4. Examples

Norm ball: $\mathcal{B} = \{ x \in \mathbb{R}^n \mid \|x - x_c\| \le r \}$ $\quad r \ge 0$, $x_c \in \mathbb{R}^n$

any norm

$x \Rightarrow x + ru$

$= \{ x + ru \in \mathbb{R}^n \mid \|u\| \le 1 \}$

Ellipsoid

$$\mathcal{E}(x_c, P) = \{ x \in \mathbb{R}^n \mid (x - x_c)^T P^{-1} (x - x_c) \le 1 \} \qquad P \succ 0$$

$\|$

$\mathcal{B}$ when $P = rI$ (special case of $\mathcal{E}$)

Q: generalize for other norms?

Aside: square root decomposition

$P \in S^n, P \succ 0$

$P = E \Lambda E^T$ where $EE^T = I = E^T E$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \qquad \sqrt{\Lambda} = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}$$

since $\sqrt{\Lambda}\sqrt{\Lambda} = \Lambda$

so $P = E \Lambda E^T = E \sqrt{\Lambda} \sqrt{\Lambda} E^T$

$= E \sqrt{\Lambda} E^T E \sqrt{\Lambda} E^T$

$= \sqrt{P} \sqrt{P}$ matrix square root

or $\sqrt{P}^2 = P$

MATLAB : sqrtm( )

valid for $P \succ 0$

$$\mathcal{E}(x_c, P) = \{ x_c + \sqrt{P} u \mid \|u\| \le 1 \}$$

ℓ_2 case: $\quad x = x_c + \sqrt{P} u$

$$\sqrt{P}^{-1}(x - x_c) = u$$

$$\|u\|_2^2 \leq 1 \iff (x-x_c)^T P^{-1} (x - x_c)$$

since $\sqrt{P}^{-1}\sqrt{P}^{-1} = P^{-1}$ (prove)

Is $B(x_c, r)$ convex?

$$B(0,1) = \{x \mid \|x\| \leq 1\}$$

$$x, y \in B(0,1) \Rightarrow \|x\| \leq 1, \ \|y\| \leq 1$$

$$z = \theta x + (1-\theta) y$$

triangle inequality

$$\|z\| = \|\theta x + (1-\theta) y\| \leq \|\theta x\| + \|(1-\theta) y\|$$

$$= |\theta| \, \|x\| + |1-\theta| \, \|y\|$$

homogeneity

(since $\theta \in [0,1]$) $= \theta \|x\| + (1-\theta) \|y\|$

$\leq \theta + (1-\theta) = 1 \qquad x, y \in B(0,1)$

Prove for $\mathcal{E}(x_c, P)$!

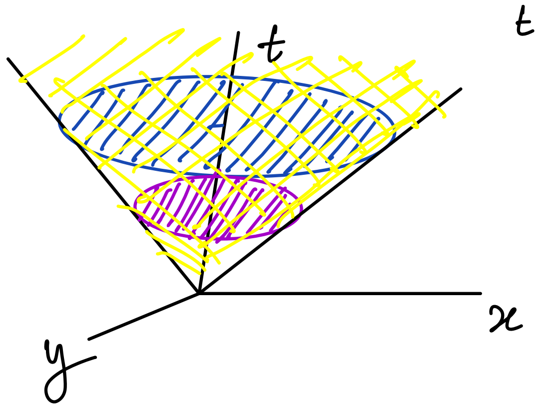
Norm Cone $\quad C = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \|x\|_2 \leq t \right\}$

$\underline{x} \in \mathbb{R}^n$

$t \in \mathbb{R}$

implies $t \geq 0$

$n = 2$



ice-cream cone

Lorentz-cone