Weak Duality

$$f_0(\tilde{x}) \geq g(\lambda, \nu) \qquad \lambda \geq 0$$

$\tilde{x}$ feasible

concave

Dual problem : $D = \max_{\substack{\lambda \geq 0 \\ (\lambda,\nu) \in \mathrm{dom}\, g}} g(\underline{\lambda}, \underline{\nu}) = -\min -g(\underline{\lambda}, \underline{\nu})$

Primal Problem : P : $\min f_0(x)$

$$f_i(x) \leq 0$$
$$h_j(x) = 0$$
$$x \in \mathcal{D}$$

Now :

$$\underbrace{\begin{matrix} f_0(\tilde{x}) \\ f_i(\tilde{x}) \leq 0 \\ h_j(\tilde{x}) = 0 \\ \tilde{x} \in \mathcal{D} \end{matrix}}_{\text{depends on } \tilde{x}} \quad \geq \quad \underbrace{\begin{matrix} g(\underline{\lambda}, \underline{\nu}) \\ \lambda \geq 0 \\ (\lambda, \nu) \in \mathrm{dom}\, g \end{matrix}}_{\text{depends on } \lambda, \nu}$$

$$\Rightarrow \quad P = \min_{\substack{f_i(x) \leq 0 \\ h_j(x) = 0 \\ x \in \mathcal{D}}} f_0(x) \quad \geq \quad \max_{\substack{\lambda \geq 0 \\ (\lambda,\nu) \in \mathrm{dom}\, g}} g(\lambda, \nu) = D$$

$$\Rightarrow \boxed{P \geq D}$$

(Weak Duality)
by definition

Note : suppose primal is unbounded below

(1) $P = -\infty \geq D \Rightarrow D = -\infty$

$$= - \min_{\lambda \geq 0} -g(\underline{\lambda}, \underline{\nu})$$

$\Rightarrow$ dual problem is infeasible

(2) $D = \infty$ (dual unbounded above)

$\Rightarrow P = \infty$ (primal infeasible)

Duality gap : $P - D \geq 0$

Eg $\min \frac{1}{2} x^T x$ least norm solution

$Ax = b \quad \cdots \quad \nu \in \mathbb{R}^p$ $\quad$ rank$(A) = p < n$

$x \in \mathbb{R}^n$

$A \in \mathbb{R}^{p \times n}$

$$L(x, \underline{\nu}) = \frac{1}{2} x^T x + \nu^T (Ax - b)$$

$$g(\nu) = \min_x L(x, \nu) = \min_x \frac{1}{2} x^T x + \nu^T (Ax - b)$$

$$\nabla_x L(x, \nu) = 0$$

or $x + A^T \nu = 0$ or $x = -A^T \nu$

so $$g(\nu) = \frac{1}{2}\nu^T A^T A \nu + \nu^T(-AA^T\nu - b)$$
$$= -\frac{1}{2}\nu^T A A^T \nu - b^T \nu$$

Dual Problem: $D = \max -\frac{1}{2}\nu^T A A^T \nu - b^T \nu \qquad \nu \in \mathbb{R}^p$

$$\nabla_\nu g(\nu) = 0 \implies -AA^T\nu - b = 0$$
$$\text{or } \nu^* = -(AA^T)^{-1} b$$

$$D = g(\nu^*) = \frac{1}{2} b^T (AA^T)^{-1} b$$

Therefore $\frac{1}{2}\tilde{x}^T\tilde{x} \geq \frac{1}{2} b^T (AA^T)^{-1} b$

for any $\tilde{x} : A\tilde{x} = b$

also $P \geq \frac{1}{2} b^T (AA^T)^{-1} b$