EE910: Digital Communication Systems-I

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Lecture #3B: Pulse Amplitude Modulation, Phase Shift Keying and Quadrature Amplitude Modulation

Pulse Amplitude Modulation (PAM)

- In digital PAM, the signal waveforms may be represented as

$$s_m(t) = A_m p(t), 1 \leq m \leq M \tag{1}$$

where $p(t)$ is a pulse of duration T and $\{A_m, 1 \leq m \leq M\}$ denotes the set of M possible amplitudes corresponding to $M = 2^k$ possible k-bit blocks of symbols.
- The signal amplitudes A_m take the discrete values $A_m = 2m - 1 - M$, $m = 1, 2, ..., M$ i.e., the amplitudes are $\pm 1, \pm 3, \pm 5, ..., \pm (M-1)$.
- The waveform $p(t)$ is a real-valued signal pulse whose shape influences the spectrum of the transmitted signal.

Pulse Amplitude Modulation (PAM)

- The energy in signal $s_m(t)$ is given by

$$\mathcal{E}_m = \int_{-\infty}^{\infty} A_m^2 p^2(t) dt = A_m^2 \mathcal{E}_p \tag{2}$$

where $\mathcal{E}_p$ is the energy in $p(t)$.
- From this,

$$\begin{aligned} \mathcal{E}_{avg} &= \frac{\mathcal{E}_p}{M} \sum_{m=1}^{M} A_m^2 \qquad (3) \\ &= \frac{2\mathcal{E}_p}{M}(1^2 + 3^2 + 5^2 + \cdots + (M-1)^2) \\ &= \frac{2\mathcal{E}_p}{M} \times \frac{M(M^2-1)}{6} \\ &= \frac{(M^2-1)\mathcal{E}_p}{3} \end{aligned}$$

Pulse Amplitude Modulation (PAM)

- Average energy per bit is given by

$$\mathcal{E}_{bavg} = \frac{(M^2 - 1)\mathcal{E}_p}{3 \log_2 M} \tag{4}$$

- Usually the PAM signals are carrier-modulated bandpass signals with lowpass equivalents of the form $A_m g(t)$, where A_m and $g(t)$ are real. In this case

$$\begin{aligned} s_m(t) &= Re\left[s_{ml}(t)e^{j2\pi f_c t}\right] \\ &= Re\left[A_m g(t)e^{2\pi f_c t}\right] = A_m g(t)\cos(2\pi f_c t) \end{aligned}$$

where f_c is the carrier frequency.

- In the generic form of PAM signaling if we substitute

$$p(t) = g(t)\cos(2\pi f_c t) \tag{5}$$

then we obtain the bandpass PAM.

Pulse Amplitude Modulation (PAM)

- For bandpass PAM we have

$$\mathcal{E}_m = \frac{A_m^2 \mathcal{E}_g}{2} \tag{6}$$

where $\mathcal{E}_g$ is the energy in $g(t)$.

- From Equations (3) and (4) we conclude

$$\begin{aligned} \mathcal{E}_{avg} &= \frac{(M^2 - 1)\mathcal{E}_g}{6} \\ \mathcal{E}_{bavg} &= \frac{(M^2 - 1)\mathcal{E}_g}{6 \log_2 M} \end{aligned} \tag{7}$$

- Clearly, PAM signals are one-dimensional ($N = 1$) since all are multiples of the same basic signals.

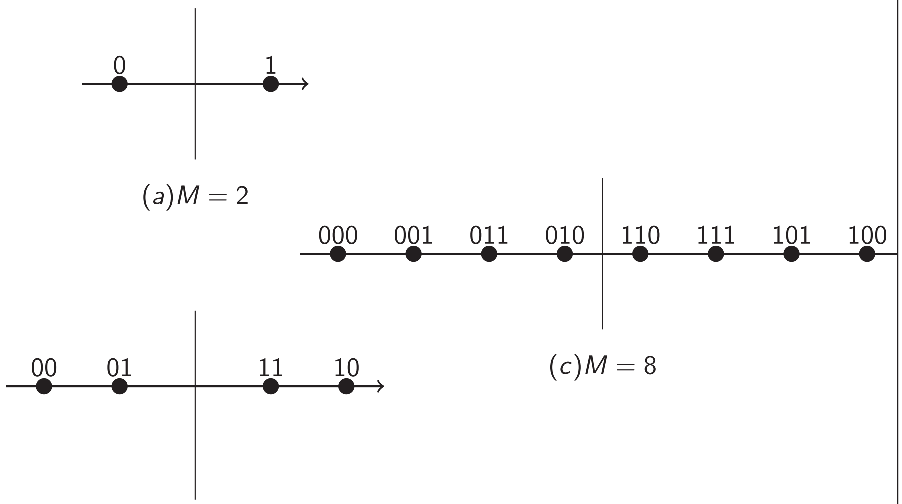
Pulse Amplitude Modulation (PAM)

- We can use
$$\phi(t) = \frac{p(t)}{\sqrt{\mathcal{E}_p}} \tag{8}$$
as the basis for the general PAM signal of the form $s_m(t) = A_m p(t)$ and
$$\phi(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g(t) \cos 2\pi f_c t \tag{9}$$
as the basis for the bandpass PAM signal given in Equation (5).
- Using these basis signals, we have
$$\begin{aligned} s_m(t) &= A_m \sqrt{\mathcal{E}_p} \phi(t) \text{ for baseband PAM} \\ s_m(t) &= A_m \sqrt{\frac{\mathcal{E}_g}{2}} \phi(t) \text{ for bandpass PAM} \end{aligned} \tag{10}$$

Pulse Amplitude Modulation (PAM)

- The one-dimensional vector representations for these signals are of the form
$$s_m = A_m \sqrt{\mathcal{E}_p}, \qquad A_m = \pm 1, \pm 3, \cdots, \pm(M-1) \tag{11}$$
$$s_m = A_m \sqrt{\frac{\mathcal{E}_g}{2}}, \qquad A_m = \pm 1, \pm 3, \cdots, \pm(M-1) \tag{12}$$

Pulse Amplitude Modulation (PAM)



$(a) M = 2$

$(c) M = 8$

$(b) M = 4$

Pulse Amplitude Modulation (PAM)

- We note that the Euclidean distance between any pair of signal points is

$$\begin{aligned} d_{mn} &= \sqrt{\|s_m - s_n\|^2} \\ &= |A_m - A_n| \sqrt{\mathcal{E}_p} \\ &= |A_m - A_n| \sqrt{\frac{\mathcal{E}_g}{2}} \end{aligned} \tag{13}$$

where the last relation corresponds to a bandpass PAM.

- For adjacent signal points $|A_m - A_n| = 2$, and hence the minimum distance of the constellation is given by

$$d_{min} = 2\sqrt{\mathcal{E}_p} = \sqrt{2\mathcal{E}_g} \tag{14}$$

Pulse Amplitude Modulation (PAM)

- The resulting expression is

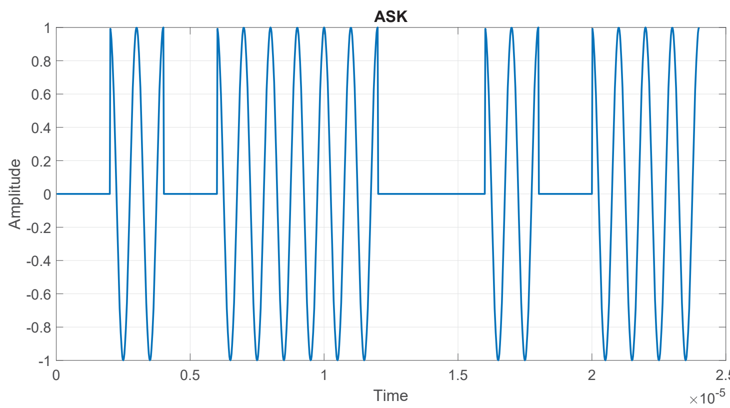
$$d_{min} = \sqrt{\frac{12 \log_2 M}{M^2 - 1} \mathcal{E}_{bavg}} \tag{15}$$

Amplitude Shift Keying (ASK)

- Can be viewed as a special case of PAM where g(t) is a sinusoid.
- Here amplitude of the carrier signal is varied according to the information sequence.
- Simplest form of ASK is on-off keying where either bursts of a carrier signal are transmitted or nothing is transmitted depending whether the input message signal is 1 or 0.

Amplitude Shift Keying (ASK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1



Phase Modulation

- In digital phase modulation, the M signal waveforms are represented as

$$
\begin{aligned}
s_m(t) &= Re\left[g(t)e^{j\frac{2\pi(m-1)}{M}}e^{j2\pi f_c t}\right], \qquad m = 1, 2, \cdots, M \quad (16)\\
&= g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right]\\
&= g(t)\cos\left(\frac{2\pi}{M}(m-1)\right)\cos 2\pi f_c t\\
&\quad -g(t)\sin\left(\frac{2\pi}{M}(m-1)\right)\sin 2\pi f_c t
\end{aligned}
$$

where $g(t)$ is the signal pulse shape and $\theta_m = 2\pi\frac{(m-1)}{M}, \quad m = 1, 2, \cdots, M$ is the M possible phases of the carrier that convey the transmitted information.

Phase Modulation

- Digital phase modulation is usually called phase-shift keying (PSK).
- We note that these signal waveforms have equal energy.

$$\mathcal{E}_{avg} = \mathcal{E}_m = \frac{1}{2}\mathcal{E}_g \tag{17}$$

and therefore,

$$\mathcal{E}_{bavg} = \frac{\mathcal{E}_g}{2\log_2 M} \tag{18}$$

Phase Modulation

- We note that $g(t)\cos 2\pi f_c T$ and $g(t)\sin 2\pi f_c t$ are orthogonal, and therefore $\phi_1(t)$ and $\phi_2(t)$ given as

$$\phi_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}}g(t)\cos 2\pi f_c t \tag{19}$$

$$\phi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}}g(t)\sin 2\pi f_c t \tag{20}$$

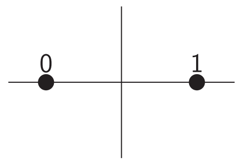
- We can write $s_m(t), 1 \leq m \leq M$, as

$$s_m(t) = \sqrt{\frac{\mathcal{E}_g}{2}}\cos\left(\frac{2\pi}{M}(m-1)\right)\phi_1(t) + \sqrt{\frac{\mathcal{E}_g}{2}}\sin\left(\frac{2\pi}{M}(m-1)\right)\phi_2(t) \tag{21}$$

Phase Modulation

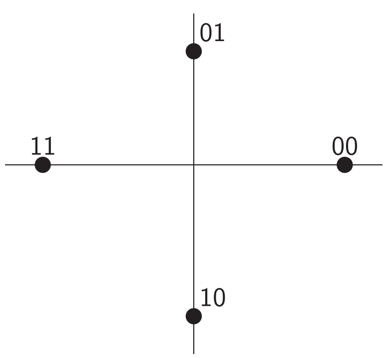
- The signal space dimensionality is $N = 2$ and the resulting vector representations are

$$s_m = \left(\sqrt{\frac{\mathcal{E}_g}{2}} \cos\left(\frac{2\pi}{M}(m-1)\right), \sqrt{\frac{\mathcal{E}_g}{2}} \sin\left(\frac{2\pi}{M}(m-1)\right) \right), m = 1, 2, \cdots, M \tag{22}$$

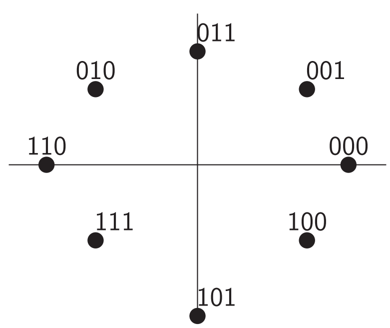


BPSK

Phase Modulation



QPSK



8PSK

Phase Modulation

- The Euclidean distance between signal points is

$$\begin{aligned} d_{mn} &= \sqrt{\|s_m - s_n\|^2} \\ &= \sqrt{\mathcal{E}_g \left[1 - \cos\left(\frac{2\pi}{M}(m-n)\right)\right]} \end{aligned} \quad (23)$$

- The minimum distance corresponding to $|m-n| = 1$ is

$$d_{min} = \sqrt{\mathcal{E}_g \left[1 - \cos\left(\frac{2\pi}{M}\right)\right]} = \sqrt{2\mathcal{E}_g \sin^2 \frac{\pi}{M}} \quad (24)$$

- Solving Equation (18) for $\mathcal{E}_g$ and substituting the result in Equation (24) result in

$$d_{min} = 2\sqrt{\left(\log_2 M \times \sin^2 \frac{\pi}{M}\right) \mathcal{E}_b} \quad (25)$$

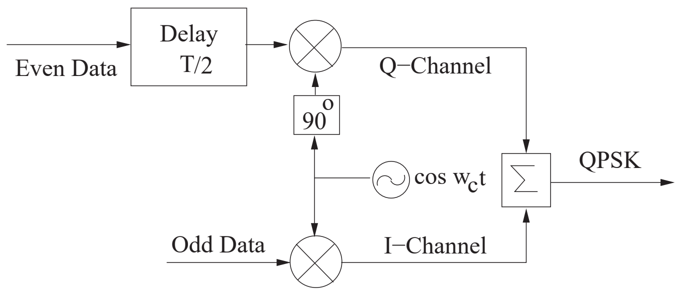
Phase Modulation

- For large values of M, we have $\sin \frac{\pi}{M} \approx \frac{\pi}{M}$, and d_{min} can be approximated by

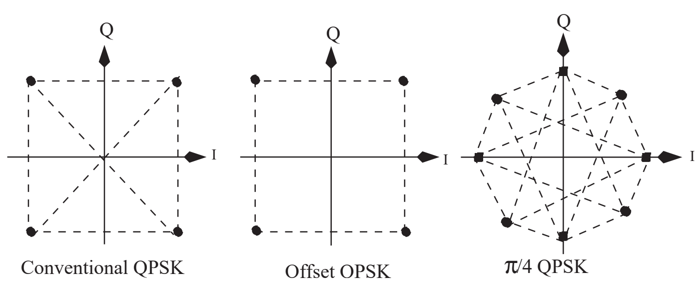
$$d_{min} \approx 2\sqrt{\frac{\pi^2 \log_2 M}{M^2} \mathcal{E}_b} \quad (26)$$

- Variants of four-phase PSK (QPSK) include offset QPSK and $\pi/4$ QPSK.
- In Offset QPSK, the phase transitions are limited to 90 degrees, the transitions on the I and Q channels are staggered.
- In $\pi/4$ QPSK the set of constellation points are toggled each symbol, so transitions through zero cannot occur.

Offset QPSK (OQPSK)

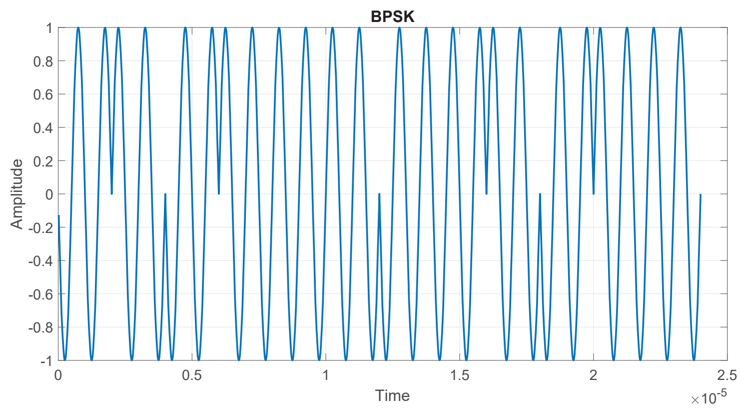


Quadrature PSK (QPSK)



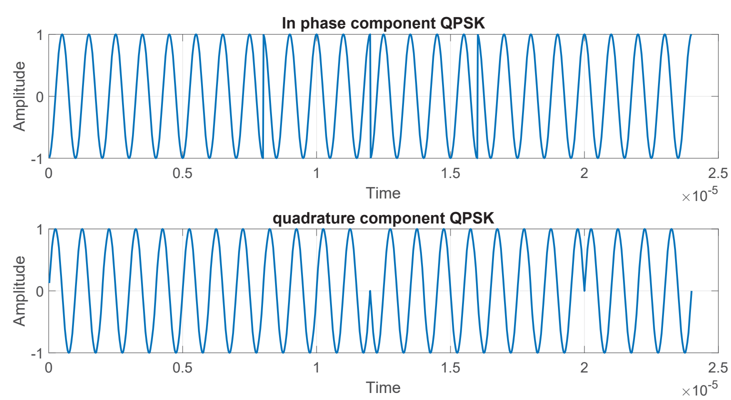
Binary Phase Shift Keying (BPSK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1



Quadrature Phase Shift Keying (QPSK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1



Quadrature Amplitude Modulation

- The bandwidth efficiency of PAM can also be obtained by simultaneously impressing two separate k-bit symbols from the information sequence on two quadrature carriers $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$.
- The resulting modulation technique is called quadrature PAM or QAM, and the corresponding signal waveforms may be expressed as

$$\begin{aligned} s_m(t) &= Re\left[(A_{mi} + jA_{mq})g(t)e^{j2\pi f_c t}\right] \quad (27) \\ &= A_{mi}g(t)\cos 2\pi f_c t - A_{mq}g(t)\sin 2\pi f_c t, \quad m = 1, 2, \cdots, M \end{aligned}$$

where A_{mi} and A_{mq} are the information-bearing signal amplitudes of the quadrature carriers and $g(t)$ is the signal pulse.

Quadrature Amplitude Modulation

- Alternatively, the QAM signal waveforms may be expressed as

$$\begin{aligned} s_m(t) &= Re\left[r_m e^{j\theta_m} e^{j2\pi f_c t}\right] \\ &= r_m \cos(2\pi f_c t + \theta_m) \end{aligned} \quad (28)$$

where $r_m = \sqrt{A_{mi}^2 + A_{mq}^2}$ and $\theta_m = tan^{-1}(A_{mi}/A_{mq})$

- Similar to the PSK case, $\phi_1(t)$ and $\phi_2(t)$ given in Equations (19) and (20) can be used as an orthonormal basis for expansion of QAM signals.
- The dimensionality of the signal space for QAM is $N = 2$.

Quadrature Amplitude Modulation

- Using this basis, we have

$$s_m(t) = A_{mi}\sqrt{\frac{\mathcal{E}_g}{2}}\phi_1(t) + A_{mq}\sqrt{\frac{\mathcal{E}_g}{2}}\phi_2(t) \tag{29}$$

which results in vector representations of the form

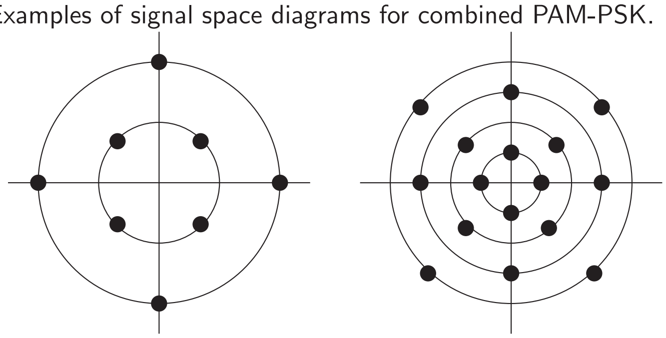
$$\begin{aligned} s_m &= (s_{m1}, s_{m2}) \\ &= \left(A_{mi}\sqrt{\frac{\mathcal{E}_g}{2}}, A_{mq}\sqrt{\frac{\mathcal{E}_g}{2}}\right) \end{aligned} \tag{30}$$

and

$$\mathcal{E}_m = \|s_m\|^2 = \frac{\mathcal{E}_g}{2}(A_{mi}^2 + A_{mq}^2) \tag{31}$$

Quadrature Amplitude Modulation

- Examples of signal space diagrams for combined PAM-PSK.



$M = 8$ $M = 16$

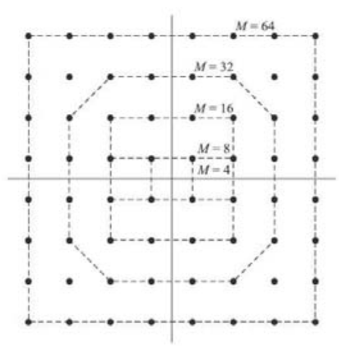
Quadrature Amplitude Modulation

- The Euclidean distance between any pair of signal vectors in QAM is

$$\begin{aligned} d_{mn} &= \sqrt{\|s_m - s_n\|^2} \\ &= \sqrt{\frac{\mathcal{E}_g}{2}\left[(A_{mi}^2 - A_{ni}^2) + (A_{mq}^2 - A_{nq}^2)\right]} \end{aligned} \tag{32}$$

- In the special case where the signal amplitudes take the set of discrete values $\{(2m-1-M), \quad m = 1, 2, \cdots, M\}$, the signal space diagram is rectangular.

Quadrature Amplitude Modulation



- In this case, the Euclidean distance between adjacent points, i.e., the minimum distance, is

$$d_{min} = \sqrt{2\mathcal{E}_g} \tag{33}$$

which is the same result as for PAM.

Quadrature Amplitude Modulation

- In the special case of a rectangular constellation with $M = 2^{2k_1}$, i.e., $M = 4, 16, 64, 256, \cdots$, and with amplitudes of $\pm 1, \pm 3, \cdots, \pm(\sqrt{M} - 1)$ on both directions, from equation (31) we have

$$\begin{aligned} \mathcal{E}_{avg} &= \frac{1}{M}\frac{\mathcal{E}_g}{2}\sum_{m=1}^{\sqrt{M}}\sum_{n=1}^{\sqrt{M}}(A_m^2 + A_n^2) \\ &= \frac{\mathcal{E}_g}{2M} \times \frac{2M(M-1)}{3} = \frac{(M-1)}{3}\mathcal{E}_g \end{aligned} \tag{34}$$

- Thus

$$\mathcal{E}_{bavg} = \frac{M-1}{3\log_2 M}\mathcal{E}_g \tag{35}$$

- Using equation (33), we have

$$d_{min} = \sqrt{\frac{6\log_2 M}{M-1}\mathcal{E}_{bavg}} \tag{36}$$

Quadrature Amplitude Modulation

- From the discussion of bandpass PAM, PSK, and QAM, it is clear that all these signaling schemes are of the general form

$$s_m(t) = Re\left[A_m g(t) e^{j2\pi f_c t}\right], \quad m = 1, 2, \cdots, M \tag{37}$$

where A_m is determined by the signaling scheme.

- The structure of the modulator for this general class of signaling schemes is shown below