Water filling example

Demonstration of how KKT conditions help us understand the problem better

Equalization of communication channels

OFDMA
TDMA
ZF-MIMO
⋮

max (sum rate)

total power ≤ budget

(no ICI)

$$\text{rate}_i = \log(1+\mathrm{SNR}_i) = \log(1+\gamma_i p_i) \qquad \gamma_i = \frac{g_i}{N_0} \geq 0$$

$$\max_{p} \sum_{i=1}^{n} \log(1+\gamma_i p_i)$$

$$\sum_{i=1}^{n} p_i \leq P \longleftarrow \text{power budget}$$

$$p_i \geq 0$$

Note : convex
affine constraints (Slater's not needed)
feasible (e.g. $p_i = 0 \ \forall i$)
not unbounded below

$\Rightarrow$ P = D & KKT

* Different ways of writing dual
* Let us keep $p_i \geq 0$ implicit (not dualize)
  — why? dual is a 1-dim (scalar) problem

$$\min_{p \geq 0} -\sum_{i=1}^{n} \log(1+p_i \gamma_i) \quad = \quad \min f(p)$$
$$\sum_{i=1}^{n} p_i \leq P \qquad\qquad \sum p_i \leq P$$

objective $\quad f(p) = -\sum_{i=1}^{n} \log(1+p_i\gamma_i)$

$$\mathrm{dom} f = \mathbb{R}^n_+$$

<u>KKT</u>

(a) Stationarity

$$p^* = \arg\min_{p \geq 0} L(p,\lambda) = -\sum \log(1+p_i\gamma_i) + \lambda^*\left(\sum p_i - P\right)$$

$$p^* = \arg\min_{p \geq 0} \sum_{i=1}^{n}\left[-\log(1+p_i\gamma_i) + \lambda^* p_i\right]$$

$$\Rightarrow \quad p_i^* = \arg\min_{p_i \geq 0} -\log(1+p_i\gamma_i) + \lambda^* p_i$$

split into
n subproblems

(a) suppose $\lambda^* = 0$ then

$$p_i^* = \underset{p_i \geq 0}{\text{argmin}} \underbrace{-\log(1+p_i r_i)}_{\to -\infty} \quad \& \ p_i^* \to \infty$$

so unbounded below so $\sum p_i^* > P$ ✗

$\Rightarrow \lambda^* > 0$

(b) $\lambda^* > 0$

b(i) suppose : $p_i^* > 0$ $\quad \frac{d}{dp_i}\left(-\log(1+p_i r_i) + \lambda^* p_i\right) = 0$

or $\lambda^* = \dfrac{r_i}{1+p_i^* r_i}$ $\quad p_i^* = \dfrac{1}{\lambda^*} - \dfrac{1}{r_i} > 0$

b(ii) in case $\dfrac{1}{\lambda^*} - \dfrac{1}{r_i} \leq 0$ then $p_i^* = 0$

(a) $p_i^* = \max\left\{0, \dfrac{1}{\lambda^*} - \dfrac{1}{p_i^*}\right\} = \left[\dfrac{1}{\lambda^*} - \dfrac{1}{r_i}\right]_+$

KKT

(b) $p^* \geq 0$, $\sum p_i^* \leq P$

(c) $\lambda^* > 0$

(d) $\lambda^*\left(\sum_{i=1}^{n} p_i^* - P\right) = 0 \Rightarrow \sum p_i^* = P$

solve KKT?

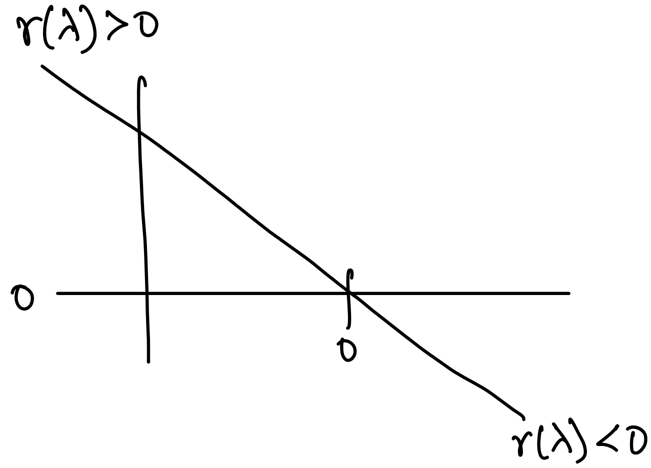
$$r(\lambda) = \sum_{i=1}^{n} \left[\frac{1}{\lambda} - \frac{1}{\gamma_i^*} \right]_+ - P = 0$$

λ^* is root of $r(\lambda)$

Intuition:

λ small $\Rightarrow r(\lambda) > 0$ | λ large $r(\lambda) < 0$

eg. $\lambda = \max_i \gamma_i$



$r(\lambda)$ decreasing

Bisection

$r(\lambda)$ decreasing

$\lambda \in [\lambda_{min}, \lambda_{max}]$

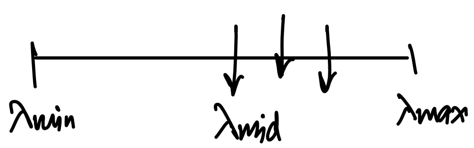
$B = \lambda_{max} - \lambda_{min}$

while $\lambda_{max} - \lambda_{min} > \epsilon$

$\lambda_{mid} = \frac{\lambda_{min} + \lambda_{mid}}{2}$

if $r(\lambda_{mid}) > 0$ $\lambda_{min} = \lambda_{mid}$

else $\lambda_{max} = \lambda_{mid}$



K rounds

$\lambda_{max} - \lambda_{min} \sim \frac{B}{2^k} = \epsilon$

$K = \log_2(B/\epsilon)$

iteration complexity