Assignment – 2 - Solution

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eMasters in Communication Systems, IITK
EE901: Probability and Random Processes
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Q1. In an experiment, two dice are rolled. Consider the sigma-algebra containing all possible subsets of the sample space. Let X be the larger of the two numbers shown. Compute the list of intervals in $\mathbb{R}$ the events in sigma algebra map to.

Solution:

The total set of possible outcomes = $6^2 = 36$

X is the larger of the two numbers displayed on the dice.

Hence, the sample space/Ω =

σ-algebra	Value of X
(1,1)	1
(1,2),(2,2),(2,1)	2
(1,3), (2,3),(3,3),(3,2),(3,1)	3
(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1)	4
(1,5),(2,5),(3,5),(4,5),(5,5),(5,4),(5,3),(5,2),(5,1)	5
(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,5),(6,4),(6,3),(6,2),(6,1)	6

Q2: For the previous question, compute the corresponding events to the X ∈ [2; 4] and X ∈ [1].

Solution:

The events corresponding to X ∈ [2; 4] :
{(1,2),(2,2),(2,1),(1,3),(2,3),(3,3),(3,2),(3,1), (1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1)}

The events corresponding to X ∈ [1] :
{(1,1)}

Q3: Let Ω be the sample space. Find and plot the CDF of X where:

(a) X(ω) = c (c is a constant).

(b) X(ω) = 1 for ω ∈ A and X(ω) = 2 otherwise, with P(A) = $\frac{1}{3}$

Solution:

(a)

x	$E_b = \{\omega: X(\omega) \in B\}$	$\mathbb{P}_X(B)$
$x < c$	ϕ	0
$x \geq c$	Ω	1

The CDF of X is:

$$F_X(x) = \begin{cases} 0, if\ x < c \\ 1, if\ x \geq c \end{cases}$$

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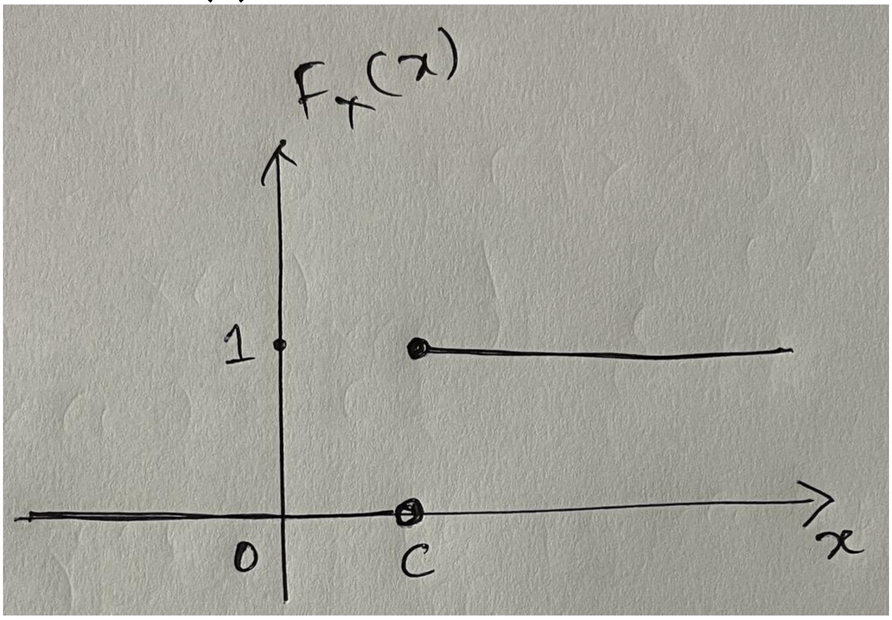
CDF Plot of X(ω) = c



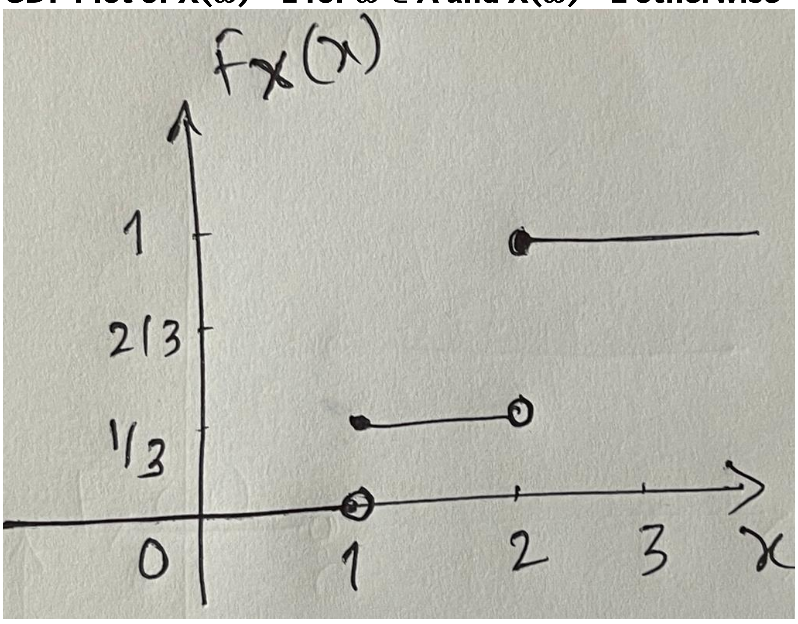
(b)

x	$E_b = \{\omega: X(\omega) \in B\}$	$\mathbb{P}_X(B)$
$x < 1$	ϕ	0
$1 \leq x \leq 2$		$\frac{1}{3}$
$x \geq 2$		1

The CDF of X is:

$$F_X(x) = \begin{cases} 0, if\ x < 1 \\ \frac{1}{3}, if\ 1 \leq x < 2 \\ 1, if\ x \geq 2 \end{cases}$$

CDF Plot of X(ω) = 1 for $\omega \in$ A and X(ω) = 2 otherwise



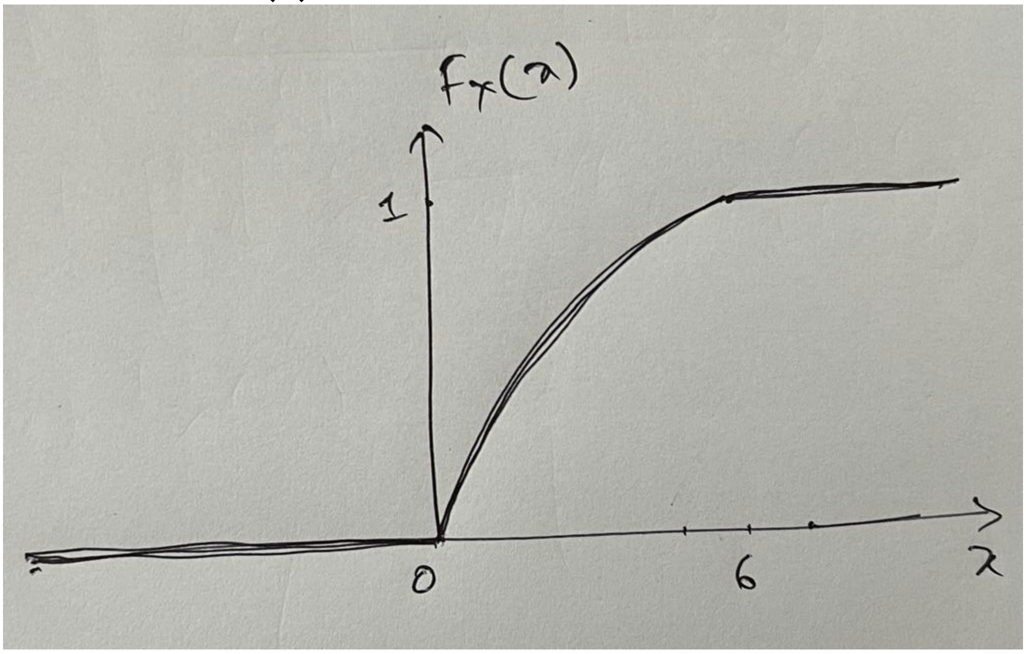
Q4: Let Ω= [0, 1] be the sample space and let P be a uniform probability measure on it such that P((a, b)) = b-a. Find and plot the CDF of X where X(Ω) = 6ω^2

Solution:
CDF of X is:

$$F_X(x) = P[X(\omega) \leq x] = P\left[6\omega^2 \leq x, \omega \in [0,1]\right] = P\left[\omega \leq \sqrt{\frac{x}{6}}, w \in [0,1]\right]$$

Therefore, $F_X(x) = \begin{cases} 0, \ if\ x \leq 0 \\ \frac{\sqrt{x}}{6}, if\ 0 < x < 6 \\ 1, if\ x \geq 6 \end{cases}$

The CDF Plot of X(Ω) = 6ω²



Q5: Consider the experiment of tossing a coin three times. Let X be the random variable giving the number of heads obtained. We assume that the tosses are independent, and the probability of a head is $\frac{1}{3}$.

(a) What is the range of X?
(b) Compute its PMF?
(c) Compute the probability that X<2

Solution:
(a) Range of X

P(H) = $\frac{1}{3}$ ➔ P(T) $= 1 - \frac{1}{3} = \frac{2}{3}$

Event	Value of X	Probability
{TTT}	X=0	$\frac{2}{3} * \frac{2}{3} * \frac{2}{3} = \frac{8}{27}$
{HTT,THT, TTH}	X=1	$3 * \frac{1}{3} * \frac{2}{3} * \frac{2}{3} = \frac{4}{9}$
{HHT, HTH, THH}	X=2	$3 * \frac{1}{3} * \frac{1}{3} * \frac{2}{3} = \frac{2}{3}$
{HHH}	X=3	$\frac{1}{3} * \frac{1}{3} * \frac{1}{3} = \frac{1}{27}$

So, the outcomes of X are = {0,1,2,3} and hence the range of X is = [0,3]

(b) PMF

$$\boldsymbol{p_X(x) = \begin{cases} \frac{8}{27}, if\ x = 0 \\ \frac{4}{9}, if\ x = 1 \\ \frac{2}{9}, if\ x = 2 \\ \frac{1}{27}, if\ x = 3 \end{cases}}$$

(c) Probability of X<2

$$\boldsymbol{P[X < 2] = \frac{8}{27} + \frac{4}{9} = \frac{20}{27}}$$

Q6: The PDF of a continuous random variable X is given by

$$\boldsymbol{f_X(x) = \begin{cases} \frac{1}{4}, 0 < x \leq 2 \\ \frac{3}{4}, 2 < x \leq 4 \\ 0, otherwise \end{cases}}$$

Find the corresponding CDF $F_x(x)$. Sketch $f_X(x)$ and $F_X(x)$.

Solution:

Let's look at the area covered by the PDF across the Ω/sample space that should be 1

$$\int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{4}(2-0) + \frac{3}{4}(4-2) + 0 = \frac{4}{2} = 2$$

Hence, the given function is not complaint to PDF principle.

However, if we divide each value of $f_X(x)$ by 2, the area can be complaint to 1.

Therefore,

$$\boldsymbol{f_X(x) = \begin{cases} \frac{1}{8}, 0 < x \leq 2 \\ \frac{3}{8}, 2 < x \leq 4 \\ 0, otherwise \end{cases}}$$

CDF, then is:

$$F_X(x) = \begin{cases} 0, if\ x < 0 \\ \int_0^x \frac{1}{8}dx = \frac{x}{8}, if\ 0 < x \leq 2 \\ \int_0^2 \frac{1}{8}dx + \int_2^x \frac{3}{8}dx = \frac{1}{8} * [2-0] + \frac{3}{8}[x-2] = \frac{3x}{8} - \frac{1}{2}, if\ 2 \leq x < 4 \\ 1, x \geq 4 \end{cases}$$

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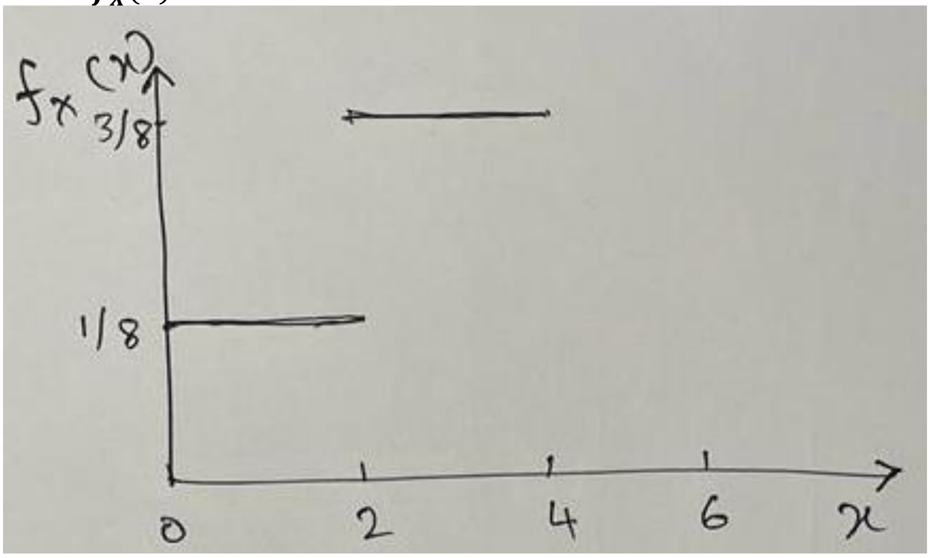
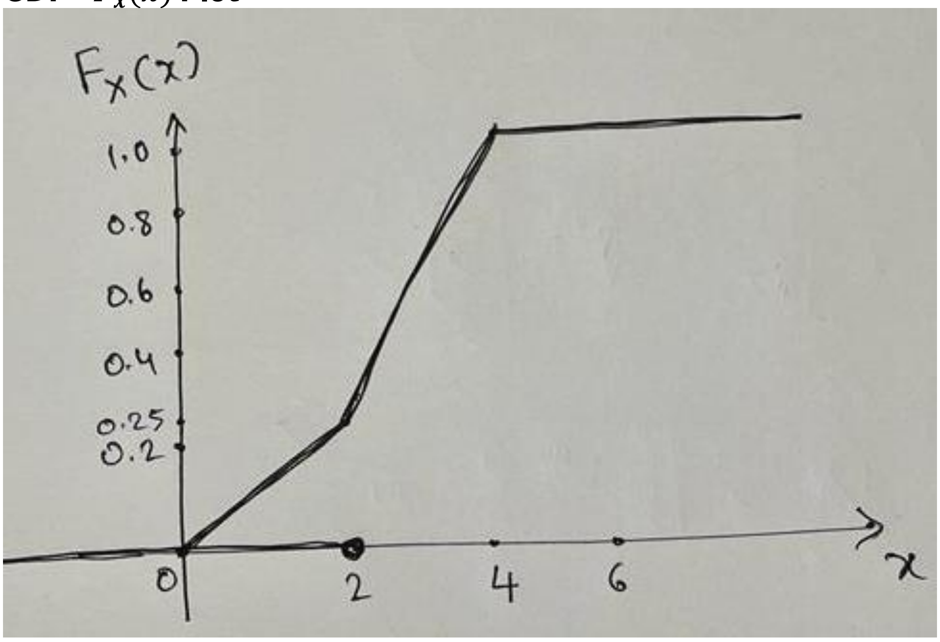
PDF - $f_X(x)$ Plot



CDF - $F_X(x)$ Plot



Q7: The PDF of a random variable X is given by:

$$f_X(x) = \begin{cases} \alpha x, 0 < x < 0.5 \\ 0, otherwise \end{cases}$$

Where α is a constant. Find the value of α.

Solution

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 => \int_0^{0.5} \alpha x dx = \alpha \left[\frac{x^2}{2}\right]_0^{0.5} = \alpha \left[\frac{0.5^2}{2}\right] = \alpha \left[\left(\frac{1}{2}\right)^2 * \frac{1}{2}\right]$$

$$\therefore \alpha * \frac{1}{8} = 1 => \alpha = 8$$

Q8: In an exam, the question paper consists of four multiple-choice questions. A student who does not know any answer, randomly guesses the answers.

(a) What is the probability of getting exactly 3 questions correct?

(b) What is the probability that you will write at least 2 questions correctly?

Solution:

(a) Probability of Getting 3 Questions Correct

If X is the student's answer, it has two outcomes – correct and wrong for each of the question out of 3 questions. We can apply binomial RV here. Assuming 4 multiple choices for each question, the probability of "correct" outcome is ¼ and "wrong" outcome is ¾.

The binomial distribution $P(x) = {}^{n}_{x}Cp^{x}(1-p)^{n-x}$

Therefore, the probability of getting exactly 3 questions correct is :

P[X=3]= ${}^{4}_{3}C(\frac{1}{4})^{3}(\frac{3}{4})^{1} = \frac{3}{64}$

(b) Probability of Getting at least 2 Questions Correct

= Probability of getting 2, 3, and 4 questions correct

⇨ $\boldsymbol{P[X \geq 2] = {}^{4}_{2}C\left(\frac{1}{4}\right)^{2}(\frac{3}{4})^{4-2} + {}^{4}_{3}C\left(\frac{1}{4}\right)^{3}(\frac{3}{4})^{4-3} + {}^{4}_{4}C\left(\frac{1}{4}\right)^{4}(\frac{3}{4})^{4-4} = \frac{67}{256}}$

Q9: The amount of time a person waits for a bus is uniformly distributed between zero and 15 minutes. What is the probability that a person waits less than 10 minutes?

Solution:

Let X be the number of minutes a person waits for the bus. The PDF of X, then, is

$$f_X(x) = \frac{1}{(15-0)} = \frac{1}{15}$$

The probability that a person waits for less than 10 minutes is:

$$P[X \leq 10] = 10 * f_X(x) = 10 * \frac{1}{15} = \frac{2}{3}$$

Q10: The time when a bulb of a specific company fails is an exponential random variable with mean .5 years. What is the probability that a bulb will fail in the third year.

Solution:

To solve this problem, we need to use the cumulative distribution function (CDF) of the exponential distribution. The CDF gives us the probability that a random variable is less than or equal to a certain value.

The exponential distribution with mean (μ) has the probability density function (PDF):

f(x) = (1/μ) * exp(-x/μ) - where x is the time (in years) and exp denotes the exponential function.

The CDF (F(x)) is the integral of the PDF from 0 to x:

F(x) = $\int_0^x f(t)dt$ = $\int_0^x \frac{1}{\mu} * e^{-\frac{t}{\mu}}dt$

Given that the mean (μ) is 0.5 years, the parameter λ (rate parameter) of the exponential distribution can be calculated as λ = 1/μ = 1/0.5 = 2.

Now, we want to find the probability that a bulb will fail in the third year, which means x = 3. So, we need to evaluate the CDF at x = 3:

F(3) = $\int_0^3 \left(\frac{1}{2}\right) * e^{-\frac{t}{2}}$

To find this integral, we can use integration by parts or look up the CDF of the exponential distribution.

The CDF of the exponential distribution with rate parameter λ is given by:
F(x) = 1 - $e^{-\lambda x}$

Substituting our value of λ = 2, we get:
F(x) = 1 - e^{-2x}

Now, for x = 3:
F(3) = 1 - $e^{-(2*3)}$
F(3) = 1 - e^{-6}

Using a calculator, we find that F(3) ≈ 0.9975. So, the probability that a bulb will fail in the third year is approximately 0.9975, or 99.75%.

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