Complete solution to $A\mathbf{x} = \mathbf{b}$

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Applied Linear Algebra for Wireless Communications

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  - Systematically calculated $N(A)$ of matrix A

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- Discuss the following today
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Complete Solution to $A\mathbf{x} = b$[1]

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- One way is to create an augmented matrix $[A\ \mathbf{b}]$

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  - By reversing signs of $3, 2,$ and 4 which gives $(-3, 1, 0, 0)$ and $(-2, 0, -4, 1)$

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- Complete solution $\mathbf{x}_p + \mathbf{x}_n$ to $A\mathbf{x} = b$

Complete solution
one x_p
many x_n

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Example1 to find complete solution to $Ax = b$ (1)

- Consider an example system of equations

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- Complete solution is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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  - There are no free variables or special solutions
  - Nullspace $N(A)$ contains only the zero vector $\mathbf{x} = 0$
  - If $A\mathbf{x} = \mathbf{b}$ has a solution (it might not) then it has only one solution

Example2 to find complete solution to $Ax = b$ (1)

- Consider another system $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} x + y + z &= 3 \\ x + 2y - z &= 4 \end{aligned}$$

- Has $n = 3$ unknowns but only $m = 2$ equations

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