1. Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $\mathbb{R}_+ \subseteq \text{dom } f$, and its running average, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)dt \tag{1}$$

with dom $F = \mathbb{R}_{++}$. Show that $F(x)$ is convex if $f(x)$ is convex.

Solution: It is given that $f(x)$ is convex. Therefore, it must hold that the affine transform $f(sx)$ is convex. Hence,

$$w(x) = \int_0^1 f(sx)ds \tag{2}$$

is also convex, as it is the limit of a sum of convex functions. Next, let us substitute $t = sx$ so that $dt = xds$. Hence

$$w(x) = \frac{1}{x}\int_0^x f(t)dt \tag{3}$$

is also convex. Note that this approach allows us to generalize the problem statement by following the steps. For instance, since $f(x)$ is convex, we also have that $h(s)f(sx)$ is convex for any $h(s) > 0$, or any other transformation. Similarly each step can be generalized.

2. Show that the running average of the non-differentiable convex function $f(x)$ is also convex. Since f is not differentiable, you must use the zeroth order condition to prove the convexity of F.

Solution: Running average of the function f is

$$F(\mathbf{x}) = \frac{1}{x}\int_0^x f(\tau)d\tau.$$

By changing variable as $\tau = xt$, we write

$$F(\mathbf{x}) = \int_0^1 f(xt)dt.$$

For any $0 < \theta < 1$,

$$\begin{aligned}
F(\theta x + (1-\theta)y) &= \int_0^1 f(\theta xt + (1-\theta)yt)dt \\
&\leq \int_0^1 (\theta f(xt) + (1-\theta)f(yt))dt \\
&= \theta \int_0^1 f(xt)dt + (1-\theta)\int_0^1 f(yt)dt \\
&= \theta F(x) + (1-\theta)F(y)
\end{aligned}$$

3. Show that the function $f(\mathbf{x})$ is convex if and only if the function $f(\mathbf{a}+t\mathbf{b})$ is convex for all $\mathbf{a}+t\mathbf{b} \in \text{dom } f$ and t.

Solution: If $f(\mathbf{x})$ is convex then, $f(\mathbf{a}+t\mathbf{b})$ is convex because affine transformation preserves convexity. Now assume $g(t) = f(\mathbf{a}+t\mathbf{b})$ is convex in t, i.e., it holds that for any $\theta \in [0,1]$:

$$
\begin{aligned}
g(\theta t_1 + (1-\theta)t_2) &\leq \theta g(t_1) + (1-\theta) g(t_2) && (4)\\
\Rightarrow f(\mathbf{a} + (\theta t_1 + (1-\theta)t_2)\mathbf{b}) &\leq \theta f(\mathbf{a}+t_1\mathbf{b}) + (1-\theta) f(\mathbf{a}+t_2\mathbf{b}) && (5)\\
\Rightarrow f(\theta(\mathbf{a}+t_1\mathbf{b}) + (1-\theta)(\mathbf{a}+t_2\mathbf{b})) &\leq \theta f(\mathbf{a}+t_1\mathbf{b}) + (1-\theta) f(\mathbf{a}+t_2\mathbf{b}) && (6)
\end{aligned}
$$

for all $\mathbf{a}$ and $\mathbf{b}$ and $t_1, t_2 \in \mathbb{R}$.

Since the statement is true for all $\mathbf{a}$, $\mathbf{b}$, t_1, and t_2. Let us consider two arbitrary points $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$. Choose $\mathbf{a} = \mathbf{x}$, $\mathbf{b} = \mathbf{y} - \mathbf{x}$, $t_1 = 0$, and $t_2 = 1$, then it can be seen that $\mathbf{x} = \mathbf{a} + t_1\mathbf{b}$ and $\mathbf{y} = \mathbf{a} + t_2\mathbf{b}$, so that we get:

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \quad (7)$$

This statement also holds for all $\mathbf{x}$, y, and $\theta \in [0,1]$, implying that f is convex.

4. Show that the harmonic function: $f(\mathbf{x}) = \left(\sum_{i=1}^n x_i^a\right)^{1/a}$ for $a < 1$, and $a \neq 0$ is concave.

Solution: Denote $w = \sum_{i=1}^n x_i^a$. Elements of Hessian matrix are,

$$
\begin{aligned}
[\nabla^2 f(\mathbf{x})]_{ii} &= (1-a) w^{1/a-2} x_i^{2a-2} \left(1 - x_i^{-a} w\right)\\
[\nabla^2 f(\mathbf{x})]_{ij} &= (1-a) w^{1/a-2} x_i^{a-1} x_j^{a-1}.
\end{aligned}
$$

Now for any vector $\mathbf{u} \in \mathbb{R}^n$,

$$\mathbf{u}^T \nabla^2 f(\mathbf{x}) \mathbf{u} = \frac{(1-a)}{w^{2-1/a}} \left[\left(\sum_{i=1}^n \frac{u_i}{x_i^{1-a}} \right)^2 - w \sum_{i=1}^n \frac{u_i^2}{x_i^{2-a}} \right] \quad (8)$$

Now set $g_i = x_i^{a/2}$ and $b_i = u_i x_i^{a/2-1}$ $\forall 1 \leq i \leq n$. Hence the expression in (8) can be written as

$$\mathbf{u}^T \nabla^2 f(\mathbf{x}) \mathbf{u} = \frac{(1-a)}{w^{2-1/a}} \left[\left(\mathbf{a}^T \mathbf{b}\right)^2 - \|\mathbf{a}\|_2^2 \|\mathbf{b}\|_2^2 \right] \leq 0.$$

The above result follows from Cauchy-Schwarz inequality.

5. Show that the entropy function $H(\mathbf{x}) = -\sum_{i=1}^n x_i \log(x_i)$ with dom $H = \{\mathbf{x} \in \mathbb{R}_{++}^n | \mathbf{1}^T\mathbf{x} = 1\}$ is concave.

Solution: The Hessian of $H(\mathbf{x})$ is $\nabla^2 H(\mathbf{x}) = -\text{diag}\{1/x_1, ..., 1/x_n\} \prec 0$ for $x_i > 0$.