Please submitted by Saturday, 29 July 2022, 11am, right before the discussion hour.

1. Attempt all problems. There is no penalty for submitting incorrect attempts
2. However, plagiarism will result in serious penalties, such as an F grade.

1. Using concavity of the logarithm, show that $x^\theta y^{1-\theta} \leq \theta x + (1-\theta)y$.
2. Show that the harmonic mean $f(x) = \left(\sum_{i=1}^n 1/x_i\right)^{-1}$ is concave.
3. Prove the reverse Jensen's inequality for a convex f with dom $f = \mathbb{R}^n$, $\lambda_i > 0$ and $\lambda_1 - \sum_{i=2}^n \lambda_i = 1$
$$f(\lambda_1 \mathbf{x}_1 - \lambda_2 \mathbf{x}_2 - \ldots - \lambda_n \mathbf{x}_n) \geq \lambda_1 f(\mathbf{x}_1) - \lambda_2 f(\mathbf{x}_2) - \ldots - \lambda_n f(\mathbf{x}_n) \tag{2}$$
4. Give an example of a function $f(\mathbf{x})$ whose epigraph is (a) half-space, (b) norm cone, and (c) polyhedron.
5. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{++}$ be two vectors. We need to show that the Itakura-Saito distance, defined as
$$D_{IS}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n \left(\frac{x_i}{y_i} - \log\left(\frac{x_i}{y_i}\right) - 1\right) \tag{5}$$
is always positive, using the following steps:
   (a) Show that for a convex differentiable function f, the Bregman divergence,
   $$D(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - f(\mathbf{y}) - \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y}) \tag{6}$$
   is always non-negative.
   (b) Show that for the convex function $f(\mathbf{x}) = -\sum_{i=1}^n \log(x_i)$, it holds that $D(\mathbf{x}, \mathbf{y}) = D_{IS}(\mathbf{x}, \mathbf{y})$.
   (c) Along similar lines, prove that the generalized KL divergence
   $$D_{KL}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n \left(x_i \log\left(\frac{x_i}{y_i}\right) - x_i + y_i\right) \tag{7}$$
   is always positive.