# Solutions of Tutorial-2

## Problem set 2.2

**7** If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

**8** If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

**12** Elimination leads to this upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + \phantom{3}z &= 8 & & x = 2 \\ y + 3z &= 4 \quad \text{gives} & & y = 1 \quad \text{If a zero is at the start of row 2 or row 3,} \\ 8z &= 8 & & z = 1 \quad \text{that avoids a row operation.} \end{aligned}$$

**24** Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$. (You could notice that the determinant $a^2 - 2a$ is zero for $a = 2$ and $a = 0$.)

## Problem set 2.3

**1** $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

**6** Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

**17** The parabola $y = a + bx + cx^2$ goes through the 3 given points when $\begin{aligned} a + 2b + 4c &= \phantom{1}8 \\ a + 3b + 9c &= 14 \end{aligned}$.

Then $a = 2, b = 1$, and $c = 1$. This matrix with columns $(1, 1, 1), (1, 2, 3), (1, 4, 9)$ is a "Vandermonde matrix."

**24** $\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 4 & 1 & \mathbf{17} \end{bmatrix} \to \begin{bmatrix} 2 & 3 & \mathbf{1} \\ 0 & -5 & \mathbf{15} \end{bmatrix}$. The triangular system is $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -5x_2 &= 15 \end{aligned}$

Back substitution gives $x_1 = 5$ and $x_2 = -3$.

**25** The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3. Then the last equation is $0 = 0$ and the system has infinitely many solutions.

**Problem set 2.4**

**5** (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$

**6** $(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

**27** (a) (row 3 of $A$) $\cdot$ (column 1 or 2 of $B$) and (row 3 of $A$) $\cdot$ (column 2 of $B$) are all zero.

(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix} \begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ : **both upper**.

**32** $A$ times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

**Problem set 2.5**

**6** (a) Multiply $AB = AC$ by $A^{-1}$ to find $B = C$ (since $A$ is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

**7** (a) In $A\boldsymbol{x} = (1, 0, 0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

**15** If $A$ has a column of zeros, so does $BA$. Then $BA = I$ is impossible. There is no $A^{-1}$.

**Problem set 2.7**

**2** $(AB)^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = B^{\mathrm{T}}A^{\mathrm{T}}$. This answer is different from $A^{\mathrm{T}}B^{\mathrm{T}}$ (except when $AB = BA$ and transposing gives $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$).

**3** (a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$.

(b) If $U$ is upper triangular, so is $U^{-1}$: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

**5** (a) $x^{\mathrm{T}}Ay = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5$

(b) This is the row $x^{\mathrm{T}}A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ times $y$.

(c) This is also the row $x^{\mathrm{T}}$ times $Ay = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

**16** $A^2 - B^2$ (but not $(A+B)(A-B)$, this is different) and also $ABA$ are symmetric if $A$ and $B$ are symmetric.

**39** Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give $q_1^{\mathrm{T}}q_1 = 1$ and $q_2^{\mathrm{T}}q_2 = 1$: *unit vectors*

(b) The off-diagonal entry is $q_1^{\mathrm{T}}q_2 = 0$ (and in general $q_i^{\mathrm{T}}q_j = 0$)

(c) The leading example for $Q$ is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.