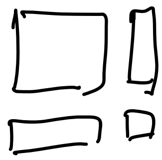
Schur's Complement

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$



$$A = A^T \qquad C = C^T$$

$$X = X^T \in S^n$$

$$S_1 = C - B^T A^{-1} B \qquad A \text{ invertible}$$

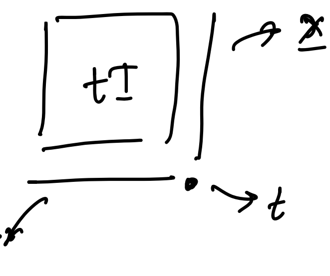
$$S_2 = A - B C^{-1} B^T \qquad C \text{ invertible}$$

Key result

(a) $X \succ 0 \iff \begin{matrix} S_1 \succ 0 \\ A \succ 0 \end{matrix} \iff \begin{matrix} S_2 \succ 0 \\ C \succ 0 \end{matrix}$

(b) $\det(X) = \det(A)\det(S_1) = \det(C)\det(S_2)$

Eg $\quad D = \begin{bmatrix} tI & \underline{x} \\ \underline{x}^T & t \end{bmatrix}$



$$A = tI \qquad B = \underline{x} \qquad C = t \qquad t > 0$$

$$S_1 = t - x^T t^{-1} x = t - \frac{x^T x}{t}$$

so $\quad D \succeq 0 \iff \begin{matrix} x^T x \leq t^2 \\ t > 0 \end{matrix}$ or $\begin{matrix} \|x\| \leq t \\ \text{(SOC)} \end{matrix}$