Vectors - inner product, length and their angles

Rohit Budhiraja, IITK

Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Introduced vector, their addition subtraction, and linear combination
- Discussed vectors in different dimensions and their visual interpretation
- Will discuss the following today
  - length and inner products
  - concept of angle between two vectors
  - Cauchy Schwartz inequality
- Reference for today's class - Chap 1.2 of the book

Inner product of two vectors

- Dot/inner product of $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ is the number $\mathbf{v} \cdot \mathbf{w}$:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

- Example 1

$$\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

- If inner product between two vectors is zero – vectors are perpendicular
- Inner product $\mathbf{w} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w}$. Order of $\mathbf{v}$ and $\mathbf{w}$ makes no difference
- Inner product with itself - $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$

$$\mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14$$

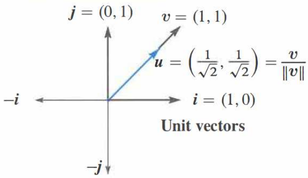
Lengths and unit vectors (1)

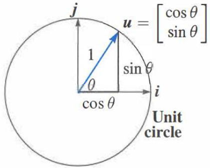
- Length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is the square root of $\mathbf{v} \cdot \mathbf{v}$ i.e.,

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

- In two dimensions, the length is $\sqrt{v_1^2 + v_2^2}$
- In three dimensions, the length is $\sqrt{v_1^2 + v_2^2 + v_3^3}$
- Word unit indicates that some measurement equals one
  - e..g, unit circle has radius one
- A unit vector $\mathbf{u}$ is a vector whose length equals one i.e., $\mathbf{u} \cdot \mathbf{u} = 1$
- Standard unit vectors along the x and y axes are :$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Lengths and unit vectors (2)

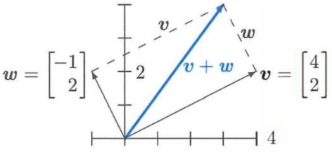




- Unit vector $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ is a unit vector in the same direction as $\mathbf{v}$
- Unit vector that makes an angle θ with x axis is $\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$
  - Obtained by rotating unit vector $\mathbf{i} = (1,0)$
- At any angle, components $\cos\theta$ and $\sin\theta$ produce $\mathbf{u} \cdot \mathbf{u} = 1$

Inner product for perpendicular vectors (1)

- Inner product is $\mathbf{v} \cdot \mathbf{w} = 0$ when $\mathbf{v}$ is perpendicular to $\mathbf{w}$. We now prove it
- Pythagoras law for the sides of a right triangle is $a^2 + b^2 = c^2$
- When $\mathbf{v}$ and $\mathbf{w}$ are perpendicular, they form two sides of a right triangle
- Third side is $\mathbf{v} + \mathbf{w}$



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Inner product for perpendicular vectors (2)

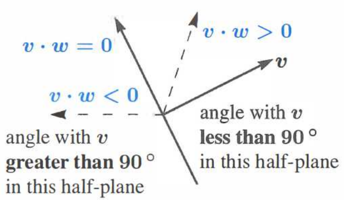
- By using the Pythagoras theorem, we have

$$\begin{aligned} \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 &= \|\mathbf{v} + \mathbf{w}\|^2 \\ \left(v_1^2 + v_2^2\right) + \left(w_1^2 + w_2^2\right) &= \left(v_1 + w_1\right)^2 + \left(v_2 + w_2\right)^2 \\ v_1 w_1 + v_2 w_2 &= 0 \end{aligned}$$

- If $\mathbf{v} \cdot \mathbf{w}$ is not zero, it may be positive or negative
  - Sign of $\mathbf{v} \cdot \mathbf{w}$ tells us whether we are above or below a right angle
- Angle is less than 90^0 when $\mathbf{v} \cdot \mathbf{w}$ is $+$ve and vice-versa

Angle Between Two Vectors

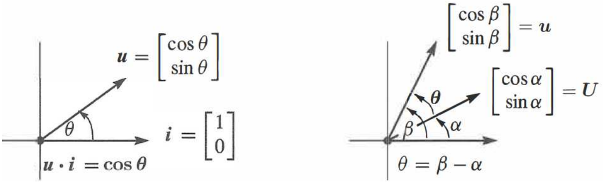
- Consider vectors $\mathbf{v} = (3, 1)$ and $\mathbf{w} = (1, 3)$, angle is less than 90^o
  - because $\mathbf{v} \cdot \mathbf{w} = 6$ is positive



- Borderline is where vectors are perpendicular to $\mathbf{v}$ i.e, $\mathbf{v} \cdot \mathbf{w} = 0$
  - This is dividing line between positive and negative values of inner product
  - $\mathbf{w} = (1, -3)$ is perpendicular to $\mathbf{v} = (3, 1)$

Inner product between two unit vectors (1)

- For two unit vectors, inner product reveals the exact angle θ between them
- Figure shows this clearly for vector $\mathbf{u} = (\cos\theta, \sin\theta)$ and $\mathbf{i} = (1, 0)$



- Inner product $\mathbf{u} \cdot \mathbf{i} = \cos\theta$. Both are unit vectors
- We rotate $\mathbf{i} = (1, 0)$ to $(\cos\alpha, \sin\alpha)$, and $\mathbf{u} = (\cos\beta, \sin\beta)$ with $\beta = \theta + \alpha$
- Inner product now becomes

$$\mathbf{u} \cdot \mathbf{i} = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\beta - \alpha) = \cos\theta$$

Inner product between two unit vectors (2)

- What if **v** and **w** are not unit vectors - divide by their lengths

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \cos\theta$$

- We know that

$$\begin{aligned} -1 \leq \cos\theta &\leq 1 \\ \Rightarrow |\cos\theta| &\leq 1 \\ \frac{|\mathbf{v} \cdot \mathbf{w}|}{\|\mathbf{v}\|\|\mathbf{w}\|} &= |\cos\theta| \\ |\mathbf{v} \cdot \mathbf{w}| &\leq \|\mathbf{v}\|\|\mathbf{w}\| \end{aligned}$$

- Above inequality is famous Cauchy Schwartz inequality

Review of important ideas

- Inner product $\mathbf{v} \cdot \mathbf{w}$ multiplies each component v_i by w_i and adds all $v_i w_i$
- Length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is the square root of $\mathbf{v} \cdot \mathbf{v}$
  - then $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$ is a unit vector : length 1
- Dot/inner product $\mathbf{v} \cdot \mathbf{w} = 0$ when $\mathbf{w}$ and $\mathbf{v}$ are orthogonal
- Cosine of θ (the angle between any nonzero $\mathbf{v}$ and $\mathbf{w}$) never exceeds 1:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

- This leads to Cauchy Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$$

- End of Section 1.2 please solve problems at the end