EE910: Digital Communication Systems-I

Adrish Banerjee

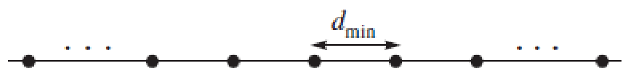
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

May 16, 2022

Lecture #6A: Optimal Detection and Error Probability for ASK or PAM and PSK Signalling

Optimal Detection and Error Probability for ASK or PAM Signalling

- The constellation for an ASK Signalling scheme is shown as



- In this constellation the minimum distance between any two points is d_{min} which is given by

$$d_{min} = \sqrt{\frac{12 \log_2 M}{M^2 - 1} \mathcal{E}_{bavg}} \qquad (1)$$

Optimal Detection and Error Probability for ASK or PAM Signalling

- The constellation points are located at $\{\pm\frac{1}{2}d_{min}, \pm\frac{3}{2}d_{min}, ..., \pm\frac{M-1}{2}d_{min}\}$
- In this ASK constellation, there are M-2 inner points and 2 outer points.

Optimal Detection and Error Probability for ASK or PAM Signalling

- Let us denote the error probabilities of inner points and outer points by P_{ei} and P_{eo}, respectively.
- Since n is a zero-mean Gaussian random variable with variance $\frac{1}{2}N_0$, we have

$$P_{ei} = P\left[|n| > \frac{1}{2}d_{min}\right] = 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \tag{2}$$

and for outer points

$$P_{eo} = \frac{1}{2}P_{ei} = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \tag{3}$$

Optimal Detection and Error Probability for ASK or PAM Signalling

- The symbol error probability is given by

$$\begin{aligned} P_e &= \frac{1}{M}\sum_{m=1}^{M} P[error|m\ sent] \\ &= \frac{1}{M}\left[2(M-2)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) + 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)\right] \\ &= \frac{2(M-1)}{M}Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \end{aligned} \tag{4}$$

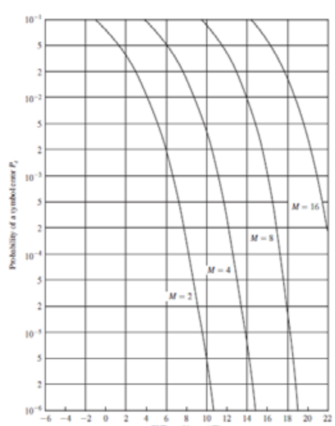
Substituting for d_{min} from Equation (1) we get

$$\begin{aligned} P_e &= 2\left(1-\frac{1}{M}\right)Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \\ &\approx 2Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{\mathcal{E}_{bavg}}{N_0}}\right) \qquad \text{for large } M \end{aligned} \tag{5}$$

Optimal Detection and Error Probability for ASK or PAM Signalling

- The average SNR/bit $\frac{\mathcal{E}_{bavg}}{N_0}$ is scaled by $\frac{6 \log_2 M}{M^2-1}$
- To keep the error probability constant as M increases, the SNR/bit must increase.
- For increasing the transmission rate by 1 bit, one would need 6 dB more power

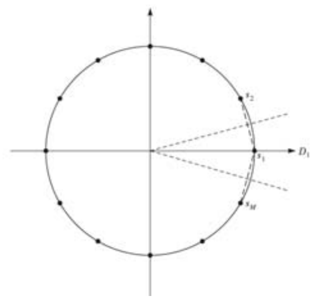
Plots of the error probability of baseband PAM/ASK.



- Increasing M deteriorates the performance, and for large M the distance between curves corresponding to M and 2M is roughly 6 dB.

Optimal Detection and Error Probability for PSK Signalling

- The constellation for an M-ary PSK Signalling is shown below



- In this constellation, the decision region D_1 is also shown.
- The decision regions are based on the minimum-distance detection rule.

Optimal Detection and Error Probability for PSK Signalling

- By symmetry of the constellation, the error probability of the system is equal to the error probability when $s_1 = (\sqrt{\mathcal{E}}, 0)$ is transmitted.
- The received vector $\mathbf{r}$ is given by

$$\mathbf{r} = (r_1, r_2) = (\sqrt{\mathcal{E}} + n_1, n_2) \tag{6}$$

Optimal Detection and Error Probability for PSK Signalling

- It is seen that r_1 and r_2 are independent Gaussian random variables with variance $\sigma^2 = \frac{1}{2}N_0$ and means $\sqrt{\mathcal{E}}$ and 0, respectively; hence

$$p(r_1, r_2) = \frac{1}{\pi N_0} e^{-\frac{(r_1 - \sqrt{\mathcal{E}})^2 + r_2^2}{N_0}} \tag{7}$$

- We introduce polar coordinates transformations of (r_1, r_2) as

$$V = \sqrt{r_1^2 + r_2^2} \tag{8}$$

$$\Theta = arctan \frac{r_2}{r_1} \tag{9}$$

Optimal Detection and Error Probability for PSK Signalling

- The joint PDF of V and Θ can be derived as

$$p_{V,\Theta}(\nu, \theta) = \frac{\nu}{\pi N_0} e^{-\frac{\nu^2 + \mathcal{E} - 2\sqrt{\mathcal{E}}\nu \cos\theta}{N_0}} \tag{10}$$

- Integrating over ν, we derive the marginal PDF of Θ as

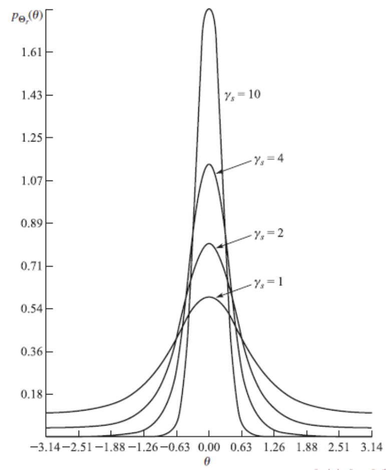
$$\begin{aligned} p_\Theta(\theta) &= \int_0^\infty p_{V,\Theta}(\nu, \theta) d\nu \\ &= \frac{1}{2\pi} e^{-\gamma_s sin^2\theta} \int_0^\infty \nu e^{-\frac{(\nu - \sqrt{2\gamma_s}\cos\theta)^2}{2}} d\nu \end{aligned} \tag{11}$$

- We have defined the symbol SNR or SNR per symbol as

$$\gamma_s = \frac{\mathcal{E}}{N_0} \tag{12}$$

Optimal Detection and Error Probability for PSK Signalling

- This figure illustrates $p_\Theta(\theta)$ for several values of γ_s



Optimal Detection and Error Probability for PSK Signalling

- Note that $p_\Theta(\theta)$ becomes narrower and more peaked about $\theta = 0$ as γ_s increases.
- The decision region D_1 can be described as $D_1 = \{\theta : \frac{-\pi}{M} < \theta \leq \frac{\pi}{M}\}$
- The message error probability is given by

$$P_e = 1 - \int_{-\pi/M}^{\pi/M} p_\Theta(\theta) d\theta \tag{13}$$

Optimal Detection and Error Probability for PSK Signalling

- In general, the integral of $p_\Theta(\theta)$ does not reduce to a simple form and must be evaluated numerically, except for M=2 and M=4.
- For binary phase modulation, the two signals $s_1(t)$ and $s_2(t)$ are antipodal, and hence the error probability is

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \tag{14}$$

Optimal Detection and Error Probability for PSK Signalling

- When M=4, we have two binary phase-modulation signals in phase quadrature. Hence, the bit error probability is identical to Binary phase modulation.
- The symbol error probability for M=4 is determined by noting that

$$P_c = (1 - P_b)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2 \tag{15}$$

where P_c is the probability of a correct decision for the 2-bit symbol.

Optimal Detection and Error Probability for PSK Signalling

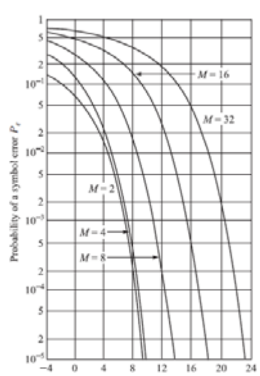
- Therefore, the symbol error probability for M=4 is

$$P_e = 1 - P_c$$
$$= 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right] \quad (16)$$

- For $M > 4$, the symbol error probability P_e is obtained by numerically integrating the equation

$$P_e = 1 - \int_{-\pi/M}^{\pi/M} p_\Theta(\theta) d\theta \quad (17)$$

Error probability as a function of the SNR per bit for $M = 2, 4, 8, 16$, and 32.



- The graph clearly illustrates the penalty in SNR per bit as M increases beyond $M = 4$.

Optimal Detection and Error Probability for PSK Signalling

- For large values of M, doubling the number of phases requires an additional 6 dB/bit to achieve the same performance.
- An approximation to the error probability for large values of M and for large SNR may be obtained by first approximating $p_\Theta(\theta)$.
- For $\frac{\mathcal{E}}{N_0} \gg 1$ and $|\theta| \le \frac{1}{2}\pi$, $p_\Theta(\theta)$ is well approximated as

$$p_\Theta(\theta) \approx \sqrt{\frac{\gamma_s}{\pi}} \cos\theta e^{-\gamma_s sin^2\theta} \tag{18}$$

- By substituting for $p_\theta(\theta)$ in equation $P_e = 1 - \int_{-\pi/M}^{\pi/M} p_\Theta(\theta) d\theta$ and performing the change in variable from θ to $u = \sqrt{\gamma_s}\sin\theta$ we find that

Optimal Detection and Error Probability for PSK Signalling

- Probability of symbol error

$$\begin{aligned} P_e &\approx 1 - \int_{\frac{-\pi}{M}}^{\frac{\pi}{M}} \sqrt{\frac{\gamma_s}{\pi}} \cos\theta e^{-\gamma_s sin^2\theta} d\theta \\ &\approx \frac{2}{\sqrt{\pi}} \int_{\sqrt{2\gamma_s}sin(\pi/M)}^{\infty} e^{-u^2} du \\ &= 2Q\Big(\sqrt{2\gamma_s}sin(\frac{\pi}{M})\Big) \\ &= 2Q\left(\sqrt{(2\log_2 M)sin^2(\frac{\pi}{M})\frac{\mathcal{E}_b}{N_0}}\right) \end{aligned} \tag{19}$$

where we have used the definition of the SNR per bit as

$$\frac{\mathcal{E}_b}{N_0} = \frac{\mathcal{E}}{N_0 \log_2 M} = \frac{\gamma_s}{\log_2 M} \tag{20}$$

Optimal Detection and Error Probability for PSK Signalling

- Note that this approximation to the error probability is good for all values of M.
- For example, when $M = 2$ and $M = 4$, we have $P_e = 2Q(\sqrt{2\gamma_b})$
- For the case when M is large, we can use the approximation $sin\frac{\pi}{M} \approx \frac{\pi}{M}$ to find another approximation to error probability for large M as

$$P_e \approx 2Q\left(\sqrt{\frac{2\pi^2 \log_2 M}{M^2}\frac{\mathcal{E}_b}{N_0}}\right) \quad \textit{for large } M \qquad (21)$$

- From this equation, it is clear that doubling M reduces the effective SNR per bit by 6 dB.

Optimal Detection and Error Probability for PSK Signalling

- The equivalent bit error probability for M-ary PSK is rather tedious to derive due to its dependence on the mapping of k-bit symbols into the corresponding signal phases.
- Since the most probable errors due to noise result in the erroneous selection of an adjacent phase to the true phase, most k-bit symbol errors contain only a single-bit error.
- Hence, the equivalent bit error probability for M-ary PSK is well approximated as

$$P_b \approx \frac{1}{k} P_e \qquad (22)$$