Systematic way to solve linear equations

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed the following in the last lecture
  - solution of "linear" equations'
- Discuss the following today
  - systematic way to solve "linear" equations'
- Reference for today's class - Chap 2.2 of the book

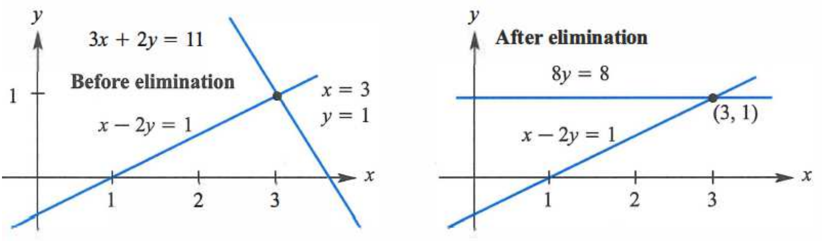
The Idea of Elimination (1)

- Systematic way to solve linear equations is called "elimination"
- We understand it using a 2 by 2 example

$$\textbf{Before} \quad \begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \qquad \textbf{After} \quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned} \qquad \begin{aligned} &(\textit{multiply equation } 1 \textit{ by } 3) \\ &(\textit{subtract to eliminate } 3x) \end{aligned}$$

- Before elimination, x and y appear in both equations
- After elimination, the first unknown x has disappeared from 2nd equation
- System is solved from bottom up (back substitution) $x = 3$ and $y = 1$
- Important point: Original equations have the same solution $x = 3$ and $y = 1$

The Idea of Elimination (2)



- Figure shows two systems as a pair of lines, intersecting at point $(3,1)$
- After elimination, lines still meet at the same point equations

The Idea of Elimination (3)

$$\textbf{Before} \quad \begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \qquad \textbf{After} \quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned} \quad \begin{aligned} &(\textit{multiply equation } 1 \textit{ by } 3) \\ &(\textit{subtract to eliminate } 3x) \end{aligned}$$

- Ask ourself how that multiplier $l = 3$ was found.
  - First equation contains $1x$: first pivot was 1 (coefficient of x)
- Second equation contains $3x$, so the multiplier was 3
- Then subtraction $3x - 3x$ produced the zero and the triangle
- Pivot = first nonzero in the row that does the elimination
- Multiplier = (entry to eliminate) divided by (pivot) = $\frac{3}{1}$
- Pivots are on the diagonal of the triangle after elimination

Breakdown of Elimination

Before $\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$ **After** $\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$ $\begin{aligned} &(\textit{multiply equation } 1 \textit{ by } 3) \\ &(\textit{subtract to eliminate } 3x) \end{aligned}$

- We could have solved those equations for x and y without reading this book
- It is an extremely humble problem, but we stay with it a little longer
- Even for a 2 by 2 system, elimination might break down
- By understanding possible breakdown (when we can't find a full set of pivots)
  - we will understand the whole process of elimination
- Normally, elimination produces pivots that take us to solution
- But failure is possible – at some point, method might ask us to divide by zero
  - We can't do it – process has to stop

Breakdown of Elimination - first example (1)

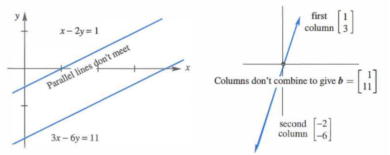
- There might be a way to adjust and continue or failure may be unavoidable
- Consider the following example

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned} \qquad \begin{array}{c}\text{Subtract 3 times} \\ \text{eqn. 1 from eqn. 2}\end{array} \qquad \boxed{\begin{aligned} x - 2y &= 1 \\ \mathbf{0}y &= 8. \end{aligned}}$$

- There is no solution to $0y = 8$
  - Normally we divide the right side 8 by the second pivot
  - But this system has no second pivot (Zero is never allowed as a pivot!)
- If there is no solution, elimination will discover that fact
  - For example, by reaching an equation like $0y = 8$

Breakdown of Elimination - first example (2)

- Row and column pictures in figure show why failure was unavoidable



- Row picture of failure shows parallel lines-which never meet
  - Solution must lie on both lines – with no meeting point, eqs have no solution
- Column picture shows the two columns (1, 3) and (-2, -6) in same direction
  - All combinations of columns lie along a line
- But the column from the right side is in a different direction (1, 11)
- No combination of columns can produce this right side – no solution

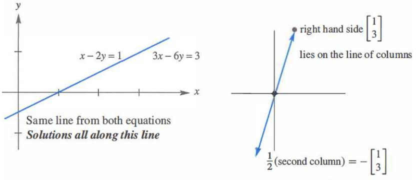
Breakdown of Elimination - second example (1)

- Instead of no solution, second example has infinitely many solutions

$x - 2y = 1$	Subtract 3 times	$x - 2y = 1$	Still only
$3x - 6y = 3$	eqn. 1 from eqn. 2	$\mathbf{0y = 0.}$	**one pivot.**

- Every y satisfies $0y = 0$
  - There is really only one equation $x - 2y = 1$
- Unknown y is "free"
  - After y is freely chosen, x is determined as $x = 1 + 2y$

Breakdown of Elimination - second example (2)



- In the row picture, the parallel lines have become the same line
  - Every point on the line satisfies both eqs – we have a whole line of solutions
- In the column picture, $b = (1, 3)$ is now the same as column 1
  - So we can choose $x = 1$ and $y = 0$ and also $x = 0$ and $y = \frac{-1}{2}$
- Every (x, y) that solves the row problem also solves the column problem

Breakdown of Elimination - third example

- Failure: for n equations we do not get n pivots
- Elimination leads to an equation
  - $0 \neq 0$ (no solution) or $0 = 0$ (many solutions)
- Success comes with n pivots. But we may have to exchange the n equations
- Temporary failure (zero in pivot). A row exchange produces two pivots:

Permutation	$0x + 2y = 4$ $3x - 2y = 5$	Exchange the two equations	$3x - 2y = 5$ $2y = 4.$

- Last equation gives $y = 2$, and then the first equation gives $x = 3$
- Row picture is normal (two intersecting lines)
- Column picture is also normal (column vectors not in the same direction)
- Pivots 3 and 2 are normal-but a row exchange is required

Breakdown of Elimination -conclusion

- Examples 1 and 2 are singular-there is no second pivot
- Example 3 is nonsingular there is a full set of pivots and exactly one solution
- Singular equations have no solution or infinitely many solutions
- Pivots must be nonzero because we have to divide by them

Three Equations in Three Unknowns (1)

- To understand Gaussian elimination, we consider a 3 by 3 system

$$\begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

- First pivot is the boldface 2, and below that pivot we want to eliminate 4
  - First multiplier is $4/2 = 2$. Multiply pivot equation by $l_{21} = 2$ and subtract
  - Subtraction removes $4x$ from the second equation
- Step 1: Subtract 2 times Eq. (1) from Eq. (2)
  - New equation is $y + z = 4$
- We also eliminate -2x from equation 3-still using the first pivot
- Quick way is to add equation 1 to equation 3
  - Rule in this book is to subtract rather than add

Three Equations in Three Unknowns (2)

- Systematic pattern has multiplier $l_{31} = -2/2 = -1$
- Step 2: Subtract -1 times Eq. 1 from Eq. 3
  - This leaves $y + 5z = 12$
- Two new equations involve only y and z. Second pivot (in boldface) is 1

$$x \textbf{ is eliminated} \qquad \begin{aligned} \mathbf{1}y + 1z &= 4 \\ 1y + 5z &= 12 \end{aligned}$$

- We have reached a 2 by 2 system. Final step eliminates y to make it 1 by 1:
- Step 3: Subtract Eq 2_{new} from 3_{new} Multiplier is $1/1 = 1$. Then $4z = 8$

Three Equations in Three Unknowns (3)

- Original $A\mathbf{x} = \mathbf{b}$ has been converted into an upper triangular $U\mathbf{x} = \mathbf{c}$

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \qquad \text{has become} \qquad \begin{aligned} 2x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8 \end{aligned}$$

- Goal is achieved-forward elimination is complete from A to U
- Notice the pivots 2, 1, 4 along the diagonal of U
- Pivots 1 and 4 were hidden in the original system
- Elimination brought them out. $U\mathbf{x} = \mathbf{c}$ is ready for back substitution

Elimination from A to U

- For a an n by n problem, elimination proceeds in the same way
  - Column 1. Use the first equation to create zeros below the first pivot
  - Column 2. Use the new equation 2 to create zeros below the second pivot
  - Columns 3 to n. Keep going to find all n pivots and the upper triangular U

$$\text{After column 2 we have } \begin{bmatrix} \mathbf{x} & x & x & x \\ 0 & \mathbf{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}. \qquad \text{We want } \begin{bmatrix} \mathbf{x} & x & x & x \\ & \mathbf{x} & x & x \\ & & \mathbf{x} & x \\ & & & \mathbf{x} \end{bmatrix}.$$

- The result of forward elimination is an upper triangular system
- It is nonsingular if there is a full set of n pivots (never zero!)

Review of key ideas

- A linear system $A\mathbf{x} = \mathbf{b}$ becomes upper triangular ($U\mathbf{x} = \mathbf{c}$) after elimination
- We subtract l_{ij} times equation j from equation i, to make (i, j) entry zero
- Multipher is $l_{ij} = \frac{\text{entry to eliminate in row i}}{\text{pivot in row j}}$. Pivots can not be zero!
- When zero is in pivot position, exchange rows if there is a nonzero below it
- Upper triangular $U\mathbf{x} = \mathbf{c}$ is solved by back substitution (starting at bottom)
- When breakdown is permanent, $A\mathbf{x} = \mathbf{b}$ has no solution or infinitely many.