EE908 Assignment-3 Solution

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eMasters in Communication Systems, IITK
EE908: Optimization in SPCOM
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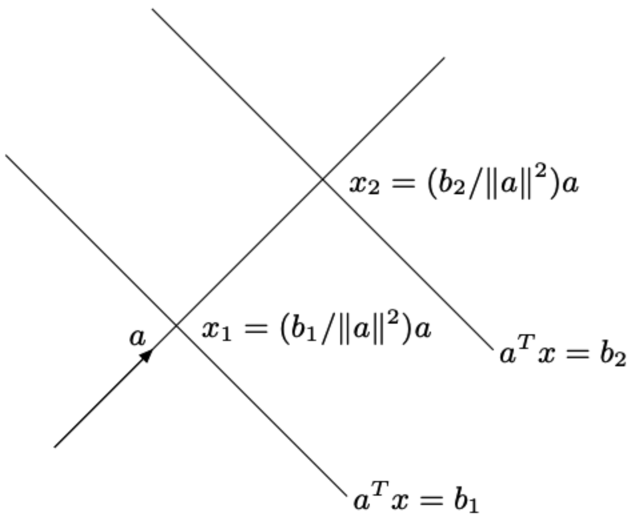
Q1. What is the minimum distance between two parallel half-spaces $\{x \in \mathbb{R}^n \mid a^T x \leq b_1\}$ and $\{x \in \mathbb{R}^n | a^T x \geq b_2 \}$
Solution:



The minimum distance occurs when the first half-space is $\{x \in \mathbb{R}^n \mid a^T x = b_1\}$ and the second-half space is $\{x \in \mathbb{R}^n | a^T x = b_2 \}$

The distance between two parallel hyperplanes = Distance between two points x_1 and $x_2$where the hyperplane intersects the line through the origin and parallel to the normal vector a.
The minimum distance is: $\|x_1 - x_2\|_2 = \frac{|b_1 - b_2|}{\|a\|_2}$

Q2. Is the following set affine: $\{x \in \mathcal{R}^n : \|x - x_1\|_1 \leq \|x - x_2\|_1\}$?
Solution:
$\|x - x_1\|_1 = |x - x_1|$ – Manhattan distance

$$\sum_{i=}^{n} |x_i - x_{1i}| \leq \sum_{i=}^{n} |x_i - x_{2i}|$$

If $x_1 = x_2$, set is affine as $\sum_{i=}^{n} |x_i - x_{1i}| = \sum_{i=}^{n} |x_i - x_{1i}|$
If $x_1 \neq x_2$ set may not necessarily affine.

Q3. Is the following set affine: $\{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$?
Solution:
Say $\mathcal{H} = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$
Let x_1 and x_2 be two points in the set $\mathcal{H} \Rightarrow x_1, x_2 \in \mathbb{R}^n$ and $\|x_1\|_\infty \leq 1$ and $\|x_2\|_\infty \leq 1$
$x_i = \theta x_1 + (1 - \theta) x_2, \theta \in \mathbb{R}^n$
$\Rightarrow \|x_i\|_\infty = \|\theta x_1 + (1 - \theta) x_2\|_\infty$

$\|x_i\|_\infty = \max|\theta x_1 + (1-\theta)x_2| \leq \theta|x_1| + (1-\theta)|x_2| \leq \theta + 1 - \theta \leq 1$
This shows that every point on the line segment between $x_1 and\ x_2$ lies within or on the boundary of set $\{x \in \mathcal{R}^n : \|x\|_\infty \leq 1\}$.
Hence it is an affine set.

Q4. Given θ, consider the set $\mathcal{S} = \{x \in \mathbb{R}^n | \|x-a\|_2 \leq \theta\|x-b\|_2\}$ for $a \neq b$. Show that $\mathcal{S}$ is half-space for $\theta = 1$, convex for $\theta < 1$. Give an example to prove that $\mathcal{S}$ can be non-convex for $\theta > 1$.
Solution:
If, $\boldsymbol{\theta} = \mathbf{1}$:
$\Rightarrow \mathcal{S} = \{x \in \mathbb{R}^n : \|x-a\|_2 \leq \|x-b\|_2, a \neq b\}$
This represents all points x such that the Euclidean distance from x to a is less than or equal to that of x to b. Geometrically, this represents the half-space on one side of the hyperplane perpendicular to the line segment connecting a and b.
QED
Thus $\mathcal{S}$ is half-space.
If, $\boldsymbol{\theta} < \mathbf{1}$:
Let's consider two points x_1, x_2 in $\mathcal{S} \Rightarrow \|x_1 - a\|_2 \leq \theta\|x_1 - b\|_2$ and $\|x_2 - a\|_2 \leq \theta\|x_2 - b\|_2$
To be convex, for any $t \in [0,1]$ consider these two points in a line $tx_1 + (1-t)x_2$
Using triangle inequality:
$\|tx_1 + (1-t)x_2 - a\|_2 \leq t\|x_1 - a\|_2 + (1-t)\|x_2 - a\|_2 \leq t\theta\|x_1 - b\|_2 + (1-t)\theta\|x_2 - b\|_2$
$= \theta(t\|x_1 - b\|_2 + (1-t)\|x_2 - b\|_2 = \theta\|tx_1 + (1-t)x_2 - b\|_2$
$\therefore tx_1 + (1-t)x_2$ also belongs to $\mathcal{S} \Rightarrow \mathcal{S}$ is convex for $\theta < 1$
QED
$\mathcal{S}$ can be non-convex for $\boldsymbol{\theta} > \mathbf{1}$:
Let's consider to points in $a = (0,0)$ and $b = (2,0)$ and $\theta = 2$ in $\mathcal{S} \in \{x \in \mathbb{R}^2 : \|x\|_2 \leq 2\|x - (2,0)\|_2\}$
Expanding the inequality,

$$\mathcal{S} = \left\{x \in \mathbb{R}^2 : \|x\|_2 \leq 2\sqrt{(x_1 - 2)^2 + x_2^2}\right\}$$

This geometrically represents the region enclosed by the circle at (2,0) with radius twice the distance from (2,0) to the origin which includes the origin with it. This region is non-convex as it contains a hole – the origin.
QED

Q5. Show that the intersection of two convex cones is a convex cone.
Solution:
$x, y \in \mathcal{C}$, then $\mathcal{C}$ convex cone $\Leftrightarrow \theta_1 x + \theta_2 y \in C$, $\theta_1, \theta_2 > 0$
If $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$,
- $x, y \in \mathcal{C} \Rightarrow \theta_1 x + \theta_2 y \in \mathcal{C}$
- $\theta_1 x + \theta_2 y \in \mathcal{C}_1 \cap \mathcal{C}_2$
- $\theta_1 x + \theta_2 y \in \mathcal{C}_1 and\ \theta_1 x + \theta_2 y \in \mathcal{C}_2 for\ any\ \theta_1, \theta_2 > 0$

Hence $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$ is convex cone.
QED

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