Live Interaction #1:

1st October 2023

E-masters Communication Systems

Detection for Wireless

- **Binary hypothesis testing**:

$$\mathcal{H}_0 : \underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\text{NULL HYPOTHESIS}}$$

$$\mathcal{H}_1 : \underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\text{ALTERNATIVE HYPOTHESIS}}$$

- How to choose between $\mathcal{H}_0, \mathcal{H}_1$?

$$p\left(\bar{\mathbf{y}}; \mathcal{H}_0\right) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} y^2(i)}$$

$$p\left(\bar{\mathbf{y}}; \mathcal{H}_1\right) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} \left(y(i) - s(i)\right)^2}$$

- **Maximum Likelihood (ML) detector**.
- Choose $\mathscr{H}_0$ if

$$\frac{p\left(\bar{\mathbf{y}};\mathscr{H}_0\right)}{p\left(\bar{\mathbf{y}};\mathscr{H}_1\right)} \geq 1$$

$$\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} y^2(i)} \geq \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}$$

$$\sum_{i=1}^{N} y^2(i) \leq \sum_{i=1}^{N}\left(y(i)-s(i)\right)^2$$

- Choose $\mathscr{H}_0$ if

$$\underbrace{\frac{p\left(\bar{\mathbf{y}};\mathscr{H}_0\right)}{p\left(\bar{\mathbf{y}};\mathscr{H}_1\right)} \geq \tilde{\gamma}}_{\textbf{Likelihood Ratio Test (LRT)}}$$

$$\frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} y^2(i)}}{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}} \geq \tilde{\gamma}$$

$$\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N} y^2(i) \geq \ln\tilde{\gamma}$$

$$\text{Choose } \mathscr{H}_0 \text{ if: } \bar{\mathbf{s}}^T\bar{\mathbf{y}} \leq \gamma$$

$$\text{Choose } \mathscr{H}_1 \text{ if: } \bar{\mathbf{s}}^T\bar{\mathbf{y}} > \gamma$$

$$\gamma = \frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2}$$

- What is value of γ for ML detector?

$$\gamma = \frac{\|\bar{\mathbf{s}}\|^2}{2}$$

- $\bar{\mathbf{s}}^T \bar{\mathbf{y}}$: **MATCHED FILTER**.
- Performance of detector?
- P_{FA}: **Probability of False Alarm**.
- Underlying hypothesis is $\mathscr{H}_0$, but decision is $\mathscr{H}_1$
- P_D: **Probability of Detection**.
- Underlying hypothesis is $\mathscr{H}_1$, decision is $\mathscr{H}_1$
- P_{FA}
- Under $\mathscr{H}_0 : \bar{\mathbf{y}} = \bar{\mathbf{v}}$, **but** $\bar{\mathbf{s}}^T \bar{\mathbf{y}} > \gamma \Rightarrow \bar{\mathbf{s}}^T \bar{\mathbf{v}} > \gamma$

$$\bar{\mathbf{s}}^T \bar{\mathbf{v}} \sim \mathcal{N}\left(0, \sigma^2 \|\bar{\mathbf{s}}\|^2\right)$$

$$\underbrace{P_{FA} = Q\left(\frac{\gamma}{\sigma \|\bar{\mathbf{s}}\|}\right)}_{\text{Probability of False Alarm}}$$

- P_D:
- Under $\mathscr{H}_1 : \bar{\mathbf{y}} = \bar{\mathbf{s}} + \bar{\mathbf{v}}$, **but** $\bar{\mathbf{s}}^T \bar{\mathbf{y}} > \gamma \Rightarrow \bar{\mathbf{s}}^T (\bar{\mathbf{s}} + \bar{\mathbf{v}}) > \gamma$

$$P_D = \underbrace{Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)}_{\text{Probability of Detection}}$$

$$Q(x) = \Pr(X \geq x)$$

- $$\gamma = -\infty$$
  $$P_{FA} = 1$$
  $$P_D = 1$$
- $$\gamma = \infty$$
  $$P_{FA} = 0$$
  $$P_D = 0$$
- Given a value of P_{FA}, what is the best P_D we can achieve?
- P_D vs P_{FA} plot is known as the ***Receiver Operating Characteristic***.

$$\bar{\mathbf{s}} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \sigma^2 = 4, \gamma = 2$$

- P_D and P_{FA}

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$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{2}{2\times 2}\right) = Q\left(\frac{1}{2}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{2-4}{2\times 2}\right) = Q\left(-\frac{1}{2}\right)$$

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