EE910: Digital Communication Systems-I

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Lecture #5A: Optimal Detection for a vector AWGN channel

Optimal detection for a General Vector Channel

- The additive white Gaussian noise (AWGN) channel model is a channel whose sole effect is addition of a white Gaussian noise process to the transmitted signal.
- The channel is mathematically described by,

$$r(t) = s_m(t) + n(t) \tag{1}$$

  where $s_m(t)$ is the transmitted signal and $n(t)$ is a zero-mean white Gaussian noise process with power spectral density of $N_0/2$; and $r(t)$ is the received waveform.
- The receiver makes an optimal decision about which message m, $1 \leq m \leq M$ was transmitted based on the decision rule that minimizes the probability of disagreement between the transmitted message m and the detected message $\hat{m}$ given by

$$P_e = P[\hat{m} \neq m] \tag{2}$$

Optimal detection for a General Vector Channel

- The general vector channel is given by

$$\mathbf{r} = \mathbf{s}_m + \mathbf{n} \tag{3}$$

  where all vectors are N-dimensional real vectors.
- The vectors $\mathbf{s}_m$ is chosen from a set of possible signal vectors $\{\mathbf{s}_m, 1 \leq m \leq M\}$ according to prior probabilities P_m.
- Let the decision function employed at the receiver by $g(\mathbf{r})$. If $g(\mathbf{r}) = \hat{m}$, then the probability of a correct decision, given that $\mathbf{r}$ is received, is given by

$$P[\mathit{correct\ decision}|\mathbf{r}] = P[\hat{m}\ \mathit{sent}|\mathbf{r}]. \tag{4}$$

- Therefore the probability of a correct decision is

$$P[\mathit{correct\ decision}] = \int P[\hat{m}\ \mathit{sent}|\mathbf{r}]p(\mathbf{r})d\mathbf{r} \tag{5}$$

MAP and ML receivers

- The optimal detection rule is the one that upon observing $\mathbf{r}$ decides in favor of the message m that maximizes $P[m|\mathbf{r}]$, i.e.,

$$\hat{m} = g_{opt}(\mathbf{r}) = \underset{1 \le m \le M}{\arg\max}\, P[m|\mathbf{r}] = \underset{1 \le m \le M}{\arg\max}\, P[\mathbf{s}_m|\mathbf{r}] \tag{6}$$

- The optimal decision rule given in (6) is known as the maximum a posteriori probability rule, or MAP rule.
- The MAP receiver can be simplified to

$$\hat{m} = \underset{1 \le m \le M}{\arg\max}\, \frac{P_m p(\mathbf{r}|\mathbf{s}_m)}{p(\mathbf{r})} \tag{7}$$

MAP and ML receivers

- Since $p(\mathbf{r})$ is independent of m and for all m remains the same, (7) equivalent to $\hat{m} = \underset{1 \le m \le M}{\arg\max}\, P_m p(\mathbf{r}|\mathbf{s}_m)$.
- If the messages are equiprobable, the optimal decision rule reduces to $\hat{m} = \underset{1 \le m \le M}{\arg\max}\, p(\mathbf{r}|\mathbf{s}_m)$ and the receiver is known as maximum likelihood receiver or ML receiver.

The Error Probability

- The region $\mathbf{D}_m$, $1 \leq m \leq M$, is called the decision region for message m; and $\mathbf{D}_m$ is the set of all outputs of the channel that are mapped into message m by the detector.
- For a MAP detector we have

$$\mathbf{D}_m = \{\mathbf{r} \in \mathcal{R}^N \ : P[m|\mathbf{r}] > P[m'|\mathbf{r}], \text{ for all } 1 \leq m' \leq M \text{ and } m' \neq m\} \quad (8)$$

- An error occurs when $\mathbf{s}_m$ is transmitted but the received $\mathbf{r}$ is not in $\mathbf{D}_m$.

The Error Probability

- The symbol error probability of a receiver is thus given by,

$$P_e = \sum_{m=1}^{M} P_m P[\mathbf{r} \notin \mathbf{D}_m | \mathbf{s}_m \; sent] = \sum_{m=1}^{M} P_m P_{e|m} \quad (9)$$

where $P_{e|m}$ denotes the error probability when message m is transmitted and is given by

$$P_{e|m} = \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s_m}) d\mathbf{r}$$

.

- Now P_e can be written as,

$$P_e = \sum_{m=1}^{M} P_m \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(\mathbf{r}|\mathbf{s_m}) d\mathbf{r}$$

which is the symbol error probability.

The Error Probability

- The bit error probability is denoted by P_b and is the error probability in transmission of a single bit.
- We can bound the bit error probability by noting that a symbol error occurs when at least one bit is in error, and the event of a symbol error is the union of the events of the errors in the $k = log_2 M$ bits representing that symbol.
- Therefore we can write $P_b \leq P_e \leq kP_b$

Preprocessing at the Receiver

- The receiver passes $\mathbf{r}$ through G and supplies the detector with $\rho = G(\mathbf{r})$.
- Since G is invertible and the detector has access to ρ, it can apply G^{-1} to ρ to obtain $G^{-1}(\rho) = G^{-1}(G(\mathbf{r})) = \mathbf{r}$. The detector now has access to both ρ and $\mathbf{r}$.
- Thus the optimal detection rule can be written as,

$$\hat{m} = \underset{1\leq m\leq M}{\arg\max}\, P_m p(\mathbf{r}, \rho|\mathbf{s}_m) = \underset{1\leq m\leq M}{\arg\max}\, P_m p(\mathbf{r}|\mathbf{s}_m)p(\rho|\mathbf{r}) = \underset{1\leq m\leq M}{\arg\max}\, P_m p(\mathbf{r}|\mathbf{s}_m)$$

- Thus it is clear that the optimal detector based on the observation of ρ makes the same decision as the optimal detector based on the observation of $\mathbf{r}$.