EE910: Digital Communication Systems-I

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Lecture #2A: Review of signals

Signals: Introduction

- A signal, $x(t)$, is defined to be a function of time ($t \in \mathcal{R}$).
- Signals in engineering systems are typically described with five different mathematical classifications:
  - Deterministic or random
  - Energy or power
  - Periodic or aperiodic
  - Complex or real
  - Continuous time or discrete time

Deterministic vs. random signal

- A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time.
- Deterministic signals can be defined exactly by a mathematical formula.
- In contrast, there is uncertainty with respect to the value of a random signal at some instant of time.
- Random signals are modeled in probabilistic terms.

Energy signal

- The energy, E_x, of a signal $x(t)$ is given by

$$E_x = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad (1)$$

- $x(t)$ is called an energy signal when $E_x < \infty$.
- Energy signals are normally associated with finite duration waveforms.

Energy signal

- Example:

$$x(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \qquad E_x = 1 \quad (2)$$

Power signal

- A signal is called a power signal if it does not have finite energy.
- The signal power, P_x, is given by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{3}$$

Note that if $E_x < \infty$, then $P_x = 0$ and if $P_x > 0$, then $E_x = \infty$.

Power signal

- Example:

$$\begin{aligned} x(t) &= \cos(2\pi f_c t) \\ P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos^2(2\pi f_c t)\, dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_c t)\right) dt = \frac{1}{2} \end{aligned} \tag{4}$$

Periodic vs. Aperiodic signal

- A periodic signal is one that repeats itself in time.
- $x(t)$ is a periodic signal when

$$x(t) = x(t + T_0) \quad \forall t \quad \text{and for some } T_0 \neq 0 \tag{5}$$

- The signal period is given by

$$T = \min(|T_0|) \tag{6}$$

- The fundamental frequency is given by

$$f_T = \frac{1}{T} \tag{7}$$

- Most periodic signals are power signals

Periodic vs. Aperiodic signal

- Example:

$$x(t) = \cos(2\pi f_m t) \quad T_0 = \frac{n}{f_m} \quad T = \frac{1}{f_m} \tag{8}$$

- An aperiodic signal is defined to be a signal that is not periodic.

Complex signal vs. real signal

- We define a complex signal and a complex exponential as

$$z(t) = x(t) + jy(t) \quad e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{9}$$

where $x(t)$ and $y(t)$ are both real signals.

- A magnitude ($\alpha(t)$) and phase ($\theta(t)$) representation of a complex signal is also commonly used

$$z(t) = \alpha(t)e^{j\theta(t)} \tag{10}$$

where

$$\alpha(t) = |z(t)| = \sqrt{x^2(t) + y^2(t)} \quad \theta(t) = \arg(z(t)) = \tan^{-1}(y(t), x(t)) \tag{11}$$

- The complex conjugate operation is defined as

$$z^*(t) = x(t) - jy(t) = \alpha(t)e^{-j\theta(t)} \tag{12}$$

Complex signal vs real signal

- Some important formulas for analyzing complex signals are

$$\begin{array}{ll}
|z(t)|^2 = \alpha(t)^2 = z(t)z^*(t) = x^2(t) + y^2(t) & \cos(\theta)^2 + \sin(\theta)^2 = 1 \\
\Re[z(t)] = x(t) = \alpha(t)\cos(\theta(t)) = \frac{1}{2}\left[z(t) + z^*(t)\right] & \cos(\theta) = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right] \\
\Im[z(t)] = y(t) = \alpha(t)\sin(\theta(t)) = \frac{1}{2j}\left[z(t) - z^*(t)\right] & \sin(\theta) = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]
\end{array}$$

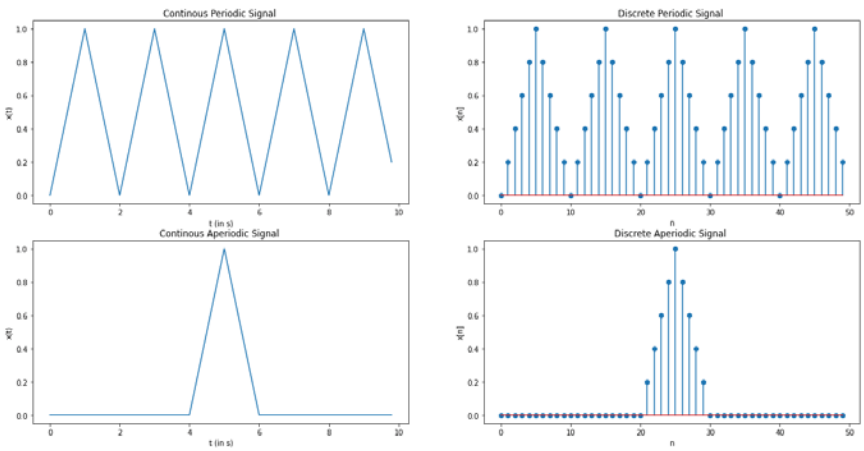
- Example:

$$\exp\left[j2\pi f_m t\right] = \cos\left(2\pi f_m t\right) + j\sin\left(2\pi f_m t\right) \tag{13}$$

Continuous Time Signals vs. Discrete Time Signals

- A signal, $x(t)$, is defined to be a continuous time signal if the domain of the function defining the signal contains intervals of the real line.
- A signal, $x(t)$, is defined to be a discrete time signal if the domain of the signal is a countable subset of the real line.
- Often a discrete signal is denoted by $x(k)$, where k is an integer and a discrete signal often arises from (uniform) sampling of a continuous time signal, e.g., $x(k) = x(kT_s)$, where T_s is the sampling period.

Continuous Time Signals vs. Discrete Time Signals



Fourier Series

- Signal analysis can be completed in either the time or frequency domains.
- If $x(t)$ is periodic with period T, then $x(t)$ can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left[\frac{j2\pi nt}{T}\right] = \sum_{n=-\infty}^{\infty} x_n \exp\left[j2\pi f_T nt\right] \tag{14}$$

where $f_T = 1/T$ and

$$x_n = \frac{1}{T}\int_0^T x(t) \exp\left[-j2\pi f_T nt\right] dt \tag{15}$$

Fourier Series

- Example:

$$x(t) = \cos\left(2\pi f_m t\right) \tag{16}$$

- For this signal $T = 1/f_m$ and the only nonzero Fourier coefficients are $x_1 = 0.5, x_{-1} = 0.5$.
- Therefore

$$x(t) = \frac{1}{2}\exp\left[j2\pi f_T t\right] + \frac{1}{2}\exp\left[-j2\pi f_T t\right] \tag{17}$$

Parseval Theorem

- Parseval's theorem states that the power of a signal can be calculated using either the time or the frequency domain representation of the signal and the two results are identical.

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2 \qquad (18)$$

Parseval Theorem

- Example: For $x(t) = \cos(2\pi f_m t)$ computing the power in the frequency domain, we get

$$P_x = |x_{-1}|^2 + |x_1|^2 = (0.5)^2 + (0.5)^2 = 0.5 \qquad (19)$$

- Similarly, computing the power in the time domain, we get

$$P_x = \frac{1}{T}\int_0^T |\cos(2\pi f_m t)|^2 dt = 0.5 \qquad (20)$$

Fourier Transform

- If $x(t)$ is an energy signal, then the Fourier transform is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \mathcal{F}\{x(t)\} \tag{21}$$

- $X(f)$ is in general complex and gives the frequency domain representation of $x(t)$.
- The inverse Fourier transform is

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \mathcal{F}^{-1}\{X(f)\} \tag{22}$$

Fourier Transform

- Example: The Fourier transform of

$$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{23}$$

is given as

$$X(f) = \int_0^T e^{-j2\pi ft} dt = \left. \frac{\exp[-j2\pi ft]}{-j2\pi f} \right|_0^T = T \exp\left[j\pi fT\right] \operatorname{sinc}(fT) \tag{24}$$

Rayleigh's Energy Theorem

- According to Rayleigh's Energy Theorem, we have

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{25}$$

- Example: For the Fourier transform pair of

$$x(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \quad X(f) = T \exp\left[j\pi f T\right] \operatorname{sinc}(fT) \tag{26}$$

the energy is most easily computed in the time domain

$$E_x = \int_0^T |x(t)|^2 dt = T \tag{27}$$

Correlation Function

- The correlation function of a signal $x(t)$ is

$$V_x(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt \tag{28}$$

- Three important characteristics of the correlation function are
  - $V_x(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = E_x$
  - $V_x(\tau) = V_x^*(-\tau)$
  - $|V_x(\tau)| < V_x(0)$

.

Correlation Function

- Example: For the pulse

$$x(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{29}$$

the correlation function is

$$V_x(\tau) = \begin{cases} T\left(1 - \frac{|\tau|}{T}\right) & |\tau| \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{30}$$

Energy Spectrum

- The energy spectrum of a signal $x(t)$ is given by

$$G_x(f) = X(f)X^*(f) = |X(f)|^2 \tag{31}$$

- The energy spectral density is the Fourier transform of the correlation function, i.e.,

$$G_x(f) = \mathcal{F}\{V_x(\tau)\} \tag{32}$$

- The energy spectrum is a functional description of how the energy in the signal $x(t)$ is distributed as a function of frequency.
- Properties of the energy spectral density:

$$G_x(f) \geq 0 \quad \forall f \quad \text{(Energy in a signal cannot be negative valued)}$$
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} G_x(f) df \tag{33}$$

Energy Spectrum

- Example: For the Fourier transform pair of

$$x(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \qquad X(f) = T \exp\left[-j\pi f T\right] \text{sinc}\left(fT\right) \quad (34)$$

the energy spectrum is

$$G_x(f) = T^2 \left(\text{sinc}\left(fT\right)\right)^2 \quad (35)$$

Bandwidth of the signal

- Bandwidth most often refers to the amount of positive frequency spectrum that a signal occupies.
- If a signal $x(t)$ has an energy spectrum $G_x(f)$, then B_X is determined as

$$10 \log \left(\max_f G_x(f) \right) = X + 10 \log \left(G_x \left(B_X \right) \right) \quad (36)$$

where $G_x\left(B_X\right) > G_x(f)$ for $|f| > B_X$
- A signal has a relative bandwidth B_X, if the energy spectrum is at least X dB down from the peak at all frequencies at or above B_X Hz.
- Often used values for X in engineering practice are the 3-dB bandwidth and the 40-dB bandwidth.

Bandwidth of the signal

- If a signal $x(t)$ has an energy spectrum $G_x(f)$, then B_P is determined as

$$P = \frac{\int_{-B_P}^{B_P} G_x(f)df}{E_x} \tag{37}$$

- In words, a signal has an integral bandwidth B_P if the percent of the total energy in the interval $[-B_P, B_P]$ is equal to P%.
- Often used values for P in engineering practice are 98% and 99%.

Bandwidth of the signal

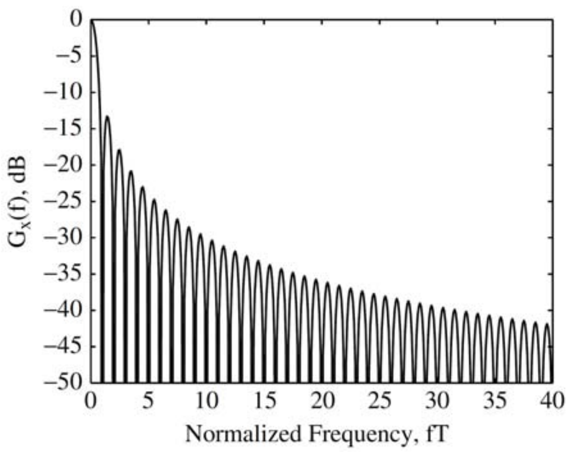
- Example: For the rectangular pulse

$$x(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{38}$$

The energy spectrum of this signal is given as

$$G_x(f) = |X(f)|^2 = T^2 \left(\text{sinc}\,(fT)\right)^2 \tag{39}$$

Bandwidth of the signal



Bandwidth of the signal

- The 3-dB bandwidth is given by $B_3 = 0.442/T$
- The 40-dB bandwidth is given by $B_{40} = 31.54/T$
- Integrating the power spectrum gives a 98% energy bandwidth of $B_{98} = 5.25/T$.