SDP Dual form

$$\min \ c^T x \qquad \text{or} \qquad \min \ c^T x$$
$$\lambda_{max}(F(x)) \le 0 \qquad\qquad F(\underset{\sim}{x}) \preceq 0$$

(not in standard convex form)

Matrix Dual variable: $Y \in S^n$

$$L(x, Y) = c^T x + \langle F(x), Y \rangle$$
$$= c^T x + \sum_{i,j} [F(x)]_{ij} Y_{ij}$$
$$= c^T x + Tr(GY) + \sum_{i=1}^{n} x_i Tr(F_i Y)$$

observe: $L(x,Y)$ affine in x & Y

$$\min_x L(x, Y) = \sum_{i=1}^{n} \min_{x_i} x_i (c_i + Tr(F_i Y)) + Tr(GY)$$

$$\min_{x_i} x_i (c_i + Tr(F_i Y)) = \begin{cases} 0 & c_i + Tr(F_i Y) \geq 0 \\ -\infty & o/w \end{cases}$$

(dual)

$$\max \ Tr(GY)$$
$$c_i + Tr(F_i Y) = 0$$
$$Y \succeq 0 \qquad \text{P.S.D.}$$

Dual of SDP is also an SDP

$$y = \mathrm{vec}(Y) \in \mathbb{R}^{n^2}$$

$$\mathrm{Tr}(GY) = g^T y \qquad g = \mathrm{vec}(G)$$

$$\mathrm{Tr}(F_i Y) = f_i^T y$$

(i,j)-th entry

$$\text{Then } Y = \underbrace{\sum_{i,j} 1_{ij} Y_{ij} \succeq 0}_{\text{LMI form}} \qquad 1_{ij} = \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix}$$

$$\begin{aligned} &\max\ g^T y \\ &f_i^T y + c_i = 0 \\ &H(y) = \sum 1_{ij} Y_{ij} \succeq 0 \end{aligned}$$

also LMI