EE901
Probability and Random Processes

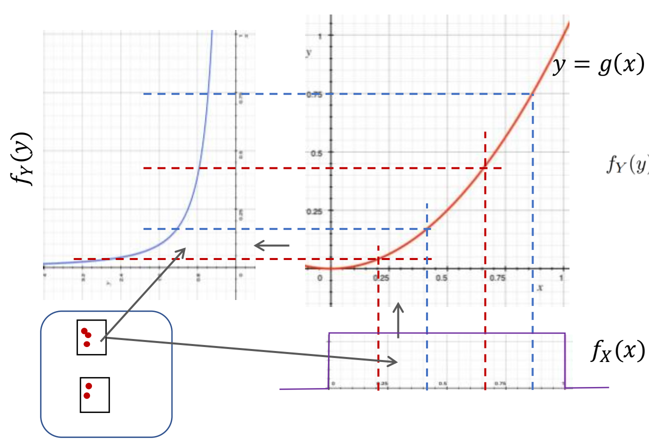
Module 5
Functions of Random Variables

Abhishek Gupta

Electrical Engineering
IIT Kanpur

1

Transformation of RVs



$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

2

Transformation of CRV with Function g

g is a monotonically increasing function

g is a monotonically decreasing function

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} \qquad f_Y(y) = \frac{f_X(g^{-1}(y))}{-g'(g^{-1}(y))}$$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

What if g is not monotonic?

3

Transformation of CRV with g

What if g is not monotonic?

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

Multiple roots of $y = g(x)$ exists - $g^{-1}(y)$ is not unique.

Example

$$y = g(x) = x^2$$

For positive y, $g(x) = y$ consists of two roots

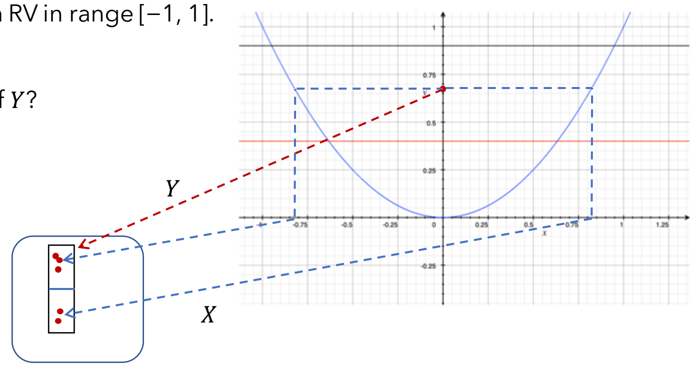
$$g_1^{-1}(y) = \sqrt{y} \; and \; g_2^{-1}(y) = -\sqrt{y}$$

4

Transformation of CRV with $g(x) = x^2$

Let X be a uniform RV in range $[-1, 1]$.
Let $Y = X^2$

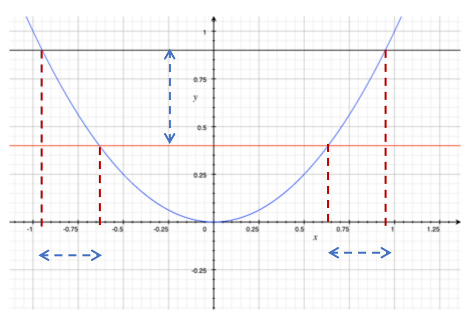
What is the PDF of Y?



Transformation of CRV with $g(x) = x^2$

Let X be a uniform RV in range $[-1, 1]$.
Let $Y = X^2$

What is the PDF of Y?



$$\mathbb{P}(Y \in [0.4, 0.9])$$

$$= \mathbb{P}(X \in [\sqrt{0.4}, \sqrt{0.9}] \cup [-\sqrt{0.9}, -\sqrt{0.4}])$$
$$= \mathbb{P}(X \in [\sqrt{0.4}, \sqrt{0.9}]) + \mathbb{P}(X \in [-\sqrt{0.9}, -\sqrt{0.4}])$$

PDF of Transformed CRV

Case: $g(x)$ consists of one root.

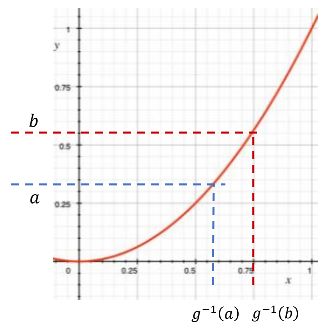
$$B = (a, b)$$

$$\{Y \in (a,b)\} = \{\omega : a < Y(\omega) < b\}$$

$$= \{\omega : a < g(X(\omega)) < b\}$$

$$= \{\omega : g^{-1}(a) < X(\omega) < g^{-1}(b)\}$$

$$\int_a^b f_Y(y)dy = \int_{g^{-1}(a)}^{g^{-1}(b)} f_X(x)dx$$



7

PDF of Transformed CRV

$g(x)$ consists of two roots $g_1^{-1}(y)$ *and* $g_2^{-1}(y)$

$$B = (a, b)$$

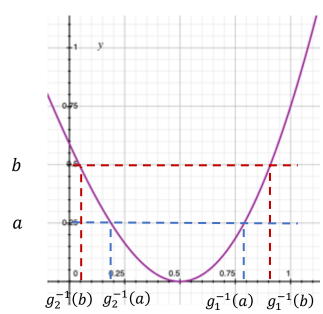
$$\{Y \in (a,b)\} = \{\omega : a < Y(\omega) < b\}$$

$$= \{\omega : a < g(X(\omega)) < b\}$$

$$= \{\omega : g_1^{-1}(a) < X(\omega) < g_1^{-1}(b)\}$$

$$\cup \{\omega : g_2^{-1}(b) < X(\omega) < g_2^{-1}(a)\}$$

$$\int_a^b f_Y(y)\mathrm{d}y = \int_{g_1^{-1}(a)}^{g_1^{-1}(b)} f_X(x)\mathrm{d}x + \int_{g_2^{-1}(b)}^{g_2^{-1}(a)} f_X(x)\mathrm{d}x$$

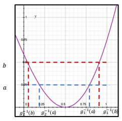


8

PDF of Transformed CRV

$g(x)$ consists of two roots $g_1^{-1}(y)$ *and* $g_2^{-1}(y)$

$$\int_a^b f_Y(y)\mathrm{d}y = \int_{g_1^{-1}(a)}^{g_1^{-1}(b)} f_X(x)\mathrm{d}x \quad + \int_{g_2^{-1}(b)}^{g_2^{-1}(a)} f_X(x)\mathrm{d}x$$



$$z = g(x) \qquad x = g_1^{-1}(z)$$
$$\mathrm{d}z = g'(x)\mathrm{d}x$$
$$= g'(g_1^{-1}(z))\mathrm{d}x$$

$$z = g(x) \qquad x = g_2^{-1}(z)$$
$$\mathrm{d}z = g'(x)\mathrm{d}x$$
$$= g'(g_2^{-1}(z))\mathrm{d}x$$

$$\int_a^b f_Y(y)dy = \int_a^b f_X(g_1^{-1}(z))\frac{\mathrm{d}z}{g'(g_1^{-1}(z))} \quad - \int_a^b f_X(g_2^{-1}(z))\frac{\mathrm{d}z}{g'(g_2^{-1}(z))}$$

9

Transformation of CRV with Function g

$g(x)$ consists of two roots $g_1^{-1}(y)$ *and* $g_2^{-1}(y)$

$$\int_a^b f_Y(y)dy = \int_a^b f_X(g_1^{-1}(z))\frac{\mathrm{d}z}{g'(g_1^{-1}(z))} \quad - \int_a^b f_X(g_2^{-1}(z))\frac{\mathrm{d}z}{g'(g_2^{-1}(z))}$$

$$\int_a^b f_Y(y)dy = \int_a^b f_X(g_1^{-1}(z))\frac{\mathrm{d}z}{g'(g_1^{-1}(z))} \quad + \int_a^b f_X(g_2^{-1}(z))\frac{\mathrm{d}z}{|g'(g_2^{-1}(z))|}$$

10

Example: Transformation with $g(x) = x^2$

Let X be a uniform RV in range $[-1, 1]$. Let $Y = X^2$. What is the PDF of Y?

$Y = g(X) = X^2$. $g^{-1}(y)$ has two values:

Example: Transformation with $g(x) = x^2$

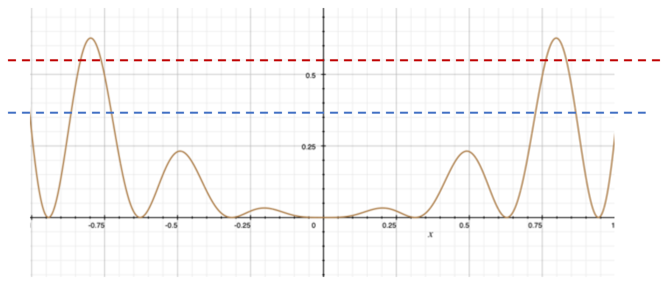
$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

Let X be a uniform RV in range $[-1, 1]$. $f_X(x) = \frac{1}{2}\mathbb{1}(-1 \le x \le 1)$

$$f_Y(y) = \frac{\mathbb{1}(-1 \le \sqrt{y} \le 1)}{4\sqrt{y}} + \frac{\mathbb{1}(-1 \le -\sqrt{y} \le 1)}{4\sqrt{y}} = \frac{\mathbb{1}(0 \le y \le 1)}{2\sqrt{y}}$$

Transformation of CRV with Function g

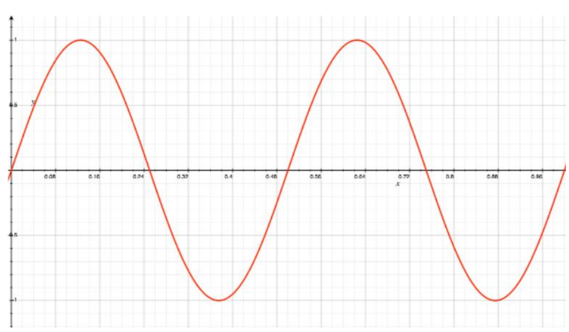
$g(x)$ consists of n roots at $x_1 = g_1^{-1}(y),\ x_2 = g_2^{-1}(y),\ \cdots\ x_n = g_n^{-1}(y)$



13

Example: $g = \sin(4\pi X)$

- Let X be a uniform RV in range [0, 1]. Let $Y = \sin 4\pi X$. What is the PDF of Y?
- Y takes value in range [-1,1]. Let us compute density of Y at $y = 0.5$.



14

Example: $g = \sin(4\pi X)$

- Y takes value in range [-1,1]. Let us compute density of Y at $y = 0.5$.
- $\sin(4\pi x) = y$ has 4 roots at y=0.5. First root is at $4\pi x = \frac{\pi}{6} \rightarrow\ x = \frac{1}{24}$.
- Rest roots are at

$4\pi x\ = \pi - \frac{\pi}{6}$

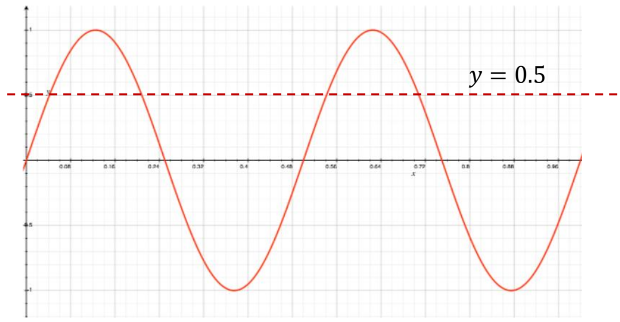
$x = \frac{5}{24}$

$4\pi x\ = 2\pi + \frac{\pi}{6}$

$x = \frac{13}{24}$

$4\pi x\ = 3\pi - \frac{\pi}{6}$

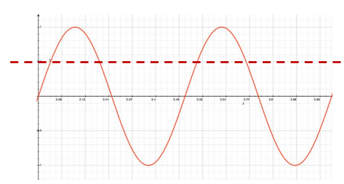
$x = \frac{17}{24}$



Example: $g = \sin(4\pi X)$

- Roots at $x = \frac{1}{24}, x = \frac{5}{24}, x = \frac{13}{24}, x\ = \frac{17}{24}$

$$f_Y(y) = \sum_{i=1}^{4} \frac{f_X(x)}{|g'(x)|}\Big|_{x=g_i^{-1}(y)}$$



X is a uniform RV in range [0, 1].

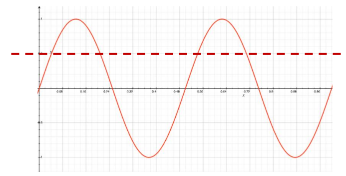
$$f_X(x) = 1(0 \leq x \leq 1)$$

$$g(x) = \sin 4\pi x \qquad g'(x) = 4\pi\cos(4\pi x)$$

$$\frac{f_X(x)}{g'(x)}\Big|_{x=1/24} = \frac{1}{4\pi\cos(4\pi x)}\Big|_{x=1/24} = \frac{1}{2\pi\sqrt{3}}$$

Example: $g = \sin(4\pi X)$

- Roots at $x = \frac{1}{24}, x = \frac{5}{24}, x = \frac{13}{24}, x = \frac{17}{24}$



$$f_Y(y) = \sum_{i=1}^{4} \frac{f_X(x)}{|g'(x)|}\Big|_{x=g_i^{-1}(y)}$$

$$\frac{f_X(x)}{|g'(x)|}\Big|_{x=\frac{1}{24}} = \frac{1}{2\pi\sqrt{3}}$$

$$\frac{f_X(x)}{|g'(x)|}\Big|_{x=\frac{5}{24}} = \frac{f_X(x)}{|g'(x)|}\Big|_{x=\frac{13}{24}} = \frac{f_X(x)}{|g'(x)|}\Big|_{x=\frac{17}{24}} = \frac{1}{2\pi\sqrt{3}}$$

17

Expectation of Transformed RV

If X is a CRV, its expectation is given as

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\mathrm{d}x$$

If g is a function, then the expectation of $g(X)$ is

$$\mathbb{E}_X\left[g(X)\right] = \int_{\mathbb{R}} g(x) f_X(x)\mathrm{d}x$$

If we see $g(X)$ as a random variable, then its expectation can be calculated as

$$\mathbb{E}_Y\left[Y\right] = \int_{\mathbb{R}} y f_Y(y)\mathrm{d}y$$

18

Expectation of Transformed RV

$$\mathbb{E}_Y[Y] = \int_{\mathbb{R}} y f_Y(y) \mathrm{d}y$$

$$y = g(x) \to \mathrm{d}y = g'(x)dx$$

$$\mathbb{E}_Y[Y] = \int g(x) f_Y(g(x)) g'(x) \mathrm{d}x$$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

$$\begin{aligned} \mathbb{E}_Y[Y] &= \int g(x) \frac{f_X(x)}{g'(x)} g'(x) \mathrm{d}x \\ &= \int g(x) f_X(x) \mathrm{d}x \end{aligned}$$

19

CRV TO DRV Transformation

Let U be a uniform RV in [2,3] and let Y be defined as $Y = g(U) = \begin{cases} 1 & \text{if } U \leq 2.5 \\ 3 & \text{if } U > 2.5 \end{cases}$

20

CRV TO DRV Transformation

Let U be a uniform RV in [2,3] and let Y be defined as $Y = g(U) = \begin{cases} 1 & \text{if } U \leq 2.5 \\ 3 & \text{if } U > 2.5 \end{cases}$

21

CRV TO DRV Transformation

Let U be a uniform RV in [2,3] and let Y be defined as $Y = g(U) = \begin{cases} 1 & \text{if } U \leq 2.5 \\ 3 & \text{if } U > 2.5 \end{cases}$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \begin{cases} 0 & \text{if } y < 1 \\ 0.5 & \text{if } 1 \leq y < 3 \\ 1 & \text{if } y \geq 3 \end{cases}$$

22