Live Interaction #7:

19th November 2023

E-masters Communication Systems

Estimation for Wireless

- **LMMSE – Linear Minimum Mean Square Error**:
- MMSE:

$$\min E\left\{\left|\hat{\mathbf{h}}-\bar{\mathbf{h}}\right|^2\right\}$$

- $\hat{\mathbf{h}}$: can be **linear or non-linear estimator**.
- When we constrain $\hat{\mathbf{h}}$ to be a ***linear estimator*** it becomes LMMSE.

$$\hat{\mathbf{h}}=\mathbf{C}\bar{\mathbf{y}}$$

$$\min E\left\{\left|\mathbf{C}\bar{\mathbf{y}}-\bar{\mathbf{h}}\right|^2\right\}$$

- When $\bar{\mathbf{h}}, \bar{\mathbf{y}}$ are zero-mean, what is the expression for the LMMSE estimate?

$$\hat{\mathbf{h}}=\mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$\underbrace{E\{\bar{\mathbf{h}}\}=0, E\{\bar{\mathbf{y}}\}=0}_{\text{zero-mean quantities}}$$

$$\mathbf{R}_{hy}=E\{\bar{\mathbf{h}}\bar{\mathbf{y}}^T\}$$

$$\mathbf{R}_{yy}=E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

- **Linear model:**

$$\bar{\mathbf{y}}=\mathbf{X}\bar{\mathbf{h}}+\bar{\mathbf{v}}$$

- What is the LMMSE estimator?

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\} = \left(\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}\right)$$

$$\mathbf{R}_{hy} = \sigma_h^2 \mathbf{X}^T = E\left\{\bar{\mathbf{h}}\left(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}\right)^T\right\}$$

$$\hat{\mathbf{h}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$= \sigma_h^2 \mathbf{X}^T\left(\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}\right)^{-1}\bar{\mathbf{y}}$$

$$= \sigma_h^2\left(\sigma_h^2 \mathbf{X}^T\mathbf{X} + \sigma^2 \mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \left(\mathbf{X}^T\mathbf{X} + \frac{\sigma^2}{\sigma_h^2}\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

- The above estimator is the LMMSE estimate for any **arbitrary distribution** with zero-mean.
- **Non-zero mean**:

$$\hat{\mathbf{h}} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\left(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y\right) + \bar{\boldsymbol{\mu}}_h$$

- **Scalar parameter:**

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} h + \bar{\mathbf{v}}$$

$$\bar{\mathbf{y}} = \bar{\mathbf{x}}h + \bar{\mathbf{v}}$$

- $E\{h\} = \mu_h,\ E\{h^2\} = \sigma_h^2$

$$\hat{h} = \frac{\frac{\frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\|\bar{\mathbf{x}}\|^2}}{\frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}} + \frac{\mu_h}{\sigma_h^2}}{\frac{1}{\frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}} + \frac{1}{\sigma_h^2}}$$

- ***Weighted linear combination*** of **ML estimate** and **prior estimate**.

$$= \frac{\frac{\textbf{ML Estimate}}{\textbf{MSE of ML}} + \frac{\textbf{Prior Mean}}{\textbf{Prior variance}}}{\frac{1}{\textbf{MSE of ML}} + \frac{1}{\textbf{Prior variance}}}$$

$$= \theta \times \textbf{ML Estimate} + (1-\theta) \times \textbf{Prior Mean}$$

$$\theta = \frac{\frac{1}{\textbf{MSE of ML}}}{\frac{1}{\textbf{MSE of ML}} + \frac{1}{\textbf{Prior variance}}}$$

$$1-\theta = \frac{\frac{1}{\textbf{Prior variance}}}{\frac{1}{\textbf{MSE of ML}} + \frac{1}{\textbf{Prior variance}}}$$

- $\textbf{Prior variance} \rightarrow \infty$

$$\theta = 1, 1-\theta = 0$$

$$\textbf{LMMSE Estimate} = \textbf{ML Estimate}$$

- Prior is not providing any information – **Non-informative prior**.
- $\textbf{MSE of ML} \rightarrow \infty. \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$

$$\theta = 0, 1-\theta = 1$$

LMMSE Estimate = Prior Mean

- **Non-informative model.**

$$= \underbrace{\left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}}_{\text{Stable estimate}}$$

- **Assignment #7 deadline: 25th November 11:59 AM.**
- **Live interaction 25th November 6:00-7:00 PM.**
- **Assignment #8 deadline: 25th November 11:59 PM.**
- **Assignment #7, #8 Discussion: 26th November 12:30 PM – 1:00 PM.**
- **Quiz #4 26th November: 1:15-2:00 PM.**