Eigenvalues and Eigenvectors

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed the concept of determinants in last lecture
- Discuss the concept of eigen values and eigenvectors today
    - Chapters 6.1, 6.2 of the book

Eigenvalues and Eigenvectors

- Almost all vectors change direction when they are multiple by A i.e., $A\mathbf{x} = \mathbf{y}$
- Certain exceptional vectors $\mathbf{x}$ are in the same direction as Ax

$$A\mathbf{x} = \lambda\mathbf{x}$$

- $\mathbf{x}$ is an eigenvector and λ is eigenvalue
  - λ tell us whether the eigenvector is shrunk or stretched or left unchanged
- When A is squared, eigenvectors stay same. The eigenvalues are squared

$$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow AA\mathbf{x} = \lambda A\mathbf{x} \Rightarrow A^2\mathbf{x} = \lambda^2\mathbf{x}$$

Calculation of eigenvalues

- We have $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$
  - Eigenvectors make up the nullspace of $A - \lambda I$
- Eigenvalues: Number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular

$$\det(A - \lambda I) = 0$$

- If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$ then

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda.$$

- So eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$

Calculation of eigenvectors

- There is nothing exceptional about $\lambda = 0$
  - Like every other number zero might be an eigenvalue and it might be not
- If $\lambda = 0$ then A is singular, and nullspace contains eigenvectors corresponding to $\lambda = 0$
- If $\lambda \neq 0$ then A is invertible, and we shift A by λI to make it singular
- Now find the eigenvectors: solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ separately for $\lambda = 0, 5$

$$(A - 0I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$(A - 5I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Determinant and Trace

- Sum of the n eigenvalues equals the sum of the n diagonal entries

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = trace = a_{11} + a_{22} + \cdots + a_{nn}$$

- Product of the n eigenvalues equals the determinant
- Imaginory eigenvalues: $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$
- They lead to complex eigenvectors

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = -i \begin{bmatrix} i \\ 1 \end{bmatrix}$$

- Real matrices can easily have complex eigenvalues and eigenvectors
- Eigenvalues i and $-i$ also illustrate two special properties of Q:
  - Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$
  - Q is a skew-symmetric matrix ($Q^T = -Q$)so each λ is pure imaginary

Eigenvalues of AB and A+ B

- If A and B have eigenvalues λ and β , then what about eigenvalue of AB?

$$AB\mathbf{x} = A\beta\mathbf{x} = \beta A\mathbf{x} = \beta\lambda\mathbf{x}$$

- Above proof is false
- Mistake is to assume that A and B have same eigenvector $\mathbf{x}$
    - Usually they don't - eigenvectors of A are not generally eigenvectors of B
- For the same reason, the eigenvalues of $A + B$ are generally not $\lambda + \beta$
- False proof suggest what is true
    - Suppose $\mathbf{x}$ is really an eigenvector for both A and B, then we do have

$$AB\mathbf{x} = \lambda\beta\mathbf{x} \text{ and } BA\mathbf{x} = \beta\lambda\mathbf{x}$$

- A and B share the same n independent eigenvectors if and only if $AB = BA$

Diagonalizing a Matrix (1)

- Suppose $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$
- Put them into the columns of an eigenvector matrix X

A times X $$AX = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}$$

- Trick is to split this matrix in RHS

X times Λ $$\begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = X\Lambda.$$

- We have $AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$

Diagonalizing a Matrix (2)

- Matrix X has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent
- Without n independent eigenvectors, we can't diagonalize
- Some matrices have too few eigenvectors, they cannot be diagonalized
  - One example $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$
  - Its eigenvalues are 0 and 0. Nothing is special about $\lambda = 0$
  - Problem is repetition of λ, eigenvectors of A are multiples of $(1, 1)$
  - There is no second eigenvector, so A cannot be diagonalized
- Remember there is no connection between invertibility and diagonalizability
  - Invertibility is concerned with the eigenvalues $\lambda = 0$ or $\lambda \neq 0$
  - Diagonalizability is concerned with eigenvectors (too few or enough for X)

Diagonalizing a Matrix (3)

- Eigenvectors corresponding to distinct eigenvalues are linearly independent

$$\begin{aligned} \text{Proof: Let } c_1\mathbf{x}_1 + c_2\mathbf{x}_2 &= \mathbf{0} \\ c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 &= \mathbf{0} \Rightarrow c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 = \mathbf{0} \qquad (1) \\ c_1\lambda_2\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 &= \mathbf{0} \qquad (2) \end{aligned}$$

- Subtracting (1)-(2) we have $c_1(\lambda_1 - \lambda_2)\mathbf{x}_1 = \mathbf{0}$. Therefore $c_1 = 0$
- Similarly, $c_2 = 0$, so $\mathbf{x}_1$ and $\mathbf{x}_2$ should be linearly independent
- An $n \times n$ matrix that has n different eigenvalues, must be diagonalizable