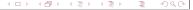
EE910: Digital Communication Systems-I

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May 9, 2022



Lecture #5A: Optimal Detection for a vector AWGN channel



Optimal detection for a General Vector Channel

- The additive white Gaussian noise (AWGN) channel model is a channel whose sole effect is addition of a white Gaussian noise process to the transmitted signal.
- The channel is mathematically described by,

$$r(t) = s_m(t) + n(t) \tag{1}$$

where $s_m(t)$ is the transmitted signal and n(t) is a zero-mean white Gaussian noise process with power spectral density of $N_0/2$; and r(t) is the received waveform.

• The receiver makes an optimal decision about which message m, $1 \le m \le M$ was transmitted based on the decision rule that minimizes the probability of disagreement between the transmitted message m and the detected message \hat{m} given by

$$P_{\rm e} = P[\hat{m} \neq m] \tag{2}$$

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Optimal detection for a General Vector Channel

• The general vector channel is given by

$$\mathbf{r} = \mathbf{s}_m + \mathbf{n} \tag{3}$$

where all vectors are N-dimensional real vectors.

- The vectors \mathbf{s}_m is chosen from a set of possible signal vectors $\{\mathbf{s}_m, 1 \leq m \leq M\}$ according to prior probabilities P_m .
- Let the decision function employed at the receiver by $g(\mathbf{r})$. If $g(\mathbf{r}) = \hat{m}$, then the probability of a correct decision, given that \mathbf{r} is received, is given by

$$P[correct \ decision | \mathbf{r}] = P[\hat{m} \ sent | \mathbf{r}]. \tag{4}$$

• Therefore the probability of a correct decision is

$$P[correct\ decision] = \int P[\hat{m}\ sent|\mathbf{r}]p(\mathbf{r})d\mathbf{r}$$
 (5)

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MAP and ML receivers

• The optimal detection rule is the one that upon observing \mathbf{r} decides in favor of the message m that maximizes $P[m|\mathbf{r}]$, i.e.,

$$\hat{m} = g_{opt}(\mathbf{r}) = \underset{1 \le m \le M}{\text{arg max}} P[m|\mathbf{r}] = \underset{1 \le m \le M}{\text{arg max}} P[\mathbf{s}_m|\mathbf{r}]$$
(6)

- The optimal decision rule given in (6) is known as the maximum a posteriori probability rule, or MAP rule.
- The MAP receiver can be simplified to

$$\hat{m} = \underset{1 \le m \le M}{\arg \max} \frac{P_m p(\mathbf{r}|\mathbf{s}_m)}{p(\mathbf{r})}$$
(7)

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MAP and ML receivers

- Since $p(\mathbf{r})$ is independent of m and for all m remains the same, (7) equivalent to $\hat{m} = \underset{1 \leq m \leq M}{\operatorname{arg max}} P_m p(\mathbf{r}|\mathbf{s}_m)$.
- If the messages are equiprobable, the optimal decision rule reduces to $\hat{m} = \arg\max_{1 \le m \le M} p(\mathbf{r}|\mathbf{s}_m)$ and the receiver is known as maximum

likelihood receiver or ML receiver.



The Error Probability

- The region \mathbf{D}_m , $1 \leq m \leq M$, is called the decision region for message m; and \mathbf{D}_m is the set of all outputs of the channel that are mapped into message m by the detector.
- For a MAP detector we have

$$\mathbf{D}_m = \{ \mathbf{r} \in \mathcal{R}^N : P[m|\mathbf{r}] > P[m'|\mathbf{r}], \text{ for all } 1 \le m' \le M \text{ and } m' \ne m \} \quad (8)$$

• An error occurs when \mathbf{s}_m is transmitted but the received \mathbf{r} is not in \mathbf{D}_m .

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The Error Probability

• The symbol error probability of a receiver is thus given by,

$$P_{e} = \sum_{m=1}^{M} P_{m} P[\mathbf{r} \notin \mathbf{D}_{m} | \mathbf{s}_{m} \operatorname{sent}] = \sum_{m=1}^{M} P_{m} P_{e|m}$$
 (9)

where $P_{e|m}$ denotes the error probability when message m is transmitted and is given by

$$P_{e|m} = \sum_{1 \le m' \le M, m' \ne m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s_m}) d\mathbf{r}$$

• Now P_e can be written as,

$$P_{e} = \sum_{m=1}^{M} P_{m} \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s_{m}}) d\mathbf{r}$$

which is the symbol error probability.

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The Error Probability

- The bit error probability is denoted by P_b and is the error probability in transmission of a single bit.
- We can bound the bit error probability by noting that a symbol error occurs when at least one bit is in error, and the event of a symbol error is the union of the events of the errors in the $k = log_2M$ bits representing that symbol.
- Therefore we can write $P_b \leq P_e \leq kP_b$



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Preprocessing at the Receiver

- The receiver passes ${\bf r}$ through ${\bf G}$ and supplies the detector with $\rho = {\bf G}({\bf r})$.
- Since G is invertible and the detector has access to ρ , it can apply G^{-1} to ρ to obtain $G^{-1}(\rho) = G^{-1}(G(\mathbf{r})) = \mathbf{r}$. The detector now has access to both ρ and \mathbf{r} .
- Thus the optimal detection rule can be written as,

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max} \ P_m p(\mathbf{r}, \rho | \mathbf{s}_m) = \underset{1 \leq m \leq M}{\arg\max} \ P_m p(\mathbf{r} | \mathbf{s}_m) p(\rho | \mathbf{r}) = \underset{1 \leq m \leq M}{\arg\max} \ P_m p(\mathbf{r} | \mathbf{s}_m)$$

ullet Thus it is clear that the optimal detector based on the observation of ho makes the same decision as the optimal detector based on the observation of ${\bf r}$.

