EE901
Probability and Random Processes

Module 5
Functions of Random Variables

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Transformation of Continuous Random Variables

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Example: Linear $g(x)$

- Let X be a CRV with CDF $F_X(x)$. Let $Y = 2X + 3$
- Compute the CDF of RV Y.

$F_Y(y) = \mathbb{P}(Y \leq y)$

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Example: Linear $g(x)$

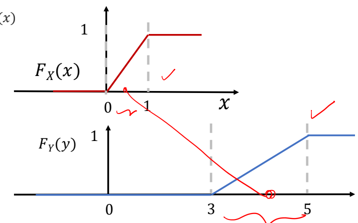
- Let X be a CRV with CDF $F_X(x)$

  $X \sim \text{Unif}(0,1)$ ~ 0.5

  $Y = 2X + 3$

  $F_Y(y) = F_X\left(\frac{y-3}{2}\right)$

  $Y \sim \text{Unif}(3,5)$ 4



$\mathbb{E}[X] = 0.5$ $\qquad$ $\mathbb{E}[Y] = \mathbb{E}[2X+3] = 2\mathbb{E}[X] + 3 = 2 \times 0.5 + 3 = 4$

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CDF of Function of CRV

- Let X be a CRV with CDF $F_X(x)$.
- Let $Y = g(X)$
- g is monotonically increasing function

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \le y] \\ &= \mathbb{P}[g(X) \le y] \\ &= \mathbb{P}[X \le g^{-1}(y)] = F_X(g^{-1}(y)) \end{aligned}$$

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Example: $g(x) = x^2$

- Suppose X is a uniform RV defined as $X \sim \text{Unif}(-1,1)$ and $Y = X^2$
- CDF of Y is

$$\begin{aligned} F_Y(y) &= \mathbb{P}[Y \le y] = \mathbb{P}[\{\omega : Y(w) \le y\}] \\ &= \mathbb{P}[\{\omega : X^2(\omega) \le y\}]. \end{aligned}$$

If $y \ge 0$

$$\begin{aligned} &= \mathbb{P}[\{\omega : -\sqrt{y} \le X(\omega) \le \sqrt{y}\}] \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

$\mathbb{P}[\{a \le X \le b\}]$

$= F_X(b) - F_X(a)$

$(-\infty, a] \cup (a, b] = (-\infty, b]$

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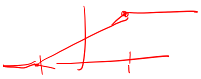
Example: $g(x) = x^2$

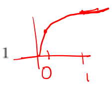
- Suppose X is a uniform RV defined as $X \sim$ Unif $(-1, 1)$ and $Y = X^2$
- CDF of Y is



$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq -1 \\ \frac{1}{2} + \frac{x}{2}, & \text{if } -1 < x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0, & \text{if } y \leq 0. \\ \sqrt{(y)}, & \text{if } 0 < y \leq 1 \\ 1, & \text{if } y > 1 \end{cases}$$



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CDF of a Function of CRV

- For a general function g, you need to collect all intervals of X which corresponds to $g(X) \leq y$ and add their probabilities.

$\{Y \leq y\} = \{g(X(\omega)) \leq y\}$ I $\{\omega, X(\omega) < g^{-1}(y)$

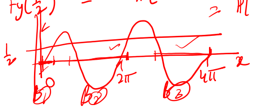
$Y = \sin(x)$ $X \sim (0, 4\pi)$

$F_Y(\frac{1}{2}) = \mathbb{P}[\sin(x) \leq 0.5]$

$= \mathbb{P}[X \in B_1] + \mathbb{P}[X \in B_2] + \mathbb{P}[X \in B_3]$

$B_1 = (a_1, b_1)$

$\mathbb{P}(X \in B_1) = F_X(b_1) - F_X(a_1)$



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CDF of a Function of CRV

- For a general function g, you need to collect all intervals of X which corresponds to $g(X) \leq y$ and add their probabilities.
- If the transformed variable Y is also continuous RV, calculating PDF $F_Y(y)$ may be easier.
- From PDF, the CDF or probability of any Borel set can be computed
- For $B = (a, b)$

$$\mathbb{P}[Y \in B] = \int_a^b f_Y(y)\mathrm{d}y$$

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PDF of a Function of CRV

- Let X has density function $f_X(x)$.
- $Y = 2X + 3$
- What is its density? Assume $Y = 2X + 3$ has density function $f_Y(y)$
- For $B = (a, b)$

$$\mathbb{P}[Y \in B] = \int_a^b f_Y(y)\mathrm{d}y$$

$$\begin{aligned}&\{Y \in B\}\\ &= \{\omega : Y(\omega) \in B\}\\ &= \{\omega : 2X(\omega) + 3 \in B\}\\ &= \{\omega : a < (2X(\omega) + 3) < b\}\\ &= \{\omega : (a-3)/2 < X(\omega) < (b-3)/2\}\end{aligned}$$

$$\mathbb{P}[Y \in B] = \int_{\frac{a-3}{2}}^{\frac{b-3}{2}} f_X(x)\mathrm{d}x$$

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PDF of a Function of CRV

$$\mathbb{P}[Y \in B] = \int_a^b f_Y(y)dy = \int_{\frac{a-3}{2}}^{\frac{b-3}{2}} f_X(x)dx$$

- In RHS, let

$$z = 2x + 3$$
$$\mathrm{d}z = 2\mathrm{d}x$$

Upper limit

$$x = (b - 3)/2 \to z = 2(b - 3)/2 + 3 = b$$

Lower limit

$$x = (a - 3)/2 \to z = 2(a - 3)/2 + 3 = a$$

$$\int_{\frac{a-3}{2}}^{\frac{b-3}{2}} f_X(x)dx = \int_a^b f_X\left(\frac{z-3}{2}\right)\frac{\mathrm{d}z}{2} = \int_a^b \frac{1}{2} f_X\left(\frac{z-3}{2}\right)dz$$

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PDF of a Function of CRV

$$\int_a^b f_Y(y)dy = \int_a^b \frac{1}{2} f_X\left(\frac{z-3}{2}\right)dz \checkmark$$

$$\int_a^{a+\varepsilon} f(x)\,dx = \varepsilon \cdot f(a) = f(a)\int_a^{a+\varepsilon} dx = f(a)\cdot\varepsilon$$

It is true for any a and b. Take $b = a + \epsilon$.

$$f_Y(a)\epsilon = \frac{1}{2} f_X\left(\frac{a-3}{2}\right)\epsilon$$

Hence

$$\boxed{f_Y(a) = \frac{1}{2} f_X\left(\frac{a-3}{2}\right)}$$

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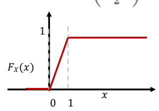
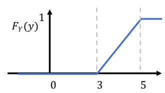
Example: Linear Transformation

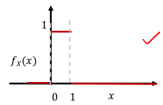
Let $X \sim \text{Unif}(0,1)$. Let $Y = 2X + 3$. $f_X(x) = \mathbb{1}(0 \le x \le 1)$

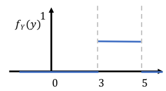
$$F_Y(y) = F_X\left(\frac{y-3}{2}\right)$$





$$f_Y(a) = \frac{1}{2} f_X\left(\frac{a-3}{2}\right)$$
$$= \frac{1}{2}\mathbb{1}\left(0 \le \frac{a-3}{2} \le 1\right)$$
$$= \frac{1}{2}\mathbb{1}(3 \le a \le 5)$$





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Transformation of CRV with increasing g

Suppose g is a monotonically increasing function

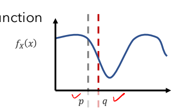
$B = (a, b)$

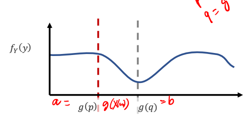
$\{Y \in (a,b)\} = \{\omega : a < Y(\omega) < b\}$

$= \{\omega : a < g(X(\omega)) < b\}$

$= \{\omega : g^{-1}(a) < X(\omega) < g^{-1}(b)\}$

$$\int_a^b f_Y(y)dy = \int_{g^{-1}(a)}^{g^{-1}(b)} f_X(x)dx$$





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Transformation of CRV with increasing g

Suppose g is a monotonically increasing function

$$\int_a^b f_Y(y)dy = \int_{g^{-1}(a)}^{g^{-1}(b)} f_X(x)dx$$

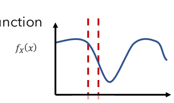
$$= \int_a^b f_X(g^{-1}(z))\frac{dz}{g'(x)}$$

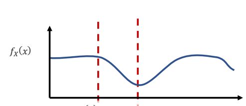
$$\int_a^b f_Y(y)dy = \int_a^b f_X(g^{-1}(z))\frac{dz}{g'(g^{-1}(z))}$$

$z = g(x)$

$dz = g'(x)dx$

$g'(g^{-1}(z))$





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Transformation of CRV with increasing g

Suppose g is a monotonically increasing function

$$\int_a^b f_Y(y)dy = \int_a^b f_X(g^{-1}(z))\frac{\mathrm{d}z}{g'(g^{-1}(z))}$$

$$f_Y(y) = \underbrace{\frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}}_{\boxed{\frac{f_X(x)}{g'(x)} \text{ evaluated at } x = g^{-1}(y)}}$$

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Example: Linear Transformation

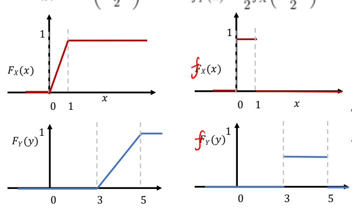
Let $X \sim$ Unif $(0, 1)$. Let $Y = 2X + 3$. $f_X(x) = 1\,(0 \le x \le 1)$

$y = g(x) = 2x+3$

$x = \frac{y-3}{2}$

$g^{-1}(y)$

$$F_Y(y) = F_X\left(\frac{y-3}{2}\right) \qquad f_Y(a) = \frac{1}{2} f_X\left(\frac{a-3}{2}\right) ✓$$



$Y = g(X) = 2X + 3.$

$g'(x) = 2, \quad g^{-1}(y) = \frac{y-3}{2}$

$f_Y(y) = \frac{f_X(\frac{y-3}{2})}{2}$

$\frac{f_X(x)}{g'(x)}$

$x = g^{-1}(y)$

$\frac{y-3}{2}$

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Transformation of CRV with decreasing g

Suppose g is a monotonically decreasing function

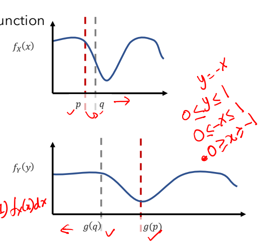
$B = (a, b)$

$\{Y \in (a, b)\} = \{\omega : a < Y(\omega) < b\}$

$= \{\omega : a < g(X(\omega)) < b\}$

$= \{\omega : g^{-1}(b) < X(\omega) < g^{-1}(a)\}$

$$\int_a^b f_Y(y)dy = \int_{g^{-1}(b)}^{g^{-1}(a)} f_X(x)dx = -\int_{g^{-1}(a)}^{g^{-1}(b)} (-1) f_X(x)dx$$



$y = -x$

$0 \le y \le 1$

$0 \le -x \le 1$

$0 \ge x \ge -1$

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Transformation of CRV with Function g

g is a monotonically increasing function

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

$g' > 0$

g is a monotonically decreasing function

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{-g'(g^{-1}(y))}$$

$-g'(x) > 0$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

Consider the behaviour of g in the range of X only.

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Transformation with $g(x) = x^2$

- Exponential RV $X \sim \mathrm{Exp}(\lambda)$
- $Y = X^2$

$f_X(x) = \lambda e^{-\lambda x} \mathbb{1}(x \geq 0)$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

x^2

$f_Y(y) = \frac{f_X(x)}{g'(x)} \Big|_{x = g^{-1}(y)} = \frac{\lambda e^{-\lambda x}}{2x} \Big|_{x=\sqrt{y}} = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}} \mathbb{1}(y \geq 0)$

$y = x^2$

$g^{-1}(y) = \sqrt{y}$

$g'(x) = 2x$

< 0

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Example: $g(x) = 2x^3 + 3$

$$Y = 2X^3 + 3 = g(X)$$

$$g^{-1}(y) = \left(\frac{y-3}{2}\right)^{\frac{1}{3}}$$

$$g'(x) = 6x^2$$

$$g'(g^{-1}(y)) = 6\left(\frac{y-3}{2}\right)^{\frac{2}{3}}$$

$$f_Y(y) = \frac{f_X\left(\left(\frac{y-3}{2}\right)^{\frac{1}{3}}\right)}{\left|6\left(\frac{y-3}{2}\right)^{\frac{2}{3}}\right|}$$

$$X = \text{Unif}[-1, 1]$$

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for} \quad -1 \le x \le 1 \\ 0 & \text{for} \quad otherwise \end{cases}$$

$$= \mathbb{1}(-1 \le x \le 1)$$

$$\mathbb{1}\left(1 \le \left(\frac{y-3}{2}\right)^{1/3} \le 1\right)$$

$$= \begin{cases} \frac{1}{12}\left(\frac{y-3}{2}\right)^{-\frac{2}{3}} & ; \quad 1 \le x \le 5 \\ 0 & ; \quad otherwise \end{cases}$$

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