Determinants And Their Calculation

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Applied Linear Algebra for Wireless Communications

Determinant

- Determinant of a square matrix is a single number
  - Number contains an amazing amount of information about the matrix
- It tells immediately whether the matrix is invertible.
  - Determinant is zero when the matrix has no inverse
- When A is invertible, determinant of A^{-1} is $1/(\det A)$
- Determinant is denoted either as $\det A$ or $|A|$
- Determinants have three basic properties, which we can use to compute $|A|$
- When A is a 2 by 2 matrix, rules 1, 2, 3 lead to the answer we expect:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Properties of determinants: 1-2

- 1. Determinant of $n \times n$ identity matrix is 1

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{ and } \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1$$

- 2. Determinant changes sign when two rows are exchanged

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{.Both sides equal } bc - ad.$$

- Because of this rule, we can find det P for any permutation matrix
  - Just exchange rows of I until we reach P
  - $|P| = +I$ for an even number of row exchanges and $|P| = -I$ for an odd number

Properties of determinants: 3

- 3. Determinant is a linear function of each row separately
  - **Multiply** row 1 by any number t:

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

  - **Add** row 1 of A to of A':

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

- Multiplication rule does not mean that det $2I = 2$ det I

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4$$

Properties of determinants: 3

- This is just like area and volume
  - Expand a rectangle by 2 and its area increases by 4
  - Expand an n-dimensional box by t and its volume increases by t^n

Properties of determinants: 4-5

- 4. If two rows of A are equal, then $|A| = 0$

$$\text{Check 2 by 2: } \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

- Rule 4 follows from rule 2. Exchange the two equal rows
  - $|A|$ is supposed to change sign
  - But also $|A|$ has to stay the same, because A is not changed
  - The only number with $-|A| = |A|$ is 0
- 5. Subtracting a multiple of one row from another row does not change $|A|$

$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Properties of determinants: 6-7

- 6. A matrix with a row of zeros has $|A| = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$$

  - Add some other row to the zero row – determinant is not changed (rule 5)
  - But the matrix now has two equal rows. So $|A| = 0$ (rule 4)
- 7. If A is triangular then $|A| = a_{11}a_{22}\cdots a_{nn} =$ product of diagonal entries

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \text{ and also } \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad$$

  - Suppose all diagonal entries are nonzero. Remove off-diagonal entries by elimination and make it diagonal
  - By rule 5 the determinant is not changed-and now the matrix is diagonal

Properties of determinants: 6-7

- We accordingly have

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \begin{aligned} &= a_{11} a_{22} a_{33} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= a_{11} a_{22} a_{33} \end{aligned}$$

Properties of determinants:: 8-10

- 8. If A is singular then det $A = 0$. If A is invertible then det $A \neq 0$
- Elimination goes from A to U
    - If A is singular then U has a zero row then rules give $|A| = |U| = 0$.
- If A is invertible then U has the pivots along its diagonal
    - Product of nonzero pivots (using rule 7) gives a nonzero determinant:

$$|A| = \pm|U| = \pm \mathit{product\ of\ pivots}$$

- 9. Determinant of AB is det$A$ times det$B$: $|AB| = |A||B|$
- 10. Transpose A^T has the same determinant as A.

Check: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ since both sides equal $ad - bc$.

Determinant calculation using co-factors (1)

- We give a formula for calculating determinant
- We begin with 3×3 example

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

- Cofactors along row 1 are $C_{1j} = (-1)^{1+j}|M_{1j}|$
- Cross out row 1 and column j to get a submatrix M_{1j} of size $n-1$
  - $C_{11} = 1(a_{22}a_{33} - a_{23}a_{32}),\ C_{12} = -1(a_{21}a_{33} - a_{23}a_{31}),\ C_{13} = 1(a_{21}a_{32} - a_{22}a_{31})$
  - $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
- Cofactor expansion is $|A| = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$.

Determinant calculation using co-factors (2)

- Determinant is the dot product of any row i of A with its cofactors:

$$|A| = a_{i1}C_{i1} + \cdots + a_{in}C_{in}.$$

- Cofactor C_{ij} (order $n-1$, without row i and column j) includes its correct sign

$$C_{ij} = (-1)^{i+j}\det M_{ij}$$