EE901
Probability and Random Processes

Module -3
Distribution of Random Variables

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Distribution of Random Variables

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Distribution of a Random Variable

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X: \Omega \to \mathbb{R}$ be a random variable. The probability law of the random variable X given as
For any set B in the Borel algebra $\mathcal{B}$

$$\mathbb{P}_X(B) \triangleq \Pr[X \in B] = \mathbb{P}(\{\omega: X(\omega) \in B\})$$

This represents the probability of the event consisting of those outcomes which correspond to X taking a value in set B.

Need to specify for every possible set B in Borel algebra $\mathcal{B}$. Lots of work!

We will see that it is sufficient to specify it for sets of the form $B_x = (-\infty, x]$ for every value of x.

This denotes that probability that X takes value less than or equal to x and will be a function of x.

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Cumulative Distribution Function (CDF)

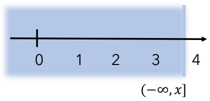
The CDF of a random variable X is defined as the probability that X takes value less than or equal to x

$$F_X(x) = \mathbb{P}(\{\omega : X(\omega) \le x\})$$

$$= \mathbb{P}(\{\omega : X(\omega) \in (-\infty, x]\})$$

$$= \mathbb{P}_X((-\infty, x])$$

- It is nothing but the probability Law $\mathbb{P}_X(B_x)$ of a random variable X for $B_x = (-\infty, x]$.
- CDF at x can be seen as the probability mass of the interval $(-\infty, x]$.



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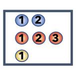
Example: Pick a Ball



$X(\omega)$ = the number of the ball in ω.

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$	$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x < 1$	$\mathbb{P}_X(B) = 0$
$x = 1$	$\mathbb{P}_X(B) = 1/2$
$1 < x < 2$	$\mathbb{P}_X(B) = 1/2$
$x = 2$	$\mathbb{P}_X(B) = 5/6$
$2 < x < 3$	$\mathbb{P}_X(B) = 5/6$
$x = 3$	$\mathbb{P}_X(B) = 1$
$3 < x$	$\mathbb{P}_X(B) = 1$

CDF is $F_X(x) = \mathbb{P}_X((-\infty, x])$

$$= \begin{cases} 0 & x < 1 \\ 1/2 & x = 1 \\ 1/2 & 1 < x < 2 \\ 5/6 & x = 2 \\ 5/6 & 2 < x < 3 \\ 1 & x = 3 \\ 1 & 3 < x \end{cases}$$

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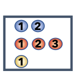
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$	$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

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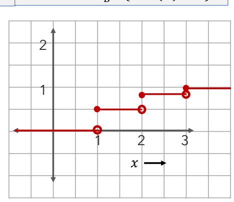

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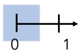
Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.



x

$B = (-\infty, x]$.

x	$E_B = \{\omega : X(\omega) \in B\}$	$\mathbb{P}_X(B)$
$x < 0$	ϕ	0
$x = 0$	ϕ	0
$0 < x < 4$	$\left(0, \frac{x}{4}\right)$	$\frac{x}{4}$
$x = 4$	$(0,1)$	1
$x > 4$	$(0,1)$	1

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Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.

x	$\mathbb{P}_X(B)$
$x < 0$	0
$x = 0$	0
$0 < x < 4$	$\frac{x}{4}$
$x = 4$	1
$x > 4$	1

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ \frac{x}{4} & 0 < x < 4 \\ 1 & x = 4 \\ 1 & x > 4 \end{cases}$$

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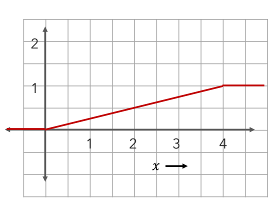
Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ \frac{x}{4} & 0 < x < 4 \\ 1 & x = 4 \\ 1 & x > 4 \end{cases}$$



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Properties of CDF

1. $$F_X(\infty) = 1$$

Proof:

Recall $F_X(x) = \mathbb{P}(E_x)$ where $E_x = \{\omega : X(\omega) \le x\}$

$$E_\infty = \{\omega : X(\omega) \le \infty\}$$

Since for every outcome, $X(\omega) < \infty$, therefore,

$$E_\infty = \Omega$$

$$F_X(\infty) = \mathbb{P}(\Omega) = 1$$

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Properties of CDF

2. $$F_X(-\infty) = 0$$

Proof:

Recall $F_X(x) = \mathbb{P}(E_x)$ where $E_x = \{\omega : X(\omega) \le x\}$

$$E_{-\infty} = \{\omega : X(\omega) \le -\infty\}$$

Since for every outcome, $X(\omega) > -\infty$, therefore,

$$E_{-\infty} = \phi$$
$$F_X(\infty) = \mathbb{P}(\phi) = 0$$

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Properties of CDF

3. $$\mathbb{P}(X > x) = 1 - F_X(x)$$

Proof:

Note that $\{X \le x\}$ is a short form of saying $\{\omega : X(\omega) \le x\}$. Therefore $\{X \le x\}$ is an event.

Now, the event $\{X \le x\}$ and the event $\{X > x\}$ are disjoint. Their union is Ω.

Therefore, from finite additivity property of probability

$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$ for disjoint events A_1 and A_2

$$\mathbb{P}(\{X \le x\}) + \mathbb{P}(\{X > x\}) = \mathbb{P}(\Omega) = 1$$
$$\mathbb{P}(\{X > x\}) = 1 - \mathbb{P}(\{X \le x\})$$

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Properties of CDF

4. $F_X(x)$ is monotonically increasing. i.e.
If $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$

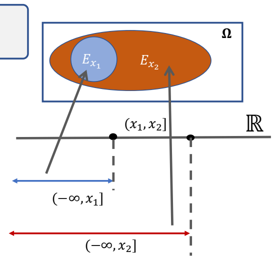
Proof:

Recall $F_X(x_1) = \mathbb{P}(E_{x_1})$

where $E_{x_1} = \{\omega: X(\omega) \le x_1\} = \{X \le x_1\}$

Recall $F_X(x_2) = \mathbb{P}(E_{x_2})$

where $E_{x_2} = \{X \le x_2\}$



Now, E_{x_1} is a subset of E_{x_2}.

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Properties of CDF

$F_X(x)$ is monotonically increasing. i.e.
If $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$

Proof:

Recall $F_X(x_1) = \mathbb{P}(E_{x_1})$

where $E_{x_1} = \{\omega: X(\omega) \le x_1\} = \{X \le x_1\}$

Recall $F_X(x_2) = \mathbb{P}(E_{x_2})$

where $E_{x_2} = \{X \le x_2\}$

Now, E_{x_1} is a subset of E_{x_2}.

Property of Probability: Monotonicity
If A ⊂ B, then $\mathbb{P}(A) \le \mathbb{P}(B)$

$\mathbb{P}(E_{x_1}) \le \mathbb{P}(E_{x_2})$

$F_X(x_1) \le F_X(x_2)$

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Properties of CDF

5. If $x_1 < x_2$, then
$\mathbb{P}(X \in (x_1, x_2]) = F_X(x_2) - F_X(x_1)$

Proof:

Recall $F_X(x_1) = \mathbb{P}(E_{x_1})$

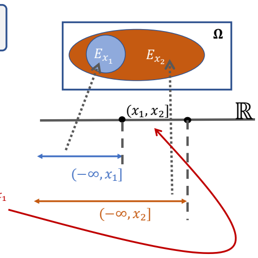
where $E_{x_1} = \{\omega: X(\omega) \le x_1\} = \{X \le x_1\}$

Recall $F_X(x_2) = \mathbb{P}(E_{x_2})$

where $E_{x_2} = \{X \le x_2\}$

The event $\{x_1 < X \le x_2\}$ is equivalent to $E_{x_2} \backslash E_{x_1}$

i.e. $\{X \le x_2\}$ is the union of $\{X \le x_1\}$ and the event $\{x_1 < X \le x_2\}$.



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Properties of CDF

5. If $x_1 < x_2$, then $\mathbb{P}(X \in (x_1, x_2]) = F_X(x_2) - F_X(x_1)$

Proof:

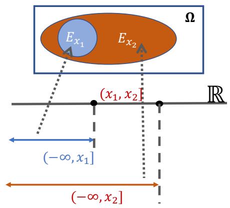
Now, $\{X \le x_2\}$ is the union of $\{X \le x_1\}$ and the event $\{x_1 < X \le x_2\}$.

Therefore, from finite additivity property of probability

$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$

for disjoint events A_1 and A_2

$$\mathbb{P}(\{X \le x_1\}) + \mathbb{P}(\{x_1 < X \le x_2\}) = \mathbb{P}(\{X \le x_2\})$$
$$F_X(x_1) + \mathbb{P}(\{x_1 < X \le x_2\}) = F_X(x_2)$$



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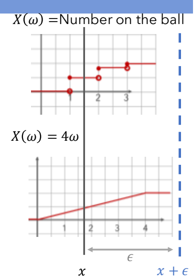
Properties of CDF

5. $F_X(x)$ is right continuous

$\lim_{\epsilon \to 0} F_X(x+\epsilon) = F_X(x)$



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Properties of CDF

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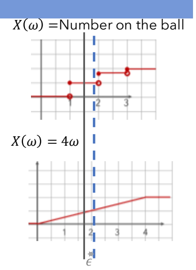

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Properties of CDF

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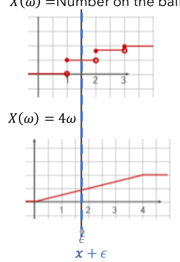

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Properties of CDF

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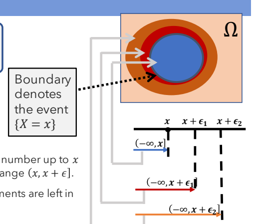

Proof:

$$F_X(x) = \mathbb{P}(\{\omega : X(\omega) \le x\})$$

Similarly,

$$F_X(x+\epsilon) = \mathbb{P}(\{\omega : X(\omega) \le x+\epsilon\})$$

$F_X(x+\epsilon)$ denotes the probability of outcomes that map to a number up to x (including x) along with some additional numbers from the range $(x, x+\epsilon]$.

As $\epsilon \to 0$, the additional range shrinks and eventually no elements are left in it.

This means that as $\epsilon \to 0$,

$F_X(x+\epsilon)$ will denote the probability of outcomes that map to a number up to x (including x) only. $=$ the same as $F_X(x)$

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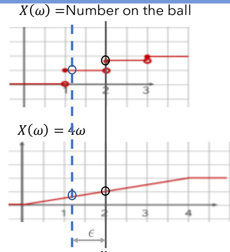
Properties of CDF

6. $F_X(x)$ may not be left continuous
$\lim_{\epsilon \to 0} F_X(x-\epsilon) = F_X(x^-) \neq F_X(x)$



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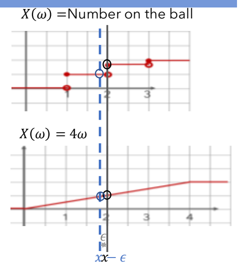
Properties of CDF

6. $F_X(x)$ may not be left continuous
$\lim_{\epsilon \to 0} F_X(x-\epsilon) = F_X(x^-) \neq F_X(x)$

$F_X(x^-) \leq F_X(x)$

$F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\})$



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Properties of CDF

6. $F_X(x)$ may not be left continuous
$\lim_{\epsilon \to 0} F_X(x-\epsilon) = F_X(x^-) \neq F_X(x)$

Proof:

$$F_X(x) = \mathbb{P}(\{\omega : X(\omega) \leq x\})$$
$$F_X(x-\epsilon) = \mathbb{P}(\{\omega : X(\omega) \leq x-\epsilon\})$$

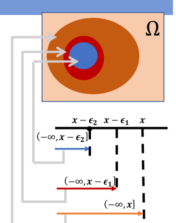
$F_X(x-\epsilon)$ denotes the probability of outcomes that map to a number up to $x-\epsilon$

As $\epsilon \to 0$, the upper limit comes closer to x, however, it will never include outcomes that map to x.

This means that as $\epsilon \to 0$,

$F_X(x-\epsilon)$ will denote the probability of outcomes that map to a number up to x (excluding x).

It is not the same as $F_X(x)$, if there are some outcomes with non-zero probability that map to $\{x\}$



$\boxed{F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\})}$

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Probability Law in terms of CDF

$B = (-\infty, x]$	$\mathbb{P}_X((-\infty, x]) = F_X(x)$	CDF
$B = (x, \infty)$	$\mathbb{P}_X((x, \infty)) = 1 - F_X(x)$	CCDF
$B = (x, y]$	$\mathbb{P}_X((x, y]) = F_X(y) - F_X(x)$	
$B = [x]$	$\mathbb{P}_X([x]) = F_X(x) - F_X(x^-)$	
$B = (x, y)$	$\mathbb{P}_X((x, y)) = ?$	

$(-\infty, x] \cup (x, y] = (-\infty, y]$

$P(A) + P(B) = P(A \cup B)$

$F_X(x) + \mathbb{P}_X((x,y]) = F_X(y)$

$\mathbb{P}_X((x,y]) = F_X(y) - F_X(x)$

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$B = (x, y)$ | $\mathbb{P}_X((x, y)) = ?$

$(x,y) \cup [y] = (x,y]$

$\mathbb{P}((x,y)) + \mathbb{P}([y]) = \mathbb{P}((x,y])$

$F_X(y) - F_X(y^-)$ $\quad F_X(y) - F_X(x)$

$\mathbb{P}((x,y)) = F_X(y) - F_X(x) - (F_X(y) - F_X(y^-))$

$= F_X(y^-) - F_X(x)$

$\mathbb{P}((x,y]) = F_X(y) - F_X(x)$

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Probability Law in terms of CDF

$B = (-\infty, x]$	$\mathbb{P}_X((-\infty, x]) = F_X(x)$	CDF
$B = (x, \infty)$	$\mathbb{P}_X((x, \infty)) = 1 - F_X(x)$	CCDF
$B = (x, y]$	$\mathbb{P}_X((x, y]) = F_X(y) - F_X(x)$	
$B = [x]$	$\mathbb{P}_X([x]) = F_X(x) - F_X(x^-)$	
$B = (x, y)$	$\mathbb{P}_X((x, y)) = F_X(y^-) - F_X(x)$	
$B = \{x, y\}$	$\mathbb{P}_X(\{x, y\}) = \mathbb{P}_X([x]) + \mathbb{P}_X([y])$	
$B = (a, b] \cup (c, d]$	$\mathbb{P}_X(B) = \mathbb{P}_X((a, b]) + \mathbb{P}_X((c, d])$	

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