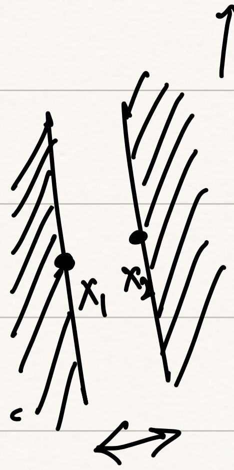
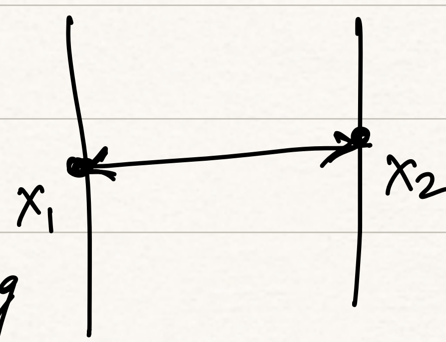
A3 1. $H_1 = \{x \mid a^T x \le b_1\}$ and $H_2 = \{x \mid a^T x \ge b_2\}$



non-overlapping



$$x_1 \in H_1 \Rightarrow a^T x_1 \le b_1$$

$$x_2 \in H_2 \Rightarrow a^T x_2 \ge b_2$$

$$-a^T x_2 \le -b_2$$

$$+ \quad a^T x_1 \le b_1$$

$$(\ ) \le a^T(x_1 - x_2) \le b_1 - b_2$$

Apply Cauchy-Schwarz inequality

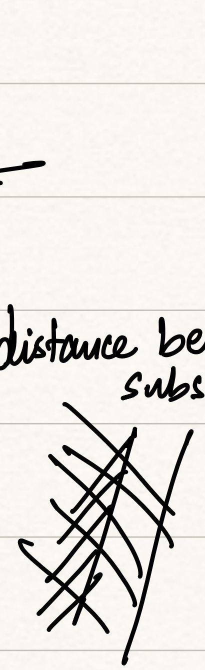
$$b_1 - b_2 \ge a^T(x_1 - x_2) \ge -\|a\| \|x_1 - x_2\|$$

$$\Rightarrow \|a\| \|x_1 - x_2\| \ge b_2 - b_1$$

$$\|x_1 - x_2\| \ge \frac{b_2 - b_1}{\|a\|_2}$$

distance between subspaces

only meaningful when $b_2 > b_1$



Alternative

$$-a^T x_1 \geq -b_1$$
$$a^T x_2 \geq b_2$$

$$(\;) \geq a^T(x_2 - x_1) \geq b_2 - b_1 \qquad a^T(x_2 - x_1) \leq \|a\|\,\|x_2 - x_1\|$$

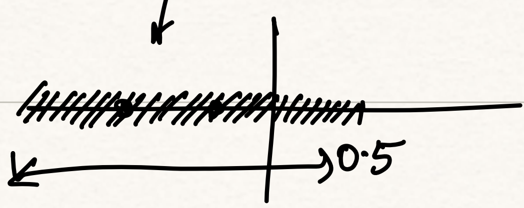
$$\Rightarrow \quad \|x_2 - x_1\| \geq \frac{b_2 - b_1}{\|a\|}$$

2. $\{x \mid \|x - x_1\|_1 \leq \|x - x_2\|_1\}$ affine?

$$\{x \in \mathbb{R} \mid |x - x_1| \leq |x - x_2|\}$$

$x_1 = 0, \; x_2 = 1$

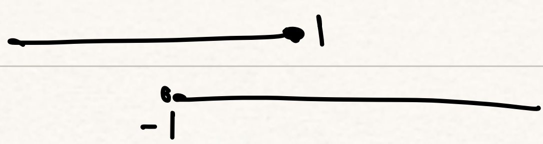
$$\{x \in \mathbb{R} \mid |x| \leq |x - 1|\}$$

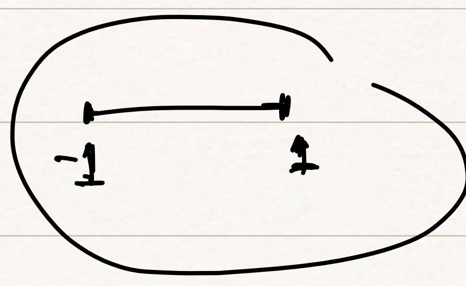


not affine

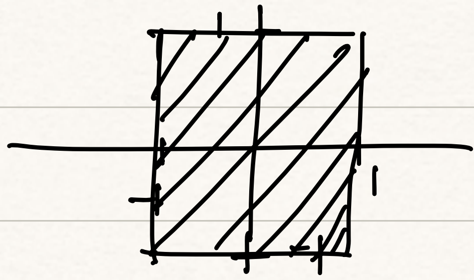
3 $\{x \in \mathbb{R} \mid \|x\|_\infty \leq 1\}$

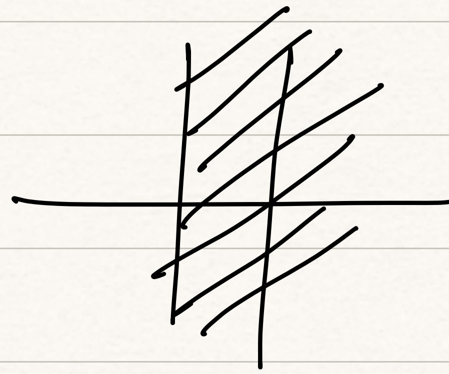
$$|x| \leq 1$$





$\|x\|_\infty \leq 1 \iff \max_{1 \leq i \leq 2} |x_i| \leq 1$



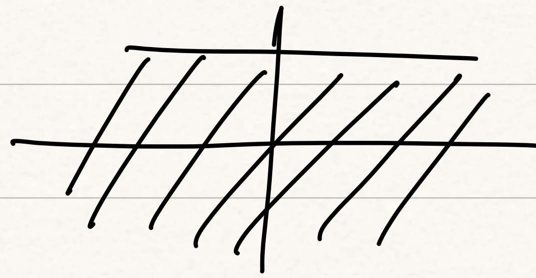


$\{x_1 \leq 1\}$

$\cap \; \{x_1 \geq -1\}$

$\cap \; \{x_2 \leq 1\}$

$\cap \; \{x_2 \geq -1\}$



$\|x\|_\infty = \max_i |x_i| \leq 1$

$\underline{x} \in \mathbb{R}^n$

$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underline{x} \iff |x_i| \leq 1 \quad \forall \; i = 1, 2 \dots n$

$x_i \leq 1 \;\; \& \;\; x_i \geq -1 \quad \forall \; i = 1, 2 \dots n$

$$\underbrace{\bigcap_{i=1}^{n} \{x_i \mid x_i \leq 1\}}_{} \cap \underbrace{\bigcap_{i=1}^{n} \{x_i \mid x_i \geq -1\}}_{}$$

2n sets

4. $S = \{ x \in \mathbb{R}^n \mid \|x-a\|_2 \le \theta \|x-b\|_2 \}$

$a \ne b$

$\theta = 1$

$$\|x-a\|_2 \le \|x-b\|$$

$$\Leftrightarrow \quad \|x-a\|_2^2 \le \|x-b\|_2^2$$

$$\cancel{x^Tx} - 2a^Tx + a^Ta \le \cancel{x^Tx} - 2b^Tx + b^Tb$$

$$\Leftrightarrow \quad 2(b-a)^Tx \le b^Tb - a^Ta$$

half-space

$\theta < 1$

$$\{ x \mid \|x-a\|_2^2 - \theta^2 \|x-b\|_2^2 \le 0 \}$$

$$x^Tx - 2a^Tx + a^Ta - \theta^2(x^Tx - 2b^Tx + b^Tb)$$

$$= \boxed{(1-\theta^2)\, x^Tx - 2(a-\theta^2 b)^T x + a^Ta - \theta^2 b^Tb}$$

$Q = 2(1-\theta^2)I$

$$(1-\theta^2)\|x - c\|_2^2 \le d$$

$$(1-\theta^2)(x^Tx - 2c^Tx + c^Tc) \le d$$

$$2(a-\theta^2 b) = 2(1-\theta^2)c \Rightarrow c = \frac{a-\theta^2 b}{1-\theta^2}$$

$$d - (1-\theta^2)c^Tc = \theta^2 b^Tb - a^Ta$$

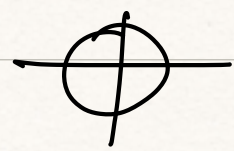
$$d = \theta^2 b^Tb - a^Ta + \cancel{(1-\theta^2)} \frac{(a-\theta^2 b)^T(a-\theta^2 b)}{(1-\theta^2)^{\cancel{2}}}$$

$$= \frac{\theta^2}{1-\theta^2}\|a-b\|^2$$

$$\Rightarrow \quad (1-\theta^2)\left\| x - \frac{a-\theta^2 b}{1-\theta^2} \right\|_2^2 \le \frac{\theta^2 \|a-b\|^2}{1-\theta^2}$$

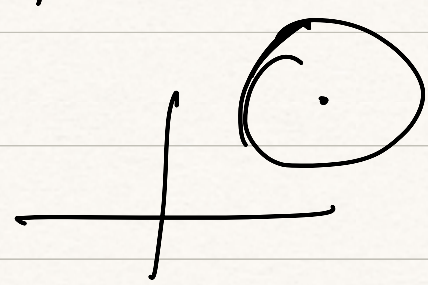
scaled & shifted ℓ_2 norm ball

$\|x\| \le 1$



$\|x-c\|_2 \le 1$

(h) $\|x-c\|_2 \le d$



In general: $\underbrace{\{ \frac{1}{2}x^TQx + q^Tx + r \le 0 \}}_{\text{convex when } Q \succeq 0 \text{ p.s.d}}$

when $\underline{Q = 2(1-\theta^2)I \succeq 0}$

$\theta > 1$ here $Q = 2(1-\theta^2)I \preceq 0$

not p.s.d hence not convex

counterexample

$n=1 \quad a=0 \quad b=1 \quad \theta=2$

$S = \{x \in \mathbb{R} \mid |x| \leq 2|x-1|\}$

e.g. $0, 3$ ✓ ✓ $\qquad 0 \leq 2$ ✓

$3 \leq 4$ ✓

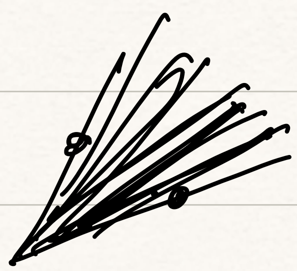
but $\quad 1 \leq 0$ ✗

$0, 3 \in S$ but $1 \notin S$

5. intersection of convex cones

C convex cone $\qquad \alpha \underline{x} + \beta y \in C \quad \forall \alpha, \beta \geq 0$

$x, y \in C$



C_1, C_2 convex cone

$C = C_1 \cap C_2$

$\underline{x, y \in C} = C_1 \cap C_2 \Rightarrow x, y \in C_1 \;.\; x, y \in C_2$

$\Downarrow \qquad\qquad \Downarrow$

$\alpha \underline{x} + \beta \underline{y} \in \overset{C}{\overset{\shortparallel}{C_1 \cap C_2}} \Leftarrow \alpha \underline{x} + \beta \underline{y} \in C_1, \quad \alpha \underline{x} + \beta \underline{y} \in C_2$