Fourier transform representation of CT aperiodic signals – Section 4.1

A large class of *aperiodic CT signals* can be represented by the *CT Fourier transform (CTFT)*.

The (CT) Fourier transform (or spectrum) of $x(t)$ is

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt.}$$

$x(t)$ can be reconstructed from its spectrum using the inverse Fourier transform

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega.}$$

The above two equations are referred to as the *Fourier transform pair* with the first one being the *analysis equation* and the second being the *synthesis equation*.

Notation:

$$\begin{aligned} X(j\omega) &= \mathcal{F}\{x(t)\} \\ x(t) &= \mathcal{F}^{-1}\{X(j\omega)\} \end{aligned}$$

$x(t)$ and $X(j\omega)$ form a Fourier transform pair, denoted by

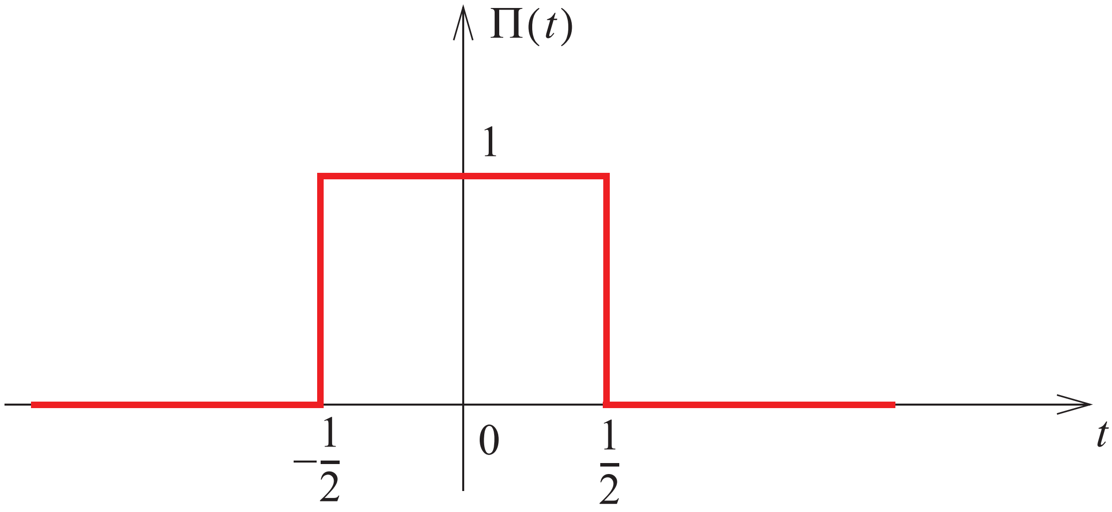
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

Often, we will use the simpler notation

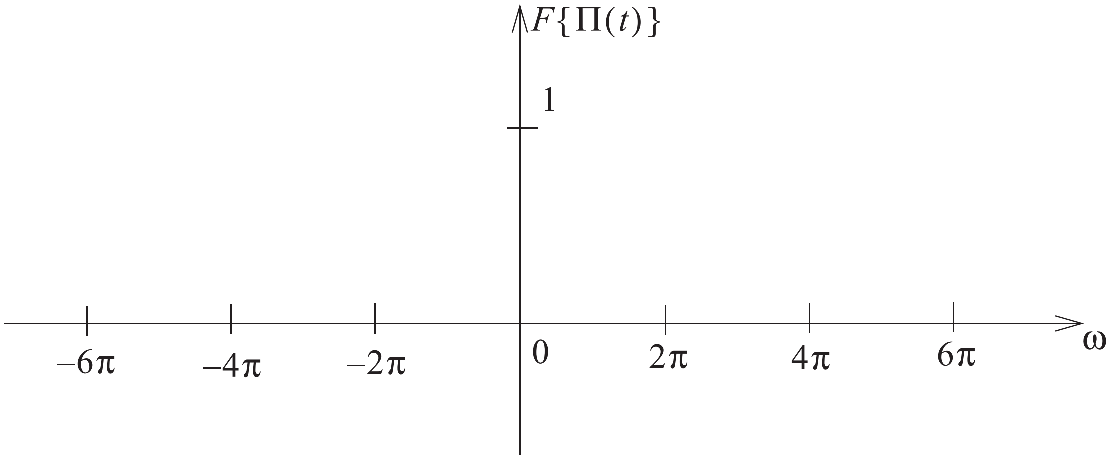
$$x(t) \longleftrightarrow X(j\omega)$$

Example:

$$\mathrm{rect}(t) \text{ or } \Pi(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \\ 1/2, & |t| = 1/2 \end{cases}$$



$\mathrm{rect}(t)$



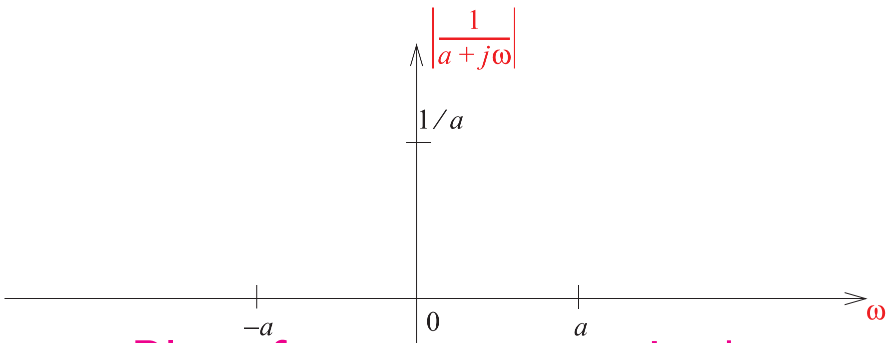
Fourier transform of $\mathrm{rect}(t)$

Example: $x(t) = e^{-at}u(t), a > 0$. We want to show that

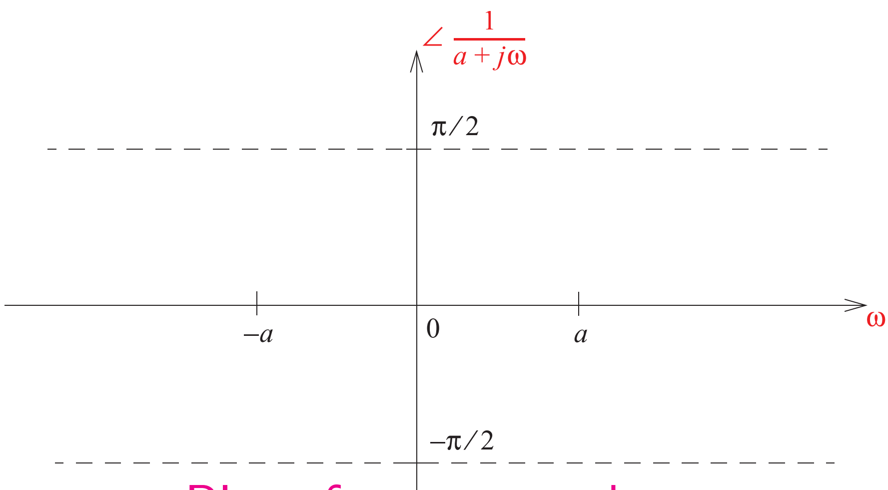
$$e^{-at}u(t) \longleftrightarrow \frac{1}{a + j\omega}, a > 0 .$$

Since the above FT is complex-valued, it is customary to plot its magnitude and phase, i.e.,

$$\begin{aligned} \left|\frac{1}{a+j\omega}\right| &= \frac{1}{\sqrt{a^2+\omega^2}} \\ \angle\frac{1}{a+j\omega} &= -\tan^{-1}\left(\frac{\omega}{a}\right). \end{aligned}$$



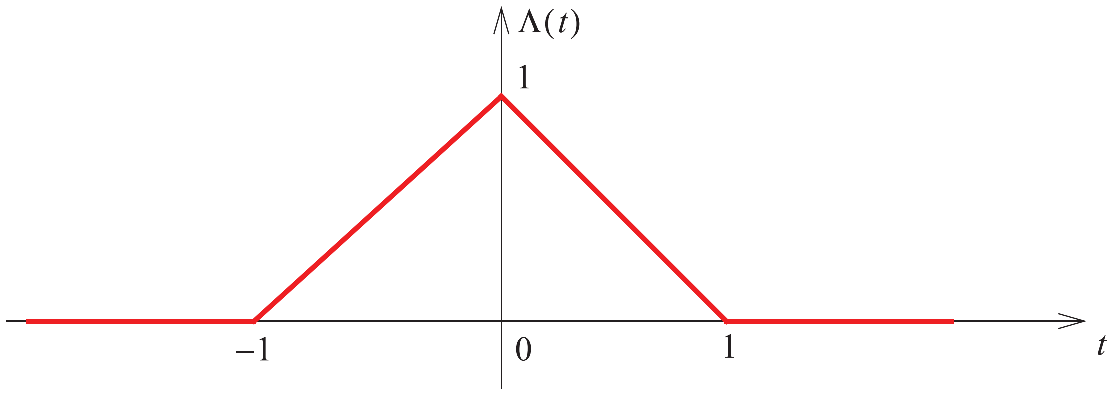
Plot of spectrum magnitude



Plot of spectrum phase

Example: triangle function

$$\wedge(t) = \begin{cases} 1 - |t|, & -1 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$



triangle function

Exercise: Show that

$$\wedge(t) \xleftrightarrow{\mathcal{F}} \text{sinc}^2(\frac{\omega}{2\pi})$$

Properties of CT Fourier transform – Section 4.3

Table 4.1 on p. 328 summarizes many CTFT properties.

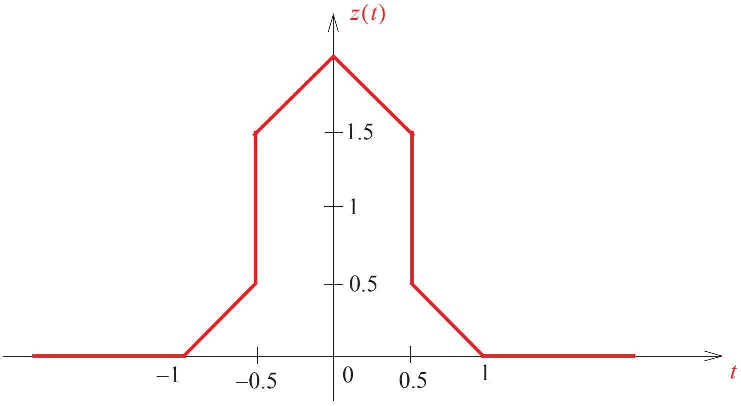
1. *Linearity – Section 4.3.1*

   Let $x(t) \longleftrightarrow X(j\omega)$, $y(t) \longleftrightarrow Y(j\omega)$. Then,

$$z(t) = ax(t) + by(t)$$
$$\longleftrightarrow Z(j\omega) = aX(j\omega) + bY(j\omega) \ .$$

Proof:

Example: What is the CTFT of the signal, $z(t)$, shown in the figure below?



Signal $z(t)$

2. *Time shift – Section 4.3.2*

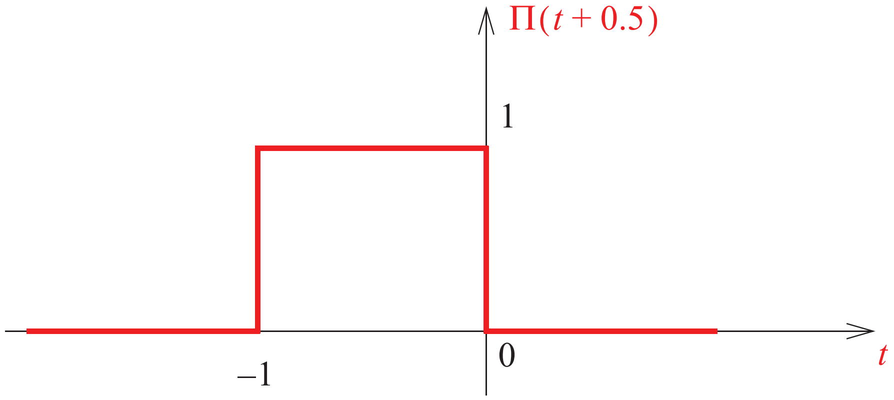
If $x(t) \longleftrightarrow X(j\omega)$, then

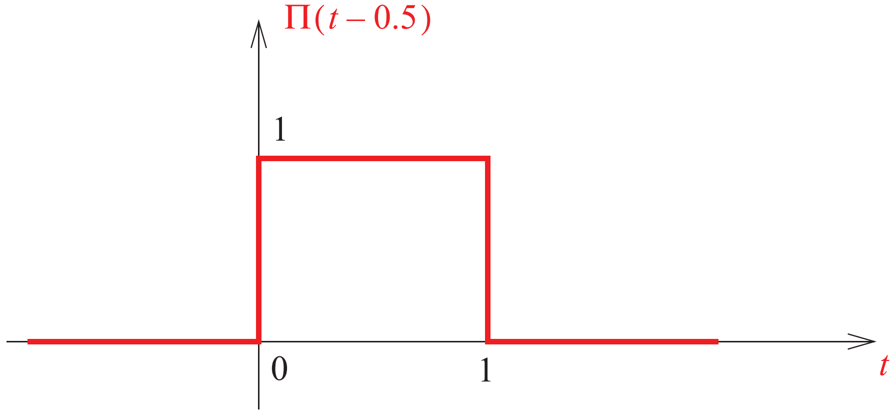
$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega) \ .$$

Proof:

Note that when a signal is delayed by t_0, the *spectrum amplitude* is *unchanged* whereas the spectrum *phase* is changed by $-\omega t_0$.

Example: What are the CTFT's of the signals shown below?





Shifted rect signals

3. *Time scaling – Section 4.3.5*

If $x(t) \leftrightarrow X(j\omega)$, then

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

where a is a non-zero real constant.

The time scaling property states that if a signal is *compressed in time* by a factor a, then its spectrum is *expanded in frequency* by the same factor a and vice-versa.

This property is an example of the *inverse relationship* between the time and frequency domains.

Example: Since

$$\Pi(t) \leftrightarrow \mathrm{sinc}\left(\frac{\omega}{2\pi}\right),$$

then

$$\Pi(2t) \leftrightarrow \frac{1}{2}\,\mathrm{sinc}\left(\frac{\omega}{4\pi}\right).$$

Note also that if $a = -1$ (corresponding to a time reversal) in the time scaling property, we have

$$x(-t) \leftrightarrow X(-j\omega)$$

i.e. the spectrum is also reversed.

Combining the time shift and scaling properties

What is the Fourier transform of $x(at-b)$?

4. *Conjugation – Section 4.3.5*

If $x(t) \leftrightarrow X(j\omega)$, then

$$x^*(t) \leftrightarrow X^*(-j\omega) \; .$$

As a result, if $x(t)$ is real, we have

$$X(-j\omega) = X^*(j\omega)$$

i.e. the spectrum *magnitude* is an *even* function of ω and the spectrum *phase* is an *odd* function of ω.

More generally, we can summarize the relationship between a signal and its spectrum as follows:

5. *Convolution – Section 4.4*

$$y(t) = x(t) * h(t) \leftrightarrow Y(j\omega) = X(j\omega)\, H(j\omega)\ .$$

Proof:

Application: Since $\wedge(t) = \Pi(t) * \Pi(t)$,

$$\begin{aligned} \mathcal{F}\{\wedge(t)\} &= \mathcal{F}\{\Pi(t)\}\mathcal{F}\{\Pi(t)\} \\ &= \operatorname{sinc}^2\left(\frac{\omega}{2\pi}\right) \end{aligned}$$

6. *Differentiation & Integration – Section 4.3.4*

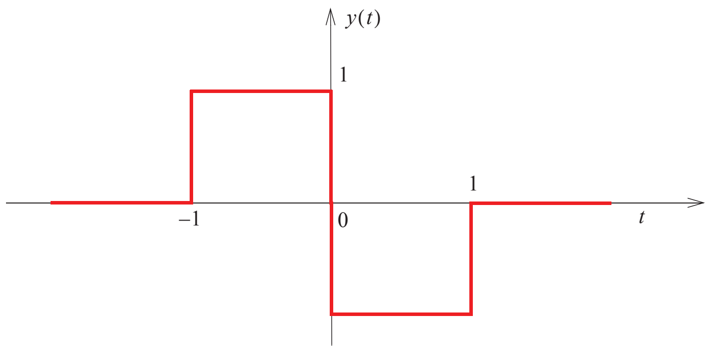
If $x(t) \leftrightarrow X(j\omega)$, then

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)\ .$$

Proof:

This result indicates that *differentiation accentuates the high frequency components* in the signal.

Application: Consider $y(t)$ as shown below



Since $y(t) = \frac{d}{dt} \wedge (t)$, we have

$$\begin{aligned} \mathcal{F}\{y(t)\} &= j\omega\ \mathcal{F}\{\wedge(t)\} \\ &= j\omega\ \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right) \end{aligned}$$

If $x(t) \leftrightarrow X(j\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X\left(j\omega\right) + \pi X(0)\delta(\omega)\ .$$

We see that in contrast to differentiation, *integration attenuates the high frequency components* in the signal.

7. *Area property*
   If $x(t) \leftrightarrow X(j\omega)$, then

$$\int_{-\infty}^{\infty} x(t)dt = X\left(0\right)\ .$$

This result follows directly from the definition of the Fourier transform, i.e.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt.$$

Also,

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi\ x\,(0)$$

since

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega.$$

Example: Evaluate $\int_{-\infty}^{\infty} \mathrm{sinc}(x)dx$.

This can be obtained as follows:

8. *Duality*

Due to the similarity between the FT analysis and synthesis equations, we have a "duality" relationship.

If $x(t) \leftrightarrow X(j\omega)$, then

$$X\left(jt\right) \leftrightarrow 2\pi x(-\omega) \ .$$

Proof:

Example: Determine the inverse FT of $\Pi(\omega)$.

Since

$$\Pi(t) \leftrightarrow \mathrm{sinc}\left(\frac{\omega}{2\pi}\right),$$

then

$$\mathrm{sinc}\left(\frac{t}{2\pi}\right) \leftrightarrow 2\pi\Pi(-\omega) = 2\pi\Pi(\omega)$$

or equivalently

$$\frac{1}{2\pi}\,\mathrm{sinc}\left(\frac{t}{2\pi}\right) \leftrightarrow \Pi(\omega).$$

Duality can also be useful in suggesting new properties of the FT.

Example:

9. *Parseval's relation*

   If $x(t) \leftrightarrow X(j\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \ .$$

Remarks:

(a) The LHS is the total energy in the signal $x(t)$.

(b) $|X(j\omega)|^2$ describes how the energy in $x(t)$ is distributed as a function of frequency. It is commonly called the *energy density spectrum* of $x(t)$.

Proof:

Example: Evaluate $\int_{-\infty}^{\infty} \mathrm{sinc}^2 x \; dx$.

CT unit impulse function – pp. 32–38, pp. 92-93

The CT unit impulse function is also known as the *Dirac* delta function, $\delta(t)$. Contrast with the Kronecker delta function, δ_{ij}. The introduction of $\delta(t)$ allows us to look at the FT of periodic signals.

$\delta(t)$ is defined by the sifting property, namely

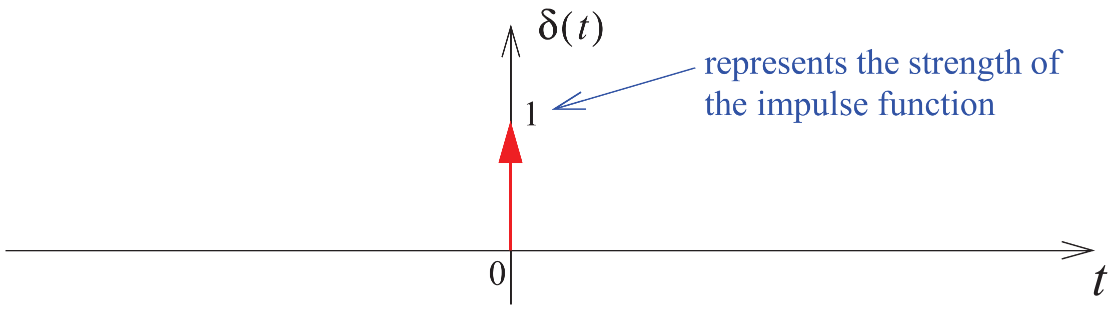
$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

if $f(t)$ is continuous at $t = 0$.

Properties of the Dirac delta function

1.

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



2.

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\; dt &= \int_{-\infty}^{\infty} \delta(\tau) f(\tau + t_0)\; d\tau \\ &= f(t_0)\end{aligned}$$

The first line is obtained using $\tau = t - t_0$.

3.

$$\begin{aligned}f(t) * \delta(t) &= \int_{-\infty}^{\infty} \delta(\tau) f(t-\tau)\; d\tau \\ &= f(t)\end{aligned}$$

The above is referred to as the *replication* property of $\delta(t)$.

4.

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

Proof:

5. What is the Fourier transform of $\delta(t)$?

$\mathcal{F}\{\delta(t)\} =$

6. What is the inverse FT of $\delta(\omega)$?

$$\begin{aligned} \mathcal{F}^{-1}\{\delta(\omega)\} &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi} \end{aligned}$$

Thus, $1 \leftrightarrow 2\pi\delta(\omega)$.

7. It follows from the last result that

$$\int_{-\infty}^{\infty} e^{-j\omega t}\,dt = 2\pi\delta(\omega)$$

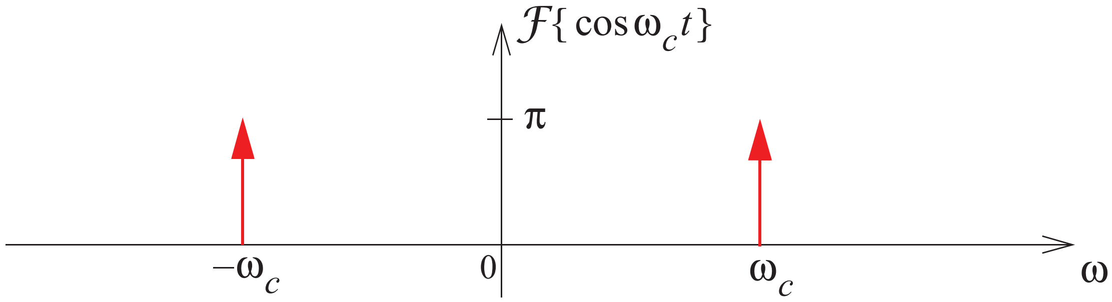
Therefore,

$$\begin{aligned} 2\pi\delta(\omega-\omega_c) &= \int_{-\infty}^{\infty} e^{-j(\omega-\omega_c)t}\,dt \\ &= \int_{-\infty}^{\infty} e^{j\omega_c t}e^{-j\omega t}\,dt \\ &= \mathcal{F}\left\{e^{j\omega_c t}\right\} \end{aligned}$$

We thus have $e^{j\omega_c t} \leftrightarrow 2\pi\delta(\omega - \omega_c)$. This result is useful in examining the FT of a periodic signal.

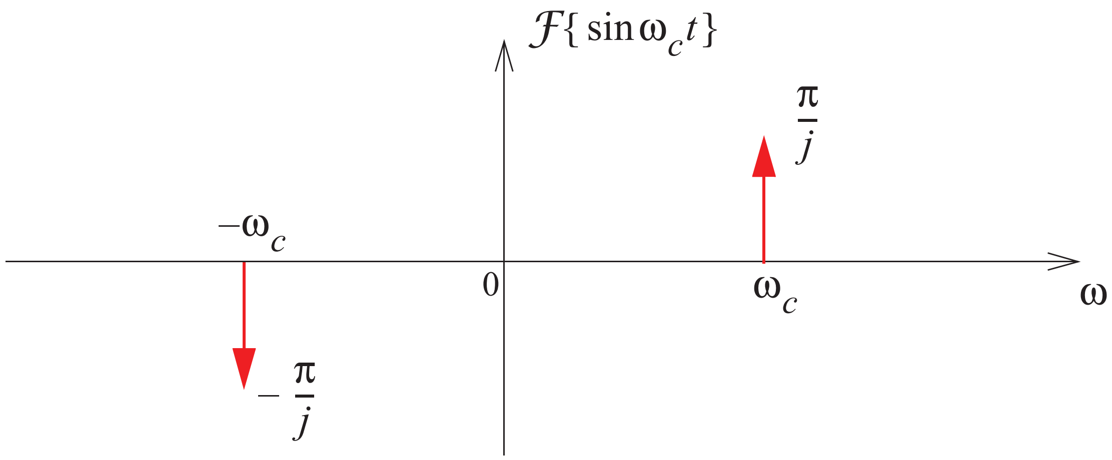
8. Using the last result, we can write

$$\begin{aligned} \cos \omega_c t &= \frac{1}{2}\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] \\ &\leftrightarrow \frac{1}{2}\left[2\pi\delta(\omega - \omega_c) + 2\pi\delta(\omega + \omega_c)\right] \\ &= \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c) \end{aligned}$$



Similarly, we have

$$\begin{aligned}\sin\omega_c t &= \frac{1}{2j}\left[e^{j\omega_c t} - e^{-j\omega_c t}\right]\\ &\leftrightarrow \frac{\pi}{j}\left[\delta(\omega-\omega_c) - \delta(\omega+\omega_c)\right]\end{aligned}$$



Fourier transform of periodic signals – Section 4.2

Recall from our discussion of FS representation: A periodic signal, $\tilde{x}(t)$, with fundamental period T can be represented by

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k \; e^{jk\left(\frac{2\pi}{T}\right)t}$$

where the (possibly complex) Fourier coefficients $\{a_k\}$ are given by

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) \; e^{-jk\left(\frac{2\pi}{T}\right)t} \; dt,$$

$$k = 0, \pm 1, \pm 2, \ldots$$

$$\text{Let} \quad x(t) \quad = \quad \begin{cases} \tilde{x}(t), & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $x(t)$ is simply one basic period of $\tilde{x}(t)$.

Then, with $\omega_0 \triangleq \frac{2\pi}{T}$, we can write

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\ e^{-jk\omega_0 t}\ dt \\ &= \frac{1}{T}X\left(jk\omega_0\right). \end{aligned}$$

The last line follows since

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\ dt.$$

In other words, a_k is equal to $\frac{1}{T}$ multiplied by the FT of $x(t)$ evaluated at $\omega = k\omega_0$.

Therefore,

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{T}X\left(jk\omega_0\right)\ e^{jk\omega_0 t} \\ &\leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(jk\omega_0\right)\ 2\pi\delta\left(\omega - k\omega_0\right) \end{aligned}$$

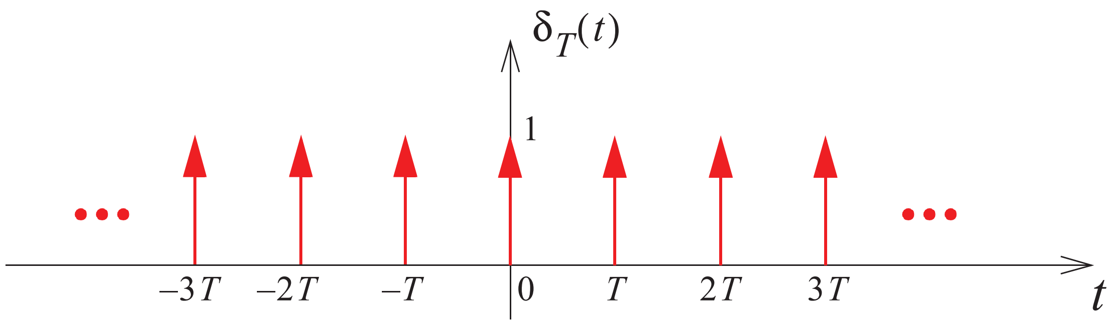
In summary, the FT of a periodic signal consists of a series of impulses located at frequencies which are multiples of the fundamental frequency ω_0. The strength of the impulse at the kth harmonic frequency $k\omega_0$ is $2\pi a_k$.

Example: What is the FT of the impulse train or comb function?

Recall that the comb function is given by

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

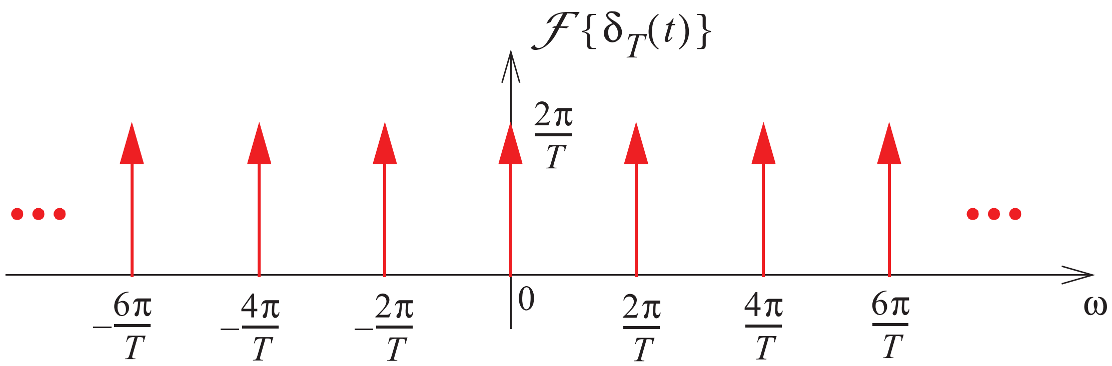
and is shown in the figure below.



In this example, "$x(t)$" $= \delta(t)$ so that $X(j\omega) = 1$.

Therefore,

$$\delta_T(t) \leftrightarrow \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right).$$



We will see that the comb function is very useful in discussing *sampling*.