eMasters in Communication Systems

Prof. Aditya Jagannatham

Elective Module:

Detection for Wireless Communication

Chapter 3

Generalized ML Detection

Detection

- **Mathematical Model**

$\mathcal{H}_0$:

$$y(1) = s_0(1) + v(1)$$

$$y(2) = s_0(2) + v(2)$$

$$\vdots$$

$$y(N) = s_0(N) + v(N)$$

$s_0(i)$

signal corresponding to $\mathcal{H}_0$.

iid Gaussian noise samples

mean $= 0$

var $= \sigma^2$

Detection

- **Mathematical Model**

$\mathcal{H}_0$:

$$y(1) = s_0(1) + v(1)$$
$$y(2) = s_0(2) + v(2)$$
$$\vdots$$
$$y(N) = s_0(N) + v(N)$$

$\bar{s}_0 = \text{Signal for } \mathcal{H}_0$

Detection

- **Mathematical Model**

$$\mathcal{H}_0: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \overset{\bar{s}_0}{\begin{bmatrix} s_0(1) \\ s_0(2) \\ \vdots \\ s_0(N) \end{bmatrix}} + \overset{\bar{v}}{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}$$

$$\bar{y} = \bar{s}_0 + \bar{v} \ ; \ \mathcal{H}_0$$

Detection

- **Mathematical Model**

$$\mathcal{H}_0: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s_0(1) \\ s_0(2) \\ \vdots \\ s_0(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\bar{y} = \bar{s}_0 + \bar{v}$$

Detection

- **Mathematical Model**

Signal for $\mathcal{H}_1$.

$\mathcal{H}_1$:

$$y(1) = s_1(1) + v(1)$$

$$y(2) = s_1(2) + v(2)$$

$$\vdots$$

$$y(N) = s_1(N) + v(N)$$

Detection

- **Mathematical Model**

$$\mathcal{H}_1:$$

$$y(1) = s_1(1) + v(1)$$

$$y(2) = s_1(2) + v(2)$$

$$\vdots$$

$$y(N) = s_1(N) + v(N)$$

Detection

- **Mathematical Model**

$$\mathcal{H}_1: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \overset{\bar{s}_1}{\begin{bmatrix} s_1(1) \\ s_1(2) \\ \vdots \\ s_1(N) \end{bmatrix}} + \overset{\bar{v}}{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}$$

$$\bar{y} = \bar{s}_1 + \bar{v} \; ; \; \mathcal{H}_1$$

Detection

- **Mathematical Model**

$$\bar{y} = \bar{s}_1 + \bar{v}$$

$$\mathcal{H}_1: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s_1(1) \\ s_1(2) \\ \vdots \\ s_1(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

Detection

i.i.d. independent identically distributed.

$v(1), v(2), \ldots, v(N)$

- Noise samples in $\bar{\mathbf{v}}$ are i.i.d. Gaussian, mean 0 variance is σ^2.
- $\bar{\mathbf{s}}_0, \bar{\mathbf{s}}_1$ are ***known signals***

$\mathcal{H}_0$ NULL Hypothesis

$\mathcal{H}_1$ ALTERNATIVE Hypothesis.

Detection

- Write it in the compact form

$\mathcal{H}_0$: $\bar{y} = \bar{s}_0 + \bar{v}$

$\mathcal{H}_1$: $\bar{y} = \bar{s}_1 + \bar{v}$

Detection

Binary hypothesis Testing
What is the LRT ?

- Write it in the compact form

$$\mathcal{H}_0 : \bar{\mathbf{y}} = \bar{\mathbf{s}}_0 + \bar{\mathbf{v}}$$

$$\mathcal{H}_1 : \bar{\mathbf{y}} = \bar{\mathbf{s}}_1 + \bar{\mathbf{v}}$$

Detection

- Define

$$\tilde{\mathbf{y}} = \bar{y} - \bar{s}_0$$

Detection

- Define

$$\bar{\mathbf{y}} - \bar{\mathbf{s}}_0 = \tilde{\mathbf{y}}$$

Detection

- Write it in the compact form

$$\mathcal{H}_0: \bar{\mathbf{y}} = \bar{\mathbf{s}}_0 + \bar{\mathbf{v}}$$

$\Rightarrow$ $\bar{y} - \bar{s}_0 = \bar{v}$

$\tilde{y} = \bar{v}$

$\Rightarrow$

Detection

- Write it in the compact form

$$\mathcal{H}_0 : \bar{\mathbf{y}} = \bar{\mathbf{s}}_0 + \bar{\mathbf{v}}$$

$$\Rightarrow \bar{\mathbf{y}} - \bar{\mathbf{s}}_0 = \bar{\mathbf{v}}$$

$$\Rightarrow \tilde{\mathbf{y}} = \bar{\mathbf{v}} \; ; \; \mathcal{H}_0$$

$\bar{y} - \bar{s}_0$

Detection

$\bar{s} = \bar{s}_1 - \bar{s}_0$

- Write it in the compact form

$$\mathcal{H}_1 : \bar{\mathbf{y}} = \bar{\mathbf{s}}_1 + \bar{\mathbf{v}}$$

$$\Rightarrow \quad \bar{y} - \bar{s}_0 = \bar{s}_1 - \bar{s}_0 + \bar{v}$$

$$\tilde{y} = \bar{s}. + \bar{v}$$

$$\Rightarrow$$

Detection

- Write it in the compact form

$$\mathcal{H}_1 : \bar{\mathbf{y}} = \bar{\mathbf{s}}_1 + \bar{\mathbf{v}}$$

$$\Rightarrow \bar{\mathbf{y}} - \bar{\mathbf{s}}_0 = \bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0 + \bar{\mathbf{v}}$$

$$\Rightarrow \tilde{\mathbf{y}} = \bar{\mathbf{s}} + \bar{\mathbf{v}}$$

$\bar{s} = \bar{s}_1 - \bar{s}_0$

Detection

- Write it in the compact form

$$\left.\begin{aligned} \mathcal{H}_0: &\quad \tilde{y} = \bar{v} \\ \mathcal{H}_1: &\quad \tilde{y} = \bar{s} + \bar{v} \end{aligned}\right\}$$

Original Signal.
Detection problem.

$\bar{y} \rightarrow \tilde{y}$

$\bar{s} \rightarrow \bar{s}_1 - \bar{s}_0$

Detection

- Write it in the compact form

$$\mathcal{H}_0: \tilde{\mathbf{y}} = \bar{\mathbf{v}}$$

$$\mathcal{H}_1: \tilde{\mathbf{y}} = \bar{\mathbf{s}} + \bar{\mathbf{v}}$$

Likelihood

- Choose $\mathcal{H}_0$ if

$$\bar{s}^T \tilde{y} \le \gamma$$

$$(\bar{s}_1 - \bar{s}_0)^T \tilde{y} \le \gamma$$

$$(\bar{s}_1 - \bar{s}_0)^T (\bar{y} - \bar{s}_0) \le \gamma$$

Likelihood

- Choose $\mathcal{H}_0$ if

LRT

$$\bar{\mathbf{s}}^T \tilde{\mathbf{y}} \le \gamma$$

$$\underbrace{(\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0)^T \tilde{\mathbf{y}}}_{\text{Test Statistic.}} \le \gamma$$

Likelihood

- Choose $\mathcal{H}_1$ if

$$\bar{s}^T \tilde{y} > \gamma$$

$$(\bar{s}_1 - \bar{s}_0)^T (\bar{y} - \bar{s}_0) > \gamma$$

Likelihood

- Choose $\mathcal{H}_1$ if

$$\bar{\mathbf{s}}^T \tilde{\mathbf{y}} > \gamma$$

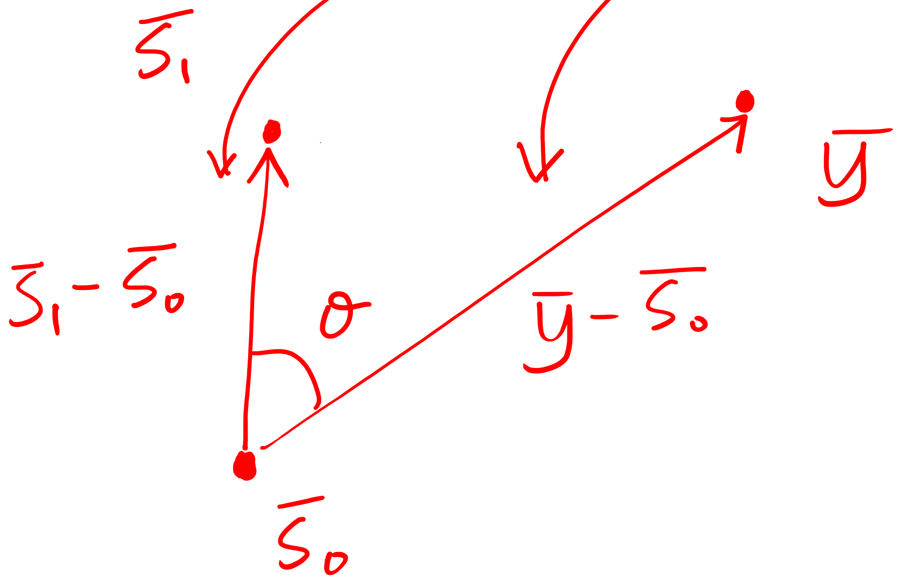
$$(\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0)^T \tilde{\mathbf{y}} > \gamma$$

Generalized Signal Detection problem.

Figure: Intuition

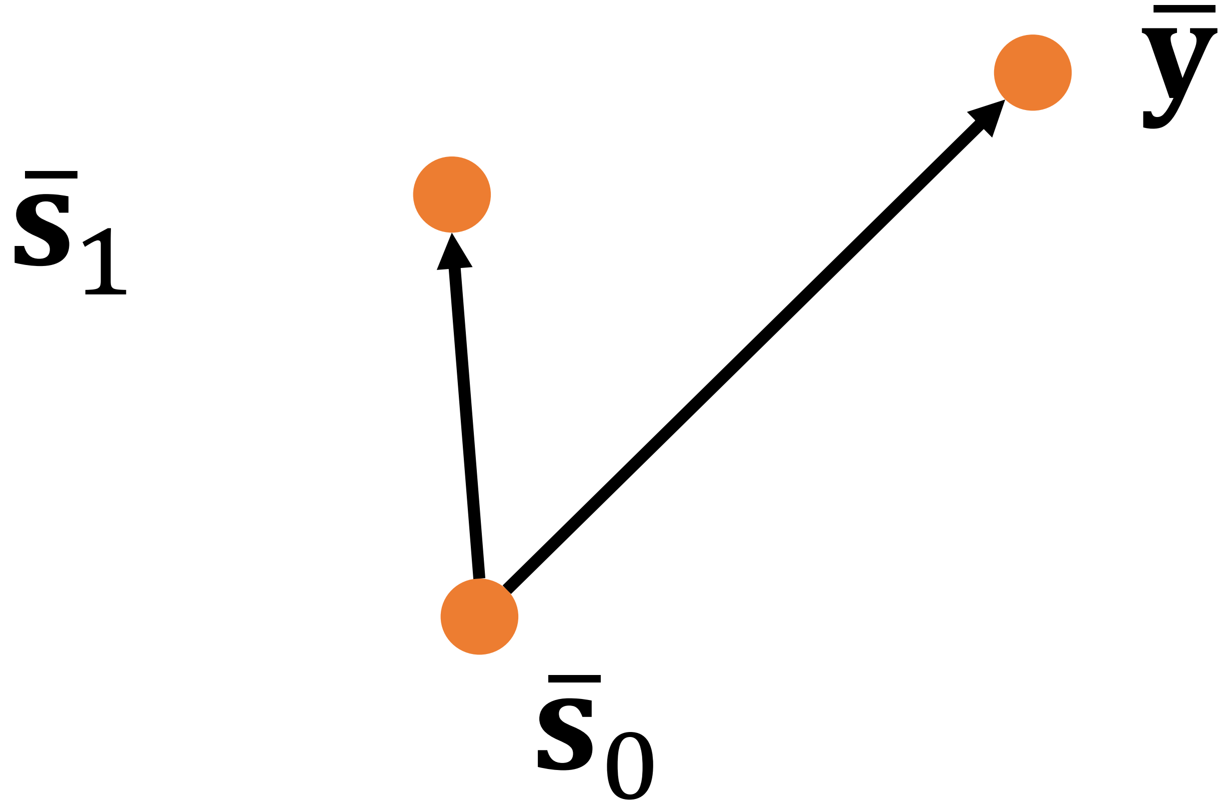
$$(\bar{s}_1-\bar{s}_0)^T(\bar{y}-\bar{s}_0)$$

$$= |\bar{s}_1-\bar{s}_0|\,|\bar{y}-\bar{s}_0|\cos\theta.$$



Test statistic = Dot product or inner product between these 2 vectors.

Figure: Intuition



Performance

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\gamma}{\sigma\|\bar{s}_1 - \bar{s}_0\|}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\gamma - \|\bar{s}_1 - \bar{s}_0\|^2}{\sigma\|\bar{s}_1 - \bar{s}_0\|}\right)$$

Performance

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|^2}{\sigma\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}\right)$$

Performance ML

$\tilde{y} = \bar{y} - \bar{s}_0$

- For ML set

$$\gamma = \frac{\|\bar{\mathbf{s}}\|^2}{2} = \frac{\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|^2}{2}$$

$\mathcal{H}_0$ if $(\bar{s}_1 - \bar{s}_0)^T \tilde{y} < \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$

$\mathcal{H}_1$ if $(\bar{s}_1 - \bar{s}_0)^T \tilde{y} > \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$

Performance ML

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}{2\sigma}\right)$$

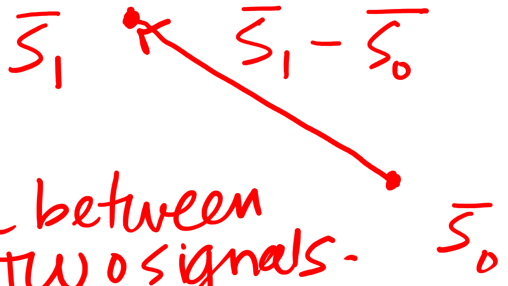
Probability of Miss Detection $= 1 - P_D$.

$$P_{MD} = Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}{2\sigma}\right)$$

Performance ML

- Therefore, P_e *is*

$\|\bar{s}_1 - \bar{s}_0\|$ = Distance between $\bar{s}_1, \bar{s}_0$



distance between two signals.

$$Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}_1 - \bar{s}_0\|}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

To minimize P_e maximize distance between signals.

Performance ML

- Therefore, P_e *is*

$$\Pr(\mathcal{H}_0)\Pr(\mathcal{H}_1|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\Pr(\mathcal{H}_0|\mathcal{H}_1)$$

$$\frac{1}{2}P_{FA} + \frac{1}{2}P_{MD} = Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}{2\sigma}\right)$$

$$= Q\left(\frac{d}{2\sigma}\right)$$

$d = \|\bar{s}_1 - \bar{s}_0\|.$

Performance ML

- Note that d is the distance between the points $\bar{s}_1, \bar{s}_0$

$$d = \|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|$$

$\bar{s}_1$

$d = \|\bar{s}_1 - \bar{s}_0\| \ldots$

$\bar{s}_0$

Example

ASK : $\{0, A\}$

- Consider **Binary Phase Shift Keying (BPSK)**

180°

0°.

$\{A, A\}$

Binary constellation

1 bit/symbol.

- $s = A$

N = 1

S_0.

$$\mathcal{H}_0 : y = -A + v$$
$$\mathcal{H}_1 : y = A + v$$

S_1

$S_0 = -A$

$S_1 = A$.

Example

- Therefore, P_e is

$\sigma^2 = \frac{N_0}{2}$

$$Q\left(\frac{\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}{2\sigma}\right) = Q\left(\frac{\|A-(-A)\|}{2\sqrt{\frac{N_0}{2}}}\right)$$

$$= Q\left(\frac{2A}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2}{N_0}}\right)$$

Example

- Therefore, P_e is

Probability of error of BPSK.

$$Q\left(\frac{\|\bar{\mathbf{s}}_1 - \bar{\mathbf{s}}_0\|}{2\sigma}\right) = Q\left(\frac{2A}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2A^2}{N_0}}\right)$$

Example

- Each symbol carries one bit
- Let E_b be the ***energy per bit***
- Let each symbol be **equiprobable**

$$\Pr(-A) = \Pr(A) = \frac{1}{2}$$

Example

$$E_b = \frac{1}{2}A^2 + \frac{1}{2}(-A)^2 = \frac{1}{2}A^2 + \frac{1}{2}A^2 = A^2$$

$$\Rightarrow A = \sqrt{E_b}$$

$$\Rightarrow A^2 = E_b.$$

Example

$$E_b = \frac{1}{2} \times (-A)^2 + \frac{1}{2} \times A^2$$

$$\Rightarrow A^2 = E_b$$

Example

- Therefore, P_e is

$$Q\left(\sqrt{\frac{2A^2}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Bit Error Rate (BER) OF BPSK.

- This is termed as **bit error rate**

Example

- Therefore, P_e is

$$Q\left(\sqrt{\frac{2A^2}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- This is termed as bit error rate (BER)

ASK vs BPSK

For same E_b.
BER of BPSK is lower!

BER of ASK	BER of BPSK
$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

ASK vs BPSK

BER of ASK	BER of BPSK
$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

TO achieve same BER with BPSK, we need half the E_b as that of ASK!

Example

- For same E_b BPSK has lower BER!

- In fact, BPSK is 3 dB **more efficient** than ASK
  - BPSK Needs half the E_b for same BER i.e. 3 dB less!

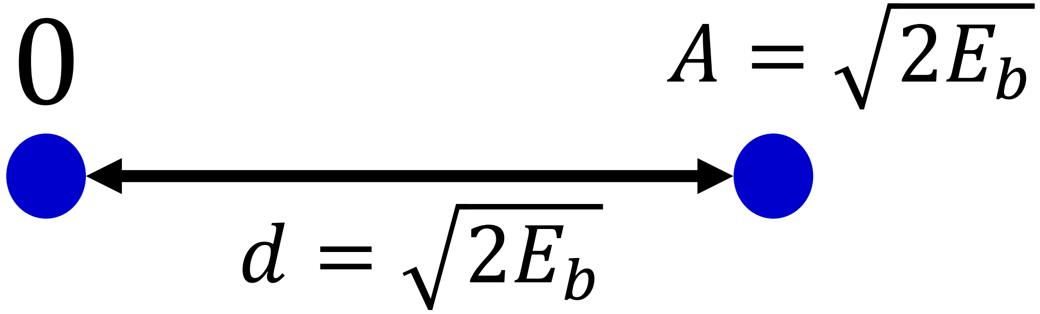
ASK Distance Properties

intuition?

$d = \sqrt{2E_b}$

$0 \longleftrightarrow A = \sqrt{2E_b}.$

ASK Distance Properties



BPSK Distance Properties

$d = 2\sqrt{E_b}$.

$A = -\sqrt{E_b}$ $\qquad\qquad$ $A = \sqrt{E_b}$

BPSK Distance Properties

$$A = -\sqrt{E_b} \qquad\qquad A = \sqrt{E_b}$$

$$d = 2\sqrt{E_b}$$

BPSK vs ASK Distance

- For same E_b distance of BPSK is $2\sqrt{E_b}$ $> \sqrt{2E_b}$ = distance of ASK!

- While that of ASK is only $\sqrt{2E_b}$

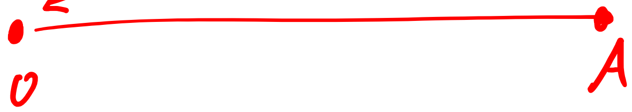
BPSK vs ASK Distance

Symmetric about 0
$\Rightarrow$ Energy = min!

- BPSK is **ANTIPODAL**, i.e., centered around zero

  - This **maximizes distance** of same average power

NOT symmetric about 0
$\Rightarrow$ Energy is higher



constellation has to be symmetric about 0 to minimize P_e.

Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.