Convex Functions

$\boxed{\text{Zeroth Order Condition}}$

f is convex

required
(a) dom f is convex set
(b) for $x, y \in \text{dom} f$

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

for $\theta \in [0, 1]$

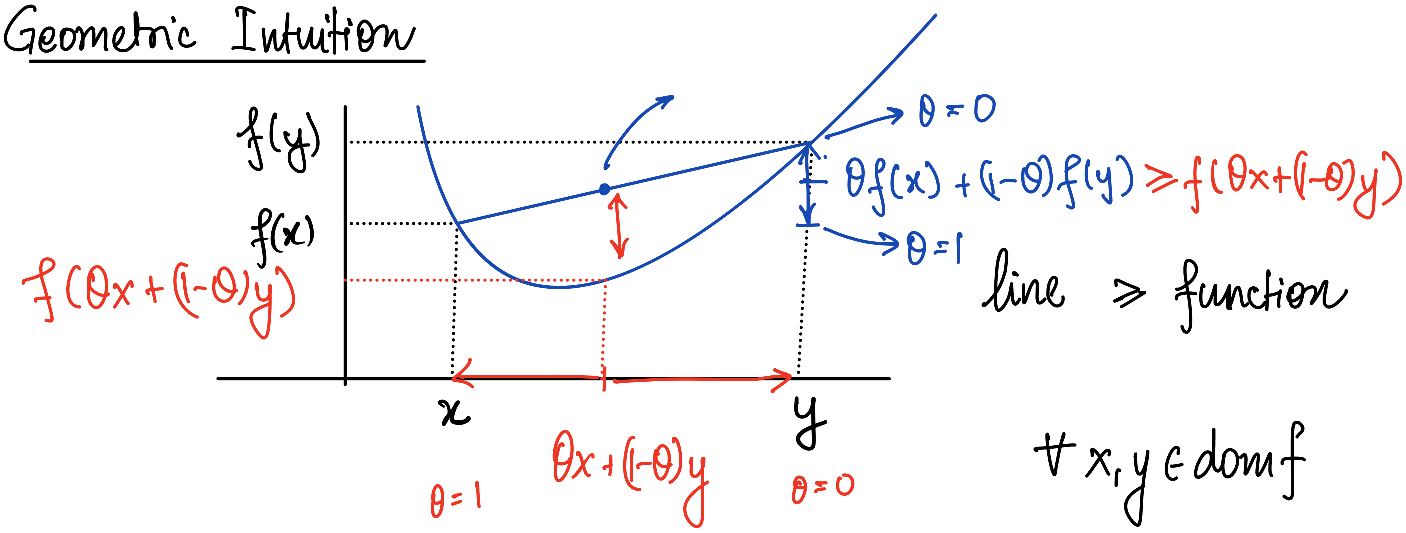
$$x, y \in \text{dom} f \xrightarrow[\text{(convex)}]{} \theta x + (1-\theta) y \in \text{dom} f$$

Eg: $f(x) = 1/x \qquad x \neq 0$

not convex

since $\text{dom} f = \mathbb{R} \setminus \{0\} \leftarrow$ not a convex set

Geometric Intuition



line $\geq$ function

$\forall x, y \in \text{dom} f$

f concave $\Leftrightarrow -f$ convex

(a) dom f is convex set
(b) $f(\theta x + (1-\theta) y) \geq \theta f(x) + (1-\theta) f(y)$

Eg: $f(x) = \|\underline{x}\|$ convex?

(a) $\text{dom} f = \mathbb{R}^n$ convex set

(b) $$f(\theta x + (1-\theta)y) = \|\theta x + (1-\theta)y\|$$
$$\leq \|\theta x\| + \|(1-\theta)y\| \qquad \text{triangle Ineq}$$
$$= |\theta|\,\|x\| + |1-\theta|\,\|y\| \qquad \text{homogeneity}$$
$$= \theta\|x\| + (1-\theta)\|y\| \qquad 0 \leq \theta \leq 1$$
$$= \theta f(x) + (1-\theta) f(y)$$

$\Rightarrow$ f convex

Eg $f(\underline{x}) = \max\{x_1, x_2, \ldots x_n\} = \max\limits_{1\le i\le n}\{x_i\}$

$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \text{dom} f = \mathbb{R}^n$

$\underline{x}, y \in \mathbb{R}^n \qquad z = \theta\underline{x} + (1-\theta)y$

To show: $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$

$$\max_i \{\theta x_i + (1-\theta)y_i\} \le \theta \max_i\{x_i\} + (1-\theta)\max_i\{y_i\}$$

suppose $j_* = \arg\max\limits_i \{\theta x_i + (1-\theta)y_i\}$

or $\max\limits_i \{\theta x_i + (1-\theta)y_i\} = \theta x_{j_*} + (1-\theta)y_{j_*}$

$$\boxed{\text{Also} \quad \begin{aligned} x_{j_*} &\le \max_i\{x_i\} \\ y_{j_*} &\le \max_i\{y_i\} \end{aligned}}$$

$\le \max\limits_i\{\theta x_i\} + \max\limits_i\{\theta y_i\}$

$= \theta \max\limits_i\{x_i\} + (1-\theta)\max\limits_i\{y_i\}$

Jensen's inequality: $f(\sum\limits_i \theta_i x_i) \le \sum \theta_i f(x_i)$

$f(EX) \le Ef(X)$