EE910: Digital Communication Systems-I

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Lecture #5D: Correlation Receiver and Matched Filter

The Correlation Receiver for AWGN Channels

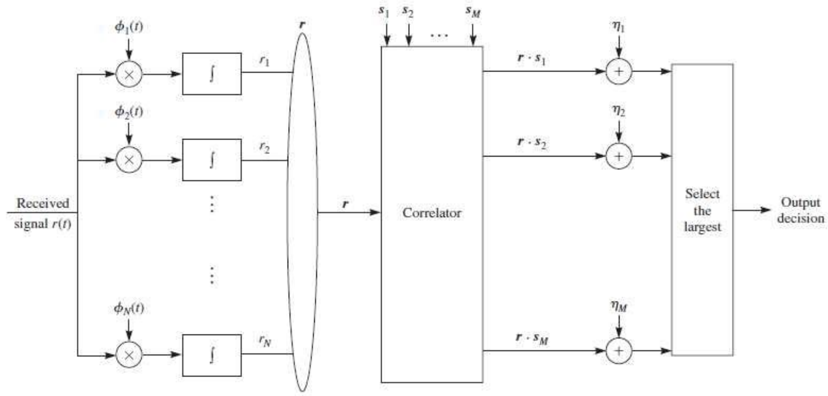

Figure: The structure of a correlation receiver with N correlators

Correlation Receiver

- An optimal receiver for the AWGN channel implements the MAP decision rule given by $\arg\max_{1 \leq m \leq M} [\eta_m + \mathbf{r}.\mathbf{s}_m]$.
- $\mathbf{r}$ is derived at the receiver from the received signal $r(t)$ using the relation $r_j = \int_{-\infty}^{\infty} r(t)\phi_j(t)dt$.

Correlation Receiver

- An alternative implementation of the optimal detector is possible looking at the optimal detection rule

$$\hat{m} = \arg \max_{1 \le m \le M} \left[\eta_m + \int_{-\infty}^{\infty} r(t) s_m(t) dt \right] \quad \text{where} \quad \eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} \mathcal{E}_m \tag{1}$$

- Typically $N < M$, so the previous implementation of correlator receiver is preferred.

The Matched Filter Receiver

- An alternative representation of optimal receiver is called matched filter receiver.
- In correlation receiver implementation, we compute quantities of the form

$$r_x = \int_{-\infty}^{\infty} r(t) x(t) dt$$

where x(t) is either $\phi_j(t)$ or $s_m(t)$.

The Matched Filter Receiver

- A filter is called matched filter if its impulse response h(t) is matched to x(t), i.e. h(t)=x(T-t), where T is chosen such that the filter is causal.
- If the input $r(t)$ is applied to the matched filter, its output, denoted by $y(t)$ is given by,

$$y(t) = r(t) \star h(t) = \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} r(\tau)x(T-t+\tau)d\tau$$

- $r_x = y(T) = \int_{-\infty}^{\infty} r(\tau)x(\tau)d\tau$

The Matched Filter Receiver

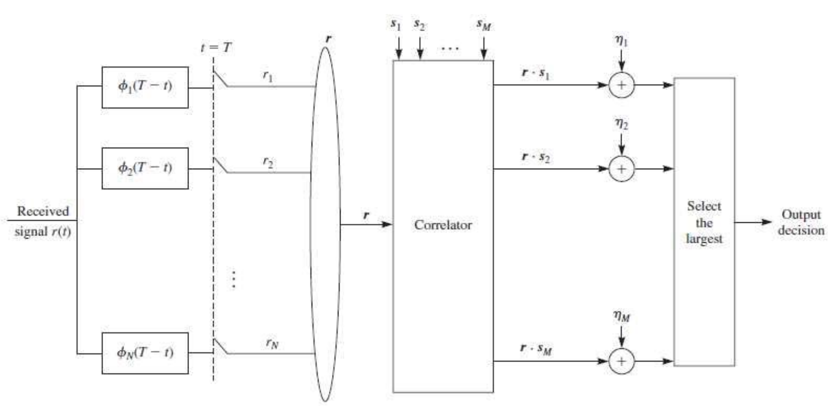

Figure: The structure of a matched filter receiver with N correlators

The Matched Filter Receiver

- The output of the correlator r_x can be obtained by sampling the output of the matched filter at exactly time $t = T$.

Frequency Domain Interpretation of the Matched Filter

- The matched filter to any signal $s(t)$, is $h(t) = s(T - t)$. The Fourier transform of this relationship is $H(f) = S^*(f)e^{-j2\pi fT}$.
- The matched filter has a frequency response that is the complex conjugate of the transmitted signal spectrum multiplied by the phase factor $e^{-j2\pi fT}$, which represents a sampling delay of T.
- The magnitude response $|H(f)| = |S(f)|$ of the matched filter is identical to the transmitted signal spectrum.
- The phase of $H(f)$ is the negative of the phase of S(f) shifted by $2\pi fT$.

Frequency Domain Interpretation of the Matched Filter

- Assume $r(t) = s(t) + n(t)$ is passed through a filter with impulse response $h(t)$ and frequency response $H(f)$.
- The output, denoted by $y(t) = y_s(t) + \nu(t)$, is sampled at some time T.
- The output consists of a signal part, $y_s(t)$, whose Fourier transform is $H(f)S(f)$ and a noise part, $\nu(t)$, whose power spectral density is $\frac{N_0}{2} \mid H(f) \mid^2$

Frequency Domain Interpretation of the Matched Filter

- Sampling at time T, we get

$$y_s(T) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}\,df \tag{2}$$

and

$$Var[\nu(T)] = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}\mathcal{E}_h, \tag{3}$$

where $\mathcal{E}_h$ is the energy in $h(t)$.
- We define the SNR at the output of the filter H(f) as

$$SNR_0 = \frac{y_s^2(T)}{Var[\nu(T)]}$$

Frequency Domain Interpretation of the Matched Filter

- Applying Cauchy-Schwartz inequality, we get

$$\begin{aligned} y_s(T) &= \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}df \quad (4) \\ &\leq \left(\int_{-\infty}^{\infty} |H(f)|^2 df\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |S(f)e^{j2\pi fT}|^2 df\right)^{1/2} \\ &= \sqrt{\mathcal{E}_h}\sqrt{\mathcal{E}_s} \end{aligned}$$

  with equality if and only if $H(f) = \alpha S^*(f)e^{-j2\pi fT}$ for some complex α.
- SNR can be written as

$$SNR_0 \leq \frac{\mathcal{E}_h \mathcal{E}_s}{\frac{N_0}{2}\mathcal{E}_h} = \frac{2\mathcal{E}_s}{N_0} \quad (5)$$

Frequency Domain Interpretation of the Matched Filter

- The filter H(f) that maximizes the signal-to-noise ratio at its output must satisfy the relation $H(f) = S^*(f)e^{-j2\pi fT}$, i.e. it is the matched filter
- Maximum possible signal-to-noise ratio at the output is $\frac{2\mathcal{E}_s}{N_0}$

Matched Filter

- Consider the signal

$$s(t) = \begin{cases} (A/T)t\cos 2\pi f_c t & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

1. Determine the impulse response of the matched filter for the signal.
2. Determine the output of the matched filter at $t = T$.
3. Suppose the signal $s(t)$ is passed through a correlator that correlates the input $s(t)$ with $s(t)$. Determine the value of the correlator output at $t = T$. Compare your result with that in part 2.

Matched Filter

- The impulse response of the matched filter is given by

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos\left(2\pi f_c(T-t)\right) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Matched Filter

- The output of the matched filter at $t = T$ is :

$$
\begin{aligned}
g(T) &= h(t) * s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \qquad (8)\\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau\\
&= \frac{A^2}{T^2}\int_0^T \nu^2\cos^2(2\pi f_c\nu)d\nu\\
&= \frac{A^2}{T^2}\left[\frac{\nu^3}{6}+\left(\frac{\nu^2}{4\times 2\pi f_c}-\frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c\nu)+\frac{\nu\cos(4\pi f_c\nu)}{4(2\pi f_c)^2}\right]\Bigg|_0^T\\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6}+\left(\frac{T^2}{4\times 2\pi f_c}-\frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T)+\frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}
$$

Matched Filter

- The output of the correlator at $t = T$ is :

$$
\begin{aligned}
q(T) &= \int_0^T s^2(\tau)d\tau\\
&= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)\,d\tau
\end{aligned} \qquad (9)
$$

- However, this is the same expression with the case of the output of the matched filter sampled at $t = T$.
- Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.