Live Interaction #4:

E-masters Communication Systems

Estimation for Wireless

- **MIMO Channel Estimation**:

$$\bar{\mathbf{y}}(k) = \mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{v}}(k)$$

- Pilots

$$\bar{\mathbf{y}}(1) = \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1)$$
$$\bar{\mathbf{y}}(2) = \mathbf{H}\bar{\mathbf{x}}(2) + \bar{\mathbf{v}}(2)$$
$$\vdots$$
$$\bar{\mathbf{y}}(N) = \mathbf{H}\bar{\mathbf{x}}(N) + \bar{\mathbf{v}}(N)$$

- Write in matrix form:

$$\underbrace{[\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \dots \quad \bar{\mathbf{y}}(N)]}_{\mathbf{Y}} =$$
$$\mathbf{H}\,\underbrace{[\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \dots \quad \bar{\mathbf{x}}(N)]}_{\mathbf{X}}$$
$$+\,\underbrace{[\bar{\mathbf{v}}(1) \quad \bar{\mathbf{v}}(2) \quad \dots \quad \bar{\mathbf{v}}(N)]}_{\mathbf{V}}$$

$$\underbrace{\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{V}}_{\substack{\text{MIMO Channel} \\ \text{Estimation model}}}$$

- **MIMO Channel Estimate:**

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^H\left(\mathbf{X}\mathbf{X}^H\right)^{-1}$$

- $\mathbf{X}$: ***Wide matrix***. Number of columns is greater than number of rows.

$$\mathbf{X} \times \mathbf{X}^H (\mathbf{X}\mathbf{X}^H)^{-1} = \mathbf{I}$$

- Example:

$$\mathbf{X} = \underbrace{\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{r\times N = 2\times 4}$$

$$\mathbf{Y} = \underbrace{\begin{bmatrix} 2 & -3 & 1 & -2 \\ -1 & -2 & 2 & 3 \end{bmatrix}}_{t\times N = 2\times 4}$$

- What is the size of the MIMO system?
- Find the MIMO Channel estimate?

$$\mathbf{X}\mathbf{X}^H = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(\mathbf{X}\mathbf{X}^H)^{-1} = \frac{1}{4}\mathbf{I}$$

$$\mathbf{Y}\mathbf{X}^H (\mathbf{X}\mathbf{X}^H)^{-1}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -3 & 1 & -2 \\ -1 & -2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\hat{\mathbf{H}} = \frac{1}{4}\begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$$

- **Equalization**:
- We perform equalization when we have ISI.

$$y(k) = h(0)x(k) + h(1)x(k-1) + \cdots$$

$$+h(L-1)x(k-L+1)+v(k)$$

- Output $y(k)$ depends also on the past symbols
- To suppress this ISI, we need equalization

$$y(k+1)=h(0)x(k+1)+h(1)x(k)+v(k+1)$$
$$y(k)=h(0)x(k)+h(1)x(k-1)+v(k)$$

- We can write this in matrix form as

$$\underbrace{\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}}_{\bar{\mathbf{y}}}=\underbrace{\begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix}}_{\mathbf{H}}\underbrace{\begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}}_{\bar{\mathbf{x}}}+\underbrace{\begin{bmatrix} v(k+1) \\ v(k) \end{bmatrix}}_{\bar{\mathbf{v}}}$$

- Equalizer: linearly combine the symbols

$$c_0 y(k+1)+c_1 y(k)$$
$$=\begin{bmatrix} c_0 & c_1 \end{bmatrix}\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}=\bar{\mathbf{c}}^T\bar{\mathbf{y}}$$
$$\bar{\mathbf{c}}^T\bar{\mathbf{y}}=\bar{\mathbf{c}}^T(\mathbf{H}\bar{\mathbf{x}}+\bar{\mathbf{v}})$$
$$=\bar{\mathbf{c}}^T\mathbf{H}\bar{\mathbf{x}}+\bar{\mathbf{c}}^T\bar{\mathbf{v}}$$
$$=\bar{\mathbf{c}}^T\mathbf{H}\begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}+\bar{\mathbf{c}}^T\bar{\mathbf{v}}$$
$$\bar{\mathbf{c}}^T\mathbf{H}=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$
$$\Rightarrow \mathbf{H}^T\bar{\mathbf{c}}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Solve the approximation problem

$$\min\left\|\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}-\mathbf{H}^T\bar{\mathbf{c}}\right\|^2$$

- Solution to the Least Squares problem above is given as

$$\bar{\mathbf{c}} = \left(\mathbf{H}^{T^T}\mathbf{H}^T\right)^{-1}\mathbf{H}^{T^T}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\underbrace{\bar{\mathbf{c}} = (\mathbf{H}\mathbf{H}^T)^{-1}\mathbf{H}\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{Zero–Forcing Equalizer}}$$

- Homework:

$$y(k) = x(k) + \frac{1}{5}x(k-1) + v(k)$$

- Determine the **equalizer** $\bar{\mathbf{c}}$ for this.

$$\mathbf{H} = \begin{bmatrix}1 & \frac{1}{5} & 0\\ 0 & 1 & \frac{1}{5}\end{bmatrix}$$

$$(\mathbf{H}\mathbf{H}^T)^{-1}\mathbf{H}\begin{bmatrix}0\\1\\0\end{bmatrix} = ?$$