EE910: Digital Communication Systems-I

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Lecture #3C: Orthogonal, Bi-orthogonal and Simplex Signaling

Orthogonal Signaling

- Orthogonal signals are defined as a set of equal energy signals $s_m(t), 1 \leq m \leq M$, such that

$$\langle s_m(t), s_n(t) \rangle = 0, \quad m \neq n \text{ and } 1 \leq m, n \leq M \tag{1}$$

- Thus we have

$$\langle s_m(t), s_n(t) \rangle = \begin{cases} \mathcal{E} & m = n \\ 0 & m \neq n \end{cases} \qquad 1 \leq m, n \leq M \tag{2}$$

- The signals are linearly independent and hence N = M.
- The orthonormal set $\{\phi_j(t), 1 \leq i \leq N\}$ given by

$$\phi_j(t) = \frac{s_j(t)}{\sqrt{\mathcal{E}}}, \quad 1 \leq j \leq N \tag{3}$$

can be used as an orthonormal basis for representation of $\{s_m(t), 1 \leq m \leq M\}$.

Orthogonal Signaling

- The resulting vector representation of the signals will be

$$\begin{aligned} s_1 &= (\sqrt{\mathcal{E}}, 0, 0, \cdots, 0) \\ s_2 &= (0, \sqrt{\mathcal{E}}, 0, \cdots, 0) \\ \vdots &= \vdots \\ s_M &= (0, 0, 0, \cdots, \sqrt{\mathcal{E}}) \end{aligned} \tag{4}$$

- From Equation (5) it is seen that for all $m \neq n$ we have

$$d_{min} = \sqrt{2\mathcal{E}} \tag{5}$$

- Using the relation

$$\mathcal{E}_b = \frac{\mathcal{E}}{\log_2 M} \tag{6}$$

we conclude that

$$d_{min} = \sqrt{2 \log_2 M \mathcal{E}_b} \tag{7}$$

Frequency-Shift Keying (FSK)

- A special case of orthogonal signals.
- Let us consider the construction of orthogonal signal waveforms that differ in frequency and are represented as

$$\begin{aligned} s_m(t) &= Re\left[s_{ml}(t)e^{j2\pi f_c t}\right], \quad 1 \le m \le M, \;\; 0 \le t \le T \\ &= \sqrt{\frac{2\mathcal{E}}{T}} \cos(2\pi f_c t + 2\pi m \triangle f t) \end{aligned} \tag{8}$$

where

$$s_{ml}(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{j2\pi m \triangle f t}, \quad 1 \le m \le M, \;\; 0 \le t \le T \tag{9}$$

- The coefficient $\sqrt{\frac{2\mathcal{E}}{T}}$ is introduced to guarantee that each signal has an energy equal to $\mathcal{E}$.

Frequency-Shift Keying (FSK)

- For this set of signals to be orthogonal, we need to have

$$Re\left[\int_0^T s_{ml}(t)s_{nl}(t)dt\right] = 0 \tag{10}$$

for all $m \neq n$.
- We have

$$\begin{aligned} \langle s_{ml}(t), s_{nl}(t)\rangle &= \frac{2\mathcal{E}}{T}\int_0^T e^{j2\pi(m-n)\triangle f t}dt \\ &= \frac{2\mathcal{E}\sin(\pi T(m-n)\triangle f)}{\pi T(m-n)\triangle f} e^{j\pi T(m-n)\triangle f} \end{aligned} \tag{11}$$

Frequency-Shift Keying (FSK)

- Also

$$
\begin{aligned}
Re\left[\langle s_{ml}(t), s_{nl}(t)\rangle\right] &= \frac{2\mathcal{E}\sin(\pi T(m-n)\triangle f)}{\pi T(m-n)\triangle f}\cos(\pi T(m-n)\triangle f) \\
&= \frac{2\mathcal{E}\sin(2\pi T(m-n)\triangle f)}{2\pi T(m-n)\triangle f} \qquad (12) \\
&= 2\mathcal{E} sinc(2T(m-n)\triangle f)
\end{aligned}
$$

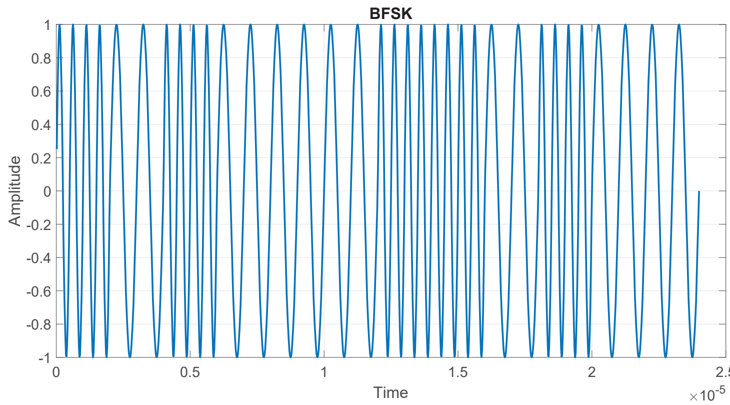
- From Equation (13) we observe that $s_m(t)$ and $s_n(t)$ are orthogonal for all $m \neq n$ if and only if $sinc(2T(m-n)\triangle f) = 0$ for all $m \neq n$.
- This is the case if $\triangle f = k/2T$ for some positive integer k.

Frequency-Shift Keying (FSK)

- The minimum frequency separation $\triangle f$ that guarantees orthogonality is $\triangle f = 1/2T$.
- Note that $\triangle f = \frac{1}{2T}$ is the minimum frequency separation that guarantees $\langle s_{ml}(t), s_{nl}(t)\rangle = 0$.
- This guarantees the orthogonality of the baseband, as well as the bandpass, frequency-modulated signals.

Frequency-Shift Keying (FSK)

Information Sequence: 0 1 0 1 1 1 0 0 1 0 1 1



Hadamard signals

- Hadamard signals are orthogonal signals which are constructed from Hadamard matrices.
- Hadamard matrices H_n are $2^n \times 2^n$ matrices for $n = 1, 2, \cdots$ defined by the following recursive relation

$$\begin{aligned} H_0 &= [1] \\ H_{n+1} &= \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix} \end{aligned} \tag{13}$$

- With this definition we have

$$\begin{aligned} H_1 &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ H_2 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{aligned} \tag{14}$$

Hadamard Signals

- $$H_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \quad (15)$$
- Hadamard matrices are symmetric matrices whose rows (and, by symmetry, columns) are orthogonal.
- Using these matrices, we can generate orthogonal signals.

Hadamard Signals

- For instance, using H_2 would result in the set of signals

$$\begin{aligned} s_1 &= \begin{bmatrix} \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} \end{bmatrix} \quad (16) \\ s_2 &= \begin{bmatrix} \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} \end{bmatrix} \\ s_3 &= \begin{bmatrix} \sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} \end{bmatrix} \\ s_4 &= \begin{bmatrix} \sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & -\sqrt{\mathcal{E}} & \sqrt{\mathcal{E}} \end{bmatrix} \end{aligned}$$

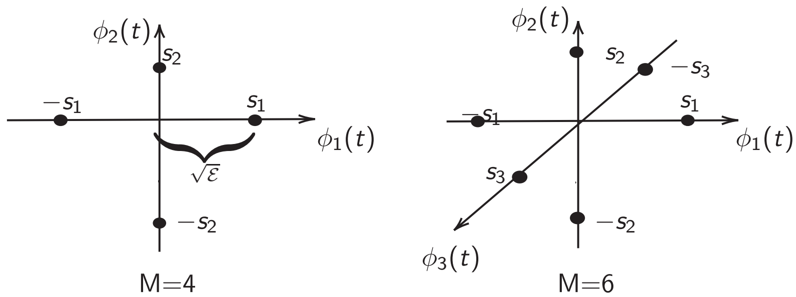
- This set of orthogonal signals may be used to modulate any four-dimensional orthonormal basis $\{\phi_j(t)\}_{j=1}^4$ to generate signals of the form

$$s_m(t) = \sum_{j=1}^{4} s_{mj}\phi_j(t), \qquad 1 \le m \le 4 \quad (17)$$

- Note that the energy in each signal is $4\mathcal{E}$, and each signal carries 2 bits of information, hence $\mathcal{E}_b = 2\mathcal{E}$.

Biorthogonal Signaling

- A set of M biorthogonal signals can be constructed from $\frac{1}{2}M$ orthogonal signals by simply including the negatives of the orthogonal signals.
- Thus, we require $N = \frac{1}{2}M$ dimensions for the construction of a set of M biorthogonal signals.
- Figure illustrates the biorthogonal signals for M = 4 and 6.



Biorthogonal Signaling

- All signals are equidistant from s_i, i.e.

$$\|s_i - s_k\| = \sqrt{2E_s} = d_{\min} \tag{18}$$

except one signal point which is the reflection through the origin, and is farther away

$$\|s_i - (-s_i)\| = 2\sqrt{E_s} \tag{19}$$

Simplex Signaling

- Suppose we have a set of M orthogonal waveforms $\{s_m(t)\}$or, equivalently, their vector representation s_m, their mean is given by

$$\bar{s} = \frac{1}{M}\sum_{m=1}^{M} s_m \tag{20}$$

- Now, let us construct another set of M signals by subtracting the mean from each of the M orthogonal signals.
- Thus,

$$s'_m = s_m - \bar{s}, \qquad m = 1, 2, \cdots, M \tag{21}$$

Simplex Signaling

- The resulting signal waveforms are called simplex signals and have the following properties. First, the energy per waveform is

$$\begin{aligned} \|s'_m\|^2 &= \|s_m - \bar{s}\|^2 \\ &= \mathcal{E} - \frac{2}{M}\mathcal{E} + \frac{1}{M}\mathcal{E} \\ &= \mathcal{E}\left(1 - \frac{1}{M}\right) \end{aligned} \tag{22}$$

- Second, the cross-correlation of any pair of signals is

$$\begin{aligned} Re\left[\rho_{mn}\right] &= \frac{s'_m \cdot s'_n}{\|s'_m\|\|s'_n\|} \\ &= \frac{-1/M}{1 - 1/M} = -\frac{1}{M-1} \end{aligned} \tag{23}$$

Simplex Signaling

- Hence, the set of simplex waveforms is equally correlated and requires less energy, by the factor $1 - 1/M$, than the set of orthogonal waveforms.
- Since only the origin was translated, the distance between any pair of signal points is maintained at $d = \sqrt{2\mathcal{E}}$, which is the same as the distance between any pair of orthogonal signals.

Simplex Signaling