

eMasters in Communication Systems

**Prof. Aditya
Jagannatham**



Elective Module:

**Estimation for Wireless
Communication**



Chapter 8

*Minimum Mean
Square Error*

MMSE Estimation



MMSE Estimation

- **MMSE = Minimum Mean Squared Error.**

ML = Maximum Likelihood.



MMSE Estimation

- Consider **observation vector** $\bar{\mathbf{y}}$ $N \times 1$
- Unknown parameter** vector $\bar{\mathbf{h}}$ $M \times 1$

$$\bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

$$\bar{\mathbf{h}} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}$$



MMSE Estimation

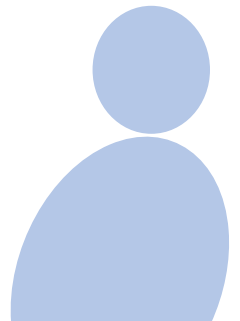
- However, in this, $\bar{\mathbf{y}}, \bar{\mathbf{h}}$ are **random** in nature

BAYESIAN
PHILOSOPHY.

RANDOM.

ML: \mathbf{h} Deterministic
unknown

Fundamental.



MMSE Estimation *minimize*

- The cost-function for the **MMSE estimate** of $\bar{\mathbf{h}}$ is given as

$$ML: \min \|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{h}}\|^2$$

LEAST SQUARES

minimum Mean Square Error

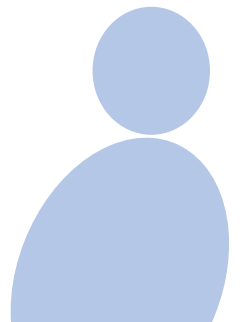
MMSE :

$$\min E \left\{ \|\hat{\mathbf{h}} - \bar{\mathbf{h}}\|^2 \right\}$$

Error

Square Error

MEAN SQUARE
Error



MMSE Estimation

- The cost-function for the **MMSE estimate** of $\bar{\mathbf{h}}$ is given as

MMSE cost function

$$\min E \left\{ \|\hat{\mathbf{h}} - \bar{\mathbf{h}}\|^2 \right\}$$



MMSE Estimation

- The expression for the **MMSE estimate** is

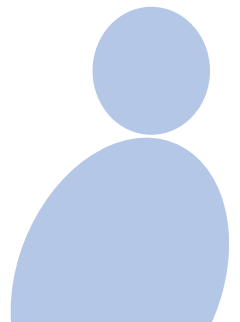
$$\hat{h} = E \{ h | \bar{y} \}$$

Simple
But challenging
in practice!

in general
fairly difficult
to evaluate!

General Expression
for MMSE Estimate

conditional expectation
of h given \bar{y} .

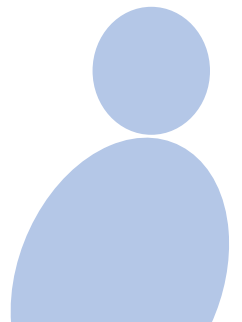


MMSE Estimation

- The expression for the **MMSE estimate** is

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\}$$

General Expression



MMSE Estimation

zero mean
Gaussian.
Random vectors.

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, this can be simplified as

zero mean.

$$E\{\bar{\mathbf{h}}\} = 0 \quad E\{\bar{\mathbf{y}}\} = 0$$

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \frac{\mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \bar{\mathbf{y}}}{E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\}}.$$

Note: MMSE only
when $\bar{\mathbf{h}}, \bar{\mathbf{y}}$ are
jointly Gaussian

MMSE Estimation

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, this can be simplified as

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

*MMSE Estimate
 $\bar{\mathbf{h}}, \bar{\mathbf{y}}$ are zero mean Gaussian.*



MMSE Estimation

- \mathbf{R}_{yy} is $\text{Covariance matrix of } \bar{\mathbf{y}}$

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\} \quad N \times N$$
$$= E\left\{ \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} [y(1) \ y(2) \ \dots \ y(N)] \right\}$$



MMSE Estimation

- \mathbf{R}_{yy} is

covariance matrix
 $\bar{\mathbf{y}}$

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$



MMSE Estimation

- \mathbf{R}_{hy} is

CROSS covariance of $\bar{\mathbf{h}}, \bar{\mathbf{y}}$
 $M \times N$
zero mean

$$\mathbf{R}_{hy} = E\{\bar{\mathbf{h}} \bar{\mathbf{y}}^T\}$$
$$= E\left\{ \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} [y(1) \ y(2) \ \dots \ y(N)] \right\}$$

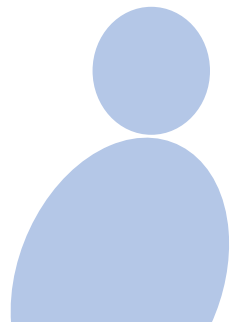


MMSE Estimation

- \mathbf{R}_{hy} is

$$\mathbf{R}_{hy} = E\{\bar{\mathbf{h}}\bar{y}^T\}$$

$$\hat{\mathbf{h}} = \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \bar{y}.$$



MMSE Estimation

Multiple input-
Single output.

- Consider now the MISO channel estimation model

$$\begin{array}{c} N \times 1 \\ \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} \end{array} = \begin{array}{c} \underbrace{X \sim N \times M} \\ \begin{bmatrix} \bar{x}^T(1) \\ \bar{x}^T(2) \\ \vdots \\ \bar{x}^T(N) \end{bmatrix} \end{array} \begin{array}{c} M \times 1 \\ \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} \end{array} + \begin{array}{c} N \times 1 \\ \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix} \end{array}$$
$$\bar{y} = X \bar{h} + \bar{v}$$

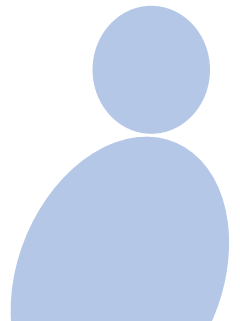
MMSE Estimation

- Consider now the MISO channel estimation model

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{\mathbf{y}}} = \underbrace{\begin{bmatrix} \bar{\mathbf{x}}^T(1) \\ \bar{\mathbf{x}}^T(2) \\ \vdots \\ \bar{\mathbf{x}}^T(N) \end{bmatrix}}_{\mathbf{X}} \bar{\mathbf{h}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{\mathbf{v}}}$$

Gaussian *Gaussian* *Gaussian*

$$\bar{\mathbf{y}} = \mathbf{X} \bar{\mathbf{h}} + \bar{\mathbf{v}}$$



MMSE Estimation

Linear
Estimation
model.

$$\bar{y}$$

$$=$$

$$X$$

$$\bar{h}$$

$$+$$

$$\bar{v}$$

$$:$$

Linear
Gaussian
model.

Linear-combination
of Gaussian Random
Vectors.

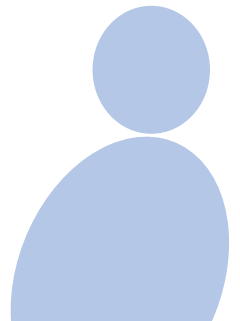
Unknown.

$\Rightarrow \bar{y}$ is also Gaussian.

$\Rightarrow \bar{y}, \bar{h}$ Gaussian.

Gaussian

Gaussian



MMSE Estimation

- We have

$$\bar{\mathbf{h}} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}$$

unknown parameter vector



MMSE Estimation

- We have

$$\bar{\mathbf{h}} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}$$



MMSE Estimation

independent-
identically
distributed -

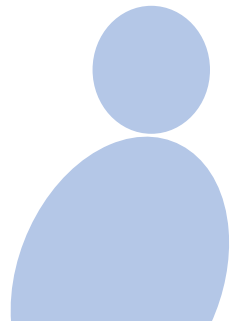
- Let the elements h_i be zero-mean i.i.d. Gaussian of variance σ_h^2

$$E\{h_i\} = 0$$

$$E\{h_i^2\} = \sigma_h^2$$

variance

$$i \neq j \quad E\{h_i h_j\} = E\{h_i\} E\{h_j\} = 0$$

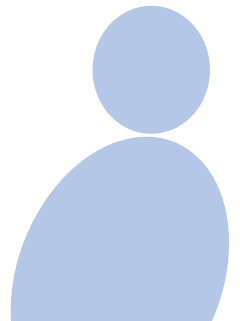


MMSE Estimation

- Then,

Covariance matrix of $\bar{\mathbf{h}}$?
MxM

$$\mathbf{R}_{hh} = E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\} = E\left\{ \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} [h_1 \ h_2 \ \dots \ h_m] \right\}$$
$$= E\left\{ \begin{bmatrix} h_1^2 & h_1 h_2 & & \\ h_2 h_1 & h_2^2 & & \\ \vdots & & \ddots & \\ & & & h_m^2 \end{bmatrix} \right\}$$



MMSE Estimation

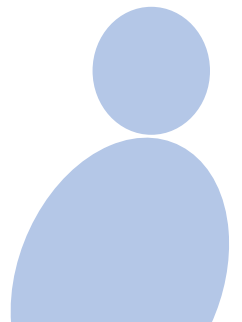
$$\mathbf{R}_{hh} =$$

$$\begin{bmatrix} \sigma_h^2 & 0 & \dots & 0 \\ 0 & \sigma_h^2 & & \\ \vdots & & \ddots & \\ 0 & & & \sigma_h^2 \end{bmatrix}$$

covariance matrix
of \underline{h}

$$\mathbf{R}_{hh} = \sigma_h^2 \mathbf{I}$$

$M \times M$



MMSE Estimation

- Then,

$$\begin{aligned}\mathbf{R}_{hh} &= E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\} \\ &= E\left\{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} [h_1 \quad h_2 \quad \dots \quad h_M]\right\}\end{aligned}$$



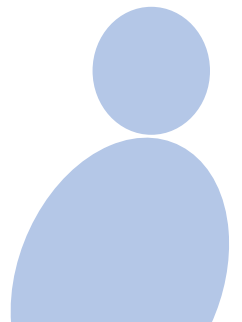
MMSE Estimation

- Then,

$$\mathbf{R}_{hh} = E \left\{ \begin{bmatrix} h_1^2 & h_1 h_2 \\ h_2 h_1 & h_2^2 \end{bmatrix} \right\}$$

Diagonal elements h_i^2
 $E\{h_i^2\} = \sigma_h^2$

Off diagonal elements $h_i h_j$
 $E\{h_i h_j\} = 0$



MMSE Estimation

- Then,

$$\mathbf{R}_{hh} = \begin{bmatrix} \sigma_h^2 & 0 & & \\ 0 & \sigma_h^2 & & \\ & & \ddots & \\ & & & \sigma_h^2 \end{bmatrix} = \sigma_h^2 \mathbf{I}$$



MMSE Estimation

- Similarly, let the elements $v(i)$ be zero-mean i.i.d. Gaussian of variance σ^2

$$\bar{V} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

iid zero mean
Gaussian
variance σ^2



MMSE Estimation

- Then,

$$\mathbf{R}_{vv} = E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\} = \sigma^2 \mathbf{I}$$

NxN matrix



MMSE Estimation

- Then,

Covariance matrix of $\bar{\mathbf{V}}$

$$\mathbf{R}_{vv} = E\{\bar{\mathbf{V}}\bar{\mathbf{V}}^H\} = \sigma^2 \mathbf{I}$$



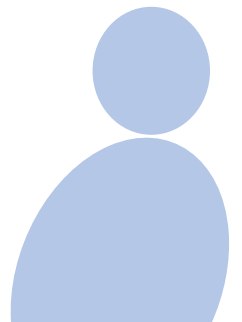
MMSE Estimation

- Consider now the MISO channel estimation model

We need R_{yy} , R_{hy} ?

How to evaluate?

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

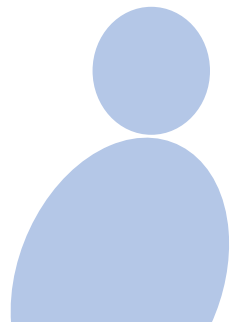


MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

$$\begin{aligned} &= E\{(X\bar{\mathbf{h}} + \bar{\mathbf{v}})(X\bar{\mathbf{h}} + \bar{\mathbf{v}})^T\} \\ &= E\{X\bar{\mathbf{h}}\bar{\mathbf{h}}^T X^T + \bar{\mathbf{v}}\bar{\mathbf{h}}^T X^T + X\bar{\mathbf{h}}\bar{\mathbf{v}}^T + \bar{\mathbf{v}}\bar{\mathbf{v}}^T\} \\ &= X E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\} X^T + E\{\bar{\mathbf{v}}\bar{\mathbf{h}}^T\} X^T \\ &\quad + X E\{\bar{\mathbf{h}}\bar{\mathbf{v}}^T\} + E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\} \end{aligned}$$



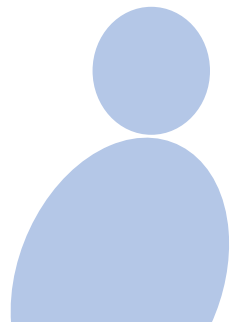
MMSE Estimation

$$E\{\bar{h}\bar{v}^T\}=? \quad E\{\bar{v}\bar{h}^T\}=?$$

\bar{v} : Thermal noise @ Receiver
 \bar{h} : Random channel because
of Scattering environment

$\Rightarrow \bar{h}, \bar{v}$ are independent

$$\Rightarrow \begin{aligned} E\{\bar{h}\bar{v}^T\} &= 0 \\ E\{\bar{v}\bar{h}^T\} &= 0 \end{aligned}$$



MMSE Estimation

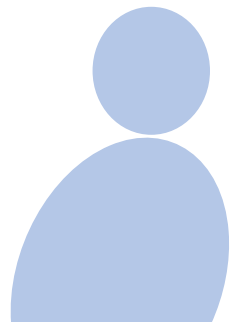
- It follows that

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

$$= \mathbf{X} E\{\mathbf{h}\mathbf{h}^T\} \mathbf{X}^T + E\{\mathbf{v}\mathbf{v}^T\}$$

$$= \mathbf{X} \sigma_h^2 \mathbf{I} \mathbf{X}^T + \sigma^2 \mathbf{I}$$

$$= \sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I}$$



MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

=



MMSE Estimation

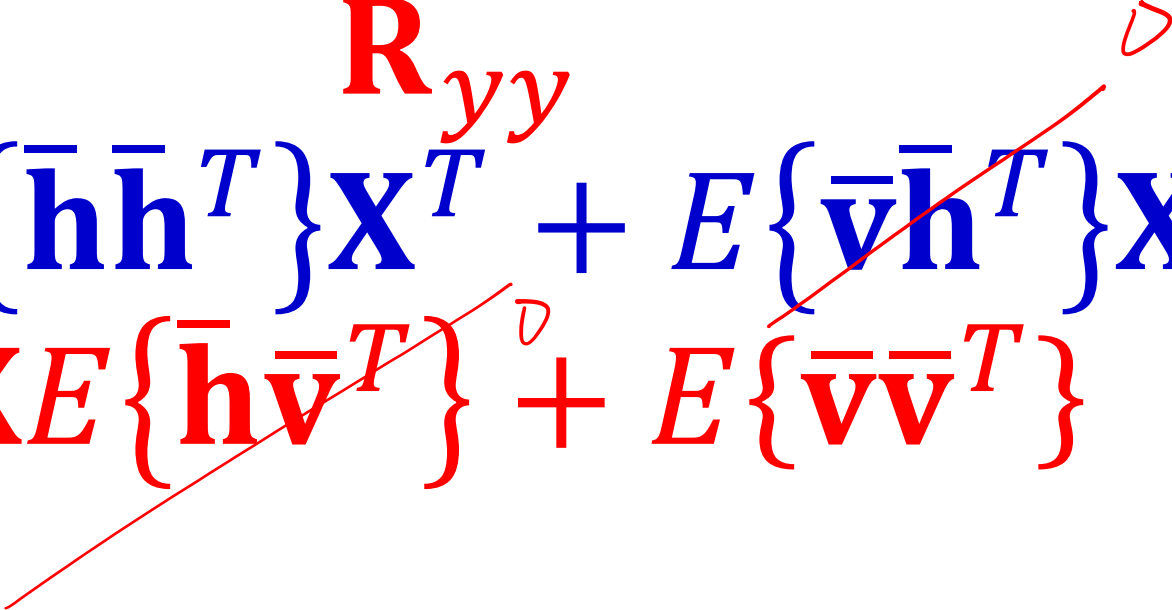
- It follows that

$$\begin{aligned}\mathbf{R}_{yy} &= E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\} \\ &= E\left\{(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}})(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}})^T\right\} \\ &= E\{\mathbf{X}\bar{\mathbf{h}}\bar{\mathbf{h}}^T\mathbf{X}^T\} + E\{\bar{\mathbf{v}}\bar{\mathbf{h}}^T\mathbf{X}^T\} \\ &\quad + E\{\mathbf{X}\bar{\mathbf{h}}\bar{\mathbf{v}}^T\} + E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\}\end{aligned}$$



MMSE Estimation

- It follows that

$$\begin{aligned} & \mathbf{R}_{yy} \\ = & \mathbf{X} E\{\bar{\mathbf{h}} \bar{\mathbf{h}}^T\} \mathbf{X}^T + E\{\bar{\mathbf{v}} \bar{\mathbf{h}}^T\} \mathbf{X}^T \\ & + \mathbf{X} E\{\bar{\mathbf{h}} \bar{\mathbf{v}}^T\} + E\{\bar{\mathbf{v}} \bar{\mathbf{v}}^T\} \end{aligned}$$




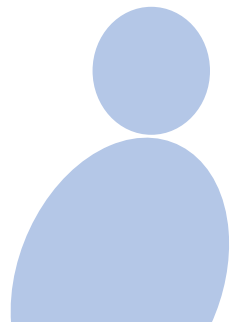
MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = \sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}$$

covariance matrix of $\bar{\mathbf{y}}$

$N \times N$ matrix.



MMSE Estimation

cross covariance matrix
 \bar{h}, \bar{y}

- Also

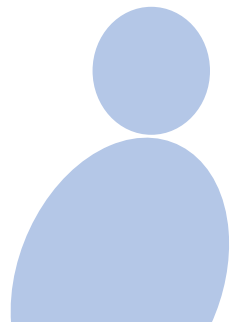
$$\mathbf{R}_{hy} = E\{\bar{h}\bar{y}^T\}$$

$$= E\{\bar{h}(X\bar{h} + \bar{v})^T\}$$

$$= E\{\bar{h}(\bar{h}^T X^T + \bar{v}^T)\}$$

$$= E\{\bar{h}\bar{h}^T\}X^T + \cancel{E\{\bar{h}\bar{v}^T\}}$$

$$= \sigma_h^2 X^T$$

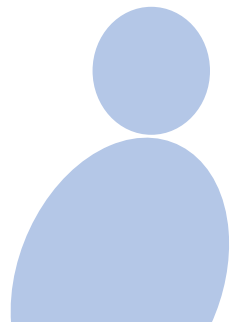


MMSE Estimation

- Also

$M \times N$. Wide matrix

$$\mathbf{R}_{hy} = \sigma_h^2 \mathbf{X}^T$$



MMSE Estimation

- Also

$$\begin{aligned}\mathbf{R}_{hy} &= E\{\bar{\mathbf{h}}\bar{\mathbf{y}}^T\} \\ &= E\left\{\bar{\mathbf{h}}(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}})^T\right\} \\ &= E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\mathbf{X}^T\} + \{\bar{\mathbf{h}}\bar{\mathbf{v}}^T\}\end{aligned}$$



MMSE Estimation

- Also

$$\begin{aligned}\mathbf{R}_{hy} &= E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\}\mathbf{X}^T + \{\bar{\mathbf{h}}\bar{\mathbf{v}}^T\} \\ &= \sigma_h^2 \mathbf{X}^T\end{aligned}$$



MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

MMSE Estimate

$$= \frac{\sigma_h^2 X^T (\sigma_h^2 X X^T + \sigma^2 \mathbf{I})^{-1} \bar{\mathbf{y}}}{\text{MMSE Estimate}}$$

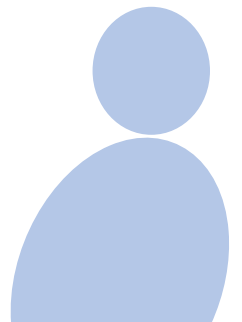


MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$= \sigma_h^2 \mathbf{X}^T \overbrace{(\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I})}^{Q_2}{}^{-1} \bar{\mathbf{y}}$$



MMSE Estimation

- Note

$$\sigma_h^2 \mathbf{X}^T \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{X}^T = \sigma_h^2 \mathbf{X}^T \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{X}^T$$

$$\Rightarrow \mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I}) = (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I}) \mathbf{X}^T$$

$$\Rightarrow \underbrace{(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T}_{Q_1} = \underbrace{\mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1}}_{Q_2}$$

MMSE Estimation

- Note

$$\begin{aligned} & \overbrace{(\sigma_h^2 X^T X + \sigma^2 I)^{-1} X^T}^{Q_1} \\ &= X^T \underbrace{(\sigma_h^2 X X^T + \sigma^2 I)^{-1}}_{Q_2} \end{aligned}$$



MMSE Estimation

- Note

$$\sigma_h^2 \mathbf{X}^T \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{X}^T = \sigma_h^2 \mathbf{X}^T \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{X}^T$$

$$\mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I}) = (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I}) \mathbf{X}^T$$



MMSE Estimation

- Note

$$(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T = \mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1}$$



MMSE Estimation

- Therefore, the MMSE estimate is given as

Form 1.

N x N.

$$\hat{\mathbf{h}} = \sigma_h^2 \mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1} \bar{\mathbf{y}}$$

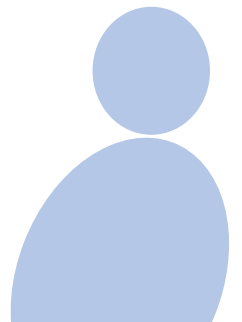
=

$$\sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

M x M matrix

Typically, $M \ll N$

Form 2



MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} =$$



MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\begin{aligned}\hat{\mathbf{h}} &= \sigma_h^2 (\sigma_h^2 X^T X + \sigma^2 I)^{-1} X^T \mathbf{y} \\ &= (X^T X + \frac{\sigma^2}{\sigma_h^2} I)^{-1} X^T \mathbf{y} \quad \text{SNR} = \frac{\sigma_h^2}{\sigma^2} \\ \hat{\mathbf{h}} &= (X^T X + \frac{1}{\text{SNR}} I)^{-1} X^T \mathbf{y}\end{aligned}$$

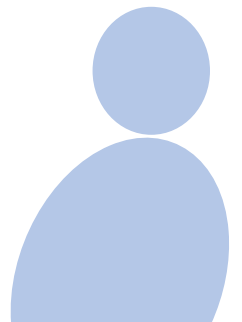


MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \sigma_h^2 \mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1} \bar{\mathbf{y}}$$

$$= \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$



MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\begin{aligned}\hat{\mathbf{h}} &= \sigma_h^2 \left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}} \\ &= \left(\mathbf{X}^T \mathbf{X} + \frac{\sigma^2}{\sigma_h^2} \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}}\end{aligned}$$



MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

$\hat{\mathbf{h}} \xrightarrow{M \times 1} = \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}}$

$\mathbf{X}^T \mathbf{X} \xrightarrow{M \times M}$ $\mathbf{X}^T \bar{\mathbf{y}} \xrightarrow{N \times 1}$

MMSE Estimate

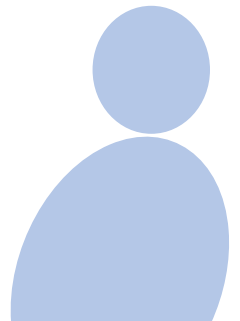
MMSE Estimation- High SNR

- At high $SNR \rightarrow \infty \Rightarrow \frac{1}{SNR} \rightarrow 0$

$$\hat{h} = \left(X^T X + \frac{1}{SNR} I \right)^{-1} X^T \bar{y}$$

$$\rightarrow \left(X^T X \right)^{-1} X^T \bar{y}$$

ML Estimate
Maximum Likelihood
Estimate !



MMSE Estimation- High SNR

- At high $SNR \rightarrow \infty \Rightarrow \frac{1}{SNR} \rightarrow 0$

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

MMSE

$$\rightarrow \underbrace{(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \bar{\mathbf{y}}}_{ML}$$

MMSE Estimation- High SNR

- Therefore **MMSE estimate** approaches the ML Estimate.
at high SNR !!
-



MMSE Estimation- High SNR

- Therefore **MMSE estimate** approaches the **ML Estimate**.
at high SNR!!
-



MMSE Estimation Example

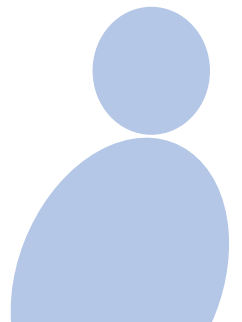
Example:

Consider

$$\bar{\mathbf{y}} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

Handwritten annotations:
A red line connects $\bar{\mathbf{y}}$ to 4×1 and $N=4$.
Another red line connects \mathbf{X} to $M=2$ and 4×2 .

What is $\hat{\mathbf{h}}$ when $SNR = -6dB = \frac{1}{4}$



MMSE Estimation Example

- The **MMSE estimate** can be calculated as follows

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\mathbf{I}$$



MMSE Estimation Example

- The **MMSE estimate** can be calculated as follows

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$



MMSE Estimation Example

- $SNR = -6 \text{ dB} \approx \frac{1}{4}$

$$\begin{aligned}\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} &= 4 \mathbf{I} + \frac{1}{1/4} \mathbf{I} \\ &= 4 \mathbf{I} + 4 \mathbf{I} \\ &= 8 \mathbf{I}\end{aligned}$$



MMSE Estimation Example

- $SNR = -6 \text{ dB} \approx \frac{1}{4}$

$$\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8 \mathbf{I}$$



MMSE Estimation Example

- $SNR = -6 \text{ dB} \approx \frac{1}{4}$


$$\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$



MMSE Estimation Example

$$\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} = 8 \mathbf{I}$$
$$\left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} = (8 \mathbf{I})^{-1} = \frac{1}{8} \mathbf{I}$$

$\frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$





MMSE Estimation Example

$$\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



MMSE Estimation Example

$$\left(\mathbf{X}^T \mathbf{X} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \mathbf{X}^T =$$

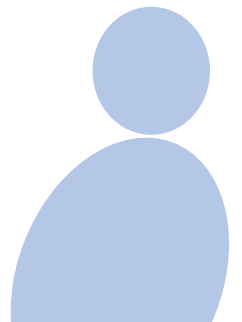
$$= \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$



MMSE Estimation Example

$$\begin{aligned} & \left(\mathbf{X}^T \mathbf{X} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \mathbf{X}^T \\ &= \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{aligned}$$



MMSE Estimation Example

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T \mathbf{X} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

MMSE Estimate of $\bar{\mathbf{h}}$

$\hat{\mathbf{h}}$

$$= \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$

MMSE Estimation Example

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T \mathbf{X} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

$$= \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \underbrace{\hat{\mathbf{h}}}_{\text{red squiggle}} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$



MMSE Covariance

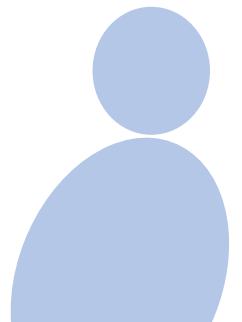
- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, the error covariance is

$$\hat{\mathbf{h}} - \bar{\mathbf{h}} = \bar{\mathbf{e}}$$
$$E\{\bar{\mathbf{e}}\bar{\mathbf{e}}^T\}$$

covariance
matrix of error

Error Covariance

$$E\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\} =$$
$$= R_{hh} - R_{hy} R_{yy}^{-1} R_{yh}.$$



MMSE Covariance

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, the error covariance is

$$\begin{aligned} & E \left\{ (\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T \right\} \\ &= \mathbf{R}_{hh} - \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{R}_{yh} \end{aligned}$$



MMSE Covariance

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, the error covariance is

$$\begin{aligned} & E \left\{ (\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T \right\} \\ &= \mathbf{R}_{hh} - \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{R}_{yh} \end{aligned}$$

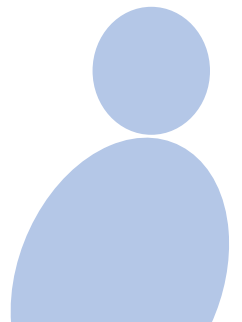


MMSE Error Covariance

- Note that

$$\mathbf{R}_{yh} = \mathbf{R}_{hy}^T$$

$$\begin{aligned} = R_{yh} &= E\{\bar{y}\bar{h}^T\} = E\{(\bar{h}\bar{y}^T)^T\} \\ &= (E\{\bar{h}\bar{y}^T\})^T = R_{hy}^T \\ &= (\sigma_h^2 X^T)^T = \sigma_h^2 X \end{aligned}$$



MMSE Error Covariance

- Note that

$$\mathbf{R}_{yh} = \mathbf{R}_{hy}^T = (\sigma_h^2 \mathbf{X}^T)^T = \sigma_h^2 \mathbf{X}$$



MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\begin{aligned} & \text{Error Covariance} \\ & \mathbf{R}_{hh} - \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{R}_{yh} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 X^T (\sigma_h^2 X X^T + \sigma^2 \mathbf{I})^{-1} \sigma_h^2 X \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 (\sigma_h^2 X^T X + \sigma^2 \mathbf{I})^{-1} \sigma_h^2 X^T X \end{aligned}$$



MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\begin{aligned} & \mathbf{R}_{hh} - \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{R}_{yh} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 \mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1} \sigma_h^2 \mathbf{X} \end{aligned}$$



MMSE Error Covariance

- Note

$$\mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1}$$

Handwritten red annotations: A bracket above the term $\sigma^2 \mathbf{I}$ is labeled $N \times N$.

$$= (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \mathbf{X}^T$$

Handwritten red annotations: An arrow points from the label $M \times M$ to the term $\sigma^2 \mathbf{I}$.



MMSE Error Covariance

- Note

$$\mathbf{X}^T \left(\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I} \right)^{-1}$$

$$= \left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{X}^T$$



MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\begin{aligned} & \mathbf{R}_{hh} - \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{R}_{yh} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 \mathbf{X}^T (\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I})^{-1} \sigma_h^2 \mathbf{X} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \sigma_h^2 \mathbf{X}^T \mathbf{X} \end{aligned}$$



MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\begin{aligned} & \mathbf{R}_{hh} - \mathbf{R}_{hy} \mathbf{R}_{yy}^{-1} \mathbf{R}_{yh} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 \mathbf{X}^T \left(\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I} \right)^{-1} \sigma_h^2 \mathbf{X} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 \left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} \right)^{-1} \sigma_h^2 \mathbf{X}^T \mathbf{X} \end{aligned}$$

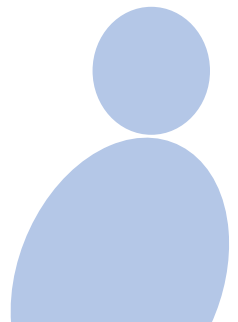


MMSE Error Covariance

Add & Subtract $\sigma^2 \mathbf{I}$

- This can be simplified as

$$\begin{aligned} & \sigma_h^2 \mathbf{I} - \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \sigma_h^2 \mathbf{X}^T \mathbf{X} \\ &= \sigma_h^2 \mathbf{I} - \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} (\underbrace{\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} - \sigma^2 \mathbf{I}}_{\text{Add & Subtract } \sigma^2 \mathbf{I}}) \\ &= \cancel{\sigma_h^2 \mathbf{I}} - \cancel{\sigma_h^2 \mathbf{I}} + \sigma_h^2 \sigma^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \\ &= \sigma_h^2 \sigma^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} \end{aligned}$$



MMSE Error Covariance

- This can be simplified as

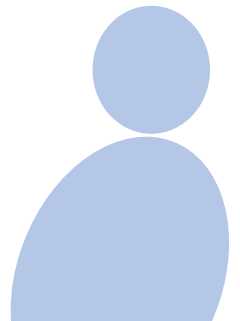
$$\text{SNR} = \frac{\sigma_h^2}{\sigma^2}$$

$$= \sigma_h^2 \sigma^2 (\sigma_h^2 X^T X + \sigma^2 I)^{-1}$$

$$= \sigma^2 \left(X^T X + \frac{\sigma^2}{\sigma_h^2} I \right)^{-1}$$

$$= \sigma^2 \left(X^T X + \frac{1}{\text{SNR}} I \right)$$

Error Covariance.



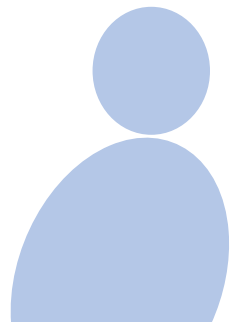
MMSE Error Covariance

- This can be simplified as

$$\sigma_h^2 \mathbf{I} - \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1} (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} - \sigma^2 \mathbf{I})$$

$$= \sigma_h^2 \mathbf{I} - \sigma_h^2 \mathbf{I} + \sigma^2 \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1}$$

$$= \sigma^2 \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1}$$



MMSE Error Covariance

- This can be simplified as

$$\sigma^2 \sigma_h^2 (\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I})^{-1}$$

$$= \left(\frac{1}{\sigma^2} \mathbf{X}^T \mathbf{X} + \frac{1}{\sigma_h^2} \mathbf{I} \right)^{-1}$$
$$= \sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$



MMSE Error Covariance

- MSE is *Minimum Mean Square Error?*

$$= E\{\|\hat{\mathbf{h}} - \bar{\mathbf{h}}\|^2\}$$

$$= \text{Tr} \left\{ \sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \right\}$$



MMSE Error Covariance

- MSE is

$$Tr \left\{ \sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \right\}$$



MMSE Error Covariance Example

- MSE is

$$\text{Tr} \left\{ \sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \right\}$$



MMSE Estimation Example

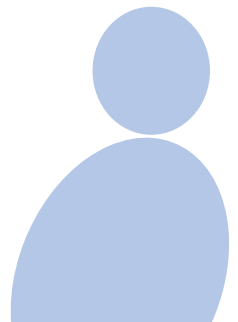
Example:

$$10 \log_{10} \sigma^2 = 3 \Rightarrow \sigma^2 = 10^{0.3} \approx 2$$

Consider

$$\bar{\mathbf{y}} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{4}$$

What is Error cov and MSE when $SNR = -6dB$ and $\sigma^2 = 3dB = 2$

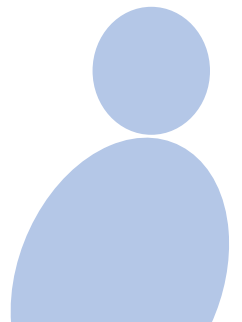


MMSE Error Covariance

- This can be simplified as

$$\sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$

$$= 2 \left(4\mathbf{I} + 4\mathbf{I} \right)^{-1} = 2 \cdot \times \frac{1}{8} \mathbf{I} = \frac{1}{4} \mathbf{I}$$



MMSE Error Covariance

- This can be simplified as

$$\sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$
$$= 2(4\mathbf{I} + 4\mathbf{I})^{-1} = \frac{1}{4} \mathbf{I}$$



MMSE Error Covariance

- MSE is

$$\begin{aligned} \text{Tr} \left\{ \frac{1}{4} \mathbf{I} \right\} & \overset{2 \times 2}{=} \text{Tr} \left\{ \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \right\} \\ &= 2 \times \frac{1}{4} = \frac{1}{2}. \end{aligned}$$



MMSE Error Covariance

- MSE is

$$\text{Tr} \left\{ \frac{1}{4} \mathbf{I} \right\} = 2 \times \frac{1}{4} = \frac{1}{2}$$

Minimum mean
Square Error.



Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.

