EE910: Digital Communication Systems-I

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

April 18, 2022

Lecture #2F: A brief introduction to random processes

Random processes

- The mean and the autocorrelation of a random $X(t)$ are defined as

$$m_x(t) = E[X(t)] \tag{1}$$

$$R_x(t_1, t_2) = E[X(t_1)X^*(t_2)] \tag{2}$$

- The cross-correlation of two random process $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y^*(t_2)] \tag{3}$$

- We have $R_X(t_2, t_1) = R_X^*(t_2, t_1)$, i.e., $R_X(t_1, t_2)$ is Hermitian.
- For cross-correlation $R_{XY}(t_1, t_2) = R_{XY}^*(t_1, t_2)$

Wide-Sense Stationary Random Processes

- Random process $X(t)$ is wide-sense stationary (WSS) if its mean is constant and autocorrelation is a function of time difference i.e

$$R_X(t_1, t_2) = R_X(\tau) \tag{4}$$

where $\tau = t_1 - t_2$
- Two processes $X(t)$ and $Y(t)$ are jointly WSS if both $X(t)$ and $Y(t)$ are WSS and $R_{XY}(t_1, t_2) = R_{XY}(\tau)$
- A complex process is WSS if its real and imaginary parts are WSS

WSS Processes

- For a WSS process, the power spectrum is the Fourier transform of the autocorrelation function $R_X(\tau)$

$$S_X(f) = \mathfrak{F}[R_X(\tau)] \tag{5}$$

- Cross spectral density of two jointly WSS processes is defined as

$$S_{XY}(f) = \mathfrak{F}[R_{XY}(\tau)] \tag{6}$$

- The CSD satisfies the following property:

$$S_{XY}(f) = S^*_{XY}(f) \tag{7}$$

WSS Processes

- If $X(t)$ and $Y(t)$ are jointly WSS random processes, then $Z(t) = aX(t) + bY(t)$ is a WSS process with

$$R_Z(\tau) = |a|^2 R_X(\tau) + |b|^2 R_Y(\tau) + ab^* R_{XY}(\tau) + ba^* R_{YX}(\tau) \tag{8}$$

$$S_Z(f) = |a|^2 S_X(f) + |b|^2 S_Y(f) + 2Re[ab^* S_{XY}(f)] \tag{9}$$

WSS Processes

- When a WSS process $X(t)$ passes through an LTI system with impulse response $h(t)$ and transfer function $H(f) = \mathfrak{F}[h(t)]$, the output processes $Y(t)$ and $X(t)$ are jointly WSS and the following relations hold

$$\begin{aligned} m_Y &= m_X \int_{-\infty}^{\infty} h(t)dt && (10) \\ R_{XY}(\tau) &= R_X(t) * h^*(-\tau) && (11) \\ R_Y(\tau) &= R_X(t) * h(\tau) * h^*(-\tau) && (12) \\ m_Y &= m_X H(0) && (13) \\ S_{XY}(f) &= S_X H^*(f) && (14) \\ S_Y(f) &= S_X(f)|H(f)|^2 && (15) \end{aligned}$$

WSS Processes

- The power in a WSS process $X(t)$ is the sum of the power at all frequencies

$$P_X = E[|X(t)|^2] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df \qquad (16)$$

Gaussian Random Processes

- A real random process $X(t)$ is Gaussian if for all positive integers n and for all $(t_1, t_2, \ldots, t_n)$,the random vector $(X(t_1), X(t_2), \ldots, X(t_n))^t$ is Gaussian random vector.
- Two real random processes $X(t)$ and $Y(t)$ are jointly Gaussian if for all positive integers n, m and all $(t_1, t_2, \ldots, t_n)$, and $(t_1^{'}, t_2^{'}, \ldots, t_m^{'})$ the random vector

$$(X(t_1), X(t_2), \ldots, X(t_n), Y(t_1^{'}), Y(t_2^{'}), \ldots, Y(t_m^{'}))^t \tag{17}$$

is a Gaussian vector.

White Processes

- A process is called a white process if its power spectral density is constant for all frequencies

$$S_X(f) = \frac{N_o}{2} \tag{18}$$

- Power in a white process is infinite, indicating that white process cannot exists as a physical process

Discrete-Time Random Processes

- PSD of a WSS discrete time random process is defined as the discrete-time Fourier transform of its autocorrelation function

$$S_X(f) = \sum_{m=-\infty}^{\infty} R_X(m) \exp^{-j2\pi fm} \tag{19}$$

- The power in discrete time random process is given by

$$P = E[|X(n)|^2] = R_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) df \tag{20}$$

Cyclostationary Random Processes

- A random process $X(t)$ is cyclostationary if its mean and autocorrelation are periodic functions with same period T_o

$$m_X(t + T_o) = m_X(t) \tag{21}$$

$$R_X(t_1 + T_o, t_2 + T_o) = R_X(t_1, t_2) \tag{22}$$

- Average autocorrelation function defined as

$$\overline{R_X}(t) = \frac{1}{T_o} \int_0^{T_o} R_X(t + \tau, t) \tag{23}$$

Proper and Circular Random Processes

- For a Complex Random process $Z(t) = X(t) + jY(t)$, the covariance and the pseudo covariance are

$$C_Z(t+\tau, t) = E[Z(t+\tau)Z^*(t)] \tag{24}$$

$$\tilde{C}_Z(t+\tau, t) = E[Z(t+\tau)Z(t)] \tag{25}$$

which can be written as

$$C_Z(t+\tau, t) = C_X(t+\tau, t) + C_Y(t+\tau, t) + j(C_{XY}(t+\tau, t) - C_YX(t+\tau, t)) \tag{26}$$

$$\tilde{C}_Z(t+\tau, t) = C_X(t+\tau, t) - C_Y(t+\tau, t) + j(C_{XY}(t+\tau, t) + C_YX(t+\tau, t)) \tag{27}$$

- A complex random process $Z(t)$ is proper if its pseudocovariance is zero

Bandpass and Lowpass Random Processes

- For a bandpass process the power spectral density is located around frequencies $\pm f_o$ and for lowpass processes the PSD is located around zero frequency
- The in-phase and quadrature components of a bandpass random process $X(t)$ is

$$X_i(t) = X(t)\cos 2\pi f_o t + \hat{X}(t)\sin 2\pi f_o t \tag{28}$$

$$X_q(t) = \hat{X}(t)\cos 2\pi f_o t - X(t)\sin 2\pi f_o t \tag{29}$$

Bandpass and Lowpass Random Processes

- $X_i(t)$ and $X_q(t)$ are jointly WSS zero-mean random process
- $X_i(t)$ and $X_q(t)$ have the same power spectral density
- $X_i(t)$ and $X_q(t)$ are both lowpass process
- We define lowpass equivalent process $X_l(t)$ as

$$X_l(t) = X_i(t) + jX_q(t) \tag{30}$$

Bandpass and Lowpass Random Processes

- Since $X(t)$ by assumption is zero mean so is $\hat{X}(t)$, its Hilbert transform. Therefore $X_i(t)$ and $X_q(t)$ are both zero-mean process
- Autocorrelation of $X_i(t)$ is

$$\begin{aligned} R_{x_i}(t+\tau, t) &= E[(X_i(t+\tau)X_i(t)] \\ &= E[X(t+\tau)\cos 2\pi f_o(t+\tau) + \hat{X}(t+\tau)\sin 2\pi f_o(t+\tau)) \\ &\times X(t)\cos 2\pi f_o t + \hat{X}(t)\sin 2\pi f_o t] \end{aligned} \tag{31}$$

on solving it we get

$$R_{X_i}(\tau) = R_x(\tau)\cos(2\pi f_o \tau) + \hat{R}_X(\tau)\sin(2\pi f_o \tau) \tag{32}$$

Bandpass and Lowpass Random Processes

- Similarly we have

$$R_{X_q}(\tau) = R_{X_i}(\tau) = R_x(\tau)\cos(2\pi f_o \tau) + \hat{R}_X(\tau)\sin(2\pi f_o \tau) \tag{33}$$

$$R_{X_i X_q}(\tau) = -R_{X_q X_i}(\tau) = R_x(\tau)\sin(2\pi f_o \tau) - \hat{R}_X(\tau)\cos(2\pi f_o \tau) \tag{34}$$

- Power Spectral densities would be

$$S_{X_i} = S_{X_q} = \begin{cases} S_X(f + f_o) + S_X(f - f_o) & |f| < f_o \\ 0 & otherwise \end{cases} \tag{35}$$

$$S_{X_i X_q} = -S_{X_i X_q} = \begin{cases} j\left\lfloor S_X(f + f_o) - S_X(f - f_o) \right\rfloor & |f| < f_o \\ 0 & otherwise \end{cases} \tag{36}$$

Bandpass and Lowpass Random Processes

- The complex process $X_l(t) = X_i(t) + jX_q(t)$ as the lowpass equivalent of $X(t)$. Since $X_i(t)$ and $X_q(t)$ are both lowpass processes, we conclude that $X_l(t)$ is also a lowpass process

$$\begin{aligned} R_{X_l}(\tau) =& 2R_{X_i}(\tau) + 2jR_{X_q X_i}(\tau) \\ =& 2[R_X(\tau) + j\hat{R}_X(\tau)]e^{-j2\pi f_o t} \end{aligned} \tag{37}$$

- Taking Fourier transform on both sides we have

$$S_{X_l} = \begin{cases} 4S_X(f + f_o) & |f| < f_o \\ 0 & otherwise \end{cases} \tag{38}$$

$$S_X(f) = \frac{1}{4}[S_{X_l}(f - f_o) + S_{X_l}(f + f_o)] \tag{39}$$