Live Interaction #5:

29th October 2023

E-masters Communication Systems

Detection for Wireless

- **Linear discriminant analysis:**

$$\mathcal{H}_0 : \mathcal{N}(\bar{\boldsymbol{\mu}}_0, \mathbf{R})$$
$$\mathcal{H}_1 : \mathcal{N}(\bar{\boldsymbol{\mu}}_1, \mathbf{R})$$

- **Likelihoods**:

$$p(\bar{\mathbf{x}}; \mathcal{H}_0) = \frac{1}{\sqrt{(2\pi)^N |\mathbf{R}|}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_0)^T \mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_0)}$$
$$p(\bar{\mathbf{x}}; \mathcal{H}_1) = \frac{1}{\sqrt{(2\pi)^N |\mathbf{R}|}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_1)^T \mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_1)}$$

- Choose $\mathcal{H}_0$ if

$$p(\bar{\mathbf{x}}; \mathcal{H}_0) \geq p(\bar{\mathbf{x}}; \mathcal{H}_1)$$
$$\Rightarrow \frac{\cancel{1}}{\cancel{\sqrt{(2\pi)^N |\mathbf{R}|}}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_0)^T \mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_0)}$$
$$\geq \frac{\cancel{1}}{\cancel{\sqrt{(2\pi)^N |\mathbf{R}|}}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_1)^T \mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_1)}$$
$$\Rightarrow (\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_0)^T \mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_0) \leq (\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_1)^T \mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}_1)$$
$$\Rightarrow \bar{\mathbf{h}}^T (\bar{\mathbf{x}} - \tilde{\boldsymbol{\mu}})^T \geq 0$$
$$\bar{\mathbf{h}} = \mathbf{R}^{-1}(\bar{\boldsymbol{\mu}}_0 - \bar{\boldsymbol{\mu}}_1)$$

$$\tilde{\boldsymbol{\mu}} = \frac{\bar{\boldsymbol{\mu}}_0 + \bar{\boldsymbol{\mu}}_1}{2}$$

- Problem: Find the **optimal detector**, probability of error

$$\mathcal{C}_0 \sim \mathcal{N}\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}\right), \mathcal{C}_1 \sim \mathcal{N}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}\right)$$

$$(\bar{\boldsymbol{\mu}}_0 - \bar{\boldsymbol{\mu}}_1) = \begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$\bar{\mathbf{h}} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -20 \\ 18 \end{bmatrix}$$

$$\tilde{\boldsymbol{\mu}} = \frac{1}{2}\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

- Choose $\mathcal{H}_0$ if

$$\bar{\mathbf{h}}^T(\bar{\mathbf{x}} - \tilde{\boldsymbol{\mu}}) \geq 0$$

$$\Rightarrow \begin{bmatrix} -20 & 18 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \geq 0$$

$$\Rightarrow -20x_1 + 18x_2 \geq 19$$

- Choose $\mathcal{H}_0$ if

$$\Rightarrow -20x_1 + 18x_2 \geq 19$$

- Choose $\mathcal{H}_1$ if

$$\Rightarrow -20x_1 + 18x_2 < 19$$

- **Probability of error:**

$$Q\left(\frac{1}{2}\sqrt{(\bar{\boldsymbol{\mu}}_1 - \bar{\boldsymbol{\mu}}_0)^T \mathbf{R}^{-1} (\bar{\boldsymbol{\mu}}_1 - \bar{\boldsymbol{\mu}}_0)}\right)$$

- Probability of error for our example

$$Q\left(\frac{1}{2}\sqrt{\begin{bmatrix} 5 & -3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}}\right)$$

$$= Q\left(\frac{1}{2}\sqrt{154}\right)$$

- **Optimal signalling**: How do design

$$\bar{\boldsymbol{\mu}}_1 - \bar{\boldsymbol{\mu}}_0 = \bar{\mathbf{s}}$$

$$\max \bar{\mathbf{s}}^T \mathbf{R}^{-1} \bar{\mathbf{s}}$$

$$\|\bar{\mathbf{s}}\|^2 = 1$$

- Choose $\bar{\mathbf{s}}$ such that
- Unit-norm eigenvector corresponding to ***minimum eigenvalue*** of $\mathbf{R}$.

$$\mathbf{R} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix}}_{\boldsymbol{\Lambda}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\mathbf{U}^T}$$

$$\underbrace{\bar{\mathbf{s}}_{\text{opt}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}}_{\substack{\text{Eigenvector for minimum}\\ \text{eigenvalue of } \mathbf{R}}}$$

- This signal gives the **lowest probability of error**.
- **Homework:**
- Consider the covariance matrix

$$\mathbf{R} = \begin{bmatrix}5 & -1\\-1 & 5\end{bmatrix}$$

- Determine the signal gives the **lowest probability of error**.
- **Assignment #5 deadline 4th November Saturday 11:59 PM.**
- **Live interaction 5th November Sunday 4:30-5:30 PM.**