# Assignment – 4 - Solution

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**eMasters in Communication Systems, IITK**
**EE901: Probability and Random Processes**
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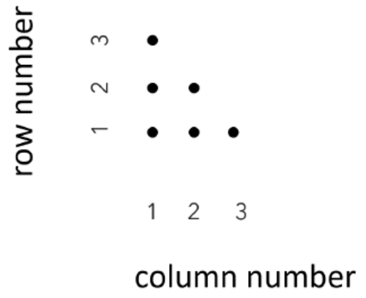
**Q1:** Consider the following experiment where a marble is randomly picked from the following arrangement.




Let X and Y denote the row and column number of the picked marble. What is the joint distribution of the X and Y? What is the conditional PMF of Y given X?

**Q1 Solution:**

**Joint distribution of X and Y:**

Joint Probability of X and Y - $p_{XY}(x,y) = \begin{cases} \frac{1}{6}\ (x,y) = (1,1),(1,2),(1,3),(2,1),(2,2),(3,1) \\ 0, Otherwise \end{cases}$

**Condition PMF of Y given X**

$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$ = {Joint Probability of X and Y}/Marginal Probability of X

Marginal Probability - $p_X(x) = \begin{cases} \frac{3}{6}, if\ x = 1 \\ \frac{2}{6}, if\ x = 2 \\ \frac{1}{6}, if\ x = 3 \end{cases}$

Therefore,

$$p_{Y|X}(y|x=1) = \begin{cases} \frac{p_{X,Y}(1,1)}{p_X(1)} = \frac{1/6}{3/6} = \frac{1}{3}, for\ y = 1 \\ \frac{p_{X,Y}(1,2)}{p_X(1)} = \frac{1/6}{3/6} = \frac{1}{3}, for\ y = 2 \\ \frac{p_{X,Y}(1,3)}{p_X(1)} = \frac{1/6}{3/6} = \frac{1}{3}, for\ y = 3 \end{cases}$$

$$p_{Y|X}(y|x=2) = \begin{cases} \frac{p_{X,Y}(2,1)}{p_X(2)} = \frac{1/6}{2/6} = \frac{1}{2}, for\ y = 1 \\ \frac{p_{X,Y}(2,2)}{p_X(2)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}, for\ y = 2 \\ 0, Otherwise \end{cases}$$

$$p_{Y|X}(y|x=3) = \begin{cases} \dfrac{p_{X,Y}(3,1)}{p_X(3)} = \dfrac{1/6}{1/6} = 1, for\ y = 1 \\ 0, Otherwise \end{cases}$$

**Q2:** Consider the following joint probability density function.

$$f_{X,Y}(x,y) = e^{-x} * \frac{1}{x} * 1(0 \le y \le x), x \ge 0$$

a) Compute the marginal PDFs of X and Y.
b) Compute the conditional PDF of Y given X : $p_{Y|X}(y|x)$

**Q2 Solution:**

Joint PDF of X and Y:

$$f_{X,Y}(x,y) = e^{-x} * \frac{1}{x} * 1(0 \le y \le x), x \ge 0$$

**(a) Marginal PDFs of X and Y**

Given joint PDF of two variables, the marginal PDF of a variable (x) is the integration of the other variable (y) over all its possible values.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\infty}^{\infty} e^{-x} * \frac{1}{x} * 1(0 \le y \le x)dy = \int_{0}^{x} e^{-x} * \frac{1}{x} dy = \frac{e^{-x}}{x} |y|_0^x = \frac{e^{-x}}{x}[x-0]$$

$$\therefore \boldsymbol{f_X(x) = e^{-x}, x \ge 0}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{0}^{\infty} e^{-x} * \frac{1}{x} * 1(y \le x \ge \infty)dx = \int_{y}^{\infty} e^{-x} * \frac{1}{x} dx$$

Let -x = t ➔ dx=-dt

$$f_Y(y) = \int_{y}^{\infty} \frac{e^t}{-t}(-dt) = \int_{y}^{\infty} \frac{e^t}{t} dt$$

The above integrand can't be represented using elementary function(s) for solving.
Representing the exponential function using Taylor series

$$e^t = \frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!}$$

And solving,

$$f_Y(y) = \int_{y}^{\infty} \frac{1}{t} * \left(\frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) dt = \int_{y}^{\infty} \left(\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots + \frac{t^n}{(n+1)!}\right) dt$$

$$= \left|\ln(t) + 0 + \frac{t}{1*1!} + \frac{t^2}{2*2!} + \frac{t^3}{3*3!} + \cdots \frac{t^n}{n*n!}\right|_y^{\infty} = \ln(-x) + \sum_{n=1}^{\infty} \frac{(-x)^n}{n*n!}$$

**(b) Conditional PDF of Y given X - $p_{Y|X}(y|x)$**

$$\boldsymbol{f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = e^{-x} * \frac{1}{x} * 1(0 \le y \le x) * \frac{1}{e^{-x}} = \frac{1}{x} * 1(0 \le y \le x)}$$

**Therefore, given X, Y is a uniform RV ~ Unif[0,X]**

**Q3:** Consider an exponential RV X with parameter $\lambda$. Compute the conditional CDF of Y=X+2 conditioned on X>2.

**Q3 Solution:**

$X = Exp(\lambda), f_X(x) = \lambda e^{-\lambda x}\, \mathbb{1}\,(x \geq 0)$

CDF of (Y | (X>2)) = $F_Y\big((Y \leq y \mid X > 2)\big) = \mathbb{P}\big((Y = X+2) \leq y \mid (X > 2)\big) = \frac{Joint\ Probability\ (X,Y)}{Marginal\ Probability\ (X)}$

$$= \frac{\mathbb{P}[((X+2) \leq y) \cap (X > 2)]}{\mathbb{P}[X > 2]} = \frac{\mathbb{P}[(X \leq (y-2)) \cap (X > 2)]}{\mathbb{P}[X > 2]} = \frac{\mathbb{P}[X \leq (y-2)]}{\mathbb{P}[X > 2]} = \frac{\int_2^{(y-2)} f_X(x)dx}{\int_2^{\infty} f_X(x)dx}$$

$$= \frac{\int_2^{(y-2)} \lambda e^{-\lambda x}\, dx}{\int_2^{\infty} \lambda e^{-\lambda x}\, dx} = \frac{\lambda \left|-e^{-\lambda x}\right|_2^{y-2}}{\lambda \left|-e^{-\lambda x}\right|_2^{\infty}} = \frac{e^{-(y-2)\lambda} - e^{-2\lambda}}{e^{-\infty\lambda} - e^{-2\lambda}} = \frac{e^{-(y-2)\lambda} - e^{-2\lambda}}{-e^{-2\lambda}} = \frac{e^{-2\lambda} - e^{-(y-2)\lambda}}{e^{-2\lambda}}$$

$$= (1 - e^{-((y-2)+2)\lambda}) = (1 - e^{-\lambda(y-4)})$$

$$\therefore \boldsymbol{F_Y\big((Y \leq y \mid X > 2)\big) = \big(1 - e^{-\lambda(y-4)}\big)}\, \mathbb{1}\,(y > 4)$$

**Q4:** Let N be a Poisson random variable with parameter $\lambda$. Conditioned on N, U is a uniform continuous RV between 0 and N. Compute the expected value of Z=UN. Compute the co-variance between U and N.

**Q4 Solution:**

- $N \sim Poiss(\lambda),\ PMF - p_N(i) = \frac{e^{-\lambda}\lambda^i}{i!},\ \mathbb{E}[N] = \lambda$
- $U \sim Uniform(0, N),\ PDF - f_U(u) = \frac{1}{N}\mathbb{1}\,(0 \leq x \leq N),\ \mathbb{E}\,[U] = \frac{N+0}{2} = \frac{N}{2}$

Expected value of Z=UN, given N=n is:

$$\mathbb{E}_{Z|N}[Z \mid N = N] = \mathbb{E}_{Z|N}[U.N \mid N = N] = \mathbb{E}_{Z|N}[U.N\,] = N * \mathbb{E}_U[U] = N * \frac{N}{2} = \frac{N^2}{2}$$

**Covariance between U and N**

$Cov(U,N) = \mathbb{E}[(U - \mathbb{E}[U])(N - \mathbb{E}[N])] = \mathbb{E}[UN] - \mathbb{E}[U|N]\mathbb{E}[N]$

Let's compute the expected values of $\mathbb{E}[UN], \mathbb{E}[U|N], \mathbb{E}[N]$

**Expected value of Z**

$\mathbb{E}_{U,N}[UN] = \mathbb{E}_{U,N}[Z] = \iint Z f_{U,N}(u,n)dudn$ - (1)

From conditional PDFs of Two RVs, we know that:

$f_{U|N}(u \mid N = n) = \frac{f_{U,N}(u,n)}{f_N(N=n)} \Rightarrow f_{U,N}(u,n) = f_{U|N}(u \mid N = n) * f_N(N = n)$

Substituting $f_{U,N}(u,n)$ in equation (1),

Expected Value of Z = $\mathbb{E}_{U,N}[UN] = \mathbb{E}_{U,N}[Z] = \iint Z f_{U|N}(u \mid N = n) * f_N(N = n)dudn$

$$\mathbb{E}_{U,N}[Z] = \int \left(\int Z f_{U|N}(u \mid N = n)du\right) f_N(N = n)dn = \mathbb{E}_N[\mathbb{E}_{U|N}[Z]]$$

$$\mathbb{E}_{U|N}[Z] = \mathbb{E}_{U|N}[U.N = N] = N.\mathbb{E}_U[U] = N * \frac{N}{2} = \frac{N^2}{2}$$

$$\therefore \mathbb{E}_{\mathbf{U,N}}[\mathbf{Z}] = \mathbb{E}_N\left[\mathbb{E}_{U|N}[\boldsymbol{Z}]\right] = \mathbb{E}_N\left[\frac{\boldsymbol{N^2}}{\mathbf{2}}\right] = \frac{\mathbf{1}}{\mathbf{2}} * \mathbb{E}_N[\boldsymbol{N^2}] = \frac{\mathbf{1}}{\mathbf{2}} * \boldsymbol{Var}\big(\boldsymbol{Poiss}(\boldsymbol{\lambda})\big) = \frac{(\boldsymbol{\lambda} + \boldsymbol{\lambda^2})}{\mathbf{2}}$$

(Note that the Var(Poiss($\lambda$)) or second moment is $= \lambda + \lambda^2$)

$\mathbb{E}[U] = \mathbb{E}[\mathbb{E}[U \mid N]]$

$\mathbb{E}[U \mid N] = \frac{N}{2} \Rightarrow \mathbb{E}[\mathbb{E}[U \mid N]] = \mathbb{E}[\frac{N}{2}] = \frac{1}{2} * \mathbb{E}[N] = \frac{1}{2} * \lambda$ (Expected value of Poiss($\lambda$) is $\lambda$)

$$\therefore \boldsymbol{Cov(U,N)} = \mathbb{E}[\boldsymbol{UN}] - \mathbb{E}[\boldsymbol{U} \mid \boldsymbol{N}]\mathbb{E}[\boldsymbol{N}] = \left[\frac{(\boldsymbol{\lambda}+\boldsymbol{\lambda}^2)}{\mathbf{2}} - \left[\frac{\boldsymbol{\lambda}}{\mathbf{2}} * \boldsymbol{\lambda}\right]\right] = \frac{\boldsymbol{\lambda}}{\mathbf{2}}$$

**Q5:** Let X=U+V and Y=U+W where U,V, and W are independent Gaussian RVs with $\mathcal{N}(0,1)$. Compute the correlation between X and Y.

**Q5 Solution:**

U,V,W are independent Gaussian RVs with $\mathcal{N}(0,1)$. As linear combination of Gaussian RVs is also Gaussian, X and Y also will be Gaussian RVs with their mean and variance being the same linear combination.

- ⇨ X = $\mathcal{N}\big((0+0),(1+1)\big) = \mathcal{N}(0,2)$ – Mean/Expected value -0 and variance = 2
- ⇨ Y = $\mathcal{N}\big((0+0),(1+1)\big) = \mathcal{N}(0,2)$ – Mean/Expected value -0 and variance = 2

Corr(X,Y)=$\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

Cov(X,Y)=Cov((U+V),(U+W))=$\mathbb{E}[(U+V)(U+W)] - \mathbb{E}[(U+V)]\mathbb{E}[(U+W)]$

$= \mathbb{E}[U^2 + UW + VU + VW] - 0.0$

$= \mathbb{E}[U^2] + \mathbb{E}[UW] + \mathbb{E}[VU] + \mathbb{E}[VW] + 0$

$= \mathbb{E}[U^2] + \mathbb{E}[U]\mathbb{E}[W] + \mathbb{E}[V]\mathbb{E}[U] + \mathbb{E}[V]\mathbb{E}[W] + 0$

$\boldsymbol{Cov(X,Y)} = \mathbb{E}[\boldsymbol{U^2}] + \mathbf{0.0} + \mathbf{0.0} + \mathbf{0.0} + \mathbf{0} = \mathbb{E}[\boldsymbol{U^2}] = \boldsymbol{Var(U)} = \mathbf{1}$

$$\therefore Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{1}{\sqrt{2*2}} = \frac{1}{2}$$

**Q6:** Consider a bus trip where the number of passengers is random. Let the average number of passengers be 40. Compute the upper bound on the probability that the number of passengers will exceed 50.

Now let us add additional information that the variance of number of passengers is 4. What will be the upper bound on the probability of the event that the number of passengers is NOT between 35 and 45 using Chebyshev inequality?

**Q6 Solution:**

Let the RV representing the passengers be $P_B$.

Given:

$- \mathbb{E}[P_B] = 40$

$- \sigma^2 = 4$

The upper bound on the probability that $P_B$ will exceed t=50 is nothing but Markov's Inequality -

$\mathbb{P}(P_B > (t=50)) \leq \frac{\mathbb{E}[P_B]}{(t=50)} = \frac{40}{50} = \frac{4}{5}$

Number of passengers is NOT between 35 and 45 = $35 \geq P_B \geq 45$ ➔ $[(P_B - \mathbb{E}[P_B]) = -5] \geq P_B \geq [(P_B - \mathbb{E}[P_B]) = 5]$ ➔ $|P_B - \mathbb{E}[P_B]| \geq 5$

The upper bound probability that $P_B$ is NOT between 35 and 45 can be computed using Chebyshev's Inequality - $\mathbb{P}(|\boldsymbol{P_B} - \mathbb{E}[\boldsymbol{P_B}]| > \boldsymbol{t}) \leq \frac{\boldsymbol{\sigma^2}}{\boldsymbol{t^2}} \leq \frac{\mathbf{4}}{\mathbf{5^2}} = \frac{\mathbf{4}}{\mathbf{25}}$

**Q7:** Let us assume that each person likes the chocolate ice-cream flavor with probability $p = 0.7$. Let $X_i$ denote the indicator that $i^{th}$ person likes it which is independent of others' choice.



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However, the value of $p$ is not known. A survey is to be conducted on $n$ persons to know the value of $p$ by calculating

$$p' = \frac{\sum_{i=1}^{n} X_i}{n}$$

Assume $n$ to be large enough for the central limit theorem to be valid. What should be the value of $n$ such that the difference between $p'$ and $p$ is within 0.01 with at least 80% probability?
Hint : $\mathbb{P}(|Z| < 1.28) \approx 0.8$ for any $Z \sim \mathcal{N}(0,1)$ ( standard Gaussian RV)

**Q7 Solution:**

$X_i$ is a Bernoulli RV
➔ $\mathbb{E}[X_i] = p = 0.7$
$Var(X_i) = \sigma^2 = p * (1-p) = 0.7 * 0.3 = 0.21$
Given : $\mathbb{P}([|p' - p| \leq 0.01]) \geq 0.8$
Standardizing the deviation $|p' - p|$ by dividing with $\frac{\sigma}{\sqrt{n}}$

➔ $\mathbb{P}\left(\left[\frac{|p' - \mathbb{E}[X_i]|}{\left(\frac{\sigma}{\sqrt{n}}\right)} \leq \frac{0.01}{\frac{\sigma}{\sqrt{n}}}\right]\right) \geq 0.8$

$\frac{|p' - \mathbb{E}[X_i]|}{\frac{\sigma}{\sqrt{n}}} = |Z|$ – Z-score of Gaussian distribution – standard normal distribution

Substituting $\frac{0.01}{\frac{\sigma}{\sqrt{n}}} = z$

$\therefore \mathbb{P}([|Z|] \leq z) \geq 0.8$

Given $\mathbb{P}(|Z| \leq 1.28) \approx 0.8$

$$\therefore 1.28 \leq \frac{0.01}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \sqrt{n} * \frac{0.01}{\sigma} = \frac{\sqrt{n} * 0.01}{\sqrt{Var(X_i)}}$$

$1.28 \leq \frac{\sqrt{n}*0.01}{\sqrt{Var(X_i)}}$ ➔ $\frac{\sqrt{n}*0.01}{\sqrt{Var(X_i)}} \geq 1.28$ ➔ $\sqrt{n} \geq \frac{1.28}{0.01} * \sqrt{Var(X_i)}$

➔ $\boldsymbol{n \geq (128)^2 * Var(X_i) = \ 16384 * 0.21 \cong 3441}$

**Q8:** Consider a random process $Y(t) = X\cos(Xt)$ where X is a uniform RV between 0 and $2\pi$. Compute the mean and auto-correlation of $Y(t)$.

**Q8 Solution:**

Mean of $Y(t) = \mathbb{E}[Y(t)] = \int_0^{2\pi} Y(t) f_X(x) dx$

$X = Uniform(0,2\pi)$ ➔ $f_X(x) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$

$$\therefore \mathbb{E}[Y(t)] = \int_0^{2\pi} x\cos(xt) * \frac{1}{2\pi} dx = \frac{1}{2\pi} \int_0^{2\pi} x\cos(xt) dx$$

The integral of $x\cos(xt)$ with respect to x involves integration by parts as it is a product of two functions. The integration by parts formula is:
$\int u\, dv = uv - \int v\, du$

Here, let's say $u = x$ and $dv = \cos(xt)\, dx$
➔ $du = dx$ and $v = \int \cos(xt)\, dx = \frac{1}{t}\sin(xt)$

$\therefore \frac{1}{2\pi}\int_0^{2\pi} x\cos(xt) dx = \frac{1}{2\pi}\left[\left|x * \left(\frac{1}{t}\sin(xt)\right)\right|_0^{2\pi} - \int_0^{2\pi} \frac{1}{t}\sin(xt)\, dx\right]$

$= \frac{1}{2\pi}\left|\frac{x\sin(xt)}{t} - \left(-\frac{1}{t^2}\cos(xt)\right)\right|_0^{2\pi} = \left|\frac{tx\sin(xt)}{t^2} + \frac{\cos(xt)}{t^2}\right|_0^{2\pi}$

$= \frac{1}{2\pi}\left[\frac{t2\pi\sin(t2\pi)}{t^2} + \frac{\cos(t2\pi)}{t^2} - \frac{t*0*\sin(0*t)}{t^2} - \frac{\cos(0*t)}{t^2}\right]$

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$$= \frac{1}{2\pi}\left[\frac{t2\pi sin(t2\pi)}{t^2} + \frac{\cos(t2\pi)}{t^2} - \frac{1}{t^2}\right] = \frac{1}{2\pi}\left[\frac{t2\pi sin(t2\pi) + \cos(t2\pi) - 1}{t^2}\right]$$
$$= \left[\frac{2\pi t sin(2\pi t) + \cos(2\pi t) - 1}{2\pi t^2}\right]$$

Auto-correlation of $Y(t) = Xcos(Xt)$

$$= R_Y(t_1, t_2) = \mathbb{E}[Y(t_1)Y(t_2)]$$
$$= \mathbb{E}[xcos(xt_1)xcos(xt_2)] = \mathbb{E}[x^2 \cos(xt_1)\cos(xt_2)]$$

Substituting $cos(xt_1)\cos(xt_2) = \frac{1}{2}[\cos(xt_1 + xt_2) + \cos(xt_1 - xt_2)]$

$$= \mathbb{E}\left[x^2\left(\frac{1}{2}\left[\cos\left(x(t_1+t_2)\right) + \cos\left(x(t_1 - t_2)\right)\right]\right)\right] = \frac{1}{2}\mathbb{E}\left[x^2 cos\left(x(t_1+t_2)\right) + x^2\cos\left(x(t_1-t_2)\right)\right]$$
$$= \frac{1}{2}\left[\mathbb{E}\left[x^2 cos\left(x(t_1+t_2)\right)\right] + \mathbb{E}\left[x^2 \cos\left(x(t_1-t_2)\right)\right]\right]$$
$$= \frac{1}{2}\left[\int_0^{2\pi} x^2\cos\left(x(t_1+t_2)\right) f_X(x)dx + \int_0^{2\pi} x^2\cos\left(x(t_1-t_2)\right) f_X(x)dx\right]$$
$$= \frac{1}{2}\left[\int_0^{2\pi} \frac{1}{2\pi} x^2\cos\left(x(t_1+t_2)\right) dx + \int_0^{2\pi} \frac{1}{2\pi} x^2\cos\left(x(t_1-t_2)\right) dx\right]$$
$$= \frac{1}{4\pi}\left[\int_0^{2\pi} x^2\cos\left(x(t_1+t_2)\right) dx + \int_0^{2\pi} x^2\cos\left(x(t_1-t_2)\right) dx\right]$$

Solving using integration by parts,

$$= \frac{(4\pi^2(t_1+t_2)^2 - 2)\sin\left(2\pi(t_1+t_2)\right) + 4\pi tcos(2\pi(t_1+t_2)))}{4\pi(t_1+t_2)^3} +$$
$$\frac{(4\pi^2(t_1-t_2)^2 - 2)\sin\left(2\pi(t_1-t_2)\right) + 4\pi tcos(2\pi(t_1-t_2)))}{4\pi(t_1-t_2)^3}$$

**Q9:** Consider sample space $\Omega = \{1,2,4,5\}$ with all outcomes equi-probable. Consider a random process $X(\omega, t)$ as

$\omega = 1, then\ X(\omega, t) = \sin(t)$
$\omega = 2, then\ X(\omega, t) = \cos(t)$
$\omega = 4\ or\ \omega = 5\ , then\ X(\omega, t) = e^{-t}$

What is the probability that $X(t = 0) = 1$?

**Q9 Solution:**

| $\omega$ | $X(\omega, 0)$ | $P(\omega)$ |
|---|---|---|
| {1} | X(1,0) = Sin(0) = 0 | 1/4 |
| {2} | X(2,0) = cos(0) = 1 | 1/4 |
| {4} | X(4,0) = $e^{-0}$ = 1 | 1/4 |
| {5} | X(5,0) = $e^{-0}$ = 1 | 1/4 |

$$\therefore \mathbb{P}(X(t=0) = 1) = \mathbb{P}[\omega \in \{2,4,5\}] = P(\omega = 2) + P(\omega = 4) + P(\omega = 5) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

**Q10:** A random process $X(t)$ takes the following three realizations with equal probability $m_1(t) = 1, m_2(t) = -3, and\ m_3(t) = 0$. What is the second order finite distribution of this process?

**Q10 Solution:**

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The $n^{th}$ order FDD – finite dimensional distribution – of random process is defined as the joint distribution of $X(\omega, t_1), X(\omega, t_2), X(\omega, t_3), \ldots, X(\omega, t_n)$ and denoted using:

$F_{X_1,X_2,X_3,\ldots X_n}(x_1, x_2, x_3, \ldots x_n) = \mathbb{P}(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3, \ldots, X_n \le x_n)$

$P_{X(t_1),X(t_2),\ldots X(t_n)}(x_1, x_2, \ldots x_n) = \mathbb{P}(X(t_1) \le x_1, X(t_2) \le x_2, X(t_3) \le x_3, \ldots, X(t_n) \le x_n)$

So the second order FDD of $X(t)$

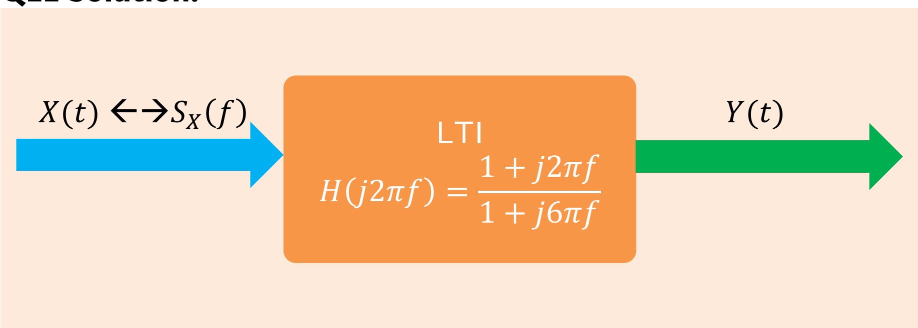
$$P_{X(t_1),X(t_2)}(x_1, x_2) = \mathbb{P}(X(t_1) \le x_1, X(t_2) \le x_2) = \begin{cases} \frac{1}{3}, & x_1 = x_2 = 1 \\ \frac{1}{3}, & x_1 = x_2 = -3 \\ \frac{1}{3}, & x_1 = x_2 = 0 \\ 0, & Otherwise \end{cases}$$

**Q11:** Let $X(t)$ be a WSS random process with PSD given as $S_X(f) = 1\ (|f| < f_0)$. $X(t)$ is applied as input to LTI system with frequency response:

$$H(j2\pi f) = \frac{1 + j2\pi f}{1 + 3j2\pi f}$$

What will be the PSD of the output waveform?

**Q11 Solution:**




$$\boldsymbol{Y(t) = S_X(f) * |H(j2\pi f|^2}$$

$$\boldsymbol{|H(j2\pi f)|^2 = H(j2\pi f) * H(j2\pi f)^* \ (conjugate) = \frac{(1 + j2\pi f)}{(1 + j6\pi f)} * \frac{(1 - j2\pi f)}{(1 - j6\pi f)} = \frac{(1^2 - (j2\pi f)^2)}{(1^2 - (j6\pi f)^2)}}$$

$$\boldsymbol{= \frac{(1 - (-1) * 4\pi^2 f^2)}{(1 - (-1) * 36\pi^2 f^2)} = \frac{(1 + 4\pi^2 f^2)}{(1 + 36\pi^2 f^2)}}$$

$$\boldsymbol{\therefore Y(t) = s_X(f) * |H(j2\pi f)|^2 = \frac{(1 + 4\pi^2 f^2)}{(1 + 36\pi^2 f^2)}\ 1(|f| < f_0)}$$

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