Live Interaction #6:

5th November 2023

E-masters Communication Systems

Estimation for Wireless

- **MMSE**: ***Minimum Mean Square Error***.

$$\min E\left\{\left\|\hat{\mathbf{h}}-\bar{\mathbf{h}}\right\|^2\right\}$$

- $\bar{\mathbf{h}}$: **Random quantity**.

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\}$$

- Simplest case: $\bar{\mathbf{h}}$,$\bar{\mathbf{y}}$ to be **jointly Gaussian** and zero-mean

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\left(\bar{\mathbf{y}}-\bar{\boldsymbol{\mu}}_y\right) + \bar{\boldsymbol{\mu}}_h$$

$$\underbrace{\mathbf{R}_{hy} = E\left\{\left(\bar{\mathbf{h}}-\bar{\boldsymbol{\mu}}_h\right)\left(\bar{\mathbf{y}}-\bar{\boldsymbol{\mu}}_y\right)^T\right\}}_{\text{cross}-\text{covariance}}$$

$$\underbrace{\mathbf{R}_{yy} = E\left\{\left(\bar{\mathbf{y}}-\bar{\boldsymbol{\mu}}_y\right)\left(\bar{\mathbf{y}}-\bar{\boldsymbol{\mu}}_y\right)^T\right\}}_{\text{covariance}}$$

- Linear Model:

$$\underbrace{\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}}_{\text{Linear model}}$$

- **MMSE Estimate**:

$$\underbrace{\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}}_{\text{MMSE Estimate}}$$

$$SNR = \frac{\sigma_h^2}{\sigma^2}$$

$$E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\} = \sigma_h^2\mathbf{I}$$

$$E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\} = \sigma^2\mathbf{I}$$

- MMSE estimate $SNR \to \infty$.

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \cancel{\frac{1}{SNR}\mathbf{I}}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}$$

$$= \underbrace{(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}}_{\text{ML Estimate}}$$

- Error covariance matrix:

$$E\left\{(\hat{\mathbf{h}} - \bar{\mathbf{h}})(\hat{\mathbf{h}} - \bar{\mathbf{h}})^T\right\} = \mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

$$= \sigma^2\left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

$$= \sigma^2\left(\mathbf{X}^T\mathbf{X} + \frac{\sigma^2}{\sigma_h^2}\mathbf{I}\right)^{-1}$$

$$= \left(\frac{1}{\sigma^2}\mathbf{X}^T\mathbf{X} + \frac{1}{\sigma_h^2}\mathbf{I}\right)^{-1}$$

- Noise power $\sigma^2 \to \infty$?

$$\sigma_h^2\mathbf{I}$$

- **Example**:

$$\mathbf{X} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}, \bar{\mathbf{y}} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\sigma^2 = 2, \sigma_h^2 = \frac{1}{4} \Rightarrow SNR = \frac{\sigma_h^2}{\sigma^2} = \frac{\frac{1}{4}}{2} = \frac{1}{8}$$

$$\Rightarrow \frac{1}{SNR} = 8$$

- **MMSE Estimate, Error covariance, MSE**?

$$\mathbf{X}^T\mathbf{X} = 4\mathbf{I}$$

$$\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I} = 4\mathbf{I} + 8\mathbf{I} = 12\mathbf{I}$$

$$\left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}$$

$$= \frac{1}{12}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\hat{\mathbf{h}} = \frac{1}{12}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$$

$$\text{Error covariance} = \sigma^2 \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

$$= 2 \times \frac{1}{12}\mathbf{I} = \frac{1}{6}\mathbf{I}$$

$$\text{MSE} = \text{Tr}\left\{\frac{1}{6}\mathbf{I}\right\} = \frac{1}{3}$$

- **Assignment 6 deadline: 16th November 11:59 PM.**
- **Assignment 5,6 discussion: 18th November 12:30-1:00 PM.**
- **Quiz 3: 19th November 11:45-12:30 PM.**
- **Live Interaction #7: 19th November 12:40-1:30 PM.**