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OPERATIONS RESEARCH

About the Author

P. Sankara Iyer is presently Retired Reader in Mathematics at Alagappa University, Karaikudi in Tamil Nadu. After holding the position of a Lecturer in Mathematics at Alagappa College from 1966 to 1981, he became Professor in 1981. He then joined Alagappa University in 1985 and retired as Reader in Mathematics in the year 2001. He is M.Sc. (Mathematics), M.Phil. (Mathematics) and Ph.D. (Computer Applications in Operations Research). He has done a Certificate course in Computer Programming and a Certificate course in Evaluation Methodology and Examination by A.I.U. His teaching experience is vast—B.Sc. (19 yrs), M.Sc. (19 yrs), M.C.A. (15 yrs) and M.Phil. (18 yrs). He has taught all subjects at B.Sc. level; Operations Research, Differential Equations, Differential Geometry, Classical Mechanics, Space Mechanics, Quantum Mechanics, Algebra, Statistics and Numerical Analysis at M.Sc. level; Resource Management Techniques (OR) and Computer based Numerical Methods at M.C.A. level; and Algebra at M.Phil. level. His study material for M.Sc. Mathematics (Operations Research and Differential Equations), M.Sc. Industrial Mathematics (Operations Research)—2 books, M.C.A. (Resource Management Techniques) and M.B.A. (Operations Research) of Alagappa University book are very popular among the students. He is a member of (i) Operations Research Society of India, (ii) Ramanujan Mathematical Society and (iii) Forum for Advancement of Science and Human Development.

Prof. Sankara Iyer's primary area of research is Operations Research: Fractional Programming (Computer Applications). He has published seven research papers and guided more than 50 students for M.Phil. Research Project. He has attended twelve conferences and seminars in Algebra, Operations Research, Analysis Statistics and Stochastic Processes. He has been the Cultural Coordinator of Alagappa University for 15 years, Director of Social Service in Alagappa College for 10 Years, Director of Fine Arts in Alagappa College for 8 Years and Secretary of Staff Club for 6 Years. He is musically inclined and has composed and tuned nearly 200 songs and given public performances. He has written and directed social dramas (besides acting as the main protagonist), promoting national integration, communal harmony and eradication of dowry. Tamil Isai Sangam, Karaikudi conferred the title 'MUTHAMIZH SELVAR' (year 2001), in appreciation of his service in the field of literature, music, and drama. He gives a musical discourse every year on the life of Saint Thyagaraja during Thyagaraja Aradhana Festival at Tiruvaiyaru in Thanjavur District, Tamil Nadu.

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OPERATIONS RESEARCH

P. Sankara Iyer
Reader in Mathematics (Retd)
Alagappa University
Karaikudi, Tamil Nadu

Tata McGraw-Hill Publishing Company Limited
NEW DELHI

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New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

The McGraw·Hill Companies

Tata McGraw-Hill

Published by Tata McGraw-Hill Publishing Company Limited,
7 West Patel Nagar, New Delhi 110 008.

ISBN (13): 978-0-07-066902-4
ISBN (10): 0-07-066902-3

Managing Director: *Ajay Shukla*
General Manager: Publishing—SEM & Tech Ed: *Vibha Mahajan*
Sponsoring Editor: *Shalini Jha*
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General Manager: Marketing—Higher Education & School: *Michael J Cruz*
Product Manager: SEM & Tech Ed: *Biju Ganesan*

Controller—Production: *Rajender P Ghansela*
Asst. General Manager—Production: *B L Dogra*

Typeset at Text-o-Graphics, B-1/56, Arawali Appartment, Sector 34, Noida 201 301, and printed at
Lalit Offset, 219, F.I.E., Patpar Ganj Industrial Area, Delhi - 110 092

Cover Design: Rashtriya Printers

RZLCRRAXRQBLY

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Foreword

I had the golden opportunity of being a student of Prof. P. Sankara Iyer, the author of this book. He is a model teacher and we (all his students) consider him as our great Guru and Guide. I feel proud to write this foreword for a book written by my beloved teacher.

Mathematics is the source of all knowledge. It is a vast ocean and the author of this book has dived deep into a particular branch of this subject, namely Operations Research.

The contents of this book have been arranged on the principle of gradation. The chapters are in a logical sequence: Introduction, Linear Programming, Integer Programming, Transportation, Assignment, Decision Theory, Games Theory, Dynamic Programming, Sequencing, Queueing, Inventory, Replacement, PERT and CPM, and Simulation.

There are two distinctive features that make this book stand apart. The first one is that the examples and exercises provided in each chapter are indeed a boon to the students and they can assess their understanding of each chapter clearly. Another unique feature is that the entire book has been based on a self-taught new method, culminating from Prof. Sankara Iyer's rich teaching experience of 35 years. Also, while structuring this book, he has kept both the instructor and the student in his purview, and hence it will prove to be a valuable resource for them.

The book covers the syllabi of almost all Indian universities. Therefore not only students of Mathematics but also students of M.C.A., M.B.A., C.A. and I.C.W.A. will benefit from this book.

I earnestly wish that the student community and faculty members derive maximum benefit from this magnificent creation.

Dr V. VEMBAIYAN
Principal,
Alagappa Govt. Arts College,
Karaikudi, Tamil Nadu.

Preface

In recent years, Operations Research—the mathematical analysis of a process, used in making decisions—has become a very popular branch of mathematics studied by undergraduate and postgraduate students. This interdisciplinary branch of Applied Mathematics has wide applications in various types of real life situations. A large number of research papers have been published on this subject and a great deal of research is being carried out.

Purpose

The primary objective of writing this book is to provide the students with a basic knowledge of the processes involved in Operations Research and the enormous practical applications of the techniques to daily life problems. Presently, there are limited good books in the market, with even fewer offering well illustrated problems and also, the style of presentation in foreign books is not very helpful to our Indian students—this generates a need for a single book that covers all the topics of this subject in order to cater to the requirement of students in various universities. This book fulfills the purpose.

Users

This book is primarily meant for the course on Operations Research to be studied by the engineering students of all branches. It offers coverage of different topics pertaining to the syllabus prescribed for the students of Mathematics and Statistics (UG and PG), M.C.A., M.B.A., M.Com., B.E. and B.Tech. of Indian Universities and also, C.A. and I.C.W.A. This book is a fine reference for competitive (UG and PG levels) and I.A.S. examinations.

Highlights

What makes this book unique is its lucid presentation of the subject aided by plenty of solved and unsolved problems. Brief yet adequate theory clearly explained in simple language supported by worked out examples (with step by step solution) and exercise problems facilitate quick learning by the students. The chapters have been structured in a planned manner so that the students can study the subject independently.

It is a comprehensive text which helps students in understanding problem solving methods based on model formulation, solution procedure and analysis. The salient features of the book are

(i) Provides sound knowledge of the OR techniques in managerial decision making.

(ii) Incorporates important chapters like Sequencing Problems, Replacement Models, Project Scheduling, Artificial Variables and Bounded Variable Techniques.

(iii) Numerous solved examples give an overall view of the applications of OR techniques. Solved examples from various university question papers are an effective tool for examinations.
(iv) To clarify concepts, figures and tables are used exhaustively.
(v) To carry out self evaluation, students can solve multitude of exercises that are provided in each chapter along with their answers.

The Sigma Format

This book is based on the Sigma Format that offers a clear understanding of fundamental theory supported with exhaustive examples, solved using a step-by-step approach. To get a better grasp on the subject, lots of exercises along with answers are provided in the chapters. This well structured format has proven to be a handy resource material for students as well as teachers.

Chapters Organisation

In the first chapter, the importance of the study of OR and its various applications have been discussed. Chapter 2 describes the method of mathematical formulation of a real life problem (in the form of a Linear Programming Problem) and the graphical method of solving an L.P.P having two decision variables. Simplex Method, an important method of solving a general L.P.P has been discussed in detail in chapter 3. Various modifications of simplex method such as Big M method, Two Phase Method, Dual Simplex Method, Principle of Duality, and Revised Simplex Method have been covered in the subsequent chapters 4, 5, 6 and 7. When the variables are restricted to assume values within certain given limits, we have to apply Bounded Variable Technique described in chapter 8. Another restriction that the variables can take only integer values, leads to Integer Programming which is given in chapter 9. Also, a study of the effect on the optimal solution due to changes in the values of the parameters, which is called Sensitivity Analysis, has been discussed in chapter 10.

Then follow chapters 11, 12, 13, 14, 15 and 16 dealing with applications of OR techniques namely Transportation and Assignment Problems, Decision Theory, Game Theory, Dynamic Programming and Sequencing of Jobs. Another set of applications have been provided in detail in chapters 17 (Queueing theory), 18 (Inventory models), 19 (Replacement theory), and 20 (P.E.R.T - C.P.M). Simulation Techniques of solving different types of problems have been taken up for discussion in the last chapter (chapter 21).

Acknowledgements

I would like to express my gratitude to all the authors who were a source of inspiration while writing this book. I wish to express my indebtedness to my teachers and well wishers. I wish to acknowledge my sincere thanks to late Dr M K Venkataraman for his inspiration and encouragement in this project. My thanks are also due to my wife and children, and M.Phil. Scholar, R G N Nagarajan of Alagappa University for their continuous help and cooperation in fulfilling the project. I express my deepest sense of gratitude to the following reviewers, whose suggestions have helped me bring the text to its present form.

J S Oberoi
Baba Banda Singh Bahadur Institute of Engineering,
Fatehgarh sahib, Punjab

S K Tiwari
National Institute of Technology, Jalandhar

C M Agarwal
Maulana Azad National Institute of Technology, Bhopal

Rajiv Lochan
The Institute of Chartered Financial Analysts of India, Jaipur

S R Kapoor
Dept. of Electrical Engineering
Kota Engineering College, Kota

Sinha Roy
Dept. of Mathematics
Future Institute of Technology, Kolkata

Shila Das
Dept. of Statistics
Dibrugarh University, Assam

Chakradhar Dash
N M Institute of Technology
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Reena Pant
Mumbai University, Mumbai

Arun Kumar,
M H Saboo Siddik College of Engineering, Mumbai

S Srinivasan
Crescent Engineering College, Chennai

S Nadarajan
Dept. of Mechanical Engineering
GCT, Coimbatore

A Rajesh
Dept. of Mechanical Engineering
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Coimbatore

A Ramakrishna Rao
Adi Shankara Instiute of Engineering and Technology,
Kalady

K R Divakara Roy
Dept. of Mechanical Engineering,
College of Engineering
Andhra University, Visakhapatnam

T. Chandrashekar
Dept. of Mechanical Engineering,
M S Ramaiah Institute of Technology, Bengaluru

I am truly grateful to the publishers at McGraw-Hill Education India for coming forward to publish this book and for their efficient management of the project. I am sure that this book will receive good response from the teachers and the taught. Suggestions from the readers for the improvement of the book are most welcome and will be duly acknowledged and incorporated.

P. Sankara Iyer
Email id: psankaraiyer@yahoo.com

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1

Operations Research

CONCEPT REVIEW

1.1 WHAT IS OPERATIONS RESEARCH?

The term Operations Research (OR) relates to military operations during the second World War. Scientists used various techniques to deal with strategic and tactical problems during the war. After the war military OR group scientists tried to apply OR techniques to civilian problems relating to business, industry, and research development.

During the 1950s educational institutions introduced OR in their curricula. Today service organizations such as airlines, railways, hospitals, libraries and banks employ OR to improve their efficiency. OR is a science which provides mathematical techniques and algorithms for solving decision problems. It is also an art in the sense that it deals with collecting data for model construction, validation of the model and implementation of the solution obtained.

In other words, OR is a scientific approach to problem solving for executive management. The problems involve integrated systems of men, machines and materials.

1.2 CHARACTERISTICS OF OR

An important characteristic of OR is that it deals with a problem as a whole. Before arriving at a decision it takes into consideration all possible interactions between various parts or departments of an organization. For example, if a problem on inventory management is taken, the production department will be interested in continuous production, so as to reduce set-up cost and other maintenance costs. But this will give rise to problems in finance, marketing and other departments. Hence it is necessary to analyse the factors affecting all parts of the organization and prepare a mathematical model. A solution to this model will optimize the profits of the organization.

Another characteristic is that OR involves a team of experts from different scientific and engineering disciplines. After a careful analysis of the problem, a suitable mathematical model is prepared and this model is solved using scientific techniques. Thus a decision or solution is obtained which gives an optimal solution to the problem.

The importance of OR has been felt very much in industries. Inventory management, replacement of equipments, deterministic and probabilistic decision-making in resource allocation, production planning, sequencing of jobs, assignments of jobs to machines and so many other problems are solved using OR techniques.

1.3 PHASES OF OR

The first step is formulation of the problem. We have to identify the objective, the decision variables and the constraints involving the variables. After that a mathematical model has to be constructed, with an objective function to be optimized and constraints in the form of equalities or inequalities. The solution of the model can be obtained by analytic, iterative or Monte-Carlo method depending on the structure of the model. These mathematical models are absolutely free from the influence of qualitative or emotional human factors which affect decision-making.

1.4 SCOPE OF OR

Some of the industrial and business problems which can be analyzed by OR technique include

1. ***Finance and Accounting*** Investment and portfolio management, auditing, cash flow analysis, capital budgeting etc.
2. ***Marketing*** Product mix, marketing, planning, sales, allocation and assignment, advertising and media planning etc.
3. ***Purchasing and procurement*** Optimal buying, re-ordering, transportation, replacement etc.
4. ***Production*** Warehouse, distribution, transportation, production planning, blending, inventory control, maintenance, project scheduling, allocation of resources, etc.
5. ***Management*** Manpower planning, project management, etc.

1.5 DRAWBACKS AND DIFFICULTIES OF OR

There are certain aspects of OR which are difficult and which need careful approach. First, formulation of the problem is very important. The problem must be properly selected and accurately defined, which involves some difficulties. Data collection may also consume a large portion of time and money. Also OR analysis is mainly based on past observations. There is no guarantee that the same set of conditions would continue in future also. Again decisions must be made at the right time and any delay may cause loss or damage. Also, the cost involved in the project is an important factor. It is the responsibility of OR experts to translate the concepts and ideas into simple procedures easily comprehensible to managers and workers alike. They should also ensure that the new proposals are properly implemented. Only then OR study will be useful to humanity.

2

Linear Programming

CONCEPT REVIEW

2.1 INTRODUCTION

Linear programming is one of the various techniques of optimization (Maximizing the gain and minimizing the loss). It deals with problems of allocating limited resources to obtain a desired optimal solution, subject to a set of given conditions (constraints).

A linear programming problem (LPP) consists of an objective function which is to be optimized and a set of constraints to be satisfied by the variables. Usually the problem is of the form

Maximize (or Minimize)

$$Z = C_1x_1 + C_2x_2 + C_3x_3 + \dots + C_nx_n$$

subject to the constraints

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ (\geq, \leq, \text{ or } =)\ b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\geq, \leq, \text{ or } =)\ b_2 \\ &\ \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\geq, \leq, \text{ or } =)\ b_m \\ &x_1, x_2, x_3, \dots, x_n > 0. \end{aligned}$$

The objective function is Z and the variables $x_1, x_2, \dots, x_n$ are called *decision variables*. The constraints are linear inequalities or equalities and they correspond to the conditions on the use of the available resources. The objective function may correspond to total cost, profit, total quantity to be produced and other similar objectives.

2.2 FORMULATION

Given a situation, describing the availability of resources and conditions and the object to be optimized, we have to convert it into a mathematical model. This process is called *formulation*. It consists of the following steps:

Step 1 Defining the decision variables and writing the objective function.

Step 2 Expressing the conditions of availability of the resources in the form of linear inequalities or equalities using the decision variables.

Example 2.1 (Production Allocation Problem) *A company manufacturers three items A, B and C. These items are processed on three machines M_1, M_2 and M_3. The time required for each product in each machine is given below. Also the total time of availability of each machine is given.*

Table 2.1

Machine	Time per unit (hours) A	B	C	Availability of the machine hr/day
M_1	1	1.5	2	18
M_2	2	1	1	20
M_3	1	2	2	16

The company gets a profit of Rs 20, Rs 30 and Rs 40 per unit of A, B and C respectively. Determine the number of units of each product to be manufactured per day in order that the total profit is maximum.

Formulation The objective is to determine the number of units to be manufactured. Hence the decision variables are x_1, x_2, x_3 corresponding to the number of units of A, B and C respectively. The profits are Rs 20, Rs 30 and Rs 40 per unit of the products. Thus the objective function becomes

$$Z = 20x_1 + 30x_2 + 40x_3$$

which is to be maximized.

Considering the machine M_1 whose availability is 18 hrs we have the condition that the total time taken by M_1 should not exceed 18 hrs. Therefore we get a constraint

$$(1)x_1 + (1.5)x_2 + (2)x_3 \le 18$$

$$\text{i.e., } x_1 + 3/2\, x_2 + 2x_3 \le 18$$

Similarly for M_2 and M_3 we get

$$2x_1 + x_2 + x_3 \le 20$$

$$x_1 + 2x_2 + 2x_3 \le 16$$

Also the number of units cannot assume negative value and this leads to the constraints $x_1 \ge 0$, $x_2 \ge 0$ and $x_3 \ge 0$.

Now the process of formulation is complete. The linear programming problem has the form.

Maximize $Z = 20x_1 + 30x_2 + 40x_3$ subject to the constraints

$$x_1 + 3/2x_2 + 2x_3 \le 18$$
$$2x_1 + x_2 + x_3 \le 20$$
$$x_1 + 2x_2 + 2x_3 \le 16$$
$$x_1, x_2, x_3 \ge 0.$$

Example 2.2 (Advertising media selection problem) A company wants to work out a plan for advertising its product in three different media, TV, Radio and Newspaper. The details are given by the following table:

Table 2.2

	TV Rs	Radio Rs	Newspaper Rs
Cost of advertising	4000	3000	1000
Number of customers	20000	30000	20000
Impact rate	.8	.6	.5

The company cannot spend more than Rs 80000 for advertising. Advertisement should be given atleast once in TV. It is required to determine the number of times the advertisement should be given in each medium so that the rate exposure is maximum.

[Rate exposure = (impact rate) (number of customers)]

Formulation It is required to determine the number of times the advertisement is to be given in each medium. Therefore we take the decision variables x_1, x_2 and x_3 to represent the frequency in TV, Radio and Newspaper respectively.

The rate exposure of one advertisement in TV is given by (0.8)(20000). In Radio it is given by (0.6)(30000) and in Newspaper it is (0.5)(20000). Therefore the total rate exposure is

$$(0.8)(20000)x_1 + (0.6)(30000)x_2 + (0.5)(20000)x_3$$

and this is to be maximized. Hence the objective function is

$$Z = 16000x_1 + 18000x_2 + 10000x_3$$

The total cost of advertising is

$4000x_1 + 3000x_2 + 1000x_3$ and this should not exceed Rs 80000. This leads to the constraint

$$4000x_1 + 3000x_2 + 1000x_3 \le 80000$$

Also at least one advertisement has to be given in TV and this gives another constraint $x_1 \ge 1$.

Thus the LPP is of the form

Maximize $Z = 16000x_1 + 18000x_2 + 10000x_3$, subject to the constraints

$$4000x_1 + 3000x_2 + 1000x_3 \le 80000$$
$$x_1 \ge 1$$
$$x_2, x_3 \ge 0.$$

Example 2.3 A soft drink plant has two bottling machines A and B manufacturing 1 litre and 2 litre bottles. The following table gives the manufacturing data:

Machine	*1 litre bottle*	*2 litre bottle*
A	100/min	60/min
B	50/min	80/min

The machines can be run 8 hours per day, 5 days per week. Weekly production of the drink cannot exceed 3,00,000 litre and the demand for 1 litre and 2 litre bottles are 25,000 and 8000 per week respectively. The manufacturer gets a profit of 50 paise on 1 litre bottle and 75 paise on 2 litre bottle. It is required to find the number of bottles to be manufactured in each type so as to maximize the total profit.

Formulation Let us assume that x_1 units of 1 litre bottle and x_2 units of 2 litre bottle be produced weekly. The total profit is Rs $(0.5)x_1 + (0.75)x_2$ and this has to be maximized. Thus we get the objective function $Z = (0.5)x_1 + (0.75)x_2$.

The machines run for 5 days per week at the rate of 8 hrs per day. Hence the total time they run per week is 40 hrs or 2400 min. On machine A, 100 units of 1 litre and 60 units of 2 litre bottles are produced per minute. Therefore we get the constraint $x_1/100 + x_2/60 \le 2400$.

Similarly, on machine B, we get the constraint

$$x_1/50 + x_2/80 \le 2400.$$

Total weekly production cannot exceed 3,00,000

$$x_1 + x_2 \le 300000.$$

Considering the demand for the items in the market, we get the constraints,

$$x_1 \ge 25000 \text{ and } x_2 \ge 8000.$$

Thus the formulated LPP is

Maximize $Z = (0.5)x_1 + (0.75)x_2$ subject to the constraints

$$x_1/100 + x_2/60 \le 2400 \text{ (or) } 3x_1 + 5x_2 \le 720000$$
$$x_1/50 + x_2/80 \le 2400 \text{ (or) } 8x_1 + 5x_2 \le 960000$$
$$x_1 + x_2 \le 300000$$
$$x_1 \ge 25000$$
$$x_2 \ge 8000.$$

Example 2.4 (Diet Problem) In our daily diet, proteins, fats and carbohydrates are required with a minimum of 5, 2 and 3 units respectively. We consider the foodstuff consisting of bread, butter and milk. The yields of these nutritional requirements and the cost per unit of the food items are given below:

Table 2.3

Food	*Yield per unit*			*Cost per unit of food*
	Protein	*Fat*	*Carbohydrate*	
Bread	4	1	2	12
Butter	3	2	1	60
Milk	3	2	1	7
Minimum requirement	5	2	3	

It is required to determine the combination of the food items such that minimum requirement is satisfied and the total cost of the food is minimum.

Formulation Let us assume that the numbers of units of bread, butter and milk to be consumed are x_1, x_2 and x_3 respectively. The total cost of the food is $12x_1 + 60x_2 + 7x_3$ and this has to be minimized. Hence the objective function is

$$Z = 12x_1 + 60x_2 + 7x_3.$$

Considering the minimum requirement of protein per day we get

$$4x_1 + 3x_2 + 3x_3 \ge 5.$$

Similarly, we get the constraints for the minimum requirement of fat and carbohydrates as

$$x_1 + 2x_2 + 2x_3 \ge 2 \text{ and}$$
$$2x_1 + x_2 + x_3 \ge 3.$$

Thus the formulation of LPP is

Minimize $Z = 12x_1 + 60x_2 + 7x_3$ subject to the constraints

$$4x_1 + 3x_2 + 3x_3 \ge 5$$
$$x_1 + 2x_2 + 2x_3 \ge 2$$
$$2x_1 + x_2 + x_3 \ge 3$$
$$x_1, x_2, x_3 \ge 0.$$

Example 2.5 (Blending Problem) A firm manufactures an alloy having the following specifications:

(i) Specific gravity ≤ 0.95
(ii) Chromium $\geq 10\%$
(iii) Melting point $\geq 420°C$.

Details of raw materials *A*, *B* and *C* and their properties are given below:

Table 2.4

Property	*Raw Material* *A*	*B*	*C*
Specific gravity	0.92	0.96	1.05
Chromium	8%	12%	15%
Melting point	380°C	400°C	480°C

Cost of the raw materials are Rs 100, Rs 300 and Rs 50 per ton for *A*, *B* and *C* respectively. It is required to determine the proportion in which the raw materials are to be used to manufacture the alloy such that the total cost is minimum.

Formulation It is assumed that x_1, x_2 and x_3 tons of raw materials *A*, *B* and *C* be used for making the alloy. The total cost of the raw materials is $100x_1 + 300x_2 + 50x_3$ and this has to be minimized.

Total specific gravity is $0.92x_1 + 0.96x_2 + 1.05x_3$ and this should be less than 0.95. Therefore we get a constraint $0.92x_1 + 0.96x_2 + 1.05x_3 < 0.95$.

It is given that the chromium content should be greater than 10%. Hence we must have $8x_1 + 12x_2 + 15x_3 > 10$.

Also the melting point of the alloy has to be greater than 420°C. This constraint can be written as

$$380x_1 + 400x_2 + 480x_3 \geq 420.$$

Thus the LPP can be written as

Minimize $Z = 100x_1 + 300x_2 + 50x_3$

subject to the constraints

$$0.92x_1 + 0.96x_2 + 1.05x_3 \leq 0.95$$
$$8x_1 + 12x_2 + 15x_3 \geq 10.$$
$$380x_1 + 400x_2 + 480x_3 \geq 420$$
$$x_1, x_2, x_3 \geq 0$$

Example 2.6 Three grades of coal *A*, *B* and *C* contain phosphorus and ash as impurities. For a particular industrial process a maximum of 100 tons of coal is required which should contain not more than 3% of ash and not more than 0.03% of phosphorus. The percentage of impurities and the profits from each grade are given below

Table 2.5

Coal	*Phosphorus (%)*	*Ash (%)*	*Profit (Rs/ton)*
A	0.02	3	12
B	0.04	2	15
C	0.03	5	14

Find the proportions in which the three grades are to be used in order to maximize the profit.

Formulation Let x_1, x_2, x_3 be the quantities in tons of grades *A*, *B* and *C* respectively. Then the objective function (profit) to be maximized is

$$Z = 12x_1 + 15x_2 + 14x_3$$

Phosphorus content should not exceed 0.03%. Therefore we get a constraint

$$0.02x_1 + 0.04x_2 + 0.03x_3 \leq 0.03\ (x_1 + x_2 + x_3)$$

Equivalently, $2x_1 + 4x_2 + 3x_3 \leq 3\ (x_1 + x_2 + x_3)$

or $-x_1 + x_2 \leq 0$

Considering the constraint on the ash contents, we get

$$3x_1 + 2x_2 + 5x_3 \leq 3\ (x_1 + x_2 + x_3)$$

or $-x_2 + 2x_3 \leq 0$

Also the total quantity required is not more than 100. So we have $x_1 + x_2 + x_3 \leq 100$.

Thus the formulated LPP is

Minimize $Z = 12x_1 + 15x_2 + 14x_3$ subject to the constraints

$$-x_1 + x_2 \leq 0$$
$$-x_2 + 2x_3 \leq 0$$
$$x_1 + x_2 + x_3 \leq 100$$
$$x_1, x_2, x_3 \geq 0$$

Example 2.7 A farmer cultivates tomato, carrot and potato in his farm of area 100 acres.

Average yield per acre is 2000 kg of tomato, 2000 kg of carrot and 1000 kg of potato. The cost of the fertilizer is Rs 3 per kg and the requirement of fertilizer is 100 kg each for tomato and carrot and 60 kg for potato per acre. Labour required is 5 man-days for tomatoes and potatoes and 6 man-days for carrot per acre. A total of 400 man-days are available at Rs 60 per man-day.

Formulate this problem as a LPP so as to maximize the farmer's profit if he can sell the items at Rs 5 per kg for tomato, Rs 8 per kg for carrot and Rs 10 per kg for potato.

Formulation Let us assume that the farmer cultivates tomato, carrot and potato in x_1, x_2 and x_3 acres respectively. Then the total sale of item is Rs $(2000x_1\,(5) + 2000x_2\,(8) + 1000x_3\,(10))$.

Total expenditure for fertilizer is

$$\text{Rs } 3\,(100x_1 + 100x_2 + 60x_3)$$

Labour cost = Rs 60 $(5x_1 + 6x_2 + 5x_3)$

The total profit P = total sales – total cost.

$$P = \text{Rs } (10000x_1 + 16000x_2 + 10000x_3) - (300x_1 + 300x_2 + 180x_3) - (300x_1 + 360x_2 + 300x_3)$$

i.e. P = Rs $(9400x_1 + 15340x_2 + 9520x_3)$

which is to be maximized.

Total area ≤ 100 acres

$\therefore \quad x_1 + x_2 + x_3 \le 100$

Total labour is restricted to 400 man-days

i.e., $5x_1 + 6x_2 + 5x_3 \le 400$

Thus the LPP is

Maximize $P = 9400x_1 + 15340x_2 + 9520x_3$

subject to the constraints

$$x_1 + x_2 + x_3 \le 100$$
$$5x_1 + 6x_2 + 5x_3 \le 400$$
$$x_1, x_2, x_3 \ge 0$$

EXERCISES

(Formulate into LPP)

1. Three articles A, B and C have weight, volume and cost as given below: The total weight cannot exceed 2000 units and total volume cannot exceed 2500 units. Find the number of articles to be selected from each type such that the total cost is maximum.

	Weight	*Volume*	*Cost (Rs)*
A	4	9	5
B	8	7	6
C	2	4	3

2. An electronic company produces three types of parts for washing machine. It purchases unfinished articles from a foundry and does the jobs of drilling, shaping and polishing. The selling prices of the parts A, B and C are Rs 8, Rs 10 and Rs 14 respectively. Their purchase costs are Rs 5, Rs 6 and Rs 10 respectively. The company has only one machine for each job. Machine costs per hour are Rs 20 for drilling, Rs 30 for shaping and Rs 30 for polishing. The capacities of each machine (parts per hour) are given below:

Machine	*Capacity per hour* *A*	*B*	*C*
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

It is required to determine how many parts of each type should be produced per hour in order to maximize the profit.

3. A furniture manufacturing company plans to make chairs and tables. The requirements are given below:

Item	*Requirement* *Timber (in board ft)*	*Man-hours*	*Profit (Rs)*
Chair	5	10	45
Table	20	15	80

Totally 400 board feet of timber and 450 man-hours are available. Determine the number of chairs and tables to be manufactured in order to maximize the profit.

4. A firm manufactures four different machine parts *A*, *B*, *C* and *D* using copper and zinc. The requirement of copper and zinc for each part and their availability and the profit earned from each part are given below:

Item	*Requirement (kg)*		*Profit (Rs)*
	Copper	*Zinc*	
A	5	2	12
B	4	3	8
C	2	8	14
D	1	1	10
Availability	100	75	

How many of each part should be manufactured in order to maximize the profit?

5. *A* company produces three types of products *A*, *B* and *C*. The production department produces components sufficient to make 50 units of *A*, 25 units of *B* and 30 units of *C* per day. In the assembly department, only 100 man hours are available to assemble the products daily. The assembly times for *A*, *B* and *C* are 0.8 hr, 1.7 hrs and 2.5 hrs respectively and the profits on them are Rs 12, Rs 20 and Rs 45 respectively. The company has an order commitment of 20 units of *A* and a total of 15 units of *B* and *C*. How many of each product are to be produced so that the profit is maximum?

6. A company sells two products *A* and *B* making a profit of Rs 40 and Rs 30 per unit respectively. The total capacity of the production process is 30000 man-hours. It takes three hours to produce one unit of *A* and one hour to produce one unit of *B*. It has been found that a maximum of 8000 units of *A* and 12000 units of *B* can be sold in the market. Find the number of units of *A* and that of *B* to be produced so as to get maximum profit.

7. An oil refinery has to decide on the optimal mix of two possible blending processes. The input and output per production run are given below.

Process	*Input*		*Output*	
	Crude A	*Crude B*	*Gasoline X*	*Gasoline Y*
1	5	3	5	8
2	4	5	4	4

The maximum quantity of crude *A* and *B* available are 200 units and 150 units respectively. At least 100 units of *X* and 80 units of *Y* are required in the market. The profit per production run from process 1 and process 2 are Rs 300 and Rs 400 respectively. Determine the optimal number of production runs of process 1 and process 2.

8. M.K.V. Foods company is developing a diet supplement called Hi-Pro. The specifications and the calorie, protein and vitamin content of three basic foods are given below:

Nutritional elements	*Units of nutritional elements per 100 gm*			
	Basic foods			*Hi-Pro*
	F_1	F_2	F_3	*specifications*
Calories	350	250	200	300
Proteins	250	300	150	200
Vitamin A	100	150	75	100
Vitamin C	75	125	150	100
Cost per 100 g (Rs)	1.50	2.00	1.20	

Determine the quantities of F_1, F_2, F_3 to be used to meet the requirements with minimum cost.

9. A sports goods company manufactures two types of soccer balls *X* and *Y*. There are two types of employees—semi-skilled and skilled. The time required for each type of ball and the time available per week are given below:

Type of employee	*Time required (hrs)*		*Time available (hrs per week)*
	Ball X	*Ball Y*	
Semi-skilled	2	3	80
Skilled	4	6	150

The wages for semi-skilled and skilled employees are Rs 6 and Rs 10 per hour respectively. At least 15 balls of type *X* and at least 10 balls of type *Y* must be manufactured in order to meet with the weekly demand. Determine the number of balls of each type to be manufactured so that the cost of production is minimum.

10. A media specialist has to decide in the allocation of advertising in three media 1, 2 and 3. For advertising once, the costs are Rs 1000, Rs 750 and Rs 500 respectively. The total annual allotment of funds is Rs 20000. Media 1 is a monthly magazine and it is desired to advertise not more than once in one issue. At least six messages should appear in media 2 while the number of advertisements in media 3 must be between 4 and 8. The expected effective audiences for one advertisement in each media are 80,000, 60,000 and 45,000 respectively. Determine the optimal allocation that would maximize the total effective audience.
11. A company produces refrigerators in unit I and heaters in unit II. The two products are produced and sold on a weekly basis. The weekly production should not exceed 25 in unit I and 36 in unit II. Only 60 workers can be employed at the maximum. A refrigerator requires 2 man-weeks of labour and a heater requires 1 man-week of labour. The profit is Rs 1200 per refrigerator and Rs 800 per heater. Determine the number of refrigerators and heaters to be produced per week so that the total profit is maximum.
12. Old hens can be bought for Rs 25 each but young ones cost Rs 50 each. The old hens lay 3 eggs per week and the young ones 5 eggs per week. The cost of an egg is Rs 2. The cost of feeding a hen is Rs 5 per week. If there are only Rs 600 for purchasing the hens and if it is not possible to house more than 20 hens at a time, find how many of each kind should be purchased in order to get maximum profit per week.
13. A mining company is taking a certain kind of ore from two mines *A* and *B*. The ore is divided into three groups g_1, g_2 and g_3 based on the quality. Every week the company has to supply 240 tons of g_1, 160 tons of g_2 and 440 tons of g_3. The cost per day of running mine *A* is Rs 5000 and of running mine *B* is Rs 3000. Each day *A* will produce 60 tons of g_1, 20 tons of g_2 and 40 tons of g_3. The corresponding figures for *B* are 20, 20 and 80. How many days per week should *A* and *B* work in order to minimize the total cost of production.
14. An automobile manufacturing company has three plants *A*, *B* and *C* for production. Plant *A* produces 40 scooters and 35 motorcycles per week. Plant *B* produces 65 scooters only and plant *C* produces 53 motorcycles only per week. The cost of operating plants *A*, *B* and *C* are respectively Rs 20,000, Rs 18,000 and Rs 1,60,000 per week. The company has to fulfill a production scheme of 1500 scooters and 1100 motorcycles in 20 weeks. Determine the production plan of minimum cost.
15. A soap manufacturing company has two factories *A* and *B* each producing toilet soap, washing soap and washing powder. The number of units produced per day are as follows:

Item	*Plant A*	*Plant B*
Toilet soap	1500	1200
Washing soap	3000	2000
Washing powder pkts	2000	4000

There is a monthly demand of 25,000 toilet soaps, 40,000 washing soap and 45,000 packets of washing powder. The operating costs are Rs 8000 and Rs 6000 per day for plants *A* and *B* respectively. Determine how many days each plant should work in one month so as to minimize the production cost.

16. A company produces both exterior and interior house paints. Two raw materials *A* and *B* are used for manufacturing the paints.

The daily requirements of A and B in tons for the two types of paints are given below

	Requirement per ton of paint		Availability (tons)
	Exterior	*Interior*	
A	1	2	6
B	2	1	8

The factory has capacity to produce a maximum of 3 tons of exterior and two tons of interior paints per day. The price per ton of exterior and interior paints are Rs 15,000 and Rs 10,000 respectively. Determine how much quantity of these paints should be produced per day in order to maximize the sales.

17. A company produces washing machines and TV sets. The weekly production cannot exceed 30 washing machines and 25 TV sets. Two workers are required for a washing machine and one worker for a TV per week. Totally there are 80 workers in the company. The profits are Rs 500 and Rs 300 on washing machine and TV respectively. How many of each item should the company produce in order to maximize the profit.
18. A company has two plants each of which produces two products A and B. Each plant can work up to 16 hours per day. In plant I it takes 3 hours to prepare and pack 1000 units of A and 1 hour to prepare and pack one unit of B. In plant II it takes 2 hours to prepare and pack 1000 units of A and 1.5 hour to prepare and pack one unit of B. In plant I it costs Rs 15,000 to prepare and pack 1000 units of A and Rs 28,000 to prepare and pack one unit of B whereas these costs are Rs 18,000 and Rs 26,000 respectively in plant II. The company must produce at least 10000 units of A and 8 units of B. Determine the production schedule so as to minimize the total production cost.
19. An engineering company plans to diversify its operations during the year 2007–2008. An amount of Rs 5,15,000 has been allotted for this purpose during 2007 and an amount of Rs 650,000 for the year 2008. There are 5 investment projects under consideration. The estimated returns and expected expenditure of each project in two years are as follows:

Project	*Estimated returns (in thousands)*	*Cash expenditure (in thousands)*	
		Year 2007	*Year 2008*
A	240	120	320
B	390	550	594
C	80	118	202
D	150	250	340
E	182	324	474

Assuming that the returns are directly proportional to the investment, determine the proportions of investment in each project.

20. An advertisement company wishes to plan its advertising strategy in three different media, TV, Radio and Magazine. Following data has been obtained from market survey.

	TV	*Radio*	*Magazine 1*	*Magazine 2*
Cost of an advertising unit (in Rs)	30,000	20,000	15,000	10,000
No. of customers reached per unit	2,00,000	6,00,000	1,50,000	1,00,000
No. of female customers reached per unit	1,50,000	4,00,000	70,000	50,000

The company wants to spend not more than Rs 4,50,000 on advertising. Further requirements are:

(i) At least 10,00,000 exposures take place among female customers
(ii) Advertising in magazines is limited to Rs 1,50,000
(iii) At least 3 advertising units be bought on magazine 1 and 2 units on magazine 2.

The number of units in TV and Radio should be each between 5 and 10. Determine the number of units to be bought in TV, Radio, Magazine 1 and Magazine 2, such that the maximum number of customers are reached.

ANSWERS

1. Maximize $Z = 5x_1 + 6x_2 + 3x_3$ subject to the constraints
$$4x_1 + 8x_2 + 2x_3 \le 2000$$
$$9x_1 + 7x_2 + 4x_3 \le 2500$$
$$x_1, x_2, x_3 \ge 0$$
2. Maximize $P = (0.25)x_1 + (1.00)x_2 + (0.95)x_3$ subject to the constraints
$$8x_1 + 5x_2 + 8x_3 \le 200$$
$$4x_1 + 5x_2 + 5x_3 \le 100$$
$$3x_1 + 4x_2 + 3x_3 \le 120$$
$$x_1, x_2, x_3 \ge 0$$
3. Maximize $P = 45x_1 + 80x_2$ subject to the constraints
$$5x_1 + 20x_2 \le 400$$
$$10x_1 + 15x_2 \le 450$$
$$x_1, x_2 \ge 0$$
4. Maximize $Z = 12x_1 + 8x_2 + 14x_3 + 10x_4$ subject to the constraints
$$5x_1 + 4x_2 + 2x_3 + x_4 \le 100$$
$$2x_1 + 3x_2 + 8x_3 + x_4 \le 75$$
$$x_1, x_2, x_3, x_4 \ge 0$$
5. Maximize $P = 12x_1 + 20x_2 + 45x_3$ subject to the constraints
$$0.8x_1 + 1.7x_2 + 2.5x_3 \le 100$$
$$x_1 \le 50$$
$$x_2 \le 25$$
$$x_3 \le 30$$
$$x_4 \ge 20$$
$$x_2 + x_3 \ge 15$$
$$x_1, x_2, x_3 \ge 0$$
6. Maximize $Z = 40x_1 + 30x_2$ subject to the constraints
$$3x_1 + x_2 \le 3000$$
$$x_1 \le 8000$$
$$x_2 \le 12{,}000$$
$$x_1, x_2 \ge 0$$
7. Maximize $Z = 300x_1 + 400x_2$ subject to the constraints
$$5x_1 + 4x_2 \le 200$$
$$3x_1 + 5x_2 \le 150$$
$$5x_1 + 4x_2 \ge 100$$
$$8x_1 + 4x_2 \ge 80$$
$$x_1, x_2 \ge 0$$
8. Minimize $C = (1.50)x_1 + (2.00)x_2 + (1.20)x_3$ subject to the constraints
$$350x_1 + 250x_2 + 200x_3 \ge 300$$
$$250x_1 + 300x_2 + 150x_3 \ge 200$$
$$100x_1 + 150x_2 + 75x_3 \ge 100$$
$$75x_1 + 125x_2 + 150x_3 \ge 100$$
$$x_1, x_2, x_3 \ge 0$$
9. Minimize $C = 52x_1 + 78x_2$ subject to the constraints
$$2x_1 + 3x_2 \le 80$$
$$4x_1 + 6x_2 \le 150$$
$$x_1 \ge 15$$
$$x_2 \ge 10$$
$$x_1, x_2 \ge 0$$
10. Maximize $Z = 80000x_1 + 60000x_2 + 45000x_3$ subject to the constraints
$$100x_1 + 750x_2 + 500x_3 \le 20000$$
$$x_1 \le 12$$
$$x_2 \ge 6$$
$$4 \le x_3 \le 8$$
$$x_1, x_2, x_3 \ge 0$$
11. Maximize $Z = 1200x_1 + 800x_2$ subject to the constraints
$$2x_1 + x_2 \le 60$$
$$x_1 \le 25$$
$$x_2 \le 36$$
$$x_1, x_2 \ge 0$$
12. Maximize $P = x_1 + 5x_2$ subject to the constraints
$$x_1 + x_2 \le 20$$
$$25x_1 + 50x_2 \le 600$$
$$x_1, x_2 \ge 0$$
13. Minimize $C = 5000x_1 + 3000x_2$ subject to the constraints
$$3x_1 + x_2 \ge 12$$
$$x_1 + x_2 \ge 8$$
$$x_1 + 2x_2 \ge 11$$
$$x_1, x_2 \ge 0$$
14. Minimize $C = 20000x_1 + 18000x_2 + 16000x_3$ subject to the constraints
$$40x_1 + 65x_2 \ge 1500$$
$$35x_1 + 53x_3 \ge 1100$$

$x_1 \le 20$
$x_2 \le 20$
$x_3 \le 20$
$x_1, x_2, x_3 \ge 0$

15. Minimize $C = 8000x_1 + 6000x_2$ subject to the constraints
$15x_1 + 12x_2 \ge 250$
$3x_1 + 2x_2 \ge 40$
$2x_1 + 4x_2 \ge 45$
$x_1, x_2 \ge 0$

16. Let x_1 tons of exterior paint and x_2 tons of interior paint be produced. The total sales is $Z = 15000x_1 + 10000x_2$ which is to be maximized. Considering the availability of raw materials we get
$x_1 + 2x_2 \le 6$ for A and
$2x_1 + x_2 \le 8$ for B
Also, on the capacity of factory we have, $x_1 \le 3$ and $x_2 \le 2$
Thus the LPP is
Maximize $Z = 15000x_1 + 10000x_2$ subject to the constraints
$x_1 + 2x_2 \le 6$
$2x_1 + x_2 \le 8$
$x_1 \le 3$
$x_2 \le 2$
$x_1, x_2 \ge 0$

17. Let the company produce x_1 washing machines and x_2 TV sets
$x_1 \le 30$ and $x_2 \le 25$.
The constraint on the workers is
$2x_1 + x_2 \le 80$
The total profit is $Z = 500x_1 + 300x_2$
The required LPP. is
Maximize $Z = 500\,x_1 + 300\,x_2$ subject to the constraints
$2x_1 + x_2 \le 80$
$x_1 \le 30$
$x_2 \le 25$
$x_1 \ge 0, x_2 \ge 0$

18. Let x_1 units of A (unit of 1000) and x_2 units of B be produced in plant I. Let x_3 units (unit of 1000) of A and x_4 units of B be produced in plant 2. The time constraints are
$3x_1 + x_2 \le 16$
$2x_3 + 1.5x_4 \le 16$
Constraints on minimum daily production are
$x_1 + x_3 \ge 10$
$x_2 + x_4 \ge 8$
Total cost is $Z = 15000\,x_1 + 28000\,x_2 + 18000x_3 + 26000\,x_4$
Thus the required formulation is
Minimize $Z = 15000\,x_1 + 28000\,x_2 + 18000x_3 + 26000x_4$ subject to the constraints
$3x_1 + x_2 \le 16$
$2x_3 + 1.5x_4 \le 16$
$x_1 + x_3 \ge 10$
$x_2 + x_4 \ge 8$
$x_1, x_2, x_3, x_4 \ge 0$

19. Let x_1, x_2, x_3, x_4 and x_5 represent the proportions of investment in the projects A, B, C, D and E respectively.
Net returns to be maximized is given by
$240x_1 + 390x_2 + 80x_3 + 150x_4 + 182x_5$
The budget constraints are
$120x_1 + 550x_2 + 118x_3 + 250x_4 + 324x_5 \le 515$ for 2007
and
$320x_1 + 594x_2 + 202x_3 + 340x_4 + 474x_5 \le 650$ for 2008
Also, each proportion has to be less than 1.
Therefore the formulated LPP is
Maximize $Z = 240x_1 + 390x_2 + 80x_3 + 150x_4 + 182x_5$ subject to the constraints
$120x_1 + 550x_2 + 118x_3 + 250x_4 + 324x_5 \le 515$
$320x_1 + 594x_2 + 202x_3 + 340x_4 + 474x_5 \le 650$
$x_1 \le 1, x_2 \le 1, x_3 \le 1, x_4 \le 1, x_5 \le 1$
$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0$

20. Let x_1, x_2, x_3 and x_4 represent the number of advertising units in TV , Radio, Magazine 1 and Magazine 2 respectively. The objective is to maximize the total number of potential customers given by
$10^5\,(2x_1 + 6x_2 + 1.5x_3 + x_4)$
The constraint of the advertising budget is
$30000x_1 + 20000x_2 + 15000x_3 + 10000x_4 \le 450000$
or $30x_1 + 20x_2 + 15x_3 + 10x_4 \le 450$
The constraint on the number of female customers reached is given by

$$150000x_1 + 400000x_2 + 70000x_3 + 50000x_4 \geq 1000000$$

or $\quad 15x_1 + 40x_2 + 7x_3 + 5x_4 \geq 100$

For advertising in the magazines, we have

$$15000x_1 + 10000x_2 \leq 150000$$

or $\quad 3x_1 + 2x_2 \leq 30$

Also, $x_3 \geq 3$, $x_4 \geq 2$ and

$$5 \leq x_1 \leq 10,\ 5 \leq x_2 \leq 10$$

Thus the required LPP is

Maximize $Z = 10^5\ (2x_1 + 6x_2 + 1.5x_3 + x_4)$

subject to the constraints

$$30x_1 + 20x_2 + 15x_3 + 10x_4 \leq 450$$
$$15x_1 + 40x_2 + 7x_3 + 5x_4 \geq 100$$
$$3x_1 + 2x_2 \leq 30$$
$$x_3 \geq 3,\ x_4 \geq 2,\ x_1 \geq 5,\ x_1 \leq 10$$
$$x_2 \geq 5,\ x_2 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

2.3 GRAPHICAL METHOD OF SOLUTION

Linear programming problems involving two variables can be easily solved by a graphical procedure.

We know that any first degree equation in x and y represents a straight line in the xy plane. If we consider the equation $2x + 3y = 6$, any point (x_1, y_1) satisfying this equation lies on this line. On the other hand, if $2x_1 + 3y_1 < 6$ then the point (x_1, y_1) lies in the region which contains the origin.

Also, $2x_1 + 3y_1 > 6$ implies that the point (x_1, y_1) lies on that side of the line not containing

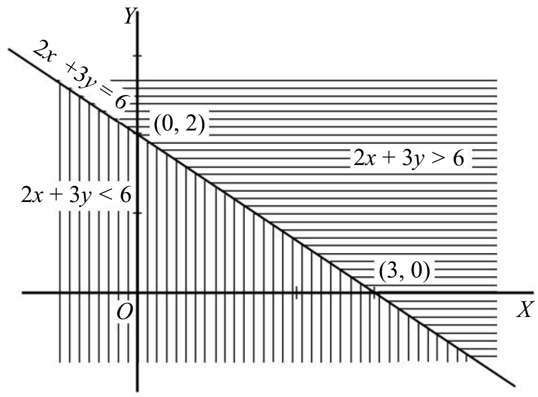


Fig. 2.1 *Regions represented by constraints*

the origin. Thus all the points satisfying the inequality $2x + 3y < 6$ lie on that side of the line $2x + 3y = 6$, which contains the origin and $2x + 3y > 6$ represents the set of all points lying on the other side which does not contain the origin.

Based on this fact we can find the region, represented by a constraint inequality. For each constraint we obtain a region. The intersection of all these regions represents the set of points which satisfy all the constraints and it is called *feasible region*. This region contains the set of all feasible solutions of the LPP and it is in the form of a polygon.

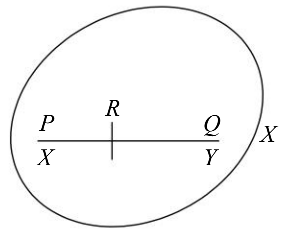


Fig. 2.2

2.3.1 Convex Set

A subset $X \subseteq R^2$ is said to be convex if for any two points P and Q in X the line segment PQ is completely contained in X.

Any point R on the line joining P and Q is given by

$$\lambda x + (1 - \lambda)y;\ 0 \leq \lambda \leq 1 \text{(convex combination)}$$

where x and y are the points (x_1, x_2) and (y_1, y_2) respectively.

R has co-ordinates

$$[\lambda x_1 + (1 - \lambda)y_1 \quad \lambda x_2 + (1 - \lambda)y_2]$$

X is convex if this point

$$\lambda x + (1 - \lambda)y \in X \quad 0 \leq \lambda \leq 1$$

where x and y are any two points of X.

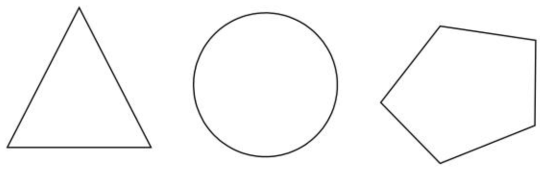


Fig. 2.3 *Convex Sets*

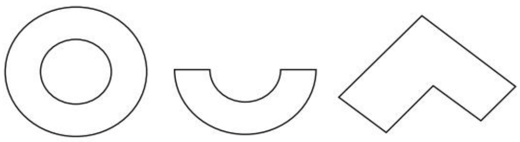


Fig. 2.4 *Non-convex Sets*

If a point in X cannot be expressed as a convex combination of any two points of X then it is called an **extreme point** of X.

Theorem 1: The intersection of two convex sets is also a convex set.

Proof: Let X_1 and X_2 be two convex sets and let $X = X_1 \cap X_2$.

Take $x_1, x_2 \,\varepsilon\, X$. Therefore $x_1, x_2 \,\varepsilon\, X_1$ as well as $x_1, x_2 \,\varepsilon\, X_2$.

Now, $x_1, x_2 \,\varepsilon\, X_1$ and X_1 is a convex set. Hence $\lambda x_1 + (1 - \lambda)x_2 \,\varepsilon\, X_1$ where $0 \le \lambda \le 1$.

Again $x_1, x_2 \,\varepsilon\, X_2$ and X_2 is convex. Hence $\lambda x_1 + (1 - \lambda)x_2 \,\varepsilon\, X_2$ where $0 \le \lambda \le 1$.

Thus $\lambda x_1 + (1 - \lambda)x_2 \,\varepsilon\, X_1 \cap X_2$.

i.e. $\lambda x_1 + (1 - \lambda)x_2 \,\varepsilon\, X \qquad (0 \le \lambda \le 1)$

Therefore we find that

$x_1, x_2 \,\varepsilon\, X \to \lambda x_1 + (1 - \lambda)x_2 \,\varepsilon\, X \; (0 \le \lambda \le 1)$

which implies that X is convex.

Theorem 2: Let S be a convex set in the plane R^2 bounded by lines. Then a linear function

$$Z = c_1x_1 + c_2x_2$$

where $x_1, x_2 \,\varepsilon\, S$ and c_1, c_2 are scalars, attains its extreme value (maximum or minimum) at the vertices of S.

Proof: Take any two points $A(x_1, x_2)$ and $B(x_1', x_2')$ of S. Then any point on the line segment AB is given by a convex combination of the coordinates of A and B. Let $P(\xi, \eta)$ be any point on the line segment AB.

$$\therefore \quad \xi = \lambda x_1 + (1 - \lambda)x_1'$$
$$\eta = \lambda x_2 + (1 - \lambda)x_2'$$

The value of Z at A is given by $(0 \le \lambda \le 1)$

$$z_1 = c_1x_1 + c_2x_2$$

Similarly, the value of Z at B is given by

$$z_2 = c_1x_1' + c_2x_2'$$

Now the value of Z at $P(\xi, \eta)$ is given by

$$\begin{aligned}\bar{z} &= c_1\xi + c_2\eta \\ &= c_1(\lambda x_1 + (1 - \lambda)x_1') + c_2(\lambda x_2 + (1 - \lambda)x_2') \\ &= \lambda(c_1x_1 + c_2x_2) + (1 - \lambda)(c_1x_1' + c_2x_2') \\ &= \lambda z_1 + (1 - \lambda)z_2 \\ &= \lambda(z_1 - z_2) + z_2 \qquad (0 \le \lambda \le 1)\end{aligned}$$

If $z_1 > z_2$ then $z_2 \le \bar{z} \le z_1$ and if $z_2 > z_1$ then $z_1 \le \bar{z} \le z_2$. In any case $\bar{z}$ lies between the values of z at A and B where A and B are any two points of S.

In case P is an interior point of S, it lies on a line segment AB with A and B on two edges of S. Hence the extreme value of Z cannot occur at P

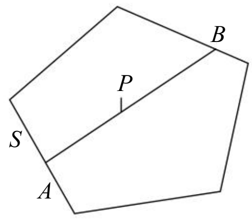


Fig. 2.5

In case P lies on an edge of S, it lies on a segment AB with A and B as two vertices and therefore z lies between the values at the vertices. Hence the extreme value of z cannot be obtained at a point on an edge of S.

Thus the extreme value of Z is attained only at one vertex of S.

2.3.2 Graphical Method

Let us consider an LPP of the form

Maximize $Z = c_1x_1 + c_2x_2$ subject to the constraints

$$a_ix_1 + b_ix_2 \le k_i \; (i = 1, 2, 3, \ldots, n)$$
$$x_1, x_2 > 0$$

First we plot the lines $a_ix_1 + b_ix_2 \le k_i$ $(i = 1, 2, 3, \ldots, n)$ in the X_1X_2 plane and mark the regions represented by the given inequalities. We have to consider only the I quadrant since $x_1, x_2 \ge 0$.

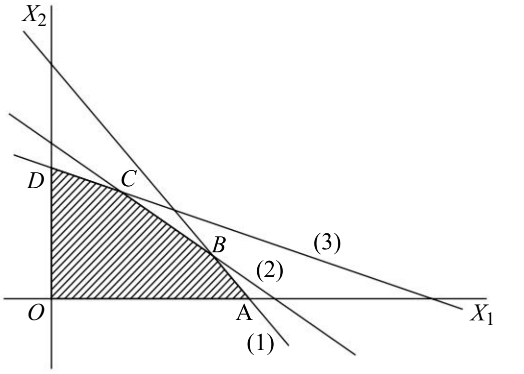


Fig. 2.6 *Model with 3 constraints*

Each constraint represents a region and the region common to all these (intersection of the sets) gives the feasible region of the given LPP. Find out the coordinates of the vertices of the feasible region by solving the equations of the lines. Substitute the coordinates of the vertices, one by one, in the objective function. Choose the vertex which gives the maximum value of Z. This gives the solution of the LPP.

Note:

1. If the constraint is of $\geq$ type we have to shade the region not containing the origin.
2. We have to find the value of Z only at the vertices since the maximum or minimum of Z is attained only at a vertex (Theorem 2.2).
3. If a feasible region does not exist then the given LPP has no solution and if the feasible region is not a closed polygon then the problem has unbounded solution.
4. If two vertices give the same maximum (or minimum) value then every point on the edge joining the vertices gives the same value and the given problem has infinite number of optimal solutions.

Example 2.8 (Solve by graphical method)

Maximize $Z = 6x_1 + 9x_2$ subject to the constraints

$$x_1 + x_2 \leq 12$$
$$x_1 + 5x_2 \leq 45$$
$$3x_1 + x_2 \leq 30$$
$$x_1, x_2 \geq 0$$

Solution We plot the lines $x_1 + x_2 = 12$, $x_1 + 5x_2 = 45$ and $3x_1 + x_2 = 30$ and mark the regions represented by (1), (2) and (3) using different shadings. The region common to all (intersection of the shaded areas) represents the set of all points forming the feasible solutions to the given LPP.

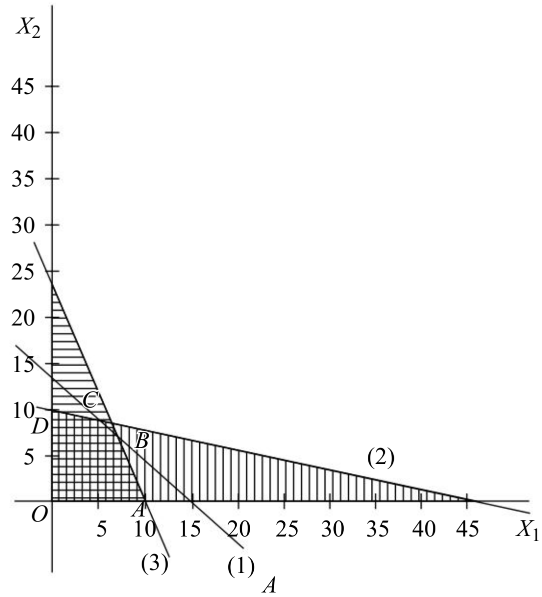


Fig. 2.7

The region corresponding to (1) is shaded with slant lines, that corresponding to (2) is shaded with vertical lines, and that corresponding to (3) is shaded with horizontal lines. We find that the feasible region is the polygon $OABCD$. O is the origin (0,0), A is the point (10,0) and D is the point (0,9). B and C are to be determined. B is the point of intersection of (1) and (3). Solving the equations $x_1 + x_2 = 12$ and $3x_1 + x_2 = 30$, we get $x_1 = 9$ and $x_2 = 3$. Thus B is the point (9,3). Now we find C by solving (1) and (2).

Solving the equations

$x_1 + x_2 = 12$ and $x_1 + 5x_2 = 45$, we get $x_1 = 15/4$ and $x_2 = 33/4$.

Thus C is the point (15/4, 33/4).

Now, the objective function is $Z = 6x_1 + 9x_2$

The value of Z at O is $Z_0 = 0$.

The value of Z at A is

$$Z_A = 6(10) + 9(0) = 60$$

Similarly,

$$Z_B = 6(9) + 9(3) = 81$$
$$Z_C = 6(15/4) + 9(33/4) = 387/4$$
$$Z_D = 6(0) + 9(9) = 81.$$

The maximum value of Z occurs at $C(15/4, 33/4)$ and the value is $Z^* = 387/4$.

Hence the optimal solution of the given LPP is $x_1 = 15/4$, $x_2 = 33/4$, $Z^* = 387/4$.

Example 2.9 Solve graphically

Maximize $Z = 40x_1 + 100x_2$ subject to the constraints

$$2x_1 + x_2 \le 500$$
$$2x_1 + 5x_2 \le 1000 \qquad x_1, x_2 \ge 0$$

Solution The lines and regions represented by constraints (1) and (2) are shown in the figure. The feasible region is given by the polygon $OABC$. O is the point (0,0), A is (250,0) C is (0,200). B can be found by solving the equations $2x_1 + x_2 = 500$ and $2x_1 + 5x_2 = 1000$. We obtain B as the point (375/2,125).

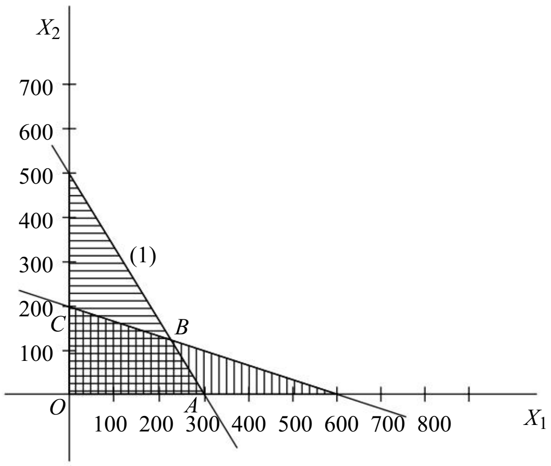


Fig. 2.8

Now $Z = 40x_1 + 100x_2$

$$Z_0 = 0,\ Z_A = 40(250) + 100(0) = 10{,}000$$
$$Z_B = 40(375/2) + 100(125) = 20{,}000$$
$$Z_C = 40(0) + 100(200) = 20{,}000$$

We find that both B and C give the maximum value 20,000. Therefore there is no unique optimal solution and any point on the line BC can be taken as an optimal solution with $Z^* = 20{,}000$.

Note: If one of the constraint equation is parallel to the objective function then we get infinite number of solutions.

Example 2.10 Solve graphically

Maximize $Z = 3x_1 + 4x_2$ subject to the constraints

$$-3x_1 + 2x_2 \le 6$$
$$3x_1 + x_2 \ge 6$$
$$x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

Solution The regions represented by $-3x_1 + 2x_2 \le 6$ (1) is shaded using vertical lines. Since $x_1 \ge 0$ the negative side of the X_1 axis is not considered. Horizontal lines represent the region corresponding to $3x_1 + x_2 \ge 6$ (2) which is $\ge$ type. The region given by $x_1 + x_2 \le 8$ (3) is represented by slant lines. The feasible region is $ABCD$.

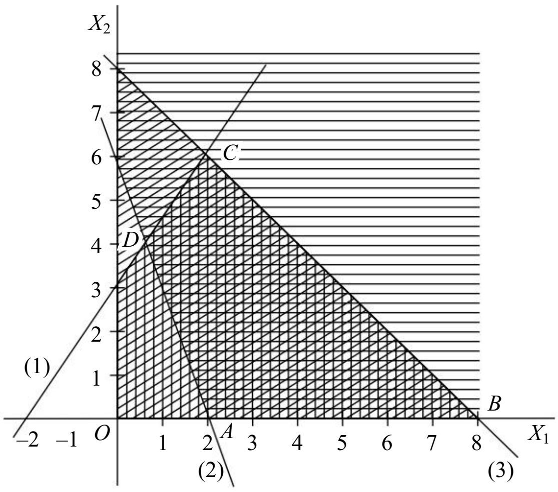


Fig. 2.9

A is (2,0) B is (8,0) C is (2,6) which is obtained by solving (1) and (3) with equality sign. D is (2/3,4) which is obtained by solving

$$-3x_1 + 2x_2 = 6 \text{ and } 3x_1 + x_2 = 6.$$

Now $Z = 3x_1 + 4x_2$

$$Z_A = 3(2) + 4(0) = 6$$
$$Z_B = 3(8) + 4(0) = 24$$
$$Z_C = 3(2) + 4(6) = 30$$
$$Z_D = 3(2/3) + 4(4) = 18.$$

The maximum value of Z occurs at C (2, 6).

Hence the solution is $x_1 = 2$, $x_2 = 6$, $Z^* = 30$.

Example 2.11 Solve graphically

Minimize $Z = 5x_1 + 2x_2$ subject to the constraints

$$3x_1 + x_2 \geq 3$$
$$3x_1 - 2x_2 \leq 6$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Solution The following graph represents the regions corresponding to the constraint inequalities and the feasible region of the given LPP.

The feasible region is given by the polygon *ABCDE*. Solving the equations $3x_1 - 2x_2 = 6$, $x_1 + x_2 = 4$ we get $x_1 = 14/5$, $x_2 = 6/5$, C is $(14/5, 6/5)$.

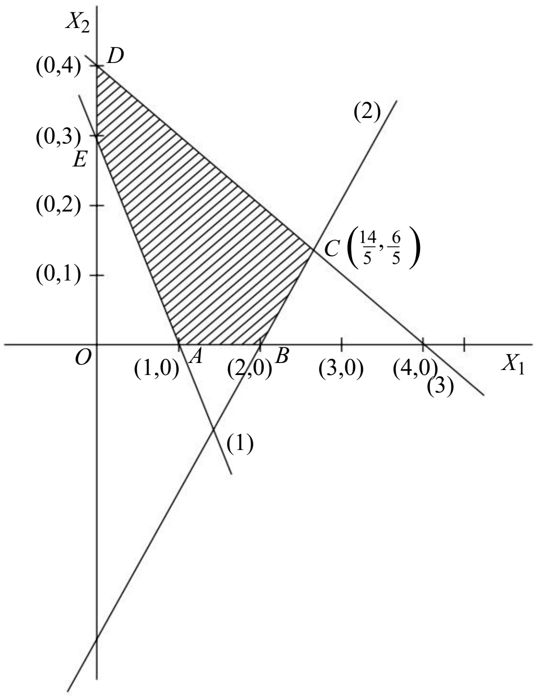


Fig. 2.10

Thus we have the vertices A (1, 0), B (2, 0), C (14/5, 6/5), D (0, 4) and E (0, 3)

Now $Z = 5x_1 + 2x_2$

$Z_A = 5(1) + 2(0) = 5;$

$Z_B = 5(2) + 2(0) = 10;$

$Z_C = 5(14/5) + 2(6/5) = 82/5;$

$Z_D = 5(0) + 2(4) = 8;$

$Z_E = 5(0) + 2(3) = 6.$

Minimum value is 5.

Hence the solution is $x_1 = 1$, $x_2 = 0$, $Z^* = 5$.

Example 2.12 Solve graphically

Minimize $Z = 5x_1 + 4x_2$ subject to the constraints

$$x_1 - 2x_2 \leq 1$$
$$x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0.$$

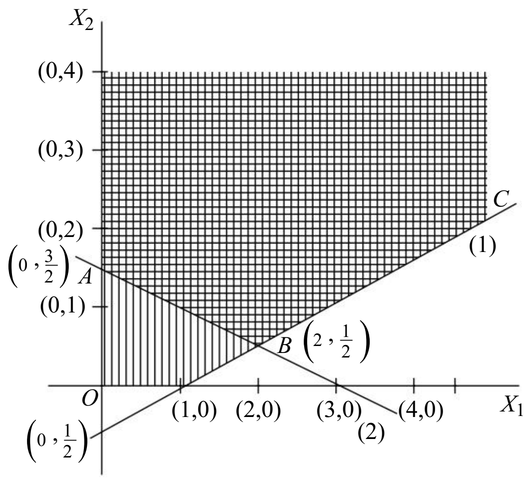


Fig. 2.11

Solution Draw the lines $x_1 - 2x_2 = 1$ and $x_1 + 2x_2 = 3$ and mark the regions $x_1 - 2x_2 \leq 1$ and $x_1 + 2x_2 \geq 3$. We find that the common region (feasible region) is not closed above. The feasible region is unbounded. We have only two vertices A and B. A is the point (0, 3/2) and by solving the equations we get B as the point (2, ½).

Now $Z = 5x_1 + 4x_2,$

$Z_A = 5(0) + 4(3/2) = 6,$

$Z_B = 5(2) + 4(1/2) = 12.$

The minimum value is 6. Therefore the solution of the given LPP is

$$x_1 = 0\ x_2 = 3/2, Z^* = 6.$$

Note: If the question is given as "Maximize $Z = 5x_1 + 4x_2$ this problem has unbounded solutions.

Example 2.13 Solve graphically

Maximize $Z = 2x_1 + 3x_2$ subject to the constraints

$$2x_1 + x_2 \le 2$$
$$3x_1 + 4x_2 \ge 12$$
$$x_1, x_2 \ge 0.$$

Solution The following graph gives the regions represented by the constraints.

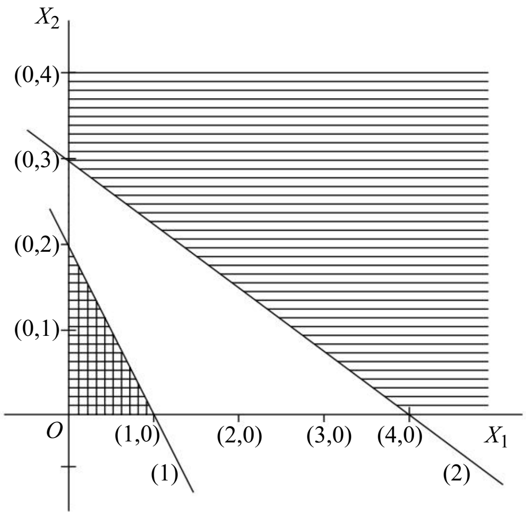


Fig. 2.12

From the graph we find that there is no common region between the two. That is to say that there is no point (x_1, x_2) which satisfies both the constraints. Hence there is no feasible solution. Thus the given LPP has no solution.

Example 2.14 Solve graphically

Maximize $Z = 10x_1 + 15x_2$ subject to the constraints

$$2x_1 + x_2 \le 26$$
$$2x_1 + 4x_2 \le 56 \text{ or } (x_1 + 2x_2 \le 28)$$
$$-x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0.$$

Solution The graph showing the feasible region is given below.

The feasible region (solution space) is enclosed by the polygon *OABCD*.

B is obtained by solving $2x_1 + x_2 = 26$ and $x_1 + 2x_2 = 28$.

B is (8, 10).

C is given by $x_1 + 2x_2 = 28$ and $-x_1 + x_2 = 5$.

C is (6, 11)

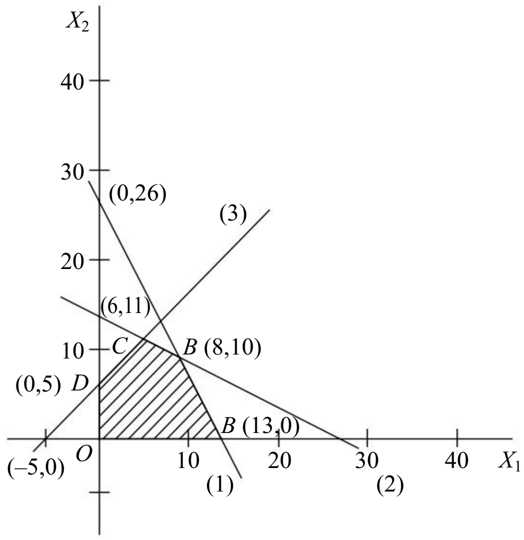


Fig. 2.13

O is (0,0); *A* is (13, 0); *D* is (0, 5).

Now $Z = 10x_1 + 15x_2$

$$Z_O = 10(0) + 15(0) = 0$$
$$Z_A = 10(13) + 15(0) = 130$$
$$Z_B = 10(8) + 15(10) = 230 \to \text{Maximum}$$
$$Z_C = 10(6) + 15(11) = 225$$
$$Z_D = 10(0) + 15(5) = 75.$$

Hence the solution is $x_1 = 8$ $x_2 = 10$ $Z^* = 230$.

Example 2.15 A manufacturer of furniture makes chairs and tables. Processing of these products is done on two machines *A* and *B*. A chair requires 2 hours on machine *A* and 6 hours on *B*. *A* table requires 5 hours on *A* and 6 hours on *B*. There are 16 hours of time available on machine *A* and 30 hours on machine *B*. At the most 4 chairs are to be manufactured. The manufacturer gets a profit of Rs 50 on a chair and Rs 100 on a table. Determine the number of chairs and tables to be manufactured so as to maximize the profit.

Solution First we formulate this into an LPP. Let x_1 chairs and x_2 tables be produced. The total profit is $P = 50x_1 + 100x_2$ which is the objective function to be maximized. Considering the time taken by x_1 chairs and x_2 tables on machine *A* we get the constraint $2x_1 + 5x_2 < 16$.

The next constraint on the time taken on machine B is given by

$$6x_1 + 6x_2 \leq 30 \text{ or } (x_1 + x_2 \leq 5)$$

Also, $x_1 \leq 4$.

Finally the non-negativity constraints are $x_1 \geq 0$ and $x_2 \geq 0$.

Hence we get the LPP

Maximize $P = 50x_1 + 100x_2$ subject to the constraints.

$$2x_1 + 5x_2 \leq 16$$
$$x_1 + x_2 \leq 5$$
$$x_1 \leq 4$$
$$x_1, x_2 \geq 0.$$

Let us represent the problem graphically.

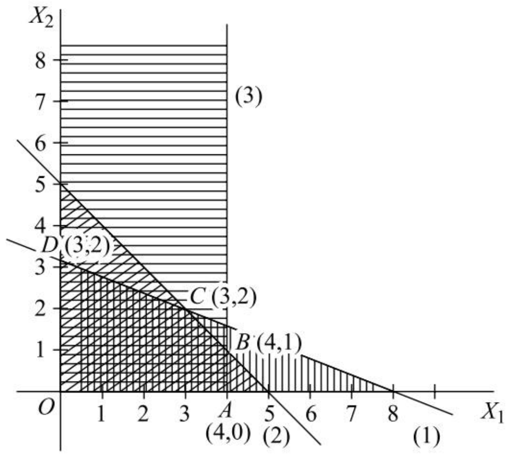


Fig. 2.14

The solution space is represented by the shaded region $OABCD$. The point B is obtained by solving the equations (2) and (3). B is (4, 1). Similarly solving (1) and (2) we get the point C (3, 2).

Now, $P = 50x_1 + 100x_2$

$P_O = 0$ $P_A = 50(4) + 0 = 200$

$P_B = 50(4) + 100(1) = 300$

$P_C = 50(3) + 100(2) = 350 \rightarrow$ Maximum

$P_D = 0 + 100(3) = 300.$

Hence the solution is $x_1 = 3$, $x_2 = 2$, $P^* = 350$.

3 chairs and 2 tables are to be manufactured in order to get the maximum profit of Rs 350.

Example 2.16 A company combines factors X and Y to form a product which must weigh not less than 50 kg. At least 20 kg of X and not more than 40 kg of Y can be used. The cost of X is Rs 10 per kg and that of Y is Rs 25 per kg. Determine how much of X and Y are to be used to minimize the total cost.

Solution

Formulation

Let us assume that x_1 kg of X and x_2 kg of Y are used to form the product. At least 20 kg of X and not more than 40 kg of Y are to be used. Therefore we have $x_1 \geq 20$ and $x_2 \leq 40$.

Also the weight of the product must be not less than 50 kg. This implies that

$$x_1 + x_2 \geq 50.$$

The total cost of the product is $10x_1 + 25x_2$ which is to be minimum.

Thus the LPP is

Minimize $C = 10x_1 + 25x_2$ subject to the constraints

$$x_1 \geq 20, \quad x_2 \leq 40, \quad x_1 + x_2 \geq 50, \quad x_1, x_2 \geq 0$$

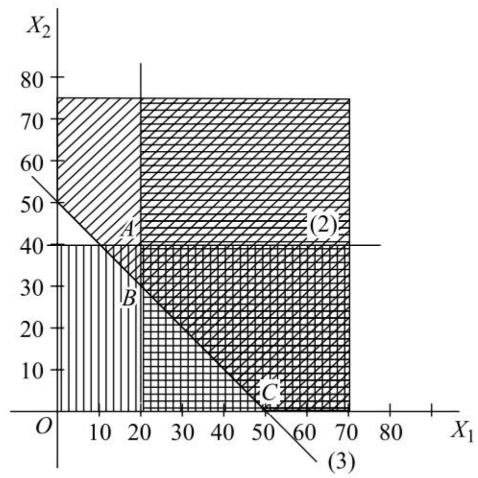


Fig. 2.15 *Graphical representation*

A is the point (20, 40), B is (20, 30) and C is (50, 0).

$C = 10x_1 + 25x_2$

$C_A = 10(20) + 25(40) = 1200$

$C_B = 10(20) + 25(30) = 950$

$C_C = 10(50) + 25(0) = 500 \rightarrow$ Minimum.

Thus the optimal solution is $x_1 = 50$ $x_2 = 0$ $C^* = 500$.

Therefore 50 kg of X alone has to be used in order to minimize the total cost and the minimum cost is Rs 500.

EXERCISES

Solve the following LP problems graphically:

1. Maximize $Z = 3x_1 + 4x_2$ subject to the constraints
$$2x_1 + x_2 \le 40$$
$$2x_1 + 5x_2 \le 180$$
$$x_1, x_2 \ge 0$$
2. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints
$$-2x_1 + x_2 \le 1$$
$$x_1 \le 2$$
$$x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0$$
3. Maximize $Z = 3x_1 + 4x_2$ subject to the constraints
$$x_1 + x_2 \le 450$$
$$2x_1 + x_2 \le 600$$
$$x_1, x_2 \ge 0$$
4. Maximize $Z = 15x_1 + 10x_2$ subject to the constraints
$$2x_1 + 3x_2 \le 180$$
$$x_1 \le 60$$
$$x_2 \le 40$$
$$x_1, x_2 \ge 0$$
5. Maximize $Z = 2x_1 + x_2$ subject to the constraints
$$x_1 + 2x_2 \le 10$$
$$x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 2$$
$$x_1 - 2x_2 \le 1$$
$$x_1, x_2 \ge 0.$$
6. A pineapple firm produces canned pineapple and canned juice and the requirement and availability of the resources are given below: The profit per unit of canned pineapple and one unit of canned juice are Rs 2 and Rs 5 respectively. Determine the production schedule which maximizes the profit.

	Canned pine-apple	*Canned juice*	*Available resources*
Labour (man hours)	2	3	12
Equipment (machine hours)	3	2	12
Raw material (unit)	2	2	9

7. A company manufactures two types of printed circuits A and B. The following table gives the requirements of transistors, resistors and capacitors for each type and other particulars. How many circuits of each type should the company produce from the available stock in order to earn maximum profit.

	Circuit		
	A	*B*	*Stock available*
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit (Rs)	5	8	

8. Maximize $Z = x_1 - 2x_2$ subject to the constraints
$$-x_1 + x_2 \le 1$$
$$3x_1 + 2x_2 \ge 12$$
$$0 \le x_1 \le 5$$
$$2 \le x_2 \le 4$$
9. Maximize $Z = 7x_1 + 3x_2$ subject to the constraints
$$x_1 + 2x_2 \ge 3$$
$$x_1 + x_2 \le 4$$
$$0 \le x_1 \le 5/2$$
$$0 \le x_2 \le 5/2$$
10. Minimize $Z = 60x_1 + 40x_2$ subject to the constraints

$$3x_1 + x_2 \geq 40$$
$$2x_1 + 5x_2 \geq 44$$
$$3x_1 + 3x_2 \geq 40$$
$$x_1, x_2 \geq 0$$

11. Minimize $Z = 7x_1 + 8x_2$ subject to the constraints
$$3x_1 + x_2 \geq 9$$
$$x_1 + 3x_2 \geq 9$$
$$x_1, x_2 \geq 0$$

12. Find the maximum and minimum values of $Z = 5x_1 + 3x_2$ subject to the constraints
$$x_1 + x_2 \leq 6$$
$$2x_1 + 3x_2 \geq 6$$
$$0 \leq x_1 \leq 3$$
$$x_2 \geq 0$$

13. Find the maximum and minimum values of $Z = 20x_1 + 10x_2$ subject to the constraints
$$x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

14. Minimize Z = $80x_1 + 120x_2$ subject to the constraints
$$x_1 + x_2 \leq 9$$
$$2x_1 + 5x_2 \leq 36$$
$$0 \leq x_1 \leq 2$$
$$0 \leq x_2 \leq 3$$

15. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints
$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

16. Maximize $Z = 6x_1 + 4x_2$ subject to the constraints
$$x_1 + 2x_2 \leq 2$$
$$2x_1 + 5x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

17. A person requires 10,12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per jar. A dry product contains 1, 2 and 4 units of A, B and C per packet. If the liquid product is Rs 3 per jar and the cost of the dry product is Rs 2 per packet find out how many of each should be purchased in order to minimize the total cost?

18. Maximize $Z = 6x_1 + 4x_2$ subject to the constraints
$$x_1 - x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

19. Minimize $Z = 2x_1 - 10x_2$ subject to the constraints
$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \geq -5 \quad (-x_1 + 5x_2 \leq 5)$$
$$0 \leq x_1 \leq 2$$
$$x_2 \geq 0$$

20. Maximize $Z = 40x_1 + 80x_2$ subject to the constraints
$$2x_1 + 3x_2 \leq 48$$
$$0 \leq x_1 \leq 15$$
$$0 \leq x_2 \leq 10$$

21. Maximize $Z = 40x_1 + 100\,x_2$ subject to the constraints
$$12x_1 + 6x_2 \leq 3000$$
$$4x_1 + 10x_2 \leq 2000$$
$$2x_1 + 3x_2 \leq 900$$
$$x_1, x_2 \geq 0$$

22. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints
$$-2x_1 + 3x_2 \leq 9$$
$$3x_1 - 2x_2 \geq -20$$
$$x_1, x_2 \geq 0$$

23. Maximize $Z = 10x_1 + 15x_2$ subject to the constraints
$$2x_1 + x_2 \leq 26$$
$$x_1 + 2x_2 \leq 28$$
$$-x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

24. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints
$$5x_1 + x_2 \geq 10$$
$$x_1 + x_2 \geq 6$$
$$x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

25. Maximize $Z = -x_1 + 2x_2$ subject to the constraints
$$-x_1 + 3x_2 \leq 10$$

$$x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \ge 0$$

26. Maximize $Z = 20x_1 + 10\,x_2$ subject to the constraints
$$x_1 + 2x_2 \le 40$$
$$3x_1 + x_2 \ge 30$$
$$4x_1 + 3x_2 \ge 60$$
$$x_1, x_2 \ge 0$$

27. Maximize $Z = 2x_1 + 3x_2$ subject to the constraints
$$x_1 + x_2 \le 30$$
$$12 \ge x_2 \ge 3$$
$$x_1 - x_2 \ge 0$$
$$x_1 \ge 0$$

28. Minimize $Z = 200x_1 + 400x_2$ subject to the constraints
$$x_1 + 3x_2 \ge 400$$
$$x_1 + 2x_2 \le 350$$
$$x_1 + x_2 \ge 200$$
$$x_1, x_2 \ge 0$$

29. Maximize $Z = 50x_1 + 30x_2$ subject to the constraints
$$2x_1 + x_2 \ge 18$$
$$x_1 + x_2 \ge 12$$
$$3x_1 + 2x_2 \le 34$$
$$x_1, x_2 \ge 0$$

30. Maximize $Z = x_1 + 3x_2$ subject to the constraints
$$x_1 + x_2 \le 7$$
$$x_2 \le 5$$
$$3x_1 + 10x_2 \ge 30$$
$$x_1, x_2 \ge 0$$

ANSWERS

1. $x_1 = 5/2, x_2 = 35, Z^* = 147.5$
2. $x_1 = 2, x_2 = 1, Z^* = 8$
3. $x_1 = 0, x_2 = 450, Z^* = 1800$
4. $x_1 = 60, x_2 = 20, Z^* = 1100$
5. $x_1 = 4, x_2 = 2, Z^* = 10$
6. $x_1 = 0, x_2 = 4, Z^* = 20$
7. $x_1 = 2, x_2 = 9, Z^* = 82$
8. $x_1 = 5, x_2 = 2, Z^* = 1$
9. $x_1 = 5/2, x_2 = 3/2, Z^* = 22$
10. $x_1 = 12, x_2 = 4, Z^* = 880$
11. $x_1 = 9/4, x_2 = 9/4, Z^* = 135/4$
12. $x_1 = 0, x_2 = 2$, Min $Z = 6$
    $x_1 = 3, x_2 = 3$, Max $Z = 24$
13. $x_1 = 6, x_2 = 12$, Min $Z = 240$.
    $x_1 = 40, x_2 = 0$, Max $Z = 800$.
14. $x_1 = 2, x_2 = 3, Z^* = 520$.
15. The solution is unbounded.
16. The problem has no solution.
17. Liquid 1, dry 5, Min cost = 13
18. There are infinite number of solutions. $Z^* = 36$
    Note that the objective function $6x_1 + 4x_2$ is parallel to $3x_1 + 2x_2 = 18$.
19. There are infinite number of solutions. $Z^* = -10$
    ($2x_1 - 10x_2$ is parallel to $x_1 - 5x_2 = -5$)
20. $x_1 = 9, x_2 = 10, Z^* = 1160$
21. There are infinite number of solutions $Z^* = 20000$.
22. There is no solution.
23. $x_1 = 8, x_2 = 10, Z^* = 230$
24. $x_1 = 1, x_2 = 5, Z^* = 13$
25. $x_1 = 2, x_2 = 0, Z^* = -2$
26. $x_1 = 6, x_2 = 12, Z^* = 240$
27. $x_1 = 18, x_2 = 12, Z^* = 72$
28. $x_1 = 100, x_2 = 100, Z^* = 60000$
29. $x_1 = 10, x_2 = 2, Z^* = 560$
30. $x_1 = 2, x_2 = 5, Z^* = 17$

3

Simplex Method

CONCEPT REVIEW

3.1 INTRODUCTION

If a set of simultaneous equations is given, in general we can solve them and obtain the values of the variables when the number of equations is equal to the number of variables. The solution is unique if the matrix of the coefficients is non-singular. For example consider two equations in two variables:

$$4x_1 + x_2 = 9$$
$$x_1 + 2x_2 = 4$$

The solution is $x_1 = 2$, $x_2 = 1$, we find that

$$\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$$

Now if we consider the equations

$$x_1 + x_2 = 3$$
$$2x_1 + 2x_2 = 6$$

we find that they are not independent. There exists an infinite number of solutions in this case: $(x_1 = 1, x_2 = 2)$, $(x_1 = 2, x_2 = 1)$, $(x_1 = 0, x_2 = 3)$ and so on.

But the equations

$$x_1 + x_2 = 3$$
$$2x_1 + 2x_2 = 5$$

are inconsistent and they have no solution.

Similar examples can be given in the case of three variables also.

But if the number of equations is less than the number of variables then the variables in excess are called *non-basic variables* and in the process of solving the equations they are treated as constants. The variables taken for solution are called *basic variables*. For example, consider the equations

$$2x_1 + x_2 + x_3 = 14$$
$$x_1 + 2x_2 + x_3 = 16$$

Since we have two equations we can solve for two variables, keeping the third one as constant. The *basic solution* with x_1 and x_2 as basic variables is given by solving

$$2x_1 + x_2 = 14 - x_3$$
$$x_1 + 2x_2 = 16 - x_3$$

we get $x_1 = 4 - x_3/3$; $x_2 = 6 - x_3/3$

Taking x_1 and x_3 as basic variables, we get

$$2x_1 + x_3 = 14 - x_2$$
$$x_1 + x_3 = 16 - 2x_2$$

The corresponding basic solution is

$$x_1 = -2 + x_2;\ x_3 = 18 - 3x_2$$

Also taking x_2 and x_3 as basic we obtain the solution

$x_2 = 2 + x_1,\ x_3 = 12 - 3x_1$

Thus if we have n variables and m equations ($n > m$) then we have m basic variables and $n - m$ non-basic variables. In the example given above there are three variables and two equations. The two basic variables are selected in $3C_2 = 3$ ways and accordingly we get three basic solutions. For m equations in n variables ($n > m$) we get, nC_m basic solutions.

We can represent the equations, given in the example, in a matrix form.

The equations are

$$2x_1 + x_2 + x_3 = 14$$
$$x_1 + 2x_2 + x_3 = 16$$

In matrix form we write,

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

The first solution with x_1 and x_2 as basic variables can be written in the form

$$x_1 + 0x_2 + x_3/3 = 4$$
$$0x_1 + x_2 + x_3/3 = 6$$

In the matrix form it can be written as

$$\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Here the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

corresponds to the basic variables (x_1, x_2) and it is called *basis matrix.*

The solution having x_1 and x_3 as basic variables can be written as

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 18 \end{pmatrix}$$

and the solution with x_2 and x_3 as basic variables is given by

$$\begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

We assign zero value to the non-basic variables and express the three basic solutions as

(i) $x_1 = 4,\ x_2 = 6,\ x_3 = 0$

(ii) $x_1 = -2,\ x_2 = 0,\ x_3 = 18$

(iii) $x_1 = 0,\ x_2 = 2,\ x_3 = 12$

In a linear programming problem if a set of values of the variables satisfy all the constraints then it is called a *feasible solution*. If a basic solution is also a feasible solution then it is called a *basic feasible solution.* A basic feasible solution which optimizes the objective function is called the *optimal solution*. In an LPP the variables should satisfy the non-negativity constraints also [i.e. $x_i \ge 0\ \forall i$]. In the above example the second solution is basic but not feasible, since $x_1 = -2$ (<0).

3.2 SIMPLEX METHOD

3.2.1 Standard form of a given LPP

Consider the LPP

Maximize $Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n$ subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$
$$x_1\ x_2 \ldots x_n \ge 0$$

In order to solve this LPP we have to express the problem in the standard form. The standard form of LPP has the following characteristic properties:

(i) The objective function should be of maximization type. If it is given, as

Minimize $Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n$

we change it to maximization type by taking

$-Z = -C_1x_1 - C_2x_2 - \ldots - C_nx_n$

and write

Minimize $W = -C_1x_1 - C_2x_2 - \ldots - C_nx_n$

where $W = -Z$

Max (W) = – Min (– W) and

Min (W) = – Max (– W)

We find that Min(Z) = – Max (– Z)

= – Max (W)

and thus Min(Z) can be derived from Max W.

(ii) All the constants b_j ($j = 1, 2, 3, 4, \ldots, m$) must be positive. If there is a negative constant then multiply the constraint inequality on both sides by –1. For example the constraint $3x_1 + 2x_2 + 5x_3 \leq -2$ can be written as $-3x_1 - 2x_2 - 5x_3 \geq 2$
[when multiplied by –1, the ≤ sign changes to ≥ and vice versa].

(iii) All the variables must be non-negative. If one variable, say x_r, is negative then put $x_r' = -x_r$ and replace x_r with $-x_r'$ in the problem. Now $x_r' \geq 0$. Again if a variable x_r is unrestricted in sign (it can assume either +ve or –ve value) then put $x_r = u_r - v_r$, where both u_r and v_r are non-negative. $u_r > v_r \to x_r$ is +ve and $u_r < v_r \to x_r$ is –ve.

(iv) Finally all the constraints must be equalities. If a constraint inequality is of upper bound type (≤) then we add a new variable s_j called *slack variable* and convert it into equality. The constraint
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$ can be changed into
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1$
Similarly, in case of lower bound type constraints (≥) we subtract a new variable s_j called *surplus variable* and obtain the equality. Accordingly, the constraint
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$ can be changed into
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - s_1 = b_1$
It should be noted that all the variables should satisfy the non-negativity constraint.

Example 3.1 Rewrite the following LPP in the standard form:

Maximize $Z = 2x_1 + 3x_2 + 4x_3$ subject to the constraints

$x_1 + 3x_2 + 5x_3 \leq 10$
$3x_1 - x_2 - 2x_3 \leq -4$
$x_1 \leq 0 \quad x_2, x_3 \geq 0$

Solution Multiplying the second constraint by –1, we get

$-3x_1 + x_2 + 2x_3 \geq 4$

Given $x_1 \leq 0$. Therefore put $x_1 = -x_1'$

Thus the problem becomes

Maximize $Z = -2x_1' + 3x_2 + 4x_3$ subject to the constraints

$-x_1' + 3x_2 + 5x_3 \leq 10$
$3x_1' + x_2 + 2x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$

Finally, we change the constraints to equalities. By introducing a slack variable s_1 and a surplus variable s_2 we get the standard form as

Maximize $Z = -2x_1' + 3x_2 + 4x_3$ subject to the constraints

$-x_1' + 3x_2 + 5x_3 + s_1 = 10$
$3x_1' + x_2 + 2x_3 - s_2 = 4$
$x_1, x_2, x_3, s_1, s_2 \geq 0$

Example 3.2 Change the following LPP into standard form:

Minimize $Z = 2x_1 + 2x_2 + 4x_3 + x_4$ subject to the constraints

$3x_1 + 2x_2 + x_3 + 9x_4 \leq 16$
$7x_1 + 2x_2 + 4x_3 + x_4 \leq 24$
$x_1, x_2, x_3 \geq 0$ x_4 unrestricted.

Solution The objective function has to be written as

Maximize $W = -Z = -2x_1 - 2x_2 - 4x_3 - x_4$

Since x_4 is unrestricted, write

$x_4 = u_4 - v_4$

The problem becomes

Maximize $W = -2x_1 - 2x_2 - 4x_3 - (u_4 - v_4)$ subject to the constraints

$3x_1 + 2x_2 + x_3 + 9(u_4 - v_4) \leq 16$
$7x_1 + 2x_2 + 4x_3 + u_4 - v_4 \leq 24$
$x_1, x_2, x_3, u_4, v_4 \geq 0$

Introducing slack variables s_1 and s_2 we get the standard form as

Maximize $W = -2x_1 - 2x_2 - 4x_3 - u_4 + v_4$ subject to the constraints

$3x_1 + 2x_2 + x_3 + 9u_4 - 9v_4 + s_1 = 16$
$7x_1 + 2x_2 + 4x_3 + u_4 - v_4 + s_2 = 24$
$x_1, x_2, x_3, u_4, v_4, s_1, s_2 \geq 0$

Note: If a constraint is given in the form of equality then we can express it as two inequalities, one (≤) type and the other (≥) type.

Example 3.3 Rewrite in the standard form:

Maximize $Z = 5x_1 + 2x_2 + x_3$ subject to the constraints

$2x_1 + x_2 - x_3 = 6$
$x_1 + x_2 + x_3 \leq 8$
$x_1, x_2, x_3 \geq 0$

Solution The first constraint is given as

$$2x_1 + x_2 - x_3 = 6$$

This can be written as two inequalities

$$2x_1 + x_2 - x_3 \leq 6$$
$$2x_1 + x_2 - x_3 \geq 6$$

Introducing slack and surplus variables. We get,

$$2x_1 + x_2 - x_3 + s_1 = 6$$
$$2x_1 + x_2 - x_3 - s_2 = 6$$

Adding a slack variable in the second constraint we get

$$x_1 + x_2 + x_3 + s_3 = 8$$

Thus the standard form of the given LPP is

Maximize $Z = 5x_1 + 2x_2 + x_3$ subject to the constraints

$$2x_1 - x_2 - x_3 + s_1 = 6$$
$$2x_1 + x_2 - x_3 - s_2 = 6$$
$$x_1 + x_2 + x_3 + s_3 = 8$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geqq 0$$

Note: An LPP is said to be in *canonical form* if

(i) the objective function is of maximization type
(ii) all the constraints are ($\leq$) type
(iii) all the variables are non-negative.

3.2.2 Simplex Algorithm

Consider the LPP (in the standard form)

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3$ subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + s_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + s_2 = b_2$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0.$$

Since we have two equations we can have two basic variables. Taking s_1 and s_2 as the basic variables we have a solution

$$s_1 = b_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3$$
$$s_2 = b_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3.$$

Assigning zero values to the non-basic variables x_1, x_2 and x_3 we obtain a basic feasible solution.

$$s_1 = b_1 \text{ and } s_2 = b_2$$

The objective function can be written as

$$Z = c_1x_1 + c_2x_2 + c_3x_3 + c_4s_1 + c_5s_2$$

Therefore for $s_1 = b_1$ and $s_2 = b_2$ we get the value of Z as $Z_0 = c_4b_1 + c_5b_2$. We want to find out other basic feasible solutions which would increase the value of Z. We shall take a non-basic variable into the basis and examine whether the new basic feasible solution gives an increased value for Z.

Substituting for s_1 and s_2 in Z, we get

$$\begin{aligned} Z &= c_1x_1 + c_2x_2 + c_3x_3 + c_4(b_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3) + c_5(b_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3) \\ &= c_4b_1 + c_5b_2 - \{c_4a_{11} + c_5a_{21} - c_1\}x_1 - \{c_4a_{12} + c_5a_{22} - c_2\}x_2 - \{c_4a_{13} + c_5a_{23} - c_3\}x_3. \end{aligned}$$

s_1 and s_2 are the current basic variables and c_4 and c_5 are their cost co-efficients in the objective function. Taking

$$X_B = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \text{ and } C_B = (c_4, c_5)$$

We have
$$\begin{aligned} C_BX_B &= c_4s_1 + c_5s_2 \\ &= c_4b_1 + c_5b_2 = Z_o \end{aligned}$$

$$A_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \quad A_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \quad A_3 = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

of the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$$\begin{aligned} \therefore\ c_4a_{11} + c_5a_{21} &= C_BA_1 \\ c_4a_{12} + c_5a_{22} &= C_BA_2 \\ c_4a_{13} + c_5a_{23} &= C_BA_3 \end{aligned}$$

We denote these values by Z_1, Z_2 and Z_3 respectively.

Thus

$$\begin{aligned} Z &= Z_o - (Z_1 - c_1)x_1 - (Z_2 - c_2)x_2 - (Z_3 - c_3)x_3 \\ &= Z_o - \sum_{j=1}^{3} (Z_j - c_j)x_j \end{aligned}$$

For some j, if $Z_j - c_j$ is negative then the value of Z will increase if the corresponding non-basic variable x_j becomes a basic variable. The most negative value of $Z_j - c_j$ will indicate the variable which will cause maximum increase in Z, when included in the basis.

Suppose $Z_1 - c_1$ is the most negative then x_1 should be taken into the basis. The present value of x_1 (non-basic) is zero. Let us assign a value θ to x_1. Then we get the solution

$$x_1 = \theta,\ x_2 = 0,\ x_3 = 0,\ s_1 = b_1 - a_{11}\theta \text{ and}$$
$$s_2 = b_2 - a_{21}\theta$$

Since x_1 has been taken into the basis and the basis can consist of only two variables, one of the current basic variables should leave the basis and become non-basic. We have to determine which of the variables s_1 and s_2 should become non-basic and assume zero value

$$s_1 = 0 \rightarrow \ b_1 - a_{11}\theta = 0$$
$$\rightarrow \ \theta = b_1/a_{11}$$
$$s_2 = 0 \rightarrow \ b_2 - a_{21}\theta = 0$$
$$\rightarrow \ \theta = b_2 / a_{21}$$

If a_{11} is negative then θ becomes negative and same is the case if a_{21} is negative. Therefore let us assume that both a_{11} and a_{21} are positive.

Let $b_1/a_{11} < b_2/a_{21}$. We take $\theta = b_1/a_{11}$ and s_1 becomes zero. On the other hand, if we take $\theta = b_2/a_{21}$ then

$$s_1 = b_1 - a_{11}\theta$$
$$= b_1 - a_{11}\, b_2/a_{21}$$
$$= (a_{21}b_1 - a_{11}b_2) / a_{21} < 0$$
$$\rightarrow \leftarrow \text{[since } b_1/a_{11} < b_2/a_{21}]$$

Hence we take the minimum ratio as θ. Now s_1 becomes non-basic with zero value

$$x_1 = \theta, x_2 = 0, x_3 = 0, s_1 = 0, s_2 = b_2 - a_{21}\theta$$

is the new solution which gives

$$Z = Z_0 - (Z_1 - c_1)\theta > Z_0 \text{ since } Z_1 - c_1 < 0$$

Note: If $Z_j - c_j \geq 0$ for all j then it is not possible to increase Z and the current solution itself is optimal.

While taking the ratio we consider only the a_{ij} which is positive. If all the a_{ij} are negative, Z can be increased to any value and in this case the solution is unbounded.

In the above example we assumed that both a_{11} and a_{21} are positive and obtained the value of θ as the minimum of b_1/a_{11} and b_2/a_{21}. If a_{11} is negative, and a_{21} is positive we have to take $\theta = b_2/a_{21}$ as it is the only possibility.

3.2.3 Simplex Tableau

The simplex algorithm can be carried out conveniently by representing the given objective function, constraint equalities, basic variables, basic solution, etc. in a tabular form known as *simplex tableau* or *simplex table*.

Consider an LPP in the standard form:

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m = b_m$$
$$x_1, x_2, x_3 \ldots x_n, s_1, s_2, \ldots s_m \geq 0$$

We find that $(s_1, s_2, \ldots, s_m)$ can be taken as the basic vector B and their coefficients in the objective function are zeroes, i.e. $C_B = (0, 0, 0 \ldots 0)$.

The simplex table representing the initial solution is given below.

Table 3.1

		x_1	x_2	x_3	....	x_s	....	x_n	s_1	s_2	....	....	s_m	*Const*
C_B	B	c_1	c_2	c_3	....	c_s	....	c_n	0	0	....	....	0	0
0	s_1	a_{11}	a_{12}	a_{13}	....	a_{1s}	....	a_{1n}	1	0	....	....	0	b_1
0	s_2	a_{21}	a_{22}	a_{23}	....	a_{2s}	....	a_{2n}	0	1	....	....	0	b_2
0	s_3	a_{31}	a_{32}	a_{33}		a_{3s}		a_{3n}	0	0	1		0	b_3
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0	s_r	a_{r1}	a_{r2}	a_{r3}	.	a_{rs}	.	a_{rn}	.	.	.	.	.	b_r
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0	s_m	a_{m1}	a_{m2}	a_{m3}	.	a_{ms}	.	a_{mn}	0	0	.	.	1	
	$Z_j - c_j$	$-c_1$	$-c_2$	$-c_3$	....	$-c_s$	....	$-c_n$	0	0	....	....	0	0

The starting solution is

$$s_1 = b_1,$$
$$s_2 = b_2, \ldots s_m = b_m,$$
$$Z_0 = 0$$

The column corresponding to the basic variables form an identity matrix I_m.

$Z_j = C_B\, A_j = 0 \ \forall\ j$ since $C_B = (0,0,0,\ldots,0)$ in this table.

Therefore $Z_j - c_j = -c_j$. $Z_j - c_j$ is called *relative cost factor* and is denoted by $\bar{c}_j$. Thus $\bar{c}_j = -c_j$ in the starting table.

3.2.4 Testing for Optimality

In the starting table if all the relative cost factors are non-negative ($\bar{c}_j \geq 0\ \forall\ j$) then it is not possible to increase Z further and the current solution itself is the optimal solution.

Suppose there exist some $\bar{c}_j < 0$. Then select the most negative of them, say $\bar{c}_s$. The corresponding column A_s is called the *pivot column* and the corresponding variable (non-basic) enters the basis and becomes a basic variable for the improved solution. Thus x_s is the entering variable.

Now find the ratios b_i/a_{is} for all $a_{is} > 0$ in the A_s column and choose the minimum ratio, say b_r/a_{rs}. Then the corresponding r^{th} row is called the *pivot row* and the element a_{rs} is called the *pivot element* which indicates that the variable s_r leaves the basis and becomes non-basic in the improved solution.

Since x_s becomes a basic variable, the pivot column should have all the elements equal to zero except the pivot element which should become unity. In order to obtain this, we make elementary row transformations. This process is called *iteration*. In the modified table x_B contains x_s as a basic variable in place of s_r and C_B contains c_s in place of 0. Thus an improved solution is obtained.

Once again we find $Z_j - c_j$ and examine whether the current solution is optimal. If there is possibility of improving further proceed to find out the entering variable and leaving variable and modify the table. This process is repeated until we obtain the optimal table in which $\bar{c}_j \geq 0\ \forall\ j$ and the corresponding solution is the optimal solution.

Example 3.4 Solve by simplex method

Maximize $Z = 3x_1 + 2x_2$ subject to the constraints

$$-2x_1 + x_2 \leq 1$$
$$x_1 \leq 2$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution The standard form of the LPP is

Maximize $Z = 3x_1 + 2x_2$ subject to

$$-2x_1 + x_2 + s_1 = 1$$
$$x_1 + s_2 = 2$$
$$x_1 + x_2 + s_3 = 3$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

The starting simplex table is

Table 3.2

Z		x_1	x_2	s_1	s_2	s_3	Const
C_B	B	3	2	0	0	0	0
0	s_1	–2	1	1	0	0	1
0	s_2	$\boxed{1}$	0	0	1	0	2 ←
0	s_3	1	1	0	0	1	3
	$Z_j - c_j$	–3 ↑	–2	0	0	0	0

Most negative $Z_j - c_j$ is –3 and hence A_1 is the pivot column. x_1 enters the basis. In the pivot column the negative element –2 should not be taken for finding the ratio. The other two ratios are 2/1 and 3/1 and the minimum ratio is 2/1 which corresponds to the 2^{nd} row and hence the 2^{nd} row is the pivot row. Therefore s_2 is the leaving variable and the pivot element is 1. Since x_1 enters the basis the first column should be transformed into

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

using row transformations.

We make the transformations $R_1' = R_1 - (-2)/(1)\, R_2$ i.e. $R_1 = R_1 + 2R_2$ and this changes –2 to 0. Also we take $R_3' = R_3 - (1)/(1)\, R_2$ i.e. $R_3' = R_3 - R_2$ which reduces 1 to 0. The pivot element is 1 and needs no change. The new table is

Table 3.3

Z		x_1	x_2	s_1	s_2	s_3	Const
C_B	B	3	2	0	0	0	0
0	s_1	0	1	1	2	0	5
3	x_1	1	0	0	1	0	2
0	s_3	0	$\boxed{1}$	0	–1	1	1 ←
	$Z_j - c_j$	0	–2 ↑	0	3	0	6

We have obtained an improved solution $s_1 = 5$, $x_1 = 2$, $s_3 = 1$, $Z = 6$. There is one negative $\bar{c}_j = -2$. Hence this solution is not optimal and it can be improved further.

The pivot column is the x_2 column. Hence x_2 enters the basis. Taking the ratios 5/1, 2/0 and 1/1 we find that the minimum is 1 corresponding to s_3 row and therefore s_3 leaves the basis. The pivot column should be changed to

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

by row transformations. Since there is already 0 as the 2nd element, the x_1 row needs no change.

Take $R_1' = R_1 - R_3$. This reduces 1 to 0. Also the pivot element itself is 1.

The new table is

Table 3.4

Z		x_1	x_2	s_1	s_2	s_3	*Const*
C_B	B	3	2	0	0	0	0
0	s_1	0	0	1	3	–1	4
3	x_1	1	0	0	1	0	2
2	x_2	0	1	0	–1	1	1
	$Z_j - c_j$	0	0	0	1	2	8

Here all the $Z_j - c_j$ are non-negative and hence the current solution is optimal. The optimal solution is

$$s_1 = 4, x_1 = 2, x_2 = 1$$

Since the coefficient of the slack variable s_1 is zero in the objective function it need not be taken into account. We give the solution in terms of the decision variables x_1 and x_2 only. Thus the optimal solution is

$$x_1 = 2, x_2 = 1, Z^* = 8$$

Example 3.5 Maximize $Z = x_1 - x_2 + 3x_3$ subject to the constraints

$$x_1 + x_2 + x_3 \le 10$$
$$2x_1 - x_3 \le 3$$
$$2x_1 - 2x_2 + 3x_3 \le 0$$
$$x_1, x_2, x_3 \ge 0$$

Solution The given LPP can be expressed in the standard form as follows:

Maximize $Z = x_1 - x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$ subject to

$$x_1 + x_2 + x_3 + s_1 = 10$$
$$2x_1 - x_3 + s_2 = 3$$
$$2x_1 - 2x_2 + 3x_3 + s_3 = 0$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

The starting simplex table is

Table 3.5

Z		1	–1	3	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	*Const*
0	s_1	1	1	1	1	0	0	10
0	s_2	2	0	–1	0	1	0	3
0	s_3	2	–2	$\boxed{3}$	0	0	1	0 ←
	$Z_j - c_j$	–1	1	– 3 ↑	0	0	0	0

The most negative $\bar{c}_j$ is –3 and therefore x_3 enters the basis. Minimum ratio is obtained as Min (10/1, 0/3) = 0 corresponding to s_3. Hence s_3 leaves the basis. The necessary row transformations are given by

$$R_1' = R_1 - (1/3)\,R_3;\quad R_2' = R_2 + (1/3)R_3;$$
$$R_3' = (1/3)R_3$$

The new table is (I Iteration)

Table 3.6

Z		1	–1	3	0	0	0	0
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	*Const*
0	s_1	1/3	$\boxed{5/3}$	0	1	0	– 1/3	10 ←
0	s_2	8/3	– 2/3	0	0	1	1/3	2
3	x_3	2/3	– 2/3	1	0	0	1/3	0
	$Z_j - c_j$	1	– 1 ↑	0	0	0	1	0

Pivot element is 5/3. x_2 enters the basis and s_1 leaves.

Performing $R_1' = R_1$ (3/5) we get (II Iteration)

$$R_2' = R_2 - (-2/3)/(5/3)R_1$$
$$\text{(i.e. } R_2' = R_2 + (2/5)R_1\text{)}$$
$$R_3' = R_3 - (-2/3)/(5/3)R_1$$
$$\text{(i.e. } R_3' = R_3 + (2/5)R_1\text{)}$$

Table 3.7

Z		1	–1	3	0	0	0	0
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const
– 1	x_2	1/5	1	0	3/5	0	– 1/5	6
0	s_2	14/5	0	0	2/5	1	1/5	6
3	x_3	4/5	0	1	2/5	0	1/5	4
	$Z_j–c_j$	6/5	0	0	3/5	0	4/5	6

All the relative cost factors are non-negative. Hence this solution is optimal.

The solution is

$$x_1 = 0, x_2 = 6, x_3 = 4, Z^* = 6$$

Note: One of the decision variables $x_1 = 0$. This is called a *degenerate solution*.

Example 3.6 Maximize $Z = 2x_1 + 2x_2 + 4x_3$ subject to the constraints

$$2x_1 + 3x_2 + x_3 \le 300$$
$$x_1 + x_2 + 3x_3 \le 300$$
$$x_1 + 3x_2 + x_3 \le 240$$
$$x_1, x_2, x_3 \ge 0$$

Solution The standard form of the LPP is

Maximize $Z = 2x_1+2x_2+4x_3+0s_1+0s_2+0s_3$ subject to

$$2x_1 + 3x_2 + x_3 + s_1 = 300$$
$$x_1 + x_2 + 3x_3 + s_2 = 300$$
$$x_1 + 3x_2 + x_3 + s_3 = 240$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

The starting table is

Table 3.8

Z		2	2	4	0	0	0	0
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const
0	s_1	2	3	1	1	0	0	300
0	s_2	1	1	$\boxed{3}$	0	1	0	300 ←
0	s_3	1	3	1	0	0	1	240
	$Z_j–c_j$	–2	–2	– 4 ↑	0	0	0	0

x_3 enters the basis since – 4 is the most negative relative cost factor. The minimum among the ratios {300/1, 300/3, 240/1} is 100 which corresponds to s_2 and hence s_2 leaves the basis. Take $R_1' = R_1 – (1/3)\ R_2$; $R_2' = (1/3)R_2$; $R_3' = R_3 – (1/3)R_2$. We get the table (I iteration)

Table 3.9

Z		2	2	4	0	0	0		
C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	$\boxed{5/3}$	8/3	0	1	–1/3	0	200	←120
4	x_3	1/3	1/3	1	0	1/3	0	100	300
0	s_3	2/3	8/3	0	0	–1/3	1	140	210
	$Z_j–c_j$	–2/3 ↑	–2/3	0	0	4/3	0	400	

x_1 and x_2 both have the same relative cost factor and hence the selection of entering variable is at our choice. Let us choose x_1 to enter the basis. The minimum ratio is 120 and hence s_1 leaves the basis.

Perform the operations

$$R_1' = (3/5)R_1;\ R_2' = R_2 – (1/5)R_1,$$
$$R_3' = R_3 – (2/5)R_1$$

We get the II iteration

Table 3.10

Z		2	2	4	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const
2	x_1	1	8/5	0	3/5	– 1/5	0	120
4	x_3	0	– 1/5	1	– 1/5	2/5	0	60
0	s_3	0	8/5	0	– 2/5	– 1/5	1	60
	$Z_j – c_j$	0	2/5	0	2/5	6/5	0	480

Optimality is reached. The optimal solution is

$$x_1 = 120, x_2 = 0, x_3 = 60, Z^* = 480$$

(This is a degenerate solution since $x_2 = 0$)

Example 3.7 Maximize $Z = 2x_1+4x_2+x_3$ subject to the constraints

$$x_1 + 3x_2 \le 4$$
$$2x_1 + x_2 \le 3$$
$$x_2 + 4x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

Solution Standard form

Maximize $Z = 2x_1 + 4x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$ subject to the constraints

$$x_1 + 3x_2 + s_1 = 4$$
$$2x_1 + x_2 + s_2 = 3$$
$$x_2 + 4x_3 + s_3 = 3$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

Initial table

Table 3.11

Z		2	4	1	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	1	$\boxed{3}$	0	1	0	0	4	4/3
0	s_2	2	1	0	0	1	0	3	3/1
0	s_3	0	1	4	0	0	1	3	3/1
	$Z_j - c_j$	– 2	– 4	– 1	0	0	0	0	

x_2 enters and s_1 leaves

$R_1' = R_1/3$; $R_2' = R_2 - (1/3)R_1$;

$R_3' = R_3 - (1/3)R_1$

Table 3.12

I Iteration

Z		2	4	1	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
4	x_2	1/3	1	0	1/3	0	0	4/3	–
0	s_2	5/3	0	0	– 1/3	1	0	5/3	–
0	s_3	– 1/3	0	$\boxed{4}$	– 1/3	0	1	5/3	5/12
	$Z_j - c_j$	– 2/3	0	–1	4/3	0	0	16/3	

$R_3' = R_3/4$

Table 3.13

II Iteration

4	x_2	1/3	1	0	1/3	0	0	4/3	4
0	s_2	$\boxed{5/3}$	0	0	– 1/3	1	0	5/3	1
1	x_3	–1/12	0	1	–1/12	0	1/4	5/12	–
	$Z_j - c_j$	–5/12	0	0	4/3	0	1/4	69/12	

$R_1' = R_1 - (1/5)R_2$, $R_2' = R_2(3/5)$

$R_3' = R_3 + (1/20)R_2$

Table 3.14

III Iteration

4	x_2	0	1	0	2/5	–1/5	0	1
2	x_1	1	0	0	– 1/5	3/5	0	1
1	x_3	0	0	1	–1/10	1/20	1/4	1/2
	$Z_j - c_j$	0	0	0	11/10	9/20	1/4	13/2

All $C_j \geq 0$. Optimality is reached. The solution is

$x_1 = 1$, $x_2 = 1$, $x_3 = 1/2$, $Z^* = 13/2$

This is a non-degenerate solution.

Note: If we divide the third constraint of the problem by 4, we get

$$(1/4)x_2 + x_3 + (1/4)s_3 = 3/4$$

We get x_3 itself as one of the starting basic variables instead of s_3. As a result we can solve the problem in two iterations.

3.2.5 Step by Step Procedure of Simplex Algorithm

Step 1 Rewrite the given LPP in the standard form.

Step 2 Form the simplex table giving the coefficients of all the variables along the corresponding columns (as given below)

Table 3.15

Z		c_1	c_2	…	c_n	0	0	…	0	0
Coefficient of basic variables C_B	Basic variables B	x_1	x_2		x_n	s_1	s_2		s_m	Const
0	s_1	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	b_1
0	s_2	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	b_2
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
0	s_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	b_m

Step 3 Write down the relative cost factor $Z_j - c_j$ for each column. If all the relative cost factors are non-negative, stop. The current solution is optimal. Otherwise choose the most negative $Z_j - c_j$. The corresponding non-basic variable (say x_s) enters the basis. This column is the pivot column.

Step 4 Find the ratios b_i/a_{is} for positive a_{is} in the pivot column. Choose the minimum ratio, say b_r/a_{rs}. Then r^{th} row is the pivot row and the corresponding basic variable leaves the basis. a_{rs} is the pivot element.

Step 5 In the pivot column, all the elements except the pivot element must be reduced to zero by suitable row transformations. For the i^{th} row, make the transformation

$$R_i' = R_i - (a_{is}/a_{rs})R_r \quad i = 1\ 2\ 3\ \ldots..\ m\ (i \neq r)$$

For the pivot row $R_r' = R_r/a_{rs}$

Carry out the operations for all the rows, including the constants.

Step 6 In the new table include x_r as a basic variable and c_r as its coefficient, in place of the leaving variable. Find the relative cost factors $Z_j - c_j$. This completes one iteration.

Step 7 Examine all the $Z_j - C_j$. If all of them are non-negative, stop. The current solution is optimal. If there exists negative $Z_j - C_j$, repeat the procedure from step 3.

Step 8 Once the optimality is reached write the solution, by giving the values of the basic variables, from the constants column and the optimum Z or Z^* is given by the constant of the $Z_j - c_j$ row. We can check the value of Z^* by substituting the values of the optimal basic variables in the objective function.

Example 3.8 Maximize $Z = x_1 + 2x_2 + x_3$ subject to the constraints

$$2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_i \ge 0 \quad i = 1, 2, 3$$

Solution Standard form of the LPP

Maximize $Z = x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$ subject to the constraints

$$2x_1 + x_2 - x_3 + s_1 = 2$$
$$2x_1 - x_2 + 5x_3 + s_2 = 6$$
$$4x_1 + x_2 + x_3 + s_3 = 6$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

It should be noted that the constraint $-2x_1 + x_2 - 5x_3 \ge -6$ has to be changed as

$2x_1 - x_2 + 5x_3 \le 6$ and then s_2 is added, since the constant should be non-negative.

Starting table

Table 3.16

Z		1	2	1	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	2	$\boxed{1}$	–1	1	0	0	2 ←	2/1
0	s_2	2	–1	5	0	1	0	6	–
0	s_3	4	1 ↑	1	0	0	1	6	6/1
	$Z_j – c_j$	–1	–2	–1	0	0	0	0	

x_2 enters the basis and s_1 leaves.

$$R_1' = R_1;\ R_2' = R_2 + R_1;\ R_3' = R_3 - R_1$$

Table 3.17

I Iteration

Z		1	2	1	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
2	x_2	2	1	–1	1	0	0	2	
0	s_2	4	0	4	1	1	0	8	8/4=2
0	s_3	2	0	$\boxed{2}$ ↑	–1	0	1	4 ←	4/2=2
	$Z_j – c_j$	3	0	–3	2	0	0	4	

$$R_1' = R_1 + R_3/2,\ R_2' = R_2 - 2R_3,\ R_3' = R_3/2$$

Table 3.18

II Iteration

2	x_2	3	1	0	1/2	0	1/2	4
0	S_2	0	0	0	3	1	–2	0
1	X_3	1	0	1	–1/2	0	1/2	2
	$Z_j – c_j$	6	0	0	1/2	0	3/2	10

All $Z_j - c_j \ge 0$. Optimality is reached. The optimal solution is

$x_1 = 0;\ x_2 = 4;\ x_3 = 2;\ Z^* = 10$

Example 3.9 Maximize $Z = 12x_1 + 15x_2 + 14x_3$ subject to the constraints

$$-x_1 + x_2 \le 0$$
$$-x_2 + 2x_3 \le 0$$
$$x_1 + x_2 + x_3 \le 100$$
x_1 unrestricted.
$$x_2, x_3 \ge 0$$

Solution Given that x_1 is unrestricted. Hence we take $x_1 = u_1 - v_1$ where $u_1 \ge 0$ and $v_1 \ge 0$. The standard form of the problem is

Maximize $Z = 12(u_1 - v_1) + 15x_2 + 14x_3$ subject to the constraints

$$-(u_1 - v_1) + x_2 + s_1 = 0$$
$$-x_2 + 2x_3 + s_2 = 0$$
$$(u_1 - v_1) + x_2 + x_3 + s_3 = 100$$
$$u_1, v_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

The starting table is

Table 3.19

	Z	12	–12	15	14	0	0	0	0	
C_B	B	u_1	v_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	–1	1	$\boxed{1}$	0	1	0	0	0	0
0	s_2	0	0	–1	2	0	1	0	0	–
0	s_3	1	–1	1	1	0	0	1	100	100
	$Z_j–c_j$	–12	12	–15	–14	0	0	0	0	

$R_1' = R_1, R_2' = R_2 + R_1, R_3' = R_3 - R_1$

Table 3.20

I Iteration

15	x_2	– 1	1	1	0	1	0	0	0	–
0	s_2	– 1	1	0	2	1	1	0	0	–
0	s_3	$\boxed{2}$	– 2	0	1	– 1	0	1	100	50
	$Z_j–c_j$	–27	27	0	–14	15	0	0	0	

$R_1' = R_1+(1/2)R_3, R_2' = R_2 + (1/2)R_3,$
$R_3' = (1/2)R_3$

Table 3.21

II Iteration

15	x_2	0	0	1	1/2	1/2	0	1/2	50	100
0	s_2	0	0	0	$\boxed{5/2}$	1/2	1	1/2	50	100/3
12	u_1	1	– 1	0	1/2	–1/2	0	1/2	50	100
	$Z_j–c_j$	0	0	0	–1/2	3/2	0	27/2	1350	

$R_1' = R_1 - (1/5)\ R_2, R_2' = R_2(2/5),$
$R_3' = R_3 - (1/5)R_2$

Table 3.22

III Iteration

15	x_2	0	0	1	0	2/5	–1/5	2/5	40	–
14	x_3	0	0	0	1	1/5	2/5	1/5	20	–
12	u_1	1	–1	0	0	–3/5	–1/5	2/5	40	–
	$Z_j–c_j$	0	0	0	0	8/5	1/5	68/5	1360	–

All $Z_j - c_j \geq 0$. Optimal solution reached. The optimal solution is

$$u_1 = 40,\ v_1 = 0,\ x_2 = 40,\ x_3 = 20,\ Z^* = 1360$$

i.e. $x_1 = 40, x_2 = 40, x_3 = 20, Z^* = 1360.$

Example 3.10 Minimize $Z = x_1 – 3x_2 + 2x_3$ subject to the constraints

$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Solution We have to change the problem into maximization type. Take $W = –Z$.

Maximize $W = –x_1 + 3x_2 - 2x_3$

Therefore, the standard form is

Maximize $W = –x_1 + 3x_2 - 2x_3$ subject to the constraints

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 + 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

The initial table is

Table 3.23

	Z	–1	3	–2	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	3	–1	2	1	0	0	7	–
0	s_2	–2	$\boxed{4}$	0	0	1	0	12	3
0	s_3	–4	3	8	0	0	1	10	10/3
	$Z_j–c_j$	1	–3	2	0	0	0	0	–

$R_1' = R_1 + (1/4)R_2, R_2' = R_2/4,$
$R_3' = R_3 - (3/4)R_2$

Table 3.24

I Iteration

0	s_1	$\boxed{5/2}$	0	2	1	1/4	0	10	4
3	x_2	–1/2	1	0	0	1/4	0	3	–
0	s_3	–5/2	0	8	0	–3/4	1	1	–
	$Z_j–c_j$	–1/2	0	2	0	3/4	0	9	–

$R_1' = R_1(2/5), R_2' = R_2 + R_1/5,$
$R_3' = R_3 + R_1$

Table 3.25

II Iteration

Z		-1	3	-2	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
-1	x_1	1	0	4/5	2/5	1/10	0	4	–
3	x_2	0	1	2/5	1/5	3/10	0	5	–
0	s_3	0	0	10	1	$-1/2$	1	11	–
	Z_j-c_j	0	0	12/5	1/5	4/5	0	11	–

All $Z_j - c_j \geq 0$. The optimal solution is reached.
The optimal solution is

$$x_1 = 4, x_2 = 5, x_3 = 0, W^* = 11$$

The solution of the given LPP is

$$x_1 = 4, x_2 = 5, x_3 = 0, Z^* = -11$$

Example 3.11 An auto parts company produces three types of parts A, B, C. The capacity of the machines and the number of machine hours required for one unit of each part is given below

Table 3.26

Machine type	*Available machine hours per week*	*Productivity in machine hours per unit*		
		A	*B*	*C*
Hobbling	250	8	2	3
Shaping	150	4	3	0
Grinding	50	2	–	1

The profit is Rs 20, Rs 6 and Rs 8 for one unit of A, B and C respectively. Find out how much of each part the company should produce per week in order to maximize the profit.

Solution Formulation

Maximize $Z = 20x_1 + 6x_2 + 8x_3$ subject to the constraints

$$8x_1 + 2x_2 + 3x_3 \leq 250$$
$$4x_1 + 3x_2 \leq 150$$
$$2x_1 + x_3 \leq 50$$
$$x_1, x_2, x_3 \geq 0$$

The standard form is

Maximize $Z = 20x_1+6x_2+8x_3$ subject to the constraints

$$8x_1 + 2x_2 + 3x_3 + s_1 = 250$$
$$4x_1 + 3x_2 + s_2 = 150$$
$$2x_1 + x_3 + s_3 = 50$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

The starting simplex table is

Table 3.27

Z		20	6	8	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	8	2	3	1	0	0	250	250/8
0	s_2	4	3	0	0	1	0	150	150/4
0	s_3	$\boxed{2}$	0	1	0	0	1	50	25
	Z_j-c_j	-20	-6	-8	0	0	0	0	0

$R_1' = R_1 - 4R_3$, $R_2' = R_2 - 2R_3$,
$R_3' = R_3/2$

Table 3.28

I Iteration

0	s_1	0	2	-1	1	0	-4	50	50/2
0	s_2	0	$\boxed{3}$	-2	0	1	-2	50	50/3
20	x_1	1	0	1/2	0	0	1/2	25	–
	Z_j-c_j	0	-6	2	0	0	10	500	–

$R_1' = R_1 - (2/3)R_2$, $R_2' = R_2/3$, $R_3' = R_3$

Table 3.29

II Iteration

0	s_1	0	0	1/3	1	$-2/3$	$-8/3$	50/3	50
6	x_2	0	1	$-2/3$	0	1/3	$-2/3$	50/3	–
20	x_1	1	0	$\boxed{1/2}$	0	0	1/2	25	50
	Z_j-c_j	0	0	-2	0	2	6	600	–

$R_1' = R_1 - (2/3)R_3$, $R_2' = R_2 + (4/3)R_3$,
$R_3' = 2R_3$

Table 3.30

III Iteration

0	s_1	$-2/3$	0	0	1	$-2/3$	-3	0	–
6	x_2	4/3	1	0	0	1/3	0	50	–
8	x_3	2	0	1	0	0	1	50	–
	Z_j-c_j	4	0	0	0	2	8	700	–

The optimal solution is reached.

The optimal solution is $x_1 = 0$, $x_2 = 50$, $x_3 = 50$, $Z^* = 700$

The company has to produce 50 units of part *B* and 50 units of part *C* and need not produce part *A*. The maximum profit is Rs 700.

Example 3.12 Maximize $Z = 2x_1 + 3x_2 + 4x_3$ subject to the constraints

$$-x_1 - 5x_2 - 9x_3 \leq 2$$
$$3x_1 - x_2 + x_3 \leq 10$$
$$2x_1 + 3x_2 - 7x_3 \leq 0$$
$$x_1, x_2, x_3 \geq 0$$

Solution The standard form of the LPP is

Maximize $Z = 2x_1 + 3x_2 + 4x_3$ subject to the constraints

$$-x_1 - 5x_2 - 9x_3 + s_1 = 2$$
$$3x_1 - x_2 + x_3 + s_2 = 10$$
$$2x_1 + 3x_2 - 7x_3 + s_3 = 0$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

The starting table is

Table 3.31

Z		2	3	4	0	0	0	0	
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	Const	Ratio
0	s_1	–1	–5	–9	1	0	0	2	–
0	s_2	3	–1	$\boxed{1}$	0	1	0	10 ←	10
0	s_3	2	3	–7 ↑	0	0	1	0	–
	$Z_j–c_j$	–2	–3	–4	0	0	0	0	–

$R_1' = R_1 + 9R_2$, $R_2' = R_2$, $R_3' = R_3 + 7R_2$

Table 3.32

I Iteration

0	s_1	26	–14	0	1	9	0	92	–
4	x_3	3	–1	1	0	1	0	10	–
0	s_3	23	–4 ↑	0	0	7	1	70	–
	$Z_j–c_j$	10	–7	0	0	4	0	40	–

All the elements of the pivot column are negative. Therefore the problem has unbounded solution.

SHORT ANSWER QUESTIONS

1. Define feasible solution
2. Define optimal solution
3. Define slack variable
4. Define surplus variable
5. Define unrestricted variable
6. Define degenerate solution
7. When do you say that an LPP has an unbounded solution?
8. When do you say that an LPP has infinite number of solutions?
9. Give the matrix form of an LPP.
10. State the characteristics of the standard form of an LPP?
11. How is the pivot column selected?
12. How is the pivot row selected?
13. How do you find out whether a solution is optimal or not.
14. What is the test of optimality?
15. Fill in the blanks:
    (i) The number of basic solutions to a system of 3 simultaneous equations in 4 unknowns is __________. (*Ans*: $4C_3 = 4$)
    (ii) The set of all feasible solutions to an LPP is a _________ set. (*Ans*: convex)
    (iii) If none of the decision variables is zero then the solution is called a ________ solution. (*Ans*: non-degenerate)
    (iv) If some decision variable is zero the solution is called a ___________. (*Ans*: degenerate)
    (v) The entering variable is the non-basic variable corresponding to _________. (*Ans* : most negative $Z_j - c_j$)
    (vi) The entering variable column is called a_________. (*Ans*: pivot column)

(vii) The leaving variable row is called a __________. (*Ans*: pivot row)

(viii) A constraint of ≤ type is changed to equality by adding a _________. (*Ans*: slack variable)

(ix) A constraint of ≥ type is reduced to equality by introducing a _________. (*Ans*: surplus variable)

(x) A slack variable is introduced in ≤ inequality constraint to obtain a __________. (*Ans*: basic solution)

16. State whether True or False

(i) The main components of an LPP include the decision variables, constraints and the objective function. (*Ans*: True)

(ii) A feasible solution satisfies all the constraints. (*Ans*: True)

(iii) Graphical method is useful when the problem involves two variables. (*Ans*: True)

(iv) Optimal solution occurs at one of the vertices of the feasible region. (*Ans*: True)

(v) An LPP may have more than one optimal solution. (*Ans*: True)

(vi) Every basic solution is a feasible solution. (*Ans*: False)

(vii) An optimal solution is a feasible solution. (*Ans*: True)

(viii) Every feasible solution is optimal (*Ans*: False)

(ix) In an LPP of standard form all the variable are non-negative (*Ans*: True)

(x) In the simplex iteration the pivot element cannot be zero or negative. (*Ans*: True)

EXERCISES

Rewrite the following LP problems in the standard form:

1. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints

$$-2x_1 + x_2 \leq 1$$
$$x_1 \leq 2$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

2. Maximize $Z = 5x_1 + 7x_2$ subject to the constraints

$$x_1 + x_2 \leq 8$$
$$3x_1 + 4x_2 \leq 20$$
$$6x_1 + 7x_2 \leq 24$$
$$x_1, x_2 \geq 0$$

3. Minimize $Z = 2x_1 + 3x_2 + x_3$ subject to the constraints

$$x_1 - 2x_2 + 3x_3 \geq -4$$
$$2x_1 + 5x_2 - 7x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

4. Maximize $Z = 4x_1 + 2x_2 + 6x_3$ subject to the constraints

$$2x_1 + 3x_2 + 2x_3 \geq 6$$
$$3x_1 + 4x_2 + x_3 \leq 8$$
$$x_1, x_2 \geq 0, x_3 \leq 0$$

5. Maximize $Z = x_1 + 2x_2 + 5x_3 + x_4$ subject to the constraints

$$2x_1 + x_2 + 3x_3 - x_4 \leq 10$$
$$x_1 + 3x_2 + x_3 + 2x_4 \leq 16$$
$$x_1 \text{ unrestricted}, x_2, x_3, x_4 \geq 0$$

Solve the following using simplex method:

6. Maximize $Z = 4x_1 + 10x_2$ subject to the constraints

$$2x_1 + x_2 \leq 50$$
$$2x_1 + 5x_2 \leq 100$$
$$2x_1 + 3x_2 \leq 90$$
$$x_1, x_2 \geq 0$$

7. Maximize $Z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints

$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 2x_2 + x_3 \leq 430$$
$$x_1, x_2, x_3 \geq 0$$

8. Maximize $Z = 6x_1 + 9x_2$ subject to the constraints

$x_1 + x_2 \leq 12$
$x_1 + 5x_2 \leq 44$
$3x_1 + x_2 \leq 30$
$x_1, x_2 \geq 0$

9. Minimize $Z = 8x_1 - 2x_2$ subject to the constraints
$-4x_1 + 2x_2 \leq 1$
$5x_1 - 4x_2 \leq 3$
$x_1, x_2 \geq 0$

10. Maximize $Z = 300x_1 + 200x_2$ subject to the constraints
$5x_1 + 2x_2 \leq 180$
$x_1 + x_2 \leq 45$
$x_1, x_2 \geq 0$

11. Maximize $Z = 100x_1 + 200x_2 + 50x_3$ subject to the constraints
$x_1 + x_2 + 2x_3 \leq 200$
$10x_1 + 8x_2 + 5x_3 \leq 2000$
$2x_1 + x_2 \leq 100$
$x_1, x_2, x_3 \geq 0$

12. Maximize $Z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints
$x_1 + 2x_2 + x_3 \leq 43$
$3x_1 + 2x_3 \leq 46$
$x_1 + 4x_2 \leq 42$
$x_1, x_2, x_3 \geq 0$

13. A firm manufactures two products A and B using wood, plastic and steel. The availability and requirements are given below.

	Availability (kg)	*Requirement for one unit* A	B
Wood	240	1	3
Plastic	370	3	4
Steel	180	2	1

The company gets profits of Rs 4 and Rs 6 on one unit of A and B respectively. Determine the number of units of A and B to be produced in order to get maximum profit.

14. Maximize $Z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints
$x_1 + x_2 + x_3 \leq 9$
$2x_1 + 3x_2 + 5x_3 \leq 30$
$2x_1 - x_2 - x_3 \leq 8$
$x_1, x_2, x_3 \geq 0$

15. Maximize $Z = 5x_1 + 3x_2 + 7x_3$ subject to the constraints
$x_1 + x_2 + 2x_3 \leq 22$
$3x_1 + 2x_2 + x_3 \leq 26$
$x_1 + x_2 + x_3 \leq 18$
$x_1, x_2, x_3 \geq 0$

16. Minimize $Z = 3x_1 - 2x_2 - x_3$ subject to the constraints
$-x_1 + 2x_2 + 3x_3 \leq 7$
$4x_1 - 2x_3 \leq 12$
$3x_1 + 8x_2 + 4x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$

17. Maximize $Z = 9x_1 + 2x_2 + 5x_3$ subject to the constraints
$2x_1 + 3x_2 - 5x_3 \leq 12$
$2x_1 - x_2 + 3x_3 \leq 3$
$3x_1 + x_2 - 2x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$

18. Maximize $Z = 2x + 3y + 4z$ subject to the constraints
$3x + 2z \leq 41$
$2x + y + z \leq 35$
$2y + 3z \leq 30$
$x, y, z \geq 0$

19. Maximize $Z = x_1 + 4x_2 + 5x_3$ subject to the constraints
$3x_1 + 6x_2 + 3x_3 \leq 22$
$x_1 + 2x_2 + 3x_3 \leq 14$
$3x_1 + 2x_2 \leq 14$
$x_1, x_2, x_3 \geq 0$

20. A farmer has a 1000 acres of land on which he grows corn, wheat or soyabeans. The particulars are given below.

Crop	*Cost of preparation per acre (Rs)*	*Number of man-days of work per acre*	*Profit per acre (Rs)*
Corn	100	7	30
Wheat	120	10	40
Soyabeans	70	8	20

If the farmer has Rs 1,00,000 for preparation and can count on 8000 man-days of work how many acres should be allotted to each crop in order to maximize profit?

21. Three grades of coal A, B and C contain phosphorus and ash as impurities. For a particular industrial process a maximum of

100 tons of coal is required which should contain not more than 3 per cent of ash and not more than 0.03 per cent of phosphorus. The percentage of impurities and the profits from each grade are given below.

Coal	*Phosphorus %*	*Ash %*	*Profit Rs/ton*
A	.02	3	12
B	.04	2	15
C	.03	5	14

Find the proportions in which the three grades are to be used in order to maximize the profit.

22. Maximize $Z = 2x_1 + 3x_2$ subject to the constraints
$$-x_1 + 2x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9$$
$$x_1, x_2 \text{ unrestricted}$$

23. Minimize $Z = 3x + 2y$ subject to the constraints
$$-2x + 3y \le 9$$
$$-x + 5y \le 20$$
$$x, y \ge 0$$

24. Maximize $Z = 6x_1 + 8x_2 + 4x_3 + 3x_4$ subject to the constraints
$$2x_1 + 3x_2 + x_3 \le 900$$
$$x_2 + 2x_3 \le 600$$
$$3x_1 + 2x_2 + 2x_3 \le 1200$$
$$3x_3 + x_4 = 100$$
$$x_1, x_2, x_3, x_4 \ge 0$$
[*Hint*: x_4 can be taken as an initial basic variable]

25. Maximize $Z = 5x_1 + 2x_2$ subject to the constraints
$$2x_1 + x_2 \le 8$$
$$x_2 \le 4$$
$$x_1 \le 3$$
$$x_1 \ge 0, x_2 \text{ unrestricted.}$$

26. Maximize $Z = 4x_1 + 3x_2$ subject to the constraints
$$2x_1 + x_2 \le 1000$$
$$x_1 + x_2 \le 800$$
$$x_1, x_2 \ge 0$$

27. Maximize $Z = 2x_1 + 3x_2$ subject to the constraints
$$-x_1 + 2x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9$$
$$x_1, x_2 \ge 0$$

28. Maximize $Z = 4x_1 + 10x_2$ subject to the constraints
$$2x_1 + x_2 \le 10$$
$$2x_1 + 5x_2 \le 20$$
$$2x_1 + 3x_2 \le 18$$
$$x_1, x_2 \ge 0$$

29. Maximize $Z = 2x_1 + x_2$ subject to the constraints
$$x_1 - 2x_2 \le 1$$
$$x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 10$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \ge 0$$

30. Maximize $Z = 4x_1 + 6x_2$ subject to the constraints
$$x_1 + 3x_2 \le 240$$
$$3x_1 + 4x_2 \le 270$$
$$2x_1 + x_2 \le 180$$
$$x_1, x_2 \ge 0$$

31. Maximize $Z = 30x_1 + 40x_2 + 20x_3$ subject to the constraints
$$10x_1 + 12x_2 + 7x_3 \le 10000$$
$$7x_1 + 10x_2 + 8x_3 \le 8000$$
$$x_1 + x_2 + x_3 \le 1000$$
$$x_1, x_2, x_3 \ge 0$$

32. Maximize $Z = 2x_1 + 4x_2 + 3x_3$ subject to the constraints
$$3x_1 + 4x_2 + 2x_3 \le 60$$
$$2x_1 + x_2 + 2x_3 \le 40$$
$$x_1 + 3x_2 + 2x_3 \le 80$$
$$x_1, x_2, x_3 \ge 0$$

33. Maximize $Z = 10x_1 + x_2 + 2x_3$ subject to the constraints
$$x_1 + x_2 - 2x_3 \le 10$$
$$4x_1 + x_2 + x_3 \le 20$$
$$x_1, x_2, x_3 \ge 0$$

34. Maximize $Z = x_1 + x_2 + 3x_3$ subject to the constraints
$$3x_1 + 2x_2 + x_3 \le 3$$
$$2x_1 + x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

35. Maximize $Z = 3x_1 + 5x_2 + 4x_3$ subject to the constraints

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0$$

36. Maximize $Z = 4x_1 + x_2 + 3x_3 + 5x_4$ subject to the constraints

$$4x_1 - 6x_2 - 5x_3 - 4x_4 \ge -20$$
$$-3x_1 - 3x_2 + 4x_3 + x_4 \le 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20$$
$$x_1, x_2, x_3, x_4 \ge 0$$

37. Maximize $Z = 2x_1 + x_2 - 3x_3 + 5x_4$ subject to the constraints

$$x_1 + 7x_2 + 3x_3 + 7x_4 \le 46$$
$$3x_1 - x_2 + x_3 + 2x_4 \le 8$$
$$2x_1 + 3x_2 - x_3 + x_4 \le 10$$
$$x_1, x_2, x_3, x_4 \ge 0$$

38. Maximize $Z = 3x_1 + 4x_2 + x_3 + 5x_4$ subject to the constraints

$$8x_1 + 3x_2 + 2x_3 + 2x_4 \le 10$$
$$2x_1 + 5x_2 + x_3 + 4x_4 \le 5$$
$$x_1 + 2x_2 + 5x_3 + x_4 \le 6$$
$$x_1, x_2, x_3, x_4 \ge 0$$

39. Maximize $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$ subject to the constraints

$$2x_1 + x_2 + 5x_3 + 6x_4 \le 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 \le 24$$
$$7x_1 + x_4 \le 70$$
$$x_1, x_2, x_3, x_4 \ge 0$$

40. Maximize $Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$ subject to the constraints

$$x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0$$

ANSWERS

1. Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$ subject to the constraints

$$-2x_1 + x_2 + s_1 = 1$$
$$x_1 + s_2 = 2$$
$$x_1 + x_2 + s_3 = 3$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0.$$

2. Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$ subject to the constraints

$$x_1 + x_2 + s_1 = 8$$
$$3x_1 + 4x_2 + s_2 = 20$$
$$6x_1 + 7x_2 + s_3 = 24$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0$$

3. Maximize $(-Z) = -2x_1 - 3x_2 - x_3$ subject to the constraints

$$-x_1 + 2x_2 - 3x_3 + s_1 = 4$$
$$2x_1 + 5x_2 - 7x_3 + s_2 = 8$$
$$x_1, x_2, x_3, s_1, s_2 \ge 0$$

4. Maximize $Z = 4x_1 + 2x_2 - 6x_3' + 0s_1 + 0s_2 + 0s_3$ subject to the constraints

$$2x_1 + 3x_2 - 2x_3' - s_1 = 6$$
$$3x_1 + 4x_2 - x_3' + s_2 = 8$$
$$x_1, x_2, x_3', s_1, s_2 \ge 0 \quad (x_3' = -x_3)$$

5. Maximize $Z = u_1 - v_1 + 2x_2 + 5x_3 + x_4 + 0s_1 + 0s_2$ subject to the constraints

$$2u_1 - 2v_1 + x_2 + 3x_3 - x_4 + s_1 = 10$$
$$u_1 - v_1 + 3x_2 + x_3 + 2x_4 + s_2 = 16$$
$$u_1, v_1, x_2, x_3, x_4, s_1, s_2 \ge 0 \quad (x_1 = u_1 - v_1)$$

6. $x_1 = 0, x_2 = 20, Z^* = 200$
7. $x_1 = 0, x_2 = 100, x_3 = 230, Z^* = 1350$
8. $x_1 = 4, x_2 = 8, Z^* = 96$
9. $x_1 = 0, x_2 = 1/2, Z^* = -1$
10. $x_1 = 30, x_2 = 15, Z^* = 12000$
11. $x_1 = 0, x_2 = 100, x_3 = 50, Z^* = 22500$
12. $x_1 = 0, x_2 = 10, x_3 = 23, Z^* = 135$
13. $x_1 = 30, x_2 = 70, Z^* = 540$
14. $x_1 = 5, x_2 = 0, x_3 = 4, Z^* = 35$
15. $x_1 = 6, x_2 = 0, x_3 = 8, Z^* = 86$
16. $x_1 = 0, x_2 = 5/4, x_3 = 0, Z^* = -5/2$
17. $x_1 = 0, x_2 = 12, x_3 = 5, Z^* = 49$
18. $x = 11, y = 9, z = 4, Z^* = 65$
19. $x_1 = 0, x_2 = 2, x_3 = 10/3, Z^* = 74/3$
20. Maximize $Z = 30x_1 + 40x_2 + 20x_3$ subject to the constraints

$$10x_1 + 12x_2 + 7x_3 \le 10000$$

$7x_1 + 10x_2 + 8x_3 \leq 8000$
$x_1 + x_2 + x_3 \leq 1000$
$x_1, x_2, x_3 \geq 0$

Also $x_1 = 250, x_2 = 625, x_3 = 0, Z^* = 32500$

21. Maximize $Z = 12x_1 + 15x_2 + 14x_3$ subject to the constraints

$-x_1 + x_2 \leq 0$
$-x_2 + 2x_3 \leq 0$
$x_1 + x_2 + x_3 \leq 100$
$x_1, x_2, x_3 \geq 0$

Also $x_1 = 40, x_2 = 40, x_3 = 20, Z^* = 1360$

22. $x_1 = 9/2, x_2 = 3/2, Z^* = 27/2$
23. Unbounded solution
24. $x_1 = 1800/7, x_2 = 900/7, x_3 = 0, x_4 = 100, Z^* = 20100/7$
25. $x_1 = 3, x_2 = 2, Z^* = 19$
26. $x_1 = 200, x_2 = 600, Z^* = 2600$
27. Unbounded solution
28. $x_1 = 15/4, x_2 = 5/2, Z^* = 40$
29. $x_1 = 4, x_2 = 2, Z^* = 10$
30. $x_1 = 30, x_2 = 70, Z^* = 540$
31. $x_1 = 250, x_2 = 625, x_3 = 0, Z^* = 32500$
32. $x_1 = 0, x_2 = 20/3, x_3 = 50/3, Z^* = 250/3$
33. $x_1 = 10, x_2 = 0, x_3 = 0, Z^* = 50$
34. $x_1 = 0, x_2 = 0, x_3 = 1, Z^* = 3$
35. $x_1 = 89/41, x_2 = 50/41, x_3 = 64/41, Z^* = 765/41$
36. Unbounded solution
37. $x_1 = 0, x_2 = 12/7, x_3 = 0, x_4 = 34/7, Z^* = 26$
38. $x_1 = 15/14, x_2 = 0, x_3 = 0, x_4 = 5/7, Z^* = 95/14$
39. $x_1 = 4, x_2 = 12, x_3 = 0, x_4 = 0, Z^* = 132$
40. $x_1 = 8, x_2 = 0, x_3 = 3, x_4 = 0, Z^* = 31$

4

Artificial Variables

CONCEPT REVIEW

BIG *M* METHOD AND TWO-PHASE METHOD

4.1 ARTIFICIAL VARIABLES

In the previous chapter we considered LP problem involving upper bound constraints (≤ type) only. These constraints were changed to equalities by introducing slack variables. These slack variables become the initial basic variables and provide a feasible solution to start with. Now we consider problems having lower bound constraints (≥ type) and equality constraints. The inequality of ≥ type can be changed to equality by subtracting a surplus variable. But its coefficient is –1 and hence it has a negative value, and therefore it gives rise to an infeasible solution. Thus a surplus variable cannot become a starting basic variable. In order to obtain an initial basic feasible solution we introduce another variable called *artificial variable*.

For example, consider the constraint

$$2x_1 + 3x_2 + 4x_3 \geq 10$$

Introducing a surplus variable s we get

$$2x_1 + 3x_2 + 4x_3 - s = 10$$

$$\therefore \quad s = 2x_1 + 3x_2 + 4x_3 - 10$$

Assigning zero values to the non-basic variables we get $s = -10$ (negative value)

Therefore we introduce an artificial variable R and write

$$2x_1 + 3x_2 + 4x_3 - s + R = 10$$

$$\therefore \quad R = 10 - 2x_1 - 3x_2 - 4x_3 + s$$

Now R becomes a starting basic variable with a value 10.

4.2 BIG *M* METHOD (CHARNE'S PENALTY METHOD)

The purpose of introducing the artificial variable is just to obtain an initial basic feasible solution. But already we have equality with a surplus variable. Adding one more variable to only one side of this equality causes violation of the equality constraint. In order to overcome this difficulty we need to get rid of these variables and not to allow them to appear as optimal basic variables. If at all an artificial variable becomes a basic variable in the final table, its value must be zero. This will ensure a solution to the problem. To achieve this, these artificial variables are assigned a large penalty cost coefficient ($-M$) in the objective function to be maximized. However, if there is an artificial variable in the optimal basis with positive value then it indicates that the given problem has no solution.

Example 4.1 Maximize $Z = 3x_1 - x_2$ subject to the constraints

$$2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 3$$
$$x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution The standard form of the problem is

Maximize $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - MR$

subject to the constraints

$$2x_1 + x_2 - s_1 + R = 2$$
$$x_1 + 3x_2 + s_2 = 3$$
$$x_2 + s_3 = 4$$
$$x_1, x_2, s_1, s_2, s_3, R \geq 0$$

Here s_1 is a surplus variable, s_2 and s_3 are slack variables and R is an artificial variable with a penalty cost $-M$ in the objective function. M is a very large number tending to infinity. The starting simplex table is

Table 4.1

Z		x_1	x_2	s_1	s_2	s_3	R	Cons	Ratio
C_B	B	3	–1	0	0	0	–M	0	
–M	R	$\boxed{2}$	1	–1	0	0	1	2	← 1
0	s_2	1	3	0	1	0	0	3	3
0	s_3	0 ↑	1	0	0	1	0	4	–
	$Z_j - c_j$	–2M–3	–M + 1	M	0	0	0	–2M	

$R_1' = (1/2)\, R_1$,
$R_2 = R_2 - (1/2)\, R_1$,
$R_3' = R_3$

Table 4.2

I Iteration

3	x_1	1	1/2	–1/2	0	0	1/2	1	–
0	s_2	0	5/2	$\boxed{1/2}$	1	0	–1/2	2	← 4
0	s_3	0	1	0 ↑	0	1	0	4	–
	$Z_j - c_j$	0	5/2	–3/2	0	0	3/2+M	3	

$R_1' = R_1 + R_2$,
$R_2' = 2R_2$,
$R_3' = R_3$

Table 4.3

II Iteration

Z		x_1	x_2	s_1	s_2	s_3	R	Cons	Ratio
C_B	B	3	–1	0	0	0	–M	0	
3	x_1	1	3	0	1	0	0	3	–
0	s_1	0	5	1	2	0	–1	4	–
0	s_3	0	1	0	0	1	0	4	–
	$Z_j - c_j$	0	10	0	3	0	M	9	–

All $\bar{c}_j \geq 0$ and therefore optimality is reached. Optimal solution is $x_1 = 3$, $x_2 = 0$, $Z^* = 9$

Example 4.2 Minimize $Z = 5x_1 + 4x_2 + 3x_3$ subject to the constraints

$$x_1 + x_2 + x_3 = 100$$
$$x_1 \leq 20$$
$$x_2 \geq 30$$
$$x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$

Solution Here the first constraint is an equality and hence we introduce an artificial variable only. In the 2nd and 4th constraint slack variables are introduced. In the third constraint we have to introduce a surplus variable and an artificial variable.

The standard form of the LPP is

Maximize $W = (-Z) = -5x_1 - 4x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3 - MR_1 - MR_2$ subject to the constraints

$$x_1 + x_2 + x_3 + R_1 = 100$$
$$x_1 + s_1 = 20$$
$$x_2 - s_2 + R_2 = 30$$
$$x_3 + s_3 = 40$$
$$x_1, x_2, x_3, s_1, s_2, s_3, R_1, R_2 \geq 0$$

The initial table is

Table 4.4

Z		x_1	x_2	x_3	s_1	s_2	s_3	R_1	R_2	*Const*	*Ratio*
C_B	*B*	–5	–4	–3	0	0	0	$-M$	$-M$	0	
$-M$	R_1	1	1	1	0	0	0	1	0	100	100
0	s_1	1	0	0	1	0	0	0	0	20	–
–M	R_2	0	$\boxed{1}$	0	0	–1	0	0	1	30	30
0	s_3	0	0	1	0	0	1	0	0	40	–
	$Z_j - c_j$	$-M+5$	$-2M+4$	$-M+3$	0	M	0	0	0	$-130M$	

$R_1' = R_1 - R_3$ No other change

Table 4.5

I Iteration

$-M$	R_1	1	0	1	0	$\boxed{1}$	0	1	–1	70	70
0	s_1	1	0	0	1	0	0	0	0	20	–
–4	x_2	0	1	0	0	–1	0	0	1	30	–
0	s_3	0	0	1	0	0	1	0	0	40	–
	$Z_j - c_j$	$-M+5$	0	$-M+3$	0	$-M+4$	0	0	$2M-4$	$-120-70M$	–

s_2 is chosen to enter the basis in order to get rid of R_1 : $R_3' = R_3 + R_1$

Table 4.6

II Iteration

0	s_2	1	0	1	0	1	0	1	–1	70	70
0	s_1	1	0	0	1	0	0	0	0	20	–
–4	x_2	1	1	1	0	0	0	1	0	100	100
0	s_3	0	0	$\boxed{1}$	0	0	1	0	0	40	40
	$Z_j - c_j$	1	0	–1	0	0	0	$-4+M$	M	–400	–

$R_1' = R_1 - R_4$ $R_3' = R_3 - R_4$

Table 4.7

III Iteration

0	s_2	1	0	0	0	1	–1	1	–1	30	–
0	s_1	1	0	0	1	0	0	0	0	20	–
–4	x_2	1	1	0	0	0	–1	1	0	60	–
–3	x_3	0	0	1	0	0	1	0	0	40	–
	$Z_j - c_j$	1	0	0	0	0	1	$-4+M$	M	–360	–

The optimal solution is

$x_1 = 0$, $x_2 = 60$, $x_3 = 40$, $Z^* = -W^* = 360$

Example 4.3 Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$ subject to constraints

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 \geq 20$$
$$x_1 + 2x_2 + x_3 + x_4 \geq 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution Introducing surplus variables and artificial variables, we get the standard form of the LPP as

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 + 0s_1 + 0s_2 - MR_1 - MR_2 - MR_3$ subject to constraints

$$x_1 + 2x_2 + 3x_3 + R_1 = 15$$
$$2x_1 + x_2 + 5x_3 - s_1 + R_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 - s_2 + R_3 = 10$$
$$x_1, x_2, x_3, x_4, s_1, s_2, R_1, R_2, R_3 \geq 0$$

Note: In the simplex table the column corresponding to x_4 is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and therefore we can take x_4 itself as a starting basic variable. R_3 need not be introduced.

Table 4.8

Z		x_1	x_2	x_3	x_4	s_1	s_2	R_1	R_2	R_3	Const	Ratio
C_B	B	1	2	3	–1	0	0	–M	–M	–M	0	
$-M$	R_1	1	2	3	0	0	0	1	0	0	15	15/3
$-M$	R_2	2	1	$\boxed{5}$	0	–1	0	0	1	0	20 ←	20/5
$-M$	R_3	1	2	1 ↑	1	0	–1	0	0	1	10	10/1
	$Z_j - c_j$	$-4M - 1$	$-5M - 2$	$-9M - 3$	$-M + 1$	M	M	0	0	0	$-45M$	

$R_1' = R_1 - (3/5)\,R_2$, $R_2' = (1/5)\,R_2$, $R_3' = R_3 - (1/5)R_2$

Table 4.9

I Iteration

$-M$	R_1	–1/5	$\boxed{7/5}$	0	0	3/5	0	1	–3/5	0	3 ←	15/7
3	X_3	2/5	1/5	1	0	–1/5	0	0	1/5	0	4	20
$-M$	R_3	3/5	9/5 ↑	0	1	1/5	–1	0	–1/5	1	6	30/9
	$Z_j - c_j$	$(-2M+1)/5$	$(-16M - 7)/5$	0	$-M + 1$	$(-4M - 3)/5$	M	0	$(9M + 3)/5$	0	$-9M + 12$	

$R_1' = (5/7)\,R_1$, $R_2' = R_2 - (1/7)R_1$, $R_3' = R_3 - (9/7)R_1$

Table 4.10

II Iteration

2	x_2	–1/7	1	0	0	3/7	0	5/7	–3/7	0	15/7	
3	x_3	3/7	0	1	0	–2/7	0	–1/7	2/7	0	25/7	
$-M$	R_3	6/7	0	0	$\boxed{1}$ ↑	–4/7	–1	–9/7	4/7	1	15/7 ←	
	$Z_j - c_j$	$-6M/7$	0	0	$-M+1$	$4M/7$	M	$(16M+7)/7$	$3M/7$	0	$-15M/7+15$	

x_4 enters, R_3 leaves (no change)

Table 4.11

III Iteration

Z		x_1	x_2	x_3	x_4	s_1	s_2	R_1	R_2	R_3	Const	Ratio
C_B	B	1	2	3	–1	0	0	–M	–M	–M	0	
2	x_2	–1/7	1	0	0	3/7	0	5/7	–3/7	0	15/7	–
3	x_3	3/7	0	1	0	–2/7	0	–1/7	2/7	0	25/7	25/3
–1	x_4	$\boxed{6/7}$ ↑	0	0	1	–4/7	–1	–9/7	4/7	1	15/7 ←	15/6
	$Z_j - C_j$		0	0	0	4/7	1	$(16M + 7)/7$	$–4/7 + M$	$–1 + M$	90/7	

$R_1' = R_1 + (1/6)R_3$, $R_2' = R_2 - (1/2)R_3$, $R_3' = (7/6)R_3$

Table 4.12

IV Iteration

2	x_2	0	1	0	1/6	1/3	–1/6	1/2	–1/3	1/6	5/2	
3	x_3	0	0	1	–1/2	0	1/2	1/2	0	–1/2	5/2	
1	x_1	1	0	0	7/6	–2/3	–7/6	–3/2	2/3	7/6	5/2	
	$Z_j - c_j$	0	0	0	1	0	0	$1 + M$	M	M	15	

The optimal solution is

$$x_1 = 5/2,\ x_2 = 5/2,\ x_3 = 5/2,\ x_4 = 0,\ Z^* = 15$$

Note: When an artificial variable leaves the basis we may drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example 4.4 Maximize $Z = x_1 + 4x_2$ subject to the constraints

$$3x_1 + x_2 \le 3$$
$$2x_1 + 3x_2 \le 6$$
$$4x_1 + 5x_2 \ge 20$$
$$x_1, x_2 \ge 0$$

Solution Standard form of the problem is

Maximize $Z = x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - MR$ subject to the constraints

$$3x_1 + x_2 + s_1 = 3$$
$$2x_1 + 3x_2 + s_2 = 6$$
$$4x_1 + 5x_2 - s_3 + R = 20$$
$$x_1, x_2, s_1, s_2, s_3, R \ge 0$$

The starting table is

Table 4.13

Z		x_1	x_2	s_1	s_2	s_3	R	Const	Ratio
C_B	B	1	4	0	0	0	–M	0	–
0	s_1	3	1	1	0	0	0	3	3/1
0	s_2	2	$\boxed{3}$	0	1	0	0	6 ←	6/3
–M	R	4	5 ↑	0	0	–1	1	20	20/5
	$Z_j - c_j$	$–4M – 1$	$–5M – 4$	0	0	M	0	–20M	–

$R_1' = R_1 - (1/3)R_2$, $R_2' = (1/3)R_2$,
$R_3' = R_3 - (5/3)R_2$

Table 4.14

I Iteration

0	s_1	$\boxed{7/3}$ ↑	0	1	–1/3	0	0	1 ←	3/7
4	x_2	2/3	1	0	1/3	0	0	2	3
–M	R	2/3	0	0	–5/3	–1	1	10	15
	$Z_j - c_j$	$–(2M+5)/3$	0	0	$(5M+4)/3$	M	0	$–10M+8$	–

$R_1' = (3/7)R_1$, $R_2' = R_2 - (2/7)R_1$,
$R_3' = R_3 - (2/7)R_1$

Table 4.15

II Iteration

	Z	x_1	x_2	s_1	s_2	s_3	R	Const	Ratio
C_B	B	1	4	0	0	0	–M	0	
1	x_1	1	0	3/7	–1/7	0	0	3/7	–
4	x_2	0	1	–2/7	3/7	0	0	12/7	–
–M	R	0	0	–2/7	–11/7	–1	1	68/7	–
	$Z_j – c_j$	0	0	(2M–5)/7	(11M+11)/7	M	0	(51–68M)/7	–

Optimality is reached. But the artificial variable R is present in the optimal basis with a value 68/7. Hence the problem has no feasible solution and hence no solution.

4.3 TWO-PHASE METHOD

The two-phase method is another method to solve LP problems involving artificial variables. This method consists of two phases. In the first phase we consider an auxiliary LPP with the objective function.

$$W = 0x_1 + 0x_2 + \dots + 0x_n - R_1 - R_2 \dots$$

where $R_1, R_2, \dots$ are all artificial variables. We maximize W using simplex method under the given constraints. Two cases arise.

Case (I) Max $W < 0$ and at least one artificial variable appears in the optimal basis with a positive value. In this case the given LPP does not possess any feasible solution.

Case (II) Max $W = 0$ and no artificial variable appears in the optimal basis or Max $W = 0$ and some artificial variable appears at zero level in the optimal basis. In this case the problem has a solution and we may proceed to phase II.

In phase II we use the optimal solution of phase I as the starting solution. Assign the actual costs to the objective variables. If there is an artificial variable at zero level, assign zero cost to that artificial variable. Apply simplex method and obtain the optimal solution. We may omit all the artificial variables from the table which are non-basic at the end of phase I.

Example 4.5 Solve using, two-phase simplex method

Maximize $Z = 5x_1 + 8x_2$ subject to the constraints

$$3x_1 + 2x_2 \geq 3$$
$$x_1 + 4x_2 \geq 4$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Solution Assigning zero costs to the slack and surplus variables and –1 to the artificial variables in the objective function we get the standard form

Maximize $Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3 - R_1 - R_2$ subject to

$$3x_1 + 2x_2 - s_1 + R_1 = 3$$
$$x_1 + 4x_2 - s_2 + R_2 = 4$$
$$x_1 + x_2 + s_3 = 5$$
$$x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0$$

Phase I Consider the auxiliary LPP with the objective function $W = -R_1 - R_2$ subject to the given constraints. The starting table is

Table 4.16

	W	x_1	x_2	s_1	s_2	s_3	R_1	R_2	Const	Ratio
C_B	X_B	0	0	0	0	0	–1	–1	0	
–1	R_1	3	2	–1	0	0	1	0	3	3/2
–1	R_2	1	$\boxed{4}$	0	–1	0	0	1	4 ←	4/4
0	S_3	1	1 ↑	0	0	1	0	0	5	5/1
	$\bar{c}_j$	–4	–6	1	1	0	0	0	–7	

$$R_1' = R_1 - (1/2)R_2$$
$$R_2' = (1/4)R_2$$
$$R_3' = R_3 - (1/4)R_2$$

Table 4.17

I Iteration

–1	R_1	$\boxed{5/2}$	0	–1	1/2	0	1	–1/2	1 ←	2/5
0	x_2	1/4	1	0	–1/4	0	0	1/4	1	4
0	s_3	3/4 ↑	0	0	1/4	1	0	–1/4	4	16/3
	$\bar{c}_j$	–5/2	0	1	–1/2	0	0	3/2	–1	

$R_1' = (2/5)\ R_1$
$R_2' = R_2 - (1/10)R_1$
$R_3' = R_3 - (3/10)R_1$

Table 4.18

II Iteration

	W	x_1	x_2	s_1	s_2	s_3	R_1	R_2	Const	Ratio
C_B	X_B	0	0	0	0	0	–1	–1	0	
0	X_1	1	0	–2/5	1/5	0	2/5	1/5	2/5	
0	X_2	0	1	1/10	–3/10	0	–1/10	3/10	9/10	
0	s_3	0	0	3/10	1/10	1	–3/10	–1/10	37/10	
	$\bar{c}_j$	0	0	0	0	0	1	1	0	

Optimality is reached. There is no artificial variable in the optimal basis. We proceed to phase II, omitting R_1 and R_2 from the table.

Phase II Maximize $Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$

The starting table is

Table 4.19

Z		x_1	x_2	s_1	s_2	s_3	Const	Ratio
C_B	B	5	8	0	0	0	0	
5	x_1	1	0	–2/5	$\boxed{1/5}$	0	2/5	2
8	x_2	0	1	1/10	–3/10	0	9/10	–
0	s_3	0	0	3/10	1/10	1	37/10	37
	$Z_j - c_j$	0	0	–6/5	–7/5	0	46/5	

$R_1' = 5R_1$, $R_2' = R_2 + (3/2)R_1$,
$R_3' = R_3 - (1/2)R_1$

Table 4.20

I Iteration

0	S_2	5	0	–2	1	0	2	–
8	x_2	3/2	1	–1/2	0	0	3/2	–
0	s_3	–1/2	0	$\boxed{1/2}$	0	1	7/2	7
	$Z_j - c_j$	7	0	–4	0	0	12	

$R_1' = R_1 + 4R_3$, $R_2' = R_2 + R_3$, $R_3' = 2R_3$

Table 4.21

II Iteration

0	s_2	3	0	0	1	4	16	
8	x_2	1	1	0	0	1	5	
0	s_1	–1	0	1	0	2	7	
	$Z_j - c_j$	3	0	0	0	8	40	Optimal

The solution is $x_1 = 0$, $x_2 = 5$, $Z^* = 40$.

Example 4.6 Solve using the two-phase method

Minimize $Z = 5x_1 - 6x_2 - 7x_3$ subject to the constraints

$$x_1 + 5x_2 - 3x_3 \geq 15$$
$$5x_1 - 6x_2 + 10x_3 \leq 20$$
$$x_1 + x_2 + x_3 = 5$$

Solution The standard form of the problem is

Maximize $Y = (-Z) = -5x_1 + 6x_2 + 7x_3 + 0s_1 + 0s_2 - R_1 - R_2$ subject to the constraints

$$x_1 + 5x_1 - 3x_3 - s_1 + R_1 = 15$$
$$5x_1 - 6x_2 + 10x_3 + s_2 = 20$$
$$x_1 + x_2 + x_3 + R_2 = 5$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$$

Phase I Consider the auxiliary LPP with the objective function $W = -R_1 - R_2$ subject to the given constraints. The starting table is

Table 4.22

W		x_1	x_2	x_3	s_1	s_2	R_1	R_2	Const	Ratio
C_B	B	0	0	0	0	0	–1	–1	0	
–1	R_1	1	$\boxed{5}$	–3	–1	0	1	0	15	15/5
0	S_2	5	–6	10	0	1	0	0	20	–
–1	R_2	1	1	1	0	0	0	1	5	5/1
	$\bar{c}_j$	–2	–6	2	1	0	0	0	–20	

$R_1' = (1/5)R_1$, $R_2' = R_2 + (6/5)R_1$,
$R_3' = R_3 - (1/5)R_1$

Table 4.23

I Iteration

0	x_2	1/5	1	–3/5	–1/5	0	1/5	0	3	–
0	S_2	31/5	0	32/5	–6/5	1	6/5	0	38	190/32
–1	R_2	4/5	0	$\boxed{8/5}$	1/5	0	–1/5	1	2	10/8
	$\bar{c}_j$	–4/5	0	–8/5	–1/5	0	1/5	0	–2	

$R_1' = R_1+(3/8)R_3$, $R_2' = R_2 - 4R_3$,
$R_3' = (5/8)R_3$

Table 4.24

II Iteration

W		x_1	x_2	x_3	s_1	s_2	R_1	R_2	Const	Ratio
C_B	B	0	0	0	0	0	–1	–1	0	
0	x_2	1/2	1	0	–1/8	0	1/8	3/8	15/4	
0	s_2	3	0	0	–2	1	2	–4	30	
0	x_3	1/2	0	1	1/8	0	–1/8	5/8	5/4	
	$\bar{c}_j$	0	0	0	0	0	1	1	0	

Optimality is reached. There is no artificial variable in the optimal basis. Hence we proceed to phase II omitting $R_1$and R_2 from the table.

Phase II Maximize $Y = -5x_1 + 6x_2 + 7x_3 + 0s_1 + 0s_2$

The starting table is

Table 4.25

Y		x_1	x_2	x_3	s_1	s_2	Const	Ratio
C_B	B	–5	6	7	0	0	0	–
6	x_2	1/2	1	0	–1/8	0	15/4	–
0	s_2	3	0	0	–2	1	30	–
7	x_3	1/2	0	1	1/8	0	5/4	–
	$\bar{c}_j$	23/2	0	0	1/8	0	125/4	–

Optimality is reached.

Solution

$$x_1 = 0,\ x_2 = 15/4,\ x_3 = 5/4,$$
$$Z^* = -Y^* = -125/4$$

Example 4.7 (Solve using the two-phase method)

Maximize $Z = -4x_1 - 3x_2 - 9x_3$ subject to the constraints

$$2x_1 + 4x_2 + 6x_3 \geq 15$$
$$6x_1 + x_2 + 6x_3 \geq 12$$
$$x_1, x_2, x_3 \geq 0$$

Solution Introducing surplus variables with zero costs and artificial variables with costs –1 in the objective function we obtain the standard form of the problem as

Maximize $Z = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2 - R_1 - R_2$ subject to the constraints

$$2x_1 + 4x_2 + 6x_3 - s_1 + R_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + R_2 = 12$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$$

Phase I The objective function is $W = -R_1 - R_2$ with the given constraints. The starting table is

Table 4.26

W		x_1	x_2	x_3	s_1	s_2	R_1	R_2	Const	Ratio
C_B	B	0	0	0	0	0	–1	–1	0	
–1	R_1	2	4	6	–1	0	1	0	15	15/6
–1	R_2	6	1	$\boxed{6}$	0	–1	0	1	12	← 12/6
	$\bar{c}_j$	–8	–5	–12 ↑	1	1	0	0	–27	

$R_1' = R_1 - R_2$, $R_2' = (1/6)R_2$

Table 4.27

I Iteration

–1	R_1	–4	$\boxed{3}$	0	–1	1	1	–1	3	← 3/3
0	x_3	1	1/6	1	0	–1/6	0	1/6	2	12
	$\bar{c}_j$	4	–3 ↑	0	1	–1	0	1	–3	

$R_1' = (1/3)R_1$, $R_2' = R_2 - (1/18)R_1$

Table 4.28

II Iteration

0	x_2	–4/3	1	0	–1/3	1/3	1/3	–1/3	1
0	x_3	11/9	0	1	1/18	–2/9	–1/18	2/9	11/6
	$\bar{c}_j$	0	0	0	0	0	1	1	0

The optimality is reached. No artificial variable appears in the optimal basis. Hence we proceed to phase II

Phase II $Z = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2$

The starting table is

Table 4.29

Z		x_1	x_2	x_3	s_1	s_2	Const	Ratio
C_B	B	–4	–3	–9	0	0	0	
–3	x_2	–4/3	1	0	–1/3	1/3	1	–
–9	x_3	$\boxed{11/9}$	0	1	1/18	–2/9	11/6	← 9/6
	$\bar{c}_j$	–3	0	0	1/2	1	–39/2	

$$R_1' = R_1 + (12/11)R_2,\ R_2' = (9/11)R_2$$

Table 4.30

I Iteration

Z		x_1	x_2	x_3	s_1	s_2	Const	Ratio
C_B	B	–4	–3	–9	0	0	0	
–3	x_2	0	1	12/11	–3/1	1/11	3	
–4	x_1	1	0	9/11	1/22	–2/11	3/2	
	$\bar{c}_j$	0	0	27/11	7/11	5/11	–15	

The optimality is reached. Solution of the given LPP is

$$x_1 = 3/2,\ x_2 = 3,\ x_3 = 0,\ Z^* = -15$$

Example 4.8 (Solve using the two-phase method)

Maximize $Z = 5x_1 + 3x_2$ subject to the constraints

$$2x_1 + x_2 \le 1$$
$$x_1 + 4x_2 \ge 6$$
$$x_1, x_2 \ge 0$$

Solution The standard form of the LPP is

Maximize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 - R_1$ subject to the constraints

$$2x_1 + x_2 + s_1 = 1$$
$$x_1 + 4x_2 - s_2 + R_1 = 6$$
$$x_1, x_2, s_1, s_2, R_1 \ge 0$$

The objective function of the auxiliary LPP is

$$W = -R_1$$

The starting table is

Table 4.31

W		x_1	x_2	s_1	s_2	R_1	Const	Ratio
C_B	B	0	0	0	0	–1	0	
0	s_1	2	$\boxed{1}$	1	0	0	1	← 1/1
–1	R_1	1	4	0	–1	1	6	6/4
	$\bar{c}_j$	–1	–4	0	1	0	–6	

$$R_1' = R_1,\ R_2' = R_2 - 4R_1$$

Table 4.32

I Iteration

0	x_2	2	1	1	0	0	1	
–1	R_1	–7	0	–4	–1	1	2	
	$\bar{c}_j$	7	0	4	1	0	–2	

All $Z_j - C_j \ge 0$. The optimality is reached. But the optimal basis has an artificial variable R_1 with a value 2. Hence the given LPP has no solution.

SHORT ANSWER QUESTIONS

1. Define surplus variable.
2. Explain: artificial variable, big-*M* method, two-phase method.
3. Explain how the two-phase method helps to find out the feasibility of an LPP.
4. Explain the advantages of the two-phase method over the big-*M* method.

Fill in the blanks

5. An artificial variable is introduced in an equality constraint to obtain a ___________. (*Ans*: basic solution)
6. Artificial variable can be dropped from the simplex iteration table, when it becomes _________. (*Ans*: non-basic)

7. An LPP has no solution if an artificial variable appears in the optimal basis with ____________. (*Ans*: positive value)

8. The number of basic solutions to an LPP is __________. (*Ans*: finite)

EXERCISES

Solve using big-*M* method

1. Minimize $Z = 5x_1 - 6x_2 - 7x_3$ subject to the constraints
$$x_1 + 5x_2 - 3x_3 \geq 15$$
$$5x_1 - 6x_2 + 10x_3 \leq 20$$
$$x_1 + x_2 + x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

2. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints
$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

3. Minimize $Z = 4x_1 + 3x_2$ subject to the constraints
$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

4. Maximize $Z = 2x_1 + x_2 + x_3$ subject to the constraints
$$4x_1 + 6x_2 + 3x_3 \leq 8$$
$$3x_1 - 6x_2 - 4x_3 \leq 1$$
$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

5. Minimize $Z = 2x_1 + 3x_2$ subject to the constraints
$$x_1 + x_2 \geq 5$$
$$x_1 + 2x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

6. Maximize $Z = 3x_1 + 2x_2$ subject to the constraints
$$2x_1 + x_2 \leq 1$$
$$3x_1 + 4x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

7. Minimize $Z = x_1 + 2x_2 + x_3$ subject to the constraints
$$x_1 + 1/2x_2 + 1/2x_3 \leq 1$$
$$(3/2)x_1 + 2x_2 + x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

8. Maximize $Z = 5x_1 - 2x_2 + 3x_3$ subject to the constraints
$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solve using the two-phase method.

9. Minimize $Z = 12x_1 + 20x_2$ subject to the constraints
$$6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

10. Minimize $Z = 2x_1 + 4x_2 + x_3$ subject to the constraints
$$x_1 + 2x_2 - x_3 \leq 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

11. Minimize $Z = x_1 + x_2 + 2x_3$ subject to the constraints
$$x_1 + x_2 + x_3 \leq 9$$
$$2x_1 - 3x_2 + 3x_3 = 1$$
$$-3x_1 + 6x_2 - 4x_3 = 3$$
$$x_1, x_2, x_3 \geq 0$$

Solve the following LPP

12. Minimize $Z = 2x_1 + x_2 - x_3 - x_4$ subject to the constraints
$$x_1 - x_2 + 2x_3 - x_4 = 2$$
$$2x_1 + x_2 - 3x_3 + x_4 = 6$$
$$x_1 + x_2 + x_3 + x_4 = 7$$
$$x_1, x_2, x_3, x_4 \geq 0$$

13. Minimize $Z = 3x_1 + 2x_2$ subject to the constraints

$$7x_1 + 2x_2 \geq 14$$
$$-2x_1 - x_2 \leq -6$$
$$x_1 + x_2 \geq 5$$
$$x_1, x_2 \geq 0$$

14. Minimize $Z = 4x_1 + x_2$ subject to the constraints

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$i_1, x_2 \geq 0$$

15. Maximize $Z = 4x_1 + 5x_2 + 2x_3$ subject to the constraints

$$2x_1 + x_2 + x_3 \leq 10$$
$$x_1 + 3x_2 + x_3 = 12$$
$$x_1 + x_2 + x_3 = 6$$
$$x_1, x_2, x_3 \geq 0$$

16. Maximize $Z = 3x_1 + x_2 + 2x_3$ subject to the constraints

$$4x_1 + x_2 + 2x_3 = 9$$
$$8x_1 + x_2 - 4x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

17. Minimize $Z = 2x_1 + 9x_2 + x_3$ subject to the constraints

$$x_1 + 4x_2 + 2x_3 \geq 5$$
$$3x_1 + x_2 + 2x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

18. Maximize $Z = 5x_1 - 2x_2 + 3x_3$ subject to the constraints

$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

19. Maximize $Z = 6x_1 + 4x_2$ subject to the constraints

$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

20. Maximize $Z = 2x_1 + x_2 + 3x_3$ subject to the constraints

$$x_1 + x_2 + 2x_3 \leq 5$$
$$2x_1 + 3x_2 + 4x_3 = 12$$
$$x_1, x_2, x_3 \geq 0$$

21. Maximize $Z = 4x_1 + 2x_2$ subject to the constraints

$$3x_1 + x_2 \geq 27$$
$$-x_1 - x_2 \leq 21$$
$$x_1 + 2x_2 \geq 30$$
$$x_1, x_2 \geq 0$$

22. Maximize $Z = 2x_1 + x_2$ subject to the constraints

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

23. Minimize $Z = 4x_1 + 2x_2$ subject to the constraints

$$3x_1 + x_2 \geq 27$$
$$-x_1 - x_2 \leq 21$$
$$x_1 + 2x_2 \geq 30$$
$$x_1, x_2 \geq 0$$

24. Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix by simplex method.
25. A company has two factories and each produces three products A, B, C from common raw material. Production per hour, costs per hour, and orders received for the products are given below.

	Product			*Operating Cost Per hour (Rs)*
	X	*Y*	*Z*	
Plant I	2	4	3	9
Plant II	4	3	2	10
Order received	50	24	60	

Find the number of production hours required to fulfill the orders at minimum cost.

26. Maximize $Z = 8x_2$ subject to the constraints

$$x_1 - x_2 \geq 0$$
$$2x_1 + 3x_2 \leq -6$$
x_1, x_2 unrestricted

27. Minimize $Z = 3x_1 + 2x_2$ subject to the constraints

$3x_1 + x_2 \geq 12$
$x_1 + x_2 \geq 8$
$x_1 + 2x_2 \geq 11$
$x_1, x_2 \geq 0$

28. Maximize $Z = 2x_1 + 3x_2 - 5x_3$ subject to the constraints
$x_1 + x_2 + x_3 = 7$
$2x_1 - 5x_2 + x_3 \geq 10$
$x_1, x_2, x_3 \geq 0$

29. Minimize $Z = 3x_1 + 2x_2$ subject to the constraints
$x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 3$
$x_1 - x_2 = 1$
$x_1, x_2 \geq 0$

30. Maximize $Z = 2x_1 + x_2$ subject to the constraints
$3x_1 + x_2 = 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 3$
$x_1, x_2 \geq 0$

31. Maximize $Z = 5x_1 + 8x_2$ subject to the constraints
$3x_1 + 2x_2 \geq 3$
$x_1 + 4x_2 \geq 4$
$x_1 + x_2 \leq 5$
$x_1, x_2 \geq 0$

32. Minimize $Z = -2x_1 - x_2$ subject to the constraints
$x_1 + x_2 \geq 2$
$x_1 + x_2 \leq 4$
$x_1, x_2 \geq 0$

33. Minimize $Z = 12x_1 + 20x_2$ subject to the constraints
$6x_1 + 8x_2 \geq 100$
$7x_1 + 12x_2 \geq 120$
$x_1, x_2 \geq 0$

34. Maximize $Z = x_1 + 2x_2 + x_3$ subject to the constraints
$2x_1 + x_2 - x_3 \leq 2$
$-2x_1 + x_2 - 5x_3 \geq -6$
$4x_1 + x_2 + x_3 \leq 6$
$x_1, x_2, x_3 \geq 0$

35. Maximize $Z = 2x_1 + 3x_2 + 5x_3$ subject to the constraints
$3x_1 + 10x_2 + 5x_3 \leq 15$
$33x_1 - 10x_2 + 9x_3 \leq 33$
$x_1 + 2x_2 + x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$

Solve using the two-phase method

36. Maximize $Z = 3x_1 + 2x_2 + 2x_3$ subject to the constraints
$5x_1 + 7x_2 + 4x_3 \leq 7$
$-4x_1 + 7x_2 + 5x_3 \geq -2$
$3x_1 + 4x_2 - 6x_3 \geq 29/7$
$x_1, x_2, x_3 \geq 0$

37. Maximize $Z = 5x_1 + 3x_2$ subject to the constraints
$x_1 + x_2 \leq 6$
$2x_1 + 3x_2 \geq 3$
$x_1 \geq 3$
$x_2 \geq 3$
$x_1, x_2 \geq 0$

38. Minimize $Z = 20x_1 + 10x_2$ subject to the constraints
$x_1 + 2x_2 \leq 40$
$3x_1 + x_2 \geq 30$
$4x_1 + 3x_2 \geq 60$
$x_1, x_2 \geq 0$

39. Maximize $Z = 2x_1 + 3x_2$ subject to the constraints
$x_1 + x_2 \leq 30$
$x_1 - x_2 \geq 0$
$x_1 \leq 20, \quad x_2 \geq 3, \quad x_2 \leq 12$
$x_1, x_2 \geq 0$

40. Maximize $Z = -3x_1 + 6x_2$ subject to the constraints
$x_1 + 2x_2 + 1 \geq 0$
$-4x_1 + x_2 + 23 \geq 0$
$2x_1 + x_2 - 4 \geq 0$
$x_1 - 4x_2 + 13 \geq 0$
$x_1 - x_2 + 1 \geq 0$
$x_1, x_2 \geq 0$

ANSWERS

1. $x_1 = 0$, $x_2 = 15/4$, $x_3 = 5/4$, $Z^* = -125/4$
2. No solution
3. $x_1 = 4$, $x_2 = 2$, $Z^* = 22$
4. $x_1 = 9/7$, $x_2 = 10/21$, $x_3 = 0$, $Z^* = 64/21$
5. $x_1 = 4$, $x_2 = 1$, $Z^* = 11$
6. $x_1 = 0$, $x_2 = 1$, $Z^* = 2$
7. No solution
8. $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$, $Z^* = \frac{85}{3}$
9. $x_1 = 15$, $x_2 = 5/4$, $Z^* = 205$
10. $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $Z^* = 1$
11. $x_1 = 0$, $x_2 = 13/6$, $x_3 = 5/2$, $Z^* = 43/6$
12. $x_1 = 3$, $x_2 = 0$, $x_3 = 1$, $x_4 = 3$, $Z^* = 2$
13. $x_1 = 1$, $x_2 = 4$, $Z^* = 11$
14. $x_1 = 3/5$, $x_2 = 6/5$, $Z^* = 18/5$
15. $x_1 = 3$, $x_2 = 3$, $Z^* = 27$
16. $x_1 = 0$, $x_2 = 0$, $x_3 = 3/2$, $Z^* = 3$
17. $x_1 = 0$, $x_2 = 0$, $x_3 = 5/2$, $Z^* = 5/2$
18. $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$, $Z^* = 85/3$
19. $x_1 = 8$, $x_2 = 0$, $Z^* = 48$

In exercise No. 19, we find $Z_j - C_j = 0$ corresponding to the non-basic variable x_2 in the optimal table. Hence we can bring x_2 into the basis and obtain another optimal solution $x_1 = 12/5$, $x_2 = 42/5$, $Z^* = 48$. Since there are two optimal solutions it means that there are infinite number of solutions.

20. $x_1 = 3$, $x_2 = 2$, $x_3 = 0$, $Z^* = 8$
21. Unbounded solution
22. $x_1 = 3/5$, $x_2 = 3$, $Z^* = 21/5$
23. $x_1 = 24/5$, $x_2 = 63/5$, $Z^* = 222/5$
24. $x_1 = 15$, $x_2 = 5/4$, $Z^* = 205$
25. $x_1 = 35/2$, $x_2 = 15/4$, $Z^* = 195$
26. $x_1 = -6/5$, $x_2 = -6/5$, $Z^* = -48/5$
27. $x_1 = 2$, $x_2 = 6$, $Z^* = 18$
28. $x_1 = 45/7$, $x_2 = 4/7$, $x_3 = 0$, $Z^* = 102/7$
29. $x_1 = 3/2$, $x_2 = 1/2$, $Z^* = 11/2$
30. $x_1 = 3/5$, $x_2 = 6/5$, $Z^* = 12/5$
31. $x_1 = 0$, $x_2 = 5$, $Z^* = 40$
32. $x_1 = 4$, $x_2 = 0$, $Z^* = -8$
33. $x_1 = 15$, $x_2 = 5/4$, $Z^* = 205$
34. $x_1 = 0$, $x_2 = 4$, $x_3 = 2$, $Z^* = 10$
35. No solution
36. $x_1 = 1$, $x_2 = 2/7$, $x_3 = 0$, $Z^* = 25/7$
37. $x_1 = 3$, $x_2 = 3$, $Z^* = 24$
38. $x_1 = 6$, $x_2 = 12$, $Z^* = 240$
39. $x_1 = 18$, $x_2 = 12$, $Z^* = 72$
40. $x_1 = 3$, $x_2 = 4$, $Z^* = 15$

5

Dual Simplex Method

CONCEPT REVIEW

5.1 DUAL SIMPLEX METHOD

The simplex method starts with an initial basic feasible solution and moves towards optimality. If the constants b_i are all non-negative ($b_i \geq 0$) the solution is feasible. Simplex method is terminated when an optimal solution is reached ($Z_j - c_j \geq 0$ for all j).

In certain cases some constants b_i may be negative (infeasibility), but at the same time optimality conditions are satisfied ($Z_j - c_j \geq 0$ for all j) in the starting table. In other words we start with an infeasible but optimal solution. Here regular simplex method cannot be used. Instead we apply dual simplex method and work towards feasibility. Finally we arrive at an optimal and feasible solution [$Z_j - c_j \geq 0$ and $b_{\text{i}} \geq 0 \;\; \forall \;\; i, j$].

5.1.1 Dual Simplex Algorithm

Step 1 Rewrite the LPP in the standard form. Convert $\geq$ type constraints to $\leq$ type by multiplying both sides by -1. Introduce slack variables, form the simplex table and obtain the initial basic solution.

Step 2 If all the constants are non-negative and $Z_j - c_j \geq 0 \;\; \forall \;\; j$ the solution is optimal. Stop.

Step 3 If all the constants are non-negative and some $Z_j - c_j < 0$ then apply simplex method for optimization.

Step 4 If $b_i < 0$ for some i and $Z_j - c_j < 0$ for some j then there exists no solution. Stop.

Step 5 If $b_i < 0$ for some i and $Z_j - c_j \geq 0 \; \forall \; j$ then choose the most negative b_i, say b_r. The corresponding row is the pivot row and the corresponding basic variable leaves the basis.

Step 6 Examine the elements of the pivot row. If all the $a_{rj} \geq 0$ there exists no solution. Stop.

Step 7 If some $a_{rj} < 0$ then find the ratios $|(Z_j - c_j)/a_{rj}|$ where $a_{rj} < 0$ and select the minimum among them, say $|Z_s - c_s / a_{rs}|$, then the s^{th} column is the pivot column and the corresponding non-basic variable enters the basis. a_{rs} is called the pivot element.

Step 8 Perform row operations, making the element a_{rs} as 1 and the other elements of the pivot column, zero. We get the improved solution.

Repeat steps 5 to 8 till optimality is reached [i.e. $b_i > 0 \;\; \forall \, i$ and $Z_j - c_j \geq 0 \;\; \forall \;\; j$].

Example 5.1 Apply the dual simplex method to solve:

Minimize $Z = 2x_1 + x_2$ subject to the constraints

$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution Converting the objective function to maximization type and the constraints to ≤ type we get

Maximize $W = -Z = -2x_1 - x_2$ subject to the constraints

$$-3x_1 - x_2 \leq -3$$
$$-4x_1 - 3x_2 \leq -6$$
$$-x_1 - 2x_2 \leq -3$$
$$x_1, x_2 \geq 0$$

The standard form is

Maximize $W = -2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$ subject to the constraints

$$-3x_1 - x_2 + s_1 = -3$$
$$-4x_1 - 3x_2 + s_2 = -6$$
$$-x_1 - 2x_2 + s_3 = -3$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

The initial simplex table is

Table 5.1

Z		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	–2	–1	0	0	0	
0	s_1	–3	–1	1	0	0	–3
0	s_2	–4	$\boxed{-3}$	0	1	0	–6 ←
0	s_3	–1	–2 ↑	0	0	1	–3
	$Z_j - c_j$	2	1	0	0	0	0

Note: We use the notation X_B to represent the values of the current basic variables.

All $Z_j - c_j \geq 0$ and all the $b_i < 0$ the solution is not optimal. The most negative b_i is –6. Hence the corresponding basic variable s_2 leaves the basis.

$$\text{Min}\left\{\left|\frac{Z_j - c_j}{a_{2j}}\right| : a_{2j} < 0\right\}$$

$= \min\{|2/(-4)|, |1/(-3)|\}$

$= \min\{1/2, 1/3\} = 1/3$

Hence the corresponding variable x_2 enters the basis. –3 is the pivot element. Make the row transformations

$R_1 - (1/3)R_2,\ R_2/(-3),\ R_3 - (2/3)R_2$

Table 5.2

I Iteration

Z		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	–2	–1	0	0	0	
0	s_1	$\boxed{-5/3}$	0	1	–1/3	0	–1 ←
–1	x_2	4/3	1	0	–1/3	0	2
0	s_3	5/3 ↑	0	0	–2/3	1	1
	$Z_j - c_j$	2/3	0	0	1/3	0	–2

s_1 leaves the basis.

Min $\{|(2/3)/(-5/3)|, |(1/3)/(-1/3)|\}$

$= \text{Min}\{2/5, 1\} = 2/5$

Hence x_1 enters the basis. The pivot element is –5/3.

The necessary row transformations are $R_1(-3/5)$, $R_2 + (4/5)R_1$, $R_3 + R_1$

The resulting table is

Table 5.3

Z		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	–2	–1	0	0	0	
–2	x_1	1	0	–3/5	1/5	0	3/5
–1	x_2	0	1	4/5	–3/5	0	6/5
0	s_3	0	0	1	–1	1	0
	$Z_j - c_j$	0	0	2/5	1/5	0	–12/5

$b_i \geq 0\ \forall\ i$, $Z_j - c_j \geq 0\ \forall\ j$. Hence optimality is reached. The solution is $x_1 = 3/5$, $x_2 = 6/5$, $Z^* = 12/5$.

Example 5.2 Solve using the dual simplex method

Maximize $Z = -3x_1 - 2x_2$ subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution Rewrite the problem as

Maximize $Z = -3x_1 - 2x_2$ subject to the constraints

$$-x_1 - x_2 \le -1$$
$$x_1 + x_2 \le 7$$
$$-x_1 - 2x_2 \le -10$$
$$x_2 \le 3$$
$$x_1, x_2 \ge 0$$

The standard form is

Maximize $Z = -3x_1 - 2x_2$ subject to the constraints

$$-x_1 - x_2 + s_1 = -1$$
$$x_1 + x_2 + s_2 = 7$$
$$-x_1 - 2x_2 + s_3 = -10$$
$$x_2 + s_4 = 3$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$$

The starting table is

Table 5.4

Z		x_1	x_2	s_1	s_2	s_3	s_4	X_B
C_B	B	–3	–2	0	0	0	0	
0	s_1	–1	–1	1	0	0	0	–1
0	s_2	1	1	0	1	0	0	7
0	s_3	–1	$\boxed{-2}$	0	0	1	0	–10 ←
0	s_4	0	1	0	0	0	1	3
	$Z_j–c_j$	3	2 ↑	0	0	0	0	0

s_3 leaves the basis

Min {|3/(–1)|, |2/(–2)|} = Min {3, 1} = 1.

The corresponding variable x_2 is the entering variable and the pivot element is –2. The necessary row transformations are $R_1 - (1/2)R_3$, $R_2 + (1/2)R_3$, $R_3/(-2)$, $R_4 + (1/2)R_3$.

We get the revised table

Table 5.5

Z		x_1	x_2	s_1	s_2	s_3	s_4	X_B
C_B	B	–3	–2	0	0	0	0	
0	s_1	–1/2	0	1	0	–1/2	0	4
0	s_2	1/2	0	0	1	1/2	0	2
–2	x_2	1/2	1	0	0	–1/2	0	5
0	s_4	$\boxed{-1/2}$	0	0	0	1/2	1	–2 ←
	$Z_j - c_j$	2 ↑	0	0	0	1	0	–10

s_4 leaves and x_1 enters the basis. The pivot element is –1/2. Performing the transformations $R_1 - R_4$, $R_2 + R_4$, $R_3 + R_4$, $R_4(-2)$, we get the new table

Table 5.6

Z		x_1	x_2	s_1	s_2	s_3	s_4	X_B
C_B	B	–3	–2	0	0	0	0	
0	s_1	0	0	1	0	–1	–1	6
0	s_2	0	0	0	1	1	1	0
–2	x_2	0	1	0	0	0	1	3
–3	x_1	1	0	0	0	–1	–2	4
	$Z_j–c_j$	0	0	0	0	3	4	–18

Optimality reached. The optimal solution is $x_1 = 4$, $x_2 = 3$, $Z^* = -18$.

Example 5.3 Using the dual simplex method solve:

Minimize $Z = x_1 + 2x_2 + 3x_3$ subject to the constraints

$$x_1 - x_2 + x_3 \ge 4$$
$$x_1 + x_2 + 2x_3 \le 8$$
$$x_2 - x_3 \ge 2,\ x_1, x_2, x_3 \ge 0$$

Solution The standard form of the problem is

Maximize $W = -Z = -x_1 - 2x_2 - 3x_3$ subject to the constraints

$$-x_1 + x_2 - x_3 + s_1 = -4$$
$$x_1 + x_2 + 2x_3 + s_2 = 8$$
$$-x_2 + x_3 + s_3 = -2$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

The initial table is

Table 5.7

W		x_1	x_2	x_3	s_1	s_2	s_3	X_B
C_B	B	–1	–2	–3	0	0	0	
0	s_1	$\boxed{-1}$	1	–1	1	0	0	–4 ←
0	s_2	1	1	2	0	1	0	8
0	s_3	0	–1	1	0	0	1	–2
	$\bar{c}_j$	1 ↑	2	3	0	0	0	0

$R_1(-1)$, $R_2 + R_1$

–1	x_1	1	–1	1	–1	0	0	4
0	s_2	0	2	1	1	1	0	4
0	s_3	0	$\boxed{-1}$	1	0	0	1	–2 ←
	$\bar{c}_j$	0	3 ↑	2	1	0	0	–4

$R_1 - R_3$, $R_2 + 2R_3$, $R_3(-1)$

W		x_1	x_2	x_3	s_1	s_2	s_3	X_B
–1	x_1	1	0	0	–1	0	–1	6
0	s_2	0	0	3	1	1	2	0
–2	x_2	0	1	–1	0	0	–1	2
	$\bar{c}_j$	0	0	5	1	0	3	–10

Optimality is reached. The optimal solution is $x_1 = 6, x_2 = 2, x_3 = 0, W^* = -10, Z^* = 10$

Example 5.4 Solve using the dual simplex method

Minimize $Z = x_1 + x_2$ subject to the constraints

$$2x_1 + x_2 \geq 2$$
$$-x_1 - x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Solution Rewriting the problem, we get

Maximize $W = -Z = -x_1 - x_2$ subject to the constraints

$$-2x_1 - x_2 \leq -2$$
$$x_1 + x_2 \leq -1$$
$$x_1, x_2 \geq 0$$

The standard form is

Maximize $W = -x_1 - x_2 + 0s_1 + 0s_2$
subject to the constraints

$$-2x_1 - x_2 + s_1 = -2$$
$$x_1 + x_2 + s_2 = -1$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The starting simplex table is

Table 5.8

W		x_1	x_2	s_1	s_2	X_B
C_B	B	–1	–1	0	0	
0	s_1	$\boxed{-2}$	–1	1	0	–2 ←
0	s_2	1 ↑	1	0	1	–1
	$\bar{c}_j$	1	1	0	0	0

x_1 enters and s_1 leaves $R_1/(-2)$, $R_2 + (1/2)R_1$

–1	x_1	1	1/2	–1/2	0	1
0	s_2	0	1/2	1/2	1	–2 ←
	$\bar{c}_j$	0	1/2	1/2	0	–1

We see that the pivot row corresponds to the basic variable s_2. Hence s_2 leaves the basis. In order to determine the entering variable we need to take the ratio of the $\bar{c}_j$'s with the negative elements of the pivot row. Since there is no negative element in the pivot row we cannot find the ratios. Hence there is no feasible solution to the given problem.

SHORT ANSWER QUESTIONS

1. What is the difference between the regular simplex method and the dual simplex method.
2. State the optimality condition in the dual simplex method.

EXERCISES

Applying the dual simplex method solve:

1. Minimize $Z = 2x_1 + 2x_2 + 4x_3$
   subject to the constraints
   $$2x_1 + 3x_2 + 5x_3 \geq 2$$
   $$3x_1 + x_2 + 7x_3 \leq 3$$
   $$x_1 + 4x_2 + 6x_3 \leq 5$$
   $$x_1, x_2, x_3 \geq 0$$
2. Maximize $Z = -3x_1 - x_2$ subject to the constraints
   $$x_1 + x_2 \geq 1$$
   $$2x_1 + 3x_2 \geq 2$$
   $$x_1, x_2 \geq 0$$
3. Minimize $Z = 20x_1 + 16x_2$ subject to the constraints
   $$x_1 + x_2 \geq 12$$
   $$2x_1 + x_2 \geq 17$$
   $$x_1 \geq 5/2$$
   $$x_2 \geq 6$$
   $$x_1, x_2 \geq 0$$

4. Minimize $Z = 10x_1 + 6x_2 + 2x_3$ subject to the constraints
$$-x_1 + x_2 + x_3 \geq 1$$
$$3x_1 + x_2 - x_3 \geq 2$$
$$x_1, x_2 \geq 0$$
5. Maximize $Z = -4x_1 - 6x_2 - 18x_3$ subject to the constraints
$$x_1 + 3x_3 \geq 3$$
$$x_2 + 2x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$
6. Maximize $Z = -2x_1 - x_3$ subject to the constraints
$$x_1 + x_2 - x_3 \geq 5$$
$$x_1 - 2x_2 + 4x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$
7. Minimize $Z = x_1 + 2x_2$ subject to the constraints
$$2x_1 + x_2 \geq 4$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$
8. Minimize $Z = 3x_1 + 5x_2 + 4x_3$ subject to the constraints
$$-2x_1 - x_2 + 5x_3 \geq 2$$
$$3x_1 + 2x_2 + 4x_3 \geq 16$$
$$x_1, x_2, x_3 \geq 0$$
9. Maximize $Z = -3x_1 - x_2$ subject to the constraints
$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$
10. Minimize $Z = 2x_1 + 2x_2 + 4x_3$ subject to the constraints
$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$
11. Minimize $Z = 2x_1 + x_2$ subject to the constraints
$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$
12. Minimize $Z = 12x_1 + 20x_2$ subject to the constraints
$$6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$
13. Maximize $Z = -2x_1 - 3x_2$ subject to the constraints
$$x_1 + x_2 \geq 5$$
$$x_1 + 2x_2 \geq 6$$
$$x_1, x_2 \geq 0$$
14. Minimize $Z = 4x_1 + 2x_2$ subject to the constraints
$$3x_1 + x_2 \geq 27$$
$$x_1 + x_2 \geq 21$$
$$x_1 + 2x_2 \geq 30$$
$$x_1, x_2 \geq 0$$
15. Maximize $Z = -5x_1 - 4x_2 - 3x_3$ subject to the constraints
$$x_1 + x_2 + x_3 \geq 100$$
$$x_1 \leq 20, x_2 \geq 30, x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$
16. Minimize $Z = 2x_1 + 9x_2 + x_3$ subject to the constraints
$$x_1 + 4x_2 + 2x_3 \geq 5$$
$$3x_1 + x_2 + 2x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

ANSWERS

1. $x_1 = 0, x_2 = 2/3, x_3 = 0\ Z^* = 4/3$
2. $x_1 = 0, x_2 = 1, Z^* = -1$
3. $x_1 = 5, x_2 = 7, Z^* = 212$
4. $x_1 = 1/4, x_2 = 5/4, x_3 = 0, Z^* = 10$
5. $x_1 = 0, x_2 = 3, x_3 = 1, Z^* = -36$
6. $x_1 = 0, x_2 = 14, x_3 = 9, Z^* = -9$
7. $x_1 = 0, x_2 = 2, Z^* = 4$
8. $x_1 = 0, x_2 = 0, x_3 = 4, Z^* = 16$
9. $x_1 = 0, x_2 = 1, Z^* = -1$
10. $x_1 = 0, x_2 = 2/3, x_3 = 0, Z^* = 4/3$
11. $x_1 = 0, x_2 = 2, Z^* = 2$
12. $x_1 = 15, x_2 = 5/4, Z^* = 205$
13. $x_1 = 4, x_2 = 1, Z^* = -11$
14. $x_1 = 3, x_2 = 18, Z^* = 48$
15. $x_1 = 0, x_2 = 60, x_3 = 40, Z^* = -360$
16. $x_1 = 0, x_2 = 0, x_3 = 5/2, Z^* = 5/2$

6

Duality

CONCEPT REVIEW

Given any LPP we can associate another LPP with it. The given problem is called *primal* and the newly constructed problem is called *dual*. If the primal has n variables and m constraints then the dual has m variables and n constraints. The primal and dual are so related that we can get the solution of the primal from the solution of the dual. Also if the number of constraints in the primal is greater than the number of variables ($m > n$) then in the dual the number of constraints n will be less than the number of variables m. It is easier to solve the dual with less number of constraints (manipulation work is reduced). Hence we prefer to solve the dual and from the solution of the dual we can obtain the solution of the given primal.

In an LPP if the objective is maximization, the constraints are $\le$ type and the variables ≥ 0 it is called the *standard type of problem.* If the objective is minimization then the problem is of standard type if the constraints are $\ge$ type and the variables ≥ 0. We give below the rules for constructing the dual problem, given a primal.

6.1 RULES FOR WRITING THE DUAL OF A PRIMAL

	Primal		Dual
(i)	Maximization Z	$\rightarrow$	Minimization W
(ii)	Minimization Z	$\rightarrow$	Maximization W
(iii)	Coefficients of the objective function	$\rightarrow$	Constants
(iv)	Constants	$\rightarrow$	Coefficients of the objective function
(v)	n variables $x_1, x_2, \ldots, x_n$	$\rightarrow$	n constraints
(vi)	m constraints	$\rightarrow$	m variables $y_1, y_2, \ldots, y_m$
(vii)	Matrix of constraint coefficients A	$\rightarrow$	A^T
(viii)	i^{th} variable ≥ 0	$\rightarrow$	i^{th} constraint is of standard type
	i^{th} variable ≤ 0	$\rightarrow$	i^{th} constraint not of standard type
	i^{th} variable unrestricted	$\rightarrow$	i^{th} constraint is an equality
(ix)	j^{th} constraint: standard type	$\rightarrow$	j^{th} variable ≥ 0

j^{th} constraint: not of standard type $\rightarrow$ j^{th} variable ≤ 0

j^{th} constraint: equality $\rightarrow$ j^{th} variable unrestricted.

These rules can be easily understood from the following examples.

Example 6.1 Write the dual of the LPP

Maximize $Z = x_1 + 2x_2 + 3x_3$ subject to the constraints

$$3x_1 + x_2 + x_3 \leq 12$$
$$x_1 + 2x_2 + 4x_3 \leq 20$$
$$2x_1 + 5x_2 - x_3 \leq 18$$
$$x_1, x_2, x_3 \geq 0$$

Solution The constants become objective coefficients maximize $\rightarrow$ minimize. $x_i \rightarrow y_j$.

Hence the objective of the dual is

Minimize $W = 12y_1 + 20y_2 + 18y_3$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 5 & -1 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 4 & -1 \end{pmatrix}$$

Objective coefficients 1, 2, 3 become constants. All the variables in the primal are ≥ 0 and hence all the constraints are of standard type $\geq$ (dual is minimization). Therefore the constraints are

$$3y_1 + y_2 + 2y_3 \geq 1$$
$$y_1 + 2y_2 + 5y_3 \geq 2$$
$$y_1 + 4y_2 - y_3 \geq 3$$

Again all the constraints of the primal are of standard type $\leq$ (maximization problem) and hence all the variables of the dual are ≥ 0. $y_1, y_2, y_3 \geq 0$.

Thus the dual of the given primal is

Minimize $W = 12y_1 + 20y_2 + 18y_3$ subject to the constraints

$$3y_1 + y_2 + 2y_3 \geq 1$$
$$y_1 + 2y_2 + 5y_3 \geq 2$$
$$y_1 + 4y_2 - y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

Example 6.2 Write the dual of

Maximize $Z = 2x_1 + 4x_2 + 5x_3$ subject to the constraints

$$2x_1 + x_2 + x_3 \leq 6$$
$$3x_1 + 2x_2 - x_3 \geq 4$$
$$5x_1 - 4x_2 + 2x_3 \leq 12$$
$$x_1 + 3x_2 + 3x_3 \leq 16$$
$$x_1 \leq 0, x_2, x_3 \geq 0$$

Solution The objective function of the dual is

Minimize $W = 6y_1 + 4y_2 + 12y_3 + 16y_4$

The first variable $x_1 \leq 0$. Hence the first constraint of the dual is not of the standard type

$\therefore$ $2y_1 + 3y_2 + 5y_3 + y_4 \leq 2$

$x_2, x_3 \geq 0 \rightarrow$. The other constraints are of standard type

$\therefore$ $y_1 + 2y_2 - 4y_3 + 3y_4 \geq 4$

$y_1 - y_2 + 2y_3 + 3y_4 \geq 5$

Again the first constraint is $\leq$ type (standard)

Hence, $y_1 \geq 0$

But the second constraint is $\geq$ type (not standard). Therefore the 2^{nd} variable $y_2 \leq 0$. Since the other constraints are of standard type ($\leq$) y_3, $y_4 \geq 0$.

Hence the dual problem is

Minimize $W = 6y_1 + 4y_2 + 12y_3 + 16y_4$ subject to the constraints

$$2y_1 + 3y_2 + 5y_3 + y_4 \leq 2$$
$$y_1 + 2y_2 - 4y_3 + 3y_4 \geq 4$$
$$y_1 - y_2 + 2y_3 + 3y_4 \geq 5$$
$$y_1 \geq 0, y_2 \leq 0, y_3, y_4 \geq 0$$

Note: In order to solve the dual we have to reduce it to the standard form.

Example 6.3 Write the dual of the LPP

Minimize $Z = 2x_1 + 5x_2 + x_3$ subject to the constraints

$$9x_1 - 3x_2 + 5x_3 \geq 7$$
$$6x_1 - 4x_2 - 3x_3 \leq 2$$
$$x_1, x_2 \geq 0$$

$\therefore$ x_3 unrestricted

Solution Minimize $\rightarrow$ Maximize

Constants $\rightarrow$ Objective coefficients

The objective function of the dual is

Maximize $W = 7y_1 + 2y_2$

The first constraint is of $\geq$ type (standard)

$\therefore$ $y_1 \geq 0$

The second constraint is $\leq$ type (not standard)

$\therefore$ $y_2 \leq 0$

$x_1, x_2 \geq 0 \rightarrow$ the first two constraints of the dual are of the standard type ($\leq$ for maximization).

Therefore the first two constraints are

$$9y_1 + 6y_2 \leq 2$$
$$-3y_1 - 4y_2 \leq 5$$

x_3 is unrestricted $\rightarrow$ the third constraint of the dual is equality

i.e. $5y_1 - 3y_2 = 1$

Thus the dual problem is

Maximize $W = 7y_1 + 2y_2$ subject to the constraints

$$9y_1 + 6y_2 \le 2$$
$$-3y_1 - 4y_2 \le 5$$
$$5y_1 - 3y_2 = 1$$
$$y_1 \ge 0,\ y_2 \le 0$$

Example 6.4 Write the dual of

Maximize $Z = 5x_1 + 6x_2$ subject to the constraints

$$x_1 + 2x_2 = 5$$
$$-x_1 + 5x_2 \ge 3$$

x_1 unrestricted, $x_2 \ge 0$.

Solution The objective function of the dual, is

Minimize $W = 5y_1 + 3y_2$

The first constraint is equality. Hence the corresponding variable of the dual is unrestricted, i.e. y_1 is unrestricted.

The second constraint is of $\ge$ type (not of standard type) and hence $y_2 \le 0$.

x_1 is unrestricted and hence the first constraint of the dual is an equality. $y_1 - y_2 = 5$.

$x_2 \ge 0$ $\rightarrow$ the second constraint of the dual is of standard type $\ge$ (minimization)

$$2y_1 + 5y_2 \ge 6$$

Thus the dual problem is

Minimize $W = 5y_1 + 3y_2$ subject to the constraints

$$y_1 - y_2 = 5$$
$$2y_1 + 5y_2 \ge 6$$

$\therefore$ y_1 is unrestricted. $y_2 \le 0$.

Example 6.5 Write the dual of the following LPP

Minimize $Z = 4x_1 + 5x_2 - 3x_3$ subject to the constraints

$$x_1 + x_2 + x_3 = 22$$
$$3x_1 + 5x_2 - 2x_3 \le 65$$
$$x_1 + 7x_2 + 4x_3 \ge 120$$
$$x_1, x_2 \ge 0$$

$\therefore$ x_3 unrestricted.

Solution The objective function of the dual is

Maximize $W = 22y_1 + 65y_2 + 120y_3$

$x_1, x_2 \ge 0$ $\rightarrow$, the first two constraints of the dual, are of standard type $\le$ (maximization)

i.e. $y_1 + 3y_2 + y_3 \le 4$

$$y_1 + 5y_2 + 7y_3 \le 5$$

x_3 unrestricted $\rightarrow$ third constraint is an equality

$$y_1 - 2y_2 + 4y_3 = -3 \text{ or } -y_1 + 2y_2 - 4y_3 = 3$$

The first constraint of the primal is an equality

$\therefore$ y_1 is unrestricted.

The second constraint is of $\le$ type (not standard)

$\therefore$ $y_2 \le 0$

The third constraint is $\ge$ (standard)

$\therefore$ $y_3 \ge 0$

Hence the dual is

Maximize $W = 22y_1 + 65y_2 + 120y_3$ subject to the constraints

$$y_1 + 3y_2 + y_3 \le 4$$
$$y_1 + 5y_2 + 7y_3 \le 5$$
$$y_1 - 2y_2 + 4y_3 = -3\ (-y_1 + 2y_2 - 4y_3 = 3)$$

y_1 is unrestricted, $y_2 \le 0$, $y_3 \ge 0$.

Example 6.6 Using the previous example verify that the dual of the dual is the primal.

Solution Given primal is

Minimize $Z = 4x_1 + 5x_2 - 3x_3$ subject to the constraints

$$x_1 + x_2 + x_3 = 22$$
$$3x_1 + 5x_2 - 2x_3 \le 65$$
$$x_1 + 7x_2 + 4x_3 \ge 120$$

$x_1, x_2 \ge 0$, x_3 unrestricted

The dual problem is

Maximize $W = 22y_1 + 65y_2 + 120y_3$ subject to the constraints

$$y_1 + 3y_2 + y_3 \le 4$$
$$y_1 + 5y_2 + 7y_3 \le 5$$
$$y_1 - 2y_2 + 4y_3 = -3$$

$\therefore$ y_1 is unrestricted, $y_2 \le 0$, $y_3 \ge 0$

Now we write the dual of the dual, considering it as the primal.

For the dual the objective function is

Minimize $Z = 4z_1 + 5z_2 - 3z_3$

y_1 unrestricted $\rightarrow$ first constraint is an equality

$$z_1 + z_2 + z_3 = 22$$

$y_2 \le 0$ $\rightarrow$ second constraint is not of the standard type. Hence it should be $\le$

$$3z_1 + 5z_2 - 2z_3 \le 65$$

$y_3 \ge 0$ $\rightarrow$ the third constraint is of standard type $\ge$

$$z_1 + 7z_2 + 4z_3 \ge 120$$

The first two constraints of the primal are $\le$ (standard)

$\therefore$ $z_1 \ge 0,\ z_2 \ge 0$

The third constraint is an equality. Hence in the dual z_3 is unrestricted.

Thus the dual is

Maximize $Z = 4z_1 + 5z_2 - 3z_3$ subject to the constraints

$$z_1 + z_2 + z_3 = 22$$
$$3z_1 + 5z_2 - 2x_3 \leq 65$$
$$z_1 + 7z_2 + 4z_3 \geq 120$$
$$z_1, z_2 \geq 0 \quad z_3 \text{ unrestricted.}$$

Changing the symbols of the variables

$$z_1 \to x_1, z_2 \to x_2, z_3 \to x_3, Z \to Z$$

We find that this is the same as the given primal.

Hence the result.

6.2 DUALITY THEOREMS

Theorem 1: The dual of the dual is the primal.

Proof: Let the given primal problem be written in the canonical form (standard type), as

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\ x, x_2, x_3, \dots, x_n \geq 0 & \qquad (1) \end{aligned}$$

The dual of this problem is

Minimize $W = b_1y_1 + b_2y_2 + \dots + b_my_m$ subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m &\geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m &\geq c_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m &\geq c_n \\ y_1, y_2, \dots y_m \geq 0 & \qquad (2) \end{aligned}$$

Now consider (2) as the primal and write its dual as

Maximize $U = c_1t_1 + c_2t_2 + \dots + c_nt_n$ subject to the constraints

$$\begin{aligned} a_{11}t_1 + a_{12}t_2 + \dots + a_{1n}t_n &\leq b_1 \\ a_{21}t_1 + a_{22}t_2 + \dots + a_{2n}t_n &\leq b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}t_1 + a_{m2}t_2 + \dots + a_{mn}t_n &\leq b_m \\ t_1, t_2, \dots t_n \geq 0 & \qquad (3) \end{aligned}$$

Writing Z for U and $x_1, x_2 \dots x_n$ instead of $t_1, t_2 \dots t_n$

We get

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\ x_1, x_2, \dots x_n &\geq 0 \end{aligned}$$

which is the given primal.

Hence the dual of the dual is the primal.

Theorem 2: The value of the objective function Z for any feasible solution of the primal is less than or equal to the value of the objective function W for any feasible solution of the dual.

Proof: Let the primal problem be

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\ x_1, x_2, \dots, x_n \geq 0 & \qquad (1) \end{aligned}$$

The dual problem is

Minimize $W = b_1y_1 + b_2y_2 + \dots + b_my_m$ subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m &\geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m &\geq c_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{1n}y_1 + a_{2n}y_1 + \dots + a_{mn}y_m &\geq c_n \\ y_1, y_2, \dots, y_m \geq 0 & \qquad (2) \end{aligned}$$

Multiplying the first constraint of (1) by y_1, the second constraint by y_2, etc. and adding all of them we get

$$\begin{aligned} &(a_{11}x_1y_1 + a_{12}x_2y_1 + \dots + a_{1n}x_ny_1) + \\ &(a_{21}x_1y_2 + a_{22}x_2y_2 + \dots + a_{2n}x_ny_2) + \\ &\vdots \qquad \vdots \qquad \vdots \\ &(a_{m1}x_1y_m + a_{m2}x_2y_m + \dots + a_{mn}x_ny_m) \\ &\leq b_1y_1 + b_2y_2 + \dots + b_my_m \qquad (3) \end{aligned}$$

Similarly, multiplying the first constraint of (2) by x_1, the second constraint by x_2, etc. and adding them all we get

$$\begin{aligned} &(a_{11}x_1y_1 + a_{21}x_1y_2 + \dots + a_{m1}x_1y_m)^+ \\ &(a_{12}x_2y_1 + a_{22}x_2y_2 + \dots + a_{m2}x_2\, y_m)^+ \\ &(a_{1n}x_ny_1 + a_{2n}x_ny_2 + \dots + a_{mn}x_ny_m) \\ &\geq c_1x_1 + c_2x_2 + \dots + c_nx_n \qquad (4) \end{aligned}$$

The sums on the LHS of (3) and (4) are equal

$\therefore \quad c_1x_1 + c_2x_2 + \dots c_nx_n \leq a_{11}x_1y_1 + \dots + a_{mn}x_ny_m$

$\leq b_1y_1 + b_2y_2 + \dots + b_my_m$

$\therefore \quad Z \leq W$

i.e. any feasible solution to the primal is less than or equal to any feasible solution to the dual.

Note: If $W \to \infty$ (unbounded solution to the dual) then the primal cannot have a feasible solution. Also if $Z \to \infty$ (unbounded solution to the primal) then the dual cannot have a feasible solution.

Primal unbounded $\to$ Dual infeasible

Dual unbounded $\to$ Primal infeasible.

Theorem 3 (Fundamental theorem): If both primal and dual have optimal solutions then their optimum values must be equal.

i.e. $Z^* = W^*$.

Proof: Let Z^* and W^* be the optimum values of Z and W respectively. The primal is a maximization problem and the dual is a minimization problem. Hence for any feasible solutions Z and W of the primal and dual we have $Z \le Z^*$ and $W^* \le W$.

But we have proved that $Z \le W$.

Since Z^* and W^* are also feasible solutions we find that $Z^* \le W^*$

Thus $Z \le Z^* \le W^* \le W$.

Suppose $Z^* < W^*$. Then there is a value of Z, say Z_1 such that $Z_1 \le W^*$ and $Z^* < Z_1$. But Z_1 is a feasible solution of the primal and max $Z = Z^*$.

Hence $Z_1 \le Z^*$. But this is a contradiction.

Hence $Z^* = W^*$

6.3 PRINCIPLE OF DUALITY

Each variable x_i of the primal corresponds to a constraint c_i of the dual. This constraint contains a slack or artificial variable (s_i or R_i) of the dual which becomes a starting basic variable. In the optimal table of the dual note down the relative cost factor of s_i (say α) or R_i (say $M - \beta$). This corresponds to the value of the primal variable x_i in the optimal solution of the primal. If x_i corresponds to s_i, then $x_i = \alpha$ If x_i corresponds to R_i then the value of x_i in the optimal solution is $x_i = \beta$. This fact can be stated as follows. "In the solution of the dual find the relative cost factors of the starting basic variables in the optimal table. In case of a slack variable it represents the optimal value of the corresponding primal variable. In case of an artificial variable delete M and change the sign. It gives the optimal value of the corresponding primal variable. This is called *principle of duality.* It helps us to give the solution of the primal from the solution of the dual without solving the primal directly. Also $Z^* = W^*$.

Example 6.7 Solve the following problem by solving its dual

Maximize $Z = x_1 + 2x_2$ subject to the constraints

$$-x_1 + 3x_2 \le 10$$
$$x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 2$$
$$x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Solution The dual of the given problem is

Minimize $W = 10y_1 + 6y_2 + 2y_3 + 6y_4 + 4y_5$ subject to the constraints

$$-y_1 + y_2 + y_3 + y_4 + 2y_5 \ge 1$$
$$3y_1 + y_2 - y_3 + 3y_4 + y_5 \ge 2$$
$$y_1, y_2, y_3, y_4, y_5 \ge 0$$

The standard form of the dual is

Maximize $W_1 = -W = -10y_1 - 6y_2 - 2y_3 - 6y_4 - 4y_5$ subject to the constraints

$$-y_1 + y_2 + y_3 + y_4 + 2y_5 - s_1 + R_1 = 1$$
$$3y_1 + y_2 - y_3 + 3y_4 + y_5 - s_2 + R_2 = 2$$
$$y_1, y_2, y_3, y_4, y_5, s_1, s_2, R_1, R_2 \ge 0$$

The simplex table (Big M method) is

Table 6.1

W_1		y_1	y_2	y_3	y_4	y_5	s_1	s_2	R_1	R_2	X_B
C_B	B	-10	-6	-2	-6	-4	0	0	$-M$	$-M$	
$-M$	R_1	-1	1	1	1	2	-1	0	1	0	1
$-M$	R_2	3	1	-1	$\boxed{3}$	1	0	-1	0	1	2
	$\bar{c}_j$	$10-2M$	$6-2M$	2	$6-4M$	$4-3M$	M	M	0	0	$-3M$

$R_1 - (1/3)R_2$ $\quad R_2/3$

		y_1	y_2	y_3	y_4	y_5	s_1	s_2	R_1	R_2	X_B
C_B	B	-10	-6	-2	-6	-4	0	0	$-M$	$-M$	
$-M$	R_1	-2	$2/3$	$4/3$	0	$\boxed{5/3}$	-1	$1/3$	1	$-1/3$ ←	$1/3$
-6	y_4	1	$1/3$	$-1/3$	1	$1/3$ ↑	0	$-1/3$	0	$1/3$	$2/3$
	$\bar{c}_j$	$4 + 2M$	$4 - (2M/3)$	$4 - (4M/3)$	0	$2 - (5M/3)$	M	$2 - (M/3)$	0	$4M/3 - 2$	$-4 - M/3$

$R_1(3/5)$, $R_2 - (1/5)R_1$

-4	y_5	$-6/5$	$2/5$	$4/5$	0	1	$-3/5$	$1/5$	$3/5$	$-1/5$	$1/5$
-6	y_4	$7/5$	$1/5$	$-3/5$	1	0	$1/5$	$-2/5$	$-1/5$	$2/5$	$3/5$
	$\bar{c}_j$	$32/5$	$16/5$	$12/5$	0	0	$6/5$	$8/5$	$M - 6/5$	$M - 8/5$	$-22/5$

The optimality is reached. Relative cost factors of the starting basic variables R_1 and R_2 in the optimal table are $M - 6/5$ and $M - 8/5$. Deleting M and changing the sign we get 6/5, 8/5. Variables corresponding to the constraints are x_1 and x_2. Therefore, $x_1 = 6/5$ and $x_2 = 8/5$ are the values of x_1 and x_2 giving optimal solution of the primal. Thus $x_1 = 6/5$, $x_2 = 8/5$, $Z^* = W^* = -W_1^* = 22/5$ is the solution of the given LPP.

Example 6.8 Using the duality theory solve:

Maximize $Z = 3x_1 + 4x_2$ subject to the constraints

$$x_1 - x_2 \le 1$$
$$x_1 + x_2 \ge 4$$
$$x_1 - 3x_2 \le 3$$
$$x_1, x_2 \ge 0$$

Solution The given problem (primal) is

Maximize $Z = 3x_1 + 4x_2$ subject to the constraints

$$x_1 - x_2 \le 1$$
$$x_1 + x_2 \ge 4$$
$$x_1 - 3x_2 \le 3$$
$$x_1, x_2 \ge 0$$

The dual is

Minimize $W = y_1 + 4y_2 + 3y_3$ subject to the constraints

$$y_1 + y_2 + y_3 \ge 3$$
$$-y_1 + y_2 - 3y_3 \ge 4$$
$$y_1 \ge 0,\ y_2 \le 0,\ y_3 \ge 0$$

Take $y_2' = -y_2$

The standard form of the dual is

Maximize $W_1 = -y_1 + 4y_2' - 3y_3 + 0s_1 + 0s_2 - MR_1 - MR_2$ subject to the constraints

$$y_1 - y_2' + y_3 - s_1 + R_1 = 3$$
$$-y_1 - y_2' - 3y_3 - s_2 + R_2 = 4$$
$$y_1, y_2', y_3 \ge 0$$

The simplex table is

Table 6.2

W_1		y_1	y_2'	y_3	s_1	s_2	R_1	R_2	X_B
C_B	B	-1	4	-3	0	0	$-M$	$-M$	
$-M$	R_1	1	-1	1	-1	0	1	0	3
$-M$	R_2	-1	-1	-3	0	-1	0	1	4
	$\bar{c}_j$	1	$2M - 4$	$2M + 3$	M	M	0	0	$-7M$

The current solution is optimal. But R_1 and R_2 are present in the optimal basis at + ve level. Hence $W_1 \to -\infty$, which shows that the primal problem has no feasible solution.

Example 6.9 Using duality theory solve

Maximize $Z = 4x_1 + 3x_2$ subject to the constraints

$$2x_1 + x_2 \le 10$$
$$x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

Solution The dual problem is

Minimize $W = 10y_1 + 8y_2$ subject to the constraints

$$2y_1 + y_2 \ge 4$$
$$y_1 + y_2 \ge 3$$
$$y_1, y_2 \ge 0$$

The standard form is

Maximize $W_1 = -10y_1 - 8y_2 + 0s_1 + 0s_2 - MR_1 - MR_2$ subject to the constraints

$$2y_1 + y_2 - s_1 + R_1 = 4$$
$$y_1 + y_2 - s_2 + R_2 = 3$$
$$y_1, y_2, s_1, s_2, R_1, R_2 \geq 0$$

The simplex table is

Table 6.3

W_1		y_1	y_2	s_1	s_2	R_1	R_2	X_B
C_B	B	-10	-8	0	0	$-M$	$-M$	0
$-M$	R_1	$\boxed{2}$	1	-1	0	1	0 ←	4
$-M$	R_2	1 ↑	1	0	-1	0	1	3
	$\bar{c}_j$	$10 - 3M$	$8 - 2M$	M	M	0	0	$-7M$

$R_1/2,\ R_2 - (1/2)\ R_1$

-10	y_1	1	1/2	$-1/2$	0	1/2	0	2
$-M$	R_2	0	1/2	$\boxed{1/2}$ ↑	-1	$-1/2$	1	1
	$\bar{c}_j$	0	$3 - M/2$	$5 - M/2$	M	$(3M/2) - 5$	0	$-M - 20$

$R_1 - R_2 \quad (R_2)2$

-10	y_1	1	0	-1	1	1	-1 ←	1
-8	y_2	0	1	1	-2	-1	2	2
	$\bar{c}_j$	0	0	2	6	$M - 2$	$M - 6$	-26

The optimality is reached. R_1 and R_2 are the starting basic variables. The relative cost factors of R_1 and R_2 in the optimal table are $M - 2$, $M - 6$. Deleting M and the negative sign we get 2 and 6. R_1 appears in the first constraint and R_2 in the second constraint which correspond to x_1 and x_2 of the primal.

Hence the solution is $x_1 = 2, x_2 = 6, Z^* = 4(2) + 3(6) = 26$.

Note: The solution of the dual is

$$y_1 = 1, y_2 = 2, W^* = -W_1^* = 26, Z^* = W^*$$

Example 6.10 Solve using the principle of duality

Minimize $Z = 5x_1 + 2x_2$ subject to the constraints

$$x_1 + 2x_2 \geq 5$$
$$2x_1 - x_2 \geq 12$$
$$x_1 + 3x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

Solution The dual of the given problem is

Maximize $W = 5y_1 + 12y_2 + 4y_3$ subject to the constraints

$$y_1 + 2y_2 + y_3 \leq 5$$
$$2y_1 - y_2 + 3y_3 \leq 2$$
$$y_1, y_2, y_3 \geq 0$$

The standard form of the dual is

Maximize $W = 5y_1 + 12y_2 + 4y_3 + 0s_1 + 0s_2$ subject to the constraints

$$y_1 + 2y_2 + y_3 + s_1 = 5$$
$$2y_1 - y_2 + 3y_3 + s_2 = 2$$
$$y_1, y_2, y_3, s_1, s_2 \geq 0$$

The simplex table is

Table 6.4

W_1		y_1	y_2	y_3	s_1	s_2	X_B
C_B	B	5	12	4	0	0	0
0	s_1	1	$\boxed{2}$	1	1	0 ←	5
0	s_2	2	-1 ↑	3	0	1	2
	$\bar{c}_j$	-5	-12	-4	0	0	0

$R_1/2 \quad R_2 + (1/2)R_1$

12	y_2	1/2	1	1/2	1/2	0	5/2
0	s_2	5/2	0	7/2	1/2	1	9/2
	$\bar{c}_j$	1	0	2	6	0	30

The optimality is reached.

The relative cost factors of s_1 and s_2 in the optimal table are 6 and 0 respectively. The primal variables corresponding to s_1 and s_2 are x_1 and x_2 respectively. Hence the optimal solution of the primal is

$$x_1 = 6, x_2 = 0, Z^* = 5(6) + 2(0) = 30$$

Note: The solution of the dual is

$$y_1 = 0, y_2 = 5/2\ y_3 = 0, W^* = 30.$$

Example 6.11 Obtain the solution by solving its dual

Maximize $Z = 3x_1 + 2x_2$ subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution The dual problem is

Minimize $W = y_1 + 7y_2 + 10y_3 + 3y_4$ subject to the constraints

$$y_1 + y_2 + y_3 \geq 3$$
$$y_1 + y_2 + 2y_3 + y_4 \geq 2$$
$$y_1 \leq 0,\ y_2,\ y_3,\ y_4 \geq 0$$

The standard form is

Maximize $W_1 = y_1' - 7y_2 - 10y_3 - 3y_4 + 0s_1 + 0s_2 - MR_1 - MR_2$ ($y_1' = -y_1$) subject to the constraints

$$-y_1' + y_2 + y_3 - s_1 + R_1 = 3$$
$$-y_1' + y_2 + 2y_3 + y_4 - s_2 + R_2 = 2$$
$$y_1',\ y_2,\ y_3,\ y_4,\ s_1,\ s_2,\ R_1,\ R_2 \geq 0$$

The simplex table is

Table 6.5

W_1		y_1'	y_2	y_3	y_4	s_1	s_2	R_1	R_2	X_B
C_B	B	1	–7	–10	–3	0	0	–M	–M	
–M	R_1	–1	1	1	0	–1	0	1	0	3
–M	R_2	–1	1	$\boxed{2}$	1	0	–1	0	1	2
	$\bar{c}_j$	$2M - 1$	$7 - 2M$	$10 - 3M$	$3 - M$	M	M	0	0	$-5M$

$R_1 - 1/2\ R_2$, $R_2/2$

–M	R_1	–1/2	1/2	0	–1/2	–1	$\boxed{1/2}$	1	–1/2	2
–10	y_3	–1/2	1/2	1	1/2	0	–1/2	0	1/2	1
	$\bar{c}_j$	$4 + (M/2)$	$2 - (M/2)$	0	$(M/2) - 2$	M	$5 - M/2$	0	$(3M/2) - 5$	$-10 - 2M$

$R_1(2)$, $R_2 + R_1$

0	S_2	–1	1	0	–1	–2	1	2	– 1	4
–10	y_3	–1	$\boxed{1}$	1	0	–1	0	1	0	3
	$\bar{c}_j$	4	– 3	0	3	10	0	$M - 10$	M	–30

$R_1 - R_2$

0	s_2	0	0	–1	–1	–1	1	1	– 1	1
–7	y_2	–1	1	1	0	–1	0	1	0	3
	$\bar{c}_j$	6	0	3	3	7	0	$M - 7$	M	–21

The optimality is reached. The starting basic variables R_1 and R_2 have relative cost factors $M - 7$ and M. The corresponding variables in the primal are x_1 and x_2. Deleting M and changing the sign we get the solution,

$$x_1 = 7,\ x_2 = 0,\ Z^* = 21$$

Note: For the dual we get $y_1' = 0$, $y_2 = 3$, $y_3 = 0$, $y_4 = 0$, $W^* = -W_1^* = 21$

6.4 ECONOMIC INTERPRETATION OF THE DUAL

The interpretation of the dual variables from the cost or economic point of view helps us in making decisions on business processes. The objective function represents the profit to be maximized while the constraints give the limits of the resources. The dual objective is to minimize the cost of the resources by considering many alternatives. The worth of the alternative uses is called *accounting* or *shadow prices* of the resources. The shadow prices are determined along with minimization objective. The management of the business will be interested in finding the shadow prices so as to minimize the expenditure associated with existing capacities of the various resources.

6.5 COMPLEMENTARY SLACKNESS THEOREM

A pair of primal and dual feasible solutions x_i $(i = 1, 2, 3\ldots.n)$ and y_j $(j = 1, 2, 3\ldots m)$ are optimal to their respective problems if and only if the following conditions are satisfied:

$$y_i\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right) = y_i s_i = 0 \qquad i = 1, 2, 3, ..., m$$

$$x_j\left(\sum_{i=1}^{m} a_{ij}y_i - c_j\right) = x_j v_j = 0 \qquad j = 1, 2, 3, ..., n$$

where s_i and v_j are slack and surplus variables associated with the primal and dual constraints.

Proof: In the optimal table if a primal variable x_j has $Z_j - c_j > 0$ then x_j is non-basic and hence $x_j = 0$.

Also in the optimal dual table, if a dual variable y_i has positive value then the i^{th} primal constraint $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ must become an equation since the corresponding slack variable is zero.

If we assume that v_j is the surplus variable for the j^{th} dual constraint and s_i is the slack variable for the i^{th} primal constraint then

$$v_j = Z_j - c_j = \sum_{i=1}^{m} a_{ij}y_i - c_j \qquad (1)$$

and $$s_i = b_i - \sum_{i=1}^{n} a_{ij}x_j \qquad (2)$$

In other words, if

$v_j > 0$ then $x_j = 0$ and if $y_i > 0$ then $s_i = 0$.

Therefore for the optimal solutions of the primal and dual

$v_j x_j = 0$ and $y_i s_i = 0$ for all i and j.

This result can be restated as

$$x_j\left[\sum_{i=1}^{m} a_{ij}y_i - c_j\right] = 0 \text{ and } y_i\left[b_i - \sum_{j=1}^{n} a_{ij}x_j\right] = 0$$

Hence the result.

SHORT ANSWER QUESTIONS

1. What is duality? Explain.
2. State the fundamental theorem of duality.
3. State the complementary slackness theorem.
4. Explain how you will write the dual of a given primal.
5. What is the principle of duality?
6. Fill in the blanks:
   (i) If the primal has n variables and m constraints then the dual has __________, __________. (*Ans:* m variables, and n constraints)
   (ii) If the k^{th} constraint of the primal is an equality then the dual variable y_k is __________. (*Ans:* unrestricted)
   (iii) If the primal is of maximization type and the variable $x_k \le 0$ then the corresponding k^{th} constraint of the dual is ______________. (*Ans:* $\le$ type)
   (iv) If the primal is of maximization type and the k^{th} constraint is of $\le$ type then the dual variable y_k is ________. (*Ans:* ≥ 0 type)
   (v) If the primal has unbounded solution then the dual has __________. (*Ans:* infeasible solution)
   (vi) If the dual has unbounded solution then the primal has __________. (*Ans:* no feasible solution)

(vii) If a primal variable has positive optimum value then the corresponding dual constraint is __________ at the optimum. (*Ans:* an equality)

EXERCISES

1. Write the dual of the LPP
   Maximize $Z = 2x_1 + 5x_2 + 6x_3$
   subject to the constraints
   $$x_1 + 6x_2 - x_3 \le 3$$
   $$2x_1 - 3x_2 + 7x_3 \le 6$$
   $$x_1, x_2, x_3 \ge 0$$
2. Write the dual of
   Maximize $Z = 3x_1 + 17x_2 + 9x_3$ subject to the constraints
   $$x_1 - x_2 + x_3 \ge 3$$
   $$-3x_1 + 2x_2 \le 1$$
   $$x_1, x_2, x_3 \ge 0$$
3. Write the dual of
   Minimize $Z = 10x_1 - 6x_2 - 8x_3$ subject to the constraints
   $$x_1 - 3x_2 + x_3 = 5$$
   $$-2x_1 + x_2 + 3x_3 = 8$$
   $$x_1, x_2, x_3 \ge 0$$
4. Write the dual of
   Minimize $Z = 5x_1 + 3x_2 + 2x_3$ subject to the constraints
   $$2x_1 + x_2 + x_3 = 6$$
   $$x_1 - 3x_2 + 2x_3 \ge 8$$
   $$3x_1 + x_2 + 5x_3 \le 12$$
   $x_1 \ge 0$, $x_2 \le 0$, x_3 unrestricted
5. Write the dual of
   Maximize $Z = 5x_1 + x_2$ subject to the constraints
   $$3x_1 + x_2 \ge 6$$
   $$x_1 - 2x_2 \ge 2$$
   $$2x_1 + 5x_2 \ge 10$$
   x_1 unrestricted, $x_2 \ge 0$
6. Obtain the solution of the following problems by solving their duals.

   (i) Maximize $Z = 2x_1 + x_2$ subject to the constraints
   $$x_1 + 2x_2 \le 10$$
   $$x_1 + x_2 \le 6$$
   $$x_1 - x_2 \le 2$$
   $$x_1 - 2x_2 \le 1$$
   $$x_1, x_2 \ge 0$$

   (ii) Minimize $Z = 2x_1 + 2x_2$ subject to the constraints
   $$2x_1 + 4x_2 \ge 1$$
   $$x_1 + 2x_2 \ge 1$$
   $$2x_1 + x_2 \ge 1$$
   $$x_1, x_2 \ge 0$$

   (iii) Minimize $Z = x_1 - x_2 + x_3$ subject to the constraints
   $$x_1 - x_3 \ge 4$$
   $$x_1 - x_2 + 2x_3 \ge 3$$
   $$x_1, x_2, x_3 \ge 0$$

   (iv) Minimize $Z = -x_1 + 7x_2 + 10x_3 + 3x_4$ subject to the constraints
   $$-x_1 + x_2 + x_3 \ge 3$$
   $$-x_1 + x_2 + 2x_3 + x_4 \ge 2$$
   $$x_1, x_2, x_3, x_4 \ge 0$$

   (v) Minimize $Z = 4x_1 + 2x_2 + 3x_3$ subject to the constraints
   $$2x_1 + 4x_3 \ge 5$$
   $$2x_1 + 3x_2 + x_3 \ge 4$$
   $$x_1, x_2, x_3 \ge 0$$

   (vi) Maximize $Z = 5x_1 + 12x_2 + 4x_3$ subject to the constraints
   $$x_1 + 2x_2 + x_3 \le 5$$
   $$2x_1 - x_2 + 3x_3 = 2$$
   $$x_1, x_2, x_3 \ge 0$$

   (vii) Maximize $Z = 2x_1 + 3x_2$ subject to the constraints
   $$-x_1 + 2x_2 \le 4$$
   $$x_1 + x_2 \le 6$$
   $$x_1 + 3x_2 \le 9$$
   x_1, x_2 unrestricted

   (viii) Maximize $z = 2x_1 + x_2$ subject to the constraints
   $$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

(ix) Minimize $Z = 3x_1 + 2x_2$ subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

(x) Minimize $Z = x_1 + 2x_2 + 3x_3$ subject to the constraints

$$x_1 - x_2 + x_3 \geq 4$$
$$x_1 + x_2 + 2x_3 \leq 8$$
$$x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

(xi) Minimize $Z = 40x_1 + 200x_2$ subject to the constraints

$$x_1 + 10x_2 \geq 40$$
$$3x_1 + 10x_2 \geq 60$$
$$4x_1 + 5x_2 \geq 40$$
$$x_1, x_2 \geq 0$$

(xii) Maximize $Z = 7x_1 + 5x_2$ subject to the constraints

$$3x_1 + x_2 \leq 48$$
$$2x_1 + x_2 \leq 40$$
$$x_1, x_2 \geq 0$$

(xiii) Minimize $Z = 10x_1 + 8x_2$ subject to the constraints

$$4x_1 + 2x_2 \geq 5$$
$$2x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

(xiv) Minimize $Z = 11x_1 + 10x_2 + 8x_3$ subject to the constraints

$$x_1 \geq 40$$
$$x_1 + 2x_2 \geq 96$$
$$x_1 + x_2 + 2x_3 \geq 100$$
$$x_1, x_2, x_3 \geq 0$$

(xv) Maximize $Z = x_1 + 6x_2$ subject to the constraints

$$x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

(xvi) Maximize $Z = 6x_1 - 2x_2 + 3x_3$ subject to the constraints

$$2x_1 - x_2 + 2x_3 \leq 2$$
$$x_1 + 4x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

(xvii) Maximize $Z = 4x_1 + 2x_2 + 6x_3$ subject to the constraints

$$4x_1 - x_2 \leq 9$$
$$-x_1 + x_2 + 2x_3 \leq 8$$
$$-3x_1 + x_2 + 4x_3 \leq 12$$
$$1 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 5,\ 0 \leq x_3 \leq 2$$

(xviii) Minimize $Z = 2x_1 + 6x_2$ subject to the constraints

$$x_1 + 2x_2 + x_3 \geq 0$$
$$x_2 - x_3 \geq 2$$
$$x_1 + 6x_2 + 3x_3 \geq -5$$
$$x_1, x_2 \geq 0 \quad x_3 \text{ unrestricted}$$

ANSWERS

1. Minimize $W = 3y_1 + 6y_2$ subject to the constraints

$$y_1 + 2y_2 \geq 2$$
$$6y_1 - 3y_2 \geq 5$$
$$-y_1 + 7y_2 \geq 6$$
$$y_1, y_2 \geq 0$$

2. Minimize $W = 3y_1 + y_2$ subject to the constraints

$$y_1 - 3y_2 \geq 3$$
$$-y_1 \geq 17$$
$$y_1 + 2y_3 \geq 9$$
$$y_1 \leq 0,\ y_2 \geq 0$$

3. Maximize $W = 5y_1 + 8y_2$ subject to the constraints

$$y_1 - 2y_2 \leq 10$$
$$3y_1 - y_2 \geq 6$$
$$-y_1 - 3y_2 \geq 8$$
$$y_1, y_2 \text{ unrestricted}$$

4. Maximize $W = 6y_1 + 8y_2 + 12y_3$ subject to the constraints

$2y_1 + y_2 + 3y_3 \leq 5$
$y_1 - 3y_2 + y_3 \geq 3$
$y_1 + 2y_2 + 5y_3 = 2$
y_1 unrestricted $y_2 \geq 0$, $y_3 \leq 0$

5. Minimize $W = 6y_1 + 2y_2 + 10y_3$ subject to the constraints
$3y_1 + y_2 + 2y_3 = 5$
$y_1 - 2y_2 + 5y_3 \geq 1$
$y_1, y_2, y_3 \leq 0$

6. (i) $x_1 = 4, x_2 = 2, Z^* = 10$
(ii) $x_1 = 1/3, x_2 = 1/3, Z^* = 4/3$
(iii) No solution
(iv) $x_1 = 0, x_2 = 3, x_3 = 0, x_4 = 0, Z^* = 21$
(v) $x_1 = 0, x_2 = 11/12, \ x_3 = 5/4 \ \ Z^* = 67/12$
(vi) $x_1 = 9/5, x_2 = 8/5, x_3 = 0, Z^* = 141/5$
(vii) $x_1 = 9/2, x_2 = 3/2, Z^* = 27/2$
(viii) $x_1 = 3/5, x_2 = 6/5, Z^* = 12/5$
(ix) $x_1 = 4, x_2 = 3, Z^* = 18$
(x) $x_1 = 6, x_2 = 2, x_3 = 0, Z^* = 10$
(xi) $x_1 = 10, x_2 = 3, Z^* = 1000$
(xii) $x_1 = 0, x_2 = 40, Z^* = 200$
(xiii) $x_1 = 1, x_2 = 1/2, Z^* = 14$
(xiv) $x_1 = 40, x_2 = 28, x_3 = 16, Z^* = 848$
(xv) $x_1 = 3/2, x_2 = 1/2, Z^* = 9/2$
(xvi) $x_1 = 4, x_2 = 6, x_3 = 0, Z^* = 12$
(xvii) $x_1 = 3, x_2 = 5, x_3 = 2, Z^* = 34$
(xviii) $x_1 = 0, x_2 = 2/3, x_3 = -4/3, Z^* = 4$

7

Revised Simplex Method

CONCEPT REVIEW

If we solve an LPP using computers, it requires large space in the memory of the computer to store the entire table. This is not possible in the case of large-scale problems. Revised simplex method is a modification of the regular simplex method and it computes and stores only the relevant information required for the current iteration and hence it is more economical on the computer. In the revised simplex method we need only the following during each iteration:

(i) the net evaluations $Z_j - c_j$ to determine the entering variable,

(ii) the pivot column, and

(iii) the current basic variables and their values in order to determine the leaving variable.

In other words, we need to compute only X_B, C_B, B^{-1} and Z. We have to compute B^{-1} at each iteration from the B^{-1} of the previous iteration.

7.1 STEP-BY-STEP PROCEDURE

Step 1 Express the given LPP in the standard form, taking the objective function also as one of the constraints, as follows:

$$
\begin{aligned}
&Z - c_1x_1 - c_2x_2 - \dots - c_nx_n - 0x_{n+1} \\
&\qquad - 0x_{n+2} - \dots - 0x_{n+m} = 0 \\
&a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \dots + a_{2}x_n + x_{n+2} = b_2 \\
&\vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m \\
&x_1, x_2, \dots, x_n\ x_{n+1}, x_{n+2} \dots x_{n+m} = 0
\end{aligned}
$$

Take Z as x_0 and c_j as $-a_{0j}$

$j = 1, 2, 3, \dots, (n + m)$.

The constraints can be expressed in matrix form as

$$
\begin{pmatrix}
1 & a_{01} & a_{02} \dots a_{0n} & a_{0,n+1} & \dots a_{0,n+m} \\
0 & a_{11} & a_{12} \dots a_{1n} & 1 & \dots & 0 \\
0 & a_{21} & a_{22} \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \\
0 & a_{m1} & a_{m2} \dots a_{mn} & 0 & \dots & 1
\end{pmatrix}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n+m} \end{pmatrix}
= \begin{pmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix} 1 & a_0 \\ 0 & A \end{pmatrix}
\begin{pmatrix} x_0 \\ x \end{pmatrix}
= \begin{pmatrix} 0 \\ b \end{pmatrix}
$$

Step 2 Obtain an initial basic feasible solution. Initial basis matrix $B = I_m$. Corresponding to each column a_j of the matrix A, a new $(m + 1)$ component vector is defined as

$$[a_{0j}, a_{1j}, a_{2j} \ldots a_{mj}] = [-c_j, a_{1j}, a_{2j} \ldots a_{mj}]$$

$$\therefore \qquad a_j^{(1)} = [-c_j\ a_j]$$

Similarly, $\qquad b^{(1)} = [0, b_1, b_2 \ldots b_m]^T = [0\ b]^T$

The column corresponding to Z (or x_0) is the $(m + 1)$ component unit vector denoted by $e^{(1)}$. The basis matrix B_1 of order $(m + 1)$ can be written as

$$B_1 = [\, e_1^{(1)}\, B_1^{(1)}\, B_2^{(1)} \ldots B_m^{(1)}]$$

Taking $e^{(1)}$ as $B_0^{(1)}$ we can write B_1 as

$$B_1 = \begin{pmatrix} 1 & -C_B \\ 0 & B \end{pmatrix}$$

$$= \begin{array}{ccccc} B_0^{(1)} & B_1^{(1)} & B_2^{(1)} & \ldots & B_m^{(1)} \end{array}$$
$$\begin{pmatrix} 1 & -C_{B1} & -C_{B2} \cdots - C_{Bm} \\ 0 & B_{11} & B_{12} \cdots - B_{1m} \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & B_{m1} & B_{m2} \quad B_{mn} \end{pmatrix}$$

Since B is invertible, we get

$$B_1^{-1} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

The initial simplex table is

Basic Variable B	$B_0^{(1)}(=Z)$	$B_1^{(1)}(=s_1)$	$B_2^{(1)}(=s_2)$	… …	$B_m^{(1)}(=s_m)$	*Solution values* $X_B^{(1)}$	$Y_K^{(1)}$
Z	1	0	0	…	0	0	$Z_k - c_k$
s_1	0	1	0	…	0	b_1	y_{1k}
s_2	0	0	1	…	0	b_2	y_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	…	$\vdots$	$\vdots$	$\vdots$
s_m	0	0	0	…	1	b_m	y_{mk}

Step 3 Select the entering variable (Pivot column)

For each non-basic variable calculate

$$Z_j - c_j = C_B B_1^{-1}\, a_j^{(1)} - c_j$$

If all the $Z_j - c_j = 0$ current solution is optimal.

If some $Z_j - c_j < 0$ then select

$$Z_k - c_k = \text{Max}\,\{|\, Z_j - c_j \,| : Z_j - c_j < 0\,\}$$

Step 4 Select the leaving variable (Pivot row)

Calculate $y_k^{(1)} = B_1^{-1} a_k^{(1)}$

where $a_k^{(1)} = [-c_k\, a_k]$

Find the ratio

$$x_{Br}/y_{rk} = \text{Min}\,\{x_{Bi}/y_{ik} : y_{ik} > 0\,\}$$

Thus the vector $B_r^{(1)}$ leaves the basis.

Step 5 Update the current solution

Update the initial table by introducing the new basic variable x_k $(= a_k^{(1)})$ into the basis and removing the variable x_r $(= B_r^{(1)})$ from the basis.

Repeat step 3 to step 5 till an optimal solution is reached or there is an indication of unbounded solution ($y_{ik} \le 0$ for all i)

Example 7.1 Solve the following LPP using the revised simplex method.

Maximize $\quad Z = 2x_1 + x_2$

subject to the constraints

$$3x_1 + 4x_2 \le 6$$
$$6x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0$$

Solution Standard form of the LPP is

$$Z - 2x_1 - x_2 = 0$$
$$3x_1 + 4x_2 + s_1 = 6$$
$$6x_1 + x_2 + s_2 = 3$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The matrix form is

$$\begin{array}{ccccc} e^{(1)} & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array}$$

$$\left(\begin{array}{c|cccc} 1 & -2 & -1 & 0 & 0 \\ \hline 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{array}\right) \begin{pmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -C \\ 0 & A \end{pmatrix} \begin{pmatrix} Z \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{array}{ccc} B_0^{(1)} & B_1^{(1)} & B_2^{(1)} \end{array}$$

Basic matrix $B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{I}$

$$B_1^{-1} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_B = [0 \quad 0]$$

Initial table is

Basic variables B	$B_0^{(1)}$ Z	$B_1^{(1)}$ s_1	$B_2^{(1)}$ s_2	$X_B^{(1)}$	$Y_k^{(1)}$
Z	1	0	0	0	$Z_k - c_k$
s_1	0	1	0	6	
s_2	0	0	1	3	

Non-basic variables

$a_1^{(1)}$ x_1	$a_2^{(1)}$ x_2
-2	-1
3	4
6	1

I Iteration

$Z_j - c_j =$ (First row of B_1^{-1}) (non-basic columns)

$$= (1 \quad 0 \quad 0)\begin{pmatrix} -2 & -1 \\ 3 & 4 \\ 6 & 1 \end{pmatrix} = (-2, \quad -1)$$

$\therefore$ x_1 enters the basis

$\therefore$ $k = 1$

$$x_{ik}/y_{ik} = \text{Min } \{x_{Bi}/y_{ik}, y_{ik} > 0\}$$
$$= \text{Min } \{x_{B1}/y_{11}, x_{B2}/y_{21}\}$$

Now $y_k^{(1)} = B_1^{-1} a_k^{(1)}$

$$= a_k^{(1)} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} \text{ for } k = 1$$

$$x_B^{(1)} = B^{-1} b$$

$$= b = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

$$x_{Br}/y_{rk} = \text{Min } \{6/3 \ 3/6\}$$
$$= 3/6 = x_{B2}/y_{21}$$

$\therefore$ $B_2^{(1)}$ leaves the basis

We get
$$\begin{array}{cccc} B_1^{(1)} & B_2^{(1)} & X_B^{(1)} & y_1^{(1)}(Z_k - c_k) \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 6 & 3 \\ 0 & 1 & 3 & 6 \end{pmatrix}$$

Make row operations

$$R_1' = R_1 + (2/6)\, R_3,$$
$$R_2' = R_2 - (3/6)\, R_3,$$
$$R_3' = R_3/6$$

The new table is

Basic variables B	$B_0^{(1)}$ Z	$B_1^{(1)}$ s_1	$B_2^{(1)}$ x_2	$X_B^{(1)}(b)$	$y_2^{(1)}\,(Z_k - c_k)$	$X_B^{(1)}/y_2^{(1)}$
Z	1	0	1/3	1	– 2/3	– 2
s_1	0	1	– 1/2	9/2	7/2	9/7
x_1	0	0	1/6	1/2	1/6	6/2

Non-basic variables

$a_4^{(1)}$	$a_2^{(1)}$
s_2	x_2
0	–1
0	4
1	1

$$a_2^{(1)} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \qquad a_4^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

New basic matrix

$$B_1^{-1} = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{pmatrix}$$

II Iteration

$Z_j - c_j$ = (first row of B_1^1) (non-basic columns)

$$= \begin{pmatrix} 1 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 4 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/3, & 1/3 \end{pmatrix} \quad (k = 2)$$

$$Z_k - c_k = -2/3$$

$\therefore$ $a_2^{(1)}$ enters the basis, i.e. x_2 enters the basis.

Now $y_2^{(1)} = B_1^{-1} a_2^{(1)}$

$$= \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 7/2 \\ 1/6 \end{pmatrix}$$

$$X_{Br}/y_{rk} = \text{Min } \{(9/2)/(7/2),\ (1/2)/(1/6)\}$$

$$= 9/7 \ (r = 1)$$

$\therefore$ $B_1^{(1)}$ leaves the basis

We get

$$\begin{array}{cccc} B_1^{(1)} & B_2^{(1)} & X_B^{(1)} & Y_2^{(1)} \end{array}$$

$$\begin{pmatrix} 0 & 1/3 & 1 & -2/3 \\ 1 & -1/2 & 9/2 & 7/2 \\ 0 & 1/6 & 1/2 & 1/6 \end{pmatrix}$$

Make the row operations

$$R_1^1 = R_1 + (4/21)\, R_2,$$

$$R_2^1 = (2/7) R_2,$$

$$R_3^1 = R_3 - (1/21) R_2$$

The new table is

Basic variables B	$B_0^{(1)}$ Z	$B_1^{(1)}$ x_2	$B_2^{(1)}$ x_1	$X_B^{(1)}(b)$
Z	1	4/21	5/21	13/7
x_2	0	2/7	–1/7	9/7
x_1	0	–2/21	4/21	2/7

Non-basic variables

$a_4^{(1)}$	$a_3^{(1)}$
s_2	s_1
0	0
0	1
1	0

$$a_3^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad a_4^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The new basic matrix

$$B_1^{-1} = \begin{pmatrix} 1 & 4/21 & 5/21 \\ 0 & 2/7 & -1/7 \\ 0 & -2/21 & 4/21 \end{pmatrix}$$

III Iteration

$$Z_j - c_j = (\text{First row of } B_1^{-1})\ [\text{column } a_j^{(1)} \text{ (non-basic)}]$$

$$= \begin{pmatrix} 1 & 4/21 & 5/21 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4/21 & 5/21 \end{pmatrix}$$

$Z_j - c_j \geq 0$ for all j

Hence the current solution is optimal.

The optimal solution is

$$x_1 = 2/7,\ x_2 = 9/7,\ Z^* = 13/7$$

Example 7.2 Using the revised simplex method

Maximize $Z = 3x_1 + 5x_2$

subject to the constraints

$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

Solution Standard form of the LPP is

$$Z - 3x_1 - 5x_2 = 0$$
$$x_1 + s_1 = 4$$
$$x_2 + s_2 = 6$$
$$3x_1 + 2x_2 + s_3 = 18$$
$$x_1, x_2, s_1, s_2, s_3 = 0$$

The matrix form is

$$\begin{array}{cccccc} e^{(1)} & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$

$$\begin{pmatrix} 1 & -3 & -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 6 \\ 18 \end{pmatrix}$$

where $e^{(1)} = B_0^{(1)}$ $a_3^{(1)} = B_1^{(1)}$ $a_4^{(1)} = B_2^{(1)}$ $a_5^{(1)} = B_3^{(1)}$

The basis matrix

$$B = \begin{array}{c} \begin{array}{cccc} B_0^{(1)} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$B_1^{-1} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix} \qquad B = \begin{array}{c} \begin{array}{ccc} B_1^{(1)} & B_2^{(1)} & B_3^{(1)} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$C_B = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

The initial simplex table is

Basic variables B	$B_0^{(1)}$ Z	$B_1^{(1)}$ s_1	$B_2^{(1)}$ s_2	$B_3^{(1)}$ s_3	$X_B^{(1)}$
Z	1	0	0	0	0
s_1	0	1	0	0	4
s_2	0	0	1	0	6
s_3	0	0	0	1	18

I Iteration

$$Z_j - c_j = (\text{First row of } B_1^{-1})\ (\text{non-basic column})$$

$$= (1\ 0\ 0\ 0) \begin{pmatrix} -3 & -5 \\ 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -3, & -5 \end{pmatrix}$$

$a_2^{(1)}$ (x_2) enters the basis $k = 2$

$$y_k^{(1)} = B_1^{-1} a_k^{(1)}$$

$$= a_k^{(1)} = a_2^{(1)} = \begin{pmatrix} -5 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$X_B^{(1)} = B_1^{-1} b = b = \begin{pmatrix} 0 \\ 4 \\ 6 \\ 18 \end{pmatrix}$$

$$X_{Br}/y_{rk} = \text{Min}\{X_{Bi}/y_{i2},\ \ y_{i2} > 0\}$$
$$= \text{Min}\{X_{B1}/y_{12},\ X_{B2}/y_{22},\ X_{B3}/y_{32}\}$$
$$= \text{Min}\{4/0,\ 6/1,\ 18/2\}$$
$$= 6 = x_{B2}/y_{22}$$

Thus $B_2^{(1)}$ (s_2) leaves the basis.

We get

$$\begin{array}{ccccc} B_1^{(1)} & B_2^{(1)} & B_3^{(1)} & X_B^{(1)} & Y_2^{(1)} \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & -5 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 6 & 1 \\ 0 & 0 & 1 & 18 & 2 \end{pmatrix}$$

Make row transformations

$R_1^1 = R_1 + 5R_3,\quad R_2^1 = R_2,\ R_3^1 = R_3,\ R_4^1 = R_4 - 2R_3$

Basic variables B	$B_0^{(1)}$ Z	$B_1^{(1)}$ s_1	$B_2^{(1)}$ x_2	$B_3^{(1)}$ s_3	$X_B^{(1)}$	$Y_1^{(1)}$	$X_B^{(1)}/y_1^{(1)}$
Z	1	0	5	0	30	– 3	–
s_1	0	1	0	0	4	0	–
x_2	0	0	1	0	6	1	6/1 = 6
s_3	0	0	0	1	6	3	6/3 = 2

Non-basic variables

$a_1^{(1)}$ (x_1)	$a_4^{(1)}$ (s_2)
–3	0
1	0
0	1
3	0

New basic matrix is

$$B_1^{-1} = \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

II Iteration

$Z_j - c_j =$ (First row of B_1^{-1}) (non-basic columns)

$$= \begin{pmatrix} 1 & 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 1 & 0 \\ 0 & 1 \\ 3 & 0 \end{pmatrix}$$

$$= (-3 \quad 5) \quad (K = 1)$$

$\therefore$ The vector $a_1^{(1)}$ (x_1) enters the basis

$$X_{Br}/y_{rk} = \text{Min}\{x_{Bi}/y_{i1},\ y_{i1} > 0\}$$
$$= \text{Min}\{x_{B1}/y_{11},\ x_{B2}/y_{21},\ x_{B3}/y_{31}\}$$
$$= \text{Min}\{4/1, 6/0,\ 6/3\}$$
$$= 6/3 = 2\ (r = 3)$$

$\therefore$ $B_3^{(1)}$ leaves the basis (s_3 leaves the basis)

We get

$$\begin{array}{ccccc} B_1^{(1)} & B_2^{(1)} & B_3^{(1)} & X_B^{(1)} & Y_1^{(1)} \end{array}$$
$$\begin{pmatrix} 0 & 5 & 0 & 30 & -3 \\ 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & -2 & 1 & 6 & 3 \end{pmatrix}$$

Make row transformations

$R_1^1 = R_1 + R_4$,

$R_2^1 = R_2 - (1/3)R_4$,

$R_3^1 = R_3$,

$R_4^1 = (1/3)R_4$

The improved solution is

Basic variables B	$B_0^{(1)}$ Z	$B_1^{(1)}$ s_1	$B_2^{(1)}$ x_2	$B_3^{(1)}$ x_1	$X_B^{(1)}$ (b)
Z	1	0	3	1	36
s_1	0	1	2/3	– 1/3	2
x_2	0	0	1	0	6
x_1	0	0	– 2/3	1/3	2

Non-basic variables

$a_4^{(1)}$	$a_5^{(1)}$
s_2	s_3
0	0
0	0
1	0
0	1

The new basic matrix is

$$B_1^{-1} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2/3 & 1/3 \end{pmatrix}$$

III Iteration

$Z_j - c_j =$ (First row of B_1^{-1}) (non-basic columns)

$$= \begin{pmatrix} 1 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3, & 1 \end{pmatrix}$$

Hence the current solution is optimal. The optimal solution is

$$x_1 = 2, x_2 = 6, Z^* = 36$$

Example 7.3 Using the revised simplex method

Maximize $Z = 4x_1 - x_2 - 2x_3$

subject to the constraints

$$2x_1 - 3x_2 + 2x_3 \le 12$$
$$-5x_1 + 2x_2 + 3x_3 \ge 4$$
$$-3x_1 + 2x_3 = 1$$
$$x_1, x_2, x_3 \ge 0$$

Solution By introducing slack, surplus and artificial variables, we obtain the standard form as

Maximize

$$Z = 4x_1 - x_2 - 2x_3 + 0s_1 + 0s_2 - MR_1 - MR_2$$

subject to the constraints

$$2x_1 - 3x_2 + 2x_3 + s_1 + 0s_2 + 0R_1 + 0R_2 = 12$$
$$-5x_1 + 2x_2 + 3x_3 + 0s_1 - s_2 + R_1 + 0R_2 = 4$$
$$-3x_1 + 0x_2 + 2x_3 + 0s_1 + 0s_2 + 0R_1 + R_2 = 1$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \ge 0$$

We have

$$C = \begin{pmatrix} 4 & -1 & -2 & 0 & 0 & -M & -M \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -3 & 2 & 1 & 0 & 0 & 0 \\ -5 & 2 & 3 & 0 & -1 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 12 \\ 4 \\ 1 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 1 & -C \\ 0 & A \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccccccc} e^{(1)} & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} & a_6^{(1)} & a_7^{(1)} \\ Z & x_1 & x_2 & x_3 & s_1 & s_2 & R_1 & R_2 \end{array} \\ B_1^{-1} = & \begin{pmatrix} 1 & -4 & 1 & 2 & 0 & 0 & M & M \\ 0 & 2 & -3 & 2 & 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 3 & 0 & -1 & 1 & 0 \\ 0 & -3 & 0 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$b_1 = \begin{pmatrix} 0 \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

$$B_1^{-1} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccc} e^{(1)} & a_4^{(1)} & a_6^{(1)} & a_7^{(1)} \end{array} \\ = & \begin{pmatrix} 1 & 0 & -M & -M \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$Z_j - c_j = \begin{pmatrix} 1 & 0 & -M & -M \end{pmatrix} \begin{array}{c} \begin{array}{cccc} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_5^{(1)} \end{array} \\ \begin{pmatrix} -4 & 1 & 2 & 0 \\ 2 & -3 & 2 & 0 \\ -5 & 2 & 3 & -1 \\ -3 & 0 & 2 & 0 \end{pmatrix} \end{array}$$

$$= (-4 + 8M, 1 - 2M, 2 - 5M, M)$$

$2 - 5M$ is most negative:

Hence x_3 enters the basis

$$y_3^{(1)} = B_1^{-1}a_3^{(1)}$$

$$= \begin{pmatrix} 1 & 0 & -M & -M \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5M+2 \\ 2 \\ 3 \\ 2 \end{pmatrix}$$

$$X_B^{(1)} = B_1^{-1}b_1$$

$$= \begin{pmatrix} 1 & 0 & -M & -M \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5M \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

$$X_{Br}/y_{rk} = X_{Br}/Y_{r3}$$

$$= \text{Min}\left(12/2 \quad 4/3 \quad 1/2\right) = 1/2$$

$\therefore \quad r = 3 \qquad \therefore R_2$ leaves the basis

We get

$$\begin{array}{c|ccc|cc} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} & X_B^{(1)} & Y_3^{(1)} \\ \hline & 0 & -M & -M & -5M & -5M+2 \\ \hline s_1 & 1 & 0 & 0 & 12 & 2 \\ R_1 & 0 & 1 & 0 & 4 & 3 \\ R_2 & 0 & 0 & 1 & 1 & \boxed{2} \end{array}$$

Introduce x_3 and drop R_2. $R_0 \ + (5M/2 - 1)\ R_3$.

$R_1 - R_3$, $R_2 - (3/2)R_3$, $R_3/2$

We get

$$B_1^{-1} = \begin{pmatrix} 1 & 0 & -M & (3M-2)/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$Z_j - c_j = \begin{pmatrix} 1 & 0 & -M & (3M-2)/2 \end{pmatrix} \begin{pmatrix} -4 & 1 & 0 & M \\ 2 & -3 & 0 & 0 \\ -5 & 2 & -1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

$$= \left[M/2 - 1, \quad -2M+1, \quad M, \quad (5M-2)/2\right]$$

x_2 enters the basis

$$y_2^{(1)} = B_1^{-1}\, a_2^{(1)}$$

$$= \begin{pmatrix} 1 & 0 & -M & (3M-2)/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2M+1 \\ -3 \\ 2 \\ 0 \end{pmatrix}$$

$$X_B^{(1)} = B_1^{-1}b_1$$

$$= \begin{pmatrix} 1 & 0 & -M & (3M-2)/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-5M-2)/2 \\ 11 \\ 5/2 \\ 1/2 \end{pmatrix}$$

$X_{Br}/Y_{r2} = (\,-\,, 5/4, \quad \infty)$ Minimum is 5/4

$\therefore \quad R_1$ leaves the basis.

We get

$$\begin{array}{c|ccc|cc} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} & X_B^{(1)} & Y_2^{(1)} \\ \hline & 0 & -M & (3M-2)/2 & (-5M-2)/2 & -2M+1 \\ \hline s_1 & 1 & 0 & -1 & 11 & -3 \\ R_1 & 0 & 1 & -3/2 & 5/2 & \boxed{2} \\ x_3 & 0 & 0 & 1/2 & 1/2 & 0 \end{array}$$

Introduce x_2 and drop R_1.
$R_0 + (M - 1/2)\,R_2,\ R_1 + (3/2)R_2,\ R_2/2$

$$B_1^{-1} = \begin{pmatrix} 1 & 0 & -1/2 & -1/4 \\ 0 & 1 & 3/2 & -13/4 \\ 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$Z_j - c_j = (1 \quad C_B B^{-1}) \quad (a_1^{(1)} \quad a_5^{(1)} \quad a_6^{(1)} \quad a_7^{(1)})$$

$$= (1 \quad 0 \quad -1/2 \quad -1/4)\begin{pmatrix} -4 & 0 & M & M \\ 2 & 0 & 0 & 0 \\ -5 & -1 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

$$= (-3/4,\ 1/2,\ M - 1/2,\ M - 1/4)$$

x_1 enters the basis

$$y_1^{(1)} = B_1^{-1}\, a_1^{(1)}$$

$$= \begin{pmatrix} 1 & 0 & -1/2 & -1/4 \\ 0 & 1 & 3/2 & -13/4 \\ 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \\ -5 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -3/4 \\ 17/4 \\ -1/4 \\ -3/2 \end{pmatrix}$$

$$X_B^{(1)} = B_1^{-1} b_1$$

$$= \begin{pmatrix} 1 & 0 & -1/2 & -1/4 \\ 0 & 1 & 3/2 & -13/4 \\ 0 & 0 & 1/2 & -3/4 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -9/4 \\ 59/4 \\ 5/4 \\ 1/2 \end{pmatrix}$$

X_{Br}/Y_{r1} = (–, (59/4)/(17/4), –)
s_1 leaves the basis

We get

$$\begin{array}{c|ccccc} & B_1^{(1)} & B_2^{(1)} & B_3^{(1)} & X_B^{(1)} & Y_1^{(1)} \\ \hline & 0 & -1/2 & -1/4 & -9/4 & -3/4 \\ \hline s_1 & 1 & 3/2 & -13/4 & 59/4 & \boxed{17/4} \\ x_2 & 0 & 1/2 & -3/4 & 5/4 & -1/4 \\ x_3 & 0 & 0 & 1/2 & 1/2 & -3/2 \end{array}$$

Introduce x_1 and drop s_1. $R_0 + (3/17)R_1$
$R_1(4/17),\quad R_2 + (1/17)R_1,\quad R_3 + (6/17)R_1$
We get

$$B_1^{-1} = \begin{pmatrix} 1 & 3/17 & -4/17 & -14/17 \\ 0 & 4/17 & 6/17 & -13/17 \\ 0 & 1/17 & 10/17 & -16/17 \\ 0 & 6/17 & 9/17 & -11/17 \end{pmatrix}$$

$$Z_j - c_j = (1 \quad 3/17 \quad -4/17 \quad -14/17)\begin{pmatrix} 0 & 0 & M & M \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (3/17 \quad 4/17 \quad M - 4/17 \quad M - 14/17)$$

All are non-negative.

∴ Current solution is optimal.

$$B_1^{-1} b_1 = \begin{pmatrix} 1 & 3/17 & -4/17 & -14/17 \\ 0 & 4/17 & 6/17 & -13/17 \\ 0 & 1/17 & 10/17 & -16/17 \\ 0 & 6/17 & 9/17 & -11/17 \end{pmatrix}\begin{pmatrix} 0 \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -6/17 \\ 59/17 \\ 36/17 \\ 97/17 \end{pmatrix}$$

∴ Solution is
$x_1 = 59/17,\ x_2 = 36/17,\ x_3 = 97/17,\ Z^* = -6/17.$

7.2 ADVANTAGES OF REVISED SIMPLEX METHOD OVER THE ORDINARY SIMPLEX METHOD

1. In the case of LP problems having larger number of variables than the number of

constraints the amount of computations is very much reduced.

2. When working with computers, in the revised simplex method, only the basic variables, inverse of the basis matrix and the constants are to be stored in the memory.
3. Since calculations are carried out on a column only if it is ready to enter the basis computing errors are minimum.
4. The data can be stored accurately and compactly since the revised simplex method works only with the original data.

EXERCISES

1. Discuss the advantages of revised simplex method over the ordinary simplex method.

Solve the following LPP using the revised simplex method.

2. Maximize $Z = x_1 + x_2$
subject to the constraints
$$3x_1 + 2x_2 \le 6$$
$$x_1 + 4x_2 \le 4$$
$$x_1, x_2 \ge 0$$
3. Maximize $Z = 6x_1 - 2x_2 + 3x_3$
subject to the constraints
$$2x_1 - x_2 + 2x_3 \le 2$$
$$x_1 + 4x_3 \le 4$$
$$x_1, x_2, x_3 \ge 0$$
4. Maximize $Z = x_1 + x_2$
subject to the constraints
$$2x_1 + 5x_2 \le 6$$
$$x_1 + x_2 \ge 2$$
$$x_1, x_2 \ge 0$$
5. Minimize $Z = x_1 + 2x_2$
subject to the constraints
$$2x_1 + 5x_2 \ge 6$$
$$x_1 + x_2 \ge 2$$
$$x_1, x_2 \ge 0$$
6. Maximize $Z = x_1 + 2x_2$
subject to the constraints
$$x_1 + x_2 \le 3$$
$$x_1 + 2x_2 \le 5$$
$$3x_1 + x_2 \le 6$$
$$x_1, x_2 \ge 0$$
7. Maximize $Z = 2x_1 + x_2$
subject to the constraints
$$3x_1 + 4x_2 \le 6$$
$$6x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0$$
8. Maximize $Z = 3x_1 + 5x_2$
subject to the constraints
$$x_1 \le 4$$
$$x_2 \le 6$$
$$3x_1 + 2x_2 \le 18$$
$$x_1, x_2 \ge 0$$
9. Minimize $Z = 2x_1 + x_2$
subject to the constraints
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$
10. Maximize $Z = x_1 + x_2 + 3x_3$
subject to the constraints
$$3x_1 + 2x_2 + x_3 \le 3$$
$$2x_1 + x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$
11. Maximize $Z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints
$$6x_1 + 5x_2 + 3x_3 \le 26$$
$$4x_1 + 2x_2 + 5x_3 \le 7$$
$$x_1, x_2, x_3 \ge 0$$
12. Maximize $Z = 2x_1 + 3x_2$
subject to the constraints
$$x_2 - x_1 \ge 0$$
$$x_1 \le 4$$
$$x_1, x_2 \ge 0$$
13. Maximize $Z = 5x_1 + 3x_2$
subject to the constraints

$$4x_1 + 5x_2 \leq 10$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

14. Maximize $Z = 3x_1 + 2x_2 + 5x_3$
subject to the constraints
$$x_1 + 2x_2 + x_3 \geq 430$$
$$3x_1 + 2x_2 \leq 460$$
$$x_1 + 4x_3 \leq 420$$
$$x_1, x_2, x_3 \geq 0$$

15. Minimize $Z = 12x_1 + 20x_2$
subject to the constraints
$$3x_1 + 4x_2 \geq 50$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

ANSWERS

2. $x_1 = 8/5, x_2 = 3/5, Z^* = 11/5$
3. $x_1 = 4, x_2 = 6, x_3 = 0 \quad Z^* = 12$
4. $x_1 = 3, x_2 = 0, Z^* = 3$
5. $x_1 = 4/3, x_2 = 2/3, Z^* = 8/3$
6. $x_1 = 1, x_2 = 2, Z^* = 5$
7. $x_1 = 2/7, x_2 = 9/7, Z^* = 13/7$
8. $x_1 = 2, x_2 = 6, Z^* = 36$
9. $x_1 = 3/5, x_2 = 6/5, Z^* = 12/5$
10. $x_1 = 0, x_2 = 0, x_3 = 1, Z^* = 3$
11. $x_1 = 0, x_2 = 7/2, x_3 = 0, Z^* = 161/2$
12. Unbounded solution
13. $x_1 = 28/17, x_2 = 15/17, Z^* = 185/17$
14. $x_1 = 0, x_2 = 100, x_3 = 230, Z^* = 1350$
15. $x_1 = 15, x_2 = 5/4, Z^* = 205$

8

Bounded Variable Technique

CONCEPT REVIEW

In general the variables of a linear programming problem have to satisfy the non-negativity constraint $x_j \geq 0$. In certain LP problems, some or all of the variables may have lower and upper limits to their values. Such a problem can be written as

Maximize $Z = CX$

subject to $AX = b$

$$l_j \leq x_j \leq u_j \qquad (j = 1, 2, 3, \ldots, n)$$

where l_j and u_j are the lower and upper limits of the variable x_j. We may convert these constraints to equalities by introducing slack and surplus variables as follows:

$$x_j + x_j' = u_j$$
$$x_j - x_j'' = l_j$$

This gives rise to $3n$ variables and $m + 2n$ constraints. This problem can be reduced in size as follows.

$x_j - x_j'' = l_j$ can be written as

$x_j = l_j + x_j''$ for lower bound constraint

This eliminates x_j from all the other constraints and the problem can be solved as usual. If we take $x_j = u_j - x_j'$ for upper bound constraint then a difficulty arises. There is no guarantee that $u_j - x_j'$ is non-negative and hence the problem may become infeasible. Therefore we use a special technique for upper bound constraints.

Instead of taking the constraint

$$x_j + x_j' = u_j$$

We modify the feasibility condition of the simplex method. In the regular simplex method all non-basic variables are assigned zero values. But in the upper bound technique a basic variable becomes non-basic with its upper bound value. Also when a non-basic variable enters the basis its value should not exceed its upper bound.

8.1 STEP-BY-STEP PROCEDURE OF BOUNDED VARIABLE TECHNIQUE

Step 1 Write the given LPP in the standard form by introducing slack and surplus variables.

Step 2 If any variable x_j has lower bound it should be substituted with $x_j = l_j + x_j'$ where $x_j' \geq 0$.

Step 3 Obtain an initial basic solution.

Step 4 If $Z_j - c_j = 0 \ \forall\ j$, then the current solution is optimal. Otherwise choose the most negative

$Z_j - c_j$ say $Z_r - c_r$. Then the corresponding non-basic variable x_r enters the basis.

Step 5 Compute the quantities

$$\theta_1 = \underset{i}{\text{Min}} \{X_{Bi}/a_{ir}, a_{ir} > 0\}$$

$$\theta_2 = \underset{i}{\text{Min}} \{u_i - X_{Bi}/-a_{ir}, a_{ir} < 0\}$$

$\theta = \text{Min}\{\theta_1, \theta_2, u_r\}$ where u_r is the upper bound of the entering variable x_r

[If $a_{ir} \geq 0 \ \forall\ i$ then $\theta_2 = \infty$]

Step 6 If $\theta = \theta_1$, corresponding to X_{Bk}/a_{kr} then x_k leaves the basis. Regular row operations can be performed.

If $\theta = \theta_2$ corresponding to X_{Bk}/a_{kr} then x_k leaves the basis, x_k becomes non-basic at its upper bound. Substitute

$$x_k = u_k - x_k' \qquad (0 \leq x_k' \leq u_k)$$

Introduce x_r and drop x_k and perform row operations. Substitute $(X_{Bi})^1 = X_{Bi} - a_{ik}u_k$ and

$$Z_0' = Z_0 - (Z_k - c_k)\, u_k$$

If $\theta = u_r$, then x_r is substituted at its upper bound. i.e. $x_r = u_r - x_r'$ and x_r remains non-basic. $(0 \leq x_r' \leq u_r)$.

Step 7 Go to step 4 and repeat the procedure till optimality is reached.

Example 8.1 Use bounded variable technique and solve:

Maximize $Z = x_1 + 3x_2 - 2x_3$

subject to the constraints

$$x_2 - 2x_3 \leq 1$$

$$2x_1 + x_2 + 2x_3 \leq 8$$

$$0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 3,\ 0 \leq x_3 \leq 2$$

Solution The standard form of the LPP is

Maximize $Z = x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2$

subject to the constraints

$$0x_1 + x_2 - 2x_3 + s_1 = 1$$

$$2x_1 + x_2 + 2x_3 + s_2 = 8$$

$$0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 3,\ 0 \leq x_3 \leq 2, 0 \leq s_1 \leq \infty,\ 0 \leq s_2 \leq \infty$$

The initial table is

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	1	3	–2	0	0	
0	s_1	0	1	–2	1	0	1
0	s_2	2	1	2	0	1	8
	$Z_j - c_j$	–1	–3	2	0	0	0
	u_j	1	3	2	∞	∞	

I Iteration x_2 enters the basis

$\theta_1 = \min\{X_{B1}/a_{12}, X_{B2}/a_{22}\} = \min\{1/1, 8/1\}$

$= 1$ corresponding to s_1

$\theta_2 = \infty$

$\theta_3 = u_2 = 3$

$\theta = \min\{\theta_1, \theta_2, \theta_3\} = \min\{1, \infty, 3\} = 1$

$\therefore$ s_1 leaves the basis

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	1	3	–2	0	0	
0	s_1	0	$\boxed{1}$	–2	1	0	1←
0	s_2	2	1	2	0	1	8
	$Z_j - c_j$	–1	–3 ↑	2	0	0	0
	u_j	1	3	2	∞	∞	

Introduce x_2 and drop s_1 . $R_2 - R_1$

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	1	3	–2	0	0	
3	x_1	0	1	–2	1	0	1
0	s_2	2	0	4	–1	1	7
	$Z_j - c_j$	–1	0	–4↑	3	0	3
	u_j	1	3	2	∞	∞	

II Iteration x_3 enters the basis

$\theta_1 = \min\{7/4\} = 7/4$ corresponding to s_2

$\theta_2 = \min\{u_2 - X_{B1}/-a_{13}\} = (3-1)/-(-2)$

$= 2/2 = 1$ corresponding to x_2

$\theta_3 = u_3 = 2$

$\theta = \min\{\theta_1, \theta_2, \theta_3\} = \min\{7/4, 1, 2\} = 1$

$\theta = \theta_2$

$\therefore$ x_2 leaves the basis.

Introduce x_3 and drop x_2 $\quad R_1/-2, \quad R_2 + 2R_1$

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	1	3	–2	0	0	
–2	x_3	0	–1/2	1	–1/2	0	–1/2
0	s_2	2	2	0	1	1	9
	$Z_j - c_j$	–1	–2	0	1	0	1
	u_j	1	3	2	∞	∞	

Since x_2 is non-basic it must be substituted at its upper bound by using

$$x_2 = u_2 - x_2' = 3 - x_2'$$
$$X_{B1}' = X_{B1} - a_{12}u_2 = -1/2 - (-1/2)3 = 1$$
$$X_{B2}' = X_{B2} - a_{22}u_2 = 9 - 2(3) = 3$$
$$Z_0' = Z_0 - (Z_2 - C_2)u_2 = 1 - (-2)3 = 7$$

Thus the simplex table is updated as follows

		x_1	x_2'	x_3	s_1	s_2	X_B
C_B	B	1	3	–2	0	0	
–2	x_3	0	1/2	1	–1/2	0	1
0	s_2	2	–2	0	1	1	3
	$Z_j - c_j$	–1↑	2	0	1	0	7
	u_j	1	3	2	∞	∞	

x_1 enters the basis

$\theta_1 = \min\{3/2, 1/0\} = 3/2$

$\theta_2 = \infty$

$\theta_3 = u_1 = 1$

$$\theta = \min\left\{\frac{3}{2}, \infty, 1\right\} = 1$$

$\theta = u_1$

Therefore x_1 is substituted at its upper bound but it remains non-basic

$$\therefore\ x_1 = u_1 - x_1' = 1 - x_1'$$
$$X_{B1}' = X_{B1} - a_{11}u_1 = 1 - 0(1) = 1$$
$$X_{B2}' = X_{B2} - a_{21}u_1 = 3 - 2(1) = 1$$
$$Z_0' = Z_0 - (Z_1 - c_1)u_1 = 7 - (-1)1 = 8$$

The updated table is

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	1	3	–2	0	0	
–2	x_3	0	1/2	1	–1/2	0	1
0	s_2	–2	–2	0	1	1	1
	$Z_j - c_j$	1	2	0	1	0	8
	u_j	1	3	2	∞	∞	

$Z_j - c_j \geq 0\ \forall\ j$. Hence the current solution is optimal

$x_1' = 0, \quad x_2' = 0, \quad x_3 = 1, \quad Z^* = 8$

But $x_1 = u_1 - x_1' = 1 - 0 = 1$

$x_2 = u_2 - x_2' = 3 - 0 = 3$

$\therefore$ Solution is $x_1 = 3, x_2 = 3, x_3 = 1, Z^* = 8$

Example 8.2 Solve using the bounded variable technique:

Maximize $Z = 3x_1 + 2x_2$

subject to the constraints

$$x_1 - 3x_2 \leq 3$$
$$x_1 - 2x_2 \leq 4$$
$$2x_1 + x_2 \leq 20$$
$$x_1 + 3x_2 \leq 30$$
$$-x_1 + x_2 \leq 6$$
$$0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 6$$

Solution Introducing slack variables, we get the standard form and the initial table as follows:

Maximize

$$Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$$

		x_1	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	3	2	0	0	0	0	0	
0	s_1	$\boxed{1}$	–3	1	0	0	0	0	3
0	s_2	1	–2	0	1	0	0	0	4
0	s_3	2	1	0	0	1	0	0	20
0	s_4	1	3	0	0	0	1	0	30
0	s_5	–1	1	0	0	0	0	1	6
	$Z_j - c_j$	–3↑	–2	0	0	0	0	0	0
	u_j	8	6	∞	∞	∞	∞	∞	

I Iteration x_1 enters the basis

$\theta_1 = \min\{3/1 \quad 4/1 \quad 20/2 \quad 30/1\}$

$= 3$ corresponding to s_1

$\theta_2 = \min\{(\infty - 6)/-(-1)\} = \infty$

$\theta_3 = u_1 = 8$

$\theta = \min\{3, \infty, 8\}$

$= 3$ corresponding to s_1

$\therefore\ s_1$ leaves the basis

		x_1	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	3	2	0	0	0	0	0	
3	x_1	1	–3	1	0	0	0	0	3
0	s_2	0	$\boxed{1}$	–1	1	0	0	0	1
0	s_3	0	7	–2	0	1	0	0	14
0	s_4	0	6	–1	0	0	1	0	27
0	s_5	0	–2	1	0	0	0	1	9
	$Z_j - c_j$	–0	–11	3	0	0	0	0	9
	u_j	8	6	∞	∞	∞	∞	∞	

$R_2 - R_1, R_3 - 2R_1, R_4 - R_1, R_5 + R_1$

II Iteration x_2 enters the basis.

θ_1 = min {1/1 14/7 27/6}

= 1 corresponding to s_2

θ_2 = min {(8 – 3)/ – (–3), ∞} = 5/3

$\theta_3 = u_2 = 6$

θ = min {1, 5/3, 6}

= 1 corresponding to s_2

∴ s_2 leaves the basis

		x_1	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	3	2	0	0	0	0	0	
3	x_1	1	0	(–2)	3	0	0	0	6
0	x_2	0	1	–1	1	0	0	0	1
0	s_3	0	0	5	–7	1	0	0	7
0	s_4	0	0	5	–6	0	1	0	21
0	s_5	0	0	–1	2	0	0	1	11
	$Z_j - c_j$	0	0	–8↑	11	0	0	0	20
	u_j	8	6	∞	∞	∞	∞	∞	

$R_1 + 3R_2, R_3 - 7R_2, R_4 - 6R_2, R_5 + 2R_2$

III Iteration s_1 enters

θ_1 = min {7/5, 21/5} = 7/5 corresponding to s_3

θ_2 = min {(8 – 6)/2, (6 – 1)/1, ∞}

= 1 corresponding to x_1

$\theta_3 = \infty$

θ = min {7/5, 1, ∞} = θ_2 = 1 corresponding to x_1

∴ x_1 leaves the basis

$R_1/(-2), R_2 - R_1/2, R_3 + (5/2)R_1, R_4 + (5/2)R_1, R_5 - R_1/2$

		x_1	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	3	2	0	0	0	0	0	
0	s_1	–1/2	0	1	–3/2	0	0	0	–3
2	x_2	–1/2	1	0	–1/2	0	0	0	–2
0	s_3	5/2	0	0	1/2	1	0	0	22
0	s_4	5/2	0	0	3/2	0	1	0	36
0	s_5	–1/2	0	0	1/2	0	0	1	8
	$Z_j - c_j$	–4	0	0	–1	0	0	0	–4
	u_j	8	6	0	0	0	0	0	

x_1 becomes non-basic. It must be substituted at its upper bound. Hence we get,

$x_1 = 8 - x_1'$

$X'_{B1} = X_{B1} - a_{11}u_1 = -3 - (-1/2)8 = 1$

$X'_{B2} = X_{B2} - a_{21}u_1 = -2 - (-1/2)8 = 2$

$X'_{B3} = X_{B3} - a_{31}u_1 = 22 - (5/2)8 = 2$

$X'_{B4} = X_{B4} - a_{41}u_1 = 36 - (5/2)8 = 16$

$X'_{B5} = X_{B5} - a_{51}u_1 = 8 - (-1/2)8 = 12$

$Z'_0 = Z_0 - (Z_1 - c_1)u_1 = -4 - (-4)8 = 28$

The updated table is

		x_1'	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	–3	2	0	0	0	0	0	
0	s_1	1/2	0	1	–3/2	0	0	0	1
2	x_2	1/2	1	0	–1/2	0	0	0	2
0	s_3	–5/2	0	0	$\boxed{1/2}$	1	0	0	2
0	s_4	–5/2	0	0	3/2	0	1	0	16
0	s_5	1/2	0	0	1/2	0	0	1	12
		–4	0	0	–1↑	0	0	0	28

IV Iteration s_2 enters the basis

θ_1 = min {2/(1/2), 16/(3/2), 12/(1/2)}

= min{ 4, 32/3, 24} = 4 corresponding to s_3

θ_2 = min {∞, (6 – 2)/1/2}

= 8 corresponding to x_2

$\theta_3 = \infty$

θ = min {4, 8, ∞ } = θ_1 = 4

∴ s_3 leaves the basis

$R_1 + 3R_3, R_2 + R_3, R_3(2), R_4 - 3R_3, R_5 - R_3$

		x_1'	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	–3	2	0	0	0	0	0	
0	s_1	–7	0	1	0	3	0	0	7
2	x_2	$\boxed{-2}$	1	0	0	1	0	0	4
0	s_2	–5	0	0	1	2	0	0	4
0	s_4	5	0	0	0	–3	1	0	10
0	s_5	3	0	0	0	–1	0	1	10
	$Z_j - c_j$	–1 ↑	0	0	0	2	0	0	32
	u_j	8	6	∞	∞	∞	∞	∞	

V Iteration x_1' enters the basis

θ_1 = min {10/5, 10/3} = 2 corresponding to s_4

θ_2 = min {∞, (6 – 4)/2, ∞}

= 1 corresponding to x_2

$\theta_3 = 8$

θ_1 = min {2, 1, 8} = θ_2

= 1 corresponding to x_2

∴ x_2 leaves the basis

$R_1 - (7/2)R_2$, $R_2/(-2)$, $R_3 - (5/2)R_2$,

$R_4 + (5/2)R_2$, $R_5 + (3/2)R_2$

		x_1'	x_2	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	–3	2	0	0	0	0	0	
0	s_1	0	–7/2	1	0	–1/2	0	0	–7
–3	x_1'	1	–1/2	0	0	–1/2	0	0	–2
0	s_2	0	–5/2	0	1	–1/2	0	0	–6
0	s_4	0	5/2	0	0	–1/2	1	0	20
0	s_5	0	3/2	0	0	1/2	0	1	16
	$Z_j - c_j$	0	–1/2 ↑	0	0	3/2	0	0	30
	u_j	8	6	0	0	0	0	0	

Since x_2 becomes non-basic, it must be substituted at its upper bound. Hence we get

$x_2 = 6 - x_2'$

$X_{B1}' = X_{B1} - a_{12}u_2 = -7 - (-7/2)6 = 14$

$X_{B2}' = X_{B2} - a_{22}u_2 = -2 - (-1/2)6 = 1$

$X_{B3}' = X_{B3} - a_{32}u_2 = -6 - (-5/2)6 = 9$

$X_{B4}' = X_{B4} - a_{42}u_2 = 20 - (5/2)6 = 5$

$X_{B5}' = X_{B5} - a_{52}u_2 = 16 - (3/2)6 = 7$

$Z_0' = Z_0 - (Z_2 - c_2)u_2 = 30 - (-1/2)6 = 33$

Updating the table we get,

		x_1'	x_2'	s_1	s_2	s_3	s_4	s_5	X_B
C_B	B	–3	–2	0	0	0	0	0	
0	s_1	0	7/2	1	0	–1/2	0	0	14
–3	x_1'	1	1/2	0	0	–1/2	0	0	1
0	s_2	0	5/2	0	1	–1/2	0	0	9
0	s_4	0	–5/2	0	0	–1/2	1	0	5
0	s_5	0	–3/2	0	0	1/2	0	1	7
	$Z_j - c_j$	0	1/2	0	0	3/2	0	0	33
	u_j	8	6	∞	∞	∞	∞	∞	

The optimality reached since $Z_j - c_j \geq 0 \ \forall\ j$

The solution is $x_1' = 1$, $x_2' = 0$, $Z^* = 33$

$x_1 = 8 - 1$, $x_2 = 6 - 0$, $Z^* = 33$

$x_1 = 7$, $x_2 = 6$, $Z^* = 33$

Example 8.3 Solve using the bounded variable technique

Maximize $Z = 3x_1 + 5x_2 + 3x_3$

subject to the constraints

$$x_1 + 2x_2 + 2x_3 \leq 14$$

$$2x_1 + 4x_3 + 3x_3 \leq 23$$

$$0 \leq x_1 \leq 4, \quad 2 \leq x_2 \leq 5, \quad 0 \leq x_3 \leq 3$$

Solution The variable x_2 has a lower bound 2. Hence we introduce a variable y_2 where $y_2 = x_2 - 2$ or $x_2 = y_2 + 2$ $(0 \leq y_2 \leq 3)$. Introducing y_2 and slack variables we get the starting table as:

		x_1	y_2	x_3	s_1	s_2	X_B
C_B	B	3	5	3	0	0	
0	s_1	1	2	2	1	0	10
0	s_2	2	4	3	0	1	15
	$Z_j - c_j$	–3	–5	–3	0	0	10
	u_j	4	3	3	∞	∞	

I Iteration y_2 enters the basis

θ_1 = min {10/2, 15/4} = 15/4

$\theta_2 = \infty$

$\theta_3 = u_2 = 3$

θ = min {θ_1, θ_2, θ_3]

= min {15/4, ∞ , 3}

$\theta = \theta_3 = 3$

Therefore y_2 is substituted at its upper bound and remains non-basic. We get

$y_2 = u_2 - y_2' = 3 - y_2'$ $(0 \le y_2' \le 3)$
$X_{B1}' = X_{B1} - a_{12}u_2 = 10 - 2(3) = 4$
$X_{B2}' = X_{B2} - a_{22}u_2 = 15 - 4(3) = 3$
$Z_0' = Z_0 - (Z_2 - c_2)u_2 = 10 - (-5)3 = 25$

The updated table is as follows.

		x_1	y_2'	x_3	s_1	s_2	X_B
C_B	B	3	–5	3	0	0	
0	s_1	1	–2	2	1	0	4
0	s_2	2	–4	$\boxed{3}$	0	1	3 ←
	$Z_j - c_j$	–3	5	–3 ↑	0	0	25
	u_j	4	3	3	∞	∞	

There is a tie for the entering variable. Take x_3 as the entering variable.

II Iteration

$\theta_1 = \min\{4/2, 3/3\} = 1$ corresponding to s_2

$\theta_2 = \infty$

$\theta_3 = 3$

$\theta_1 = \min\{1, \infty, 3\} = 1$ corresponding to s_2

$\therefore$ s_2 leaves the basis.

$R_1 - 2/3R_2,\ R_2/3$

		x_1	y_2'	x_3	s_1	s_2	X_B
C_B	B	3	–5	3	0	0	
0	s_1	–1/3	2/3	0	1	–2/3	2
3	x_3	$\boxed{2/3}$	–4/3	1	0	1/3	1 ←
	$Z_j - c_j$	–1 ↑	1	0	0	1	28
	u_j	4	3	3	∞	∞	

x_1 enters the basis

III Iteration

$\theta_1 = \min\{1/(2/3)\} = 3/2$ corresponding to x_3

$\theta_2 = \min\{\infty\} = \infty$

$\theta_3 = 4$

$\theta\ = \min\left\{\frac{3}{2}, \infty, 4\right\} = 3/2$ corresponding to x_3

x_3 leaves the basis

$R_1 + R_2/2,\ R_2\ (3/2).$

We get

		x_1	y_2'	x_3	s_1	s_2	X_B
C_B	B	3	–5	3	0	0	
0	s_1	0	0	1/2	1	–1/2	5/2
3	x_1	1	$\boxed{-2}$	3/2	0	1/2	3/2 ←
	$Z_j - c_j$	0	–1 ↑	3/2	0	3/2	59/2
	u_j	4	3	3	∞	∞	

IV Iteration y_2' enters the basis

$\theta_1 = \infty,$

$\theta_2 = (4 - 3/2)/2 = 5/4$

$\theta_3 = 3$

$\theta_1 = \min\{\infty, 5/4, 3\} = 5/4$

$= \theta_2$ corresponding to x_1

$\therefore$ x_1 leaves the basis

$R_2/(-2)$ is the only operation.

We get

		x_1	y_2¢	x_3	s_1	s_2	X_B
C_B	B	3	–5	3	0	0	
0	s_1	0	0	1/2	1	–1/2	5/2
–5	y_2'	–1/2	1	–3/4	0	–1/4	–3/4
	$Z_j - c_j$	–1/2	0	3/4	0	5/4	115/4
	u_j	4	3	3	∞	∞	

Since $\theta = \theta_2$ corresponding to x_1, we find that the leaving variable x_1 should be substituted at its upper bound 4.

$\therefore$ $x_1 = 4 - x_1'$

$X_{B1}' = X_{B1} - a_{11}u_1 = 5/2 - 0.4 = 5/2$

$X_{B2}' = X_{B2} - a_{21}u_1 = -3/4 - (-1/2)4 = 5/4$

$Z_0' = Z_0 - (Z_1 - c_1)\ u_1 = 115/4 - (-1/2)4 = 123/4.$

The updated table is

		x_1'	y_2'	x_3	s_1	s_2	X_B
C_B	B	–3	–5	3	0	0	
0	s_1	0	0	1/2	1	–1/2	5/2
–5	y_2'	1/2	1	–3/4	0	–1/4	5/4
	$Z_j - c_j$	1/2	0	3/4	0	5/4	123/4

The optimality reached.
The solution is

$x_1' = 0 \qquad y_2' = 5/4$

$x_1 = 4 - 0 \qquad y_2 = 3 - 5/4$

$x_1 = 4 \qquad y_2 = 7/4$

$$x_2 = 2 + y_2 = 15/4$$

$$x_1 = 4,\ x_2 = 15/4,\ x_3 = 0,\ Z^* = 123/4$$

Solve the following LPP using the bounded variable technique.

EXERCISES

1. Maximize $Z = x_2 + 3x_3$
   subject to the constraints
   $$x_1 + x_2 + x_3 \le 10$$
   $$x_1 - 2x_3 \le 0$$
   $$2x_2 - x_3 \le 10$$
   $0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ x_3 \ge 0$
2. Maximize $Z = 4x_1 + 4x_2 + 3x_3$
   subject to the constraints
   $$-x_1 + x_2 + 3x_3 \le 15$$
   $$-x_2 + x_3 \le 4$$
   $$2x_1 + x_2 - x_3 \le 6$$
   $$x_1 - x_2 + 2x_3 \le 10$$
   $0 \le x_1 \le 8,\ x_2 \ge 0,\ 0 \le x_3 \le 4$
3. Maximize $z = 4x_1 + 2x_2 + 6x_3$
   subject to the constraints
   $$4x_1 - x_2 \le 9$$
   $$-x_1 + x_2 + 2x_3 \le 8$$
   $$-3x_1 + x_2 + 4x_3 \le 12$$
   $1 \le x_1 \le 3,\ 0 \le x_2 \le 5,\ 0 \le x_3 \le 2$
4. Maximize $Z = 3x_1 + 5x_2 + 2x_3$
   subject to the constraints
   $$x_1 + x_2 + 2x_3 \le 14$$
   $$2x_1 + 4x_2 + 3x_3 \le 34$$
   $0 \le x_1 \le 4,\ 7 \le x_2 \le 10,\ 0 \le x_3 \le 3$
5. Maximize $Z = 4x_1 + 5x_2$
   subject to the constraints
   $$2x_1 + 3x_2 \le 12$$
   $$2x_1 + x_2 \le 6$$
   $x_1 \ge 0,\ 1 \le x_2 \le 3$

ANSWERS

1. $x_1 = 20/3,\ x_2 = 0,\ x_3 = 10/3,\ Z^* = 10$
2. $x_1 = 17/5,\ x_2 = 16/5,\ x_3 = 4,\ Z^* = 192/5$
3. $x_1 = 3,\ x_2 = 5,\ x_3 = 2,\ Z^* = 34$
4. $x_1 = 4,\ x_2 = 35/4,\ x_3 = 0,\ Z^* = 223/4$
5. $x_1 = 3/2,\ x_2 = 3,\ Z^* = 21$

9

Integer Programming

CONCEPT REVIEW

In linear programming each variable is allowed to take any non-negative real (fractional) value. But there are certain problems in which the variables cannot assume fractional values. For example if x_1 represents the number of workers and x_2 represents the number of machines then their values can be only integers. It is meaningless to take 8.7 workers or 3.2 machines. Hence in such situations the variables must have only integer values. In a problem if all the decision variables are required to assume non-negative integer values it is called *pure integer programming*. If, on the other hand, only some of the variables are restricted to assume non-negative integer values it is called *mixed integer programming* problem. Problems involving allocation of goods, men, machines, etc. are integer programming problems.

Suppose that the given pure integer programming problem is

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$x_1, x_2, x_3, \ldots, x_n \geq 0$ and are all integers.

We may be tempted to solve this problem by the simplex method, ignoring the integer restriction and then round off the non-integral values of the variables in the optimal solution. But then the integer valued solution may not satisfy all the constraints, leading to an infeasible solution. For example, consider the LPP.

Maximize $Z = x_1 + x_2$

subject to the constraints

$$3x_1 + 4x_2 \leq 14$$

$$x_1, x_2 \geq 0 \quad \text{and are integers.}$$

The optimal solution of this problem is found (by graphical method) to be

$x_1 = 14/3, \quad x_2 = 0, \quad Z^* = 14/3$

If we round off the solution as $x_1 = 5$, $x_2 = 0$ then the constraint $3x_1 + 4x_2 \leq 14$ is not satisfied. On the other hand if we take $x_1 = 4$ then $Z^* = 4$ which is not optimal.

Thus we find that it is necessary to develop a systematic and efficient method to solve an integer programming problem (IPP).

9.1 GOMARY'S CUTTING PLANE METHOD

In this method we generate new constraints so as to ensure integral solution of the given LPP. These additional constraints do not cut off that portion of the original feasible region which contains a feasible integer point. Also it cuts off the current non-integer solution of the LPP.

First we solve the LPP, ignoring the integer requirement, by the usual simplex method. If the optimal solution contains integer values for the variables then it is itself the solution of the IPP. If some of the variables have fractional values then choose the variable having the largest fractional part, say x_r. The corresponding r^{th} constraint can be written as

$$b_r = x_r + a_{r1}x_1 + a_{r2}x_2 + \dots$$

$$= x_r + \sum_{j=1}^{n} a_{rj}x_j \qquad (j \neq r)$$

If $[b_r]$ denotes the integral part of b_r and $[a_{rj}]$ denotes the integral part of a_{rj}, then we can write

$$b_r = [b_r] + f_r,\ 0 < f_r < 1 \text{ and}$$

$$a_{rj} = [a_{rj}] + f_{rj},\ 0 < f_{rj} < 1$$

Therefore we get

$$[b_r] + f_r = x_r + \Sigma\,\{[a_{rj}] + f_{rj}\}\,x_j \qquad (j \neq r)$$

$$\therefore\ f_r + \{[b_r] - x_r - \Sigma\,[a_{rj}]\,x_j\} = \Sigma\, f_{rj}\,x_j \quad (j \neq r)$$

Since $\{[b_r] - x_r - \Sigma\,[a_{rj}]x_j\} \geq 0$ we have

$$f_r \leq \Sigma\, f_{rj}\,x_j$$

Introducing a slack variable, we get

$$f_r + s = \Sigma f_{rj}\,x_j$$

i.e. $\qquad s - \Sigma\, f_{rj}\,x_j = -f_r$

This is a new constraint called *Gomary's constraint*, which represents a cutting plane.

9.2 GOMARY'S ALGORITHM (STEP-BY-STEP PROCEDURE)

Step 1 Solve the given LPP using the simplex method ignoring the integer constraint.

Step 2 If the optimal values of all the variables are integers it is itself the solution of the LPP. Stop.

Step 3 If some variables assume fractional values, choose the variable having the largest fractional part, say x_r. Write the Gomary's constraint in the form

$$s - \Sigma\, f_{rj}\,x_j = -f_r\ \{0 < f_r < 1,\ 0 < f_{rj} < 1\}$$

Step 4 Add the new constraint at the bottom of the simplex table and find a new optimal solution using the dual simplex method. If the new solution gives integer values to the variables it is the required solution. Otherwise go to step 3. This process is repeated until all the variables assume integer values.

Example 9.1 Solve the following IPP using Gomary's cutting plane method

Maximize $Z = x_1 + x_2$

subject to the constraints

$$3x_1 + 2x_2 \leq 5$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Solution Applying simplex method we get the initial table (standard form)

		x_1	x_2	s_1	s_2	X_B
C_B	B	1	1	0	0	
0	s_1	$\boxed{3}$	2	1	0	5 ←
0	s_2	0 ↑	1	0	1	2
	$Z_j - c_j$	–1	–1	0	0	0
						$R_1/3$
1	x_1	1	2/3	1/3	0	5/3
0	s_2	0	$\boxed{1}$ ↑	0	1	2 ←
	$Z_j - c_j$	0	–1/3	1/3	0	5/3
						$R_1 - (2/3)R_2$
1	x_1	1	0	1/3	–2/3	1/3
1	x_2	0	1	0	1	2
	$Z_j - c_j$	0	0	1/3	1/3	7/3

In the optimal solution obtained x_2 assumes integer value. But x_1 has a fractional value. Hence we generate a Gomary's cut. The constraint corresponding to x_1 is

$$x_1 + (1/3)s_1 - (2/3)\, s_2 = 1/3$$
$$1/3 = x_1 + (1/3)s_1 - (2/3)s_2$$
$$(0 + 1/3) = (1 + 0)x_1 + (0 + 1/3)s_1 + (-1 + 1/3)s_2$$

i.e., $1/3 < (1/3)\, s_1 + (1/3)\, s_2$

Using a slack variable s, we get

$$1/3 + s = (1/3)s_1 + (1/3)s_2$$

i.e., $s - (1/3)s_1 - (1/3)s_2 = -1/3$

Introducing this constraint, we get the table as

		x_1	x_2	s_1	s_2	s	X_B
C_B	B	1	1	0	0	0	
1	x_1	1	0	1/3	–2/3	0	1/3
1	x_2	0	1	0	1	0	2
0	s	0	0	$\boxed{-1/3}$ ↑	–1/3	1	–1/3 ←
	$Z_j - c_j$	0	0	1/3	1/3	0	7/3

Apply the dual simplex method.

s leaves the basis and s_1 enters

$R_1 + R_3$, $R_3(-3)$.

The new table is

		x_1	x_2	s_1	s_2	s	X_B
C_B	B	1	1	0	0	0	
1	x_1	1	0	0	–1	1	0
1	x_2	0	1	0	1	0	2
0	s_1	0	0	1	1	–3	1
	$Z_j - c_j$	0	0	0	0	1	2

Optimal solution of IPP is

$x_1 = 0$, $x_2 = 2$, $Z^* = 2$

Example 9.2 Solve the IPP

Maximize $Z = x_1 + 2x_2$

subject to the constraints

$$x_1 + x_2 \le 7$$
$$2x_1 \le 11$$
$$2x_2 \le 7$$
$$x_1, x_2 \ge 0 \text{ and are integers.}$$

Solution After introducing the slack variables we get the initial table as

		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	1	2	0	0	0	
0	s_1	1	1	1	0	0	7
0	s_2	2	0	0	1	0	11
0	s_3	0	$\boxed{2}$ ↑	0	0	1	7 ←
	$Z_j - c_j$	–1	–2	0	0	0	0
						$R_1 - R_3/2$	$R_3/2$
0	s_1	$\boxed{1}$	0	1	0	–1/2	7/2 ←
0	s_2	2	0	0	1	0	11
2	x_2	0 ↑	1	0	0	1/2	7/2
	$Z_j - c_j$	–1	0	0	0	1	7
							$R_2 - 2R_1$
1	x_1	1	0	1	0	–1/2	7/2
0	s_2	0	0	–2	1	1	4
2	x_2	0	1	0	0	1/2	7/2
	$Z_j - c_j$	0	0	1	0	1/2	21/2

Optimality reached. The variables x_1 and x_2 have fractional values 7/2 and 7/2 each. Both have fractional part 1/2. We can select one variable at our choice. Let us take the value $x_2 = 7/2$. The corresponding constraint is

$$x_2 + (1/2)s_3 = 7/2$$
$$7/2 = x_2 + (1/2)s_3$$
$$3 + (1/2) = (1 + 0)x_2 + (0 + 1/2)s_3$$
$$1/2 < 1/2s_3$$

Introducing slack variable s

$$1/2 + s = (1/2)s_3$$
$$s = (1/2)s_3 - 1/2$$

$(-1/2)s_3 + s = -1/2$ is the Gomary's constraint.

The new table is

		x_1	x_2	s_1	s_2	s_3	s	X_B
C_B	B	1	2	0	0	0	0	
1	x_1	1	0	1	0	–1/2	0	7/2
0	s_2	0	0	–2	1	1	0	4
2	x_2	0	1	0	0	1/2	0	7/2
0	s	0	0	0	0	$\boxed{-1/2}$ ↑	1	–1/2 ←
	$Z_j - c_j$	0	0	1	0	1/2	0	21/2

s leaves and s_3 enters the basis

$R_1 - R_4$, $R_2 + 2(R_4)$, $R_3 + R_4$, $R_4(-2)$

		x_1	x_2	s_1	s_2	s_3	s	X_B	
C_B	B	1	2	0	0	0	0		
1	x_1	1	0	1	0	0	–1	4	
0	s_2	0	0	–2	1	0	2	3	
2	x_2	0	1	0	0	0	1	3	
0	s_3	0	0	0	0	1	–2	1	
	$Z_j - c_j$	0	0	1	0	0	1	10	(Optimal)

The solution of the IPP is

$x_1 = 4$, $x_2 = 3$, $Z^* = 10$

Example 9.3 Solve the IPP

Maximize $Z = x_1 + 4x_2$

subject to the constraints

$$2x_1 + 4x_2 \le 7$$
$$5x_1 + 3x_2 \le 15$$
$$x_1, x_2 \ge 0 \text{ and are integers.}$$

Solution The initial simplex table is

		x_1	x_2	s_1	s_2	X_B
C_B	B	1	4	0	0	
0	s_1	2	4	1	0	7 ←
0	s_2	$\boxed{5}$	3 ↑	0	1	15
	$Z_j - c_j$	–1	–4	0	0	0
				$R_1/4$	$R_2 - (3/4)R_1$	
4	x_2	1/2	1	1/4	0	7/4
0	s_2	7/2	0	–3/4	1	39/4
	$Z_j - c_j$	1	0	1	0	7

Both constants have the same fractional part 3/4. Choose the x_2 row. We have

$$(1/2)x_1 + x_2 + (1/4)s_1 = 7/4$$
$$7/4 = (1/2)x_1 + x_2 + (1/4)s_1$$
$$1 + 3/4 = (0 + 1/2)\, x_1 + x_2 + (0 + 1/4)\, s_1$$
$$3/4 < \left(\frac{1}{2}\right)x_1 + \left(\frac{1}{4}\right)s_1$$
$$3/4 + s = (1/2)x_1 + (1/4)\, s_1$$

i.e., $s - (1/2)x_1 - (1/4)s_1 = -3/4$ is the Gomary constraint.

The new table is

		x_1	x_2	s_1	s_2	s	X_B
C_B	B	1	4	0	0	0	
4	x_2	1/2	1	1/4	0	0	7/4
0	s_2	7/2	0	– 3/4	1	0	39/4
0	s	$\boxed{-1/2}$	0	– 1/4	0	1	– 3/4 ←
	$Z_j - c_j$	1 ↑	0	1	0	0	7

Applying the dual simplex method, we find that s leaves and x_1 enters the basis

$R_1 + R_3$, $R_2 + 7R_3$, $R_3(-2)$ gives

		x_1	x_2	s_1	s_2	s	X_B
C_B	B	1	4	0	0	0	
4	x_2	0	1	0	0	1	1
0	s_2	0	0	– 5/2	1	7	9/2
1	x_1	1	0	1/2	0	– 2	3/2
	$Z_j - c_j$	0	0	1/2	0	2	11/2

Choose the x_1 row for constructing the Gomary constraint

$1 + (1/2) = x_1 + (1/2)s_1 - 2s$

$1/2 < 1/2\, s_1$. Introduce slack variable t,

$1/2 + t = 1/2s_1$

$t - 1/2s_1 = -1/2$ is the Gomary cut.

The new table is

		x_1	x_2	s_1	s_2	s	t	X_B
C_B	B	1	4	0	0	0	0	
4	x_2	0	1	0	0	1	0	1
0	s_2	0	0	– 5/2	1	7	0	9/2
1	x_1	1	0	1/2	0	– 2	0	3/2
0	t	0	0	$\boxed{-1/2}$	0	0	1	– 1/2 ←
	$Z_j - c_j$	0	0	1/2 ↑	0	2	0	11/2

By the dual simplex method, t leaves and s_1 enters the basis.

$R_2 - 5R_4$, $R_3 + R_4$, $R_4(-2)$

		x_1	x_2	s_1	s_2	s	t	X_B	
C_B	B	1	4	0	0	0	0		
4	x_2	0	1	0	0	1	0	1	
0	s_2	0	0	0	1	7	–5	7	
1	x_1	1	0	0	0	–2	1	1	
0	s_1	0	0	1	0	0	–2	1	
	$Z_j - c_j$	0	0	0	0	2	1	5	(Optimal)

Solution of the given IPP is

$x_1 = 1, \quad x_2 = 1, \quad Z^* = 5$

9.3 MIXED INTEGER PROGRAMMING

In the previous section we discussed Gomary's method to solve IPP where all the variables including slack variables were restricted to take integer values. In the mixed integer programming, only some of the decision and slack variables are restricted to assume integer values.

Ignoring the integer restriction, we solve the LPP. In the optimal table suppose x_r has the largest fractional part. Then the corresponding constraint can be written as

$$b_r = x_r + \Sigma\, a_{rj}\, x_j + \Sigma\, a_{rj}\, x_j\, (j \neq r)$$
$$(a_{rj} \geq 0) \qquad\qquad (a_{rj} < 0)$$

$$[b_r] + f_r = (1 + 0)\, x_r + \Sigma\, \{[a_{rj}] + f_{rj}\}\, x_j\, (a_{rj} \geq 0) + \Sigma\{[a_{rj}] + f_{rj}\}x_j\, (a_{rj} < 0).$$

Rewriting this equation such that all the integer coefficients appear on the RHS, we get

$$\Sigma^+ f_{rj}\, x_j + \Sigma^- f_{rj}\, x_j = f_r + [b_r] - x_r - a_{rj}{}^+ - a_{rj}{}^-$$
$$= f_r + I\ (0 < f_r < 1)$$

where I is the integer value. LHS is positive or negative according as $f_r + I$ is positive or negative.

Case (I) Let $f_r + I$ be positive.

Then $\Sigma f_{rj}{}^+ x_j + \Sigma f_{rj}{}^- x_j \geq f_r$

Since $\Sigma f_{rj}{}^+ x_j \geq \Sigma f_{rj}{}^+ x_j + \Sigma f_{rj}^- x_j$

we have $\Sigma f_{rj}{}^+ x_j \geq f_r$ (1)

Case (II) Let $f_r + I$ be negative.

Then $\Sigma f_{rj}{}^+ x_j + \Sigma f_{rj}{}^- x_j \leq -1 + f_r$

Now $\Sigma f_{rj}{}^- x_j \leq \Sigma f_{rj}{}^+ x_j + \Sigma f_{rj}{}^- x_j$

we have $f_{rj}{}^- x_j \leq -1 + f_r$

Multiply both sides by the negative number $(f_r/f_r - 1)$.

We get $(f_r/f_r - 1)\, \Sigma f_{rj}{}^- x_j \geq f_r$ (2)

Combining (1) and (2) we find that any feasible solution must satisfy the inequality,

$$\Sigma f_{rj}{}^+ x_j + (f_r/f_r - 1)\, \Sigma f_{rj}{}^- x_j \geq f_r$$

Introducing a slack variable s,

$$s + f_r = \Sigma f_{rj}{}^+ x_j + (f_r/f_r - 1)\, \Sigma f_{rj}{}^- x_j$$

or $s - \Sigma f_{rj}{}^+ x_j - (f_r/f_r - 1)\, \Sigma f_{rj}{}^- x_j = -f_r \quad 0 < f_r < 1$

This is the Gomary's constraint.

Adding this new constraint to the table find a new optimal solution using dual simplex method.

Example 9.4 Solve the following mixed integer programming problem:

Maximize $Z = -3x_1 + x_2 + 3x_3$

subject to the constraints

$$-x_1 + 2x_2 + x_3 \leq 4$$
$$2x_2 - (3/2)x_3 \leq 1$$
$$x_1 - 3x_2 + 2x_3 \leq 3$$
$$x_1, x_2, x_3 \geq 0\ x_3 \text{ is an integer.}$$

Solution The standard form is

Maximize $Z = -3x_1 + x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$-x_1 + 2x_2 + x_3 + s_1 = 4$$
$$4x_2 - 3x_3 + s_2 = 2$$
$$x_1 - 3x_2 + 2x_3 + s_3 = 3$$
$$x_1, x_2, x_3, s_1, s_2, s_3 = 0 \quad x_3 \text{ is an integer.}$$

The simplex table is

		x_1	x_2	x_3	s_1	s_2	s_3	X_B
C_B	B	–3	1	3	0	0	0	
0	s_1	–1	2	1	1	0	0	4
0	s_2	0	4	–3	0	1	0	2
0	s_3	1	–3	$\boxed{2}$ ↑	0	0	1	3 ←
	$Z_j - c_j$	3	–1	–3	0	0	0	0

$R_1 - R_3/2,\ R_2 + (3/2)R_3,\ R_3/2$

(Contd)

		x_1	x_2	x_3	s_1	s_2	s_3	X_B
C_B	B	–3	1	3	0	0	0	
0	s_1	–3/2	$\boxed{7/2}$	0	1	0	–1/2	5/2 ←
0	s_2	3/2	–1/2	0	0	1	3/2	13/2
3	x_3	1/2	–3/2 ↑	1	0	0	1/2	3/2
	$Z_j - c_j$	9/2	–11/2	0	0	0	3/2	9/2
				$R_1(2/7)$,	$R_2 + R_1/7$,	$R_3 + 3R_1/7$		
1	x_2	–3/7	1	0	2/7	0	–1/7	5/7
0	s_2	9/7	0	0	1/7	1	10/7	48/7
3	x_3	–1/7	0	1	3/7	0	2/7	18/7
	$Z_j - c_j$	15/7	0	0	11/7	0	5/7	59/7

x_3 must be an integer. The corresponding constraint is

$18/7 = -(1/7)x_1 + x_3 + (3/7)s_1 + (2/7)s_3$

$2 + (4/7) = -(1/7)x_1 + x_3 + (3/7)s_1 + (2/7)s_3$

$f_r = 4/7$. The coefficient of x_1 is –1/7.

Applying the rule we get

$4/7 \leq (3/7)s_1 + (2/7)s_3 + [(4/7) / \{(4/7) - 1\}] (-1/7)x_1$

$\leq (3/7)s_1 + (2/7)s_3 + (4/21)x_1$

Introducing slack variable s

$4/7 + s = (4/21)x_1 + (3/7)s_1 + (2/7)s_3$

The Gomary constraint is

$-(4/21)x_1 - (3/7)s_1 - (2/7)s_3 + s = -4/7$

Adding this new constraint to the table we get

		x_1	x_2	x_3	s_1	s_2	s_3	s	X_B
C_B	B	–3	1	3	0	0	0	0	
1	x_2	–3/7	1	0	2/7	0	–1/7	0	5/7
0	s_2	9/7	0	0	1/7	1	10/7	0	48/7
3	x_3	–1/7	0	1	3/7	0	2/7	0	18/7
0	s	–4/21	0	0	–3/7	0	$\boxed{-2/7}$ ↑	1	–4/7 ←
	$Z_j - c_j$	15/7	0	0	11/7	0	5/7	0	59/7
					$R_1 - R_4/2$,	$R_2 + 5R_4$,	$R_3 + R_4$,	$R_4(-7/2)$	

Apply dual simplex method. s leaves and s_3 enters the basis. We get

		x_1	x_2	x_3	s_1	s_2	s_3	s	X_B	
C_B	B	–3	1	3	0	0	0	0		
1	x_2	–1/3	1	0	1/2	0	0	–1/2	1	
0	s_2	1/3	0	0	–2	1	0	5	4	
3	x_3	–1/3	0	1	0	0	0	1	2	
0	s_3	2/3	0	0	3/2	0	1	–7/2	2	
	$Z_j - c_j$	5/3	0	0	1/2	0	0	5/2	7	(Optimal)

The solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $Z^* = 7$

Example 9.5 Solve the following mixed IPP

Maximize $Z = x_1 + x_2$

subject to the constraints

$2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 = 0$ x_1 is an integer

Solution The starting table is

		x_1	x_2	s_1	s_2	X_B	
C_B	B	1	1	0	0		
0	s_1	2	$\boxed{5}$	1	0	16 ←	
0	s_2	6	5 ↑	0	1	30	
	$Z_j - c_j$	–1	–1	0	0	0	
				$R_1/5$,	$R_2 - R_1$		
1	x_2	2/5	1	1/5	0	16/5	
0	s_2	$\boxed{4}$ ↑	0	–1	1	14 ←	
	$Z_j - c_j$	–3/5	0	1/5	0	16/5	
				$R_1 - R_2/10$,	$R_2/4$		
1	x_2	0	1	3/10	–1/10	9/5	
1	x_1	1	0	–1/4	1/4	7/2	
	$Z_j - c_j$	0	0	1/20	3/20	53/10	(Optimal)

Since x_1 must have an integer value we take the corresponding constraint.

$x_1 - (1/4)s_1 + (1/4)s_2 = 7/2$

$7/2 = x_1 - (1/4)\, s_1 + (1/4)s_2$

$3 + (1/2) = x_1 - (1/4)s_1 + (1/4)s_2$

$1/2 \leq [(1/2)/((1/2) - 1)]\,(-1/4)s_1 + (1/4)s_2$

$\leq (1/4)s_1 + (1/4)s_2$

Introduce s.

$1/2 + s = (1/4)\, s_1 + (1/4)s_2$

$s - (1/4)s_1 - (1/4)s_2 = -1/2$ is the Gomary constraint.

Adding this constraint to the table we get

		x_1	x_2	s_1	s_2	s	X_B
C_B	B	1	1	0	0	0	
1	x_2	0	1	3/10	–1/10	0	9/5
1	x_1	1	0	–1/4	1/4	0	7/2
0	s	0	0	$\boxed{-1/4}$ ↑	–1/4	1	–1/2 ←
	$Z_j - c_j$	0	0	1/20	3/20	0	53/10

Apply the dual simplex method

s leaves, s_1 enters.

Perform the operations

$R_1 + (6/5)\, R_3,\ R_2 - R_3,\ R_3(-4)$.

We get

		x_1	x_2	s_1	s_2	s	X_B
C_B	B	1	1	0	0	0	
1	x_2	0	1	0	–2/5	6/5	6/5
1	x_1	1	0	0	1/2	–1	4
0	s	0	0	1	1	–4	2
	$Z_j - c_j$	0	0	0	1/10	1/5	26/5

The optimal solution is

$x_1 = 4,\ x_2 = 6/5,\ Z^* = 26/5$

EXERCISES

Solve the following Integer Programming Problems

1. Maximize $Z = 2x_1 + 2x_2$
   subject to the constraints
   $5x_1 + 3x_2 \le 8$
   $2x_1 + 4x_2 \le 8$
   $x_1, x_2 \ge 0$ and are integers.
2. Minimize $Z = -2x_1 - 3x_2$
   subject to the constraints
   $2x_1 + 2x_2 \le 7$
   $x_1 \le 2$
   $x_2 \le 2$
   $x_1, x_2 \ge 0$ and are integers.
3. Maximize $Z = x_1 + x_2$
   subject to the constraints
   $2x_1 + 5x_2 \le 16$
   $6x_1 + 5x_2 \le 30$
   $x_1, x_2 \ge 0$ and are integers.
4. Maximize $Z = x_1 + 2x_2$
   subject to the constraints
   $x_1 + x_2 \le 7$
   $2x_1 \le 11$
   $2x_2 \le 7$
   $x_1, x_2 \ge 0$ and are integers.
5. Maximize $Z = 3x_1 + 4x_2$
   subject to the constraints
   $3x_1 + 2x_2 \le 8$
   $x_1 + 4x_2 \le 10$
   $x_1, x_2 \ge 0$ and are integers.
6. Maximize $Z = 7x_1 + 9x_2$
   subject to the constraints
   $-x_1 + 3x_2 \le 6$
   $7x_1 + x_2 \le 35$
   $x_1, x_2 \ge 0$ and are integers.
7. Maximize $Z = 9x_1 + 10x_2$
   subject to the constraints
   $4x_1 + 3x_2 \ge 40$
   $x_1 \le 9$
   $x_2 \le 8$
   $x_1, x_2 \ge 0$ and are integers.
8. Maximize $Z = 2x_1 + 20x_2 - 10x_3$
   subject to the constraints
   $2x_1 + 20x_2 + 4x_3 \le 15$
   $6x_1 + 20x_2 + 4x_3 = 20$
   $x_1, x_2, x_3 \ge 0$ and are integers.
9. Minimize $Z = x_1 - 3x_2$
   subject to the constraints

$$x_1 + x_2 \leq 5$$
$$-2x_1 + 4x_2 \leq 11$$
$$x_1, x_2 \geq 0 \quad x_2 \text{ is an integer.}$$

10. Minimize $Z = 10x_1 + 9x_2$
subject to the constraints
$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0 \quad x_1 \text{ is an integer.}$$

11. Maximize $Z = 4x_1 + 6x_2 + 2x_3$
subject to the constraints
$$4x_1 - 4x_2 \leq 5$$
$$-x_1 + 6x_2 \leq 5$$
$$-x_1 + x_2 + x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0 \; x_1, x_3 \text{ are integers.}$$

12. Maximize $Z = 20x_1 + 17x_2$
subject to the constraints
$$4x_1 + 3x_2 \leq 7$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

13. Maximize $Z = 4x_1 + 3x_2$
subject to the constraints
$$x_1 + 2x_2 \leq 4$$
$$2x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

14. Maximize $Z = 3x_1 + 2x_2$
subject to the constraints
$$2x_1 + x_2 \leq 60$$
$$x_1 \leq 25$$
$$x_2 \leq 35$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

15. Maximize $Z = 3x_1 + 5x_2$
subject to the constraints
$$2x_1 + 4x_2 \leq 25$$
$$x_1 \leq 8$$
$$x_2 \leq 5$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

16. Minimize $Z = 3x_1 + 2.5x_2$
subject to the constraints
$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

17. Maximize $Z = 2x_1 + 3x_2$
subject to the constraints
$$x_1 + 3x_2 \leq 9$$
$$3x_1 + x_2 \leq 7$$
$$x_1 - x_2 \leq 1$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

18. Maximize $Z = 7x_1 + 6x_2$
subject to the constraints
$$2x_1 + 3x_2 \leq 12$$
$$6x_1 + 5x_2 \leq 30$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

19. Maximize $Z = 5x_1 + 4x_2$
subject to the constraints
$$x_1 + x_2 \geq 2$$
$$5x_1 + 3x_2 \leq 15$$
$$3x_1 + 5x_2 \leq 15$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

20. Maximize $Z = 3x_1 + 2x_2 + 5x_3$
subject to the constraints
$$5x_1 + 3x_2 + 7x_3 \leq 28$$
$$4x_1 + 5x_2 + 5x_3 \leq 30$$
$$x_1, x_2, x_3 \geq 0 \text{ and are integers.}$$

21. Maximize $Z = 3x_1 + 6x_2 + 8x_3$
subject to the constraints
$$5x_1 + 4x_2 + 8x_3 \leq 24$$
$$2x_1 + 4x_2 - x_3 \leq 7$$
$$x_1, x_2, x_3 \geq 0 \text{ and } x_3 \text{ is an integer.}$$

ANSWERS

1. $x_1 = 1, x_2 = 1, Z^* = 4$ $[x_1 = 0, x_2 = 2, Z^* = 4]$
2. $x_1 = 1, x_2 = 2, Z^* = -8$
3. $x_1 = 3, x_2 = 2, Z^* = 5$, $[x_1 = 4, x_2 = 1, Z^* = 5]$
4. $x_1 = 4, x_2 = 3, Z^* = 10$
5. $x_1 = 0, x_2 = 3, Z^* = 12$
6. $x_1 = 4, x_2 = 3, Z^* = 55$
7. $x_1 = 9, x_2 = 2, Z^* = 101$
8. $x_1 = 2, x_2 = 0, x_3 = 2, Z^* = -16$
9. $x_1 = 1/2, x_2 = 3, Z^* = -17/2$
10. $x_1 = 8, x_2 = 5/3, Z^* = 95$

11. $x_1 = 2, x_2 = 1, x_3 = 6, Z^* = 26$
12. $x_1 = 1, x_2 = 1, Z^* = 37$
13. $x_1 = 3, x_2 = 0, Z^* = 12$
14. $x_1 = 13, x_2 = 34, Z^* = 107$
15. $x_1 = 8, x_2 = 2, Z^* = 34$
16. $x_1 = 14, x_2 = 4, Z^* = 52$
17. $x_1 = 0, x_2 = 3, Z^* = 9$
18. $x_1 = 5, x_2 = 0, Z^* = 35$
19. $x_1 = 3, x_2 = 0, Z^* = 15$
20. $x_1 = 0, x_2 = 0, x_3 = 4, Z^* = 20$
21. $x_1 = 0, x_2 = 2, x_3 = 2, Z^* = 28$

9.4 BRANCH AND BOUND METHOD

Branch and bound method can be used to solve all integer as well as mixed integer programming problems. We divide the entire feasible solution space into smaller parts and then find an optimal solution for each sub-problem. We start by imposing feasible upper and lower bounds for the decision variables in each sub-problem and thus the number of iterations are reduced by discarding those sub-problems having non-integer and infeasible solutions. The integer valued solution which gives the best optimum value for the objective function is selected as the optimal solution of the given IPP.

A disadvantage of this method is that it requires the optimal solution of each sub-problem. Whenever the optimum value of the objective function of a sub-problem is less than (for maximization problem) that of the best available solution (already obtained) then this sub-problem is called fathomed and is dropped from consideration.

9.4.1 Step-by-step Procedure

Step 1 Solve the given LPP ignoring the integer restrictions. If the optimal solution satisfies the integer restrictions then the optimal integer solution is obtained. Stop. Otherwise go to step 2.

Step 2 Let the optimum Z be Z_1. Let x_k be the basic variable having largest fractional value. We add two mutually exclusive constraints.

$$x_k \le [x_k] \quad \text{and} \quad x_k \ge [x_k] + 1$$

This gives rise to two sub-problems

(1) Maximize $Z = \sum_{j=1}^{n} c_j x_j$

subject to $\sum a_{ij} x_j = b_i$

$x_k \le [x_k]$

$x_j \ge 0$

(2) Maximize $Z = \sum c_j x_j$

subject to $\sum a_{ij} x_j = b_i$

$x_k \ge [x_k] + 1$

$x_j \ge 0$

Step 3 Obtain the optimal solutions of the sub-problems (1) and (2). Optimum values of Z obtained be Z_2 and Z_3. The best integer solution value becomes the lower bound say Z_L.

Step 4 If any sub-problem has an infeasible solution delete it. If a sub-problem has feasible but non-integer solution return to step 2. If a sub-problem yields an integer solution, equal to Z_1, then optimal solution is reached. If this value is not equal to the upper bound but greater than Z_L then take this value as the new upper bound. Return to step 2. Finally if it is less than Z_L, then terminate this branch.

Step 5 This procedure of branching and bounding continues until no further sub-problem remains to be examined. Now the integer solution corresponding to the current lower bound is the optimal integer solution of the given problem.

Example 9.6 Use the branch and bound method:

Maximize $Z = 2x_1 + 2x_2$

subject to the constraints

$5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$ and are integers

Solution The standard form of the problem is

Maximize $Z = 2x_1 + 2x_2 + 0s_1 + 0s_2$

subject to the constraints

$5x_1 + 3x_2 + s_1 = 8$

$$x_1 + 2x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Z		x_1	x_2	s_1	s_2	X_B
C_B	B	2	2	0	0	
0	s_1	$\boxed{5}$	3	1	0	8 ←
0	s_2	1 ↑	2	0	1	4
	$\bar{c}_j$	–2	–2	0	0	0
		$R_1/5,\ R_2 - R_1/5$				
2	x_1	1	3/5	1/5	0	8/5
0	s_2	0	$\boxed{7/5}$	–1/5	1	12/5 ←
	$\bar{c}_j$	0	–4/5 ↑	2/5	0	16/5
		$R_1 - (3/7)R_2,\ R_2(5/7)$				
2	x_1	1	0	2/7	–3/7	4/7
2	x_2	0	1	–1/7	5/7	12/7
	$\bar{c}_j$	0	0	2/7	4/7	32/7 (Optimal)

The non-integer optimal solution is

$$x_1 = 4/7,\ x_2 = 12/7,\ Z^* = 32/7$$

x_2 has the maximum fractional part 5/7

$$x_2 = 12/7, \quad 1 < x_2 < 2$$

Take two constraints

$$x_2 \leq 1 \quad \text{and} \quad x_2 \geq 2$$

Sub-problem (1)

Maximize $Z = 2x_1 + 2x_2$

subject to the constraints

$$5x_1 + 3x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Sub-problem (2)

Maximize $Z = 2x_1 + 2x_2$

subject to the constraints

$$5x_1 + 3x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Solution of sub-problem (1) (by graphical method)

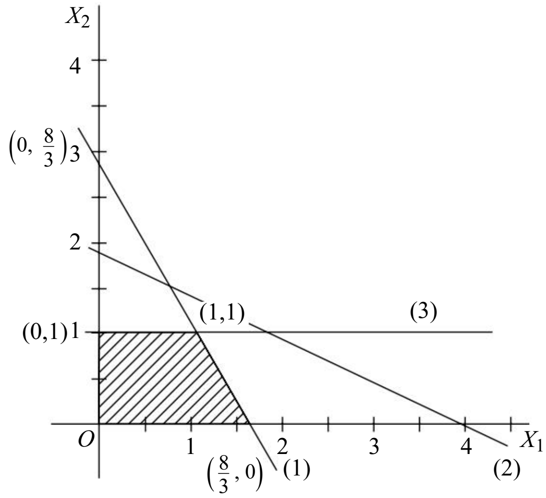


Fig. 9.1

The solution is $x_1 = 1, x_2 = 1, Z^* = 4$ which is an integer solution. The lower bound of Z is 4. No sub-problem is to be taken. It is fathomed.

Solution of the sub-problem (2)

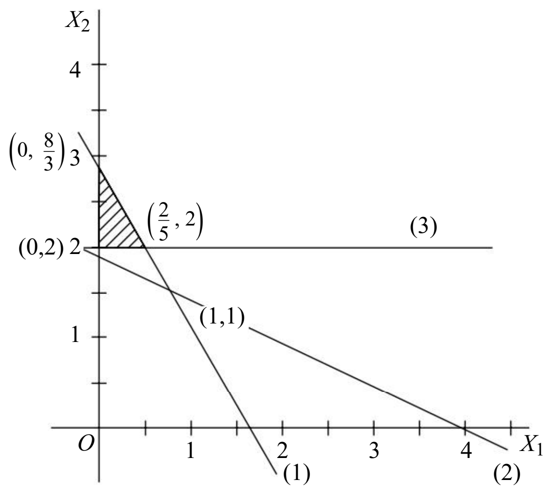


Fig. 9.2

The optimal integer solution is

$x_1 = 0, x_2 = 2, Z^* = 4$. It is fathomed. Hence the integer solution is given by

$x_1 = 1, x_2 = 1, Z^* = 4$. (or) $x_1 = 0, x_2 = 2, Z^* = 4$

Example 9.7 (Use branch and bound method)

Minimize $Z = 3x_1 + 2.5\,x_2$

subject to the constraints

$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$

x_1, x_2 are non-negative integers.

Solution We obtain a non-integer optimal solution by graphical method.

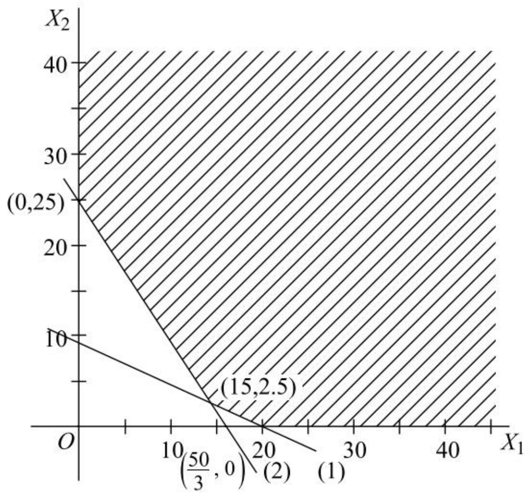


Fig. 9.3

The solution is

$x_1 = 15$, $x_2 = 2.5$, $Z_1 = 51.25$

$Z_1 = 51.25$ is the lower bound. Z value in the subsequent sub-problems cannot be less than 51.25.

$x_2(= 2.5)$ is the only non-integer value. Hence we get two sub-problems (1) and (2).

Sub-problem (1)

Minimize $Z = 3x_1 + 2.5x_2$

subject to the constraints

$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$
$$x_2 \leq 2$$

x_1, x_2 are non-negative integers.

Sub-problem (2)

Minimize $Z = 3x_1 + 2.5x_2$

subject to the constraints

$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$
$$x_2 \geq 3$$

x_1, x_2 are non-negative integers.

Solution of sub-problem (1) (Graphical method)

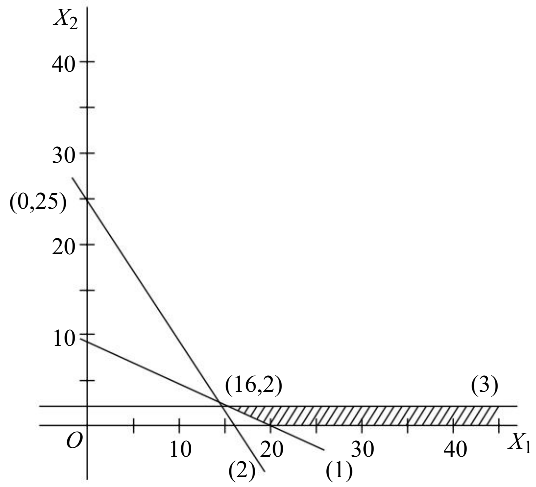


Fig. 9.4

The solution is $x_1 = 16$, $x_2 = 2$, $Z_2 = 53$

Solution of sub-problem (2) (Graphical method)

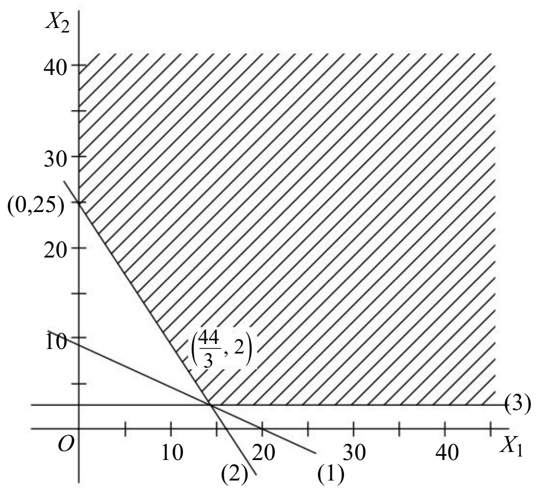


Fig. 9.5

The solution is $x_1 = 44/3$, $x_2 = 3$, $Z_3 = 51.5$

Since the solution of sub-problem (1) is all integer no more sub-division is necessary.

$Z_3 = 51.5$ is the new lower bound.

Also $Z_3 < Z_2$.

Hence we have to subdivide problem (2) into problem (3) and (4).

Sub-problem (3)

Minimize $Z = 3x_1 + 2.5x_2$

subject to the constraints

$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$
$$x_2 \geq 3$$
$$x_1 \leq 14$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Sub-problem (4)

Minimize $Z = 3x_1 + 2.5x_2$

subject to the constraints

$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$
$$x_2 \geq 3$$
$$x_1 \geq 15$$
$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Solution of problem (3)

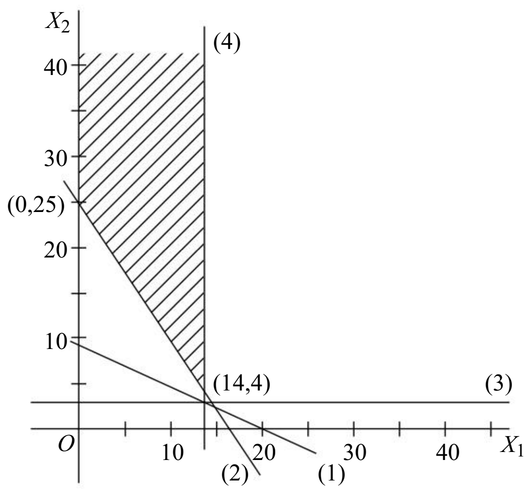


Fig. 9.6

Solution is $x_1 = 14$, $x_2 = 4$, $Z_4 = 52$

Solution of Sub-problem (4)

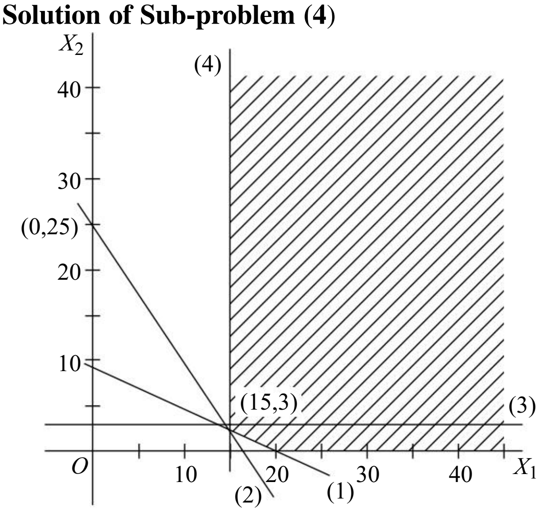


Fig. 9.7

The solution is $x_1 = 15$, $x_2 = 3$, $Z_5 = 52.5$

Both problems (3) and (4) have all integer solutions and therefore branch and bound algorithm is terminated.

The optimal all integer solution is given by sub-problem (3), which is $x_1 = 14$, $x_2 = 4$, $Z_4 = 52$

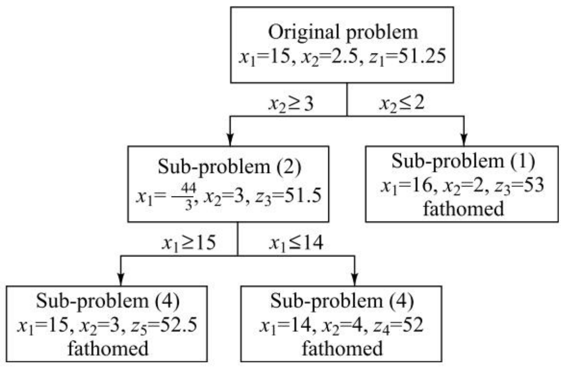


Fig. 9.8

Example 9.8 Branch and Bound Method

Maximize $Z = 7x_1 + 9x_2$

subject to the constraints

$$-x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 35$$
$$x_1 \leq 7$$
$$x_2 \leq 7$$
$$x_1, x_2 \text{ are non-negative integers.}$$

Solution $x_1 = 9/2, x_2 = 7/2, Z_1 = 63$

Sub-problem (1)

Maximize $Z = 7x_1 + 9x_2$

subject to the constraints

$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$x_2 \le 7$$
$$x_1 \le 4$$

x_1, x_2 are non-negative integers.

The solution is $x_1 = 4, x_2 = 10/3, Z_2 = 58$

Sub-problem (2)

Maximize $Z = 7x_1 + 9x_2$

subject to the constraints

$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$x_2 \le 7$$
$$x_1 \ge 5$$
$$x_1 \le 7$$

x_1, x_2 are non-negative integers.

The solution is

$x_1 = 5, x_2 = 0, Z_3 = 35$ (fathomed).

Sub-problem (3)

Maximize $Z = 7x_1 + 9x_2$

subject to the constraints

$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$x_1 \le 4$$
$$x_2 \le 3$$

x_1, x_2 are non-negative integers.

The solution is $x_1 = 4, x_2 = 3, Z_4 = 55$

Sub-problem (4)

Maximize $Z = 7x_1 + 9x_2$

subject to the constraints

$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$x_1 \le 4$$
$$x_2 \ge 4$$
$$x_2 \le 7$$

This has no solution (infeasible)

$\therefore$ The optimal integer solution is

$x_1 = 4, \quad x_2 = 3, \quad Z_4 = 55$

EXERCISES

Using the branch and bound technique solve the following IPP:

1. Maximize $Z = 3x_1 + 4x_2$
   subject to the constraints
   $$3x_1 + 2x_2 \le 8$$
   $$x_1 + 4x_2 \le 10$$
   $x_1, x_2 \ge 0$ and are integers.
2. Maximize $Z = 110x_1 + 100x_2$
   subject to the constraints
   $$6x_1 + 5x_2 \le 29$$
   $$2x_1 + 7x_2 \le 24$$
   $x_1, x_2 \ge 0$ and are integers.
3. Maximize $Z = 2x_1 + 3x_2$
   subject to the constraints
   $$x_1 + 3x_2 \le 9$$
   $$3x_1 + x_2 \le 7$$
   $$x_1 - x_2 \le 1$$
   $x_1, x_2 \ge 0$ and are integers.
4. Minimize $Z = 9x_1 + 10x_2$
   subject to the constraints
   $$x_1 \le 9$$
   $$x_2 \le 8$$
   $$4x_1 + 3x_2 \ge 40$$
   $x_1, x_2 \ge 0$ and are integers.
5. Minimize $Z = x_1 + 4x_2$
   subject to the constraints
   $$2x_1 + 4x_2 \le 7$$
   $$5x_1 + 3x_2 \le 15$$
   $x_1, x_2 \ge 0$ and are integers.
6. Maximize $Z = x_1 + x_2$
   subject to the constraints
   $$4x_1 - x_2 \le 10$$

$$2x_1 + 5x_2 \le 10$$
$$2x_1 - 3x_2 \le 6$$
$$x_1, x_2 \ge 0 \text{ and are integers.}$$

7. Maximize $Z = x_1 + 2x_2$
   subject to the constraints
   $$2x_1 \le 7$$
   $$x_1 + x_2 \ge 7$$
   $$2x_1 \ge 11$$
   $$x_1, x_2 \ge 0 \text{ and are integers.}$$
8. Maximize $Z = 4x_1 + 3x_2$
   subject to the constraints
   $$x_1 + 2x_2 \le 4$$
   $$2x_1 + x_2 \le 6$$
   $$x_1, x_2 \ge 0 \text{ and are integers.}$$
9. Maximize $Z = 3x_1 + 2x_2$
   subject to the constraints
   $$2x_1 + x_2 \le 60$$
   $$x_1 \le 25$$
   $$x_2 \le 35$$
   $$x_1, x_2 \ge 0 \text{ and are integers.}$$
10. Maximize $Z = 20x_1 + 17x_2$
    subject to the constraints
    $$4x_1 + 3x_2 \le 7$$
    $$x_1 + x_2 \le 4$$
    $$x_1, x_2 \ge 0 \text{ and are integers.}$$

ANSWERS

1. $x_1 = 0, x_2 = 3, Z^* = 12$
2. $x_1 = 4,\ x_2 = 1, Z^* = 540$
3. $x_1 = 0,\ x_2 = 3, Z^* = 9$
4. $x_1 = 9,\ x_2 = 2, Z^* = 101$
5. $x_1 = 1,\ x_2 = 1, Z^* = 5$
6. $x_1 = 2,\ x_2 = 1, Z^* = 3$
7. $x_1 = 4, x_2 = 3, Z^* = 10$
8. $x_1 = 3,\ x_2 = 0,\ Z^* = 12$
9. $x_1 = 13, x_2 = 34, Z^* = 107$
10. $x_1 = 1, x_2 = 1, Z^* = 37$

10

Sensitivity Analysis

CONCEPT REVIEW

In a linear programming problem we have the values of cost coefficient c_j, availability of resources b_i and constraint coefficient a_{ij} which contribute very much in arriving at an optimal solution. If there is any change or error in any of these parameters it gives rise to a change in the optimal solution. Also it may sometimes be required to analyse the effect of changes in these parameters on the optimal solution. Such changes may be discrete or continuous. A study of the effect on the optimal solution due to discrete changes in the parameters is called *sensitivity analysis* or *post-optimality analysis*. In the case of continuous changes it is called *parametric programming*. Here we shall discuss the sensitivity analysis only. In most of the cases it is not necessary to solve the problem again. Some simple calculations applied to the current optimal solution will give the desired result.

There are five types of changes which are possible in an LPP:

(i) change in c_j
(ii) change in b_i
(iii) change in a_{ij}
(iv) addition of a new variable
(v) addition of a new constraint

10.1 DISCRETE CHANGES IN c_j

Let the cost coefficient c_k be changed to $c_k + \Delta c_k$. Two cases arise:

Case (I) c_k does not belong to C_B.

In this case the relative cost factor $\bar{c}_k$ becomes $z_k - (c_k + \Delta c_k)$. Hence the current solution continues to be optimal if

$$Z_k - (c_k + \Delta c_k) \geq 0 \quad \text{or} \quad Z_k - C_k \geq \Delta c_k$$

Case (II) $c_k \in C_B$

$$\text{Now } \bar{c}_j{}' = \sum_{i=1}^{m} C_{Bi} a_{ij} - c_j{}'$$

$$\bar{c}_j{}' = \sum_{i \neq k} C_{Bi} a_{ij} + (c_k + \Delta c_k) a_{kj} - c_j$$

$$= \sum_{i=1}^{m} C_{Bi} a_{ij} + \Delta c_k a_{kj} - c_j$$

$$= Z_j - c_j + \Delta c_k a_{kj}$$

Hence the current solution continues to be optimal if $Z_j - c_j + \Delta c_k a_{kj} \geq 0$

i.e. $\Delta c_k \, a_{kj} \geq -(Z_j - c_j)$

i.e. $\Delta c_k \geq -(Z_j - c_j)/a_{kj}$

i.e. $\underset{a_{kj}>0}{\text{Max}} \{-(Z_j - c_j)/a_{kj}\} \le \Delta c_k \le \underset{a_{kj}<0}{\text{Min}} \{-(Z_j - c_j)/a_{kj}\}$

Example 10.1 Consider the LPP

Minimize $Z = 3x_1 + 5x_2$ subject to

$$3x_1 + 2x_2 \le 18$$
$$x_1 \le 14$$
$$x_2 \le 6$$
$$x_1, x_2 \ge 0$$

Discuss the change in c_j on the optimal solution of the problem.

Solution The optimal solution of the problem (using the simplex method) is

$$x_1 = 2, x_2 = 6, Z^* = 36$$

The optimal table is

Table 10.1

		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	3	5	0	0	0	
3	x_1	1	0	1/3	0	–2/3	2
0	s_2	0	0	–2/3	1	4/3	0
5	x_2	0	1	0	0	1	6
	$\bar{c}_j$	0	0	1	0	3	36

For the basic variable $x_1 (k = 1)$ we get the condition

$\underset{a_{1j}>0}{\text{Max}} \{-(Z_j - c_j)/a_{1j}\} \le \Delta c_1 \le \underset{a_{1j}<0}{\text{Min}} \{-(Z_j - c_j)/a_{1j}\}$

$-1/(1/3) \le \Delta c_1 \le -3/(-2/3)$ $\qquad a_{13} = 1/3$

$-3 \le \Delta c_1 \le 9/2$ $\qquad a_{15} = -2/3$

But $a_{13} = 1/3$ and $a_{15} = -2/3$ correspond to the non-basic variables s_1 and s_3 and hence the current optimal solution remains the same so long as

$$c_1 \le 3 + 9/2 \text{ and } c_1 \ge 3 - 3 \text{ since } c_1 = 3$$

i.e. $15/2 \ge c_1 \ge 0$

For the other basic variable x_2, $k = 3$ we get

$-3/1 \le \Delta c_2 \le -1/-0$

$-3 \le \Delta c_2 \le \infty$ and $c_2 = 5$

$\therefore$ $c_2 \le 5 + \infty$ and $c_2 \ge 5 - 3$

i.e. $2 \le c_2 \le \infty$

Thus the optimal solution is unaltered if $0 \le c_1 \le 15/2$ and $2 \le c_2 < \infty$

Example 10.2 Consider the LPP

Maximize $Z = 4x_1 + 6x_2 + 2x_3$ subject to the constraints

$$x_1 + x_2 + x_3 \le 3$$
$$x_1 + 4x_2 + 7x_3 \le 9$$
$$x_1, x_2, x_3 \ge 0$$

Find the effect on the optimality of the current solution when the objective function is changed to $3x_1 + 8x_2 + 5x_3$

Solution The optimal solution table obtained by the simplex method is

Table 10.2

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	4	6	2	0	0	
4	x_1	1	0	–1	4/3	–1/3	1
6	x_2	0	1	2	–1/3	1/3	2
	$\bar{c}_j$	0	0	6	10/3	2/3	16

Replacing the objective function with the new one, we have

Table 10.3

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	3	8	5	0	0	
3	x_1	1	0	–1	4/3	–1/3	1
8	x	0	1	2	–1/3	1/3	2
	$\bar{c}_j$	0	0	8	4/3	5/3	19

$\bar{c}_j = Z_j - c_j \ge 0 \ \forall j$. Therefore the optimal solution remains the same

$$x_1 = 1, x_2 = 2, Z^* = 19$$

Note: We find that changes in the objective coefficients c_j do not affect the feasibility of the LPP.

Example 10.3 Consider the LPP

Maximize $Z = 2x_1 + x_2 + 4x_3 - x_4$ subject to the constraints

$$x_1 + 2x_2 + x_3 - 3x_4 \le 8$$
$$-x_2 + x_3 + 2x_4 \le 0$$
$$2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Discuss the effect of changing $(c_1\ c_2\ c_3\ c_4)$ to $(1\ 2\ 3\ 4)$ on the optimal solution.

Solution The optimal table obtained by the simplex method is

Table 10.4

		x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B
C_B	B	2	1	4	–1	0	0	0	
2	x_1	1	0	3	1	1	2	0	8
1	x_2	0	1	–1	–2	0	–1	0	0
0	s_3	0	0	–4	2	–2	3	1	5
	$\bar{c}_j$	0	0	1	1	2	3	0	16

For the new objective function the table becomes

Table 10.5

		x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B
C_B	B	1	2	3	4	0	0	0	
1	x_1	1	0	3	1	1	2	0	8
2	x_2	0	1	–1	–2	0	–1	0	0
0	s_3	0	0	–4	$\boxed{2}$	–2	3	1	5 ←
	$\bar{c}_j$	0	0	–2	–7 ↑	1	0	0	8

$R_1 - (1/2)R_3,\ R_2 + R_3,\ R_3/2$

The solution is not optimal. In the iteration x_4 enters and s_3 leaves the basis

Table 10.6

I Iteration

		x_1	x_2	x_3	x_4	s_1	s_2	s_3	X_B
C_B	B	1	2	3	4	0	0	0	
1	x_1	1	0	$\boxed{5}$	0	2	1/2	–1/2	← 11/2
2	x_2	0	1	–5	0	–2	2	1	5
4	x_4	0	0	–2	1	–1	3/2	1/2	5/2
	$\bar{c}_j$	0	0	–16 ↑	0	–6	21/2	7/2	51/2

$R_1/5,\ R_2 + R_1,\ R_3 + (2/5)R_1$

Table 10.7

II Iteration

3	x_3	1/5	0	1	0	2/5	1/10	–1/10	11/10
2	x_2	1	1	0	0	0	5/2	1/2	21/2
4	x_4	2/5	0	0	1	–1/5	17/5	3/10	47/10
	$\bar{c}_j$	16/5	0	0	0	2/5	189/10	19/10	431/10

This solution is optimal. The new optimal solution is

$$x_1 = 0,\ x_2 = 21/2,\ x_3 = 11/10,\ x_4 = 47/10,$$
$$Z^* = 431/10$$

10.2 DISCRETE CHANGES IN b_i

We know that the optimal solution is given by $X_B = B^{-1}\,b$ where B^{-1} is the matrix of coefficients corresponding to the slack variables in the optimal table.

If the resource b_k is changed to $b_k + \Delta b_k$ then $b' = (b_1, b_2, \ldots.. b_k + \Delta b_k, \ldots, b_m)^T$

$$\therefore\quad b' = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_k \\ . \\ b_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Delta b_k \\ 0 \\ 0 \end{pmatrix} = b + \Delta b$$

$$\text{Now}\quad X_B' = B^{-1}(b + \Delta b)$$
$$= B^{-1}\,b + B^{-1}\,\Delta b$$
$$= X_B + B^{-1}\begin{pmatrix} 0 \\ 0 \\ . \\ \Delta b_k \\ 0 \\ 0 \end{pmatrix}$$
$$= X_B + \beta_k(\Delta b_k)$$

where β_k is the k^{th} row of B^{-1}

The new value of Z is

$$Z' = C_B X_B'$$
$$= C_B X_B + C_B B^{-1}\Delta b_k$$
$$= Z + C_B B^{-1}(\Delta b_k)$$

Note: If we get b_i negative then we can apply the dual simplex method.

Example 10.4 Consider the LPP

Maximize $Z = 4x_1 + 6x_2 + 2x_3$ subject to the constraints

$$x_1 + x_2 + x_3 \le 3$$
$$x_1 + 4x_2 + 7x_3 \le 9$$
$$x_1, x_2, x_3 \ge 0$$

Discuss the effect on the optimal solution when the constants are changed from (3, 9) to (9, 6).

Solution The optimal solution obtained by the simplex method is

Table 10.8

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	4	6	2	0	0	
4	x_1	1	0	–1	4/3	–1/3	1
6	x_2	0	1	2	–1/3	1/3	2
	$\bar{c}_j$	0	0	6	10/3	2/3	16

$x_1 = 1, x_2 = 2, x_3 = 0, Z^* = 16$

If new values of the constants are (9, 6) then in the optimal table the new values of x_1 and x_2 are given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$$

Since x_2 is negative we apply the dual simplex method

Table 10.9

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	4	6	2	0	0	
4	x_1	1	0	–1	4/3	–1/3	10
6	x_2	0	1	2	$\boxed{-1/3}$	1/3	–1
	$\bar{c}_j$	0	0	6	10/3	2/3	34

$R_1 + 4R_2, R_2(-3)$

4	x_1	1	4	7	0	1	6
0	S_1	0	–3	–6	1	–1	3
	$\bar{c}_j$	0	10	26	0	4	24

The new optimal solution is $x_1 = 6$, $x_2 = 0$, $x_3 = 0$, $Z^* = 24$.

Example 10.5 Consider the LPP

Maximize $Z = 5x_1 + 12x_2 + 4x_3$ subject to the constraints

$$x_1 + 2x_2 \le 5$$
$$5x_1 - x_2 + 2x_3 = 2$$
$$x_1, x_2, x_3 \ge 0$$

Analyze the effect of changing the requirement vector from $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ on the optimal solution.

Solution By using the big M method the following optimal solution is obtained

Table 10.10

		x_1	x_2	x_3	s_1	s_2	X_B
C_B	B	5	12	4	0	–M	
12	x_2	0	1	–1/5	2/5	–1/5	8/5
5	x_1	1	0	7/5	1/5	2/5	9/5
	$\bar{c}_j$	0	0	3/5	29/5	M–2/5	141/5

The solution is $x_1 = 9/5$, $x_2 = 8/5$, $x_3 = 0$, $Z^* = 141/5$

If the requirement vector becomes $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then

$$X_B = B^{-1}b = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 11/5 \end{pmatrix}$$

Since $x_1 \ge 0$ and $x_2 \ge 0$ this solution is feasible.

The new optimal solution is

$$x_1 = 11/5, x_2 = 12/5, x_3 = 0$$
$$Z^* = 5(11/5) + 12(12/5) + 4(0) = 199/5.$$

10.3 CHANGES IN THE MATRIX ELEMENTS a_{ij}

Let the coefficient a_{rk} be changed to $a_{rk} + \Delta a_{rk}$. Two cases arise:

(i) column a_k is a non-basic column.

(ii) column a_k is basic.

Case (I) If a_k is non-basic (the column a_k is not in the basis matrix) we find $X_B = B^{-1}b$ and hence the only effect of the change is on the optimality condition. The optimality condition is $Z_k' - c_k \ge 0$, where $Z_k' = C_B B^{-1} a_k'$

$$\therefore \quad Z_k' = C_B B^{-1}(a_k + \Delta a_k)$$
$$= C_B B^{-1} a_k + C_B B^{-1}(\Delta a_{rk})$$
$$=> Z_k + C_B \beta_k (\Delta a_{rk})$$
$$Z_k' - c_k \ge 0 => Z_k + C_B\beta_k (\Delta a_{rk}) - c_k \ge 0$$

i.e. Max $\{-(Z_k - c_k)/C_B\beta_k;\ C_B b_k > 0\} \le \Delta a_{rk} \le$ Min $\{-(Z_k - c_k)/C_B\beta_k;\ C_B b_k > 0\}$

If this condition is satisfied then the current solution continues to be optimal.

Case (II) If a_k is basic, we define

$$\lambda = (Z_j - c_j)\,\beta_{kk} - y_{kj}\, C_B\beta_k$$

The condition for the current optimal solution to remain optimal, is

$$\max_{\lambda > 0} \{-(Z_j - c_j)/\lambda\} \le \Delta a_{rk} = \min_{\lambda < 0} \{-(Z_j - c_j)/\lambda\}$$

Example 10.6 Consider the LPP

Maximize $Z = 5x_1 + 12x_2 + 4x_3$ subject to the constraints

$$x_1 + 2x_2 + x_3 \le 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \ge 0$$

Discuss the effect of changing the column

$$a_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ to } \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

on the optimal solution.

Solution Applying big-M method we get the optimal solution as

Table 10.11

		x_1	x_2	x_3	s_1	R_1	X_B
C_B	B	5	12	4	0	$-M$	
12	x_2	0	1	–1/5	2/5	–1/5	8/5
5	x_1	1	0	7/5	1/5	2/5	9/5
	$\bar{c}_j$	0	0	3/5	29/5	M–2/5	141/5

When the x_3 column is changed to $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ the new 3rd column is

$$\begin{pmatrix} a_{13}' \\ a_{23}' \end{pmatrix} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -12/5 \\ -1/5 \end{pmatrix}$$

The table becomes

Table 10.12

		x_1	x_2	x_3	s_1	R_1	X_B
C_B	B	5	12	4	0	$-M$	
12	x_2	0	1	–12/5	2/5	–1/5	8/5
5	x_1	1	0	–1/5	1/5	2/5	9/5
	C_j	0	0	–169/5	29/5	M–2/5	141/5

All the elements of the pivot column are negative and hence the problem has unbounded solution.

10.4 ADDITION OF A VARIABLE

Suppose we add a new variable with coefficients column a_{n+1} and cost c_{n+1} in the objective function.

We compute

$$y_{n+1} = B^{-1} a_{n+1}$$

and $$Z_{n+1} - c_{n+1} = C_B\, y_{n+1} - c_{n+1}$$

If $Z_{n+1} - c_{n+1} \ge 0$ the current solution continues to be optimal.

If $Z_{n+1} - c_{n+1} < 0$ apply the simplex method and obtain a new optimal solution.

Example 10.7 Consider the LPP

Maximize $Z = 3x_1 + 5x_2$ subject to the constraints

$$3x_1 + 2x_2 \le 18$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_1, x_2 \ge 0$$

Discuss the change in the optimal solution when a new variable x_3 is introduced with cost coefficient 2 and

$$a_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution Introducing slack variables s_1, s_2, s_3 and applying the simplex method we get the optimal solution.

Table 10.13

		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	3	5	0	0	0	
3	x_1	1	0	1/3	0	–2/3	2
0	s_2	0	0	–2/3	1	4/3	0
5	x_2	0	1	0	0	1	6
	C_j	0	0	1	0	3	36

The solution is $x_1 = 2$, $x_2 = 6$, $Z^* = 36$

After the addition of a new variable x_3 the problem becomes

Maximize $Z = 3x_1 + 5x_2 + 2x_3$ subject to the constraints

$$3x_1 + 2x_2 + x_3 \le 18$$
$$x_1 + x_3 \le 4$$
$$x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

The coefficient column of x_3 in the optimal table is given by

$$y_3 = B^{-1}a_3$$

$$= \begin{pmatrix} 1/3 & 0 & -2/3 \\ -2/3 & 1 & 4/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 5/3 \\ 1 \end{pmatrix}$$

$$Z_3 - c_3 = C_B y_3 - c_3$$

$$= \begin{pmatrix} 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} -1/3 \\ 5/3 \\ 1 \end{pmatrix} - 2 > 0$$

Hence the current solution itself is optimal.

10.5 ADDITION OF A CONSTRAINT

If a new constraint $a_{m+1,1}\ x_1 + a_{m+1,2}\ x_2 + \ldots + a_{m+1,n}\ x_n \leq b_{m+1}$ is added to the system of constraints of an LPP then two cases arise.

Case (I) The optimal solution of the original problem satisfies the new constraint. In this case the current optimal solution continues to be optimal.

Case (II) The optimal solution X_B of the original problem does not satisfy the new constraint. In this case we obtain a new optimal solution to the modified problem. Let B be the basis matrix for the original problem and B_1 be the basis matrix for the new problem with $m + 1$ constraints.

$$\therefore \quad B_1 = \begin{pmatrix} B & 0 \\ \alpha & 1 \end{pmatrix}$$

where $\alpha = (a_{m+1,1}\ a_{m+1,2} \ldots a_{m+1,n})$ and the 2nd column $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to the slack or surplus variable added in the new constraint.

$$\text{Now} \quad B_1^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -\alpha B^{-1} & 1 \end{pmatrix}$$

$$\text{and} \quad y_j^* = B_1^{-1} a_j^*$$

$$= \begin{pmatrix} B^{-1} & 0 \\ -\alpha B^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_j \\ a_{m+1,\ j} \end{pmatrix}$$

$$= \begin{pmatrix} B^{-1}a_j \\ -\alpha B^{-1} \quad a_j + a_{m+1,\ j} \end{pmatrix}$$

$$= \begin{pmatrix} y_j \\ a_{m+1,\ j} - \alpha y_j \end{pmatrix}$$

Now $Z_j^* - c_j = Z^* y_j^* - c_j$

$$= \begin{bmatrix} C_B & C_{Bm+1} \end{bmatrix} \begin{pmatrix} y_j \\ a_{m+1,\ j} - \alpha y_j \end{pmatrix} - c_j$$

$$= C_B y_j + C_{Bm+1}\ a_{m+1,\ j} - C_{B\ m+1}\ \alpha\, y_j - c_j$$

where C_{Bm+1} is the cost associated with the new variable introduced in the basis. Since the new variable is a slack or surplus variable its coefficient in the objective function is zero. Therefore, $C_{Bm+1} = 0$

$$\therefore \quad Z_j^* - c_j = C_B y_j - c_j = Z_j - c_j$$

Therefore the values of $\bar{c}_j$ are the same for the new problem also.

$$\text{Also} \quad Z^* = \begin{bmatrix} C_B & 0 \end{bmatrix} \begin{pmatrix} X_B \\ s \end{pmatrix} = C_B X_B = Z$$

Thus the optimum value remains unchanged. Since the optimal solution of the original problem does not satisfy the new constraints the slack or surplus variable possesses negative value. Hence we have to use the dual simplex method and obtain the new optimal solution.

Note: If the new constraint is an equation then an artificial variable appears in the basis at a positive value and Big-M method can be applied.

Example 10.8 Consider the LPP

Maximize $Z \leq 3x_1 + 5x_2$ subject to the constraints

$$3x_1 + 2x_2 \leq 18$$
$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Discuss the effect of adding a new constraint $2x_1 + x_2 \leq 8$ to the given set of constraints.

Solution The given problem has an optimal simplex table as given below.

Table 10.14

		x_1	x_2	s_1	s_2	s_3	X_B
C_B	B	3	5	0	0	0	
3	x_1	1	0	1/3	0	–2/3	2
0	s_2	0	0	–2/3	1	4/3	0
5	x_2	0	1	0	0	1	6
	$\bar{c}_j$	0	0	1	0	3	36

The optimal solution is $x_1 = 2$, $x_2 = 6$, $Z^* = 36$.

The new constraint is $2x_1 + x_2 \le 8$.

This constraint is not satisfied by the current solution. Hence we add a slack variable s_4 to this constraint. The new table is

Table 10.15

		x_1	x_2	s_1	s_2	s_3	s_4	X_B
C_B	B	3	5	0	0	0	0	
3	x_1	1	0	1/3	0	–2/3	0	2
0	s_2	0	0	–2/3	1	4/3	0	0
5	x_2	0	1	0	0	1	0	6
0	s_4	2	1	0	0	0	1	8
	$\bar{c}_j$	0	0	1	0	3	0	36

The basis matrix has been disturbed due to the 4th row. a_{41} and a_{42} coefficients must be reduced to zero. Take

$$R_4^1 = R_4 - R_3 - 2R_1$$

The table becomes

Table 10.16

		x_1	x_2	s_1	s_2	s_3	s_4	X_B
C_B	B	3	5	0	0	0	0	
3	x_1	1	0	1/3	0	–2/3	0	2
0	s_2	0	0	–2/3	1	4/3	0	0
5	x_2	0	1	0	0	1	0	6
0	s_4	0	0	$\boxed{-2/3}$	0	1/3	1	–2
	$\bar{c}_j$	0	0	1	0	3	0	36

s_4 leaves and s_1 enters the basis. (Apply the dual simplex method)

Take $R_1^{\,1} = R_1 + (1/2)R_4$

$R_2^{\,1} = R_2 - R_4$

$R_4^{\,1} = R_4\,(-3/2)$

The new table becomes

Table 10.17

		x_1	x_2	s_1	s_2	s_3	s_4	X_B
C_B	B	3	5	0	0	0	0	
3	x_1	1	0	0	0	–1/2	1/2	1
0	s_2	0	0	0	1	1	–1	2
5	x_2	0	1	0	0	1	0	6
0	s_1	0	0	1	0	–1/2	–3/2	3
	C_j	0	0	0	0	7/2	3/2	33

The new optimal solution is

$x_1 = 1$, $x_2 = 6$, $Z^* = 33$

Note that the introduction of a new constraint has reduced the optimum Z from 36 to 33.

EXERCISES

1. In the LPP
   Maximize $Z = 15x_1 + 45x_2$ subject to the constraints
   $$x_1 + 16x_2 \le 250$$
   $$5x_1 + 2x_2 \le 162$$
   $$x_2 \le 50$$
   $$x_1, x_2 \ge 0$$
   If c_2 is kept fixed at 45 determine how much c_1 can be changed without affecting the optimal solution?
2. For the LPP
   Maximize $Z = -x_1 + 2x_2 - x_3$ subject to the constraints
   $$3x_1 + x_2 - x_3 \le 10$$
   $$-x_1 + 4x_2 + x_3 \ge 6$$
   $$x_2 + x_3 \le 4$$
   $$x_1, x_2, x_3 \ge 0$$

Determine the ranges for discrete changes in b_1 and b_2 in order to maintain the optimality of the current solution.

3. Consider the problem,
Maximize $Z = 3x_1 + 5x_2$ subject to the constraints
$$3x_1 + 2x_2 \le 18$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_1, x_2 \ge 0$$
Discuss the effect on the optimality of the solution when the objective function is changed to $3x_1 + x_2$.

4. Consider the LPP
Maximize $Z = 3x_1 + 5x_2$ subject to the constraints
$$x_1 \le 4$$
$$3x_1 + 2x_2 \le 18$$
$$x_1, x_2 \ge 0$$
Solve the LPP. If a new variable x_3 is added to the problem with column $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and cost coefficient 7 find the change in the optimal solution.

5. In the problem
Maximize $Z = 5x_1 + 12x_2 + 4x_3$ subject to the constraints
$$x_1 + 2x_2 + x_3 \le 5$$
$$2x_1 + x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \ge 0$$
Discuss the change in the optimal solution when a_3 is changed from $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

6. Given the LPP
Minimize $Z = 3x_1 + 8x_2$ subject to the constraints
$$x_1 + x_2 = 200$$
$$x_1 \le 80$$
$$x_2 \ge 60$$
$$x_1, x_2 \ge 0$$
discuss the effect on the optimality by adding a new variable x_3 with column coefficients $(2,2,2)^T$ and coefficient in the objective function, 5.

7. Solve the LPP
Maximize $Z = 4x_1 + 6x_2 + 2x_3$ subject to the constraints
$$x_1 + x_2 + x_3 \le 3$$
$$x_1 + 4x_2 + 7x_3 \le 9$$
$$x_1, x_2, x_3 \ge 0$$
(i) What is the new optimal solution when c_3 is increased from 2 to 12?
(ii) Find the effect of changing the objective function to $Z = 2x_1 + 8x_2 + 4x_3$

8. Consider the LPP
Maximize $Z = 45x_1 + 100x_2 + 30x_3 + 50x_4$ subject to the constraints
$$7x_1 + 10x_2 + 4x_3 + 9x_4 \le 1200$$
$$3x_1 + 40x_2 + x_3 + x_4 \le 800$$
$$x_1, x_2, x_3, x_4 \ge 0$$
Find the change in the optimal solution when a new variable x_5 is added with cost coefficient 120 and column $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$

9. Solve the LPP
Maximize $Z = 3x_1 + 5x_2$ subject to the constraints
$$x_1 \le 4$$
$$3x_1 + 2x_2 \le 18$$
$$x_1, x_2 \ge 0$$
Discuss the change in the optimal solution when
(i) a constraint $x_2 \le 10$ is added
(ii) a constraint $x_2 \le 6$ is added.

10. Solve
Maximize $Z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints
$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_2 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$x_1, x_2, x_3 \ge 0$$
Find the new optimal solution when x_1 column is changed to $(1,1,6)^T$, $c_2 = 1$ and $c_3 = 3$.

ANSWERS

1. $15 - (195/16) \leq c_1 \leq 15 + 195/16$.
2. $b_2 < 10$ $-5/2 \leq b_3 \leq 6$
3. The original optimal solution is
   $x_1 = 2, x_2 = 6, Z^* = 36$
   The new optimal solution is
   $x_1 = 2, x_2 = 6, Z^* = 12$
4. The optimal solution of the original problem is
   $x_1 = 0, x_2 = 9, Z^* = 45$
   For the modified problem, the solution is
   $x_1 = 0, x_2 = 5, x_3 = 4, Z^* = 53$
5. There is no change in the current optimal solution.
6. The original optimal solution is
   $x_1 = 80, x_2 = 120, Z^* = 1200$
   The optimal solution of the new problem is
   $x_1 = 20, x_2 = 120, x_3 = 70, Z^* = 1120$
7. The original optimal solution is
   $x_1 = 1, x_2 = 2, x_3 = 0, Z^* = 16$
   If c_3 becomes 12, the solution changes to
   $x_1 = 2, x_2 = 0, x_3 = 1, Z^* = 20$
   If the new objective function is $2x_1 + 8x_2 + 4x_3$ the solution becomes
   $x_1 = 1, x_2 = 2, x_3 = 0, Z^* = 18$
8. The original optimal solution is
   $x_1 = 0, x_2 = 40/3, x_3 = 800/3, x_4 = 0,$
   $Z^* = 28000/3$
   The new optimal solution is
   $x_1 = 0, x_2 = 0, x_3 = 400/3, x_4 = 0, x_5 = 200/3,$
   $Z^* = 12000$
9. The original optimal solution is $x_1 = 0, x_2 = 9,$
   $Z^* = 45$
   (i) There is no change in the optimal solution.
   (ii) The new solution is $x_1 = 2, x_2 = 6, Z^* = 36$.
10. The original optimal solution is $x_1 = 0,$
    $x_2 = 100, x_3 = 230, Z^* = 1350$
    The new solution is $x_1 = 4, x_2 = 99, x_3 = 228,$
    $Z^* = 795$

11

Transportation Problem

CONCEPT REVIEW

Transportation problem deals with the transportation of commodity from different sources to different destinations.

Given m sources and n destinations, the total quantity available at each source, the total quantity required at each destination, and the costs of transportation of unit commodity from each source to each destination, it is required to determine the quantity of commodity to be transported from each source to different destinations such that the total cost of transportation is minimum. This is transportation problem.

11.1 MATHEMATICAL FORMULATION

Suppose there are m sources $S_1, S_2, \ldots, S_m$ and n destinations $D_1, D_2, \ldots, D_n$. Let the cost of transporting one unit of commodity from S_i to D_j be c_{ij} and let x_{ij} be the quantity to be transported from S_i to D_j. Then the problem is

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

The total availability at the source S_i is a_i

i.e. $x_{i1} + x_{i2} + x_{i3} + \ldots + x_{in} = a_i$ (1)

$(i = 1, 2, 3, \ldots, m)$

Similarly, the total demand at the destination D_j is b_j.

i.e. $x_{1j} + x_{2j} + x_{3j} + \ldots x_{mj} = b_j$ (2)

$(j = 1, 2, 3, \ldots, n)$

Also the total availability $\sum_{i=1}^{m} a_i$ and the total demand $\sum_{j=1}^{n} b_j$ must be equal

i.e. $$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad (3)$$

These are the constraints to be satisfied. Also there is the non-negativity constraint

$x_{ij} \geq 0$ (4)

We form a rectangular transportation table where the quantity x_{ij} is put in the $(i, j)^{\text{th}}$ cell. The unit cost c_{ij} is written in a corner of the $(i, j)^{\text{th}}$ cell.

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ then the problem is called a *balanced problem*. In order to obtain a feasible solution we need a balanced problem.

Table 11.1

		D_1	D_2		D_j		D_n	Availability
	S_1	x_{11} c_{11}	x_{12} c_{12}		x_{1j} c_{1j}		x_{1n} c_{1n}	a_1
	S_2	x_{21} c_{21}	x_{22} c_{22}		x_{2j} c_{2j}		x_{2n} c_{2n}	a_2
	S_3	x_{31} c_{31}	x_{32} c_{32}		x_{3j} c_{3j}		x_{3n} x_{3n}	a_3
Source								
	S_i	x_{i1} c_{i1}	x_{i2} c_{i2}		x_{ij} c_{ij}		x_{in} c_{in}	a_i
	S_m	x_{m1} c_{m1}	x_{m2} c_{m2}		x_{mj} c_{mj}		x_{mn} c_{mn}	a_m
Demand		b_1	b_2		B_j		b_n	Total

$$\left(\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j\right)$$

A set of non-negative values x_{ij} which represents the quantity of commodity transported from the i^{th} source to the j^{th} destination ($i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$) which satisfy the constraints (1) (2) (3) and (4) is called a *feasible solution* to the transportation problem.

While filling up each row we have $(n - 1)$ degrees of freedom and for each column we have $(m - 1)$ degrees of freedom. Hence out of the mn cells of the table we get $mn - (m - 1)(n - 1)$ independent allotments. In other words the maximum independent allotments of quantity is

$$mn - (m - 1)(n - 1) = m + n - 1$$

Thus there are $(m + n - 1)$ values of $x_{ij} > 0$ the remaining cells have $x_{ij} = 0$. These $(m + n - 1)$ cells with $x_{ij} > 0$ are called *basic cells*. In order to solve the transportation problem we must have $(m + n - 1)$ basic cells. If there are less than $(m + n - 1)$ basic cells, it is called a degenerate problem.

To solve a non-degenerate transportation problem we need to start with a basic feasible solution, then test the initial solution for optimality, improve that solution if possible and finally we arrive at an optimal solution.

11.2 INITIAL BASIC FEASIBLE SOLUTION

There are different methods of obtaining an initial solution of a given transportation problem.

11.2.1 North-West Corner Rule

Prepare the transportation table, consider the north-west corner cell (1, 1) of the table. The availability at the source S_1 is a_1 and the demand at the destination D_1 is b_1. Choose the minimum of a_1 and b_1, say a_1. Allot the quantity a_1 to the (1, 1) cell. This is the maximum quantity that can be allotted to the (1, 1) cell. Having allotted a_1 to the (1, 1) cell, we find that the total availability at S_1 is exhausted. Hence no more allotment can be made in the first row. Therefore $x_{11} = a_1$ and $x_{12} = x_{13} = \ldots = x_{1n} = 0$. Now the demand at D_1 becomes $b_1 - a_1$. In the table the north-west corner cell is (2, 1). As before choose the minimum of $b_1 - a_1$ and a_2 and allot that quantity in the (2, 1) cell. Suppose $b_1 - a_1$ is the minimum then allot $b_1 - a_1$ to the (2, 1) cell and assign zero values to the remaining cells of the first column. Now the availability at S_2 becomes $a_2 - (b_1 - a_1)$. In the new table (2, 2) cell becomes the north-west corner cell and allot maximum possible quantity to this cell. Proceeding like this, after a finite number of steps we arrive at an initial basic feasible solution. In order that it is a non-degenerate solution we must have $(m + n - 1)$ basic cells.

Example 11.1 Obtain an initial basic feasible solution, using the north-west corner rule for the following transportation problem:

	D_1	D_2	D_3	D_4	Availability
S_1	6	8	8	5	30
S_2	5	11	9	7	40
S_3	8	9	7	13	50
Demand	35	28	32	25	120

Solution In the table $\Sigma a_i = 30 + 40 + 50 = 120$

$\Sigma b_j = 35 + 28 + 32 + 25 = 120$

$\therefore \quad \Sigma a_i = \Sigma b_j$

It is a balanced problem.

Min $(a_1, b_1) = \min(30, 35) = 30$.

Allot 30 units to the north-west corner cell (1, 1). Thus the availability at S_1 is exhausted. No more allotment can be made in the first row.

	D_1	D_2	D_3	D_4	
S_1	30 [6]	– [8]	– [8]	– [5]	~~30~~
S_2	[5]	[11]	[9]	[7]	40
S_3	[8]	[9]	[7]	[13]	50
	~~35~~ 5	28	32	25	

Now (2, 1) cell is the north-west corner cell. The maximum quantity that can be allotted is the minimum of 40 and 5. Thus we can assign 5 units to the (2, 1) cell and no more allotment is possible in the first column.

	D_1	D_2	D_3	D_4	
S_1	30 [6]	– [8]	– [8]	– [5]	~~30~~
S_2	5 [5]	[11]	[9]	[7]	~~40~~ 35
S_3	[8]	[9]	[7]	[13]	50
	~~35~~	28	32	25	

Now choose the north-west corner cell of the reduced table which is the (2, 2) cell. Allot 28 units which is the minimum of (35, 28). We get

	D_1	D_2	D_3	D_4	
S_1	30 [6]	– [8]	– [8]	– [5]	~~30~~
S_2	5 [5]	28 [11]	[9]	[7]	~~40~~ ~~35~~ 7
S_3	– [8]	– [9]	[7]	[13]	50
	~~35~~	~~28~~	32	25	

Next allotment is min(32, 7) to the (2, 3) cell. Therefore we allot 7 units to the (2, 3) cell. The reduced table is

	D_1	D_2	D_3	D_4	
S_1	30 [6]	– [8]	– [8]	– [5]	~~30~~
S_2	5 [5]	28 [11]	7 [9]	– [7]	~~40~~
S_3	– [8]	– [9]	[7]	[13]	50
	~~35~~	~~28~~	~~32~~ 25	25	

Now allot 25 units to the (3, 3) cell and the remaining 25 to the (3, 4) cell. We obtain a basic feasible solution as

	D_1	D_2	D_3	D_4	
S_1	30 [6]	– [8]	– [8]	– [5]	~~30~~
S_2	5 [5]	28 [11]	7 [9]	– [7]	~~40~~
S_3	– [8]	– [9]	25 [7]	25 [13]	~~50~~
	~~35~~	~~28~~	~~32~~ ~~25~~	~~25~~	

					(Rs)
$S_1 \to D_1$	30 units	:	cost	30×6	= 180
$S_2 \to D_1$	5 units	:	cost	5×5	= 25
$S_2 \to D_2$	28 units	:	cost	28×11	= 308
$S_2 \to D_3$	7 units	:	cost	7×9	= 63
$S_3 \to D_3$	25 units	:	cost	25×7	= 175
$S_3 \to D_4$	25 units	:	cost	25×13	= 325
				Total cost	= Rs 1076

Here $x_{11} = 30$, $x_{21} = 5$, $x_{22} = 28$, $x_{23} = 7$, $x_{33} = 25$, and $x_{34} = 25$.

All other x_{ij} are zeroes.

Also $m + n - 1 = 3 + 4 - 1 = 6$. We have 6 basic cells.

Example 11.2 Find an initial basic feasible solution to the following transportation problem using north-west corner rule:

		D_1	D_2	D_3	Availability
Origin	O_1	2	7	4	5
	O_2	3	3	1	8
	O_3	5	4	7	7
	O_4	1	6	2	14
Requirements		7	9	18	34

Solution As per north-west corner rule the first allotment goes to (1, 1) cell. The maximum quantity that can be allotted is min (7, 5) = 5. With this allotment the table becomes

	D_1	D_2	D_3	
O_1	5 [2]	– [7]	– [4]	~~5~~
O_2	[3]	[3]	[1]	8
O_3	[5]	[4]	[7]	7
O_4	[1]	[6]	[2]	14
	~~7~~ 2	9	18	

The next allotment is for the (2, 1) cell and min(8, 2) = 2 and hence we can allot 2 units there. Now the table reduces to

	D_1	D_2	D_3	
O_1	5 2	– 7	– 4	~~5~~
O_2	2 3	3	1	~~8~~ 6
O_3	– 5	4	7	7
O_4	– 1	6	2	14
	~~7~~ ~~2~~	9	18	

Now we consider the cell (2, 2) which is the north-west corner cell and allot 6 units there since min(6, 9) = 6. The resulting table is

	D_1	D_2	D_3	
O_1	5 2	– 7	– 4	~~5~~
O_2	2 3	6 3	– 1	~~8~~ ~~6~~
O_3	– 5	4	7	7
O_4	– 1	6	2	14
	~~7~~ ~~2~~	~~9~~ 3	18	

The north-west corner of the table is the (3, 2) cell for which a maximum of 3 units can be allotted. We get the table

	D_1	D_2	D_3	
O_1	5 2	– 7	– 4	~~5~~
O_2	2 3	6 3	– 1	~~8~~ ~~6~~
O_3	– 5	3 4	7	~~7~~ 4
O_4	– 1	6	2	14
	~~7~~ ~~2~~	~~9~~ ~~3~~	18	

Finally we assign 4 units to the (3, 3) cell which leads to the solution satisfying all the constraints. The resulting initial basic feasible solution is

	D_1	D_2	D_3	
O_1	5 2	– 7	– 4	~~5~~
O_2	2 3	6 3	– 1	~~8~~ ~~6~~
O_3	– 5	3 4	4 7	~~7~~ ~~4~~
O_4	– 1	– 6	14 2	~~14~~
	~~7~~ ~~2~~	~~9~~ ~~3~~	~~18~~	

					(Rs)
$O_1 \to D_1$	5 units	:	cost	5×2 =	10
$O_2 \to D_1$	2 units	:	cost	2×3 =	6
$O_2 \to D_2$	6 units	:	cost	6×3 =	18
$O_3 \to D_2$	3 units	:	cost	3×4 =	12
$O_3 \to D_3$	4 units	:	cost	4×7 =	28
$O_4 \to D_3$	14 units	:	cost	14×2 =	28

Total cost = Rs 102

11.2.2 Row Minima Method

In the transportation table consider the cell having the least cost in the first row and allot the maximum possible quantity to that cell. Then we take the cell containing the next higher cost and allot the maximum possible quantity. If there is a tie between the cells choose the cell of your choice. Repeat the process till the availability a_1 is exhausted. Now take the second row and allot the quantities starting from the cell having the least cost. After exhausting a_2 consider the third row and so on. Finally we get the initial solution when all the rows have been filled up.

Example 11.3 Obtain an initial basic feasible solution to the following transportation problem using the row minima method.

	D_1	D_2	D_3	D_4	Availability
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

Solution $\Sigma a_i = \Sigma b_j = 35$. Hence it is a balanced problem. The least cost in the first row is 1. Allot min (14, 15) = 14 units to the (1, 3) cell. We get

	D_1	D_2	D_3	D_4	
O_1	– 6	– 7	14 1	– 5	~~14~~
O_2	8	9	2	7	16
O_3	4	3	6	2	5
	6	10	~~15~~ 1	4	

In the 2nd row the least cost is 2 and we can allot 1 unit [min (16, 1)] to the (2, 3) cell. This gives

	D_1	D_2	D_3	D_4	
O_1	– 6	– 4	14 1	– 5	~~14~~
O_2	8	9	1 2	7	~~16~~ 15
O_3	4	3	6	2	5
	6	10	~~15~~ ~~1~~	4	

The next higher cost in the 2nd row is 7 and 4 units can be allotted there. We get

	D_1	D_2	D_3	D_4	
O_1	– 6	– 4	14 1	– 5	~~14~~
O_2	8	9	1 2	4 7	~~16~~ ~~15~~ 11
O_3	4	3	– 6	– 2	5
	6	10	~~15~~ ~~1~~	~~4~~	

The next higher cost in the 2nd row is 8 and we allot 6 units to the cell (2, 1)

	D_1	D_2	D_3	D_4	
O_1	– 6	– 4	14 1	– 5	~~14~~
O_2	6 8	9	1 2	4 7	~~16~~ ~~15~~ ~~11~~ 5
O_3	– 4	3	– 6	– 2	~~5~~
	~~6~~	10	~~15~~ ~~1~~	~~4~~	

Now we can allot 5 units to the (2, 2) cell and 5 units to the (3, 2) cell which gives the initial solution. The resulting solution is

	D_1	D_2	D_3	D_4	
O_1	– 6	– 4	14 1	– 5	~~14~~
O_2	6 8	5 9	1 2	4 7	~~16~~ ~~15~~ ~~11~~ ~~5~~
O_3	– 4	5 3	– 6	– 2	~~5~~
	~~6~~	~~10~~ ~~5~~	~~15~~ ~~1~~	~~4~~	

				(Rs)
$O_1 \to D_3$	14 units	:	cost	$14 \times 1 =$ 14
$O_2 \to D_1$	6 units	:	cost	$6 \times 8 =$ 48
$O_2 \to D_2$	5 units	:	cost	$5 \times 9 =$ 45
$O_2 \to D_3$	1 unit	:	cost	$1 \times 2 =$ 2
$O_2 \to D_4$	4 units	:	cost	$4 \times 7 =$ 28
$O_3 \to D_2$	5 units	:	cost	$5 \times 3 =$ 15
				Total = Rs 152

Also $m + n - 1 = 6$ and we have 6 basic cells. Hence it is a non-degenerate solution.

11.2.3 Column Minima Method

This method is the same as the row minima method except that we consider the columns one by one instead of rows. Choose the first column and allot the maximum possible quantity to the cell having the least cost. Then consider the cell with the next higher cost and allot the available quantity. Having exhausted the first column go to the 2nd column and repeat the process. Finally after making the allotments in the last column we obtain an initial basic feasible solution.

Example 11.4 Obtain an initial basic feasible solution to the following transportation problem using the column minima method.

	D_1	D_2	D_3	Availability
S_1	6	10	15	2
S_2	4	6	16	5
S_3	12	5	8	9
Requirements	1	8	7	16 Total

Solution $\Sigma a_i = \Sigma b_j = 16$. Therefore the problem is a balanced one. We can find a basic feasible solution. Take the first column. The least cost is 4 in the (2, 1) cell. We make the allotment 1 in that cell. We get the reduced table.

	D_1	D_2	D_3	
S_1	– 6	– 10	– 15	2
S_2	1 4	6	16	~~5~~ 4
S_3	– 12	5	8	9
	~~1~~	8	7	

Consider the 2nd column. The least cost is 5 in the (3, 2) cell. An allotment of 8 units can be made there. The resulting table is

	D_1	D_2	D_3	
S_1	– 6	– 10	– 15	2
S_2	1 4	– 6	16	~~5~~ 4
S_3	– 12	8 5	8	~~9~~ 1
	~~1~~	~~8~~	7	

In the third column the least cost is 8 and an allotment of 1 unit can be made there. The table becomes

	D_1	D_2	D_3	
S_1	– 6	– 10	– 15	2
S_2	1 4	– 6	16	~~5~~ 4
S_3	– 12	8 5	1 8	~~9~~ ~~1~~
	~~1~~	~~8~~	~~7~~ 6	

We can allot 2 units for (1, 3) cell and 4 units for (2, 3) cell. This completes the allocation. There are $m + n - 1 = 5$ basic cells. The initial basic feasible solution obtained is

	D_1	D_2	D_3	
S_1	– / 6	– / 10	2 / 15	$\cancel{2}$
S_2	1 / 4	– / 6	4 / 16	$\cancel{5}$ $\cancel{4}$
S_3	– / 12	8 / 5	1 / 8	$\cancel{9}$ $\cancel{1}$
	$\cancel{1}$	$\cancel{8}$	$\cancel{7}$ $\cancel{6}$ $\cancel{2}$ $\cancel{4}$	

(Rs)

$S_1 \to D_3$	2 units	:	cost	2×15	=	30
$S_2 \to D_1$	1 unit	:	cost	1×4	=	4
$S_2 \to D_3$	4 units	:	cost	4×16	=	64
$S_3 \to D_2$	8 units	:	cost	8×5	=	40
$S_3 \to D_3$	1 unit	:	cost	1×8	=	8

Total cost = Rs 146

11.2.4 Least Cost Method (Matrix Minima Method)

Choose the cell in the table having the least cost altogether and allot the maximum possible quantity to that cell. Then consider the cell having the next higher cost in the table and make the allotment there. Use your choice when there is a tie between cells. Make the necessary alterations in a_i and b_j accordingly. Repeating the process for a finite number of times we arrive at an initial basic feasible solution.

Example 11.5 Obtain an initial basic feasible solution to the following transportation problem using the least cost rule

	D_1	D_2	D_3	
S_1	1	2	6	7
S_2	8	4	2	12
S_3	3	7	5	11
	10	10	10	

Solution $\Sigma a_i = \Sigma b_j = 30$. Hence the problem is balanced. The least cost in the table is 1 in the (1, 1) cell. Hence we make the first allotment of 7 units there. The reduced table is

	D_1	D_2	D_3	
S_1	7 / 1	– / 2	– / 6	$\cancel{7}$
S_2	8	4	2	12
S_3	3	7	5	11
	$\cancel{10}$ 3	10	10	

The next higher cost is 2 in the (2, 3) cell where we can allot 10 units. Now the table becomes

	D_1	D_2	D_3	
S_1	7 / 1	– / 2	– / 6	$\cancel{7}$
S_2	8	4	10 / 2	$\cancel{12}$ 2
S_3	3	7	– / 5	11
	$\cancel{10}$ 3	10	10	

We can assign 3 units to the (3, 1) cell having the next higher cost 3. Therefore we have

	D_1	D_2	D_3	
S_1	7 / 1	– / 2	– / 6	$\cancel{7}$
S_2	– / 8	4	10 / 2	$\cancel{12}$ 2
S_3	3 / 3	7	– / 5	$\cancel{11}$ 8
	$\cancel{10}$ $\cancel{3}$	10	$\cancel{10}$	

Now allot 2 units to the (2, 2) cell having the next higher cost 4. Finally allot the remaining 8 units to the (3, 2) cell. The initial solution obtained is

	D_1	D_2	D_3
S_1	7 / 1	– / 2	– / 6
S_2	– / 8	2 / 4	10 / 2
S_3	3 / 3	8 / 7	– / 5

(Rs)

$S_1 \to D_1$	:	7 units	cost	7×1	=	7
$S_2 \to D_2$	:	2 units	cost	2×4	=	8
$S_2 \to D_3$	:	10 units	cost	10×2	=	20
$S_3 \to D_1$	:	3 units	cost	3×3	=	9
$S_3 \to D_2$	:	8 units	cost	8×7	=	56

Total cost = Rs 100

11.2.5 Vogel's Approximation Method (VAM)

Vogel's approximation method gives a very good initial solution. In many cases this initial solution itself becomes an optimal solution.

Take the first row and note down the difference between the least and the next higher cost in that row. Then take the second row and note down the difference between the least and the next higher cost there. Repeat this for all the remaining rows.

Then consider the columns one by one and note down the difference in each column. Equal costs should be taken as a single cost in any row or column. We get $(m + n)$ such values. Now choose the largest among these numbers. If the largest difference corresponds to the i^{th} row then make an allotment in the i^{th} row in the cell having the least cost in that row, say (i, j)th cell. Allot the minimum of a_i and b_j to that cell. After making necessary modifications in the availability and demand constraints, consider the row or column having the largest difference in the reduced table. Repeat this process till all the available quantity are exhausted and the demands are satisfied. If any two rows or columns have the same maximum difference then break the tie at your choice. Finally we obtain an initial basic feasible solution.

Example 11.6 Obtain an initial basic feasible solution to the following transportation problem using Vogels approximation method:

	D_1	D_2	D_3	D_4	
S_1	2	3	11	7	6
S_2	1	5	6	1	4
S_3	5	8	15	9	10
	8	6	3	3	

Solution The transportation table is

	D_1	D_2	D_3	D_4		
S_1	2	3	11	7	6	(1)
S_2	1	5	6	1	4	(4)
S_3	5	8	15	9	10	(3)
	8	6	3	3		
	(1)	(2)	(5)	(6)		

In the first row the difference between the least and the next higher cost is 3 – 2 = (1). In the second row the difference is 5 – 1 = (4). In the third row the difference is 8 – 5 = (3). Similarly, the differences in the columns are (1), (2), (5) and (6). The maximum of all these is (6) in the 4$^{\text{th}}$ column. In the 4$^{\text{th}}$ column the least cost is 1 in (2, 4) cell. Hence we allot 3 units to this cell and cancel the other cells of the 4$^{\text{th}}$ column. The reduced table is

	D_1	D_2	D_3	D_4		
S_1	– 2	– 3	– 11	– 7	6	(1)
S_2	1	5	6	3 / 1	$\cancel{4}$ 1	(4)
S_3	5	8	15	– 9	10	(3)
	8	6	3	$\cancel{3}$		
	(1)	(2)	(5)			

The difference between the least and the next higher cost is noted down for each row and column of the reduced table. The maximum difference is (5) in the third column. The least cost in the third column is 6 in the (2, 3) cell for which 1 unit can be assigned. With this allotment the revised table is given below:

	D_1	D_2	D_3	D_4		
S_1	2	3	11	– 7	6	(1)
S_2	– 1	– 5	1 / 6	3 / 1	$\cancel{4}$	
S_3	5	8	15	– 9	10	(3)
	8	6	$\cancel{3}$ 2	$\cancel{3}$		
	(2)	(5)	(4)			

(5) is the maximum difference corresponding to the second column and the least cost is 3 in (1, 2) cell. Hence we can allot 6 units to this cell Note that $a_1 = b_2 = 6$. The resulting table is

	D_1	D_2	D_3	D_4	
S_1	– 2	6 / 3	– 11	– 7	$\cancel{6}$
S_2	– 1	– 5	1 / 6	3 / 1	$\cancel{4}$
S_3	5	– 8	15	– 9	10
	8	$\cancel{6}$	$\cancel{3}$ 2		

Now allot 8 units to the (3, 1) cell and the remaining 2 units to the (3, 3) cell. This yields the initial solution to the given problem as given below

	D_1	D_2	D_3	D_3
S_1	– 2	6 / 3	– 11	– 7
S_2	– 1	– 5	1 / 6	3 / 1
S_3	8 / 5	– 8	2 / 15	– 9

$S_1 \to D_2$: 6 units
$S_2 \to D_3$: 1 unit

$S_2 \to D_4$: 3 units
$S_3 \to D_1$: 8 units
$S_3 \to D_3$: 2 units

Total cost is 18 + 6 + 3 + 40 + 30 = Rs 97.

Note that there are only 5 basic cells but $m + n - 1 = 6$. Hence it is a degenerate solution.

EXERCISES

Obtain an initial basic feasible solution using north west corner rule:

1.

	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	

2.

	M_1	M_2	M_3	M_4	Availability
F_1	6	8	8	5	30
F_2	5	11	9	7	40
F_3	8	9	7	13	50
Demand	35	28	32	25	

3.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	

4.

	D_1	D_2	D_3	D_4	Supply
O_1	2	3	11	7	6
O_2	1	0	6	1	1
O_3	5	8	15	9	10
Demand	7	5	3	2	

5. Obtain an initial basic feasible solution using the row minima method

	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
	4	2	2	

Obtain an initial basic feasible solution using least cost rule.

6.

	D_1	D_2	D_3	
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
	7	9	18	

7.

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

8.

	W_1	W_2	W_3	W_4	Supply
A	1	2	1	4	30
B	3	3	2	1	50
C	4	2	5	9	20
Demand	20	40	30	10	

Apply VAM to obtain an initial solution:

9.

	D_1	D_2	D_3	Supply
A	2	7	4	5
B	3	3	1	8
C	5	4	7	7
D	1	6	2	14
Demand	7	9	18	

10.

	D_1	D_2	D_3	D_4	Supply
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400
Demand	200	225	275	250	

11.

	W_1	W_2	W_3	W_4	Supply
A	5	1	3	3	34
B	3	3	5	4	15
C	6	4	4	3	12
D	4	1	4	2	19
Demand	21	25	17	17	

12.

	C_1	C_2	C_3	C_4	Supply
P_1	2	3	11	7	6
P_2	1	0	6	1	1
P_3	5	8	15	9	10
Demand	7	5	3	2	

ANSWERS

1. $O_1 - D_1 : 5$; $O_2 - D_1 : 2$; $O_2 - D_2 : 6$; $O_3 - D_2 : 3$; $O_3 - D_3 : 4$; $O_4 - D_3 : 14$
   Total cost = Rs 102
2. $F_1 - M_1$: 30; $F_2 - M_1 : 5$; $F_2 - M_2$: 28; $F_2 - M_3 : 7$; $F_3 - M_3 : 25$; $F_3 - M_4 : 25$
   Total cost = Rs 1076
3. $O_1 - D_1 : 6$; $O_1 - D_2 : 8$; $O_2 - D_2 : 2$; $O_2 - D_3 : 14$ $O_3 - D_3 : 1$; $O_3 - D_4 : 4$
   Total cost = Rs 128
4. $O_1 - D_1 : 6$; $O_2 - D_1 : 1$; $O_3 - D_2 : 5$; $O_3 - D_3 : 3$; $O_3 - D_4 : 2$
   Total cost = Rs 116
5. I $-B$: 1; II $- A$: 2; II $- B$: 1; III $- A$: 2 III $- C$: 2 Total cost = Rs 855
6. $O_1 - D_2 : 2$; $O_1 - D_3 : 3$; $O_2 - D_3 : 8$; $O_3 - D_2 : 7$; $O_4 - D_1 : 7$; $O_4 - D_3 : 7$
   Total cost = Rs 83
7. $O_1 - D_2 : 6$; $O_2 - D_3 : 2$; $O_2 - D_4 : 6$; $O_3 - D_1 : 4$; $O_3 - D_3 : 6$;
   Total cost = Rs 28
8. $A - W_1 : 20$; $A - W_3 : 10$; $B - W_2 : 20$; $B - W_3 : 20$; $B - W_4 : 10$; $C - W_2 : 20$
   Total cost = Rs 180
9. $A - D_1 : 5$; $B - D_3 : 8$; $C - D_2 : 7$ $D - D_1 : 2$; $D - D_2 : 2$; $D - D_3 : 10$
   Total cost = Rs 80
10. $S_1 - D_1 : 200$; $S_1 - D_2 : 50$; $S_2 - D_2 : 175$; $S_2 - D_4 = 125$; $S_2 - D_3 : 275$; $S_3 - D_4 : 125$
    Total cost = Rs 12075
11. $A - W_2 : 25$; $A - W_3 : 9$; $B - W_1 : 15$ $C - W_1 = 4$; $C - W_3 : 8$; $D - W_1 : 2$ $D - W_4 : 17$ Total cost = Rs 195
12. $P_1 - C_1 : 1$; $P_1 - C_2 : 5$; $P_2 - C_4 : 1$ $P_3 - C_1$: 6; $P_3 - C_3 : 3$; $P_3 - C_4 : 1$
    Total cost = Rs 102

11.3 TESTING FOR OPTIMALITY

We can find an initial solution to a transportation problem, using any one of the methods described previously. The next step is to examine whether the solution obtained is optimal or not.

We introduce *simplex multipliers (dual variables)* $u_i (i = 1, 2, 3, \ldots, m)$ and $v_j (j = 1, 2, 3, \ldots, n)$ for the m rows and n columns with the condition that they should satisfy the equations

$u_i + v_j = c_{ij}$ where c_{ij} is the cost of transportation in the (i,j)th cell which is a basic (occupied) cell. For $(m + n - 1)$ basic cells we get $(m + n - 1)$ equations and there are $(m + n)$ variables $u_1, u_2, \ldots, u_m, v_1, v_2 \ldots, v_n$. To solve these equations we have to reduce the number of variables to $(m + n - 1)$. Hence we assign zero value to one of the simplex multipliers. In order to make the procedure easy, we assign zero value to the simplex multiplier which corresponds to the row or column having the maximum number of basic cells.

Now we have $(m + n - 1)$ variables and they can be solved and the values of u_1 and v_j (for each i and j) can be obtained.

Next we calculate the relative cost factors $\bar{c}_{ij}$ for each non-basic cell using the formula

$$\bar{c}_{ij} = c_{ij} - u_i - v_j$$

If all the $\bar{c}_{ij}$ are non-negative then the current solution is optimal and the total transportation cost obtained is the least and cannot be reduced further. On the other hand, if some $\bar{c}_{ij}$ are negative then it indicates that the current solution is not optimal and that it can be improved further.

11.4 OPTIMIZATION—MODI METHOD (MODIFIED DISTRIBUTION METHOD)

Select the most negative $\bar{c}_{ij}$. The corresponding (i, j) cell is the entering cell for the improved solution. Let us allot a quantity θ to this cell. Draw a loop (figure having horizontal and vertical sides) having one vertex in the entering cell and all the remaining vertices in basic cells. Add and subtract θ from the allotments in these basic cells in order to balance the availability and demand constraints. Now the entering cell has allotment θ and the other vertices of the loop have allotments $x_{ij} + \theta$ and $x_{ij} - \theta$. Of the cells having allotments of the form $x_{ij} - \theta$, choose the cell having the least x_{ij}. Let it be x_{pq}. If we take θ as x_{pq}, then the allotment in this cell becomes zero and hence this cell becomes non-basic (leaves the basis). With the value of θ as x_{pq}, we get a modified table with revised allotments. Again we test this solution for optimality and proceed as before.

This procedure is repeated till an optimal solution is obtained.

Example 11.7 Solve the following transportation problem

	D_1	D_2	D_3	D_4	Availability
S_1	6	8	8	5	30
S_2	5	11	9	7	40
S_3	8	9	7	13	50
Demand	35	28	35	25	120

balanced

Solution We obtain an initial basic feasible solution using the VAM method. The difference between the least and the next higher costs of each row and column are noted down.

	D_1	D_2	D_3	D_4		
S_1	6	8	8	5	30	(1)
S_2	5	11	9	7	40	(2)
S_3	8	9	7	13	50	(1)
	35	28	32	25		
	(1)	(1)	(1)	(2)		

The second row and fourth column have the maximum difference 2. Select one of them, say 2^{nd} row. Allot 35 units to the cell (2, 1) having the least cost. Adjust the constraints and note down the differences in the next table. We get

	D_1	D_2	D_3	D_4		
S_1	– / 6	8	8	5	30	(3)
S_2	35 / 5	11	9	7	~~40~~ 5	(2)
S_3	– / 8	9	7	13	50	(2)
	~~35~~	28	32	25		
		(1)	(1)	(2)		

The first row has the maximum difference. Allot 25 units to the (1, 4) cell. The revised table is

	D_1	D_2	D_3	D_4		
S_1	– / 6	8	8	25 / 5	~~30~~ 5	(0)
S_2	35 / 5	11	9	– / 7	~~40~~ 5	(2)
S_3	– / 8	9	7	– / 13	50	(2)
	~~35~~	28	32	~~25~~		
		(1)	(1)			

Select the third row with the difference 2 and allot 32 units to the (3, 3) cell. The table becomes

	D_1	D_2	D_3	D_4	
S_1	– / 6	5 / 8	– / 8	25 / 5	~~30~~ ~~5~~
S_2	35 / 5	5 / 11	– / 9	– / 7	~~40~~ ~~5~~
S_3	– / 8	18 / 9	32 / 7	– / 13	~~50~~ ~~18~~
	~~35~~	~~28~~	~~32~~	~~25~~	

The 2^{nd} column is automatically filled up with the remaining quantity. This is an initial basic feasible solution with the total cost of Rs 781.

Introduce simplex multipliers u_1, u_2, u_3 for the rows and v_1, v_2, v_3, v_4 for the columns. Form the equations $u_i + v_j = c_{ij}$ for the basic cells. We have

$u_1 + v_2 = 8$ $\quad$ $u_2 + v_2 = 11$
$u_1 + v_4 = 5$ $\quad$ $u_3 + v_2 = 9$
$u_2 + v_1 = 5$ $\quad$ $u_3 + v_3 = 7$

Since the 2nd column contains the maximum number of basic cells we take $v_2 = 0$. Substituting $v_2 = 0$ in the above equations and solving them we get

$u_1 = 8$ $\quad$ $u_2 = 11$ $\quad$ $u_3 = 9$ $\quad$ $v_1 = -6$
$v_2 = 0$ $\quad$ $v_3 = -2$ $\quad$ $v_4 = -3$

Using these values we calculate $\bar{c}_{ij} = c_{ij} - u_i - v_j$ for the non-basic cells.

$\bar{c}_{11} = c_{11} - u_1 - v_1 = 4;$ $\quad$ $\bar{c}_{13} = c_{13} - u_1 - v_3 = 2$
$\bar{c}_{23} = c_{23} - u_2 - v_3 = 0;$ $\quad$ $\bar{c}_{24} = c_{24} - u_2 - v_4 = -1$
$\bar{c}_{31} = c_{31} - u_3 - v_1 = 5;$ $\quad$ $\bar{c}_{34} = c_{34} - u_3 - v_4 = 7$

We find $\bar{c}_{24}$ is negative and hence (2, 4) is the entering cell. Introduce θ in (2, 4) cell and form a loop as shown below.

	D_1	D_2	D_3	D_4
S_1	– 6	$5+\theta$ 8	– 8	$25-\theta$ 5
S_2	35 5	$5-\theta$ 11	– 9	θ 7
S_3	– 8	18 9	32 7	– 13

The vertices (2, 2) and (1, 4) of the loop have $5 - \theta$, $25 - \theta$. The least quantity is 5 and hence $\theta = 5$ is the maximum value of θ such that all the entries are non-negative (feasible). Thus, taking $\theta = 5$ we find that (2, 2) cell becomes non-basic and (2, 4) cell becomes basic with an allotment of 5. The improved solution is

	D_1	D_2	D_3	D_4	
S_1	– 6	10 8	8	20 5	$u_1 = -2$
S_2	35 5	11	9	5 7	$u_2 = 0$
S_3	8	18 9	32 7	13	$u_3 = -1$
	$v_1 = 5$	$v_2 = 10$	$v_3 = 8$	$v_4 = 7$	

We test whether this is an optimal solution. Write the equations $c_{ij} = u_i + v_j$ for the basic cells. We get

$u_1 + v_2 = 8$ $\quad$ $u_2 + v_4 = 7$
$u_1 + v_4 = 5$ $\quad$ $u_3 + v_2 = 9$
$u_2 + v_1 = 5$ $\quad$ $u_3 + v_3 = 7$

Taking $u_2 = 0$ and solving these equations we obtain

$u_1 = -2$ $\quad$ $u_2 = 0$ $\quad$ $u_3 = -1$ $\quad$ $v_1 = 5$
$v_2 = 10$ $\quad$ $v_3 = 8$ $\quad$ $v_4 = 7$

Using these values we calculate the relative cost factors $\bar{c}_{ij}$ for the non basic cells.

$\bar{c}_{11} = c_{11} - u_1 - v_1 = 3$ $\quad$ $\bar{c}_{13} = c_{13} - u_1 - v_3 = 2$
$\bar{c}_{23} = c_{23} - u_2 - v_3 = 1$ $\quad$ $\bar{c}_{24} = c_{24} - u_2 - v_4 = 0$
$\bar{c}_{31} = c_{31} - u_3 - v_1 = 4$ $\quad$ $\bar{c}_{34} = c_{34} - u_3 - v_4 = 7$

Since all the relative cost factors are non negative the current solution is optimal. The optimal solution is

				(Rs)
$S_1 \to D_2$	:	10 units	cost	80
$S_1 \to D_4$	:	20 units	cost	100
$S_2 \to D_1$	:	35 units	cost	175
$S_2 \to D_4$	:	5 units	cost	35
$S_3 \to D_2$	:	18 units	cost	162
$S_3 \to D_3$	:	32 units	cost	224

Total cost = 776

Example 11.8 A company has three factories A, B and C which supply units to warehouses X, Y and Z every month. The capacities of the factories are 60, 70 and 80 units at A, B and C respectively. The requirements of X, Y and Z per month are 50, 80 and 80 units respectively. Transportation cost per unit in rupees are given in the following table. Find out the minimum cost of transportation

	X	Y	Z
A	8	7	5
B	6	8	9
C	9	6	5

Solution The transportation table for the given data is

	X	Y	Z	Availability	
A	8	7	5	60	
B	6	8	9	70	
C	9	6	5	80	
Requirement	50	80	80	210	balanced

We find an initial basic feasible solution by least cost rule. We get the initial solution as follows:

The least cost is 5 in (1, 3) cell and (3, 3) cell. Select (1, 3) cell and allot 60 unit. We have,

	X	Y	Z	
A	– 8	– 7	60 5	~~60~~
B	6	8	9	70
C	9	6	5	80
	50	80	~~80~~ 20	

The least cost in the reduced table is 5 in (3, 3) cell. Allot 20 units there. We get

	X	Y	Z	
A	– 8	– 7	60 5	~~60~~
B	6	8	– 9	70
C	9	6	20 5	~~80~~ 60
	50	80	~~80~~ ~~20~~	

(2, 1) cell and (3, 2) cell both have the minimum cost 6. Select (2, 1) cell for an allotment of 50 units. The table becomes

	X	Y	Z	
A	– 8	– 7	60 5	~~60~~
B	50 6	8	– 9	~~70~~ 20
C	– 9	6	20 5	~~80~~ 60
	~~50~~	80	~~80~~ ~~20~~	

Now we can allot 20 units to (2, 2) cell and 60 units to (3, 2) cell. The initial solution is

	X	Y	Z	
A	– 8	– 7	60 5	$u_1 = -2$
B	50 6	20 8	– 9	$u_2 = 0$
C	– 9	60 6	20 5	$u_3 = -2$
	$v_1 = 6$	$v_2 = 8$	$v_3 = 7$	

$m + n - 1 = 5$ and we have 5 basic cells. Defining simplex multipliers u_i and v_j for the rows and columns we get the equations for the basic cells,

$$u_1 + v_3 = 5 \qquad u_3 + v_2 = 6$$
$$u_2 + v_1 = 6 \qquad u_3 + v_3 = 5$$
$$u_2 + v_2 = 8$$

Assign zero value to u_2 we get the solutions

$$u_1 = -2 \qquad v_1 = 6$$
$$u_2 = 0 \qquad v_2 = 8$$
$$u_3 = -2 \qquad v_3 = 7$$

The relative cost factors for the non-basic cells are

$$\bar{c}_{11} = c_{11} - u_1 - v_1 = 4 \qquad \bar{c}_{12} = c_{12} - u_1 - v_2 = 1$$
$$\bar{c}_{23} = c_{23} - u_2 - v_3 = 2 \qquad \bar{c}_{31} = c_{31} - u_3 - v_1 = 5$$

All $\bar{c}_{ij} \geq 0$. Hence this solution is itself optimal. The optimal solution is

			(Rs)
$A \to Z$	:	60 units Cost	300
$B \to X$	:	50 units Cost	300
$B \to Y$	:	20 units Cost	160
$C \to Y$	:	60 units Cost	360
$C \to Z$	:	20 units Cost	100

Total cost = Rs 1220

11.5 DEGENERACY

As we have discussed earlier, the number of basic cells in a transportation table of m rows and n columns, should be $m + n - 1$. If we get less than $(m + n - 1)$ basic cells optimality test cannot be carried out. This is called a *degenerate solution*.

In order to overcome this difficulty and to solve the degeneracy, we need $(m + n - 1)$ basic cells. Suppose we have only $(m + n - 2)$ basic cells, then we select a non-basic cell which does not form a loop with the existing basic cells. Allot an infinitesimal quantity $\in$ to this selected cell and make it a basic cell. This infinitesimal quantity will not affect the other allotments when it is added or subtracted. In other words it is only a dummy allotment. But it helps us to carry out the iteration. Now we have $(m + n - 1)$ basic cells and we can perform optimality test and proceed with the iteration if necessary. Finally $\in$ is taken as zero.

Example 11.9 Solve the following transportation problem:

	X	Y	Z	Availability
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
	50	80	80	210 balanced

Solution Applying VAM method we obtain an initial basic feasible solution, given below

	X	Y	Z
A	– 8	– 7	60 3
B	50 3	8	20 9
C	– 11	80 3	– 5

We have 4 basic cells but we should have (3 + 3 – 1 = 5) 5 basic cells. Select any non-basic cell from the set {(1, 2), (2, 2), (3, 1) and (3, 3)} which does not form a loop with the existing basic cells. Let us choose (2, 2) cell and make it a basic cell by allotting an infinitesimal quantity $\in$ to it. Now we get an initial basic feasible solution having 5 basic cells.

Introduce simplex multipliers u_1, u_2, u_3 for the rows and v_1, v_2, v_3 for the columns. We get 5 equations $u_i + v_j = c_{ij}$ corresponding to the basic cells.

$u_1 + v_3 = 3$ $\qquad$ $u_2 + v_3 = 9$

$u_2 + v_1 = 3$ $\qquad$ $u_3 + v_2 = 3$

$u_2 + v_2 = 8$

	X	Y	Z	
A	8	7	60 3	$u_1 = -6$
B	50 3	$\in$ 8	20 9	$u_2 = 0$
C	– 11	80 3	5	$u_3 = -5$
	$v_1 = 3$	$v_2 = 8$	$v_3 = 9$	

Set $u_2 = 0$. We get the solution

$u_1 = -6$ $\qquad$ $v_1 = 3$

$u_2 = 0$ $\qquad$ $v_2 = 8$

$u_3 = -5$ $\qquad$ $v_3 = 9$

The relative cost factors for the non-basic cells are $C_{11} = 8 + 6 - 3 = 11$

$\bar{c}_{12} = 7 + 6 - 8 = 5$ $\qquad$ $\bar{c}_{31} = 11 + 5 - 3 = 13$

$\bar{c}_{33} = 5 + 5 - 9 = 1$

$\bar{c}_{ij}$ are all non negative. Hence this solution is optimal. Take $\in$ to be zero. The optimal solution is

			(Rs)
$A \to Z$	:	60 units	Cost 180
$B \to X$	:	50 units	Cost 150
$B \to Z$	:	20 units	Cost 180
$C \to Y$	:	80 units	Cost 240
			Total Cost = 750

Example 11.10 Solve the following TP

Market

		M_1	M_2	M_3	M_4	a_i
Warehouse	W_1	8	10	7	6	50
	W_2	12	9	4	7	40
	W_3	9	11	10	8	30
	b_j	25	32	40	23	120

balanced

Solution An initial basic feasible solution obtained by the Least Cost Rule is given below

	M_1	M_2	M_3	M_4
W_1	25 8	2 10	7	23 6
W_2	12	9	40 4	7
W_3	9	30 11	10	8

Here $m + n - 1 = 4 + 3 - 1 = 6$. But we have only 5 basic cells. Hence we introduce one more basic cell which does not form a loop with these basic cells. Let us take the cell (2, 4) and allot an infinitesimal quantity $\in$ to it. Now we have

	M_1	M_2	M_3	M_4	
W_1	25 8	2 10	7	23 6	$u_1 = 0$
W_2	12	9	40 4	$\in$ 7	$u_2 = 1$
W_3	9	30 11	10	8	$u_3 = 1$
	$v_1 = 8$	$v_2 = 10$	$v_3 = 3$	$v_3 = 6$	

Using simplex multipliers u_1, u_2, u_3 for rows and v_1, v_2, v_3, v_4 for columns we get the equations $u_i + v_j = c_{ij}$

$u_1 + v_1 = 8$ $\qquad$ $u_2 + v_3 = 4$

$u_1 + v_2 = 10$ $\qquad$ $u_2 + v_4 = 7$

$u_1 + v_4 = 6$ $\qquad$ $u_3 + v_2 = 11$

for the basic cells.

Setting $u_1 = 0$, we get the following solution

$u_1 = 0$ $\qquad$ $v_1 = 8$

$u_2 = 1$ $\qquad$ $v_2 = 10$

$u_3 = 1$ $\qquad$ $v_3 = 3$

$v_4 = 6$

The relative cost factors for the non-basic cells are given by

$\bar{c}_{13} = 7 - 0 - 3 = 4$ $\quad\bar{c}_{31} = 9 - 1 - 8 = 0$
$\bar{c}_{21} = 12 - 1 - 8 = 3$ $\quad\bar{c}_{33} = 10 - 1 - 3 = 6$
$\bar{c}_{22} = 9 - 1 - 10 = -2$ $\quad\bar{c}_{34} = 8 - 1 - 6 = 1$

Since $\bar{c}_{22}$ is negative (2, 2) cell enters the basis. Allot a quantity θ to this cell.

	M_1	M_2	M_3	M_4
W_1	25 / 8	$2-\theta$ / 10	7	$23+\theta$ / 5
W_2	12	θ / 9	40 / 4	$\in -\theta$ / 7
W_3	9	30 / 11	10	8

A loop is formed with cells (1, 2) (2, 2) (2, 4) and (1, 4). We find that $\theta = \in$. Substituting $\theta = \in$ we get the improved solution as given below.

	M_1	M_2	M_3	M_4	
W_1	25 / 8	2 / 10	7	23 / 6	$u_1 = 10$
W_2	12	$\in$ / 9	40 / 4	7	$u_2 = 9$
W_3	9	30 / 11	10	8	$u_3 = 11$
	$v_1 = -2$	$v_2 = 0$	$v_3 = -5$	$v_4 = -4$	

The relative cost factors for the non-basic cells are

$\bar{c}_{13} = 7 - 10 + 5 = 2$ $\quad\bar{c}_{31} = 9 - 11 + 2 = 0$
$\bar{c}_{21} = 12 - 9 + 2 = 5$ $\quad\bar{c}_{33} = 10 - 11 + 5 = 4$
$\bar{c}_{24} = 7 - 9 + 4 = 2$ $\quad\bar{c}_{34} = 8 - 11 + 4 = 1.$

Since all the $\bar{c}_{ij}$ are non negative, the current solution is optimal. The optimal solution is

	(Rs)
$W_1 \to M_1$: 25 units	Cost 200
$W_1 \to M_2$: 2 units	Cost 20
$W_1 \to M_4$: 23 units	Cost 138
$W_2 \to M_3$: 40 units	Cost 160
$W_3 \to M_2$: 30 units	Cost 330
	Total Cost = Rs 848

11.6 UNBALANCED TRANSPORTATION PROBLEM

In order that a transportation problem should yield a feasible solution it is necessary that the total quantity available at the sources must be equal to the total quantity required at the destinations

$$\text{i.e.}\quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

If this condition is not satisfied, then the problem is called an *unbalanced transportation problem*. We solve this type of problem by changing it to a balanced problem.

When the availability is greater than the requirement, we introduce a dummy destination in the table. The cost of transportation from each source to this destination is assumed to be zero. The excess of availability is assigned as the requirement at this dummy destination. Now the problem is balanced and it can be solved.

When the requirement is more than the availability, we introduce a dummy source in the table. The cost of transportation from this source to each destination is taken as zero. The excess of demand is assigned as the availability at this dummy source. Thus the problem is changed to a balanced one and can be solved in the usual way.

Example 11.11 Solve the following transportation problem

	D_1	D_2	D_3	D_4		Supply
S_1	4	2	3	2	6	8
S_2	5	4	5	2	1	12
S_3	6	5	4	7	3	14
Demand	4	4	6	8	8	

Solution Here $\Sigma a_i = 34$, $\Sigma b_j = 30$

Therefore the problem is unbalanced. The total supply is 4 units in excess of total demand and hence we introduce a dummy destination D_0 with demand 4 and change the problem into a balanced problem. The table is

	D_0	D_1	D_2	D_3	D_4	D_5	Supply
S_1	0	4	2	3	2	6	8
S_2	0	5	4	5	2	1	12
S_3	0	6	5	4	7	3	14
Demand	4	4	4	6	8	8	34

We obtain an initial basic feasible solution by VAM as given below.

	D_0	D_1	D_2	D_3	D_4	D_5
S_1	0	∈ 4	4 2	3	4 2	6
S_2	0	5	4	5	4 2	8 1
S_3	4 0	4 6	5	6 4	7	3

$m + n - 1 = 6 + 3 - 1 = 8$. But we have only 7 basic cells. Hence this solution is degenerate. Let us choose the cell (1, 1) which doest not form a loop with the basic cells. Allot an infinitesimal quantity $\in$ to this cell. Now have 8 basic cells.

Introduce the simplex multipliers u_1, u_2, u_3 for the rows and v_0, v_1, v_2, v_3, v_4 and v_5 for the columns. Form the equations $u_i + v_j = c_{ij}$ for the basic cells.

We get
$u_1 + v_1 = 4$ $\quad$ $u_2 + v_5 = 1$
$u_1 + v_2 = 2$ $\quad$ $u_3 + v_0 = 0$
$u_1 + v_4 = 2$ $\quad$ $u_3 + v_1 = 6$
$u_2 + v_4 = 2$ $\quad$ $u_3 + v_3 = 4$

Setting $u_1 = 0$. We get the solution.

$u_1 = 0$ $\quad$ $v_0 = -2$
$u_2 = 0$ $\quad$ $v_1 = 4$
$u_3 = 2$ $\quad$ $v_2 = 2$
$v_3 = 2$
$v_4 = 2$
$v_5 = 1$

	D_0	D_1	D_2	D_3	D_4	D_5	
S_1	0	∈ 4	4 2	3	4 2	6	$u_1 = 0$
S_2	0	5	4	5	4 2	8 1	$u_2 = 0$
S_3	4 0	4 6	5	6 4	7	3	$u_3 = 2$
	$v_0 = -2$	$v_1 = 4$	$v_2 = 2$	$v_3 = 2$	$v_4 = 2$	$v_5 = 1$	

The relative cost factors for the non basic cells are

$\bar{c}_{10} = 0 - 0 + 2 = 2$ $\quad$ $\bar{c}_{22} = 4 - 0 - 2 = 2$
$\bar{c}_{13} = 3 - 0 - 2 = 1$ $\quad$ $\bar{c}_{23} = 5 - 0 - 2 = 3$
$\bar{c}_{15} = 6 - 0 - 1 = 5$ $\quad$ $\bar{c}_{32} = 5 - 2 - 2 = 1$
$\bar{c}_{20} = 0 - 0 + 2 = 2$ $\quad$ $\bar{c}_{34} = 7 - 2 - 2 = 3$
$\bar{c}_{21} = 5 - 0 - 4 = 1$ $\quad$ $\bar{c}_{35} = 3 - 2 - 1 = 0$

We find that all $\bar{c}_{ij} \geq 0$. Hence the current solution is optimal. The optimal solution is

	(Rs)
$S_1 \to D_2$: 4 units	Cost 8
$S_1 \to D_4$: 4 units	Cost 8
$S_2 \to D_4$: 4 units	Cost 8
$S_2 \to D_5$: 8 units	Cost 8
$S_3 \to D_1$: 4 units	Cost 24
$S_3 \to D_3$: 6 units	Cost 24

Total cost = Rs 80 $\quad (\in = 0)$

Example 11.12 Solve the following transportation problem

A company operates three coal mines A, B, C which provide 400, 500, and 700 tonnes respectively per week. Orders for 500, 400, 300, 300 and 600 tonnes per week have been received from customers C_1, C_2, C_3, C_4, C_5 respectively. Transportation costs in rupees per tonne from each mine to each customer are given below

	C_1	C_2	C_3	C_4	C_5	Supply
A	4	16	1	16	14	400
B	18	10	8	12	12	500
C	6	1	4	13	2	700
Demand	500	400	300	300	600	

For each tonne of coal demanded but not supplied there is a loss of one rupee per tonne for the company. Find the weekly shipping (transporting) schedule which minimizes the total expenses.

Solution Here, the total supply is 1600 units. But the total demand is 2100 units. Hence it is an unbalanced problem. We introduce a dummy mine D which provides 500 tonnes per week. The loss of one rupee per tonne is taken as the transportation cost from this mine D to each customer. The balanced transportation table is given below.

	C_1	C_2	C_3	C_4	C_5	Supply
A	4	16	1	16	14	400
B	18	10	8	12	12	500
C	6	1	4	13	2	700
D	1	1	1	1	1	500
Demand	500	400	300	300	600	2100

We find an initial basic feasible solution by VAM as shown below:

	C_0	C_1	C_2	C_3	C_4	Supply
A	100 4	16	1	16	14	$u_1 = 4$
B	18	400 10	8	12	100 12	$u_2 = 16$
C	200 6	1	4	13	500 2	$u_3 = 1$
D	200 1	1	1	300 1	1	$u_4 = 1$
Demand	$v_1 = 0$	$v_2 = -6$	$v_3 = -3$	$v_4 = 0$	$v_5 = -4$	

$m + n - 1 = 5 + 4 - 1 = 8$ and we have 8 basic cells. Therefore it is a non-degenerate feasible solution.

To test for optimality we introduce simplex multipliers u_i ($i = 1, 2, 3, 4$) and v_j($j = 1, 2, 3, 4, 5$) and form the equations, $u_i + v_j = c_{ij}$. We get the equations

$u_1 + v_1 = 4 \qquad u_3 + v_1 = 6$
$u_1 + v_3 = 1 \qquad u_3 + v_5 = 2$
$u_2 + v_2 = 10 \qquad u_4 + v_1 = 1$
$u_2 + v_5 = 12 \qquad u_4 + v_4 = 1$

Take $v_1 = 0$

The values of u_i and v_j are

$u_1 = 4 \qquad u_2 = 16$
$u_3 = 6 \qquad u_4 = 1$
$v_1 = 0 \qquad v_2 = -6$
$v_3 = -3 \qquad v_4 = 0$
$v_5 = -4$

The relative cost factors for the non-basic cells are

$\bar{c}_{12} = 16 - 4 + 6 = 18 \qquad \bar{c}_{24} = 12 - 16 - 0 = -4$
$\bar{c}_{14} = 16 - 4 - 0 = 12 \qquad \bar{c}_{32} = 1 - 6 + 6 = 1$
$\bar{c}_{15} = 14 - 4 + 4 = 14 \qquad \bar{c}_{33} = 4 - 6 + 3 = 1$
$\bar{c}_{21} = 18 - 16 - 0 = 2 \qquad \bar{c}_{34} = 13 - 6 - 0 = 7$
$\bar{c}_{23} = 8 - 16 + 3 = -5 \qquad \bar{c}_{42} = 1 - 1 + 6 = 6$
$\bar{c}_{43} = 1 - 1 + 3 = 3$

(2, 3) cell has the most negative relative cost factor –5. Hence (2, 3) cell enters the basis. We allot a quantity θ to the (2, 3) cell and form a loop having all other vertices at basic cells as shown below:

	C_1	C_2	C_3	C_4	C_5
A	$100+\theta$ 4	16	$300-\theta$ 1	16	14
B	18	400 10	θ 8	12	$100-\theta$ 12
C	$200-\theta$ 6	1	4	13	$500+\theta$ 2
D	200 1	1	1	300 1	1

Of the cells with allotment of the form $x_{ij} - \theta$ (2, 5) cell has the least entry 100. Therefore $\theta = 100$ and (2, 5) cell becomes non-basic. The revised distribution is

	C_1	C_2	C_3	C_4	C_5	
A	200 4	16	200 1	16	14	$u_1 = 4$
B	18	400 10	100 8	12	12	$u_2 = 11$
C	100 6	1	4	13	600 2	$u_3 = 6$
D	200 1	1	1	300 1	1	$u_4 = 1$
	$v_1 = 0$	$v_2 = -1$	$v_3 = -3$	$v_4 = 0$	$v_5 = -4$	

Introducing simplex multipliers u_i and v_j for the basic cells we get the equations

$u_1 + v_1 = 4 \qquad u_3 + v_5 = 2$
$u_1 + v_3 = 1 \qquad u_4 + v_1 = 1$
$u_2 + v_2 = 10 \qquad u_4 + v_4 = 1$
$u_2 + v_3 = 8$
$u_3 + v_1 = 6$

Take $v_1 = 0$

The values of u_i and v_j are

$u_1 = 4 \qquad u_2 = 11$
$u_3 = 6 \qquad u_4 = 1$
$v_1 = 0 \qquad v_2 = -1$
$v_3 = -3 \qquad v_4 = 0$
$v_5 = -4$

The relative cost factors for the non-basic cells are

$\bar{c}_{12} = 16 - 4 + 1 = 13 \qquad \bar{c}_{14} = 16 - 4 - 0 = 12$
$\bar{c}_{15} = 14 - 4 + 4 = 14 \qquad \bar{c}_{21} = 18 - 11 - 0 = 7$
$\bar{c}_{24} = 12 - 11 - 0 = 1 \qquad \bar{c}_{32} = 1 - 6 + 1 = -4$
$\bar{c}_{33} = 4 - 6 + 3 = 1 \qquad \bar{c}_{34} = 13 - 6 - 0 = 7$
$\bar{c}_{42} = 1 - 1 + 1 = 1 \qquad \bar{c}_{43} = 1 - 1 + 3 = 3$
$\bar{c}_{45} = 1 - 1 + 4 = 4$

Since $\bar{c}_{32} = -4$, (3, 2) cell enters the basis. We allot a quantity θ to this cell and form a loop with this cell at one vertex and basic cells at the other vertices as given below

	C_1	C_2	C_3	C_4	C_5
A	$200+\theta$ 4	16	$200-\theta$ 1	16	14
B	18	$400-\theta$ 10	$100+\theta$ 8	12	12
C	$100-\theta$ 6	θ 1	4	13	600 2
D	200 1	1	1	300 1	1

From the entries we find that $\theta = 100$ and consequently (3, 1) cell becomes non-basic. The improved solution is

	C_1	C_2	C_3	C_4	C_5	
A	300 / 4	16	100 / 1	16	14	$u_1 = 0$
B	18	300 / 10	200 / 8	12	12	$u_2 = 7$
C	6	100 / 1	4	13	600 / 2	$u_3 = -2$
D	200 / 1	1	1	300 / 1	1	$u_4 = -3$
	$v_1 = 4$	$v_2 = 3$	$v_3 = 1$	$v_4 = 4$	$v_5 = 4$	

Defining simplex multipliers u_i and v_j and forming the equations

$$u_i + v_j = c_{ij}$$

for the basic cells we obtain

$u_1 + v_1 = 4$ $\quad u_3 + v_2 = 1$
$u_1 + v_3 = 1$ $\quad u_3 + v_5 = 2$
$u_2 + v_2 = 10$ $\quad u_4 + v_1 = 1$
$u_2 + v_3 = 8$ $\quad u_4 + v_4 = 1$

Take $u_1 = 0$ and solve the equations. We get

$u_1 = 0 \quad u_2 = 7 \quad u_3 = -2 \quad u_4 = -3$
$v_1 = 4 \quad v_2 = 3 \quad v_3 = 1 \quad v_4 = 4 \quad v_5 = 4$

The relative cost factors for the non-basic cells are

$\bar{c}_{12} = 13 \quad \bar{c}_{14} = 12 \quad \bar{c}_{15} = 10 \quad \bar{c}_{21} = 7$
$\bar{c}_{24} = 1 \quad \bar{c}_{25} = 1 \quad \bar{c}_{31} = 4 \quad \bar{c}_{33} = 5$
$\bar{c}_{34} = 11 \quad \bar{c}_{42} = 1 \quad \bar{c}_{43} = 3 \quad \bar{c}_{45} = 0$

Hence the current solution is optimal.

$A \rightarrow C_1$: 300 units $\quad A \rightarrow C_3$: 100 units
$B \rightarrow C_2$: 300 units $\quad B \rightarrow C_3$: 200 units
$C \rightarrow C_2$: 100 units $\quad C \rightarrow C_5$: 600 units

Minimum cost is 1200 + 100 + 3000 + 1600 + 100 + 1200 + (200 + 300) = Rs 7700.

11.7 MAXIMIZATION TYPE

In general, transportation problem is motivated towards minimizing the transportation cost of goods from various sources to different destinations. But in certain cases the objective may be to maximize the profit or total output. To solve such type of problems of maximization we change the sign of the costs to negative and then apply minimization algorithm.

If it is found to be inconvenient to deal with negative figures we can subtract all the costs in the table from the highest cost given and then apply minimization process.

Example 11.13 Solve the following transportation problem to maximize the total profit. (The entries denote profits from sale of unit product.)

		Market				
		M_1	M_2	M_3	M_4	Supply
	S_1	15	51	42	33	23
Source	S_2	80	42	26	81	44
	S_3	90	40	66	60	33
Demand		23	31	16	30	100 balanced

Solution First we convert the problem to minimization problem by subtracting each element from the highest figure 90. We get the following table:

	M_1	M_2	M_3	M_4	Supply
S_1	75	39	48	57	23
S_2	10	48	64	9	44
S_3	0	50	24	30	33
Demand	23	31	16	30	

We obtain an initial basic feasible solution by VAM as given below

$(m + n - 1 = 4 + 3 - 1 = 6)$

	M_1	M_2	M_3	M_4	
S_1	– / 75	23 / 39	– / 48	– / 57	$u_1 = -9$
S_2	6 / 10	8 / 48	– / 64	30 / 9	$u_2 = 0$
S_3	17 / 0	– / 50	16 / 24	– / 30	$u_3 = -10$
	$v_1 = 10$	$v_2 = 48$	$v_3 = 34$	$v_4 = 9$	

Writing the equations $u_i + v_j = c_{ij}$ for the basic cells we get

$u_1 + v_2 = 39 \quad u_2 + v_4 = 9 \quad$ where $u_2 = 0$
$u_2 + v_1 = 10 \quad u_3 + v_1 = 0$
$u_2 + v_2 = 48 \quad u_3 + v_3 = 24$

The solutions are

$u_1 = -9 \quad v_1 = 10 \quad u_2 = 0 \quad v_2 = 48$
$u_3 = -10 \quad v_3 = 34 \quad v_4 = 9$

The relative cost factors for the non-basic cells are

$\bar{c}_{11} = 75 + 9 - 10 = 74$
$\bar{c}_{13} = 48 + 9 - 34 = 23$
$\bar{c}_{14} = 57 + 9 - 9 = 57$
$\bar{c}_{23} = 64 - 0 - 34 = 30$
$\bar{c}_{32} = 50 + 10 - 48 = 12$
$\bar{c}_{34} = 30 + 10 - 9 = 31$

$\bar{c}_{ij} \geq 0$ for all i, j. Hence the current solution is optimal and the optimal solution is

(Rs)

$S_1 \to M_2$: 23 units Profit $23 \times 51 = 1173$
$S_2 \to M_1$: 6 units Profit $6 \times 80 = 480$
$S_2 \to M_2$: 8 units Profit $8 \times 42 = 336$
$S_2 \to M_4$: 30 units Profit $30 \times 81 = 2430$
$S_3 \to M_1$: 17 units Profit $17 \times 90 = 1530$
$S_3 \to M_3$: 16 units Profit $16 \times 66 = 1056$

Total Profit = Rs 7005

EXERCISES

1. Solve the following transportation problems:

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	3	2	3	4	1	100
O_2	4	1	2	4	2	125
O_3	1	0	5	3	2	75
Demand	100	60	40	75	25	

2. A company has three mines A, B and C and five factories F_1, F_2, F_3, F_4 and F_5. The mines can supply 80, 100 and 140 tonnes of ore daily and the requirements of the factories are 40, 50, 70 and 80 respectively. The following table gives the unit transportation cost of ore.

	F_1	F_2	F_3	F_4	F_5
A	4	2	3	2	6
B	5	4	5	2	1
C	6	5	4	7	3

Give a distribution plan to minimize the total cost of transportation.

3. Solve the TP

	X	Y	Z	Availability
P	6	4	12	1
Q	10	6	5	8
R	15	16	8	7
Requirement	2	5	9	

4. Solve the TP

	W_1	W_2	W_3	W_4	Supply
F_1	2	3	4	5	400
F_2	3	2	3	4	500
F_3	4	3	3	4	600
F_4	6	4	4	5	700
	700	600	500	400	

5. Solve the TP

	D_1	D_2	D_3	D_4	Supply
A	6	4	1	5	14
B	8	9	2	7	16
C	4	3	6	2	5
Demand	6	10	15	4	

6. Solve the TP

	P	Q	R	S	Demand
A	8	9	6	3	170
B	6	11	5	10	200
C	3	8	7	9	180
Stock	150	160	110	130	(Take the transpose)

7. Solve the TP

	W_1	W_2	W_3	W_4	Supply
F_1	11	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Demand	5	8	7	14	

8. Solve the TP

	D_1	D_2	D_3	D_4	Availability
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirement	5	8	7	15	

9. Solve the TP

	A_1	A_2	A_3	A_4	A_5	A_6	Demand
D_1	5	3	7	3	8	5	3
D_2	5	6	12	5	7	11	4
D_3	2	8	3	4	8	2	2
D_4	9	6	10	5	10	9	8
Stock	3	3	6	2	1	2	(Take the transpose)

10. Solve the TP

	D_1	D_2	D_3	D_4	Stock
S_1	2	3	4	2	30
S_2	3	2	1	5	15
S_3	4	2	3	2	25
S_4	3	4	2	5	30
Demand	10	25	37	28	

11. Solve

	M_1	M_2	M_3	M_4	M_5	Supply
I	2	3	5	7	5	17
II	4	1	2	1	6	13
III	2	8	6	1	3	16
IV	5	3	7	2	4	20
	16	20	13	12	5	

12. Solve

	P	Q	R	S	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Demand	7	12	17	9	

13. Solve

	D_1	D_2	D_3	Supply
S_1	2	7	4	5
S_2	3	3	7	8
S_3	5	4	1	7
S_4	1	6	2	14
Demand	7	9	18	

14. Solve

	D_1	D_2	D_3	D_3	Supply
S_1	5	3	6	4	30
S_2	3	4	7	8	15
S_3	9	6	5	8	15
Demand	10	25	18	7	

15. Solve

	A	B	C	
X	4	8	8	76
Y	16	24	16	82
Z	8	16	24	77
	72	102	41	(unbalanced)

16. Solve

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6	6	3	8
S_2	4	7	7	6	5	5
S_3	8	4	6	6	4	9
Demand	4	4	5	4	8	(unbalanced)

17. Solve

	W_1	W_2	W_3	W_4	Supply
F_1	11	20	7	8	50
F_2	21	16	10	12	40
F_3	8	12	18	9	70
Demand	30	25	35	40	(Unbalanced)

18. Solve

	A	B	C	D	E	F	
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
O_3	6	5	9	11	3	11	2
O_4	6	8	11	2	2	10	9
	4	4	6	2	4	2	

19. Solve the following transportation problem to maximize the total profit.

	A	B	C	D	Availability
S_1	40	25	22	33	100
S_2	44	35	30	30	30
S_3	38	38	28	30	70
Demand	40	20	60	30	(unbalanced)

20. A company has four plants manufacturing the same product and five sales centres. The cost of manufacturing, cost of raw materials and capacities of the plants are given below.

	Plant			
	1	*2*	*3*	*4*
Manufacturing cost per unit	12	10	8	8
Cost of raw material per unit	8	7	7	5
Capacity	100	200	120	80

The sales prices, transportation costs and demands are given in the following table

Sales Centre	*Transportation cost per unit*				*Unit sales price*	*Demand*
	1	*2*	*3*	*4*		
A	4	7	4	3	30	80
B	8	9	7	8	32	120
C	2	7	6	10	28	150
D	10	7	5	8	34	70
E	2	5	8	9	30	90

Formulate this problem into a transportation problem to maximize profit and find the optimal solution.

21.

	W_1	W_2	W_3	W_4	W_5	Availability
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Demand	30	30	15	20	15	100

22.

	1	2	3	4	5	Availability
A	4	3	1	2	6	80
B	5	2	3	4	5	60
C	3	5	6	3	2	40
D	2	4	4	5	3	20
Demand	60	60	30	40	10	200

23.

	1	2	3	4	Availability
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	43

24.

	1	2	3	Availability
A	7	3	2	2
B	2	1	3	3
C	3	4	6	5
Demand	4	1	5	10

25.

	A	*B*	*C*	*D*	*E*	Availability
P	4	1	2	6	9	100
Q	6	4	3	5	7	120
R	5	2	6	4	8	120
Demand	40	50	70	90	90	340

26.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Supply
W_1	5	6	4	3	7	5	4	70
W_2	9	4	3	4	3	2	1	40
W_3	8	4	2	5	4	8	3	100
Demand	10	20	45	40	20	35	30	210

27.

	D_1	D_2	D_3	D_4	Availability
S_1	27	23	31	69	150
S_2	10	45	40	32	40
S_3	30	54	35	57	80
Demand	90	70	50	60	270

28.

	D_1	D_2	D_3	D_4	Availability
S_1	13	25	12	21	18
S_2	18	23	14	9	27
S_3	23	15	12	16	21
Demand	14	12	23	17	66

29.

	M_1	M_2	M_3	Availability
W_1	16	20	12	200
W_2	14	8	18	160
W_3	26	24	16	90
Demand	180	120	150	450

30.

	D_1	D_2	D_3	D_4	Availability
S_1	6	3	5	4	22
S_2	5	9	2	7	15
S_3	5	7	8	6	8
Demand	7	12	17	9	45

ANSWERS

1. $O_1-D_1: 25$; $O_1-D_4: 50$; $O_1-D_5: 25$
   $O_2-D_2: 60$; $O_2-D_3: 40$; $O_2-D_4: 25$
   O_3-D_1:75
   Total cost = Rs 615
2. $A-F_2: 50$; $A-F_4: 30$; $B-F_4: 50$
   $B-F_5$:50; $C-F_1: 40$; $C-F_3: 20$;
   $C-F_5: 30$
   Total cost = Rs 920
3. $P-X: 1$; $Q-X: 1$; $Q-$ Y : 5
   $Q-$ Z : 2; $R-Z=7$
   Total cost = Rs 142
4. $F_1-W_1: 400$; $F_2-W_2: 500$
   $F_3-W_1: 300$; $F_3-W_2: 100$
   $F_3-W_3: 200$; $F_4-W_3: 300$
   $F_4-W_4: 400$;
   Total cost = Rs 142
5. $A-D_1: 4$; $A-D_2: 10$; $B-D_1: 1$;
   $B-D_3: 15$; $C-D_1: 1$; $C-D_4: 4$;
   Total cost = Rs 114
6. $P-C: 150$; $Q-A: 40$; $Q-B: 90$
   $Q-C: 30$; $R-B: 110$; $S-A: 130$
   Total cost = Rs 2980
7. $F_1-W_1: 5$; $F_2-W_2: 2$; $F_3-W_2: 6$
   $F_1-W_4: 2$; $F_2-W_3: 7$; $F_3-W_4: 12$
   Total cost = Rs 743
8. $P_1-D_1: 5$; $P_2-D_2: 3$; $P_3-D_2: 5$
   $P_1-D_4: 2$; $P_2-D_3: 7$; $P_3-D_4: 13$
   Total cost = Rs 201
9. $A_1-D_2: 3$; $A_2-D_1: 1$; $A_2-D_4: 2$
   $A_3-D_3: 2$; $A_3-D_4: 4$; $A_4-D_4: 2$
   $A_5-D_2: 1$; $A_6-D_1: 2$
   Total cost = Rs 103
10. $S_1-D_1: 2$; $S_1-D_4: 28$; $S_2-D_3: 15$
    $S_3-D_2: 25$; $S_4-D_1: 8$; $S_4-D_3: 22$
    Total cost = Rs 193
11. I $-M_1: 16$; I $-M_2: 1$; II $-M_3: 13$
    III $-M_4: 12$; III $-M_5: 4$; IV $-M_2: 19$
    IV $-M_5: 1$
    Total cost = Rs 146
12. $A-P: 7$; $A-Q: 4$; $A-R: 2$
    $A-S: 9$; $B-R: 15$; $C-Q: 8$
    Total cost = Rs 186
13. S_1-D_1-5; $S_2-D_2: 8$; $S_3-D_2=1$
    $S_3-D_3: 6$; $S_4-D_1: 2$; $S_4-D_3: 12$
    Total cost = Rs 70
14. $S_1-D_2: 20$; $S_1-D_3: 3$; $S_1-D_4=7$
    $S_2-D_1: 10$; $S_2-D_2: 5$; $S_3-D_3: 15$
    Total cost = Rs 231
15. $X-B: 76$; $Y-B: 21$; $Y-C: 41$
    $Z-A: 72$; $Z-B: 5$
    Total cost = Rs 2424
16. $S_1-D_5: 8$; $S_2-D_1: 4$; $S_2-D_4: 1$
    $S_3-D_2: 4$; $S_3-D_3: 2$; $S_3-D_4: 3$
    Total cost = Rs 92
17. $F_1-W_3: 25$; $F_1-W_4: 25$; $F_2-W_3: 10$
    $F_3-W_1: 30$; $F_3-W_2: 25$; $F_3-W_4: 15$
    Total cost = Rs 1150
18. $O_1-C: 5$; $O_2-B: 4$; $O_2 \to F: 2$
    $O_3-A: 1$; $O_3-C: 1$; $O_4-A: 3$
    $O_4-D: 2$; $O_4-E: 4$
    Total cost = Rs 112
19. $S_1-A: 20$; $S_1-D: 30$; $S_2-A: 20$
    $S_2-C: 10$; $S_3-B: 20$; $S_3-C: 50$;
    Max. Profit = Rs 5130
20. Profit matrix

	1	2	3	4	5	Demand
A	6	6	11	15	0	80
B	4	6	10	12	0	120
C	6	4	7	6	0	150
D	4	10	14	14	0	70
E	8	8	7	9	0	90
Supply	100	200	120	80	10	

    $B-2: 70$; $B-3: 50$; $C-1: 100$
    $C-2: 40$; $D-3: 70$; $E-2: 90$;
21. $F_1-W_1: 5$; $F_1-W_3: 15$; $F_1-W_4: 20$
    $F_2-W_2: 30$; $F_3-W_1: 15$; $F_3-W_5: 5$
    $F_4-W_1: 10$
    Total cost = Rs 510
22. $A-1: 10$; $A-3: 30$; $A-4: 40$
    $B-2: 60$; $C-1: 30$; $C-5: 10$;
    $D-1: 20$; Total cost = Rs 420
23. I – 4 : 11; II – 1 : 6; II – 2 : 3
    II – 4 : 4; III – 2 : 7; III – 3 : 12
    Total cost = Rs 796

24. $A-3:2$; $B-2:1$; $B-3:2$;
$C-1:4$; $C-3:1$
Total cost = Rs 29
25. $P-A:10$; $P-B:50$; $P-C:40$
$Q-C:30$; $Q-E:90$; $R-A:30$
$R-D:90$; Total cost = Rs 1400
26. $W_1-S_1:10$; $W_1-S_4:40$; $W_1-S_7:10$
$W_2-S_7:5$; $W_2-S_6:35$; $W_3-S_2:20$
$W_3-S_3:45$; $W_3-S_5:20$; $W_3-S_7:15$
Total cost = Rs 580
27. $S_1-D_1:30$; $S_1-D_2:70$; $S_1-D_3:50$
$S_2-D_4:40$; $S_3-D_1:60$; $S_3-D_4:20$
Total cost = Rs 8190
28. $S_1-D_1:14$; $S_1-D_3:4$; $S_2-D_3:10$
$S_2-D_4:17$; $S_3-D_2:12$; $S_3-D_3:9$
Total cost = Rs 811
29. $W_1-M_1:140$; $W_1-M_3:60$
$W_2-M_1:40$; $W_2-M_2:120$
$W_3-M_3:90$; Total cost = Rs 5920
30. $S_1-D_1:7$; $S_1-D_2:4$; $S_1-D_3:2$
$S_1-D_4:9$; $S_2-D_3:15$; S_3-D_2: 8
Total cost = Rs 186

12

Assignment Problems

CONCEPT REVIEW

12.1 WHAT IS AN ASSIGNMENT PROBLEM?

Given n jobs, n workers, and the time taken by each worker to do each job, the objective is to allot the different jobs to the workers such that (i) no worker is given more than one job (ii) no job is allotted to more than one worker and (iii) the total time taken to complete all the jobs is minimum. This is called an *assignment problem.*

The problem can also have cost or profit instead of time taken for each job. In that case we have to maximize the total profit or minimize the total cost. In other words we have to optimize the objective assigning the jobs to the workers suitably. This is the same as transportation problem except that here the availability at each source and the demand at each destination are taken to be one.

12.2 MATHEMATICAL FORMULATION

We are given n jobs $J_1, J_2, J_3, \ldots, J_n$ and n workers $W_1, W_2, W_3, \ldots, W_n$. The time taken by the i^{th} worker to do the j^{th} job (Cost of the assignment) is given as c_{ij}, and x_{ij} denotes the assignment. The problem can be written in the form

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_j$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1 \; (i = 1, 2, 3, \ldots, n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \; (j = 1, 2, 3, \ldots, n)$$

$x_{ij} = 0$ or 1 for all i and j.

We take $x_{ij} = 1$ if i^{th} worker gets j^{th} job.

$x_{ij} = 0$ otherwise

We may note that in the table of assignment there would be one and only one assignment in each row or column. In other words, if $x_{ij} = 1$ then there is no other assignment in the i^{th} row and no other assignment in the j^{th} column.

The data is given in the form of a table (matrix) as given below:

	J_1	J_2	J_3	...	...	...	J_n
W_1	c_{11}	c_{12}	c_{13}	.	.	.	c_{1n}
W_2	c_{21}	c_{22}	c_{23}	.	.	.	c_{2n}
W_3	c_{31}	c_{32}	c_{33}	.	.	.	c_{3n}
.							
.							
.							
W_n	c_{n1}	c_{n2}	c_{n3}				c_{nn}

12.3 METHOD OF SOLVING AN AP

First of all the problem can be solved by finding a list of all possible solutions which means that we have to evaluate $n!$ assignments and choose the optimal one from among them. When n is large, this method involves a large number of calculations and hence it is not suitable for manual calculations.

Another method of solving an AP is by the simplex method. We can treat it as a 0 – 1 integer programming problem. There are n^2 decision variables and $2n$ equalities. This again is difficult to solve when n becomes large.

An assignment problem can be taken as a transportation problem wherein we have n basic cells. But we must have $2n - 1$ $(n + n - 1)$ basic cells. Hence we have to introduce $n - 1$ dummy allotments in order to apply the transportation algorithm. Thus it gives rise to computational difficulties.

The Hungarian method which we are going to discuss, is the best choice for solving an AP. The algorithm is very easy to apply and there is no difficulty in manual calculations.

12.4 STEP-BY-STEP PROCEDURE OF HUNGARIAN METHOD (MINIMIZATION PROBLEM)

Step 1 In the assignment table subtract the least element of the first row from all the elements in the first row. Subtract the least element of the second row from all the elements of the second row. Repeat this for all the rows one by one. Having finished all the rows, subtract the least element of each column from all the elements of that column.

Step 2 Draw horizontal and vertical lines to cover all the zeroes obtained in step 1. If the minimum number of lines required to cover all the zeroes is equal to n (number of jobs) then optimal solution is reached. Go to step 4.

Step 3 If the minimum number of lines required to cover all the zeroes is less than n, it means that the current solution is not optimal. Examine the elements not covered by any of the lines drawn. Select the least among these elements and subtract it from all the elements not covered by the lines. Add this least element to every element lying at the intersections of the lines. This gives rise to a new table. Again draw vertical and horizontal lines to pass through the zeroes. Check whether the minimum number of lines required, is equal to n. Repeat step 3 till an optimal table is reached.

Step 4 **(Obtaining the optimal solution)** In the optimal table examine the first row. If there is only one zero in the first row, square it and cancel all the other zeroes in that column. If there are more than one zero in the first row, pass on to the second row. If there is only one zero in the second row square it and cancel all the other zeroes in that column. In case the second row contains more than one zero leave it and pass on to the third row. After exhausting all the rows, take the columns one by one and repeat the procedure of squaring the zeroes. If there is a tie use your choice and square the zeroes. Finally we get the optimal solution with n zeroes (each row having exactly one zero and each column having exactly one zero). A zero in the (i, j) cell indicates that i^{th} worker is allotted with the j^{th} job. This cell is a *basic cell*. In the original assignment table add all the times (costs) of the basic cells. This gives the total minimum time required (minimum cost) to complete all the jobs.

Example 12.1 Solve the following assignment problem.

Jobs

Workers		J_1	J_2	J_3	J_4
	W_1	10	15	24	30
	W_2	16	20	28	10
	W_3	12	18	30	16
	W_4	9	24	32	18

Solution

Step 1 Subtract the least element of each row from all the elements of the respective row. We get

	J_1	J_2	J_3	J_4
W_1	0	5	14	20
W_2	6	10	18	0
W_3	0	6	18	4
W_4	0	15	23	9

Subtract the least element of each column from all the elements of the respective column. We have

	J_1	J_2	J_3	J_4
W_1	0	0	0	20
W_2	6	5	4	0
W_3	0	1	4	4
W_4	0	10	9	9

Step 2 All the zeroes can be covered by a minimum of 3 lines as shown below.

	J_1	J_2	J_3	J_4
W_1	0	0	0	20
W_2	6	5	4	0
W_3	0	1	4	4
W_4	0	10	9	9

Since the number of lines is 3 < 4 we go to step 3.

Step 3 The least element not covered by any line is 1. Subtract 1 from all the elements not on the lines and add 1 to the elements at the points of intersection of the lines. We obtain

	J_1	J_2	J_3	J_4
W_1	1	0	0	20
W_2	7	5	4	0
W_3	0	0	3	3
W_4	0	9	8	8

Now we find that the minimum number of lines required to cover all the zeroes is 4 ($n = 4$). Hence an optimal solution is reached.

Step 4 In the first row there are two zeroes. Therefore take the second row and there is only one zero in (2, 4) cell. Squaring it we have

	J_1	J_2	J_3	J_4
W_1	1	0	0	20
W_2	7	5	4	$\boxed{0}$
W_3	0	0	3	3
W_4	0	9	8	8

The third row has two zeroes. Go to the fourth row. Square the only zero in (4, 1) cell and cancel the zero in the first column (3, 1) cell. We have

	J_1	J_2	J_3	J_4
W_1	1	0	0	20
W_2	7	5	4	$\boxed{0}$
W_3	$\cancel{0}$	0	3	3
W_4	$\boxed{0}$	9	8	8

Now examine the columns. In the first column one zero has been squared already. In the second column there are two zeroes. Go to the third column. There is only one zero in the third column (1, 3) cell. Square it and cancel the zero in (1, 2) cell. This gives

	J_1	J_2	J_3	J_4
W_1	1	$\cancel{0}$	$\boxed{0}$	20
W_2	7	5	4	$\boxed{0}$
W_3	$\cancel{0}$	0	3	3
W_4	$\boxed{0}$	9	8	8

Now there is only one zero in the third row (or second column) (3, 2) cell.

Squaring that zero also we get the optimal solution table.

	J_1	J_2	J_3	J_4
W_1	1	$\cancel{0}$	$\boxed{0}$	20
W_2	7	5	4	$\boxed{0}$
W_3	$\cancel{0}$	$\boxed{0}$	3	3
W_4	$\boxed{0}$	9	8	8

The optimal assignment is

$W_1 \to J_3$: 24

$W_2 \to J_4$: 10

$W_3 \to J_2$: 18

$W_4 \to J_1$: 9

Total = 61 Optimum cost = 61

Example 12.2 The assignment costs of four operators to four machines are given in the following table.

Operators

Machines		I	II	III	IV
	A	10	5	13	15
	B	3	9	18	3
	C	10	7	3	2
	D	5	11	9	7

Find the optimal assignment using the Hungarian method.

Solution The assignment table is

	I	II	III	IV
A	10	5	13	15
B	3	9	18	3
C	10	7	3	2
D	5	11	9	7

Subtracting the least element of each row from all the elements of the same row, we get

	I	II	III	IV
A	5	0	8	10
B	0	6	15	0
C	8	5	1	0
D	0	6	4	2

Subtracting the least element of each column from all the elements of the same column, we get

	I	II	III	IV
A	5	0	7	10
B	0	6	14	0
C	8	5	0	0
D	0	6	3	2

Drawing the minimum number of lines to cover all the zeroes we have

	I	II	III	IV
A	5	0	7	10
B	0	6	14	0
C	8	5	0	0
D	0	6	3	2

We find that the minimum number of lines required to cover all the zeroes is 4 which is the order of the matrix. Hence optimality is reached.

The optimal solution is obtained as follows. There is only one zero in the first row (1, 2) cell square it. There are two zeroes in the 2nd and 3rd row. But there is only one zero in the 4th row (4, 1) cell. Square it and cancel the zero in the (2, 1) cell. We have

	I	II	III	IV
A	5	$\boxed{0}$	7	10
B	$\cancel{0}$	6	14	0
C	8	5	0	0
D	$\boxed{0}$	6	3	2

Now consider the unassigned columns. In the 3rd column there is only one zero in the (3, 3) cell.

Square it and cancel the zero in the (3, 4) cell. This gives rise to only one zero in the fourth column (2, 4) cell. Square it. Thus the optimal assignment is

	I	II	III	IV
A	5	$\boxed{0}$	7	10
B	$\cancel{0}$	6	14	$\boxed{0}$
C	8	5	$\boxed{0}$	$\cancel{0}$
D	$\boxed{0}$	6	3	2

The optimal solution is

$A \to$ II : 5
$B \to$ IV : 3
$C \to$ III : 3
$D \to$ I : 5
Total = 16

Minimum total cost = Rs 16.

Example 12.3 Solve the following AP:

Machines

		M_1	M_2	M_3	M_4
Jobs	J_1	18	26	17	11
	J_2	13	28	14	26
	J_3	38	19	18	15
	J_4	19	26	24	10

Solution Subtracting the least element of each row and column from all the elements of the respective row and column we get the reduced matrix as follows.

	M_1	M_2	M_3	M_4
J_1	7	11	5	0
J_2	0	11	0	13
J_3	23	0	2	0
J_4	9	12	13	0

The minimum number of lines required to cover all the zeroes is 3 as shown below:

	M_1	M_2	M_3	M_4
J_1	7	11	5	0
J_2	0	11	0	13
J_3	23	0	2	0
J_4	9	12	13	0

Since the number of lines is less than 4, optimality is not reached.

Select the least element 5 not covered by the lines and subtract from all the elements not on the lines and add it to the elements at the points of intersection of the lines. The resulting table with the minimum number of lines covering all the zeroes is

	M_1	M_2	M_3	M_4
J_1	2	6	0	0
J_2	0	11	0	18
J_3	23	0	2	5
J_4	4	7	8	0

Since the number of lines is 4, optimality is reached.

Square the zero in (3, 2) cell and the zero in the (4, 4) cell and cancel (1, 4) cell. Next square the zero in (2, 1) cell and cancel (2, 3) cell. Square the zero in (1, 3) cell. This gives the optimal solution as given below.

	M_1	M_2	M_3	M_4
J_1	2	6	$\boxed{0}$	$\cancel{0}$
J_2	$\boxed{0}$	11	$\cancel{0}$	18
J_3	23	$\boxed{0}$	2	5
J_4	4	7	8	$\boxed{0}$

$J_1 \to M_3$: 17
$J_2 \to M_1$: 13
$J_3 \to M_2$: 19
$J_4 \to M_4$: 10
Total = 59

Example 12.4 Solve the following AP

Machines

		1	2	3	4	5
Operators	*A*	5	5	–	2	6
	B	7	4	2	3	4
	C	9	3	5	–	3
	D	7	2	6	7	2
	E	6	5	7	9	1

Solution Here we find that the operator *A* cannot operate the machine 3 and the operator *C* cannot operate the machine 4. Hence we assume that the time taken by *A* on 3 and by *C* on 4 are ∞ each.

Therefore the table is

	1	2	3	4	5
A	5	5	∞	2	6
B	7	4	2	3	4
C	9	3	5	∞	3
D	7	2	6	7	2
E	6	5	7	9	1

Subtracting the least element from all the elements in each row we get

	1	2	3	4	5
A	3	3	∞	0	4
B	5	2	0	1	2
C	6	0	2	∞	0
D	5	0	4	5	0
E	5	4	6	8	0

Subtracting the least element in each column from all the elements of that column, we obtain

	1	2	3	4	5
A	0	3	∞	0	4
B	2	2	0	1	2
C	3	0	2	∞	0
D	2	0	4	5	0
E	2	4	6	8	0

Let us try to cover all the zeroes by the minimum number of vertical and horizontal lines. We find that just 4 lines are sufficient to cover all the zeroes, as shown below:

	1	2	3	4	5
A	0	3	∞	0	4
B	2	2	0	1	2
C	3	0	2	∞	0
D	2	0	4	5	0
E	2	4	6	8	0

Since the number of rows is 5 and we have only 4 lines this solution is not optimal.

Now, the least element not covered by any line, is 1. Subtract 1 from all the elements not on the lines and add 1 to the elements at the points of intersection of lines. We get

	1	2	3	4	5
A	0	4	∞	0	5
B	1	2	0	0	2
C	2	0	2	∞	0
D	1	0	4	4	0
E	1	4	6	7	0

Again we find that 4 lines are enough to cover all the zeroes. Hence we repeat the step. The smallest element not covered by any line is 1. Subtract 1 from all the elements not on the lines and add 1 to the elements at the intersection of the lines. Now we have

	1	2	3	4	5
A	0	5	∞	0	6
B	1	3	0	0	3
C	1	0	1	∞	0
D	0	0	3	3	0
E	0	4	5	6	0

Now the minimum number of lines required to cover all the zeroes is 5 and hence optimality is reached.

All the rows contain more than one zero. The third column contains only one zero. Square the zero in (2, 3) cell and cancel the zero in (2, 4) cell. Now fourth column has only one zero in (1, 4) cell. Square it and cancel the zero in (1, 1) cell. We have

	1	2	3	4	5
A	$\cancel{0}$	5	∞	$\boxed{0}$	6
B	1	3	$\boxed{0}$	$\cancel{0}$	3
C	1	0	1	∞	0
D	0	0	3	3	0
E	0	4	5	6	0

Now there is a tie in each row and column. We use our choice.

Square the zero in the (3, 5) cell and cancel the other zeroes in the 3rd row and the 5th column. Now the 2nd column contains only one zero in the (4, 2) cell. Square it and cancel the zero in (4, 1) cell. The remaining only zero in (5, 1) can be squared now. This gives the optimal assignment

	1	2	3	4	5
A	$\cancel{0}$	5	∞	$\boxed{0}$	6
B	1	3	$\boxed{0}$	$\cancel{0}$	3
C	1	$\cancel{0}$	1	∞	$\boxed{0}$
D	$\cancel{0}$	$\boxed{0}$	3	3	$\cancel{0}$
E	$\boxed{0}$	4	5	6	$\cancel{0}$

The optimal solution is

$A \rightarrow 4$ time 2
$B \rightarrow 3$ time 2
$C \rightarrow 5$ time 3
$D \rightarrow 2$ time 2
$E \rightarrow 1$ time 6
Total time = 15

For any other choice also we shall get the same total time 15 even though the assignments may be different.

12.5 MAXIMIZATION ASSIGNMENT PROBLEM

Sometimes we may have to deal with an assignment problem in which the objective is to maximize the profit, maximize the sales or maximize the total output of the machines. Since max $Z = -\min(-Z)$, we multiply all the elements of the table by –1 and then apply the Hungarian method. Also we can apply another method. Subtract all the elements of the matrix from the largest element in the matrix and form a new table to apply the Hungarian method.

Example 12.5 A company has 4 salesmen *A*, *B*, *C* and *D*. These salesmen are to be allotted 4 districts 1, 2, 3 and 4. The estimated profit per day for each salesman in each district is given in the following table.

	1	2	3	4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

What is the optimal assignment which will yield maximum profit?

Solution The largest element in the table is 16. Subtracting all the elements from 16 we get the table

	1	2	3	4
A	0	6	2	5
B	2	5	1	1
C	1	1	3	4
D	3	4	2	1

Subtract the least element of each row from all the elements of that row. We get

	1	2	3	4
A	0	6	2	5
B	1	4	0	0
C	0	0	2	3
D	2	3	1	0

Each column has at least one zero. Minimum number of lines required to cover all the zeroes is 4 (order of the matrix). Therefore optimality is reached.

Square the only zero in (1, 1) cell of the first row and cancel the zero in (3, 1) cell. Square the zero in (3, 2) cell. Square the only zero of the 4th row in (4, 4) cell and cancel the zero in (2, 4) cell. Now square the zero in (2, 3) cell. We obtain the optimal assignment as follows.

	1	2	3	4
A	$\boxed{0}$	6	2	5
B	1	4	$\boxed{0}$	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	2	3
D	2	3	1	$\boxed{0}$

			(Rs)
A	$\to 1$	Profit	16
B	$\to 3$	Profit	15
C	$\to 2$	Profit	15
D	$\to 4$	Profit	15

Total Profit = Rs 61.

12.6 UNBALANCED ASSIGNMENT PROBLEM

If the number of jobs and the number of workers (machines) are not equal it is called an *unbalanced assignment problem*. As in the case of transportation problem, we introduce a dummy job or a dummy worker in order to balance the problem. We assign the cost zero for every cell in the newly introduced row or column. Now we can apply Hungarian method and solve this balanced assignment problem.

Example 12.6 Three jobs are to be done by 4 machines. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Job	M_1	M_2	M_3	M_4
J_1	18	24	28	32
J_2	8	13	17	19
J_3	10	15	19	22

What are the job assignments which will minimize the total cost?

Solution It is an unbalanced problem with 3 jobs and 4 machines. Therefore we introduce a dummy job J_4 having all the costs zero in the 4^{th} row. We have the balanced problem

	M_1	M_2	M_3	M_4
J_1	18	24	28	32
J_2	8	13	17	19
J_3	10	15	19	22
J_4	0	0	0	0

Subtracting the least element from all the elements of each row we get

	M_1	M_2	M_3	M_4
J_1	0	6	10	14
J_2	0	5	9	11
J_3	0	5	9	12
J_4	0	0	0	0

We find that only two lines are necessary to cover all the zeroes and hence it is not optimal. Subtract the least element 5, not covered by the lines from all the elements not on the line and add 5 to the point of intersection of the lines we get

	M_1	M_2	M_3	M_4
J_1	0	1	5	9
J_2	0	0	4	6
J_3	0	0	4	7
J_4	5	0	0	0

Again we find that all the zeroes can be covered by only 3 lines and hence optimality is not reached.

Subtract the least element 4 from all the elements not on the lines and add 4 to the points of intersection of the lines. We have the new table

	M_1	M_2	M_3	M_4
J_1	0	1	1	5
J_2	0	0	0	2
J_3	0	0	0	3
J_4	9	4	0	0

This table requires at least 4 lines to cover all the zeroes and hence it gives optimal solution.

Square the zero in (1, 1) cell and cancel the other zeroes in the first column. Square the zero in (4, 4) cell and cancel the zero in the (4, 3) cell. We get

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	1	1	5
J_2	$\cancel{0}$	0	0	2
J_3	$\cancel{0}$	0	0	3
J_4	9	4	$\cancel{0}$	$\boxed{0}$

Now there is a tie. Square the zeroes in (2, 3) and (3, 2) cells or square the zeroes in (2, 2) and

(3, 3) cells. We get two optimal solutions given below:

(i)

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	1	1	5
J_2	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
J_3	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	3
J_4	9	4	$\cancel{0}$	$\boxed{0}$

			(Rs)
J_1	$\to M_1$	Cost	18
J_2	$\to M_3$	Cost	17
J_3	$\to M_2$	Cost	15

Total cost = Rs 50.

M_4 remains idle

(ii)

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	1	1	5
J_2	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
J_3	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
J_4	9	4	$\cancel{0}$	$\boxed{0}$

			(Rs)
J_1	$\to M_1$	Cost	18
J_2	$\to M_2$	Cost	13
J_3	$\to M_3$	Cost	19

Total cost = Rs 50.

M_4 remains idle

12.7 TRAVELLING SALESMAN PROBLEM

A travelling salesman goes from one city to another, visiting each city only once and returns to the starting city after visiting all the other cities. The cost of his travel from each city to every other city is given. The objective is to give a route such that the total cost is minimum.

Thus the travelling salesman problem is also an assignment problem in which (i) going from a city to itself is not allowed. Hence (i, i) cells are given the cost ∞ (ii) the travel should be a cycle passing through all the cities and no city to be visited twice. With these restrictions we solve the problem as an assignment problem. If the optimal solution does not satisfy the route (cycle) condition we make adjustments in the assignment and make it a cycle with minimum possible increase in the total cost.

Example 12.7 Solve the travelling salesman problem with the following cost matrix:

Cities

Cities	A	B	C	D
A	∞	46	16	40
B	41	∞	50	40
C	82	32	∞	60
D	40	40	36	∞

Solution The given cost matrix is

	A	B	C	D
A	∞	46	16	40
B	41	∞	50	40
C	82	32	∞	60
D	40	40	36	∞

Subtracting the least element of each row from all the elements of that row, we get

	A	B	C	D
A	∞	30	0	24
B	1	∞	10	0
C	50	0	∞	28
D	4	4	0	∞

Subtracting the least element of each column from the other elements of that column, we get

	A	B	C	D
A	∞	30	0	24
B	0	∞	10	0
C	49	0	∞	28
D	3	4	0	∞

We note that the minimum number of lines required to cover all the zeroes is only 3 whereas the order of the matrix is 4. Hence optimality is not reached.

Subtract 3 from all the elements not on the lines and add 3 to the points of intersection of the lines. This gives,

	A	B	C	D
A	∞	27	0	21
B	0	∞	13	0
C	49	0	∞	28
D	0	1	0	∞

Optimality is reached. Square the zero in (1, 3) cell and the zero in (3, 2) cell. Now square the zero in (2, 4) cell and cancel the (2, 1) cell. Finally square the zero in (4, 1) cell. The optimal solution is reached.

	A	B	C	D
A	∞	27	0	21
B	0	∞	13	0
C	49	0	∞	28
D	0	1	0	∞

	(Rs)
$A \to C$	16
$B \to D$	40
$C \to B$	32
$D \to A$	40

Total Cost = Rs 128

We get $A \to C,\ C \to B,\ B \to D,\ D \to A$

or $A \to C \to B \to D \to A$

which is a cycle. Hence the solution is $A \to C \to B \to D \to A$

with minimum total cost Rs 128.

Example 12.8 A salesman has to visit five cities A, B, C, D and E. The distances (in 100 km) between any two cities are given in the following table. The salesman starts from A and has to come back to A after visiting all the other cities in a cycle. Which route he has to select in order that the total distance travelled by him is minimum?

From	A	B	C	D	E
A	–	4	7	3	4
B	4	–	6	3	4
C	7	6	–	7	5
D	3	3	7	–	7
E	4	4	5	7	–

Solution Assign ∞ value to the diagonal cells and consider the assignment problem

	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

Subtracting the least element of each row from all the elements of that row we obtain the table,

	A	B	C	D	E
A	∞	1	4	0	1
B	1	∞	3	0	1
C	2	1	∞	2	0
D	0	0	4	∞	4
E	0	0	1	3	∞

Subtracting the least element in each column from all the elements of that column we get

	A	B	C	D	E
A	∞	1	3	0	1
B	1	∞	2	0	1
C	2	1	∞	2	0
D	0	0	3	∞	4
E	0	0	0	3	∞

Just with 4 lines all the zeroes have been covered (4 < 5). Hence the table is not optimal.

Subtract 1 from all the elements not on the lines and add 1 to the elements at the points of intersection of the lines. The resulting table is

∞	0	2	0	0
0	∞	1	0	0
2	1	∞	3	0
0	0	3	∞	4
0	0	0	4	∞

Since the minimum number of lines required to cover all the zeroes is 5 we have reached optimality.

Square the zero in (3, 5) cell in the 3rd row and cancel the other zeroes in the 5th column. Square the zero in (5, 3) cell in the 3rd column and cancel the other zeroes in the 5th row. We have

	A	B	C	D	E
A	∞	0	2	0	$\cancel{0}$
B	0	∞	1	0	$\cancel{0}$
C	2	1	∞	3	$\boxed{0}$
D	0	0	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

Now make arbitrary assignments. Square the zero in (1, 4) cell. Then we can square zero in (2, 1) cell and finally square the zero in (4, 2). We get an optimal solution to the assignment problem.

	A	B	C	D	E
A	∞	$\cancel{0}$	2	$\boxed{0}$	$\cancel{0}$
B	$\boxed{0}$	∞	1	$\cancel{0}$	$\cancel{0}$
C	2	1	∞	3	$\boxed{0}$
D	$\cancel{0}$	$\boxed{0}$	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

$A \to D$

$B \to A$

$C \to E$

$D \to B$

$E \to C$

We get

$$A \to D \to B \to A, \quad C \to E \to C$$

It is not a cycle. Hence we have to modify the solution. The smallest non-zero element is 1 in (2, 3) and (3, 2) cells. Let us make an assignment to (2, 3) cell. Now we get

$$A \to D \quad B \to C \quad C \to E \quad D \to B \quad E \to A$$

i.e. $\quad A \to D \to B \to C \to E \to A$

which is the optimal route and the total distance is

$$3 + 3 + 6 + 5 + 4 = 21 \text{ (2100 km)}$$

EXERCISES

1. Solve the AP

	J_1	J_2	J_3	J_4	J_5
A	20	15	25	25	29
B	13	19	30	13	19
C	20	17	14	12	15
D	14	20	20	16	24
E	14	16	19	11	22

2.

	J_1	J_2	J_3	J_4
W_1	20	13	7	5
W_2	25	18	13	10
W_3	31	23	18	15
W_4	45	40	23	21

3.

	W_1	W_2	W_3	W_4	W_5
J_1	10	15	13	15	16
J_2	13	9	18	13	10
J_3	10	9	12	12	12
J_4	15	11	9	9	12
J_5	11	9	10	14	12

4.

	I	II	III	IV	V
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

5.

	J_1	J_2	J_3	J_4
W_1	5	7	11	6
W_2	8	5	9	6
W_3	4	7	10	7
W_4	10	4	8	3

6.

	Machines			
Job	I	II	III	IV
1	12	30	21	15
2	18	33	9	31
3	44	25	24	21
4	23	30	28	14

7. The profit per day of 4 salesmen in 4 districts are given below. Find an optimal assignment of the districts to the salesmen so as to maximize the total profit.

Districts

Salesman	1	2	3	4
A	14	8	12	9
B	12	9	13	13
C	13	13	11	10
D	11	10	12	13

8. The efficiency of 5 machines on each of 5 jobs is given below. Determine an assignment schedule of the jobs to the machines such that the total efficiency is maximum.

Job

Machine	1	2	3	4	5
I	62	78	50	101	82
II	70	85	60	75	55
III	88	96	118	85	71
IV	48	64	87	77	80
V	60	70	98	66	83

9. There are 5 jobs and 4 machines. The expected profits on each job on each machine is given below. Determine an optimal assignment of the machines to the jobs so that the total profit is maximum.

Job

Machine	1	2	3	4	5
I	62	78	50	101	82
II	71	84	61	73	59
III	87	92	111	71	81
IV	48	64	87	77	80

Hint: Introduce a dummy machine V with zero profit for each job. Then convert into the minimization type.

10. Solve the AP and find the minimum cost

	W_1	W_2	W_3	W_4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

11. Solve the AP

Jobs

Workers	I	II	III	IV
W_1	12	9	12	9
W_2	15	–	13	20
W_3	4	8	10	6

12. Solve

	M_1	M_2	M_3	M_4
J_1	60	51	32	32
J_2	48	–	37	43
J_3	39	26	–	33
J_4	40	–	51	30

13. Solve

Machine

Operators	M_1	M_2	M_3	M_4
O_1	5	5	–	2
O_2	7	4	2	3
O_3	9	3	5	–
O_4	7	2	6	7

14. Solve the following travelling salesman problem:

	A	B	C	D
A	–	4	7	3
B	4	–	6	3
C	7	6	–	7
D	3	3	7	–

15. Solve the following travelling salesman problem

	A	B	C	D	E	F
A	–	5	12	6	4	8
B	6	–	10	5	4	3
C	8	7	–	6	3	11
D	5	4	11	–	5	8
E	5	2	7	8	–	4
F	6	3	11	5	4	–

16. Solve the following AP.

	A	B	C
1	120	100	80
2	80	90	110
3	110	140	120

17.

	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

18.

	P	Q	R	S	T
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

19.

	I	II	III	IV	V
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

20.

	A	B	C	D	E
1	160	130	175	190	200
2	135	120	130	160	175
3	140	110	155	170	185
4	50	50	90	80	110
5	55	35	70	80	105

21.

	A	B	C	D
I	200	200	400	100
II	300	100	300	300
III	400	100	100	500
IV	200	200	400	200

22.

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

23.

	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

24.

	Job				
Machine	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	–	10	9
M_3	–	11	14	11	7
M_4	14	8	12	7	8

25.

	1	2	3	4
A	10	24	30	15
B	16	22	28	12
C	12	20	32	10
D	9	26	34	16

26.

	Job			
	1	2	3	4
A	4	7	5	6
B	–	8	7	4
C	3	–	5	3
D	6	6	4	2

27. The expected sales by four salesmen in each of the four zones are given below

	1	2	3	4
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

Determine the optimal assignment so as to maximize the sales.

28. Maximization type

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

29. Maximization type

	1	2	3	4	5
A	2	8	∞	5	6
B	0	1	8	2	6
C	5	6	1	4	∞
D	7	4	8	2	3
E	5	4	0	6	7

30. A company has 4 plants each of which can produce any one of the 4 products A, B, C, or D. Production costs and sales revenue of each plant are given below:

Production Cost

Plant

Product		1	2	3	4
	A	49	60	45	61
	B	55	63	45	49
	C	52	62	49	68
	D	55	64	48	66

Sales Revenue

Plant

Product		1	2	3	4
	A	50	68	49	62
	B	60	70	51	74
	C	55	67	53	70
	D	58	65	54	68

Determine which product should each plant produce in order to maximize the profit (Profit = Revenue – Cost)

ANSWERS

1. $A \to J_2$, $B \to J_5$, $C \to J_3$, $D \to J_1$, $E \to J_4$ Total cost = 73.
2. $W_1 \to J_1$, $W_2 \to J_2$, $W_3 \to J_4$, $W_4 \to J_3$ Total cost = 76.
3. $J_1 \to W_1$, $J_2 \to W_5$, $J_3 \to W_2$, $J_4 \to W_4$, $J_5 \to W_3$ Total cost = 48
4. $A \to V$, $B \to$ I, $C \to$ IV, $D \to$ III, $E \to$ II Total cost = 9.
5. $W_1 \to J_1$, $W_2 \to J_2$, $W_3 \to J_3$, $W_4 \to J_4$ Total cost = 23.
6. $1 \to$ I, $2 \to$ III, $3 \to$ II, $4 \to$ IV Total cost = 60.
7. $A \to 1$, $B \to 3$, $C \to 2$, $D \to 4$ Total profit = 53.
8. I $\to 4$, II $\to 2$, III $\to 1$, IV $\to 5$, V $\to 3$ Total efficiency = 452
9. I $\to 4$, II $\to 2$, III $\to 3$, IV $\to 5$ Job 1 remains unassigned. Total profit = 376
10. $A \to W_1$, $B \to W_3$, $C \to W_2$ Total cost = 50
11. $W_1 \to$ II, $W_2 \to$ III, $W_3 \to$ I Total cost = 26
12. $J_1 \to M_4$, $J_2 \to M_3$, $J_3 \to M_2$, $J_4 \to M_1$ Total cost = 135
13. $O_1 \to M_4, O_2 \to M_3, O_3 \to M_2, O_4 \to M_1$ Total cost = 14
14. $A \to C \to B \to D \to A$ Total cost = 19
15. $A \to B \to F \to E \to C \to D \to A$ Total cost = 30
16. $1 - C$ $2 - B$ $3 - A$ Total time = 280 hours
17. A – II B – I C – V D – III E – IV Total time = 23 hours
18. $A - T$ $B - R$ $C - S$ $D - P$ $E - Q$ Total time = 399 hours
19. A – III B – V C – I D – IV E – II Total time = 13 hours
20. $1 - E$ $2 - C$ $3 - B$ $4 - A$ $5 - D$ Total time = 570 hours
21. I – D II – B III – C IV– A Total time = 7 hours
22. A – I B – III C – II D – IV Total time = 41 hours
23. $1 - B$ $2 - C$ $3 - D$ $4 - A$ (or) $1 - C$ $2 - D$ $3 - B$ $4 - A$ Total time = 17 hours
24. $M_1 - A$ $M_2 - B$ $M_3 - E$ $M_4 - D$ (C unassigned) Total = 32

25. $A-2$ $B-3$ $C-4$ $D-1$ (or)
$A-3$ $B-2$ $C-4$ $D-1$ (or)
$A-3$ $B-4$ $C-2$ $D-1$
Total cost = 71

26. $A-2$ $B-4$ $C-1$ $D-3$ (or)
$A-3$ $B-4$ $C-1$ $D-2$
Total time = 18 hours

27. $A-1$ $B-3$ $C-2$ $D-4$ (or)
$A-1$ $B-2$ $C-3$ $D-4$
Total sales = 99

28. $1-B$ $2-A$ $3-E$ $4-C$
$5-D$ (or)
$1-B$ $2-E$ $3-A$ $4-C$
$5-D$ Total = 191

29. $A-1$ $B-2$ $C-4$ $D-5$
$E-3$ Total = 40

30. $A-2$ $B-4$ $C-1$ $D-3$
Total profit = 42

13

Decision Theory

CONCEPT REVIEW

13.1 WHAT IS DECISION THEORY?

Decision theory is the process of logical and quantitative analysis of various factors involved in a problem of decision-making and it helps us in making the best possible decision. There may be a number of courses of action (*strategies*) before us. The problem is to choose the best out of them so as to maximize the gain or (minimize the loss). There are different models of decision-making. Mostly we make decisions by analyzing the past situations and the availability of resources at present.

Linear programming problems are examples of decision under certainty. There is only one possible outcome and the process is well defined. But when there is uncertainty about the outcome then the decision-maker has no knowledge about the possibilities of the occurrence of different outcomes and he has to decide which course of action is to be taken. Also, the results or outcomes of the action depend on certain factors beyond our control. They are called *events* which occur according to the laws of nature. Under these circumstances we have to choose the strategy which results in optimum pay-off (gain).

13.2 STEPS IN DECISION MAKING

1. Make a list of all possible events which may occur.
2. Determine all the courses of action that can be taken in the situation.
3. Determine the pay-off for each combination of action and event.
4. Choose the best course of action which results in maximum pay-off.

There are several different criteria for decision-making.

13.3 MAXIMIN AND MAXIMAX CRITERIA

Maximin criterion is based on the assumption that the worst possibility is going to happen. We consider each strategy and select the minimum possible pay off corresponding to that strategy. From among these we select the maximum one (best among the worst) and choose the corresponding action.

Maximax criterion is based on the optimistic outlook. We select the best among the best. Select the maximum possible pay-off for each strategy and choose the maximum from among these. The strategy giving this pay-off is selected as the course of action.

Example 13.1 A business man has three alternatives, X, Y and Z, each of which gives rise to four possible events A, B, C and D. The pay-off (in Rs) are given below.

Alternative (Action)	*Pay-off*			
	A	*B*	*C*	*D*
X	8	0	–10	6
Y	–4	12	18	–2
Z	14	6	0	8

Find the best possible alternative using (i) Maximin criterion (ii) maximax criterion

Solution The following table gives the minimum and maximum pay-off corresponding to each alternative

Alternative	*Minimum Pay-off*	*Maximum Pay-off*
X	–10	8
Y	–4	18
Z	0	14

Maximum among the minimum pay-offs is 0. Hence under maximin criterion Z is to be chosen.

Maximum among the maximum pay-offs is 18. Hence under maximax criterion Y is to be chosen.

13.4 HURWICZ CRITERION AND LAPLACE CRITERION

In decision-making we cannot be completely optimistic nor pessimistic and therefore we have to adopt a mixture of both. In Hurwicz criterion we take into account both minimum and maximum pay-off for each alternative and assign them some weight or probability. We find the sum of these weighted pay-offs for each alternative. The alternative corresponding to the maximum sum is selected. The procedure is explained by the following steps.

Step 1 Choose a weight α (degree of optimism) for the maximum pay-off and a weight $1 - \alpha$ (degree of pessimism) for the minimum pay-off ($0 < \alpha < 1$).

Step 2 Determine $h = \alpha$ (maximum) + $(1 - \alpha)$ minimum for each alternative.

***Step* 3** Select the alternative corresponding to the maximum h.

The Laplace criterion assigns equal probability to all the events. In other words, we take the average pay-off corresponding to each action (expected pay off). The action corresponding to the maximum expected pay-off is selected.

Example 13.2 A company wants to introduce a new product in place of an old one. It is to be decided whether the price is to be fixed as very high, moderate or slightly increased (H, M or S). Three possible outcomes are expected, viz. increase in sales, no change in sales or decrease in sales (I, N or D). The expected sales are given in the following table (in lakhs of rupees)

Strategies	*Events*		
	I	*N*	*D*
H	70	30	15
M	50	45	0
S	30	30	30

Which alternative should be chosen according to (i) Maximin criterion (ii) Hurwicz criterion with $\alpha = 0.6$ and (iii) Laplace criterion?

Solution

(i) *Maximin criterion*

	I	*N*	*D*	Minimum
H	70	30	15	15
M	50	45	0	0
S	30	30	30	30 →

Maximin value is 30. Hence the company should adopt the strategy S.

(ii) *Hurwicz criterion* ($\alpha = 0.6$)

	I	*N*	*D*	*Max*	*Min*	*H*
H	70	30	15	70	15	70(0.6) + 15(0.4) = 48→
M	50	45	0	50	0	50(0.6) + 0(0.4) = 30
S	30	30	30	30	30	30(0.6) + 30(0.4) = 30

According to the Hurwicz criterion, strategy *H* is to be adopted.

(iii) *Laplace criterion*

	I	*N*	*D*	*Expected Pay-off*
H	70	30	15	1/3 (70+30+15) = 38.3 →
M	50	45	0	1/3 (50+45+0) = 31.6
S	30	30	30	1/3 (30+30+30) = 30

Since the largest return is from *H*, the company should adopt strategy *H*.

13.5 MINIMAX REGRET CRITERION (SAVAGE CRITERION)

This method is based on the regret of the decision-maker after he has taken a decision, which leads to loss of pay-off. He wants to minimize this loss. For each event we define the i^{th} regret as the difference between the maximum pay-off and the i^{th} pay-off. This process is repeated for each event. Determine the maximum regret for each alternative. From among these values choose the minimum one and the corresponding alternative is the best one.

Example 13.3 Apply the minimax regret criterion for the following decision problem.

Alternatives	*Events and pay-offs*		
	A	*B*	*C*
S_1	700	300	150
S_2	500	450	200
S_3	300	300	100

Solution Consider the event *A*. For S_1, we get the maximum pay-off 700. For S_2 and S_3 we get regrets 200 and 400 respectively. Similarly, we find the regrets for the events *B* and *C*. The regret pay-offs for each action-event combination are given below:

Action	*Pay-off*			*Regret Pay-off*			*Maximum regret*
	A	*B*	*C*	*A*	*B*	*C*	
S_1	700	300	150	0	150	50	150 →
S_2	500	450	200	200	0	0	200
S_3	300	300	100	400	150	100	400
Max. pay-off	700	450	200				

Among the maximum regrets, we find that the value corresponding to S_1, is the minimum. Hence according to the Minimax regret criterion, the alternative S_1 is to be chosen.

Example 13.4 Mr Girish wants to invest Rs 10,000 in one of the three options *A*, *B* and *C*. The pay-off for his investment depends on the nature of the economy (inflation, recession or no change). The possible returns under each economic situation are given below:

Strategy	*Nature of Economy*		
	Inflation: E_1	*Recession:* E_2	*No change:* E_3
A	2000	1200	1500
B	3000	800	1000
C	2500	1000	1800

What course of action has he to take according to

(i) The pessimistic criterion (Maximin)
(ii) The optimistic criterion (Maximax)
(iii) The equally likely criterion (Laplace)
(iv) The regret criterion.

Solution

(i) *Maximin criterion*

	E_1	E_2	E_3	*Min*
A	2000	1200	1500	1200 →
B	3000	800	1000	800
C	2500	1000	1800	1000

Maximum among the minimum pay-offs is 1200. Hence the corresponding option *A* is to be chosen.

(ii) *Maximax criterion*

	E_1	E_2	E_3	*Max*
A	2000	1200	1500	2000
B	3000	800	1000	3000 →
C	2500	1000	1800	2500

Maximum among the maximum pay-offs is 3000. Therefore option *B* is to be chosen.

(iii) *Laplace criterion*

	E_1	E_2	E_3	*Expected Pay-off*
A	2000	1200	1500	1/3(2000 + 1200 + 1500) = 1566
B	3000	800	1000	1/3(3000 + 800 + 1000) =1600
C	2500	1000	1800	1/3(2500 + 1000 + 1800) =1766 →

Alternative *C* has the maximum expected pay-off. Hence alternative *C* is to be chosen.

(iv) *Regret criterion*

				Regret pay-off			*Maximum regret*
	E_1	E_2	E_3	E_1	E_2	E_3	
A	2000	1200	1500	1000	0	300	1000
B	3000	800	1000	0	400	800	800
C	2500	1000	1800	500	200	0	500 →
Max	3000	1200	1800				

Among the maximum regrets, we find that 500 is the minimum. Therefore alternative *C* is to be chosen.

Example 13.5 Apply the Hurwicz criterion (α = 0.7) and Minimax regret criterion to solve the decision problem with the following pay-off table.

Strategy	*Events*			
	1	*2*	*3*	*4*
X	8	0	–10	6
Y	–4	12	18	–2
Z	14	6	0	8

Solution *Hurwicz Criterion* (α = 0.7)

	Pay-off				*Max*	*Min*	*h*
	1	*2*	*3*	*4*			
X	8	0	–10	6	8	–10	8(0.7) – 10(0.3) = 2.6
Y	–4	12	18	–2	18	–4	18(0.7) – 4(0.3) = 11.4 →
Z	14	6	0	8	14	0	14(0.7) + 0 (0.3) = 9.8

According to Hurwicz criterion, alternative *Y* is to be chosen.

Minimax Regret Criterion

	Pay-off				*Regrets*				*Max Regret*
	1	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*h*
X	8	0	–10	6	6	12	28	2	28
Y	–4	12	18	–2	18	0	0	10	18 →
Z	14	6	0	8	0	6	18	0	18 →
Max	14	12	18	8					

We find that the minimax regret occurs for both *Y* and *Z*. Hence either alternative *Y* or *Z* is to be chosen as the course of action.

13.6 DECISION-MAKING UNDER RISK

13.6.1 Expected Monetary Value Criterion

In many cases the decision-maker cannot predict the outcome of his action. In such situations we can assign probability values to the likely occurrence of each event. For each combination of action and event we assign a probability. This probability is calculated on the basis of past experience. We calculate the conditional pay-off for each combination of action and event. The Expected Monetary Value (EMV) of an action is the sum of all the expected conditional pay-offs associated with that action. The action which has the maximum EMV has to be chosen. The step-by-step procedure is as follows.

1. Determine the conditional pay-off for each combination of act and event.
2. Calculate the expected conditional pay-offs by multiplying the conditional pay-offs by the corresponding probabilities.
3. Find the EMV for each action by adding all these expected conditional pay-offs.
4. Choose the action corresponding to the maximum EMV.

Example 13.6 A shopkeeper buys vegetable puff at Rs 2 and sells it at Rs 5. Unsold puffs are given to poor people free of cost. The following is the sales details during the past 100 days.

Daily sales of puffs	10	11	12	13
No. of days sold	15	20	40	25

Apply EMV criterion to determine how many puffs the shopkeeper has to stock everyday in order to maximize his profit.

Solution We have 4 events

E_1 : Demand for 10 puffs
E_2 : Demand for 11 puffs
E_3 : Demand for 12 puffs
E_4 : Demand for 13 puffs

The courses of action are

A_1 : Stock 10 puffs
A_2 : Stock 11 puffs
A_3 : Stock 12 puffs
A_4 : Stock 13 puffs

Stocking less than 10 or more than 13 need not be considered as they have zero probability of sales. The profit on each puff sold is Rs 3. Suppose that the shopkeeper stocks 10 puffs and all of them are sold. Then he gets a profit of Rs 30. At the same time if he stocks 12 puffs and only 10 are sold then he loses Rs 4 on unsold puffs and hence his profit is only Rs 26 (The cost of 2 unsold puffs is Rs 4). Thus we can find the profit for each combination of action and event.

The profit (pay-off) for each combination is given by the following table:

Conditional pay-off

Actions (stock)		*Conditional pay-off Events (Demand)* 10 E_1	11 E_2	12 E_3	13 E_4
10	A_1	30	30	30	30
11	A_2	28	33	33	33
12	A_3	26	31	36	36
13	A_4	24	29	34	39

From the sales details in the past 100 days we find that the probabilities of demand are

10 puffs: 15/100 = 0.15
11 puffs: 20/100 = 0.20
12 puffs: 40/100 = 0.40
13 puffs: 25/100 = 0.25

Multiplying these probabilities with the pay-offs we get the expected conditional pay-offs as given below:

Prob.	*E_1 0.15*	*E_2 0.20*	*E_3 0.40*	*E_4 0.25*	*EMV (Total) Rs*
A_1	30 (.15)	30 (.20)	30 (.40)	30 (.25)	4.5 + 6 + 12 + 7.5 = 30
A_2	28 (.15)	33 (.20)	33 (.40)	33 (.25)	4.2 + 6.6 + 13.2 + 8.25 = 32.25
A_3	26 (.15)	31 (.20)	36 (.40)	36 (.25)	3.9 + 6.2 + 14.4 + 9 = 33.5
A_4	24 (.15)	29 (.20)	34 (.40)	39 (.25)	3.6 + 5.8 + 13.6 + 9.75 = 32.75

Since the EMV for A_3 is maximum, the shopkeeper should stock 12 puffs at the beginning of each day.

Example 13.7 A flower merchant purchases roses at Rs 10 per dozen and sells them at Rs 30. Unsold flowers are donated to a temple. The daily demand for roses has the following probability distribution:

Demand (in dozen)	7	8	9	10
Probability	0.1	0.2	0.4	0.3

Using EMV criterion determine how many dozens should he purchase in order to maximize the profit. What is the maximum expected profit?

Solution Here the events and actions are

E_1: Demand for 7 dozens with prob 0.1
E_2: Demand for 8 dozens with prob 0.2
E_3: Demand for 9 dozens with prob 0.4
E_4: Demand for 10 dozens with prob 0.3
A_1: Purchase 7 dozens
A_2: Purchase 8 dozens
A_3: Purchase 9 dozens
A_4: Purchase 10 dozens

The conditional pay-off table is as follows:

	Demand			
	7 E_1	8 E_2	9 E_3	10 E_4
A_1	140	140	140	140
A_2	130	160	160	160
A_3	120	150	180	180
A_4	110	140	170	200

Expected conditional pay-offs are given by

Prob.	E_1 0.1	E_2 0.2	E_3 0.4	E_4 0.3
A_1	14	28	56	42
A_2	13	32	64	48
A_3	12	30	72	54
A_4	11	28	68	60

$$\left[\begin{array}{l}\text{Profit per dozen} = \text{Rs } 20\\ \text{Loss per unsold dozen} = \text{Rs } 10\end{array}\right]$$

EMV of A_1 = Rs 140

EMV of A_2 = Rs 157

EMV of A_3 = Rs 168 $\rightarrow$

EMV of A_4 = Rs 167

We find that A_3 has the maximum EMV and hence A_3 is to be chosen. In other words the merchant should purchase 9 dozens of roses everyday and the maximum expected profit is Rs 168.

Example 13.8 A fruit merchant purchases orange at Rs 50 per basket and sells them at Rs 80. Unsold oranges are disposed off at Rs 20 per basket the next day. Past record of sales for 120 days is as given below:

Baskets sold	15	16	17	18
No. of days	12	24	48	36

Find how many baskets the merchant should purchase everyday in order to maximize his profit.

Solution Here unsold baskets have a salvage value of Rs 20 each. Therefore the loss per basket is 50 – 20 = 30. Thus the profit is given by
30 (no. of baskets sold) – 30 (no. of baskets unsold)

The conditional pay-off table is

		Demand / *Prob.*	E_1 15 0.1	E_2 16 0.2	E_3 17 0.4	E_4 18 0.3
Purchase	15	A_1	450	450	450	450
	16	A_2	420	480	480	480
	17	A_3	390	450	510	510
	18	A_4	360	420	480	540

Multiplying the pay-off by the corresponding probability we get the following table giving expected conditional pay-off:

	E_1	E_2	E_3	E_4	*EMV*
A_1	45	90	180	135	450
A_2	42	96	192	144	474
A_3	39	90	204	153	486 $\rightarrow$
A_4	36	84	192	162	474

Since the highest EMV is 486, the course of action to be chosen is A_3. Thus the merchant should purchase 17 baskets everyday.

13.7 EXPECTED OPPORTUNITY LOSS CRITERION (EOL)

This method is the same as that of EMV criterion except that, here we choose the act which leads to minimum loss. EOL is the amount of pay-off that is lost by not selecting the course of action that has the greatest pay-off for the event that actually occurs. For a given event find the difference between the maximum conditional pay-off and the pay-offs corresponding to different actions. This is called *Conditional Opportunity Loss* (COL). Multiplying these values by the respective probabilities we obtain the EOL. The sum of the expected COL of an action is the EOL. The action yielding the minimum EOL is chosen. The steps of this procedure are

1. Determine the conditional pay-off for each act event combination.

2. For each event find the COL values by subtracting the pay-offs of the actions from the maximum value.
3. For each action calculate the expected COL values by multiplying with the corresponding probabilities and find their sum (EOL).
4. Select the act having the minimum EOL.

Example 13.9 A newspaper boy purchases magazines at Rs 3 each and sells them at Rs 5 each. He cannot return the unsold magazines. The probability distribution of the demand for the magazine is given below.

Demand	16	17	18	19	20
Probability	0.10	0.15	0.20	0.25	0.30

Determine how many copies of magazine should he purchase (Apply EOL criterion).

Solution

The events are

E_1: Demand for 16

E_2: Demand for 17

E_3: Demand for 18

E_4: Demand for 19

E_5: Demand for 20

The courses of action are

A_1: Purchase 16

A_2: Purchase 17

A_3: Purchase 18

A_4: Purchase 19

A_5: Purchase 20

Conditional pay-off table

			Demand				
			E_1 16	E_2 17	E_3 18	E_4 19	E_5 20
Purchase	A_1	16	32	32	32	32	32
	A_2	17	29	34	34	34	34
	A_3	18	26	31	36	36	36
	A_4	19	23	28	33	38	38
	A_5	20	20	25	30	35	40
Max			32	34	36	38	40

COL table

	E_1	E_2	E_3	E_4	E_5
A_1	0	2	4	6	8
A_2	3	0	2	4	6
A_3	6	3	0	2	4
A_4	9	6	3	0	2
A_5	12	9	6	3	0
Prob.	0.10	0.15	0.20	0.25	0.30

Expected COL

	E_1	E_2	E_3	E_4	E_5	*EOL*
A_1	0	0.3	0.8	1.5	2.4	5
A_2	0.3	0	0.4	1	1.8	3.5
A_3	0.6	0.45	0	0.5	1.2	2.75 →
A_4	0.9	0.9	0.6	0	0.6	3
A_5	1.2	1.35	1.2	0.75	0	4.5

The course of action A_3 has the minimum EOL. Therefore it is advisable that the newspaper boy should purchase 18 copies of the magazine in order to minimize his loss.

Example 13.10 The sales of butter during 200 days has the following distribution:

Demand in kg	14	18	22	24	30	40
No. of days	10	20	20	80	40	30

The cost of butter is Rs 80 per kg and it is sold at Rs 120 per kg. Unsold stock is disposed off at the rate of Rs 60 per kg the next day. Apply EMV criterion and EOL criterion to determine how much of butter is to be stocked at the beginning of each day.

Solution (i) The conditional pay-off table is as follows: (Loss = 80 – 60 = Rs 20 per kg)

			14 E_1	18 E_2	22 E_3	24 E_4	30 E_5	40 E_6
Stock	14	A_1	560	560	560	560	560	560
	18	A_2	480	720	720	720	720	720
	22	A_3	400	640	880	880	880	880
	24	A_4	360	600	840	960	960	960
	30	A_5	240	480	720	840	1200	1200
	40	A_6	40	280	520	640	1000	1600
Prob.			0.05	0.10	0.10	0.40	0.20	0.15

Expected conditional pay-off table

	E_1	E_2	E_3	E_4	E_5	E_6	EMV
A_1	28	56	56	224	112	84	560
A_2	24	72	72	288	144	108	708
A_3	20	34	88	352	176	132	832
A_4	18	60	84	384	192	144	882
A_5	12	48	72	336	240	180	888
A_6	2	28	52	256	200	240	778

A_5 has the maximum EMV. Hence according to EMV criterion 30 kg of butter is to be stocked everyday.

(ii) Conditional pay-off table

		14 E_1	18 E_2	22 E_3	24 E_4	30 E_5	40 E_6
14	A_1	560	560	560	560	560	560
18	A_2	480	720	720	720	720	720
22	A_3	400	640	880	880	880	880
24	A_4	360	600	840	960	960	960
30	A_5	240	480	720	840	1200	1200
40	A_6	40	280	520	640	1000	1600
Max		560	720	880	960	1200	1600

COL table

	E_1	E_2	E_3	E_4	E_5	E_6
A_1	0	160	320	400	640	1040
A_2	80	0	160	240	480	880
A_3	160	80	0	80	320	720
A_4	200	120	40	0	240	640
A_5	320	240	160	120	0	400
A_6	520	440	360	320	200	0
Prob.	0.05	0.10	0.10	0.40	0.20	0.15

The expected COL table is given below.

	E_1	E_2	E_3	E_4	E_5	E_6	EOL
A_1	0	16	32	160	128	156	492
A_2	4	0	16	96	96	132	344
A_3	8	8	0	32	64	108	220
A_4	10	12	4	0	48	96	170
A_5	16	24	16	48	0	60	164 →
A_6	26	44	36	128	40	0	274

The minimum EOL corresponds to A_5. Hence according to EOL criterion 30 kg of butter is to be stocked.

Note: We find that both EMV and EOL criteria are consistent and they give identically the same results.

EXERCISES

1. Apply Maximax, Maximin and Laplace criteria for the following decision problems:

(a)

		Events			
		E_1	E_2	E_3	E_4
Acts	A_1	400	–10	600	1800
	A_2	2000	500	40	0
	A_3	2000	1500	–200	100

(b)

	E_1	E_2	E_3	E_4
A_1	4	0	–5	3
A_2	–2	6	9	1
A_3	7	3	2	4

2. Apply Hurwicz the criterion with $\alpha = 0.6$ for the following problems:

(a)

		Strategies			
		A	B	C	D
Events	E_1	3	4	4	5
	E_2	6	2	1	3
	E_3	1	8	8	4

(b)

	Events		
Alternatives	I	II	III
A	3	3	7
B	4	3	2
C	2	5	3

3. Solve the following problems using EMV criterion:
   (a) A business man buys an item at Rs 8 and sells it at Rs 12. Unsold items are useless. The following table gives the sales details for the past 100 days.

Number of items sold	25	26	27	28	29
No. of days	10	10	25	40	15

   Determine how many items he has to purchase in order to maximize his profit.

   (b) The cost of a cake is Rs 2 and it is sold at Rs 4. Unsold cakes are given free of cost to poor people at the end of the day. The probability distribution of the demand for the cakes is given below.

Demand	15	16	17	18	19
Prob.	0.1	0.2	0.3	0.25	0.15

   Determine the number of cakes to be purchased in order to get maximum profit.

   (c) A farmer wants to decide which of the three crops rice, wheat or corn he should cultivate in his 100 acre farm. The profits from the crops depend on the nature of rainfall: high, moderate or low. The following table gives the pay off from the crops under different conditions (events).

		Profit		
		Rice	*Wheat*	*Corn*
Rainfall	High	7000	2500	4000
	Moderate	3500	3500	4000
	Low	1000	4000	3000

   The probabilities of high, moderate and low rainfall are 0.2, 0.3 and 0.5 respectively. Determine which crop should be cultivated.

4. Apply EOL criterion for solving the following decision problems:
   (a) The pay-offs of three acts A_1, A_2, A_3 and the events E_1, E_2, E_3 are given below:

	E_1	E_2	E_3
A_1	25	400	650
A_2	–10	440	740
A_3	–125	400	750

   The probabilities of the events E_1, E_2, E_3 are 0.1, 0.7 and 0.2 respectively. Determine which act can be chosen as the best.

   (b) A company is proposing manufacture of three products *X*, *Y* and *Z*. The demand for each product may be good, moderate or poor. The probabilities of the states of nature are given below.

	Nature of demand		
Product	*Good*	*Moderate*	*Poor*
X	0.7	0.2	0.1
Y	0.5	0.3	0.2
Z	0.4	0.5	0.1

   The estimated profit or loss in thousands of rupees under the three states is given below.

	Good	*Moderate*	*Poor*
X	30	20	10
Y	60	30	20
Z	40	10	–15

   Give your advice to the company about the choice of the product.

   (c) A boat manufacturer finds that the distribution of the demand for the boat is as follows:

Demand	0	1	2	3	4	5	6
Prob.	0.14	0.27	0.27	0.18	0.09	0.09	0.01

   Each boat costs Rs 7000 and he sells it at Rs 10,000. Unsold boats are disposed off at Rs 6000 each at the end of the season. How many boats should he stock?

5. Apply Minimax Regret Criterion

	E_1	E_2	E_3
A_1	70	30	15
A_2	50	45	10
A_3	30	30	20

6. Apply Minimax Regret Criterion

	E_1	E_2	E_3	E_4
A_1	40	–1	60	180
A_2	200	50	4	0
A_3	200	150	–20	10

7. Apply EMV criterion

	A_1	A_2	A_3	*Prob.*
E_1	25	–10	–125	0.1
E_2	400	440	400	0.7
E_3	650	740	750	0.2

8. A pharmacist purchases a particular vaccine on Monday each week. Unsold vaccine becomes useless the next week. The cost is Rs 2 per dose and the pharmacist sells it at Rs 4 per dose. The demand for the medicine over the past 50 weeks is given below:

Doses per week	20	25	50	60
No. of weeks	5	15	25	5

Using EMV. criterion determine how many doses he should buy every week.

9. The following table gives the information on the sales of cakes in a bakery.

Sales per day	25	26	27	28
No. of days	10	30	50	10

A cake costs Rs 3 and sells for Rs 5. Apply EOL criterion and determine how many cakes should be purchased in order to maximize the profit (unsold cakes are given to poor people free of cost).

10. The cost of manufacturing an item is Rs 25 and the selling price is Rs 30. Unsold items are sold at Rs 20 per item at the end of the week. The following table shows the weekly sales and the number of weeks from past experience.

Weekly sales	less than 3	4	5	6	7	greater than 7
No. of weeks	0	10	20	40	30	0

Find the optimum number of items to be manufactured per week.

11. The probability distribution of monthly sales of an item is as follows:

Monthly sales	0	1	2	3	4	5	6
Probabilities	0.01	0.06	0.25	0.30	0.22	0.10	0.06

Unsold items are stored at a cost of Rs 30 per unit, per month and the loss due to shortage is Rs 70 per unit. Determine the optimum stock per month.

12. The cost of storing a TV for a week is Rs 50. Loss due to shortage is Rs 200 per TV. The probability distribution of the demand is given below:

Weekly demand	0	1	2	3	4	5
Probabilities	0.05	0.1	0.2	0.3	0.2	0.15

How many TV sets should the dealer order per week?

ANSWERS

1. (a) Maximax A_2, A_3; Maximin A_2; Laplace A_3
   (b) Maximax A_2; Maximin A_3; Laplace A_3
2. (a) E_3 (b) A
3. (a) 27 items (b) 17 cakes (c) wheat
4. (a) A_2 (b) Y (c) 3
5. Strategy A_1 is to be adopted.
6. Optimal strategy A_1
7. Optimal strategy A_2
8. He should buy 50 doses every week.
9. Ether 26 or 27 cakes should be purchased.
10. 6 items per week.
11. 4 items to be stocked per month.
12. 4 TV sets per week.

14

Theory of Games

CONCEPT REVIEW

14.1 INTRODUCTION

Theory of games deals with situation in which two or more competitors are engaged in decision-making activities involving profit. The competitors are called *players*. Theory of games is one type of decision-making technique in which one player's course of action (strategy) depends on all possible strategies of the opponent. Each player will act so as to maximize his gain or minimize his loss. Different strategies adopted by the players lead to different pay-offs. The strategy which optimizes a player's pay-off is called *optimal strategy*. When each player plays his optimal strategy the resulting pay-off is called the *value of the game*. In general there are two types of strategies, viz. *pure strategy* and *mixed strategy*. Pure strategy is a decision rule used by a player to select the particular course of action. Each player knows all strategies out of which he selects the one, irrespective of the other's choice, which optimizes his pay-off. When both players are left to guess the course with some probability it is called mixed strategy. It is a probabilistic method of deciding the optimal strategy.

14.2 TWO-PERSON ZERO SUM GAMES

A game with two players A and B in which a gain for A is equal to the loss for B (total sum is zero) is called a *two-person zero sum game*. This game is represented by the pay-off (gain) matrix of one player A. For B the same pay-off matrix applies with the signs of the values changed. The strategies of A are denoted by $A_1, A_2, \ldots, A_m$ and those of B are $B_1, B_2, \ldots, B_n$. For each combination of A_i ($i = 1, 2, \ldots, m$) and B_j ($j = 1, 2, 3, \ldots, n$) we have a pay-off a_{ij}. Generally we assume that a_{ij} denotes the pay-off for A. Of course B's pay-off is $-a_{ij}$. Pay of matrix is of the following form:

$$\begin{array}{cc} & B\text{'s strategies} \\ \begin{array}{r} \\ A_1 \\ A\text{'s strategies} \quad A_2 \\ \cdots \\ A_m \end{array} & \begin{pmatrix} B_1 & B_2 & B_3 & \cdots\cdots & B_n \\ a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{pmatrix} \end{array}$$

Some important characteristics of two-person zero sum game are

(i) There are exactly two players and each player has a finite number of courses of action (strategies).
(ii) The given matrix refers to the pay-off values for A.
(iii) Each player knows all possible pay-offs for himself as well as for the other player.

14.3 GAMES WITH SADDLE POINT (PURE STRATEGY)

Maximin and Minimax Principle

Each row of the pay-off matrix represents the pay-off for A, corresponding to different strategies of B. In the first row we find the pay-off values of A_1 corresponding to $B_1, B_2, \ldots, B_n$. A wants to ensure the minimum profit. Hence the minimum of the I row is taken. Similarly for A_2 the minimum of the II row is chosen and thus we note down the minimum pay-off in each row. Thus A is assured of these pay-offs for his actions $A_1, A_2, A_3, \ldots, A_m$. Now he selects the maximum among these (maximin) value $\underline{v}$ and plays the corresponding strategy. In case of B, each column gives his loss corresponding to different strategies of A. He wants to minimize his loss. But A adopts the strategy giving maximum gain. B cannot lose more than the maximum pay-off of each column (corresponding to $B_1, B_2, \ldots, B_n$). In order to minimize his loss B has to choose the minimum among these maximum pay-offs $\overline{v}$. This value is the minimax value and he chooses to play the corresponding strategy.

If the maximin and the minimax values are equal then the game is said to have *Saddle point* and the corresponding pay-off is called the value of the game v. ($\underline{v} = \overline{v} = v$). The step-by-step procedure is given below.

(i) Note down the minimum pay-off in each row.
(ii) Select the maximum among these and square it. It is the maximin value.
(iii) Note down the maximum pay-off in each column.
(iv) Select the minimum among these and square it. It is the minimax value.
(v) If the maximin and minimax values are equal then mark the position of that pay-off in the matrix. This element represents the value of the game and it is called the *saddle point of the game.*

This procedure is called solving the game with the given pay-off matrix.

Note: In general maximin $\underline{v} \le v \le$ minimax $\overline{v}$

If $\underline{v} = 0 = \overline{v}$ the game is said to be *fair.*

If $\underline{v} = v = \overline{v}$ the game is said to *determinable* (stable).

Example 14.1 Solve the game with the pay-off matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \\ \hline 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{array}$$

Solution

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{Row minimum} \\ A_1 & 1 & 7 & 3 & 4 & 1 \\ A_2 & 5 & 6 & \mathbf{4} & 5 & \mathbf{4} \\ A_3 & 7 & 2 & 0 & 3 & 0 \\ \text{Col. max} & 7 & 7 & \mathbf{4} & 5 & \end{array}$$

Maximin value = 4 = Minimax value. $\underline{v} = v = \overline{v}$. The value of the game is 4. Saddle point is (2, 3). Optimal strategy of A is A_2 and that of B is B_3.

Example 14.2 Solve the following game:

	B_1	B_2	B_3	B_4
A_1	20	15	12	35
A_2	25	14	8	10
A_3	40	2	10	5
A_4	–5	4	11	0

Solution Row minimum and column maximum values are given below

	B_1	B_2	B_3	B_4	Row minimum	
A_1	20	15	**12**	35	**12**	maximin
A_2	25	14	8	10	8	
A_3	40	2	10	5	0	
A_4	–5	4	11	0	– 5	
Col. max	40	15	**12**	35	$\underline{v} = v = \overline{v} = 12$	
			minimax			

We find that maximin = minimax =12. Therefore the value of the game is 12 and the saddle point is (1, 3). The optimal strategies are A_1 of A and B_3 of B.

Example 14.3 If the following game is determinable find the limits for the value of λ.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{pmatrix} \lambda & 6 & 4 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{pmatrix} \end{array}$$

Solution Ignoring the value of λ the maximin and minimax values are

	B_1	B_2	B_3	Row minimum	
A_1	λ	6	4	**4**	**maximin**
A_2	–1	λ	–7	–7	
A_3	–2	4	λ	–2	
Col. max	–1	6	4		
	minimax				

maximin value = 4

minimax value = –1

If the game is determinable then $\underline{v} = v = \overline{v}$.

Here $\overline{v} = 4$ and $\underline{v} = -1$

$\therefore$ λ = lies between –1 and 4.

Example 14.4 If the saddle point is (2, 2) in the following pay-off matrix find the range of the values of p and q.

$$\begin{pmatrix} 2 & 6 & 5 \\ 10 & 7 & q \\ 5 & p & 8 \end{pmatrix}$$

Solution Ignoring p and q we find the maximin and minimax values.

	B_1	B_2	B_3	Row minimum	
A_1	2	6	5	2	
A_2	10	**7**	q	7	**maximin**
A_3	5	p	8	5	
Col. max	10	**7**	8		
		minimax			

We find that $\underline{v} = \overline{v} = 7$. Given that the saddle point is at (2, 2). Therefore $v = 7$ and $p \leq 7$ and $q \geq 7$.

14.4 GAMES WITHOUT SADDLE POINTS (MIXED STRATEGY)

When there is no saddle point the players select the strategies with some probabilities. A selects his strategies $A_1, A_2, A_3, \ldots, A_m$ with probabilities $p_1, p_2, \ldots, p_m$ respectively, where $p_1 + p_2 + p_3 + \ldots + p_m = 1$.

B selects his strategies $B_1, B_2, B_3, \ldots, B_n$ with probabilities $q_1, q_2, q_3, \ldots, q_n$ where $q_1 + q_2 + \ldots + q_n = 1$. For each strategy of A we can find the expected gain.

14.5 FORMULA FOR FINDING THE VALUE OF THE GAME IN CASE OF 2 × 2 GAMES WITHOUT SADDLE POINT

Let the pay-off matrix for A be

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}$$

The strategies and probabilities are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

A selects A_1 with prob p_1 and A_2 with prob p_2. Corresponding to B's strategy B_1, the expected gain of A is $E_1(p) = a_{11} p_1 + a_{21} p_2$.

Corresponding to B's strategy B_2 the expected gain of A is $E_2(p) = a_{12}p_1 + a_{22}p_2$.

Similarly, corresponding to A's strategy A_1 the expected loss of B is $E_1(q) = a_{11}q_1 + a_{12}q_2$.

Corresponding to A_2, expected loss of B is $E_2(q) = a_{21}q_1 + a_{22}q_2$.

If the value of the game is v, then

$$E_1(p) \geq v \text{ and } E_2(p) \geq v$$
$$E_1(p) \leq v \text{ and } E_2(q) \leq v$$

For optimal strategy we should have $E_1(p) = E_2(p) = v$ and $E_1(q) = E_2(q) = v$.

$$\therefore \quad a_{11}p_1 + a_{21}p_2 = a_{12}p_1 + a_{22}p_2.$$
$$p_1(a_{11} - a_{12}) = p_2(a_{22} - a_{21})$$
$$p_1/p_2 = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}$$

or
$$p_2/p_1 = \frac{a_{11} - a_{12}}{a_{22} - a_{21}}$$

Also
$$p_1 + p_2 = 1.$$
$$p_2/p_1 + 1 = \frac{a_{11} - a_{12}}{a_{22} - a_{21}} + 1$$
$$(p_1 + p_2)/p_1 = \frac{a_{11} - a_{12} + a_{22} - a_{21}}{a_{22} - a_{21}}$$
$$1/p_1 = \frac{(a_{11} + a_{22}) - (a_{12} + a_{21})}{(a_{22} - a_{21})}$$
$$\therefore \quad p_1 = \frac{(a_{22} - a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{1}$$
$$\therefore \quad p_2 = 1 - p_1 = \frac{(a_{11} - a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{2}$$

Again taking $E_1(q) = E_2(q)$ we get

$$q_1/q_2 = \frac{(a_{22} - a_{12})}{(a_{11} - a_{21})}$$

and
$$q_1 = \frac{(a_{22} - a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{3}$$
$$q_2 = \frac{(a_{11} - a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{4}$$

The value of the game is given by

$$v = a_{11}p_1 + a_{21}p_2$$
$$= \frac{a_{11}(a_{22} - a_{21}) + a_{21}(a_{11} - a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$
$$= \frac{(a_{11}a_{22} - a_{21}a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{5}$$

These formulae are important and are used frequently in solving games without saddle point.

Example 14.5 Solve the following game

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} 8 & -3 \\ -3 & 1 \end{pmatrix} \end{array}$$

Solution First let us find whether the matrix has a saddle point.

	B_1	B_2	Row min
A_1	8	–3	–3
A_2	–3	1	–3
Col. max	8	1	

Since there is no saddle point we apply the method of mixed strategy.

$$p_1 = \frac{(a_{22} - a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$
$$= \frac{1 + 3}{(8 + 1) - (-3 - 3)} = 4/15$$
$$p_2 = 1 - p_1 = 11/15$$
$$q_1 = \frac{(a_{22} - a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$
$$= \frac{(1 + 3)}{(8 + 1) - (-3 - 3)} = 4/15$$
$$q_2 = 1 - q_1 = 11/15$$

Value of the game is

$$\frac{(a_{11}a_{22} - a_{12}a_{21})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{8 - 9}{15} = -1/15$$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ 4/15 & 11/15 \end{pmatrix}$$
$$S_B = \begin{pmatrix} B_1 & B_2 \\ 4/15 & 11/15 \end{pmatrix}$$

Example 14.6 Solve the following game

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} \end{array}$$

Solution

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \quad \text{Row min} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} \begin{array}{c} 1 \\ 3 \end{array} \\ \text{Col. max} & \begin{array}{cc} 5 & 4 \end{array} \end{array}$$

No saddle point.

Now take $D = (a_{11} + a_{22}) - (a_{12} + a_{21})$

$= (5 + 4) - (3 + 1) = 5$

$p_1 = (a_{22} - a_{21})/D = (4 - 3)/5 = 1/5$; $p_2 = 4/5$

$q_1 = (a_{22} - a_{12})/D = (4 - 1)/5 = 3/5$; $q_2 = 2/5$

Value $= (a_{11}a_{22} - a_{12}a_{21})/D = (20 - 3)/D = 17/5$

Optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ 1/5 & 4/5 \end{pmatrix} \quad S_B = \begin{pmatrix} B_1 & B_2 \\ 3/5 & 2/5 \end{pmatrix}$$

Example 14.7 In a game of matching biased coins, player A gets Rs 8 if two heads turn up and Re 1 if two tails turn up. Player B gets Rs 3 if it is otherwise. Solve the game.

Solution A's pay-offs are 8 for (HH), 1 for (TT), – 3 for (HT), and – 3 for (TH)

The matrix is

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 8 & -3 \\ -3 & 1 \end{pmatrix} \end{array}$$

It is a game without saddle point.

$D = a_{11} + a_{22} - (a_{12} + a_{21})$

$= (8 + 1) - (-3 - 3) = 15$

$p_1 = (a_{22} - a_{21})/D = (1 + 3)/15 = 4/15$;

$p_2 = 11/15$

$q_1 = (a_{22} - a_{12})/D = (1 + 3)/15 = 4/15$

$q_2 = 11/15$

$v = (a_{11}a_{22} - a_{12}a_{21})/D = (8 - 9)/15$

$= -1/15$

$$S_A = \begin{pmatrix} H & T \\ 4/15 & 11/15 \end{pmatrix}$$

and $$S_B = \begin{pmatrix} H & T \\ 4/15 & 11/15 \end{pmatrix}$$

14.6 DOMINANCE PROPERTY

Dominance property is used to reduce the size of the pay-off matrix. In the pay-off matrix if each element of the i^{th} row is less than or equal to the corresponding element of some other row (say j^{th} row) then we say that the j^{th} row dominates over the i^{th} row and the i^{th} row can be deleted from the pay-off matrix, since it implies that A's pay-off is less for the strategy A_i than for A_j whatever be the strategy of B. A_i is an inferior strategy.

Similarly, if every element of the r^{th} column is greater than or equal to the corresponding element of some other column (say s^{th} column) then B will never play the r^{th} strategy as it involves more loss. We say that s^{th} column dominates over the r^{th} coloumn and r^{th} column can be deleted. B_r is an inferior strategy.

Also if a strategy $k(A_i$ or $B_j)$ is inferior to a convex combination of two or more of other pure strategies then the row or column corresponding to A_i or B_j can be deleted.

Note: A convex combination of A_r and A_s is $(\lambda_1 a_{r1} + \lambda_2 a_{s1}, \lambda_1 a_{r2} + \lambda_2 a_{s2}, \ldots, \lambda_1 a_{rm} + \lambda_2 a_{sn})$, where $\lambda_1 + \lambda_2 = 1$; $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$.

Using this principle of dominance we can reduce the pay-off matrix to a smaller one and then solve.

Example 14.8 Solve the following game

$$\begin{array}{c c} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & 7 & 2 \\ 0 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix} \end{array}$$

The given pay-off matrix is

$$\begin{array}{c c} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \quad \text{Row min} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & 7 & 2 \\ 0 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix} \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \\ \text{Col. max} & \begin{array}{ccc} 5 & 7 & 7 \end{array} \end{array}$$

The game has no saddle point. We find that every element in the III column is greater than the corresponding element in column I. Therefore, column III is inferior and it can be deleted.

Now we have

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & 1 & 7 \\ A_2 & 0 & 2 \\ A_3 & 5 & 1 \end{array}$$

In this matrix every element of row II is less than the corresponding element of row I. Therefore, row II is inferior and it can be deleted. The resulting pay-off matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & 1 & 7 \\ A_3 & 5 & 1 \end{array}$$

Considering the mixed strategy we get

$D = (1 + 1) - (5 + 7) = 2 - 12 = -10.$

$p_1 = (a_{22} - a_{21})/D = (1 - 5)/D$

$= -4/-10 = 2/5; \; p_3 = 3/5$

$q_1 = (a_{22} - a_{12})/D = (1 - 7)/D$

$= -6/-10 = 3/5;$

$q_2 = 2/5$

Value of the game is $(a_{11}a_{22} - a_{12}a_{21})/D$

$= (1 - 35)/-10$

$= -34/-10 = \mathbf{17/5}$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/5 & 0 & 3/5 \end{pmatrix}$$

and $$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 3/5 & 2/5 & 0 \end{pmatrix}$$

Example 14.9 Solve

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 3 & 2 & 4 & 0 \\ A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 8 \end{array}$$

Solution The given pay-off matrix is

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{Row min} \\ A_1 & 3 & 2 & 4 & 0 & 0 \\ A_2 & 3 & 4 & 2 & 4 & 2 \\ A_3 & 4 & 2 & 4 & 0 & 0 \\ A_4 & 0 & 4 & 0 & 8 & 0 \\ \text{Col. max} & 4 & 4 & 4 & 8 & \end{array}$$

There is no saddle point.

Row III dominates over row I and hence row I can be deleted. Thus we get

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 8 \end{array}$$

Here we find that column I is dominated by column III and therefore we can delete column I. The resulting matrix is

$$\begin{array}{c|ccc} & B_2 & B_3 & B_4 \\ A_2 & 4 & 2 & 4 \\ A_3 & 2 & 4 & 0 \\ A_4 & 4 & 0 & 8 \end{array}$$

We find that column I is inferior to a convex combination of columns II and III.

$4 \geq (2 + 4)/2, \; 2 \geq (4 + 0)/2,$

$4 \geq (0 + 8)/2, \; (\lambda_1 = \lambda_2 = 1/2)$

$\therefore$ column 1 can be deleted and the resulting matrix is

$$\begin{array}{c|cc} & B_3 & B_4 \\ A_2 & 2 & 4 \\ A_3 & 4 & 0 \\ A_4 & 0 & 8 \end{array}$$

Again in this matrix, row I is dominated by a convex combination of rows II and III.

$2 \leq (4 + 0); \; 4 \leq (0 + 8)/2$

and hence row I can be deleted.

Finally we get

$$\begin{array}{c|cc} & B_3 & B_4 \\ A_3 & 4 & 0 \\ A_4 & 0 & 8 \end{array}$$

$D = (4 + 8) - (0 + 0) = 12$

$p_3 = 8/12 = 2/3, \; p_4 = 1/3$

$q_3 = 8/12 = 2/3, \; q_4 = 1/3$

Value of the game is

$v = 32 / 12 = \mathbf{8/3}$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Example 14.10 Solve

$$A = \begin{array}{c} B \\ \begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix} \end{array}$$

Solution The given pay-off matrix is

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{Row min} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \\ \text{Col. max} & \begin{array}{ccc} 6 & 7 & 7 \end{array} & \text{No saddle point} \end{array}$$

Row II dominates over row III and therefore row III can be deleted.

We get

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \end{pmatrix} \end{array}$$

Column I dominates over column III and hence column III can be deleted.

We get

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 1 & 7 \\ 6 & 2 \end{pmatrix} \end{array}$$

Applying mixed strategy we have

$$D = (1 + 2) - (6 + 7) = -10$$
$$p_1 = (2 - 6)/-10 = -4/-10 = 2/5;$$
$$p_2 = 3/5$$
$$q_1 = (2 - 7)/-10 = -5/-10 = 1/2;$$
$$q_2 = 1/2$$

Value of the game is

$$v = ((1)(2) - (6)(7))/-10 = \mathbf{4}$$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/5 & 3/5 & 0 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

Example 14.11 Solve

$$A \quad \begin{array}{c} B \\ \begin{pmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{pmatrix} \end{array}$$

Solution Given matrix is

$$\begin{array}{ccc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} & \text{Row min} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{pmatrix} & \begin{array}{c} -10 \\ 1 \\ 1 \\ -1 \end{array} \\ \text{Col. max} & \begin{array}{cccc} 8 & 7 & 15 & 4 \end{array} & \end{array}$$

It has no saddle point

R_3 dominates R_1. R_1 can be deleted.

We get

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{pmatrix} \end{array}$$

R_2 dominates over R_1. R_1 can be deleted.

We have

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_3 \\ A_4 \end{array} & \begin{pmatrix} 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{pmatrix} \end{array}$$

C_4 dominates over C_2. C_2 can be deleted.

Thus we have

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_3 \\ A_4 \end{array} & \begin{pmatrix} 8 & 15 & 1 \\ 3 & -1 & 4 \end{pmatrix} \end{array}$$

C_1 dominated by a convex combination C_2 and C_3.

$8 \geq (15 + 1)/2 \quad (\lambda_1 = \lambda_2 = 1/2)$

$3 \geq (-1 + 4)/2$

Hence C_1 is deleted.

Finally we get

$$\begin{array}{cc} & \begin{array}{cc} B_3 & B_4 \end{array} \\ \begin{array}{c} A_3 \\ A_4 \end{array} & \begin{pmatrix} 15 & 1 \\ -1 & 4 \end{pmatrix} \end{array}$$

$D = (15 + 4) - (-1 + 1) = 19$
$p_3 = (4 + 1)/19 = 5/19$; $p_4 = 14/19$
$q_3 = (4 - 1)/19 = 3/19$; $q_4 = 16/19$
Value $v = (60 + 1)/19 = 61/19$

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 5/19 & 14/19 \end{pmatrix}$$

and $$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 3/19 & 16/19 \end{pmatrix}$$

14.7 GRAPHICAL METHOD

The graphical method can be applied to solve the game whose pay-off matrix is of the form $2 \times n$ or $m \times 2$. In this case one of the players has only two undominated pure strategies. Using the graphical method we can reduce the given pay-off matrix to 2×2 type which can be solved using mixed strategies.

14.7.1 Procedure for 2 × *n* Matrix

Let the $2 \times n$ pay-off matrix be

$$\begin{array}{cc} & \begin{array}{cccc} B_1, & B_2, & B_3\ldots, & B_n \end{array} & \text{Prob.} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} a_{11}, & a_{12}, & a_{13},\ldots, & a_{1n} \\ a_{21}, & a_{22}, & a_{23},\ldots, & a_{2n} \end{pmatrix} & \begin{array}{c} p_1 \\ p_2 \end{array} \\ \text{Prob.} & \begin{array}{cccc} q_1, & q_2, & q_3, \ldots, & q_n \end{array} & \end{array}$$

A's expected pay-off corresponding to each strategy of B are given by

***B*'s strategy**	***A*'s expected pay-off**
B_1	$a_{11}p_1 + a_{21}p_2$
B_2	$a_{12}p_1 + a_{22}p_2$
B_3	$a_{13}p_1 + a_{23}p_2$
$\vdots$	$\vdots$
B_n	$a_{1n}p_1 + a_{2n}p_2$

B chooses strategy B_j for which $E_j(p_1)$ is minimum.

$$\therefore \quad v = \min_j \{E_j(p_1)\} \quad (j = 1, 2, \ldots, n)$$

Now A chooses p_1 and p_2 in such a manner that v is as large as possible. In order to determine this value we shall plot each $E_j(p_1)$ as a straight line of the form

$$E_j(p_1) = a_{1j}p_1 + a_{2j}p_2$$
$$= a_{1j}p_1 + a_{2j}(1 - p_1)$$
$$= (a_{1j} - a_{2j})p_1 + a_{2j}$$

Consider two vertical axes corresponding to $p_1 = 0$ and $p_1 = 1$. These axes are two parallel lines at a distance 1 unit apart. For $p_1 = 0$ we get the point $(0, a_{2j})$ on the 0 axis and for $p_1 = 1$ we get the point $(1, a_{1j})$ on the 1 axis. Joining these two points we get the line corresponding to $j = 1, 2, 3, \ldots, n$. Determine the lower envelope of all these lines. The highest point of this envelope represents the maximum among the minimum expected pay-off (maximin value). The lines which intersect at this point give the optimal strategies of B.

Example 14.12 Solve the following game graphically

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix} \end{array}$$

Solution

Let $$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$$

We have the expected pay-off as follows:

B's move

$$B_1 \to E_1(p_1) = (a_{11} - a_{21})p_1 + a_{21}$$
$$= (2 - 1)p_1 + 1$$

i.e., $$E_1(p_1) = p_1 + 1 \qquad (1)$$

Similarly, $$B_2 \to E_2(p_1) = (a_{12} - a_{22})p_1 + a_{22}$$
$$= (1 - 0)p_1 + 0 = p_1$$
$$E_2(p_1) = p_1 \qquad (2)$$
$$B_3 \to E_3(p_1) = (a_{13} - a_{23})p_1 + a_{23}$$
$$= (0 - 3)p_1 + 3$$
$$E_3(p_1) = -3p_1 + 3 \qquad (3)$$
$$B_4 \to E_4(p_1) = (a_{14} - a_{24})p_1 + a_{24}$$
$$= (-2 - 2)p_1 + 2$$
$$E_4(p_1) = -4p_1 + 2 \qquad (4)$$

Take two vertical axes $p_1 = 0$ and $p_1 = 1$. $E_1(p_1)$ is represented by the line joining $E_1(0)$ and $E_1(1)$.
$E_1(0) = 0 + 1 = 1$, which is the point (0, 1)
$E_1(1) = 1 + 1 = 2$, which is the point (1, 2).

Thus $E_1(p_1)$ is represented by the line joining the points (0, 1) on the 0 axis and (1, 2) on the 1 axis. Now $E_2(p_1)$ is the line joining (0, 0) and (1, 1)

$E_3(p_1)$ is the line joining (0, 3) and (1, 0)

$E_4(p_1)$ is the line joining (0, 2) and (1, –2)

Let us represent these 4 lines in the graph as given below.

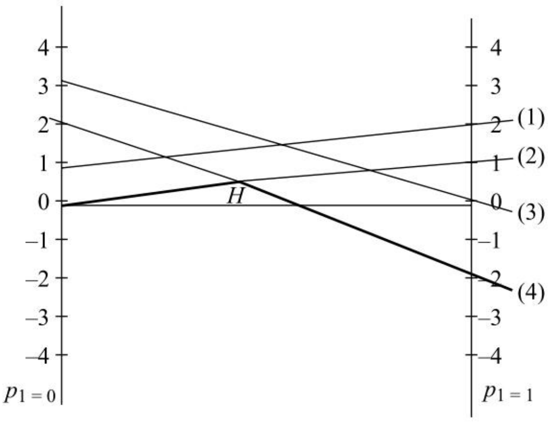


Fig. 14.1

From the graph we find that the highest point of the lower envelope is H which is the intersection of $E_2(p_1)$ and $E_4(p_1)$. Therefore B_2 and B_4 are the optimal strategies of B. This leads to the 2 × 2 pay-off matrix.

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \end{array}$$

We have $D = (1 + 2) - (0 - 2) = 5$

$$p_1 = (2 - 0)/5 = 2/5,\ p_2 = 3/5$$

$$q_2 = (2 + 2)/5 = 4/5,\ q_4 = 1/5$$

Value of the game

$$v = (2 - 0)/5 = 2/5$$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{pmatrix}$$

Example 14.13 Solve the following game using the graphical method

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{pmatrix} \end{array}$$

Solution

Let $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$

$$E_1(p_1) = (2 - 4)p_1 + 4 = -2p_1 + 4 \quad (1)$$

$$E_2(p_1) = (2 - 3)p_1 + 3 = -p_1 + 3 \quad (2)$$

$$E_3(p_1) = (3 - 2)p_1 + 2 = p_1 + 2 \quad (3)$$

$$E_4(p_1) = (-2 - 6)p_1 + 6 = -8p_1 + 6 \quad (4)$$

$E_1(0) = 4$

$E_1(1) = 2$ Line joining (0, 4) and (1, 2)

$E_2(0) = 3$

$E_2(1) = 2$ Line joining (0, 3) and (1, 2)

$E_3(0) = 2$

$E_3(1) = 3$ Line joining (0, 2) and (1, 3)

$E_4(0) = 6$

$E_4(1) = -2$ Line joining (0, 6) and (1, –2)

The graph is

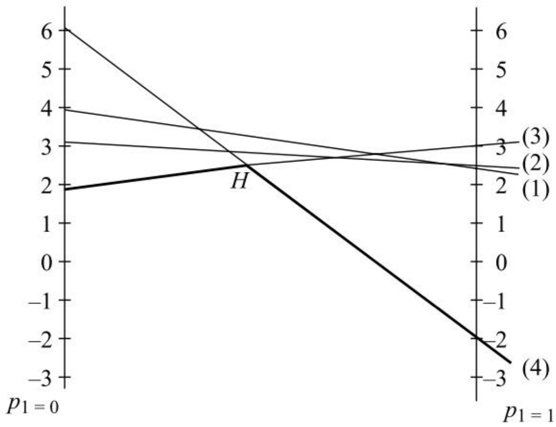


Fig. 14.2

In the graph the highest point of the lower envelope is H which is the intersection of (3) and (4). Therefore B_3 and B_4 are the optimal strategies. We get the pay-off matrix.

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_1 & 3 & -2 \\ A_2 & 2 & 6 \end{array}$$

$$D = (6 + 3) - (2 - 2) = 9$$

$$p_1 = (6 - 2)/9 = 4/9,\ p_2 = 5/9$$

$$q_3 = (6 + 2)/9 = 8/9,\ q_4 = 1/9$$

$$V = (18 + 4)/9 = \mathbf{22/9}$$

$$S_A = \begin{pmatrix} A_1 & A_2 \\ 4/9 & 5/9 \end{pmatrix}$$

and $$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 8/9 & 1/9 \end{pmatrix}$$

14.7.2 Procedure for $m \times 2$ Matrix

Let the pay-off matrix be

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ A_m \\ \text{Prob.} \end{array}
\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{m1} & a_{m2} \end{pmatrix} \\ \begin{array}{cc} q_1 & q_2 \end{array} \end{array}
\begin{array}{c} \text{Prob.} \\ p_1 \\ p_2 \\ \cdot \\ \cdot \\ p_m \\ \ \end{array}$$

B's expected loss corresponding to each strategy of *A* are given by the following table:

***A*'s strategy**	***B*'s expected loss**
A_1	$E_1(q_1) = a_{11}q_1 + a_{12}q_2$
A_2	$E_2(q_1) = a_{21}q_1 + a_{22}q_2$
A_3	$E_3(q_1) = a_{31}q_1 + a_{32}q_2$
$\vdots$	$\vdots$
A_m	$E_m(q_1) = a_{m1}q_1 + a_{m2}q_2$

A chooses the strategy for which $E_i(q_1)$ is maximum

$$V = \max_i \{E_i(q_1)\} \quad i = 1, 2, 3, \ldots, m.$$

B chooses q_1 and q_2 such that *V* is as small as possible. Hence we have to find the minimax value.

Draw two vertical lines $q_1 = 0$ and $q_1 = 1$ and draw lines joining the pair of points. $E_i(0)$ and $E_i(1)$ $(i = 1, 2, 3,\ldots, m)$. Find the upper envelope of all these lines. The lowest point *H* of this envelope represents the minimax value. The lines intersecting at *H* represent the optimal strategies of *A*.

Example 14.14 Solve the following game

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} B \\ \begin{pmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{pmatrix} \end{array}$$

Solution The given matrix is

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ \text{Prob.} \end{array}
\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{pmatrix} \\ \begin{array}{cc} q_1 & q_2 \end{array} \end{array}
\begin{array}{c} \text{Prob.} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \ \end{array}$$

$$E_1(q_1) = a_{11}q_1 + a_{12}q_2 = -6q_1 + 7(1 - q_1) = -13q_1 + 7 \quad (1)$$

$$E_2(q_1) = a_{21}q_1 + a_{22}q_2 = 4q_1 - 5(1 - q_1) = 9q_1 - 5 \quad (2)$$

$$E_3(q_1) = a_{31}q_1 + a_{32}q_2 = -q_1 - 2(1 - q_1) = q_1 - 2 \quad (3)$$

$$E_4(q_1) = a_{41}q_1 + a_{42}q_2 = -2q_1 + 5(1 - q_1) = -7q_1 + 5 \quad (4)$$

$$E_5(q_1) = a_{51}q_1 + a_{52}q_2 = 7q_1 - 6(1 - q_1) = 13q_1 - 6 \quad (5)$$

$E_1(0) = 7$,

$E_1(1) = -6$ Line joining (0, 7) and (1, – 6)

$E_2(0) = -5$,

$E_2(1) = 4$ Line joining (0, – 5) and (1, 4)

$E_3(0) = -2$,

$E_3(1) = -1$ Line joining (0, – 2) and (1, – 1)

$E_4(0) = 5$,

$E_4(1) = -2$ Line joining (0, 5) and (1, – 2)

$E_5(0) = -6$,

$E_5(1) = 7$ Line joining (0, – 6) and (1, 7)

The graph is

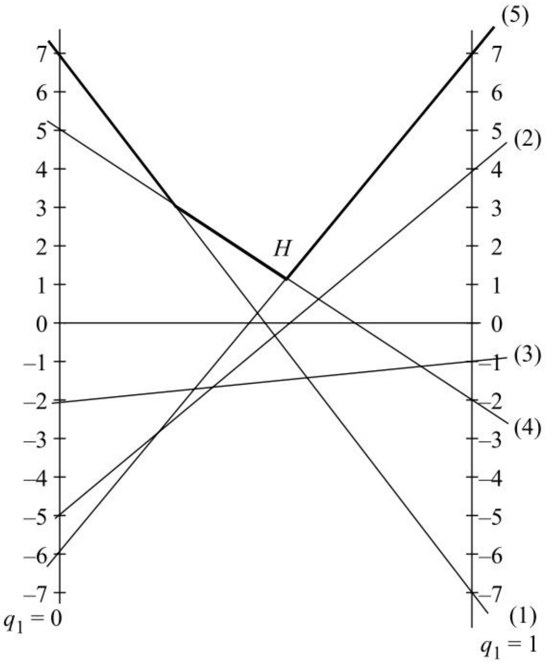


Fig. 14.3

The minimax point H is the intersection (4) and (5). Therefore the optimal strategies of A are A_4 and A_5.

We have the pay-off matrix

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_4 \\ A_5 \end{array} & \begin{pmatrix} -2 & 5 \\ 7 & -6 \end{pmatrix} \end{array}$$

$D = (-2-6) - (7+5) = -20$

$p_4 = (-6-7)/-20 = 13/20$

$p_5 = 7/20$

$q_1 = (-6-5)/-20$

$= 11/20$

$q_2 = 9/20$

$v = (12-35)/-20 = 23/20$

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & 13/20 & 7/20 \end{pmatrix}$$

and $S_B = \begin{pmatrix} B_1 & B_2 \\ 11/20 & 9/20 \end{pmatrix}$

Example 14.15 Solve the game

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{pmatrix} \end{array}$$

Solution Let the player B play the mixed strategy

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

B's expected pay-offs corresponding to A's pure strategies are given by

$A_1 \to E_1(q_1) = 4q_1 - 3,$

$E_1(0) = -3,\ E_1(1) = 1$

$A_2 \to E_2(q_1) = -2q_1 + 5,$

$E_2(0) = 5,\ E_2(1) = 3$

$A_3 \to E_3(q_1) = -7q_1 + 6,$

$E_3(0) = 6,\ E_3(1) = -1$

$A_4 \to E_4(q_1) = 3q_1 + 1,$

$E_4(0) = 1,\ E_4(1) = 4$

$A_5 \to E_5(q_1) = 2,$

$E_5(0) = 2,\ E_5(1) = 2$

$A_6 \to E_6(q_1) = -5q_1,$

$E_6(0) = 0,\ E_6(1) = -5$

In the graph we draw the lines joining

(0, – 3) and (1, 1) (1)

(0, 5) and (1, 3) (2)

(0, 6) and (1, – 1) (3)

(0, 1) and (1, 4) (4)

(0, 2) and (1, 2) (5)

(0, 0) and (1, – 5) (6)

The graph is

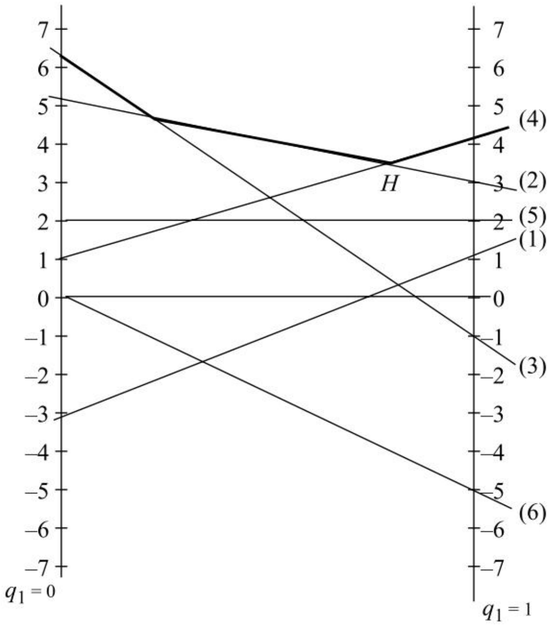


Fig. 14.4

The lowest point of the upper envelope is H which is the intersection of (2) and (4). Hence A's optimal strategies are A_2 and A_4. We have the pay-off matrix

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_2 \\ A_4 \end{array} & \begin{pmatrix} 3 & 5 \\ 4 & 1 \end{pmatrix} \end{array}$$

$$D = (3 + 1) - (4 + 5) = -5$$
$$p_2 = (1 - 4) / -5 = 3/5, \; p_4 = 2/5$$
$$q_1 = (1 - 5) / -5 = 4/5, \; q_2 = 1/5$$
$$v = (3 - 20) / -5 = 17/5$$

The optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{pmatrix}$$

and $S_B = \begin{pmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{pmatrix}$

14.8 ALGEBRAIC METHOD

In a game with $m \times n$ type pay-off matrix if there is no saddle point and if it is not possible to reduce the matrix to 2×2 type using dominance property then the algebraic method is used.

Let the pay-off matrix be

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot\cdot & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

Assume that the optimal strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_m \\ p_1 & p_2 & p_3 & \cdots & p_m \end{pmatrix}$$

and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_n \\ q_1 & q_2 & q_3 & \cdots & q_n \end{pmatrix}$

If the value of the game is v then

$$\left.\begin{array}{l} a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + \ldots + a_{m1}p_m \geq v \\ a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + \ldots + a_{m2}p_m \geq v \\ \quad . \quad . \quad . \quad . \\ a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m \geq v \end{array}\right\} \text{Maximin principle}$$

Also

$$\left.\begin{array}{l} a_{11}q_1 + a_{12}q_2 + \ldots + a_{1n}q_n \leq v \\ a_{21}q_1 + a_{22}q_2 + \ldots + a_{2n}q_n \leq v \\ \quad . \quad . \quad . \\ \quad . \quad . \quad . \\ a_{m1}q_1 + a_{m2}q_2 + \ldots + a_{mn}q_n \leq v \end{array}\right\} \text{Minimax principle}$$

In order to obtain the value of the game, we convert these inequalities into equations. Thus we get

$$\begin{array}{l} a_{11}p_1 + a_{21}p_2 + \ldots + a_{m1}p_m = v \\ a_{12}p_1 + a_{22}p_2 + \ldots + a_{m2}p_m = v \\ \quad \vdots \qquad \vdots \qquad \vdots \quad \vdots \\ a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m = v \\ a_{11}q_1 + a_{12}q_2 + \ldots + a_{1n}q_n = v \\ a_{21}q_1 + a_{22}q_2 + \ldots + a_{2n}q_n = v \\ \quad \vdots \qquad \vdots \qquad \vdots \quad \vdots \\ a_{m1}q_1 + a_{m2}q_2 + \ldots + a_{mn}q_n = v \end{array}$$

Solving these equations we obtain the values of $p_1, p_2, \ldots, p_m$; $q_1, q_2, \ldots, q_n$ in terms of v.

Using the equations

$$p_1 + p_2 + \ldots + p_m = 1$$

and $\quad q_1 + q_2 + \ldots + q_n = 1$

We can obtain the values of p's and q's and v

Example 14.16 Solve the game

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{pmatrix} \end{array}$$

applying algebraic method.

Solution

Let $\quad S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$

and $\quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$

We have
$$-p_1 + p_2 + 3p_3 = v$$
$$2p_1 - 2p_2 + 4p_3 = v$$
$$p_1 + 2p_2 - 3p_3 = v$$
$$-q_1 + 2q_2 + q_3 = v$$
$$q_1 - 2q_2 + 2q_3 = v$$
$$3q_1 + 4q_2 - 3q_3 = v$$

Solving these simultaneous equations we get

$$p_1 = 17v/30;\ p_2 = 20v/30;\ p_3 = 9v/30$$
$$p_1 + p_2 + p_3 = 1 \Rightarrow v = 30/46$$

or $\quad v = 15/23$

$$p_1 = 17/46,\ p_2 = 20/46,\ p_3 = 9/46$$

Similarly, we get

$$q_1 = 7v/15;\ q_2 = 6v/15;\ q_3 = 10v/15$$
$$q_1 + q_2 + q_3 = 1 \Rightarrow v = 15/23$$
$$q_1 = 7/23,\ q_2 = 6/23,\ q_3 = 10/23$$

Therefore

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 17/46 & 10/23 & 9/46 \end{pmatrix}$$

and $\quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 7/23 & 6/23 & 10/23 \end{pmatrix}$

and value of the game is $v = 15/23$

14.9 LINEAR PROGRAMMING METHOD

Problems which are solved using the algebraic method can be solved using linear programming method also.

For the given $m \times n$ pay-off matrix we assume

$$S_A = \begin{pmatrix} A_1 & A_2 & \ldots & A_m \\ p_1 & p_2 & \ldots & p_m \end{pmatrix}$$

and $\quad S_B = \begin{pmatrix} B_1 & B_2 & \ldots & B_n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix}$

Using maximin criterion for A we get

$$a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + \ldots + a_{m1}p_m \geq v$$
$$a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + \ldots + a_{m2}p_m \geq v$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m \geq v$$

where $p_1 + p_2 + p_3 + \ldots + p_m = 1,\ p_i \geq 0$

Dividing both sides by v, we get

$$a_{11}p_1/v + a_{21}p_2/v + \ldots + a_{m1}p_m/v \geq 1$$
$$a_{12}p_1/v + a_{22}p_2/v + \ldots + a_{m2}p_m/v \geq 1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{1n}\,p_1/v + a_{2n}\,p_2/v + \ldots + a_{mn}p_m/v \geq 1$$
$$p_1/v + p_2/v + \ldots\ p_m/v = 1/v$$

Denote $p_1/v, p_2/v, \ldots, p_m/v$ by $x_1\ x_2, \ldots, x_m$.

We get the LPP,

$$\left.\begin{array}{l} \text{Minimize } 1/v = Z = x_1 + x_2 + x_3 + \ldots + x_m \\ \qquad\qquad (\text{Max } v \Leftrightarrow \text{Min } 1/v) \\ \text{subject to } a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + \ldots \\ \qquad\qquad + a_{m1}x_m \geq 1 \\ \qquad a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m \geq 1 \\ \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \qquad a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{mn}x_m \geq 1 \\ \qquad x_1, x_2, \ldots, x_m \geq 0 \end{array}\right\} \text{I}$$

Similarly, considering minimax criterion for B we get

$$a_{11}q_1/v + a_{12}q_2/v + \ldots + a_{1n}q_n/v \leq 1$$
$$a_{21}q_1/v + a_{22}q_2/v + \ldots + a_{2n}q_n/v \leq 1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}q_1/v + a_{m2}q_2/v + \ldots\ a_{mn}q_n/v \leq 1$$
$$q_1/v + q_2/v + \ldots\ q_n/v = 1/v$$

Denote $\quad q_1/v, q_2/v, \ldots, q_n/v$ by $y_1, y_2, y_3, \ldots, y_n$

We get the LPP

$$\left.\begin{array}{l} \text{Maximize } 1/v = W = y_1 + y_2 + y_3 + \ldots + y_n \\ \qquad\qquad (\text{Min } v \Leftrightarrow \text{Max } 1/v) \\ \text{subject to } a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \leq 1 \\ \qquad a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \leq 1 \\ \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \qquad a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \leq 1 \\ \qquad y_1, y_2, \ldots, y_n \geq 0 \end{array}\right\} \text{II}$$

We find that I and II are dual problems. Therefore we can solve one of these and obtain the solution of the other using the principle of duality. The values of $x_1, x_2, \ldots, x_m$ give A's strategies and $y_1, y_2, \ldots, y_n$ yield B's strategies and the value of the game can be determined.

Example 14.17 Solve using the LPP method

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{pmatrix} \end{array}$$

Solution The given pay-off matrix is

$$\begin{array}{cccc} & B_1 & B_2 & B_3 \\ A_1 & 9 & 1 & 4 \\ A_2 & 0 & 6 & 3 \\ A_3 & 5 & 2 & 8 \end{array}$$

The matrix has no saddle point.

Take $\quad S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$

and $\quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$

Considering the pay-off for A the LPP is

$$\left.\begin{array}{l} \text{Minimize } Z = 1/v = x_1 + x_2 + x_3 \\ \text{subject to the constraints} \\ \qquad 9x_1 + 5x_3 \ge 1 \\ \qquad x_1 + 6x_2 + 2x_3 \ge 1 \\ \qquad 4x_1 + 3x_2 + 8x_3 \ge 1 \\ \qquad x_1,\ x_2,\ x_3 \ge 0 \end{array}\right\} \text{I}$$

The corresponding LPP for B's pay-off is

$$\left.\begin{array}{l} \text{Maximize } W = 1/v = y_1 + y_2 + y_3 \\ \text{subject to the constraints} \\ \qquad 9y_1 + y_2 + 4y_3 \le 1 \\ \qquad 6y_2 + 3y_3 \le 1 \\ \qquad 5y_1 + 2y_2 + 8y_3 \le 1 \\ \qquad y_1, y_2, y_3 \ge 0 \end{array}\right\} \text{II}$$

This is the dual of problem I. Let us solve problem II. The standard form is

Maximize $W = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$9y_1 + y_2 + 4y_3 + s_1 = 1$$
$$6y_2 + 3y_3 + s_2 = 1$$
$$5y_1 + 2y_2 + 8y_3 + s_3 = 1$$
$$y_1, y_2, y_3, s_1, s_2, s_3 \ge 0$$

The simplex algorithm is

C_B	B	1 y_1	1 y_2	1 y_3	0 s_1	0 s_2	0 s_3	X_B	
0	s_1	**9**	1	4	1	0	0	1	$\leftarrow R_1/9$
0	s_2	0	6	3	0	1	0	1	
0	s_3	5	2	8	0	0	1	1	$R_3 - 5/9R_1$
	$\bar{C}_j$	$-1\uparrow$	-1	-1	0	0	0	0	
1	y_1	1	1/9	4/9	1/9	0	0	1/9	$R_1 - 1/54R_2$
0	s_2	0	**6**	3	0	1	0	1	$\leftarrow R_2/6$
0	s_3	0	13/9	52/9	$-5/9$	0	1	4/9	$R_3 - 13/54\ R_2$
	$\bar{C}_j$	0	$-8/9\uparrow$	$-5/9$	1/9	0	0	1/9	
1	y_1	1	0	7/18	1/9	$-1/54$	0	5/54	$R_1 - 1/13R_3$
1	y_2	0	1	1/2	0	1/6	0	1/6	$R_2 - 9/19\ R_3$
0	s_3	0	0	**91/18**	$-5/9$	$-13/54$	1	11/54	$R_3 \times 18/91R_3$
	$\bar{C}_j$	0	0	$-1/9\uparrow$	1/9	4/27	0	7/27	
1	y_1	1	0	0	2/13	0	$-1/13$	1/13	
1	y_2	0	1	0	5/91	4/21	$-9/91$	40/273	
1	y_3	0	0	1	$-10/91$	$-1/21$	18/91	11/273	
	$\bar{C}_j$	0	0	0	9/91	1/7	2/91	24/91	

Optimality reached $W^* = 24/91$

$\therefore\ V = 91/24$

$y_1 = 1/13 \Rightarrow q_1 = v/13 = 7/24$

$y_2 = 40/273 \Rightarrow q_2 = 40v/273 = 5/9$

$y_3 = 11/273 \Rightarrow q_3 = 11v/273 = 11/72$

By principle of duality we get (from the $\bar{C}_j$ of the optimal table)

$$x_1 = \frac{9}{91} \Rightarrow p_1 = \frac{9}{91} \times v = 3/8$$

$$x_2 = \frac{1}{7} \Rightarrow p_1 = \frac{1}{7} \times v = 13/24$$

$$x_3 = \frac{2}{91} \Rightarrow p_3 = \frac{2}{91} \times v = 1/12$$

Thus we get

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 3/8 & 13/24 & 1/12 \end{pmatrix}$$

and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 7/24 & 5/9 & 11/72 \end{pmatrix}$

$v = 91/24$

EXERCISES

1. Solve the games with the following pay-off matrices

(a) $\begin{pmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{pmatrix}$

(d.) $\begin{pmatrix} 2 & 6 & 2 \\ -1 & 2 & -7 \\ -2 & 4 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 6 & 1 & 3 \\ 0 & 9 & 7 \\ 2 & 3 & 4 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 3 & 2 & 7 & 4 \\ 3 & 4 & 1 & 5 & 6 \\ 6 & 5 & 7 & 6 & 5 \\ 2 & 0 & 6 & 3 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} -3 & 4 & 2 & 9 \\ 7 & 8 & 6 & 10 \\ 6 & 2 & 4 & -1 \end{pmatrix}$

(i) $\begin{pmatrix} 10 & 4 & 2 & 9 & 1 \\ 7 & 6 & 5 & 7 & 8 \\ 3 & 5 & 4 & 4 & 9 \\ 6 & 7 & 3 & 3 & 2 \end{pmatrix}$

(j) $\begin{pmatrix} -1 & 0 & 0 & 5 & 3 \\ 3 & 2 & 2 & 2 & 2 \\ -4 & -3 & 0 & 2 & 6 \\ 5 & 3 & -4 & 2 & 6 \end{pmatrix}$

(k) $\begin{pmatrix} 3 & -5 & 0 & 6 \\ -4 & -2 & 1 & 2 \\ 5 & 4 & 2 & 3 \end{pmatrix}$

(l) $\begin{pmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{pmatrix}$

(m) $\begin{pmatrix} -5 & 3 & 1 & 10 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{pmatrix}$

(n) $\begin{pmatrix} 30 & 40 & -80 \\ 0 & 15 & -20 \\ 90 & 20 & 50 \end{pmatrix}$

(o) $\begin{pmatrix} 2 & -2 & 4 & 1 \\ 6 & 1 & 12 & 3 \\ -3 & 2 & 0 & 6 \\ 2 & -3 & 7 & 1 \end{pmatrix}$

(p) $\begin{pmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{pmatrix}$

(q) $\begin{pmatrix} 8 & 10 & 9 & 14 \\ 10 & 11 & 8 & 12 \\ 13 & 12 & 14 & 13 \end{pmatrix}$

(r) $\begin{pmatrix} 2 & 4 & 3 & 8 & 4 \\ 5 & 6 & 3 & 7 & 8 \\ 6 & 7 & 9 & 8 & 7 \\ 4 & 2 & 6 & 4 & 3 \end{pmatrix}$

(s) $\begin{pmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{pmatrix}$

2. Use graphical method to solve

(a) $\begin{pmatrix} 3 & 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -4 & 6 & -3 & 5 \\ -3 & 4 & -4 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -6 & 10 \\ -3 & -2 \\ 3 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} -2 & 5 \\ -5 & 3 \\ 0 & -2 \\ -3 & 0 \\ 1 & -4 \end{pmatrix}$

(f) $\begin{pmatrix} 2 & 4 \\ 2 & 3 \\ 3 & 2 \\ -2 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 3 & -1 & 4 & 2 & -5 \\ -3 & 5 & 6 & 1 & 2 & 0 \end{pmatrix}$

(i) $\begin{pmatrix} -2 & -4 & 3 & 4 \\ -6 & -5 & 2 & 1 \end{pmatrix}$

(j) $\begin{pmatrix} 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \end{pmatrix}$

(k) $\begin{pmatrix} 19 & 6 & 7 & 5 \\ 7 & 3 & 14 & 6 \\ 12 & 8 & 18 & 4 \\ 8 & 7 & 13 & -1 \end{pmatrix}$

Hint: Reduce using dominance property.

3. Use algebraic method to solve

(a) $\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 8 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ -1 & -2 & 3 \end{pmatrix}$

4. Use the LPP method to solve

(a) $\begin{pmatrix} 3 & -4 & 2 \\ 1 & -3 & -7 \\ -2 & 4 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 2 & 2 \\ 3 & -1 & 3 \\ 4 & 4 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{pmatrix}$

ANSWERS

1. (a). Saddle point (2, 3).
   Value of the game = 4
   (b) Saddle points (1, 1), and (1, 3).
   Value of the game = 1
   (c) Saddle point (2, 2).
   Value of the game = 5
   (d) Saddle points (1, 1), and (1, 3). Value of the game = 2
   (e) Saddle point (3, 3).
   Value = 6
   (f) Saddle point (2, 3).
   Value = 7
   (g) Saddle point (3, 2).
   Value = 5
   (h) Saddle point (2, 3).
   Value = 6
   (i) Saddle point (2, 3).
   Value = 5
   (j) Saddle point (2, 3).
   Value of the game = 2
   (k) Saddle point (3, 3).
   Value of the game = 2
   (l) Saddle point (1, 1) and (1, 3).
   Value = –2
   (m) Saddle point (2, 3).
   Value = 4
   (n) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 1/5 & 0 & 4/5 \end{pmatrix}$ and
   $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 13/15 & 2/15 \end{pmatrix}$
   Value of the game: 24
   (o) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$ and
   $S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 1/10 & 9/10 & 0 & 0 \end{pmatrix}$
   Value: 3/2
   (p) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 12/18 & 6/18 \end{pmatrix}$
   $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 13/18 & 5/18 \end{pmatrix}$
   Value: 10/3
   (q) Saddle point (3, 2). Value = 12
   (r) Saddle point (3, 1). Value = 6
   (s) Saddle point (1, 3). Value = 6
2. (a) $S_A = \begin{pmatrix} A_1 & A_2 \\ 3/7 & 4/7 \end{pmatrix}$ and
   $S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 1/17 & 0 & 0 & 6/7 \end{pmatrix}$ $v = 5/7$

(b) $S_A = \begin{pmatrix} A_1 & A_2 \\ 2/3 & 1/3 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 1/3 & 2/3 & 0 & 0 \end{pmatrix}$ $v = 2/3$

(c) $S_A = \begin{pmatrix} A_1 & A_2 \\ 4/9 & 5/9 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 4/9 & 0 & 0 & 5/9 \end{pmatrix}$ $v = -7/9$

(d) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 4/7 & 3/7 & 0 & 0 \end{pmatrix}$ and

$S_B = \begin{pmatrix} A_1 & A_2 \\ 5/7 & 2/7 \end{pmatrix}$ $v = 34/7$

(e) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 5/12 & 0 & 0 & 0 & 7/12 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 \\ 3/4 & 1/4 \end{pmatrix}$ $v = -1/4$

(f) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 1/3 & 0 & 2/3 & 0 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 \\ 2/3 & 1/3 \end{pmatrix}$ $v = 8/3$

(g) $S_A = \begin{pmatrix} A_1 & A_2 \\ 3/11 & 8/11 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 2/11 & 9/11 \end{pmatrix}$ Value = 49/11

(h) $S_A = \begin{pmatrix} A_1 & A_2 \\ 4/5 & 1/5 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{pmatrix}$

Value = 17/5

(i) $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/3 & 2/3 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 1/5 & 4/5 & 0 & 0 \end{pmatrix}$

Value = –18/5

(j) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 3/5 & 0 & 2/5 & 0 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{pmatrix}$ Value = 17/5

(k) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 3/4 & 1/4 & 0 & 0 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix}$

Value = 21/4

3. (a) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 6/17 & 7/17 & 4/17 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 2/17 & 9/17 & 6/17 \end{pmatrix}$

Value = 5/17

(b) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 3/8 & 13/24 & 1/12 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 7/24 & 5/9 & 11/72 \end{pmatrix}$ $v = 91/24$

(c) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$ $v = 8/3$

(d) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$ Value = 1

(e) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 4/7 & 3/7 & 0 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 5/7 & 2/7 & 0 \end{pmatrix}$ $v = 1/7$

4. (a) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 6/13 & 0 & 7/13 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 8/13 & 5/13 & 0 \end{pmatrix}$ $v = 4/13$

(b) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 12/25 & 9/25 & 4/25 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 19/50 & 15/50 & 8/25 \end{pmatrix}$ $v = 18/25$

(c) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 6/11 & 3/11 & 2/11 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 5/22 & 8/22 & 9/22 \end{pmatrix}$ $v = 17/11$

(d) $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 6/13 & 3/13 & 4/13 \end{pmatrix}$ and

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 6/13 & 4/13 & 3/13 \end{pmatrix}$ $v = -1/13$

15

Dynamic Programming

CONCEPT REVIEW

15.1 INTRODUCTION

Dynamic programming is a mathematical technique of optimization using multistage decision process. It is a systematic procedure for determining the combination of decisions which maximize the overall objective. Decisions regarding a certain problem are optimized in stages rather than simultaneously. The original problem is split into sub-problems such that the outcome of each depends on the result of the previous one. The solution is obtained in an orderly manner by going from one stage to the next and the complete solution is obtained at the final stage.

Decision-making process consists of selecting a combination of plans from a large number of alternatives. All the combinations must be specifically known beforehand. Then only we can select the optimal policy. If we deal with the problem as a whole it involves lot of computation work and time. Certain combinations may not be feasible. In such a situation dynamic programming becomes very useful. We break the problem into sub-problems (*stages*). Only one stage is considered at a time and the infeasible combinations are eliminated. The solution is obtained by moving from one stage to the next and is completed at the final stage. At each stage there are many alternatives and the selection of one feasible alternative is called *stage decision*. The variables which specify the condition of the process and summarize the current status (state) are called *state variables*. At any stage there could be a finite or infinite number of states.

Bellman's principle of optimality states that "an optimal policy has the property that whatever the initial stages and decisions are, the remaining decisions must constitute an optimal policy with regards to the state resulting from the first decision". This implies that a wrong decision taken at one stage does not prevent from taking optimum decisions for the remaining stages.

15.2 DYNAMIC PROGRAMMING METHOD

Example 15.1 (Salesmen allocation problem) Consider the problem of allocating 9 salesmen to three zones in such a manner that the total profits of the company are maximized. The following table gives the details about the zones, number of salesmen and the corresponding profits.

	Profits (in thousands of rupees)		
No. of salesmen	*Zone 1*	*Zone 2*	*Zone 3*
0	30	35	42
1	45	45	54
2	60	52	60
3	70	64	70
4	79	72	82
5	90	82	95
6	98	93	102
7	105	98	110
8	100	109	110
9	90	100	110

In this example the stages are three zones and the number of salesmen is the state variable.

Stage I (Zone I)

x	0	1	2	3	4	5	6	7	8	9
$f_1(x)$ (Profit)	30	45	60	70	79	90	98	105	100	90

This table gives the profit corresponding to different number of salesmen allocated to zone 1.

$$F_1(A) = f_1(x).$$

Stage II (Zones 1 and 2)

If S denotes the total profits from each combination we have the general form

$$S = f_2(x) + F_1(A - x)$$

where A is the total number of salesmen.

We have to maximize

$$F_2(A) = \text{Max } \{f_2(x) + F_1(A - x)\}.$$

This equation can be applied to determine the optimal distribution of any number of salesmen ($A = 9, 8, 7, 6, 5, \ldots$) as shown in the following table.

Zone 2 \ *Zone 1*		*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*No. of Salesmen*
		30	*45*	*60*	*70*	*79*	*90*	*98*	*105*	*100*	*90*	*Profits*
0	35	**65**	**80**	**95**	**105**	114	**125**	133	140	135	125	
1	45	75	90	**105**	**115**	124	**135**	**143**	150	145		
2	52	82	97	112	122	131	142	150	157			
3	64	94	109	124	134	**143**	**154**	162				
4	72	102	117	132	142	151	162					
5	82	112	127	142	152	161						
6	93	123	138	153	**163**							
7	98	128	143	158								
8	109	130	145									
9	100	130										

The maximum profit for each value of A are represented in bold.

The above table shows the profits for all combinations. For a given number of salesmen, the profits for different combinations can be found along the diagonal. The maximum profits for various combinations are given below:

A	0	1	2	3	4	5	6	7	8	9
S	65	80	95	105	115	125	135	143	154	163
	0+0	0+1	0+2	1+2 or 0+3	1+3	0+5	1+5	3+4 or 1+6	3+5	6+3

We have

$$F_2(A) = \text{Max } \{f_2(x) + F_1(A - x)\}$$

Now we go to the next stage and consider the distribution of 9 salesmen in 3 zones.

$$S = f_3(x) + F_2(A - x)$$

$$F_3(A) = \text{Max } \{f_3(x) + F_2(A - x)\}$$

Stage III

Zone 3		*Zone 1 + Zone 2* 0 / 65	1 / 80	2 / 95	3 / 105	4 / 115	5 / 125	6 / 135	7 / 143	8 / 154	9 / 163
0	42	**107**	**122**	**137**	147	157	167	177	185	196	205
1	54	119	134	**149**	**159**	**169**	**179**	189	197	208	
2	60	125	140	155	165	175	185	195	203		
3	70	135	150	165	175	185	195	205			
4	82	147	162	177	187	197	207				
5	95	160	175	**190**	**200**	**210**					
6	102	167	182	197	207						
7	110	175	190	205							
8	110	175	190								
9	110	175									

$A = (x_1 + x_2) + x_3$

The maximum profits for different values of A are

A	0	1	2	3	4	5	6	7	8	9
S	107	122	137	149	159	169	179	190	200	210
	0+0	0+1	0+2	1+2	1+3	1+4	1+5	5+2	5+3	5+4

Thus for $\boldsymbol{A = 9}$. We find that the profit is maximum for $x_1 + x_2 = 4$ and $x_3 = 5$. From stage II we find that for $A = 4$, the profit has max value 115 with $\boldsymbol{x_1 = 3\, x_2 = 1}$. Therefore the optimal distribution of salesmen is

$$\left.\begin{array}{l} \text{Zone 1} \longrightarrow 3 \\ \text{Zone 2} \longrightarrow 1 \\ \text{Zone 3} \longrightarrow 5 \end{array}\right\} \text{Max profit} = \text{Rs } 2,10,000$$

Example 15.2 (Production allocation problem) The owner of four fruit shops has purchased six boxes of apple. The quantity in demand and the profits are different at these stores. The following table gives the total profit at each store for various numbers of boxes allotted.

Number of boxes	*Stores* 1	2	3	4
0	0	0	0	0
1	4	2	6	2
2	6	4	8	3
3	7	6	8	4
4	7	8	8	4
5	7	9	8	4
6	7	10	8	4

Find the mode of allocation of the six boxes to the stores so as to maximize the profit.

Solution Let us take stage I as the allocation of the boxes to 1 store

Stage I (Store 1)

X	0	1	2	3	4	5	6
$f_1(x)$ profit	0	4	6	7	7	7	7

Stage II (Stores 1 and 2) Total profit $P = F_2(A) = Max\,\{f_2(x) + F_1(A - x)\}$ where A denotes the total number of boxes allocated. Distribution of different number of boxes at stores 1 and 2 is as follows:

Store 2		*Store 1* 0 / 0	1 / 4	2 / 6	3 / 7	4 / 7	5 / 7	6 / 7	*No. of boxes Profit*
0	0	**0**	**4**	**6**	7	7	7	7	
1	2	2	**6**	**8**	9	9	9		
2	4	4	**8**	**10**	11	11			
3	6	6	**10**	**12**	13				
4	8	8	**12**	**14**					
5	9	9	13						
6	10	10							

The maximum profits for different no of allocations to stores 1 and 2 are

A	0	1	2	3	4	5	6
P	0	4	6	8	10	12	14
$x_1 + x_2$	0+0	0+1	0+2 1+1	1+2 2+1	2+2 3+1	3+2 4+1	4+2

We have $F_2(A) = \text{Max}\,[f_2(x) + F_1(A - x)]$

Stage III (Stores 1, 2 and 3)

$P = F_3\,(A) = \text{Max}\,\{f_3(x) + F_2\,(A - x)\}$

Store 3		*Store 1 + Store 2*						
		0	*1*	*2*	*3*	*4*	*5*	*6*
		0	*4*	*6*	*8*	*10*	*12*	*14*
0	0	**0**	4	6	8	10	12	14
1	6	**6**	**10**	**12**	**14**	**16**	**18**	
2	8	8	**12**	**14**	**16**	**18**		
3	8	8	12	14	16			
4	8	8	12	14				
5	8	8	12					
6	8	8						

The maximum profits for different allocations for stores 1, 2 and 3 are

A	0	1	2	3	4	5	6
P	0	6	10	12	14	16	18
	0+0	1+0	1+1	1+2 2+1	1+3 2+2	1+4 2+3	1+5 2+4

Stage IV (Stores 1, 2, 3 and 4)

$P = F_4(A) = \text{Max}\,\{f_4(x) + F_3(A - x)\}$

Store 4		*Store 1 + Store 2 + Store 3*						
		0	*1*	*2*	*3*	*4*	*5*	*6*
		0	*6*	*10*	*12*	*14*	*16*	*18*
0	0	**0**	**6**	**10**	**12**	**14**	**16**	**18**
1	2	2	8	**12**	**14**	**16**	**18**	
2	3	3	9	13	15	17		
3	4	4	10	14	16			
4	4	4	10	14				
5	4	4	10					
6	4	4						

The maximum profits for different values of *A* are

A	0	1	2	3	4	5	6
P	0	6	10	12	14	16	18
	0+0	0+1	0+2	0+3 1+2	0+4 1+3	0+5 1+4	0+6 1+5

For four stores taken together, we find that the max profit is 18, for 6 baskets.

The optimal distributions are

Store 1	*Store 2*	*Store 3*	*Store 4*
2	3	1	0
1	4	1	0
2	2	2	0
1	3	2	0
2	2	1	1
1	3	1	1
2	1	2	1
1	2	2	1

Example 15.3 (Capital budgeting problem) A company produces automobile parts, bicycle parts, and sewing machine parts. An amount of Rs 20,000 has been allotted for expanding production facilities. In autoparts and bicycle parts sections production can be increased either by addition of new machines or by replacing old inefficient machines by automatic machines. In sewing machine parts section, additional amount can be invested only by adding new machine. The cost of addition and replacement of machines and the corresponding returns are given below: Select a set of expansion plans which may yield maximum return.

Alternatives	*Auto parts*		*Bicycle parts*		*Sewing machine*	
	Cost	*Return*	*Cost*	*Return*	*Cost*	*Return*
No expansion	0	0	0	0	0	0
Add new achines	4000	8000	8000	12000	2000	8000
Replacement	6000	10000	12000	18000	–	–

Solution

Stage I Each department can be considered as a stage. The capital amount is the state variable. Let us take auto parts section as stage I.

State x_1	Alternatives 1 $C_{11} = 0$ $R_{11} = 0$	2 $C_{12} = 4$ $R_{12} = 8$	3 $C_{13} = 6$ $R_{13} = 10$	Optimal Solution Return $F_1(A)$	Decision
0	0	–	–	0	1
2	0	–	–	0	1
4	0	8	–	8	2
6	0	8	10	10	3
8	0	8	10	10	3
10	0	8	10	10	3
12	0	8	10	10	3
14	0	8	10	10	3
16	0	8	10	10	3
18	0	8	10	10	3
20	0	8	10	10	3

Stage II We consider auto parts and bicycle parts

$$F_2(A) = \text{Max } \{f_2(x) + F_1(A - x)\}$$

State x_2	Alternatives 1 $C_{21} = 0$ $R_{21} = 0$	2 $C_{22} = 8$ $R_{22} = 12$	3 $C_{23} = 12$ $R_{23} = 18$	Optimal Solution Return $F_2(A)$	Decision
0	0+0=0	–	–	0	1
2	0+0=0	–	–	0	1
4	0+8=8	–	–	8	1
6	0+10=10	–	–	10	1
8	0+10=10	12+0=12	–	12	2
10	0+10=10	12+0=12	–	12	2
12	0+10=10	12+8=20	18+0=18	20	2
14	0+10=10	12+10=22	18+0=18	22	2
16	0+10=10	12+10=22	18+8=26	26	3
18	0+10=10	12+10=22	18+10=28	28	3
20	0+10=10	12+10=22	18+10=28	28	3

Stage III We consider all the three sections together for stage III. The computations are given below.

$$F_3(A) = \text{Max } \{f_3(x) + F_2(A - x)\}$$

State x_3	Alternatives 1 $C_{31} = 0$ $R_{31} = 0$	2 $C_{32} = 2$ $R_{32} = 8$	Optimal Solution Return $F_3(A)$	Decision
0	0 + 0 = 0	–	0	1
2	0 + 0 = 0	8 + 0 = 8	8	2
4	0 + 8 = 8	8 + 0 = 8	8	1, 2
6	0 + 10 = 10	8 + 8 = 16	16	2
8	0 + 12 = 12	8 + 10 = 18	18	2
10	0 + 12 = 12	8 + 12 = 20	20	2
12	0 + 20 = 20	8 + 12 = 20	20	1, 2
14	0 + 22 = 22	8 + 20 = 28	28	2
16	0 + 26 = 26	8 + 22 = 30	30	2
18	0 + 28 = 28	8 + 26 = 34	34	2
20	0 + 28 = 28	8 + 28 = 36	36	2

For stage III, optimal decision is the **2nd alternative** with maximum return of Rs 36,000 (C_{32} = 2). For the remaining amount 18 we find that in stage II optimal decision is the **3rd alternative** with maximum return of Rs 28,000 (C_{23} = 12). For the remaining amount 6 we find that in stage I optimal decision is the **3rd alternative** with maximum return of Rs 10,000 (C_{13} = 6).

Thus for the allocation of Rs 20,000, the optimal policy of expansion is

1. Replacing old machines in auto parts
2. Replacing old machines in bicycle parts and
3. Adding new machines in sewing machines parts

The optimum returns is Rs 36,000.

Example 15.4 (Advertising media problem) A company is interested in selecting the advertising media for its product. The frequency of advertising per week and the corresponding expected sales in thousands of rupees are given below:

Frequency per week	Sales (in thousands of rupees) TV	Radio	Newspaper
1	220	150	100
2	275	250	175
3	325	300	225
4	350	320	250

The costs of advertising in newspaper, radio and television are Rs 500, Rs 1000 and Rs 2000 respectively per appearance. The budget provides Rs 4000 per week for advertisement. Determine the optimal combination of advertising media and frequency.

Solution Let us consider TV as stage I. The costs are C_{11} C_{12} C_{13} C_{14} for the frequencies and the corresponding returns are R_{11} R_{12} R_{13} R_{14}.

State	*Frequency*					
x_1	*1* $C_{11} = 2000$ $R_{11} = 220$	*2* $C_{12} = 4000$ $R_{12} = 275$	*3* $C_{13} = 6000$ $R_{13} = 325$	*4* $C_{14} = 8000$ $R_{14} = 350$	*Return*	*Decision*
500	–	–	–	–	0	0
1000	–	–	–	–	0	0
1500	–	–	–	–	0	0
2000	220	–	–	–	220	1
2500	220	–	–	–	220	1
3000	220	–	–	–	220	1
3500	220	–	–	–	220	1
4000	220	275	–	–	275	2

$F_1(A) = \text{Max } f_1(x)$

Now let us take TV and Radio

Stage II

$F_2(A) = \text{Max } \{f_2(x) + F_1(A - x)\}$

State	*Frequency*					*Optimal*	
x_2	*0*	*1* $C_{21} = 1000$ $R_{21} = 150$	*2* $C_{22} = 2000$ $R_{22} = 250$	*3* $C_{23} = 3000$ $R_{23} = 300$	*4* $C_{24} = 4000$ $R_{24} = 320$	*Return*	*Decision*
500	0	–	–	–	–	0	0
1000	0	150	–	–	–	150	1
1500	0	150	–	–	–	150	1
2000	220	150 + 0	250	–	–	250	2
2500	220	150 + 0	250 + 0	–	–	250	2
3000	220	150 + 220	250 + 0	300	–	370	1
3500	220	150 + 220	250 + 0	300 + 0	–	370	1
4000	275	150 + 220	250 + 220	300 + 0	320	470	2

Stage III

TV, Radio and Newspaper

$F_3(A) = \text{Max } \{f_3(x) + F_2(A - x)\}$

State x_3	*Frequency* 0	1 $C_{31} = 500$ $R_{31} = 100$	2 $C_{32} = 1000$ $R_{32} = 175$	3 $C_{33} = 1500$ $R_{33} = 225$	4 $C_{34} = 2000$ $R_{34} = 250$	*Optimal* Return $F_3(A)$	Decision
500	0	100	–	–	–	100	1
1000	150	100 + 0	175	–	–	175	2
1500	150	100 + 150	175 + 0	225	–	250	1
2000	250	100 + 150	175 + 150	225 + 0	250	325	2
2500	250	100 + 250	175 + 150	225 + 150	250 + 0	375	3
3000	370	100 + 250	175 + 250	225 + 150	250 + 150	425	2
3500	370	100 + 370	175 + 250	225 + 250	250 + 150	475	3
4000	470	100 + 370	175 + 370	225 + 250	250 + 250	545	2

Thus the maximum return is 5,45,000. The optimal policy is: Twice in Newspaper, once in Radio and once in TV per week.

Example 15.5 (Cargo loading problem) A truck can carry a total of 10 tons of a product. Three types of product are available for transport. Their weight and values are given below. Determine the loading such that the total value of the cargo is maximum (at least one unit of each type must be loaded).

Type	*Value (Rs)*	*Weight (tons)*
A	20	1
B	50	2
C	60	2

Solution We have to determine how many units of *A*, *B* and *C* is to be loaded. We take the loading of type *C* as stage I.

State x_1 *(Weight)*	*No. of units* 1 $C_{11} = 60$ $W = 2$	2 $C_{12} = 120$ $W = 4$	3 $C_{13} = 180$ $W = 6$	*Optimal* Value $F_1(A)$	Decision
2	60	–	–	60	1
3	60	–	–	60	1
4	60	120	–	120	2
5	60	120	–	120	2
6	60	120	180	180	3
7	60	120	180	180	3

One unit in each of *A* and *B* must be taken. Hence x_1 cannot exceed 7 tons. We take *C* and *B* for stage II

$F_2(A) = \text{Max } \{f_2(x) + F(A - x)\}$

Stage II

State *(Weight)*	*No. of units* 1 $C_{21} = 50$ $W = 2$	2 $C_{22} = 100$ $W = 4$	3 $C_{23} = 150$ $W = 6$	*Optimal* Value $F_2(A)$	Decision
4	50 + 60	100	–	110	1
5	50 + 60	100	–	110	1
6	50 + 120	100 + 60	150	170	1
7	50 + 120	100 + 60	150	170	1
8	50 + 180	100 + 120	150 + 60	230	1
9	50 + 180	100 + 120	150 + 60	230	1

Stage III (C, B and A are taken)

$F_3(A) = \text{Max } \{f_3(x) + F_2(A - x)\}$

State	No. of units						Optimal	
x_3	1 $C_{31} = 20$ $W = 1$	2 $C_{32} = 40$ $W = 2$	3 $C_{33} = 60$ $W = 3$	4 $C_{34} = 80$ $W = 4$	5 $C_{35} = 100$ $W = 5$	6 $C_{36} = 120$ $W = 6$	$F_3(A)$	Decision
5	20 + 110	–	–	–	–	–	130	1
6	20 + 110	40 + 110	–	–	–	–	150	2
7	20 + 170	40 + 110	60 + 110	–	–	–	190	1
8	20 + 170	40 + 170	60 + 110	80 + 110	–	–	210	2
9	20 + 230	40 + 170	60 + 170	80 + 110	100 + 110	–	250	1
10	20 + 230	40 + 230	60 + 170	80 + 170	100 + 110	120 + 110	270	2

We find that the maximum value is **Rs 270**.

The loading is ***A* – 2 units, *B* – 1 unit, *C* – 3 units.**

Weight **2 + 2 + 6 = 10 tons**

15.3 APPLICATIONS TO MATHEMATICAL PROBLEMS

Example 15.6 (Optimal sub-division problem) Divide a positive number c into n parts such that their product is maximum.

Solution The problem can be taken as

Maximize $Z = y_1\, y_2\, y_3 \ldots y_n$ where

$y_1 + y_2 + y_3 + \ldots + y_n = c \;(y_i \geq 0\; i = 1, 2,\ldots, n)$

Each i can be considered as a stage. Here y_i is continuous. Hence the optimal decisions at each stage are obtained by the method of maxima and minima of differential calculus.

For $n = 1$ (one stage) y_1 = c only one part

$\therefore\; \boldsymbol{f_1(c) = c}$

For $n = 2$ (two stage) take $y_1 = x$, $y_2 = c - x$

$$f_2(c) = \text{Max } y_1 y_2 = \text{Max } \{x.f_1(c - x)\}$$
$$= \textbf{Max}\{\boldsymbol{x\,(c - x)}\}$$

Now $u = x\,(c - x) = cx - x^2$

$\dfrac{du}{dx} = c - 2x = 0 \Rightarrow x = c/2$. Therefore, $c - x = c/2$ when u is maximum

$\therefore\;$ Max $x(c - x) = c/2.\; c/2 = (c/2)^2$

$\therefore\; f_2(c) = (c/2)^2$. $\boldsymbol{y_1 = y_2 = c/2}$

For $n = 3$

$$f_3(c) = \text{Max } y_1\, y_2\, y_3$$
$$= \text{Max } \{x.f_2(c - x)\}$$
$$= \text{Max } \{x.[(c - x)/2]^2\}$$

Take $u = x[(c - x)/2]^2 = \dfrac{1}{4}x(c - x)^2$

$$\frac{du}{dx} = \frac{1}{4}[x\,2\,(c - x)\,(-1) + (c - x)^2]$$
$$= \frac{1}{4}[(c - x)^2 - 2x\,(c - x)]$$
$$= \frac{1}{4}[(c - x)\,(c - 3x)]$$

$\dfrac{du}{dx} = 0 \Rightarrow \boldsymbol{x = c/3}$ or $\boldsymbol{x = c}$

For max u, $x = c/3$. $\therefore$ Therefore $c - x = 2c/3$

Max $u = \dfrac{1}{4}\, c/3.\; 4\,(c/3)^2 = (c/3)^3$

$\therefore\;$ $\boldsymbol{f_3(c) = (c/3)^3}$ and $\boldsymbol{y_1 = y_2 = y_3 = c/3}$.

Assume that the optimal policy for $n = m$ is $(c/m, c/m, \ldots, c/m)$

$f_m(c) = (c/m)^m$

For $n = m + 1$, $f_{m+1}(c) = \text{Max}\{x\,f_m(c - x)\}$

$$= \text{Max } [x.\,\{(c - x)/m\}^m]$$
$$u = x\,(c - x)^m / m^m$$
$$\frac{du}{dx} = (1/m^m)\,\{x.m\,(c - x)^{m-1}.\,(-1) + (c - x)^m\}$$
$$= (1/m^m)\,(c - x)^{m-1}\,[c - x - mx]$$
$$= (1/m^m)\,(c - x)^{m-1}\,[c - (m + 1)x]$$

u is maximum when $\boldsymbol{x = c/(m + 1)}$

$(c - x)^m/m^m = (mc)^m/(m + 1)^m m^m = c^m/(m + 1)^m$

$$\therefore\; f_{m+1}(c) = (c/m + 1).\; c^m/(m + 1)^m$$
$$= c^{m+1}/(m + 1)^{m+1}$$

∴ The subdivision is $c/(m + 1)$, $c/(m + 1)$, …. $c/(m + 1)$ and $f_{m+1}(c) = [c/(m + 1)]^{m+1}$

This shows that the result is true for $\boldsymbol{n = m + 1}$. Also we have seen that it is true for $n = 1, 2, 3$. Hence by induction, it is true for any value of n.

∴ The optimal subdivision is

$(c/n, c/n, \ldots, c/n)$ and the product is $(c/n)^n$.

Example 15.7 Factorize a positive number b into n factors such that their sum is minimum.

We have to find $y_1, y_2, y_3, \ldots, y_n$ such that $y_1, y_2, y_3, \ldots, y_n = b$ and their sum

$Z = y_1 + y_2 + y_3 + \ldots + y_n$ is minimum.

Solution One stage problem

(n = 1)

$$y_1 = b$$

$$\therefore f_1(b) = \min_{y_1 = b} \{y_1\} = b$$

Two stage problem

(n = 2)

$$b = y_1 y_2.$$

$$y_1 = x,\ y_2 = b/x$$

$$f_2(b) = \min (y_1 + y_2)$$

$$= \min (x + f_1(b/x)\}$$

$$= \min (x + b/x)$$

$$u = x + b/x$$

$$\frac{du}{dx} = 1 - b/x^2 = 0$$

$$\Rightarrow \quad x^2 = b \text{ or } x = \sqrt{b} \text{ when } u \text{ is minimum.}$$

$$\therefore f_2(b) = \min \{x + b/x\}$$

$$= \sqrt{b} + \sqrt{b} = 2\sqrt{b} = 2b^{\frac{1}{2}}$$

The optimal policy is $\{b^{1/2}, b^{1/2}\}$ and $f_2(b) = 2b^{1/2}$

(n = 3)

$$b = y_1 y_2 y_3$$

$$f_3(b) = \min \{x + f_2(b/x)\}$$

$$= \min \{x + 2.(b/x)^{1/2}\}$$

$$u = x + 2.b^{1/2}/x^{1/2}$$

$$\frac{du}{dx} = 1 + 2b^{1/2}(-1/2)x^{-3/2}$$

when u is minimum

$$\frac{du}{dx} = 1 - b^{1/2} / x^{3/2} = 0$$

$$\Rightarrow x^{3/2} = b^{1/2} \text{ or } x = b^{1/3}$$

$$\therefore f_3(b) = b^{1/3} + 2\, b^{1/2}/b^{1/6}$$

$$= b^{1/3} + 2b^{1/3} = 3b^{1/3}$$

The optimal policy is $\{b^{1/3}\, b^{1/3}\, b^{1/3}\}$ and $f_3(b) = \boldsymbol{3b^{1/3}}$

Now let us assume that the result is true for $\boldsymbol{n = m}$

$$\therefore \quad b = y_1\, y_2\, y_3 \ldots y_m$$

Optimal policy is $\{b^{1/m}\, b^{1/m} \ldots b^{1/m}\}$ and $\boldsymbol{f_m(b) = mb^{1/m}}$

$$f_{m+1}(b) = \min \{x + f_m(b/x)\}$$

$$= \min \{x + m\,(b/x)^{1/m}\}$$

Take $u = x + m\,.\,b^{1/m}/x^{1/m}$

$$\frac{du}{dx} = 1 + m\, b^{1/m} \{(-1/m)\, x^{-1/m - 1}\}$$

$$= 1 - b^{1/m} / x^{1 + 1/m} = 1 - b^{1/m} / x^{(m+1)/m}$$

$$\frac{du}{dx} = 0 \Rightarrow x^{(m+1)/m} = b^{1/m} \text{ or } x = b^{1/m+1}$$

when u is minimum

$$\therefore \quad f_{m+1}(b) = b^{1/m+1} + m\,.\,b^{1/m}/b^{1/m(m+1)}$$

$$= b^{1/m+1} + m\,.\,b^{1/m - 1/m(m+1)}$$

$$= b^{1/m+1} + m\, b^{1/m+1}$$

$$= (m + 1)\, b^{1/m+1}$$

∴ The optimal policy is

$$\{ b^{1/m+1}, b^{1/m+1}, \ldots, b^{1/m+1}\}.$$

Thus we find that the result is true for $n = m + 1$ also. We have seen that the result is true for $n = 1, 2, 3 \ldots$ Therefore by principle of induction, we find that it is true for all values of n.

∴ The optimal factorization is

$$b^{1/n}\, b^{1/n} \ldots b^{1/n} \text{ and } z = \boldsymbol{n\, b^{1/n}}$$

Example 15.8 If $p_1 + p_2 + p_3 + \ldots + p_n = 1$, show that the sum

$z = p_1 \log p_1 + p_2 \log p_2 + \ldots + p_n \log p_n\ (p_i \geq 0)$ is minimum when $p_1 = p_2 = p_3 = p_3 \ldots p_n = 1/n$.

Solution For $n = 1$, $p_1 = 1$ and $\boldsymbol{z = p_1 \log p_1 f_1(1) = 1 \log 1 = 0}$ trivial case

For $n = 2$, take $p_1 = x$ and $p_2 = 1 - x$

$$z = f_2(1) = \min \{p_1 \log p_1 + p_2 \log p_2\}$$

$$= \min \{x \log x + (1 - x) \log (1 - x)\ \}$$

Take $u = x \log x + (1 - x) \log (1 - x)$

$$\frac{du}{dx} = x.1/x + \log x + (1 - x)\,(-1) / (1 - x) + (-1) \log (1 - x)$$

$$= 1 + \log x - 1 - \log (1 - x)$$

$$\frac{du}{dx} = 0 \Rightarrow \log x - \log(1-x) = 0$$

$\Rightarrow$ $\log x = \log(1-x)$

$\Rightarrow$ $x = 1 - x$ or $x = ½$ when u is minimum

$\therefore$ $f_2(1) = \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2} = 2(1/2 \log \frac{1}{2})$

$\therefore$ $p_1 = p_2 = \frac{1}{2}$ and min $z = 2(1/2 \log \frac{1}{2})$

Let us assume that the result is true for $n = m$.

$\therefore$ $p_1 + p_2 + p_3 + \ldots + p_m = 1$ and $z = p_1 \log p_1 + p_2 \log p_2 + \ldots + p_m \log p_m$ is minimum when $p_1 = p_2 = \ldots = p_m = 1/m$ or $f_m(1) = \boldsymbol{m(1/m \log 1/m)}$.

Now $f_{m+1}(1) = \min\{x \log x + f_m(1-x)\}$

$= \min\{x \log x + m\,(1-x)/m \log(1-x)/m\}$

Take $u = x \log x + m \,.\, (1-x)/m \log(1-x)/m$

$$\frac{du}{dx} = x \,.\, 1/x + \log x + (1-x)(-1)/(1-x) + \log(1-x)/m\,(-1)$$

$= \log x - \log(1-x)/m$

$$\frac{du}{dx} = 0 \Rightarrow \log x = \log(1-x)/m$$

$\Rightarrow$ $x = 1/(m+1)$ when u is minimum.

$\therefore f_{m+1}(1) = 1/(m+1).\log 1/(m+1) + m[\{1 - 1/(m+1)\}/m]\log\{1 - 1/(m+1)\}/m$

$= 1/(m+1)\log 1/(m+1) + [m/(m+1)]\log 1/(m+1)$

$= \boldsymbol{(m+1)\,[\,1/(m+1)\log\{1/(m+1)\}]}$

Therefore, the result is true for $n = m + 1$. But we have seen that it is true for $n = 1, 2$. Hence by induction principle the result is true for all n. Thus the optimal policy ($1/n$ $1/n$ $1/n$ … $1/n$) and $\boldsymbol{f_n(1) = n\,[1/n \log(1/n)]}$

Example 15.9 Solve the following problem using dynamic programming.

Minimize $z = x_1^2 + x_2^2 + x_3^2$ subject to the constraints

$x_1 + x_2 + x_3 = 15$, $x_1, x_2, x_3 \geq 0$

Solution

Stage I $f_1(s_1) = f_1(x_1) = x_1^2 = 15^2$

Stage II $x_1 + x_2 = 15$. Therefore, $s_2 = 15$

$f_2(s_2) = \min\{x_2^2 + (15 - x_2)^2\}$

Take $u = x_2^2 + (15 - x_2)^2$

$$\frac{du}{dx} = 2x_2 + 2(15 - x_2)(-1) = 4x_2 - 30$$

If u is minimum $\frac{du}{dx} = 0 \Rightarrow x_2 = 15/2$

$\therefore$ $x_1 = 15/2$

$\boldsymbol{f_2(15) = (15/2)^2 + (15/2)^2 = 2\,(15/2)^2}$

Stage III $x_1 + x_2 + x_3 = 15$

$f_3(s_3) = \text{Min}\{x_3^2 + f_2(15 - x_3)\}$

$= \text{Min}\{x_3^2 + 2\,((15 - x_3)/2)^2\}$

Take $u = x_3^2 + 2.\,((15 - x_3)/2)^2$

$$\frac{du}{dx} = 2x_3 + 2.\,2(15 - x_3)/4.\,(-1)$$

$= 2x_3 - (15 - x_3) = 3x_3 - 15$

If u is minimum $\frac{du}{dx} = 0$. $\therefore$ $\boldsymbol{x_3 = 5}$

$\therefore$ $f_3(s_3) = f_3(15)$

$= 5^2 + 2\,.\,(10/2)^2 = 3\,(5^2) = 75$

$\boldsymbol{x_1 = x_2 = x_3 = 5}$

Min z = **75** when $\boldsymbol{x_1 = x_2 = x_3 = 5}$

15.4 SOLUTION OF LINEAR PROGRAMMING PROBLEMS

Given the LPP Maximize $z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$ subject to the constraint

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_j \quad i = 1, 2, 3, \ldots m$$

$x_j \geq 0$ $j = 1, 2, 3, \ldots$ n

we consider it as n stage problem. Each j denotes a stage. $b_1, b_2 \ldots b_m$ are the resources constituting the states. Let $(B_{1j}, B_{2j}, \ldots, B_{mj})$ be the state of the system at j^{th} stage and $f_j(B_{1j}, B_{2j}, \ldots, B_{mj})$ be the optimum value of the objective function at j^{th} stage for the stages $B_{1j}, B_{2j}, \ldots, B_{mj}$.

We use computational procedure, starting from stage 1. The general equation is

$$\boldsymbol{f_j(B_{1j}, B_{2j}, \ldots) = \text{Max}\{c_jx_j + f_{j-1}(B_{1j-1}, B_{2j-1}, \ldots, B_{mj-1})\}}$$

Example 15.10 Solve using dynamic programming

Maximize $z = 3x_1 + 4x_2$

subject to the constraints

$$2x_1 + 5x_2 \geq 120$$
$$2x_1 + x_2 \leq 40$$
$$x_1 , x_2 \geq 0$$

Solution Here we have two stages for two variables.

Stage I

$$f_1 (B_{11}\ B_{21}) = \text{Max } (c_1\ x_1)$$
$$= \text{Max } (3x_1)$$
$$= 3 \text{ Max}(x_1)$$

From the constraints we find that

$$x_1 \leq (120 - 5x_2)/\ 2 \text{ and}$$

and $\quad x_1 \leq (40 - x_2)/\ 2$

In order to satisfy both the constraints

$x_1 \leq \min \{(120 - 5x_2)/\ 2, (40 - x_2)/\ 2\}$

$\therefore\ f_1 (B_{11}\ B_{21}) = 3 \min \{(120 - 5x_2)/2, (40 - x_2)/\ 2\}$

Stage II

$$f_2 (B_{12}\ B_{22}) = \text{Max } \{ f_1(B_{11}\ B_{21}) + 4x_2\}$$
$$= \text{Max } [3 \min \{(120 - 5x_2)/2, (40 - x_2)/2\} + 4x_2]$$

$$= \max \begin{cases} 3(120 - 5x_2)/2 + 4x_2 \text{ if} \\ \quad (120 - 5x_2)/2 \leq (40 - x_2)/2 \\ 3(40 - x_2)/2 + 4x_2 \text{ if} \\ \quad (120 - 5x_2)/2 \geq (40 - x_2)/2 \end{cases}$$

Now $(120 - 5x_2)/2 \leq (40 - x_2)/2$

$\Rightarrow\ 120 - 5x_2 \leq 40 - x_2$

$\Rightarrow\ 80 \leq 4x_2 \Rightarrow x_2 \geq 20$

$(120 - 5x_2)/2 \geq (40 - x_2)/2$

$\Rightarrow\ x_2 \leq 20$

When $x_2 = 20$

$3\ (120 - 5x_2)/2 + 4x_2$

$= 3\ (40 - x_2)/2 + 4x_2 = 110$

$\therefore\ f_2 (120, 40) = 110\ x_2^* = 20$

$x_1^* = (40 - x_2^*)/\ 2 = 10$

Solution is $\mathbf{x_1 = 10, x_2 = 20, z^* = 110}$

Example 15.11 Maximize $z = 5x_1 + 2x_2$

Subject to $x_1 + 2x_2 \leq 43$

$$x_1 \leq 23$$
$$x_1, x_2 \geq 0$$

Solution

Stage I

$$f_1 (B_{11}\ B_{21}) = \text{Max } (5x_1)$$
$$= 5 \max (x_1)$$
$$= 5 \min \{ 43 - 2x_2, 23\}$$

Stage II

$$f_2 (B_{12}, B_{22}) = \max [5 \min \{43 - 2x_2, 23\} + 2x_2]$$

$$= \text{Max} \begin{cases} 5(43 - 2x_2) + 2x_2 \text{ if } 43 - 2x_2 \leq 23 \\ 5(23) + 2x_2 \text{ if } 43 - 2x_2 \geq 23 \end{cases}$$

$$= \text{Max} \begin{cases} 215 - 8x_2 \text{ if } x_2 \geq 10 \\ 115 + 2x_2 \text{ if } x_2 \leq 10 \end{cases}$$

Both the values become equal when $x_2 = 10$

$\therefore\ f_2 (B_{12}\ B_{22}) = (215 - 80)$

or $\quad (115 + 20) = \mathbf{135}$

$\therefore\ \boldsymbol{f_2\ (43\ ,\ 23) = 135\ x_2^* = 10}$

$x_1^* = 43 - 2x_2 = 23$

$\therefore$ **Solution is $\mathbf{x_1 = 23, x_2 = 10\ z^* = 135}$**

Example 15.12 Maximize $z = 3x_1 + 2x_2$

subject to the constraints

$$x_1 + x_2 \leq 300$$
$$2x_1 + 3x_2 \leq 800$$
$$x_1, x_2 \geq 0$$

Solution We apply the backward procedure, taking stage I with x_2.

$$f_2 (B_{12}, B_{22}) = \max (2x_2)$$
$$= 2 \max (x_2)$$
$$= 2 [\min \{ 300 - x_1, (800 - 2x_1)/3\}]$$

Stage II

$$f_1 (B_{11}\ B_{21}) = \max [3x_1 + 2x_2]$$
$$= \max [3x_1 + 2 \min\{300 - x_1, (800 - 2x_1)/3\}]$$

$$= \max \begin{cases} 3x_1 + 2(300 - x_1) \text{ if} \\ \quad 300 - x_1 \leq (800 - 2x_1)/3 \\ 3x_1 + 2(800 - 2x_1)/3 \text{ if} \\ \quad 300 - x_1 \geq (800 - 2x_1)/3 \end{cases}$$

$$= \max \begin{cases} x_1 + 600 \text{ if } 100 \le x_1 \\ (5x_1 + 1600)/3 \text{ if } 100 \ge x_1 \end{cases}$$

Both the values are equal if $x_1 = 100$

$\therefore \quad f_1 (B_{11} B_{21}) = 100 + 600$ or $(500 + 1600) / 3$ for $x_1 = 100$

f_1 (300, 800) = 700. $x_1^* = 100$

$\boldsymbol{x_2^* = 300 - x_1^* = 200}$

$\therefore$ The solution is $\boldsymbol{z^* = 700\ x_1 = 100\ x_2 = 200}$

Example 15.13 Maximize $Z = x_1 + 9x_2$

subject to the constraints

$$2x_1 + x_2 \le 25$$
$$x_2 \le 11$$
$$x_1, x_2 \ge 0$$

Solution (Backward Procedure)

Stage I

$$\begin{aligned} f_2 (B_{12} B_{22}) &= \max (9x_2) \\ &= 9 \max(x_2) \\ &= 9 \min \{25 - 2x_1, 11\} \end{aligned}$$

Stage II

$f_1 (B_{11}, B_{21}) = \max \{x_1 + 9x_2\}$

$= \max [x_1 + 9 \min \{25 - 2x_1, 11\}]$

$$= \max \begin{cases} x_1 + 9(25 - 2x_1) \text{ if } 25 - 2x_1 \le 11 \\ x_1 + 9.11 \qquad \text{if } 25 - 2x_1 \ge 11 \end{cases}$$

$$= \max \begin{cases} 225 - 17x_1 \text{ if } x_1 \ge 7 \\ x_1 + 99 \qquad \text{if } x_1 \le 7 \end{cases}$$

$\therefore \quad f_1 (25, 11) = 106,\ x_1^* = 7$

$x_2^* = 11$

$\therefore$ The solution is $x_1 = 7$, $x_2 = 11$, $z^* = 106$

EXERCISES

1. A corporation runs three plants. Each plant is requested to submit its proposals giving total cost C and revenue R for each proposal. Budget is Rs 5 lakh for allocation to all the three plants. The following table gives the details:

Proposal	*Plant 1*		*Plant 2*		*Plant 3*	
	C_1	R_1	C_2	R_2	C_3	R_3
1	0	0	0	0	0	0
2	1	5	2	8	1	3
3	2	6	3	9	–	–
4	–	–	4	12	–	–

Determine the allocation so as to maximize the revenue.

2. Six workers are to be allotted to 3 types of jobs. The following table gives the expected returns from the various allotments.

No. of workers	*Job*		
	1	*2*	*3*
0	0	0	0
1	25	20	33
2	42	38	43
3	55	54	47
4	63	65	50
5	69	73	52
6	74	80	53

Determine the optimal allotment.

3. A contractor has to complete four projects. He has six foremen and each project is to be allotted to at least one foreman. The following table gives the time required and the number of foremen to complete the projects. Determine the minimum time required for completing all the projects with six foremen.

No. of foremen	Project			
	A	B	C	D
1	15	17	19	21
2	13	15	18	18
3	12	13	17	18

4. (Cargo loading problem). A truck can carry a total of 10 tons of a product. Three types of product are to be loaded. Their weights and values are given below:

Product	Weight (tons)	Value
T_1	2	65
T_2	3	80
T_3	1	30

Determine how many units of T_1, T_2 and T_3 are to be loaded in order to maximize the total value. [At least one unit of each type must be selected.]

5. The costs of advertisement in newspaper radio and TV are Rs 5000, Rs 10,000 and Rs 20,000 respectively per appearance. The company has a budget of Rs 40,000 for advertisement. The following table gives the frequency and the corresponding sales:

Frequency per month	Sales		
	Newspaper	Radio	TV
1	125	180	300
2	225	290	350
3	260	340	450
4	300	370	500

Determine the optimal combination of the advertising media and frequency.

6. (Salesmen Problem). A company has five salesmen, who are to be allocated to three marketing zones. The expected returns for different number of salesmen in different zones are given below.

No. of salesmen	Zones		
	1	2	3
1	45	30	35
2	58	45	45
3	82	70	64
4	93	79	72
5	101	90	82

Determine the optimal allocation.

7. An investor has Rs 6000 to invest. This amount can be invested in any of the three ventures *A*, *B* and *C*. But he must invest in units of Rs 1000. The following table gives the returns from the investment in the ventures.

Investment	Return		
	A	B	C
0	0	0	0
1	0.5	1.5	1.2
2	1.0	2.0	2.4
3	3.0	2.2	2.5
4	3.1	2.3	2.6
5	3.2	2.4	2.7
6	3.3	2.5	2.8

Find the optimal investment policy.

8. Apply dynamic programming to solve.

(a) Minimize $z = x_1^2 + x_2^2 + x_3^2$
subject to the constraints
$$x_1\, x_2\, x_3 = 24$$
$$x_1, x_2, x_3 \geq 0.$$

(b) Maximize $z = x_1\, x_2\, x_3$
subject to the constraints
$$x_1 + x_2 + x_3 = 12$$
$$x_1, x_2, x_3 \geq 0.$$

(c) Maximize $z = 2x_1 + 3x_2 + 4x_3$
subject to the constraints
$$x_1 + x_2 + x_3 = 18$$
$$x_1, x_2, x_3 \geq 0.$$

(d) Minimize $z = x_1^2 + x_2^2 + x_3^2$
subject to the constraints
$$x_1 + x_2 + x_3 \geq 12$$
$$x_1, x_2, x_3 \geq 0.$$

9. Solve the following LPP using dynamic programming

(a) Maximize $z = 3x_1 + 5x_2$
subject to the constraints
$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

(b) Maximize $z = 5x_1 + 10x_2$
subject to the constraints

$$10x_1 + 5x_2 \le 250$$
$$4x_1 + 10x_2 \le 200$$
$$2x_1 + 3x_2 \le 900$$
$$x_1, x_2 \ge 0$$

(c) Maximize $z = 3x_1 + 4x_2$
subject to the constraints

$$2x_1 + x_2 \le 6$$
$$2x_1 + 3x_2 \le 9$$
$$x_1, x_2 \ge 0$$

(d) Maximize $z = 4x_1 + 3x_2$
subject to the constraints

$$2x_1 + x_2 \le 2$$
$$-x_1 + x_2 \le 1$$
$$x_1, x_2 \ge 0$$

10. A student has five days at his disposal to revise the subject before examination. The course is divided into four sections. He may study a section for one day, two days, three days, etc. or not at all. The expected grade points he will get for different alternative arrangements are as follows:

Study	*Sections*		
	I	*II*	*III*
0	0	1	0
1	1	1	1
2	1	3	3
3	3	4	3

How should be distribute the available days to the different sections so that he may get maximum grade points?

11. Consider the problem of designing electronic devices to carry five power cells, each of which must be located within three electronic systems. If the power fails in one system then it will be powered on an auxiliary basis by the cells of the remaining systems. The probability of power failure in a particular system depends on the number of cells originally assigned to it. Estimated probabilities of power failure for the systems are given below:

Power cells	*Probability of power failure*		
	System 1	*System 2*	*System 3*
1	0.50	0.60	0.40
2	0.15	0.20	0.25
3	0.04	0.10	0.10
4	0.02	0.05	0.05
5	0.01	0.02	0.01

Determine how many cells should be assigned to each system so as to maximize the overall system reliability (least probability of total power failure).

12. The following table gives the number of votes (in thousands) gained by a political party in three districts corresponding to various allocation of six party workers in each district.

Workers	*Districts*		
	D_1	D_2	D_3
0	0	0	0
1	25	20	33
2	42	38	43
3	55	54	47
4	63	65	50
5	69	73	52
6	74	80	53

Determine the number of workers to be assigned to each district so as to maximize the number of votes gained.

13. A company has eight salesmen working in three areas A_1, A_2, A_3. The sales (in thousands of rupees) by each salesman in the three areas are given below. Determine the optimum allocation of salesmen in order to maximize the sales.

Number of salesmen	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
A_1	15	22	30	38	45	48	54	60	65
A_2	26	35	40	46	55	62	70	76	83
A_3	30	38	44	50	60	65	72	80	85

14. World Health Council wants to determine the number of medical teams to be allocated among three countries C_1, C_2 and C_3. It has

five medical teams. The measure of effectiveness used, is additional man years of life. The following table gives the estimated additional man years of life (in thousands) for each country against each possible allocation of medical teams.

Number of Medical teams	*Additional man-years of life (in thousands)*		
	C_1	C_2	C_3
0	0	0	0
1	45	20	50
2	70	45	70
3	90	75	80
4	105	110	100
5	120	150	130

Determine how many teams are to be allocated to each country to get maximum effectiveness.

15. An investor wants to invest Rs 60,000 in any of the three ventures A, B and C, in units of Rs 10,000. The following table shows the return from the investment in the ventures (in thousands of rupees)

Amount invested	*Return from venture*		
	A	B	C
0	0	0	0
1	5	15	12
2	10	20	24
3	30	22	25
4	31	23	26
5	32	24	27
6	33	25	28

Determine the optimal investment policy.

16. Find the minimum value of $Z = x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$
subject to the condition
$$x_1, x_2, x_3, \ldots, x_n = C \; (C \neq 0)$$

17. Find the maximum value of
$$Z = a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n$$
where $x_1 + x_2 + x_3 + \ldots x_n = c$ and
$$a_1 \le a_2 \le a_3 \le \ldots \le a_n$$

18. Minimize $Z = x_1^2 + x_2^2 + x_3^2$
subject to the condition
$$x_1 + x_2 + x_3 \ge 10$$
$$x_1, x_2, x_3 \ge 0$$

19. Find the maximum value of
$$Z = x_1^2 + 2x_2^2 + 4x_3$$
subject to the condition
$$x_1 + 2x_2 + x_3 \le 8$$
$$x_1, x_2, x_3 \ge 0$$

20. Find the minimum value of $Z = x_1^2 + 2x_2^2 + 4x_3$
subject to the condition
$$x_1 + 2x_2 + x_3 \ge 8$$
$$x_1, x_2, x_3 \ge 0$$

21. Minimize $Z = xyz$
subject to the condition
$$x + y + z = 5$$
$$x_1, x_2, x_3 \ge 0$$

22. Solve the LPP
Maximize $Z = 8x + 7y$ subject to the constraints
$$2x + y \le 8$$
$$5x + 2y \le 15$$
$$x, y \ge 0$$

23. Solve the LPP
Maximize $Z = 3x + y$ subject to the constraints
$$2x + y \le 6$$
$$x \le 2$$
$$y \le 4$$
$$x, y \ge 0$$

24. Solve the LPP
Maximize $Z = 3x + 2y$ subject to the constraints
$$-2x + y \le 1$$
$$x \le 2$$
$$x + y \le 3$$
$$x, y \ge 0$$

ANSWERS

1. Maximum revenue = 17 lakh

   Optimal allocation to plants 1, 2 and 3 $\left.\right\}$ 2, 4, 1; 2, 3, 2 or 3, 2, 2

2. Maximum return = 129

   Optimal allocation to jobs 1, 2 and 3 $\left.\right\}$ 2, 3, 1

3. Table (left out in the question)
   Minimum time required = 67

No. of foremen:	*A*	*B*	*C*	*D*
	1	2	1	2
or	2	1	1	2

4. T_1 – 3 units, T_2 – 1 unit, T_3 – 1 unit, value = 305

5. | *News-paper* | *Radio* | *TV* | *Total sales = 705* |
   |---|---|---|---|
   | Frequency → 2 | 1 | 1 | |

6. Zone 1 = 3, Zone 2 = 2, Zone 3 = 0
   Total returns = 177
7. *A* – 3000, *B* – 1000, *C* – 2000
   Total returns = 6.9
8. (a) $x_1^* = x_2^* = x_3^* = 24^{1/3}$; $Z^* = 3.24^{2/3}$
   (b) $x_1^* = x_2^* = x_3^* = 4$; $Z^* = 64$
   (c) $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 18$; $Z^* = 72$
   (d) $x_1^* = x_2^* = x_3^* = 4$; $Z^* = 48$
9. (a). $x_1^* = 2$, $x_2^* = 6$; $Z^* = 36$
   (b) $x_1^* = 18.75$, $x_2^* = 12.5$; $Z^* = 218.75$
   (c) $x_1^* = 9/4$, $x_2^* = 3/2$; $Z^* = 51/4$
   (d) $x_1^* = 1/3$, $x_2^* = 4/3$; $Z^* = 16/3$
10. Section I = 1 day, Section II = 0, Section III = 2 days
    Required maximum grade points = 5
11. System 1 = 3 cells, System 2 = 1 cell, System 3 = 1 cell
    Smallest probability of total failure is 0.0096
12. $D_1 : 2$, $D_2 : 3$, $D_3 : 1$ Total votes = 129
13. $A_1 : 3$, $A_2 : 1$, $A_3 : 4$ Total sales = 133
14. $C_1 : 1$, $C_2 : 3$, $C_3 : 1$ Total = 170
15. $A : 3$, $B : 1$, $C : 2$ Total return = 69
16. $X_1 = X_2 = X_3 = \ldots = X_n = C^{1/n}$
    Min value : $nc^{2/n}$
17. $X_1 = X_2 = X_3 = \ldots = X_{n-1} = 0$, $X_n = C$
    Max value : $a_n c$
18. $X_1 = X_2 = X_3 = 10/3$; Min $Z = 100/3$
19. $X_1 = 8$, $X_2 = 0$, $X_3 = 0$; Max $Z = 64$
20. $X_1 = X_2 = X_3 = 2$; Min $Z = 20$
21. $X = 0$, $Y = 0$, $Z = 5$; Min $Z = 0$
22. $X = 0$, $Y = 15/2$; Max $Z = 52.5$
23. $X = 2$, $Y = 2$; Max $Z = 8$
24. $X = 2$, $Y = 1$; Max $Z = 8$

16

Sequencing Problems

CONCEPT REVIEW

16.1 INTRODUCTION

If there are n jobs to be performed, one at a time, on each of m machines, the sequence of the machines in which each job should be performed is given and the time required by the jobs on each machine is also given, then the problem is to find the sequence (order) of the jobs which minimizes the total time taken from the starting of the first job on the first machine till the completion of the last job on the last machine. This is referred to as the *sequencing problem*. In the case when there are three jobs and three machines the total number of possible sequences will be $(3!)^3 = 216$. Hence it requires lot of computational time. Thus the sequencing technique becomes very useful in that it reduces computational work and time.

Here, the processing times on different machines are exactly known and are independent of the order of processing. The time taken by the jobs in moving from one machine to another is negligible. Once a job has begun on a machine, it must be completed before another job can begin on the same machine. Only one job can be processed on a given machine at a time. The order of completion of jobs is independent of the sequence of jobs.

16.2 PROCESSING n JOBS THROUGH TWO MACHINES

The characteristics of this problem are

(i) there are only two machines, A and B

(ii) each job is processed in the order A, B

(iii) the processing times A_1 $A_2 \ldots A_n$ on A and B_1 $B_2 \ldots B_n$ on B are known.

Job	*Processing time*	
	A	B
1	A_1	B_1
2	A_2	B_2
3	A_3	B_3
.	.	.
.	.	.
n	A_n	B_n

We have to determine the sequence of jobs which minimizes the total time elapsed from the start of the first job to the completion of last job.

The step-by-step procedure due to S.M. Johnson and R. Bellman is as follows.

Step 1 Examine the columns (showing the processing times) on machines A and B and select the smallest among all of them: min $\{A_i, B_i\}$

Step 2

(i) If the minimum value belongs to column A schedule this job first on machine A. Suppose this value falls in column B then schedule this job last on machine A.

(ii) If there are two equal minimum values one in each column, schedule the one in the first column as the first job and the other in the second column as the last.

(iii) If both equal minimum values are in the first column select the job corresponding to the smallest subscript first.

(iv) If the equal values occur in the second column select the job corresponding to the largest subscript first.

Step 3 Delete the assigned job. Continue the process of placing the jobs next to first or next to last till all the jobs are assigned. The resulting sequence will minimize the total time T.

Note:

Idle time for the machine A (M_1)

= Total time – Time when the last job is finished on A.

Idle time for the machine B (M_2)

= Time at which the first job is finished on $A + \Sigma$ [Time of starting of the j^{th} job on B – Time when $(j - 1)^{th}$ job is finished on B]

Example 16.1 Find the sequence of jobs that minimizes the total elapsed time to complete the following jobs on two machines.

Job	*1*	*2*	*3*	*4*	*5*	*6*
Machine A	3	12	5	2	9	11
Machine B	8	10	9	6	3	1

Solution The given table is

Job	*Machines*	
	A	*B*
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3
6	11	1

We find that the smallest value of time is 1 unit for job 6 in the 2^{nd} column B. Therefore we schedule job 6 as last.

					6

The reduced processing times are

Job	*A*	*B*
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3

Now the smallest value is 2 for job 4 on A. Hence we schedule job 4 as the first on A. We have

4					6

The reduced table is

Job	*A*	*B*
1	3	8
2	12	10
3	5	9
5	9	3

Now there are two equal minimum values 3 one for job 1 on A and the other for job 5 on B. By the rule job 1 is scheduled next to job 4 and job 5 before job 6. Thus we get

4	1			5	6

The reduced table is

Job	*A*	*B*
2	12	10
3	5	9

Here the minimum is 5 on *A* for job 3. Hence job 3 is scheduled next to job 1 on *A*. Naturally job 2 is assigned next to job 3. Thus we get the optimal sequence.

4	1	3	2	5	6

The following table gives the details of time taken.

Job	*A*		*B*	
	Time-in	*Time-out*	*Time-in*	*Time-out*
4	0	2	2	8
1	2	5	8	16
3	5	10	16	25
2	10	22	25	35
5	22	31	35	38
6	31	42	42	43

Idle time for machine *A* is 43 – 42 = **1 unit.**
Idle time for *B* is 2 + (42 – 38) = **6 units.**

Example 16.2 Determine the optimal sequencing to complete the following tasks on two machines.

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Machine 1	2	5	4	9	6	8	7	5	4
Machine 2	6	8	7	4	3	9	3	8	11

Solution The given table is:

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Machine 1	2	5	4	9	6	8	7	5	4
Machine 2	6	8	7	4	3	9	3	8	11

The smallest processing time is 2 corresponding to the task *A* on machine Machine 1.

∴ We assign the task *A* to Machine 1 first.

A								

Now we have 8 tasks with the processing times as shown below:

Task	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Machine 1	5	4	9	6	8	7	5	4
Machine 2	8	7	4	3	9	3	8	11

In this table the minimum time is 3 on Machine 2 corresponding to *E* and *G*. Since the processing time of *E* on Machine 1 is less than that of *G* on Machine 1, *E* is placed as the last and *G* is processed before *E*. Thus we have

A							*G*	*E*

We have the following table for the remaining jobs

Task	*B*	*C*	*D*	*F*	*H*	*I*
Machine 1	5	4	9	8	5	4
Machine 2	8	7	4	9	8	11

Now the least time is 4 which corresponds to tasks *C* and *I* on Machine 1 and *D* on Machine 2. Therefore *C* will be taken next to *A* and *I* will be taken up next to *C* and *D* at last before *G*. Thus the sequence is

A	*C*	*I*				*D*	*G*	*E*

The reduced table is

Task	*B*	*F*	*H*
Machine 1	5	8	5
Machine 2	8	9	8

The smallest time is 5 on Machine 1 for *B* and *H*. The corresponding times of these jobs on Machine 2 are the same, we can select either of the jobs for 4th and 5th places. Finally *F* is assigned the 6th place. Thus the optimal sequence is

A	*C*	*I*	*B*	*H*	*F*	*D*	*G*	*E*

or

A	*C*	*I*	*H*	*B*	*F*	*D*	*G*	*E*

The minimum time elapsed for completing all the jobs.

Task	*Machine 1*		*Machine 2*	
	Time-in	*Time-out*	*Time-in*	*Time-out*
A	0	2	2	8
C	2	6	8	15
I	6	10	15	26
B	10	15	26	34
H	15	20	34	42
F	20	28	42	51
D	28	37	51	55
G	37	44	55	58
E	44	50	58	61

Total time taken to complete all the jobs is **61 hours.**

Idle time for Machine 1 is 61 – 50 = 11 hours

Idle time for Machine 2 is 2 + 58 – 58 = 2 hours

Example 16.3 Determine the optimal sequencing of the following 7 jobs on two machines Machine 1 and Machine 2.

Job	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Machine 1	3	12	15	6	10	11	9
Machine 2	8	10	10	6	12	1	3

Solution: The given table is

Job	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Machine 1	3	12	15	6	10	11	9
Machine 2	8	10	10	6	12	1	3

The least processing time is 1 on Machine 2 for job 6. Assign job 6, the last place. We get

						6

Next we see that there are two equal values 3 one for job 1 on Machine 1 and the other for job 7 on Machine 2.

Job	*1*	*2*	*3*	*4*	*5*	*7*
Machine 1	3	12	15	6	10	9
Machine 2	8	10	10	6	12	3

Take job 1 as the first and job 7 as the last but one. Therefore we have

1					7	6

The reduced set of processing times is

Job	*2*	*3*	*4*	*5*
Machine 1	12	15	6	10
Machine 2	10	10	6	12

Here we have two equal minimum values 6 for the same job 4 on both Machine 1 and Machine 2. We can put job 4 next to job 1 or before job 7. Let us choose job 4 next to job 1.

1	4				7	6

Now we have the reduced table

Job	*2*	*3*	*5*
Machine 1	12	15	10
Machine 2	10	10	12

Here we have 10 as the minimum value for job 5 on Machine 1 and job 2 and job 3 on Machine 2. By the rule job 5 is processed next to job 4, job 2 before job 7 and job 3 before job 2. This gives the optimal sequence

1	4	5	3	2	7	6

The elapsed time according to this sequence is given below:

Job	*Machine 1*		*Machine 2*		*Idle time for*
	Time-in	*Time-out*	*Time-in*	*Time-out*	*Machine 2*
1	0	3	3	11	3
4	3	9	11	17	0
5	9	19	19	31	2
3	19	34	34	44	3
2	34	46	46	56	2
7	46	55	56	59	0
6	55	66	66	67	7
					17 hrs

Total time taken is 67 hrs

Idle time for Machine 1 is 67 – 66 = 1 hr

Idle time for Machine 2 is 17 hrs

16.3 PROCESSING *n* JOBS ON THREE MACHINES

For processing *n* jobs on three machines say *A*, *B* and *C*, the previous method can be extended.

Extension is possible with either one or both of the following conditions. If neither condition holds good this method fails. The conditions are

(i) the minimum time on machine A must be greater than or equal to the maximum time on B.

(ii) the minimum time on C must be greater than or equal to the maximum time on B.

One or both of these conditions must hold in order to apply the method.

The procedure is given by the following steps:

(i) Introduce two dummy machines G and H with the corresponding processing times given by

$$G_j = A_j + B_j \quad j = 1, 2, 3, \ldots, n$$
$$H_j = B_j + C_j \quad j = 1, 2, 3, \ldots, n$$

(The processing times of the corresponding jobs are added)

(ii) Consider the machines G and H and apply the previous method taking the order GH.

Example 16.4 Determine the sequence which minimizes the total time for processing five jobs on three machines A, B and C. The following table gives the processing times.

Job	*1*	*2*	*3*	*4*	*5*
Machine A	8	10	6	7	11
Machine B	5	6	2	3	4
Machine C	4	9	8	6	5

Solution In the table we find that

min $A_j = 6$ min $C_j = 4$

max $B_j = 6$

Since min $A_j \geq$ max B_j condition (i) is satisfied.

Now take $G_j = A_j + B_j$ ($j = 1, 2, 3, 4, 5$)

$H_j = B_j + C_j$ ($j = 1, 2, 3, 4, 5$)

We have

Job	*Machine*	
	G	*H*
1	13	9
2	16	15
3	8	10
4	10	9
5	15	9

The minimum value is 8 for job 3 on G. Hence job 3 is processed first

3				

The reduced columns are

Job	*G*	*H*
1	13	9
2	16	15
4	10	9
5	15	9

Here the minimum is 9 for 3 jobs 1, 4 and 5 on H (decreasing order of the processing times on G are 15, 13, 10). We place job 4 at the last, job 1 next and job 5 before job 1. Job 2 is processed next to job 3. Thus we get the sequence

3	2	5	1	4

The elapsed time according to this sequence is given below.

Job	*Machine*					
	A		*B*		*C*	
	Time-in	*Time-out*	*Time-in*	*Time-out*	*Time-in*	*Time-out*
3	0	6	6	8	8	16
2	6	16	16	22	22	31
5	16	27	27	31	31	36
1	27	35	35	40	40	44
4	35	42	42	45	45	51

Total time elapsed 51 hrs

Idle time for $A = (45 - 42) + (51 - 45) = 9$ hrs

Idle time for $B = 6 + 8 + 5 + 4 + 2 + 6 = 31$ hrs

Idle time for $C = 8 + 6 + 0 + 4 + 1 = 19$ hrs

Example 16.5 The processing times of six jobs on three machines M_1, M_2, M_3 are given below.

Job	M_1	M_2	M_3
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Determine the optimal sequence of jobs.

Solution Here we find that minimum $A_j = 2$ minimum $C_j = 8$; maximum $B_j = 8$; condition (ii) is satisfied.

Take $G_j = A_j + B_j$ (j = 1, 2, 3, 4, 5, 6)

$H_j = B_j + C_j$

We have the table giving processing times on G and H as given below:

Job	*1*	*2*	*3*	*4*	*5*	*6*
G	11	18	9	8	12	12
H	21	20	13	18	11	14

The minimum time is 8 for job 4 on G. Hence job 4 is processed first.

4					

The reduced table is

Job	*1*	*2*	*3*	*5*	*6*
G	11	18	9	12	12
H	21	20	13	11	14

Now the minimum is 9 for job 3 on G. Hence job 3 is processed next to job 4.

4	3				

The reduced table is

Job	*1*	*2*	*5*	*6*
G	11	18	12	12
H	21	20	11	14

Here the minimum is 11 for job 1 on G and for job 5 on H. Process job 1 next to job 3 and job 5 last in the sequence. We get

4	3	1			5

The reduced table is

Job	2	6
G	18	12
H	20	14

The minimum 12 is for job 6 on G. Therefore job 6 is processed next to 1. The optimal sequence becomes

4	3	1	6	2	5

The elapsed time for completing all the jobs is given by

Job	*M_1*		*M_2*		*M_3*	
	Time-in	*Time-out*	*Time-in*	*Time-out*	*Time-in*	*Time-out*
4	0	2	2	8	8	20
3	2	7	8	12	20	29
1	7	10	12	20	29	42
6	10	21	21	22	42	55
2	21	33	33	39	55	69
5	33	42	42	45	69	77

The total time taken to complete all the job is 77 hrs

The idle time for M_1 is 77 – 42 = 35 hrs

The idle time for M_2 is 2 + 0 + 0 + 11 + 11 + 3 + 32 = 59 hrs

The idle time for M_3 is 8 hrs

16.4 PROCESSING *n* JOBS THROUGH *m* MACHINES

Let there be n jobs $A_1, A_2, A_3 \ldots A_n$ to be processed through m machines $M_1, M_2 \ldots M_m$ in the order $M_1 M_2 M_3 \ldots M_m$. The optimal solution to this problem can be found if either or both of the following conditions are satisfied.

(i) Minimum $A_{i1} \geq \max A_{ij} \begin{cases} i = 1, 2, 3, \ldots, n \\ j = 2, 3 \ldots, m-1 \end{cases}$

(ii) Minimum $A_{im} \geq \max A_{ij} \begin{cases} i = 1, 2, \ldots, n \\ j = 2, 3, \ldots, m-1 \end{cases}$

or minimum processing time on M_1 or M_m must be not less than the maximum processing time on any of the remaining $m - 2$ machines. The table is

		Machines				
		M_1	*M_2*	*M_3*	...	*M_m*
Jobs	A_1	A_{11}	A_{12}	A_{13}	...	A_{1m}
	A_2	A_{21}	A_{22}	A_{23}	...	A_{2m}
	A_3	A_{31}	A_{32}	A_{33}	...	A_{3m}
	.	.	.	.	.	.
	.	.	.	.	.	.
	.	.	.	.	.	.
	A_n	A_{n1}	A_{n2}	A_{n3}	...	A_{nm}

If the condition is satisfied then we can solve the problem as follows.

Step 1 Examine whether the condition is satisfied

Step 2 Convert the m machine problem to a two machine problem.

Introduce two dummy machines G and H with the processing times

$G_i = A_{i1} + A_{i2} + A_{i3} + \ldots. A_{im-1}$ $(i = 1, 2, 3, \ldots, n)$

$H_i = A_{i2} + A_{i3} + A_{i4} + \ldots. A_{im}$ $(i = 1, 2, 3, \ldots, n)$

That is processing time of a job on G, is the sum of the processing times of the job on M_1, $M_2, \ldots, M_{m-1}$ and processing time of a job on H, is the sum of the processing times of the job on M_2, $M_3, \ldots M_m$.

Step 3 The new processing times so obtained can be used for solving two machine – n job problem as before.

Example 16.6 Solve the following sequencing problem of four jobs on five machines

Job	*Machines*				
	M_1	M_2	M_3	M_4	M_5
A	7	5	2	3	9
B	6	6	4	5	10
C	5	4	5	6	8
D	8	3	3	2	6

Solution In the given table min $A_{i1} = 5$ min $A_{i5} = 6$ max $A_{ij} = 6$

$\therefore$ The condition is satisfied.

Take $G_i = A_{i1} + A_{i2} + A_{i3} + A_{i4}$

$H_i = A_{i2} + A_{i3} + A_{i4} + A_{i5}$

We have the new table for two machines G and H.

Job	*G*	*H*
A	17	19
B	21	25
C	20	23
D	16	14

Minimum is 14 corresponding to the job D on H. Therefore D is selected for processing last.

			D

The new table is

Job	*G*	*H*
A	17	19
B	21	25
C	20	23

The minimum is 17 for job A on G. Therefore, A is processed first.

A			*D*

In the reduced table the minimum is 20 for C on G. Hence C is processed next to A. Thus the sequence is

A	*C*	*B*	*D*

Total elapsed time is given by the following table.

Job	M_1		M_2		M_3		M_4		M_5	
	Time-in	*Time-out*	*Time-in*	*Time-out*	*Time-in*	*Time-out*	*Time-in*	*Time-out*	*Time-in*	*Time-out*
A	0	7	7	12	12	14	14	17	17	26
C	7	12	12	16	16	21	21	27	27	35
B	12	18	18	24	24	28	28	33	35	45
D	18	26	26	29	29	32	33	35	45	51

Minimum time required to complete all the jobs is 51 hrs.

Idle time for M_1 : 51 – 26 = 25 hrs

Idle time for M_2 : 7 + 2 + 2 + (51 – 29) = 33 hrs

Idle time for M_3 : 12 + 2 + 3 + 1 + (51 – 32)

= 37 hrs

Idle time for M_4 : 17 + 1 = 18 hrs

Example 16.7 Solve the following sequencing problem.

		Machines						
		M_1	M_2	M_3	M_4	M_5	M_6	M_7
Jobs	*A*	20	10	9	4	12	9	40
	B	22	8	11	8	10	10	30
	C	12	7	10	7	9	12	32
	D	30	6	5	6	10	11	28

Solution In the table min $A_{i1} = 12$ min $A_{i7} = 28$ max $A_{ij} = 12$

Condition is satisfied.

Take $G_i = A_{i1} + A_{i2} + A_{i3} + A_{i4} + A_{i5} + A_{i6}$

$H_i = A_{i2} + A_{i3} + A_{i4} + A_{i5} + A_{i6} + A_{i7}$

The table of processing times on G and H is

Job	Machines	
	G	H
A	64	84
B	69	77
C	57	77
D	68	66

The minimum is 57 for job C on G. Hence job C is to be processed first

C			

Deleting C from the table, the minimum of the remaining processing times is 64 for job A on G. $\therefore$ job A is processed next to C.

C	A		

In the remaining table (after deleting C and A) the minimum is 66 for D on H. Hence job D is assigned the last place. Thus we get the sequence

C	A	B	D

The total elapsed time is given by

Job	M_1		M_2		M_3		M_4		M_5		M_6		M_7	
	Time-in	Time-out	Time-in	Time-out	Time-in	Time-out	Time-in	Time-out	Time-in	Time-out	Time-in	Time-out	Time-in	Time-out
C	0	12	12	19	19	29	29	36	36	45	45	57	57	89
A	12	32	32	42	42	51	51	55	55	67	67	76	89	129
B	32	54	54	62	62	73	73	81	81	91	91	101	129	159
D	54	84	84	90	90	95	95	101	101	111	111	122	159	187

The total time elapsed is 187 hrs.

Idle time for:

M_1 : 187 – 84 = 103

M_2 : 12 + 13 + 12 + 22 + 97 = 156

M_3 : 19 + 13 + 11 + 17 + 92 = 152

M_4 : 29 + 15 + 18 + 14 + 86 = 162

M_5 : 36 + 10 + 14 + 10 + 76 = 146

M_6 : 45 + 10 + 15 + 10 + 65 = 145

M_7 : 57

EXERCISES

Solve the following sequencing problems:

1. Five jobs on two machines:

Job	1	2	3	4	5
M_1	5	1	9	3	10
M_2	2	6	7	8	4

2. Six jobs on two machines

Job	1	2	3	4	5	6
M_1	5	9	4	7	8	6
M_2	7	4	8	3	9	5

3. Seven jobs on two machines

Job	1	2	3	4	5	6	7
M_1	5	7	3	4	6	7	12
M_2	2	6	7	5	9	5	8

4. Six jobs on three machines

Job	1	2	3	4	5	6
M_1	18	12	29	35	43	37
M_2	7	12	11	2	6	12
M_3	19	12	23	47	28	36

5. Six jobs on three machines

Job	*1*	*2*	*3*	*4*	*5*	*6*
M_1	8	3	7	2	5	1
M_2	3	4	5	2	1	6
M_3	8	7	6	9	10	9

6. Four jobs on six machines

Job	*Machines*					
	M_1	M_2	M_3	M_4	M_5	M_6
A	18	8	7	2	10	25
B	17	6	9	6	8	19
C	11	5	8	5	7	15
D	20	4	3	4	8	12

7. Five jobs on four machines

Job	*Machines*			
	M_1	M_2	M_3	M_4
A	10	3	5	14
B	12	2	6	7
C	8	4	4	12
D	15	1	7	8
E	16	5	3	10

8. Five jobs on two machines

Job	*1*	*2*	*3*	*4*	*5*
M_1	3	8	5	7	4
M_2	4	10	6	5	8

9. Five jobs on two machines

Job	*1*	*2*	*3*	*4*	*5*
M_1	5	4	8	7	6
M_2	3	9	2	4	10

10. Eight jobs on two machines

Job	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
M_1	14	26	17	11	9	26	18	15
M_2	21	15	16	21	22	12	13	25

11. Seven jobs on three machines

Job	*A*	*B*	*C*	*D*	*E*	*F*	*G*
M_1	3	8	7	4	9	8	7
M_2	4	3	2	5	1	4	3
M_3	6	7	5	11	5	6	12

12. Five jobs on three machines

Job	*1*	*2*	*3*	*4*	*5*
M_1	3	9	6	5	4
M_2	4	5	1	2	3
M_3	8	9	5	7	10

13. Six jobs on three machines

Job	*1*	*2*	*3*	*4*	*5*	*6*
M_1	12	8	7	11	10	5
M_2	3	4	2	5	5	4
M_3	7	10	9	6	10	4

14. Four jobs on four machines

Job	*1*	*2*	*3*	*4*
M_1	15	5	5	15
M_2	12	2	10	12
M_3	16	2	4	16
M_4	18	3	4	18

15. Seven jobs on three machines

Job	*1*	*2*	*3*	*4*	*5*	*6*	*7*
M_1	12	6	5	11	5	7	6
M_2	7	8	9	4	7	8	3
M_3	3	4	1	5	2	3	4

16. Five jobs on four machines

Job	*A*	*B*	*C*	*D*	*E*
M_1	11	13	9	16	17
M_2	4	3	5	2	6
M_3	6	7	5	8	4
M_4	15	8	13	9	11

ANSWERS

1. | 2 | 4 | 3 | 5 | 1 |
   |---|---|---|---|---|

   Total time = 30 units
   Idle time for $M_1 = 2$
   Idle time for $M_2 = 3$

2. | 3 | 1 | 6 | 5 | 2 | 4 |
   |---|---|---|---|---|---|

   Total time = 42 units
   Idle time for $M_1 = 3$
   Idle time for $M_2 = 6$

3. | 3 | 4 | 5 | 7 | 2 | 6 | 1 |
   |---|---|---|---|---|---|---|

   Total time = 46 units
   Idle time for $M_1 = 3$
   Idle time for $M_2 = 5$

4. | 2 | 1 | 3 | 4 | 6 | 5 |
   |---|---|---|---|---|---|

   Total time = 209 units
   Idle time for $M_1 = 34$
   Idle time for $M_2 = 159$
   Idle time for $M_3 = 44$

5. | 4 | 5 | 2 | 6 | 1 | 3 |
   |---|---|---|---|---|---|

   Total time = 53 units
   Idle time for $M_1 = 27$
   Idle time for $M_2 = 32$
   Idle time for $M_3 = 4$

6. | C | A | B | D |
   |---|---|---|---|

   Total time = 112 units
   Idle time for $M_1 = 46$
   Idle time for $M_2 = 99$
   Idle time for $M_3 = 85$
   Idle time for $M_4 = 95$
   Idle time for $M_5 = 79$
   Idle time for $M_6 = 41$

7. | C | A | E | D | B |
   |---|---|---|---|---|

   Total time = 76 units
   Idle time for $M_1 = 15$
   Idle time for $M_2 = 61$
   Idle time for $M_3 = 51$
   Idle time for $M_4 = 25$

8. | 1 | 5 | 3 | 4 | 2 |
   |---|---|---|---|---|

   Total time = 37 units
   Idle time for M_1 = 10 units
   Idle time for M_2 = 4 units

9. | 2 | 5 | 4 | 1 | 3 |
   |---|---|---|---|---|

   Total time = 32 units
   Idle time for M_1 = 2 units
   Idle time for M_2 = 4 units

10. | E | D | A | H | C | B | G | F |
    |---|---|---|---|---|---|---|---|

    Total time = 154 units
    Idle time for M_1 = 18 units
    Idle time for M_2 = 9 units

11. | A | D | G | F | B | C | E |
    |---|---|---|---|---|---|---|

    Total time = 59 units
    Idle time for M_1 = 13 units
    Idle time for M_2 = 37 units
    Idle time for M_3 = 7 units

12. | 4 | 1 | 5 | 2 | 3 |
    |---|---|---|---|---|

    Total time = 46 units
    Idle time for M_1 = 19 units
    Idle time for M_2 = 31 units
    Idle time for M_3 = 7 units

13. | 3 | 2 | 5 | 4 | 1 | 6 |
    |---|---|---|---|---|---|

    Total time = 62 units
    Idle time for M_1 = 9 units
    Idle time for M_2 = 39 units
    Idle time for M_3 = 16 units

14. | C | B | D | A |
    |---|---|---|---|

    (or)

C	B	A	D

    Total time = 86 units

Idle time for M_1 = 25 units
Idle time for M_2 = 74 units
Idle time for M_3 = 63 units
Idle time for M_4 = 25 units

15. | 3 | 5 | 6 | 2 | 1 | 4 | 7 |
|---|---|---|---|---|---|---|

(or)

3	5	2	6	1	4	7

Total time = 59 units
Idle time for M_1 = 7 units
Idle time for M_2 = 13 units
Idle time for M_3 = 37 units

16. | C | A | E | D | B |
|---|---|---|---|---|

Total time = 83 units
Idle time for M_1 = 18 units
Idle time for M_2 = 63 units
Idle time for M_3 = 53 units
Idle time for M_4 = 29 units

17

Queueing Theory

CONCEPT REVIEW

17.1 INTRODUCTION

In our everyday life people are seen waiting at a railway ticket booking office, a doctor's clinic, post office, bank, petrol pump and many other places for getting service. When there is too much demand on the facilities available, the customers have to wait and hence they form a queue or waiting line. The problem is to reduce congestion and delay in serving the customer. A unit that requires some service to be performed, is called a *customer*. The system which offers service is called the *service facility*. Customers arrive at a service counter and are attended by servers. After the service is completed, the customer leaves the system.

17.2 CHARACTERISTICS OF A QUEUEING SYSTEM

A queueing system can be completely described by the following characteristics.

(i) Input process This gives the mode of arrival of the customers into the system. Generally the customers arrive in a random manner. Hence the distribution of inter-arrival time follows some probability law. We assume that the customers arrive in a 'Poisson' fashion. That is to say, the rate of arrival is a poisson variate. Mean (average) arrival rate is taken as λ.

(ii) Queue discipline This is the law according to which the customers are served. The simplest rule is *First-in First-out* (FIFO) where the customers are served according to the order of their arrival. Generally this rule is followed almost everywhere, like ration shops, ticket booking, etc., Another rule is *Last-in First-out* (LIFO). Here the unit which arrives last is served first as in the case of a passenger who gets into a crowded bus as the last man, gets down first or an item which is dumped into a godown, as the last one, is taken out first. *The third rule is Service in Random Order* (SIRO) where the customers are selected for service at random irrespective of their arrival time.

(iii) Service Mechanism This is the facility available in the system to serve the customers. In certain cases one server many find it difficult when there is a large number of customers in a shop. Then the number of servers may be increased in order to reduce the waiting time of the customers lest the business should be lost. The number of servers (service channels) depends on the number of customers and their waiting time in the system. In a multichannel system (as in the case of railway or airlines ticket booking) many customers are served simultaneously. Also the customers may be served in batches of fixed size as seen in cinema hall, marriage dinner party, etc. It is called *bulk bsrvice*.

(iv) Capacity of the system This is the capacity of the queue or waiting line. It is the number as to how many customers can be there in the queue. Generally a queue is of infinite capacity. In some systems the customers are admitted into a waiting room whose capacity is limited. When the waiting line reaches a certain length no further customers are allowed until space becomes available by a service completion.

17.3 SYMBOLS AND ASSUMPTIONS

The following symbols are generally used:

n – number of customers in the system both waiting and in service

λ – average number of customers arriving per unit time (arrival rate)

μ – average number of customers served per unit time (service rate)

ρ – $\dfrac{\lambda}{\mu}$ (traffic density)

c – number of servers

$E(n)$ – average number of customers in the system $L(s)$.

$E(m)$ – average number of customers in the queue $L(q)$

$P_n(t)$ – Probability that there are n customers in the system at time t.

$N(t)$ – Total number of arrivals up to time t.

We make the following assumptions:

(i) Probability that one arrival occurs during a small interval Δt is

$$\lambda(\Delta t) + 0(\Delta t)$$

(ii) Δt is very small such that the probability of more than one arrival in Δt is of order Δt. $0(\Delta t)$ represents terms such that

$$\underset{\Delta t \to 0}{lt} \frac{0(\Delta t)}{(\Delta t)} \to 0$$

(iii) In short $(\Delta t)^2$ and higher powers of Δt are negligible.

(iv) The number of arrivals in the intervals are statistically independent. The process has independent increments.

(v) Similarly probability that one service is completed during Δt is

$$\mu(\Delta t) + 0(\Delta t)$$

$$\underset{\Delta t \to 0}{lt} \frac{0(\Delta t)}{(\Delta t)} \to 0$$

17.4 TRANSIENT AND STEADY STATES

A queueing system is said to be in transient state if the operating characteristics like input, output, mean queue length, etc. are dependent on time. If these are independent of time then the system is said to be in steady state. For example $P_n(t)$ is a function of time t in transient state. But $P_n(t)$ is constant for any value of t if it is in steady state.

17.5 MODEL I ($M/M/1/\infty$ QUEUEING MODEL)

In the notation $M/M/1/\infty$, M denotes poisson arrival rate and exponential service rate, 1 denotes the number of servers (single server system) and ∞ is the capacity of the system. It is understood that service is done on FIFO basis. Mean arrival rate is λ and mean service rate is μ.

17.5.1 Formula for Steady State Probability P_n

The probability that there are n customers at time $(t + \Delta t)$ is given by $P_n(t + \Delta t)$. This probability is the sum of the probabilities of four mutually exclusive cases (service completion and departure are the same)

$$P_n(t + \Delta t) = P_n(t)P \text{ (no arrival in } \Delta t). \; P\text{(no departure in } \Delta t)$$
$$+ \; P_n(t).P \text{ (one arrival in } \Delta t) \; P\text{(one departure in } \Delta t)$$

$+\ P_{n+1}(t).P$ (no arrival in Δt) P (one departure in Δt)

$+\ P_{n-1}(t).P$ (one arrival in Δt) P (no departure in Δt)

$$= P_n(t)[1 - \lambda(\Delta t) + 0\Delta t][1 - \mu(\Delta t) + 0(\Delta t)]$$
$$+ P_n(t)[\lambda(\Delta t) + 0\Delta t][\mu(\Delta t) + 0(\Delta t)]$$
$$+ P_{n+1}(t)[1 - \lambda(\Delta t) + 0\Delta t][\mu(\Delta t) + 0(\Delta t)]$$
$$+ P_{n-1}(t)[\lambda(\Delta t) + 0\Delta t][1 - \mu(\Delta t) + 0(\Delta t)]. \qquad (n \geq 1)$$

i.e. $P_n(t+\Delta t) = P_n(t)[1 - \lambda(\Delta t) - \mu(\Delta t) + 0(\Delta t)]$
$$+ P_n(t)[0(\Delta t)]$$
$$+ P_{n+1}(t)[\mu(\Delta t) + 0(\Delta t)]$$
$$+ P_{n-1}(t)[\lambda(\Delta t) + 0(\Delta t)]$$
$$= P_n(t) - (\lambda + \mu)P_n(t)(\Delta t) + \mu P_{n+1}(\Delta t) + \lambda P_{n-1}(t)(\Delta t) + 0(\Delta t)$$

$\therefore\ P_n(t+\Delta t) - P_n(t) = -(\lambda+\mu)P_n(t)(\Delta t) + \mu P_{n+1}(t)(\Delta t) + \lambda P_{n-1}(t)(\Delta t) + 0(\Delta t)$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t) + \frac{0(\Delta t)}{(\Delta t)}$$

Taking limits as $\Delta t \to 0$ we get

$$P_n'(t) = -(\lambda+\mu)\,P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t) \quad (1)$$

For $n = 0$, $P_{n-1}(t)$ does not exist. We take this case separately.

$P_0(t+\Delta t) = P_0(t).P$ (no arrival in (Δt))

$+ P_1(t).P$ (no arrival in (Δt)). P(one departure in (Δt))

$$= P_0(t)[1 - \lambda(\Delta t) + 0(\Delta t)]$$
$$+ P_1(t)[1 - \lambda(\Delta t) + 0(\Delta t)][\mu(\Delta t) + 0(\Delta t)]$$
$$= P_0(t) - \lambda P_0(t)(\Delta t) + \mu P_1(\Delta t) + 0(\Delta t)]$$
$$P_0(t+\Delta t) - P_0(t) = -\lambda P_0 t(\Delta t) + \mu P_1(t)(\Delta t) + 0(\Delta t)$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0 t + \mu P_1(t) + \frac{0(\Delta t)}{(\Delta t)}.$$

Taking limits as $\Delta t \to 0$ we get

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad (2)$$

For steady state probability $P_n(t)$ and $P_0(t)$ are independent of time. Hence

$$P_n'(t) = P_0'(t) = 0$$

Therefore deleting the time t, we get

$$0 = -(\lambda+\mu)P_n + \mu P_{n+1} + \lambda P_{n-1} \text{ from (1)}$$

i.e. $\mu P_n{+}1 = (\lambda+\mu)P_n - \lambda P_{n-1}$

$$P_{n+1} = \frac{(\lambda+\mu)}{\mu}P_n - \frac{\lambda}{\mu}P_{n-1} \quad n \geq 1 \quad (3)$$

Also from (2) we have

$$0 = -\lambda P_0 + \mu P_1 \quad \text{or} \quad P_1 = \frac{\lambda}{\mu}P_0$$

i.e. $P_1 = \rho P_0$

Substituting $n = 1, 2, 3, \ldots$ successively in the recurrence relation (3) with

$$P_1 = \frac{\lambda}{\mu}P_0$$

we get the value of P_n for all n, in terms of P_0

For $n = 1$

$$P_2 = \frac{(\lambda+\mu)}{\mu}P_1 - \frac{\lambda}{\mu}P_0$$
$$= \frac{(\lambda+\mu)}{\mu}\frac{\lambda}{\mu}P_0 - \frac{\lambda}{\mu}P_0$$
$$= \frac{\lambda}{\mu}\left[\frac{(\lambda+\mu)}{\mu} - 1\right]P_0$$
$$= \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$\therefore\ P_2 = \rho^2 P_0$

For $n = 2$

$$P_3 = \frac{(\lambda+\mu)}{\mu}P_2 - \frac{\lambda}{\mu}P_1$$
$$= \left(\frac{(\lambda+\mu)}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 P_0 - \frac{\lambda}{\mu}.\frac{\lambda}{\mu}P_0$$
$$= \left(\frac{\lambda}{\mu}\right)^2 P_0\left(\frac{(\lambda+\mu)}{\mu} - 1\right)$$
$$= \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$\therefore\ P_3 = \rho^2 P_0$

Thus we get $\therefore P_4 = \rho^4 P_0 \quad P_5 = \rho^5 P_0$ and so on

$\therefore\ P_n = \rho^n P_0$

Now the total probability is 1

$\therefore\ P_0 + P_1 + P_3 + \ldots + P_n + \ldots \text{ to } \infty = 1$

i.e. $P_0 + \rho P_0 + \rho^2 P_0 + \rho^3 P_0 + \ldots \text{ to } \infty = 1$

$$P_0[1 + \rho + \rho^2 + \rho^3 + \ldots \infty] = 1$$

$$P_0\left(\frac{1}{1-\rho}\right) = 1 \Rightarrow P_0 = 1-\rho \qquad (\rho < 1)$$

Thus we get the expression for the steady state probability, as $P_n = \rho^n(1-\rho)$ where $\rho = \dfrac{\lambda}{\mu}$.

Note: The formula for P_0 is true only when $(\rho < 1)$ or $\lambda < \mu$ (condition for the existence of sum to infinity of a G.P.)

17.5.2 Important Formulae

(i) Probability that the number of customers in the system is greater than or equal to *n*

$$\begin{aligned}
P(N \geq n) &= \sum_{N=n}^{\infty} P_n \\
&= \sum_{N=n}^{\infty} p^n(1-\rho) \\
&= (1-\rho)[\rho^n + \rho^{n+1} + \rho^{n+2} + \ldots \text{ to } \infty] \\
&= (1-\rho)[1 + \rho + \rho^2 + \ldots + \rho^{n-1} + \rho^n \\
&\quad + \rho^{n+1} + \ldots \text{ to } \infty - (1 + \rho + \rho^2 + \rho^3 \\
&\quad + \ldots + \rho^{n-1})] \\
&= (1-\rho)\left[\frac{1}{1-\rho} - \frac{1-\rho^n}{1-\rho}\right] \\
&= (1-\rho)\left[\frac{1-(1-\rho^n)}{1-\rho}\right] \\
&= \rho^n
\end{aligned}$$

(ii) Average number of customers in the system

$$E(n) = \sum n.P_n \qquad \left[E(X) = \sum XP(X)\right]$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} n.\rho^n(1-\rho) \\
&= \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1} \\
&= \rho(1-\rho)[1 + 2\rho + 3\rho^2 + \ldots \text{ to } \infty] \\
&= \rho(1-\rho)(1-\rho)^{-2} \text{ (Binomial series)} \\
&= \frac{\rho(1-\rho)}{(1-\rho)^2} = \frac{\rho}{(1-\rho)} \text{ or } \frac{\lambda}{\mu-\lambda}
\end{aligned}$$

$$\therefore \quad E(n) = \frac{\rho}{1-\rho} \quad \text{[Also denoted by } L(s)]$$

(iii) Average number of customers in the queue *E*(*m*)

$m = n - 1$ (number of customers in the queue excluding one customer in service)

$$\begin{aligned}
E(m) &= \sum_{n=1}^{\infty} (n-1)P_n \\
&= \sum_{n=0}^{\infty} n.P_n - \sum_{n=1}^{\infty} P_n \\
&= E(n) - (\rho + \rho^2 + \rho^3 + \ldots + \infty) \\
&= E(n) - (1 - P_0) \\
&= \frac{\rho}{1-\rho} - [1-(1-\rho)] \\
&= \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} \text{ or } \frac{\lambda^2}{\mu(\mu-\lambda)}
\end{aligned}$$

i.e. $$E(m) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

[Also denoted by $L(q)$]

(iv) Expected waiting time of a customer in the system (W_s)

The expected period of time a unit has to be in the system from arrival till completion of service is given by

$$W_s = \frac{\text{Expected number of units in the system}}{\text{Arrival rate}}$$

$$\therefore \quad W_s = \frac{E(n)}{\lambda} = \frac{\lambda}{\lambda(\mu-\lambda)} = \frac{1}{\mu-\lambda}$$

(v) Expected waiting time of a customer in the queue (W_q)

The expected time a unit has to wait in the queue before entering into service is given by subtracting the service time from the total waiting time in the system.

$$W_q = W_s - \frac{1}{\mu}$$

$$= \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$\therefore \quad W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

Note: (i) Service rate = μ. Service time = $\dfrac{1}{\mu}$

(ii) Arrival rate = $\dfrac{1}{\text{inter arrival time}}$

(iii) $E(n) = \lambda.W_s$; $E(m) = \lambda W_q$. This is known as *Little's formula*

Example 17.1 A TV repair man finds that the time spent on repairing has an exponential distribution with mean 30 min per unit. The arrival of TV sets is poisson with an average of 10 sets per day of 8 hours. What is his expected idle time per day? How many sets are there on the average?

Solution

Service time per unit = 30 min = $\dfrac{1}{2}$ hr

Service rate (per hr) = 2 i.e. $\mu = 2$

Arrival rate = 10 per 8 hr

= 10/8 per hr

$\therefore \quad \lambda = 10/8 = 5/4$

P_0 = Probability that there is no unit in the system

$= 1 - \rho = 1 - \dfrac{\lambda}{\mu}$

$= 1 - 5/8 = 3/8$

$\therefore$ Idle time per day of 8 hrs = $3/8 \times 8 = 3$ hrs

Average number of units in the system = $E(n)$

$$= \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda} = \frac{5/4}{2 - 5/4} = 5/3 \text{ sets}$$

Example 17.2 In a railway yard goods trains arrive at the rate of 30 trains per day. Assuming that the service time follows exponential distribution with an average of 36 minutes, find

(i) the probability that the number of trains in the yard exceeds 10.

(ii) the average number of trains in the yard.

Solution

Arrival rate = 30 trains per day (24 hr)

$= \dfrac{30}{24}$ per hour

$\therefore \quad \lambda = \dfrac{30}{60 \times 24}$ per min

Service time = 36 min per unit

Service rate = 1/36 per min

$\therefore \quad \mu = 1/36.$

$$\rho = \frac{\lambda}{\mu} = \frac{30 \times 36}{60 \times 24 \times 1} = \frac{3}{4}$$

(i) Number of trains exceeds 10 $\Rightarrow n \geq 11$

Required probability = ρ^{11}

$$= \left(\frac{3}{4}\right)^{11}$$

(ii) Average number of trains in the yard = $E(n)$

$$\left(\frac{\rho}{1-\rho}\right) = \frac{3/4}{1/4} = 3$$

Example 17.3 In a store with one server, 9 customers arrive on an average of 5 minutes. Service is done for 10 customers in 5 minutes. Find

(i) the average number of customers in the system.

(ii) the average queue length.

(iii) the average time a customer spends in the store.

(v) the average time a customer waits before being served.

Solution 9 customers arrive in 5 min

$\therefore$ Arrival rate = 9/5 per min

$\therefore \quad \lambda = 9/5$

10 customers are served in 5 min

$\therefore$ Service rate = 10/5 per min

$\therefore \quad \mu = 2$

(i) average number of customers in the system $= E(n)$

$$= \frac{\lambda}{\mu - \lambda} = \frac{9/5}{2 - 9/5} = 9$$

(ii) average queue length = $E(m)$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu} \cdot \frac{\lambda}{(\mu - \lambda)}$$

$$= \frac{9/5}{2}.9 = 8.1$$

(iii) average time a customer spends in the system $= W_s$

$$= \frac{E(n)}{\lambda} = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{2 - 9/5} = 5 \text{ min}$$

(iv) average time a customer spends in the queue, before entering into service = W_q

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{1}{\mu} \cdot \left(\frac{\lambda}{\mu - \lambda}\right)$$

$= (1/2)9 = 9/2$ min

[W_q = Total time in the system – service time

$= 5 - \frac{1}{2} = 9/2$ min]

Example 17.4 In a telephone booth, the arrivals follow poisson distribution with an average of 9 minutes between two consecutive arrivals. The duration of a telephone call is exponential with an average of 3 min.

(i) Find the probability that a person arriving at the booth has to wait.

(ii) Find the average queue length.

(iii) Find the fraction of the day, the phone will be in use.

(iv) The company will install a second booth if a customer has to wait for phone, for at least 4 minutes. If so, find the increase in the flow of arrivals in order that another booth will be installed.

Solution

Inter-arrival time = 9 min

$\therefore$ Arrival rate λ = 1/9 per min

Duration of a call = 3 min

$\therefore$ Service rate μ = 1/3 per min

$$\rho = \frac{\lambda}{\mu} = \frac{3}{9} = \frac{1}{3}$$

(i) A person has to wait if the system is not empty.

$\therefore$ Probability that an arrival has to wait is given by $1 - P_0$

$$1 - P_0 = 1 - (1 - \rho) = \rho = \frac{1}{3}$$

(ii) Average queue length is $E(m)$.

$$E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{1/81}{1/3(1/3 - 1/9)} = \frac{1}{6}$$

(iii) Probability that the phone is in use, is the probability that the system is busy which is given by the traffic density ρ.

$\therefore$ Fraction of time, the system is busy

$$= \rho = \frac{1}{3}$$

(iv) Let λ_1 be the arrival rate for the average waiting time in the queue to be at least 4 minutes

$$W_q = \frac{\lambda_1}{\mu(\mu - \lambda_1)}$$

$$\therefore \quad 4 = \frac{\lambda_1}{\frac{1}{3}\left(\frac{1}{3} - \lambda_1\right)} = \frac{9\lambda_1}{1 - 3\lambda_1}$$

$$\therefore \quad 4 - 12\lambda_1 = 9\lambda_1$$

$$\lambda_1 = 4/21$$

Thus a second booth will be installed if the arrival rate is 4/21.

$\therefore$ Required increase in the flow of arrivals

$$\therefore \quad = \frac{4}{21} - \frac{1}{9} = \frac{12 - 7}{63} = \frac{5}{63} \text{ per min}$$

Example 17.5 In a carwash station cars arrive for service according to poisson distribution, with mean 4 per hour. The average service time of a car is 10 min.

(i) Determine the probability that an arriving car has to wait.

(ii) Find the average time a car stays in the station.

(iii) If the parking space cannot hold more than 6 cars, find the probability that an arriving car has to wait outside.

Solution

Arrival rate = 4 per hour

λ = 4/60 per min = 1/15

Service time = 10 min per car

$\therefore$ service rate = 1/10 per min.

$$\mu = \frac{1}{10}$$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{15} = \frac{2}{3}$$

(i) The probability that an arriving car has to wait

$= 1 - P_0 = 1 - (1 - \rho)$
$= \rho$
$= 2/3$

(ii) Average time a car stays in the station

$$= W_s = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{\frac{1}{10} - \frac{1}{15}} = \frac{1}{\frac{5}{150}} = \frac{150}{5} = 30 \text{ min}$$

(iii) If the parking space is full, then there are 6 cars waiting for service and one car in service. Therefore $n = 7$. The new arrival has to wait outside if $n \geq 8$. The required probability is

$$P(n \geq 8) = \rho^8 = \left(\frac{2}{3}\right)^8$$

Example 17.6 On an average 96 patients per day require the service of an emergency clinic which can handle only one patient at a time. It takes on the average 10 minutes to give treatment to a patient. The cost of the treatment is Rs 100 per patient for 10 minutes. The cost increases at Rs 10 per minute of time reduced. How much amount should be budgeted by the clinic to reduce the queue size to 1/2?

Solution Average number of arrivals = 96 per day

$$= \frac{96}{24} = 4 \text{ per hour}$$

$\therefore \quad \lambda = 4$

Service time = 10 min per patient = 1/6 hr

$\therefore$ service rate = 6 per hour

$\mu = 6$

Queue length = $E(m)$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{16}{6(6-4)}$$

$$= \frac{4}{3}$$

$E(m)$ is to be reduced to $\frac{1}{2}$.

Take the service rate as μ^1

$$\therefore \quad 1/2 = \frac{\lambda}{\mu^1(\mu^1 - \lambda)} = \frac{16}{\mu^1(\mu^1 - 4)}$$

$$= \mu^{1^2} - 4\mu^1 - 32 = 0$$

$\mu^1 = 8$ or $\mu^1 = -4$ (not possible)

$\therefore \quad \mu^1 = 8$

$\therefore$ The revised service rate = 8

$\therefore$ Service time = 1/8 hr = 15/2 min

Decrease in service time = 10 – 15/2 = 5/2 min

Increase in cost = 5/2 × 10 = Rs 25

$\therefore$ The budget required

= Rs 100 + Rs 25
= Rs 125

Therefore in order to reduce the queue (increase the service rate to 8) the clinic has to provide Rs 125 as the cost per patient.

Example 17.7 Trucks arrive at a loading dock at an average rate of 4 trucks per hour. The loading of a truck takes 10 minutes on the average by a crew of three loadmen. The operating cost of a truck is Rs 20 per hour and the loadmen are paid at Rs 6 each per hour. Is it advisable to add another crew of three loadmen?

Solution Arrival rate of trucks = 4 per hour

$\lambda = 4$

Service time (loading time) = 10 min = 1/6 hour

$\therefore$ Service rate = 6 per hour

$\mu = 6$

Loading crew cost = 3 × 6 = Rs 18

Total time taken by truck

$$= W_s = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{6-4} = \frac{1}{2} \text{ hour}$$

Cost of waiting time

$$= \frac{1}{2} \text{(arrival rate)} \times \text{(hourly cost)}$$

$$= \frac{1}{2} \times 4 \times 20 = \text{Rs } 40$$

Total hourly cost

= crew cost + cost of waiting time

= 18 + 40 = Rs 58

If the crew becomes 6, then

Crew cost = 6 × 6 = Rs 36

Service rate = 12 per hour

$\mu = 12$

$$W_s = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{12 - 4} = \frac{1}{8} \text{ hour}$$

Cost of waiting time = 1/8 × 4 × 20 = 10

The new total hourly cost = 36 + 10 = Rs 46

Existing total cost = Rs 58 per hour

Revised total cost = Rs 46 per hour.

Therefore if the crew becomes 6, the cost is reduced from Rs 58 to Rs 46. Hence it is advisable to add three more loadmen.

17.6 MODEL 2—*M/M/1/N* (QUEUE WITH FINITE CAPACITY *N*)

In this model the capacity of the system is limited to N customers only. Maximum number of customers allowed in the system is N. If there are N customers in the system then the probability of a new arrival is zero. So long as $n < N$ this system is the same as model 1. For example, in a doctor's clinic the waiting hall for the patients has only a limited number of seats, say 20. When the waiting hall is full, a newcomer has to stand outside and he is not considered as a member of the system. The maximum value n can take, is N, whereas in the previous model n can vary from 0 to ∞.

17.6.1 Formula for P_n

$$P_N(t + \Delta t) = P_N(t).P \text{ (no departure in } \Delta t) + P_{N-1}(t)P \text{ (one arrival in } \Delta t).P \text{ (no departure in } \Delta t)$$

$$= P_N(t)[1 - \mu(\Delta t) + 0(\Delta t)] + P_{N-1}(t)[\lambda(\Delta t) + 0(\Delta t)][1 - \mu(\Delta t) + 0(\Delta t)]$$

$$= P_N(t) - \mu P_N(t)(\Delta t) + \lambda P_{N-1}(t)(\Delta t) + 0(\Delta t)$$

$$P_N(t + \Delta t) - P_N(t) = -\mu P_N(t)(\Delta t) + \lambda P_{N-1}(t)(\Delta t) + 0(\Delta t)$$

$$\frac{P_N(t + \Delta t) - P_N(t)}{\Delta t} = -\mu P_N(t) + \lambda P_{N-1}(t) + \frac{0(\Delta t)}{(\Delta t)}$$

In the limit as $(\Delta t) \to 0$

$$P'_N(t) = -\mu P_N(t) + \lambda P_{N-1}(t)$$

For steady state probability $P'_N = 0$

$$\therefore \quad 0 = -\mu P_N + \lambda P_{N-1}$$

$$\mu P_N = \lambda P_{N-1}$$

$$P_N = \frac{\lambda}{\mu} P_{N-1}$$

Also for $n < N$,

$$P_n = \frac{\lambda + \mu}{\mu} P_n - \frac{\lambda}{\mu} P_n \text{ from (3) of 17.5.1}$$

and $P_1 = \dfrac{\lambda}{\mu} P_0$

Thus for $n \le N$, we get

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \text{ or } P_n = \rho^n P_0 \quad (0 \le n \le N)$$

Now $\sum_{n=0}^{N} P_n = 1$ (Total probability)

i.e., $P_0 + \rho P_0 + \rho^2 P_0 + ... + \rho^N P_0 = 1$

$$P_0[1 + \rho + \rho^2 + ... + \rho^N] = 1$$

$$\therefore \quad P_0\left[\frac{1 - \rho^{N+1}}{1 - \rho}\right] = 1$$

$$\therefore \quad P_0 = \frac{1 - \rho}{1 - \rho^{N+1}} \quad (\rho \ne 1)$$

Note: Here the sum of $N + 1$ terms of a GP is taken. Hence ρ need not be less than 1. The number of units allowed into the system is controlled by the queue length $N - 1$ and not by λ and μ.

If $\rho = 1$, $P_0(N + 1) = 1$. Therefore $P_0 = \dfrac{1}{N+1}$

Therefore we get

$$\left.\begin{aligned} P_n &= \rho^n\left(\frac{1 - \rho}{1 - \rho^{N+1}}\right) \quad (n \le N) \\ P_0 &= \frac{1 - \rho}{1 - \rho^{N+1}} \end{aligned}\right\} \rho \ne 1$$

$$P_n = P_0 = \frac{1}{N+1} \qquad (\rho = 1)$$

17.6.2 Important formulae

(i) Average number of customers in the system

Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{N} n.P_n = \sum_{n=0}^{N} n.\rho^n P_0$$

$$= P_0.\rho \sum_{n=0}^{N} n.\rho^{n-1} = P_0.\rho \sum_{n=0}^{N} \frac{d(\rho^n)}{d\rho}$$

$$= P_0.\rho \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n$$

$$= P_0.\rho \frac{d}{d\rho}\left[\frac{1-\rho^{N+1}}{1-\rho}\right]$$

$$= P_0.\rho \left[\frac{(1-\rho)\{-(N+1)\rho^N\} - (1-\rho^{N+1})(-1)}{(1-\rho)^2}\right]$$

$$= P_0.\frac{\rho}{(1-\rho)^2}[-(N+1)\rho^N(1-\rho) + 1 - \rho^{N+1}]$$

$$= P_0.\frac{\rho}{(1-\rho)^2}[1-(N+1)\rho^N + N\rho^{N+1}]$$

$$= \frac{(1-\rho)}{1-\rho^{N+1}} \cdot \frac{\rho}{(1-\rho)^2}[1-(N+1)\rho^N + N\rho^{N+1}]$$

$$= \frac{\rho}{(1-\rho)(1-\rho^{N+1})}[1-(N+1)\rho^N + N\rho^{N+1}] \quad (\rho \neq 1)$$

If $\rho = 1$ $\quad E(n) = \sum_{n=0}^{N} n.\frac{1}{N+1}$

$$= \frac{1}{N+1}\frac{N(N+1)}{2} = \frac{N}{2}$$

(ii) Average number of customers in the queue

$$E(m) = \sum_{n=1}^{N} (n-1)P_n$$

$$= E(n) - \sum_{n=1}^{N} P_n$$

$$= E(n) - (1 - P_0)$$

$$= E(n) - \left[1 - \frac{1-\rho}{1-\rho^{N+1}}\right]$$

$$= E(n) - \left[\frac{\rho - \rho^{N+1}}{1-\rho^{N+1}}\right]$$

$$= \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})} - \frac{\rho(1-\rho^N)}{1-\rho^{N+1}}$$

$$= \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1} - (1-\rho)(1-\rho^N)]}{(1-\rho)(1-\rho^{N+1})}$$

$$= \frac{\rho}{(1-\rho)(1-\rho^{N+1})}[1-(N+1)\rho^N + N\rho^{N+1} - (1-\rho^N-\rho+\rho^{N+1})]$$

$$= \frac{\rho}{(1-\rho)(1-\rho^{N+1})}[\rho - N\rho^N + (N-1)\rho^{N+1}]$$

$$= \frac{\rho^2}{(1-\rho)(1-\rho^{N+1})}[1 - N\rho^{N-1} + (N-1)\rho^N] \quad (\rho \neq 1)$$

$$E(m) = \frac{N(N-1)}{2(N+1)} \quad (\rho = 1)$$

(iii) Expected waiting time in the system

$$W_S = \frac{E(n)}{\lambda^1} \quad \text{where } \lambda^1 = \lambda(1-P_N)$$

$$W_S = \frac{E(n)}{\lambda(1-P_N)}$$

(iv) Expected waiting time in the queue

$$W_q = \frac{E(m)}{\lambda^1} = \frac{E(m)}{\lambda(1-P_N)}$$

or $W_q = W_S - \dfrac{1}{\mu}$

Example 17.8 At a railway station only one train is handled at a time. The yard can accommodate only two trains to wait. Arrival rate of trains is 6 per hour and the railway station can handle them at the rate of 12 per hour. Find the steady state probabilities for the various number of trains in the system. Also find the average waiting time of a newly arriving train.

Solution We have

$$\lambda = 6 \text{ per hour}$$
$$\mu = 12 \text{ per hour}$$
$$\rho = \frac{6}{12} = \frac{1}{2} = 0.5$$

Maximum queue length is 2 and the maximum number of trains in the system is $N = 3$.

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.5}{1-(0.5)^4} = 0.53$$
$$P_1 = P_0\rho = (0.53)\,(0.5) = 0.27$$
$$P_2 = P_0\rho^2 = (0.53)\,(0.5)^2 = 0.13$$
$$P_3 = P_0\rho^3 = (0.53)\,(0.5)^3 = 0.07$$

These are the probabilities for the various number of trains in the system

$$\text{Now } E(n) = \sum_{n=0}^{3} n.P_n$$
$$= 0 + 1(0.27) + 2(0.13) + 3(0.07)$$
$$= 0.74$$

Average number of trains in the system = 0.74

Each train takes 1/12 hour for service.

$\therefore$ Service time for one train
= 1/12 hour = 0.085 hour

$\therefore$ A newly arriving train finds 0.74 trains in the system with service time
= 0.085 hr each.

$\therefore$ Expected waiting time
= (0.74) (0.085) hr
= 0.0629 hour = 3.8 minutes

Example 17.9 The capacity of a queueing system is 4. Inter-arrival time of the units is 20 minutes and the service time is 36 minutes per unit. Find the probability that a new arrival enters into service without waiting. Also find the average number of units in the system.

Solution Given that the inter-arrival time is 20 min

Therefore the arrival rate λ = 1/20 per min

Service time = 36 min

$\therefore$ service rate = 1/36 per min

$$\therefore \quad \rho = \frac{\lambda}{\mu} = 36/20 = 1.8$$
$$N = 4$$

(i) A new arrival can enter into service without waiting if the system is empty. The probability is

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{\rho-1}{\rho^{N+1}-1}$$
$$= \frac{1.8-1}{(1.8)^5-1} = 0.04$$

(ii) Average number of units in the system is

$$E(n) = \sum_{n=1}^{4} n.p_n = P_1 + 2P_2 + 3P_3 + 4P_4$$
$$= P_0[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4]$$
$$= (0.04)\,[(1.8) + 2(1.8)^2 + 3(1.8)^3 + 4(1.8)^4]$$
$$= (0.04)(67.77) = 2.71 \text{ (nearly 3)}$$

$\therefore$ Average number of units in the system is 3 (nearly)

Example 17.10 Patients arrive at a clinic at the rate of 30 patients per hour. The waiting hall cannot accommodate more than 14 patients. It takes 3 minutes on the average to examine a patient.

(i) Find that probability that an arriving patient need not wait.

(ii) Find the probability that an arriving patient finds a vacant seat in the hall.

(iii) What is the expected duration of time a patient spends in the clinic?

Solution

Arrival rate = λ = 30 / hour

Service rate = μ = 20 / hour

$$\rho = \frac{\lambda}{\mu} 30/20 = 1.5 \qquad (\rho > 1)$$

Capacity of the system = $N = 14$

(i) An arriving patient need not wait if the system is empty.

$$\therefore \text{ Required probability } = P_0 = \frac{\rho - 1}{\rho^{N+1} - 1}$$

$$\frac{1.5 - 1}{(1.5)^{15} - 1} = \frac{0.5}{(1.5)^{15} - 1} = 0.001$$

(ii) Since the capacity of the system is 14, an arriving patient finds a vacant seat if the number of patients in the system is less than 14. Hence the required probability is given by
$P_0 + P_1 + P_2 + ... + P_{13}$
which is equal to $1 - P_{14}$. Thus the required probability is

$$1 - P_0\rho^{14} = 1 - \left[\frac{(0.5)}{(1.5)^{15} - 1}\right](1.5)^{14}$$

$$= 1 - 0.29 = 0.71$$

(iii) Average time spent by a patient in the clinic

$$W_S = \frac{E(n)}{\lambda[1 - P_N]}$$

$$E(n) = \frac{\rho}{(1-\rho)(1-\rho^{N+1})}[1 - (N+1)\rho^N + N\rho^{N+1}]$$

$$= \frac{(1.5)}{(0.5)[(1.5)^{15} - 1]} \times [1 - 15(1.5)^{14} + 14(1.5)^{15}]$$

$$= \frac{(1.5)}{(0.5)[(437)]}[1 - 15(292) + 14(438)]$$

$$= \frac{3 \times 1753}{437} = 12$$

$$P_N = P_{14} = P_0\rho^{14} = (0.001)(292) = 0.29$$

$$\therefore \quad W_s = \frac{12}{(1.5)(0.71)} = \frac{12}{1.065} = 11.27 \text{ hrs}$$

Note: We find that W_S becomes large when $\rho > 1$

17.7 MODEL 3—(MULTICHANNEL SYSTEM) *M/M/C/∞*

In this model, instead of a single service channel, there are C parallel channels (C servers serving C customers simultaneously). It is assumed that customers arrive at an average rate of λ and each server has the same mean service rate μ. If there are more than C customers in the system, all the servers will be busy and the mean service rate is $C\mu$. If there are n customers where $n < C$ then the service rate is $n\mu$ and $C - n$ servers remain idle. Thus we find.

(i) $\mu_1 = \mu$, $\mu_2 = 2\mu$, $\mu_3 = 3\mu$ ..., $\mu_n = n\mu$ $\quad 1 < n < C$

(ii) If $n > C$ then C customers are served and $(n - C)$ customers will be in the queue and $\mu_n = c\mu \quad n \geq C$.

(iii) Arrival rate is the same $\lambda_n = \lambda$ for all n

17.7.1 Expression for P_n

The steady state probabilities are

$$P_1 = \frac{\lambda}{\mu_1} P_0$$

$$P_2 = \frac{\lambda^2}{\mu_2\mu_1} P_0$$

$$P_3 = \frac{\lambda^3}{\mu_3\mu_2\mu_1} P_0$$

.
.
.

$$P_n = \frac{\lambda^n}{\mu_n\mu_{n-1} \ldots \mu_3\mu_2\mu_1} P_0$$

Thus we have

$$P_n = \frac{\lambda^n P_0}{n\mu.(n-1).\mu(n-2)\mu \ldots 3\mu 2\mu.\mu} \qquad (1 \leq n < C)$$

$$= \frac{\lambda^n P_0}{C\mu.C\mu \ldots C\mu(C-\mathbf{1})\mu(C-\mathbf{2})\mu \ldots \mathbf{2}\mu.\mu} \qquad (n \geq C)$$

$$P_n = \frac{\lambda^n P_0}{n!\mu^n} \quad (1 \le n < C)$$

$$P_n = \frac{\lambda^n P_0}{C^{n-C} C!\mu^n} \quad (n \ge C)$$

Taking $\frac{\lambda}{\mu} = \rho$ we get

$$P_n = \frac{\rho^n}{n!} P_0 \quad (1 \le n \le C)$$

$$= \frac{\rho^n P_0}{C^{n-C} C!} \quad (n \ge C)$$

17.7.2 Expression for P_0

Total probability = 1

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\sum_{n=0}^{c-1} \frac{\rho^n}{n!} P_0 + \sum_{n=C}^{\infty} \frac{\rho^n P_0}{C^{n-C} C!} = 1$$

$$\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \sum_{n=C}^{\infty} \frac{\rho^n}{C^{n-C} C!}\right] P_0 = 1$$

$$\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^C}{C!} \sum_{n=C}^{\infty} \left(\frac{\rho}{C}\right)^{n-C}\right] P_0 = 1$$

$$\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^C}{C!} \frac{1}{\left(1 - \frac{\rho}{C}\right)}\right] P_0 = 1$$

(assuming that $\frac{\rho}{C} < 1$)

$$P_0 = \frac{1}{\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho}{C!} \cdot \frac{C\mu}{(C\mu - \lambda)}\right]}$$

Note: This result is valid only if

$$\frac{\rho}{C} < 1, \text{ or } \frac{\lambda}{C\mu} < 1 \text{ ie } \lambda < C\mu$$

In other words, mean arrival rate must be the less than the maximum service rate.

17.7.3 Important Formulae

(i) Probability that an arrival has to wait, is given by $P(n > C)$

$$P(n \ge C) = \sum_{n=C}^{\infty} P_n = \sum_{n=C}^{\infty} \frac{\rho^n P_0}{C!C^{n-C}}$$

$$= \frac{\rho^C}{C!} P_0 \sum_{n=C}^{\infty} \left(\frac{\rho}{C}\right)^{n-C}$$

$$= \frac{\rho^C}{C!} \left(\frac{C\mu}{C\mu - \lambda}\right) . P_0$$

(ii) Probability that an arrival enters into service without waiting, is given by $P(n < C)$

$P(n < C) = 1 - P(n \ge C)$

$$1 - \frac{\rho^C}{C!} \left(\frac{C\mu}{C\mu - \lambda}\right) . P_0$$

(iii) Average queue length $E(m)$

$$E(m) = \sum_{n=C}^{\infty} (n - C) P_n$$

$$= \sum_{K=0}^{\infty} K \; P_{K+C} \quad \text{where } K = n - C$$

$$= \sum_{K=0}^{\infty} K . \frac{\rho^{C+K} P_0}{C!C^K}$$

$$= \frac{\rho^C}{C!} P_0 \sum_{K=0}^{\infty} K . \frac{\rho^K}{C^K} \quad \text{Put } \frac{\rho}{C} = x$$

$$= \frac{\rho^C}{C!} P_0 \sum_{K=0}^{\infty} K \, x^K$$

$$= \frac{\rho^C}{C!} \frac{\rho}{C} P_0 \sum_{K=1}^{\infty} K \, x^{K-1}$$

$$= \frac{\rho^{C+1}}{C.C!} P_0 [1 + 2x + 3x^2 + \ldots \text{ to } \infty]$$

$$= \frac{\rho^{C+1}}{C.C!} P_0 (1 - x)^{-2}$$

$$= \frac{\rho^{C+1}}{C.C!} P_0 \frac{1}{\left(1 - \frac{\rho}{C}\right)^2}$$

$$= \frac{\rho^{C+1}}{C.C!} P_0 \frac{C^2}{(C - \rho)^2}$$

$$= \frac{\rho^{C+1} P_0}{(C-1)!(C - \rho)^2}$$

$$= \frac{\rho^{C+1} \mu^2 . P_0}{(C-1)!(C\mu - \lambda)^2}$$

(iv) Average number of customers in the system

$$E(n) = E(m) + \frac{\lambda}{\mu}$$

$$\frac{\rho^{C+1} P_0}{(C-1)!(C - \rho)^2} + \rho$$

(v) Average waiting time of a customer in the queue

$$W_q = \frac{E(m)}{\lambda} = \frac{\rho^C \mu . P_0}{(C-1)!(C\mu - \lambda)^2}$$

(vi) Average waiting time in the system

$$W_S = W_q + \frac{1}{\mu}$$

(vii) Average number of idle servers

Number of servers remaining idle = C – (average number servers in service)

$$C - [E(n) - E(m)] = C - \rho = C - \frac{\lambda}{\mu}$$

Example 17.11 A supermarket has two sales girls. The service time for each customer is 4 minutes on the average and the arrival rate is 10 per hour. Find

(i) the probability that an arrival has to wait.
(ii) the expected percentage of idle time for each girl.
(iii) the expected waiting time of a customer in the system.

Solution

Arrival rate = 10 per hour

λ = 1/6 per min

Service time = 4 min

Service rate μ = 1/4 per min

ρ = 2/3 and $C = 2$

$$P_0 = \frac{1}{\left[1 + \rho + \frac{\rho^2}{2} \cdot \frac{2\mu}{2\mu - \lambda}\right]}$$

$$= \frac{1}{1 + \frac{2}{3} + \frac{1}{2} \cdot \frac{4}{9} \frac{\frac{1}{2}}{\left(\frac{1}{2} - \frac{1}{6}\right)}}$$

$$= \frac{1}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2}$$

$$P_0 = \frac{1}{2}$$

$$P_1 = \frac{\lambda}{\mu} P_0 = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

(i) Probability that an arrival has to wait

$= P(n \geq 2)$

$= 1 - P_0 - P_1$

$$= 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(ii) Expected number of idle girls

$$= C - \frac{\lambda}{\mu}$$

$= 2 - (2/3) = 4/3$

Also, expected number of idle girls

$= 2.P_0 + 1.P_1$

$$= 2\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) = \frac{4}{3}$$

$\therefore$ Probability that a girl is idle

$$= \frac{\text{Expected number of idle girls}}{\text{Total number of girls}}$$

$$= \frac{4/3}{2} = \frac{2}{3} = 0.67$$

$\therefore$ Percentage of idle time for each girl
= 67%

(iii) Average queue length = $E(m)$
Expected waiting time in the queue

$$W_q = \frac{E(m)}{\lambda}$$

$$= \frac{\rho^C \mu P_0}{(C-\mathbf{1})!(C\mu - \lambda)^2}$$

$$= \frac{(2/3)^2.1/4.1/2}{1!(1/3)^2} = \frac{1}{2}$$

Average waiting time in the system

$$W_S = W_q + \frac{1}{\mu} = \left(\frac{1}{2}\right) + 4 = 4.5 \text{ min}$$

Example 17.12 There are 4 counters at a check post for verifying the passport of the tourists. The arrival rate of the tourists is λ and the service rate has an average $\lambda/2$. Find the expected queue length at the check post.

Solution

Given that arrival rate = λ
Service rate = $\lambda/2$

$$\therefore \rho = \frac{\lambda}{\lambda/2} = 2 \quad C = 4$$

$$P_0 = \frac{1}{\sum_{n=0}^{C-1} \frac{\rho^n}{n!} + \frac{\rho^C}{C!}\left(\frac{C\mu}{C\mu - \lambda}\right)}$$

$$= \frac{1}{1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{6} + \frac{\rho^4}{4!}\left(\frac{4.\lambda/2}{4.\lambda/2 - \lambda}\right)}$$

$$= \frac{1}{1 + 2 + 2 + 4/3 + 16/24(2/1)} = 3/23$$

Expected queue length

$$E(m) = \frac{\rho^{C+1} P_0}{(C-1)(C-\rho)^2}$$

$$= \frac{2^5}{3!(4-2)^2} \cdot \frac{3}{23} = \frac{32}{6.4} \cdot \frac{3}{23} = \frac{4}{23}$$

17.8 MODEL 4—(MULTICHANNEL SYSTEM WITH LIMITED CAPACITY) *M/M/C/N*

In this model the maximum number of customers is limited to N and there are C service channels. Therefore we have

$$\lambda_n = \begin{cases} \lambda & 0 \le n \le N \\ 0 & n > N \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \le n \le C \\ C\mu & C \le n \le N \end{cases}$$

$$P_n = \begin{cases} \frac{\rho^n}{n!} P_0 & 1 \le n \le C \\ \frac{\rho^n P_0}{C!C^{n-C}} & C \le n \le N \\ 0 & n > N \end{cases}$$

17.8.1 Expression for P_0

We have $\sum_{n=0}^{N} P_n = \sum_{n=0}^{C-1} P_n + \sum_{n=C}^{N} P_n = 1$

(total probability)

i.e. $$\sum_{n!}^{C-1} \frac{\rho^n}{n!} P_0 + \sum_{n=C}^{N} \frac{\rho^n}{C!C^{n-C}} P_0 = 1$$

$$P_0 \left[\sum_{n=0}^{C-1} \frac{\rho^n}{n!} + \frac{\rho^C}{C!} \sum_{n=C}^{N} \frac{\rho^{n-C}}{C^{n-C}}\right] = 1$$

Now $$\sum_{n=C}^{N} \frac{\rho^{n-C}}{C^{n-C}} = \sum_{n=C}^{N} \left(\frac{\rho}{C}\right)^{n-C}$$

$$= 1 + \frac{\rho}{C} + \left(\frac{\rho}{C}\right)^2 + \ldots + \left(\frac{P}{C}\right)^{N-C}$$

$$= \frac{1-\left(\frac{\rho}{C}\right)^{N-C+1}}{1-\frac{\rho}{C}}$$

[Sum of $N-C+1$ terms of the GP]

$$\therefore \quad P_0 = \frac{1}{\sum_{n=0}^{C-1}\frac{\rho^n}{n!} + \frac{\rho^c}{C!}C!\left[\frac{1-\left(\frac{\rho}{C}\right)^{N-C+1}}{1-\frac{\rho}{C}}\right]}$$

$$\left(\frac{\rho}{C} \neq 1\right)$$

Here $\frac{\rho}{C}$ need not be less than 1.

If $\rho = C$ then $\frac{\rho}{C} = \mathbf{1}$

$$\sum_{n=C}^{N}\left(\frac{\rho}{C}\right)^{n-C} = N - C + 1$$

$$P_0 = \frac{1}{\sum_{n=0}^{C-1}\frac{\rho^n}{n!} + \frac{\rho^C}{C!}(N-C+1)}$$

17.8.2 Important Formula

(i) Expected number of customers in the queue

$$E(m) = \sum_{n=C}^{N}(n-C)P_n$$

$$= \sum_{n=C}^{N}(n-C)\frac{\rho^n P_0}{C!C^{n-C}}$$

$$= \frac{\rho^C P_0}{C!}\sum_{n=C}^{N}(n-C)\left(\frac{\rho}{C}\right)^{n-C}$$

$$= \frac{\rho^C P_0}{C!}\left(\frac{\rho}{C}\right)\sum_{n=C}^{N}(n-C)\left(\frac{\rho}{C}\right)^{n-C-1}$$

$$= \frac{\rho^C P_0}{C!}\left(\frac{\rho}{C}\right)\sum_{k=0}^{N-C}K\, x^{K-1}$$

$$\left(x = \frac{\rho}{C}, K = n - C\right)$$

$$= \frac{\rho^C P_0}{C!}\frac{\rho}{C}\frac{d}{dx}\left(\sum_{k=0}^{N-C}x^K\right)$$

$$= \frac{\rho^C P_0}{C!}\frac{\rho}{C}\frac{d}{dx}\left(\frac{1-x^{K+1}}{1-x}\right) \quad K = N-C$$

$$= \frac{\rho^C P_0}{C!}\cdot\frac{\rho}{C}\left[\frac{(1-x)(K+1)x^K(-1)-(1-x^{K+1})(-1)}{(1-x)^2}\right]$$

$$E(m) = \frac{\rho^C P_0}{C!}\frac{\rho}{C}\left[\frac{1-x^{K+1}-(1-x)(K+1)x^K}{(1-x)^2}\right]$$

where $x = \frac{\rho}{C}$ and $K = N - C$

(ii) Expected number of customers in the system

$$E(n) = E(m) + \frac{\lambda}{\mu}(1-P_N)$$

(iii) Expected waiting time in the system

$$W_S = \frac{E(n)}{\lambda(1-P_N)}$$

(iv) Expected waiting time in the queue

$$W_q = W_s - \frac{1}{\mu} = \frac{E(m)}{\lambda(1-P_N)}$$

Example 17.13 There are three stalls at an automobile inspection centre. The centre can accommodate seven cars at the maximum (three in service and four in waiting). On the average one car arrives every minute and the service time is 6 minutes. Find (i) the average number of cars in the system (ii) average waiting time in the system.

Solution Given

$$\lambda = 1 \text{ per min}$$
$$\mu = 1/6 \text{ per min}$$
$$C = 3$$
$$N = 7$$
$$\rho = \frac{\lambda}{\mu} = 6$$

$$P_0 = \frac{1}{\sum_{n=0}^{2} \frac{\rho^n}{n!} + \frac{\rho^3}{3!}\left[\frac{1-\left(\frac{\rho}{3}\right)^5}{1-\frac{\rho}{3}}\right]}$$

$$P_0 = \frac{1}{1+6+\frac{36}{2}+\frac{216}{6}\left(\frac{1-32}{1-2}\right)} = \frac{1}{1141}$$

(i) Expected number of cars in the queue

$$E(m) = \frac{\rho^C}{C!}.P_0.\frac{\rho}{C}\left[\frac{1-x^{K+1}-(1-x)(K+1)x^K}{(1-x)^2}\right]$$

$$x = \frac{\rho}{C} = 2 \quad K = N - C = 4$$

$$\therefore E(m) = \frac{6^3}{3!}.\frac{1}{1141}.2.\left[\frac{1-2^5-(1-2)(5)2^4}{(1-2)^2}\right]$$

$$= \frac{216}{6}.\frac{1}{1141}.2.\frac{(1-32+80)}{1}$$

$$= \frac{72.49}{1141} = 3.09$$

$$= 3.09 = 3 \text{ cars (nearly)}$$

$\therefore$ Expected number of cars in the system

$$E(n) = E(m) + \frac{\lambda}{\mu}(1-P_N)$$

$$= 3.09 + 6(1 - P_N)$$

Now $P_N = \frac{\rho^N P_0}{C!\,C^{N-C}} = \frac{6^7}{6.3^4} = \frac{1}{1141} = 0.5$

$\therefore \quad E(n) = 3.09 + 6\,(1 - 0.5) = 6$ cars (nearly)

(ii) Average waiting time in the system

$$W_S = \frac{E(n)}{\lambda(1-P_N)} = \frac{6}{1(1-0.5)} = 12 \text{ min}$$

EXERCISES

1. Arrivals at a telephone booth are poisson with an average inter-arrival time of 10 minutes. The length of a phone call is exponential with mean 3 minutes.
   (i) Find the probability that a person arriving at the booth will have to wait.
   (ii) Find the average length of the queue.
   (iii) Find the total time a person spends in the booth.
2. In a single server queueing system the arrival rate is 25 per hour. What must be the service rate in order that a customer has to wait for not more than 10 minutes to enter into service.
3. In a repair shop having a single mechanic, customers arrive at the rate of 4 per hour. It takes the mechanic 6 minutes on the average to inspect one item. Find the
   (i) proportion of time the mechanic is idle.
   (ii) probability that there is at least one customer.
   (iii) average number of customers in the shop.
   (iv) average time spent by a customer in the shop.
4. Customers arrive at a box office window with a single server, at a mean rate of 30 per hour.

The time required to serve a customer is 90 seconds on the average. Find the average waiting time of a customer and the average number of customers in the system.

5. Customers arrive at a bank cash counter in a poisson manner at an average rate of 30 per hour. The cashier takes on an average 1.5 minutes for each cash cheque. Calculate
   (i) the percentage of time the cashier is busy.
   (ii) the average waiting time of a customer in the bank.
6. Trucks arrive at a factory for collecting furnished goods, at the rate of 10 per hour and the loading rate is 15 per hour. Transporters complain that their trucks have to wait for nearly 2 hours at the factory. Examine whether their complaint is justified. What is the probability that the loaders are idle.
7. Customers arrive at a one — window drive - in bank at the rate of 10 per hour. Average service time is 5 minutes. The space in front of the window can accommodate a maximum of 3 cars including the one under service.
   (i) Find the probability that an arriving car can directly drive to the space in front of the window for service.
   (ii) What is the probability that an arriving can will have to wait outside the indicated space?
   (iii) How long is an arriving customer expected to wait before starting service?
8. Letters arrive at the rate of 5 per hour, to a typist's table during an 8 hour day. The typing rate is on the average 8 letters per hour. If the time of the typist is valued at Rs 10 per hour determine
   (i) equipment utilization
   (ii) the period of time, an arriving letter has to wait
   (iii) average cost due to waiting and operating the typewriter.
9. In a public telephone both the arrivals are on the average 15 per hour. A call on the average takes 3 minutes. If there is only one phone in the booth. Find
   (i) the queue length at any time.
   (ii) the proportion of time, the booth is idle.
10. In a petrol pump cars arrive at an average rate of 10 per hour. If there is only one serving unit and the service time has an average of 3 minutes, find
    (i) the average number of cars in the system.
    (ii) the average waiting time in the queue.
    (iii) the probability that there are two cars in the system.
11. At a one-man saloon customers arrive at the rate of 5 per hour and the hair cutting time is on the average 10 minutes per customer. Find
    (i) the average number of customers in the saloon.
    (ii) the average number of customers waiting.
    (iii) the probability that a new arrival does not wait for service.
12. A single server coffee shop has space to accommodate only 10 customers. If a new arrival finds that the shop is full he goes to another shop. The arrival rate is 10 per hour and the service time is 5 minutes per customer Find P_0 and P_n.
13. Customers arrive at a photocopier shop at an average rate of 5 per hour and they are served at 6 per hour. There are only 5 seats available for the customers. Find the average number of customers and the average time a customer spends at the shop.
14. An insurance company has three claim adjusters. People with claims arrive at the office at the rate of 20 per 8 hour day. The mean service time is 40 minutes per claimant.
    (i) How many hours per week an adjuster is busy?
    (ii) How much time a claimant spends in the office?
15. Given an average arrival rate of 20 per hour there are two options for a customer: A single channel with service rate 22 customers per hour or a two-server channel with service rate of 11 customers per hour. Determine which is a better option.
16. A tax consulting firm has 4 service counters On the average 80 persons arrive during a

day of 8 hours. The average service time per person is 20 minutes. Find
(i) the average number of customers in the system.
(ii) the average time a customer spends in the system.
(iii) the expected number of idle servers.
(iv) the number of hours a server spends with customers during one week.

17. A barber shop has two barbers and three chairs for waiting customers. If a customer finds that there is no vacant chair he will leave the shop. Customers arrive at the rate of 5 per hour and the service time is 15 minutes per customer. Find the expected number of customers in the shop. Also find the percentage of idle time for each server.

18. In a supermarket the average arrival rate of a customer is 10 per 30 minutes. On the average it takes 2.5 minutes to serve a customer. Find the probability that there are more than 6 customers in the queue.

19. In a telephone booth people arrive at the rate of 15 per hour. The duration of a call is 3 minutes on the average. Find
(i) The expected number of customers in the booth.
(ii) The proportion of time, the booth is idle.

20. At a telephone booth, the arrivals follow poison distribution with an inter arrival time of 12 min. The duration of a call is 4 minutes on the average
(i) What is the probability that a fresh arrival will not have to wait for the phone?
(ii) What is the probability that an arrival has to wait for more than 10 min?
(iii) What is the average length of the queue?

21. People arrive at a theatre ticket booth at the rate of 25 per hour. Service time for a customer has an average of 2 min. Find the following:
(i) Mean number in the waiting line
(ii) Mean waiting time
(iii) Utilisation factor

22. In a single server bank, working from 7 am to 1 pm, the average number of customers is 54 per day. Average service time is 5 min per customer. Find the
(i) Average number of customers in the system
(ii) Average number in the queue
(iii) Average waiting time.

23. In a single server tools store operators arrive at the rate of 10 per hour. Service time is 3 minutes per operator. The rate of production is 100 units per day of 8 hours. If the cost of product is Rs 200, find the loss due to waiting.

24. In a telephone exchange there are two operators. Calls arrive at the rate of 15 per hour and the average length of a call is 5 minutes. Find the
(i) probability that a customer has to wait.
(ii) average waiting time of a customer.

25. A petrol station has two pumps. Cars arrive at the rate of 10 per hour and the average service time is 4 minutes per car. Find
(i) the probability that a customer has to wait.
(ii) the percentage of idle time.

26. There are four counters on the frontier of a country to check passports and other documents of tourists. The arrival rate is λ and the service rate is $\lambda/2$. What is the average queue length at each counter?

27. The mean arrival rate at a railway reservation centre is 24 per hour and there are 3 counters
(i) Find the probability that there are no customers if the mean service rate is 10/hr.
(ii) the average time spent by a customer in the system if there are 5 counters

28. In a tax consulting company there are four service counters. On the average 80 persons arrive during an 8 hour service day. The average service time is 20 minutes per customer. Find the
(i) average number of customers in the system.

(ii) average waiting time of a customer in the system.

(iii) number of hours, each consultant spends with the customers, per week.

ANSWERS

1. (i) 0.3 (ii) 9/70 (iii) 30/7
2. $\mu \geq \frac{1}{2}$
3. (i) 3/5 (ii) 21/25 (iii) 2/3 (iv) 10 min
4. W_q = 9/2min, $E(n)$ = 3
5. (i) 75% (ii) W_s = 6min
6. W_s = 12 min. Complaint is not justified. Probability that the loaders are idle = 1/3
7. (i) 91/216 (ii) 125/216 (iii) 25 min
8. (i) 62.5% of time (ii) 12.5 min (iii) Rs 133.30
9. (i) $L_q = 2$ (ii) $\frac{1}{4}$
10. (i) 1 (ii) 3 min (iii) 1/8
11. (i) 5 (ii) 4 (iii) 1/6
12. $P_0 = 1/6$, $P_n = 1/6(5/6)^n$
13. $E(n) = 5$, $W_s = 6$
14. (i) 22 hrs during a 5-day week (ii) 50min
15. For a single channel, W_q = 0.45 hr
    For two server channel W_q = 0.43 hrs
    Hence two server channel is better.
16. (i) 7 (ii) 40 min (iii) 1 (iv) 33 hrs
17. $E(n)$ = 3, Percentage of idle time = 37%
18. Required probability = 0.3348
19. Average number of customers = 4
    Proportion of idle time = 1/4
20. (i) 0.67 (ii) 0.63 (iii) 1.5
21. (i) 4 (ii) 9.6 min (iii) 0.833
22. (i) 3 (ii) 2.25 (iii) 20 min
23. Average waiting time is 2/5 hour. Loss due to waiting
    = (2 / 5) (100 / 8) 200 = Rs 1000
24. (i) 0.48 (ii) 3.2 min
25. (i) 0.167 (ii) 67%
26. L_q = 4/23
27. (i) P_0 = 0.019 (ii) 6 min
28. (i) 6.61 (ii) 0.66 hours
    (iii) 33.3 hours

18

Inventory Models

CONCEPT REVIEW

18.1 INTRODUCTION

Inventory is the stock of raw materials and goods required for production in a factory or finished goods for sales. If an order for a product is received we should have sufficient stock of materials required for manufacturing the item in order to avoid delay in production and supply. Also there should not be over stock of materials and goods as it involves storage cost and wastage in storing. Therefore inventory control is essential to promote business. Maintaining inventory helps to run the business smoothly and efficiently and also to provide adequate service to the customer. Inventory control is very useful to reduce the cost of transportation and storage. The basic questions are

(i) When should an order for materials be placed?

(ii) How much to be produced during each period or how much to be ordered each time?

Inventory control helps us to solve these problems to a great extent.

Also inventory control is necessary to ensure adequate and prompt supply of items to the customers and avoid shortage of stock and consequent loss.

18.2 COSTS INVOLVED IN INVENTORY CONTROL

Various costs incurred in the maintenance of inventory are as follows:

(i) **Holding cost (Storage cost)** This is the cost associated with holding (carrying) the inventory in the godown. It includes rent for godown, interest for the money locked up, insurance premium, salaries for the watchmen, deterioration (damage) cost etc. Carrying cost is calculated in two ways.

(a) Carrying cost = (Holding cost per unit for unit time) × (Average number of units in stock)

(b) Carrying cost = (Cost of storing stock of one rupee worth) × (Rupee value of the units stored)

Carrying cost (storage cost) is denoted by C_1.

(ii) **Set-up cost (Ordering cost)** Ordering cost is associated with the cost of placing orders for procurement of materials or finished

goods from suppliers. In the case of production, setup cost includes the cost involved in setting up the machines for production run. Ordering cost includes the cost of stationery, postage, telephones, travelling expenses, handling of materials etc. Ordering cost = (Cost per order or per set up) × (Number of orders or production runs)
It is denoted by C_S

(iii) **Production cost (Purchase cost)**
Production cost per unit item depends upon the length of production runs. For long production runs it is lower. Also purchase cost is less for orders of large quantity due to quantity discounts or price breaks.

(iv) **Shortage cost (Stock out cost)** If the inventory on hand is not sufficient to meet with the demand of materials or finished goods, then it results in shortage of supply. There are two cases: (i) Orders are kept pending and as soon as stock is replenished back orders are supplied. (ii) Unfilled demand (order) results in lost sales and cancellation of order. We consider only the first case. Shortage cost per unit, per unit time of delay is denoted by C_2.

(v) **Total inventory cost** The total inventory cost is denoted by C.
C = ordering cost + Holding cost + Shortage cost. If price discounts are available on purchase cost then purchase cost is also to be included in the total cost.

18.3 CHARACTERISTICS OF INVENTORY SYSTEM

(i) **Demand** The nature of demand must be known for inventory control. Demand may be deterministic or probabilistic. In the deterministic case the quantity of material required over a period of time is certain and known before hand. The demand may be uniform in many cases. But in the probabilistic case the nature of demand is uncertain and is given by some probability distribution.

(ii) **Order cycle** The period of time between two consecutive placement of orders is called order cycle.

(iii) **Lead time** When an order is placed, the supplier takes some time to supply the item. Lead time is the time between the placement of an order and the receipt of the material.

(iv) **Replenishment of stock** Replenishment of stock may occur instantaneously without lead time or uniform replenishment may occur when the goods are manufactured by the factory itself.

(v) **Planning horizon (period)** The duration of time over which a particular inventory level will be maintained is called planning horizon.

18.4 DETERMINISTIC INVENTORY MODELS WITH NO SHORTAGE

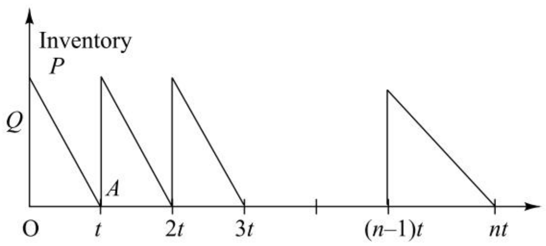
18.4.1 Economic Lot Size Model with Constant Demand

In this model demand is assumed to be constant or completely pre-determined. Our problem is to determine the quantity to be ordered in the most economical manner. This quantity is called Economic Order Quantity (EOQ). We make the following assumptions:

(i) Demand is known and is uniform. Q denotes the lot size in each production run.
(ii) D denotes the total number of units produced or supplied per unit period.
(iii) Shortages are not allowed. As soon as the level of inventory reaches zero, the inventory is replenished.
(iv) Production or supply of materials is instantaneous. In other words, replenishment of inventory is instantaneous without delay.
(v) Lead time is zero.
(vi) Set up cost per production run is C_S.
(vii) Holding cost (storage cost) is C_1. If the holding cost is expressed as a percentage I of the unit cost C of the commodity then $C_1 = IC$.

Formula for EOQ (Wilson Harris Formula)

Let us assume that the total period (say one year) is divided into n parts each of length t. Let Q be the quantity produced during each interval (run). Therefore the total quantity produced during n runs is $D = nQ$. Also $nt = 1$. The following figure illustrates the model.



If the demand rate is R then a quantity $Q = Rt$ must be produced in each run. The stock in a small time interval dt is $Rt\ dt$. The total stock during the period t is given by

$$\int_0^t Rt\ dt = R\frac{t^2}{2} = \frac{1}{2}Rt.t = \frac{1}{2}Q.t$$

$$\therefore \quad \text{Average stock} = \frac{1/2\, Qt}{t} = \frac{1}{2}Q.$$

$$\left[\text{Area of the triangle } OAP = \frac{1}{2}Qt\right.$$

$$\therefore \text{ Total stock} = \frac{1}{2}Qt$$

$$\left.\therefore \text{ Average stock} = 1/2\ Q\right]$$

$$\text{Holding cost per run} = \frac{1}{2}Q.C_1.t$$

$$= \frac{1}{2}Q.C_1.\frac{1}{n}$$

$$\text{Annual holding cost} = \frac{1}{2}QC_1\frac{1}{n}.n$$

(since there are n runs)

$$= \frac{1}{2}QC_1$$

$$\text{Set-up cost per run} = C_S$$

$$\text{Annual set-up cost} = n\ .C_S$$

$$= \frac{D}{Q}C_S \qquad (\because D = nQ)$$

Thus the total annual inventory cost is given by

$$C_A = \text{Total holding cost} + \text{Total set-up cost}$$

$$= \frac{1}{2}QC_1 + \frac{D}{Q}C_S$$

We want to determine Q such that C_A is minimum. By the principle of maxima minima the condition is $\frac{dC_A}{dQ} = 0$ and $\frac{d^2C_A}{dQ^2} > 0$

$$\text{Now} \qquad C_A = \frac{1}{2}QC_1 + \frac{D}{Q}C_S$$

$$\frac{dC_A}{dQ} = \frac{1}{2}C_1 + \left(-\frac{D}{Q^2}\right)C_S$$

$$\frac{C_1}{2} - \frac{DC_S}{Q^2} \qquad \left(\text{obviously } \frac{d^2C_A}{dQ^2} > 0\right)$$

$$\frac{dC_A}{dQ} = 0 \Rightarrow C_1Q^2 = 2DC_S$$

$$\therefore \qquad Q = \sqrt{\frac{2DC_S}{C_1}}$$

The optimum quantity is denoted by Q^* which is called Economic Order Quantity (EOQ)

$$\therefore \qquad \text{EOQ} = Q^* = \sqrt{\frac{2DC_S}{C_1}}$$

This is called the Wilson Harris formula for EOQ.

Note: If the holding cost is expressed in terms of the value of the stock held then $C_1 = IC$ and the value of $Q^* = Q^*C$.

$$= \sqrt{\frac{2DC_S}{IC}}.C = \sqrt{\frac{2DC_SC}{I}}$$

The optimum annual inventory cost is

$$C_A^* = \frac{1}{2}Q^*C_1 + \frac{D}{Q^*}C_S$$

$$= \frac{1}{2}\sqrt{\frac{2DC_S}{C_1}}.C_1 + \frac{D}{\sqrt{\frac{2DC_S}{C_1}}}C_S$$

$$= \frac{1}{2}\sqrt{2DC_SC_1} + \frac{1}{2}\sqrt{2DC_SC_1}$$

$$= \sqrt{2DC_S C_1}$$

Optimal period of one run (order cycle) is

$$t^* = \frac{Q^*}{D}$$

Number of runs per year is $n = \dfrac{D}{Q^*}$

Example 18.1 A factory requires 3600 kg of raw material for producing an item per year. The cost of placing an order is Rs 36 and the holding cost of the stock is Rs 2.50 per kg per year. Determine the EOQ.

Solution We are given that

D = 3600 kg per year

C_S = Rs 36 per order

C_1 = Rs 2.50 per kg per year

$\therefore$ Now EOQ $= Q^* = \sqrt{\dfrac{2DC_S}{C_1}}$

$$= \sqrt{\frac{2 \times 3600 \times 36}{2.5}} = 322 \text{ kg}$$

Period of one run $t^* = \dfrac{Q^*}{D} = \dfrac{322}{3600}$

= 0.089 year

= 32 days

Number of runs per year $n = \dfrac{D}{Q^*} = 11.4$

Example 18.2 The annual demand for an item is 3200 units. The unit cost is Rs 6. The inventory carrying cost is 25% per annum per unit. The cost of one procurement is Rs 150. Determine

(i) EOQ
(ii) Number of orders per year
(iii) Time between two consecutive orders
(iv) Total annual cost

Solution Given $D = 3200$, $C = 6$, $I = 0.25$

$C_1 = IC = 1.5$

$C_S = 150$

(i) EOQ $= Q^* = \sqrt{\dfrac{2DC_S}{C_1}} = \sqrt{\dfrac{2 \times 3200 \times 150}{1.5}}$

= 800 units

(ii) Number of orders $n = \dfrac{D}{Q^*} = \dfrac{3200}{800} = 4$

(iii) Time between two consecutive orders

$$= t^* = \frac{Q^*}{D} = \frac{800}{3200} = \frac{1}{4} = 3 \text{ months}$$

(iv) Total annual cost

= material cost + inventory cost

$= D \times C + \sqrt{2DC_S C_1}$

$= 3200 \times 6 + \sqrt{2 \times 3200 \times 150 \times 1.5}$

= 19200 + 1200

= Rs 20400

Example 18.3 A manufacturer has to supply 12000 units of a product per year to his customer. Shortages are not permitted and there is no lead time. The inventory holding cost is Rs 0.20 per unit per month and the set-up cost per run is Rs 350. Determine

(i) the economic lot size
(ii) the period of one run
(iii) the minimum annual inventory cost

Solution We have D = 12000 units per year

= 1000 units per month

C_S = Rs 350

C_1 = Rs 0.20 per month

(i) $Q^* = \sqrt{\dfrac{2DC_S}{C_1}} = \sqrt{\dfrac{2 \times 1000 \times 350}{0.20}}$

= 1870 units

(ii) Period of one run $t^* = \dfrac{Q^*}{D} = \dfrac{1870}{1000}$

= 1.87 months

= 56 days

(iii) Annual inventory cost

$$C_A = \sqrt{2DC_S C_1} \times 12$$

$$= \sqrt{2 \times 1000 \times 350 \times (0.2)} \times 12$$

= 374 × 12 = Rs 4490

Example 18.4 Annual requirement of machine parts for a company is 9000 units and the cost of one unit is Rs 20. It costs Rs 15 for placing an order and the storage cost is 15% of the value per year. The company is at present placing orders every month. How much would it save if it places order as per the optimal period?

Solution

Given that annual requirement $D = 9000$

Ordering cost C_S = Rs 15

Storage cost $C_1 = (15/100) \times 20$ = Rs 3 per year

$$\therefore \text{ EOQ} = Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2 \times 9000 \times 15}{3}}$$

$$= 300 \text{ units}$$

$$t^* = \frac{Q^*}{D} = \frac{300}{9000} \text{ year} = 12 \text{ days}$$

Annual inventory cost $C_A = \sqrt{2DC_SC_1}$

$$= \sqrt{2 \times 9000 \times 15 \times 3} = \text{Rs } 900$$

$\therefore$ The optimal period of one run is 12 days and the minimum annual inventory cost is Rs 900.

The company is placing orders every month.

$\therefore$ t = 30 days

Ordering cost per year = $12 \times 15 = 180$

Monthly requirement = 9000/12

= 750 units

Average inventory = 1/2(750) = 375

Storage cost = 375×3 = Rs 1125

$\therefore$ Annual inventory cost = 1125 + 180

= Rs 1305

By ordering for 300 parts after every 12 days the annual cost is Rs 900. Thus there is a saving of Rs 405 per year.

Example 18.5 The demand for an item is 100 units per day. For placing an order a cost of Rs 400 is incurred. Holding cost is Rs 0.08 per day. If the lead time is 13 days determine the EOQ and the re order point.

Solution Given D = Rs 100 per day

C_S = Rs 400

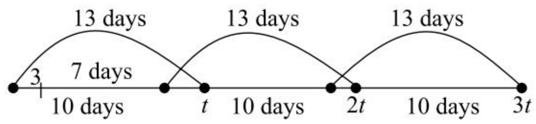
C_1 = 0.08 per day

$$\therefore \ Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2 \times 100 \times 400}{0.08}}$$

$$= 1000 \text{ units}$$

Length of a cycle is $t^* = \dfrac{Q^*}{D}$ = 1000/100

= 10 days



Lead time is 13 days. Hence an order is to be placed three days before the stock exhausts. There must be sufficient stock for 3 days.

$\therefore$ Re order level is $3 \times 100 = 300$ units

Example 18.6 A company has to supply 100 units of an item to the customers every week. The item is purchased from the supplier at Rs 60 per unit. The cost of ordering and procurement from the supplier is Rs 150 per order. The storage cost is 15% of the cost per year. Find the economic lot size and the optimum cost per week.

Solution

We have D = 100 units per week

C_S = Rs 150

C_1 = 15% per year of the cost

= $15/100 \times 60$

= Rs 9 per unit per year (52 weeks)

= Rs 9/52 per unit per week

$$\text{(i)} \ \therefore \ Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2 \times 100 \times 150}{9/52}}$$

= 416 units

(ii) Optimum total cost

= Purchase cost + Inventory cost

$= 100 \times 60 + \sqrt{2DC_SC_1}$

$= 6000 + \sqrt{2 \times 100 \times 150 \times 9/52}$

= 6000 + 72 = Rs 6072

Example 18.7 The average daily requirement of an electrical appliance is 120 units and the firm has 250 working days a year. The manufacturing cost is Rs 0.50 per part. Insurance, tax, interest etc., come to 20% of the unit cost. The set-up cost is Rs 50 per run. Determine the EOQ.

Solution

Here $D = 120 \times 250 = 30000$ units

$C_S = 50$

$C_1 = 20\%$ of the cost per part

$= 20/100 \times 0.50 = 0.1$

$$\text{EOQ} = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2 \times 30000 \times 50}{0.1}}$$

$$= 5477 \text{ parts}$$

Example 18.8 The demand for a commodity is 600 units per year. The ordering cost is Rs 80. The cost of the item is Rs 3 and the inventory holding cost is 20% of the cost per year. If the lead time is one year find

(i) The EOQ
(ii) The re-order point
(iii) Te minimum average annual cost

Solution

$D = 600$ per year

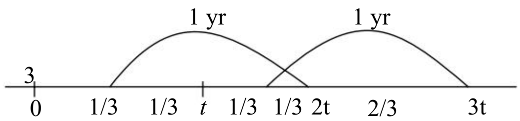
$C_S =$ Rs 80

$C_1 = IC = 20/100 \times 3 = 0.60$ per year

(i) $$Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2.600.80}{0.60}} = 400 \text{ units}$$

(ii) $$t^* = \frac{Q^*}{D} = 400/600 = 2/3 \text{ year}$$

Lead time = 1 year



Order should be placed after 1/3 year when the stock level is $1/3 \times 600 = 200$ units

$\therefore$ A quantity of 400 units is to be ordered when the inventory level is 200 units

(iii) Minimum annual inventory cost

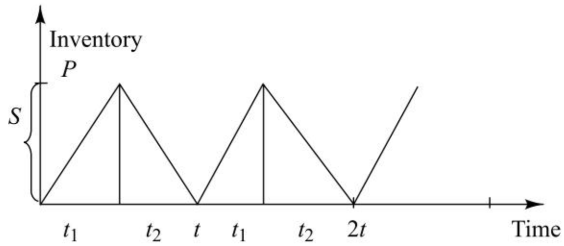
$$= \sqrt{2DC_SC_1}$$

$$= \sqrt{2.600.80(0.60)} = \text{Rs } 240$$

18.4.2 Manufacturing Model

In this model the assumptions are same as in model 1 except that the replenishment rate (manufacturing rate) is finite, say k units per unit time and demand rate (consumption rate) is r units per unit time ($k > r$). Each production run of length t is divided into parts t_1 and t_2 such that

(i) the inventory is building up at a constant rate of $(k - r)$ units per unit time during t_1
(ii) there is no production during t_2 and the inventory is decreasing at the rate of r units per unit time $(t_1 + t_2 = t)$.



This model is illustrated in the above figure. At the end of time t_1 let the inventory level be S. $\therefore$ $S = (k - r)\, t_1$. This inventory is used during the period t_2 at the rate of r $\therefore$ $S = rt_2$.

$$\therefore \quad t_1 = \frac{S}{k - r} \text{ and } t_2 = \frac{S}{r}$$

Let Q be the quantity produced per run, then

$$Q = k\, t_1$$

$$\therefore \quad S = (k - r)t_1 = kt_1 - rt_1 = Q - rt_1$$

$$\therefore \quad rt_1 = Q - S \Rightarrow t_1 = \frac{Q - S}{r}$$

$$\text{Also} \quad t_1 = \frac{Q}{k}$$

$$\therefore \quad \frac{Q}{k} = \frac{Q - S}{r} \Rightarrow Q - S = \frac{Qr}{K}$$

$$\therefore \quad S = Q - r\,t_1 = Q - r\frac{Q}{k}$$

$$= \left(\frac{k - r}{k}\right)Q$$

Now, total inventory during time t

$$= 1/2(t_1 + t_2)S = 1/2\, St$$

$$\text{Average inventory} = \frac{1}{2}\frac{St}{t} = \frac{S}{2}$$

$$\text{Inventory holding cost} = \frac{1}{2}\, SC_1$$

$$= \frac{1}{2}(k-r)\frac{Q}{k}C_1$$

Set-up cost $= n.C_s = \dfrac{C_S}{t}$

Now $\quad t = t_1 + t_2 = \dfrac{S}{k-r} + \dfrac{S}{r} = \dfrac{Sk}{r(k-r)}$

$$= \frac{(k-r)}{k}Q.\frac{k}{r(k-r)} = \frac{Q}{r}$$

Thus total inventory cost

$$C_A = \frac{1}{2}\left(\frac{k-r}{k}\right)QC_1 + \frac{r}{Q}C_S$$

For C_A to be minimum $\dfrac{dC_A}{dQ} = 0$ and $\dfrac{d^2C_A}{dQ^2} > 0$

Now $\quad \dfrac{dC_A}{dQ} = \dfrac{1}{2}\left(\dfrac{k-r}{k}\right)C_1 - \dfrac{r}{Q^2}C_S = 0$

$$\Rightarrow \frac{r}{Q^2}C_S = \frac{1}{2}\left(\frac{k-r}{k}\right)C_1$$

$$Q^* = \sqrt{2\frac{C_S}{C_1}\left(\frac{kr}{k-r}\right)}$$

Obviously $\dfrac{d^2C_A}{dQ^2} > 0 \quad \therefore \; C_A$ minimum for Q^*

$$\therefore \quad \text{EOQ} = \sqrt{\frac{2C_S}{C_1}\left(\frac{kr}{k-r}\right)}$$

$\therefore$ Optimal period of a run $t^* = \dfrac{Q^*}{r}$

$$= \sqrt{\frac{2C_S}{rC_1}\left(\frac{k}{k-r}\right)}$$

Total annual inventory cost

$$C_A^* = \frac{1}{2}\left(\frac{k-r}{k}\right)Q^*C_1 + \frac{r}{Q^*}C_S$$

$$= \frac{1}{2}\left(\frac{k-r}{k}\right)\sqrt{\frac{2C_S}{C_1}\left(\frac{kr}{k-r}\right)}\,C_1$$

$$+ r.\sqrt{\frac{C_1(k-r)}{2C_1kr}}\,C_S$$

$$= \sqrt{2C_SC_1r\left(\frac{k-r}{k}\right)}$$

No. of runs per year $n = \dfrac{1}{t^*} = \sqrt{\dfrac{rC_1}{2C_S}\left(\dfrac{k-r}{k}\right)}$

Example 18.9 A contractor has to supply 10000 bearings per day to an automobile manufacturer. He can produce 25000 bearings per day. The holding cost is Rs 2 per year and the set-up cost is Rs 180. How frequently should the production run be made?

Solution

We are given $r = 10000$, $k = 25000$

$C_S = 180$, $\quad C_S =$ Rs 2 per year

$= 0.0055$ per day

$$\text{EOQ} = Q^* = \sqrt{\frac{2C_S}{rC_1}\left(\frac{kr}{k-r}\right)}$$

$$= \sqrt{\frac{2.180}{0.0055}.\frac{25000.10000}{15000}}$$

$= 33029$ units

Period of one run $t^* = \dfrac{Q^*}{r} = \dfrac{33029}{10000} = 3.3$ days

Example 18.10 A factory manufactures a product at the rate of 100 units per day and the daily demand is 40 units. The cost of one unit is Rs 5. Holding cost is 25% of the value per unit per year. Setup cost is Rs 150 per run. Determine the economic lot size and the number of runs per year. (Take 250 days for one year.)

Solution

Hence $\quad k = 100$ per day

$r = 40$ per day

$C_1 = 5 \times 25/100 = 1.25$ per year

$= \dfrac{1.25}{250} = 0.005$ per day

$C_S =$ Rs 150

$$Q^* = \sqrt{\frac{2C_S}{rC_1}\left(\frac{k}{k-r}\right)}$$

$$= \sqrt{\frac{2.150}{0.005}\left(\frac{100.40}{60}\right)}$$

$= 2000$ units

$$t^* = \frac{Q^*}{r} = \frac{2000}{40} = 50 \text{ days}$$

$$\text{Number of runs} = \frac{250}{50} = 5$$

$$t_1 = \frac{Q^*}{k} = \frac{2000}{100} = 20 \text{ days}$$

(production period)

$\therefore \ t_2 = 30$ days

Example 18.11 An item is produced at the rate of 50 units per day. The demand is at the rate of 25 units per day. The set-up cost is Rs 100 and the holding cost is Rs 0.01 per unit per day. Find the EOQ and the minimum annual inventory cost. After how many days should production be stopped during each run?

Solution Given $k = 50$ units per day

$r = 25$ per day

$C_s =$ Rs 100

$C_1 = 0.01$ per year

(i) $$\text{EOQ} = Q^* = \sqrt{\frac{2C_S}{C_1}\left(\frac{kr}{k-r}\right)}$$

$$= \sqrt{\frac{2 \times 100}{0.01} \times \left(\frac{50 \times 25}{50-25}\right)} = 1000 \text{ units}$$

$$\text{Inventory cost} = \sqrt{2C_S C_1 r\left(\frac{k-r}{k}\right)}$$

$$= \sqrt{2.100.(0.01)25.\frac{25}{50}}$$

$=$ Rs 5 per day

(ii) Annual inventory cost $C_A^* = 5 \times 365$

$=$ Rs 1825

(iii) $$t^* = \frac{Q^*}{r} = \frac{1000}{25} = 40 \text{ days}$$

$$t_1 = \frac{Q^*}{k} = \frac{1000}{50} = 20 \text{ days}$$

$\therefore \ t_2 = 20$ days

During each run of 40 days, production continues for 20 days and there would be no production for the remaining 20 days.

18.4.3 Model with Price Breaks (Quantity Discounts)

In the previous models the cost of the material (purchase cost) was not affected by the order size since it remains constant. Often we find that quantity discounts are offered to encourage the buyers to purchase more units of an item. There may be quantity discount or price breaks. In such situations the variable purchase cost is also included in determining the EOQ. Let us assume that the cost per unit commodity is p_1 if $Q < q$ and p_2 if $Q \geq q$ where $p_1 > p_2$. Let the ordering cost be C_S and the holding cost be C_1. If the demand is D then the number of orders n is given by $n = D/Q$

The total annual cost is

$$TC_1 = Dp_1 + \frac{D}{Q}C_S + \frac{1}{2}QC_1 \quad \text{for } Q < q$$

$$TC_2 = Dp_2 + \frac{D}{Q}C_S + \frac{1}{2}QC_1 \quad \text{for } Q \geq q$$

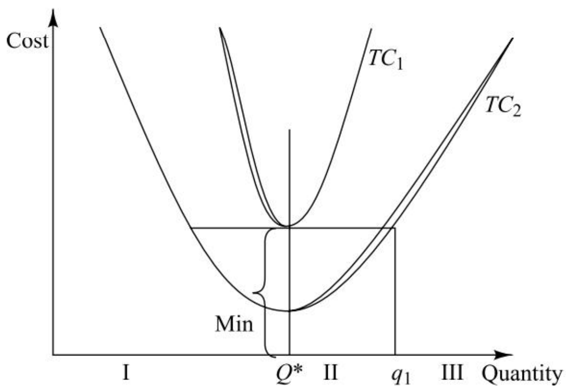
Neglecting the price break, we get the EOQ

$$Q^* = \sqrt{\frac{2DC_S}{C_1}} \qquad \text{...(1)}$$

Choose the quantity q_1 such that

$$TC_1(Q^*) = TC_2(q_1) \qquad \text{...(2)}$$

The curves representing TC_1 and TC_2 are given below:



The optimum quantity Q^o depends on the value of q (price break point). There are three zones. $0 < q < Q^*$ (I zone); $Q^* < q < q_1$ (II zone) $q \geq q_1$ (III zone)

Equation (1) gives the value of Q^* and using (2) we can find q_1. We can determine Q^o as follows

$$\begin{aligned} Q^o &= Q^* \text{ if } 0 < q < Q^* && \text{(I zone)} \\ &= q \text{ if } Q^* < q < q_1 && \text{(II zone)} \\ &= Q^* \text{ if } q \geq q_1 && \text{(III zone)} \end{aligned}$$

Example 18.12 The cost of a commodity is Rs 2 for an order less than 15 and Re 1 for orders greater than or equal to 15. Holding cost per unit is Re 1 and setup cost is Rs 10 per run and the requirement is 5 units per day. Find the EOQ.

Solution Given $p_1 = 2, \quad p_2 = 1$

$D = 5, \quad q = 15$

$C_S = 10, \quad C_1 = 1$

Ignoring the price break we have

$$Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2.5.10}{1}} = 10 \text{ units}$$

Since $q = 15$, we find that $Q^* < q$

To find q_1

$$TC_1(Q^*) = Dp_1 + \frac{D}{Q^*}C_S + \frac{Q^*}{2}C_1$$

$$= 10 + \frac{5 \times 10}{10} + \frac{10}{2}.1 = 20$$

$$TC_2(Q_1) = Dp_2 + \frac{D}{q_1}C_S + \frac{q_1}{2}.1$$

$$= 5 + \frac{5.10}{q_1} + \frac{q_1}{2}$$

Take $TC_1(Q^*) = TC_2(q_1)$

$$20 = 5 + \frac{50}{q_1} + \frac{q_1}{2}$$

$$30q_1 = 100 + q_1^2$$

$$q_1^2 - 30q_1 + 100 = 0 \Rightarrow q_1 = 26.18 \text{ or } 3.82$$

By definition q_1 is taken as the larger value

$\therefore \quad q_1 = 26.18$

$Q^* = 10 \quad q = 15 \quad q_1 = 26.18$

$\therefore \quad Q^* < q < q_1$ (II zone)

$Q^o = q = 15$

$\therefore \quad$ EOQ = 15

Example 18.13 The price break of a product is

$$\text{Unit cost} = \begin{cases} \text{Rs } 10 & q < 500 \\ \text{Rs } 9.25 & q \geq 500 \end{cases}$$

Monthly demand is 200 units. Storage cost is 2% of the unit cost per month. Find the EOQ if

(i) ordering cost is Rs 350

(ii) ordering cost is Rs 100

Solution

(i) Ordering cost $C_S =$ Rs 350

$C_1 = 10(0.02) = 0.2$

$D = 200$

$$Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2.200.350}{0.2}} = 837$$

$q = 500 \quad q < Q^*$

Hence $Q^o = Q^* = 837$

(ii) $C_S = 100 \quad C_1 = 0.2 \quad D = 200 \quad q = 500$

$p_1 = 10 \quad p_1 = 9.25$

$$Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2.200.100}{0.2}} = 447$$

$\therefore \; Q^* < q$

To find q_1

$$TC_1(Q^*) = Dp_1 + \frac{D}{Q^*}C_S + \frac{Q^*}{2}C_1$$

$$= 200 \times 10 + \frac{200}{447} \times 100 + \frac{447}{2} \times (0.2)$$

$$= 2000 + 44.7 + 44.7 = 2089$$

$$TC_2(q_1) = Dp_2 \frac{D}{q_1}C_S + \frac{q_1}{2}.C_1$$

$$= 200(9.25) + \frac{200}{q_1}.100 + \frac{q_1}{2}.(0.185)$$

$[C_1 = (9.25)(0.2) = 0.185]$

$$= 1850 + \frac{20000}{q_1} + q_1(0.09)$$

Take $\quad 2089 = 1850 + \dfrac{20000}{q_1} + 0.09q_1$

$$0.09q_1^2 - 239q_1 + 20000 = 0$$

$$q_1 = \frac{239 \pm 223}{0.18} = 2567 \text{ or } 88.9$$

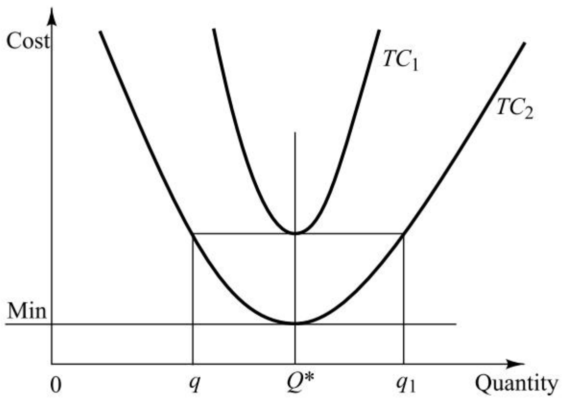
Taking $q_1 = 2567$ we find that $q < q_1$

$\therefore \quad Q^* < q < q_1 \quad$ (II zone)

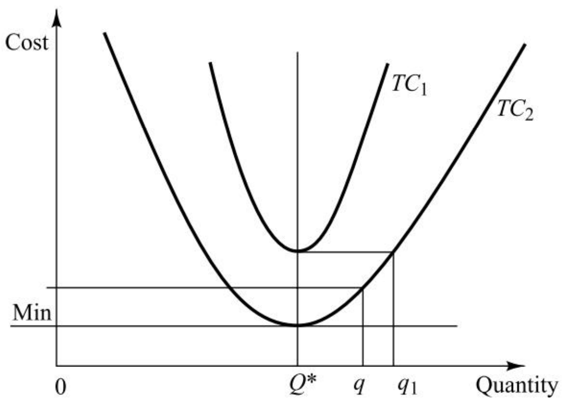
$Q^o = q = 500$

Note: The following figures illustrate the different cases.

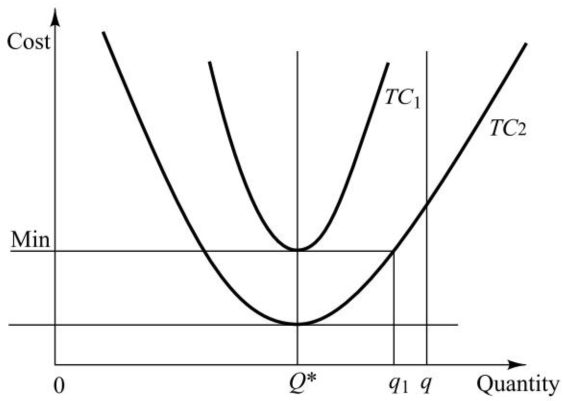
I Zone $\quad (0 < q < Q^*) \quad Q^o = Q^*$



II Zone $\quad (Q^* < q < q_1) \quad Q^o = q$



III Zone $\quad (q \geq q_1) \quad Q^o = Q^*$



Example 18.14 The annual demand of a product is 10000 units. Each unit costs Rs 100 for orders below 200 units and Rs 95 for orders of 200 and above. The annual inventory holding cost is 10% of the value and the ordering cost is Rs 5 per order. Find the EOQ.

Solution Given $D = 10000$ per year

$$C_S = 5$$

$$C_1 = 10\% = 100 \cdot \frac{10}{100} = 10$$

$$p_1 = 100, \quad p_2 = 95, \quad q = 200$$

$$Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2.10000.5}{10}} = 100 \text{ units}$$

$q = 200 \quad \therefore \quad Q^* < q$

To find q_1

$$TC_1(Q^*) = Dp_1 + \frac{D}{Q^*}C_S + \frac{Q^*}{2}C_1$$

$$= 10000 \times 100 + \frac{10000}{100} \times 5 + \frac{100}{2} \times 10$$

$$= 1001000$$

$$TC_2(q_1) = Dp_2 + \frac{D}{q_1}C_S + \frac{q_1}{2}C_1$$

$$= 10000 \times 95 + \frac{10000}{q_1} \times 5 + \frac{q_1}{2} \times 9.5$$

$$= 950000 + \frac{50000}{q_1} + \frac{19}{4} q_1$$

Take $1001000 = 950000 + \dfrac{50000}{q_1} + \dfrac{19}{4} q_1$

$$19q_1^2 - 204000q_1 + 200000 = 0$$

$$q_1 = 0.97 \text{ or } 10736$$

Taking $q_1 = 10736$ we get $Q^* < q < q_1$ (II zone)

$$Q^o = q = 200$$

Example 18.15 The monthly demand of an item is 500. The cost of the commodity is Rs 100 for orders less than 100 and Rs 95 for orders above 100 units. The holding cost is 20% of the value per month. The ordering cost is Rs 50. Find the EOQ and also the optimum monthly inventory cost.

Solution Given $D = 500$, $C_S = 50$,
$p_1 = 100$, $p_2 = 95$

$$C_1 = 100 \times \frac{20}{100} = 20,$$

$$q = 100$$

$$Q^* = \sqrt{\frac{2DC_S}{C_1}} = \sqrt{\frac{2.500.50}{20}} = 50$$

$$Q^* < q$$

$$TC_1(Q^*) = Dp_1 + \frac{D}{Q^*} C_S + \frac{Q^*}{2} C_1$$

$$= 50000 + \frac{500}{50} \times 50 + \frac{50}{2} \times 20$$

$$= 50000 + 500 + 500 = 51000$$

$$TC_2(q_1) = Dp_2 + \frac{D}{q_1} C_S + \frac{q_1}{2} C_1$$

$$= 47500 + \frac{25000}{q_1} + 10q_1$$

Taking $TC_1(Q^*) = TC_2(q_1)$

$$51000 = 47500 + \frac{25000}{q_1} + 10q_1$$

$$10q_1^2 - 3500q_1 + 25000 = 0$$

$$q_1 = 342.7 \text{ or } 7.3$$

Take $q_1 = 342.7$

Thus $Q^* < q < q_1$ (II zone)

$$Q^o = q = 100$$

Monthly optimum inventory cost is

$$TC_2(100) = 500 \times 95 + \frac{500}{100} \times 50 + \frac{100}{2} \times 19$$

$$= 47500 + 250 + 950$$

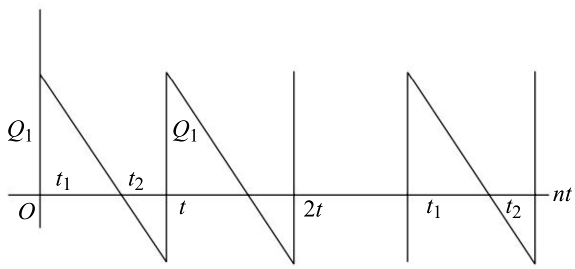
$$= \text{Rs } 48700$$

18.4.4 Model with Shortages

This model is based on all the assumptions of the I model except that the inventory system runs out of stock for a certain period of time. In other words shortages are allowed. Customers leave the orders with the supplier and this back order is supplied as soon as the stock is replenished. It involves shortage cost. Shortage cost depends upon how long the customer waits to receive supply. It is expressed in rupees per unit of time of delay. Let C_2 be the shortage cost per unit time. The period t of one run is divided into two parts t_1 and t_2. During the period t_1 the items are drawn as per requirement from the stock and during t_2 orders for the items are accumulated but not supplied due to shortage of stock ($t = t_1 + t_2$).

At the end of the period t_1 a quantity Q is produced. This Q is divided into two parts Q_1 and Q_2 such that Q_1 is the quantity stored for current use and Q_2 is exhausted to supply the back orders

This model is illustrated by the diagram given below:



$$\text{Total inventory over time } t = \frac{1}{2} Q_1 t_1$$

$$\text{Average inventory} = \frac{1/2 Q_1 t_1}{t}$$

Annual holding cost = $\dfrac{(1/2Q_1t_1)}{t} \times C_1$

Total amount of shortage during $t = \dfrac{1}{2} Q_2 t_2$

Annual shortage cost = $\dfrac{(1/2Q_2t_2)}{t} \times C_2$

Set-up cost $= nC_S = \dfrac{D}{Q} C_S$

Total annual inventory cost

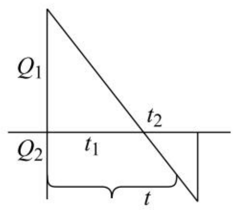
$$C_A = \frac{(1/2Q_1t_1)}{t} C_1 + \frac{(1/2Q_2t_2)}{t} C_2 + \frac{D}{Q} C_S$$

$$= \frac{1}{2}\frac{(t_1)}{t} Q_1C_1 + \frac{1}{2}\frac{(t_2)}{t} Q_2C_2 + \frac{D}{Q} C_S$$

From similar triangles

$$\frac{t_1}{t} = \frac{Q_1}{Q} \text{ and } \frac{t_2}{t} = \frac{Q_2}{Q}$$

$$\therefore\ t_1 = \frac{Q_1 t}{Q}\ ;\ t_2 = \frac{Q_2 t}{Q}$$



$$\therefore\quad C_A = \frac{1}{2}\frac{Q_1^2}{Q} C_1 + \frac{1}{2}\frac{(Q-Q_1)^2}{Q} C_2 + \frac{D}{Q} C_S$$

$$(\because\ Q_2 = Q - Q_1)$$

If C_A is to be minimum then

$$\frac{\partial C_A}{\partial Q_1} = \frac{\partial C_A}{\partial Q} = 0$$

$$\frac{\partial C_A}{\partial Q_1} = \frac{1}{2}\frac{2Q_1}{Q} C_1 + \frac{1}{2}\frac{2(Q-Q_1)}{Q}(-1)C_2 = 0$$

$$\frac{Q_1}{Q} C_1 + \frac{Q_1}{Q} C_2 - C_2 = 0$$

$$\frac{Q_1}{Q}(C_1 + C_2) = C_2 \Rightarrow Q_1 = \frac{C_2}{C_1 + C_2} Q \qquad \text{...(1)}$$

Also

$$\frac{\partial C_A}{\partial Q} = -\frac{Q_1^2}{2Q^2} C_1 + \frac{1}{2}\left[\frac{Q.2(Q-Q_1) - (Q-Q_1)^2}{Q^2}\right] C_2 - \frac{D}{Q^2} C_S = 0$$

i.e. $$-\frac{Q_1^2}{2} C_1 + Q(Q-Q_1)C_2 - \frac{1}{2}(Q-Q_1)^2 C_2 - DC_S = 0$$

$$-\frac{1}{2}Q_1^2 C_1 + C_2Q^2 - C_2QQ_1 - \frac{1}{2}Q^2C_2 + QQ_1C_2 - \frac{1}{2}Q_1^2C_2 - DC_S = 0$$

$$\frac{1}{2}Q^2C_2 = \frac{Q_1^2}{2} C_1 + \frac{Q_1^2}{2} C_2 + DC_S$$

$$Q^2C_2 = Q_1^2C_1 + Q_1^2C_2 + 2DC_S$$

$$Q^2 = \frac{2DC_S + Q_1^2C_1}{C_2} + Q_1^2$$

$$Q = \sqrt{\frac{2DC_S + Q_1^2C_1}{C_2} + Q_1^2} \qquad \text{...(2)}$$

From (1) $Q_1 = \dfrac{QC_2}{C_1 + C_2}$

$$\therefore\quad Q^2 = \frac{2DC_S}{C_2} + \frac{Q^2C_2^2}{(C_1+C_2)^2}\frac{C_1}{C_2} + \frac{Q^2C_2^2}{(C_1+C_2)^2}$$

$$= \frac{2DC_S}{C_2} + \frac{Q^2C_2^2}{(C_1+C_2)^2}\left[\frac{C_1 + C_2}{C_2}\right]$$

$$= \frac{2DC_S}{C_2} + \frac{Q^2C_2}{(C_1+C_2)}$$

$$\therefore\quad Q^2\left[1 - \frac{C_2}{C_1 + C_2}\right] = \frac{2DC_S}{C_2}$$

$$Q^2\left(\frac{C_1}{C_1 + C_2}\right) = \frac{2DC_S}{C_2}$$

$$\therefore \qquad Q^2 = \frac{2DC_S}{C_2} \cdot \frac{C_1 + C_2}{C_1}$$

$$\therefore \qquad Q^* = \sqrt{\frac{2DC_S}{C_1 C_2}(C_1 + C_2)}$$

and
$$Q_1^* = \frac{Q^* C_2}{C_1 + C_2}$$

$$= \sqrt{\frac{2DC_S(C_1 + C_2)}{C_1 C_2}} \left(\frac{C_2}{C_1 + C_2}\right)$$

$$= \sqrt{\frac{2DC_S C_2}{C_1(C_1 + C_2)}}$$

Optimum annual cost is given by

$$C_A^* = \frac{1}{2}\frac{Q_1^{*2}}{Q^*} C_1 + \frac{1}{2}\frac{(Q^* - Q_1^*)^2}{Q^*} C_2 + \frac{D}{Q^*} C_S$$

On substitution and simplification, we get

$$C_A^* = \sqrt{\frac{2DC_S C_1 C_2}{C_1 + C_2}}$$

Note: If shortages are not allowed then $C_2 = \infty$

$$Q_1^* = \frac{C_2 Q^*}{C_1 + C_2} = \frac{Q^*}{\dfrac{C_1}{C_2} + 1}$$

$$= Q^* = \sqrt{\frac{2DC_S}{C_1}}$$

which is the result obtained for Model 1.

Example 18.16 A commodity is to be supplied at the rate of 200 units per day. Ordering cost is Rs 50 and the holding cost is Rs 2 per day. The delay in supply induces a penalty of Rs 10 per unit per delay of one day. Find the optimal policy and the re-order cycle period.

Solution Given $D = 200$ units per day

$C_S = 50, \quad C_1 = 2, \quad C_2 = 10$

$$\text{EOQ} = Q^* = \sqrt{\frac{2DC_S(C_1 + C_2)}{C_1 C_2}}$$

$$= \sqrt{\frac{2 \times 200 \times 50(2 + 10)}{2 \times 10}}$$

$= 109.5$ units

$$\text{Re-order cycle period } t^* = \frac{Q^*}{D} = \frac{109.5}{200}$$

$= 1/2$ day

Example 18.17 The annual demand of an item is 10000 units. The ordering cost is Rs 10. The cost of the item is Rs 20. The holding cost is 20% of the value of the inventory per year. If the shortage cost is 25% of the value per unit per year find the EOQ and the optimal annual inventory cost.

Solution Given $D = 10000$ units

$$C_S = 10, \quad C_1 = \frac{20}{100} \times 20 = 4$$

$$C_2 = \frac{25}{100} \times 20 = 5$$

$$Q^* = \sqrt{\frac{2DC_S(C_1 + C_2)}{C_1 C_2}}$$

$$= \sqrt{\frac{2 \times 10000 \times 10(9)}{20}} = 300 \text{ units}$$

$$C_A^* = \sqrt{\frac{2DC_S.(C_1 C_2)}{C_1 + C_2}}$$

$$= \sqrt{\frac{2 \times 10000 \times 10 \times 20}{9}} = \text{Rs } 666.7$$

Example 18.18 A contractor supplies diesel engines to a truck manufacturer at the rate of 25 per day. The penalty for delay is Rs 10 per day per engine. The holding cost is Rs 16 per month. What should be the level of inventory at the beginning of each month?

Solution Here the fixed time interval is one month.

$t = 30$ days. There is no set-up cost.

Given $C_1 = \dfrac{16}{30}$ per day $\quad C_2 = 10$

$$Q = rt = 25 \times 30 = 750$$

$$Q_1^* = \frac{QC_2}{C_1 + C_2} = 750\left(\frac{10}{16/30 + 10}\right)$$

$= 712$ engines

Example 18.19 The demand of an item is uniform at the rate of 20 units per month. The fixed cost is Rs 10 per run. The production cost is Re 1 per item and the holding cost is Re 0.25 per month. If the shortage cost is Rs 1.25 per item per month determine the EOQ, the period of a run and shortage period.

Solution Given $D = r = 20$; $C_1 = 0.25$, $C_2 = 1.25$, $C_S = 10$

$$Q^* = \sqrt{\frac{2DC_S(C_1 + C_2)}{C_1C_2}}$$

$$= \sqrt{\frac{2 \times 20 \times 10(0.25 + 1.25)}{(0.25)(1.25)}} = 44$$

$$t^* = \frac{Q^*}{D} = \frac{44}{20} = 2.2 \text{ months (66 days)}$$

$$Q_1 = \frac{Q^*C_2}{C_1 + C_2} = 44 \times \frac{1.25}{1.5} = 37$$

$$\therefore \quad Q_2 = 44 - 37 = 7$$

$$t_2 = \frac{Q_2}{Q}\, t = \frac{7}{44} \times 66 = 10.5 \text{ days}$$

18.5 PROBABILISTIC MODEL

In the deterministic models, the demand is known precisely. But in most practical situations the demand is probabilistic. We know only the probability distribution of the future demand. This probability distribution is determined from the data collected from past experience.

In this probabilistic model, the demand D is probabilistic and the inventory on hand is y. There is no setup cost and the lead time is zero. C_1 is the holding cost and C_2 is the shortage cost.

The distribution of the inventory on hand is given by

$$H(y) = \begin{cases} y - D & D < y \\ 0 & D \geq y \end{cases}$$

Similarly, the distribution of the shortage inventory is given by

$$G(y) = \begin{cases} 0 & D < y \\ D - y & D \geq y \end{cases}$$

Assume that $f(D)$ is the p.d.f. of demand. The expected inventory cost is given by

$E[C(y)] = E$ (holding cost) $+ E$ (shortage cost)

$$= C_1 \int_0^\infty H(y) f(D) dD + C_2 \int_0^\infty G(y) f(D) dD$$

$$= C_1 \int_0^y (y - D) f(D) dD + C_2 \int_y^\infty (D - y) f(D) dD$$

If $E[C(y)]$ is to be minimum $\dfrac{\partial[E\{C(y)\}]}{\partial y} = 0$

$$\frac{\partial[E\{C(y)\}]}{\partial y} = 0$$

$$\Rightarrow \quad C_1 \int_0^y f(D) dD + C_2(-1) \int_y^\infty f(D) dD = 0$$

$$\Rightarrow \quad C_1 \int_0^y f(D) dD - C_2 \int_y^\infty f(D) dD = 0$$

$$\text{Now} \int_y^\infty f(D) dD = 1 - \int_0^y f(D) dD$$

$$\therefore \quad C_1 \int_0^y f(D) dD - C_2 \left[1 - \int_0^y f(D) dD\right] = 0$$

$$(C_1 + C_2) \int_0^y f(D) dD = C_2$$

$$\setminus \quad \int_0^y f(D) dD = \frac{C_2}{C_1 + C_2}$$

Thus we get $P(D \leq y^*) = \dfrac{C_2}{C_1 + C_2}$

This is the condition to be satisfied by y^* and y^* is determined accordingly.

In case the distribution of D is discrete then we have

$E[C(y)]$

$$= C_1 \sum (y - D) f(D) + C_2 \sum (D - y) f(D)$$

If $E[C(y^*)]$ is minimum then

$E[C(y^* - 1)] \geq E[C(y^*)]$ and

$E[C(y^* + 1) \geq E[C(y^*)]$

$$P[D \leq (y^* - 1)] \leq \frac{C_2}{C_1 + C_2} \text{ and}$$

$$P[D \leq y^*] \geq \frac{C_2}{C_1 + C_2}$$

$$\therefore \quad P[D \leq (y^* - 1)] \leq \frac{C_2}{C_1 + C_2} \leq P[D \leq y^*]$$

This is the condition to be satisfied by y^*.

Example 18.20 The demand of an item is given by the p.d.f. $f(D) = \begin{cases} 1/10 & 0 \leq D \leq 10 \\ 0 & D > 10 \end{cases}$

The holding cost is Re 1 per unit and shortage cost is Rs 4 per unit. Find the EOQ.

Solution Given $C_1 = 1$, $C_2 = 4$,

$$\frac{C_2}{C_1 + C_2} = \frac{4}{1+4} = \frac{4}{5} = 0.8$$

$$f(D) = \frac{1}{10}$$

$$P[D \leq y^*] = \int_0^{y^*} f(D) dD = \int_0^{y^*} \frac{1}{10} dD = \frac{1}{10} y^*$$

$$\therefore \quad \frac{y^*}{10} = 0.8$$

$$\therefore \quad y^* = 8$$

Example 18.21 The distribution of demand of an item is given by

D	0	1	2	3	4	5
$f(D)$	0.1	0.2	0.25	0.2	0.15	0.1

Holding cost is Re 1 and the shortage cost is Rs 2. Find the EOQ.

Solution

Here $C_1 = 1$ and $C_2 = 2$

The cumulative distribution of demand is

D	0	1	2	3	4	5
$P(D \leq y)$	0.1	0.3	0.55	0.75	0.9	1

$$\frac{C_2}{C_1 + C_2} = \frac{2}{1+2} = \frac{2}{3} = 0.6$$

From the table we find that

$P(D \leq 2) = 0.55 \quad P(D \leq 3) = 0.75$

$P(D \leq 2) < 0.6 < P(D \leq 3)$

$\therefore \quad y^* = 3$

Example 18.22 A bookseller purchases a weekly magazine at Rs 3 and sells at Rs 7 each. He cannot return unsold magazines. Weekly demand has the following distribution.

D	23	24	25	26	27	28	29	30	31	32
$f(D)$	.01	.03	.06	.1	.2	.25	.15	.1	.05	.05

How many magazines should be ordered every week?

Solution Given $C_1 = 3$, $C_2 = 7 - 3 = 4$

The cumulative distribution of demand is

D	23	24	25	26	27	28	29	30	31	32
$P(D \leq y)$	0.01	0.04	0.10	0.20	0.40	0.65	0.80	0.90	0.95	1

$$\frac{C_2}{C_1 + C_2} = \frac{4}{3+4} = \frac{4}{7} = 0.57$$

From the table

$P(D \leq 27) = .4 \quad P(D \leq 28) = 0.65$

$\therefore \quad P(D \leq 27) < .57 > P(D \leq 28)$

$\therefore \quad y^* = 28$

Note: If the cost, selling price and the salvage value of unsold item are given then we use the formula

$$P(D \leq y^*) = \frac{\text{Profit}}{\text{Profit + loss}}$$

Example 18.23 The sales of butter (demand for butter) has the following probability distribution.

Demand (kg)	14	15	16	17	18	19	20
Probability	0.02	0.08	0.1	0.4	0.2	0.15	0.05

The cost of butter is Rs 80 per kg and the selling price is Rs 120 per kg. Unsold stock is disposed off at Rs 60 per kg at the end of the day. Determine how much of butter is to be stocked at the beginning of each day.

Solution The cumulative distribution of demand is given by

D	14	15	16	17	18	19	20
Probability	.02	.08	.1	.4	.2	.15	.05
$P(D \le y)$	.02	.1	.2	.6	.8	.95	1

Profit per kg of butter = 120 – 80 = Rs 40
Salvage value = Rs 60

$\therefore$ Loss per kg of unsold sold butter = Rs 20

Now, for y^* the condition is

$$P(D \le y^*) = \frac{\text{Profit}}{\text{Profit} + \text{loss}}$$

$$= \frac{40}{40+20} = 0.67$$

From the table

$$P(D \le 17) = 0.6 \quad P(D \le 18) = 0.8$$

$\therefore \quad P(D \le 17) < 0.67 < P(D \le 18)$

$\therefore \quad y^* = 18$

Every day 18 kg of butter is to be stocked.

EXERCISES

1. A company purchases lubricants at the rate of Rs 42 per tin. The requirement is 1800 per year. The cost of placing an order is Rs 16 and the holding charges are 20 paise per rupee per year. Find the EOQ.
2. Raw material costs Re 1 per unit and the annual requirement is 2000 units. Ordering cost is Rs 10 and carrying cost is 16% per year per unit. Find the EOQ and optimum inventory cost per year.
3. An aircraft company requires rivets at the rate of 5000 kg per year. The cost of the rivet is Rs 20 per kg and the ordering cost is Rs 200. The holding cost is 10% per kg per year. How frequently should orders be placed? What quantity should be ordered for?
4. A company has a demand of 12000 units of an item per year. Set-up cost is Rs 400 and the holding cost is 15 paise per unit per month. Find the EOQ and the optimum inventory cost per year.
5. The daily demand of an item is 30 units. Holding cost is 5 paise per unit per day and the setup cost is Rs 100. Find the economic lot size, period of one run and the total inventory cost.
6. A company has to supply 10000 auto parts per day. They can produce 25000 parts per day. Holding cost is 20 paise per unit per year and the set-up cost is Rs 180 per run. How frequently should production be made?
7. Given $k = 24000$, $r = 12000$, $C_1 = 1.8$, $C_S = 400$, find Q^*.
8. An item is produced at the rate of 50 per day. The demand is 25 items per day. The setup cost is Rs 100 and the holding cost is Re 0.01 per item per day. Find
   (i) economic lot size
   (ii) period of one run
   (iii) minimum total inventory cost.
9. The production rate of tomato ketchup is 600 bottles per day and the demand is 150 bottles per day. The cost per bottle is Rs 6. The set-up cost is Rs 90 per run and the holding cost is 20% of the cost per annum. Find the economic lot size and the number of production runs.
10. The annual sales of a particular grade of paint is 30000 litres. The holding cost is Re 1 per litre. The setup cost is Rs 80 per run. Determine the economic lot size.
11. For a particular product the daily demand is 20 units, setup cost is Rs 100 and the holding

cost is Rs 0.20 per day. The production rate is 50 units per day. Determine the economic lot size and the period of a run.

12. A shopkeeper purchases an item at the rate of 50 units per month. The cost is Rs 6 for orders less than 200 and Rs 5.70 for orders not less than 200 units. The ordering cost is Rs 10 and the holding cost is 20% of the value per year. Find the EOQ.
13. Find the EOQ for a product for which the price breaks are as follows:

Price (Rs)	*Quantity*
10	$q < 500$
9	$q \geq 500$

The monthly demand is 200 units. The ordering cost is Rs 350 and the holding cost is 2% of the cost per month.

14. The annual demand of an item is 10000 units. The ordering cost is Rs 40 and the holding cost is 20% of the cost per unit per year. The price of the item is Rs 4 and there is a discount of 10% for orders of 1500 units or more. Find the EOQ.
15. Given the following data find the EOQ and the reorder level for an item

Annual demand	=	2500 units
Ordering cost	=	Rs 12.50
Holding cost	=	25% of the average inventory
No.of working days	=	250 per year
Lead time	=	3 days.

The cost of the item is Rs 4 per unit for orders less than 200 and Rs 3 for orders not less than 200.

16. The cost of a notebook is Rs 10 with a discount of 5% for orders of 200 or more. The demand is at the rate of 50 notebooks per month and the ordering cost is Rs 10. The holding cost is 20% of the value per year. Determine the annual minimum inventory cost and the number of orders per year.
17. A manufacturer has to supply 24000 units of his product per year. Penalty for delay in supply is Rs 0.20 per unit per month. The holding cost is Rs 0.10 per unit per month and the setup cost is Rs 350 per run. Find the economic lot size.
18. The demand of an item is at the rate of 25 units per month. The fixed cost is Rs 15 per run. The carrying cost is Rs 0.30 per item per month. If the shortage cost is Rs 1.50 per item per month determine the economic lot size and the period of a run.
19. A TV company requires speakers at the rate of 8000 units per month. The setup cost is Rs 12000 per run. The holding cost is Rs 0.30 per unit per month. Find the EOQ
    (i) when shortage are not allowed.
    (ii) when the shortage cost is Rs 1.10 per unit per month.
20. The demand for an item is 18000 units per year. The holding cost is Rs 1.20 per unit per year. The cost of production is Rs 400 and the shortage cost is Rs 5. Determine the EOQ.
21. Find the EOQ if the demand is 16 units per day, ordering cost is Rs 15, holding cost is 15 paise per unit per day and the shortage cost is 75 paise per unit per day.
22. The annual demand of an automobile part is 6000 units. The setup cost per production run is Rs 500 and the storage cost per unit per year is Rs 8. If the shortage cost is Rs 20 per unit per year find
    (i) economic lot size
    (ii) period of one run
    (iii) number of shortages per run.
23. A newspaper boy purchases newspapers at Rs 2 per copy and sells them at Rs 3. The demand for the newspaper has the following probability distribution.

Demand	10	11	12	13	14	15	16
Probability	0.05	0.15	0.40	0.20	0.10	0.05	0.05

Unsold newspapers cannot be returned. Determine how many copies should he purchase in order to minimize his loss.

24. The demand for a food commodity has the probability distribution given below:

Demand	10	11	12	13	14
Probability	0.10	0.15	0.20	0.25	0.30

The cost is Rs 30 each and the selling price is Rs 50. Unsold items become useless. Determine how many items should be prepared.

25. Some of the spare parts of a car cost Rs 200 each. These parts can be ordered only together with the car. Suppose there is a loss of Rs 4800 for each spare part not in stock. Also the probability distribution of their demand is given by

Demand	0	1	2	3	4	5	6 or more
Probability	0.9	.04	.025	.02	.01	.005	0

Determine how many spare parts should be procured.

26. The demand for an item has the p.d.f.
$$f(x) = \frac{1}{1000} \quad (4000 \le x \le 5000)$$
The holding cost is Re 1 and the shortage cost is Rs 7. Find the EOQ

27. A bakery sells cake by weight. There is a profit of Rs 4.50 per kg sold. Unsold cakes are disposed off at a loss or Rs 1.50 per kg. The distribution of demand is rectangular between 2000 and 3000 kg. Determine the optimum quantity of cake to be prepared per day.

28. The demand for an item has rectangular distribution
$$f(x) = \frac{1}{1000} \quad (5000 \le x \le 6000)$$
The holding cost is Rs 10 and the shortage cost is Rs 30. Find the EOQ.

29. Find the EOQ, reorder level and the optimum annual inventory cost given that the annual demand is 1000 units, holding cost is 0.15 ordering cost is Rs 10, and lead time is 2 years.

30. Annual demand is 2500 units. The cost of one unit is Rs 30. The ordering cost is Rs 130 and the carrying cost is 10% per year. Estimate the EOQ and also determine the number of orders per year.

31. A factor has to supply a product at the rate of 25 per day. The carrying cost is Rs 16 per month and the shortage cost is Rs 10 per unit per day. The setup cost for production run is Rs 10000. Determine how frequently should the production be started and what should be the level of inventory at the beginning of each run.

32. An auto part has a demand of 9000 units per year. The ordering cost is Rs 100 and the holding cost is Rs 2.40 per unit per year. If the production is instantaneous and no shortages are allowed find the
    (i) EOQ
    (ii) Number of orders per year
    (iii) Time between two consecutive orders
    (iv) Annual inventory cost.

33. A shopkeeper purchases an item at Rs 40 per unit. Annual demand of the item is 2000 units. The cost of placing an order is Rs 15 and the holding cost is 20% per year. Determine the EOQ.

34. The demand of an item is 400 units per week. The cost of the item is Rs 50 per unit. Ordering cost is Rs 75 and the holding cost is 7.5% of the cost of the item per year. Find the economic lot size and the optimum inventory cost.

35. An item costs Rs 235 per ton. The monthly requirement is 5 tons and the setup cost is Rs 1000. The holding cost is 10% of the value per year. Determine the EOQ.

36. A company uses rivets at the rate of 5000 kg per year. The cost of the rivet is Rs 2 per kg. The cost of placing an order is Rs 20 and the holding cost is 10% per year. How frequently should the order for rivets be placed and for how much?

37. The production rate of an item is 200 units per day and the demand is 100 units per day. The set-up cost is Rs 200 and the holding cost of one unit is Rs 0.81 per day. Find the economic lot size.

38. The cost of a hardware item is Rs 300 per ton. Annual requirements are 120 tons and the setup cost is Rs 240. If the inventory carrying cost is 12% of the value determine the period of one run.

39. The demand for an item is 18000 units per year. The setup cost is Rs 400 and the holding cost is Rs 1.20. If the shortage cost is Rs 5 determine the optimal order quantity.
40. Determine the EOQ
    (i) Annual demand: 240 units
    Set-up cost: Rs 10
    Holding cost: Rs 0.25 per month
    Shortage cost: Rs 1.25 per month
    (ii) Weekly demand: 16 units
    Set-up cost: Rs 15
    Holding cost: 15% per week
    Purchase cost: Rs 8 per unit
    Shortage cost: Rs 0.75 per week
    (iii) Monthly demand: 25 units
    Set-up cost: Rs 15
    Holding cost: Rs 0.30 per month
    Shortage cost: Rs 1.50 per month
41. The demand of an item has p.d.f
    $f(x) = 0.1 \quad 0 \le x \le 10$
    $\quad 0 \quad x > 10$
    The holding cost is Rs 0.50 per day Shortage cost is Rs 4.5 per day. Find the optimum inventory level.
42. Annual demand: 3600 units
    Ordering cost: Rs 50
    Holding cost: 20% of the unit cost
    Price break O $< q <$ 100 – Rs 20
    $Q \ge$ 100 – Rs 18

ANSWERS

1. 83
2. $Q^* = 500$; $C_A^* =$ Rs 80
3. $Q^* = 1000$; Order cycle = 2 months 12 days
4. $Q^* = 730$; $C_A^* =$ Rs 4157
5. $Q^* = 346$; Period of run = 11.5 days; $C_A^* =$ Rs 17.32
6. Production run = 10.5 days
7. $Q^* = 3266$
8. $Q^* = 1000$; Period of run = 40; $C_A^* =$ Rs 5
9. $Q^* = 3309$; No. of runs = 17 per year
10. $Q^* = 2191$
11. $Q^* = 183$; Period of run = 9 days
12. $Q^0 = q = 200$
13. $Q^0 = 837$
14. $Q^0 = 1500$
15. $Q^0 = 250$; Reorder level = 220 units
16. $Q^0 = 200$; $C_A^0 =$ Rs 494; 3 orders per year
17. $Q^* = 4583$
18. $Q^* = 17$, $t^* = 20$ days
19. (i) 25298 (ii) 28540
20. $Q^* = 3857$
21. $Q^* = 6$
22. (i) $Q^* = 1024$, (ii) $t^* = 2$ months (iii) $Q_2^* = 293$
23. $y^* = 13$
24. $y^* = 12$
25. $y^* = 2$
26. $y^* = 4875$
27. $y^* = 2750$ kg
28. $y^* = 5750$
29. $Q^* = 365$; Reorder level = 2000 units, Rs 54.80
30. $Q^* = 466$; 5.3 orders per year
31. 40 days; 943 units
32. $Q^* = 866$; No. of orders = 10.4 / year; Time between two orders = 35 days
    Annual inventory cost = Rs 2080
33. 87 units
34. $Q^* = 288$; Inventory cost = Rs 65.80 per week.
35. $Q^* = 71.458$ tons
36. $t^* = 73$ days; $Q^* = 1000$ kg
37. $Q^* = 314$
38. $t^* = 4$ months
39. $Q^* = 3856$ units
40. (i) 44 (ii) 32 (iii) 55
41. 9
42. 134

19

Replacement Models

CONCEPT REVIEW

19.1 INTRODUCTION

Replacement becomes necessary when the job-performing units such as men, machines, equipment, parts etc., lose their efficiency and effectiveness because of gradual deterioration or sudden failure or breakdown. Planned replacement of these items would reduce maintenance cost and other overhead expenses. When a machine loses its efficiency gradually the maintenance becomes very expensive. Therefore the problem is to determine the age at which it is most economical to replace the item. On the other hand, certain items such as bulbs, radio and television parts fail suddenly without giving any indication of failure and they become completely useless. These items are to be replaced immediately as and when they fail to function. Thus there are two types of situations for replacement.

(i) Replacement of items which deteriorate and whose maintenance cost increases with time.

(ii) Replacement of items which fail all on a sudden.

19.2 REPLACEMENT OF ITEMS WHICH DETERIORATE IN EFFICIENCY WITH TIME

A machine loses efficiency with time and we have to determine the best time at which we have to go for a new one. In case of a vehicle the maintenance cost is increasing as it is getting aged. These costs increase day by day if we postpone the replacement.

First we consider the situation where the maintenance costs increase with time assuming that the value of money remains the same during the period.

19.2.1 Formula for Finding the Time for Replacement

Let C be the capital cost of the machine, $S(t)$ be the scrap value of the machine after t years, $f(t)$ be the operating cost or maintenance cost at time t and n be the number of years. The annual cost of the machine at time t is given by $C - S(t) + f(t)$. The total maintenance cost during n years is given by $\sum_{t=0}^{n} f(t)$ (when t is discrete) or $\int_0^n f(t)\,dt$ (when t is continuous). Thus we find that the total cost after n years is given by

$$T = C - S(t) + \int_0^n f(t)\,dt$$

The average annual cost is given by

$$T_A = \frac{1}{n}\left[C - S(t) + \int_0^n f(t)\,dt\right]$$

We have to determine the value of n which would give minimum T_A

Now $T_A = \frac{1}{n}[C - S(t)] + \int_0^n f(t)\,dt$

$$\frac{dT_A}{dn} = -\frac{1}{n^2}[C - S(t)] - \frac{1}{n^2}\int_0^n f(t)dt + \frac{1}{n}f(n)$$

T_A is minimum $\Rightarrow \frac{dT_A}{dn} = 0$

$$\therefore \quad f(n) - \frac{[C - S(t)]}{n} - \frac{1}{n}\int_0^n f(t)dt = 0$$

$$f(n) = \frac{[C - S(t)]}{n} + \frac{1}{n}\int_0^n f(t)\,dt$$

$$= \frac{1}{n}\left[C - S(t) + \int_0^n f(t)\,dt\right]$$

$$= T_A$$

Thus T_A becomes the least at the time when

$T_A = f(n)$

Note: When t is discrete the tabular method is used to find n for which T_A is minimum.

Example 19.1 A machine costs Rs 12,200. The scrap value is Rs 200. The maintenance costs of the machine are given below:

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost	200	500	800	1200	1800	2500	3200	4000

When should the machine be replaced?

Solution Given

$C = 12{,}200$

$S(t) = 200,$

$$T_A = \frac{C - S(t) + \Sigma f(t)}{n}$$

The average annual cost is given by the following table.

Year	*f(t)*	*Σf(t)*	*S(t)*	*C – S(t)*	*T*	T_A
1	200	200	200	12000	12200	12200
2	500	700	200	12000	12700	6350
3	800	1500	200	12000	13500	4500
4	1200	2700	200	12000	14700	3675
5	1800	4500	200	12000	16500	3300
6	2500	7000	200	12000	19000	**3167**
7	3200	10200	200	12000	22200	3171
8	4000	14200	200	12000	26200	3275

From the table we find that T_A is minimum at the end of the 6th year. Hence it is profitable to replace the machine at the end of the 6th year.

Example 19.2 The maintenance cost and the resale price of a truck are given below:

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost	1000	1300	1700	2200	2900	3800	4800	6000
Resale price	4000	2000	1200	600	500	400	400	400

The purchase price of the truck is Rs 8000. Determine the time at which it is profitable to replace the truck.

Solution Given

$C = 8000$

$T = C - S(t) + \Sigma f(t)$

$$T_A = \frac{T}{n}$$

Year	*f(t)*	*Σf(t)*	*S(t)*	*C – S(t)*	*T*	T_A
1	1000	1000	4000	4000	5000	5000
2	1300	2300	2000	6000	8300	4150
3	1700	4000	1200	6800	10800	3600
4	2200	6200	600	7400	13600	3400
5	2900	9100	500	7500	16600	**3320**
6	3800	12900	400	7600	20500	3417
7	4800	17700	400	7600	25300	3614
8	6000	23700	400	7600	31300	3913

From the table we find that the average annual cost is minimum at the end of the 5th year. Hence it is profitable to replace the truck at the end of the 5th year.

Example 19.3 A machine costs Rs 900. The annual operating cost is Rs 200 for the first year and is then increasing by Rs 2000 per year for subsequent years. There is no scrap value. Determine the best age to replace the machine.

Solution Given

$C = 9000$; $S(t) = 0$

Year	$f(t)$	$\Sigma f(t)$	$S(t)$	$C - S(t)$	T	T_A
1	200	200	0	9000	9200	9200
2	2200	2400	0	9000	11400	5700
3	4200	6600	0	9000	15600	**5200**
4	6200	12800	0	9000	21800	5450
5	8200	21000	0	9000	30800	6160
6	10200	31200	0	9000	40200	6700
7	12200	43400	0	9000	52400	7486

Since T_A is minimum at the end of the 3rd year, the best time to replace the machine is at the end of the 3rd year.

Example 19.4 There are two types of autorikshaws, type *A* and type *B*. The purchase prices of these two types are Rs 7000 and Rs 9000 respectively. The resale values and the running costs for the two types are given below.

Type *A*

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running cost	1100	1300	1500	1900	2400	2900	3500	4100
Resale price	3100	1600	850	475	300	300	300	300

Type *B*

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running cost	1300	1600	1900	2500	3200	4100	5100	6200
Resale price	4100	2100	1100	600	400	400	400	400

Determine which type of autorickshaw is to be purchased.

Solution

For type *A*, *C* = Rs 7000

Year	$f(t)$	$\Sigma f(t)$	$S(t)$	$C - S(t)$	T	T_A
1	1100	1100	3100	3900	5000	5000
2	1300	2400	1600	5400	7800	3900
3	1500	3900	850	6150	10050	3350
4	1900	5800	475	6525	12325	3081
5	2400	8200	300	6700	14900	2980
6	2900	11100	300	6700	17800	**2967**
7	3500	14600	300	6700	21300	3043
8	4100	18700	300	6700	25400	3175

For type *B*, *C* = Rs 9000

Year	$f(t)$	$\Sigma f(t)$	$S(t)$	$C - S(t)$	T	T_A
1	1300	1300	4100	4900	6200	6200
2	1600	2900	2100	6900	9800	4900
3	1900	4800	1100	7900	12700	4233
4	2500	7300	600	8400	15700	3925
5	3200	10500	400	8600	19100	**3820**
6	4100	14600	400	8600	23200	3867
7	5100	19700	400	8600	28300	4043
8	6200	25900	400	8600	34500	4313

From the above tables we find that type *A* should be replaced at the end of the 6th year and type *B* should be replaced at the end of the 5th year. Also the annual average cost of *A* is less than that of *B* uniformly.

Hence it is advisable to purchase the autorikshaw of type *A*.

Example 19.5 Machine *A* costs Rs 8000. Annual operating cost is Rs 1000 for the first year and is increasing at the rate of Rs 2000 per year afterwards. The machine has no resale value. Another machine *B* costs Rs 10,000. The annual operating cost is Rs 400 for the first year and is increasing at the rate of Rs 800 every year with no resale value. If the machine *A* is one year old, is it advisable to replace *A* with *B*? If so find the appropriate time for replacement.

Solution

For machine *A*, *C* = 8000

Year	$f(t)$	$\Sigma f(t)$	$S(t)$	$C - S(t)$	T	T_A
1	1000	1000	0	8000	9000	9000
2	3000	4000	0	8000	12000	6000
3	5000	9000	0	8000	17000	**5667**
4	7000	16000	0	8000	24000	6000
5	9000	25000	0	8000	33000	6600
6	11000	36000	0	8000	44000	7333

For machine B, C = 10000

Year	$f(t)$	$\Sigma f(t)$	$S(t)$	$C - S(t)$	T	T_A
1	400	400	0	10000	10400	10400
2	1200	1600	0	10000	11600	5800
3	2000	3600	0	10000	13600	4533
4	2800	6400	0	10000	16400	4100
5	3600	10000	0	10000	20000	**4000**
6	4400	14400	0	10000	24400	4067
7	5200	19600	0	10000	29600	4229

From the tables we find that the best age for replacement of machine A is 3 years and the time for replacement of B is 5 years.

Now assume that at present machine A is one year old. Hence its cost is Rs 9000 now, Rs 6000 after one year, Rs 5667 after two years Rs 6000 after three years, Rs 6600 after four years and so on. Comparing with the average total cost of B we find that the average total cost for B does not exceed that of one year old A in the 3rd year. Hence machine A should be replaced with machine B after 3 years.

Example 19.6 Machine A costs Rs 45000 and the running cost is Rs 1000 for the first year increasing by Rs 10000 per year afterwards. Another machine B costs Rs 50000 and operating cost is Rs 2000 for the first year increasing by Rs 4000 per year subsequently. If we have machine A now, should we replace it with B? If so find the best time for replacement.

Solution For machine A, C = 45000 and for machine B, C = 50000. The following tables show the average annual costs of the two machines, taking the resale value to be zero.

Machine A, $S(t)$ = 0

Year	$f(t)$	$\Sigma f(t)$	$C - S(t)$	T	T_A
1	1000	1000	45000	46000	46000
2	11000	12000	45000	57000	28500
3	21000	33000	45000	78000	**26000**
4	31000	64000	45000	109000	27250
5	41000	105000	45000	150000	30000
6	51000	156000	45000	201000	33500
7	61000	217000	45000	262000	37429

Machine B, $S(t)$ = 0

Year	$f(t)$	$\Sigma f(t)$	$C - S(t)$	T	T_A
1	2000	2000	50000	52000	52000
2	6000	8000	50000	58000	29000
3	10000	18000	50000	68000	22667
4	14000	32000	50000	82000	20500
5	18000	50000	50000	100000	**20000**
6	22000	72000	50000	122000	20333
7	26000	98000	50000	148000	21143

We find that the average annual cost of A is lowest in the 3rd year (Rs 26000) and that of B is lowest in the 5th year (Rs 20000). Hence machine A should be replaced by machine B.

To find the time of replacement

Machine A should be replaced by machine B at the time when the total cost for the next year exceeds the lowest average annual cost (Rs 20000) of machine B.

Year	Difference between the total costs
1	46000
2	57000 – 46000 = 11000
3	78000 – 57000 = 21000
4	109000 – 78000 = 31000

Thus the total cost of machine A during the third year (Rs 21000) is more than the lowest average annual cost of machine B.

Therefore A should be replaced by B after two years.

19.2.2 Replacement of Items Whose Maintenance Costs Increase with Time and the Value of Money Also Changes with Time

As the value of money changes with time we have to calculate the present worth of the money we gain or spend a few years hence. Suppose the initial cost of a machine is C and the maintenance costs be C_j (j = 1, 2, 3 ...) for the j^{th} year. The value of money is decreasing at the rate of r% per year. Thus one rupee invested today worths $(1 + r)$ rupee next year, $(1 + r)^2$ after two years, ... $(1 + r)^n$ after n years. In other words ... $(1 + r)^{-n}$ is the present worth of one rupee spent after n years. Denoting

$(1 + r)^{-1}$ by v we say that v^n is the *present worth factor* (p.w.f) of one rupee spent after n years. $v = \frac{1}{1+r}$ is called discount rate or depreciation value per unit money. Let V_n be the p.w.f. of all the future costs.

$$V_n = \frac{C + C_1 + C_2 v + \ldots + C_n v^{n-1}}{1 - v^n}$$

The value of V_n should be minimum if we want to replace after $n - 1$ years. The condition is given by

$$\frac{C_{n-1}}{1-v} < V_n < \frac{C_n}{1-v}$$

i.e. $C_{n-1} < (1 - v)V_n < C_n$

i.e. $C_{n-1} <$

$$(1-v)\frac{(C + C_1 + C_2 v + \ldots + C_n v^{n-1})}{1-v^n} < C_n$$

$$\text{i.e. } C_{n-1} < \frac{C + C_1 + C_2 v + \ldots + C_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} < C_n$$

Denoting $\frac{C + C_1 + C_2 v + \ldots + C_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}}$ by R

we have $C_{n-1} < R < C_n$. R is called *weighted average cost.*

Note: If A and B are two machines with weighted average costs R_1 and R_2 then we say that machine A is better than B if $R_1 < R_2$

Example 19.7 A computer costs Rs 5 lakhs. Running and maintenance costs are Rs 60000 for each of the first five years, increasing by Rs 20000 per year in subsequent years. Assuming that the value of money is decreasing at 10% per year and that there is no salvage value for the computer, find the optimal period for replacing the computer.

Solution Given that money is worth 10% per year the discount factor

$$v = \frac{1}{1+0.1} = \frac{1}{1.1} = 0.9091$$

The average cost is given by the following table

$C = 500000$

Year	C_n	v^{n-1}	$C_n v^{n-1}$	$C + \Sigma C_n v^{n-1}$	Σv^n	R
1	60000	1	60000	560000	1	560000
2	60000	0.9091	54546	614546	1.9091	321904
3	60000	0.8264	49584	664130	2.7355	242782
4	60000	0.7513	45078	709208	3.4868	203398
5	60000	0.6830	40980	750188	4.1698	179910
6	80000	0.6209	49672	799860	4.7907	166961
7	100000	0.5645	56450	856310	5.3552	159903
8.	120000	0.5132	61584	917894	5.8684	156413
9	140000	0.4665	65310	983204	6.3349	155204
10	160000	0.4241	67856	1051060	6.7590	155505

From the table we find that $C_9 < R < C_{10}$ [140000 < 155204 < 160000]

Therefore the computer must be replaced after 9 years.

Example 19.8 A machine costs Rs 250000. The running and maintenance costs are Rs 120000 per year for the first five years and increasing there after by Rs 20000 per year. If the money is worth 10% per year and there is no salvage value for the machine determine the best period for replacing the machine.

Solution Given

$$v = \frac{1}{1+r} = \frac{1}{1+0.1} = \frac{1}{1.1} = 0.9091$$

Table for calculating the average cost is given below.

Year	C_n	v^{n-1}	$C_n v^{n-1}$	$C + \Sigma C_n v^{n-1}$	Σv^n	R
1	*120000*	*1*	*120000*	*370000*	*1*	*370000*
2	120000	0.9091	109092	479092	1.9091	250952
3	120000	0.8264	99168	578260	2.7355	211391
4	120000	0.7513	90156	668416	3.4868	191699
5	120000	0.6830	81960	750376	4.1698	179955
6	140000	0.6209	86926	837302	4.7907	174777
7	160000	0.5645	90320	927622	5.3552	173219
8	180000	0.5132	92376	1019998	5.8684	173812
9	200000	0.4665	93300	1113298	6.3349	175740
10	220000	0.4241	93302	1206600	6.7590	178518

Here $C_7 < R < C_8$ [160000 < 173219 < 180000]

$\therefore$ The machine should be replaced after 7 years.

Example 19.9 There are two machines A and B. A costs Rs 2500. Its running cost is Rs 400 per year for the first five years and is increasing by Rs 100 per year afterwards. B costs Rs 1250 with running cost Rs 600 per year for 6 years increasing by Rs 100 per year thereafter. If money is worth 10% per year and both machines have no scrap value, determine which machine should be purchased.

Solution Let us calculate the average cost for A.

Year	C_n	v^{n-1}	$C_n v^{n-1}$	$C + \Sigma C_n v^{n-1}$	Σv^n	R
1	400	1	400	2900	1	2900
2	400	0.9091	363.64	3263.64	1.9091	1709.45
3	400	0.8264	330.56	3594.20	2.7355	1313.84
4	400	0.7513	300.52	3894.72	3.4868	1116.93
5	400	0.6830	273.20	4167.92	4.1698	999.50
6	500	0.6209	310.45	4478.37	4.7907	948.23
7	600	0.5645	338.70	4817.07	5.3552	899.40
8	700	0.5132	359.24	5176.31	5.8684	881.92
9	800	0.4665	373.20	5549.51	6.3349	**875.86**
10	900	0.4241	381.69	5931.20	6.7590	877.35

We find that A should be replaced after 9 years.

Average cost for machine B

Year	C_n	v^{n-1}	$C_n v^{n-1}$	$C + \Sigma C_n v^{n-1}$	Σv^n	R
1	600	1	600	1850	1	1850
2	600	0. 9091	545.46	2395.46	1.9091	1254.75
3	600	0.8264	495.84	2891.30	2.7355	1056.95
4	600	0.7513	450.78	3342.08	3.4868	958.49
5	600	0.6830	409.80	3751.88	4.1698	899.77
6	600	0.6209	372.54	4124.42	4.7907	860.92
7	700	0.5645	395.15	4519.57	5.3552	843.96
8	800	0.5132	410.56	4930.13	5.8684	**840.11**
9	900	0.4665	419.85	5349.98	6.3349	844.52
10	1000	0.4241	424.10	5774.08	6.7590	854.28

The table shows that B should be replaced after 8 years.

The weighted average cost of A in 9 years (Rs 875.86) is more than that of B in 8 years (Rs 840.11). Hence it is advisable to purchase B.

Example1 9.10 A scooter costs Rs 6000. The running cost and the salvage cost are given below. If the interest rate is 10% per year find when the scooter should be replaced.

Year	1	2	3	4	5	6	7
Running cost	1200	1400	1600	1800	2000	2400	3000
Salvage value	4000	2666	2000	1500	1000	600	600

Solution The salvage value is to be taken to account only at the end of the year. Hence we multiply C_n by v^{n-1} and S_n by v^n and obtain the total cost at the end of $(n - 1)^{\text{th}}$ year as $C + \Sigma C_n v^{n-1} - S_n v^n$. For the first year we find $C + C, v^0 - S_1 v$, for the second year we find $C + (C_1 + C_2 v) - S_2 V^2$ and so on.

At the end of the n^{th} year the average cost becomes

$$\frac{C + \sum C_n v^{n-1} - S_n v^n}{\sum v^{n-1}} \qquad (n = 1, 2, 3, \ldots.)$$

The following table gives the average cost at the end of each year.

C = 6000

Year	C_n	v^{n-1}	$C_n v^{n-1}$	$\Sigma C_n v^{n-1}$	S_n	v^n	$S_n v^n$	$\Sigma C_n v^{n-1} - S_n v^n$	$C + \Sigma C_n v^{n-1} - S_n v^n$	Σv^{n-1}	R
1	1200	1	1200	1200	4000	0.9091	3636.4	– 2436.4	3563.6	1	3563.6
2	1400	0.9091	1272.7	2472.7	2666	0.8264	2203.2	269.5	6269.5	1.9091	3284.0
3	1600	0.8264	1322.2	3794.9	2000	0.7513	1502.6	2292.3	8292.3	2.7355	3031.3
4	1800	0.7513	1352.3	5147.2	1500	0.6830	1024.5	4122.7	10122.7	3.4868	2903.1
5	2000	0.6830	1366.0	6513.2	1000	0.6209	620.9	5892.3	11892.3	4.1698	2852.0
6	2400	0.6209	1490.2	8003.4	600	0.5645	338.7	7664.7	13664.7	4.7907	2852.3
7	3000	0.5645	1693.5	9696.9	600	0.5132	307.9	9389	15389	5.3552	2873.6

$C_6 < R < C_7$ [2400 < 2852.3 < 3000]

Therefore it is advisable to replace the scooter at the end of 6^{th} year.

19.3 REPLACEMENT OF ITEMS THAT FAIL COMPLETELY

There are many situations in which items do not deteriorate with time but fail all on a sudden completely. It may not be possible to predict the time of failure. Hence we make use of the probability distribution of the failure time which is obtained from past experience.

Two types of replacement policies are followed in such situations. They are

(i) Individual replacement policy
(ii) Group replacement policy

Under individual replacement policy, an item (or equipment) is replaced immediately after its failure.

Under group replacement policy, all the items are collectively replaced after a specific period of time irrespective of whether they have failed or not. If any item fails before this time it will also be replaced immediately. The optimal period of replacement is determined by calculating the minimum total cost. This total cost is calculated using

(i) Probability of failure at time t
(ii) Number of items failing during time t
(iii) Cost of group replacement
(iv) Cost of individual replacement

If $p(x)$ is the probability that an item will fail at age x then its average age at failure is

$$E(x) = \sum_{x=1}^{k} x\,p(x)$$

If there are N items in the group then the number of failures per unit time is $\dfrac{N}{E(x)}$. Suppose C_1 is the cost of replacing per unit under group replacement and C_2 be the cost of individual replacement. We wish to take a decision about group replacement. Let $f(i)$ denote the number of failures and $C(t)$ denote the total cost at time t.

Then $C(t) = NC_1 + C_2 \sum_{i=1}^{t-1} f(i)$. The optimum value of t must satisfy the inequality,

$$tf(t-1) - \sum_{i=1}^{t-1} f(i) < \frac{NC_1}{C_2} < t\,f(t) - \sum_{i=1}^{t-1} f(i)$$

$$\therefore \quad tf(t) - \sum f(i) > \frac{N.C_1}{C_2}$$

$$\text{i.e.} \quad tf(t) > \frac{N.C_1}{C_2} + \sum f(i)$$

$$\text{i.e.} \quad C_2 f(t) > \frac{NC_1 + C_2 \sum f(i)}{t} > \frac{C(t)}{t} \quad \text{(Average cost)}$$

Thus group replacement must be made at the end of period t if the cost of individual replacement for the t^{th} period is greater than the average cost till the end of t periods. If the cost of individual replacement at the end of $(t-1)^{\text{th}}$ period is less than the average cost for t periods group replacement should not be done. In general we take one period of time as one year.

Example 19.11 The probability distribution of the failure time of a certain type of electric bulb is given below:

Week	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Prob. of failure	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.0

The cost of individual replacement is Rs 4 per bulb. The cost of group replacement is Re 1 per bulb. If there are 1000 bulbs in use find the optimal replacement policy under

(i) individual replacement
(ii) group replacement

Solution

(i) We calculate the probability of failure during each week as follows:

Week	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Prob. of failure	0.05	0.08	0.12	0.18	0.25	0.20	0.08	0.04

Expected life

$E(x) = \Sigma x\, p(x)$

$= 1(0.05) + 2\,(0.08) + 3\,(0.12)$
$+ 4(0.18) + 5\,(0.25) + 6\,(0.20)$
$+ 7(0.08) + 8\,(0.04)$
$= 4.62$

Average number of failures per week

$$= \frac{1000}{4.62} = 216$$

Cost of individual replacement per week is Rs 216×4 = Rs 864

(ii) Let N_i denote the total number of replacements made at the end of the i^{th} week

$N = 1000$

$N_1 = Np_1 = 1000\,(0.05) = 50$

$N_2 = Np_2 + N_1p_1$
$= 1000\,(0.08) + 50\,(0.05) = 82$

$N_3 = Np_3 + N_1p_2 + N_2\,p_1$
$= 1000\,(0.12) + 50\,(0.08) + 82\,(0.05)$
$= 128$

$N_4 = Np_4 + N_1p_3 + N_2p_2 + N_3p_1$
$= 1000\,(0.18) + 50\,(0.12) + 82\,(0.08) + 128\,(0.05)$
$= 199$

$N_5 = Np_5 + N_1p_4 + N_2p_3 + N_3p_4 + N_4p_1$
$= 1000\,(0.25) + 50\,(0.18) + 82\,(0.12) + 128\,(0.08) + 199\,(0.05)$
$= 289$

$N_6 = Np_6 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1$
$= 1000\,(0.20) + 50\,(0.25) + 82\,(0.18) + 128\,(0.12) + 199\,(0.08) + 289\,(0.05)$
$= 272$

$N_7 = Np_7 + N_1p_6 + N_2p_5 + N_3p_4 + N_4p_3 + N_5p_2 + N_6p_1$
$= 1000\,(0.08) + 50\,(0.20) + 82\,(0.25) + 128\,(0.18) + 199\,(0.12) + 289\,(0.08) + 272\,(0.05)$
$= 194$

$N_8 = Np_8 + N_1p_7 + N_2p_6 + N_3p_5 + N_4p_4 + N_5p_3 + N_6p_2 + N_7p_1$
$= 1000\,(0.04) + 50\,(0.08) + 82\,(0.20) + 128\,(0.25) + 199\,(0.18) + 289\,(0.12) + 272\,(0.08) + 194\,(0.05)$
$= 195$

Average cost at every week end is given by

End of week	*Total cost of group replacement*	*Average cost per week*
1	$1000 + 50 \times 4 = 1200$	1200
2	$1000 + (50 + 82)4 = 1528$	764
3	$1000 + (50 + 82 + 128)4 = 2040$	680 ⟶
4	$1000 + (50 + 82 + 128 + 199)4 = 2836$	709
5	$1000 + (50 + 82 + 128 + 199 + 289)4 = 3992$	798
6	$1000 + (50 + 82 + 128 + 199 + 289 + 272)4 = 5080$	847

We find that the weekly average cost is least at the end of the 3rd week. Hence group replacement is to be made at the end of 3rd week.

Note: Comparing the two types of replacement we find that the total cost of individual replacement for 3 weeks, is 3×864 = Rs 2592.

Under group replacement the total cost of replacement for 3 weeks is Rs 2040 only. Hence group replacement at the end of every 3 weeks is less expensive than individual replacement.

Example 19.12 The following table gives the probability distribution of the failure time of a machine.

Week	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Prob. of failure	.03	.04	.05	.06	.07	.08	0.09	0.16	0.20	0.22

The cost of repairing a broken machine is Rs 200. Preventive maintenance service is done for all the 30 machines collectively at Rs 15 per machine at the end of a period T. Find T so as to minimize the cost of maintenance.

Solution Given $N = 30$. Let N_i denote the number of service jobs done at the end of the i^{th} week

$N_1 = Np_1 = 30\,(0.03) = 0.9 \cong 1$

$N_2 = Np_2 + N_1p_1 = 30(04) + 1\,(0.03)$
$= 1.23 \cong 1$

$N_3 = Np_3 + N_1p_2 + N_2\,p_1$

$$
\begin{aligned}
&= 30\ (0.05) + 1\ (0.04) + 1\ (0.03)\\
&= 1.57 \cong\\
N_4 &= Np_4 + N_1p_3 + N_2p_2 + N_3p_1\\
&= 30\ (0.06) + 1(0.05) + 1\ (0.04)\\
&\quad + 2\ (0.03)\\
&= 1.95 \cong 2\\
N_5 &= Np_5 + N_1p_4 + N_2p_3 + N_3p_4 + N_4p_1\\
&= 30\ (0.07) + 1\ (0.06) + 1\ (0.05)\\
&\quad + 2\ (0.04) + 2\ (0.03)\\
&= 2.35 \cong 2\\
N_6 &= Np_6 + N_1p_5 + N_2p_4 + N_3p_3\\
&\qquad + N_4p_2 + N_5p_1\\
&= 30\ (0.08) + 1\ (0.07) + 1\ (0.06) + 2\ (0.05)\\
&\quad + 2\ (0.04) + 2(0.03)\\
&= 2.77 \cong 3\\
N_7 &= Np_7 + N_1p_6 + N_2p_5 + N_3p_4 + N_4p_3\\
&\quad + N_5p_2 + N_6p_1\\
&= 30\ (0.09) + 1\ (0.08) + 1\ (0.07) + 2(0.06)\\
&\quad + 2\ (0.05) + 2\ (0.04) + 3\ (0.03)\\
&= 3.24 \cong 3
\end{aligned}
$$

$$
\begin{aligned}
N_8 &= Np_8 + N_1p_7 + N_2p_6 + N_3p_5 + N_4p_4\\
&\quad + N_5p_3 + N_6p_2 + N_7p_1\\
&= 30\ (0.16) + 1\ (0.09) + 1\ (0.08)\\
&\quad + 2\ (0.07) + 2\ (0.06) + 2\ (0.05)\\
&\quad + 3\ (0.04) + 3\ (0.03)\\
&= 5.54 \cong 6\\
N_9 &= Np_9 + N_1p_8 + N_2p_7 + N_3p_6 + N_4p_5\\
&\quad + N_5p_4 + N_6p_3 + N_7p_2 + N_8p_1\\
&= 30\ (0.20) + 1\ (0.16) + 1\ (0.09)\\
&\quad + 2\ (0.08) + 2\ (0.07) + 2\ (0.06)\\
&\quad + 3\ (0.05) + 3\ (0.04) + 6\ (.03)\\
&= 7.12 \cong 7\\
N_{10} &= Np_{10} + N_1p_9 + N_2p_8 + N_3p_7 + N_4p_6\\
&\quad + N_5p_5 + N_6p_4 + Np_4 + N_8p + N_9p_1\\
&= 30\ (0.22) + 1\ (0.20) + 1\ (0.16)\\
&\quad + 2\ (0.09) + 2\ (0.08) + 2\ (0.07)\\
&\quad + 3\ (0.06) + 3\ (0.05) + 6\ (0.04)\\
&\quad + 7\ (0.03)\\
&= 8.22 \cong 8
\end{aligned}
$$

End of week	*Total cost of maintenance*	*Average cost (Rs)*
1	(30 × 15) + (1)200 = 650	650
2	(30 × 15) + (1 + 1)200 = 850	425
3	(30 × 15) + (1 + 1 + 2)200 = 1250	417
4	(30 × 15) + (1 + 1 + 2 + 2)200 = 1650	412
5	(30 × 15) + (1 + 1 + 2 + 2 + 2)200 = 2050	410 ⟶
6	(30 × 15) + (1 + 1 + 2 + 2 + 2 + 3)200 = 2650	442
7	(30 × 15) + (1 + 1 + 2 + 2 + 2 + 3 + 3)200 = 3250	464
8	(30 × 15) + (1 + 1 + 2 + 2 + 2 + 3 + 3 + 6)200 = 4450	556
9	(30 × 15) + (1 + 1 + 2 + 2 + 2 + 3 + 3 + 6 + 7)200 = 5850	650
10	(30 × 15) + (1 + 1 + 2 + 2 + 2 + 3 + 3 + 6 + 7 + 8)200 = 7450	745

The average cost is minimum at the end of 5 weeks.
Hence group service should be done at the end of the 5^{th} week.

EXERCISES

1. The cost of a machine is Rs 6000. The resale value and maintenance costs every year are given below.

Year	*1*	*2*	*3*	*4*	*5*	*6*
Maintenance cost	1000	1200	1400	1800	2300	2800
Resale value	3000	1500	750	325	200	200

Determine the best time for replacing the machine.

2. A machine costs Rs 6100. The scrap value is Rs 100. The maintenance costs are given below:

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

3. The purchase price of a taxi is Rs 60000. Its resale value decreases by Rs 6000 per year. The operating cost every year is given below:

Year	*1*	*2*	*3*	*4*	*5*
Maintenance cost	1000	12000	15000	18000	20000

After 5 years the operating cost is Rs 6000k (k = 6, 7, 8, 9, 10,…) every year. What is the best replacement policy?

4. A vehicle costs Rs 50000. Running costs and resale values every year are given below:

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost	5000	6000	7000	9000	12500	16000	18000
Resale value	30000	15000	7500	3750	2000	2000	2000

Thereafter running cost increases by Rs 2000, but resale value remains constant at Rs 2000. At what time is a replacement due?

5. The cost of a machine is Rs 5000. The maintenance cost during the n^{th} year is given by $c_n = 500(n - 1)$ where n = 1, 2, 3,… Suppose the discount rate is 5% per year. After how many years will it be economical to replace the machine?

6. An equipment costs Rs 3000. The running costs are given below

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost	500	600	800	1000	1300	1600	2000

If money is worth 10% per year determine the optimal period of replacing the equipment.

7. The cost of a machine A is Rs 5000. The running costs are Rs 800 per year for each of the first five years, increasing by Rs 200 per year thereafter. Another machine B costs Rs 2500 but has running costs Rs 1200 per year for six years, increasing by Rs 200 per year thereafter. If money is worth 10% per year determine which machine should be purchased.

8. The following failure rates have been observed for an electric bulb

End of week	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Prob. of failure	0.10	0.15	0.20	0.35	0.65	0.90	0.95	1.00

The cost of replacing an individual bulb is Rs 3 and the cost of group replacement is Re 1 per bulb. Discuss whether group replacement is better than individual replacement, if there are 1000 bulbs.

9. Determine the optimal group replacement policy if there are totally 1000 bulbs and the cost of individual replacement is Rs 5 and the cost of group replacement is Rs 2 per bulb. The failure rate is given by the following table.

End of week	*1*	*2*	*3*	*4*	*5*	*6*
Prob. of failure	0.10	0.25	0.50	0.85	0.97	1.0

Find the optimal group replacement policy.

10. A computer contains 10000 resistors. The cost of replacing a resistor is Rs 2. If all the resistors are replaced at the same time it costs only Rs 0.75 per resistor. The percentage of resistors surviving at the end of each month is given by the following table:

End of month	*1*	*2*	*3*	*4*	*5*	*6*
Percentage of survival	96	90	65	35	20	0

What is the optimal replacement plan?

11. The cost of a machine is Rs 60000. The following table gives the data on running the machine.

Year	*1*	*2*	*3*	*4*	*5*
Resale value	42000	30000	20400	14400	9650
Cost of spares	4000	4270	4880	5700	6800
Cost of labour	14000	16000	18000	21000	25000

Determine the optimum period of replacement.

12. Purchase price of machine A is Rs 200000. The particulars on running cost and resale value are given below:

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost	30000	38000	46000	58000	72000	90000	110000
Resale value	100000	50000	25000	12000	8000	8000	8000

(i) What is the optimum period of replacement?

(ii) When the machine *A* is two years old, a new model machine *B* is available. The optimum period of replacement for *B* is 4 years with an average cost of Rs 72000. Should *A* be replaced with *B*? If so when?

13. For a machine the following data are available:

Year	*1*	*2*	*3*	*4*	*5*	*6*
Cost of spares	200	400	700	1000	1400	1600
Maintenance cost	1200	1200	1400	1600	2000	2600
Loss due to breakdown	600	800	700	1000	1200	1600
Resale value	6000	3000	1500	800	400	400

If the present cost of the machine is Rs 12000, find the optimum period of replacement.

14. A company is considering the purchase of a new machine at a cost of Rs 15000. The expected life of the machine is 8 years. The salvage value at the end of the life will be Rs 3000. The annual running cost is Rs 7000. Assuming the interest rate of 5%, calculate the present worth of the future cost after 8 years. The presently owned machine has an annual operating cost of Rs 5000. Its maintenance cost is Rs 1500 in the second year and increases at Rs 500 in the subsequent years. Compare the two machines to decide whether the new machine is to be purchased.

15. A manual stamper costs Rs 1000 and is expected to last two years with an operating cost of Rs 4000 per year. An automatic stamper has a cost of Rs 3000 and expected life of four years with an operating cost of Rs 3000 per year. If money is worth 10% per annum determine which stamper should be purchased.

16. A truck is priced at Rs 60000 and the running costs are Rs 6000 for each of the first four years, increasing by Rs 2000 per year during the subsequent years. If money is worth 10% per year determine when the truck is to be replaced assuming that it has no resale value.

17. If it is required to get a return of 10% per annum on the investment, which of the following plans is preferable?

	Plan A (Rs)	*Plan B (Rs)*
Cost	200000	250000
Scrap value after 15 years	150000	180000
Annual revenue	25000	30000

18. There are 10000 resistors in a computer. The cost of individual replacement of a resistor is Re 1. For group replacement the cost is Rs 0.35 only. The percentage of survival of the resistors at the end of the month *t* is given below:

Month	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Percentage of survival	100	97	90	70	30	15	0

Determine the optimal replacement plan.

19. The following failure rates have been observed for an electric bulb.

End of week	1	2	3	4	5	6	7	8	9	10	11
Prob. of failure	0.01	0.04	0.09	0.16	0.26	0.41	0.61	0.76	0.87	0.95	1.0

Individual replacement costs Rs 1.25 and group replacement costs Rs 0.50 per bulb. Determine the optimal group replacement policy.

20. The following failure rates have been observed for a certain type of fuse.

End of week	*1*	*2*	*3*	*4*	*5*
Percentage of failure	5	15	35	57	100

There are 1000 fuses. Cost of individual replacement is Rs 5 and that of group replacement is Rs 1.25 per fuse. At what intervals should the group replacement be made?

ANSWERS

1. At the end of 5^{th} year.
2. At the end of 6^{th} year.
3. At the end of first year.
4. At the end of 6^{th} year.
5. At the end of 5^{th} year.
6. At the end of 4^{th} year.
7. It is advisable to purchase machine *B*.
8. Group replacement after 3 weeks is advisable.
9. Individual replacement is cheaper Rs 1500 per week.
   Group replacement may be done after 3 weeks. Total cost = Rs 4705.
10. Optimal plan is group replacement after 2 years.
11. At the end of the 4^{th} year.
12. (i) At the end of the 5^{th} year.
    (ii) A should be replaced with *B*, when it is four years old.
13. After 5 years.
14. The present worth of the new machine after 8 years is Rs 58212. The present worth of the old machine after 8 years is Rs 44103.20. Hence the new machine need not be purchased.
15. Purchase an automatic stamper.
16. After 9 years.
17. Plan A.
18. Replace every 3 months.
19. After 6 weeks.
20. After every 2 weeks.

20

Project Scheduling (PERT and CPM)

CONCEPT REVIEW

20.1 INTRODUCTION

A project is a collection of inter-related activities (tasks) which must be completed in a specified time according to a specified sequence. A project requires resources such as workers, materials, money and facilities. Some examples of project are: construction of a bridge, highway power plant, maintenance of oil refinery or an aeroplane, marketing a new product etc. Our aim is to develop a sequence of activities of the project so that the project completion time and cost are properly balanced and kept at the optimum level. There are two techniques used for project scheduling. They are

(i) Critical Path Method (CPM) and

(ii) Programme Evaluation and Review Technique (PERT)

These two techniques are applied using network diagrams.

20.2 NETWORK DIAGRAMS

All the techniques of project completion start with the representation of the project as a network of activities. Networks are diagrams representing a sequence of activities involved in the project work. Network helps us to minimize the trouble spots, delays etc., in the various stages of the project.

20.2.1 Activity

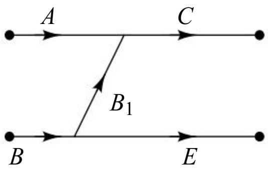
Activity is a part of the project, which takes time, effort, money and other resources. In the network an activity is represented by an arrow whose tail represents the start and head represents the end of the activity.

20.2.2 Event (Node)

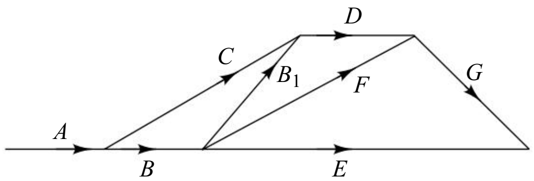
The beginning point and the end point of an activity are called *events* (*nodes*). It is represented by a number in a circle. The number of the head node is always greater than that of the tail node. A network is a combination of activities and events involved in a project.

20.2.3 Network Construction

From the given set of activities we decide the beginning and the ending activities. According to the precedence order the activities are put in a logical sequence. If an activity *A* is to be completed before starting another activity B then we write $A < B$. A is preceding B or B is succeeding A. Two activities A and B can also be performed simultaneously. Suppose A, $B < C$ and $B < E$ then it is represented by introducing a dummy activity B_1 as follows:



The network diagram of the project having the activities A, B, C, D, E, F, G where $A < B$, C; B, $C < D$; $B < E$, F; D, $F < G$ is given by



There is no predecessor for A and hence A is the beginning activity. B_1 is a dummy activity. There is no successor for E and G. Hence E and G are the final activities.

20.2.4 Fulkerson's Rule

After the network is drawn with the given activities in their order we have to assign numbers to the events (nodes). The numbering is to be done according to Fulkerson's rule which gives the following steps:

(i) The initial node which has only outgoing arrows is assigned (1).

(ii) Delete all the arrows going out from (1). We get a set of initial nodes. Assign numbers (2), (3), (4) etc. to these nodes.

(iii) Follow this method to the nodes (2) (3) (4)... and assign successive numbers to the newly obtained nodes.

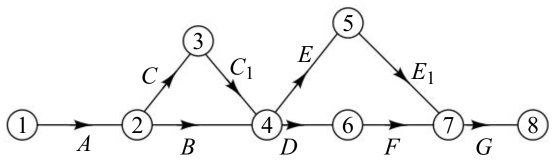
(iv) Finally we are left with a single node with no arrow going out. Assign the last number to this end node.

(v) Always the numbers of the head node must be greater than that of the tail node of each activity.

Example 20.1 Draw a network diagram for the following set of activities:

$A < B$, C; $B < D$, E; $C < D$; $D < F$; $E < G$; $F < G$

Solution

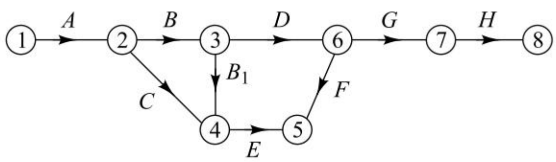


In this diagram $C < D$. Hence we introduce a dummy activity C_1. Similarly, $E < G$. Therefore another dummy activity E_1 is introduced.

Example 20.2 Draw a network diagram for the following set of activities:

$A < B$, C; $B < D$, E; $C < E$; $E < F$; D, $F < G$; $G < H$

Solution

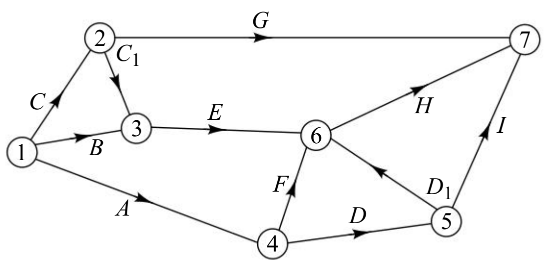


(B_1 is dummy)

Example 20.3 Draw a network for the following set of activities

Activity	A	B	C	D	E	F	G	H	I
Immediate predecessor	–	–	–	A	B,C	A	C	D,E,F	D

Solution



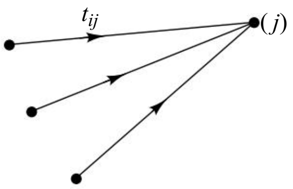
(C_1 and D_1 are dummy)

Note: The time taken for completing the activity (i, j) is given by t_{ij}. In the network diagram t_{ij} is marked by the side of the arrow representing the activity (i,j).

20.3 CRITICAL PATH METHOD

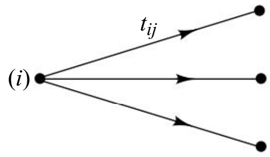
20.3.1 Time Calculation

There are some terms which are associated with the activities of a project. The Earliest Starting Time of the activity (i, j) is denoted by ES_i and the Earliest Finishing Time of the activity (i, j) is denoted by ES_j. For the activity (i, j), $ES_j = ES_i + t_{ij}$. If more than one activity end at the node (j) then $ES_j = \max\limits_{i} \{ES_i + t_{ij}\}$.



For the initial activity $ES_1 = 0$. If (n) is the last node then ES_n represents the time taken to complete the whole project. Also, $ES_n = EF_n$.

Starting from the last node we proceed backward to calculate the latest times. The Latest Starting Time of the activity (i, j) is denoted by LS_i and the Latest Finishing Time of the activity (i, j) is denoted by LS_j. For the activity (i, j), $LS_i = LS_j - t_{ij}$. If more than one activity start from (i) then $LS_i = \min\limits_{j} \{LS_j - t_{ij}\}$



For the last node (n) $ES_n = LS_n$. For the activities (i, j) and (j, k) the earliest starting time ES_j of (j, k) is the same as the earliest finishing time EF_j of the activity (i, j). Similarly, $LS_j = LF_j$.

An activity is called *critical activity* if a delay in starting that activity will cause further delay in completing the project. For a critical activity (i, j) the following equalities hold

(i) $ES_i = LS_i$
(ii) $ES_j = LS_j$
(iii) $ES_j = ES_i = LS_j - LS_i = t_{ij}$

Total float for an activity (i, j) is the difference between the maximum time available to finish the activity and the time required to complete it. [Total float is simply called as *Float.*] Total float $= LS_j - ES_i - t_{ij}$

Free float is the time by which an activity can be delayed beyond its earliest finish time without affecting the earliest start time of a succeeding activity

Free float = Total float – $(LS_j - ES_j)$

Independent Float = Total float – $(LS_i - ES_i)$

The path formed by all the critical activities is called the *critical path.*

20.3.2 Step-by-Step Procedure to Find the Critical Path

1. Draw the network diagram of the project and indicate the time for each activity.
2. Calculate the earliest starting time at each node.
3. Calculate the latest starting time at each node.
4. Prepare a table giving the time t_{ij}, ES_i, ES_j, LS_i, and LS_j for each activity.
5. Find the difference between the earliest time ES_i and the latest time LS_i. Note down the difference (float) $LS_i - ES_i$, $(LF_j - EF_j)$ for each activity.
6. Choose the critical activities with zero float and form the critical path by double line.
7. The sum Σt_{ij} of the critical activities gives the time of completion of the project. (ES_n also denotes the time of completion of the project.)

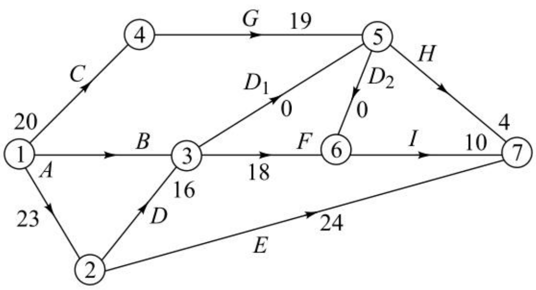
Example 20.4 A project consists of a series of jobs $A, B, C, D, E, F, G, H, I$ such that $A < D, E$; $B, D < F$; $C < G$; $B < H$; $F, G < I$

The time of completion of each job is given below:

Job	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Time (days)	23	8	20	16	24	18	19	4	10

Find the critical path and the minimum time required to complete the project.

Solution The network of the given project is as shown below:



D_1 and D_2 are dummy activities with $t_{ij} = 0$.

The earliest and latest starting times at the nodes are calculated as given below.

$ES_1 = 0 \quad ES_2 = ES_1 + t_{12} = 0+23 = 23$

$ES_3 = \text{Max}\ [ES_1 + t_{13}, ES_2 + t_{23}]$
$= \text{Max}\ [0 + 8, 23 + 16] = 39$

$ES_4 = ES_1 + t_{14} = 0 + 20 = 20$

$ES_5 = \text{Max}\ [ES_4 + t_{45}, ES_3 + t_{35}]$
$= \text{Max}\ [20 + 19, 39 + 0] = 39$

$ES_6 = \text{Max}\ [ES_3 + t_{36}, ES_5 + T_{56}]$
$= \text{Max}\ [39 + 18, 39 + 0] = 57$

$ES_7 = \text{Max}\ [ES_2 + t_{27}, ES_6 + T_{67}, ES_5 = t_{57}]$
$= \text{Max}\ [23 + 24, 57 + 10, 39 + 4] = 67$

Now,

$LS_7 = ES_7 = 67$

$LS_6 = LS_7 - t_{67} = 67 - 10 = 57$

$LS_5 = \text{Min}\ [LS_6 - t_{56}, LS_7 - t_{57}]$
$= \text{Min}\ [57 - 0, 67 - 4] = 57$

$LS_4 = LS_5 - t_{45} = 57 - 19 = 38$

$LS_3 = \text{Min}\ [LS_6 - t_{36}, LS_5 - t_{35}]$
$= \text{Min}\ [57 - 18, 57 - 0] = 39$

$LS_2 = \text{Min}\ [LS_3 - t_{23}, LS_7 - t_{27}]$
$= \text{Min}\ [39 - 16, 67 - 24] = 23$

$LS_1 = \text{Min}\ [LS_4 - t_{14}, LS_3 - t_{13}, LS_2 - t_{12}]$
$= \text{Min}\ [38 - 20, 39 - 8, 23 - 23] = 0$

To find the critical nodes we prepare the table as given below:

$[EF_j = ES_i + t_{ij};\ LS_i = LF_j - t_{ij} = LS_j - t_{ij}$
$\text{Float} = LS_i - ES_i = LS_j - ES_j]$

Job	*Time*	*Earliest time* Start ES_i	Finish ES_j	*Latest time* Start LS_j	Finish ES_j	*Float*
(1, 2)	23	0	23	0	23	0 →
(1, 3)	8	0	8	31	39	31
(1, 4)	20	0	20	18	38	18
(2, 3)	16	23	39	23	39	0 →
(2, 7)	24	23	47	43	67	20
(3, 5)	0	39	39	57	57	18
(3, 6)	18	39	57	39	57	0 →
(4, 5)	19	20	39	38	57	18
(5, 6)	0	39	39	57	57	18
(5, 7)	4	39	43	63	67	24
(6, 7)	10	57	67	57	67	0 →

From the table we find that the critical activities are (1, 2), (2, 3), (3, 6) and (6, 7). Hence the critical path is 1 – 2 – 3 – 6 – 7. The minimum time of completion of the project is 23 + 16 + 18 + 10 = 67 days.

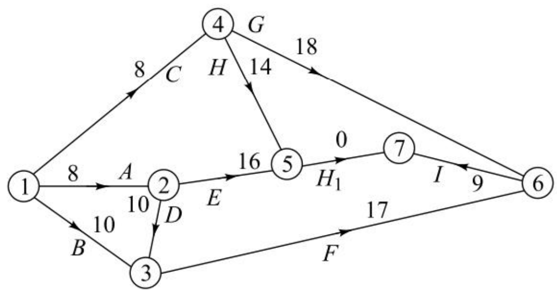
Example 20.5 A project consists of jobs *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I* such that $A < D$; $A < E$; $B < F$; $D < F$; $C < G$; $C < H$; $F < I$; $G < I$

The time taken for each job is given below.

Job	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Time (days)	8	10	8	10	16	17	18	14	9

Draw the network diagram. Find the critical path and minimum time of completion of the project.

Solution The network diagram is given below:



(H_1 is a dummy activity)

The earliest and latest times are calculated as follows:

$$
\begin{aligned}
ES_1 &= 0 \qquad ES_2 = ES_1 + t_{12} = 0 + 8 = 8\\
ES_3 &= \text{Max } [ES_1 + t_{13}, ES_2 + t_{23}]\\
&= \text{Max } [0 + 10, 8 + 10] = 18\\
ES_4 &= ES_1 + t_{14} = 0 + 8 = 8\\
ES_5 &= \text{Max } [ES_2 + t_{25}, ES_4 + t_{45}]\\
&= \text{Max } [8 + 16, 8 + 14] = 24\\
ES_6 &= \text{Max } [ES_4 + t_{46}, ES_3 + t_{36}]\\
&= \text{Max } [8 + 18, 18 + 17] = 35\\
ES_7 &= \text{Max } [ES_5 + t_{57}, ES_6 + t_{67}]\\
&= \text{Max } [24 + 0, 35 + 9] = 44\\
ES_7 &= ES_7 = 44\\
LS_6 &= LS_7 - t_{67} = 44 - 9 = 35\\
LS_5 &= LS_7 - t_{57} = 44 - 0 = 44\\
LS_4 &= \text{Min } [LS_6 - t_{46}, LS_5 - t_{45}]\\
&= \text{Min } [35 - 18, 44 - 14] = 17\\
LS_3 &= LS_6 - t_{36} = 35 - 17 = 18\\
LS_2 &= \text{Min } [LS_5 - t_{25}, LS_3 - t_{23}]\\
&= \text{Min } [44 - 16, 18 - 10] = 8\\
LS_1 &= \text{Min } [LS_4 - t_{14}, LS_3 - t_{13}, LS_2 - t_{12}]\\
&= \text{Min } [17 - 8, 18 - 10, 8 - 8] = 8
\end{aligned}
$$

The table giving the critical activities is

Job	*Time*	*Earliest time*		*Latest time*		*Float*	
		Start ES_i	*Finish* ES_j	*Start* LS_i	*Finish* LS_j		
(1, 2)	8	0	8	0	8	0	→
(1, 3)	10	0	10	8	18	8	
(1, 4)	8	0	8	9	17	9	
(2, 3)	10	8	18	8	18	0	→
(2, 5)	16	8	24	28	44	20	
(3, 6)	17	18	35	18	35	0	→
(4, 6)	18	8	26	17	35	9	
(4, 5)	14	8	22	30	44	22	
(5, 7)	0	24	24	44	44	20	
(6, 7)	9	35	44	35	44	0	→

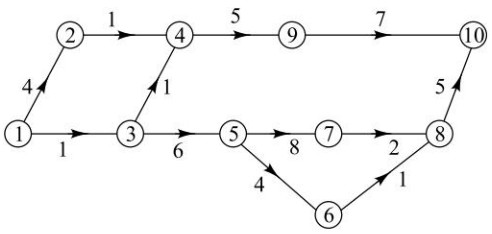
From the table we find that the critical path is 1 – 2 – 3 – 6 – 7.

Minimum time of completion is 44 units.

Example 20.6 Find the critical path of the project with the following activities

Job	*(1, 2)*	*(1, 3)*	*(2, 4)*	*(3, 4)*	*(3, 5)*	*(4, 9)*	*(5, 6)*	*(5, 7)*	*(6, 8)*	*(7, 8)*	*(8, 10)*	*(9, 10)*
Time	4	1	1	1	6	5	4	8	1	2	5	7

Solution The network diagram of the project is given below:



$$
\begin{aligned}
ES_1 &= 0\\
ES_2 &= ES_1 + t_{12} = 4\\
ES_3 &= ES_1 + t_{13} = 1\\
ES_4 &= \text{Max } [4 + 1, 1 + 1] = 5\\
ES_5 &= ES_3 + t_{35} = 7\\
ES_6 &= ES_5 + t_{56} = 11\\
ES_7 &= ES_5 + t_{57} = 15\\
ES_8 &= \text{Max } [11 + 1, 15 + 2] = 17\\
ES_9 &= ES_4 + t_{49} = 10\\
ES_{10} &= \text{Max } [10 + 7, 17 + 5] = 22
\end{aligned}
$$

Again,

$$
\begin{aligned}
LS_{10} &= 22\\
LS_9 &= LS_{10} - t_{9,10} = 15\\
LS_8 &= LS_{10} - t_{8,10} = 17\\
LS_7 &= LS_8 - t_{7,8} = 15\\
LS_6 &= LS_8 - t_{68} = 16\\
LS_5 &= \text{Min } [16 - 4, 15 - 8] = 7\\
LS_4 &= LS_9 - t_{49} = 10\\
LS_3 &= \text{Min } [10 - 1, 7 - 6] = 1\\
LS_2 &= LS_4 - t_{24} = 9\\
LS_1 &= \text{Min } [9 - 4, 1 - 1] = 0
\end{aligned}
$$

The following table gives the slack time for each activity.

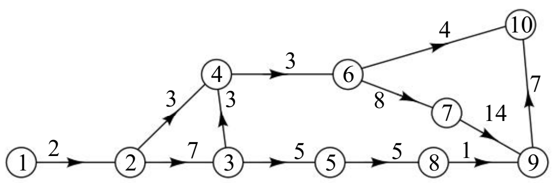
Job	Time	Earliest time		Latest time		Float	
		ES_i	ES_j	Start LS_i	Finish LS_j		
(1, 2)	4	0	4	5	9	5	
(1, 3)	1	0	1	0	1	0	→
(2, 4)	1	4	5	9	10	5	
(3, 4)	1	1	2	9	10	8	
(3, 5)	6	1	7	1	7	0	→
(4, 9)	5	5	10	10	15	5	
(5, 6)	4	7	11	12	16	5	
(5, 7)	8	7	15	7	15	0	
(6, 8)	1	11	12	16	17	5	
(7, 8)	2	15	17	15	17	0	→
(8, 10)	5	17	22	17	22	0	→
(9, 10)	7	10	17	15	22	5	

The critical path is
1 – 3 – 5 – 7 – 8 – 10
Time of completion is 22 days.

Example 20.7 Find the critical path of a project having the tasks as given below:

Job	Time	Job	Time
(1, 2)	2	(5, 8)	5
(2, 3)	7	(6, 7)	8
(2, 4)	3	(6, 10)	4
(3, 4)	3	(7, 9)	4
(3, 5)	5	(8, 9)	1
(4, 6)	3	(9, 10)	7

Solution The network diagram is



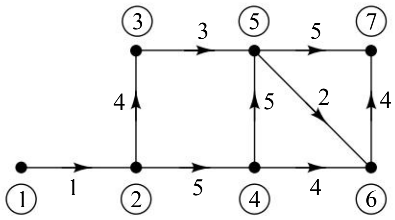
$ES_1 = 0$
$ES_2 = 0 + 2 = 2$
$ES_3 = 2 + 7 = 9$
$ES_4 = \text{Max } [5, 12] = 12$
$ES_5 = 9 + 5 = 14$
$ES_6 = 12 + 3 = 15$
$ES_7 = 15 + 8 = 23$
$ES_8 = 14 + 5 = 19$
$ES_9 = \text{Max } [23 + 4, 19 + 1] = 27$
$ES_{10} = \text{Max } [15 + 4, 27 + 7] = 34$
$LS_{10} = 34$
$LS_9 = 34 - 7 = 27$
$LS_8 = 27 - 1 = 26$
$LS_7 = 27 - 4 = 23$
$LS_6 = \text{Min } [30, 15] = 15$
$LS_5 = 26 - 5 = 21$
$LS_4 = 15 - 3 = 12$
$LS_3 = \text{Min } [16, 9] = 9$
$LS_2 = \text{Min } [2, 9] = 2$
$LS_1 = 2 - 2 = 0$

Job	Time	Earliest time		Latest time		Float	
		Start- ES_i	Finish- ES_j	Start- LS_i	Finish- LS_j		
(1, 2)	2	0	2	0	2	0	→
(2, 3)	7	2	9	2	9	0	→
(2, 4)	3	2	5	9	12	7	
(3, 4)	3	9	12	9	12	0	→
(3, 5)	5	9	14	16	21	7	
(4, 6)	3	12	15	12	15	0	→
(5, 8)	5	14	19	21	26	7	
(6, 7)	8	15	23	15	23	0	→
(6, 10)	4	15	19	30	34	15	
(7, 9)	4	23	27	23	27	0	→
(8, 9)	1	19	20	26	27	7	
(9, 10)	7	27	34	27	34	0	→

The critical activities are (1, 2), (2, 3), (3, 4), (4, 6), (6, 7), (7, 9), (9, 10). The critical path is 1 – 2 – 3 – 4 – 6 – 7 – 9 – 10

Time of completion is 34 days

Example 20.8 Find the critical path of the project represented by the following network:



Solution The earliest and latest times are calculated as follows:

$$ES_1 = 0; \quad ES_2 = 1; \quad ES_3 = 5$$
$$ES_4 = 6; \quad ES_5 = \text{Max } [8, 11] = 11$$
$$ES_6 = \text{Max } [13, 10] = 13$$
$$ES_7 = \text{Max } [16, 17] = 17$$
$$LS_7 = 17; \quad LS_6 = 13$$
$$LS_5 = \text{Min } [12, 11] = 11$$
$$LS_4 = \text{Min } [6, 9] = 6; \quad LS_3 = 8$$
$$LS_2 = \text{Min } [4, 1] = 1; \quad LS_1 = 0$$

Job	*Time*	*Earliest time*		*Latest time*		*Float*	
		Start ES_i	*Finish* ES_j	*Start* LS_i	*Finish* LS_j	LS_i–ES_i	
(1, 2)	1	0	1	0	1	0	→
(2, 3)	4	1	5	4	8	3	
(2, 4)	5	1	6	1	6	0	→
(3, 5)	3	5	8	8	11	3	
(4, 5)	5	6	11	6	11	0	→
(4, 6)	4	6	10	9	13	3	
(5, 6)	2	11	13	11	13	0	→
(5, 7)	5	11	16	12	17	1	
(6, 7)	4	13	17	13	17	0	→

The critical path is 1 – 2 – 4 – 5 – 6 – 7. Time of completion is 17 units of time.

Example 20.9 A company plans the following activities promoting its business:

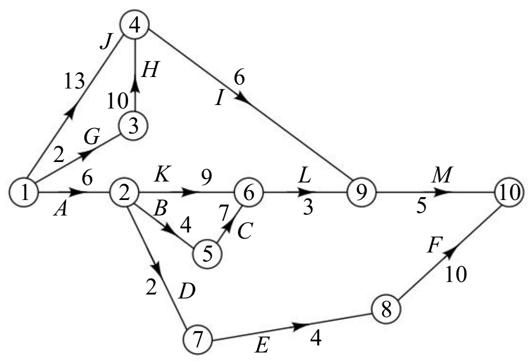
Activity	*Description*	*Time (Weeks)*	*Preceding activity*
A	Organize sales office	6	-
B	Hire salesmen	4	*A*
C	Train salesmen	7	*B*
D	Select advertising agency	2	*A*
E	Plan advertising campaign	4	*D*
F	Conduct campaign	10	*E*
G	Design package	2	-
H	Set up packing facilities	10	*G*
I	Packing initial stocks	6	*H, J*
J	Place order with manufacturer	13	-
K	Select distributors	9	*A*
L	Sell to distributors	3	*C, K*
M	Transport the stocks	5	*I, L*

(i) Draw the network diagram

(ii) Determine the critical path

(iii) Find the total float, free float and the independent float of the activities

Solution The network diagram for the given project is given below:



The earliest and latest times are calculated as follows:

$$ES_1 = 0; \quad ES_2 = 0 + 6 = 6$$
$$ES_3 = 0 + 2 = 2$$
$$ES_4 = \text{Max } [\,0 + 13,\ 2 + 10] = 13$$
$$ES_5 = 6 + 4 = 10$$
$$ES_6 = \text{Max } [\,6 + 9,\ 10 + 7] = 17$$
$$ES_7 = 6 + 2 = 8$$
$$ES_8 = 8 + 4 = 12$$
$$ES_9 = \text{Max } [13 + 6,\ 17 + 3] = 20$$
$$ES_{10} = \text{Max } [12 + 10,\ 20 + 5] = 25$$
$$LS_{10} = 25$$
$$LS_9 = 25 - 5 = 20$$
$$LS_8 = 25 - 10 = 15$$
$$LS_7 = 15 - 4 = 11$$
$$LS_6 = 20 - 3 = 17$$
$$LS_5 = 17 - 7 = 10$$
$$LS_4 = 20 - 6 = 14$$
$$LS_3 = 14 - 10 = 4$$
$$LS_2 = \text{Min } [\,10 - 4,\ 17 - 9,\ 11 - 2] = 6$$
$$LS_1 = \text{Min } [\,6 - 6,\ 4 - 2,\ 14 - 13] = 0$$

The following table gives the critical path and the float times:

Job (i, j)	t_{ij}	Earliest time		Latest time		Total float		Free float	Independent float
		ES_i	ES_j	LS_i	LS_j	$LS_i - ES_i$			
(1, 2)	6	0	6	0	6	0	→	0	0
(1, 3)	2	0	2	2	4	2		0	2
(1, 4)	13	0	13	1	14	1		0	1
(2, 5)	4	6	10	6	10	0	→	0	0
(2, 6)	9	6	15	8	17	2		2	2
(2, 7)	2	6	8	9	11	3		0	3
(3, 4)	10	2	12	4	14	2		1	0
(4, 9)	6	13	19	14	20	1		1	0
(5, 6)	7	10	17	10	17	0	→	0	0
(6, 9)	3	17	20	17	20	0	→	0	0
(7, 8)	4	8	12	11	15	3		0	0
(8, 10)	10	12	22	15	25	3		3	0
(9, 10)	5	20	25	20	25	0	→	0	0

The critical path is 1 – 2 – 5 – 6 – 9 – 10. The time of completion is 25 weeks.

Notes:

(i) The difference between slack time and float time is that slack is used for nodes whereas float is applied for activities.

(ii) Total float of an activity (i, j) is $LS_i - ES_i$ (Slack at i) or $LS_j - ES_j$ (Slack at j)

(iii) Free float = Total float – Slack time at j.

(iv) Independent float = Total float – Slack time at i

(v) Negative value of a float is considered as zero.

20.4 PERT CALCULATIONS

20.4.1 CPM and PERT

The following facts help to distinguish CPM and PERT calculations.

(i) In CPM the time of an activity is deterministic. But in PERT the time for each activity is probabilistic. We are given three different time estimates t_o = optimistic time (shortest possible time to complete the activity), t_m = most likely time (normal time the activity would take in general), t_p = pessimistic time (longest time taken for the activity if things go wrong).We find the expected time t_e for each activity using the formula

$$t_e = \frac{1}{6}\left(t_o + 4t_m + t_p\right)$$

Variance of the distribution of time for each activity is calculated using the formula

$$\sigma_{ij}^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

(ii) In CPM we find the minimum time required to complete the project without considering the cost involved. But in PERT the cost involved in each activity is also taken into account. The normal time required for each activity can be crashed to the least time required at an increased cost. We crash the activities in a systematic way and find the least possible time to complete the project at an optimum cost.

(iii) In CPM, the time calculations are activity oriented. But in PERT each node represents an event and the calculations are based on these events.

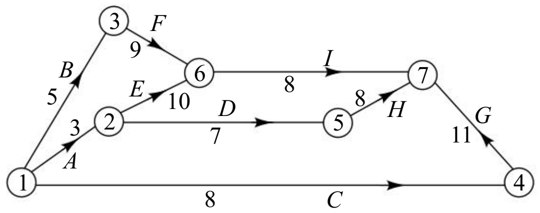
(iv) In CPM the time of completion of the project is of deterministic nature where as in PERT we can find only the probability of completion of the project within a stipulated time.

20.4.2 Time Calculations and Critical Path

Instead of calculating the starting and finishing times of an activity, we find the earliest expected time $E(\mu_i)$ of the event (i) and the latest expected time $E(L_i)$ of the event (i) $E(\mu_i)$ is found as in the case of ES_i in CPM. $E(\mu_i) = \underset{i}{\text{Max}}\,[E(\mu_i) + t_{ij}]$ Similarly $E(L_i) = \underset{j}{\text{Min}}\,[E(L_j) - t_{ij}]$. If $E(\mu_i) = E(L_i)$ then (i) is a *critical node.* The path joining the critical nodes is the *critical path.*

Example 20.10 A project is represented by the network given below and has the following data:

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Optimistic time t_o	2	3	5	5	8	7	9	3	6
Most likely time t_m	3	5	8	7	10	9	11	8	8
Pessimistic time t_p	4	7	11	9	12	11	13	13	10



Determine the following:

(i) Expected task times, t_e

(ii) Variance of the tasks, σ^2

(iii) Critical path

Solution We calculate the expected time and the variance of each task as given below:

$$t_e = \frac{1}{6}(t_o + 4t_m + t_p)\,\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

Task	t_0	t_m	t_p	t_e	σ^2
(1, 2)	2	3	4	3	0.111
(1, 3)	3	5	7	5	0.444
(1, 4)	5	8	11	8	1
(2, 5)	5	7	9	7	0.444
(2, 6)	8	10	12	10	0.444
(3, 6)	7	9	11	9	0.444
(4, 7)	9	11	13	11	0.444
(5, 7)	3	8	13	8	2.778
(6, 7)	6	8	10	8	0.444

Time calculations

$E(\mu_1) = 0$

$E(\mu_2) = 0 + 3 = 3$

$E(\mu_3) = 0 + 5 = 5$

$E(\mu_4) = 0 + 8 = 8$

$E(\mu_5) = 3 + 7 = 10$

$E(\mu_6) = \text{Max}\,[5 + 9, 3 + 10] = 14$

$E(\mu_7) = \text{Max}\,[10 + 8, 14 + 8, 8 + 11] = 22$

$E(L_7) = 22;\quad E(L_6) = 22 - 8 = 14$

$E(L_5) = 22 - 8 = 14$

$E(L_4) = 22 - 11 = 11$

$E(L_3) = 14 - 9 = 5$

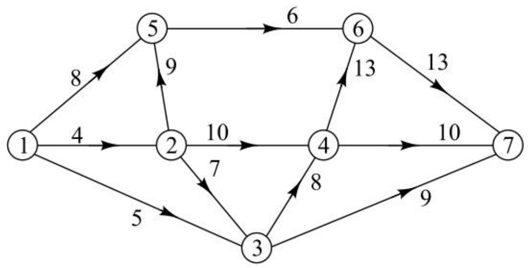
$E(L_2) = \text{Min}\,[14 - 10, 14 - 7] = 4$

$E(L_1) = \text{Min}\,[11 - 8, 5 - 5, 4 - 3] = 0$

Event	$E(\mu_i)$	$E(L_i)$	*Slack*	
1	0	0	0	→
2	3	4	1	
3	5	5	0	→
4	8	11	3	
5	10	14	4	
6	14	14	0	→
7	22	22	0	→

The critical path is 1 – 3 – 6 – 7. The expected time of completion is 22 units.

Example 20.11 Find the critical path of the project given by the following network diagram



Job	*(1, 2)*	*(1, 3)*	*(1, 5)*	*(2, 3)*	*(2, 4)*	*(2, 5)*	*(3, 4)*	*(3, 7)*	*(4, 6)*	*(4, 7)*	*(5, 6)*	*(6, 7)*
t_o	2	2	5	6	9	5	4	6	12	7	3	10
t_m	4	5	8	7	10	9	8	9	13	10	6	13
t_p	6	8	11	8	11	13	12	12	14	13	9	16

Solution The time calculations are given below:

$$t_e = \frac{1}{6}(t_o + 4t_m + t_p)$$

Job	t_o	t_m	t_p	t_e
(1, 2)	2	4	6	4
(1, 3)	2	5	8	5
(1, 5)	5	8	11	8
(2, 3)	6	7	8	7
(2, 4)	9	10	11	10
(2, 5)	5	9	13	9
(3, 4)	4	8	12	8
(3, 7)	6	9	12	9
(4, 6)	12	13	14	13
(4, 7)	7	10	13	10
(5, 6)	3	6	9	6
(6, 7)	10	13	16	13

$$\begin{aligned}
E(\mu_1) &= 0; \qquad E(\mu_2) = 4 \\
E(\mu_3) &= \text{Max } [4 + 7, 0 + 5] = 11 \\
E(\mu_4) &= \text{Max } [4 + 10, 11 + 8] = 19 \\
E(\mu_5) &= \text{Max } [0 + 8, 4 + 9] = 13 \\
E(\mu_6) &= \text{Max } [19 + 13, 13 + 6] = 32 \\
E(\mu_7) &= \text{Max } [19 + 10, 11 + 9, 32 + 13] = 45 \\
E(L_7) &= 45; \qquad E(L_6) = 32; \qquad E(L_5) = 26 \\
E(L_4) &= \text{Min } [45 - 10, 32 - 13] = 19 \\
E(L_3) &= \text{Min } [45 - 9, 19 - 8] = 11 \\
E(L_2) &= \text{Min } [11 - 7, 19 - 10, 26 - 6] = 4 \\
E(L_1) &= \text{Min } [4 - 4, 11 - 5, 26 - 8] = 0
\end{aligned}$$

Event	$E(\mu_i)$	$E(L_i)$	*Slack*	
1	0	0	0	→
2	4	4	0	→
3	11	11	0	→
4	19	19	0	→
5	13	26	13	
6	32	32	0	→
7	45	45	0	→

The critical path is 1 – 2 – 3 – 4 – 6 – 7

The expected time of completion of the project is 45 units.

20.4.3 Probability of Completion Within a Specified Time

In PERT since the time estimates are based on probability, the project may not be completed exactly in time. We can calculate the probability of completing the project within any scheduled time. It is assumed that the probability distribution of the activity times is normal distribution. The mean of the distribution is T (expected time of completion) and the variance is $\sigma^2 = \Sigma\sigma_{ij}^2$ where σ_{ij}^2 are the variances of the critical activities. We can calculate the probability of completing the project in a given time t as follows:

Define $Z_1 = \dfrac{t - T}{\sqrt{\sigma^2}}$. Now Z is a standard normal variate and hence we can use the table of area under normal curve to find $P\,[Z \le Z_1]$. This is the required probability.

Example 20.12 A project consists of the following activities and time estimates.

Activity	*Estimated duration in weeks*		
	Optimistic	*Most likely*	*Pessimistic*
(1, 2)	1	1	7
(1, 3)	1	4	7
(1, 4)	2	2	8
(2, 5)	1	1	1
(3, 5)	2	5	14
(4, 6)	2	5	8
(5, 6)	3	6	15

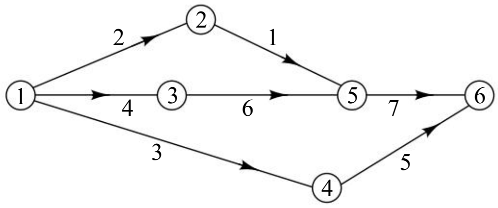
(i) Draw the network.

(ii) Find the expected time and variance for each activity.

(iii) What is the probability that the project will be completed 4 weeks earlier than the expected time.

(iv) What is the probability that the project will be completed in 19 weeks.

Solution The network is given below:



The expected time and variance of each activity is given below:

Activity	t_o	t_m	t_p	t_e	σ^2
(1, 2)	1	1	7	2	1
(1, 3)	1	4	7	4	1
(1, 4)	2	2	8	3	1
(2, 5)	1	1	1	1	0
(3, 5)	2	5	14	6	4
(4, 6)	2	5	8	5	1
(5, 6)	3	6	15	7	4

The following are the time calculations

$E(\mu_1) = 0$; $E(\mu_2) = 2$; $E(\mu_3) = 4$

$E(\mu_4) = 3$; $E(\mu_5) = \text{Max } (2 + 1, 4 + 6) = 10$

$E(\mu_6) = \text{Max } \{10 + 7, 3 + 5\} = 17$

$E(L_6) = 17$; $E(L_5) = 10$; $E(L_4) = 12$

$E(L_3) = 4$; $E(L_2) = 9$;

$E(L_1) = \text{Min } \{9 - 2, 4 - 4, 12 - 3\} = 0$

The critical events are given in the following table:

Event	$E(\mu_i)$	$E(L_i)$	*Slack*	
1	0	0	0	→
2	2	9	7	
3	4	4	0	→
4	3	12	9	
5	10	10	0	→
6	17	17	0	→

The critical events are 1, 3, 5, 6. Hence the critical path is 1 – 3 – 5 – 6.

Expected time of completion of the project is 17 weeks $\quad \therefore T = 17$

$\sigma^2 = \sigma^2_{1,3} + \sigma^2_{3,5} + \sigma^2_{5,6} = 1 + 4 + 4 = 9$

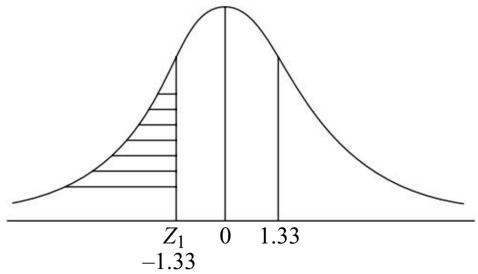
$\therefore \sigma = 3$

(iii) To find the probability that the project would be completed 4 weeks earlier:

$$\text{We have } Z = \frac{t - 17}{3}$$

$$\text{Given } t = 17 - 4 = 13$$

$$\therefore \quad Z_1 = \frac{13 - 17}{3} = -1.33$$

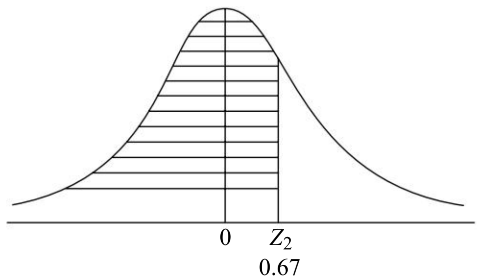


From the table of area under normal curve we have $P\,[Z \le 1.33] = 0.91 = 0.5 + 0.41$

$\therefore \quad P\,[Z \le -1.33] = 0.5 - 0.41 = 0.09$

(iv) To find the probability that the project would be completed in 19 weeks:

$$\text{We have } Z_2 = \frac{19 - 17}{3} = 0.67$$



From the table we find that

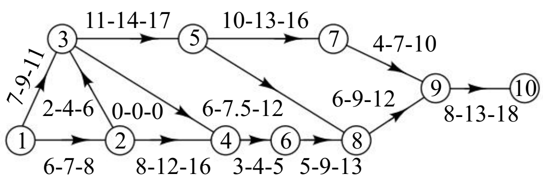
$\therefore P\,[Z \le 0.67] = 0.75$

Thus the project would be completed 4 weeks earlier with a probability of 0.09 and in 19 weeks with a probability of 0.75.

Example 20.13 For the project represented by the following network, find the probability that the project will be completed

(i) two weeks earlier than expected.

(ii) two weeks later than expected.



Solution The time estimates t_o, t_m, and t_p are given in the network diagram. The expected time and variance of each activity is given by the following table:

Activity	t_o	t_m	t_p	t_e	*Variance* σ^2
(1, 2)	6	7	8	7	0.111
(1, 3)	7	9	11	9	0.444
(2, 3)	2	4	6	4	0.444
(2, 4)	8	12	16	12	1.778
(3, 4)	0	0	0	0	0
(3, 5)	11	14	17	14	1
(4, 6)	3	4	5	4	0.111
(5, 7)	10	13	16	13	1
(5, 8)	6	7.5	12	8	1
(6, 8)	5	9	13	9	1.778
(7, 9)	4	7	10	7	1
(8, 9)	6	9	12	9	1
(9, 10)	8	13	18	13	2.778

Now let us calculate the earliest and latest times at each node.

$E(\mu_1) = 0$; $E(\mu_2) = 7$

$E(\mu_3) = \text{Max}\,[9, 11] = 11$

$E(\mu_4) = \text{Max}\,[11 + 0, 7 + 12] = 19$

$E(\mu_5) = 25$; $E(\mu_6) = 23$

$E(\mu_7) = 38$

$E(\mu_8) = \text{Max}\,[33, 32] = 33$

$E(\mu_9) = \text{Max}\,[45, 42] = 45$

$E(\mu_{10}) = 58$

$E(L_{10}) = 58$; $E(L_9) = 45$; $E(L_8) = 36$

$E(L_7) = 38$; $E(L_6) = 27$

$E(L_5) = \text{Min}\,[25, 28] = 25$

$E(L_4) = 23$; $E(L_3) = \text{Min}\,[11, 23] = 11$

$E(L_2) = \text{Min}\,[11, 7] = 7$

$E(L_1) = \text{Min}\,[0, 2] = 0$

To find the critical path:

Event	$E(\mu_i)$	$E(L_i)$	*Slack*	
1	0	0	0	→
2	7	7	0	→
3	11	11	0	→
4	19	23	4	
5	25	25	0	→
6	23	27	4	
7	38	38	0	→
8	33	36	3	
9	45	45	0	→
10	58	58	0	→

The critical path is 1 – 2 – 3 – 5 – 7 – 9 – 10

Expected time of completion of the project is 58 weeks.

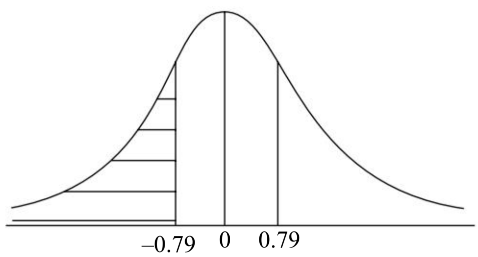
$\sigma^2 = \sigma^2_{12} + \sigma^2_{2,3} + \sigma^2_{3,5} + \sigma^2_{5,7} + \sigma^2_{7,9} + \sigma^2_{9,10}$

$= 0.111 + 0.444 + 1 + 1 + 1 + 2.778 = 6.333$

(i) $T = 58$ weeks. Given $t = 58 - 2 = 56$ weeks.

$$\sigma = \sqrt{6.333} = 2.52$$

$$Z_1 = \frac{t - T}{\sigma} = \frac{56 - 58}{2.52} = -0.79$$



From the table we find that

$P\,[Z \le 0.79] = 0.7852 = 0.5 + 0.2852$

$P\,[Z \le -0.79] = 0.50 - 0.2852 = 0.2148$

Probability that the project would be completed two weeks earlier than expected, is 0.2148

(ii) $T = 58$ weeks. Given $t = 58 + 2 = 60$ weeks.

$$Z_2 = \frac{t - T}{\sigma} = \frac{60 - 58}{2.52} = 0.79$$

From the table, $P\,[Z \le 0.79] = 0.7852$

Probability that the project would be completed two weeks later than expected, is 0.7852**.**

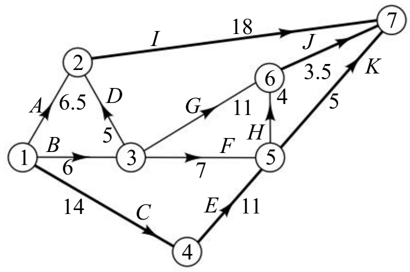
Example 20.14 The activities and their estimates of time of a project are given below:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
t_o	3	2	6	2	5	3	3	1	4	1	2
t_m	6	5	12	5	11	6	9	4	19	3	4
t_p	12	14	30	8	17	15	27	7	28	8	12

Precedence relationships are
$A < D, I$; $B < G, F$; $D < G, F$; $C < E$; $E < H, K$
$F < H, K$; $G, H < J$.

What is the probability that the project will be completed in 35 days?

Solution The network diagram is given below



The expected time and variance of each job is given by the following table:

	Job	t_o	t_m	t_p	t_e	Variance
A	(1, 2)	3	6	12	6.5	2.25
B	(1, 3)	2	5	14	6	4
C	(1, 4)	6	12	30	14	16
D	(2, 3)	2	5	8	5	1
I	(2, 7)	4	19	28	18	16
F	(3, 5)	3	6	15	7	4
G	(3, 6)	3	9	27	11	16
E	(4, 5)	5	11	17	11	4
H	(5, 6)	1	4	7	4	1
K	(5, 7)	2	4	12	5	2.78
J	(6, 7)	1	3	8	3.5	1.36

Now $E(\mu_1) = 0$; $E(\mu_2) = 6.5$
$E(\mu_3) = 11.5$; $E(\mu_4) = 14$
$E(\mu_5) = 25$; $E(\mu_6) = 29$
$E(\mu_7) = 32.5$; $E(L_7) = 32.5$
$E(L_6) = 29$; $E(L_5) = 25$;
$E(L_4) = 14$; $E(L_3) = 18$;
$E(L_2) = 13$; $E(L_1) = 0$

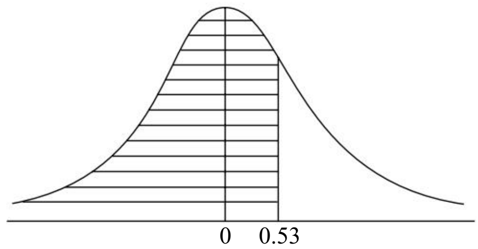
The critical path is 1 – 4 – 5 – 6 – 7

The expected time of completion of the project is 32.5 days

$$\sigma^2 = \sigma^2_{1,4} + \sigma^2_{4,5} + \sigma^2_{5,6} + \sigma^2_{6,7}$$
$$= 16 + 4 + 1 + 1.36 = 22.36$$
$$\therefore \sigma = 4.73$$

Now $T = 32.5$ days. Given $t = 35$ days.

$$Z_1 = \frac{t - T}{\sigma} = \frac{35 - 32.5}{4.73} = 0.53$$



From the table we find that
$P[Z \le 0.53] = 0.702$

$\therefore$ Probability that the project would be completed in 35 days, is 0.702.

20.4.4 Cost Considerations in PERT

In PERT calculations the cost involved in completing the project is also considered. There are two kinds of costs, *Direct cost* and *Indirect cost*. Direct costs are the costs associated with each activity such as machine cost, labour cost, etc. Direct cost varies inversely as the duration of the activity. It increases when the time of completion of the job is to be reduced (crashed) since more machines and more labour, become necessary to complete the job in less time. Indirect costs are the costs due to management services, rentals, cost of security, establishment charges and similar overhead expenditures. When the duration of the project is shortened, indirect cost decreases. Therefore there is some optimum project duration which is a balance between the direct costs increasing with reducing of project duration and the indirect costs increasing with the lengthening of the duration. This method of finding the optimum duration is called Least Cost Schedule.

For each activity a normal time and normal cost is known. Also the crash time and the corresponding crash cost are also given. In order to reduce the duration we crash the critical activities one by one and obtain the optimum duration.

Step-by-step procedure

(i) Draw the network diagram.
(ii) Determine the critical path, normal duration and total normal cost of the project.
(iii) Find the cost slope of each activity using the formula

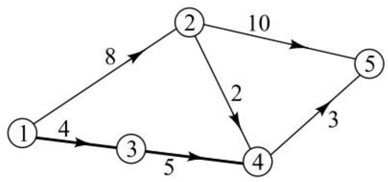
$$\textit{Cost slope} = \left(\frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}\right)$$

(iv) Crash the critical activity having the least slope.
(v) Calculate the revised total cost [Revised direct cost + new indirect cost] for the reduced duration.
(vi) Determine the new critical path and repeat the process of crashing systematically till an optimum duration is obtained.

Example 20.15 For the project represented by the following network and the table showing time and cost, find the optimum duration and cost.

Activity	*Normal*		*Crash*		ΔT	ΔC	$\frac{\Delta C}{\Delta T}$
	Time (days)	*Cost (Rs)*	*Time (days)*	*Cost (Rs)*			
(1, 2)	8	100	6	200	2	100	50
(1, 3)	4	150	2	350	2	200	100
(2, 4)	2	50	1	90	1	40	40
(2, 5)	10	100	5	400	5	300	60
(3, 4)	5	100	1	200	4	100	25
(4, 5)	3	80	1	100	2	20	10

Indirect cost = Rs 70 per day.

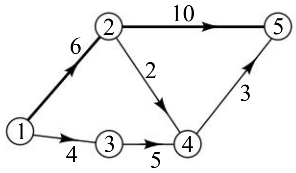


Solution We find the critical path first.

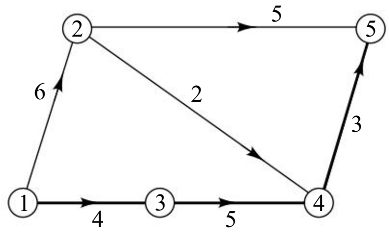
$E(\mu_1) = 0$; $E(\mu_2) = 8$; $E(\mu_3) = 4$
$E(\mu_4) = 10$; $E(\mu_5) = 18$; $E(L_5) = 18$
$E(L_4) = 15$; $E(L_3) = 10$; $E(L_2) = 8$
$E(L_1) = 0$

The critical path is 1 – 2 – 5 (Longest path)
Time of completion = 18 days
Direct cost (Normal cost) = Rs 580
Indirect cost = 18 × 70 = Rs 1260
Total cost = 580 + 1260 = Rs 1840

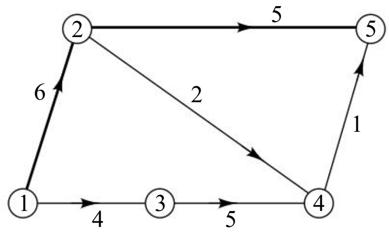
Crashing First we crash the critical activity (1, 2) having the least cost slope 50. The new network is given below:



The critical path remains the same with revised duration 16 days
Direct cost = Rs 680 (580 + 100)
Indirect cost = 16 × 70 = Rs 1120
Total cost = Rs 1800
Next we crash the critical activity (2, 5)
The new network is given below



The new critical path (longest path) is 1 – 3 – 4 – 5 and the duration becomes
12 days direct cost = Rs 980 (680 + 300)
Indirect cost = 12 × 70 = Rs 840
Total cost = Rs 1820
Next we crash the critical activity (4, 5) [which has the least slope 10] by two days. The network becomes



The critical path becomes 1 – 2 – 5 with duration 11 days.

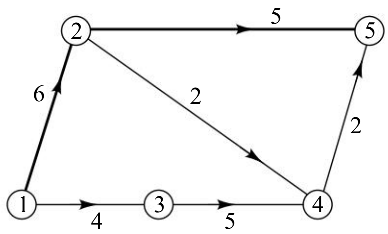
Direct cost = Rs 1000 (980 + 20)

Indirect cost = Rs 11 × 70 = Rs 770

Total cost = Rs 1770

The critical activities (1, 2) and (2, 5) have been crashed already. Hence no more crashing can be done.

Now we find that the activity (4, 5) can be extended by one day without affecting the project duration. This will save the crash cost of the activity (4, 5) by one day. Thus we get the network



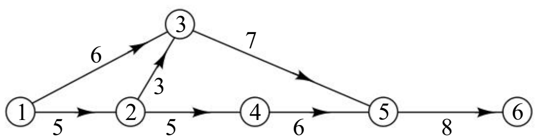
∴ optimum duration = 11 days

Total cost = 1770 – 10 = Rs 1760

Example 20.16 Find the optimum cost schedule for the project with the following data

Activity	*Normal*		*Crash*		*ΔT*	*ΔC*	*ΔC/ΔT*
	Time (days)	*Cost (Rs)*	*Time (days)*	*Cost (Rs)*			
(1, 2)	5	100	3	220	2	120	60
(1, 3)	6	120	3	195	3	75	25
(2, 3)	3	60	2	100	1	40	40
(2, 4)	5	80	3	170	2	90	45
(3, 5)	7	100	4	250	3	150	50
(4, 5)	6	120	2	240	4	120	30
(5, 6)	8	160	6	220	2	60	30
		740		Indirect cost is zero			

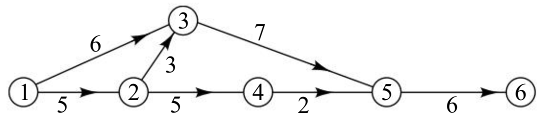
Solution The network of the project is



It is easily seen that the critical path (longest path) is 1 – 2 – 4 – 5 – 6.

And the duration is 4 days. There is no indirect cost. Hence the total cost is Rs 740.

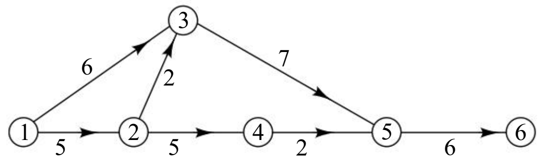
Crashing The critical activities (4, 5) and (5, 6) have the least slope 30. Hence we crash these two activities by 4 days and 2 days respectively. The network becomes



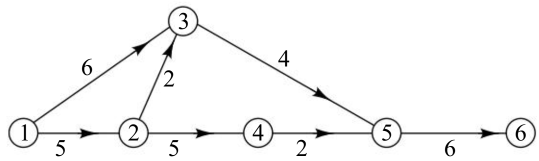
Now the critical path (longest path) is 1 – 2 – 3 – 5 – 6 with duration 21 days

Total cost becomes 740 + 120 + 60 = Rs 920.

The critical activity (2, 3) has the least slope. Hence we crash (2, 3) by one day. The new network is

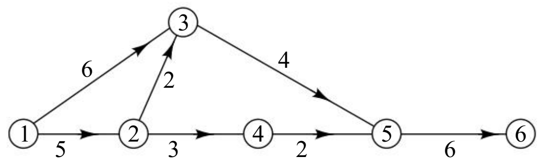


The critical path remains the same 1 – 2 – 3 – 5 – 6 with duration 20 days. The total cost is Rs 960. Next we crash the critical activity (3, 5) by 3 days. The revised net work is

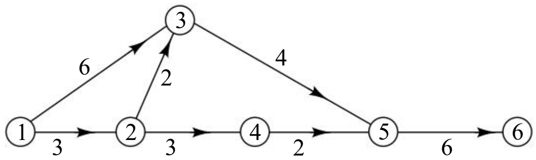


Now the critical path is 1 – 2 – 4 – 5 – 6 with duration 18 days. The total cost is Rs (960 + 150) = Rs 1110

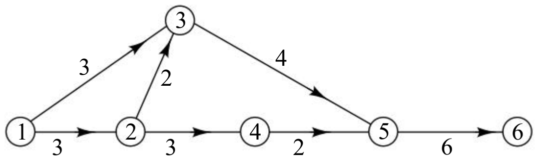
The activity (2, 4) with slope 45 is to be crashed next. The network becomes



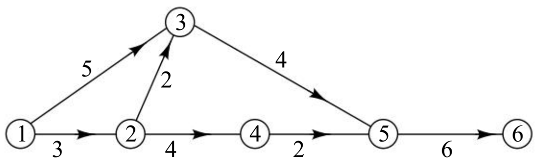
Now the critical path 1 – 2 – 3 – 5 – 6 and the duration is 17 days. The total cost is Rs 1200. Next we crash the remaining critical activity (1, 2) by 2 days. The new network is



The new critical path is 1 – 3 – 5 – 6 having duration 16 days. The total cost is Rs 1320. Next we crash (1, 3) by 3 days. The network becomes



The critical path (longest path) is 1 – 2 – 3 – 5 – 6. The duration is 15 days. Total cost is Rs 1395. Thus the minimum project duration is 15 days and the total cost is Rs 1395. However we want to decrease the cost without increasing the duration. We find that by increasing the duration of (1, 3) by 2 days we save (2 × 25) Rs 50. Also by increasing the duration of (2, 4) by one day we save Rs 45. The critical path remains the same. Thus the revised schedule of activities is given by the following network.



The optimum cost of the project is 1395 – 50 – 45 = Rs 1300

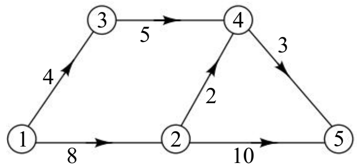
The optimum duration is 15 days.

Example 20.17 The following table gives the cost particulars of a project

Activity	*Normal*		*Crash*		*ΔT*	*ΔC*	*Slope*
	Time (days)	*Cost (Rs)*	*Time (days)*	*Cost (Rs)*			
(1, 2)	8	100	6	200	2	100	50
(1, 3)	4	150	2	350	2	200	100
(2, 4)	2	50	1	90	1	40	40
(2, 5)	10	100	5	100	5	300	60
(3, 4)	5	100	1	200	4	100	25
(4, 5)	3	80	1	100	2	20	10
		Rs 580					

If the indirect cost is Rs 100 per day, find the least cost schedule (optimum duration).

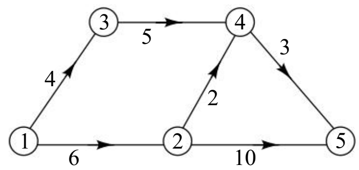
Solution The network diagram of the project is as shown below:



The critical path is 1 – 2 – 5 and the duration of the project is 18 days.

Direct cost = Rs 580
Indirect cost = Rs 1800
Total cost = Rs 2380

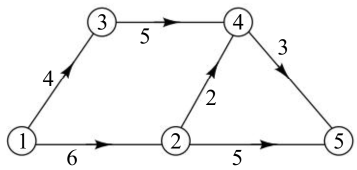
Crashing The critical activity (1, 2) having the least slope, is crashed by 2 days. The network becomes



The critical path remains the same (1 – 2 – 5) with duration 16 days.

Direct cost = Rs 680
Indirect cost = Rs 1600
Total cost = Rs 2280

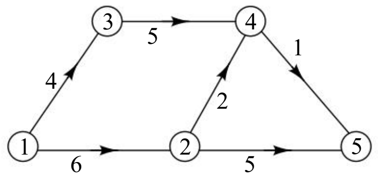
Next we crash the activity (2, 5) by 5 days. We have,



The new critical path is 1 – 3 – 4 – 5 having duration 12 days.

Direct cost = Rs 980
Indirect cost = Rs 1200
Total cost = Rs 2180

Now we crash (4, 5) by 2 days. We obtain the network



The critical path is 1 – 2 – 5 with duration 11 days.

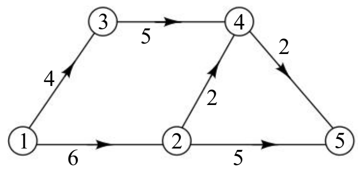
Direct cost = Rs 1000

Indirect cost= Rs 1100

Total cost = Rs 2100

We can decrease the cost by Rs 10 by increasing the duration of (4, 5) to 2

Thus the optimum cost is Rs 2090 with project duration 11 days. The network schedule is



EXERCISES

1. Construct the network for the projects whose activities and their relation ships are given below:
   (a) $A < B, C; C < G, F; D < G, F; E, F < H$
   (b) $A < C; B < D, E; C, D < F; E < G; F<H$
   (c) $A < C, D; B < E; C, E < F, G; D < H, G < I; H, I < J$
   (d) $A < B, C; D < E; B, C, E < F; F < G; E < H; G, H < I$
2. Find the critical path and time of completion of the projects described below:

(a)

Activity	*(1, 2)*	*(1, 3)*	*(1, 4)*	*(2, 5)*	*(2, 6)*	*(3, 5)*	*(4, 6)*	*(5, 6)*
Time (Days)	4	2	10	12	12	6	8	9

(b)

Activity	*(1, 2)*	*(1, 3)*	*(2, 4)*	*(3, 4)*	*(3, 5)*	*(4, 7)*	*(5, 6)*	*(5, 7)*	*(6, 8)*	*(7, 8)*
Time (Days)	4	1	1	1	6	5	4	8	1	2

(c)

Activity	*(1, 2)*	*(2, 3)*	*(3, 4)*	*(3, 7)*	*(4, 5)*	*(4, 7)*	*(5, 6)*	*(6, 7)*
Time (Days)	3	4	4	4	2	2	3	2

(d)

Activity	*(1, 2)*	*(1, 3)*	*(2, 4)*	*(3, 4)*	*(3, 5)*	*(4, 5)*	*(4, 6)*	*(5, 6)*
Time (Weeks)	6	5	10	3	4	6	2	9

(e) $A < B, C, D, E; E < F; D < G; G, F < H; B < I$

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Time (Weeks)	1	4	2	3	2	3	2	1	3

(f)

Job	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
Time	13	8	10	9	11	10	8	6	7	14	18

$A < B; B < C, E; C < D; E < F, H; D, F < G; H < I; G, I < J; J < K$

(g)

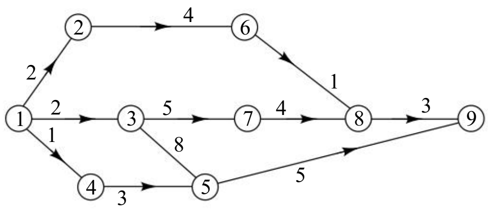
Job	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Time	30	7	10	14	10	7	21	7	12	15	30	15

$A < B, C$; $B < D, G, K$; $C < D, G$; $D < E$; $E < F$; $F < H, I, L$; $G < L$; $H, I < J$; $K < L$

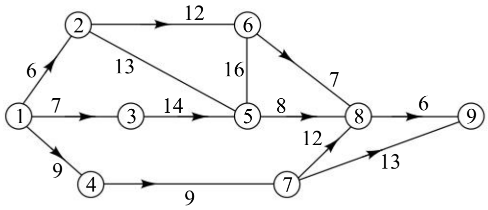
(h) $A < B, C$; $B < D, E$; $C < F, G$; $F < H$; $D < I$; $E, G < K$; $H < J$

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
Time (weeks)	10	9	7	6	12	6	8	8	4	11	7

(i)



(j)



3. A project has the following data:

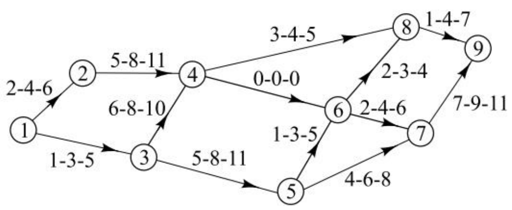
Job	t_o	t_m	t_p
(1, 2)	3	4	5
(1, 3)	1	2	3
(2, 3)	6	8	10
(2, 4)	2	5	8
(2, 5)	3	5	7
(3, 4)	6	9	12
(4, 7)	1	1	1
(5, 6)	2	5	8
(6, 7)	4	8	12

(i) Draw the network
(ii) Determine the critical path and the project length

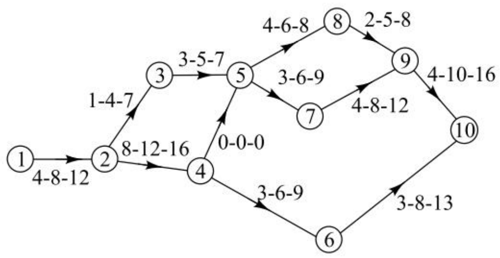
4. For the project having the following jobs and time estimates find the probability of completing the project in 66 days.

Activity	*(1, 2)*	*(2, 3)*	*(2, 4)*	*(3, 5)*	*(4, 5)*	*(4, 6)*	*(5, 7)*	*(6, 7)*	*(7, 8)*	*(8, 9)*
t_o	9	10	5	3	4	10	8	6	12	7
t_m	10	13	9.5	5	7	15	12	7.5	15	11
t_p	11	16	11	13	10	26	22	12	24	15

5. For the project represented by the following network find the probability of completing the project in 26 days.



6. The network of a project is given below:



Find the probability that the project would be completed in 48 days.

7. A project has the following data:

Activity	A	B	C	D	E	F	G	H
t_o	4	8	4	1	2	4	10	18
t_m	5	12	5	3	2	5	14	20
t_p	6	16	12	5	2	6	18	34

$A < C$; $B < D$; $A, D < E$; $B < F$; $C, E, F < G$; $G < H$.
 (i) Draw the network
 (ii) Find the critical path and the expected time of completion of the project
 (iii) What is the probability that the project would be completed in 60 days?

8. Draw the network for the following data.

Activity	A	B	C	D	E	F	G	H	I
t_o	3	14	11	2	2	10	3	4	1
t_m	7	21	14	2	3.5	14	4	4.5	2
t_p	11	28	17	2	8	21	5	8	4

$A < B, C, D$; $C < D$; $B < E$; $D, E < F$; $E < G$; $E, F < H$; $G, H < I$
What is the probability that the project would be completed in 56 days?

9. A project is represented by the following activities:

Activity	(1, 2)	(1, 3)	(1, 4)	(2, 5)	(3, 6)	(3, 7)	(4, 6)	(5, 8)	(6, 9)	(7, 8)	(8, 9)
Time	2	2	1	4	8	5	3	1	5	4	3

Find the
 (i) Critical path
 (ii) duration of the project
 (iii) the free float and the independent float for each activity

10. The following table gives the tasks and their durations of a project.

Task	A	B	C	D	E	F	G	H	I
Time	23	8	20	16	24	18	19	4	10

$A < D, E$; $B, D < F$; $C < G$; $B < H$; $F, G, H < I$
Find the
 (i) Critical path
 (ii) duration of the project
 (iii) the free float and the independent float for each activity

11. Given

Task	A	B	C	D	E	F	G	H	I	J
Time	14	22	10	16	12	10	6	8	24	16

$A < B$; $B < C, D, E$; $C < F, G$; $F, G < H$; $D, E, H < I$; $I < J$
Find the critical path, project duration and the free float for each task.

12. Find the least cost schedule for the following projects by systematic crashing of the activities.

(a)

Activity	Normal		Crash	
	Time	Cost	Time	Cost
(1, 2)	3	300	2	400
(2, 3)	3	30	3	30
(2, 4)	7	420	5	580
(2, 5)	9	720	7	810
(3, 5)	5	250	4	300
(4, 5)	0	0	0	0
(5, 6)	6	320	4	410
(6, 7)	4	400	3	470
(6, 8)	13	780	10	900
(7, 8)	10	1000	9	1200

Indirect cost = Rs 50 per week

(b)

Activity	Normal		Crash	
	Time	Cost	Time	Cost
(1, 2)	4	60	3	90
(1, 3)	2	38	1	60
(1, 4)	6	150	4	250
(2, 4)	5	150	3	250
(3, 4)	2	100	2	100
(2, 5)	7	115	5	175
(4, 5)	4	100	2	240

Indirect cost = Rs 60 per day

(c)

Activity	Normal Time	Normal Cost	Crash Time	Crash Cost
(1, 2)	6	300	4	400
(2, 3)	8	400	5	550
(2, 4)	7	400	6	500
(2, 5)	12	1000	8	1800
(3, 5)	5	500	3	800
(4, 5)	8	800	7	1000
(5, 6)	6	700	5	800

Indirect cost = Rs 50 per day

(d)

Activity	Normal Time	Normal Cost	Crash Time	Crash Cost
A	2	100	1	150
B	6	200	4	400
C	3	200	1	300
D	4	500	2	700
E	2	400	1	550
F	8	200	5	450
G	5	300	3	500

$A < D$; $B < F$; $C < E$; $D < G$; $E < F$

13. The following table gives the activities and their durations in a project.

Activity	Duration (in weeks)	Immediate Predecessor
A	4	E
B	2	A
C	1	B
E	14	–
F	2	E
G	3	F
H	2	F
I	4	C, G, H
J	12	I

Draw the network and find the critical path.

14. The following table gives the activities and their durations in their order, of a project. Draw the network and find the minimum time required for completion of the project.

Activity	Preceding Activity	Duration (in days)
A	–	9
B	–	4
C	–	7
F	B, C	8
E	A	7
F	C	5
G	E	10
H	E	8
I	D, F, H	6

15. Draw the network and find the critical path and also calculate the total float and free float for each job in a project given by the following table:

Job	Precedence Activity	Duration (in hours)
A	–	14
B	A	22
C	B	10
D	B	16
E	B	12
F	C	10
G	C	6
H	F, G	8
I	D, E, H	24
J	I	16

16. Draw the network for the project represented by the following activities and find the critical path.

Activity	A	B	C	D	E	F	G	H	I	J	K
Time (days)	12	10	15	8	20	12	6	8	16	12	8

$A < C, D, I$; $B < G, F$; $D < G, F$; $F < H, K$; $G, H < J$; $I, J, K < E$

17. Draw the network for the project whose activities are given below. Calculate the total float, free float and independent float for each activity and also the project duration.

Activity	(1, 2)	(2, 3)	(2, 4)	(3, 5)	(3, 6)	(4, 5)	(4, 7)	(5, 8)	(6, 8)	(7, 8)
Time (weeks)	3	8	12	6	3	3	8	5	3	8

18. Find the critical path of the project having the following activities.

Activity	*(1, 2)*	*(2, 3)*	*(2, 4)*	*(3, 5)*	*(4, 6)*	*(4, 7)*	*(5, 8)*	*(6, 7)*	*(7, 8)*
Time (days)	2	3	4	2	4	3	6	5	6

19. For the following project find the probability of completion in 42 days.

Activity	t_o	t_p	t_m
(1, 2)	6	10	8
(1, 3)	18	22	20
(1, 4)	26	40	33
(2, 5)	16	20	18
(2, 6)	15	25	20
(3, 6)	6	12	9
(4, 7)	8	12	10
(5, 7)	7	9	8
(6, 7)	3	5	4

20. A project has the following data
$A < B, C; B, C < D$
$B < E; D, E < F$
$D < G$

Activity	*Time estimates*		
	t_o	t_m	t_p
A	1	4	7
B	4	6	14
C	3	3	3
D	4	10	22
E	3	8	13
F	2	5	14
G	4	4	4

Find the expected time of completion. Also find the probability that the project would be completed in 30 days.

21. For the following data, draw the network, calculate the expected time of completion and find the probability of completing the project in 50 weeks.

Activity	*Time estimates*		
	t_o	t_m	t_p
A	4	6	8
B	5	7	15
C	4	8	12
D	15	20	25
E	10	18	26
F	8	9	16
G	4	8	12
H	1	2	3
I	7	6	8

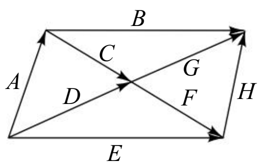
$A < B, C;\; B < D, E;$
$C < F;$
$E < G;$
$D, F < H;$
$G, H < I.$

22. Determine the critical path

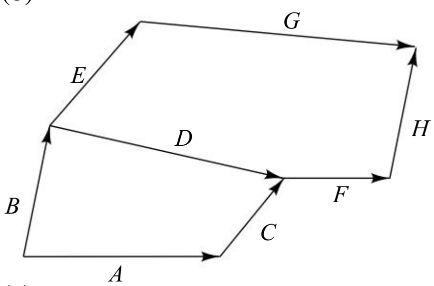
Activity	t_o	t_m	t_p
(1, 2)	3	4	5
(1, 3)	6	8	10
(1, 4)	2	3	4
(2, 5)	4	5	12
(3, 5)	5	7	9
(3, 6)	9	16	17
(4, 7)	8	12	16
(5, 9)	4	5	6
(6, 9)	7	8	15
(7, 8)	8	11	14
(8, 9)	8	10	12

ANSWERS

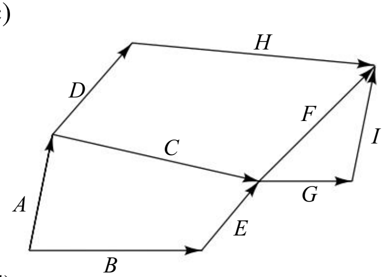
1. (a)



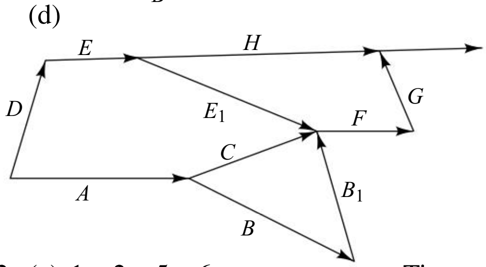
(b)



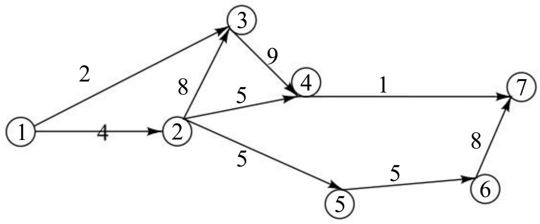
(c)



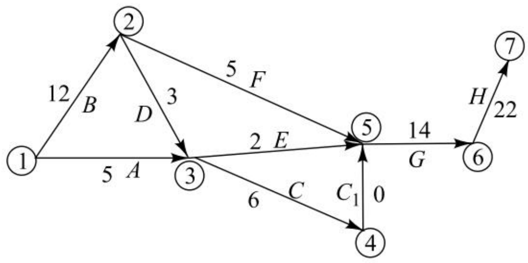
(d)



2. (a) 1 – 2 – 5 – 6; Time = 25
   (b) 1 – 3 – 5 – 7 – 8; Time = 17
   (c) 1 – 2 – 3 – 4 – 5 – 6 – 7; Time = 18
   (d) 1 – 2 – 4 – 5 – 6; Time = 31
   (e) $A - B - I$; Time = 8
   (f) $A - B - E - F - G - J - K$; Time = 82
   (g) $A - C - C_1 - D - E - F - I - I_1 - J$; Time = 98
   (h) $A - C - F - H - J$; Time = 42
   (i) 1 – 3 – 5 – 9; Time = 15
   (j) 1 – 3 – 5 – 6 – 8 – 9; Time = 50
3. 1 – 2 – 3 – 4 – 7; Time = 22

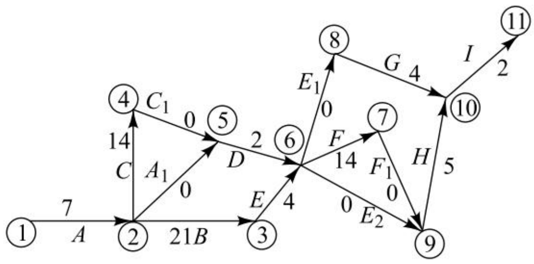


4. 0.1515
5. 0.963
6. 0.898
7.



1 – 2 – 3 – 4 – 5 – 6 – 7; Time = 57; Probability = 0.7995

8.

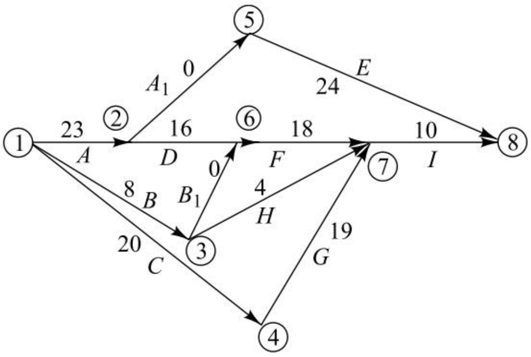


1 – 2 – 3 – 6 – 7 – 9 – 10 – 11; Time = 53; Probability = 0.8

9. 1 – 3 – 6 – 9; Duration = 15
   Free float is zero for all the activities except (5, 8), which has Free Float 4.

Independent float is zero for all activities except (1, 4) with I.F. 1 and (3, 7) with I.F. 1.

10.



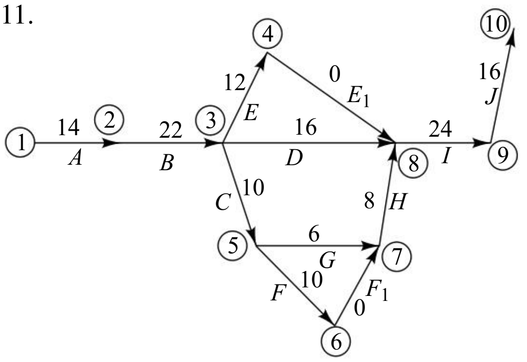
Critical path: 1 – 2 – 6 – 7 – 8

Duration: 67

Free Float: (3, 7) – 45; (4, 7) – 18; (5, 8) – 20; Zero for other activities

Independent Float: (2, 5) – 20; (3, 7) – 14; Zero for other activities

11.



Critical path: 1 – 2 – 3 – 5 – 6 – 7 – 8 – 9 – 10

Duration: 104

Free Float: (4, 8) – 16; (5, 7) – 4; Zero for other activities

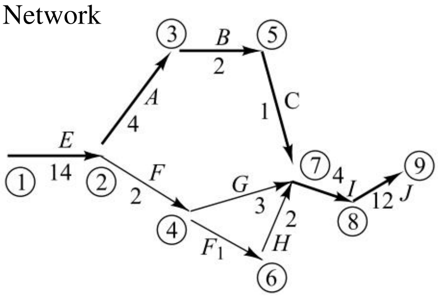
12. (a) Normal duration is 32 weeks.
 Total cost = Rs 5820
 Critical path: 1 – 2 – 5 – 6 – 7 – 8
 After crashing the critical activities, we obtain the optimal duration 29 weeks
 Optimal cost = Rs 5805

(b) Normal duration is 13 days.
 Total cost = Rs 1493
 Critical path: 1 – 2 – 4 – 5
 After crashing the critical activities, we obtain the optimal duration 8 days
 Optimal cost = Rs 1523

(c) Normal duration is 27 days.
 Total cost = Rs 5450
 Critical path: 1 – 2 – 4 – 5 – 6
 After crashing the critical activities, we obtain the optimal duration 22 days
 Optimal cost = Rs 5700

(d) Normal duration is 14 days.
 Total cost = Rs 1900
 Critical path: *B* – *F*
 After crashing the critical activities, we obtain the optimal duration 10 days
 Optimal cost = Rs 2300

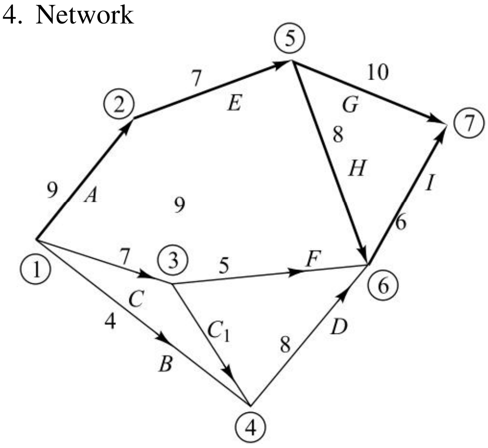
13. Network



Critical paths: *E* – *A* – *B* – *C* – *I* – *J*

Completion time: 37 weeks

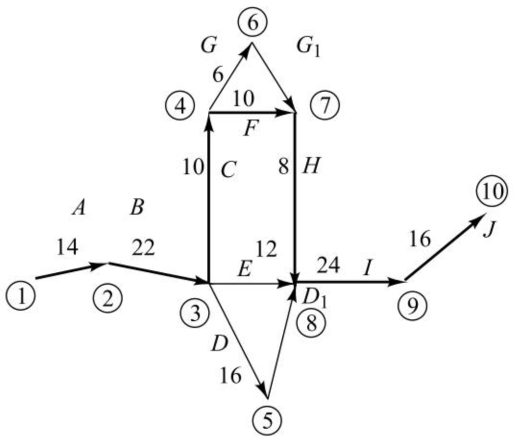
14. Network



Critical path: *A* – *E* – *H* – *I*

Time of completion 30 days.

15. Network

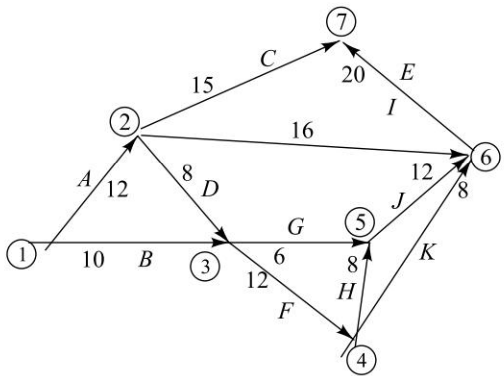


Critical path: 1 – 2 – 3 – 4 – 7 – 8 – 9 – 10
Time of completion is 104 hours

Job	*Total Float*
(3, 5)	12
(3, 8)	16
(4, 6)	4
(5, 8)	12
(6, 7)	4
For others	0

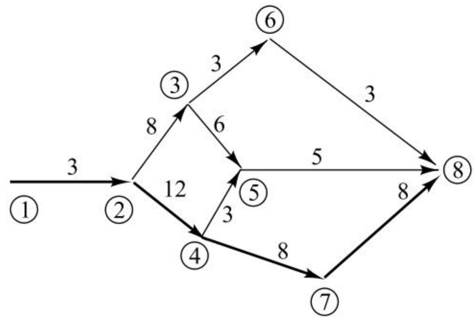
Free float is zero for all the activities.

16.



Critical path: 1 – 2 – 3 – 4 – 5 – 6 – 7
$(A – D – F – H – J – E)$
Time of completion = 72 days

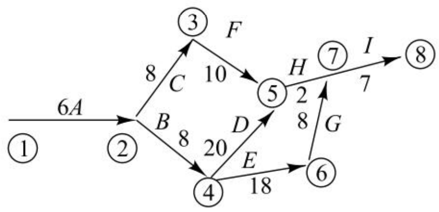
17. Network



Critical path: 1 – 2 – 4 – 7 – 8
Project duration: 31 weeks

Activity	*Total float*	*Free float*	*Independent float*
(1, 2)	0	0	0
(2, 3)	9	0	0
(2, 4)	0	0	0
(3, 5)	9	1	0
(3, 6)	14	0	0
(4, 5)	8	0	0
(4, 7)	0	0	0
(5, 8)	8	8	0
(6, 8)	14	14	0
(7, 8)	0	0	0

18. Critical path: 1 – 2 – 4 – 6 – 7 – 8
Project duration : 21 days
19. Critical path: 1 – 4 – 7
Expected time of completion: 43 days
Required probability: 0.34
20. Expected time of completion: 28 days
Required probability: 0.69
21. Network



Expected time of completion: 47 weeks
Required probability = 0.8
22. Critical path: 1 – 4 – 7 – 8 – 9
Expected time of completion: 36 days

SHORT ANSWER QUESTIONS

1. Explain the method of numbering the events of a network.
2. What is a dummy activity?
3. Define: Total float, free float and independent float.
4. Define critical path.
5. Distinguish between float and slack.
6. If the duration of any activity is changed then the project duration is affected. True or false? (*Ans*: False)
7. Write the formula to find the latest finishing time of an activity.
8. What are the differences between PERT and CPM?
9. Write the formula for finding the expected duration of an activity.
10. If the SDs of six critical activities are 5, 1, 4, 8, 9 and 3, what is σ c.p.? (*Ans:* 14)

21

Simulation

CONCEPT REVIEW

21.1 INTRODUCTION

There are certain real world problems which are very complicated in nature and it is not possible to construct mathematical models for them. Such problems can be solved by the method of simulation. The method of simulation is applied to problems which involve probability and probability distribution. An analogous physical model is constructed to represent the mathematical and logical relationship among the variables of the problem. Thus simulation is the process of designing a model of a real system and conducting experiments with this model to study the actual system. For this purpose a series of random values for the variables is generated using different methods on a computer.

A children's cycling park with various crossings and signals is a simulated model of the traffic in a city. Experiments are conducted in the laboratories on simulated models and the behaviour of the real system is predicted. Simulation experiments are done only with the model. Hence there is no interference on the system. With the help of simulation we can spot the difficulties which may arise due to the change in equipment and process of the real system. Also the simulation techniques are easier to carry out compared to the much complicated mathematical models. Moreover it is advantageous to train people on simulated models before putting their hands into the real system.

21.2 EVENT TYPE SIMULATION

In this type of simulation, the behaviour of the system is observed for a long period of time and relevant informations are collected based on these observations.

Example 21.1 As an example let us consider a queueing system with one server. We shall analyse the system in order to evaluate the quality of service by assessing the average waiting time per customer and percentage of idle time of server. There are two events, an arrival and a departure. Denote the arrivals by $a_1, a_2, a_3, \ldots$ and departures by $d_1, d_2, d_3 \ldots$. Assume that the inter arrival and service time are 1.5 min and 4 min respectively. We simulate this system for a period of 20 minutes. A customer leaves the system as soon as his service is over. Simulation starts with the arrival of the first customer a_1 at time $t = 0$. His departure time is $t = 4$. Next arrival a_2 occurs at $t = 1.5$. Since the system is busy serving the first customer, the second customer waits in the queue. At $t = 3$ third customer arrives (a_3). At $t = 4$, first customer leaves (d_1). This procedure is repeated till $t = 20$ min. The following table gives the results in detail.

Time (minutes)	*Event*	*Customer*	*Waiting time of the customer*
0.0	a_1	1	0 (c_1)
1.5	a_2	2	
3.0	a_3	3	
4.0	d_1	1	4 – 1.5 = 2.5 (c_2)
4.5	a_4	4	
6	a_5	5	
7.5	a_6	6	
8.0	d_2	2	8 – 3 = 5 (c_3)
9.0	a_7	7	
10.5	a_8	8	
12.0	d_3	3	12 – 4.5 = 7.5 (c_4)
12.0	a_9	9	
13.5	a_{10}	10	
15.0	a_{11}	11	
16.0	d_4	4	16 – 6 = 10 (c_5)
16.5	a_{12}	12	
18.0	a_{13}	13	
19.5	a_{14}	14	
20.0	d_5	5	20 – 7.5 = 12.5 (c_6)

Customers getting service after 20 min and their waiting times during the period under simulation are given below.

Customer no.	*Waiting time (min)*
7	20 – 9 = 11
8	20 – 10.5 = 9.5
9	20 – 12 = 8
10	20 – 13.5 = 6.5
11	20 – 15 = 5
12	20 – 16.5 = 3.5
13	20 – 18 = 2
14	20 – 19.5 = 0.5
Total = 46	

From this simulation table we get the average waiting time of the customer as

$$\frac{2.5+5+7.5+10+12.5+11+9.5+8+6.5+5+3.5+2+0.5}{14}$$

$$= \frac{83.5}{14} = 6 \text{ min nearly}$$

Since the service facility is always busy, the idle time of the server is zero.

21.3 MONTE–CARLO TECHNIQUE

This technique involves the selection of random observations in the simulation model. We assume a probability distribution giving the probability of the events and use random numbers for the cumulative density. These random numbers may be selected from the table of random numbers or generated using computer.

21.3.1 Step-by-Step Procedure

1. Decide the variables which influence the measure of effectiveness.
2. Determine the probability distribution for each variable.
3. Choose a suitable set of random variables.
4. Consider each random number as a decimal value of the cumulative probability.
5. Use the simulated values so generated into the formula derived from the chosen measure of effectiveness.
6. Repeat steps 4 and 5 until the sample is large enough to arrive at a satisfactory decision.

Example 21.2 The sales (in thousands of rupees) in a company follow Poisson distribution with mean 5. Using the Monte–Carlo technique estimate the sales for 10 days.

Solution The probability for k sales is given by

$P(x = k) = e^{-5}\, 5^k / k!$ $\qquad e^{-5} = 0.0067$

The cumulative probabilities for the values of k are given below:

k	Cumulative prob.	k	Cumulative prob.
0	0.0067	7	0.8666
1	0.0404	8	0.9319
2	0.1247	9	0.9682
3	0.2650	10	0.9763
4	0.4405	11	0.9845
5	0.6160	12	0.9999
6	0.7622		

Now note down 10 two-digit random numbers from the random number tables and read the corresponding values of k. These will give the sales for 10 days. The values are given below.

Random no.	Sales	Random no.	Sales
48	05	64	06
11	02	79	07
53	05	16	03
20	03	57	05
17	03		
81	07		

Example Number 53 (5300) lies in the interval (4405, 6160). Hence we take 05 corresponding to $k = 5$.

Number 79 (7900) belongs to (7622, 8666). Hence we take $k = 7$.

Example 21.3 At a sales depot the arrival of customers and the service times follow the following probability distributions.

Arrival time (min)	Prob.	Cumulative prob.	Interval
0.5	0.02	0.02	00–01
1.0	0.06	0.08	02–07
1.5	0.10	0.18	08–17
2.0	0.25	0.43	18–42
2.5	0.20	0.63	43–62
3.0	0.14	0.77	63–76
3.5	0.10	0.87	77–86
4.0	0.07	0.94	87–93
4.5	0.04	0.98	94–97
5.0	0.02	1.00	98–99

Service time (min)	Prob.	Cumulative prob.	Interval
0.5	0.12	0.12	00–11
1.0	0.21	0.33	12–32
1.5	0.36	0.69	33–68
2.0	0.19	0.88	69–87
2.5	0.07	0.95	88–94
3.0	0.05	1.00	95–99

Estimate the average waiting time and percentage of idle time of the server, by simulation, for 10 arrivals.

Solution 10 random numbers are chosen for arrival and service each and are linked to appropriate events. The results are given in the following table.

Arrival no.	Random no.	Int Arr time (min)	Arrival time (min)	Random no.	Service time	Service		Waiting time	
						Start	End	Customer	Server
1	93	4.0	4.0	78	2.0	4.0	6.0	–	4.0
2	22	2.0	6.0	76	2.0	6.0	8.0	–	–
3	53	2.5	8.5	58	1.5	8.5	10.0	–	0.5
4	64	3.0	11.5	54	1.5	11.5	13.0	–	1.5
5	39	2.0	13.5	74	2.0	13.5	15.5	–	0.5
6	07	1.0	14.5	92	2.5	15.5	18.0	1.0	–
7	10	1.5	16.0	38	1.5	18.0	19.5	2.0	–
8	63	3.0	19.0	70	2.0	19.5	21.5	0.5	–
9	76	3.0	22.0	96	3.0	22.0	25.0	–	0.5
10	35	2.0	24.0	92	2.5	25.0	27.5	1.0	–
						Total		4.5	7.0

From the above table we get,

$$\text{Average waiting time of a customer} = \frac{4.5}{10} = 0.45 \text{ min}$$

$$\text{Average idle time of the server} = \frac{7.0}{10} = 0.7 \text{ min}$$

Example 21.4 A tourist car company finds that during the past 200 days the demand for the car has the following frequency distribution.

Trips per week	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	16	24	30	60	40	30

Using random numbers simulate the demand for a period of 10 weeks.

Solution The probability distribution of the demand (trips) is given by the following table:

Trips per week	*Frequency*	*Prob.*	*Cumulative prob.*	*Interval*
0	16	0.08	0.08	00–07
1	24	0.12	0.20	08–19
2	30	0.15	0.35	20–34
3	60	0.30	0.65	35–64
4	40	0.20	0.85	65–84
5	30	0.15	1.00	85–99

Random numbers generated are 82, 95, 18, 96, 20, 84, 56, 11, 52 and 03. Using these random numbers we simulate for 10 weeks. The simulated demand is shown in the following table:

Week	*Random number*	*Demand*
1	82	4
2	95	5
3	18	1
4	96	5
5	20	2
6	84	4
7	56	3
8	11	1
9	52	3
10	03	0
	Total	28

We find that the total demand for the 10 weeks is 28. Hence the average demand is $\frac{28}{10} = 2.8$ cars per week.

Example 21.5 A company observes from past experience that the demand for a special product has the following probability distribution.

Daily demand	*5*	*10*	*15*	*20*	*25*	*30*
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Simulate the demand for the next 10 days. Also find the average demand.

Solution The probability distribution of the demand is given below:

Demand	*Prob.*	*Cumulative prob.*	*Interval*
5	0.01	0.01	00–00
10	0.20	0.21	01–20
15	0.15	0.36	21–35
20	0.50	0.86	36–85
25	0.12	0.98	86–97
30	0.02	1.00	98–99

The random numbers generated are:
69, 46, 16, 79, 64, 81, 11, 20, 60, 53.

Using these numbers, we simulate for 10 days. The simulated demand is given by the following table:

Day	*Random no.*	*Demand*
1	69	20
2	46	20
3	16	10
4	79	20
5	64	20
6	81	20
7	11	10
8	20	10
9	60	20
10	53	20
	Total	170

$$\text{Average demand} = \frac{170}{10} = 17 \text{ units}$$

EXERCISES

1. A bakery keeps stock plum cakes. Past experience shows that the daily demand of the cake has the probability distribution given below:

Daily demand	*0*	*10*	*20*	*30*	*40*	*50*
Prob.	0.01	0.20	0.15	0.50	0.12	0.02

   Using the random numbers 25, 39, 65, 76, 12, 05, 73, 89, 19, 49 simulate the demand for the next 10 days.

2. The following probabilities are given for the arrival and service times of the customers in a queuing system.

Arrival time		*Service time*	
Minutes	*Prob.*	*Minutes*	*Prob.*
2	0.15	1	0.10
4	0.23	3	0.22
6	0.35	5	0.35
8	0.17	7	0.23
10	0.10	9	0.10

   Using the random numbers 93, 14, 72, 10, 21, 81, 87, 90, 38, 71, 63, 14, 53, 64, 42, 07, 54 and 66 simulate the queue behaviour for 60 min. Estimate the average waiting time of the customer and the percentage of idle time for the server.

3. In a shop the sales of a particular soap has probability distribution $P(x = r) = e^{-\lambda} \lambda^r / r!$ (Poisson distribution) with $\lambda = 5$. Given $e^{-5} = 0.0067$ and the random numbers 49, 58, 89, 15, 12, 94, 85, 34, 07 and 53 simulate the sales for 10 days.

4. In a machine shop the time of breakdown of machines and their repair times in hours have the following frequency distributions.

Time between breakdown	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
Frequency	4	10	14	16	12	6	4	3	3	1

Repair time	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*
Frequency	2	3	8	16	14	12	8	5	3	2

   Use the random numbers 46, 64, 09, 48, 97, 22, 29, 40, 75, 10 and 09, 70, 41, 40, 37, 21, 44, 97, 51, 46 simulate the system and find out the average waiting time of the machine and the percentage of idle time for the mechanic.

ANSWERS

1.

Day	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Demand	20	30	30	30	10	10	30	40	10	30

   Average demand = 24

2.

Arrival no.	Random no.	Inter arrival time in mts	Arrival time	Random no.	Service time in mts	Service Start	Service End	Waiting time Customer	Waiting time Server
1	93	10	10	71	7	10	17	–	10
2	14	2	12	63	5	17	22	5	–
3	72	6	18	14	3	22	25	4	–
4	10	2	20	53	5	25	30	5	–
5	21	4	24	64	5	30	35	6	–
6	81	8	32	42	5	35	40	3	–
7	87	8	40	07	1	40	41	–	–
8	90	10	50	54	5	50	55	–	9
9	38	6	56	66	5	56	61	–	1
								23	20

Average waiting time of the customer = 23/9 = 2.5 min
Idle time for the server = 20 min; Percentage = 20/60 × 100 = 33%

3.

Day	1	2	3	4	5	6	7	8	9	10
Random no.	49	58	89	15	12	94	85	34	7	53
Sales	5	5	8	3	2	9	7	4	2	5

4.

Machine no.	Random no.	Inter arrival time in hrs	Time in hrs	Random No.	Repair time in hrs	Repair Start	Repair End	Waiting time Machine	Waiting time Mechanic
1	46	13	13	9	10	13	23	-	13
2	64	14	27	70	13	27	40	-	4
3	9	11	38	41	12	40	52	2	-
4	48	13	51	40	12	52	64	1	-
5	97	18	69	37	11	69	80	-	5
6	22	12	81	21	11	81	92	-	1
7	29	12	93	44	12	93	105	-	1
8	40	13	106	97	17	106	113	-	1
9	75	14	120	51	12	120	132	-	7
10	10	11	131	46	12	132	144	1	-
								4	32

Average waiting time of the machine = 4/10 = 24 min
Idle time for the mechanic = 32 hrs
Percentage = 32/144 × 100 = 22 %

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