Vectors and Linear Combinations

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Applied Linear Algebra for Wireless Communications

Agenda for today's class

- Introduce vector, addition and subtraction
- Discuss vectors in different dimensions and their visual interpretation
- Reference for today's class - Chap 1.1 of the book

Introduction to vectors (1)

- A two dimensional vector $\mathbf{v}$ is a pair of number with components v_1 and v_2

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- We write $\mathbf{v}$ and $\mathbf{w}$ as column vectors
- Vector addition: we don't add v_1 to v_2 while adding vectors
- First component of $\mathbf{v}$ and $\mathbf{w}$ stay separate from their second component:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

Introduction to vectors (2)

- Scalar multiplication - multiply each component of $\mathbf{v}$ by a scalar c

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

- Notice that the sum of $-\mathbf{v}$ and $\mathbf{v}$ is the zero vector $\mathbf{0}$,
    - Not same as the number zero
    - $\mathbf{0}$ has components 0 and 0

Linear combinations of vectors

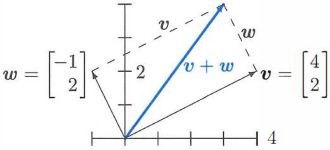
- Combine addition with scalar multiplication to produce a "linear combination" of $\mathbf{v}$ and $\mathbf{w}$
  - Multiply $\mathbf{v}$ by c and multiply $\mathbf{w}$ by b and then add them $c\mathbf{v} + d\mathbf{w}$
- Four special combinations are sum, difference, zero, and a scalar multiple c

$$
\begin{gathered}
1\mathbf{v} + 1\mathbf{w} \\
1\mathbf{v} - 1\mathbf{w} \\
0\mathbf{v} + 0\mathbf{w} \\
c\mathbf{v} + 0\mathbf{w}
\end{gathered}
$$

- Zero vector is possible combination - every time we see a space of vectors, zero vector will always be included

Visualization of vector addition

- We visualize $\mathbf{v} + \mathbf{w}$ using arrows
- A vector with two components corresponds to a point in the xy plane

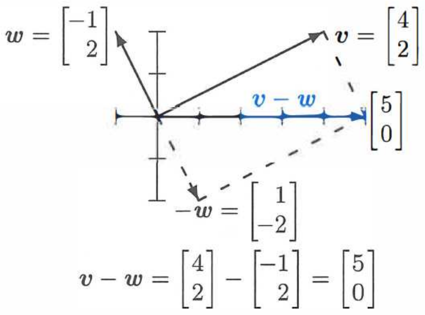


$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

- Components of $\mathbf{v}$ are coordinates of point $x = v_1$ and $y = v_2$
- Arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$

Visualization of vector subtraction

- We visualize $\mathbf{v} - \mathbf{w}$ using arrows



Vectors in Three Dimensions (1)

- Now we allow vectors to have three components (v_1, v_2, v_3)
- xy plane is replaced by three-dimensional xyz space
- Here is a typical example

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \mathbf{v} = (1, 2, 2)$$

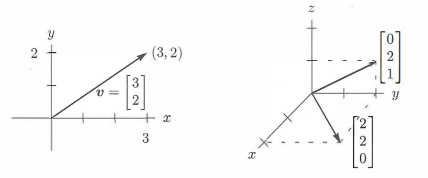
- From now on, vector $\mathbf{v}$ is also written as $\mathbf{v} = (1, 2, 2)$ to save spave
- Still column vector but with three components and temporarily lying down
- Row vector $\mathbf{v} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ is different and "Transpose" of column vector

Vectors in Three Dimensions (2)

- Linear combination of 3 vectors

$$\mathbf{u} + 4\mathbf{v} - 2\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$$

- Vector **v** corresponds to an arrow in 3D space

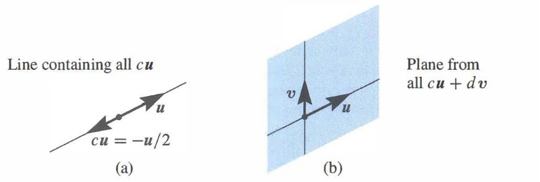


  - Arrow starts at the "origin", where xyz axes meet and coordinates are $(0, 0, 0)$
  - Arrow ends at the point with coordinates v_1, v_2, v_3

Important aspects of linear combination

- For one vector $\mathbf{u}$, only linear combinations are the multiples $c\mathbf{u}$
- For two vectors, linear combinations are $c\mathbf{u} + d\mathbf{v}$
- For three vectors, linear combinations are $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$
  - every c and d and e are allowed
- Suppose vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in three dimensional space
  - $c\mathbf{u}$ fills a line through $(0, 0, 0)$
  - $c\mathbf{u} + d\mathbf{v}$ fills a plane through $(0, 0, 0)$
  - $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fills 3D space
- Zero vector $(0, 0, 0)$ is on the line because c can be zero
- Zero vector $(0, 0, 0)$ is on the plane because c and d could both be zero

Linear combination in 3D space



- Line of vectors $c\mathbf{u}$ is infinitely long (forward and backward)
- Adding all $c\mathbf{u}$ on one line to all $d\mathbf{v}$ on the other line fills in the plane
- When we include a third vector $\mathbf{w}$, the multiples $e\mathbf{w}$ give a third line
- Suppose that third line is not in the plane of $\mathbf{u}$ and $\mathbf{v}$
  - Combining all $e\mathbf{w}$ with all $c\mathbf{u} + d\mathbf{v}$ fills up whole three-dimensional space

Review of important ideas

- A vector $\mathbf{v}$ in two-dimensional space has two components v_1 and v_2
- $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$ is calculated a component at a time
- $c\mathbf{v} = (cv_1, cv_2)$ is calculated a component at a time
- Linear combination of three vectors $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{w}$ is $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$
- Take all linear combinations of uv, or $\mathbf{u}$ and $\mathbf{v}$, or $\mathbf{u}, \mathbf{v}, \mathbf{w}$
- In three dimensions, those combinations typically fill a line, then a plane, then the whole space $\mathbf{R}^3$
- End of Section 1.1 – lots of problems at the end - please solve them