EE901
Probability and Random Processes

Module -3
Distribution of Random Variables

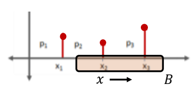
Abhishek Gupta
Electrical Engineering
IIT Kanpur

1

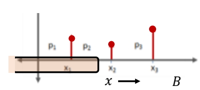
Distribution of a Discrete RV

- The distribution can be specified by PMF $p_X(x) = \mathbb{P}(\{X = x\})$ which shows the probability mass concentrated on each point x_i
- For any set B, the corresponding probability would be summation of probability mass of all those x_i's that are in the set B

$$\mathbb{P}(\{X \in B\}) = \mathbb{P}_X(B) = \sum_{x_i \in B} p_X(x_i)$$



$$\mathrm{F}_X(x) = \mathbb{P}(\{X \le x\}) = \sum_{x_i \le x} p_X(x_i)$$

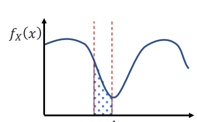


2

Distribution of Continuous RV

- The distribution can be specified by PDF $f_X(x)$
- The probability of X taking a value in the set B is given as

$$\mathbb{P}_X(B) = \int_B f_X(x)\mathrm{d}x$$



$$F_X(x) = \mathbb{P}_X\big((-\infty, x]\big) = \int_{-\infty}^{x} f_X(x)\mathrm{d}x$$

3

Examples of Random Variable Distributions

4

Examples of Discrete RV: Bernoulli

- Bernoulli random variable: $X \sim \mathrm{Bern}(p)$

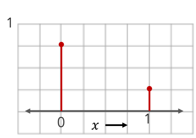
PMF	CDF
$p_X(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$	$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geqslant 1 \end{cases}$

- X is Bernoulli distributed
- X has Bernoulli distribution with parameter p.
- Applications: to represent the outcome of random experiments with success/failure outcomes.
  - Coin toss, Error in a channel, Random failure of a machine.

5

Bernoulli Random Variable

- Pick a direction with equal probability. $\Omega = \{N, W, E, S\}$
  - X is 1 if picked direction is N, otherwise it is 0
  - $X(\omega) = 1(\omega = N)$
  - X takes two values 0 and 1

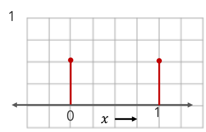


$$p_X(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

6

Bernoulli Random Variable

- Flip a coin. $\Omega = \{H, T\}$
  - X is 1 if head occurs, otherwise it is 0
  - $X(\omega) = 1(\omega = H)$
  - X takes two values 0 and 1



$$p_X(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

7

Examples of Discrete RV: Uniform

- Discrete uniform random variable $X \sim \text{Unif}(1{:}\,N)$

CDF	PMF
$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \lfloor x \rfloor / N & \text{if } 1 \le x < N \\ 1 & \text{if } x \ge N \end{cases}$	$p_X(x) = \begin{cases} 1/N & \text{if } x = 1, 2, \cdots, N \\ 0 & \text{otherwise} \end{cases}$

- X has uniform distribution with parameter N.
- Applications: to model output in random experiments with N equally likely outcomes
  - Dice roll, selection of an object.

8

Uniform Random Variable

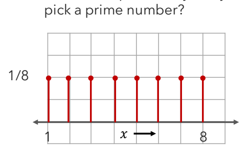
- Pick a natural number between 1 and 8. $\Omega = \{1,2,3,4,5,6,7,8\}$
  - X is equal to the picked number
  - $X(\omega) = \omega$
  - X takes value between 1 and 8
  - What is the probability that you pick a prime number?

$B = \{1, 2, 3, 5, 7\}$

$P[\{X \in B\}]$

$= \sum_{x_i \in B} P_X(x_i)$

$= \sum_{x_i =} 1/8 = \frac{1}{8} \times 5 = \frac{5}{8}$



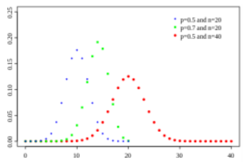
9

Examples of Discrete RV: Binomial

- Binomial random variable $X \sim \text{Binom}(p, N)$

PMF

$$p_X(i) = {}^nC_i\, p^i (1-p)^{n-i},\ 0 \le i \le N$$



- Applications: to model number of successes in N independent trials of a random experiment
  - Failure of N identical and independent machines,

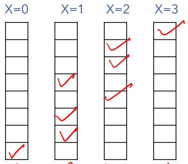
10

Binomial Random Variable

- Flip three fair "independent" coins. $\Omega = \{H, T\}$
- X is the number of heads. X takes four values: 0 1 2 3. Compute its PMF.

Outcomes X=0 X=1 X=2 X=3

HHH
THH
HTH
TTH
HHT
THT
HTT
TTT



$p_X(0) = \frac{1}{8}$ $\frac{3}{8}$ $\frac{3}{8}$ $\frac{1}{8}$

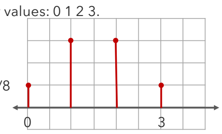
11

Binomial Random Variable

- Flip three fair "independent" coins. $\Omega = \{H, T\}$
  - X is the number of heads. X takes four values: 0 1 2 3.

Outcomes

HHH
THH
HTH
TTH
HHT
THT
HTT
TTT



$n = 3$
$p = 0.5$

${}^nC_k\, p^k (1-p)^{n-k}$

$$p_X(i) = {}^nC_i\, p^i (1-p)^{n-i},\ 0 \le i \le N$$

12

Binomial Random Variable

- Flip three fair "independent" coins. $\Omega = \{H, T\}$
  - X is number of heads.
  - X_i is the indicator that ith coin shows head.

Outcomes.	X_1	X_2	X_3	X
HHH	1	1	1	3
THH	0	1	1	2
HTH	1	0	1	2
TTH	0	0	1	1
HHT	1	1	0	2
THT	0	1	0	1
HTT	1	0	0	1
TTT	0	0	0	0

$X = X_1 + X_2 + X_3$

Each X_i is a Bernoulli random variable.

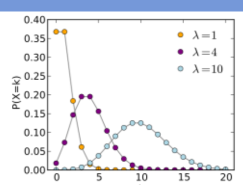
13

Examples of Discrete RV: Poisson

- Poisson random variable $X \sim \text{Poiss}(\lambda)$

PMF

$$p_X(i) = \frac{e^{-\lambda}\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots \infty$$



- Applications: to model arrivals in natural processes
  - Number of photons emitted from a light bulb.

14

Poisson Random Variable

- In a shop, the number of people arriving everyday is distributed as Poisson distribution with parameter 4. The shop keeps products only for 4 people. What is the probability that the shop runs out of supply on a particular day?
- Random experiment is about picking total number of people arriving on a particular day. $\Omega \neq \{1,2,\ldots\}$.
- $X(\omega) = \omega$. We need to calculate $\mathbb{P}(\{X > 4\})$.

$$p_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$$
$$\lambda = 4$$

15

Poisson Random Variable

- $\mathbb{P}(\{X > 4\})$.

$$p_X(i) = \frac{e^{-\lambda}\lambda^i}{i!} \qquad \lambda = 4$$

$X=5,\ X=6 \ \cdots \ X=100 \ \cdots \ \infty$

$\{X>4\}$ $= \sum_{x_i=5}^{\infty} p_X(x) = \sum_{i=5}^{\infty} \frac{e^{-4}4^i}{i!}$

$\{X\le 4\} = \{X=0\} \cup \{X=1\} \cup \{X=2\} \cup \{X=3\} \cup \{X=4\}$

$P[\{X\le 4\}] = \sum_{x=0}^{4} p_X(x) = \sum_{i=0}^{4} \frac{e^{-4}4^i}{i!}$

$\mathbb{P}[\{X>4\}] = 1 - \mathbb{P}[\{X\le 4\}] = 1 - \sum_{i=0}^{4} \frac{e^{-4}4^i}{i!}$

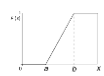
16

Examples of Continuous RV: Uniform

- Uniform random variable $X \sim$ Uniform(a,b)

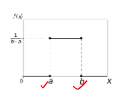
CDF

$$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$$



PDF

$$f_X(x) = \frac{1}{b-a}\mathbb{1}(a \le x \le b)$$



- Applications: to model output in experiments with equally likely and uncountable number of outcomes

17

Uniform Random Variable

- Let $X \sim$ Uniform(-1,1)
- What is the probability that $|X| > 0.3$?
- $|X| > 0.3$ means that $X > 0.3$ or $X < -0.3$.

$$\Pr(|X| > 0.3) = \mathbb{P}(\{X \in B\}) \quad \text{where } B = (-1,-0.3) \cup (0.3,1)$$

$$= \int_B f_X(x)\,dx = \int_{-1}^{-0.3} \frac{1}{2}dx + \int_{0.3}^{1} \frac{1}{2}dx = 0.7$$

$$\boxed{f_X(x) = \frac{1}{2}\mathbb{1}(-1 < x < 1)}$$

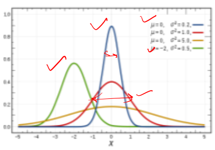
18

Examples of Continuous RV: Gaussian

- Gaussian random variable $X\ \mathcal{N}(\mu, \sigma^2)$

PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$



- Applications:
  - To model natural occurring signals or physical quantities that are expected to be the sum of many independent processes. For example, noise.
  - To approximate other distributions.

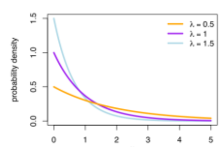
19

Examples of Continuous RV: Exponential

- Exponential random variable $X \sim \text{Exp}(\lambda)$

PDF

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{1}\,(x \geq 0)$$



- Applications:
  - To model waiting period of an arrival in an arrival process or the time for a continuous process to change state
  - Channel fading coefficient for Rayleigh fading.

20

Exponential Random Variable

- Let m be input of a communication channel. m is either 1 or -1.
- The channel adds a noise $N \sim \text{Exp}(1)$. The output at the receiver is Y such that

$$Y = m + N$$

Compute the probability that $Y > 1$ when $m = -1$.

$$\Pr(Y > 1) = \mathbb{P}(\{Y > 1\}) = \mathbb{P}(\{-1 + N > 1\})$$
$$= \mathbb{P}(\{N > 2\}) = \int_2^\infty e^{-x} \mathrm{d}x = e^{-2}$$

21

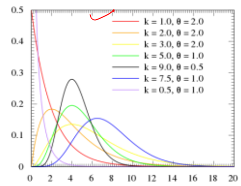
Examples of Continuous RV: Gamma

- Gamma random variable $X \sim \Gamma(\lambda, k)$:

PDF

$$f_X(x) = \frac{\lambda^k}{\Gamma(k)} e^{-\lambda x} x^{k-1} \mathbb{1}\,(x \geq 0)$$

- where λ = rate , k= shape



- Applications:
  - To model waiting period for multiple arrivals
  - Channel fading coefficient for Nakagami fading.

22

CDF without Explicit Probability Space

- At many places, we see X defined as a RV with CDF $F_X(x)$ without explicitly talking about the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Why it is so? Can we always define a probability space for given the CDF?

Let us suppose, there exists a function $F(x)$ that satisfies all the properties of a CDF, then does there exist a probability space in which we can define a random variable X, which has the same CDF as $F(x)$?

- Now, let us understand how we can construct this probability space.

23

CDF without Explicit Probability Space

Let, $\Omega = \mathbb{R}, \mathcal{F} = B(\mathbb{R}), \mathbb{P}[(-\infty, x]] \triangleq F(x)$.

Define a random variable, $X(\omega) = \omega$.

Claim: X's CDF is $F(x)$

Proof: Let $B = (-\infty, x]$. Then,

$$\begin{aligned} F_X(x) &= \mathbb{P}(\{\omega : X(\omega) \in B\}) \\ &= P(\{\omega : \omega \in B\}) \\ &= \mathbb{P}\,((-\infty, x]) = F(x) \end{aligned}$$

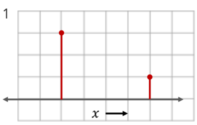
This implies, if a function $F(x)$ satisfies all the properties of CDF, we can always construct a probability space and define a random variable which has CDF $F(x)$.

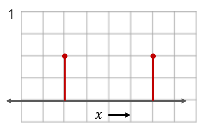
This will not be unique as there can be many random variables with the same CDF.

24

Multiple Random Variables on the same Probability Space

- Pick a direction with equal probability. $\Omega = \{N, W, E, S\}$
  - X is 1 if picked direction is N, otherwise it is 0
  - $X(\omega) = 1(\omega = N)$
  - X takes two values 0 and 1
  - Y is 1 if picked direction is N or W, otherwise it is 0
  - $Y(\omega) = 1(\omega = N \text{ or } \omega = W)$
  - Y takes two values 0 and 1



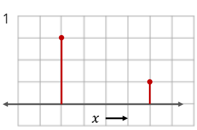


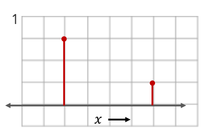
25

Multiple Random Variables on the same Probability Space

- Pick a direction with equal probability. $\Omega = \{N, W, E, S\}$
  - X is 1 if picked direction is N, otherwise it is 0
  - $X(\omega) = 1(\omega = N)$
  - X takes two values 0 and 1
  - Z is 1 if picked direction is W, otherwise it is 0
  - $Z(\omega) = 1(\omega = W)$
  - Z takes two values 0 and 1

Are these two equal?

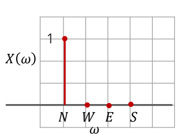


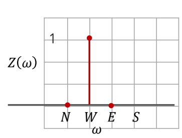


26

Equality of Random Variables

- Since random variables are functions, two random variables X and Y are equal if they are equal for every outcome in the sample space
  - X is 1 if picked direction is N, otherwise it is 0
  - $X(\omega) = 1(\omega = N)$
  - X takes two values 0 and 1
  - Z is 1 if picked direction is W, otherwise it is 0
  - $Z(\omega) = 1(\omega = W)$
  - Z takes two values 0 and 1





27