6. Sets specified in terms of matrices

$$S^n = \{ X \in \mathbb{R}^{n \times n} \mid X = X^T \}$$

$$= \{ X \in \mathbb{R}^{n \times n} \mid X_{12} = X_{21} \}$$

linear restriction

$$X_{23} = X_{32}$$

$$\vdots$$

intersection of hyperplanes in $\mathbb{R}^{n^2}$

$a^T X = b$ — vectors

$\langle A, X \rangle = b$ — matrices

$X_{12} = X_{21}$ verify

$$A = \left[\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right] \qquad b = 0$$

$S_+^n = \{X \in S^n \mid X \succcurlyeq 0\}$ $\quad$ X P.S.D.

S_+^n is convex cone

$X, Y \in S_+^n \Rightarrow X \succcurlyeq 0, \; Y \succcurlyeq 0$

$\Downarrow \qquad \Downarrow$

$u^T X u \geq 0 \; \forall u \qquad u^T Y u \geq 0 \; \forall u$

$\Downarrow$

$Z = \theta_1 X + \theta_2 Y$

first: $Z^T = \theta_1 X^T + \theta_2 Y^T = \theta_1 X + \theta_2 Y = Z$

$\Rightarrow Z \in S^n$

also

$v^T Z v = v^T(\theta_1 X + \theta_2 Y) v$

$\theta_1, \theta_2 \geq 0$

$= \theta_1 \underbrace{(v^T X v)}_{\geq 0 \; \forall v} + \theta_2 \underbrace{(v^T Y v)}_{v \geq 0 \; \forall v}$

$\geq 0 \quad \forall v$

$\Rightarrow Z \in S_+^n$

P.S.D. Cone