Matrices – Introduction

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed the following in the last lecture
  - length and dot products
  - concept of angle between two vectors
  - Cauchy Schwartz inequality
- Discuss the following today
  - matrices and linear equations,
  - Independence and dependence of vectors
- Reference for today's class - Chap 1.3 of the book

Matrices (1)

- Section starts with three vectors **u**, **v**, **w** which we will combine using matrices
- Three vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Linear combinations in three-dimensional space are $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \tag{1}$$

- Learn something important: rewrite this combination using a matrix

Matrices (2)

- Vectors **u**, **v**, **w** form the columns of matrix A

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \tag{2}$$

- Matrix A "multiplies" the vector $\mathbf{x}$ with components (x_1, x_2, x_3)
- Product $A\mathbf{x}$ is same as $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$ of three columns in (1)
- Rewriting brings a crucial first change in viewpoint
  - At first, the numbers x_1, x_2, x_3 were multiplying the vectors
  - Now the matrix is multiplying those numbers
- Matrix A acts on the vector $\mathbf{x}$: output $A\mathbf{x}$ is a combination of columns of A
- A is a "difference matrix" because $\mathbf{b}$ contains differences of input vector $\mathbf{x}$
  - Top difference is $x_1 - x_0 = x_1 - 0$

Matrices (3)

- You may already have learned about multiplying $A\mathbf{x}$, a matrix times a vector
- Usual way takes the dot product of each row with $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0).(x_1,x_2,x_3) \\ (-1,1,0).(x_1,x_2,x_3) \\ (0,-1,1).(x_1,x_2,x_3) \end{bmatrix}$$

- Dot products are same x_1 and $x_2 - x_1$ and $x_3 - x_2$ that we wrote in (2)
- New way is to work with $A\mathbf{x}$ a column at a time
  - $A\mathbf{x}$ is a linear combination of the columns of A

Linear Equations (1)

- Second change in viewpoint – up to now, numbers x_1, x_2, x_3 were known
  - Right hand side $\mathbf{b}$ was not known
- We found that vector of differences by multiplying A times $\mathbf{x}$
- Now we think of $\mathbf{b}$ as known and we calculate $\mathbf{x}$
- Old question: compute linear combination $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$ to find $\mathbf{b}$
- New question: which combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ produces a particular vector $\mathbf{b}$?
  - Inverse problem to calculate input $\mathbf{x}$ that gives the desired output $\mathbf{b} = A\mathbf{x}$
- We will now solve that system $A\mathbf{x} = \mathbf{b}$ to find x_1, x_2, x_3 :

$$A\mathbf{x} = \mathbf{b} \longrightarrow \qquad \begin{aligned} x_1 \qquad &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

Linear Equations (2)

- Solution

$$\mathbf{x} = A^{-1}\mathbf{b} \longrightarrow \quad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned} \tag{3}$$

- In this example, first equation decided $x_1 = b_1$, second one $x_2 = b_1 + b_2$
  - Equations can be solved in order (top to bottom) as A is a triangular matrix
- This matrix A is "invertible". From $\mathbf{b}$ we can recover $\mathbf{x}$. We write $\mathbf{x}$ as $A^{-1}\mathbf{b}$
- Let us repeat the solution $\mathbf{x}$ in (3). A sum matrix will appear!

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{4}$$

Linear Equations (3)

- We have

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A^{-1}\mathbf{b} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

- Above equation for solution $\mathbf{x} = (x_1, x_2, x_3)$ tells us two important facts:
  - For every $\mathbf{b}$ there is one solution to $A\mathbf{x} = \mathbf{b}$
  - The matrix A^{-1} produces $\mathbf{x} = A^{-1}\mathbf{b}$
- Next chapters ask about other equations $A\mathbf{x} = \mathbf{b}$
  - Is there a solution? How to find it?

Cyclic Differences (1)

- We now keep the same columns $\mathbf{u}$ and $\mathbf{v}$ but change $\mathbf{w}$ to a new vector $\mathbf{w}^*$:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Now the linear combinations of $\mathbf{u}$ $\mathbf{v}$ and $\mathbf{w}^*$ leads to a cyclic difference matrix

$$\mathbf{Cx} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \mathbf{b}$$

- Matrix C is not triangular – not so simple to solve for $\mathbf{x}$ when we are given $\mathbf{b}$

Cyclic Differences (2)

- Actually it is impossible to find the solution to $C\mathbf{x} = \mathbf{b}$, because
  - Three equations either have infinitely many solutions (sometimes)
  - or else no solution (usually)

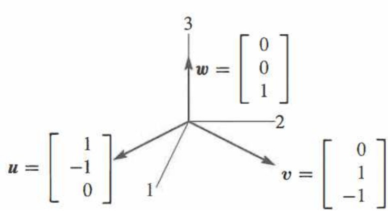
$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is solved by all vectors } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

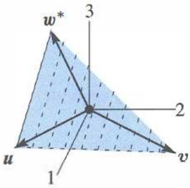
- A constant vector like $\mathbf{x} = (3, 3, 3)$ has zero differences when we go cyclically
- More likely possibility for $C\mathbf{x} = \mathbf{b}$ is no solution $\mathbf{x}$ at all:

$$\mathbf{Cx} = \mathbf{b} \quad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Cyclic Differences (3)

- Look at this example geometrically and pictorially show crucial difference
  - between $\mathbf{u}, \mathbf{v}, \mathbf{w}$ (first example) and $\mathbf{u}, \mathbf{v}, \mathbf{w}^*$ (second example)





- Figure shows those column vectors, first of the matrix A and then of C
- First two columns **u** and **v** are the same in both pictures
- Combinations of those two vectors give a two-dimensional plane

Independence and Dependence (1)

- Key question is whether the third vector is in that plane:
  - Independence: $\mathbf{w}$ is not in the plane of $\mathbf{u}$ and $\mathbf{v}$
  - Dependence: $\mathbf{w}^*$ is in the plane of $\mathbf{u}$ and $\mathbf{v}$
- Important point is that the new vector $\mathbf{w}^*$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} + \mathbf{v} + \mathbf{w}^* \quad \mathbf{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\mathbf{u} - \mathbf{v}$$

- All three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}^*$ have components adding to zero
- By including $\mathbf{w}^*$ we get no new vectors because $\mathbf{w}^*$ is already on that plane

Independence and Dependence (2)

- Combinations of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ fill the whole three-dimensional space
  - because $\mathbf{x} = A^{-1}\mathbf{b}$ in (3) gave the right combination to produce any $\mathbf{b}$
- Two matrices A and C, with third columns $\mathbf{w}$ and $\mathbf{w}^*$, allowed us discuss
  - two key words of linear algebra: independence and dependence
- Independent $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$: No combination except $0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{0}$ gives $\mathbf{b} = 0$
- Dependent $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}^*$: Other combinations like $\mathbf{u} + \mathbf{v} + \mathbf{w}^*$ gives $\mathbf{b} = 0$
- Visualize in three dimensions: three vectors lie in a plane or they don't
- Independent columns: $A\mathbf{x} = 0$ has one solution. A is an invertible matrix
- Dependent columns: $C\mathbf{x} = 0$ has many solutions. C is a singular matrix

Review of important ideas

- Matrix times vector: $A\mathbf{x}$ = combination of the columns of A.
- Solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$, when A is an invertible matrix
- Cyclic matrix C has no inverse. Its three columns lie in the same plane
  - Those dependent columns add to the zero vector
  - $C\mathbf{x} = 0$ has many solutions
- End of Section 1.3 – lots of problems at the end - please solve them