EE910: Digital Communication Systems-I

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Lecture #6C: Optimal Detection And Error Probability For Orthogonal, Bi-orthogonal and Simplex Signalling

Optimal Detection And Error Probability For Orthogonal Signalling

- Orthogonal, biorthogonal, and simplex signalling is characterized by high dimensional constellations.
- These signaling schemes are more power efficient but less bandwidth-efficient than ASK, PSK, and QAM.
- In an equal-energy orthogonal signaling scheme, $N = M$ and the vector representation of the signals is given by

$$\begin{aligned} s_1 &= (\sqrt{\mathcal{E}}, 0, ..., 0) \qquad (1) \\ s_2 &= (0, \sqrt{\mathcal{E}}, ..., 0) \\ &\vdots \\ s_M &= (0, ..., 0, \sqrt{\mathcal{E}}) \end{aligned}$$

Optimal Detection And Error Probability For Orthogonal Signalling

- For equiprobable, equal-energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross-correlation between the received vector **r** and each of the M possible transmitted signal vectors $\{s_m\}$, i.e.,

$$\hat{m} = \underset{1 \leq m \leq M}{\arg\max}\ \mathbf{r}.s_m \qquad (2)$$

- By symmetry of the constellation and by observing that the distance between any pair of signal points in the constellation is equal to $\sqrt{2\mathcal{E}}$ we conclude that the error probability is independent of the transmitted signal.
- Therefore, to evaluate the probability of error, we can suppose that the signal s_1 is transmitted.

Optimal Detection And Error Probability For Orthogonal Signalling

- With this assumption, the received signal vector is

$$\mathbf{r} = (\sqrt{\mathcal{E}} + n_1, n_2, n_3, ..., n_M) \tag{3}$$

where $\sqrt{\mathcal{E}}$ denotes the symbol energy and $n_1, n_2, ..., n_M$ are zero-mean, mutually statistically independent Gaussian random variables with equal variance $\sigma_n^2 = \frac{1}{2}N_0$.

- Let us define random variables $R_m, 1 \leq m \leq M$ as

$$R_m = \mathbf{r}.s_m \tag{4}$$

- With this definition and from equations (3) and (1), we have

$$\begin{aligned} R_1 &= \mathcal{E} + \sqrt{\mathcal{E}} n_1 \\ R_m &= \sqrt{\mathcal{E}} n_m, \quad 2 \leq m \leq M \end{aligned} \tag{5}$$

Optimal Detection And Error Probability For Orthogonal Signalling

- Since we are assuming that s_1 was transmitted, the detector makes a correct decision if $R_1 > R_m$ for $m = 2, 3, ..., M$.
- Therefore, the probability of a correct decision is given by

$$\begin{aligned} P_c &= P[R_1 > R_2, R_1 > R_3, ..., R_1 > R_M | s_1 \ sent] \\ &= P[\sqrt{\mathcal{E}} + n_1 > n_2, \sqrt{\mathcal{E}} + n_1 > n_3, ..., \sqrt{\mathcal{E}} + n_1 > n_M | s_1 \ sent] \end{aligned} \tag{6}$$

Optimal Detection And Error Probability For Orthogonal Signalling

- Events $\sqrt{\mathcal{E}} + n_1 > n_2, \sqrt{\mathcal{E}} + n_1 > n_3, ..., \sqrt{\mathcal{E}} + n_1 > n_M$ are not independent due to the existence of the random variable n_1 in all of them.
- We can however, condition on n_1 to make these events independent. Therefore,we have

$$\begin{aligned} P_c &= \int_{-\infty}^{\infty} P[n_2 < n + \sqrt{\mathcal{E}}, n_3 < n + \sqrt{\mathcal{E}}, ..., n_M < n + \sqrt{\mathcal{E}} | s_1 \ sent, \ \ n_1 = n] p_{n_1}(n) dn \\ &= \int_{-\infty}^{\infty} \left(P[n_2 < n + \sqrt{\mathcal{E}} | s_1 \ sent, \ \ n_1 = n] \right)^{M-1} p_{n_1}(n) dn \end{aligned} \tag{7}$$

- In the last step we have used the fact that $n_m's$ are i.i.d. random variables for $m = 2, 3, ..., M$.

Optimal Detection And Error Probability For Orthogonal Signalling

- We have

$$P[n_2 < n + \sqrt{\mathcal{E}} | s_1 \ sent, n_1 = n] = 1 - Q\left(\frac{n + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \tag{8}$$

- Hence,

$$P_c = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \left[1 - Q\left(\frac{n + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} e^{-\frac{n^2}{N_0}} dn \tag{9}$$

and

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - (1 - Q(x))^{M-1}] e^{-\frac{(x - \sqrt{\frac{2\mathcal{E}}{N_0}})^2}{2}} dx \tag{10}$$

where $x = \frac{n + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}}$

Optimal Detection And Error Probability For Orthogonal Signalling

- In orthogonal signaling, due to the symmetry of the constellation, the probabilities of receiving any of the messages $m = 2, 3, ..., M$ when s_1 is transmitted, are equal.
- Therefore, for any $2 \leq m \leq M$,

$$P[s_m received|s_1 \ sent] = \frac{P_e}{M-1} = \frac{P_e}{2^k - 1} \tag{11}$$

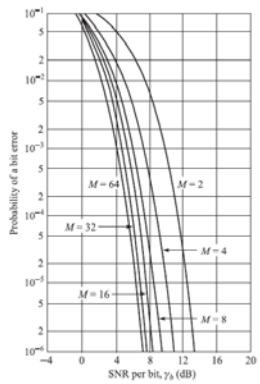
Optimal Detection And Error Probability For Orthogonal Signalling

- Let us assume that s_1 corresponds to a data sequence of length k with a 0 at the first component.
- The probability of an error at this component is the probability of detecting an s_m corresponding to a sequence with a 1 at the first component.
- Since there are 2^{k-1} such sequences, we have

$$P_b = 2^{k-1}\frac{P_e}{2^k - 1} = \frac{2^{k-1}}{2^k - 1}P_e \approx \frac{1}{2}P_e \tag{12}$$

where the last approximation is valid for $k \gg 1$

Probability of a binary digit error as a function of the SNR per bit for $M = 2, 4, 8, 16, 32, 64$



- This figure illustrates that, by increasing the number M of waveforms, one can reduce the SNR per bit required to achieve a given probability of a bit error.

Optimal Detection And Error Probability For FSK Signalling

- FSK signaling is a special case of orthogonal signaling when the frequency separation Δf is given by

$$\Delta f = \frac{l}{2T} \tag{13}$$

for a positive integer l.
- For this value of frequency separation the error probability of M-ary FSK is given by Equation (10).

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - (1 - Q(x))^{M-1}] e^{-\frac{(x - \sqrt{\frac{2\mathcal{E}}{N_0}})^2}{2}} dx \tag{14}$$

Optimal Detection And Error Probability For FSK Signalling

- Note that in the binary FSK signaling, a frequency separation that guarantees orthogonality does not minimize the error probability.
- The error probability of binary FSK is minimized when the frequency separation is of the form

$$\Delta f = \frac{0.715}{T} \tag{15}$$

Optimal Detection And Error Probability For FSK Signalling

- For binary signals, we know the probability of error can be expressed in terms of the distance d_{12} between the signal points, as

$$P_e = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right] \tag{16}$$

where the distance between the two points is given by

$$d_{12}^2 = 2\mathcal{E}_b(1-\rho) \tag{17}$$

where $\mathcal{E}_b$ is energy per information bit, and ρ correlation between two signals.

Optimal Detection And Error Probability For FSK Signalling

- The correlation between two signals in binary FSK is given by

$$\rho = \frac{\sin(2\pi\Delta f T)}{2\pi\Delta f T} \tag{18}$$

- To find the minimum value of the correlation, we set the derivative of ρ with respect to Δf equal to zero, hence we get

$$\frac{\partial \rho}{\partial \Delta f} = 0 = \frac{\cos(2\pi\Delta f T)2\pi T}{2\pi\Delta f T} - \frac{\sin(2\pi\Delta f T)}{(2\pi\Delta f T)^2}2\pi T \tag{19}$$

and therefore

$$2\pi\Delta f T = \tan(2\pi\Delta f T) \tag{20}$$

Optimal Detection And Error Probability For FSK Signalling

- Solving numerically Equation (20) of the form $x = \tan(x)$, we get $x = 4.4934$.
- Thus,

$$2\pi\Delta f T = 4.4934 \implies \Delta f = \frac{0.7151}{T} \tag{21}$$

and corresponding value of $\rho = -0.2172$.
- Putting this value of ρ in Equation (17), and using Equation (16), we get

$$P_e = Q\left[\sqrt{\frac{2\mathcal{E}_b(1-\rho)}{2N_0}}\right] = Q\left[\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}}\right] \tag{22}$$

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Optimal Detection and Error Probability for Biorthogonal Signalling

- A set of $M = 2^k$ biorthogonal signals is constructed from $\frac{1}{2}M$ orthogonal signals by including the negatives of the orthogonal signals.
- Thus, we achieve a reduction in the complexity of the demodulator for the biorthogonal signals relative to that for orthogonal signals, since the former is implemented with $\frac{1}{2}M$ cross-correlators or matched filters, whereas the latter requires M matched filters, or cross-correlators.

Optimal Detection and Error Probability for Biorthogonal Signalling

- In biorthogonal signaling $N = \frac{1}{2}M$, and the vector representation for signals are given by

$$\begin{aligned} s_1 &= -s_{N+1} = (\sqrt{\mathcal{E}}, 0, ..., 0) \qquad (23) \\ s_2 &= -s_{N+2} = (0, \sqrt{\mathcal{E}}, ..., 0) \\ &\vdots \\ s_N &= -s_{2N} = (0, ..., 0, \sqrt{\mathcal{E}}) \end{aligned}$$

Optimal Detection and Error Probability for Biorthogonal Signalling

- To evaluate the probability of error for the optimum detector, let us assume that the signal $s_1(t)$ corresponding to the vector $s_1 = (\sqrt{\mathcal{E}}, 0, ..., 0)$ was transmitted.
- Then the received signal vector is

$$\mathbf{r} = (\sqrt{\mathcal{E}} + n_1, n_2, ..., n_N) \tag{24}$$

where the $\{n_m\}$ are zero-mean, mutually statistically independent and identically distributed Gaussian random variables with variance $\sigma_n^2 = \frac{1}{2}N_0$

Optimal Detection and Error Probability for Biorthogonal Signalling

- Since all signals are equiprobable and have equal energy, the optimum detector decides in favour of the signal corresponding to the largest in magnitude of the cross-correlators.

$$C(\mathbf{r}, s_m) = \mathbf{r}.s_m, \quad 1 \le m \le \frac{1}{2}M \tag{25}$$

while the sign of this largest term is used to decide whether $s_m(t)$ or $-s_m(t)$ was transmitted.

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Optimal Detection and Error Probability for Biorthogonal Signalling

- According to this decision rule the probability of a correct decision is equal to the probability that $r_1 = \sqrt{\mathcal{E}} + n_1 > 0$ and r_1 exceeds $|r_m| = |n_m|$ for $m = 2, 3, ..., \frac{1}{2}M$.
- But

$$\begin{aligned} P[|n_m| < r_1 | r_1 > 0] &= \frac{1}{\sqrt{\pi N_0}} \int_{-r_1}^{r_1} e^{-x^2/N_0} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{r_1}{\sqrt{N_0/2}}}^{\frac{r_1}{\sqrt{N_0/2}}} e^{-x^2/2} dx \end{aligned} \tag{26}$$

Optimal Detection and Error Probability for Biorthogonal Signalling

- Then the probability of a correct decision is

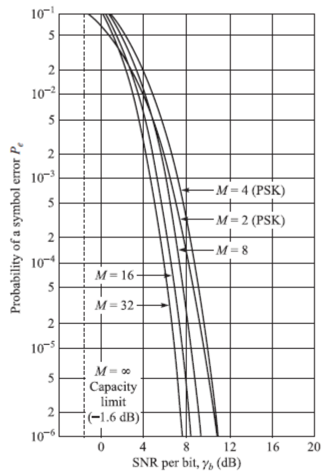
$$P_c = \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\frac{r_1}{\sqrt{N_0/2}}}^{\frac{r_1}{\sqrt{N_0/2}}} e^{-x^2/2} dx \right)^{M/2-1} p(r_1) dr_1 \tag{27}$$

- Upon substitution for p(r_1), we obtain

$$P_c = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2\mathcal{E}/N_0}}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-(\nu+\sqrt{2\mathcal{E}/N_0})}^{(\nu+\sqrt{2\mathcal{E}/N_0})} e^{-x^2/2} dx \right)^{M/2-1} e^{-\frac{\nu^2}{2}} d\nu \tag{28}$$

where we have used the PDF of r_1 as a Gaussian random variable with mean equal to $\sqrt{\mathcal{E}}$ and variance $\frac{1}{2}N_0$.
- Finally, the probability of a symbol error $P_e = 1 - P_c$.
- P_c and hence P_e may be evaluated numerically for different values of M from Equation (28)

P_e as a function of $\mathcal{E}_b/N_0$, where $\mathcal{E} = k\mathcal{E}_b$,for $M = 2, 3, 8, 16, 32$



Optimal Detection and Error Probability for Biorthogonal Signalling

- We observe that this graph is similar to that for orthogonal signals.
- However, in this case the probability of error for $M = 4$ is greater than that for $M = 2$
- This is due to the fact that we have plotted the symbol error probability P_e in the above graph
- If we plotted the equivalent bit error probability, we should find that the graphs for $M = 2$ and $M = 4$ coincide.
- As in the case of orthogonal signals, as $M \to \infty$ or $(k \to \infty)$, the minimum required $\mathcal{E}_b/N_0$ to achieve an arbitrarily small probability of error is -1.6dB, the Shannon limit.

Optimal Detection and Error Probability for Simplex Signalling

- Simplex signals are obtained from a set of orthogonal signals by shifting each signal by the average of the orthogonal signals
- Since the signals of an orthogonal signal are simply shifted by a constant vector to obtain the simplex signals, the geometry of the simplex signal, i.e., the distance between signals and the angle between lines joining signals, is exactly the same as that of the original orthogonal signals.
- Therefore, the error probability of a set of simplex signals is given by the same expression as the expression derived for orthogonal signals.
- However, since simplex signals have a lower energy, the energy in the expression for error probability should be scaled accordingly.

Optimal Detection and Error Probability for Simplex Signalling

- Therefore, the expression for the error probability in simplex signaling becomes

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - (1 - Q(x))^{M-1}\right] e^{-\frac{\left(x - \sqrt{\frac{M}{M-1} \frac{2\mathcal{E}}{N_0}}\right)^2}{2}} dx \quad (29)$$

- This indicates a relative gain of $10 \log \frac{M}{M-1}$ over orthogonal signaling.
- For simplex signals, similar to orthogonal and biorthogonal signals, the error probability decreases as M increases.