Digital Communication Systems-1

Assignment-2

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1. (a) To show that the waveforms $f_n(t)(n=1,2)$ are orthogonal, we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t)f_n(t)dt = 0, \quad m \neq n$$

Clearly:

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \int_0^4 f_1(t)f_2(t)dt \\ &= \int_0^2 f_1(t)f_2(t)dt + \int_2^4 f_1(t)f_2(t)dt \\ &= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4}\times 2 - \frac{1}{4}\times(4-2) \\ &= 0. \end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy : $\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1$ for $i = 1,2$. Hence, they are orthonormal.

2. (a) As an orthonormal set of basis functions we consider the set

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad f_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}.$$

Hence, The representation vectors are $s_1 = [2\ -1\ -1\ -1]$, $s_2 = [-2\ 1\ 1\ 0]$, $s_3 = [1\ -1\ 1\ -1]$, $s_4 = [1\ -2\ -2\ 2]$.

3. (a) For any AWGN channel, the received signal is represented as $y = x + n$, where $n \sim \mathcal{N}(0, \sigma^2)$. Hence,

$$f_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n^2}{2\sigma^2}\right).$$

For $x = A$, the mean of the output is $E[y] = A$ and variance of the output is $var[y] = \sigma^2$. Therefore, the pdf of the output y can be given by

$$f_Y(y) = f_N(y-A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-A)^2}{2\sigma^2}\right).$$

4. (d) $f_{X,Y}(x,y)$ is a PDF, hence its integral over the supporting region of x, and y is 1

$$\begin{aligned}\int_0^\infty \int_y^\infty f_{X,Y}(x,y)dxdy &= \int_0^\infty \int_y^\infty Ke^{-x-y}dxdy \\ &= K\int_0^\infty e^{-y}\int_y^\infty e^{-x}dxdy \\ &= K\int_0^\infty e^{-2y}dy = K\left(-\frac{1}{2}\right)e^{-2y}\Big|_0^\infty = K\frac{1}{2}\end{aligned}$$

Thus K should be equal to 2 .

5.

$$\begin{aligned}E[X \mid Y=-y] &= \int_y^\infty Kxe^{-x+y}dx = 2e^y\int_y^\infty xe^{-x}dx \\ &= 2e^y\left[-xe^{-x}\big|_y^\infty + \int_y^\infty e^{-x}dx\right] \\ &= 2e^y\left(ye^{-y}+e^{-y}\right) = 2(y+1).\end{aligned}$$

So, none of the given options are correct.

6. (b) Since X has mean 1 and Variance 9, hence

$$\frac{X-1}{3} \sim \mathcal{N}(0,1)$$

$$\therefore \mathbb{P}[X>7] = Q\left(\frac{X-1}{3}\right)\Bigg|_{X=7} = Q(2).$$

7. (d) Please see the lecture slides.

8. (d)

a) $\mathbb{E}[X(t)] = \sum_{k=-\infty}^{\infty}\mathbb{E}[a_k]p(t-kT) = \mu\sum_{k=-\infty}^{\infty}p(t-kT)$ which is a periodic function with period T ($\mathbb{E}[a_k] = \mu$).

$$\begin{aligned}R_X(t_1+T,t_2+T) &= \mathbb{E}[\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_kp(t_1+T-kT)a_mp(t_2+T-mT)] \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathbb{E}[a_ka_m]p(t_1-(k-1)T)p(t_2-(m-1)T) \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathbb{E}[a_ka_m]p(t_1-kT)p(t_2-mT) = R_X(t_1,t_2)\end{aligned}$$

which is a periodic function and has a period T

b) $\mathbb{E}[x(t)] = \mathbb{E}[A\cos(2\pi ft+\theta)] = \frac{1}{2\pi}\int_0^{2\pi}A\cos(2\pi f_ct+\theta)d\theta = \frac{1}{2\pi}\int_{2\pi f_ct}^{2\pi f_ct+2\pi}A\cos(\alpha)d\alpha = 0$ and we have already shown that $R_{xx}(t,t+\tau) = \frac{A^2}{2}\cos(2\pi f\tau)$ which has time period $1/f$. Hence both mean and auto-correlation are periodic functions with same period $1/f$

c)

$$\mathbb{E}[x(t)] = \mathbb{E}[a(t)\cos(2\pi ft+\theta)] = a(t)\mathbb{E}[\cos(2\pi ft+\theta)] = 0$$

and

$$
\begin{aligned}
R_{xx}(t_1+T, t_2+T) &= \mathbb{E}[a(t_1+T)a(t_2+T)\cos(2\pi f(t_1+T)+\theta)\cos(2\pi f(t_2+T)+\theta)] \\
&= a(t_1+T)a(t_2+T)\mathbb{E}[\cos(2\pi f(t_1+T)+\theta)\cos(2\pi f(t_2+T)+\theta)] \\
&= a(t_1+T)a(t_2+T)\frac{1}{2}\cos(2\pi f(t_1-t_2)) + a(t_1+T)a(t_2+T)\frac{1}{2}\mathbb{E}[\cos(2\pi f(t_1+t_2+2T)+2\theta)] \\
&= \frac{1}{2}a(t_1+T)a(t_2+T)\cos(2\pi f(t_1-t_2))
\end{aligned}
$$

which is not cyclostationary unless $a(t)$ is periodic.

9. (b)

$$
\begin{aligned}
R_{ss}(t, t+\tau) &= \mathbb{E}[s(t)s(t+\tau)] \\
&= \mathbb{E}[A^2\sin(2\pi f_c t+\theta)\sin(2\pi f_c(t+\tau)+\theta)] \\
&= \frac{A^2}{2}\mathbb{E}[-\cos(2\pi f_c(2t+\tau)+2\theta)+\cos(2\pi f_c\tau)] \\
&= \frac{A^2}{2}\mathbb{E}[-\cos(2\pi f_c(2t+\tau)+2\theta)] + \frac{A^2}{2}\cos(2\pi f_c\tau) \\
&= \frac{A^2}{2}\cos(2\pi f_c\tau)
\end{aligned}
$$

10. (d)

$$
\mathbb{E}[u(t)] = \mathbb{E}[X\cos(2\pi f_c t)+Y\cos(2\pi f_c t)] = \mathbb{E}[X]\cos(2\pi f_c t)+\mathbb{E}[Y]\sin(2\pi f_c t) = 0
$$

$$
\begin{aligned}
R_{uu}(t, t+\tau) &= \mathbb{E}[u(t)u(t+\tau)] \\
&= \mathbb{E}[(X\cos(2\pi f_c t)+Y\sin(2\pi f_c t))\,(X\cos(2\pi f_c(t+\tau))+Y\sin(2\pi f_c(t+\tau)))] \\
&= \mathbb{E}[X^2]\cos(2\pi f_c t)\cos(2\pi f_c(t+\tau)) + \mathbb{E}[Y^2]\sin(2\pi f_c t)\sin(2\pi f_c(t+\tau)) \\
&\qquad + \mathbb{E}[XY]\,(\cos(2\pi f_c t)\sin(2\pi f_c(t+\tau)) + \cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)) \\
&= \cos(2\pi f_c t)\cos(2\pi f_c(t+\tau)) + \sin(2\pi f_c t)\sin(2\pi f_c(t+\tau)) \\
&= \cos(2\pi f_c\tau)
\end{aligned}
$$

Clearly since the mean is constant and the auto-correlation function $R_{uu}(t_1, t_2) = R_{uu}(t_2-t_1)$ is only a function of $t_2 - t_1$ Hence the above process is wide sense stationary.