1. Consider the sinusoidal measurement model in Gaussian noise described as,

$$y(n) = \alpha_d + \alpha_c \cos(2\pi f_0 n) + \alpha_s \sin(2\pi f_0 n) + w(n), 0 \le n \le N-1, \tag{1}$$

where $w(n)$ is additive white Gaussian noise and $\mathrm{E}\left(|w(n)|^2\right) = \sigma_n^2$. Answer the questions that follow

(a) Formulate the LS estimation problem for parameters $\alpha_d, \alpha_c, \alpha_s$.

(b) Derive the LS estimator of $\alpha_d, \alpha_c, \alpha_s$ with a suitable approximation for large N.

Solution: Let $\mathbf{y} = [y(0)...y(N-1)]^T$, $\mathbf{c} = [1...\cos(2\pi f_0(N-1))]^T$, $\mathbf{s} = [0...\sin(2\pi f_0(N-1))]^T$, $\mathbf{w} = [w(0)...w(N-1)]^T$, and $\mathbf{x} = [\alpha_d\ \alpha_d\ \alpha_s]^T$. Now the equations in (1) can be written as $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}$, where $\mathbf{A} = [\mathbf{1}\ \ \mathbf{c}\ \ \mathbf{s}]$.

(a) LS estimation problem is,

$$\min_{\mathbf{a}} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \tag{2}$$

(b) For large N, $\mathbf{A}\mathbf{x} = \mathbf{y}$ may not have solutions since, the vector $\mathbf{y}$ may not lie in the 3-dimensional subspace $span\{\mathbf{1}, \mathbf{c}, \mathbf{s}\}$. Hence the approximate solution $\hat{\mathbf{x}}$ is for the vector $\mathbf{y}$ that has been projected orthogonally onto the subspace $span\{\mathbf{1}, \mathbf{c}, \mathbf{s}\}$. In other words, the error residue $\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}$ is orthogonal to all of the column vectors in $\mathbf{A}$ i.e., $\mathbf{A}^T(\mathbf{y} - \mathbf{A}\hat{\mathbf{x}}) = \mathbf{0}$. That implies $\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$. Other alternative method to find the estimator is by differentiating the objective function in (2) and making it equal to zero.

Upon further solving it, we can actually see that

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{1}^T \\ \mathbf{c}^T \\ \mathbf{s}^T \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{c} & \mathbf{s} \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} N & \mathbf{1}^T\mathbf{c} & \mathbf{1}^T\mathbf{s} \\ \mathbf{1}^T\mathbf{c} & \mathbf{c}^T\mathbf{c} & \mathbf{s}^T\mathbf{c} \\ \mathbf{1}^T\mathbf{s} & \mathbf{s}^T\mathbf{c} & \mathbf{s}^T\mathbf{s} \end{bmatrix} \tag{4}$$

Now, a term of the form

$$\sum_{i=0}^{N-1} \cos^2(2\pi f_0 i) = \frac{N}{2} + \frac{1}{2}\sum_{i=1}^{N-1} \cos(4\pi f_0 i) \tag{5}$$

where the last term vanishes for large N. Likewise, $\mathbf{s}^T\mathbf{s} \approx N/2$, $\mathbf{c}^T\mathbf{s} \approx 0$, $\mathbf{1}^T\mathbf{c} \approx 0$, and $\mathbf{1}^T\mathbf{s} \approx 0$.

2. Consider the two sets:

$$\mathcal{S}_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \tag{6}$$

$$\mathcal{S}_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q\} \tag{7}$$

Formulate the linear optimization problem to determine the separating hyperplane, i.e., find $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\mathbf{a}^T\mathbf{v}_i \le b \qquad i = 1, \ldots, p \tag{8}$$

$$\mathbf{a}^T\mathbf{u}_i \geq b \qquad i = 1, \ldots, q \tag{9}$$

Ensure that your problem excludes the trivial solution $\mathbf{a} = b = 0$.

Solution:

$$\min \|\mathbf{a}\| \tag{10}$$
$$\mathbf{a}^\top \mathbf{v}_i \leq 1 \tag{11}$$
$$\mathbf{a}^\top \mathbf{u}_i \geq 1 \tag{12}$$

3. Solve the following optimization problem for $\mathbf{A} \succ 0$,

$$\min\ \mathbf{c}^T\mathbf{x} \tag{13}$$
$$\text{s. t. } (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{A}(\mathbf{x} - \mathbf{x}_c) \leq 1 \tag{14}$$

Solution: From eigenvalue decomposition, $\mathbf{A} = \mathbf{U}\Sigma\mathbf{U}^T$, with change of variable $\mathbf{y} = \sqrt{\Sigma}\mathbf{U}^T(\mathbf{x} - \mathbf{x}_c)$, we can reformulate the problem as

$$\min\ \mathbf{c}^T\mathbf{U}\sqrt{\Sigma}^{-1}\mathbf{y}$$
$$\text{s. t. } \|\mathbf{y}\|_2^2 \leq 1.$$

From the Cauchy–Schwarz inequality, the objective is minimized at

$$\mathbf{y}^\star = -\frac{\sqrt{\Sigma}^{-1}\mathbf{U}^T\mathbf{c}}{\left\|\sqrt{\Sigma}^{-1}\mathbf{U}^T\mathbf{c}\right\|_2}.$$

Equivalently, at

$$\mathbf{x}^\star = \mathbf{x}_c - \frac{\mathbf{U}\Sigma^{-}1\mathbf{U}^T\mathbf{c}}{\left\|\sqrt{\Sigma}^{-1}\mathbf{U}^T\mathbf{c}\right\|_2} = \mathbf{x}_c - \frac{\mathbf{A}^{-1}\mathbf{c}}{\sqrt{\mathbf{c}^T\mathbf{A}^{-1}\mathbf{c}}}.$$