eMasters in Communication Systems

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Elective Module:

Detection for Wireless Communication

Chapter 7

Detection Over Wireless Channel

BPSK

$x \in \{-A, A\}$

- Previously we have seen **Binary Phase Shift Keying (BPSK)**

$\sigma^2 = \frac{N_0}{2}$

- $s = A$

$\mathcal{H}_0$: $y = -A + v \sim \mathcal{N}(0, \sigma^2)$

$\mathcal{H}_1$: $y = A + v$

BPSK

- Previously we have seen **Binary Phase Shift Keying (BPSK)**

Wireline AWGN channel.
Wireline channel.
Additive White Gaussian noise

- $s = A$

NULL hypothesis.

$$\mathcal{H}_0 : y = -A + v$$

$$\mathcal{H}_1 : y = A + v$$

Alternative hypothesis.

Example

$$y = x + V$$

$$E_b = E\{|x|^2\} = \text{Signal Power}$$

$$\frac{N_0}{2} = E\{|V|^2\} = \text{Noise power}$$

$$E_b/\tfrac{N_0}{2} = \rho = SNR = \text{Signal to Noise power ratio.}$$

- P_e is

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0/2}}\right) = Q\left(\sqrt{SNR}\right) = Q(\sqrt{\rho})$$

- This is termed as bit error rate (BER)

Example

- P_e is

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{SNR}\right) = Q(\sqrt{\rho})$$

$\rho = SNR = \frac{2E_b}{N_0}$

- This is termed as bit error rate (BER)

Wireless Communication

Channel is fluctuating

Fading channel.

- Consider **Binary Phase Shift Keying (BPSK)** over **wireless channel**

Fading channel coefficient

$$y = hx + v$$

Fading channel coefficient

- $s = A$

$\mathcal{H}_0$: $y = -hA + v$

$\mathcal{H}_1$: $y = hA + v$

Gain $|h|$.

complex.

Gain of wireless channel. is rapidly Fluctuating. This is termed as a Fading channel.

h = Fading coefficient is RANDOM in nature.

Wireless Communication

- Consider **Binary Phase Shift Keying (BPSK)** over **wireless channel**
- $s = A$

$$\mathcal{H}_0 : y = -hA + v$$
$$\mathcal{H}_1 : y = hA + v$$

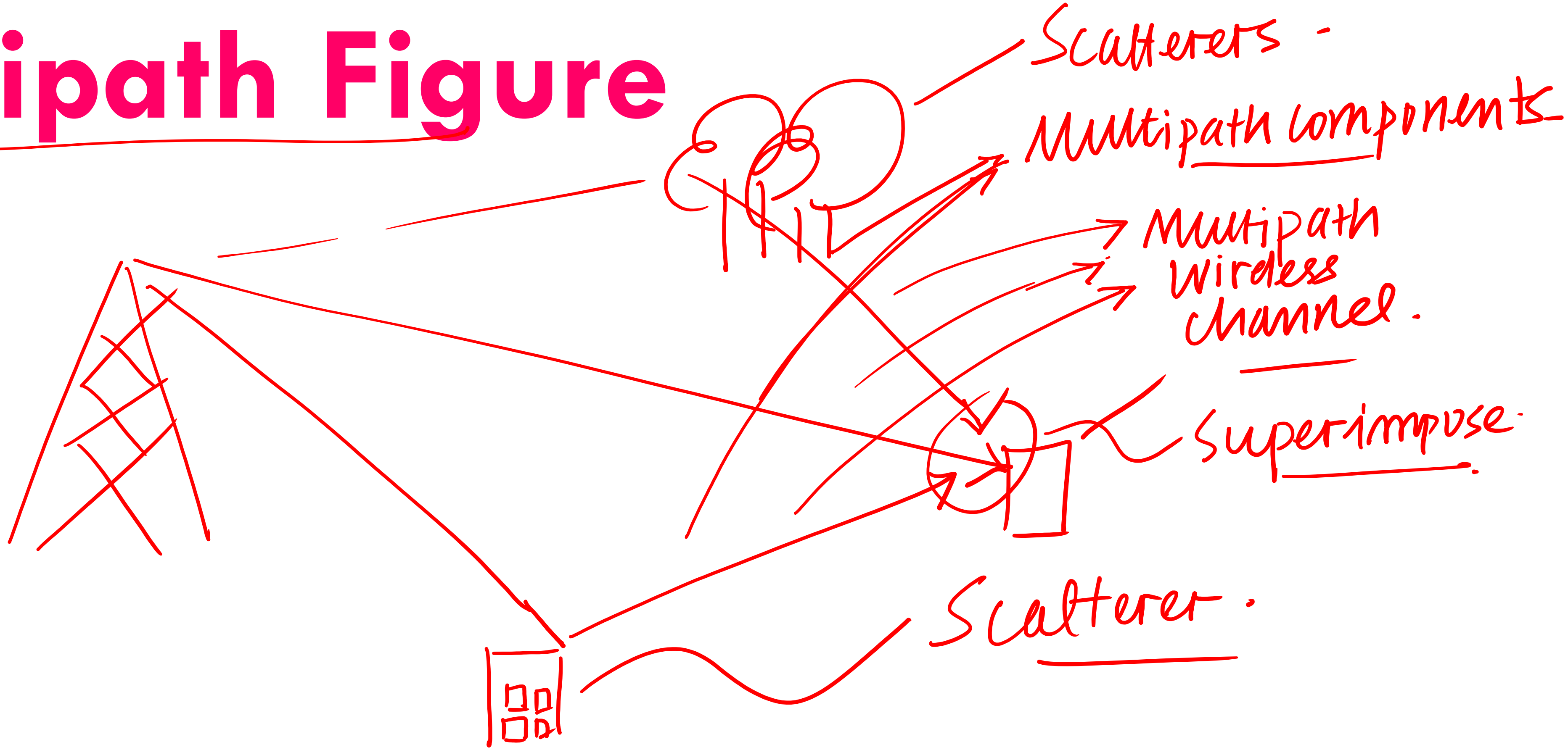
Wireless Communication

- h denote the **fading channel coefficient.**

Gain of channel.

- $|h| = a$ determines **output power.**

Multipath Figure



Wireless Communication

- a follows the Rayleigh PDF given as

Random

$$f_A(a) = \underbrace{2ae^{-a^2}, \quad a \geq 0}_{\text{Rayleigh probability density function}}$$

channel is 'Rayleigh Fading channel'!

Wireless Communication

- a follows the Rayleigh PDF given as

$$f_A(a) = 2ae^{-a^2}, a \geq 0$$

Wireless Communication

- Output SNR is given as

$$y = hx + v, x \in \{-A, A\}$$

$$SNR_o = |h|^2 \frac{E\{|x|^2\}}{E\{|v|^2\}} = a^2 \cdot SNR = a^2 \rho.$$

$$\rho = \frac{A^2}{N_0/2} = \frac{E_b}{N_0/2} = \frac{2E_b}{N_0}$$

$$SNR_o = a^2 \rho \qquad a = |h|.$$

'Fading'

Wireless Communication

- Output SNR is given as

$$E\{|x|^2\} = A^2 = E_b$$

$$SNR_o = |h|^2 \times \frac{E\{|x|^2\}}{E\{|v|^2\}} = a^2\rho$$

Wireless Communication

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

- Instantaneous BER is

$$Q\left(\sqrt{SNR_o}\right) = Q\left(\sqrt{a^2\rho}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{a^2\rho}}^{\infty} e^{-x^2/2} dx \quad = \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{\rho}}^{\infty} e^{-\frac{x^2}{2}} dx$$

Wireless Communication

- Instantaneous BER is

$$Q\left(\sqrt{SNR_o}\right) = Q\left(\sqrt{a^2\rho}\right)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{a^2\rho}}^{\infty} e^{-\frac{x^2}{2}} dx$$

Wireless Communication

- Average BER is

$$P_e = \int_0^{\infty} Q\left(\sqrt{a^2\rho}\right) f_A(a)\, da$$

$$= \int_0^{\infty} Q\left(\sqrt{a^2\rho}\right) 2a e^{-a^2}\, da$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{\rho}}^{\infty} e^{-\frac{x^2}{2}}\, 2a e^{-a^2}\, dx\, da$$

inner integral

outer integral.

Wireless Communication

- Average BER is

$$\int_0^{\infty} Q\left(\sqrt{a^2\rho}\right) 2ae^{-a^2}da$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\sqrt{a^2\rho}}^{\infty} e^{-\frac{x^2}{2}} dx \, 2ae^{-a^2} da$$

Wireless Communication

- Let us now simplify

$$\frac{x}{a\sqrt{\rho}} = u \Rightarrow$$

$$x = a\sqrt{\rho}\, u$$

$$dx = a\sqrt{\rho} \cdot du$$

Average BER

$$x/a\sqrt{\rho} = u$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{a\sqrt{\rho}}^\infty e^{-\frac{x^2}{2}} \cdot 2a e^{-a^2}\, dx\, da$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_1^\infty e^{-\frac{1}{2}a^2\rho . u^2} \cdot 2a e^{-a^2} a\sqrt{\rho}\, du . da$$

wrto u

$$= \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_0^\infty \int_1^\infty 2a^2 e^{-a^2\left(\frac{\rho u^2+2}{2}\right)}\, du\, da$$

wrto a

Average BER

interchange order of integration

inner integral

outer integral

$$P_e = \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_1^\infty \int_0^\infty 2a^2 e^{-a^2\left(\frac{\rho u^2 + 2}{2}\right)} da \cdot du$$

Wireless Communication

$$\frac{x}{a\sqrt{\rho}} = u$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \int_1^\infty a\sqrt{\rho} e^{-a^2\rho\frac{u^2}{2}} du \, 2ae^{-a^2} da$$

$$= \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_1^\infty \int_0^\infty 2a^2 e^{-a^2\left(\rho\frac{u^2}{2}+1\right)} da \, du$$

wrto a

wrto u

Average BER

$E\{a^2\}$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, a^2 e^{-\frac{a^2}{2\sigma^2}}\, da = \sigma^2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, a^2 e^{-a^2/2\sigma^2} = \sigma^3$$

$$\Rightarrow \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}}\, a^2 e^{-a^2/2\sigma^2} = \sigma^3.$$

Average BER

$$P_e = \sqrt{\rho}\int_1^{\infty}\int_0^{\infty} \frac{2}{\sqrt{2\pi}}\, a^2 e^{-a^2\left(\frac{\rho u^2+2}{2}\right)}\, da.du.$$

$$\sigma$$

$$= \sqrt{\rho}.\int_1^{\infty} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, a^2 e^{-\frac{a^2}{2}\cdot\frac{1}{\left(\frac{1}{\sqrt{\rho u^2+2}}\right)^2}}\, da}_{\sigma^3} . du.$$

$$= \sqrt{\rho}\int_1^{\infty} \left(\frac{1}{\rho u^2+2}\right)^{3/2} du$$

Wireless Communication

$$= \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_{1}^{\infty} \int_{-\infty}^{\infty} a^2 e^{-a^2\left(\rho \frac{u^2}{2} + 1\right)} da\, du$$

$$= \sqrt{\rho} \int_{1}^{\infty} \frac{1}{(2 + \rho u^2)^{\frac{3}{2}}} du$$

Wireless Communication

- Use the property

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} a^2 e^{-\frac{a^2}{2\sigma^2}} da = \sigma^2$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a^2 e^{-a^2/2\sigma^2} = \sigma^3 .$$

Wireless Communication

- Use the property

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} a^2 e^{-\frac{a^2}{2\sigma^2}} da = \sigma^2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a^2 e^{-\frac{a^2}{2\sigma^2}} da = \sigma^3$$

Wireless Communication

$$u = \sqrt{\frac{2}{\rho}}\tan\theta\,,\, du = \sqrt{\frac{2}{\rho}}\sec^2\theta\, d\theta$$

Wireless Communication

$$u = \sqrt{\frac{2}{\rho}} \tan\theta \,, du = \sqrt{\frac{2}{\rho}} \sec^2\theta \, d\theta$$

Average BER

$$u = \sqrt{\frac{2}{\rho}} \tan\theta$$

$$P_e = \sqrt{\rho} \int_{1}^{\infty} \left(\frac{1}{2+\rho u^2} \right)^{3/2} du$$

$$= \sqrt{\rho} \int_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\pi/2} \left(\frac{1}{2+2\tan^2\theta} \right)^{3/2} \sqrt{\frac{2}{\rho}} \cdot \sec^2\theta \, d\theta$$

$$= \cancel{\sqrt{\rho}} \int_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\pi/2} \frac{1}{2^{3/2}} \cdot \cos\theta \cdot \frac{\sqrt{2}}{\cancel{\sqrt{\rho}}} \, d\theta$$

Average BER

$$= \frac{1}{2}\int_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{1}{2} \sin\theta \Big|_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\pi/2}$$

$$= \frac{1}{2}\left(1 - \frac{\tan \tan^{-1}\sqrt{\frac{\rho}{2}}}{\sqrt{1+\tan^2 \tan^{-1}\sqrt{\frac{\rho}{2}}}}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{\rho/2}}{\sqrt{1+\rho/2}}\right)$$

Average BER

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\rho/2}{1+\rho/2}}\right)$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{2+\rho}}\right)$$

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{2+SNR}}\right)$$

Average BER.

Wireless Communication

$$P_e = \sqrt{\rho} \int_{1}^{\infty} \frac{1}{(2+\rho u^2)^{\frac{3}{2}}} du$$

$$= \sqrt{\rho} \int_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\frac{\pi}{2}} \left(\frac{1}{2^{\frac{3}{2}} \sec^3 \theta} \sqrt{\frac{2}{\rho}} \sec^2 \theta \right) d\theta$$

Wireless Communication

$$= \frac{1}{2}\int_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\frac{\pi}{2}} \cos\theta \, d\theta$$

$$= \frac{1}{2}\sin\theta\Big|_{\tan^{-1}\sqrt{\frac{\rho}{2}}}^{\frac{\pi}{2}} = \frac{1}{2}\left(1 - \sin\left(\tan^{-1}\sqrt{\frac{\rho}{2}}\right)\right)$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\frac{\rho}{2}}{1+\frac{\rho}{2}}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{2+\rho}}\right)$$

BER of Fading ~~wireless~~ channel.

Wireless Communication

- This can be **approximated** as

$(1+x)^{-1/2} \approx 1-\frac{1}{2}x$

x small.

at high SNR

$\frac{2}{\rho}$ very small.

$$P_e = \frac{1}{2}\left(1-\sqrt{\frac{\rho}{2+\rho}}\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{\sqrt{1+\frac{2}{\rho}}}\right)$$

$$= \frac{1}{2}\left(1-\left(1+\frac{2}{\rho}\right)^{-1/2}\right) \approx \frac{1}{2}\left(1-\left(1-\frac{1}{2}\cdot\frac{2}{\rho}\right)\right)$$

$$= \frac{1}{2}\times\frac{1}{2}\cdot\frac{2}{\rho} = \frac{1}{2\rho}.$$

Wireless Communication

high SNR approximation

- This can be approximated as

$$\frac{1}{2}\left(1-\sqrt{\frac{\rho}{2+\rho}}\right)=\frac{1}{2}\left(1-\left(\frac{2}{\rho}+1\right)^{-\frac{1}{2}}\right)$$

$$\approx\frac{1}{2}\left(1-\left(\frac{2}{\rho}+1\right)^{-\frac{1}{2}}\right)=\frac{1}{2}\left(1-\left(1-\frac{1}{2}\times\frac{2}{\rho}\right)\right)=\frac{1}{2\rho}$$

Example

- Find BER of **Wireless** and **Wireline** channels, $\mathrm{SNR} = 20\ dB$

Example

$\text{SNR} = 20\, dB =$ 100

$10\log_{10} SNR = 20$

$\Rightarrow SNR = 10^2 = 100$

Example

$$\mathrm{SNR} = 20\ dB = 10^2 = 100$$

Example

- BER of **Wireline is**

$$BER = Q\left(\sqrt{100}\right) = Q(10)$$

$$= 7.62 \times 10^{-24}$$

Example

- BER of **Wireline is**
  $Q(\sqrt{SNR}) = Q(10) = 7.62 \times 10^{-24}$

Example

- BER of **Wireless is**

$$BER = \frac{1}{2\rho} = \frac{1}{2 \times 100} = 5 \times 10^{-3}$$

Example

- BER of **Wireless is**

$$\frac{1}{2\rho} = \frac{1}{200} = 5 \times 10^{-3}$$

Example

- BER of **Wireless** is Significantly higher than **Wireline**!

Example

- BER of **Wireless** is ***significantly higher*** than **Wireline**!

FADING !!

Example

- Wireline BER decreases **exponentially**!!

decreases exponentially!

$$BER_{Wireline} = Q(\sqrt{\rho}) = Q(\sqrt{SNR}) \le \frac{1}{2} e^{-\frac{1}{2} SNR}$$

Example

Very Very rapid decrease!

- Wireline BER decreases **exponentially**!!

$$BER_{Wireline} = Q\left(\sqrt{SNR}\right) \le \frac{1}{2} e^{-\frac{1}{2}SNR}$$

e^{-x} decreases much faster than $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$

Example

- Wireless BER decreases only as $\frac{1}{SNR}$!!

very slow rate of decrease.

$BER_{Wireless} = \frac{1}{2 \times SNR} \propto \frac{1}{SNR}$

Example

- Wireless BER decreases only as $\frac{1}{SNR}$!!

$$BER_{Wireless} = \frac{1}{2 \times SNR}$$

QAM

Quadrature Amplitude Modulation.

- BER for QAM in **Wireline channel** is

$$P_e = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{N_0(M-1)}}\right)$$

QAM

M-ary QAM

M: # symbols in QAM

- BER for QAM in **Wireline channel** is

$$4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

QAM

- Let $\frac{E_S}{N_0} = SNR = \rho$
- BER is

$$P_e = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\rho}{M-1}}\right)$$

QAM

- Let $\frac{E_S}{N_0} = SNR = \rho$

$$4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\rho}{(M-1)}}\right)$$

QAM

Wireless: output SNR $= a^2\rho$. effective SNR

- For wireless, instantaneous SER is

$$P_e^{int} = 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{a^2\,\frac{3\rho}{M-1}}\right)$$

Average wrto a

$$\text{Average } P_e = 4\left(1-\frac{1}{\sqrt{M}}\right)\cdot\frac{1}{2\times\frac{3\rho}{M-1}}$$

$$= \frac{2}{3}\left(1-\frac{1}{\sqrt{M}}\right)(M-1).$$

QAM

- For wireless, instantaneous SER is

$$4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{a^2\frac{3\rho}{(M-1)}}\right)$$

QAM

- Average SER is

decreases as $\frac{1}{\rho}$.

$$P_e = \frac{2}{3\rho}\left(1-\frac{1}{\sqrt{M}}\right)(M-1) \propto \frac{1}{\rho}.$$

Symbol Error Rate (SER) of M-ary QAM in Fading wireless channel.

QAM

- Average SER is

$$4\left(1-\frac{1}{\sqrt{M}}\right)\times\frac{1}{2\times\frac{3\rho}{(M-1)}}$$

$$=\frac{2}{3\rho}\left(1-\frac{1}{\sqrt{M}}\right)(M-1)\propto\frac{1}{\rho}=\frac{1}{SNR}.$$

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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