Composition Rules

Scalar: $f(x) = h(g(x))$ or $h \circ g(x)$

$f: \mathbb{R}^n \to \mathbb{R}$

$$\frac{d^2 f}{dx^2} = h''(g(x))(g'(x))^2 + g''(x)h'(g(x))$$

Sufficient conditions: when both terms ≥ 0

$h''(\cdot) \geq 0$ → h convex

$+$ $+$

$g''(\cdot) \geq 0$, $h'(\cdot) \geq 0$ or $g''(\cdot) \leq 0$, $h'(\cdot) \leq 0$

convex + non-decreasing concave + non-increasing

Summary:

g convex, h convex, non-decreasing
g concave, h concave, non-increasing

$h \circ g(x)$ convex

– also true when f non-differentiable

Eg: $e^{g(x)}$ $h(y) = e^y$ $g(x)$ ← need this to be convex

convex, non-decreasing

$g(x)$ convex $\Rightarrow e^{g(x)}$ convex

Fail case

Eg: $h(x) = x$ $\quad$ dom $f = [1,2]$ $\leftarrow$ restricted

$g(x) = x^2$ $\quad$ dom $g = \mathbb{R}$

$f(x) = h(g(x)) = x^2$

dom f : $\{x \mid g(x) \in [1,2]\}$

$x^2 \in [1,2]$

Rule: h convex, non-increasing in $[1,2]$,

g convex

but

dom f = $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

disjoint intervals

$\Rightarrow$ dom f not convex

$\Rightarrow$ f not convex

Contradiction due to domain restrictions

Fix : allow $f(x) \in \mathbb{R} \cup \{-\infty, \infty\}$

$\mathbb{R}$: real line $\quad \{-\infty < x < \infty\}$

$\mathbb{R} \cup \{-\infty, \infty\}$: extended real line : $\{-\infty \leq x \leq \infty\}$

Extended function :

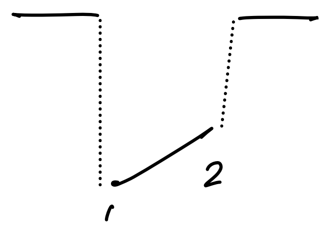
convex : $$\tilde{f}(\underline{x}) = \begin{cases} f(x) & \underline{x} \in \text{dom} f \\ \infty & x \notin \text{dom} f \end{cases}$$

concave : $$-\tilde{f}(\underline{x}) = \begin{cases} -f(x) & \underline{x} \in \text{dom} f \\ \infty & \underline{x} \notin \text{dom} f \end{cases}$$

Revised rule:

$$\left.\begin{array}{l} \tilde{g} \text{ convex}, \tilde{h} \text{ convex, non-decreasing} \\ \tilde{g} \text{ concave}, \tilde{h} \text{ concave, non-increasing} \end{array}\right\} \Rightarrow f \text{ convex}$$

Eg $\tilde{h}(x) = \begin{cases} x & x \in [1,2] \\ \infty & x \notin [1,2] \end{cases}$



not a non-decreasing function

Vector Composition

$$f(x) = h(g_1(x), g_2(x), \ldots, g_m(x)) \qquad x \in \mathbb{R}^m$$

$g_i : \mathbb{R}^n \to \mathbb{R}$

$h : \mathbb{R}^m \to \mathbb{R}$

$h(\underline{g}(\underline{x}))$

↳ vector-valued function

$\tilde{g}_i$ convex, $\tilde{h}$ convex, non-decreasing in each component

$\tilde{g}_i$ concave, $\tilde{h}$ convex, non-increasing in each component

Eg $h(\underline{x}) = \max\{x_i\}$ convex, non-decreasing in each component

$g_i(x) = e^{x_i}$ convex

$\Rightarrow \max\limits_i e^{x_i}$ convex