ is a periodic signal when

$$x(t) = x(t + T_0) \quad \forall t \quad \text{and for some } T_0 \neq 0 \tag{5}$$

- The signal period is given by

$$T = \min(|T_0|) \tag{6}$$

- The fundamental frequency is given by

$$f_T = \frac{1}{T} \tag{7}$$

- Most periodic signals are power signals



## Periodic vs. Aperiodic signal

- Example:

$$x(t) = \cos(2\pi f_m t) \quad T_0 = \frac{n}{f_m} \quad T = \frac{1}{f_m} \tag{8}$$

- An aperiodic signal is defined to be a signal that is not periodic.

## Complex signal vs. real signal

- We define a complex signal and a complex exponential as
$$z(t) = x(t) + jy(t) \quad e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{9}$$
where $x(t)$ and $y(t)$ are both real signals.
- A magnitude ($\alpha(t)$) and phase ($\theta(t)$) representation of a complex signal is also commonly used
$$z(t) = \alpha(t)e^{j\theta(t)} \tag{10}$$
where
$$\alpha(t) = |z(t)| = \sqrt{x^2(t) + y^2(t)} \quad \theta(t) = \arg(z(t)) = \tan^{-1}(y(t), x(t)) \tag{11}$$
- The complex conjugate operation is defined as
$$z^*(t) = x(t) - jy(t) = \alpha(t)e^{-j\theta(t)} \tag{12}$$



## Complex signal vs real signal

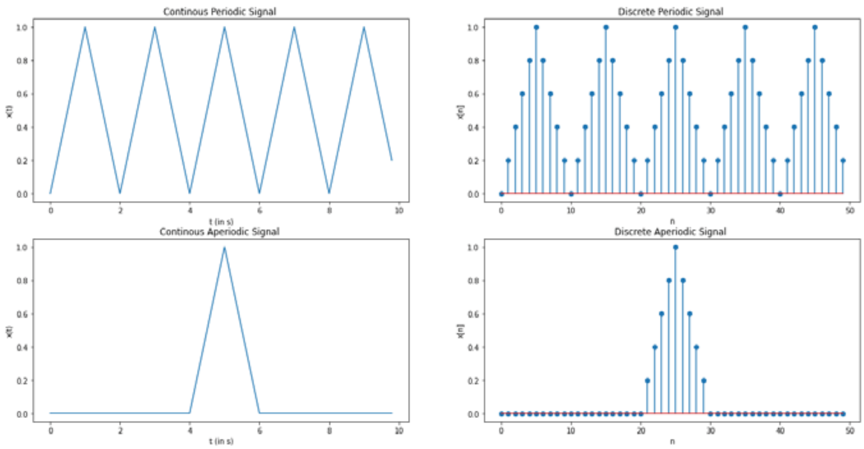
- Some important formulas for analyzing complex signals are
$$\begin{array}{ll} |z(t)|^2 = \alpha(t)^2 = z(t)z^*(t) = x^2(t) + y^2(t) & \cos(\theta)^2 + \sin(\theta)^2 = 1 \\ \Re[z(t)] = x(t) = \alpha(t)\cos(\theta(t)) = \frac{1}{2}[z(t) + z^*(t)] & \cos(\theta) = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right] \\ \Im[z(t)] = y(t) = \alpha(t)\sin(\theta(t)) = \frac{1}{2j}[z(t) - z^*(t)] & \sin(\theta) = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right] \end{array}$$
- Example:
$$\exp[j2\pi f_m t] = \cos(2\pi f_m t) + j\sin(2\pi f_m t) \tag{13}$$

## Continuous Time Signals vs. Discrete Time Signals

- A signal, $x(t)$, is defined to be a continuous time signal if the domain of the function defining the signal contains intervals of the real line.
- A signal, $x(t)$, is defined to be a discrete time signal if the domain of the signal is a countable subset of the real line.
- Often a discrete signal is denoted by $x(k)$, where $k$ is an integer and a discrete signal often arises from (uniform) sampling of a continuous time signal, e.g., $x(k) = x(kT_s)$, where $T_s$ is the sampling period.



## Continuous Time Signals vs. Discrete Time Signals



## Fourier Series

- Signal analysis can be completed in either the time or frequency domains.
- If $x(t)$ is periodic with period $T$, then $x(t)$ can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left[\frac{j2\pi nt}{T}\right] = \sum_{n=-\infty}^{\infty} x_n \exp\left[j2\pi f_T nt\right] \quad (14)$$

where $f_T = 1/T$ and

$$x_n = \frac{1}{T}\int_0^T x(t) \exp\left[-j2\pi f_T nt\right] dt \quad (15)$$



## Fourier Series

- Example:

$$x(t) = \cos\left(2\pi f_m t\right) \quad (16)$$

- For this signal $T = 1/f_m$ and the only nonzero Fourier coefficients are $x_1 = 0.5, x_{-1} = 0.5$.
- Therefore

$$x(t) = \frac{1}{2}\exp\left[j2\pi f_T t\right] + \frac{1}{2}\exp\left[-j2\pi f_T t\right] \quad (17)$$

## Parseval Theorem

- Parseval's theorem states that the power of a signal can be calculated using either the time or the frequency domain representation of the signal and the two results are identical.

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2 \tag{18}$$



## Parseval Theorem

- Example: For $x(t) = \cos(2\pi f_m t)$ computing the power in the frequency domain, we get

$$P_x = |x_{-1}|^2 + |x_1|^2 = (0.5)^2 + (0.5)^2 = 0.5 \tag{19}$$

- Similarly, computing the power in the time domain, we get

$$P_x = \frac{1}{T}\int_0^T |\cos(2\pi f_m t)|^2 dt = 0.5 \tag{20}$$

## Fourier Transform

- If $x(t)$ is an energy signal, then the Fourier transform is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \mathcal{F}\{x(t)\} \tag{21}$$

- $X(f)$ is in general complex and gives the frequency domain representation of $x(t)$.
- The inverse Fourier transform is

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \mathcal{F}^{-1}\{X(f)\} \tag{22}$$



## Fourier Transform

- Example: The Fourier transform of

$$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{23}$$

is given as

$$X(f) = \int_{0}^{T} e^{-j2\pi ft}dt = \left.\frac{\exp[-j2\pi ft]}{-j2\pi f}\right|_{0}^{T} = T\exp\left[j\pi fT\right]\text{sinc}\left(fT\right) \tag{24}$$

## Rayleigh's Energy Theorem

- According to Rayleigh's Energy Theorem, we have

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{25}$$

- Example: For the Fourier transform pair of

$$x(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \qquad X(f) = T \exp\left[j\pi f T\right] \text{sinc}\left(fT\right) \tag{26}$$

the energy is most easily computed in the time domain

$$E_x = \int_0^T |x(t)|^2 dt = T \tag{27}$$



## Correlation Function

- The correlation function of a signal $x(t)$ is

$$V_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \tag{28}$$

- Three important characteristics of the correlation function are
  - $V_x(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = E_x$
  - $V_x(\tau) = V_x^*(-\tau)$
  - $|V_x(\tau)| < V_x(0)$

.

## Correlation Function

- Example: For the pulse

$$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{29}$$

the correlation function is

$$V_x(\tau) = \begin{cases} T\left(1 - \frac{|\tau|}{T}\right) & |\tau| \le T \\ 0 & \text{elsewhere} \end{cases} \tag{30}$$



## Energy Spectrum

- The energy spectrum of a signal $x(t)$ is given by

$$G_x(f) = X(f)X^*(f) = |X(f)|^2 \tag{31}$$

- The energy spectral density is the Fourier transform of the correlation function, i.e.,

$$G_x(f) = \mathcal{F}\{V_x(\tau)\} \tag{32}$$

- The energy spectrum is a functional description of how the energy in the signal $x(t)$ is distributed as a function of frequency.
- Properties of the energy spectral density:

$$G_x(f) \ge 0 \quad \forall f \quad \text{(Energy in a signal cannot be negative valued)}$$
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} G_x(f) df \tag{33}$$

## Energy Spectrum

- Example: For the Fourier transform pair of

$$x(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \qquad X(f) = T \exp\left[-j\pi f T\right] \operatorname{sinc}(fT) \qquad (34)$$

the energy spectrum is

$$G_x(f) = T^2 \left(\operatorname{sinc}(fT)\right)^2 \qquad (35)$$



## Bandwidth of the signal

- Bandwidth most often refers to the amount of positive frequency spectrum that a signal occupies.
- If a signal $x(t)$ has an energy spectrum $G_x(f)$, then $B_X$ is determined as

$$10 \log \left( \max_f G_x(f) \right) = X + 10 \log \left( G_x\left(B_X\right)\right) \qquad (36)$$

where $G_x\left(B_X\right) > G_x(f)$ for $|f| > B_X$
- A signal has a relative bandwidth $B_X$, if the energy spectrum is at least $X$ dB down from the peak at all frequencies at or above $B_X$ Hz.
- Often used values for $X$ in engineering practice are the 3-dB bandwidth and the 40-dB bandwidth.

## Bandwidth of the signal

- If a signal $x(t)$ has an energy spectrum $G_x(f)$, then $B_P$ is determined as

$$P = \frac{\int_{-B_P}^{B_P} G_x(f)df}{E_x} \tag{37}$$

- In words, a signal has an integral bandwidth $B_P$ if the percent of the total energy in the interval $[-B_P, B_P]$ is equal to P%.
- Often used values for $P$ in engineering practice are 98% and 99%.



## Bandwidth of the signal

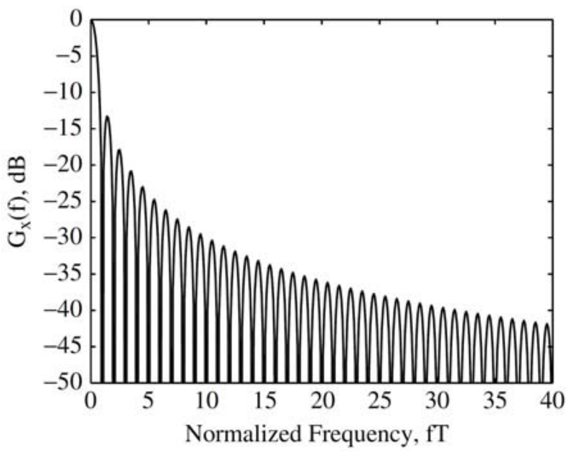
- Example: For the rectangular pulse

$$x(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{38}$$

The energy spectrum of this signal is given as

$$G_x(f) = |X(f)|^2 = T^2 \left(\text{sinc}\,(fT)\right)^2 \tag{39}$$

## Bandwidth of the signal






## Bandwidth of the signal

- The 3-dB bandwidth is given by $B_3 = 0.442/T$
- The 40-dB bandwidth is given by $B_{40} = 31.54/T$
- Integrating the power spectrum gives a 98% energy bandwidth of $B_{98} = 5.25/T$.

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**Lecture #2B: Representation of lowpass and bandpass signals**

## Commonly Used Signals

- Rectangular signal

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ \frac{1}{2} & t = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$



## Commonly Used Signals

- Sinc signal

$$\text{sinc(t)} = \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases} \quad (2)$$

## Commonly Used Signals

- Signum signal

$$sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases} \tag{3}$$



## Commonly Used Signals

- Unit step signal

$$u_{-1}(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{4}$$

## Commonly Used Signals

- Triangular signal

$$\Lambda(t) = \Pi(t) * \Pi(t) = \begin{cases} t+1 & -1 \leq t \leq 0 \\ -t+1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (5)$$



## Signals and their frequency spectrums

- Fourier transform of a real signal $x(t)$ has *Hermitian* symmetry, i.e. $X(-f) = X^*(f)$ or

$$|X(-f)| = |X(f)| \quad \text{and} \quad \angle X(-f) = -\angle X(f)$$

- The *positive spectrum* and the *negative spectrum*

$$X_+(f) = \begin{cases} X(f) & f > 0 \\ \frac{1}{2}X(0) & f = 0 \\ 0 & f < 0 \end{cases} \qquad X_-(f) = \begin{cases} X(f) & f < 0 \\ \frac{1}{2}X(0) & f = 0 \\ 0 & f > 0 \end{cases}$$

$$\begin{aligned} X_+(f) &= X(f)u_{-1}(f), \; (u_{-1}(.) \text{ is unit-step func.}) \\ X_-(f) &= X(f)u_{-1}(-f) \\ X(f) &= X_+(f) + X_-(f) = X_+(f) + X_+^*(-f) \qquad \text{(for real signal)} \end{aligned}$$

which means knowledge of $X_+(f)$ is sufficient to reconstruct $X(f)$

## Bandpass and lowpass signals

- A bandpass signal is real signal whose frequency spectrum is located around some frequency $f_0$ which is far from zero.
- A lowpass signal is a real signal whose spectrum is located around the zero frequency.



## Bandpass and lowpass signals

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# Bandpass and lowpass signals

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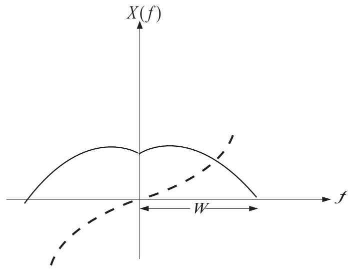


Figure: The spectrum of a real valued lowpass baseband signal

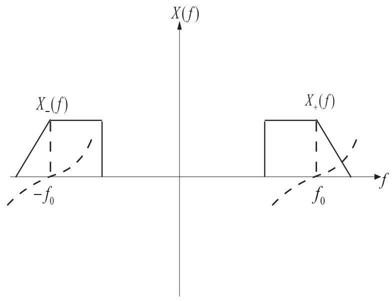


Figure: The spectrum of a real valued bandpass signal



# Lowpass equivalent of bandpass signals

- The *analytic signal* or the *pre-envelope*, corresponding to $x(t)$ denoted by the signal $x_+(t)$ whose Fourier transform is $X_+(f)$

$$\begin{aligned} x_+(t) &= \mathcal{F}^{-1}[X_+(f)] \\ &= \mathcal{F}^{-1}[X(f)u_{-1}(f)] \\ &= x(t) * \left(\frac{1}{2}\delta(t) + j\frac{1}{2\pi t}\right) \\ &= \frac{1}{2}x(t) + \frac{j}{2}\hat{x}(t) \end{aligned}$$

where, $\hat{x}(t)$ is *Hilbert transform* of $x(t)$.

## Lowpass equivalent of bandpass signals

- *Hilbert transform* of $x(t)$

$$\hat{x}(t) = \frac{1}{\pi t} * x(t)$$

and

$$\mathcal{F}[\hat{x}(t)] = -j\text{sgn}(f)X(f)$$

where

$$\text{sgn}(f) = \begin{cases} -1 & f < 0 \\ 1 & f \geq 0 \end{cases}$$



## Lowpass equivalent of bandpass signals

- $x_l(t)$ denotes the *lowpass equivalent* or *complex envelope* of $x(t)$ whose Fourier transform is

$$X_l(f) = \mathcal{F}[x_l(t)] = 2X_+(f + f_0) = 2X(f + f_0)u_{-1}(f + f_0)$$

- Applying the modulation theorem of the Fourier transform

$$\begin{aligned} x_l(t) &= \mathcal{F}^{-1}[X_l(f)] = 2x_+(t)e^{-j2\pi f_0 t} \\ &= (x(t) + j\hat{x}(t))\, e^{-j2\pi f_0 t} \\ &= (x(t)\cos 2\pi f_0 t + \hat{x}(t)\sin 2\pi f_0 t) \\ &\quad + j(\hat{x}(t)\cos 2\pi f_0 t - x(t)\sin 2\pi f_0 t) \quad \textbf{(demodulation)} \end{aligned}$$

- We can write

$$\begin{aligned} x(t) &= \text{Re}\left[x_l(t)e^{j2\pi f_0 t}\right] \quad \textbf{(modulation)} \\ \Rightarrow X(f) &= \frac{1}{2}[X_l(f - f_0) + X_l^*(-f - f_0)] \end{aligned}$$

## Lowpass equivalent of bandpass signals

- *In-phase component* and *quadrature component* of $x_l(t)$

$$x_l(t) = x_i(t) + jx_q(t))$$

where

$$\begin{aligned} x_i(t) &= x(t)\cos 2\pi f_0 t + \hat{x}(t)\sin 2\pi f_0 t \\ x_q(t) &= \hat{x}(t)\cos 2\pi f_0 t - x(t)\sin 2\pi f_0 t \end{aligned}$$

$$\Rightarrow \begin{array}{l} x(t) = x_i(t)\cos 2\pi f_0 t - x_q(t)\sin 2\pi f_0 t \\ \hat{x}(t) = x_q(t)\cos 2\pi f_0 t + x_i(t)\sin 2\pi f_0 t \end{array}$$



## Lowpass equivalent of bandpass signals

- $x_l(t)$ in polar coordinates

$$\begin{aligned} \text{envelope of } x(t) \quad r_x(t) &= \sqrt{x_i^2(t) + x_q^2(t)} \\ \text{phase of } x(t) \quad \theta_x(t) &= \arctan \frac{x_q(t)}{x_i(t)} \\ \rightarrow x_l(t) &= r_x(t)e^{j\theta_x(t)} \end{aligned}$$

$$x(t) = \text{Re}\left[r_x(t)e^{(j2\pi f_0 t + \theta_x(t))}\right]$$

$\Rightarrow$ resulting in

$$x(t) = r_x(t)\cos\left(2\pi f_0 t + \theta_x(t)\right)$$

## Energy Considerations

- The *energy* of a signal $x(t)$ is defined is

$$
\begin{aligned}
\mathcal{E}_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \int_{-\infty}^{\infty} |X(f)|^2 df \quad \textbf{(by Rayleigh's relation)} \\
&= \int_{-\infty}^{\infty} |X_+(f) + X_-(f)|^2 df \qquad (6) \\
&= \int_{-\infty}^{\infty} |X_+(f)|^2 df + \int_{-\infty}^{\infty} |X_-(f)|^2 df \quad \left(\mathbf{X_+(f)X_-(f) = 0}\right) \\
&= 2\int_{-\infty}^{\infty} |X_+(f)|^2 df \quad \left(\textbf{for real signals}\right) \\
&= 2\mathcal{E}_{x_+} \qquad (7)
\end{aligned}
$$



## Energy Considerations

recall,

$$
\begin{aligned}
X_l(f) &= 2X_+(f+f_0) \\
\Rightarrow \int_{-\infty}^{\infty} \left|\frac{X_l(f)}{2}\right|^2 df &= \int_{-\infty}^{\infty} |X_+(f)|^2 df
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\mathcal{E}_x &= 2\int_{-\infty}^{\infty} |X_+(f)|^2 df \\
&= \int_{-\infty}^{\infty} \left|\frac{X_l(f)}{2}\right|^2 df \\
&= \frac{1}{2}\mathcal{E}_{x_l}
\end{aligned}
$$

## Energy Considerations

- Define *inner product of two signals* $x(t)$ and $y(t)$ as

$$\begin{aligned}\langle x(t), y(t)\rangle &\triangleq \int_{-\infty}^{\infty} x(t)y^*(t)dt \\ &= \int_{-\infty}^{\infty} X(f)Y^*(f)df \quad \textbf{(by Parseval's relation)} \\ \Rightarrow \mathcal{E}_x &= \langle x(t), x(t)\rangle\end{aligned}$$

- If $x(t)$ and $y(t)$ are two bandpass signals with lowpass equivalents $x_l(t)$ and $y_l(t)$ with respect to the same center frequency $f_0$, then

$$\langle x(t), y(t)\rangle = \frac{1}{2}\mathrm{Re}\left[\langle x_l(t), y_l(t)\rangle\right] \tag{8}$$



## Bandpass system

- A system whose transfer function is located around a frequency $f_0$ (far from origin) or, a system whose impulse response $h(t)$ is a bandpass signal. Since $h(t)$ is bandpass,

$$h(t) = \mathrm{Re}\left[h_l(t)e^{j2\pi f_0 t}\right]$$

- If a bandpass signal $x(t)$ is passed through a bandpass system with impulse response $h(t)$ then its output $y(t)$ is also a bandpass signal

$$Y(f) = X(f)H(f)$$

# Lowpass equivalent of a bandpass system

- Spectrum of its lowpass equivalent $Y_l(f)$

$$
\begin{aligned}
Y_l(f) &= 2Y(f+f_0)u_{-1}(f+f_0) \\
&= 2H(f+f_0)X(f+f_0)u_{-1}(f+f_0) \\
&= \frac{1}{2}\left[2X(f+f_0)u_{-1}(f+f_0)\right]\left[2H(f+f_0)u_{-1}(f+f_0)\right] \textbf{(true if } f > f_0\textbf{)} \\
&= \frac{1}{2}X_l(f)H_l(f)
\end{aligned}
$$

# EE910: Digital Communication Systems-I

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**Lecture #2C: Signal space representation of waveforms**

## Signal space concepts

- The *inner product* of two generally complex valued signals $x_1(t)$ and $x_2(t)$ is defined as

$$\begin{aligned}\langle x_1(t), x_2(t)\rangle &\triangleq \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt \\ \langle x_1(t), x_2(t)\rangle &= 0 \qquad \textbf{(orthogonality)}\end{aligned}$$

- The *norm of a signal*

$$\|x(t)\| = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{\mathcal{E}_x}$$

where, $\mathcal{E}_x$ is the energy in $x(t)$.



## Signal space concepts

- A set of $m$ signals is *orthonormal* if they are
  - Orthogonal;
  - Unit norm.

## Signal space concepts

- A set of $m$ signals is *linearly independent* if no signal can be represented as a linear combination of the remaining signals.
- $$\begin{aligned} \|x_1(t)+x_2(t)\| &\leq \|x_1(t)\|+\|x_2(t)\| \quad \text{(Triangle inequality)} \\ |\langle x_1(t), x_2(t)\rangle| &\leq \|x_1(t)\|\cdot\|x_2(t)\| \quad \text{(Cauchy- Schwartz inequality)} \\ &= \sqrt{\mathcal{E}_{x_1}\mathcal{E}_{x_2}} \\ \text{equivalently} & \\ \left|\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt\right| &\leq \left|\int_{-\infty}^{\infty}|x_1(t)|^2dt\right|^{1/2}\left|\int_{-\infty}^{\infty}|x_2(t)|^2dt\right|^{1/2} \end{aligned}$$

  with equality when $x_2(t)=\alpha x_1(t)$ for some complex number $\alpha$.



## Orthogonal expansions of signals

- A set of orthonormal functions $\{\phi_n(t), \quad n=1,2,\cdots,K\}$

$$\langle\phi_n(t),\phi_m(t)\rangle=\int_{-\infty}^{\infty}\phi_n(t)\phi_m^*(t)dt=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases}$$

- Approximation of signal $s(t)$ by $\hat{s}(t)$ is

$$\hat{s}(t)=\sum_{k=1}^{K}s_k\phi_k(t)$$

- Approximation error

$$e(t)=s(t)-\hat{s}(t)$$

## Orthogonal expansions of signals

- Energy in the error signal

$$\begin{aligned}\mathcal{E}_e &= \int_{-\infty}^{\infty} |s(t) - \hat{s}(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \left| s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right|^2 dt\end{aligned}$$

the coefficients $\{s_k\}$ are selected such that the error energy $\mathcal{E}_e$ is minimized (in mean square error sense) and are given by

$$s_n = \langle s(t), \phi_n(t) \rangle = \int_{-\infty}^{\infty} s(t)\phi_n^*(t)dt, \qquad n = 1, 2, \cdots, K$$



## Gram Schmidt procedure

- A set of orthogonal signals from the set of finite energy waveforms $\{s_m(t), \quad m = 1, 2, \cdots, M\}$ is constructed as follows. choose a signal waveform randomly from the set $\{s_m(t), \quad m = 1, 2, \cdots, M\}$, $s_1(t)$

$$\begin{aligned}\phi_k(t) &= \frac{s_k(t)}{\sqrt{\int_{-\infty}^{\infty} |s_k(t)|^2 dt}} = \frac{s_k(t)}{\sqrt{\mathcal{E}_k}}, \qquad \text{For k=1} \\ \gamma_k(t) &= s_k(t) - \sum_{i=1}^{k-1} c_{ki}\phi_i(t) \\ \phi_k(t) &= \frac{\gamma_k(t)}{\sqrt{\mathcal{E}_k}} \qquad \text{For } k > 1\end{aligned}$$

## Gram Schmidt procedure

where,

$$\begin{aligned} c_{ki} &= \langle s_k(t), \phi_i(t) \rangle = \int_{-\infty}^{\infty} s_k(t)\phi_i^*(t)dt \\ \mathcal{E}_k &= \int_{-\infty}^{\infty} \gamma_k^2(t)dt \end{aligned}$$

- A signal $s_m(t)$ can be written in the term of set of orthonormal waveforms $\phi_n(t)$ as

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\phi_n(t) \qquad \text{for } m = 1, 2, \cdots, M$$



## Gram Schmidt procedure

- Series expansion of the signal represents orthogonal projection of $s_i(t)$ onto the space spanned by the N basis function.
- Expansion coefficient $s_{ik}$ can be interpreted as the projection of the $i$th signal onto the $k$th basis function.
- Each signal is represented as a point in N-dimensional signal space.
- The basis set for the signal set are the basis functions.

## Gram Schmidt procedure

- For a fixed set of basis orthonormal waveforms $\phi_n(t)$, signals $\{s_m(t)\}$ can be written equivalently as vectors

$$\mathbf{s}_m = \begin{bmatrix} s_{m1} & s_{m2} & \cdots & s_{mN} \end{bmatrix}^T \qquad \text{for } m = 1, 2, \cdots, M$$

and by orthogonality of the basis

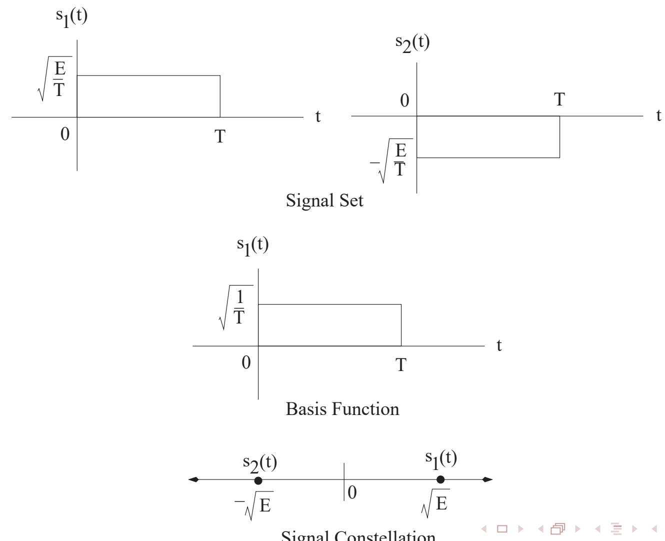
$$\langle s_k(t), s_l(t) \rangle = \langle \mathbf{s}_k, \mathbf{s}_l \rangle$$



## Gram Schmidt procedure

- Note that the functions $\{\phi_n(t)\}$ obtained from the Gram Schmidt procedure are not unique.
- For the different order of orthogonalization process of $\{s_m(t)\}$, the orthonormal waveforms $\{\phi_n(t)\}$ will be different and the corresponding vector representation of the signal $s_m(t)$, $\mathbf{s}_m$ will be different.
- The dimensionality of the signal space $N$ will not change, and the vectors $\{\mathbf{s}_m\}$ will retain their geometric configuration, i.e. their lengths and their inner products will be invariant to the choice of the orthonormal functions $\{\phi_n(t)\}$ .

## Orthonormal Basis Sets






## Gram Schmidt Procedure

- Let the set of M signals be denoted by $s_1(t), s_2(t), \cdots, s_M(t)$, defined over the interval $[0, T]$.
- First basis function is defined by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

where $E_1$ is the energy of the signal $s_1(t)$ chosen arbitrarily from the set.
- $s_1(t)$ can then be represented as

$$\begin{aligned} s_1(t) &= \sqrt{E_1}\phi_1(t) \\ &= s_{11}\phi_1(t) \end{aligned}$$

where the coefficient $s_{11} = \sqrt{E_1}$ and $\phi_1(t)$ has unit energy.

## Gram Schmidt Procedure

- Next using $s_2(t)$ we define

$$s_{21}(t) = \int_0^T s_2(t)\phi_1(t)dt$$

- Let $g_2(t)$, a function orthogonal to $\phi_1(t)$ over the interval $[0, T]$ be defined as

$$g_2(t) = s_2(t) - s_{21}(t)\phi_1(t)$$

- Second basis function can be defined as

$$\begin{aligned} \phi_2(t) &= \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} \\ &= \frac{s_2(t) - s_{21}(t)\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \end{aligned}$$

where $E_2$ is the energy of the signal $s_2(t)$.



## Gram Schmidt Procedure

- Continuing in this fashion we can define

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}(t)\phi_j(t)$$

where the coefficients $s_{ij}$ are defined as

$$s_{ij}(t) = \int_0^T s_i(t)\phi_j(t)dt \quad ,j = 1, 2, \cdots, i-1$$

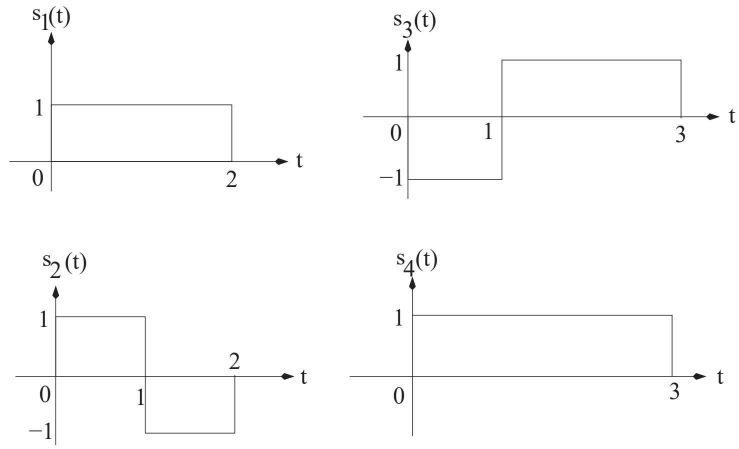
- Set of basis functions that form the orthonormal set can be defined as

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}} \quad ,i = 1, 2, \cdots, N$$

- If the signals $s_1(t), s_2(t) \cdots, s_M(t)$ forms a linearly independent set, then $N = M$, otherwise $N < M$

## Gram Schmidt Procedure

- Apply Gram-Schmidt procedure to the signals given below



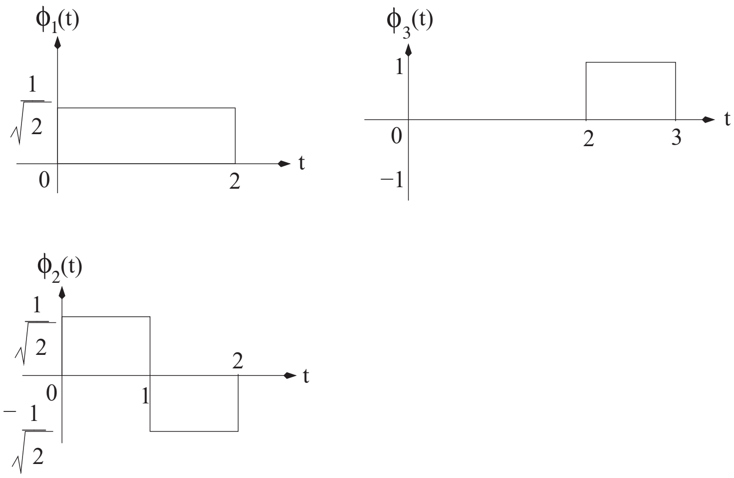



## Gram Schmidt Procedure

- Signal $s_1(t)$ has energy 2, so $\phi_1(t) = s_1(t)/\sqrt{2}$.
- $\phi_1(t)$ and $s_2(t)$ are orthogonal, so $\phi_2(t) = s_2(t)/\sqrt{2}$, where $E_2 = 2$.
- $g_3(t) = s_3(t) + \sqrt{2}\phi_2(t)$.
- $g_3(t)$ has unit energy, so $\phi_3(t) = g_3(t)$.
- $g_4(t) = s_4(t) - \sqrt{2}\phi_1(t) - \phi_3(t) = 0$.
- Thus $s_4(t)$ is linear combination of $\phi_1(t)$ and $\phi_3(t)$.
- The dimensionality of the signal set is $N = 3$.

# Gram Schmidt Procedure

Solution:



# EE910: Digital Communication Systems-I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**Lecture #2D: Some useful random variables**

## Bernoulli Random Variable

- The Bernoulli random variable is a discrete binary-valued random variable taking values 1 and 0 with probabilities $p$ and $1-p$, respectively.
- Therefore the probability mass function (PMF) for this random variable is given by

$$\mathrm{P}[X=1]=p \quad \mathrm{P}[X=0]=1-p \tag{1}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= p \\ \mathrm{VAR}[X] &= p(1-p) \end{aligned} \tag{2}$$



## The Binomial Random Variable

- The binomial random variable models the sum of $n$ independent Bernoulli random variables with common parameter $p$.
- The PMF of this random variable is given by

$$\mathrm{P}[X=k]=\binom{n}{k} p^k(1-p)^{n-k}, \quad k=0,1,\ldots,n \tag{3}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= np \\ \mathrm{VAR}[X] &= np(1-p) \end{aligned} \tag{4}$$

## The Uniform Random Variable

- The uniform random variable is a continuous random variable with PDF

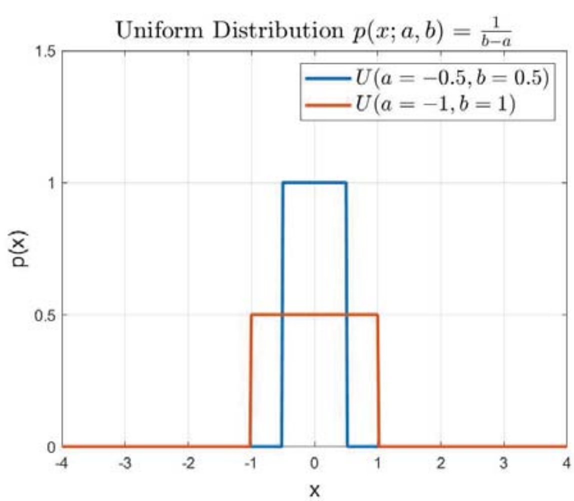
$$p(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

  where $b > a$ and the interval $[a, b]$ is the range of the random variable.
- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= \frac{b-a}{2} \\ \mathrm{VAR}[X] &= \frac{(b-a)^2}{12} \end{aligned} \tag{6}$$



## The Uniform Random Variable



## The Gaussian (Normal) Random Variable

- The Gaussian random variable is described in terms of two parameters $m \in \mathbb{R}$ and $\sigma > 0$ by the PDF

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{7}$$

- We usually use the shorthand form $\mathcal{N}\left(m, \sigma^2\right)$ to denote the PDF of Gaussian random variables and write $X \sim \mathcal{N}\left(m, \sigma^2\right)$.
- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= m \\ \mathrm{VAR}[X] &= \sigma^2 \end{aligned} \tag{8}$$



## The Gaussian (Normal) Random Variable

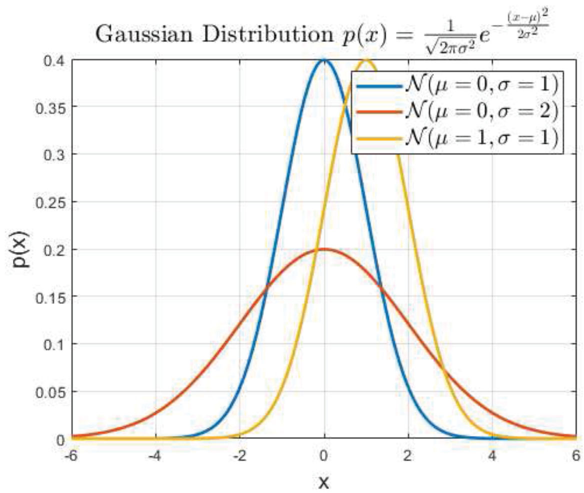
- A Gaussian random variable with $m = 0$ and $\sigma = 1$ is called a standard normal.
- A function closely related to the Gaussian random variable is the $Q$ function defined as

$$Q(x) = \mathrm{P}[\mathcal{N}(0,1) > x] = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt$$

- The CDF of a Gaussian random variable is given by

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}} dt \\ &= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}} dt \\ &= 1 - \int_{\frac{x-m}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ &= 1 - Q\left(\frac{x-m}{\sigma}\right) \end{aligned} \tag{9}$$

## The Gaussian (Normal) Random Variable






## The Rayleigh Random Variable

- If $X_1$ and $X_2$ are two iid Gaussian random variables each distributed according to $\mathcal{N}\left(0,\sigma^2\right)$, then

$$X = \sqrt{X_1^2 + X_2^2} \tag{10}$$

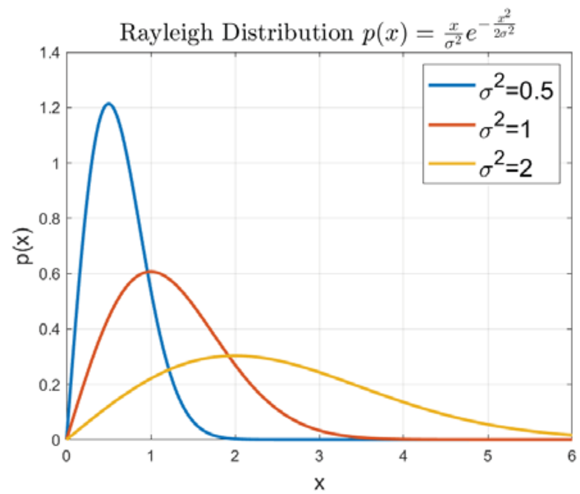
is a Rayleigh random variable.
- The PDF of a Rayleigh random variable is given by

$$p(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= \sigma\sqrt{\frac{\pi}{2}} \\ \mathrm{VAR}[X] &= \left(2 - \frac{\pi}{2}\right)\sigma^2 \end{aligned} \tag{12}$$

## The Rayleigh Random Variable






## The Ricean Random Variable

- If $X_1$ and $X_2$ are two independent Gaussian random variables distributed according to $\mathcal{N}\left(m_1, \sigma^2\right)$ and $\mathcal{N}\left(m_2, \sigma^2\right)$ (i.e., the variances are equal and the means may be different), then

$$X = \sqrt{X_1^2 + X_2^2} \tag{13}$$

is a Ricean random variable with PDF given by

$$p(x) = \begin{cases} \frac{x}{\sigma^2} I_0\left(\frac{sx}{\sigma^2}\right) e^{-\frac{x^2+s^2}{2\sigma^2}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $s = \sqrt{m_1^2 + m_2^2}$ and $I_0(x)$ is the modified Bessel function of the first kind and order zero.

## The Ricean Random Variable

- For $s = 0$, the Ricean random variable reduces to a Rayleigh random variable.
- For large $s$ the Ricean random variable can be well approximated by a Gaussian random variable.
- The CDF of a Ricean random variable can be expressed as

$$F(x) = \begin{cases} 1 - Q_1\left(\frac{s}{\sigma}, \frac{x}{\sigma}\right) & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where $Q_1(a, b) = \int_b^\infty x e^{-\frac{a^2+x^2}{2}} I_0(ax)dx$, known as Marcum Q function



## The Nakagami Random Variable

- Nakagami-m distribution is frequently used to characterize the statistics of signals transmitted through multipath fading channels.
- The PDF for this distribution is given by Nakagami (1960) as

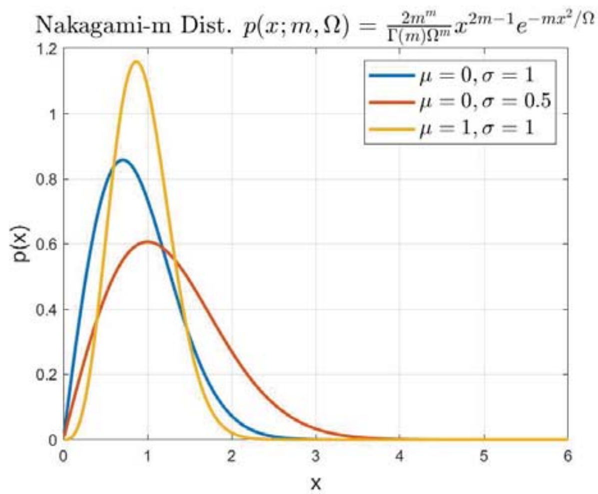
$$p(x) = \begin{cases} \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m x^{2m-1} e^{-mx^2/\Omega} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $\Omega$ is defined as

$$\Omega = \mathrm{E}\left[X^2\right] \tag{17}$$

and the parameter $m$ is defined as the ratio of moments $\left(\frac{\Omega^2}{E[(X^2-\Omega)^2]}\right)$, called the fading figure,

## The Nakagami Random Variable






## The Lognormal Random Variable

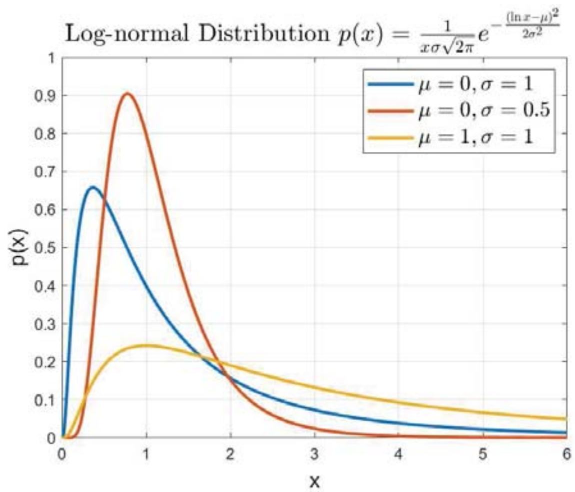
- Suppose that a random variable $Y$ is normally distributed with mean $m$ and variance $\sigma^2$. Let us define a new random variable $X$ that is related to $Y$ through the transformation $Y = \ln X$ ( or $X = e^Y$). Then the PDF of $X$ is

$$p(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}x} e^{-(\ln x - m)^2/2\sigma^2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

- The mean and variance of this random variable are given by

$$\begin{aligned} \mathrm{E}[X] &= e^{m + \frac{\sigma^2}{2}} \\ \mathrm{VAR}[X] &= e^{2m+\sigma^2}\left(e^{\sigma^2} - 1\right) \end{aligned} \tag{19}$$

## The Lognormal Random Variable






## Jointly Gaussian Random Variables

- An $n \times 1$ column random vector $\boldsymbol{X}$ with components $\{X_i, 1 \leq i \leq n\}$ is called a Gaussian vector, and its components are called jointly Gaussian random variables or, multivariate Gaussian random variables if the joint PDF of $X_i$ 's can be written as

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}(\det C)^{1/2}} e^{-\frac{1}{2}(x-m)^t C^{-1}(x-m)} \tag{20}$$

where $\boldsymbol{m}$ and $\boldsymbol{C}$ are the mean vector and covariance matrix, respectively, of $\boldsymbol{X}$ and are given by

$$\begin{aligned} m &= \mathrm{E}[\boldsymbol{X}] \\ \boldsymbol{C} &= \mathrm{E}\left[(\boldsymbol{X}-\boldsymbol{m})(\boldsymbol{X}-\boldsymbol{m})^t\right] \end{aligned} \tag{21}$$

## Jointly Gaussian Random Variables

- From this definition it is clear that

$$C_{ij} = \text{COV}\left[X_i, X_j\right] \tag{22}$$

and therefore $C$ is a symmetric matrix.

- We know that $\boldsymbol{C}$ is nonnegative definite.
- In the special case of $n = 2$, we have

$$\begin{aligned} \boldsymbol{m} &= \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \\ \boldsymbol{C} &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \end{aligned} \tag{23}$$

where

$$\rho = \frac{\text{COV}\left[X_1, X_2\right]}{\sigma_1\sigma_2} \tag{24}$$

is the correlation coefficient of the two random variables.



## Jointly Gaussian Random Variables

- In this case the PDF reduces to

$$p\left(x_1, x_2\right) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1-m_1}{\sigma_1}\right)^2 + \left(\frac{x_2-m_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-m_1}{\sigma_1}\right)\left(\frac{x_2-m_2}{\sigma_2}\right)}{2\left(1-\rho^2\right)}} \tag{25}$$

where $m_1, m_2, \sigma_1^2$ and, $\sigma_2^2$ are means and variances of the two random variables and $\rho$ is their correlation coefficient.

- For the special case when $\rho = 0$ (i.e., when the two random variables are uncorrelated), we have

$$p\left(x_1, x_2\right) = \mathcal{N}\left(m_1, \sigma_1^2\right) \times \mathcal{N}\left(m_2, \sigma_2^2\right) \tag{26}$$

- This means that the two random variables are independent, and therefore for this case independence and uncorrelatedness are equivalent.
- This property is true for general jointly Gaussian random variables.

## Jointly Gaussian Random Variables

- Linear combinations of jointly Gaussian random variables are also jointly Gaussian.
- In other words, if $\boldsymbol{X}$ is a Gaussian vector, the random vector $\boldsymbol{Y} = \boldsymbol{AX}$, where the invertible matrix $\boldsymbol{A}$ represents a linear transformation, is also a Gaussian vector whose mean and covariance matrix are given by

$$\begin{aligned} m_Y &= Am_X \\ C_Y &= AC_XA^t \end{aligned} \tag{27}$$



## Jointly Gaussian Random Variables

- In summary, jointly Gaussian random variables have the following important properties:
  - For jointly Gaussian random variables, uncorrelated is equivalent to independent.
  - Linear combinations of jointly Gaussian random variables are themselves jointly Gaussian.
  - The random variables in any subset of jointly Gaussian random variables are jointly Gaussian, and any subset of random variables conditioned on random variables in any other subset is also jointly Gaussian (all joint subsets and all conditional subsets are Gaussian).
- Any set of independent Gaussian random variables is jointly Gaussian, but this is not necessarily true for a set of dependent Gaussian random variables.

# EE910: Digital Communication Systems-I

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**Lecture #2E: Complex random variables**

## Complex Random Variable

- A complex random Z= X + jY can be considered as a pair of real random variables.
- A complex random variable can be treated as a two-dimensional random vector with components X and Y.
- The PDF of a complex random variable is defined to be the joint PDF of its real and complex parts.
- If X and Y are jointly Gaussian random variables, then Z is a complex Gaussian random variable.
- The PDF of a zero-mean complex Gaussian random variable Z with i.i.d. real and imaginary parts is given by

$$\begin{aligned} p(z) &= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \\ &= \frac{1}{2\pi\sigma^2} e^{-\frac{|z|^2}{2\sigma^2}} \end{aligned}$$



## Complex Random Vectors

- For a complex random variable Z, the mean and variance are defined by

$$E[Z] = E[X] + jE[Y] \tag{1}$$
$$Var[Z] = E[|Z|^2] - |E[Z]|^2 = Var[X] + Var[Y]$$

- A complex random vector is defined as $Z = X + jY$, where X and Y are real-valued random vectors of size n. Real-valued matrices for a complex random vector Z are defines as

$$\begin{aligned} C_X &= E[(X - E(X))(X - E[X])^t] \\ C_Y &= E[(Y - E(Y))(Y - E[Y])^t] \\ C_{XY} &= E[(X - E(X))(Y - E[Y])^t] \\ C_{YX} &= E[(Y - E(Y))(X - E[X])^t] \end{aligned} \tag{2}$$

Matrices $C_X$ and $C_Y$ are the covariance matrices of real random vectors X and Y.

- $C_{YX} = {C_{XY}}^t$

## Complex Random Vectors

- Let 2n-dimensional real vector is defines as
$$\tilde{Z} = \begin{pmatrix} X \\ Y \end{pmatrix} \tag{3}$$
then the PDF of the complex vector Z is the PDF of the real vector $\tilde{Z}$.
- The covariance matrix of $\tilde{Z}$, can be written as
$$C_{\tilde{Z}} = \begin{pmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{pmatrix}$$
-
$$\begin{aligned} C_Z &= E[(Z - E[Z])(Z - E[Z])^H] \\ \tilde{C_Z} &= E[(Z - E[Z])(Z - E[Z])^t] \end{aligned} \tag{4}$$
$C_Z$ and $\tilde{C_Z}$ are called the covariance and the pseudocovariance of the complex random vector Z, respectively.



## Complex Random Vectors

- From definition, we have
$$\begin{aligned} C_Z &= C_X + C_Y + j(C_{YX} - C_{XY}) \\ \tilde{C_Z} &= C_X - C_Y + j(C_{XY} + C_{YX}) \\ C_X &= \frac{1}{2}Re[C_Z + \tilde{C_Z}] \\ C_Y &= \frac{1}{2}Re[C_Z - \tilde{C_Z}] \\ C_{YX} &= \frac{1}{2}Im[C_Z + \tilde{C_Z}] \\ C_{XY} &= \frac{1}{2}Im[\tilde{C_Z} - C_Z] \end{aligned} \tag{5}$$

## Proper and Circularly Symmetric Random Vectors

- A complex random vector Z is called proper if its pseudocovariance is zero, i.e., if $\tilde{C}_Z = 0$.
- For a proper random vector

$$\begin{aligned} C_X &= C_Y \\ C_{XY} &= -C_{YX} \end{aligned} \tag{6}$$

- Also,

$$\begin{aligned} C_Z &= 2C_X + 2jC_{YX} \\ C_X = C_Y &= \frac{1}{2}Re[C_Z] \\ C_{YX} = -C_{XY} &= \frac{1}{2}Im[C_Z] \\ C_{\tilde{Z}} &= \begin{pmatrix} C_X & C_{XY} \\ -C_{XY} & C_X \end{pmatrix} \end{aligned} \tag{7}$$



## Proper and Circularly Symmetric Random Vectors

- For n= 1

$$\begin{aligned} Var[X] &= Var[Y] \\ Cov[X,Y] &= -Cov[Y,X] \end{aligned} \tag{8}$$

which means that Z is proper if X and Y have equal variances and are uncorrelated.

- If the complex random vector $Z = X + jY$ is Gaussian, meaning that X and Y are jointly Gaussian, then we have

$$p(z) = p(\tilde{z}) = \frac{1}{(2\pi)^n (detC_{\tilde{z}})^{\frac{1}{2}}} e^{-\frac{1}{2}(\tilde{z}-\tilde{m})^t C_{\tilde{z}}{}^{-1}(\tilde{z}-\tilde{m})} \tag{9}$$

where

$$\tilde{m} = E[\tilde{Z}]$$

## Proper and Circularly Symmetric Random Vectors

- If Z is a proper n-dimensional complex Gaussian random vector, with mean $m = E[Z]$ and nonsingular covariance matrix $C_Z$, its PDF can be written as

$$p(z) = \frac{1}{\pi^n detC_Z} e^{-\frac{1}{2}(z-m)^\dagger {C_Z}^{-1}(z-m)} \tag{10}$$

- A complex random vector Z is called circularly symmetric or circular if rotating the vector by any angle does not change its PDF.
- For complex Gaussian random vectors being zero-mean and proper is equivalent to being circular.



## Proper and Circularly Symmetric Random Vectors

- If $\mathbf{Z}$ is circular, then it is zero-mean and proper.
- Since $\boldsymbol{Z}$ and $\boldsymbol{Z}e^{j\theta}$ have the same pdf, we have $E[\boldsymbol{Z}] = E\left[\boldsymbol{Z}e^{j\theta}\right] = e^{j\theta}E[\boldsymbol{Z}]$ for all $\theta$.
- Putting $\theta = \pi$ gives $E[\boldsymbol{Z}] = \mathbf{0}$.
- We also have $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = E\left[\boldsymbol{Z}e^{j\theta}\left(\boldsymbol{Z}e^{j\theta}\right)^t\right]$ or $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = e^{2j\theta}E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right]$, for all $\theta$.
- Putting $\theta = \frac{\pi}{2}$ gives $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = \mathbf{0}$.
- Since $\boldsymbol{Z}$ is zero-mean and $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = \mathbf{0}$, we conclude that it is proper.

## Proper and Circularly Symmetric Random Vectors

- If Z is a zero-mean proper Gaussian complex vector, then Z is circular.
- If Z is a proper n-dimensional complex Gaussian random vector, with mean $m = E[Z]$ and nonsingular covariance matrix $C_Z$, its PDF can be written as

$$p(z) = \frac{1}{\pi^n det C_Z} e^{-\frac{1}{2}(z-m)^\dagger C_Z{}^{-1}(z-m)}$$

- We note that for the zero-mean proper case if $\boldsymbol{W} = e^{j\theta}\boldsymbol{Z}$, it is sufficient to show that $\det(\boldsymbol{C_W}) = \det(\boldsymbol{C_Z})$ and $\boldsymbol{w}^H \boldsymbol{C}_W^{-1} \boldsymbol{w} = \boldsymbol{z}^H \boldsymbol{C}_Z^{-1} \boldsymbol{z}$.
- But $\boldsymbol{C_W} = \left[\boldsymbol{W}\boldsymbol{W}^H\right] = E\left[e^{j\theta}\boldsymbol{Z}e^{-j\theta}\boldsymbol{Z}^H\right] = E\left[\boldsymbol{Z}\boldsymbol{Z}^H\right] = \boldsymbol{C_Z}$, hence $\det(\boldsymbol{C_W}) = \det(\boldsymbol{C_Z})$. Similarly, $\boldsymbol{w}^H \boldsymbol{C}_W^{-1} \boldsymbol{w} = e^{-j\theta}\boldsymbol{z}^H \boldsymbol{C}_Z^{-1} \boldsymbol{z} e^{j\theta} = \boldsymbol{z}^H \boldsymbol{C}_Z^{-1} \boldsymbol{z}$.
- Substituting this, we conclude that $p(\boldsymbol{w}) = p(\boldsymbol{z})$.



## Proper and Circularly Symmetric Random Vectors

- If Z is a proper complex vector, then any transform of the form $\mathbf{W} = \mathbf{AZ} + \mathbf{b}$ is also a proper complex vector.
- Since $\boldsymbol{Z}$ is proper, we have $E\left[(\boldsymbol{Z} - E(\boldsymbol{Z}))(\boldsymbol{Z} - E(\boldsymbol{Z}))^t\right] = \mathbf{0}$.
- Let $\boldsymbol{W} = \boldsymbol{AZ} + \boldsymbol{b}$, then

$$E\left[(\boldsymbol{W} - E(\boldsymbol{W}))(\boldsymbol{W} - E(\boldsymbol{W}))^t\right] = \boldsymbol{A}E\left[(\boldsymbol{Z} - E(\boldsymbol{Z}))(\boldsymbol{Z} - E(\boldsymbol{Z}))^t\right]\boldsymbol{A}^t = \mathbf{0} \tag{11}$$

- Hence $\boldsymbol{W}$ is proper.

# EE910: Digital Communication Systems-I

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**Lecture #2F: A brief introduction to random processes**

## Random processes

- The mean and the autocorrelation of a random $X(t)$ are defined as

$$m_x(t) = E[X(t)] \tag{1}$$

$$R_x(t_1, t_2) = E[X(t_1)X^*(t_2)] \tag{2}$$

- The cross-correlation of two random process $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y^*(t_2)] \tag{3}$$

- We have $R_X(t_2, t_1) = R_X^*(t_2, t_1)$, i.e., $R_X(t_1, t_2)$ is Hermitian.
- For cross-correlation $R_{XY}(t_1, t_2) = R_{XY}^*(t_1, t_2)$



## Wide-Sense Stationary Random Processes

- Random process $X(t)$ is wide-sense stationary (WSS) if its mean is constant and autocorrelation is a function of time difference i.e

$$R_X(t_1, t_2) = R_X(\tau) \tag{4}$$

where $\tau = t_1 - t_2$
- Two processes $X(t)$ and $Y(t)$ are jointly WSS if both $X(t)$ and $Y(t)$ are WSS and $R_{XY}(t_1, t_2) = R_{XY}(\tau)$
- A complex process is WSS if its real and imaginary parts are WSS

## WSS Processes

- For a WSS process, the power spectrum is the Fourier transform of the autocorrelation function $R_X(\tau)$

$$S_X(f) = \mathfrak{F}[R_X(\tau)] \tag{5}$$

- Cross spectral density of two jointly WSS processes is defined as

$$S_{XY}(f) = \mathfrak{F}[R_{XY}(\tau)] \tag{6}$$

- The CSD satisfies the following property:

$$S_{XY}(f) = S^*_{XY}(f) \tag{7}$$



## WSS Processes

- If $X(t)$ and $Y(t)$ are jointly WSS random processes, then $Z(t) = aX(t) + bY(t)$ is a WSS process with

$$R_Z(\tau) = |a|^2 R_X(\tau) + |b|^2 R_Y(\tau) + ab^* R_{XY}(\tau) + ba^* R_{YX}(\tau) \tag{8}$$

$$S_Z(f) = |a|^2 S_X(f) + |b|^2 S_Y(f) + 2Re[ab^* S_{XY}(f)] \tag{9}$$

## WSS Processes

- When a WSS process $X(t)$ passes through an LTI system with impulse response $h(t)$ and transfer function $H(f) = \mathfrak{F}[h(t)]$, the output processes $Y(t)$ and $X(t)$ are jointly WSS and the following relations hold

$$\begin{aligned} m_Y &= m_X \int_{-\infty}^{\infty} h(t)dt && (10) \\ R_{XY}(\tau) &= R_X(t) * h^*(-\tau) && (11) \\ R_Y(\tau) &= R_X(t) * h(\tau) * h^*(-\tau) && (12) \\ m_Y &= m_X H(0) && (13) \\ S_{XY}(f) &= S_X H^*(f) && (14) \\ S_Y(f) &= S_X(f)|H(f)|^2 && (15) \end{aligned}$$



## WSS Processes

- The power in a WSS process $X(t)$ is the sum of the power at all frequencies

$$P_X = E[|X(t)|^2] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df \qquad (16)$$

## Gaussian Random Processes

- A real random process $X(t)$ is Gaussian if for all positive integers n and for all $(t_1, t_2, \ldots, t_n)$,the random vector $(X(t_1), X(t_2), \ldots, X(t_n))^t$ is Gaussian random vector.
- Two real random processes $X(t)$ and $Y(t)$ are jointly Gaussian if for all positive integers $n, m$ and all $(t_1, t_2, \ldots, t_n)$, and $(t_1^{'}, t_2^{'}, \ldots, t_m^{'})$ the random vector

$$(X(t_1), X(t_2), \ldots, X(t_n), Y(t_1^{'}), Y(t_2^{'}), \ldots, Y(t_m^{'}))^t \tag{17}$$

is a Gaussian vector.



## White Processes

- A process is called a white process if its power spectral density is constant for all frequencies

$$S_X(f) = \frac{N_o}{2} \tag{18}$$

- Power in a white process is infinite, indicating that white process cannot exists as a physical process

## Discrete-Time Random Processes

- PSD of a WSS discrete time random process is defined as the discrete-time Fourier transform of its autocorrelation function

$$S_X(f) = \sum_{m=-\infty}^{\infty} R_X(m) \exp^{-j2\pi fm} \tag{19}$$

- The power in discrete time random process is given by

$$P = E[|X(n)|^2] = R_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) df \tag{20}$$



## Cyclostationary Random Processes

- A random process $X(t)$ is cyclostationary if its mean and autocorrelation are periodic functions with same period $T_o$

$$m_X(t + T_o) = m_X(t) \tag{21}$$

$$R_X(t_1 + T_o, t_2 + T_o) = R_X(t_1, t_2) \tag{22}$$

- Average autocorrelation function defined as

$$\overline{R_X}(t) = \frac{1}{T_o} \int_0^{T_o} R_X(t + \tau, t) \tag{23}$$

## Proper and Circular Random Processes

- For a Complex Random process $Z(t) = X(t) + jY(t)$, the covariance and the pseudo covariance are

$$C_Z(t+\tau, t) = E[Z(t+\tau)Z^*(t)] \tag{24}$$

$$\tilde{C}_Z(t+\tau, t) = E[Z(t+\tau)Z(t)] \tag{25}$$

which can be written as

$$C_Z(t+\tau, t) = C_X(t+\tau, t) + C_Y(t+\tau, t) + j(C_{XY}(t+\tau, t) - C_YX(t+\tau, t)) \tag{26}$$

$$\tilde{C}_Z(t+\tau, t) = C_X(t+\tau, t) - C_Y(t+\tau, t) + j(C_{XY}(t+\tau, t) + C_YX(t+\tau, t)) \tag{27}$$

- A complex random process $Z(t)$ is proper if its pseudocovariance is zero



## Bandpass and Lowpass Random Processes

- For a bandpass process the power spectral density is located around frequencies $\pm f_o$ and for lowpass processes the PSD is located around zero frequency
- The in-phase and quadrature components of a bandpass random process $X(t)$ is

$$X_i(t) = X(t)\cos 2\pi f_o t + \hat{X}(t)\sin 2\pi f_o t \tag{28}$$

$$X_q(t) = \hat{X}(t)\cos 2\pi f_o t - X(t)\sin 2\pi f_o t \tag{29}$$

## Bandpass and Lowpass Random Processes

- $X_i(t)$ and $X_q(t)$ are jointly WSS zero-mean random process
- $X_i(t)$ and $X_q(t)$ have the same power spectral density
- $X_i(t)$ and $X_q(t)$ are both lowpass process
- We define lowpass equivalent process $X_l(t)$ as

$$X_l(t) = X_i(t) + jX_q(t) \tag{30}$$



## Bandpass and Lowpass Random Processes

- Since $X(t)$ by assumption is zero mean so is $\hat{X}(t)$, its Hilbert transform. Therefore $X_i(t)$ and $X_q(t)$ are both zero-mean process
- Autocorrelation of $X_i(t)$ is

$$\begin{aligned} R_{x_i}(t+\tau, t) &= E[(X_i(t+\tau)X_i(t)] \\ &= E[X(t+\tau)\cos 2\pi f_o(t+\tau) + \hat{X}(t+\tau)\sin 2\pi f_o(t+\tau)) \\ &\times X(t)\cos 2\pi f_o t + \hat{X}(t)\sin 2\pi f_o t] \end{aligned} \tag{31}$$

on solving it we get

$$R_{X_i}(\tau) = R_x(\tau)\cos(2\pi f_o \tau) + \hat{R}_X(\tau)\sin(2\pi f_o \tau) \tag{32}$$

## Bandpass and Lowpass Random Processes

- Similarly we have

$$R_{X_q}(\tau) = R_{X_i}(\tau) = R_x(\tau)\cos(2\pi f_o \tau) + \hat{R}_X(\tau)\sin(2\pi f_o \tau) \quad (33)$$

$$R_{X_i X_q}(\tau) = -R_{X_q X_i}(\tau) = R_x(\tau)\sin(2\pi f_o \tau) - \hat{R}_X(\tau)\cos(2\pi f_o \tau) \quad (34)$$

- Power Spectral densities would be

$$S_{X_i} = S_{X_q} = \begin{cases} S_X(f + f_o) + S_X(f - f_o) & |f| < f_o \\ 0 & otherwise \end{cases} \quad (35)$$

$$S_{X_i X_q} = -S_{X_i X_q} = \begin{cases} j\left\lfloor S_X(f + f_o) - S_X(f - f_o) \right\rfloor & |f| < f_o \\ 0 & otherwise \end{cases} \quad (36)$$



## Bandpass and Lowpass Random Processes

- The complex process $X_l(t) = X_i(t) + jX_q(t)$ as the lowpass equivalent of $X(t)$. Since $X_i(t)$ and $X_q(t)$ are both lowpass processes, we conclude that $X_l(t)$ is also a lowpass process

$$\begin{aligned} R_{X_l}(\tau) =& 2R_{X_i}(\tau) + 2jR_{X_q X_i}(\tau) \\ =& 2[R_X(\tau) + j\hat{R}_X(\tau)]e^{-j2\pi f_o t} \end{aligned} \quad (37)$$

- Taking Fourier transform on both sides we have

$$S_{X_l} = \begin{cases} 4S_X(f + f_o) & |f| < f_o \\ 0 & otherwise \end{cases} \quad (38)$$

$$S_X(f) = \frac{1}{4}[S_{X_l}(f - f_o) + S_{X_l}(f + f_o)] \quad (39)$$

# EE910: Digital Communication Systems-I

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**Lecture #3A: Digital Modulation: An Introduction**

## Introduction

- In a digital communication system, the source to be transmitted is discrete both in time and amplitude
- Digital information carrying signals must be first converted to an analog waveform prior to transmission
- At the receiving end, analog signals are converted back to a digital format before presentation to the end user
- The conversion process at the transmitting end is known as modulation
- The receiving end is known as demodulation or detection
- In digital wireless communication systems, the modulating signal may be represented as a time sequence of symbols or pulses, where each symbol has M finite states.
- Each symbol represents n bits of information where $n = \log_2 M$ bits/symbol.



## Advantages of Digital Modulation over Analog

- Greater noise immunity (due to its finite process)
- Robustness to channel impairments
- Easier multiplexing of various forms of information like voice, data, video
- Security –by using coding techniques to avoid jamming
- Accommodation of digital error control codes which detect and/or correct transmission errors
- Equalization to improve the performance of over all communication link
- Supports complex signal conditioning and processing methods

## Introduction

- Factors that influence digital modulation:
  - Low BER at low received signal-to-interference noise ratio (SINR)
  - Should performs well in multi-path and fading
  - Should have high spectral efficiency
  - Essentially, good BER performance at a low SINR under conditions of co-channel interference, and fading.
- The performance of a modulation scheme is often measured in terms of its power efficiency and bandwidth efficiency.
- The power efficiency is the ability of a modulation technique to preserve the fidelity (acceptable BER) of the digital message at low power levels
- Bandwidth efficiency is a measure of how many bits per symbol that we can reliably send over the communication system.

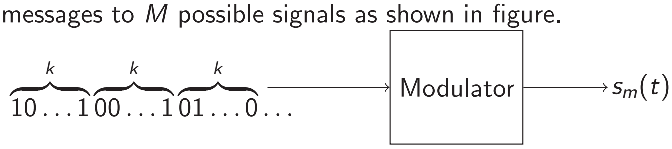


## Introduction

- The source information is normally represented as a baseband (low-pass) signal
- Because of signal attenuation, it is necessary to move the baseband signal spectrum to reside at a much higher frequency band centered at $f_c$, called the carrier frequency, in the radio spectrum
- At the receiver end, the demodulation process removes the carrier frequency to recover the baseband information signal
- We choose different carrier frequencies for different signals
- Modulation/demodulation process facilitates channel assignment and reduces interference from other transmissions
- The modulation can be classified into two categories:
  - Linear modulation
  - Nonlinear modulation
- Also modulation can be memoryless or with memory.

## Memoryless Modulation

- In a memoryless modulation scheme, the binary sequence is parsed into subsequences each of length $k$.
- Each sequence is mapped into one of the $s_m(t)$, $1 \leq m \leq 2^k$, signals regardless of the previously transmitted signals.
- This modulation scheme is equivalent to a mapping from $M = 2^k$ messages to $M$ possible signals as shown in figure.






## Modulation with memory

- In a modulation scheme with memory, the mapping is from the set of the current $k$ bits and the past $(L-1)k$ bits to the set of possible $M = 2^k$ messages.
- In this case the transmitted signal depends on the current $k$ bits as well as the most recent $L-1$ blocks of $k$ bits. This defines a finite-state machine with $2^{(L-1)k}$ states.

## Modulation with memory

- If at time instant $\ell - 1$ the modulator is in state $S_{\ell-1} \in \{1, 2, ..., 2^{(L-1)k}\}$ and the input sequence is $I_\ell \in \{1, 2, ..., 2^k\}$, then the modulator transmits the output $s_{m_\ell}(t)$ and moves to new state $S_\ell$ according to mappings

$$m_\ell = f_m(S_{\ell-1}, I_\ell) \quad (1)$$

$$S_\ell = f_s(S_{\ell-1}, I_\ell) \quad (2)$$

- Parameters $k$ and $L$ and functions $f_m(\cdot,\cdot)$ and $f_s(\cdot,\cdot)$ completely describe the modulation scheme with memory.
- Parameter $L$ is called the constraint length of modulation.



## Representation of digitally modulated signals

- The case of $L = 1$ corresponds to a memoryless modulation scheme.
- The modulator in a digital communication system maps a sequence of $k$ binary symbols which in case of equiprobable symbols carries $k$ bits of information into a set of corresponding signal waveforms $s_m(t)$, $1 \le m \le M$, where $M = 2^k$ .

## Representation of digitally modulated signals

- We assume that these signals are transmitted at every $T_s$ seconds, where $T_s$ is called the signaling interval.
- This means that in each second

$$R_s = \frac{1}{T_s} \tag{3}$$

symbols are transmitted.
- Parameter $R_s$ is called the signaling rate or symbol rate.
- Since each signal carries $k$ bits of information, the bit interval $T_b$, i.e., the interval in which 1 bit of information is transmitted, is given by

$$T_b = \frac{T_s}{k} = \frac{T}{\log_2 M} \tag{4}$$



## Representation of digitally modulated signals

- The bit rate $R$ is given by

$$R = kR_s = R_s \log_2 M \tag{5}$$

- If the energy content of $s_m(t)$ is denoted by $\mathcal{E}_m$, then the average signal energy is given by

$$\mathcal{E}_{avg} = \sum_{m=1}^{M} p_m \mathcal{E}_m \tag{6}$$

where $p_m$ indicates the probability of the $m$-th signal (message probability).

## Representation of digitally modulated signals

- In the case of equiprobable messages, $p_m = \frac{1}{M}$, and therefore,

$$\mathcal{E}_{avg} = \frac{1}{M}\sum_{m=1}^{M}\mathcal{E}_m \tag{7}$$

- If all signals have the same energy, then $\mathcal{E}_m = \mathcal{E}$ and $\mathcal{E}_{avg} = \mathcal{E}$.
- The average energy for transmission of 1 bit of information, or average energy per bit, when the signals are equiprobable is given by

$$\mathcal{E}_{bavg} = \frac{\mathcal{E}_{avg}}{k} = \frac{\mathcal{E}_{avg}}{\log_2 M} \tag{8}$$



## Representation of digitally modulated signals

- If all signals have equal energy of $\mathcal{E}$, then

$$\mathcal{E}_b = \frac{\mathcal{E}}{k} = \frac{\mathcal{E}}{\log_2 M} \tag{9}$$

- If a communication system is transmitting an average energy of $\mathcal{E}_{bavg}$ per bit, and it takes $T_b$ seconds to transmit this average energy, then the average power sent by the transmitter is

$$P_{avg} = \frac{\mathcal{E}_{bavg}}{T_b} = R\mathcal{E}_{bavg} \tag{10}$$

which for the case of equal energy signals becomes

$$P = R\mathcal{E}_b \tag{11}$$

# EE910: Digital Communication Systems-I

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**Lecture #3B: Pulse Amplitude Modulation, Phase Shift Keying and Quadrature Amplitude Modulation**

## Pulse Amplitude Modulation (PAM)

- In digital PAM, the signal waveforms may be represented as

$$s_m(t) = A_m p(t), 1 \leq m \leq M \tag{1}$$

where $p(t)$ is a pulse of duration $T$ and $\{A_m, 1 \leq m \leq M\}$ denotes the set of M possible amplitudes corresponding to $M = 2^k$ possible k-bit blocks of symbols.

- The signal amplitudes $A_m$ take the discrete values $A_m = 2m - 1 - M$, $m = 1, 2, ..., M$ i.e., the amplitudes are $\pm 1, \pm 3, \pm 5, ..., \pm(M-1)$.
- The waveform $p(t)$ is a real-valued signal pulse whose shape influences the spectrum of the transmitted signal.



## Pulse Amplitude Modulation (PAM)

- The energy in signal $s_m(t)$ is given by

$$\mathcal{E}_m = \int_{-\infty}^{\infty} A_m^2 p^2(t) dt = A_m^2 \mathcal{E}_p \tag{2}$$

where $\mathcal{E}_p$ is the energy in $p(t)$.

- From this,

$$\begin{aligned} \mathcal{E}_{avg} &= \frac{\mathcal{E}_p}{M} \sum_{m=1}^{M} A_m^2 \\ &= \frac{2\mathcal{E}_p}{M}(1^2 + 3^2 + 5^2 + \cdots + (M-1)^2) \\ &= \frac{2\mathcal{E}_p}{M} \times \frac{M(M^2-1)}{6} \\ &= \frac{(M^2-1)\mathcal{E}_p}{3} \end{aligned} \tag{3}$$

## Pulse Amplitude Modulation (PAM)

- Average energy per bit is given by

$$\mathcal{E}_{bavg} = \frac{(M^2-1)\mathcal{E}_p}{3\log_2 M} \tag{4}$$

- Usually the PAM signals are carrier-modulated bandpass signals with lowpass equivalents of the form $A_m g(t)$, where $A_m$ and $g(t)$ are real. In this case

$$\begin{aligned} s_m(t) &= Re\left[s_{ml}(t)e^{j2\pi f_c t}\right] \\ &= Re\left[A_m g(t)e^{2\pi f_c t}\right] = A_m g(t)\cos(2\pi f_c t) \end{aligned}$$

where $f_c$ is the carrier frequency.

- In the generic form of PAM signaling if we substitute

$$p(t) = g(t)\cos(2\pi f_c t) \tag{5}$$

then we obtain the bandpass PAM.



## Pulse Amplitude Modulation (PAM)

- For bandpass PAM we have

$$\mathcal{E}_m = \frac{A_m^2\mathcal{E}_g}{2} \tag{6}$$

where $\mathcal{E}_g$ is the energy in $g(t)$.

- From Equations (3) and (4) we conclude

$$\begin{aligned} \mathcal{E}_{avg} &= \frac{(M^2-1)\mathcal{E}_g}{6} \\ \mathcal{E}_{bavg} &= \frac{(M^2-1)\mathcal{E}_g}{6\log_2 M} \end{aligned} \tag{7}$$

- Clearly, PAM signals are one-dimensional ($N = 1$) since all are multiples of the same basic signals.

## Pulse Amplitude Modulation (PAM)

- We can use

$$\phi(t) = \frac{p(t)}{\sqrt{\mathcal{E}_p}} \tag{8}$$

as the basis for the general PAM signal of the form $s_m(t) = A_m p(t)$ and

$$\phi(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g(t) \cos 2\pi f_c t \tag{9}$$

as the basis for the bandpass PAM signal given in Equation (5).

- Using these basis signals, we have

$$\begin{aligned} s_m(t) &= A_m \sqrt{\mathcal{E}_p} \phi(t) \text{ for baseband PAM} \\ s_m(t) &= A_m \sqrt{\frac{\mathcal{E}_g}{2}} \phi(t) \text{ for bandpass PAM} \end{aligned} \tag{10}$$



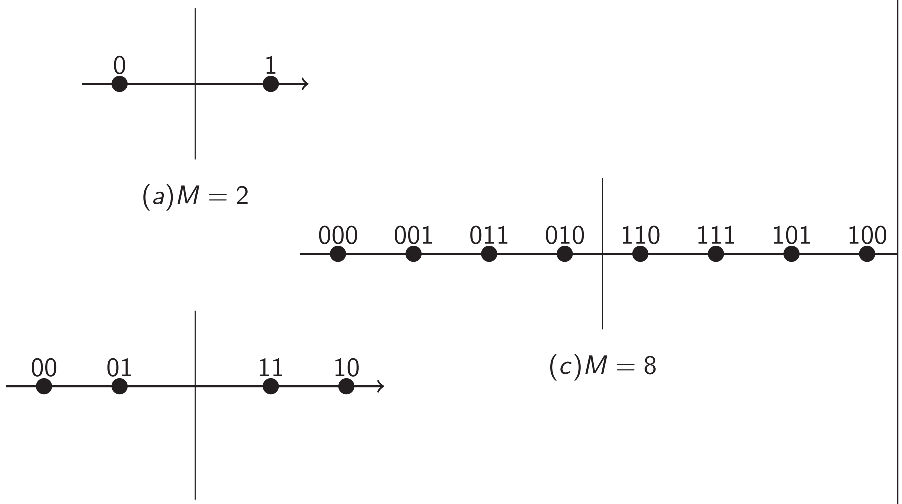
## Pulse Amplitude Modulation (PAM)

- The one-dimensional vector representations for these signals are of the form

$$s_m = A_m \sqrt{\mathcal{E}_p}, \qquad A_m = \pm 1, \pm 3, \cdots, \pm (M-1) \tag{11}$$

$$s_m = A_m \sqrt{\frac{\mathcal{E}_g}{2}}, \qquad A_m = \pm 1, \pm 3, \cdots, \pm (M-1) \tag{12}$$

## Pulse Amplitude Modulation (PAM)




$(a) M = 2$

$(c) M = 8$

$(b) M = 4$



## Pulse Amplitude Modulation (PAM)

- We note that the Euclidean distance between any pair of signal points is

$$\begin{aligned} d_{mn} &= \sqrt{\|s_m - s_n\|^2} \\ &= |A_m - A_n| \sqrt{\mathcal{E}_p} \\ &= |A_m - A_n| \sqrt{\frac{\mathcal{E}_g}{2}} \end{aligned} \tag{13}$$

where the last relation corresponds to a bandpass PAM.

- For adjacent signal points $|A_m - A_n| = 2$, and hence the minimum distance of the constellation is given by

$$d_{min} = 2\sqrt{\mathcal{E}_p} = \sqrt{2\mathcal{E}_g} \tag{14}$$

## Pulse Amplitude Modulation (PAM)

- The resulting expression is

$$d_{min} = \sqrt{\frac{12 \log_2 M}{M^2 - 1} \mathcal{E}_{bavg}} \tag{15}$$

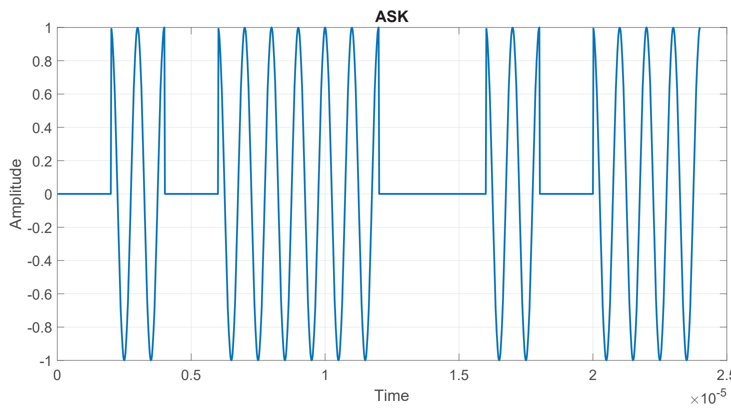


## Amplitude Shift Keying (ASK)

- Can be viewed as a special case of PAM where g(t) is a sinusoid.
- Here amplitude of the carrier signal is varied according to the information sequence.
- Simplest form of ASK is on-off keying where either bursts of a carrier signal are transmitted or nothing is transmitted depending whether the input message signal is 1 or 0.

## Amplitude Shift Keying (ASK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1






## Phase Modulation

- In digital phase modulation, the M signal waveforms are represented as

$$
\begin{aligned}
s_m(t) &= Re\left[g(t)e^{j\frac{2\pi(m-1)}{M}}e^{j2\pi f_c t}\right], \qquad m = 1, 2, \cdots, M \quad (16) \\
&= g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right] \\
&= g(t)\cos\left(\frac{2\pi}{M}(m-1)\right)\cos 2\pi f_c t \\
&\quad -g(t)\sin\left(\frac{2\pi}{M}(m-1)\right)\sin 2\pi f_c t
\end{aligned}
$$

where $g(t)$ is the signal pulse shape and $\theta_m = 2\pi\frac{(m-1)}{M}, \quad m = 1, 2, \cdots, M$ is the $M$ possible phases of the carrier that convey the transmitted information.

## Phase Modulation

- Digital phase modulation is usually called phase-shift keying (PSK).
- We note that these signal waveforms have equal energy.

$$\mathcal{E}_{avg} = \mathcal{E}_m = \frac{1}{2}\mathcal{E}_g \tag{17}$$

and therefore,

$$\mathcal{E}_{bavg} = \frac{\mathcal{E}_g}{2\log_2 M} \tag{18}$$



## Phase Modulation

- We note that $g(t)\cos 2\pi f_c T$ and $g(t)\sin 2\pi f_c t$ are orthogonal, and therefore $\phi_1(t)$ and $\phi_2(t)$ given as

$$\phi_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g(t)\cos 2\pi f_c t \tag{19}$$

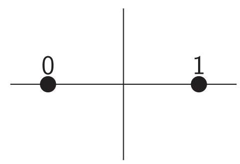
$$\phi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}} g(t)\sin 2\pi f_c t \tag{20}$$

- We can write $s_m(t), 1 \leq m \leq M$ , as

$$s_m(t) = \sqrt{\frac{\mathcal{E}_g}{2}} \cos\left(\frac{2\pi}{M}(m-1)\right)\phi_1(t) + \sqrt{\frac{\mathcal{E}_g}{2}} \sin\left(\frac{2\pi}{M}(m-1)\right)\phi_2(t) \tag{21}$$

- The signal space dimensionality is $N = 2$ and the resulting vector representations are

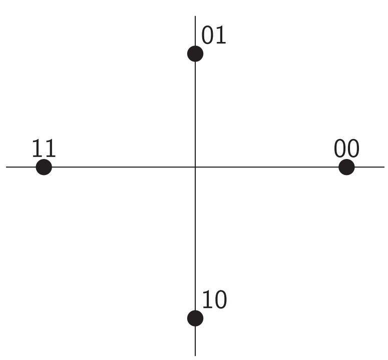
$$s_m = \left( \sqrt{\frac{\mathcal{E}_g}{2}} \cos\left(\frac{2\pi}{M}(m-1)\right), \sqrt{\frac{\mathcal{E}_g}{2}} \sin\left(\frac{2\pi}{M}(m-1)\right) \right), m = 1, 2, \cdots, M \tag{22}$$



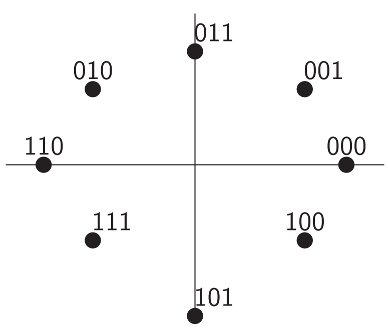

*BPSK*








*QPSK*




8*PSK*

## Phase Modulation

- The Euclidean distance between signal points is

$$\begin{aligned} d_{mn} &= \sqrt{\|s_m - s_n\|^2} \\ &= \sqrt{\mathcal{E}_g \left[1 - \cos\left(\frac{2\pi}{M}(m-n)\right)\right]} \end{aligned} \tag{23}$$

- The minimum distance corresponding to $|m-n| = 1$ is

$$d_{min} = \sqrt{\mathcal{E}_g \left[1 - \cos\left(\frac{2\pi}{M}\right)\right]} = \sqrt{2\mathcal{E}_g \sin^2 \frac{\pi}{M}} \tag{24}$$

- Solving Equation (18) for $\mathcal{E}_g$ and substituting the result in Equation (24) result in

$$d_{min} = 2\sqrt{\left(\log_2 M \times \sin^2 \frac{\pi}{M}\right) \mathcal{E}_b} \tag{25}$$



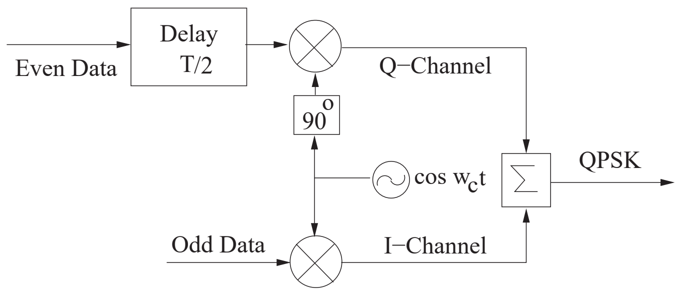
## Phase Modulation

- For large values of M, we have $\sin \frac{\pi}{M} \approx \frac{\pi}{M}$, and $d_{min}$ can be approximated by

$$d_{min} \approx 2\sqrt{\frac{\pi^2 \log_2 M}{M^2} \mathcal{E}_b} \tag{26}$$

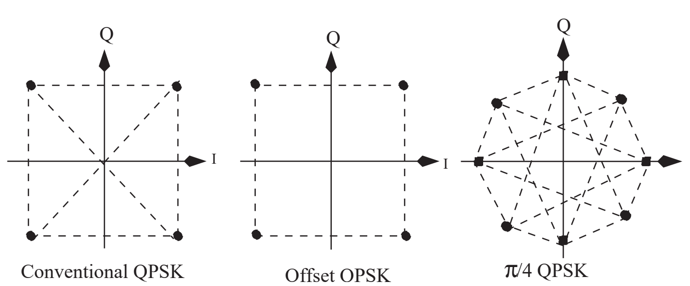
- Variants of four-phase PSK (QPSK) include offset QPSK and $\pi/4$ QPSK.
- In Offset QPSK, the phase transitions are limited to 90 degrees, the transitions on the I and Q channels are staggered.
- In $\pi/4$ QPSK the set of constellation points are toggled each symbol, so transitions through zero cannot occur.

# Offset QPSK (OQPSK)



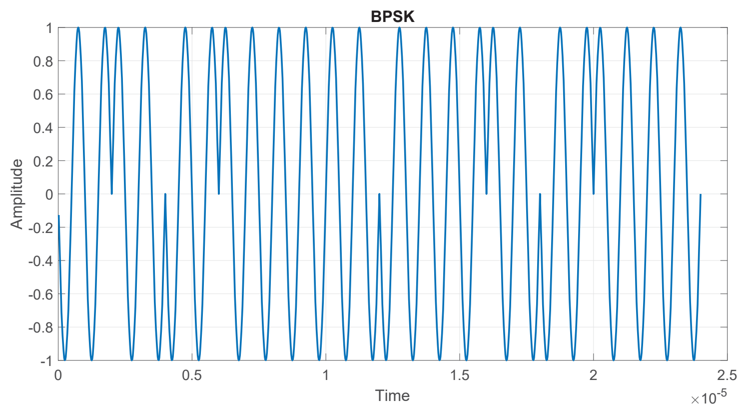



# Quadrature PSK (QPSK)



## Binary Phase Shift Keying (BPSK)

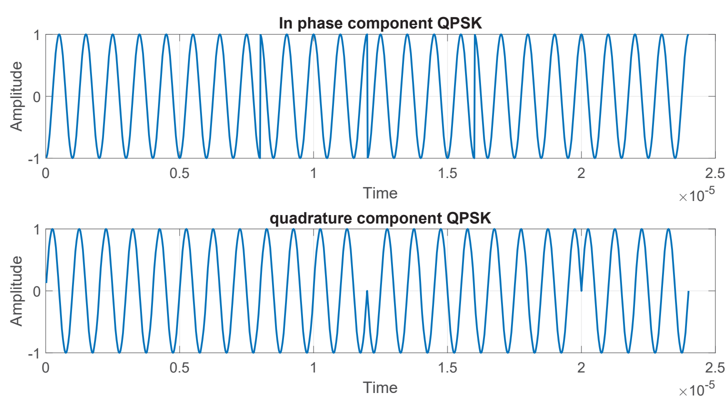
Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1






## Quadrature Phase Shift Keying (QPSK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1



## Quadrature Amplitude Modulation

- The bandwidth efficiency of PAM can also be obtained by simultaneously impressing two separate k-bit symbols from the information sequence on two quadrature carriers $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$.
- The resulting modulation technique is called quadrature PAM or QAM, and the corresponding signal waveforms may be expressed as

$$\begin{aligned} s_m(t) &= Re\left[(A_{mi} + jA_{mq})g(t)e^{j2\pi f_c t}\right] \quad (27)\\ &= A_{mi}g(t)\cos 2\pi f_c t - A_{mq}g(t)\sin 2\pi f_c t, \quad m = 1, 2, \cdots, M \end{aligned}$$

where $A_{mi}$ and $A_{mq}$ are the information-bearing signal amplitudes of the quadrature carriers and $g(t)$ is the signal pulse.



## Quadrature Amplitude Modulation

- Alternatively, the QAM signal waveforms may be expressed as

$$\begin{aligned} s_m(t) &= Re\left[r_m e^{j\theta_m} e^{j2\pi f_c t}\right] \\ &= r_m \cos(2\pi f_c t + \theta_m) \end{aligned} \quad (28)$$

where $r_m = \sqrt{A_{mi}^2 + A_{mq}^2}$ and $\theta_m = tan^{-1}(A_{mi}/A_{mq})$

- Similar to the PSK case, $\phi_1(t)$ and $\phi_2(t)$ given in Equations (19) and (20) can be used as an orthonormal basis for expansion of QAM signals.
- The dimensionality of the signal space for QAM is $N = 2$.

## Quadrature Amplitude Modulation

- Using this basis, we have

$$s_m(t) = A_{mi}\sqrt{\frac{\mathcal{E}_g}{2}}\phi_1(t) + A_{mq}\sqrt{\frac{\mathcal{E}_g}{2}}\phi_2(t) \tag{29}$$

which results in vector representations of the form

$$\begin{aligned} s_m &= (s_{m1}, s_{m2}) \\ &= \left(A_{mi}\sqrt{\frac{\mathcal{E}_g}{2}}, A_{mq}\sqrt{\frac{\mathcal{E}_g}{2}}\right) \end{aligned} \tag{30}$$

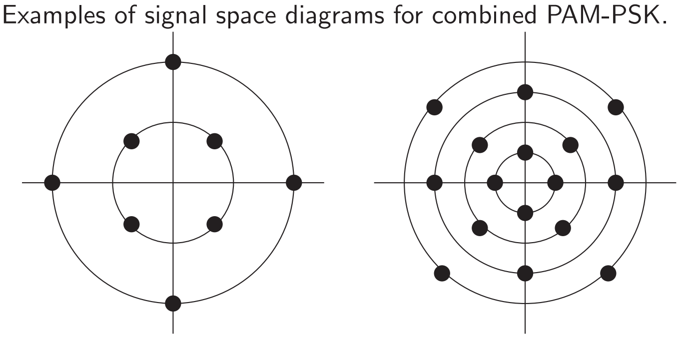
and

$$\mathcal{E}_m = \|s_m\|^2 = \frac{\mathcal{E}_g}{2}(A_{mi}^2 + A_{mq}^2) \tag{31}$$



## Quadrature Amplitude Modulation

- Examples of signal space diagrams for combined PAM-PSK.



$M = 8$ $M = 16$

## Quadrature Amplitude Modulation

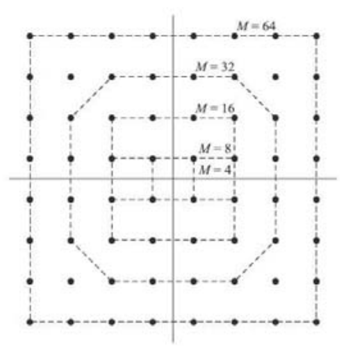
- The Euclidean distance between any pair of signal vectors in QAM is

$$
\begin{aligned}
d_{mn} &= \sqrt{\|s_m - s_n\|^2} \\
&= \sqrt{\frac{\mathcal{E}_g}{2}\left[(A_{mi}^2 - A_{ni}^2) + (A_{mq}^2 - A_{nq}^2)\right]}
\end{aligned} \quad (32)
$$

- In the special case where the signal amplitudes take the set of discrete values $\{(2m-1-M), \quad m = 1, 2, \cdots, M\}$, the signal space diagram is rectangular.



## Quadrature Amplitude Modulation




- In this case, the Euclidean distance between adjacent points, i.e., the minimum distance, is

$$
d_{min} = \sqrt{2\mathcal{E}_g} \quad (33)
$$

which is the same result as for PAM.

## Quadrature Amplitude Modulation

- In the special case of a rectangular constellation with $M = 2^{2k_1}$, i.e., $M = 4, 16, 64, 256, \cdots$, and with amplitudes of $\pm 1, \pm 3, \cdots, \pm(\sqrt{M} - 1)$ on both directions, from equation (31) we have

$$\begin{aligned}\mathcal{E}_{avg} &= \frac{1}{M}\frac{\mathcal{E}_g}{2}\sum_{m=1}^{\sqrt{M}}\sum_{n=1}^{\sqrt{M}}(A_m^2 + A_n^2) \qquad (34)\\ &= \frac{\mathcal{E}_g}{2M} \times \frac{2M(M-1)}{3} = \frac{(M-1)}{3}\mathcal{E}_g\end{aligned}$$

- Thus

$$\mathcal{E}_{bavg} = \frac{M-1}{3\log_2 M}\mathcal{E}_g \qquad (35)$$

- Using equation (33), we have

$$d_{min} = \sqrt{\frac{6\log_2 M}{M-1}\mathcal{E}_{bavg}} \qquad (36)$$

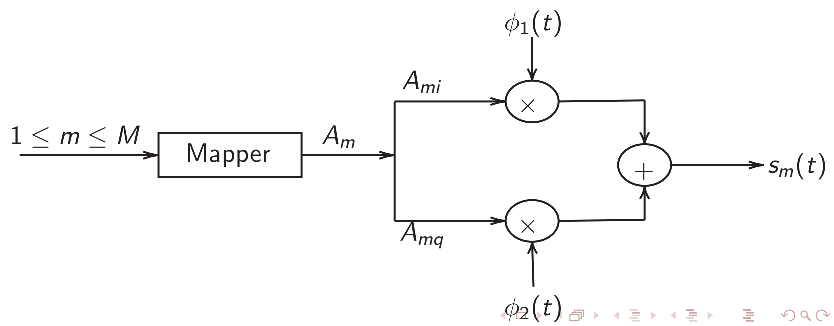


## Quadrature Amplitude Modulation

- From the discussion of bandpass PAM, PSK, and QAM, it is clear that all these signaling schemes are of the general form

$$s_m(t) = Re\left[A_m g(t) e^{j2\pi f_c t}\right], \quad m = 1, 2, \cdots, M \qquad (37)$$

where $A_m$ is determined by the signaling scheme.
- The structure of the modulator for this general class of signaling schemes is shown below



# EE910: Digital Communication Systems-I

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April 25, 2022

**Lecture #3C: Orthogonal, Bi-orthogonal and Simplex Signaling**

## Orthogonal Signaling

- Orthogonal signals are defined as a set of equal energy signals $s_m(t), 1 \leq m \leq M$, such that

$$\langle s_m(t), s_n(t) \rangle = 0, \quad m \neq n \; and \; 1 \leq m, n \leq M \tag{1}$$

- Thus we have

$$\langle s_m(t), s_n(t) \rangle = \begin{cases} \mathcal{E} & m = n \\ 0 & m \neq n \end{cases} \qquad 1 \leq m, n \leq M \tag{2}$$

- The signals are linearly independent and hence N = M.
- The orthonormal set $\{\phi_j(t), 1 \leq i \leq N\}$ given by

$$\phi_j(t) = \frac{s_j(t)}{\sqrt{\mathcal{E}}}, \quad 1 \leq j \leq N \tag{3}$$

can be used as an orthonormal basis for representation of $\{s_m(t), 1 \leq m \leq M\}$.



## Orthogonal Signaling

- The resulting vector representation of the signals will be

$$\begin{aligned} s_1 &= (\sqrt{\mathcal{E}}, 0, 0, \cdots, 0) \\ s_2 &= (0, \sqrt{\mathcal{E}}, 0, \cdots, 0) \\ \vdots &= \vdots \\ s_M &= (0, 0, 0, \cdots, \sqrt{\mathcal{E}}) \end{aligned} \tag{4}$$

- From Equation (5) it is seen that for all $m \neq n$ we have

$$d_{min} = \sqrt{2\mathcal{E}} \tag{5}$$

- Using the relation

$$\mathcal{E}_b = \frac{\mathcal{E}}{\log_2 M} \tag{6}$$

we conclude that

$$d_{min} = \sqrt{2 \log_2 M \mathcal{E}_b} \tag{7}$$

## Frequency-Shift Keying (FSK)

- A special case of orthogonal signals.
- Let us consider the construction of orthogonal signal waveforms that differ in frequency and are represented as

$$
\begin{aligned}
s_m(t) &= Re\left[s_{ml}(t)e^{j2\pi f_c t}\right], \quad 1 \le m \le M, \ \ 0 \le t \le T \\
&= \sqrt{\frac{2\mathcal{E}}{T}}\cos(2\pi f_c t + 2\pi m \triangle f t)
\end{aligned} \tag{8}
$$

where

$$
s_{ml}(t) = \sqrt{\frac{2\mathcal{E}}{T}}e^{j2\pi m \triangle f t}, \quad 1 \le m \le M, \ \ 0 \le t \le T \tag{9}
$$

- The coefficient $\sqrt{\frac{2\mathcal{E}}{T}}$ is introduced to guarantee that each signal has an energy equal to $\mathcal{E}$ .



## Frequency-Shift Keying (FSK)

- For this set of signals to be orthogonal, we need to have

$$
Re\left[\int_0^T s_{ml}(t)s_{nl}(t)dt\right] = 0 \tag{10}
$$

for all $m \neq n$.

- We have

$$
\begin{aligned}
\langle s_{ml}(t), s_{nl}(t)\rangle &= \frac{2\mathcal{E}}{T}\int_0^T e^{j2\pi(m-n)\triangle f t}dt \\
&= \frac{2\mathcal{E}\sin(\pi T(m-n)\triangle f)}{\pi T(m-n)\triangle f}e^{j\pi T(m-n)\triangle f}
\end{aligned} \tag{11}
$$

## Frequency-Shift Keying (FSK)

- Also

$$
\begin{aligned}
Re\left[\langle s_{ml}(t), s_{nl}(t)\rangle\right] &= \frac{2\mathcal{E}\sin(\pi T(m-n)\triangle f)}{\pi T(m-n)\triangle f}\cos(\pi T(m-n)\triangle f) \\
&= \frac{2\mathcal{E}\sin(2\pi T(m-n)\triangle f)}{2\pi T(m-n)\triangle f} \qquad (12) \\
&= 2\mathcal{E}\, sinc(2T(m-n)\triangle f)
\end{aligned}
$$

- From Equation (13) we observe that $s_m(t)$ and $s_n(t)$ are orthogonal for all $m \neq n$ if and only if $sinc(2T(m-n)\triangle f) = 0$ for all $m \neq n$.
- This is the case if $\triangle f = k/2T$ for some positive integer k.

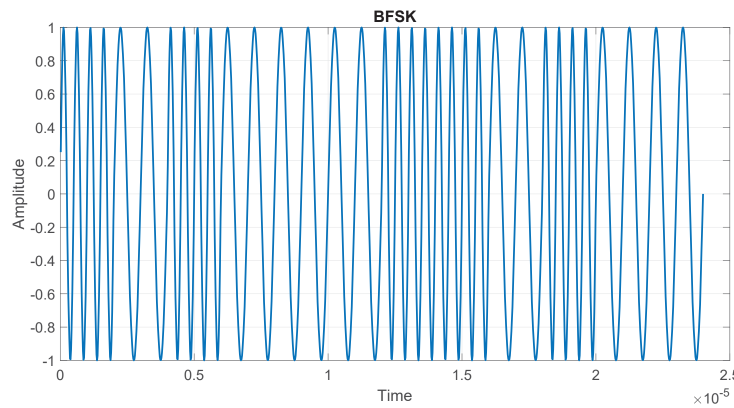


## Frequency-Shift Keying (FSK)

- The minimum frequency separation $\triangle f$ that guarantees orthogonality is $\triangle f = 1/2T$.
- Note that $\triangle f = \frac{1}{2T}$ is the minimum frequency separation that guarantees $\langle s_{ml}(t), s_{nl}(t)\rangle = 0$.
- This guarantees the orthogonality of the baseband, as well as the bandpass, frequency-modulated signals.

## Frequency-Shift Keying (FSK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1






## Hadamard signals

- Hadamard signals are orthogonal signals which are constructed from Hadamard matrices.
- Hadamard matrices $H_n$ are $2^n \times 2^n$ matrices for $n = 1, 2, \cdots$ defined by the following recursive relation

$$H_0 = [1]$$
$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix} \tag{13}$$

- With this definition we have

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{14}$$

## Hadamard Signals

- $$H_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{15}$$
- Hadamard matrices are symmetric matrices whose rows (and, by symmetry, columns) are orthogonal.
- Using these matrices, we can generate orthogonal signals.



## Hadamard Signals

- For instance, using $H_2$ would result in the set of signals

$$\begin{aligned} s_1 &= \begin{bmatrix} \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} \end{bmatrix} \\ s_2 &= \begin{bmatrix} \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} \end{bmatrix} \\ s_3 &= \begin{bmatrix} \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} \end{bmatrix} \\ s_4 &= \begin{bmatrix} \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} \end{bmatrix} \end{aligned} \tag{16}$$

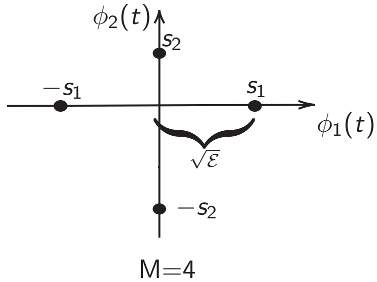
- This set of orthogonal signals may be used to modulate any four-dimensional orthonormal basis $\{\phi_j(t)\}_{j=1}^4$ to generate signals of the form

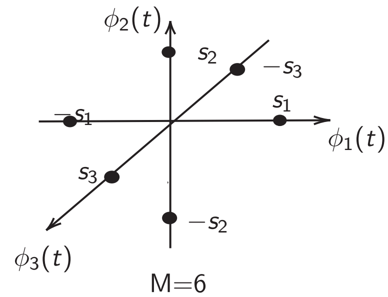
$$s_m(t) = \sum_{j=1}^{4} s_{mj}\phi_j(t), \qquad 1 \le m \le 4 \tag{17}$$

- Note that the energy in each signal is $4\mathcal{E}$ , and each signal carries 2 bits of information, hence $\mathcal{E}_b = 2\mathcal{E}$ .

# Biorthogonal Signaling

- A set of M biorthogonal signals can be constructed from $\frac{1}{2}M$ orthogonal signals by simply including the negatives of the orthogonal signals.
- Thus, we require $N = \frac{1}{2}M$ dimensions for the construction of a set of M biorthogonal signals.
- Figure illustrates the biorthogonal signals for M = 4 and 6.






# Biorthogonal Signaling

- All signals are equidistant from $s_i$, i.e.

$$\|s_i - s_k\| = \sqrt{2E_s} = d_{\min} \tag{18}$$

except one signal point which is the reflection through the origin, and is farther away

$$\|s_i - (-s_i)\| = 2\sqrt{E_s} \tag{19}$$

## Simplex Signaling

- Suppose we have a set of M orthogonal waveforms $\{s_m(t)\}$or, equivalently, their vector representation $s_m$, their mean is given by

$$\bar{s} = \frac{1}{M}\sum_{m=1}^{M} s_m \tag{20}$$

- Now, let us construct another set of M signals by subtracting the mean from each of the M orthogonal signals.
- Thus,

$$s'_m = s_m - \bar{s}, \qquad m = 1, 2, \cdots, M \tag{21}$$



## Simplex Signaling

- The resulting signal waveforms are called simplex signals and have the following properties. First, the energy per waveform is

$$\begin{aligned} \|s'_m\|^2 &= \|s_m - \bar{s}\|^2 \\ &= \mathcal{E} - \frac{2}{M}\mathcal{E} + \frac{1}{M}\mathcal{E} \\ &= \mathcal{E}\left(1 - \frac{1}{M}\right) \end{aligned} \tag{22}$$

- Second, the cross-correlation of any pair of signals is

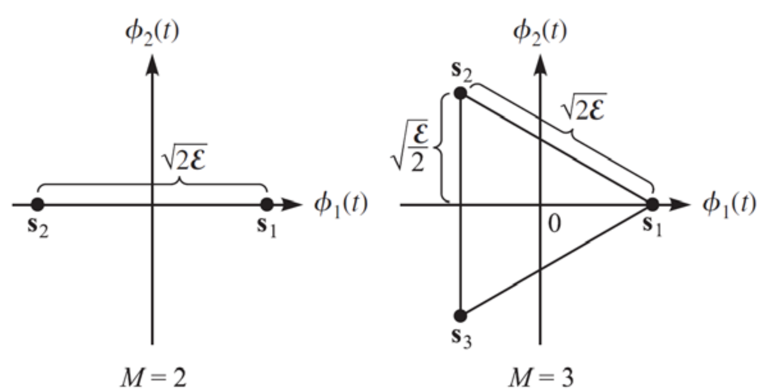
$$\begin{aligned} Re\left[\rho_{mn}\right] &= \frac{s'_m \cdot s'_n}{\|s'_m\|\|s'_n\|} \\ &= \frac{-1/M}{1 - 1/M} = -\frac{1}{M-1} \end{aligned} \tag{23}$$

# Simplex Signaling

- Hence, the set of simplex waveforms is equally correlated and requires less energy, by the factor $1 - 1/M$, than the set of orthogonal waveforms.
- Since only the origin was translated, the distance between any pair of signal points is maintained at $d = \sqrt{2\mathcal{E}}$ , which is the same as the distance between any pair of orthogonal signals.



# Simplex Signaling



# EE910: Digital Communication Systems-I

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May 2, 2022

**Lecture #4A: Continuous-Phase Frequency-Shift Keying (CPFSK)**

## Continuous-Phase Frequency-Shift Keying (CPFSK)

- We consider a class of digital modulation methods in which the phase of the signal is constrained to be continuous.
- This constraint results in a phase or frequency modulator that has memory.
- To represent a CPFSK signal, we begin with a PAM signal

$$d(t) = \sum_n I_n g(t - nT) \tag{1}$$

where $\{I_n\}$ denotes the sequence of amplitudes obtained by mapping k-bit blocks of binary digits from the information sequence $\{a_n\}$ into the amplitude levels $\pm 1, \pm 3, \cdots, \pm(M-1)$ and $g(t)$ is a rectangular pulse of amplitude $1/2T$ and duration $T$ seconds.



## Continuous-Phase Frequency-Shift Keying (CPFSK)

- The signal $d(t)$ is used to frequency-modulate the carrier. Consequently, the equivalent lowpass waveform $\upsilon(t)$ is expressed as

$$\upsilon(t) = \sqrt{\frac{2\varepsilon}{T}} e^{j\left[4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau + \phi_0\right]} \tag{2}$$

where $f_d$ is the peak frequency deviation and $\phi_0$ is the initial phase of the carrier.

- The carrier-modulated signal corresponding to Equation (2) may be expressed as

$$s(t) = \sqrt{\frac{2\varepsilon}{T}} cos\left[2\pi f_c t + \phi(t; \mathbf{I}) + \phi_0\right] \tag{3}$$

where $\phi(t; \mathbf{I})$ represents the time-varying phase of the carrier.

## Continuous-Phase Frequency-Shift Keying (CPFSK)

- We have

$$\begin{aligned}\phi(t;\mathbf{I}) &= 4\pi T f_d \int_{-\infty}^{t} d(\tau)d\tau \\ &= 4\pi T f_d \int_{-\infty}^{t} \left[\sum_n I_n g(\tau - nT)\right] d\tau\end{aligned} \tag{4}$$

- Although $d(t)$ contains discontinuities, the integral of $d(t)$ is continuous. Hence, we have a continuous-phase signal.
- The phase of the carrier in the interval $nT \leq t \leq (n+1)T$ is determined by integrating Equation (4).



## Continuous-Phase Frequency-Shift Keying (CPFSK)

- Thus,

$$\begin{aligned}\phi(t;I) &= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n q(t-nT) \\ &= \theta_n + 2\pi h I_n q(t-nT)\end{aligned} \tag{5}$$

where $h$, $\theta_n$ , and $q(t)$ are defined as

$$\begin{aligned}h &= 2f_d T \\ \theta_n &= \pi h \sum_{k=-\infty}^{n-1} I_k \\ q(t) &= \begin{cases} 0 & t < 0 \\ \frac{t}{2T} & 0 \leq t \leq T \\ \frac{1}{2} & t > T \end{cases}\end{aligned} \tag{6}$$

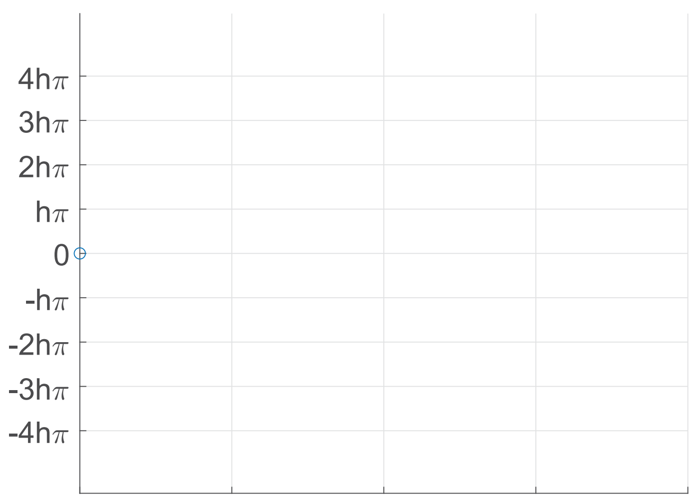
- $\theta_n$ represents the accumulation (memory) of all symbols up to time $(n-1)T$ .
- The parameter $h$ is called the modulation index.

## Continuous-Phase Modulation (CPM)

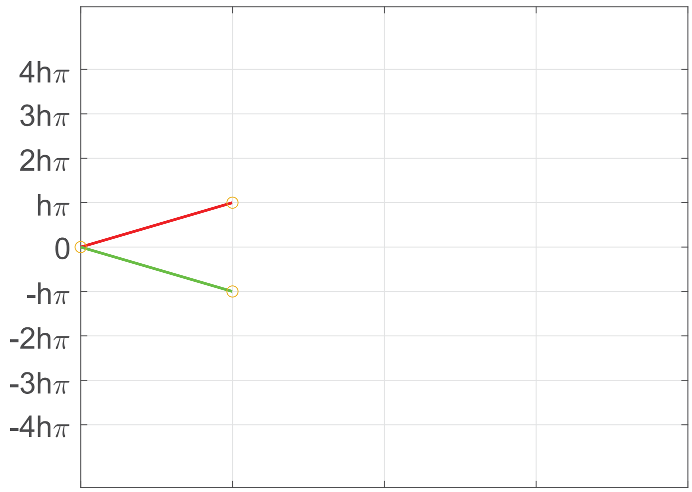
- One can sketch the set of phase trajectories $\phi(t; I)$ generated by all possible values of the information sequence $\{I_n\}$.
- Consider the case of CPFSK with binary symbols $I_n = \pm 1$, the set of phase trajectories beginning at time $t = 0$:



## Continuous-Phase Modulation (CPM)

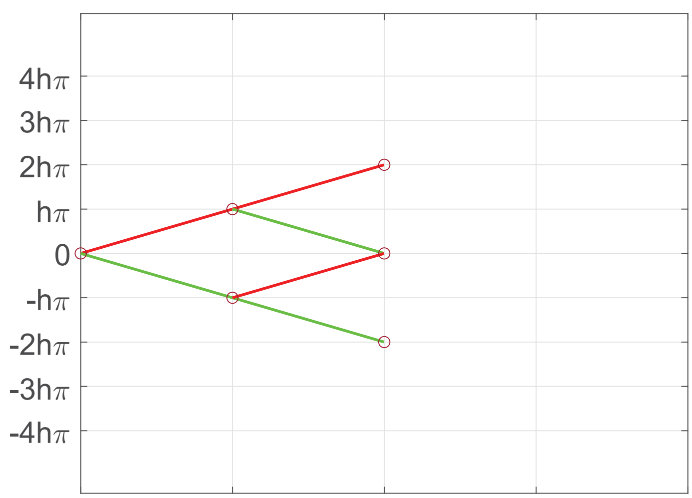


## Continuous-Phase Modulation (CPM)






## Continuous-Phase Modulation (CPM)

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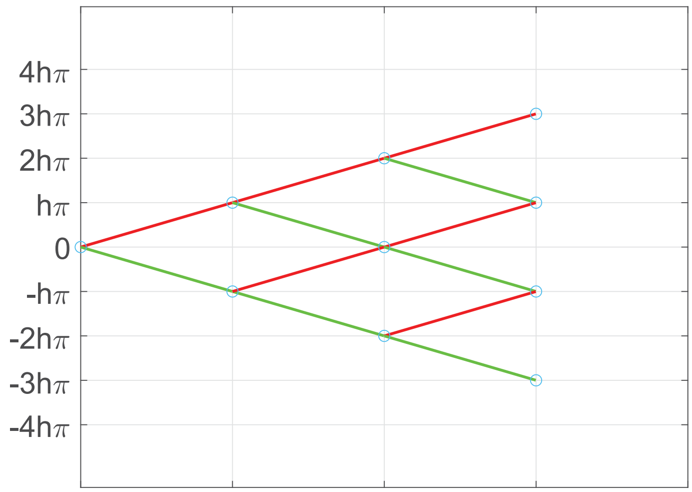
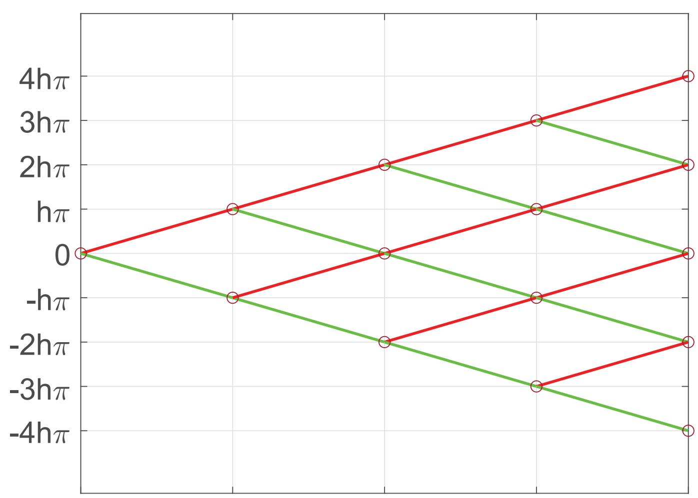
## Continuous-Phase Modulation (CPM)






## Continuous-Phase Modulation (CPM)

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EE910: Digital Communication Systems-I

## Minimum-Shift Keying (MSK)

- MSK is a special form of binary CPFSK (and, therefore, CPM) in which the modulation index $h = \frac{1}{2}$ and $g(t)$ is a rectangular pulse of duration $T$.
- The phase of the carrier in the interval $nT \leq t \leq (n+1)T$is

$$\begin{aligned}\phi(t; I) &= \frac{1}{2}\pi \sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t - nT) \\ &= \theta_n + \frac{1}{2}\pi I_n(\frac{t - nT}{T}), \qquad nT \leq t \leq (n+1)T\end{aligned} \tag{7}$$

and the modulated carrier signal is

$$\begin{aligned}s(t) &= Acos\left[2\pi f_c t + \theta_n + \frac{1}{2}\pi I_n(\frac{t - nT}{T})\right] \\ &= Acos\left[2\pi(f_c + \frac{1}{4T}I_n)t - \frac{1}{2}n\pi I_n + \theta_n\right], \qquad nT \leq t \leq (n+1)T\end{aligned} \tag{8}$$



## Minimum-Shift Keying (MSK)

- Equation (8) indicates that the binary CPFSK signal can be expressed as a sinusoid having one of two possible frequencies in the interval $nT \leq t \leq (n+1)T$ .
- If we define these frequencies as

$$\begin{aligned}f_1 &= f_c - \frac{1}{4T} \\ f_2 &= f_c + \frac{1}{4T}\end{aligned} \tag{9}$$

then the binary CPFSK signal given by Equation (8) may be written in the form

$$s_i(t) = Acos\left[2\pi f_i t + \theta_n + \frac{1}{2}n\pi(-1)^{i-1}\right], \qquad i = 1, 2 \tag{10}$$

which represents an FSK signal with frequency separation of $\triangle f = f_2 - f_1 = 1/2T$.

## Minimum-Shift Keying (MSK)

- Recall that $\Delta f = 1/2T$ is the minimum frequency separation needed to ensure orthogonality of signals $s_1(t)$ and $s_2(t)$ over a signalling interval of length T.
- This is why binary CPFSK with $h = \frac{1}{2}$ is called minimum shift keying (MSK).



## Minimum Shift Keying

- In an MSK signal, the initial state for the phase is either 0 or $\pi$ rad. Determine the terminal phase state for the following four input pairs of input data:
  1. 00
  2. 01
  3. 10
  4. 11
- We assume that the input bits 0,1 are mapped to the symbols $-1$ and 1 respectively. The terminal phase of an MSK signal at time instant $n$ is given by

$$\theta(n; \mathbf{a}) = \frac{\pi}{2} \sum_{k=0}^{k} a_k + \theta_0$$

where $\theta_0$ is the initial phase and $a_k$ is $\pm 1$ depending on the input bit at the time instant $k$.

# Minimum Shift Keying

- The following table shows $\theta(n; \mathbf{a})$ for two different values of $\theta_0(0, \pi)$, and the four input pairs of data: $\{00, 01, 10, 11\}$.

| $\theta_0$ | $b_0$ | $b_1$ | $a_0$ | $a_1$ | $\theta(n; \mathbf{a})$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $-1$ | $-1$ | $-\pi$ |
| 0 | 0 | 1 | $-1$ | 1 | 0 |
| 0 | 1 | 0 | 1 | $-1$ | 0 |
| 0 | 1 | 1 | 1 | 1 | $\pi$ |
| $\pi$ | 0 | 0 | $-1$ | $-1$ | 0 |
| $\pi$ | 0 | 1 | $-1$ | 1 | $\pi$ |
| $\pi$ | 1 | 0 | 1 | $-1$ | $\pi$ |
| $\pi$ | 1 | 1 | 1 | 1 | $2\pi$ |

# EE910: Digital Communication Systems-I

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**Lecture #4B: Continuous-Phase Modulation (CPM)**

## Continuous-Phase Modulation (CPM)

- CPFSK becomes a special case of a general class of continuous-phase modulated (CPM) signals in which the carrier phase is

$$\phi(t; I) = 2\pi \sum_{k=-\infty}^{n} I_k h_k q(t - kT), \qquad nT \leq t \leq (n+1)T \tag{1}$$

where

  - $\{I_k\}$ is the sequence of M-ary information symbols selected from the alphabet $\pm 1, \pm 3, \cdots, \pm(M-1)$,
  - $\{h_k\}$ is a sequence of modulation indices, and
  - $q(t)$ is some normalized waveform shape.



## Continuous-Phase Modulation (CPM)

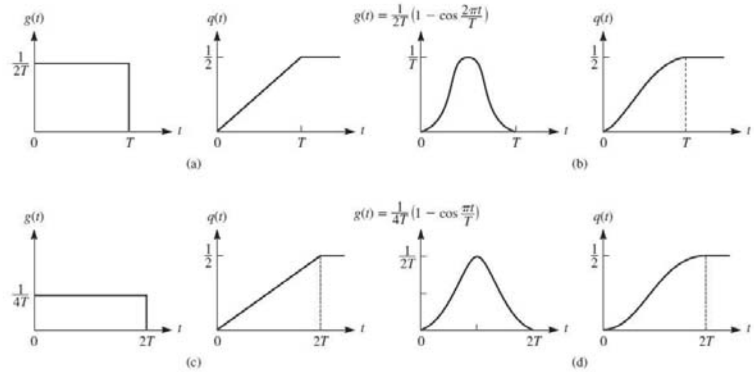
- When $h_k = h$ for all k, the modulation index is fixed for all symbols.
- When the modulation index varies from one symbol to another, the signal is called multi-h CPM.
- The waveform $q(t)$ may be represented in general as the integral of some pulse $g(t)$, i.e.

$$q(t) = \int_0^t g(\tau) d\tau \tag{2}$$

- If $g(t) = 0$ for $t > T$ , the signal is called full-response CPM.
- If $g(t) \neq 0$ for $t > T$, the modulated signal is called partial-response CPM.

# Continuous-Phase Modulation (CPM)

- Figure illustrates several pulse shapes for $g(t)$ and the corresponding $q(t)$.





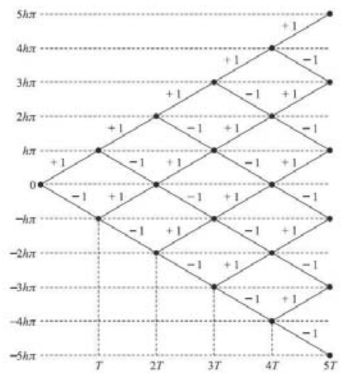

# Continuous-Phase Modulation (CPM)

- Three popular pulse shapes are given in Table

| | |
|---|---|
| LREC | $g(t) = \begin{cases} \frac{1}{2LT} & 0 \le t \le LT \\ 0 & \text{otherwise} \end{cases}$ |
| LRC | $g(t) = \begin{cases} \frac{1}{2LT}(1 - cos\frac{2\pi t}{LT}) & 0 \le t \le LT \\ 0 & \text{otherwise} \end{cases}$ |
| GMSK | $g(t) = \dfrac{Q(2\pi B(t - \frac{T}{2})) - Q(2\pi B(t + \frac{T}{2}))}{\sqrt{ln2}}$ (3) |

## Continuous-Phase Modulation (CPM)

- One can sketch the set of phase trajectories $\phi(t; I)$ generated by all possible values of the information sequence $\{I_n\}$.
- For example, in the case of CPFSK with binary symbols $I_n = \pm 1$, the set of phase trajectories beginning at time $t = 0$ is shown in Figure.






## Continuous-Phase Modulation (CPM)

- Simpler representations for the phase trajectories can be obtained by displaying only the terminal values of the signal phase at the time instants $t = nT$.
- In this case, we restrict the modulation index of the CPM signal to be rational.
- In particular, let us assume that $h = m/p$, where $m$ and $p$ are relatively prime integers, then a full-response CPM signal at the time instants $t = nT$ will have the terminal phase states

$$\Theta_s = \left\{0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \cdots, \frac{(p-1)\pi m}{p}\right\} \tag{4}$$

when m is even and

$$\Theta_s = \left\{0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \cdots, \frac{(2p-1)\pi m}{p}\right\} \tag{5}$$

when m is odd.

## Continuous-Phase Modulation (CPM)

- On the other hand, when the pulse shape extends over $L$ symbol intervals (partial-response CPM), the number of phase states may increase up to a maximum of $S_t$ , where

$$S_t = \begin{cases} pM^{L-1} & \text{even } m \\ 2pM^{L-1} & \text{odd } m \end{cases} \tag{6}$$

where M is the alphabet size.



## Continuous-Phase Modulation (CPM)

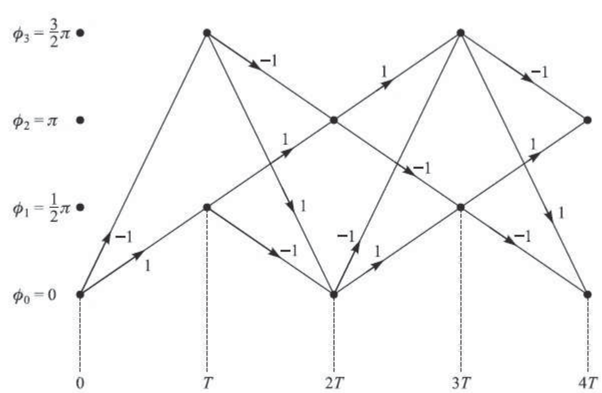
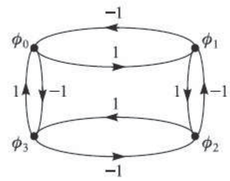



Figure: The state trellis for the binary CPFSK signal (full-response, rectangular pulse) with $h = \frac{1}{2}$

# Continuous-Phase Modulation (CPM)

- An alternative representation to the state trellis is the state diagram, which also illustrates the state transitions at the time instants $t = nT$ .
- In this representation, only the possible (terminal) phase states and their transitions are displayed.
- For example, the state diagram for the CPFSK signal with $h = \frac{1}{2}$ is shown in Figure .






# Continuous-Phase Modulation (CPM)

- Determine the number of states in the state trellis diagram for a full-response binary CPFSK with $h = \frac{2}{3}$ or $\frac{3}{4}$.
- $h = \frac{2}{3}$**:** There are no correlative states in this system, since it is a full response CPM.
- Recall, a full-response CPM signal at the time instants $t = nT$ will have the terminal phase states

$$\Theta_s = \left\{0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \cdots, \frac{(p-1)\pi m}{p}\right\}$$

  when $m$ is even
- Thus, we obtain the phase states as:

$$\Theta_s = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

## Continuous-Phase Modulation (CPM)

- $h = \frac{3}{4}$: Recall, a full-response CPM signal at the time instants $t = nT$ will have the terminal phase states $\Theta_s = \left\{0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \cdots, \frac{(2p-1)\pi m}{p}\right\}$ when $m$ is odd.
- In this case, we obtain the phase states :

$$\Theta_s = \left\{0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4} \equiv \frac{\pi}{4}, \pi, \frac{15\pi}{4} \equiv \frac{7\pi}{4}, \frac{18\pi}{4} \equiv \frac{\pi}{2}, \frac{21\pi}{4} \equiv \frac{5\pi}{4}\right\}$$



## Continuous-Phase Modulation (CPM)

- Determine the number of states in the state trellis diagram for a partial-response $L = 3$ binary CPFSK with $h = \frac{2}{3}$ or $\frac{3}{4}$.
  1. The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1}, I_{n-2}\}$ take the values $\pm 1$. Hence there are $3 \times 2 \times 2 = 12$ combined states in all.
  2. The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\{I_{n-1}, I_{n-2}\}$ take the values $\pm 1$. Hence there are $8 \times 2 \times 2 = 32$ combined states in all.

# Continuous-Phase Modulation (CPM)

- Determine the number of terminal phase states in the state trellis diagram for a full-response binary CPFSK with $h = \frac{2}{3}$ or $\frac{3}{4}$.
- We know that

$$\phi(t; \mathbf{I}) = 2\pi h \sum_{k=-\infty}^{n} I_k q(t - kT)$$

- Full response binary CPFSK ($q(T) = 1/2$) :
  1. $h = 2/3$. At the end of each bit interval the phase is : $2\pi \frac{2}{3} \frac{1}{2} \sum_{k=-\infty}^{n} I_k = \frac{2\pi}{3} \sum_{k=-\infty}^{n} I_k$. Hence the possible terminal phase states are $\{0, 2\pi/3, 4\pi/3\}$.
  2. $h = 3/4$. At the end of each bit interval the phase is : $2\pi \frac{3}{4} \frac{1}{2} \sum_{k=-\infty}^{n} I_k = \frac{3\pi}{4} \sum_{k=-\infty}^{n} I_k$. Hence the possible terminal phase states are $\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}$

# EE910: Digital Communication Systems-I

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May 9, 2022

**Lecture #5A: Optimal Detection for a vector AWGN channel**

## Optimal detection for a General Vector Channel

- The additive white Gaussian noise (AWGN) channel model is a channel whose sole effect is addition of a white Gaussian noise process to the transmitted signal.
- The channel is mathematically described by,

$$r(t) = s_m(t) + n(t) \tag{1}$$

where $s_m(t)$ is the transmitted signal and $n(t)$ is a zero-mean white Gaussian noise process with power spectral density of $N_0/2$; and $r(t)$ is the received waveform.
- The receiver makes an optimal decision about which message $m$, $1 \leq m \leq M$ was transmitted based on the decision rule that minimizes the probability of disagreement between the transmitted message $m$ and the detected message $\hat{m}$ given by

$$P_e = P[\hat{m} \neq m] \tag{2}$$



## Optimal detection for a General Vector Channel

- The general vector channel is given by

$$\mathbf{r} = \mathbf{s}_m + \mathbf{n} \tag{3}$$

where all vectors are N-dimensional real vectors.
- The vectors $\mathbf{s}_m$ is chosen from a set of possible signal vectors $\{\mathbf{s}_m, 1 \leq m \leq M\}$ according to prior probabilities $P_m$.
- Let the decision function employed at the receiver by $g(\mathbf{r})$. If $g(\mathbf{r}) = \hat{m}$, then the probability of a correct decision, given that $\mathbf{r}$ is received, is given by

$$P[\textit{correct decision}|\mathbf{r}] = P[\hat{m} \textit{ sent}|\mathbf{r}]. \tag{4}$$

- Therefore the probability of a correct decision is

$$P[\textit{correct decision}] = \int P[\hat{m} \textit{ sent}|\mathbf{r}]p(\mathbf{r})d\mathbf{r} \tag{5}$$

## MAP and ML receivers

- The optimal detection rule is the one that upon observing $\mathbf{r}$ decides in favor of the message $m$ that maximizes $P[m|\mathbf{r}]$, i.e.,

$$\hat{m} = g_{opt}(\mathbf{r}) = \underset{1 \leq m \leq M}{\arg\max}\, P[m|\mathbf{r}] = \underset{1 \leq m \leq M}{\arg\max}\, P[\mathbf{s}_m|\mathbf{r}] \tag{6}$$

- The optimal decision rule given in (6) is known as the maximum a posteriori probability rule, or MAP rule.
- The MAP receiver can be simplified to

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\, \frac{P_m p(\mathbf{r}|\mathbf{s}_m)}{p(\mathbf{r})} \tag{7}$$



## MAP and ML receivers

- Since $p(\mathbf{r})$ is independent of $m$ and for all $m$ remains the same, (7) equivalent to $\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\, P_m p(\mathbf{r}|\mathbf{s}_m)$.
- If the messages are equiprobable, the optimal decision rule reduces to $\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\, p(\mathbf{r}|\mathbf{s}_m)$ and the receiver is known as maximum likelihood receiver or ML receiver.

## The Error Probability

- The region $\mathbf{D}_m$, $1 \leq m \leq M$, is called the decision region for message $m$; and $\mathbf{D}_m$ is the set of all outputs of the channel that are mapped into message m by the detector.
- For a MAP detector we have

$$\mathbf{D}_m = \{\mathbf{r} \in \mathcal{R}^N \ : P[m|\mathbf{r}] > P[m'|\mathbf{r}], \text{ for all } 1 \leq m' \leq M \text{ and } m' \neq m\} \quad (8)$$

- An error occurs when $\mathbf{s}_m$ is transmitted but the received $\mathbf{r}$ is not in $\mathbf{D}_m$.



## The Error Probability

- The symbol error probability of a receiver is thus given by,

$$P_e = \sum_{m=1}^{M} P_m P[\mathbf{r} \notin \mathbf{D}_m | \mathbf{s}_m \ sent] = \sum_{m=1}^{M} P_m P_{e|m} \quad (9)$$

where $P_{e|m}$ denotes the error probability when message m is transmitted and is given by

$$P_{e|m} = \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s_m}) d\mathbf{r}$$

.

- Now $P_e$ can be written as,

$$P_e = \sum_{m=1}^{M} P_m \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s_m}) d\mathbf{r}$$

which is the symbol error probability.

## The Error Probability

- The bit error probability is denoted by $P_b$ and is the error probability in transmission of a single bit.
- We can bound the bit error probability by noting that a symbol error occurs when at least one bit is in error, and the event of a symbol error is the union of the events of the errors in the $k = log_2 M$ bits representing that symbol.
- Therefore we can write $P_b \leq P_e \leq kP_b$



## Preprocessing at the Receiver

- The receiver passes $\mathbf{r}$ through $G$ and supplies the detector with $\rho = G(\mathbf{r})$.
- Since $G$ is invertible and the detector has access to $\rho$, it can apply $G^{-1}$ to $\rho$ to obtain $G^{-1}(\rho) = G^{-1}(G(\mathbf{r})) = \mathbf{r}$. The detector now has access to both $\rho$ and $\mathbf{r}$.
- Thus the optimal detection rule can be written as,

$$\hat{m} = \arg\max_{1 \leq m \leq M} P_m p(\mathbf{r}, \rho | \mathbf{s}_m) = \arg\max_{1 \leq m \leq M} P_m p(\mathbf{r}|\mathbf{s}_m) p(\rho|\mathbf{r}) = \arg\max_{1 \leq m \leq M} P_m p(\mathbf{r}|\mathbf{s}_m)$$

- Thus it is clear that the optimal detector based on the observation of $\rho$ makes the same decision as the optimal detector based on the observation of $\mathbf{r}$.

# EE910: Digital Communication Systems-I

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May 9, 2022

## Lecture #5B: Waveform and vector AWGN channels

## Equivalence of Waveform and Vector AWGN channels

- The waveform AWGN channel is described by the input-output relation
$$r(t) = s_m(t) + n(t) \tag{1}$$
where $s_m(t)$ is one of the possible $M$ signals and n(t) is a zero-mean white Gaussian process with power spectral density $N_0/2$.
- Using the Gram-Schmidt procedure, we can derive an orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$ for representation of the signals.



## Equivalence of Waveform and Vector AWGN channels

- The noise process cannot be completely expanded in terms of the basis $\{\phi_j(t)\}_{j=1}^{N}$.
- One component, denoted by $n_1(t)$ is part of the noise process that can be expanded in terms of $\{\phi_j(t)\}_{j=1}^{N}$ and the the other part, denoted by $n_2(t)$, is the part that cannot be expressed in terms of this basis function.
- Thus, we can write $n_1(t) = \sum_{j=1}^{N} n_j \phi_j(t)$, where $n_j = \langle n(t), \phi_j(t) \rangle$ and $n_2(t) = n(t) - n_1(t)$.

## Equivalence of Waveform and Vector AWGN channels

- $s_m(t)$ can be written as $s_m(t) = \sum_{j=1}^{N} s_{mj}\phi_j(t)$, where $s_{mj} = \langle s_m(t), \phi_j(t)\rangle$.
- Thus (1) can be written as $r(t) = \sum_{j=1}^{N}(s_{mj} + n_j)\phi_j(t) + n_2(t)$.
- We define $r_j = s_{mj} + n_j$ where, $r_j = \langle s_m(t), \phi_j(t)\rangle + \langle n(t), \phi_j(t)\rangle = \langle r(t), \phi_j(t)\rangle$.
- Thus we have $r(t) = \sum_{j=1}^{N} r_j\phi_j(t) + n_2(t)$.



## Equivalence of Waveform and Vector AWGN channels

- $n_j$ is defined by $n_j = \int_{-\infty}^{\infty} n(t)\phi_j(t)dt$.
- The mean of $n_j$ is given as,

$$E[n_j] = E\left[\int_{-\infty}^{\infty} n(t)\phi_j(t)dt\right] = \int_{-\infty}^{\infty} E[n(t)]\phi_j(t)dt = 0, \qquad (2)$$

where the last equality holds since $n(t)$ is zero-mean.

## Equivalence of Waveform and Vector AWGN channels

- The covariance of $n_i$ and $n_j$ is,

$$\begin{aligned} COV[n_i n_j] &= E[n_i n_j] - E[n_i]E[n_j] \\ &= E\left[\int_{-\infty}^{\infty} n(t)\phi_i(t)dt \int_{-\infty}^{\infty} n(s)\phi_j(s)ds\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[n(t)n(s)]\phi_i(t)\phi_j(s)dtds \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\delta(t-s)\phi_i(t)dt\right]\phi_j(s)ds \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty}\phi_i(s)\phi_j(s)ds \\ &= \begin{cases} \frac{N_0}{2}, & i=j \\ 0, & i\neq j \end{cases} \end{aligned} \tag{3}$$



## Equivalence of Waveform and Vector AWGN channels

- $n_2(t) = n(t) - n_1(t)$, which is a linear combination of two jointly Gaussian processes, is itself a Gaussian process.
- The covariance at any given $t$ is,

$$\begin{aligned} COV[n_j n_2(t)] &= E[n_j n_2(t)] \\ &= E[n_j n(t)] - E[n_j n_1(t)] \\ &= E\left[n(t)\int_{-\infty}^{\infty} n(s)\phi_j(s)ds\right] - E\left[n_j \sum_{i=1}^{N} n_i\phi_i(t)\right] \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty}\delta(t-s)\phi_j(s)ds - \frac{N_0}{2}\phi_j(t) \\ &= \frac{N_0}{2}\phi_j(t) - \frac{N_0}{2}\phi_j(t) \\ &= 0 \end{aligned} \tag{4}$$

- The AWGN waveform channel of the form $r(t) = s_m(t) + n(t)$, $1 \leq m \leq M$ is equivalent to the N- dimensional vector channel $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$, $1 \leq m \leq M$.

## Optimal Detection for the Vector AWGN Channel

- The MAP detector for the AWGN vector channel is given by

$$
\begin{aligned}
\hat{m} &= \arg\max_{1\le m\le M} \left[P_m p(\mathbf{r}|\mathbf{s}_m)\right] \\
&= \arg\max_{1\le m\le M} P_m[p_n(\mathbf{r}-s_m)] \\
&= \arg\max_{1\le m\le M} \left[P_m\left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}}\right] \\
&= \arg\max_{1\le m\le M} \left[P_m e^{-\frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}}\right] \\
&= \arg\max_{1\le m\le M} \left[\ln P_m - \frac{\|\mathbf{r}-\mathbf{s}_m\|^2}{N_0}\right] \\
&= \arg\max_{1\le m\le M} \left[\frac{N_0}{2}\ln P_m - \frac{1}{2}\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\
&= \arg\max_{1\le m\le M} \left[\frac{N_0}{2}\ln P_m - \frac{1}{2}(\|\mathbf{r}\|^2+\|\mathbf{s}_m\|^2-2\mathbf{r}.\mathbf{s}_m)\right] \qquad (5)
\end{aligned}
$$



## Optimal Detection for the Vector AWGN Channel

- 

$$
\begin{aligned}
&\overset{(a)}{=} \arg\max_{1\le m\le M} \left[\frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m + \mathbf{r}.\mathbf{s}_m\right] \\
&\overset{(b)}{=} \arg\max_{1\le m\le M} \left[\eta_m + \mathbf{r}.\mathbf{s}_m\right] \qquad (6)
\end{aligned}
$$

where we have used the following steps in simplifying the expression: (a): $\|\mathbf{s}_m\|^2 = \mathcal{E}_m$ and (b): $\eta_m = \frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m$ as the bias term.

## Optimal Detection for the Vector AWGN Channel

- In the special case where the signals are equiprobable, i.e., $P_m = 1/M$ for all $m$,

$$\begin{aligned}\hat{m} &= \arg\max_{1\le m\le M}\left[\frac{N_0}{2}\ln P_m - \frac{1}{2}\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\ &= \arg\max_{1\le m\le M}\left[-\|\mathbf{r}-\mathbf{s}_m\|^2\right] \\ &= \arg\min_{1\le m\le M}\|\mathbf{r}-\mathbf{s}_m\| \qquad (7)\end{aligned}$$

- When the signals are both equiprobable and have equal energy, the bias terms defined as $\eta_m = \frac{N_0}{2}\ln P_m - \frac{1}{2}\mathcal{E}_m$ are independent of $m$ and can be dropped. The optimal detection rule in this case reduces to

$$\hat{m} = \arg\max_{1\le m\le M} \mathbf{r}\,.\,\mathbf{s}_m \qquad (8)$$



## Optimal Detection for the Vector AWGN Channel

- The decision region $D_m$ is given as

$$\mathbf{D}_m = \{\mathbf{r}\in\mathcal{R}^N : \mathbf{r}.\mathbf{s}_m + \eta_m > \mathbf{r}.\mathbf{s}_{m'} + \eta_{m'}, \text{for all } 1\le m'\le M \text{ and } m'\neq m\}$$

- The boundaries of the decision regions in general are hyperplanes and are of the form $\mathbf{r}.(\mathbf{s}_m - \mathbf{s}_{m'}) \ge \eta_{m'} - \eta_m$.
- The optimal MAP detection rule in an AWGN channel can be written in the form

$$\hat{m} = \arg\max_{1\le m\le M}\left[\frac{N_0}{2}\ln P_m + \int_{-\infty}^{\infty} r(t)s_m(t)dt - \frac{1}{2}\int_{-\infty}^{\infty} s_m^2(t)dt\right] \qquad (9)$$

where we have used the relation $\mathbf{r}.\mathbf{s}_m = \int_{-\infty}^{\infty} r(t)s_m(t)dt$ and $\mathcal{E} = \|\mathbf{s}\|^2 = \int_{-\infty}^{\infty} s_m^2(t)dt$.

## Optimal Detection for the Vector AWGN Channel

- The ML detector has the following form

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max} \left[ \int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} s_m^2(t) dt \right] \tag{10}$$

- The distance metric is defined as

$$D(\mathbf{r}, \mathbf{s_m}) = \|\mathbf{r} - \mathbf{s}_m\|^2 = \int_{-\infty}^{\infty} (r(t) - s_m(t))^2 dt \tag{11}$$

- Modified distance metric is defined as $D'(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r}.\mathbf{s}_m + \|s_m\|^2$



## Optimal Detection for the Vector AWGN Channel

- The correlation metric is defined as the negative of the modified distance metric and is denoted by $C(\mathbf{r}, \mathbf{s}_m)$.
- With these definitions the optimal detection rule (MAP rule) in general can be written as

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max} \left[ N_0 \ln P_m - D(\mathbf{r}, \mathbf{s}_m) \right] = \underset{1 \leq m \leq M}{\arg\max} \left[ N_0 \ln P_m + C(\mathbf{r}, \mathbf{s}_m) \right]$$

- The ML detection rule becomes $\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\, C(\mathbf{r}, \mathbf{s}_m)$

# EE910: Digital Communication Systems-I

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**Lecture #5C: Optimal Detection for Binary Antipodal Signaling**

## Optimal Detection for Binary Antipodal Signaling

- In binary antipodal signaling scheme $s_1(t) = s(t)$ and $s_2(t) = -s(t)$.
- The probabilities of messages 1 and 2 are $p$ and $1-p$, respectively.
- The vector representations of the two signals are just scalars with $s_1(t) = \sqrt{\mathcal{E}_s}$ and $s_2(t) = -\sqrt{\mathcal{E}_s}$, where $\mathcal{E}_s$ is energy in each signal and is equal to $\mathcal{E}_b$.
- The decision region $D_1$ is given as,

$$\begin{aligned} D_1 &= \left\{ r : r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln p - \frac{1}{2}\mathcal{E}_b > -r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln(1-p) - \frac{1}{2}\mathcal{E}_b \right\} \\ &= \left\{ r : r > \frac{N_0}{4\sqrt{\mathcal{E}_b}}\ln\frac{1-p}{p} \right\} \\ &= \{ r : r > r_{th} \} \end{aligned} \quad (1)$$

where the threshold is defined as $r_{th} = \frac{N_0}{4\sqrt{\mathcal{E}_b}}\ln\frac{1-p}{p}$.



## Optimal Detection for Binary Antipodal Signaling

- The error probability of this system is derived as

$$\begin{aligned} P_e &= \sum_{m=1}^{2} P_m \sum_{1\le m'\le 2, m'\neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s}_m)d\mathbf{r} \\ &= p\int_{D_2} p(r|s=\sqrt{\mathcal{E}_b})dr + (1-p)\int_{D_1} p(r|s=-\sqrt{\mathcal{E}_b})dr \\ &= p\int_{-\infty}^{r_{th}} p(r|s=\sqrt{\mathcal{E}_b})dr + (1-p)\int_{r_{th}}^{\infty} p(r|s=-\sqrt{\mathcal{E}_b})dr \\ &= p\,P\left[\mathcal{N}(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}) < r_{th}\right] + (1-p)\,P\left[\mathcal{N}(\sqrt{-\mathcal{E}_b}, \frac{N_0}{2}) > r_{th}\right] \\ &= p\,\mathcal{Q}\left(\frac{\sqrt{\mathcal{E}_b} - r_{th}}{\sqrt{\frac{N_0}{2}}}\right) + (1-p)\,\mathcal{Q}\left(\frac{\sqrt{\mathcal{E}_b} + r_{th}}{\sqrt{\frac{N_0}{2}}}\right) \end{aligned} \quad (2)$$

- In the special case where $p = \frac{1}{2}$, we have $r_{th} = 0$ and the error probability simplifies to $P_e = \mathcal{Q}\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$

## Recall

- Q function is closely related to the Gaussian random variable

$$Q(x) = P[\mathcal{N}(0,1) > x] = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt \quad (3)$$

- If X$\sim \mathcal{N}(m, \sigma^2)$, then

$$P[X > \alpha] = Q\left(\frac{\alpha - m}{\sigma}\right) \quad (4)$$

$$P[X < \alpha] = Q\left(\frac{m - \alpha}{\sigma}\right) \quad (5)$$

- Some of the important properties of the Q function:

$$Q(0) = 1/2 \qquad Q(\infty) = 0$$
$$Q(-\infty) = 1 \qquad Q(-x) = 1 - Q(x) \quad (6)$$



## Error Probability for Equiprobable Binary Signaling Schemes

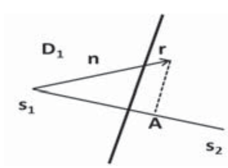



Figure: Decision regions for binary equiprobable signals

- In this case the transmitter transmits one of the two equiprobable signals $s_1(t)$ and $s_2(t)$ over the AWGN channel.
- Since the signals are equiprobable, the two decision regions are separated by the perpendicular bisector of the line connecting $s_1$ and $s_2$.

## Error Probability for Equiprobable Binary Signaling Schemes

- The error probability is given by

$$
\begin{aligned}
P_b &= P\left[\frac{\mathbf{n}.(\mathbf{s}_2 - \mathbf{s}_1)}{d_{12}} > \frac{d_{12}}{2}\right] \\
&= P\left[\mathbf{n}.(\mathbf{s}_2 - \mathbf{s}_1) > \frac{d_{12}^2}{2}\right] \qquad (7)
\end{aligned}
$$

- $\mathbf{n}.(\mathbf{s}_2 - \mathbf{s}_1)$ is a zero-mean Gaussian random variable with variance $\frac{d_{12}^2 N_0}{2}$ and thus we can write

$$
P_b = \mathcal{Q}\left(\frac{\frac{d_{12}^2}{2}}{d_{12}\sqrt{\frac{N_0}{2}}}\right) = \mathcal{Q}\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) \qquad (8)
$$



## Error Probability for Equiprobable Binary Signaling Schemes

- Since $\mathcal{Q}$ is a decreasing function, in order to minimize the error probability, the distance between signal points has to be maximized.
- The distance $d_{12}$ is obtained from $d_{12}^2 = \int_{-\infty}^{\infty}(s_1(t) - s_2(t))^2 dt$
- In the special case that the binary signals are equiprobable and have equal energy, i.e., when $\mathcal{E}_{s1} = \mathcal{E}_{s2} = \mathcal{E}$ and we get $d_{12}^2 = \mathcal{E}_{s1} + \mathcal{E}_{s2} - 2\langle s_1(t), s_2(t)\rangle = 2\mathcal{E}(1-\rho)$

## Optimal Detection for Binary Orthogonal Signaling

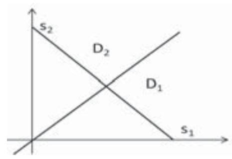
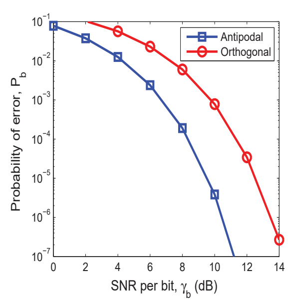



Figure: Signal constellation and decision regions for equiprobable binary orthogonal signaling

- For binary orthogonal signals we have,
$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \begin{cases} \mathcal{E}, & i=j \\ 0, & i \neq j \end{cases} \tag{9}$$
for $1 \leq i,j \leq 2$.
- Since the system is binary, $\mathcal{E}_b = \mathcal{E}$ and $\phi_j(t)$ is choosen as $\phi_j(t) = \frac{s_j(t)}{\sqrt{\mathcal{E}_b}}$ for $j = 1, 2$.



## Optimal Detection for Binary Orthogonal Signaling




- The vector representations of the signal set become $\mathbf{s}_1 = (\sqrt{\mathcal{E}_b}, 0)$ and $\mathbf{s}_2 = (0, \sqrt{\mathcal{E}_b})$
- Here $d = \sqrt{2\mathcal{E}_b}$ and $P_b = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$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# EE910: Digital Communication Systems-I

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May 9, 2022

**Lecture #5D: Correlation Receiver and Matched Filter**

## The Correlation Receiver for AWGN Channels

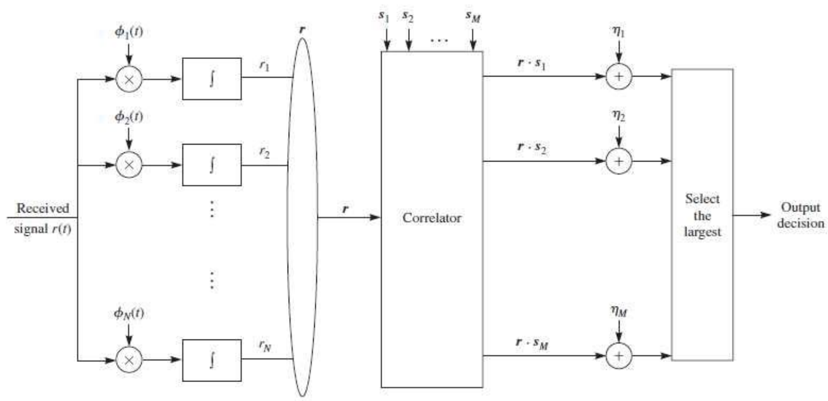


Figure: The structure of a correlation receiver with N correlators



## Correlation Receiver

- An optimal receiver for the AWGN channel implements the MAP decision rule given by $\arg\max\limits_{1\leq m\leq M}[\eta_m + \mathbf{r}.\mathbf{s}_m]$.
- $\mathbf{r}$ is derived at the receiver from the received signal $r(t)$ using the relation $r_j = \int_{-\infty}^{\infty} r(t)\phi_j(t)dt$.

## Correlation Receiver

- An alternative implementation of the optimal detector is possible looking at the optimal detection rule

$$\hat{m} = \arg \max_{1 \le m \le M} \left[ \eta_m + \int_{-\infty}^{\infty} r(t) s_m(t) dt \right] \quad \text{where} \quad \eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} \mathcal{E}_m \tag{1}$$

- Typically $N < M$, so the previous implementation of correlator receiver is preferred.



## The Matched Filter Receiver

- An alternative representation of optimal receiver is called matched filter receiver.
- In correlation receiver implementation, we compute quantities of the form

$$r_x = \int_{-\infty}^{\infty} r(t) x(t) dt$$

where x(t) is either $\phi_j(t)$ or $s_m(t)$.

## The Matched Filter Receiver

- A filter is called matched filter if its impulse response h(t) is matched to x(t), i.e. h(t)=x(T-t), where T is chosen such that the filter is causal.
- If the input $r(t)$ is applied to the matched filter, its output, denoted by $y(t)$ is given by,

$$y(t) = r(t) \star h(t) = \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} r(\tau)x(T-t+\tau)d\tau$$

- $r_x = y(T) = \int_{-\infty}^{\infty} r(\tau)x(\tau)d\tau$



## The Matched Filter Receiver

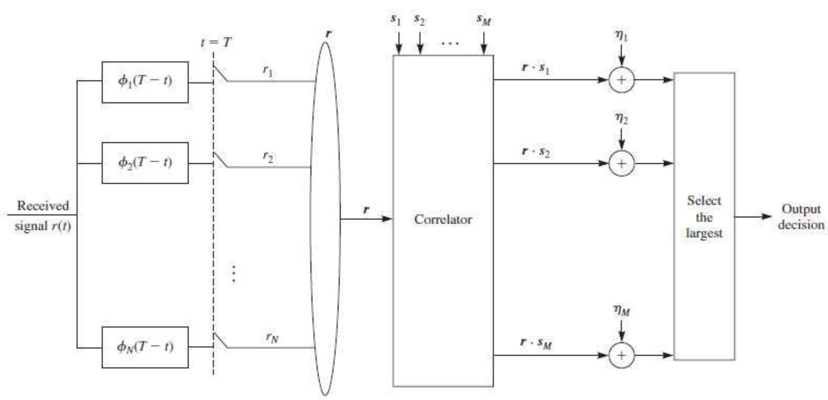


Figure: The structure of a matched filter receiver with N correlators

## The Matched Filter Receiver

- The output of the correlator $r_x$ can be obtained by sampling the output of the matched filter at exactly time $t = T$.



## Frequency Domain Interpretation of the Matched Filter

- The matched filter to any signal $s(t)$, is $h(t) = s(T - t)$. The Fourier transform of this relationship is $H(f) = S^*(f)e^{-j2\pi fT}$.
- The matched filter has a frequency response that is the complex conjugate of the transmitted signal spectrum multiplied by the phase factor $e^{-j2\pi fT}$, which represents a sampling delay of T.
- The magnitude response $|H(f)| = |S(f)|$ of the matched filter is identical to the transmitted signal spectrum.
- The phase of $H(f)$ is the negative of the phase of S(f) shifted by $2\pi fT$.

## Frequency Domain Interpretation of the Matched Filter

- Assume $r(t) = s(t) + n(t)$ is passed through a filter with impulse response $h(t)$ and frequency response $H(f)$.
- The output, denoted by $y(t) = y_s(t) + \nu(t)$, is sampled at some time T.
- The output consists of a signal part, $y_s(t)$, whose Fourier transform is $H(f)S(f)$ and a noise part, $\nu(t)$, whose power spectral density is $\frac{N_0}{2} \mid H(f) \mid^2$



## Frequency Domain Interpretation of the Matched Filter

- Sampling at time T, we get

$$y_s(T) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}\,df \tag{2}$$

and

$$Var[\nu(T)] = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}\mathcal{E}_h, \tag{3}$$

where $\mathcal{E}_h$ is the energy in $h(t)$.
- We define the SNR at the output of the filter H(f) as

$$SNR_0 = \frac{y_s^2(T)}{Var[\nu(T)]}$$

## Frequency Domain Interpretation of the Matched Filter

- Applying Cauchy-Schwartz inequality, we get

$$\begin{aligned} y_s(T) &= \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}df \\ &\leq \left(\int_{-\infty}^{\infty} |H(f)|^2 df\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |S(f)e^{j2\pi fT}|^2 df\right)^{1/2} \\ &= \sqrt{\mathcal{E}_h}\sqrt{\mathcal{E}_s} \end{aligned} \tag{4}$$

with equality if and only if $H(f) = \alpha S^*(f)e^{-j2\pi fT}$ for some complex $\alpha$.

- SNR can be written as

$$SNR_0 \leq \frac{\mathcal{E}_h\mathcal{E}_s}{\frac{N_0}{2}\mathcal{E}_h} = \frac{2\mathcal{E}_s}{N_0} \tag{5}$$



## Frequency Domain Interpretation of the Matched Filter

- The filter H(f) that maximizes the signal-to-noise ratio at its output must satisfy the relation $H(f) = S^*(f)e^{-j2\pi fT}$, i.e. it is the matched filter
- Maximum possible signal-to-noise ratio at the output is $\frac{2\mathcal{E}_s}{N_0}$

## Matched Filter

- Consider the signal

$$s(t) = \begin{cases} (A/T)t\cos 2\pi f_c t & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

1. Determine the impulse response of the matched filter for the signal.
2. Determine the output of the matched filter at $t = T$.
3. Suppose the signal $s(t)$ is passed through a correlator that correlates the input $s(t)$ with $s(t)$. Determine the value of the correlator output at $t = T$. Compare your result with that in part 2.



## Matched Filter

- The impulse response of the matched filter is given by

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos\left(2\pi f_c(T-t)\right) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

## Matched Filter

- The output of the matched filter at $t = T$ is :

$$
\begin{aligned}
g(T) &= h(t) * s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \qquad (8)\\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau\\
&= \frac{A^2}{T^2}\int_0^T \nu^2\cos^2(2\pi f_c\nu)d\nu\\
&= \frac{A^2}{T^2}\left[\frac{\nu^3}{6}+\left(\frac{\nu^2}{4\times 2\pi f_c}-\frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c\nu)+\frac{\nu\cos(4\pi f_c\nu)}{4(2\pi f_c)^2}\right]\Bigg|_0^T\\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6}+\left(\frac{T^2}{4\times 2\pi f_c}-\frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T)+\frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}
$$



## Matched Filter

- The output of the correlator at $t = T$ is :

$$
\begin{aligned}
q(T) &= \int_0^T s^2(\tau)d\tau\\
&= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)\,d\tau
\end{aligned} \qquad (9)
$$

- However, this is the same expression with the case of the output of the matched filter sampled at $t = T$.
- Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

# EE910: Digital Communication Systems-I

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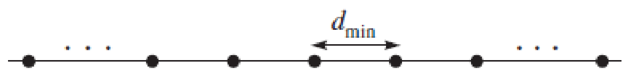
## Lecture #6A: Optimal Detection and Error Probability for ASK or PAM and PSK Signalling

# Optimal Detection and Error Probability for ASK or PAM Signalling

- The constellation for an ASK Signalling scheme is shown as




- In this constellation the minimum distance between any two points is $d_{min}$ which is given by

$$d_{min} = \sqrt{\frac{12 \log_2 M}{M^2 - 1} \mathcal{E}_{bavg}} \tag{1}$$




# Optimal Detection and Error Probability for ASK or PAM Signalling


- The constellation points are located at $\left\{ \pm\frac{1}{2}d_{min}, \pm\frac{3}{2}d_{min}, \ldots, \pm\frac{M-1}{2}d_{min} \right\}$
- In this ASK constellation, there are M-2 inner points and 2 outer points.

## Optimal Detection and Error Probability for ASK or PAM Signalling

- Let us denote the error probabilities of inner points and outer points by $P_{ei}$ and $P_{eo}$, respectively.
- Since $n$ is a zero-mean Gaussian random variable with variance $\frac{1}{2}N_0$, we have

$$P_{ei} = P\Big[|n| > \frac{1}{2}d_{min}\Big] = 2Q\Big(\frac{d_{min}}{\sqrt{2N_0}}\Big) \qquad (2)$$

and for outer points

$$P_{eo} = \frac{1}{2}P_{ei} = Q\Big(\frac{d_{min}}{\sqrt{2N_0}}\Big) \qquad (3)$$



## Optimal Detection and Error Probability for ASK or PAM Signalling

- The symbol error probability is given by

$$\begin{aligned} P_e &= \tfrac{1}{M}\sum_{m=1}^{M} P[error|m\ sent] \\ &= \tfrac{1}{M}\Big[2(M-2)Q\Big(\tfrac{d_{min}}{\sqrt{2N_0}}\Big) + 2Q\Big(\tfrac{d_{min}}{\sqrt{2N_0}}\Big)\Big] \\ &= \tfrac{2(M-1)}{M}Q\Big(\tfrac{d_{min}}{\sqrt{2N_0}}\Big) \end{aligned} \qquad (4)$$

Substituting for $d_{min}$ from Equation (1) we get

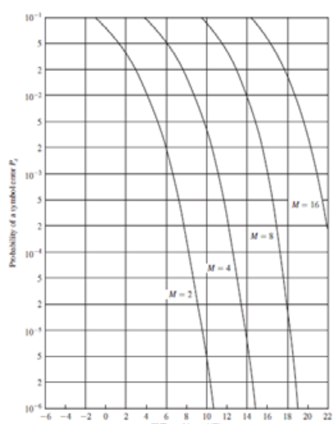
$$\begin{aligned} P_e &= 2\Big(1-\tfrac{1}{M}\Big)Q\Big(\sqrt{\tfrac{6\log_2 M}{M^2-1}\tfrac{\mathcal{E}_{bavg}}{N_0}}\Big) \\ &\approx 2Q\Big(\sqrt{\tfrac{6\log_2 M}{M^2-1}\tfrac{\mathcal{E}_{bavg}}{N_0}}\Big) \qquad for\ large\ M \end{aligned} \qquad (5)$$

## Optimal Detection and Error Probability for ASK or PAM Signalling

- The average SNR/bit $\frac{\mathcal{E}_{bavg}}{N_0}$ is scaled by $\frac{6 \log_2 M}{M^2-1}$
- To keep the error probability constant as M increases, the SNR/bit must increase.
- For increasing the transmission rate by 1 bit, one would need 6 dB more power



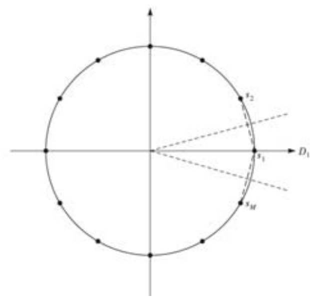
## Plots of the error probability of baseband PAM/ASK.




- Increasing M deteriorates the performance, and for large M the distance between curves corresponding to M and 2M is roughly 6 dB.

# Optimal Detection and Error Probability for PSK Signalling

- The constellation for an M-ary PSK Signalling is shown below




- In this constellation, the decision region $D_1$ is also shown.
- The decision regions are based on the minimum-distance detection rule.



# Optimal Detection and Error Probability for PSK Signalling

- By symmetry of the constellation, the error probability of the system is equal to the error probability when $s_1 = (\sqrt{\mathcal{E}}, 0)$ is transmitted.
- The received vector $\mathbf{r}$ is given by

$$\mathbf{r} = (r_1, r_2) = (\sqrt{\mathcal{E}} + n_1, n_2) \tag{6}$$

## Optimal Detection and Error Probability for PSK Signalling

- It is seen that $r_1$ and $r_2$ are independent Gaussian random variables with variance $\sigma^2 = \frac{1}{2}N_0$ and means $\sqrt{\mathcal{E}}$ and 0, respectively; hence

$$p(r_1, r_2) = \frac{1}{\pi N_0} e^{-\frac{(r_1-\sqrt{\mathcal{E}})^2 + r_2^2}{N_0}} \tag{7}$$

- We introduce polar coordinates transformations of $(r_1, r_2)$ as

$$V = \sqrt{r_1^2 + r_2^2} \tag{8}$$

$$\Theta = arctan \frac{r_2}{r_1} \tag{9}$$



## Optimal Detection and Error Probability for PSK Signalling

- The joint PDF of V and $\Theta$ can be derived as

$$p_{V,\Theta}(\nu, \theta) = \frac{\nu}{\pi N_0} e^{-\frac{\nu^2 + \mathcal{E} - 2\sqrt{\mathcal{E}}\nu \cos\theta}{N_0}} \tag{10}$$

- Integrating over $\nu$, we derive the marginal PDF of $\Theta$ as

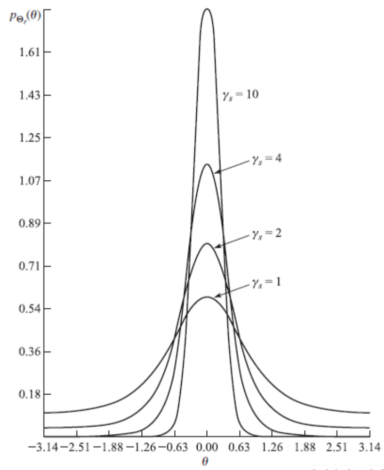
$$\begin{aligned} p_\Theta(\theta) &= \int_0^\infty p_{V,\Theta}(\nu, \theta) d\nu \\ &= \frac{1}{2\pi} e^{-\gamma_s sin^2\theta} \int_0^\infty \nu e^{-\frac{(\nu - \sqrt{2\gamma_s}\cos\theta)^2}{2}} d\nu \end{aligned} \tag{11}$$

- We have defined the symbol SNR or SNR per symbol as

$$\gamma_s = \frac{\mathcal{E}}{N_0} \tag{12}$$

## Optimal Detection and Error Probability for PSK Signalling

- This figure illustrates $p_\Theta(\theta)$ for several values of $\gamma_s$






## Optimal Detection and Error Probability for PSK Signalling

- Note that $p_\Theta(\theta)$ becomes narrower and more peaked about $\theta = 0$ as $\gamma_s$ increases.
- The decision region $D_1$ can be described as $D_1 = \{\theta : \frac{-\pi}{M} < \theta \leq \frac{\pi}{M}\}$
- The message error probability is given by

$$P_e = 1 - \int_{-\pi/M}^{\pi/M} p_\Theta(\theta) d\theta \tag{13}$$

## Optimal Detection and Error Probability for PSK Signalling

- In general, the integral of $p_\Theta(\theta)$ does not reduce to a simple form and must be evaluated numerically, except for M=2 and M=4.
- For binary phase modulation, the two signals $s_1(t)$ and $s_2(t)$ are antipodal, and hence the error probability is

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \tag{14}$$



## Optimal Detection and Error Probability for PSK Signalling

- When M=4, we have two binary phase-modulation signals in phase quadrature. Hence, the bit error probability is identical to Binary phase modulation.
- The symbol error probability for M=4 is determined by noting that

$$P_c = (1 - P_b)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2 \tag{15}$$

where $P_c$ is the probability of a correct decision for the 2-bit symbol.

## Optimal Detection and Error Probability for PSK Signalling

- Therefore, the symbol error probability for M=4 is

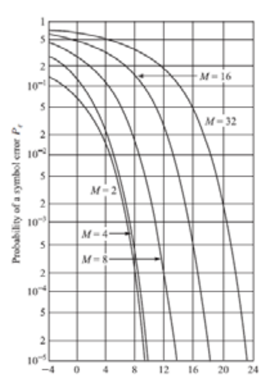
$$\begin{aligned} P_e &= 1 - P_c \\ &= 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right] \end{aligned} \tag{16}$$

- For $M > 4$, the symbol error probability $P_e$ is obtained by numerically integrating the equation

$$P_e = 1 - \int_{-\pi/M}^{\pi/M} p_{\Theta}(\theta)\, d\theta \tag{17}$$



## Error probability as a function of the SNR per bit for $M = 2, 4, 8, 16$, and 32.




- The graph clearly illustrates the penalty in SNR per bit as M increases beyond $M = 4$.

## Optimal Detection and Error Probability for PSK Signalling

- For large values of M, doubling the number of phases requires an additional 6 dB/bit to achieve the same performance.
- An approximation to the error probability for large values of M and for large SNR may be obtained by first approximating $p_\Theta(\theta)$.
- For $\frac{\mathcal{E}}{N_0} \gg 1$ and $|\theta| \leq \frac{1}{2}\pi$, $p_\Theta(\theta)$ is well approximated as

$$p_\Theta(\theta) \approx \sqrt{\frac{\gamma_s}{\pi}} \cos\theta e^{-\gamma_s sin^2\theta} \tag{18}$$

- By substituting for $p_\theta(\theta)$ in equation $P_e = 1 - \int_{-\pi/M}^{\pi/M} p_\Theta(\theta) d\theta$ and performing the change in variable from $\theta$ to $u = \sqrt{\gamma_s}\sin\theta$ we find that



## Optimal Detection and Error Probability for PSK Signalling

- Probability of symbol error

$$\begin{aligned} P_e &\approx 1 - \int_{\frac{-\pi}{M}}^{\frac{\pi}{M}} \sqrt{\frac{\gamma_s}{\pi}} \cos\theta e^{-\gamma_s sin^2\theta} d\theta \\ &\approx \frac{2}{\sqrt{\pi}} \int_{\sqrt{2\gamma_s} sin(\pi/M)}^{\infty} e^{-u^2} du \\ &= 2Q\Big(\sqrt{2\gamma_s} sin(\frac{\pi}{M})\Big) \\ &= 2Q\left(\sqrt{(2\log_2 M) sin^2(\frac{\pi}{M})\frac{\mathcal{E}_b}{N_0}}\right) \end{aligned} \tag{19}$$

where we have used the definition of the SNR per bit as

$$\frac{\mathcal{E}_b}{N_0} = \frac{\mathcal{E}}{N_0 \log_2 M} = \frac{\gamma_s}{\log_2 M} \tag{20}$$

## Optimal Detection and Error Probability for PSK Signalling

- Note that this approximation to the error probability is good for all values of M.
- For example, when $M = 2$ and $M = 4$, we have $P_e = 2Q(\sqrt{2\gamma_b})$
- For the case when M is large, we can use the approximation $sin\frac{\pi}{M} \approx \frac{\pi}{M}$ to find another approximation to error probability for large M as

$$P_e \approx 2Q\left(\sqrt{\frac{2\pi^2 \log_2 M}{M^2}\frac{\mathcal{E}_b}{N_0}}\right) \quad \textit{for large } M \tag{21}$$

- From this equation, it is clear that doubling M reduces the effective SNR per bit by 6 dB.



## Optimal Detection and Error Probability for PSK Signalling

- The equivalent bit error probability for M-ary PSK is rather tedious to derive due to its dependence on the mapping of k-bit symbols into the corresponding signal phases.
- Since the most probable errors due to noise result in the erroneous selection of an adjacent phase to the true phase, most k-bit symbol errors contain only a single-bit error.
- Hence, the equivalent bit error probability for M-ary PSK is well approximated as

$$P_b \approx \frac{1}{k}P_e \tag{22}$$

# EE910: Digital Communication Systems-I

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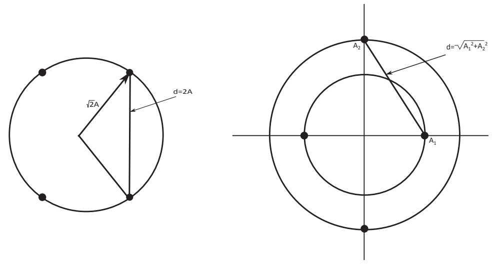
May 16, 2022

**Lecture #6B: Optimal Detection and Error Probability for QAM Signalling**

# Optimal Detection and Error Probability for QAM Signalling

- To determine the probability of error for QAM, we must specify the signal point constellation.
- Consider QAM signal sets that have $M = 4$ points as shown in figure.

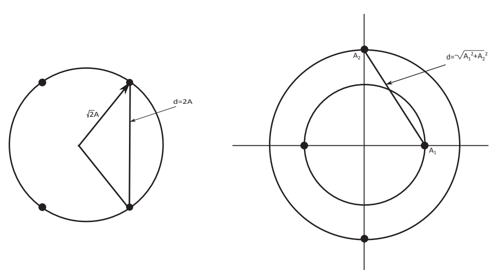



(a) (b)




# Optimal Detection and Error Probability for QAM Signalling

- Consider QAM signal sets that have $M = 4$ points as shown in figure.


(a) (b)

- The first is a four-phase modulated signal, and the second is a QAM signal with two amplitude levels, labelled $A_1$ and $A_2$, and four phases.

## Optimal Detection and Error Probability for QAM Signalling

- Impose the condition that $d_{min} = 2A$ for both signal constellations.
- Let us evaluate the average transmitted power, based on the premise that all signal points are equally probable.
- For the four-phase signal,we have

$$\mathcal{E}_{avg} = 2A^2 \tag{1}$$



## Optimal Detection and Error Probability for QAM Signalling

- For the two-amplitude, four-phase QAM, we place the points on circles of radii A and $\sqrt{3}A$. Thus, $d_{min} = 2A$, and

$$\mathcal{E}_{avg} = \frac{1}{4}\left[2(3A^2) + 2A^2\right] = 2A^2 \tag{2}$$

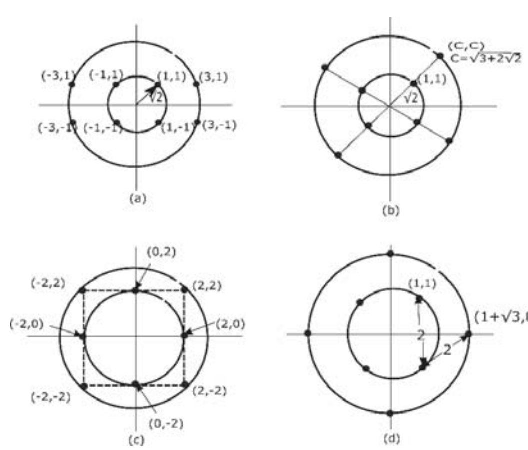
which is the average power as the $M = 4$ phase signal constellation.
- Hence, the error rate performance of the two signal sets is the same.
- There is no advantage of the two-amplitude QAM signal set over $M = 4$ phase modulation.

## Optimal Detection and Error Probability for QAM Signalling

- Let us consider $M = 8QAM$. In this case, there are many possible signal constellations.
- We shall consider the four signal constellations shown in figure (next page) all of which consist of two amplitudes and have a minimum distance between signal points of 2A.
- The coordinates $(A_{mc}, A_{ms})$ for each signal point, normalized by A, are shown in figure.



## Optimal Detection and Error Probability for QAM Signalling



## Optimal Detection and Error Probability for QAM Signalling

- Assuming that the signal points are equally probable, the average transmitted signal energy is

$$\begin{aligned}\mathcal{E}_{avg} &= \frac{1}{M}\sum_{m=1}^{M}\left(A_{mc}^2 + A_{ms}^2\right) \\ &= \frac{A^2}{M}\sum_{m=1}^{M}\left(a_{mc}^2 + a_{ms}^2\right)\end{aligned} \tag{3}$$

where $(a_{mc}, a_{ms})$ are the coordinates of the signal points, normalized by A.

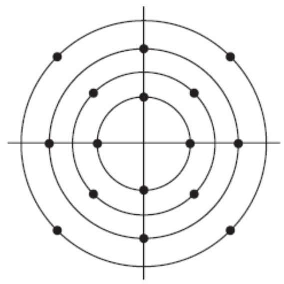


## Optimal Detection and Error Probability for QAM Signalling

- The two signal sets (a) and (c) in the figure contain signal points that fall on a rectangular grid and have $\mathcal{E}_{avg} = 6A^2$.The signal set (b) requires an average transmitted energy $\mathcal{E}_{avg} = 6.83A^2$, and (d) requires $\mathcal{E}_{avg} = 4.73A^2$
- The fourth signal set requires approximately 1 dB less energy than the first two and 1.6dB less energy than the third, to achieve the same probability of error.
- This signal constellation is known to be the best eight-point QAM constellation because it requires the least power for a given minimum distance between signal points.

# Optimal Detection and Error Probability for QAM Signalling

- For $M \geq 16$, there are many more possibilities for selecting the QAM signal points in two-dimensional space.
- For example, we may choose a circular multi amplitude constellation for $M = 16$, as shown in figure



$M = 16$



# Optimal Detection and Error Probability for QAM Signalling

- In this case, the signal points at a given amplitude level are phase-rotated by $\frac{1}{4}\pi$ relative to the signal points at adjacent amplitude levels.
- The circular 16-QAM constellation is not the best 16-point QAM signal constellation for the AWGN channel.

# Optimal Detection and Error Probability for QAM Signalling

- Rectangular QAM signal constellations have the distinct advantage of being easily generated as two PAM signals impressed on the in-phase and quadrature carriers.
- Although they are not the best M-ary QAM signal constellations for $M \geq 16$, the average transmitted power required to achieve a given minimum distance is only slightly greater than the average required power for the best M-ary QAM signal constellation.
- Thus, rectangular M-ary QAM signals are most frequently used in practice.



# Optimal Detection and Error Probability for ASK or PAM Signalling

- In the special case where k is even and the constellation is square, it is possible to derive an exact expression for the error probability.
- The minimum distance of this constellation is given by

$$d_{min} = \sqrt{\frac{6 \log_2 M}{M-1} \mathcal{E}_{bavg}} \tag{4}$$

- This constellation can be considered as two $\sqrt{M}$-ary PAM constellations in the in-phase and quadrature directions.

## Optimal Detection and Error Probability for ASK or PAM Signalling

- An error occurs if either $n_1$ or $n_2$ is large enough to cause an error in one of the two PAM signals.
- The probability of a correct detection for this QAM constellation is therefore the product of correct decision probabilities for constituent PAM systems, i.e.,

$$P_{c,M-QAM} = P^2_{c,\sqrt{M}-PAM} = \left(1 - P_{e,\sqrt{M}-PAM}\right)^2 \tag{5}$$



## Optimal Detection and Error Probability for QAM Signalling

- This results in

$$\begin{aligned} P_{e,M-QAM} &= 1 - \left(1 - P_{e,\sqrt{M}-PAM}\right)^2 \\ &= 2P_{e,\sqrt{M}-PAM}\left(1 - \frac{1}{2}P_{e,\sqrt{M}-PAM}\right) \end{aligned} \tag{6}$$

- From equation

$$\begin{aligned} P_e &= \frac{1}{M}\sum_{m=1}^{M} P[error|m\ sent] \\ &= \frac{1}{M}\left[2(M-2)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) + 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)\right] \\ &= \frac{2(M-1)}{M}Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \end{aligned} \tag{7}$$

we have

$$P_{e,\sqrt{M}-PAM} = 2\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \tag{8}$$

## Optimal Detection and Error Probability for QAM Signalling

- Substituting the value for $d_{min}$ from equation (4), we get

$$P_{e,\sqrt{M}-PAM} = 2\Big(1 - \frac{1}{\sqrt{M}}\Big) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \tag{9}$$



## Optimal Detection and Error Probability for QAM Signalling

- Substituting Equation (9) into Equation (6) yields

$$\begin{aligned} P_{e,M-QAM} &= 4\big(1 - \frac{1}{\sqrt{M}}\big) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \\ & x\left(1 - \Big(1 - \frac{1}{\sqrt{M}}\Big) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\right) \\ & \leq 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \end{aligned} \tag{10}$$

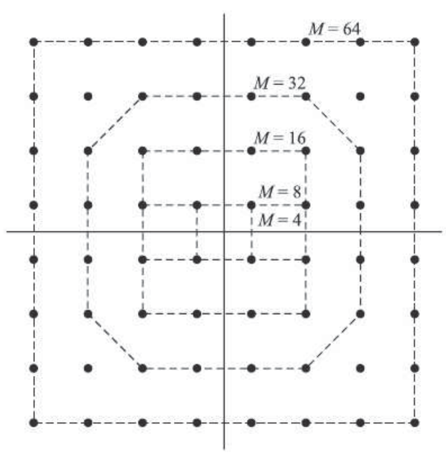
## Optimal Detection and Error Probability for QAM Signalling

- For large M and moderate to high SNR per bit, the upper bound given by above equation is quite tight.
- Although above equation is obtained for square constellations, for large M it gives a good approximation for general QAM constellations with $M = 2^k$ points which are either in the shape of a square (when k is even) or in the shape of a cross (when k is odd).

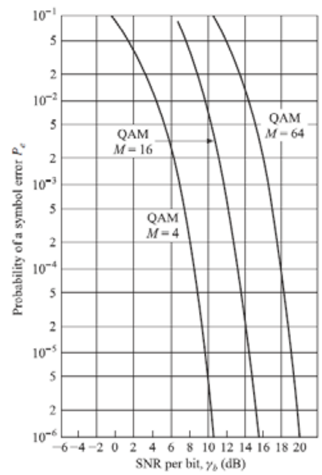


## Optimal Detection and Error Probability for QAM Signalling

- These types of constellations are illustrated in the below figure



## Error probability of M-ary QAM as a function of SNR per bit






## Optimal Detection and Error Probability for QAM Signalling

- Comparing the error performance of M-ary QAM with M-ary ASK and MPSK, we observe that unlike PAM and PSK Signalling in which in the penalty for increasing the rate was 6 dB/bit, in QAM this penalty is 3 dB/bit.
- This shows that QAM is more power efficient compared with PAM and PSK.
- The advantage of PSK is, however, its constant-envelope properties.

## Optimal Detection and Error Probability for QAM Signalling

- QPSK can be considered as $4QAM$ with a square constellation. Using Equation (10) with $M = 4$, we obtain

$$\begin{aligned} P_4 &= 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right] \\ &\leq 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \end{aligned} \tag{11}$$

- For 16-QAM with a rectangular constellation we obtain

$$\begin{aligned} P_{16} &= 3Q\left(\sqrt{\frac{4}{5}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\left[1 - \frac{3}{4}Q\left(\sqrt{\frac{4}{5}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\right] \\ &\leq 3Q\left(\sqrt{\frac{4}{5}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \end{aligned} \tag{12}$$



## Optimal Detection and Error Probability for QAM Signalling

- For nonrectangular QAM signal constellations, we may upper-bound the error probability by use of the union bound as

$$P_M \leq (M-1)Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) \tag{13}$$

where $d_{min}$ is the minimum Euclidean distance of the constellation

- This bound may be loose when M is large. In such a case, we may approximate $P_M$ by replacing $M-1$ by $N_{min}$, where $N_{min}$ is the largest number of neighbouring points that are at distance $d_{min}$ from any constellation point.

## Optimal Detection and Error Probability for QAM Signalling

- It is interesting to compare the performance of QAM with that of PSK for any given signal size M, since both types of signals are two-dimensional.
- For $M - ary$ PSK, the probability of a symbol error is approximated as

$$P_M \approx 2Q\left(\sqrt{(2\log_2 M) sin^2(\frac{\pi}{M})\frac{\mathcal{E}_b}{N_0}}\right) \tag{14}$$



## Optimal Detection and Error Probability for QAM Signalling

- For $M - ary$ QAM, we may use the expression (10).

$$\begin{aligned} P_{e,M-QAM} &= 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \\ &\quad x\left(1 - \left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right)\right) \\ &\leq 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \end{aligned} \tag{15}$$

## Optimal Detection and Error Probability for QAM Signalling

- Since the error probability is dominated by the argument of the Q function, we may simply compare the arguments of Q for the two signal formats.
- Thus, the ratio of these two arguments is

$$R_M = \frac{\frac{3}{M-1}}{2sin^2(\frac{\pi}{M})} \tag{16}$$



## Optimal Detection and Error Probability for QAM Signalling

- When M $= 4$, we have $R_M = 1$. Hence, $4 - PSK$ and $4 - QAM$ yield comparable performance for the same SNR per symbol.
- When $M > 4$, we find that $R_M > 1$, so that $M - ary$ QAM yields better performance than $M - ary$ PSK.
- The following table illustrates the SNR advantage of QAM over PSK for several values of M.

| $M$ | $10 \log \mathcal{R}_M$ |
|---|---|
| 8 | 1.65 |
| 16 | 4.20 |
| 32 | 7.02 |
| 64 | 9.95 |

# EE910: Digital Communication Systems-I

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May 16, 2022

**Lecture #6C: Optimal Detection And Error Probability For Orthogonal, Bi-orthogonal and Simplex Signalling**

## Optimal Detection And Error Probability For Orthogonal Signalling

- Orthogonal, biorthogonal, and simplex signalling is characterized by high dimensional constellations.
- These signaling schemes are more power efficient but less bandwidth-efficient than ASK, PSK, and QAM.
- In an equal-energy orthogonal signaling scheme, $N = M$ and the vector representation of the signals is given by

$$\begin{aligned} s_1 &= (\sqrt{\mathcal{E}}, 0, ..., 0) \qquad (1) \\ s_2 &= (0, \sqrt{\mathcal{E}}, ..., 0) \\ &\vdots \\ s_M &= (0, ..., 0, \sqrt{\mathcal{E}}) \end{aligned}$$



## Optimal Detection And Error Probability For Orthogonal Signalling

- For equiprobable, equal-energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross-correlation between the received vector **r** and each of the M possible transmitted signal vectors $\{s_m\}$, i.e.,

$$\hat{m} = \arg\max_{1 \leq m \leq M} \mathbf{r}.s_m \qquad (2)$$

- By symmetry of the constellation and by observing that the distance between any pair of signal points in the constellation is equal to $\sqrt{2\mathcal{E}}$ we conclude that the error probability is independent of the transmitted signal.
- Therefore, to evaluate the probability of error, we can suppose that the signal $s_1$ is transmitted.

## Optimal Detection And Error Probability For Orthogonal Signalling

- With this assumption, the received signal vector is

$$\mathbf{r} = (\sqrt{\mathcal{E}} + n_1, n_2, n_3, ..., n_M) \tag{3}$$

where $\sqrt{\mathcal{E}}$ denotes the symbol energy and $n_1, n_2, ..., n_M$ are zero-mean, mutually statistically independent Gaussian random variables with equal variance $\sigma_n^2 = \frac{1}{2}N_0$.
- Let us define random variables $R_m, 1 \leq m \leq M$ as

$$R_m = \mathbf{r}.s_m \tag{4}$$

- With this definition and from equations (3) and (1), we have

$$\begin{aligned} R_1 &= \mathcal{E} + \sqrt{\mathcal{E}} n_1 \\ R_m &= \sqrt{\mathcal{E}} n_m, \quad 2 \leq m \leq M \end{aligned} \tag{5}$$



## Optimal Detection And Error Probability For Orthogonal Signalling

- Since we are assuming that $s_1$ was transmitted, the detector makes a correct decision if $R_1 > R_m$ for $m = 2, 3, ..., M$.
- Therefore, the probability of a correct decision is given by

$$\begin{aligned} P_c &= P[R_1 > R_2, R_1 > R_3, ..., R_1 > R_M | s_1 \ sent] \\ &= P[\sqrt{\mathcal{E}} + n_1 > n_2, \sqrt{\mathcal{E}} + n_1 > n_3, ..., \sqrt{\mathcal{E}} + n_1 > n_M | s_1 \ sent] \end{aligned} \tag{6}$$

# Optimal Detection And Error Probability For Orthogonal Signalling

- Events $\sqrt{\mathcal{E}} + n_1 > n_2, \sqrt{\mathcal{E}} + n_1 > n_3, ..., \sqrt{\mathcal{E}} + n_1 > n_M$ are not independent due to the existence of the random variable $n_1$ in all of them.
- We can however, condition on $n_1$ to make these events independent. Therefore,we have

$$
\begin{aligned}
P_c &= \int_{-\infty}^{\infty} P[n_2 < n + \sqrt{\mathcal{E}}, n_3 < n + \sqrt{\mathcal{E}}, ..., n_M < n + \sqrt{\mathcal{E}} | s_1 \ sent, \ \ n_1 = n] p_{n_1}(n) dn \\
&= \int_{-\infty}^{\infty} \left( P[n_2 < n + \sqrt{\mathcal{E}} | s_1 \ sent, \ \ n_1 = n] \right)^{M-1} p_{n_1}(n) dn
\end{aligned} \tag{7}
$$

- In the last step we have used the fact that $n'_m s$ are i.i.d. random variables for $m = 2, 3, ..., M$.



# Optimal Detection And Error Probability For Orthogonal Signalling

- We have

$$
P[n_2 < n + \sqrt{\mathcal{E}} | s_1 \ sent, n_1 = n] = 1 - Q\left( \frac{n + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \tag{8}
$$

- Hence,

$$
P_c = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \left[ 1 - Q\left( \frac{n + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} e^{-\frac{n^2}{N_0}} dn \tag{9}
$$

and

$$
P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - (1 - Q(x))^{M-1}] e^{-\frac{(x - \sqrt{\frac{2\mathcal{E}}{N_0}})^2}{2}} dx \tag{10}
$$

where $x = \frac{n + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}}$

## Optimal Detection And Error Probability For Orthogonal Signalling

- In orthogonal signaling, due to the symmetry of the constellation, the probabilities of receiving any of the messages $m = 2, 3, ..., M$ when $s_1$ is transmitted, are equal.
- Therefore, for any $2 \leq m \leq M$,

$$P[s_m received|s_1\ sent] = \frac{P_e}{M-1} = \frac{P_e}{2^k - 1} \tag{11}$$



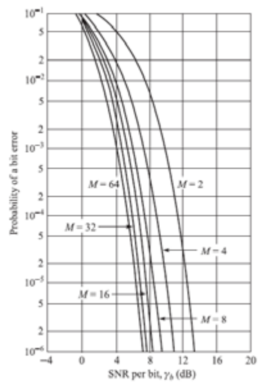
## Optimal Detection And Error Probability For Orthogonal Signalling

- Let us assume that $s_1$ corresponds to a data sequence of length k with a 0 at the first component.
- The probability of an error at this component is the probability of detecting an $s_m$ corresponding to a sequence with a 1 at the first component.
- Since there are $2^{k-1}$ such sequences, we have

$$P_b = 2^{k-1}\frac{P_e}{2^k - 1} = \frac{2^{k-1}}{2^k - 1}P_e \approx \frac{1}{2}P_e \tag{12}$$

where the last approximation is valid for $k \gg 1$

## Probability of a binary digit error as a function of the SNR per bit for $M = 2, 4, 8, 16, 32, 64$




- This figure illustrates that, by increasing the number M of waveforms, one can reduce the SNR per bit required to achieve a given probability of a bit error.



## Optimal Detection And Error Probability For FSK Signalling

- FSK signaling is a special case of orthogonal signaling when the frequency separation $\Delta f$ is given by

$$\Delta f = \frac{l}{2T} \tag{13}$$

for a positive integer l.

- For this value of frequency separation the error probability of M-ary FSK is given by Equation (10).

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - (1 - Q(x))^{M-1}] e^{-\frac{(x - \sqrt{\frac{2\mathcal{E}}{N_0}})^2}{2}} dx \tag{14}$$

## Optimal Detection And Error Probability For FSK Signalling

- Note that in the binary FSK signaling, a frequency separation that guarantees orthogonality does not minimize the error probability.
- The error probability of binary FSK is minimized when the frequency separation is of the form

$$\Delta f = \frac{0.715}{T} \tag{15}$$



## Optimal Detection And Error Probability For FSK Signalling

- For binary signals, we know the probability of error can be expressed in terms of the distance $d_{12}$ between the signal points, as

$$P_e = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right] \tag{16}$$

where the distance between the two points is given by

$$d_{12}^2 = 2\mathcal{E}_b(1-\rho) \tag{17}$$

where $\mathcal{E}_b$ is energy per information bit, and $\rho$ correlation between two signals.

## Optimal Detection And Error Probability For FSK Signalling

- The correlation between two signals in binary FSK is given by

$$\rho = \frac{\sin(2\pi\Delta f T)}{2\pi\Delta f T} \quad (18)$$

- To find the minimum value of the correlation, we set the derivative of $\rho$ with respect to $\Delta f$ equal to zero, hence we get

$$\frac{\partial\rho}{\partial\Delta f} = 0 = \frac{\cos(2\pi\Delta f T)2\pi T}{2\pi\Delta f T} - \frac{\sin(2\pi\Delta f T)}{(2\pi\Delta f T)^2}2\pi T \quad (19)$$

and therefore

$$2\pi\Delta f T = \tan(2\pi\Delta f T) \quad (20)$$



## Optimal Detection And Error Probability For FSK Signalling

- Solving numerically Equation (20) of the form $x = \tan(x)$, we get $x = 4.4934$.
- Thus,

$$2\pi\Delta f T = 4.4934 \implies \Delta f = \frac{0.7151}{T} \quad (21)$$

and corresponding value of $\rho = -0.2172$.

- Putting this value of $\rho$ in Equation (17), and using Equation (16), we get

$$P_e = Q\left[\sqrt{\frac{2\mathcal{E}_b(1-\rho)}{2N_0}}\right] = Q\left[\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}}\right] \quad (22)$$

## Optimal Detection and Error Probability for Biorthogonal Signalling

- A set of $M = 2^k$ biorthogonal signals is constructed from $\frac{1}{2}M$ orthogonal signals by including the negatives of the orthogonal signals.
- Thus, we achieve a reduction in the complexity of the demodulator for the biorthogonal signals relative to that for orthogonal signals, since the former is implemented with $\frac{1}{2}M$ cross-correlators or matched filters, whereas the latter requires M matched filters, or cross-correlators.



## Optimal Detection and Error Probability for Biorthogonal Signalling

- In biorthogonal signaling $N = \frac{1}{2}M$, and the vector representation for signals are given by

$$\begin{aligned} s_1 &= -s_{N+1} = (\sqrt{\mathcal{E}}, 0, ..., 0) \\ s_2 &= -s_{N+2} = (0, \sqrt{\mathcal{E}}, ..., 0) \\ &\vdots \\ s_N &= -s_{2N} = (0, ..., 0, \sqrt{\mathcal{E}}) \end{aligned} \quad (23)$$

## Optimal Detection and Error Probability for Biorthogonal Signalling

- To evaluate the probability of error for the optimum detector, let us assume that the signal $s_1(t)$ corresponding to the vector $s_1 = (\sqrt{\mathcal{E}}, 0, ..., 0)$ was transmitted.
- Then the received signal vector is

$$\mathbf{r} = (\sqrt{\mathcal{E}} + n_1, n_2, ..., n_N) \tag{24}$$

where the $\{n_m\}$ are zero-mean, mutually statistically independent and identically distributed Gaussian random variables with variance $\sigma_n^2 = \frac{1}{2}N_0$



## Optimal Detection and Error Probability for Biorthogonal Signalling

- Since all signals are equiprobable and have equal energy, the optimum detector decides in favour of the signal corresponding to the largest in magnitude of the cross-correlators.

$$C(\mathbf{r}, s_m) = \mathbf{r}.s_m, \quad 1 \leq m \leq \frac{1}{2}M \tag{25}$$

while the sign of this largest term is used to decide whether $s_m(t)$ or $-s_m(t)$ was transmitted.

## Optimal Detection and Error Probability for Biorthogonal Signalling

- According to this decision rule the probability of a correct decision is equal to the probability that $r_1 = \sqrt{\mathcal{E}} + n_1 > 0$ and $r_1$ exceeds $|r_m| = |n_m|$ for $m = 2, 3, ..., \frac{1}{2}M$.
- But

$$\begin{aligned} P[|n_m| < r_1 | r_1 > 0] &= \frac{1}{\sqrt{\pi N_0}} \int_{-r_1}^{r_1} e^{-x^2/N_0} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{r_1}{\sqrt{N_0/2}}}^{\frac{r_1}{\sqrt{N_0/2}}} e^{-x^2/2} dx \end{aligned} \tag{26}$$



## Optimal Detection and Error Probability for Biorthogonal Signalling

- Then the probability of a correct decision is

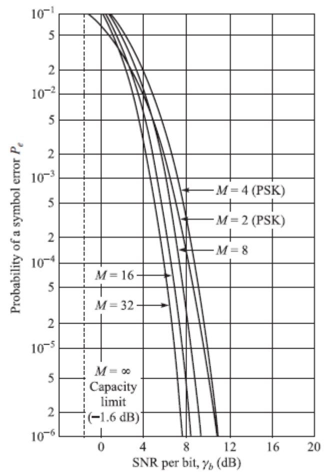
$$P_c = \int_0^\infty \left( \frac{1}{\sqrt{2\pi}} \int_{-\frac{r_1}{\sqrt{N_0/2}}}^{\frac{r_1}{\sqrt{N_0/2}}} e^{-x^2/2} dx \right)^{M/2-1} p(r_1) dr_1 \tag{27}$$

- Upon substitution for p($r_1$), we obtain

$$P_c = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2\mathcal{E}/N_0}}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-(\nu+\sqrt{2\mathcal{E}/N_0})}^{(\nu+\sqrt{2\mathcal{E}/N_0})} e^{-x^2/2} dx \right)^{M/2-1} e^{-\frac{\nu^2}{2}} d\nu \tag{28}$$

  where we have used the PDF of $r_1$ as a Gaussian random variable with mean equal to $\sqrt{\mathcal{E}}$ and variance $\frac{1}{2}N_0$.
- Finally, the probability of a symbol error $P_e = 1 - P_c$.
- $P_c$ and hence $P_e$ may be evaluated numerically for different values of M from Equation (28)

## $P_e$ as a function of $\mathcal{E}_b/N_0$, where $\mathcal{E} = k\mathcal{E}_b$,for $M = 2, 3, 8, 16, 32$






## Optimal Detection and Error Probability for Biorthogonal Signalling

- We observe that this graph is similar to that for orthogonal signals.
- However, in this case the probability of error for $M = 4$ is greater than that for $M = 2$
- This is due to the fact that we have plotted the symbol error probability $P_e$ in the above graph
- If we plotted the equivalent bit error probability, we should find that the graphs for $M = 2$ and $M = 4$ coincide.
- As in the case of orthogonal signals, as $M \to \infty$ or $(k \to \infty)$, the minimum required $\mathcal{E}_b/N_0$ to achieve an arbitrarily small probability of error is $-1.6$dB, the Shannon limit.

## Optimal Detection and Error Probability for Simplex Signalling

- Simplex signals are obtained from a set of orthogonal signals by shifting each signal by the average of the orthogonal signals
- Since the signals of an orthogonal signal are simply shifted by a constant vector to obtain the simplex signals, the geometry of the simplex signal, i.e., the distance between signals and the angle between lines joining signals, is exactly the same as that of the original orthogonal signals.
- Therefore, the error probability of a set of simplex signals is given by the same expression as the expression derived for orthogonal signals.
- However, since simplex signals have a lower energy, the energy in the expression for error probability should be scaled accordingly.



## Optimal Detection and Error Probability for Simplex Signalling

- Therefore, the expression for the error probability in simplex signaling becomes

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - (1 - Q(x))^{M-1}\right] e^{-\frac{\left(x - \sqrt{\frac{M}{M-1}\frac{2\mathcal{E}}{N_0}}\right)^2}{2}} dx \quad (29)$$

- This indicates a relative gain of $10 \log \frac{M}{M-1}$ over orthogonal signaling.
- For simplex signals, similar to orthogonal and biorthogonal signals, the error probability decreases as M increases.

# EE910: Digital Communication Systems-I

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**Lecture #7A: Noncoherent detection of carrier modulated signals**

## Noncoherent Detection

- In the detection schemes we have studied so far, we made the implicit assumption that the signals $\{s_m(t), 1 \leq m \leq M\}$ are available at the receiver.
- This assumption was in the form of either the availability of the signals themselves or the availability of an orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$.
- There are many cases where cannot make such an assumption
  - One of the cases in which such an assumption is invalid occurs when transmission over the channel introduces random changes to the signal as either a random attenuation or a random phase shift.
  - Another situation that results in imperfect knowledge of the signals at the receiver arises when the transmitter and the receiver are not perfectly synchronized.



## Noncoherent Detection

- In this case, although the receiver knows the general shape of $\{s_m(t)\}$, due to imperfect synchronization with the transmitter, it can use only signals in the form of $\{s_m(t - t_d)\}$, where $t_d$ represents the time slip between the transmitter and the receiver clocks.
- This time slip can be modelled as a random variable.
- To study the effect of random parameters of this type on the optimal receiver design and performance, we consider the transmission of a set of signals over the AWGN channel with some random parameter denoted by the random vector $\theta$.
- We assume that signals $\{s_m(t), 1 \leq m \leq M\}$ are transmitted, and the received signal r(t) can be written as

$$\mathbf{r} = s_m(t; \theta) + n(t) \tag{1}$$

where $\theta$ is in general a vector-valued random variable.

## Noncoherent Detection

- We can find an orthonormal basis for expansion of the random process $s_m(t;\theta)$ and the same orthonormal basis can be used for expansion of the white Gaussian noise process n(t).
- By using this basis, the waveform channel given in Equation (1).5-1 becomes equivalent to the vector channel

$$\mathbf{r} = s_{m,\theta} + n \tag{2}$$

for which the optimal detection rule is given by

$$\begin{aligned}\hat{m} &= \underset{1\leq m\leq M}{argmax} P_m p(\mathbf{r}|m) \\ &= \underset{1\leq m\leq M}{argmax} P_m \int p(\mathbf{r}|m,\theta)p(\theta)d\theta \\ &= \underset{1\leq m\leq M}{argmax} P_m \int p_n(\mathbf{r}-s_{m,\theta})p(\theta)d\theta\end{aligned} \tag{3}$$



## Noncoherent Detection

- Equation (3) represents the optimal decision rule and the resulting decision regions.
- The minimum error probability, when the optimal detection rule of Equation (3) is employed, is given by

$$\begin{aligned}P_e &= \sum_{m=1}^{M} P_m \int_{D_m^c} \left( \int p(\mathbf{r}|m,\theta)p(\theta)d\theta \right) d\mathbf{r} \\ &= \sum_{m=1}^{M} P_m \sum_{m'=1,m'\neq m}^{M} \int_{D_m^c} \left( \int p(\mathbf{r}|m,\theta)p(\theta)d\theta \right) d\mathbf{r}\end{aligned} \tag{4}$$

- Equations (3) and (4) are quite general and can be used for all types of uncertainties in channel parameters.

## Noncoherent Detection

- Consider a binary antipodal signalling system where equiprobable signals $s_1(t) = s(t)$ and $s_2(t) = -s(t)$ are used on an AWGN channel with noise power spectral density of $\frac{N_0}{2}$.
- Consider a channel that introduces a random gain of A which can take only nonnegative values.
- This channel can be modelled as

$$r(t) = As_m(t) + n(t) \tag{5}$$

where A is a random gain with PDF p(A) and $p(A) = 0$ for $A < 0$.



## Noncoherent Detection

- Using Equation (3), and noting that $p(\mathbf{r}|m, A) = p_n(r - As_m)$,$D_1$, the optimal decision region for $s_1(t)$ is given by

$$D_1 = r : \int_0^\infty e^{-\frac{r-A\sqrt{\mathcal{E}_b})^2}{N_0}} p(A)dA > \int_0^\infty e^{-\frac{(r+A\sqrt{\mathcal{E}_b})^2}{N_0}} p(A)dA \tag{6}$$

- Equation (6) simplifies to

$$D_1 = r : \int_0^\infty e^{-\frac{A^2\mathcal{E}_b}{N_0}} \left( e^{\frac{2rA\sqrt{\mathcal{E}_b}}{N_0}} - e^{\frac{2rA\sqrt{\mathcal{E}_b}}{N_0}} \right) p(A)dA > 0 \tag{7}$$

- Since A takes only positive values, the expression inside the paranthesis is positive if and only if $r > 0$. Therefore,

$$D_1 = \{r : r > 0\} \tag{8}$$

## Noncoherent Detection

- To compute the error probability we have

$$
\begin{aligned}
P_b &= \int_0^\infty \left( \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r+A\sqrt{\mathcal{E}_b})^2}{N_0}} dr \right) p(A)dA \\
&= \int_0^\infty Q\left( A\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) p(A)dA \qquad (9) \\
&= E\left( Q\left( A\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) \right)
\end{aligned}
$$

where the expectation is taken with respect to A.



## Noncoherent Detection

- For instance, if A takes values $\frac{1}{2}$ and 1 with equal probability, then

$$
P_b = \frac{1}{2} Q\left( \sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) + \frac{1}{2} Q\left( \sqrt{\frac{\mathcal{E}_b}{2N_0}} \right) \qquad (10)
$$

- It is important to note that in this case the average received energy per bit is $\mathcal{E}_{bavg} = \frac{1}{2}\mathcal{E}_b + \frac{1}{2}(\frac{1}{4}\mathcal{E}_b) = \frac{5}{8}\mathcal{E}_b$

## Noncoherent detection of carrier modulated signals

- For carrier modulated signals,$\{s_{ml}(t), 1 \leq m \leq M\}$ are bandpass signals with lowpass equivalents $\{s_{ml}(t), 1 \leq m \leq M\}$ where

$$s_m(t) = Re[s_{ml}(t)e^{j2\pi f_c t}] \tag{11}$$

- Received signal in the AWGN channel is given by

$$r(t) = s_m(t - t_d) + n(t) \tag{12}$$

where $t_d$ indicates the random time asynchronism between the clocks of the transmitter and receiver.



## Noncoherent detection of carrier modulated signals

- We can see that the received random process r(t) is a function of three random phenomena,
  - The message m, which is selected with probability $P_m$,
  - The random variable $t_d$, and
  - The random process n(t).
- From equations (11) and (12) we have

$$\begin{aligned} r(t) &= Re[s_{ml}(t - t_d)e^{j2\pi f_c(t-t_d)}] + n(t) \\ &= Re[s_{ml}(t - t_d)e^{-j2\pi f_c t_d} e^{j2\pi f_c t}] + n(t) \end{aligned} \tag{13}$$

- Therefore, the lowpass equivalent of $s_m(t - t_d)$ is equal to $s_{ml}(t - t_d)e^{-j2\pi f_c t_d}$

## Noncoherent detection of carrier modulated signals

- In practice $t_d \ll T_s$, where $T_s$ is the symbol duration.
- Thus the effect of a time shift of size $t_d$ on $s_{ml}(t)$ is negligible.
- However the term $e^{-j2\pi f_c t_d}$ can introduce a large phase shift $\phi = -2\pi f_c t_d$.
- Since $t_d$ is random and even small values of $t_d$ can cause large phase shifts that are folded modulo $2\pi$.
- We can model $\phi$ as a random variable uniformly distributed between 0 and $2\pi$.
- This model of the channel and detection of signals under this assumption is called noncoherent detection



## Noncoherent detection of carrier modulated signals

- In the noncoherent case

$$Re[r_l(t)e^{j2\pi f_c t}] = Re[(e^{j\phi}s_{ml}(t) + n_l(t))e^{j2\pi f_c t}] \tag{14}$$

or, in the baseband, we have

$$r_l(t) = e^{j\phi}s_{ml}(t) + n_l(t) \tag{15}$$

- Since the lowpass noise process $n_l(t)$ is circular and its statistics are independent of any rotation; hence we can ignore the effect of phase rotation on the noise component.

## Noncoherent detection of carrier modulated signals

- For the phase coherent case where the receiver knows $\phi$, it can compensate for it, and the lowpass equivalent channel will have the familiar form of
$$r_l(t) = s_{ml}(t) + n_l(t) \tag{16}$$
- In the noncoherent case,the vector equivalent of equation (16) is given by
$$\mathbf{r}_l = e^{j\phi}\mathbf{s}_{ml} + \mathbf{n}_l \tag{17}$$



## Noncoherent detection of carrier modulated signals

- The optimal detector for the baseband vector channel of equation (17) is given by
$$\hat{m} = \underset{1\leq m\leq M}{\arg\max}\ \frac{P_m}{2\pi}\int_0^{2\pi} p_{n_l}(\mathbf{r}_l - e^{j\phi}\mathbf{s}_{ml})d\phi \tag{18}$$
- Note that $n_l(t)$ is a complex baseband random process with power spectral density of $2N_0$ in the [-W,W] frequency band.
- The projections of this process on an orthonormal basis will have complex i.i.d. zero-mean Gaussian components with variance $2N_0$ (variance $N_0$ per real and imaginary components).
- Therefore we can write
$$\hat{m} = \underset{1\leq m\leq M}{\arg\max}\ \frac{P_m}{2\pi}\frac{1}{(4\pi N_0)^N}\int_0^{2\pi} e^{-\frac{||\mathbf{r}_l - e^{j\phi}\mathbf{s}_{ml}||^2}{4N_0}}d\phi \tag{19}$$

## Noncoherent detection of carrier modulated signals

- Expanding the exponent, and dropping terms that do not depend on m, and noting that $||\mathbf{s}_{ml}||^2 = 2\mathcal{E}_m$, we obtain

$$\begin{aligned}\hat{m} &= \arg\max_{1\leq m\leq M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[r_l . e^{j\phi} s_{ml}]} d\phi \\ &= \arg\max_{1\leq m\leq M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[(r_l . s_{ml}) e^{-j\phi}]} d\phi \\ &= \arg\max_{1\leq m\leq M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[|r_l . s_{ml}| e^{-j(\phi-\theta)}]} d\phi \\ &= \arg\max_{1\leq m\leq M} \frac{P_m}{2\pi} e^{-\frac{\mathcal{E}_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} |r_l . s_{ml}| cos(\phi-\theta)} d\phi\end{aligned} \quad (20)$$

where $\theta$ denotes the phase of $\mathbf{r}_l . s_{ml}$



## Noncoherent detection of carrier modulated signals

- Note that the integrand in equation (20) is a periodic function of $\phi$ with period $2\pi$, and we are integrating over a complete period; therefore $\theta$ has no effect on the result.
- Using the relation

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x cos\phi} d\phi \quad (21)$$

where $I_0(x)$ is the modified Bessel function of the first kind and order zero, we obtain

$$\hat{m} = \arg\max_{1\leq m\leq M} P_m e^{-\frac{\mathcal{E}_m}{2N_0}} I_0\left(\frac{|\mathbf{r}_l . \mathbf{s}_{ml}|}{2N_0}\right) \quad (22)$$

## Noncoherent detection of carrier modulated signals

- In general, the decision rule is given in equation (22) cannot be made simpler.
- However, in the case of equiprobable and equal-energy signals, the terms $P_m$ and $\mathcal{E}_m$ can be ignored, and the optimal detection rule becomes

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\ I_0\left(\frac{|\mathbf{r}_l.\mathbf{s}_{ml}|}{2N_0}\right) \tag{23}$$

- Since for $x > 0$, $I_0(x)$ is an increasing function of x, the decision rule in this case reduces to

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\ |\mathbf{r}_l.\mathbf{s}_{ml}| \tag{24}$$

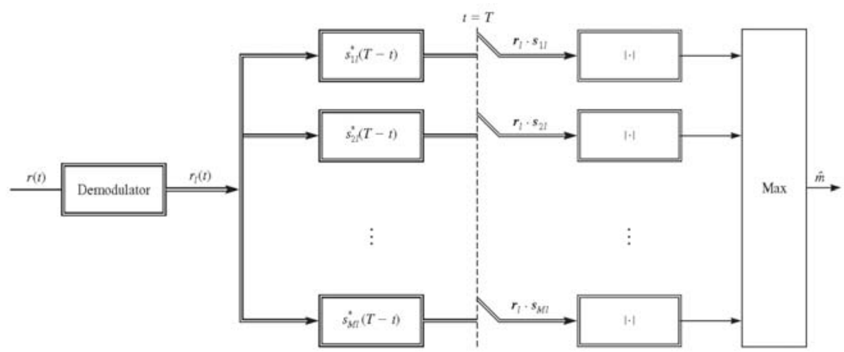


## Noncoherent detection of carrier modulated signals

- From equation (24) it is clear that an optimal noncoherent detector first demodulates the received signal, using its nonsynchronized local oscillator, to obtain $r_l(t)$, the lowpass equivalent of the received signal.
- It then correlates $r_l(t)$ with all $s_{ml}(t)$'s and chooses the one that has the maximum absolute value, or envelope.
- This detector is called envelope detector.
- Note that equation (24) can also be written as

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max} \left| \int_{-\infty}^{\infty} r_l(t) s_{ml}^*(t) dt \right| \tag{25}$$

# Envelope Detector



# EE910: Digital Communication Systems-I

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May 23, 2022

**Lecture #7B: Error probability of orthogonal signaling with noncoherent detection**

## Error probability of orthogonal signaling with noncoherent detection

- Let us assume M equiprobable, equal-energy, carrier modulated orthogonal signals are transmitted over an AWGN channel.
- These signals are noncoherently demodulated at the receiver and and then optimally detected.
- The lowpass equivalent of the signals can be written as M N-dimensional vectors ($N = M$)

$$\begin{aligned} s_1 &= (\sqrt{2\mathcal{E}_s}, 0, 0, ..., 0) \\ s_2 &= (0, \sqrt{2\mathcal{E}_s}, 0, ..., 0) \\ &\vdots \\ s_N &= (0, 0, ..., 0, \sqrt{2\mathcal{E}_s}) \end{aligned} \quad (1)$$



## Error probability of orthogonal signaling with noncoherent detection

- Without loss of generality we can assume that $s_{1l}$ is transmitted.
- Therefore the received vector will be

$$\mathbf{r}_l = e^{j\phi}\mathbf{s}_{1l} + \mathbf{n}_l \quad (2)$$

where $\mathbf{n}_l$ is a complex circular zero-mean Gaussian random vector with variance of each complex component equal to $2N_0$.

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EE910: Digital Communication Systems-I

## Error probability of orthogonal signaling with noncoherent detection

- The optimal receiver computes and compares $|\mathbf{r}_l.\mathbf{s}_{ml}|$, for all $1 \leq m \leq M$. This results in

$$\begin{aligned} |\mathbf{r}_l.\mathbf{s}_{1l}| &= |2\mathcal{E}_s e^{j\phi} + \mathbf{n}_l.\mathbf{s}_{1l}| \\ |\mathbf{r}_l.\mathbf{s}_{1l}| &= |\mathbf{n}_l.\mathbf{s}_{ml}| \quad 2 \leq m \leq M \end{aligned} \tag{3}$$



## Error probability of orthogonal signaling with noncoherent detection

- For $1 \leq m \leq M$, $\mathbf{n}_l.\mathbf{s}_{ml}$ is a circular zero-mean complex Gaussian random variable with variance $4\mathcal{E}_s N_0$ ($2\mathcal{E}_s N_0$ per real and imaginary parts).
- From equation (3) we have

$$\begin{aligned} Re[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(2\mathcal{E}_s cos\phi, 2\mathcal{E}_s N_0) \\ Im[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(2\mathcal{E}_s sin\phi, 2\mathcal{E}_s N_0) \\ Re[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(0, 2\mathcal{E}_s N_0) \quad 2 \leq m \leq M \\ Im[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(0, 2\mathcal{E}_s N_0) \quad 2 \leq m \leq M \end{aligned} \tag{4}$$

## Error probability of orthogonal signaling with noncoherent detection

- From the definition of Rayleigh and Ricean random variables, we conclude that random variables $R_m,\ 1 \leq m \leq M$ defined as

$$R_m = |\mathbf{r}_l.\mathbf{s}_{ml}| \quad 1 \leq m \leq M \tag{5}$$

are independent random variables.

- $R_1$ has a Ricean distribution with parameters $s = 2\mathcal{E}_s$ and $\sigma^2 = 2\mathcal{E}_s N_0$, and
- $R_m, 2 \leq m \leq M$, are Rayleigh random variables with parameter $\sigma^2 = 2\mathcal{E}_s N_0$.



## Error probability of orthogonal signaling with noncoherent detection

- In other words,

$$p_{R_1}(r_1) = \begin{cases} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{r_1^2+s^2}{2\sigma^2}}, & r_1 > 0 \\ 0, & otherwise \end{cases} \tag{6}$$

and

$$p_{R_m}(r_m) = \begin{cases} \frac{r_m}{\sigma^2} e^{-\frac{r_m^2}{2\sigma^2}}, & r_m > 0 \\ 0, & otherwise \end{cases} \tag{7}$$

for $2 \leq m \leq M$.

## Error probability of orthogonal signaling with noncoherent detection

- Since by assumption $\mathbf{s}_{1l}$ is transmitted, a correct decision is made at the receiver if $R_1 > R_m$ for $2 \leq m \leq M$.
- Although random variables $R_m$ for $1 \leq m \leq M$ are statistically independent, the events $R_1 > R_2, R_1 > R_3, ..., R_1 > R_M$ are not independent due to the existence of the common $R_1$
- To make them independent, we need to condition on $R_1 = r_1$ and then average over all values of $r_1$.



## Error probability of orthogonal signaling with noncoherent detection

- Therefore

$$\begin{aligned} P_c &= P[R_2 < R_1, R_3 < R_1, ..., R_M < R_1] \\ &= \int_0^\infty P[R_2 < r_1, R_3 < r_1, ..., R_M < r_1 | R_1 = r_1] p_{R_1}(r_1) dr_1 \\ &= \int_0^\infty (P[R_2 < r_1])^{M-1} p_{R_1}(r_1) dr_1 \end{aligned} \quad (8)$$

- But

$$\begin{aligned} P[R_2 < r_1] &= \int_0^{r_1} p_{R_2}(r_2) dr_2 \\ &= 1 - e^{-\frac{r_1^2}{2\sigma^2}} \end{aligned} \quad (9)$$

## Error probability of orthogonal signaling with noncoherent detection

- Using the binomial expansion, we have

$$\left(1 - e^{-\frac{r_1^2}{2\sigma^2}}\right)^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-\frac{nr_1^2}{2\sigma^2}} \tag{10}$$

- Substituting into equation (8), we get

$$\begin{aligned} P_c &= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^{\infty} e^{-\frac{nr_1^2}{2\sigma^2}} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{r_1^2+s^2}{2\sigma^2}} dr_1 \\ &= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^{\infty} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{(n+1)r_1^2+s^2}{2\sigma^2}} dr_1 \\ &= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-\frac{ns^2}{2(n+1)\sigma^2}} \int_0^{\infty} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{(n+1)r_1^2+\frac{s^2}{n+1}}{2\sigma^2}} dr_1 \end{aligned} \tag{11}$$



## Error probability of orthogonal signaling with noncoherent detection

- By introducing a change of variables

$$s' = \frac{s}{\sqrt{n+1}} \tag{12}$$

$$r' = r_1\sqrt{n+1} \tag{13}$$

the integral in equation (11) becomes

$$\begin{aligned} \int_0^{\infty} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{(n+1)r_1^2+\frac{s^2}{n+1}}{2\sigma^2}} dr_1 &= \frac{1}{n+1} \int_0^{\infty} \frac{r'}{\sigma^2} I_0(\frac{r's'}{\sigma^2}) e^{-\frac{s'^2+r'^2}{2\sigma^2}} dr' \\ &= \frac{1}{n+1} \end{aligned} \tag{14}$$

where we have used the fact that the area under a Ricean pdf is equal to 1.

## Error probability of orthogonal signaling with noncoherent detection

- Substituting Equation (14) into Equation (11) and noting that $\frac{s^2}{2\sigma^2} = \frac{4\mathcal{E}^2}{4\mathcal{E}_s N_0} = \frac{\mathcal{E}_s}{N_0}$ we obtain

$$P_c = \sum_{n=0}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} e^{-\frac{n}{n+1}\frac{\mathcal{E}_s}{N_0}} \tag{15}$$

- Then the probability of a symbol error becomes

$$P_e = \sum_{n=0}^{M-1} \frac{(-1)^{n+1}}{n+1} \binom{M-1}{n} e^{-\frac{n log_2 M}{n+1}\frac{\mathcal{E}_b}{N_0}} \tag{16}$$



## Error probability of orthogonal signaling with noncoherent detection

- For binary orthogonal signalling, including binary orthogonal FSK with noncoherent detection, Equation (16) simplifies to

$$P_b = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}} \tag{17}$$

- Comparing this result with coherent detection of binary orthogonal signals for which the error probability is given by

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) \tag{18}$$

- Using the inequality $Q(x) \leq \frac{1}{2}e^{-x^2/2}$, we conclude that $P_b(\text{noncoherent}) \geq P_b(\text{coherent})$, as expected.

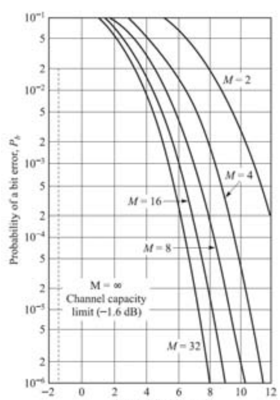
## Error probability of orthogonal signaling with noncoherent detection

- For error probabilities less than $10^{-4}$, the difference between the performance of coherent and noncoherent detection of binary orthogonal is less than 0.8 dB
- For $M > 2$, we may compute the probability of a bit error by making use of the relationship

$$P_b = \frac{2^{k-1}}{2^k - 1} P_e \tag{19}$$



## Error probability of orthogonal signaling with noncoherent detection



# EE910: Digital Communication Systems-I

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May 23, 2022

**Lecture #7C: Differential Phase Shift Keying**

## Differential PSK

- In differentially encoded PSK, the information sequence determines the relative phase, or phase transition, between adjacent symbol intervals
- In $M$-ary DPSK, the carrier phase angle of the modulator for the $n$th symbol interval is given by

$$\theta_n = \left(\theta_{n-1} + x_n \frac{2\pi}{M}\right). \quad \text{modulo } 2\pi \tag{1}$$

where $x_n$ is a modulator input symbol contained in $\{0, 1, \ldots, M-1\}$.

- The transmitted signal is the PSK waveform

$$s(t) = \left(\frac{2E_s}{T_s}\right)^{1/2} \cos\left(\omega_c t + \theta_n\right), \quad nT_s \leq t \leq (n+1)T_s. \tag{2}$$

- Thus, we implement $M$-ary PSK modulation, but with the phase differences $\delta_n = \theta_n - \theta_{n-1}$, modulo $2\pi$, defined by the symbol sequence $\{x_n\}$.



## Differential PSK

- At the receiver, due to an unknown (but assumed fixed) phase offset $\theta$ and the addition of white Gaussian noise, we observe

$$r(t) = \left(\frac{2E_s}{T_s}\right)^{1/2} \cos\left(\omega_c t + \theta_n + \theta\right) + n(t). \tag{3}$$

- For $M = 2$

| | | | | | | |
|---|---|---|---|---|---|---|
| Information Sequence | | 0 | 1 | 1 | 0 | 1 |
| Carrier Phase at Modulator Output | 0 | 0 | $\pi$ | 0 | 0 | $\pi$ |
| Carrier Phase at Demodulator Output | $\theta$ | $\theta$ | $\pi + \theta$ | $\theta$ | $\theta$ | $\pi + \theta$ |
| Phase Difference | | 0 | $\pi$ | $\pi$ | 0 | $\pi$ |
| Output | | 0 | 1 | 1 | 0 | 1 |

## Differential PSK

- Since in differential PSK the information is in the phase transitions and not in the absolute phase, the phase ambiguity from a PLL cancels between the two adjacent intervals and will have no effect on the performance of the system.
- The performance of the system is only slightly degraded due to the tendency of errors to occur in pairs, and the overall error probability is twice the error probability of a PSK system.



## Differential PSK

- A differentially encoded phase-modulated signal also allows noncoherent detection.
- Since the information is in the phase transition, we have to do the detection over a period of two symbols.
- The vector representation of the lowpass equivalent of the mth signal over a period of two symbol intervals is given by

$$\mathbf{s}_{ml} = (\sqrt{2\mathcal{E}_s} \ \sqrt{2\mathcal{E}_s} e^{j\theta_m}), \qquad 1 \leq m \leq M \tag{4}$$

where $\theta_m = \frac{2\pi(m-1)}{M}$ is the phase transition corresponding to the mth message.

## Differential PSK

- When $\mathbf{s}_{ml}$ is transmitted, the vector representation of the lowpass equivalent of the received signal on the corresponding two-symbol period is given by

$$\mathbf{r} = (r_1 \ r_2) = (\sqrt{2\mathcal{E}_s} \ \sqrt{2\mathcal{E}_s}e^{j\theta_m})e^{j\phi} + (n_{1l} \ n_{2l}), \quad 1 \leq m \leq M \quad (5)$$

where $n_{1l}$ and $n_{2l}$ are two complex valued zero-mean circular Gaussian random variables each with variance $2N_0$ (variance $N_0$ for real and imaginary components) and $\phi$ is the random phase due to noncoherent detection.

- We assume that the phase offset $\phi$ remains the same over adjacent signaling periods.
- The optimal noncoherent receiver uses the following detection rule

$$\hat{m} = \arg\max_{1\leq m\leq M} I_0\left(\frac{|\mathbf{r}_l \cdot \mathbf{s}_{ml}|}{N_0}\right) \quad (6)$$



## Differential PSK

- Thus we have

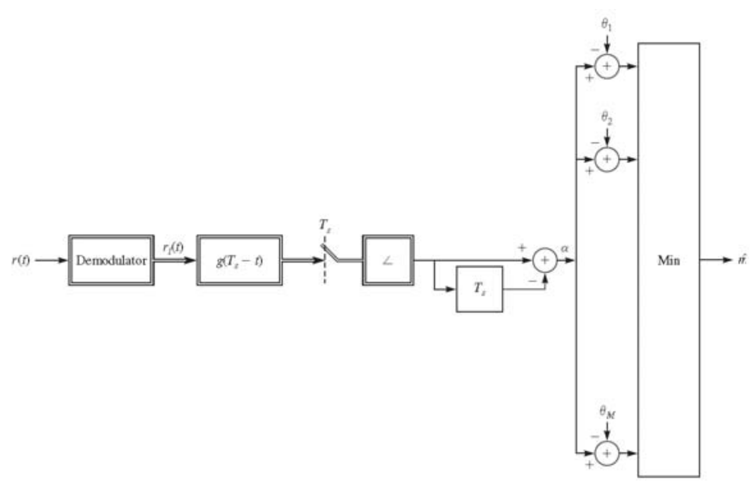
$$\begin{aligned}
\hat{m} &= \arg\max_{1\leq m\leq M} |\mathbf{r}_l.\mathbf{s}_{ml}| \\
&= \arg\max_{1\leq m\leq M} \sqrt{2\mathcal{E}_s}|\mathbf{r}_1 + r_2 e^{-j\theta_m}| \\
&= \arg\max_{1\leq m\leq M} |r_1 + r_2 e^{-j\theta_m}|^2 \\
&= \arg\max_{1\leq m\leq M} (|r_1|^2 + |r_2|^2 + 2Re[r_1^* r_2 e^{-j\theta_m}]) \\
&= \arg\max_{1\leq m\leq M} Re[r_1^* r_2 e^{-j\theta_m}] \\
&= \arg\max_{1\leq m\leq M} |r_1 r_2| cos(\angle r_2 - \angle r_1 - \theta_m) \\
&= \arg\max_{1\leq m\leq M} cos(\angle r_2 - \angle r_1 - \theta_m) \\
&= \arg\min_{1\leq m\leq M} |\angle r_2 - \angle r_1 - \theta_m|
\end{aligned} \quad (7)$$

## Differential PSK

- Note that $\alpha = \angle r_2 - \angle r_1$ is the phase difference of the received signal in two adjacent intervals
- The receiver computes this phase difference and compares it with $\theta_m = \frac{2\pi}{M}(m-1)$ for all $1 \leq m \leq M$ and selects the m for which $\theta_m$ is closest to $\alpha$, thus maximizing $cos(\alpha - \theta_m)$.
- Differentially encoded PSK signal that uses this method for demodulation detection is called differential PSK (DPSK).
- This method of detection has lower complexity in comparison with coherent detection of PSK signals and can be used in situations where the assumption that $\phi$ remains constant over two-symbol intervals is valid.



## Differential PSK



## Differential PSK

- In binary DPSK the phase difference between adjacent symbols is either 0 or $\pi$, corresponding to a 0 or 1.
- The two lowpass equivalent signals are

$$\begin{aligned} \mathbf{s}_{1l} &= (\sqrt{2\mathcal{E}_s}\ \sqrt{2\mathcal{E}_s}) \\ \mathbf{s}_{2l} &= (\sqrt{2\mathcal{E}_s}\ -\sqrt{2\mathcal{E}_s}) \end{aligned} \tag{8}$$

- These two signals are noncoherently demodulated and detected using the general approach for optimal noncoherent detection.
- Note that the two signals are orthogonal on an interval of length $2T_s$.
- Therefore, the error probability can be obtained from the expression for the error probability of binary orthogonal signaling.
- The difference is that the energy in each of the signals $s_1(t)$ and $s_2(t)$ is $2\mathcal{E}_s$.



## Differential PSK

- This is seen easily from Equation (8) which shows that the energy in lowpass equivalents is $4\mathcal{E}_s$ .
- Therefore

$$\begin{aligned} P_b &= \frac{1}{2}e^{-\frac{2\mathcal{E}_s}{2N_0}} \\ &= \frac{1}{2}e^{-\frac{\mathcal{E}_b}{N_0}} \end{aligned} \tag{9}$$

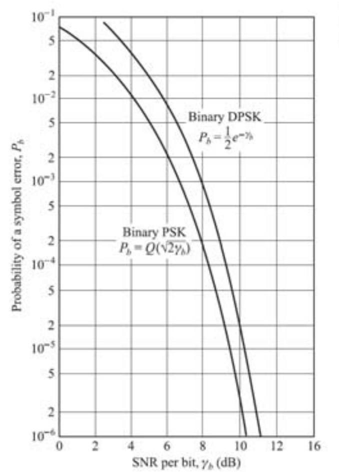
- This is the bit error probability for binary DPSK.
- Comparing this result with coherent detection of BPSK where the error probability is given by

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \tag{10}$$

we observe that by the inequality $Q(x) \leq \frac{1}{2}e^{-x^2/2}$,we have

$$P_b(\text{coherent}) \leq P_b(\text{noncoherent}) \tag{11}$$

# Differential PSK



# EE910: Digital Communication Systems-I

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May 30, 2022

**Lecture #8A: Maximum Likelihood Sequence Estimation: Viterbi Algorithm**

## Maximum Likelihood Sequence Estimation

- Any system with memory can be represented by its state diagram or trellis diagram.
- Viterbi algorithm is an computational efficient way to find maximum likelihood sequence estimate.
- We will take an example of a simple error correcting code having four states to illustrate how maximum likelihood sequence estimation works.

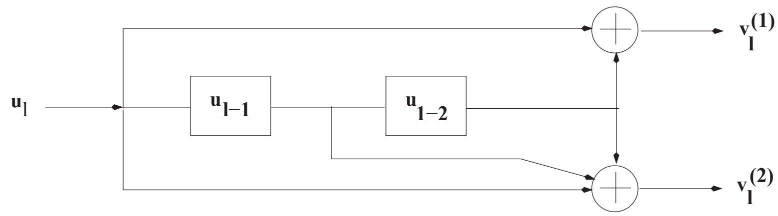


## Viterbi Decoding

- We will consider an example where each information bit is encoded into two coded bits using the error correcting code that has memory two.
- We will assume that the data is transmitted over a binary symmetric channel.

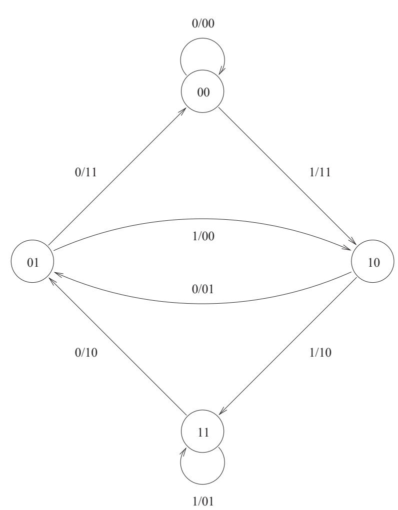
# Viterbi Decoding

Example:






# Viterbi Decoding



## Viterbi decoding

On BSC:

- Let the information sequence of length L

$$\mathbf{u} = (u_0, u_1, \cdots, u_l, \cdots, u_{L-1})$$

is encoded into code sequence of length $N \triangleq L$

$$\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_l, \cdots, \mathbf{v}_{L-1})$$

- If the code sequence $\mathbf{v}$ is transmitted over a channel, let the received sequence is,

$$\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_l, \cdots, \mathbf{r}_{L-1}),$$

where the $l^{th}$ received block is

$$\mathbf{r}_l = (r_l^{(1)}, r_l^{(2)}).$$



## Viterbi decoding

On BSC:

- A maximum likelihood decoder finds a path through the trellis that maximizes the path conditional probability

$$P(\mathbf{r}|\mathbf{v}) = \prod_{l=0}^{L} P(\mathbf{r}_l|\mathbf{v}_l)$$

where the branch conditional probability

$$P(\mathbf{r}_l|\mathbf{v}_l) = \prod_{l=1}^{2} P(r_l^{(i)}|v_l^{(i)})$$

- The bit conditional probabilities $P(r_l^{(i)}|v_l^{(i)})$ are the channel transition probabilities.

## Viterbi decoding

On BSC:

- Maximizing $P(\mathbf{r}|\mathbf{v})$ is equivalent to maximizing

$$M(\mathbf{r}|\mathbf{v}) \triangleq \log P(\mathbf{r}|\mathbf{v})$$

- $M(\mathbf{r}|\mathbf{v})$ is called the path metric.



## Viterbi decoding

- We have

$$\begin{aligned} M(\mathbf{r}|\mathbf{v}) &= \sum_{l=0}^{L} \log P(\mathbf{r}_l|\mathbf{v}_l) \\ &= \sum_{l=0}^{L} M(\mathbf{r}_l|\mathbf{v}_l), \qquad \text{(branch metrics)} \end{aligned}$$

$$\begin{aligned} M(\mathbf{r}_l|\mathbf{v}_l) &= \sum_{i=1}^{2} \log P(r_l^{(i)}|v_l^{(i)}) \\ &= \sum_{i=1}^{2} M(r_l|v_l), \qquad \text{(bit metrics)} \end{aligned}$$

## Viterbi decoding

- The partial path metric for the first $j$ branches of a path $\mathbf{v}$ is given by

$$M([\mathbf{r}|\mathbf{v}]_j) = \sum_{l=0}^{j-1} M(\mathbf{r}_l|\mathbf{v}_l)$$

- For BSC, the maximum likelihood decoder decodes the received sequence $\mathbf{r}$ into code sequence $\mathbf{v}$ that minimizes the Hamming distance $d(\mathbf{r}, \mathbf{v})$
- The Viterbi algorithm is a computationally efficient method of finding the path through the trellis with the best metric.



## Viterbi decoding

- The Viterbi decoder proceeds through the trellis level by level in search of the path with the best metric.
- At each level, the decoder compares the metric of all partial paths entering each state.
- The decoder stores the partial path entering each state with the best metric (survivor path) and eliminates all other partial paths.
- For $m \leq l \leq L$, there are total $2^m$ survivors.
- To bring the code to zero state is known as termination process.
- The number of survivors decrease during the termination process, until there is only one survivor left.
- The surviving path is the maximum likelihood path.

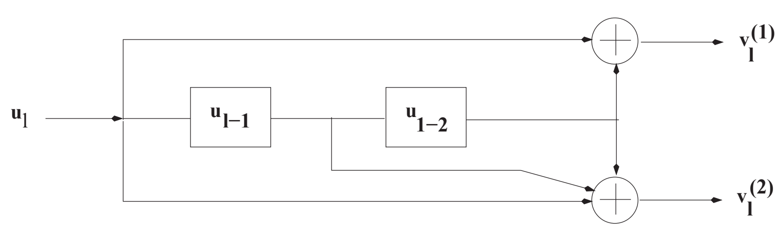
# Viterbi decoding

- Step 1: Starting at level $l = m$ in the trellis, compute the partial metric for the single path entering each $m^{th}$ level state. Store the survivor path and its metric for each state.
- Step 2: Increase time $l$ by one. Compute the partial metric for all the paths entering at the $(l+1)^{th}$ level state by adding the branch metric entering that state to the metric of the connecting survivor path at the previous $l^{th}$ level state. Store the survivor path and its metric for each state.
- Step 3: Repeat Step 2 until you are at the end of the trellis.



# Viterbi Decoding

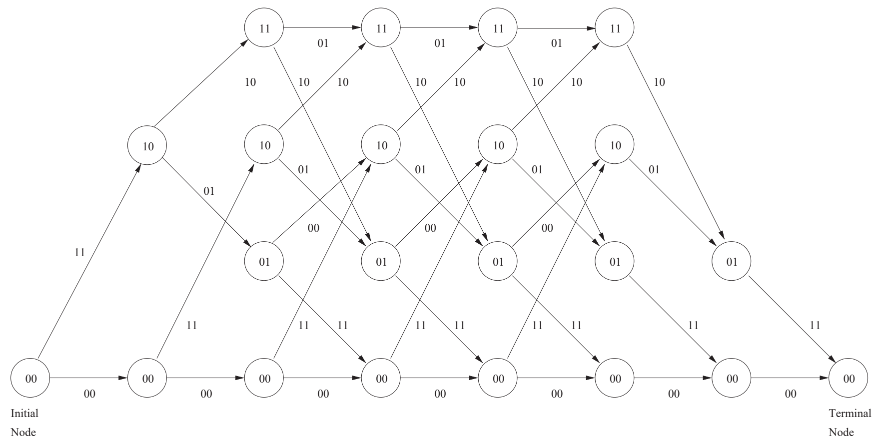
Example:




- This $(2,1,2)$ convolutional code with $L = 7$(including termination bits) is used on a BSC. The received sequence is

$$\mathbf{r} = (01,\ 11,\ 10,\ 10,\ 00,\ 11,\ 10)$$

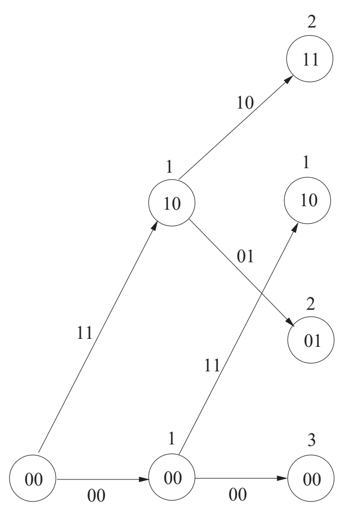
## Viterbi Decoding




Trellis diagram of $(2,1,2)$ convolutional code.



## Viterbi Decoding



Level 2

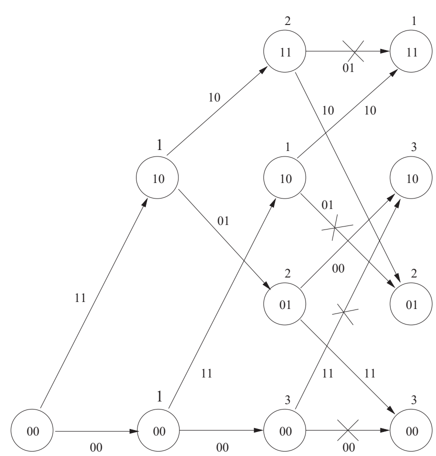
## Viterbi Decoding

$$\mathbf{r} = (01,\ 11\ )$$

$$\begin{array}{ll} \mathbf{v}_1 = (00,\ 00\ ) & d(\mathbf{v}_1, \mathbf{r}) = 3 \\ \mathbf{v}_2 = (00,\ 11\ ) & d(\mathbf{v}_2, \mathbf{r}) = 1 \\ \mathbf{v}_3 = (11,\ 01\ ) & d(\mathbf{v}_3, \mathbf{r}) = 2 \\ \mathbf{v}_4 = (11,\ 10\ ) & d(\mathbf{v}_4, \mathbf{r}) = 2 \end{array}$$



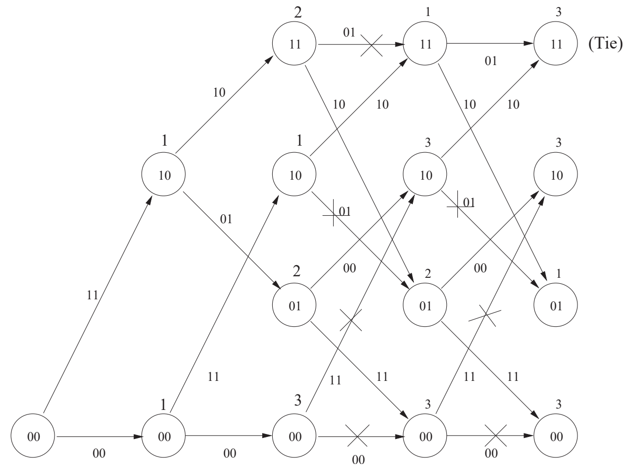
## Viterbi Decoding




$$\mathbf{r} = (01,\ \ 11\ \ 10)$$

Level 3

# Viterbi Decoding

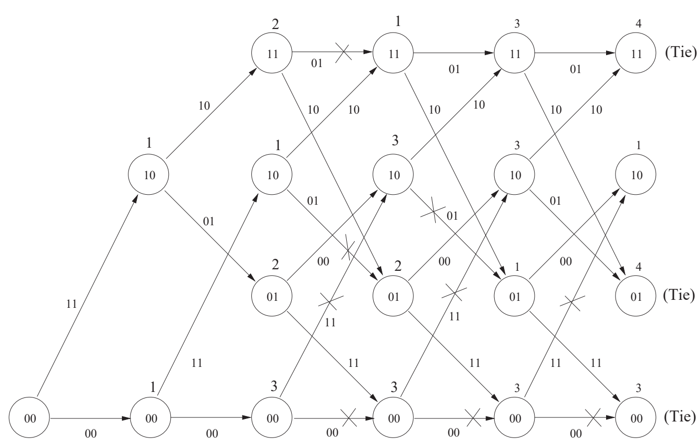



$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10)$

Level 4



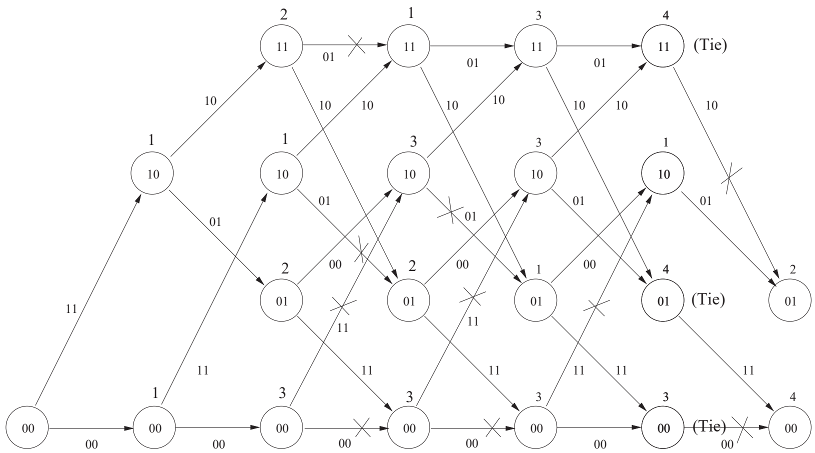
# Viterbi Decoding




$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10\ \ 00)$

Level 5

# Viterbi Decoding

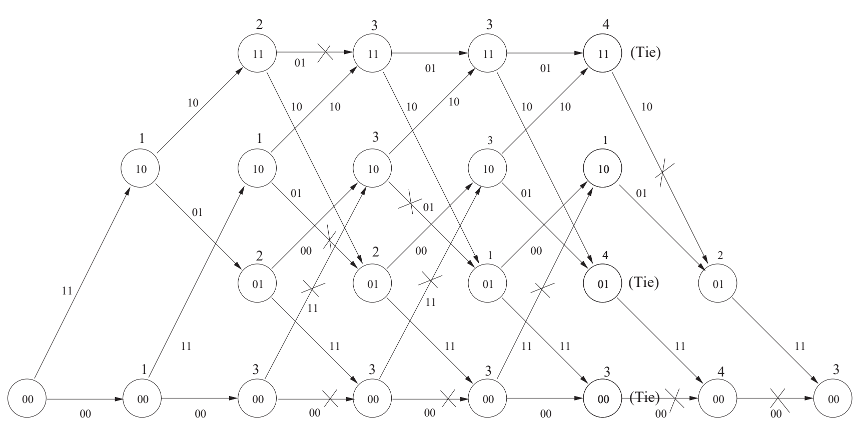


$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10\ \ 00\ \ 11)$

Level 6



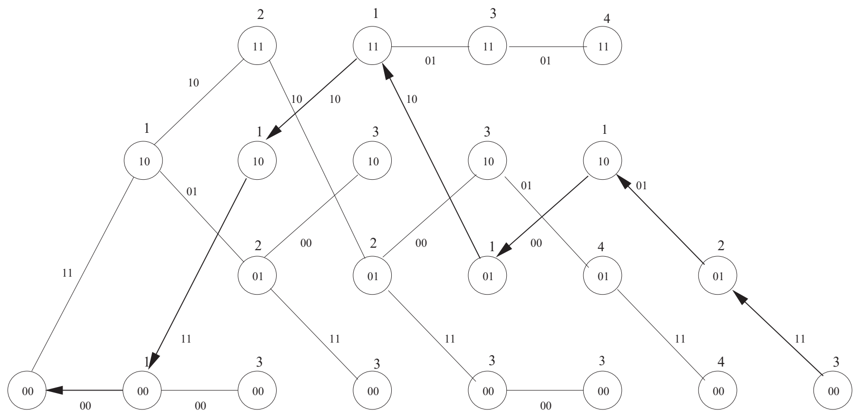
# Viterbi Decoding



$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10\ \ 00\ \ 11\ \ 10)$

Level 7

# Viterbi Decoding



$$\hat{\mathbf{v}} = (00,\ \ 11\ \ 10\ \ 10\ \ 00\ \ 01\ \ 11)$$

Level 7

# EE910: Digital Communication Systems-I

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May 30, 2022

**Lecture #8B: Optimum Receivers for CPM Signals**

## Optimum Receiver for CPM Signals

- The transmitted CPM signal may be expressed as

$$s(t) = \sqrt{\frac{2\mathcal{E}}{T}} cos[2\pi f_c t + \phi(t; I)] \tag{1}$$

- The filtered received signal for an additive Gaussian noise channel is

$$r(t) = s(t) + n(t) \tag{2}$$

where

$$n(t) = n_i(t) cos(2\pi f_c t) - n_q(t) \sin(2\pi f_c t) \tag{3}$$



## Optimum Demodulation and Detection of CPM

- The optimum receiver for CPM signal consists of a correlator followed by a maximum likelihood (ML) sequence detector.
- The ML sequence detector searches all the paths through the state trellis for minimum Euclidean distance path.
- Viterbi algorithm is used for performing this search.

## Optimum Demodulation and Detection of CPM

- The carrier phase for a CPM signal with a fixed modulation index h may be expressed as

$$\begin{aligned}\phi(t;\mathbf{I}) &= 2\pi h \sum_{k=-\infty}^{n} I_k q(t-KT) \\ &= \pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^{n} I_k q(t-KT) \\ &= \theta_n + \theta(t;I), \qquad\qquad nT \leq t \leq (n+1)T\end{aligned} \tag{4}$$

where we have assumed that q(t) = 0 for $t < 0$, q(t) = $\frac{1}{2}$ for $t \geq LT$, and

$$q(t) = \int_0^t g(\tau)d\tau \tag{5}$$



## Optimum Demodulation and Detection of CPM

- Now, when h is rational, i.e., h = m/p where m and p are relatively prime positive integers, the CPM scheme can be represented by a trellis. In this case, there are p phase states

$$\Theta_s = \left\{ \quad 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, ......, \frac{(p-1)\pi m}{p} \quad \right\} \tag{6}$$

when m is even, and 2p phase states

$$\Theta_s = \left\{ \quad 0, \frac{\pi m}{p}, ......, \frac{(2p-1)\pi m}{p} \quad \right\} \tag{7}$$

- On the other hand, if L> 1, we have an additional number of states due to the partial response character of the signal pulse g(t).

$$\theta(t;\mathbf{I}) = 2\pi h \sum_{k=n-L+1}^{n-1} I_k q(t-kT) + 2\pi h I_n q(t-KT) \tag{8}$$

## Optimum Demodulation and Detection of CPM

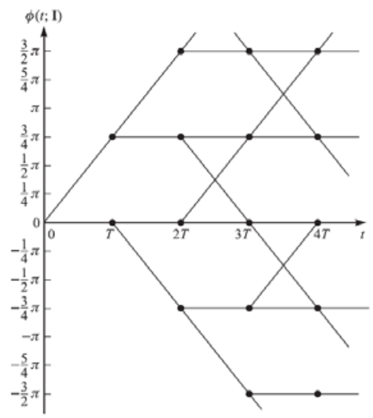
- The state of the CPM signal (or the modulator) at time t = nT may be expressed as the combined phase state and correlative state, denoted as
$$S_n = \{\theta_n, I_{n-1}, I_{n-2}, ...., I_{n-L+1}\} \tag{9}$$
- For a partial response signal pulse of length LT, where L > 1. In this case, the number of states is
$$N_s = \begin{cases} pM^{L-1} & (even \quad m) \\ 2pM^{L-1} & (odd \quad m) \end{cases} \tag{10}$$
when $h = m/p$.



## Optimum Demodulation and Detection of CPM

- Now, suppose the state of the modulator at t = nT is $S_n$. The effect of the new symbol in the time interval $nT \leq t \leq (n+1)T$ is to change the state from $S_n$ to $S_{n+1}$. Hence, at t = (n + 1)T , the state becomes
$$S_{n+1} = \{\theta_{n+1}, I_n, I_{n-1}, ...., I_{n-L+2}\} \tag{11}$$
where
$$\theta_{n+1} = \theta_n + \pi h I_{n-L+1} \tag{12}$$

# Optimum Demodulation and Detection of CPM




Phase tree for L=2 partial response CPM with h=$\frac{3}{4}$.



# Optimum Demodulation and Detection of CPM

- For CPM signals, the logarithm of the probability of the observed signal r(t) conditioned on a particular sequence of transmitted symbols $\mathbf{I}$ is proportional to the cross-correlation metric

$$\begin{aligned} CM_n(\mathbf{I}) &= \int_{-\infty}^{(n+1)T} r(t)\cos[w_c t + \phi(t;\mathbf{I})]dt \qquad (13) \\ &= CM_{n-1}(\mathbf{I}) + \int_{nT}^{(n+1)T} r(t)\cos[w_c t + \phi(t;\mathbf{I})]dt \end{aligned}$$

## Optimum Demodulation and Detection of CPM

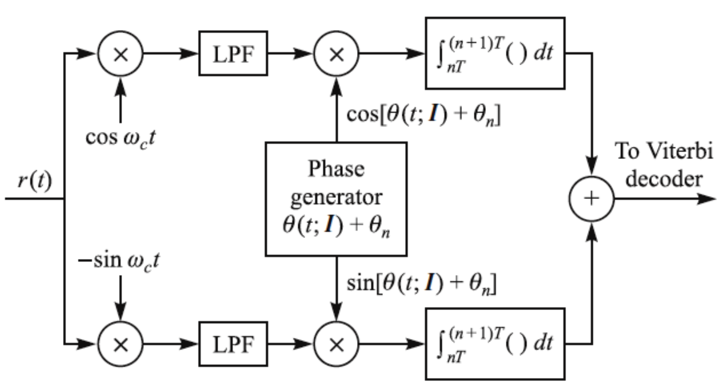
- The term $CM_{n-1}(\mathbf{I})$ represents the metrics for the surviving sequences up to time nT, and the term

$$\nu_n(\mathbf{I}; \theta_n) = \int_{nT}^{(n+1)T} r(t) \cos[w_c t + \theta(t; \mathbf{I}) + \theta_n] dt \qquad (14)$$

represents the additional increments to the metrics contributed by the signal in the time interval $nT \leq t \leq (n+1)T$.



## Performance of CPM Signals



## Performance of CPM Signals

- There are $pM^L$( or $2pM^L$) different values of $\nu_n(\mathbf{I}, \theta_n)$ computed in each signal interval.
- Each value is used to increment the metrics corresponding to $pM^{L-1}$( or $2pM^{L-1}$) surviving sequences from the previous signaling interval.
- The number of surviving sequences at each state of the Viterbi decoding process is $pM^{L-1}$ (or $2pM^{L-1}$).
- For each surviving sequence, we have M new increments of $\nu_n(\mathbf{I}, \theta_n)$ that are added to the existing metrics to yield $pM^L$( or $2pM^L$) sequences with $pM^L$( or $2pM^L$) metrics.



## Performance of CPM Signals

- This number is then reduced back to $pM^{L-1}$( or $2pM^{L-1}$) survivors with corresponding metrics by selecting the most probable sequence of the M sequences merging at each node of the trellis and discarding the other $M-1$ sequences.
- In evaluating the performance of CPM signals achieved with ML sequence detection, we must determine the minimum Euclidean distance of paths through the trellis that separate at the node at $t=0$ and remerge at a later time at the same node.

## Performance of CPM Signals

- The Euclidean distance between the two signals over an interval of length NT, where 1/T is the symbol rate, is defined as,

$$
\begin{aligned}
{d_{ij}}^2 &= \int_0^{NT} [s_i(t) - s_j(t)]^2 dt \\
&= \int_0^{NT} s_i(t)^2 dt + \int_0^{NT} s_j(t)^2 dt - 2\int_0^{NT} s_i(t)s_j(t)dt \\
&= 2N\mathcal{E} - 2\frac{2\mathcal{E}}{T}\int_0^{NT} cos[\omega_c t + \phi(t; I_i)]cos[\omega_c t + \phi(t; I_j)]dt \qquad (15) \\
&= 2N\mathcal{E} - \frac{2\mathcal{E}}{T}\int_0^{NT} cos[\phi(t; I_i) - \phi(t; I_j)]dt \\
&= \frac{2\mathcal{E}}{T}\int_0^{NT} \{1 - cos[\phi(t; I_i) - \phi(t; I_j)]\}dt
\end{aligned}
$$



## Performance of CPM Signals

- It is desirable to express the distance ${\delta_{ij}}^2$ in terms of the bit energy. Since $\mathcal{E} = \mathcal{E}_b log_2 M$, Equation (15) becomes

$$
{d_{ij}}^2 = 2\mathcal{E}_b {\delta_{ij}}^2 \qquad (16)
$$

where ${\delta_{ij}}^2$ is defined as

$$
{\delta_{ij}}^2 = \frac{log_2 M}{T}\int_0^{NT} \{1 - cos[\phi(t; I_i) - \phi(t; I_j)]\}dt \qquad (17)
$$

- Furthermore, we observe that

$$
\phi(t; I_i) - \phi(t; I_j) = \phi(t; I_i - I_j), \quad with\ \xi = I_i - I_j \qquad (18)
$$

- Thus we have

$$
{\delta_{ij}}^2 = \frac{log_2 M}{T}\int_0^{NT} \{1 - cos[\phi(t; \xi)]\}dt \qquad (19)
$$

## Performance of CPM Signals

- The error rate performances for CPM is dominated by the term corresponding to the minimum Euclidean distance, and it may be expressed as

$$P_m = K_{\delta_{\min}} Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}\delta^2_{\min}}\right) \tag{20}$$

where $K_{\delta_{\min}}$ is the number of paths having the minimum distance

- We have

$$\begin{aligned} \delta^2_{\min} &= \lim_{N\to\infty} \min_{i,j} {\delta_{ij}}^2 \\ &= \lim_{N\to\infty} \min_{i,j} {\delta_{ij}}^2 \quad \left\{\frac{log_2 M}{T}\int_0^{NT}\{1 - cos[\phi(t;\xi)]\}dt\right\} \end{aligned} \tag{21}$$

- Note that for conventional binary PSK with no memory, N = 1 and $\delta^2_{\min} = {\delta_{12}}^2 = 2$



## Performance of CPM Signals

- Since $\delta^2_{\min}$ characterizes the performance of CPM, we investigate the effect on $\delta^2_{\min}$ resulting from varying the alphabet size M, the modulation index h, and the length of the transmitted pulse in partial response CPM.
- First, we consider full response (L = 1) CPM. If we take M = 2 we note that the sequences

$$\begin{aligned} \mathbf{I}_i &= +1, -1, I_2, I_3 \\ \mathbf{I}_j &= -1, +1, I_2, I_3 \end{aligned} \tag{22}$$

which differ for k = 0, 1 and agree for k $\geq$ 2, result in two phase trajectories that merge after the second symbol.

- This corresponds to the difference sequence

$$\xi = \{2, -2, 0, 0, 0.....\} \tag{23}$$

## Performance of CPM Signals

- The Euclidean distance for this sequence is easily calculated from Equation (19), and provides an upper bound on $\delta^2_{\text{min}}$ .
- This upper bound for CPFSK with M = 2 is

$$d_B{}^2(h) = 2\Big(1 - \frac{\sin 2\pi h}{2\pi h}\Big), \qquad M = 2 \qquad (24)$$

- For $M \geq 2$ and full response CPM, the phase trajectories merge at t = 2T.

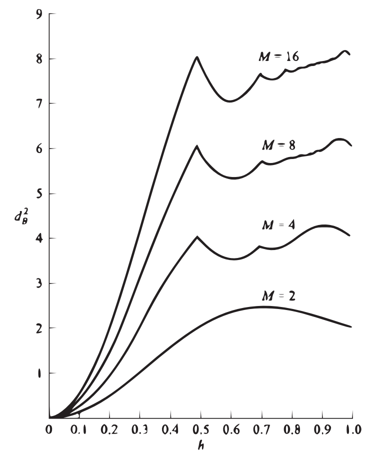


## Performance of CPM Signals

- Hence, an upper bound on $\delta^2_{\text{min}}$ can be obtained by considering the phase difference sequence
$\xi = \{\alpha, -\alpha, 0, 0, \cdots\}$ where $\alpha = \pm 2, \pm 4, \cdots, \pm 2(M-1)$
- This sequence yields the upper bound for M-ary CPFSK as

$$d_B{}^2(h) = \min_{1 \leq k \leq M-1} 2log_2\ M\Big(1 - \frac{\sin 2k\pi h}{2k\pi h}\Big), \qquad (25)$$

# Performance of CPM Signals




Upper bound $d_B{}^2(h)$ versus h for M = 2, 4, 8, 16 for full response CPM with rectangular pulses



# Performance of CPM Signals

- Large performance gains can also be achieved with maximum-likelihood sequence detection of CPM by using partial response signals.
- For example, for partial response, raised cosine pulses given by

$$g(t) = \begin{cases} \frac{1}{2LT}\left(1 - cos\frac{2\pi t}{2LT}\right) & 0 \leq t \leq LT \\ 0 & otherwise \end{cases} \tag{26}$$

  - As L increases, the distance bound $d_B{}^2(h)$ also achieves higher values.
  - The performance of CPM improves as the correlative memory L increases, but h must also be increased in order to achieve the larger values of $d_B{}^2(h)$.

- Upper bound $d_B{}^2(h)$ on the minimum distance for partial response (raised cosine pulse) binary CPM.