

eMasters in Communication Systems

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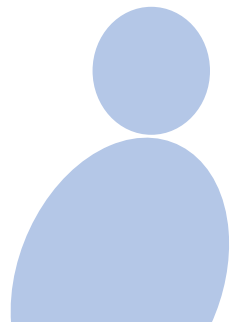
Elective Module:

**Estimation for Wireless
Communication**



Chapter 10

Online/ Sequential Estimation



Online Estimation

- Consider the **SISO channel estimation** problem.

$$y = hx + n$$

SISO channel.



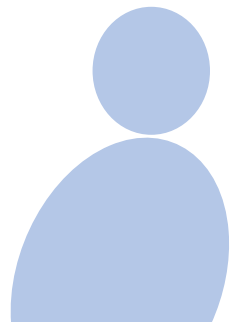
Wireless System Model

- The corresponding model is

$$y(1) = hx(1) + v(1)$$

$$y(2) = hx(2) + v(2)$$

$$y(N) = \underline{hx(N) + v(N)}.$$



Wireless System Model

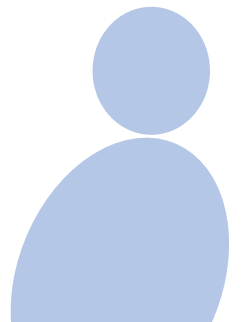
- The corresponding model is

$$y(1) = hx(1) + v(1)$$

$$y(2) = hx(2) + v(2)$$

$$\vdots$$

$$y(N) = hx(N) + v(N)$$



Wireless System Model

- The channel estimate is

$$\hat{h} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}} = \frac{\sum_{k=1}^N x(k)y(k)}{\sum_{k=1}^N x^2(k)}.$$

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}.$$



Wireless System Model

- The channel estimate is *ML Estimate*.

$$\hat{h} = \frac{\sum_{k=1}^N x(k)y(k)}{\sum_{k=1}^N x^2(k)} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}}$$



Wireless System Model

- The quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$



Wireless System Model

- The quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$



Wireless System Model

- This can also be termed as estimate at time N

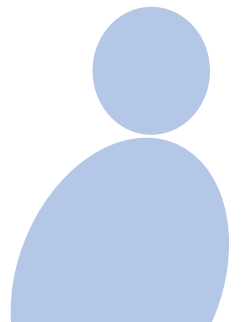
Estimate at time N .

Estimate is updated with time.

$$\hat{h}(N) = \frac{\sum_{k=1}^N x(k)y(k)}{\sum_{k=1}^N x^2(k)} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

$\hat{h}(1), \hat{h}(2), \dots, \hat{h}(N)$

Sequence of Estimates.



Wireless System Model

- Consider now the next output

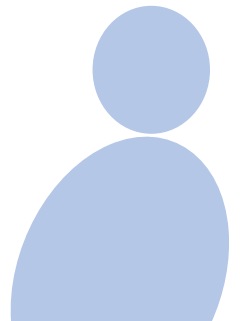
$$y(N+1) = hx(N+1) + v(N+1)$$

$y(N+1)$: output
 $x(N+1)$: input/pilot - } New set of output/input -

- Do we need to repeat estimation?

- Can we simply **update** the previous estimate?

$$\hat{h}(N) \xrightarrow{\text{update}} \hat{h}(N+1)?$$



Sequential Estimation

- This update process is termed **sequential estimation**.
- Since estimation is carried out **sequentially** as ^{$y(N+1), y(N+2), y(N+3), \dots$} outputs arrive...

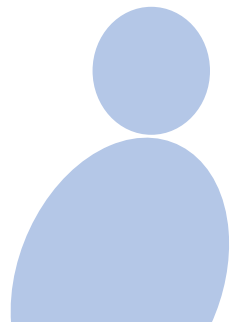
$$\hat{h}(N), \hat{h}(N+1), \hat{h}(N+2), \dots$$

SEQUENCE.



Online Estimation

- This is also termed online estimation.
 - Since estimation is being carried out continuously...
Estimator is ONLINE!
 - And never stops

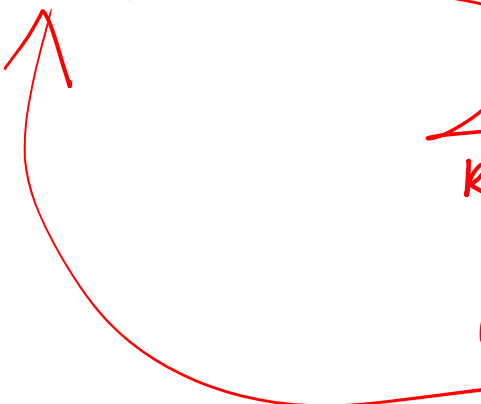


Online Estimation

- This can be achieved as follows.
- Note estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\sum_{k=1}^{N+1} x(k) y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

channel Estimate
Time: $N + 1$.





Online Estimation

- This can be achieved as follows.
- Note estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\sum_{k=1}^{N+1} x(k)y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$



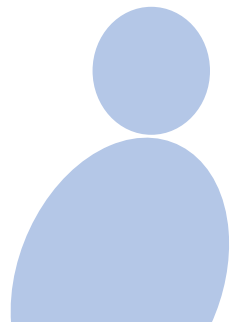
Online Estimation

- MSE at time N is

$p(N) \cdot$
MSE at time N .

$$p(N) = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

$$\Rightarrow \|\bar{\mathbf{x}}\|^2 = \frac{\sigma^2}{p(N)}.$$



Online Estimation

- MSE at time N is

$$p(N) = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2} \Rightarrow \|\bar{\mathbf{x}}\|^2 = \frac{\sigma^2}{p(N)}$$



Wireless System Model

- Therefore, it follows that

$$\hat{h}(N) = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

Estimate at time N .

$$\Rightarrow \bar{x}^T \bar{y} = \hat{h}(N) \cdot \|\bar{x}\|^2$$
$$= \frac{\sigma^2}{P(N)} \cdot \hat{h}(N).$$



Wireless System Model

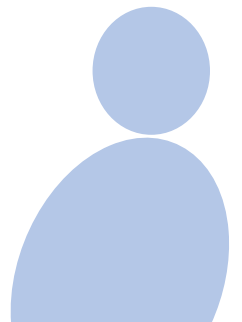
- Therefore, it follows that

$$\hat{h}(N) = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\|\bar{\mathbf{x}}\|^2}$$

$$\Rightarrow \bar{\mathbf{x}}^T \bar{\mathbf{y}} = \hat{h}(N) \frac{\sigma^2}{p(N)}$$

Estimate at
Time N

MSE at
Time N.

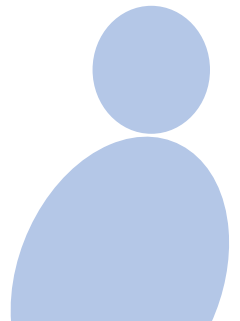


Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\sum_{k=1}^{N+1} x(k)y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sum_{k=1}^N x(k)y(k) + x(N+1)y(N+1)}{\sum_{k=1}^N x^2(k) + x^2(N+1)}.$$



Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\sum_{k=1}^{N+1} x(k)y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sum_{k=1}^N x(k)y(k) + x(N + 1)y(N + 1)}{\sum_{k=1}^N x^2(k) + x^2(N + 1)}$$



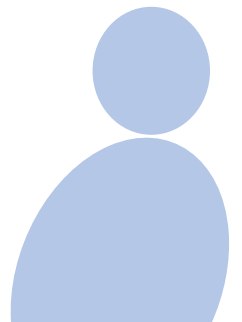
Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\sum_{k=1}^N x(k)y(k) + x(N + 1)y(N + 1)}{\sum_{k=1}^N x^2(k) + x^2(N + 1)}$$

$$= \frac{\bar{x}^T \bar{y} + x(N+1)y(N+1)}{\|\bar{x}\|^2 + x^2(N+1)}.$$

$$= \frac{\frac{\sigma^2}{p(N)} \cdot \hat{h}(N) + x(N+1)y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}.$$



Online Estimation

- Estimate at time $N + 1$ is

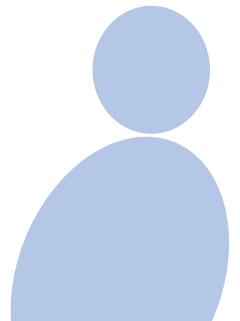
$$\begin{aligned}\hat{h}(N + 1) &= \frac{\sum_{k=1}^N x(k)y(k) + x(N + 1)y(N + 1)}{\sum_{k=1}^N x^2(k) + x^2(N + 1)} \\ &= \frac{\frac{\sigma^2}{p(N)} \hat{h}(N) + x(N + 1)y(N + 1)}{\frac{\sigma^2}{p(N)} + x^2(N + 1)}\end{aligned}$$



Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\frac{\sigma^2}{p(N)} \hat{h}(N) + x(N + 1)y(N + 1)}{\frac{\sigma^2}{p(N)} + x^2(N + 1)}$$
$$= \frac{\hat{h}(N) \left\{ \frac{\sigma^2}{p(N)} + x^2(N + 1) - x^2(N + 1) \right\} + x(N + 1)y(N + 1)}{\frac{\sigma^2}{p(N)} + x^2(N + 1)}$$



Online Estimation

- Estimate at time $N + 1$ is update .

$$\begin{aligned}\hat{h}(N+1) &= \hat{h}(N) + \frac{x(N+1)y(N+1) - \hat{h}(N)x^2(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)} e(N+1) \\ &= \hat{h}(N) + \underbrace{\frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}}_{k(N+1)} \{y(N+1) - \hat{h}(N)x(N+1)\}\end{aligned}$$



Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) =$$



Online Estimation

Estimate at time $N+1$

- Estimate at time $N+1$ is

Gain

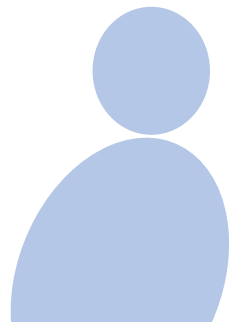
Prediction
Error

$$\hat{h}(N+1) = \hat{h}(N) + K(N+1) e(N+1)$$

Innovation.

Update Rule.

Estimate at time N .



Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \frac{\frac{\sigma^2}{p(N)} \hat{h}(N) + x(N + 1)y(N + 1)}{\frac{\sigma^2}{p(N)} + x^2(N + 1)}$$

$$= \frac{\hat{h}(N) \left(\frac{\sigma^2}{p(N)} + x^2(N + 1) - x^2(N + 1) \right) + x(N + 1)y(N + 1)}{\frac{\sigma^2}{p(N)} + x^2(N + 1)}$$



Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N + 1) = \hat{h}(N) + \frac{x(N + 1)y(N + 1) - \hat{h}(N)x^2(N + 1)}{\frac{\sigma^2}{p(N)} + x^2(N + 1)}$$

$$= \hat{h}(N) + \frac{p(N)x(N + 1)}{\sigma^2 + p(N)x^2(N + 1)} \left(y(N + 1) - \hat{h}(N)x(N + 1) \right)$$



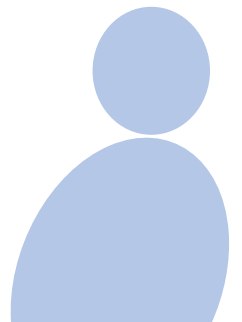
Online Estimation

- This can be expressed as

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)$$

$$k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)} \} \text{ gain.}$$

$$e(N+1) = \underline{y(N+1) - \hat{h}(N)x(N+1)} \} \text{ error.}$$



Online Estimation

- This can be expressed as

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)$$

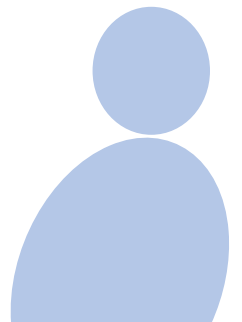
$$k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}$$

$$e(N+1) = \left(y(N+1) - \hat{h}(N)x(N+1) \right)$$

MSE at time N

Gain

error



Online Estimation

- Thus we have derived the online estimator

$$\hat{h}(N + 1) = \hat{h}(N) + k(N + 1)e(N + 1)$$

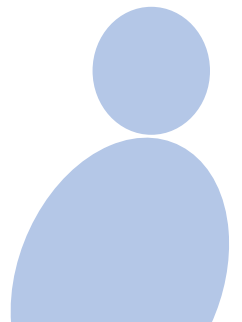


MSE Update

- The MSE can be updated as follows

MSE at time $N+1$.

$$p(N + 1) = \frac{\sigma^2}{\sum_{k=1}^{N+1} x^2(k)}$$
$$= \frac{\sigma^2}{\sum_{k=1}^N x^2(k) + x^2(N + 1)}$$



MSE Update

- The MSE can be updated as follows

$$\begin{aligned} p(N+1) &= \frac{\sigma^2}{\sum_{k=1}^{N+1} x^2(k)} \\ &= \frac{\sigma^2}{\underbrace{\sum_{k=1}^N x^2(k)}_{\|\bar{x}\|^2} + x^2(N+1)} = \frac{\sigma^2}{\frac{\sigma^2}{p(N)} + x^2(N+1)}. \end{aligned}$$

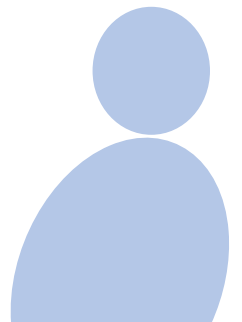


MSE Update

- The MSE can be updated as follows

$$p(N+1) = \frac{\sigma^2}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\sigma^2 p(N)}{\sigma^2 + p(N) x^2(N+1)}$$



MSE Update

- The MSE can be updated as follows

$$\begin{aligned} p(N+1) &= \left(\frac{\sigma^2}{\sigma^2 + p(N)x^2(N+1)} \right) p(N). \\ &= \left(1 - \frac{p(N)x^2(N+1)}{\sigma^2 + p(N)x^2(N+1)} \right) p(N). \\ &= \left(1 - \frac{p(N)x(N+1)}{\underbrace{\sigma^2 + p(N)x^2(N+1)}_{k(N+1)}} \cdot x(N+1) \right) p(N). \end{aligned}$$

MSE Update

- The MSE can be updated as follows

$$\underline{p(N+1) = (1 - k(N+1)x(N+1))p(N) .}$$

MSE update .



MSE Update

- The MSE can be updated as follows

$$\begin{aligned} p(N+1) &= \frac{\sigma^2}{\frac{\sigma^2}{p(N)} + x^2(N+1)} \\ &= \frac{\sigma^2 p(N)}{\sigma^2 + p(N)x^2(N+1)} \end{aligned}$$



MSE Update

- The MSE can be updated as follows

$$\begin{aligned} p(N+1) &= \frac{\sigma^2 p(N)}{\sigma^2 + p(N)x^2(N+1)} \\ &= \left(\frac{\sigma^2}{\sigma^2 + p(N)x^2(N+1)} \right) p(N) \\ &= \left(1 - \frac{p(N)x^2(N+1)}{\sigma^2 + p(N)x^2(N+1)} \right) p(N) \end{aligned}$$



MSE Update

- The MSE can be updated as follows

$$\begin{aligned} p(N+1) &= \left(1 - \frac{p(N)x^2(N+1)}{\sigma^2 + p(N)x^2(N+1)} \right) p(N) \\ &= \left(1 - \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)} x(N+1) \right) p(N) \\ &= (1 - k(N+1)x(N+1))p(N) \end{aligned}$$



MSE Update

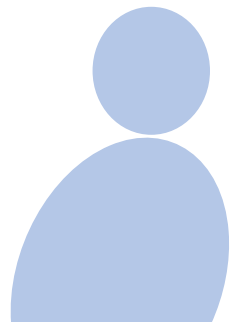
- The MSE can be updated as follows

$$p(N+1) = (1 - k(N+1)x(N+1))p(N)$$

MSE update Rule.

MSE at time $N+1$.

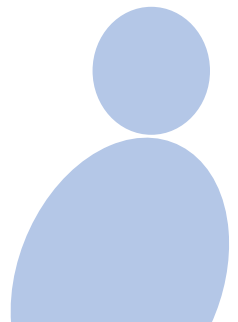
MSE at time N



Online Estimation Vector Parameter

- Consider now the MISO channel estimation problem

N outputs. $\left\{ \begin{array}{l} y(1) = \bar{\mathbf{x}}^T(1)\bar{\mathbf{h}} + v(1) \\ y(2) = \bar{\mathbf{x}}^T(2)\bar{\mathbf{h}} + v(2) \\ \vdots \\ y(N) = \bar{\mathbf{x}}^T(N)\bar{\mathbf{h}} + v(N) \end{array} \right.$



Online Estimation Vector Parameter

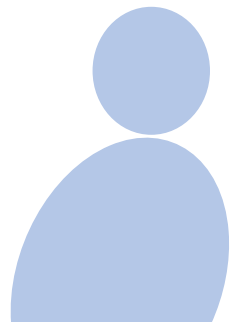
- Consider now the MISO channel estimation problem

$$y(1) = \bar{\mathbf{x}}^T(1)\bar{\mathbf{h}} + v(1)$$

$$y(2) = \bar{\mathbf{x}}^T(2)\bar{\mathbf{h}} + v(2)$$

\vdots

$$y(N) = \bar{\mathbf{x}}^T(N)\bar{\mathbf{h}} + v(N)$$



MISO Channel Model

- This can be written in the matrix form vectors

$$\bar{y} = X \bar{h} + \bar{v}$$
$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} \bar{x}^T(1) \\ \bar{x}^T(2) \\ \vdots \\ \bar{x}^T(N) \end{bmatrix} \bar{h} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$\underbrace{\hspace{10em}}_X$

$\underbrace{\hspace{10em}}_{\bar{y}} \quad \underbrace{\hspace{10em}}_{\bar{v}}$

MISO Channel Model

$N \times M$
Pilot matrix

- This can be written in the matrix form
vectors

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{\mathbf{y}}} = \underbrace{\begin{bmatrix} \bar{\mathbf{x}}^T(1) \\ \bar{\mathbf{x}}^T(2) \\ \vdots \\ \bar{\mathbf{x}}^T(N) \end{bmatrix}}_{\mathbf{X}} \bar{\mathbf{h}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{\mathbf{v}}}$$



ML Estimate

Estimate at time N .

- The LS estimate at time N is

$$\hat{\mathbf{h}}(N) = (X^T X)^{-1} X^T \bar{\mathbf{y}}$$



ML Estimate

- The LS estimate at time N is

$$\hat{\mathbf{h}}(N) = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$



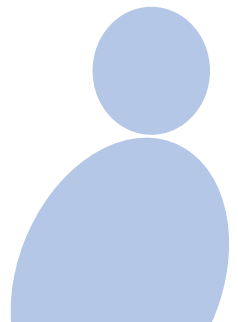
Online Estimation Vector Parameter

- Consider now a new output Time N+1.

$$y(N+1) = \bar{x}^T(N+1) \bar{h} + v(N+1)$$

Output at time N+1

Pilot vector time N+1.



Online Estimation Vector Parameter

- Consider now a new output

$$y(N + 1) = \bar{\mathbf{x}}^T(N + 1)\bar{\mathbf{h}} + v(N + 1)$$



Online Estimation Vector Parameter

- How to update $\hat{\mathbf{h}}(N)$?

→ To obtain $\hat{\mathbf{h}}(N+1)$?

Try to deduce from scalar model.



Online Estimation

- The scalar model can be modified as

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)$$

scalar parameter

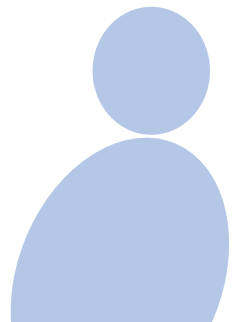
Gain vector

$M \times 1$
vector

scalar

$$\hat{h}(N+1) = \hat{h}(N) + \bar{k}(N+1) e(N+1)$$

Vector parameter.



Online Estimation

- The scalar model can be modified as

$$\underbrace{\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)}_{\text{scalar parameter}}$$

$$\underbrace{\hat{\mathbf{h}}(N+1) = \hat{\mathbf{h}}(N) + \bar{\mathbf{k}}(N+1)e(N+1)}_{\text{vector parameter}}$$



Online Estimation

- The scalar model can be modified as

$$\underbrace{k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}}_{\text{scalar parameter}}$$

$$\underbrace{\bar{k}(N+1) = \frac{p(N)\bar{x}(N+1)}{\sigma^2 + \bar{x}^T(N+1)p(N)\bar{x}(N+1)}}_{\text{Vector Parameter .}}$$



Online Estimation

- The scalar model can be modified as

$$\underbrace{k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}}_{\text{scalar parameter}}$$

$$\underbrace{\bar{\mathbf{k}}(N+1) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}}_{\text{vector parameter}}$$

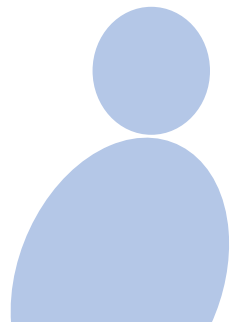


Online Estimation

- The scalar model can be modified as

$$\underbrace{e(N+1) = \left(y(N+1) - x(N+1)\hat{h}(N) \right)}_{\text{scalar parameter}}$$

$$\underbrace{e(N+1) = y(N+1) - \bar{x}^T(N+1)\hat{h}(N)}_{\text{vector parameter .}}$$

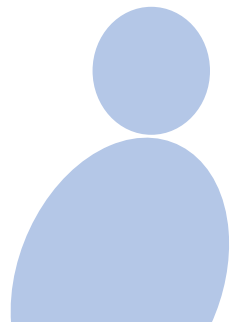


Online Estimation

- The scalar model can be modified as

$$\underbrace{e(N+1) = \left(y(N+1) - x(N+1)\hat{h}(N) \right)}_{\text{scalar parameter}}$$

$$\underbrace{e(N+1) = \left(y(N+1) - \bar{\mathbf{x}}^T(N+1)\hat{\mathbf{h}}(N) \right)}_{\text{vector parameter}}$$



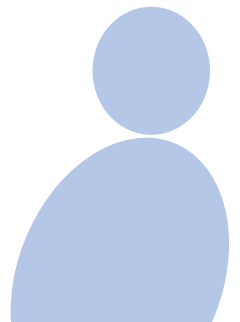
Online Estimation Vector Parameter

- Therefore, the net model is

$$\hat{\mathbf{h}}(N + 1) = \hat{\mathbf{h}}(N) + \bar{\mathbf{k}}(N + 1)e(N + 1)$$

$$\bar{\mathbf{k}}(N + 1) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N + 1)}{\sigma^2 + \bar{\mathbf{x}}^T(N + 1)\mathbf{P}(N)\bar{\mathbf{x}}(N + 1)}$$

$$e(N + 1) = \left(y(N + 1) - \bar{\mathbf{x}}^T(N + 1)\hat{\mathbf{h}}(N) \right)$$



Error Covariance Update

- The error covariance at time N is

$$\mathbf{P}(N) = \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1}$$



Error Covariance Update

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Error Covariance Update

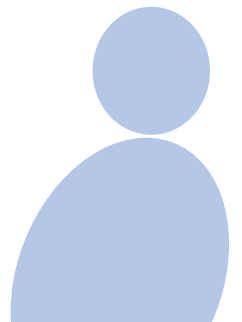
- Inspired by the scalar case, this can be updated for time N as

$$\underbrace{p(N+1) = (1 - k(N+1)x(N+1))p(N)}_{\text{scalar parameter}}$$

error covariance at time N .

$$P(N) = (I - \bar{K}(N+1)\bar{x}^T(N+1))P(N).$$

vector parameter.



Error Covariance Update

- Inspired by the scalar case, this can be updated for time $N + 1$ as

$$\mathbf{P}(N + 1) = \left(\mathbf{I} - \bar{\mathbf{k}}(N + 1)\bar{\mathbf{x}}^T(N + 1) \right) \mathbf{P}(N)$$



Online Example

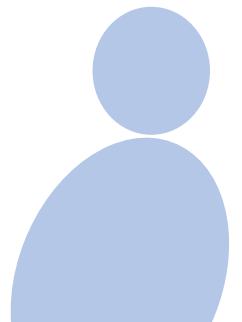
- Consider the problem

$$\bar{\mathbf{y}} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

Output vector

$N=4$

Pilot matrix



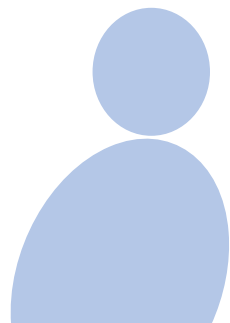
Online Example

- The ML estimate is

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{\mathbf{h}}(N) = \hat{\mathbf{h}}(4) = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \bar{\mathbf{y}} = 4\mathbf{I}$$

$$= \frac{1}{4} \cdot \mathbf{I} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$$



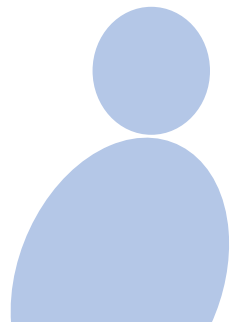
Online Example

- The ML estimate is

$$\hat{\mathbf{h}}(N) = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

$\hat{h}(4)$
Estimate at
time $N=4$.

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$



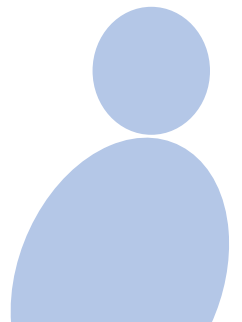
Online Example

- Let $\sigma^2 = 4$. Error covariance is

$$\mathbf{P}(N) = \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1} = 4 \cdot \frac{1}{4} \mathbf{I}$$

$$\mathbf{P}(4) = \mathbf{I}$$

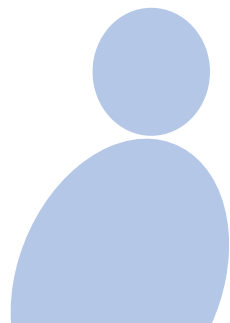
Error covariance
matrix $N=4$.



Online Example

- Let $\sigma^2 = 4$. Error covariance is

$$\mathbf{P}(N) = \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1} = \mathbf{I}$$



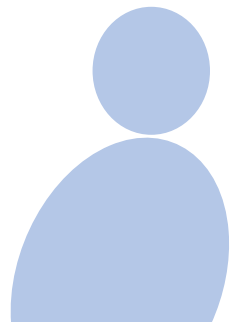
Online Example

- Consider now a new input-output

$$N+1=5$$

$$y(5) = -2, \bar{\mathbf{x}}(5) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$y(N+1) = -2 \quad \bar{\mathbf{x}}(N+1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\bar{\mathbf{k}}(5) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}$$

$$= \frac{\mathbf{I} \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix}}{4 + \begin{bmatrix} -2 & 2 \end{bmatrix}^T \cdot \mathbf{I} \begin{bmatrix} -2 \\ 2 \end{bmatrix}} = \frac{1}{12} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\bar{\mathbf{k}}(5) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}$$

scalar

No matrix inversion!

$$= \frac{\mathbf{I} \times \begin{bmatrix} -2 \\ 2 \end{bmatrix}}{4 + \begin{bmatrix} -2 & 2 \end{bmatrix} \mathbf{I} \times \begin{bmatrix} -2 \\ 2 \end{bmatrix}} = \frac{1}{12} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$e(5) = \left(y(5) - \bar{\mathbf{x}}^T(5) \hat{\mathbf{h}}(4) \right)$$

$$\begin{aligned} &= -2 - \begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} = -2 - (2 - 1) \\ &= -2 - 1 = -3 = e(5) \end{aligned}$$

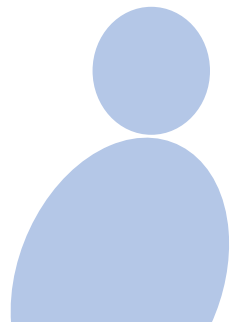


Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$e(5) = \left(y(5) - \bar{\mathbf{x}}^T(5) \hat{\mathbf{h}}(4) \right)$$

$$= -2 - \begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = -3$$

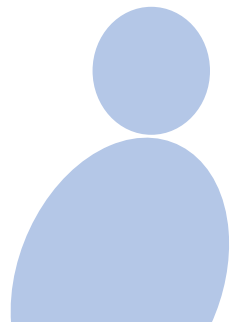


Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\hat{\mathbf{h}}(5) = \hat{\mathbf{h}}(4) + \bar{\mathbf{k}}(5)e(5)$$

$$= \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} + \frac{1}{12} \begin{bmatrix} -2 \\ 2 \end{bmatrix} (-3) = \begin{bmatrix} -1 + \frac{1}{2} \\ -\frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}$$



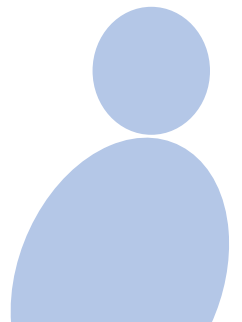
Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\hat{\mathbf{h}}(5) = \hat{\mathbf{h}}(4) + \bar{\mathbf{k}}(5)e(5)$$

update procedure has very low complexity!

$$= \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} + \frac{1}{12} \begin{bmatrix} -2 \\ 2 \end{bmatrix} (-3) = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -1 \end{bmatrix}$$



Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.

