

eMasters in Communication Systems

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Elective Module:

**Estimation for Wireless
Communication**



Chapter 11

MMSE/LMMSE

MIMO Estimation and Receivers



MIMO

$r = \#$ Receive antennas.
 $t = \#$ Transmit antennas.

- MIMO system model is given as

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}_{\substack{\bar{\mathbf{y}}(k) \\ \text{output} \\ \text{vector}}} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix}}_{\substack{\mathbf{H} \\ \text{channel} \\ \text{matrix}}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\substack{\bar{\mathbf{x}}(k) \\ \text{input} \\ \text{vector}}} + \underbrace{\begin{bmatrix} v_1(k) \\ v_2(k) \\ \vdots \\ v_r(k) \end{bmatrix}}_{\substack{\bar{\mathbf{v}}(k) \\ \text{noise} \\ \text{vector}}}$$

$r \times 1$

$r \times t$

$t \times 1$

$r \times 1$

MIMO

- MIMO system model can be represented in the **compact fashion**

$$\bar{y}(k) = H \bar{x}(k) + \bar{v}(k) .$$



MIMO

- MIMO system model can be represented in the compact fashion

$$\bar{\mathbf{y}}(k) = \mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{v}}(k)$$



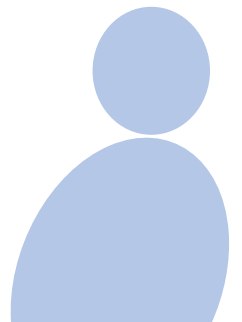
MIMO

- Consider the transmission of N pilot vectors

$$\bar{\mathbf{y}}(1) = \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1)$$

$$\bar{y}(z) = H \bar{x}(z) + \bar{v}(z)$$

$$\bar{y}(N) = H \bar{x}(N) + \bar{v}(N).$$



MIMO

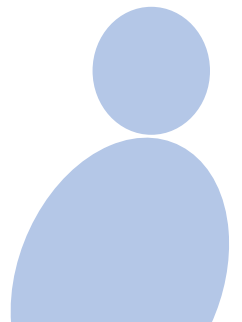
- Consider the transmission of N pilot vectors

$$\bar{\mathbf{y}}(1) = \mathbf{H}\bar{\mathbf{x}}(1) + \bar{\mathbf{v}}(1)$$

$$\bar{\mathbf{y}}(2) = \mathbf{H}\bar{\mathbf{x}}(2) + \bar{\mathbf{v}}(2)$$

\vdots

$$\bar{\mathbf{y}}(N) = \mathbf{H}\bar{\mathbf{x}}(N) + \bar{\mathbf{v}}(N)$$



MIMO

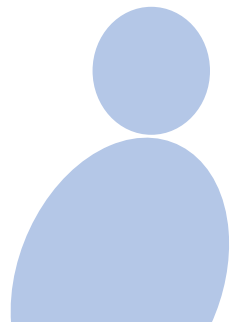
- We can concatenate them as

$$\begin{bmatrix} \bar{y}(1) & \bar{y}(2) & \dots & \bar{y}(N) \end{bmatrix}$$

$$Y \sim r \times N.$$

$Y \sim$ output matrix
 $X \sim$ Pilot matrix.

$$= H \begin{bmatrix} \bar{x}(1) & \bar{x}(2) & \dots & \bar{x}(N) \end{bmatrix} + \begin{bmatrix} \bar{v}(1) & \bar{v}(2) & \dots & \bar{v}(N) \end{bmatrix}$$
$$X \sim t \times N$$
$$V \sim r \times N.$$

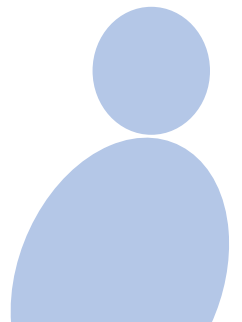


MIMO

$$Y \sim r \times N \quad X \sim t \times N \\ V \sim r \times N$$

- We can concatenate them as

$$\begin{aligned} & [\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \dots \quad \bar{\mathbf{y}}(N)] \\ & \quad \underbrace{\hspace{10em}}_{\mathbf{Y}} \\ &= \mathbf{H} [\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \dots \quad \bar{\mathbf{x}}(N)] \\ & \quad \underbrace{\hspace{10em}}_{\mathbf{X}} \\ &+ [\bar{\mathbf{v}}(1) \quad \bar{\mathbf{v}}(2) \quad \dots \quad \bar{\mathbf{v}}(N)] \\ & \quad \underbrace{\hspace{10em}}_{\mathbf{V}} \end{aligned}$$



MIMO Estimation Model

- This can be represented in the compact fashion

$$Y = HX + N.$$



MIMO Estimation Model

- This can be represented in the compact fashion

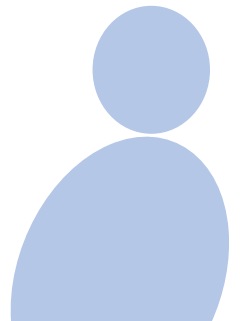
$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{V}$$

MIMO channel Estimation model.

$$\text{Pinv of } \mathbf{X} = \mathbf{X}^T (\mathbf{X}\mathbf{X}^T)^{-1} \\ \mathbf{X}^H (\mathbf{X}\mathbf{X}^H)^{-1}$$

Widematrix.

$t \times N$



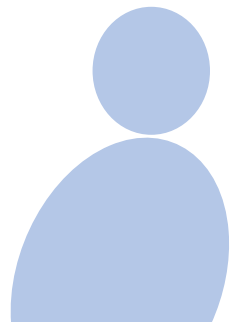
ML MIMO Estimate

- The ML MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T (\mathbf{X}\mathbf{X}^T)^{-1}$$

ML channel estimate

min. $\|\mathbf{Y} - \mathbf{H}\mathbf{X}\|_F^2$ ← Frobenius norm.



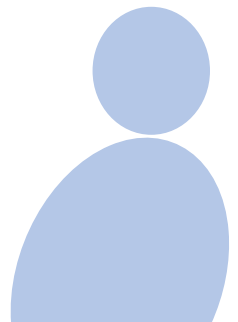
LMMSE MIMO Estimate

- The LMMSE MIMO channel estimate is *Linear minimum mean square error estimate*

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T \left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$

$$SNR = \frac{\sigma_h^2}{\sigma^2}$$

*SNR $\rightarrow \infty$
LMMSE \rightarrow ML*



MIMO Estimation Example

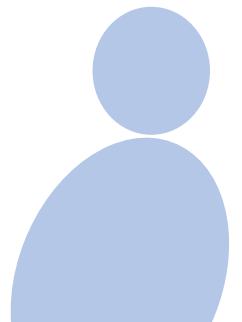
- Consider the MIMO channel estimation problem with pilot vectors

$$\bar{\mathbf{x}}(1) = [3 \quad -2]^T, \bar{\mathbf{x}}(2) = [-2 \quad 3]^T$$
$$\bar{\mathbf{x}}(3) = [4 \quad 2]^T, \bar{\mathbf{x}}(4) = [2 \quad 2]^T$$

$$N = 4$$

- What is the pilot matrix? $N=4 \quad t=2$
 2×4

$$X = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}$$



MIMO Estimation Example

- The pilot matrix is

$$\begin{aligned} \mathbf{X} &= [\bar{\mathbf{x}}(1) \quad \bar{\mathbf{x}}(2) \quad \bar{\mathbf{x}}(3) \quad \bar{\mathbf{x}}(4)] \\ &= \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix} \end{aligned}$$



MIMO Estimation Example

- The output vectors are

$$\bar{\mathbf{y}}(1) = [-2, 1, -3]^T,$$

$$\bar{\mathbf{y}}(2) = [-1, 3, 3]^T,$$

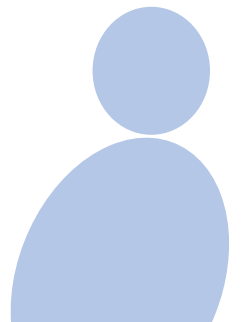
$$\bar{\mathbf{y}}(3) = [-1, -2, 2]^T,$$

$$\bar{\mathbf{y}}(4) = [-3, -1, 1]^T$$

$r=3$
 $r \times t = 3 \times 2$ MIMO system. 3×4

- What is the **output matrix**?

$$\mathbf{Y} = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}$$



MIMO Estimation Example

- The output matrix is

$$\mathbf{Y} = [\bar{\mathbf{y}}(1) \quad \bar{\mathbf{y}}(2) \quad \bar{\mathbf{y}}(3) \quad \bar{\mathbf{y}}(4)]$$
$$= \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}$$



MIMO Estimation Example

- The LMMSE channel estimate is given as follows

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T \left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$



MIMO Estimation Example

- Let us first evaluate

$$= \underbrace{\begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}}_X \underbrace{\begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}}_{X^T} = \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}$$

Handwritten red annotations: A red 'X' under the first matrix and a red 'X^T' under the second matrix.



MIMO Estimation Example

Let $SNR = -6 \text{ dB} = \frac{1}{4}$

$$\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR} \mathbf{I} \right)^{-1} = \left(\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4\mathbf{I} \right)^{-1}$$

$$= \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 37 & 0 \\ 0 & 25 \end{bmatrix}$$



MIMO Estimation Example

Let $SNR = -6 \text{ dB}$

$$\left(\mathbf{X}\mathbf{X}^T + \frac{1}{SNR} \mathbf{I} \right)^{-1} = \left(\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4\mathbf{I} \right)^{-1}$$

$$= \begin{bmatrix} \frac{1}{37} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$$



MIMO Estimation Example

- Let us now evaluate

$$= \underbrace{\begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}}_Y \underbrace{\begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}}_{X^T} = \overbrace{YX^T}^{YX^T} = \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix}$$

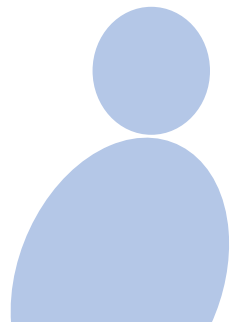


MIMO Estimation Example

- Finally, the MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T \left(\mathbf{X}\mathbf{X}^T + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1}$$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} \frac{1}{37} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} -\frac{14}{37} & -\frac{7}{25} \\ -\frac{13}{37} & \frac{1}{25} \\ -\frac{5}{37} & \frac{21}{25} \end{bmatrix}$$



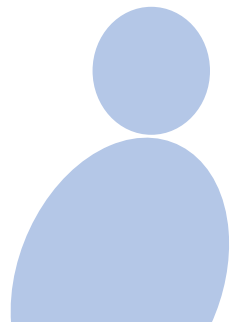
MIMO Estimation Example

- Finally, the MIMO channel estimate is

$$\hat{\mathbf{H}} = \mathbf{Y}\mathbf{X}^T \left(\mathbf{X}\mathbf{X}^T + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1}$$
$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} \frac{1}{37} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} -\frac{14}{37} & -\frac{7}{25} \\ -\frac{13}{37} & \frac{1}{25} \\ -\frac{5}{37} & \frac{21}{25} \end{bmatrix}$$

LMMSE MIMO
channel estimate.

MIMO Receivers



MIMO Receivers

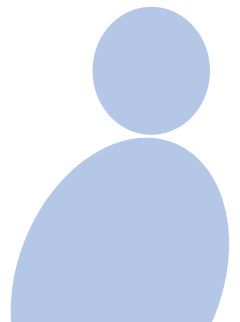
- Consider the MIMO model

$$\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{v}}$$

~~$\bar{\mathbf{y}}(k) = \mathbf{H}\bar{\mathbf{x}}(k) + \bar{\mathbf{v}}(k)$~~

Dropped index k .

- Time index k dropped for simplicity



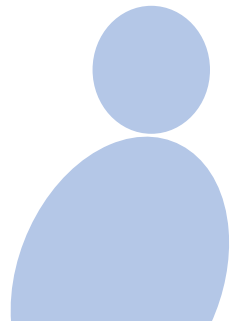
MIMO Receivers

$$\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{v}}$$

- Given $\bar{\mathbf{y}}$, $\hat{\mathbf{x}} = ?$ We know \mathbf{H} .
- How to determine $\bar{\mathbf{x}}$ given $\bar{\mathbf{y}}$?

$\hat{\mathbf{x}} = ?$

What is estimate
of transmit vector?



MIMO ZF Receiver

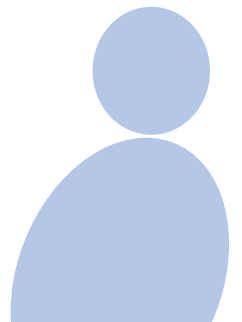
Least Squares Receiver

$$\min \|\bar{\mathbf{y}} - \mathbf{H}\bar{\mathbf{x}}\|^2$$

- This is termed the **least-squares problem**.

$$\hat{\mathbf{x}} = (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \bar{\mathbf{y}}$$

Zero Forcing (ZF) Receiver



MIMO Receiver

$$\hat{\mathbf{x}} = \underbrace{(\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H}_{\text{pseudo-inverse of } \mathbf{H}} \bar{\mathbf{y}}$$

- This is termed as the **zero-forcing (ZF) Receiver**.

$(\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H$ is pseudoinverse of \mathbf{H} .
 \mathbf{H} is tall matrix
 $r \geq t$



MIMO ZF Receiver Example

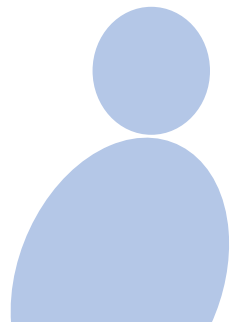
Example:

Consider

$$\bar{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

4 x 2
r=4 Receive antennas -
t=2 transmit antennas

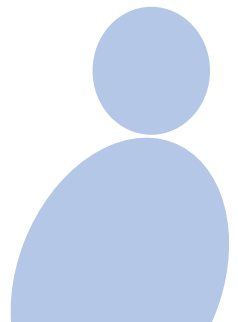
What is $\hat{\mathbf{x}}$?



MIMO ZF Receiver Example

The **ZF estimate** can be calculated as follows

$$\mathbf{H}^T \mathbf{H} = \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}}^{\mathbf{H}^T} \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}^{\mathbf{H}}$$
$$= \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$



MIMO ZF Receiver Example

$$\mathbf{H}^T \mathbf{H} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(\mathbf{H}^T \mathbf{H})^{-1} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$



MIMO ZF Receiver Example

$$\mathbf{H}^T \mathbf{H} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(\mathbf{H}^T \mathbf{H})^{-1} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$



MIMO ZF Receiver Example

$$(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T = \overbrace{\frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}}^{(\mathbf{H}^T \mathbf{H})^{-1}} \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}}^{\mathbf{H}^T}$$

$$= \underbrace{\begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{10} & -\frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix}}_{(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T}$$



MIMO ZF Receiver Example

$$\begin{aligned}\hat{\mathbf{x}} &= (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \bar{\mathbf{y}} \\ &= \begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 3 & 1 & 1 & 3 \\ -\frac{3}{10} & -\frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}}_{\bar{\mathbf{y}}} \\ &= \begin{bmatrix} \frac{3}{2} \\ 2 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \hat{\mathbf{x}} = \text{Zero Forcing Estimate}\end{aligned}$$

*ZF Receiver
Linear Receiver*

MIMO ZF Receiver Example

- Therefore, the **ZF estimate** is

$$\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

Output of ZF Receiver.

LMMSE Receiver

- Another popular MIMO receiver is the LMMSE Receiver.

ML: Deterministic
MMSE/LMMSE: Random -



LMMSE

$$\min E\{\|\underbrace{\mathbf{C}^H}_{\text{red wavy line}} \bar{\mathbf{y}} - \bar{\mathbf{x}}\|^2\}$$

$$\hat{\mathbf{x}} = \mathbf{C}^H \bar{\mathbf{y}}$$

Estimate is Linear Transformation
of output.



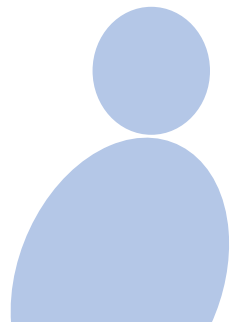
LMMSE Receiver

$$\begin{aligned} E\{x(i)\} &= 0 \\ E\{|x(i)|^2\} &= P \end{aligned}$$

- Consider the symbols to be i.i.d. mean 0, power P .

P = power of symbols -

$$E\{\bar{\mathbf{x}}\bar{\mathbf{x}}^H\} = P\mathbf{I}$$



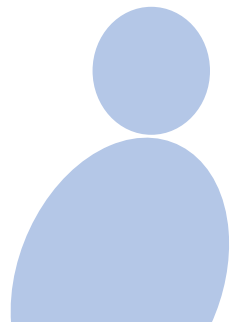
LMMSE Receiver

- The LMMSE Receiver is

$$\hat{\mathbf{x}} = \left(\mathbf{H}^H \mathbf{H} + \frac{1}{SNR} \mathbf{I} \right)^{-1} \mathbf{H}^H \bar{\mathbf{y}}$$

$$SNR = \frac{P}{N_0}$$

$$\zeta_{NR} = \frac{P}{N_0}$$



MIMO LMMSE Receiver Example

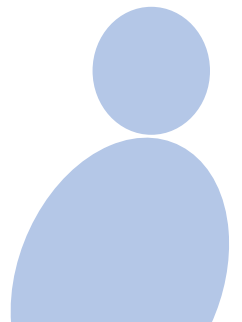
Example:

Consider

$$\bar{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Handwritten red annotations: A bracket above the matrix H is labeled "4x2".

What is $\hat{\mathbf{x}}$ when $SNR = -3dB = \frac{1}{2}$



MIMO LMMSE Receiver Example

The **LMMSE estimate** can be calculated as follows

$$\mathbf{H}^T \mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4 \mathbf{I}$$



MIMO LMMSE Receiver Example

- $SNR = -3 \text{ dB} \approx \frac{1}{2}$

$$\mathbf{H}^T \mathbf{H} + \frac{1}{SNR} \mathbf{I} = 4\mathbf{I} + 2\mathbf{I} = 6\mathbf{I}$$
$$= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$



MIMO LMMSE Receiver Example

- $SNR = -3 \text{ dB} \approx \frac{1}{2}$

$$\mathbf{H}^T \mathbf{H} + \frac{1}{SNR} \mathbf{I} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$



MIMO LMMSE Receiver Example

$$\mathbf{H}^T \mathbf{H} + \frac{1}{SNR} \mathbf{I} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\left(\mathbf{H}^T \mathbf{H} + \frac{1}{SNR} \mathbf{I} \right)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



MIMO LMMSE Receiver Example

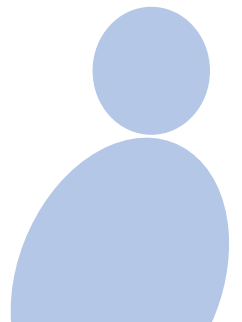
$$\mathbf{H}^T \mathbf{H} + \frac{1}{SNR} \mathbf{I} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\left(\mathbf{H}^T \mathbf{H} + \frac{1}{SNR} \mathbf{I} \right)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



MIMO LMMSE Receiver Example

$$\begin{aligned} \left(\mathbf{H}^T \mathbf{H} + \frac{1}{\text{SNR}} \mathbf{I} \right)^{-1} \mathbf{H}^T &= \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \\ &\quad \underbrace{\hspace{10em}}_{\mathbf{H}^T} \end{aligned}$$



MIMO LMMSE Receiver Example

$$\hat{\mathbf{x}} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \bar{\mathbf{y}}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

$\bar{\mathbf{y}}$ LMMSE Receiver output.

$$= \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \hat{\mathbf{x}}_{\text{LMMSE}}$$

MIMO LMMSE Receiver Example

- Therefore, the **LMMSE estimate** is

$$\hat{\mathbf{x}} = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

LMMSE Receiver

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Three options provided below for the font size.

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