Assignment – 3 - Solution

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eMasters in Communication Systems, IITK
EE901: Probability and Random Processes
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Q1: In an experiment, a dice is rolled twice. Let X be the sum of the outcomes. Find $\mathbb{E}[X]$.

Q1 Solution:

Let Y and Z be the outcomes of the first and second dice rolls – values shown by the dice after the roll.
The outcomes of the first roll – Y = {1,2,3,4,5,6} – 6 outcomes
The outcomes of the second roll – Z = {1,2,3,4,5,6} – 6 outcomes
Then, X = Y+Z
Total outcomes are 6*6 = 36
The Ω for X and the respective mapped events are:

ω	X	Probability
(1,1)	2	1/36
(1,2),(2,1)	3	2/36
(1,3),(3,1),(2,2)	4	3/36
(1,4),(4,1),(2,3),(3,2)	5	4/36
(1,5),(5,1)(2,4),(4,2),(3,3)	6	5/36
(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)	7	6/36
(2,6),(6,2),(3,5),(5,3),(4,4)	8	5/36
(3,6),(6,3),(4,5),(5,4)	9	4/36
(4,6),(6,4),(5,5)	10	3/36
(5,6),(6,5)	11	2/36
(6,6)	12	1/36

$$\mathbb{E}[X] = \sum_{i=2}^{12} x_i p(x_i)$$

$$= \frac{1}{36}(2*1+3*2+4*3+5*4+6*5+7*6+8*5+9*4+10*3+11*2+12*1) = 7$$

Q2: Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2, |x| \leq 1 \\ 0, Otherwise \end{cases}$$

1. **Find the constant c**
2. **Find $\mathbb{E}[X]$ *and Var*(X)**

Q2 Solution:

(a) To find c, we can use the PDF property $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$\therefore \int_{-\infty}^{-1} 0 + \int_{-1}^{1} cx^2 + \int_{1}^{\infty} 0 = 0 + \left|\frac{1}{(2+1)} c * x^{(2+1)}\right|_{-1}^{1} + 0 = \frac{1}{3} * c * |x^3|_{-1}^{1} = \frac{2}{3} * c = 1$$

3. **c=$\frac{3}{2}$**

(b) When PDF is given,

$$\mathbb{E}[X] = \int_{-1}^{1} x f_X(x)dx = \int_{-1}^{1} x * \frac{3}{2} * x^2 dx = \frac{3}{2}\int_{-1}^{1} x^3 dx = \frac{3}{2} * \frac{1}{(3+1)} * \left|x^{(3+1)}\right|_{-1}^{1} = \frac{3}{2} * \frac{1}{4} * [1-1] = 0$$

Var(X) = Second Central Moment = $\sigma^2 = M_\mu^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

4. $\boldsymbol{\sigma^2 = \mathbb{E}[X^2] - (0)^2 = \mathbb{E}[X^2]}$ **– The expected value of $\boldsymbol{X^2}$**
5. $\boldsymbol{\mathbb{E}[X^2] = \int_{-1}^{1} x^2 f_X(x)dx = \int_{-1}^{1} x^2 * \frac{3}{2} * x^2 dx = \frac{3}{2}\int_{-1}^{1} x^4 dx = \frac{3}{2} * \frac{1}{(4+1)} |x^{(4+1)}|_{-1}^{1}}$
6. $\boldsymbol{= \frac{3}{2} * \ (1-(-1)) = \frac{3}{5}}$

$\therefore \boldsymbol{\mathbb{E}[X^2] = \sigma^2 = Var(X) = \frac{3}{5}}$

Q3: Let X be a Gaussian random variable with parameter (μ, σ^2). Find the first four moments using the direct formula involving its PDF.

Q3 Solution:

The PDF of Gaussian random variable – $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$

The First Moment – Mean

$$\mathbb{E}[X] = M_0^1 = \mu = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx$$

Simplifying and solving the integration by substituting $x - \mu = y$

$$\int_{-\infty}^{\infty} (y+\mu) * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy = \int_{-\infty}^{\infty} y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + \int_{-\infty}^{\infty} \mu * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$$

$$= \int_{-\infty}^{\infty} y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + \mu * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$$

$$= \int_{-\infty}^{\infty} y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + \mu * 1$$

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy \text{ is Gaussian PDF with } \mu = 0 \text{ and its integral over } \infty \text{ to } -\infty \text{ equals } 1\right)$$

$\int_{-\infty}^{\infty} y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$ is an odd function (y) multiplied by an even function the result of which would be odd again. The integral of an odd function over a symmetric interval $(-\infty, \infty)$ will be 0

$\therefore \mathbb{E}[X] = \boldsymbol{0 + \mu = \mu}$

Second Moment - Variance

Second raw moment – $M_0^2 = \mathbb{E}[(X-0)^2] = \sigma^2 = \int_{-\infty}^{\infty} x^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx$

Simplifying and solving the integration by substituting $x - \mu = y$

$$\int_{-\infty}^{\infty} (y+\mu)^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy = \int_{-\infty}^{\infty} (y^2 + 2y\mu + \mu^2) * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$$

$$= \int_{-\infty}^{\infty} y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + 2 * \mu * \int_{-\infty}^{\infty} y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + \mu^2 * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$$

From the First Moment solution derivation, we know that:

$$- \left(\mu^2 * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy\right) = \mu^2 * 1 = \mu^2$$

$- \left(y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ is an odd function and its symmetric integral over the interval $(-\infty, \infty)$ will be 0

$- \left(y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ *is an even function and its integral over* $(-\infty, \infty)$ *can be 2 times that of it over* $(0, \infty)$

Simplifying above equation,

$$2 * \int_{0}^{\infty} y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + 2 * \mu * 0 + \mu^2 = 2 * \int_{0}^{\infty} y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + \mu^2$$

Defining $\frac{y^2}{2\sigma^2} = t$

➔ $\frac{d}{dt}\left(\frac{y^2}{2\sigma^2}\right) = \frac{dt}{dt} = 1 \Rightarrow \frac{1}{2\sigma^2} * \frac{d}{dt}(y^2) = \frac{1}{2\sigma^2} * 2y * \frac{dy}{dt} = 1 \Rightarrow dy = \frac{\sigma^2}{y} * dt \text{ and } y = \sqrt{2\sigma^2 * t}$

Now solving the main equation,

$$= 2 * \int_{0}^{\infty} (t * 2\sigma^2) * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{(-t)} * \frac{\sigma^2}{y} * dt + \mu^2$$

$$= 2 * \int_{0}^{\infty} (t * 2\sigma^2) * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{(-t)} * \frac{\sigma^2}{\sqrt{2\sigma^2 * t}} * dt + \mu^2 = \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} (t)^{\frac{1}{2}} * e^{(-t)} dt + \mu^2$$

The integral – $\frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} (t)^{\frac{1}{2}} * e^{(-t)} dt$ – is a gamma function of the form, $\int_0^{\infty} (t)^{(z-1)} * e^{(-t)} dt$ with z=3/2 and $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$$\therefore \mathbb{E}[X^2] = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) + \mu^2 = \frac{2\sigma^2}{\sqrt{\pi}} * \frac{\sqrt{\pi}}{2} + \mu^2 = \sigma^2 + \mu^2$$

$$\therefore \mathbb{E}[\boldsymbol{X^2}] = \boldsymbol{\sigma^2} + \boldsymbol{\mu^2}$$

Note that the second moment of a distribution is variance, and we know variance of Gaussian distribution is σ^2. But the above derivation shows the variance of Gaussian PDF as $\sigma^2 + \mu^2$.

Why?

The second moment we calculated is the raw second moment of Gaussian PDF – M_0^2 which is with respect to 0 – $\mathbb{E}[(X-0)^2]$.

σ^2 is the second central moment of Gaussian distribution – M_μ^2 which is with respect to mean $\mathbb{E}[(X-\mu)^2]$. Solving for third central moment of Gaussian distribution and substituting the value of $\mu = 0$,

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$\boldsymbol{M_\mu^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[(X-0)^2] = \sigma^2 + 0^2 = \sigma^2}$

Third Moment - Skewness

Third raw moment – $M_0^3 = \mathbb{E}[(X-0)^3] = \int_{-\infty}^{\infty} x^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx$

Simplifying and solving the integration by substituting $x - \mu = y$

$$\int_{-\infty}^{\infty} (y+\mu)^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy = \int_{-\infty}^{\infty} (y^3 + 3y^2\mu + 3y\mu^2 + \mu^3) * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$$

$$= \int_{-\infty}^{\infty} y^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + 3\mu * \int_{-\infty}^{\infty} y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy +$$

$3\mu^2 * \int_{-\infty}^{\infty} y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy + \mu^3 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy$ – (1)

From the First and Second Moments solution derivation, we know that:

- $(\mu^2 * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy) = \mu^2 * 1 = \mu^2$
- $\left(y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ is an odd function and its integral over the symmetric interval $(-\infty, \infty)$ is 0
- $\left(y^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ is an odd function and its integral over the symmetric interval $(-\infty, \infty)$ is 0
- $\left(y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ *is an even function and its integral over* $(-\infty, \infty)$ *can be 2 times that of it over* $(0, \infty)$
- $\int_{-\infty}^{\infty} y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy = \boldsymbol{\sigma^2}$

Simplifying (1) for the third raw moment,
$\boldsymbol{M_0^3 = \mathbb{E}[(X)^3] = [0] + 3\mu[\sigma^2] + 3\mu^2[0] + \mu^3 = \mu^3 + 3\mu\sigma^2 = 0^3 + 3 * 0 * \sigma^2 = 0}$

However, the third central moment is $M_\mu^3 = \mathbb{E}[(X-\mu)^3$ and solving as above would result in:

$$\boldsymbol{M_\mu^3 = \mathbb{E}[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx = \mu^3 + 3\mu\sigma^2}$$

The third moment measures the asymmetry or skewness of the distribution. Since Gaussian distribution is symmetric, its third moment is 0.

Fourth Moment – Kurtosis (Peakedness)

Fourth raw moment – $M_0^4 = \mathbb{E}[(X-0)^4] = \int_{-\infty}^{\infty} x^4 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{x^2}{2\sigma^2}\right)} dx$ – (2)

Fourth central moment – $M_\mu^4 = \mathbb{E}[(X-\mu)^4] = \int_{-\infty}^{\infty} (x-\mu)^4 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$ – (3)

$$\therefore M_\mu^4 = \mathbb{E}[(X-\mu)^4] = \int_{-\infty}^{\infty} (x-\mu)^4 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$$

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$$= \int_{-\infty}^{\infty} (x^4 - 4x^3\mu + 6x^2\mu^2 - 4x\mu^3 + \mu^4) * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$$

$$= \int_{-\infty}^{\infty} x^4 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx - \int_{-\infty}^{\infty} 4x^3\mu * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$$

$$+ \int_{-\infty}^{\infty} 6x^2\mu^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx - \int_{-\infty}^{\infty} 4x\mu^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$$

$$+ \int_{-\infty}^{\infty} \mu^4 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$$

From the First, Second, and Third Moments solution derivation above, we know that:

$-\ (\mu^2 * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy) = \mu^2 * 1 = \mu^2$

$-\ \left(y * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ is an odd function and its integral over the symmetric interval $(-\infty, \infty)$ is 0

$-\ \left(y^3 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ is an odd function and its integral over the symmetric interval $(-\infty, \infty)$ is 0

$-\ \left(y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)}\right)$ $is\ an\ even\ function\ and\ its\ integral\ over\ (-\infty, \infty)\ can\ be\ 2\ times\ that\ of\ it\ over\ (0, \infty)$

$-\ \int_{-\infty}^{\infty} y^2 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{y^2}{2\sigma^2}\right)} dy = \boldsymbol{\sigma^2}$

$-\ Simplifying\ and\ solving\ for\ \int_{-\infty}^{\infty} x^4 * \frac{1}{\sqrt{2\pi\sigma^2}} * e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx$

$\mathbb{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$
Fourth Raw Moment = $\boldsymbol{M_0^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^2}$
Applying $\boldsymbol{\mu = 0}$, for 4th Central moment ➔ $\boldsymbol{M_\mu^4 = \mathbb{E}[(X-\mu)^4] = 3\sigma^2}$

Q4: Let X be an exponential random variable with parameter (λ). Find all of its moments using its MGF.

Q4 Solution

PDF of exponential RV is $f_X(x) = \lambda e^{-\lambda x}\ (x \geq 0)$ – $(\lambda\ is\ represented\ as\ \frac{1}{\beta} sometimes)$

$\therefore The\ MGF = M_X(t) = \mathbb{E}[e^{tx}] = \int_{-\infty}^{\infty} M_X(t) * f_X(x) = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x} dx$ – Exponential RV is defined for $x \geq 0$

➔ $M_X(t) = \lambda \int_0^{\infty} e^{(t-\lambda)x} dx = \lambda * \frac{1}{(t-\lambda)} \left| e^{(t-\lambda)x} \right|_0^{\infty} = \frac{\lambda}{(t-\lambda)}[0-1] = -\frac{\lambda}{(t-\lambda)} = \frac{\lambda}{(\lambda-t)}$ for $t < \lambda$

Each moment of the distribution is found by deriving derivative of MGF successively.

1. **First Moment – Mean**

$$\mathbb{E}[X] = M_X'(t)|_{t=0} = \frac{d}{dt}(MGF)|_{t=0} = \frac{d}{dt}\left[\frac{\lambda}{(\lambda-t)}\right]_{t=0} = \lambda * (-1) * \frac{1}{(\lambda - t)^2} * (-1)|_{t=0}$$

$\therefore$ **First Moment - Mean** $= \frac{\lambda}{(\lambda-t)^2}|_{t=0} = \frac{1}{\lambda}$

2. **Second Moment – Variance**

$$\mathbb{E}[X^2] = M_X''(t)|_{t=0} = \frac{d}{dt}\left(M_X'(t)\right)|_{t=0} = \frac{d}{dt}\left[\frac{\lambda}{(\lambda-t)^2}\right]_{t=0} = \lambda * (-2) * \frac{1}{(\lambda-t)^3} * (-1)|_{t=0}$$

$\therefore$ **Second Moment – Variance** = $\frac{2\lambda}{(\lambda-t)^3}|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$

3. **Third Moment – Skewness**

$$\mathbb{E}[X^3] = M_X'''(t)|_{t=0} = \frac{d}{dt}\left(M_X''(t)\right)|_{t=0} = \frac{d}{dt}\left[\frac{2\lambda}{(\lambda-t)^3}\right]_{t=0} = 2\lambda * (-3) * \frac{1}{(\lambda-t)^4} * (-1)|_{t=0}$$

$\therefore$ **Third Moment – Skewness** = $\frac{6\lambda}{(\lambda-t)^4}|_{t=0} = \frac{6\lambda}{\lambda^4} = \frac{6}{\lambda^3}$ **– As the distribution is NOT symmetric, this moment – Skewness is NOT zero**

4. **Fourth Moment – Kurtosis**

$$\mathbb{E}[X^4] = M_X''''(t)|_{t=0} = \frac{d}{dt}\left(M_X'''(t)\right)|_{t=0} = \frac{d}{dt}\left[\frac{6\lambda}{(\lambda-t)^4}\right]_{t=0}$$

$$= 6\lambda * (-4) * \frac{1}{(\lambda-t)^5} * (-1)|_{t=0}$$

$\therefore$ **Fourth Moment – Kurtosis** = $\frac{24\lambda}{(\lambda-t)^5}|_{t=0} = \frac{24\lambda}{\lambda^5} = \frac{24}{\lambda^4}$

Q5: Let X be a uniformly distributed integer taking value between 1 and 55. Let Y = modulus(X,8). Find the PMF of Y.

Q5 Solution:

$$X = (1, 55) \rightarrow \Omega = \{1, 2, 3, 5, \ldots, 55\} \rightarrow p_X(x) = \frac{1}{55}$$

PMF of X

$$p_X(x) = \begin{cases} \frac{1}{55}, x = 1,2,3, \ldots 55 \\ 0, otherwise \end{cases}$$

$Y = X \; modulo \; 8 \; (remainder \; of \; 8)$

Events Mapping Across X and Y

Y	X
0	{8,16,24,32,40,48} – 6 Outcomes
1	{1,9,17,25,33,41,49} - 7 Outcomes
2	{2,10,18,26,34,42,50} - 7 Outcomes
3	{3,11,19,27,35,43,51} - 7 Outcomes
4	{4,12,20,28,36,44,52} - 7 Outcomes
5	{5,13,21,29,37,45,53} - 7 Outcomes
6	{6,14,22,30,38,46,54} - 7 Outcomes
7	{7,15,23,31,39,47,55} - 7 Outcomes
	Total – 55 Outcomes

PMF of Y

$$p_Y(y) = \begin{cases} \frac{6}{55}, & y = 0 \\ \frac{7}{55}, & y = 1, 2, 3, 4, 5, 6, 7 \\ 0, Otherwise \end{cases}$$

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Q6: Let X be a continuous random variable with uniform distribution between 0 and 1. Compute distribution of Y = 1/X in terms of PDF and CDF.

Q6 Solution:

$X = Unif[0,1]$ ➔ $f_X(x) = \frac{1}{(b-a)} = \frac{1}{(1-0)} = \frac{1}{1} = 1,\ a \le x \le b$ ➔ $f_X(x) = 1, a \le x \le b$

➔ $f_X\left(g^{-1}(y)\right) = 1$

$Y = \frac{1}{X}$ = g(X) ➔ $g^{-1}(y) = \frac{1}{y}$ ➔ $g'(x) = \frac{d}{dy}\left(g(x)\right) = \frac{d}{dy}\left(\frac{1}{x}\right) = \left(-\frac{1}{x^2}\right)$

X Limits :[0,1] ➔ Y Limits : When X=0 ➔ $Y = \frac{1}{0} = \infty$, and X=1 ➔ $Y = \frac{1}{1} = 1$ ➔ $[1, \infty]$

➔ $g'\left(g^{-1}(y)\right) = \left(-\frac{1}{\left(\frac{1}{y}\right)^2}\right) = -y^2$

$\therefore f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{|g'(g^{-1}(y))|} = \frac{1}{|-y^2|} = \frac{1}{y^2}$

PDF of Y

$$f_Y(y) = \begin{cases} \frac{1}{y^2}, & 1 \le y \le \infty \\ 0, Otherwise \end{cases}$$

CDF of Y

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy = \int_{-\infty}^{1} \frac{1}{y^2}dy + \int_{1}^{y} \frac{1}{y^2}dy = 0 + \int_{1}^{y} \frac{1}{y^2}dy = \left|-\frac{1}{y}\right|_1^y = -\left[\frac{1}{y} - 1\right] = \left[1 - \frac{1}{y}\right]$$

$$F_Y(y) = \begin{cases} \left(1 - \frac{1}{y}\right), & y \ge 1 \\ 0, & Otherwise \end{cases}$$

Q7: Let X be a continuous random variable with PDF given by $f_X(x) = \frac{1}{2}e^{-|x|}, for\ all\ x \in \mathbb{R}$, if $Y = X^2$, find $F_Y(y)$

Q7 Solution:

$X \in \mathbb{R} - [0, \infty]$ ➔ $Y \in [0, \infty]$

When PDF of X is given, CDF of X= $\int_{-\infty}^{x} f_X(x)dx$

$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} * e^{-|x|}dx = 2\int_{0}^{\sqrt{y}} \frac{1}{2} * e^{-x}dx = 2 * \frac{1}{2}\int_{0}^{\sqrt{y}} e^{-x}dx$$

$$= [-e^{-x}]_0^{\sqrt{y}} = [-e^{-\sqrt{y}} - (-e^{-0}] = \left[\mathbf{1 - e^{-\sqrt{y}}}\right]$$

Therefore

$$\boldsymbol{F_Y(y)} = \begin{cases} \boldsymbol{(1 - e^{-\sqrt{y}}), y > 0} \\ \boldsymbol{0, Otherwise} \end{cases}$$

Q8: Let X be a discrete random variable with range $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$ with equal probability. Find $\mathbb{E}[sin\ (X)]$.

Q8 Solution:

For a discrete RV, $\mathbb{E}[g(x)] = \sum_{x_i} g(x_i) * P_X(x_i)$

$$g(x) = sin(x), P_X(x) = \frac{1}{5}$$

$$\therefore \sum_{x_i \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}} x_i * P_X(x_i) = \frac{1}{5} * \left[\sin(0) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin(\pi)\right]$$

$$= \frac{1}{5} * \left[0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0\right] = \frac{1}{5} * \left[\frac{2 + 2\sqrt{2}}{2}\right] = \frac{1}{5} * \frac{2}{2} * (1 + \sqrt{2}) = \frac{1}{5} * (1 + \sqrt{2}) = \frac{1 + \sqrt{2}}{5}$$

Q9: Let X be an exponential random variable with parameter (1). Find $\mathbb{E}\left[\frac{1}{(1+X)}\right]$

Q9 Solution:

The PDF of Exponential RV $X(\lambda)$ - $f_X(x) = \lambda e^{-\lambda x}, x \geq 0$

$$\therefore f_X(x) = \begin{cases} 1 * e^{-1*x} = e^{-x}, x \geq 0 \\ 0, Otherwise \end{cases}$$

$$g(x) = \frac{1}{(1+x)}$$

$$\mathbb{E}[\frac{1}{(1+x)} = \int_{-\infty}^{\infty} g(x) * f_X(x) dx = \int_{-\infty}^{0} \frac{1}{(1+x)} * e^{-x} dx + \int_{0}^{\infty} \frac{1}{(1+x)} * e^{-x} dx$$

$$= 0 + \int_{0}^{\infty} \frac{1}{(1+x)} * e^{-x} dx = \int_{0}^{\infty} \frac{e^{-x}}{(1+x)} dx$$

- $\int_0^\infty \frac{e^{-x}}{(1+x)} dx$ does not have simple closed-form expression in terms of elementary function for this solution.
- The solution can be arrived at using either numerical methods or special integral functions.
- The special integral function that can be used for expressing this integral is "Exponential Integral function" and is written as:

$$\boldsymbol{Ei(x) = \int_{0}^{\infty} \frac{e^{-x}}{(1+x)} dx = Ei(-1) = \sim 0.5963}$$

Q10: Let X be the random variable representing the value of the number rolled of a fair 4-sided die.
(a) Write down the moment generating function for X
(b) Use this moment generating function to compute the first and second moments of X

Q10 Solution:

(a)

X= Roll of 4-sided fair die D_4 ➔ $\Omega_X = \{1,2,3,4\}$

The probability distribution/mass function of X is:

$$p_X(x) = P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = \frac{1}{4}$$

The MGF of X = $M_X(t) = \mathbb{E}[e^{tx}] = \sum_{x \in \{1,2,3,4\}} e^{tx} * p_X(x)$

➔ $M_X(t) = \sum_{x \in \{1,2,3,4\}} e^{tx} * \frac{1}{4} = \frac{1}{4} * [e^t + e^{2t} + e^{3t} + e^{4t}] = \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t})$

(b)

First moment = $M'_X(t)|_t = 0 = \frac{d}{dt}\left[\frac{1}{4} * (e^t + e^{2t} + e^{3t} + e^{4t})\right]$

$$= \frac{1}{4} * (e^t + 2e^{2t} + 3e^{3t} + 4e^{4t})|_{t=0}$$

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$= \frac{1}{4} * (1 + 2 + 3 + 4) = \frac{10}{4} = \frac{5}{2}$

$\boldsymbol{M_X^1(t) = \frac{5}{2}}$

Second moment = $M_X''(t)|_{t=0} = \frac{d}{dt}[M_X'(t)]_{t=0} = \frac{d}{dt}\left[\frac{1}{4} * (e^t + 2e^{2t} + 3e^{3t} + 4e^{4t}\right]_{t=0}$

$= \frac{1}{4} * [e^t + 4e^{2t} + 9e^{3t} + 16e^{4t}]_{t=0} = \frac{1}{4} * [1 + 4 + 9 + 16] = \frac{1}{4} * 30 = \frac{15}{2}$

$\boldsymbol{M_X^2(t) = \frac{15}{2}}$

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