Solutions of Tutorial-5

Problem set 4.3

12 (a) $\boldsymbol{a} = (1, \dots, 1)$ has $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = m$, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = b_1 + \cdots + b_m$. Therefore $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/m$ is the **mean** of the b's (their average value)

(b) $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$ and $\|e\|^2 = (b_1 - \text{ mean })^2 + \cdots + (b_m - \text{ mean })^2 =$ **variance** (denoted by $\boldsymbol{\sigma}^2$).

(c) $\boldsymbol{p} = (3,3,3)$ and $\boldsymbol{e} = (-2,-1,3)$ $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{e} = 0$. Projection matrix $P = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

13 $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. This tells us: When the components of $A\boldsymbol{x} - \boldsymbol{b}$ add to zero, so do the components of $\widehat{\boldsymbol{x}} - \boldsymbol{x}$: Unbiased.

17 $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

Problem set 4.4

1 (a) *Independent* (b) *Independent* and *orthogonal* (c) *Independent* and *orthonormal*.

For orthonormal vectors, (a) becomes $(1,0)$, $(0,1)$ and (b) is $(.6,.8)$, $(.8,-.6)$.

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries $1^2, 2^2, 3^2 = 1, 4, 9$.

5 *Orthogonal* vectors are $(1,-1,0)$ and $(1,1,-1)$. *Orthonormal* after dividing by their lengths : $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

10 (a) If $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are *orthonormal* then the dot product of $\boldsymbol{q}_1$ with $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3 = \boldsymbol{0}$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$. This proves : *Independent* q's

(b) $Q\boldsymbol{x} = \boldsymbol{0}$ leads to $Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{0}$ which says $\boldsymbol{x} = \boldsymbol{0}$.

12 (a) Orthonormal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1) = x_1$

(b) Orthogonal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1)$. Therefore $x_1 = \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1$

(c) x_1 is the first component of A^{-1} times $\boldsymbol{b}$ (A is 3 by 3 and invertible).

20 (a) *True* because $Q^{\mathrm{T}}Q = I$ leads to $(Q^{-1})(Q^{-1}) = I$.

(b) *True*. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$. Also $||Q\boldsymbol{x}||^2 = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$.

Problem set 5.1

3 (a) *False*: $\det(I + I)$ is not $1 + 1$ (except when $n = 1$) (b) *True*: The product rule extends to ABC (use it twice) (c) *False*: $\det(4A)$ is $4^n \det A$

(d) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q^{\mathrm{T}}||Q| = |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so its determinant can't blow up as $n \to \infty$.

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* just $-\det DC$. If n is even then $\det CD = \det DC$ and we can have an invertible CD.

19 For triangular matrices, just multiply the diagonal entries: $\det(U) = 6, \det(U^{-1}) = \frac{1}{6}$, and $\det(U^2) = 36$. 2 by 2 matrix: $\det(U) = ad, \det(U^2) = a^2d^2$. If $ad \neq 0$ then $\det(U^{-1}) = 1/ad$.

22 $\det(A) = 3, \det(A^{-1}) = \frac{1}{3}, \det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. The (singular !) matrices are

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$