Conjugate Functions

$$\underset{\text{convex}}{f^*(y)} = \max_{x \in \mathrm{dom} f} \underbrace{\langle y, x \rangle - f(x)}_{\text{affine in } y}$$

$$\Rightarrow \quad f^*(y) \geq \langle y, x \rangle - f(x) \quad \forall x$$

$$\Rightarrow \quad f(x) + f^*(y) \geq \langle y, x \rangle$$

Eg. $f(x) = \frac{1}{2} x^T x$

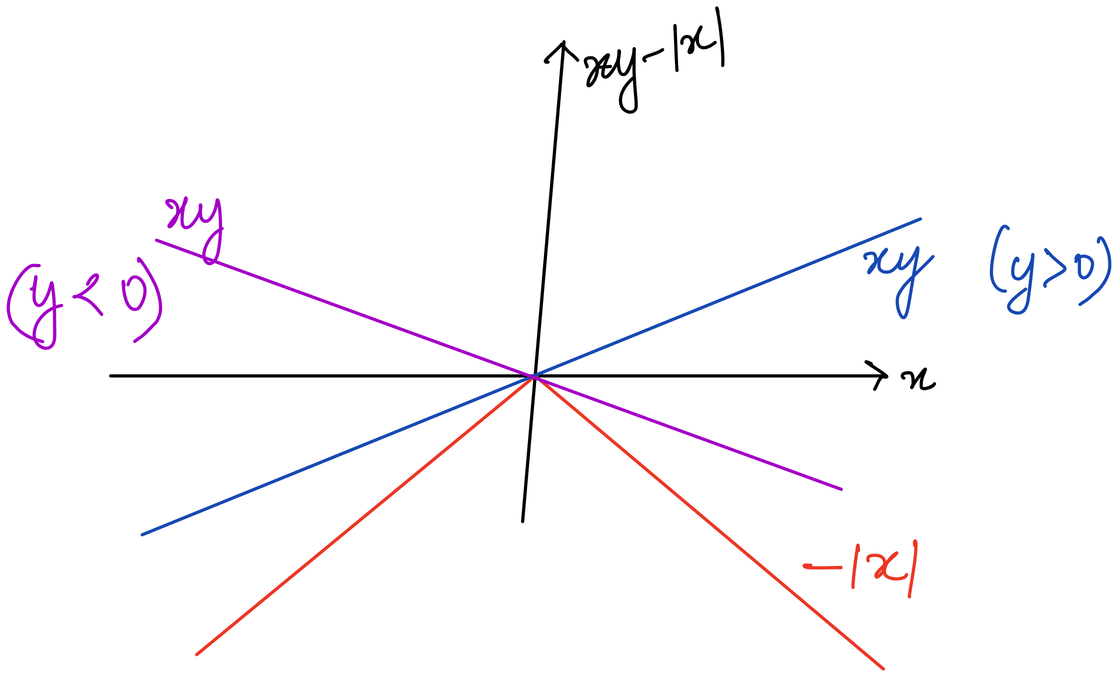
$$f^*(y) = \max_x \; y^T x - \frac{1}{2} x^T x \underset{y=x}{=} \frac{1}{2} y^T y$$

$$\frac{1}{2} x^T x + \frac{1}{2} y^T y \geq x^T y$$

Eg $\quad f(x) = \|x\|_2$

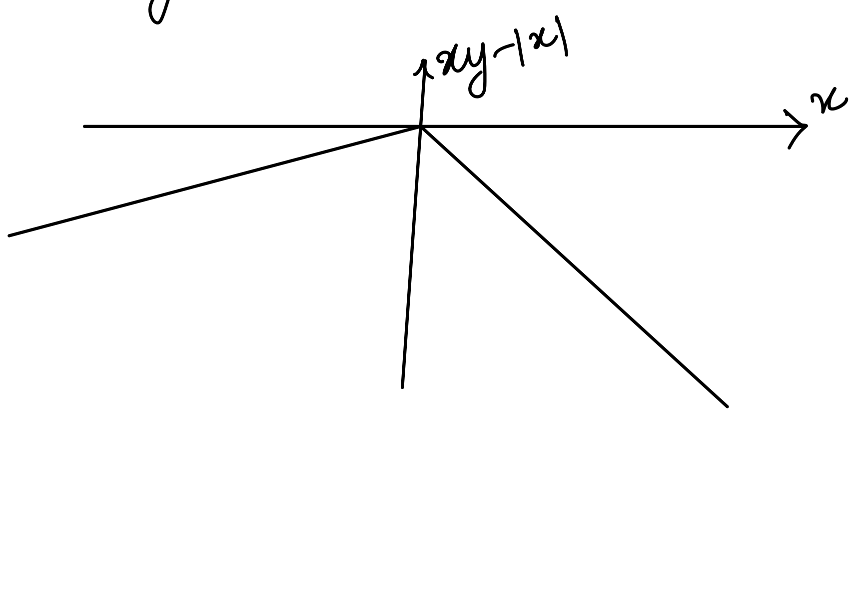
$$f(y^*) = \max_x \; x^T y - \|x\|_2$$

$n=1$ case $\quad f^*(y) = \max_x \; xy - \underline{|x|}$

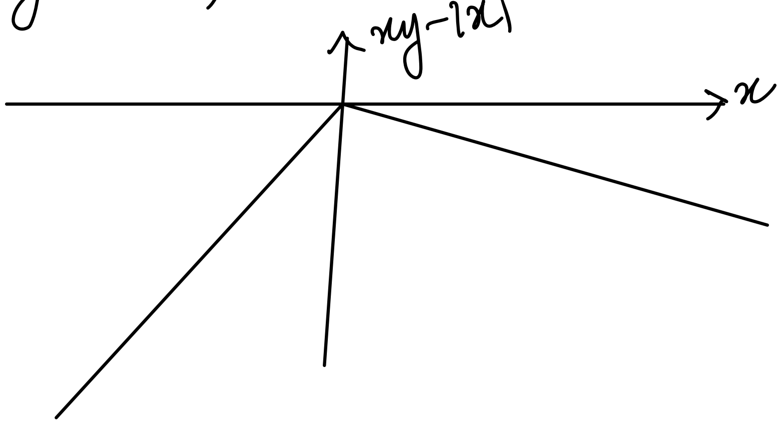



Further split: $y$ small or large

case 1(a) $\quad y > 0$, small

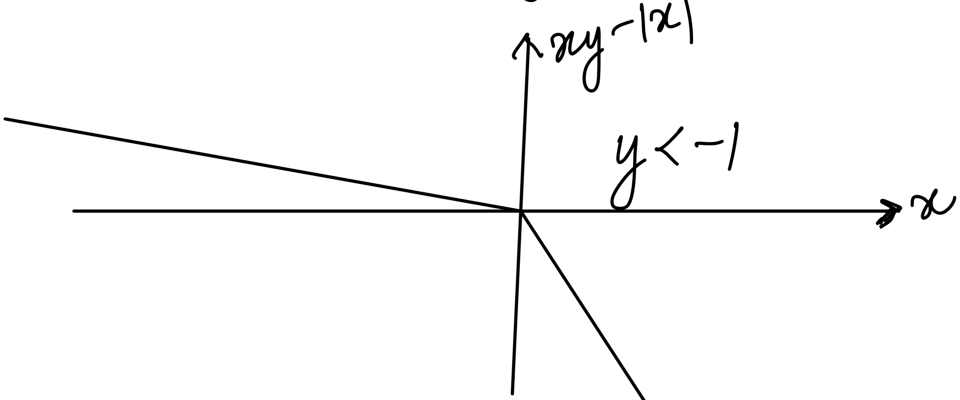


case 2(a) $y < 0$, small

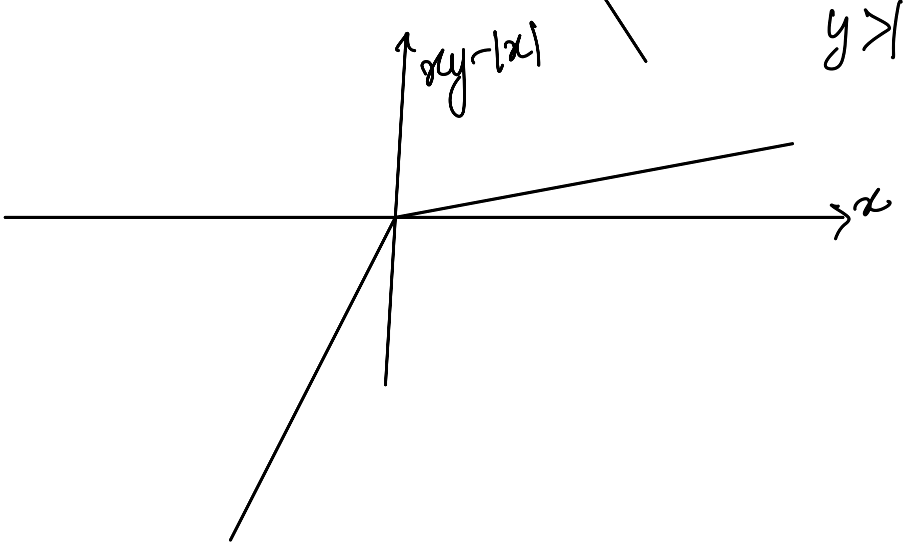



both cases: max at 0 $\quad f^*(y) = 0$

how small: $|y| < 1$




$f^*(y) = -\infty$ for $|y| > 1$




$$f^*(y) = \left.\begin{cases} 0 & |y| \leq 1 \\ -\infty & |y| > 1 \end{cases}\right\} \quad \mathrm{dom} f = \{y \mid |y| \leq 1\}$$

General case

case (a) $||y|| \leq 1$

$$x^T y \leq ||x|| \, ||y||$$

$$\Rightarrow x^T y - ||x|| \leq ||x|| (||y|| - 1) \leq 0$$

$$\Rightarrow \max_x \; x^T y - ||x|| \leq 0$$

Also $x = 0$ then $x^T y - ||x|| = 0$

$$\Rightarrow \max_x \; x^T y - ||x|| = 0 \quad \text{when } ||y|| \leq 1$$

case (b) $||y|| > 1$

let us consider special case when

$$x = \frac{\alpha y}{||y||}$$

$$x^T y - ||x|| = \frac{\alpha y^T y}{||y||} - \alpha = \alpha(||y||_2 - 1)$$

so as $\alpha \to \infty$ $\quad x^T y - ||x|| \to \infty$

$$f^*(y) = \begin{cases} 0 & ||y|| \leq 1 \\ \infty & ||y|| > 1 \end{cases}$$