eMasters in Communication Systems

Prof. Aditya Jagannatham

Elective Module:

Estimation for Wireless Communication

Chapter 2

Channel Estimation

Most important!!

Wireless System Model

output symbol.

- The wireless system can be modeled as follows

Transmit/input symbol.

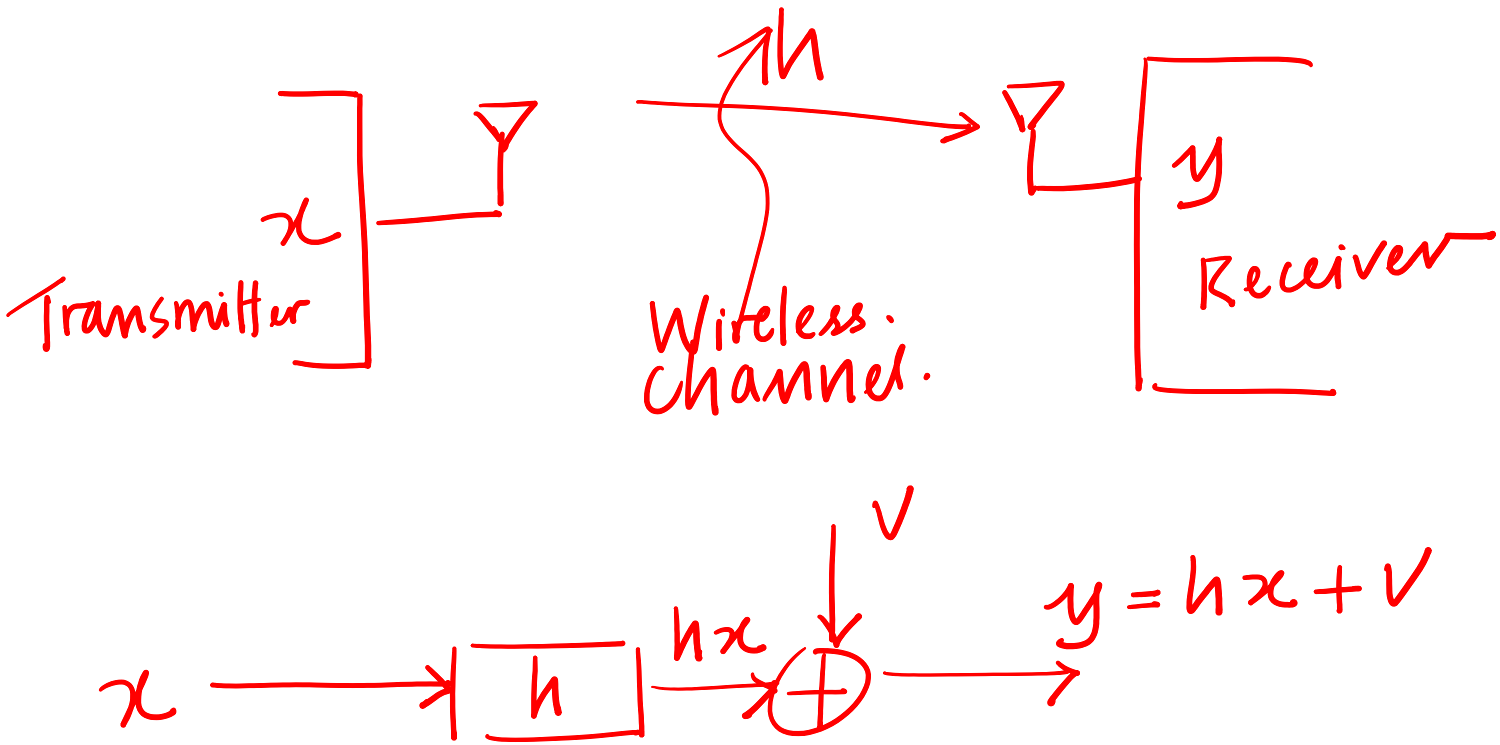
$$y = hx + v$$

Noise

channel coefficient is Necessary For decoding at Receiver

- h = Channel coefficient
- x = Transmit symbol
- y = Received symbol n = noise

Wireless System Schematic



Wireless System Model

- The channel coefficient h is unknown
- Estimating this is termed ***channel estimation***.

HOW?

Wireless System Model

- This is achieved via transmission of training or pilot symbols

Fixed Symbols Purely for the purpose of channel Est.

$$\underbrace{x(1), x(2), \ldots, x(N)}_{\text{Training symbols. Pilots}}$$

Wireless System Model

Fixed symbols

- ***Pilot symbols*** are transmitted purely for the purpose of ***channel estimation***.

Predetermined

Known

Wireless System Model

- The corresponding outputs are given as

$$y(1) = hx(1) + v(1)$$

$$y(2) = hx(2) + v(2)$$

$$\vdots$$

$$y(N) = hx(N) + v(N).$$

Wireless System Model

- The corresponding outputs are given as

$$\text{Outputs:}\left\{\begin{array}{c} y(1) = hx(1) + v(1) \\ y(2) = hx(2) + v(2) \\ \vdots \\ y(N) = hx(N) + v(N) \end{array}\right.$$

Wireless System Model

output vector

$$\bar{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} \sim N \times 1.$$

Pilot vector

$$\bar{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \sim N \times 1$$

$$\underbrace{x(1), x(2), \ldots, x(N)}_{\text{Pilot Symbols}}$$

$$\underbrace{y(1), y(2), \ldots, y(N)}_{\text{Outputs.}}$$

Wireless System Model

- Consider now the kth observation

kth output

kth pilot

$$y(k) = hx(k) + v(k)$$

$\mathcal{N}(hx(k), \sigma^2)$

Gaussian

Mean $= hx(k)$

var $= \sigma^2$

Gaussian

mean $= 0$

var $= \sigma^2$

$\mathcal{N}(0, \sigma^2)$

- PDF of $y(k)$ is given as follows

$$\underbrace{f_{Y(k)}(y(k)) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y(k)-hx(k))^2}{2\sigma^2}}}_{\text{PDF of } y(k).}$$

Wireless System Model

$$y(k) = hx(k) + v(k)$$

- Observe $y(k)$ is Gaussian with mean $hx(k)$ variance σ^2
- Hence, PDF is

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(y(k) - hx(k)\right)^2}$$

Wireless System Model

$$y(k) = hx(k) + v(k)$$

- Observe $y(k)$ is Gaussian with mean $hx(k)$ variance σ^2
- Hence, PDF is

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y(k)-hx(k)\right)^2}{2\sigma^2}}$$

Wireless System Model

Noise samples i.i.d.

- Joint PDF of observations is

$$= f_{Y(1)}(y(1)) \times \cdots \times f_{Y(N)}(y(N))$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y(1)-hx(1))^2} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y(N)-hx(N))^2}}_{\text{Joint PDF}}$$

Wireless System Model

- Joint PDF of observations is

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y(1)-hx(1)\right)^2}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y(N)-hx(N)\right)^2}{2\sigma^2}}$$

Multiplication of individual PDFs.

Wireless System Model

- Joint PDF of observations is

$$f_{\bar{Y}}(\bar{y}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2}$$

Wireless System Model

- Joint PDF of observations is

$$\underbrace{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2}}_{p(\bar{y};\,h)}$$

Likelihood as a function of h

To compute Estimate of h maximize Likelihood.

This is MLE.

Wireless System Model

- This is the likelihood $p(\bar{\mathbf{y}};h)$

constant

−ve constant

minimize

$$p(\bar{\mathbf{y}};h) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2}$$

minimize

- To maximize the likelihood, minimize

$$\equiv \sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2$$

Equivalent Cost Function.

Wireless System Model

- This is the likelihood $p(\bar{\mathbf{y}};h)$

$$p(\bar{\mathbf{y}};h) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2}$$

- To maximize the likelihood, minimize

$$\sum_{k=1}^{N}\left(y(k) - hx(k)\right)^2$$

Wireless System Model

- This can be achieved as follows

Take Derivative wrto h
& Set equal to 0.

$$\frac{d}{dh}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2 = 0$$

$$\Rightarrow \sum_{k=1}^{N}\cancel{2}\left(y(k)-hx(k)\right)\left(-x(k)\right) = 0$$

$$\Rightarrow \sum_{k=1}^{N} y(k)x(k) = \sum_{k=1}^{N} hx^2(k)$$

Wireless System Model

inner product

$\bar{x}^T \bar{y}$

$$\Rightarrow \hat{h} = \frac{\sum_{k=1}^{N} y(k)\, x(k)}{\sum_{k=1}^{N} x^2(k)}$$

$\|\bar{x}\|^2$

MLE

Maximum Likelihood Estimate

Channel Estimate

$$\boxed{\hat{h} = \frac{\bar{x}^T \bar{y}}{\bar{x}^T \bar{x}} = \frac{\bar{x}^T y}{\|\bar{x}\|^2}}$$

Wireless System Model

- This can be achieved as follows

$$\frac{d}{dh}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2$$

$$=\sum_{k=1}^{N}-2x(k)\left(y(k)-hx(k)\right)=0$$

Wireless System Model

$$\sum_{k=1}^{N} -2x(k)\big(y(k) - hx(k)\big) = 0$$

$$\Rightarrow \hat{h} = \frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)}$$

Wireless System Model

- This can be expressed as follows

$$\hat{h} = \frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} = \frac{\bar{x}^T \bar{y}}{\bar{x}^T \bar{x}}$$

$$= \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

Wireless System Model

- This can be expressed as follows

$$\hat{h} = \frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}}$$

Wireless System Model

- This can be expressed as follows

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \qquad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}.$$

Wireless System Model

- This can be expressed as follows

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Wireless System Model

- For complex quantities, the channel estimate is given as

$\bar{x}$ = complex
$\bar{y}$ = complex

$$\hat{h} = \frac{\sum_{k=1}^{N} x^*(k)\,y(k)}{\sum_{k=1}^{N} |x(k)|^2} = \frac{\bar{x}^H \bar{y}}{\bar{x}^H \bar{x}}$$

$$= \frac{\bar{x}^H \bar{x}}{\|\bar{x}\|^2}$$

Wireless System Model

- For complex quantities, the channel estimate is given as

$$\hat{h} = \frac{\sum_{k=1}^{N} x^*(k)y(k)}{\sum_{k=1}^{N} |x(k)|^2} = \frac{\bar{\mathbf{x}}^H \bar{\mathbf{y}}}{\bar{\mathbf{x}}^H \bar{\mathbf{x}}}$$

Properties of MLE

Gaussian

$$\hat{h} = \frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} = \underbrace{\sum_{k=1}^{N} \overbrace{\frac{x(k)}{\|x\|^2}}^{c_k} \cdot y(k)}_{\text{Linear combination of } y(k)}$$

- $\hat{h}$ is a linear combination of **Gaussian RVs**
  - Hence, it is **Gaussian**

$y(k) \sim$ Gaussian

$\Rightarrow \hat{h}$ is Gaussian RV

Properties of MLE

- What is mean of $\hat{h}$

$$E\{\hat{h}\} = E\left\{\frac{\sum_{k=1}^{N} x(k)\,y(k)}{\sum_{k=1}^{N} x^2(k)}\right\}$$

$$= \frac{\sum_{k=1}^{N} x(k)\,E\{y(k)\}}{\sum_{k=1}^{N} x^2(k)} = \frac{\sum_{k=1}^{N} x(k)\,E\{h\,x(k)+v(k)\}}{\sum_{k=1}^{N} x^2(k)}$$

$$= \frac{\sum_{k=1}^{N} x(k)\,h\,x(k)}{\sum_{k=1}^{N} x^2(k)}$$

Properties of MLE

$$E\{\hat{h}\} = h \cdot \cancel{\frac{\sum_{k=1}^{N} x^2(k)}{\sum_{k=1}^{N} x^2(k)}} = h$$

UNBIASED

$$E\{\hat{h}\} = h$$

$\Rightarrow$ Estimate is unbiased!!

Properties of MLE

- We now explore properties of the ML Estimate

$$E\{\hat{h}\} = E\left\{\frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)}\right\} = \frac{\sum_{k=1}^{N} x(k)E\{y(k)\}}{\sum_{k=1}^{N} x^2(k)}$$

$$= \frac{\sum_{k=1}^{N} x(k)E\{hx(k) + v(k)\}}{\sum_{k=1}^{N} x^2(k)}$$

Properties of MLE

$$E\{\hat{h}\} = \frac{\sum_{k=1}^{N} x(k) h x(k)}{\sum_{k=1}^{N} x^2(k)}$$

$$= h \frac{\sum_{k=1}^{N} x^2(k)}{\sum_{k=1}^{N} x^2(k)} = h$$

Properties of MLE

- Therefore

$$E\{\hat{h}\} = h$$

- This is termed an **unbiased estimate**

Properties of MLE

- What about MSE?

Variance

$$E\left\{\left(\hat{h}-h\right)^2\right\} = ?$$

mean

$$\frac{E\{(X-\mu_X)^2\}}{\text{Variance.}}$$

MSE = Mean Square Error

- This is also variance
- This can be found as follows

Properties of MLE

$$E\left\{\left(\hat{h}-h\right)^2\right\} = E\left\{\left(\frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} - h\right)^2\right\}$$

$$= E\left\{\left(\frac{\sum_{k=1}^{N} x(k)y(k) - h\sum_{k=1}^{N} x^2(k)}{\sum_{k=1}^{N} x^2(k)}\right)^2\right\}$$

$$= \frac{E\left\{\left(\sum_{k=1}^{N} x(k)\overbrace{\left(y(k)-hx(k)\right)}^{v(k)}\right)^2\right\}}{\|x\|^4}$$

Properties of MLE

$V(i)$ iid
independent
identically distributed
Gaussian
$\mathcal{N}(0, \sigma^2)$

$$= \frac{E\left\{\left(\sum_{k=1}^{N} x(k)V(k)\right)^2\right\}}{\|x\|^4}$$

$$= \frac{E\left\{\left(\sum_{k=1}^{N} x(k)V(k)\right)\left(\sum_{l=1}^{N} x(l)V(l)\right)\right\}}{\|x\|^4}$$

$$= \frac{\sum_{k=1}^{N}\sum_{l=1}^{N} x(k)x(l)\,\underbrace{E\{V(k)V(l)\}}_{\sigma^2\delta(k-l)}}{\|x\|^4}$$

Properties of MLE

$$MSE = E\left\{(\hat{h}-h)^2\right\} = \frac{\sum_{k=1}^{N}\sum_{l=1}^{N}\sigma^2 x(k)x(l)\delta(k-l)}{\|\bar{x}\|^4}$$

$$= \frac{\sigma^2 \sum_{k=1}^{N} x^2(k)}{\|\bar{x}\|^4} = \frac{\sigma^2\|\bar{x}\|^2}{\|\bar{x}\|^4}$$

$$= \frac{\sigma^2}{\|\bar{x}\|^2} \Big\}$$ Variance
MSE

Properties of MLE

$$E\left\{\left(\hat{h}-h\right)^2\right\} = E\left\{\left(\frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} - h\right)^2\right\}$$

$$= E\left\{\left(\sum_{k=1}^{N} \frac{x(k)\big(y(k) - hx(k)\big)}{x^2(k)}\right)^2\right\}$$

$$= \frac{E\left\{\left(\sum_{k=1}^{N} x(k)v(k)\right)^2\right\}}{\left(\sum_{k=1}^{N} x^2(k)\right)^2}$$

Properties of MLE

$$\frac{E\left\{\left(\sum_{k=1}^{N} x(k)v(k)\right)^2\right\}}{\left(\sum_{k=1}^{N} x^2(k)\right)^2}$$

$$= \frac{1}{\left(\sum_{k=1}^{N} x^2(k)\right)^2} E\left\{\left(\sum_{k=1}^{N} x(k)v(k)\right)\left(\sum_{l=1}^{N} x(l)v(l)\right)\right\}$$

$$= \frac{1}{\left(\sum_{k=1}^{N} x^2(k)\right)^2} \sum_{k=1}^{N}\sum_{l=1}^{N} x(k)x(l)E\{v(l)v(k)\}$$

Properties of MLE

$$\frac{1}{\left(\sum_{k=1}^{N} x^2(k)\right)^2} \sum_{k=1}^{N} \sum_{l=1}^{N} x(k)x(l)\sigma^2 \delta(k-l)$$

$$= \frac{\sigma^2}{\left(\sum_{k=1}^{N} x^2(k)\right)^2} \sum_{k=1}^{N} x^2(k)$$

$$= \frac{\sigma^2}{\sum_{k=1}^{N} x^2(k)} = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

Properties of MLE

Mean Square Error

- Therefore, MSE decreases as

$$MSE = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2} \propto \frac{1}{\|\bar{\mathbf{x}}\|^2}$$

decreases as $\frac{1}{\|\bar{x}\|^2}$

$\|\bar{x}\|^2 =$ Energy of pilot

Properties of MLE

- Therefore, $\hat{h}$ is Gaussian with mean h and variance $\frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$

Unbiased.

$$\hat{h} \sim \mathcal{N}\left(h, \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}\right)$$

channel Estimate

Example

Pilots
input/pilot vector

- $\bar{\mathbf{x}} = [1 \quad -1 \quad 1 \quad -1]^T$

output vector

- $\bar{\mathbf{y}} = [2 \quad -3 \quad -2 \quad 1]^T$

N = 4 pilots.

- What is the MLE of the channel coefficient h?

$$\hat{h} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2} = \frac{[1 \ -1 \ 1 \ -1]\begin{bmatrix} 2 \\ -3 \\ -2 \\ 1 \end{bmatrix}}{1+1+1+1}$$

Example

- The channel estimate is,

$$\hat{h} = \frac{2+3-2-1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\hat{h} = \frac{1}{2} \rightarrow \text{Maximum Likelihood Estimate}$$

MLE.

Example

- The channel estimate is,

$$\hat{h} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}} = \frac{2}{4} = \frac{1}{2}$$

Example

- Given i.i.d. Gaussian noise with zero-mean and variance $\sigma^2 = 2$
- Variance of the maximum likelihood $\hat{h}$ is,

$$\frac{\sigma^2}{\|x\|^2} = \frac{2}{4} = \frac{1}{2}$$

Example

MSE = Mean Squared Error

- Given i.i.d. Gaussian noise with zero-mean and variance $\sigma^2 = 2$
- Variance of the maximum likelihood $\hat{h}$ is,

$$\frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2} = \frac{2}{4} = \frac{1}{2}$$

Estimate of channel.

$y = hx + v$

$\hat{x} = \frac{y}{\hat{h}}$

EQUALIZATION

Equalization

information symbol

$$y = hx + n$$

$$\hat{x} = \frac{y}{h}$$ ideal.

h is unknown

- Estimate h
- $\hat{x} = \frac{y}{\hat{h}}$ Practical Equalization with $\hat{h}$

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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