Approximation Problems

ℓ_1-norm minimization $\qquad \min_{x \in \mathbb{R}^n} \|Ax-b\|_1$

$b \in \mathbb{R}^m$

$$= \min_{\underline{x}} \sum_{i=1}^{m} |a_i^T x - b_i|$$

LP? (epigraph trick)

$$= \min_{(\underline{x},\underline{r})} \sum r_i$$

$r \in \mathbb{R}^m$ $\qquad |a_i^T x - b_i| \le r_i \qquad i=1,2..m$

$$-r_i \le a_i^T x - b_i \le r_i$$

$$\min_{(x,r)} \mathbb{1}^T r \qquad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$-r \le Ax-b \le r$$

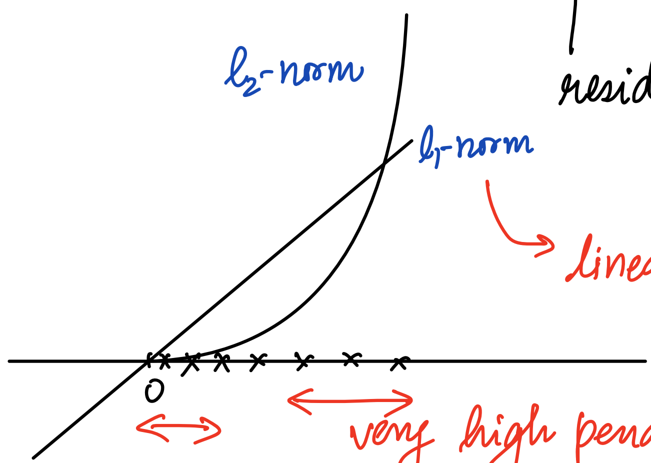
entrywise

Contrast with L.S. ? $\quad$ (ℓ_2- norm minimization)

(L.S.)
$$\min \sum_{i=1}^{m} r_i^2$$
$$-r_i \leq a_i^T x - b_i \leq r_i$$

← same constraint

residuals (bounds on error)



linear penalty (ℓ_1)

allows large residuals also

very high penalty (ℓ_2)

lower penalty (ℓ_2)

minimize # points with large residuals

ℓ_∞-norm minimization

$$\min_x \|Ax-b\|_\infty = \min_x \max_{1\leq i\leq m} |a_i^T x - b_i|$$

(epigraph trick)

$r \in \mathbb{R}$

$$= \min_{(x,r)} r$$
$$\max_{1\leq i\leq m} |a_i^T x - b_i| \leq r$$

$$= \min r$$
$$|a_i^T x - b_i| \leq r \quad \forall\ i = 1, \cdots, m$$
$$-r \leq a_i^T x - b_i \leq r$$

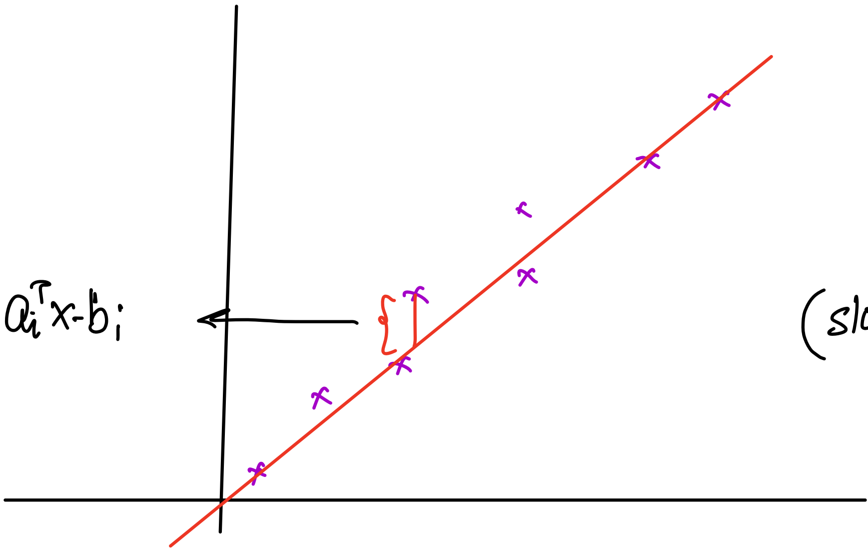
(minimize largest residuals)

Eg: Line fitting in $\mathbb{R}^2$ $(n=2)$

$r = Ax - b$ $\quad \underline{r} \in \mathbb{R}^m$ $\quad \underline{x} \in \mathbb{R}^2$

$A \in \mathbb{R}^{m \times 2}$ $\quad$ m points

$a_i^T x - b_i$



line fitting

(slope, abscissa) $\to x \in \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & a_1 \\ 1 & \vdots \\ 1 & \\ 1 & a_m \end{bmatrix}$$

$\underline{r} = Ax - b$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

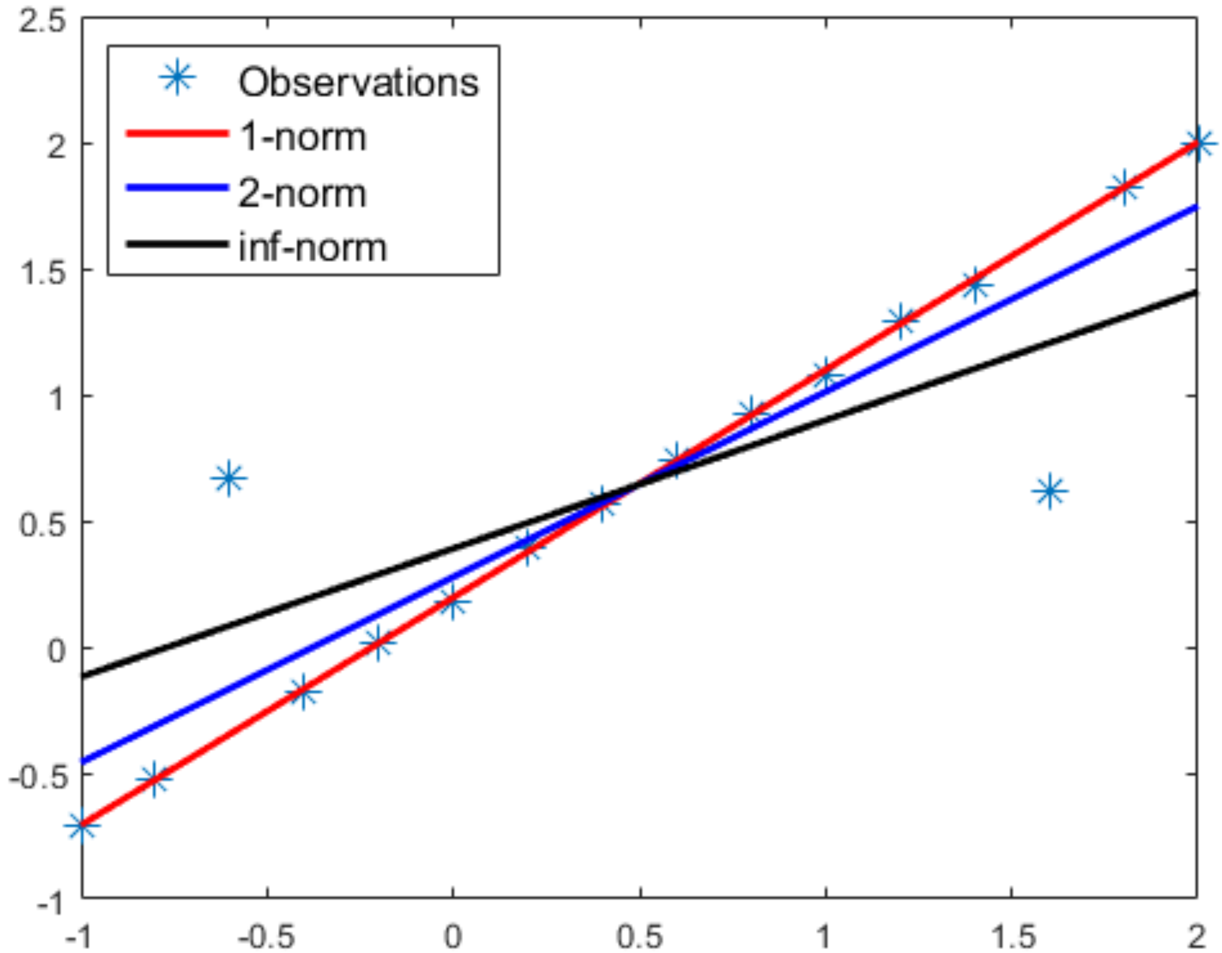
Given m points (A, b)

Goal: find $x \in \mathbb{R}^2$ $\quad p = 1, 2, \infty$

$$\min_x \|Ax - b\|_p$$

l_1	l_2	l_∞
$\sum r_i$	$\sum r_i^2$	$\max_i \lvert r_i \rvert$

$\underline{r} = Ax - b$



Observations
1-norm
2-norm
inf-norm
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2