Karush-Kuhn-Tucker (KKT) conditions

<u>Assume</u> : (1) x^* primal optimum

(2) (λ^*, ν^*) dual optimum

(3) $P = D$ strong duality

$\Rightarrow$ KKT conditions hold

$$D = g(\lambda^*, \nu^*) = \min_{x \in \mathcal{D}} L(x, \lambda^*, \nu^*)$$ definition

$$\overset{1}{\leq} L(x^*, \lambda^*, \nu^*)$$

$$= f_0(x^*) + \underbrace{\sum_{i=1}^{m} \lambda_i^* f_i(x^*)}_{2} + \underbrace{\sum_{j=1}^{p} \nu_j^* h_j(x^*)}_{3}$$

$$= f_0(x^*) = P$$

But $P = D$ from strong duality

KKT conditions

1. Primal feasibility $f_i(x^*) \leq 0$, $h_j(x^*) = 0$

2. Dual feasibility $\lambda_i^* \geq 0$

3. Complementary slackness : $\lambda_i^* f_i(x^*) = 0$

$\Rightarrow$ either $\lambda_i^* = 0$ or $f_i(x^*) = 0$

soln. on the boundary

4. Stationarity condition

$$x^* = \arg\min_{x \in \mathcal{D}} L(x, \lambda^*, \nu^*)$$

unconstrained case : $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$

for convex problems

(1) $\left.\begin{array}{l}\text{optimum } (x^*, \lambda^*, \nu^*) \\ P = D\end{array}\right] \Leftrightarrow$ KKT conditions

Note : convex $\not\Rightarrow$ KKT or $P=D$ ...

<u>Slater's Thm</u>

$\left.\begin{array}{l} f_0, f_i \text{ convex} \\ h_j \text{ affine} \\ P \text{ finite} \\ \text{Slater's condition} \end{array}\right] \Rightarrow \begin{array}{l} P = D \\ \lambda^* \text{ finite} \\ \text{KKT conditions} \end{array}$

$\exists \hat{x}$ s.t. $f_i(\hat{x}) < 0 \quad \forall i = 1 \ldots m$

for non-linear f_i

Eg $P = \min_{x} \frac{1}{2}x^T P x + q^T x$ convex $(P \succeq 0)$

$Ax = b$

$(f_i = 0)$

$b \in \mathcal{R}(A) \Rightarrow$ feasible

$+$ not unbounded below $\Big] \Rightarrow P$ finite

$\Rightarrow P = D$ & (1) $Ax^* = b$

(2) $\nabla L(x, \nu) = 0 = Px^* + q + A^T \nu^*$

$L(x, \nu) = \frac{1}{2}x^T P x + q^T x + \nu^T (Ax - b)$

solve KKT ?

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

$\Rightarrow (x^*, \nu^*)$