eMasters in Communication Systems

Prof. Aditya Jagannatham

Elective Module:

Detection for Wireless Communication

Chapter 9

GLRT

GLRT
Generalized Likelihood
Ratio Test

GLRT

- GLRT stands for Generalized Likelihood Ratio Test

GLRT

GLRT

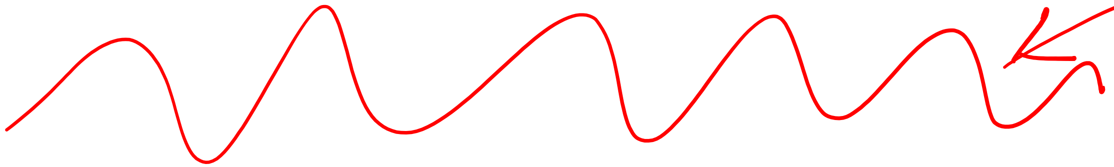
- GLRT stands for **Generalized Likelihood Ratio Test.**

GLRT

- This is for Detection of Signals with

UNKNOWN PARAMETERS:

Ex: Signal detection with unknown carrier frequency



f_c is unknown

$f_c \in [B_L, B_H]$

GLRT

Occurs frequently in practice.

- This is for Detection of Signals with **Unknown Parameters.**

GLRT

- Consider the ***binary hypothesis testing*** problem

NULL hypothesis
Signal Absent

Noise vector

unknown parameter
Scaling factor
EX: carrier Amplitude.
EX: channel coefficient
complex A.

$$\mathcal{H}_0 : \bar{y} = \bar{v}$$

$$\mathcal{H}_1 : \bar{y} = A\bar{s} + \bar{v}$$

$$\bar{v} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

Gaussian iid
mean = 0
var = σ^2

Alternative Hypothesis

known signal.

$$\bar{s} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix}$$

known signal. Vector

GLRT

Modified signal detection problem.

- Consider the ***binary hypothesis testing*** problem

unknown Amplitude.

$$\mathcal{H}_0 : \bar{\mathbf{y}} = \bar{\mathbf{v}}$$

$$\mathcal{H}_1 : \bar{\mathbf{y}} = A\bar{\mathbf{s}} + \bar{\mathbf{v}}$$

GLRT

- $\bar{s}$ is a known signal.
- A is an unknown scaling factor

GLRT

- $\bar{\mathbf{s}}$ is a **known signal**.
- A is an unknown ***scaling factor***.

GLRT

- **Likelihood Ratio Test (LRT)** for this problem.
- Choose $\mathcal{H}_1$ if

Likelihood corresponding to $\mathcal{H}_1$ with unknown parameter A

$$\frac{p(\bar{y}; A, \mathcal{H}_1)}{p(\bar{y}; \mathcal{H}_0)} > \tilde{\gamma}$$

Likelihood corresponding to $\mathcal{H}_0$

Choose H_1 if:

$$\frac{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}}} e^{-\frac{\|\bar{y}-A\bar{s}\|^2}{2\sigma^2}}}{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}}} e^{-\frac{\|\bar{y}\|^2}{2\sigma^2}}} > \tilde{\gamma}$$

Taking ln on both sides:

$$\|\bar{y}\|^2 - \|\bar{y} - A\bar{s}\|^2 > 2\sigma^2 \ln\tilde{\gamma}$$

$$\Rightarrow \cancel{\|\bar{y}\|^2} - \left(\cancel{\|\bar{y}\|^2} + A^2\|\bar{s}\|^2 - 2A\bar{s}^T\bar{y}\right) > 2\sigma^2 \ln\tilde{\gamma}$$

$$\Rightarrow -A^2 \|\bar{s}\|^2 + 2A\bar{s}^T\bar{y} > 2\sigma^2 \ln \tilde{\gamma}$$

$\Rightarrow$ choose $\mathcal{H}_1$ if

$$\underline{2A\bar{s}^T\bar{y} - A^2\|\bar{s}\|^2 > 2\sigma^2 \ln \tilde{\gamma}}$$

how to choose A?

GLRT

- **Likelihood Ratio Test (LRT)** for this problem.
- Choose $\mathcal{H}_1$ if

$$\frac{p(\bar{\mathbf{y}}; A, \mathcal{H}_1)}{p(\bar{\mathbf{y}}; \mathcal{H}_0)} > \tilde{\gamma}$$

$$\Rightarrow \frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\|\bar{\mathbf{y}}-A\bar{\mathbf{s}}\|^2}}{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\|\bar{\mathbf{y}}\|^2}} > \tilde{\gamma}$$

GLRT

$$\Rightarrow \frac{1}{2\sigma^2}\left(\|\bar{\mathbf{y}}\|^2 - \|\bar{\mathbf{y}} - A\bar{\mathbf{s}}\|^2\right) > \ln\tilde{\gamma}$$

$$\Rightarrow 2A\,\bar{\mathbf{s}}^T\bar{\mathbf{y}} - A^2\|\bar{\mathbf{s}}\|^2 > 2\sigma^2\ln\tilde{\gamma}$$

GLRT

unknown scaling factor

- What value of A to choose?

GLRT

- Choose the value of A such that…

choose Maximum Likelihood (ML) Estimate of A

$$\hat{A} = \arg\max \; p(\bar{y}; A, \mathcal{H}_1)$$

choose value of parameter corresponding that maximizes likelihood to unknown parameter.

GLRT

- Choose the value of A such that…

$$\underbrace{\hat{A} = \arg\max p(\bar{\mathbf{y}}; A, \mathcal{H}_1)}_{\text{ML Estimate of A}}$$

GLRT

- The LRT is modified as

Likelihood Ratio Test

Choose $\mathcal{H}_1$ if $\dfrac{p(\bar{y};\hat{A}_{ML},\mathcal{H}_1)}{p(\bar{y};\mathcal{H}_0)} > \tilde{\gamma}$

GLRT

- The LRT is modified as

$$\frac{p(\bar{\mathbf{y}}; \hat{A}_{ML}, \mathcal{H}_1)}{p(\bar{\mathbf{y}}; \mathcal{H}_0)} \geq \tilde{\gamma}$$

Substitute ML Estimate in LRT

This is termed GLRT.

GLRT

- This is termed GLRT i.e. Generalized Likelihood Ratio Test.

GLRT

- This is termed GLRT – **Generalized Likelihood Ratio Test.**

GLRT

- **ML estimate** of A can be found as follows

$$\hat{A} = \arg\max \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\|\bar{y}-A\bar{s}\|^2}{2\sigma^2}}$$

$$= \arg\min \frac{\|\bar{y}-A\bar{s}\|^2}{2\sigma^2}$$

$$= \arg\min \|\bar{y}-A\bar{s}\|^2$$

Least Squares (LS) Problem.

GLRT

$$\bar{s}^T\bar{s} = \|\bar{s}\|^2$$

$$\hat{A} = \arg\min \|\bar{y} - A\bar{s}\|^2$$

$$(\bar{s}^T\bar{s})^{-1}\bar{s} = \text{Pseudoinverse of } \bar{s}$$

$$\hat{A} = \underbrace{(\bar{s}^T\bar{s})^{-1}\bar{s}^T}\bar{y}$$

$$\hat{A} = \frac{\bar{s}^T\bar{y}}{\|\bar{s}\|^2}$$

ML Estimate of A.

GLRT

- **ML estimate** of A can be found as follows

$$\arg\max \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\|\bar{\mathbf{y}}-A\bar{\mathbf{s}}\|^2}$$

$$\equiv \arg\min \|\bar{\mathbf{y}} - A\bar{\mathbf{s}}\|^2$$

GLRT

$$\hat{A} = \frac{\bar{\mathbf{s}}^T \bar{\mathbf{y}}}{\|\bar{\mathbf{s}}\|^2}$$

Substitute in LRT
to obtain GLRT

GLRT

- **GLRT reduces to**

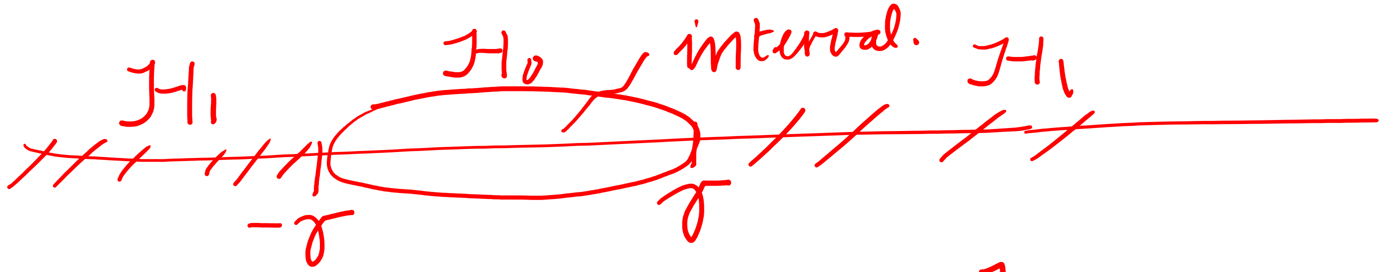
$$2\hat{A}\,\bar{s}^T\bar{y} - \hat{A}^2\|\bar{s}\|^2 > 2\sigma^2 \ln\tilde{\gamma}$$

$$\Rightarrow 2\cdot\frac{\bar{s}^T\bar{y}}{\|\bar{s}\|^2}\cdot\bar{s}^T\bar{y} - \frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^{\cancel{4}\,2}}\cdot\|\bar{s}\|^{\cancel{2}} > 2\sigma^2\ln\tilde{\gamma}$$

$$\Rightarrow 2\frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^2} - \frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^2} > 2\sigma^2\ln\tilde{\gamma}$$

$$\Rightarrow \frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^2} > 2\sigma^2\ln\tilde{\gamma}$$

GLRT



$$\Rightarrow \quad (\bar{s}^T\bar{y})^2 > 2\sigma^2 \ln\tilde{\gamma}\,\|\bar{s}\|^2$$

Matched Filter

$$\Rightarrow \quad \bar{s}^T\bar{y} > \gamma \quad \text{or} \quad \bar{s}^T\bar{y} < -\gamma$$

where $\gamma = \|\bar{s}\|\sqrt{2\sigma^2\ln\tilde{\gamma}}$

Choose H_1 if $|\bar{s}^T\bar{y}| > \gamma$

$\bar{s}^T\bar{y} > \gamma$ or $\bar{s}^T\bar{y} < -\gamma$

Choose H_0 if $-\gamma < \bar{s}^T\bar{y} < \gamma$

GLRT

$|\bar{s}^H\bar{y}|^2$ — For complex Signals.

Generalized Matched Filter

$|\bar{s}^T\bar{y}|$

$(\bar{s}^T\bar{y})^2$

Energy output of matched Filter

Unknown parameter = Scaling factor A.

A +ve
-ve.

Matched Filter + Energy detector

Sign of scaling factor NOT known

A = complex. then phase is UNKNOWN.

GLRT

$$2\hat{A}\,\bar{\mathbf{s}}^T\bar{\mathbf{y}} - \hat{A}^2\|\bar{\mathbf{s}}\|^2 \geq 2\sigma^2 \ln\tilde{\gamma}$$

GLRT

$$\Rightarrow 2\left(\frac{\bar{\mathbf{s}}^T\bar{\mathbf{y}}}{\|\bar{\mathbf{s}}\|^2}\right)\bar{\mathbf{s}}^T\bar{\mathbf{y}} - \left(\frac{\bar{\mathbf{s}}^T\bar{\mathbf{y}}}{\|\bar{\mathbf{s}}\|^2}\right)^2 \|\bar{\mathbf{s}}\|^2 \geq 2\sigma^2 \ln\tilde{\gamma}$$

$$\Rightarrow 2\frac{(\bar{\mathbf{s}}^T\bar{\mathbf{y}})^2}{\|\bar{\mathbf{s}}\|^2} - \frac{(\bar{\mathbf{s}}^T\bar{\mathbf{y}})^2}{\|\bar{\mathbf{s}}\|^2} \geq 2\sigma^2 \ln\tilde{\gamma}$$

GLRT

$$\Rightarrow \frac{(\bar{\mathbf{s}}^T \bar{\mathbf{y}})^2}{\|\bar{\mathbf{s}}\|^2} \geq 2\sigma^2 \ln \tilde{\gamma}$$

$$\Rightarrow (\bar{\mathbf{s}}^T \bar{\mathbf{y}})^2 \geq \|\bar{\mathbf{s}}\|^2 2\sigma^2 \ln \tilde{\gamma}$$

GLRT

$$\Rightarrow |\bar{\mathbf{s}}^T\bar{\mathbf{y}}| \geq \underbrace{\sqrt{\|\bar{\mathbf{s}}\|^2 2\sigma^2 \ln\tilde{\gamma}}}_{\gamma}$$

$$\Rightarrow |\bar{\mathbf{s}}^T\bar{\mathbf{y}}| \geq \gamma$$

$$\Rightarrow \underbrace{\bar{\mathbf{s}}^T\bar{\mathbf{y}} \geq \gamma \text{ or } \bar{\mathbf{s}}^T\bar{\mathbf{y}} \leq -\gamma}_{\text{choose } \mathcal{H}_1}$$

choose $\mathcal{H}_0$ if $-\gamma \leqslant \bar{s}^T\bar{y} \leqslant \gamma$

GLRT — Performance

- P_{FA} can be found as follows

under $\mathcal{H}_0$, probability decision is signal present

$$= \Pr\left(\bar{s}^T\bar{y} > \gamma \text{ or } \bar{s}^T\bar{y} < -\gamma ; \mathcal{H}_0\right) \quad \bar{v} \sim \mathcal{N}(0, \sigma^2 I)$$

$$\mathcal{H}_0 : \quad \bar{y} = \bar{v}$$ — contains iid Gaussian noise samples. mean $= 0$ var $= \sigma^2$

$$\bar{s}^T\bar{y} = \bar{s}^T\bar{v} \sim \mathcal{N}(0, \sigma^2\|\bar{s}\|^2).$$

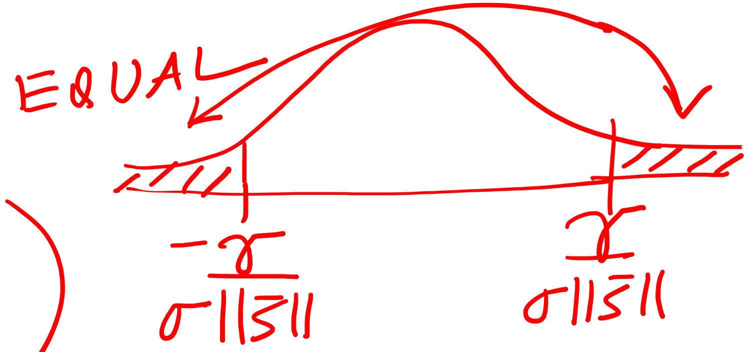
$$E\{\bar{s}^T\bar{v}\} = \bar{s}^T E\{\bar{v}\} = 0$$

$$E\{(\bar{s}^T\bar{v})^2\} = E\{\bar{s}^T\bar{v} \cdot \bar{v}^T\bar{s}\}$$

$$= \bar{s}^T E\{\bar{v}\bar{v}^T\}\bar{s}$$

$$= \bar{s}^T \sigma^2 I \cdot \bar{s} = \sigma^2\|\bar{s}\|^2$$

GLRT



$$\Pr\left(\bar{s}^T\bar{v} > \gamma \text{ or } \bar{s}^T\bar{v} < -\gamma\right)$$

$$= \Pr\left(\mathcal{N}(0,\sigma^2\|\bar{s}\|^2) > \gamma \text{ or } \mathcal{N}(0,\sigma^2\|\bar{s}\|^2) < -\gamma\right)$$

$$= \Pr\left(\underbrace{\frac{\mathcal{N}(0,\sigma^2\|\bar{s}\|^2)}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} > \frac{\gamma}{\sigma\|\bar{s}\|} \text{ or } \underbrace{\frac{\mathcal{N}(0,\sigma^2\|\bar{s}\|^2)}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} < \frac{-\gamma}{\sigma\|\bar{s}\|}\right)$$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{\gamma}{\sigma\|\bar{s}\|} \text{ or } \mathcal{N}(0,1) < \frac{-\gamma}{\sigma\|\bar{s}\|}\right)$$

$$= 2\,\overbrace{Q}^{\text{CCDF of standard Normal}}\left(\frac{\gamma}{\sigma\|\bar{s}\|}\right)$$

GLRT

GLRT

GLRT

- P_{FA} can be found as follows

$$P_{FA} = \Pr(\bar{\mathbf{s}}^T\bar{\mathbf{y}} \geq \gamma \text{ or } \bar{\mathbf{s}}^T\bar{\mathbf{y}} \leq -\gamma) : \mathcal{H}_0$$

$$\Rightarrow \Pr(\bar{\mathbf{s}}^T\bar{\mathbf{v}} \geq \gamma \text{ or } \bar{\mathbf{s}}^T\bar{\mathbf{v}} \leq -\gamma)$$

GLRT

- Under $\mathcal{H}_0$

$$\boldsymbol{\mathcal{H}}_0 : \bar{\mathbf{s}}^T \bar{\mathbf{v}} \sim \mathcal{N}(0, \sigma^2 \|\bar{\mathbf{s}}\|^2)$$

$$\Pr(\bar{\mathbf{s}}^T \bar{\mathbf{v}} \geq \gamma \text{ or } \bar{\mathbf{s}}^T \bar{\mathbf{v}} \leq -\gamma)$$

GLRT

$$= \Pr\left(\frac{\bar{\mathbf{s}}^T\bar{\mathbf{v}}}{\sigma\|\bar{\mathbf{s}}\|} \geq \frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|} \text{ or } \frac{\bar{\mathbf{s}}^T\bar{\mathbf{v}}}{\sigma\|\bar{\mathbf{s}}\|} \leq -\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$P_{FA} = 2Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

GLRT

- P_D can be found as follows

Under $\mathcal{H}_1$ decision is signal present

$$\bar{s}^T\bar{y} = \bar{s}^T(A\bar{s}+\bar{v})$$

$$= \underbrace{A\|\bar{s}\|^2}_{\text{mean}} + \bar{s}^T\bar{v} \sim \mathcal{N}(0, \sigma^2\|\bar{s}\|^2)$$

$$\bar{s}^T\bar{y} \sim \mathcal{N}(A\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2)$$

GLRT

$$P_D = \Pr\left(\bar{s}^T \bar{y} > \gamma \text{ or } \bar{s}^T \bar{y} < -\gamma ; \mathcal{H}_1\right)$$

$$= \Pr\left(\mathcal{N}(A\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2) > \gamma \text{ or } \mathcal{N}(A\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2) < -\gamma\right)$$

$$= \Pr\left(\overbrace{\frac{\mathcal{N}(A\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2) - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}}^{\mathcal{N}(0,1)} > \frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|} \text{ or } \underbrace{\frac{\mathcal{N}(A\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2) - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} < \frac{-\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

GLRT

$$= \Pr\left(\overbrace{\mathcal{N}(0,1) > \frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}}^{Q\left(\frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)} \text{ or } \mathcal{N}(0,1) < \frac{-\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|} \right)$$

$$= Q\left(\frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) + 1 - Q\left(\frac{-\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$$P_D = Q\left(\frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) + Q\left(\frac{\gamma + A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

GLRT

- P_D can be found as follows

$$P_D = \Pr(\bar{\mathbf{s}}^T \bar{\mathbf{y}} \geq \gamma \ \text{ or } \bar{\mathbf{s}}^T \bar{\mathbf{y}} \leq -\gamma) : \mathcal{H}_1$$

$$\Rightarrow \Pr(\bar{\mathbf{s}}^T (A\bar{\mathbf{s}} + \bar{\mathbf{v}}) \geq \gamma \ \text{ or } \bar{\mathbf{s}}^T (A\bar{\mathbf{s}} + \bar{\mathbf{v}}) \leq -\gamma)$$

GLRT

- Under $\mathcal{H}_1$

$$\boldsymbol{\mathcal{H}}_1 : \bar{\mathbf{s}}^T \bar{\mathbf{y}}$$
$$\sim \mathcal{N}(A\|\bar{\mathbf{s}}\|^2, \sigma^2\|\bar{\mathbf{s}}\|^2)$$

$$\Pr(\bar{\mathbf{s}}^T \bar{\mathbf{y}} \geq \gamma \ \text{ or } \bar{\mathbf{s}}^T \bar{\mathbf{y}} \leq -\gamma)$$

GLRT

- Under $\mathcal{H}_1$

$$= \Pr\left(\frac{\bar{\mathbf{s}}^T\bar{\mathbf{y}} - A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|} \geq \frac{\gamma - A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|} \text{ or } \frac{\bar{\mathbf{s}}^T\bar{\mathbf{y}} - A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|} \leq \frac{-\gamma - A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$\underbrace{P_D = Q\left(\frac{\gamma - A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) + Q\left(\frac{\gamma + A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)}_{\text{Probability of Detection GLRT.}}$$

GLRT

- ROC is

$$P_{FA} = 2Q\left(\frac{\gamma}{\sigma\|s\|}\right)$$

$$\Rightarrow \gamma = \sigma\|s\|\, Q^{-1}\left(\frac{P_{FA}}{2}\right)$$

GLRT

$$P_D = Q\left(\frac{\sigma\|\bar{s}\|\, Q^{-1}\left(\frac{P_{FA}}{2}\right) - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) + Q\left(\frac{\sigma\|\bar{s}\|\, Q^{-1}\left(\frac{P_{FA}}{2}\right) + A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$$= Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - \frac{A\|\bar{s}\|}{\sigma}\right) + Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + \frac{A\|\bar{s}\|}{\sigma}\right)$$

GLRT

P_D as a function of P_{FA}

Receiver operating characteristic

ROC :

GLRT

- ROC is

$$P_{FA} = 2Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$\Rightarrow \gamma = \sigma\|\bar{\mathbf{s}}\|Q^{-1}\left(\frac{P_{FA}}{2}\right)$$

GLRT

- ROC is

$$P_D = Q\left(\frac{\sigma\|\bar{\mathbf{s}}\|Q^{-1}\left(\frac{P_{FA}}{2}\right) - A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) + Q\left(\frac{\sigma\|\bar{\mathbf{s}}\|Q^{-1}\left(\frac{P_{FA}}{2}\right) + A\|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$P_D = Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - \frac{A\|\bar{\mathbf{s}}\|}{\sigma}\right) + Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + \frac{A\|\bar{\mathbf{s}}\|}{\sigma}\right)$$

ROC of GLRT with unknown scaling factor

Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.