EE901
Probability and Random Processes

Module -2
Random Variables

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Random Variables

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Random Variables

- It is easier to represent outcomes by assigning numbers to them

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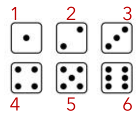
Random Variables

- It is easier to represent outcomes by assigning numbers to them
- Sometimes it is intuitive
- There is a difference between number 1 and the outcome 1.
  - Outcome is a physical activity, it includes information regarding what number is on the top, what number is on the north side, where the dice falls, how many turns it makes.
  - Number 1 is just a number.



$\Omega = \{1,2,3,4,5,6\}$

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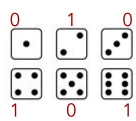
Random Variables

- It is easier to represent outcomes by assigning numbers to them
- Or we can take any arbitrary assignment.



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Random Variables

- It is easier to represent outcomes by assigning numbers to them
- Sometimes, the number is all we can see.
- For example, a random generator or a slot machine
- We can differentiate between different outcomes only based on this observed number.

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Random Variables

- A random variable X is a function from the sample space to real line

$$X: \Omega \to \mathbb{R}$$
$$\begin{array}{cc} \omega & X(\omega) \\ H & \to 1 \\ T & \to 0 \end{array}$$

- For every outcome ω, we have a $X(\omega)$

$$X(H) = 1$$
$$X(T) = 0$$

$\Omega = \{H, T\}$

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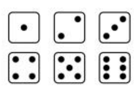
Random Variables

- A random variable X is a function from the sample space to real line

$$X: \Omega \to \mathbb{R}$$
$$\begin{array}{cc} \omega & X(\omega) \\ 1 & \to 1 \\ 2 & \to 0 \\ 3 & \to 1 \\ 4 & \to 0 \\ 5 & \to 1 \\ 6 & \to 0 \end{array}$$



$\Omega = \{1,2,3,4,5,6\}$

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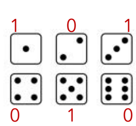
Events under Random Variable Map

- A random variable X is a function

$$X: \Omega \to \mathbb{R}$$
$$\begin{array}{c|c} \omega & X(\omega) \\ 2 & 0 \end{array}$$

Where does any event in sigma algebra map to?

$$\begin{array}{l|l} \mathrm{E} = \{1,2\} & \{0,1\} \\ \mathrm{E} = \{1,5\} & \{1\} \\ \mathrm{E} = \{2,4\} & \{0\} \\ \mathrm{E} = \phi & \{\} \\ \mathrm{E} = \Omega & \{0,1\} \\ \mathcal{F} & \{\{\},\{0\},\{1\},\{0,1\}\} \end{array}$$



$\Omega = \{1,2,3,4,5,6\}$
$\mathcal{F}$ =Collections of all subsets of Ω

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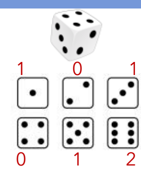
Events under Random Variable Map

- A random variable X is a function

$$X: \Omega \to \mathbb{R}$$

Where does any event in sigma algebra maps to?

$E = \{1,2\}$	$\{0,1\}$
$E = \{1,5\}$	$\{1\}$
$E = \{2,4\}$	$\{0\}$
$E = \{6\}$	$\{2\}$
$E = \{1,3,5,6\}$	$\{1,2\}$
$E = \phi$	$\{\}$
$E = \Omega$	$\{0,1,2\}$
$\mathcal{F}$	$\{\{\},\{0\},\{1\},\{0,1\},\{1,2\},\{0,2\},\{0,1,2\}\}$



$\Omega = \{1,2,3,4,5,6\}$
$\mathcal{F}$ =Collections of all subsets of Ω

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Events under Random Variable Map

- A random variable X is a function

$$X: \Omega \to \mathbb{R}$$

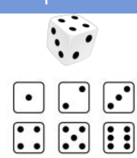
Where does any event in sigma algebra maps to?

$\mathcal{F}$

$\mathcal{B}$: Borel algebra on $\mathbb{R}$

Collection of all open, closed intervals of $\mathbb{R}$ and their intersections and countable unions …

Includes
$(0, x)$,
$[x]$,
(a, b)



$\Omega = \{1,2,3,4,5,6\}$
$\mathcal{F}$ =Collections of all subsets of Ω

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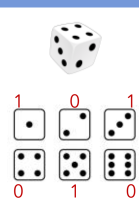
Events under Random Variable Map

$$X: \Omega \to \mathbb{R}$$
$$\mathcal{F} \;\to\; \mathcal{B}$$

Any set B in $\mathcal{B}$ represents a set of values X can take.

Let us take $B = [1]$.
$X \in [1]$
or
$X = 1$?
What does the above represent?



$\Omega = \{1,2,3,4,5,6\}$

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Events under Random Variable Map

- We see the value of the random variable at the output.

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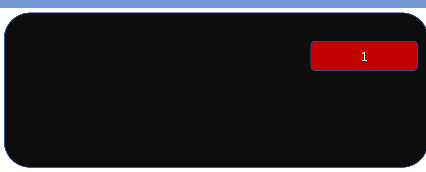
Events under Random Variable Map



- What does $X = 1$ represent?
- When does 1 occur at the output?
  - When dice roll shows 1,3 or 5.

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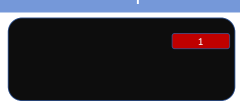
Events under Random Variable Map

- What does $X = 1$ represent?
- When does 1 occur?
  - When dice roll shows 1,3 or 5.



- The set of outcomes is $\mathrm{E} = \{\omega\colon X(\omega) = 1\}$.
- $\mathrm{E} = \{\omega\colon X(\omega) = 1\} = \{1, 3, 5\}$
- E is an event. We will also use event $\{X = 1\}$ to denote E.
- Similarly, what does $X = 0$ represent? $X \in [0]$
- What does $X = 0$ or 1 represent? $X \in \{0,1\}$.

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Events under Random Variable Map

- What does $X \in (0.5,1.2)$ represent?
- When does X takes a value in (0.5,1.2)?
  - When dice roll shows 1,3 or 5.
- The set of outcomes is $\mathrm{E} = \{\omega\colon X(\omega) \in (0.5,1.2)\}$.
- $\mathrm{E} = \{1, 3, 5\}$. E is an event.
- For any set (a, b), there is an equivalent event for $X \in (a, b)$
- Similarly, for any set B, there is an equivalent event for $X \in \mathrm{B}$

$$\mathrm{E}_B = \{\omega\colon X(\omega) \in B\}.$$

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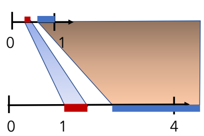
Example: Pick a Number

- Pick a number in (0,1)

Probability space $((0,1), \mathcal{B}(0,1), \mathbb{P})$



Define random variable X as 4 times the chosen number i.e.

$$X(\omega) = 4\omega \text{ for each } \omega \in \Omega.$$

What will $X > 2$? In other words, what does $X \in (2, \infty)$ represent?

Compute, for what values of ω, $X(\omega) > 2$?

$$X(\omega) > 2$$
$$4\omega > 2$$
$$\omega > \frac{2}{4} = 0.5$$

Equivalent event is $\mathrm{E} = (0.5, 1)$.

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Probability Law of Random Variables

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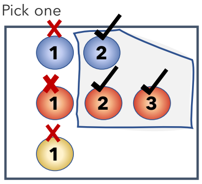
Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



- Define random variable X as the number written on the ball in an outcome.
- What does $X \in (1.5, 3.5)$ represent?
- What are those ω's for which $X(\omega) \in (1.5, 3.5)$?

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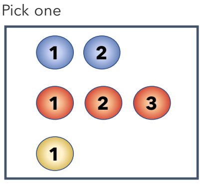
Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



- Define random variable X as the number on the ball in an outcome.
- What does $X \in (1.5, 3.5)$ represent?
- What are those ω's for which $X(\omega) \in (1.5, 3.5)$?
- What is the probability that $X \in (1.5, 3.5)$?

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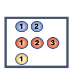
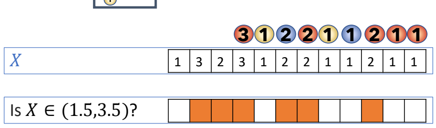


What is the probability that $X \in (1.5, 3.5)$?



$n = 12$

$n' = 6$

Relative frequency of $X \in (1.5, 3.5)$? Is equal to the probability of picking a ball with number 2 or 3.

$$\Pr[X \in (1.5, 3.5)] = \mathbb{P}(\{\omega : X(\omega) \in (1.5, 3.5)\}) = \frac{3}{6}$$

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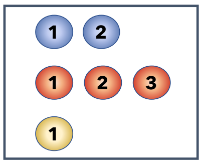
Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



- Define random variable X as the number on the ball in an outcome.
- What is the probability that $X \in (1.5,3.5)$?

$\Pr[X \in (1.5,3.5)] = \mathbb{P}(\{\mathbb{P}\omega: X(\omega) \in (1.5,3.5)\}) = \frac{3}{6}$

- For any set B

$\Pr[X \in B] = \mathbb{P}(\{\omega: X(\omega) \in B\})$

$\mathbb{P}_X(B)$

Known as the probability Law of the random variable X.

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Probability Law of a Random Variable

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X: \Omega \to \mathbb{R}$ be a random variable. The probability law of the random variable X given as

For any set B in the Borel field $\mathcal{B}$

$$\mathbb{P}_X(B) \triangleq \Pr[X \in B] = \mathbb{P}(\{\omega: X(\omega) \in B\})$$

This corresponds to the probability of the event consisting of those outcomes which correspond to X taking a value in set B.

If we perform the random experiment and observe the value of X, it will denote the probability that X takes a value from set B.

For example if $B = (-\infty, x)$, this will denote that probability that X takes value less than x.

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Probability Law of a Random Variable

$X: \Omega \to \mathbb{R}$

For any set B in the Borel field $\mathcal{B}$

$$\mathbb{P}_X(B) \triangleq \Pr[X \in B] = \mathbb{P}(\{\omega: X(\omega) \in B\})$$

This is valid probability measure for the probability space $(\mathbb{R}, \mathcal{B}, \mathbb{P}_X)$.
It assigns a measure or size to each set B. This measure is equal to $\mathbb{P}_X(B)$.

Measure of a physical entity can mean its actual size (such as length or area) or its mass or its weight or number of elements in it

$\mathbb{P}_X(B)$ can be seen as a type of mass present in the set B. Let us call it the probability mass in B.

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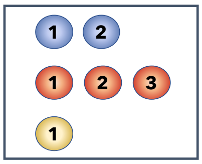
Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



- Define random variable X as

  $X(\omega)$ = the number on the ball in ω.

- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$
$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

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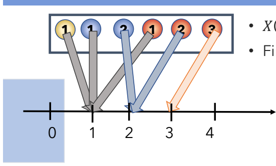
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$
$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x < 1$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$
$\mathrm{E}_B = \{\ \phi\ \}$
$\mathbb{P}_X(B) = 0$

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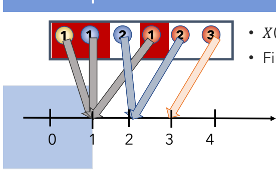
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$
$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x = 1$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$
$\mathrm{E}_B = \{$ 1 , 1 , 1 $\}$
$\mathbb{P}_X(B) = \frac{3}{6} = \frac{1}{2}$

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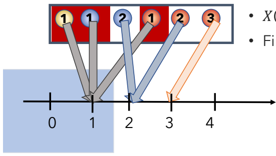
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$2 > x > 1$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$\mathrm{E}_B = \{$ 1 , 1 , 1 $\}$

$\mathbb{P}_X(B) = \frac{3}{6} = \frac{1}{2}$

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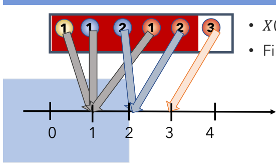
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x = 2$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$\mathrm{E}_B = \{$ 1, 1 2 , 1 2 $\}$

$\mathbb{P}_X(B) = \frac{5}{6}$

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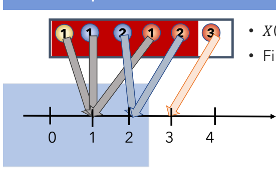
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
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$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$3 > x > 2$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$\mathrm{E}_B = \{$ 1, 1 2 , 1 2 $\}$

$\mathbb{P}_X(B) = \frac{5}{6}$

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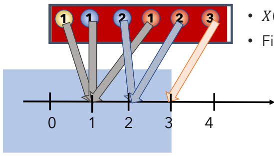
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x = 3$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$\mathrm{E}_B = \{1, 1, 2, 1, 2, 3\}$

$\mathbb{P}_X(B) = 1$

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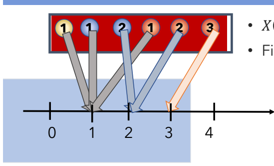
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.
- Find the probability law of X?

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x > 3$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$\mathrm{E}_B = \{1, 1, 2, 1, 2, 3\}$

$\mathbb{P}_X(B) = 1$

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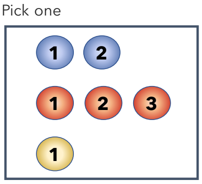
Example: Pick a Ball

- A bag full of balls
- Each ball has a color and number
- Pick one



- Define random variable X as

$X(\omega)$ = the number of the ball in ω.

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$

$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x < 1$	$\mathbb{P}_X(B) = 0$
$x = 1$	$\mathbb{P}_X(B) = 1/2$
$1 < x < 2$	$\mathbb{P}_X(B) = 1/2$
$x = 2$	$\mathbb{P}_X(B) = 5/6$
$2 < x < 3$	$\mathbb{P}_X(B) = 5/6$
$x = 3$	$\mathbb{P}_X(B) = 1$
$3 < x$	$\mathbb{P}_X(B) = 1$

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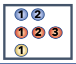
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$ $\quad$ $\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

$B = (-\infty, x]$.

$x < 1$	$\mathbb{P}_X(B) = 0$
$x = 1$	$\mathbb{P}_X(B) = 1/2$
$1 < x < 2$	$\mathbb{P}_X(B) = 1/2$
$x = 2$	$\mathbb{P}_X(B) = 5/6$
$2 < x < 3$	$\mathbb{P}_X(B) = 5/6$
$x = 3$	$\mathbb{P}_X(B) = 1$
$3 < x$	$\mathbb{P}_X(B) = 1$

Other values of B

$B = [1.5]$ $\quad$ $\mathrm{E}_B = \{\omega : X(\omega) = 1.5\} = \{\}, \mathbb{P}_X(B) = 0$

$B = [1]$ $\quad$ $\mathrm{E}_B = \{\omega : X(\omega) = 1\} = \{1,1,1\}, \mathbb{P}_X(B) = 0.5$

$B = (0,5),$ $\quad$ $\mathrm{E}_B = \Omega, \mathbb{P}_X(B) = 1$

$B = (-\infty, \infty),$ $\quad$ $\mathrm{E}_B = \Omega, \mathbb{P}_X(B) = 1$

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Additional Requirements for Random Variables

- Probability law

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$ $\quad$ $\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

- What if for some B, E_B does not exist in sigma algebra $\mathcal{F}$?
- Probability of E_B is not defined that.
- We require the random variable should be such that for each x, $\mathrm{E}_{(-\infty,x]}$ should be in $\mathcal{F}$.

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Additional Requirements for Random Variables

- Probability law

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$ $\quad$ $\mathrm{E}_B = \{\omega : X(\omega) \in B\}$

- We require the random variable should be such that for each x, $\mathrm{E}_{(-\infty,x]}$ should be in $\mathcal{F}$.
- We also require that $\Pr[X = -\infty]$ and $\Pr[X = \infty]$ should be zero.

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Random Variable Definition

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
a random variable X is a function

$$X : \Omega \to \mathbb{R}$$

such that

- for each x, $\mathrm{E}_{(-\infty,x]} = \{\omega : X(\omega) \in (-\infty, x]\}$ should be in $\mathcal{F}$ and
- $\Pr[X = -\infty\,]$ and $\Pr[X = \infty]$ should be zero.

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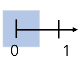
Example: Pick a Number

Pick a number in (0,1)   Probability space $(\Omega, \mathcal{B}(0,1), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.



$B = (-\infty, x]$.

E_B
$= \{\omega : X(\omega) \in B\}$
$= \{\omega : 4\omega \in (-\infty, x]\}$
$= \{\omega : 4\omega \le x\}$
$= \left\{\omega : \omega \le \frac{x}{4}\right\}$
$= \left\{\omega : \omega \le \frac{x}{4}, 0 < \omega < 1\right\}$
$= \left(-\infty, \frac{x}{4}\right] \cap (0,1)$

x	$\mathrm{E}_B = \{\omega : X(\omega) \in B\}$	$\mathbb{P}_X(B)$

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Distribution of a Random Variable

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable. The probability law of the random variable X given as
For any set B in the Borel field $\mathcal{B}$

$$\mathbb{P}_X(B) \triangleq \Pr[X \in B] = \mathbb{P}(\{\omega : X(\omega) \in B\})$$

This represents the probability of the event consisting of those outcomes which correspond to X taking a value in set B.

Need to specify for every possible set B in Borel algebra $\mathcal{B}$. Lots of work!

We will see that it is sufficient to specify it for sets of the form $B_x = (-\infty, x)$ for every value of x.

This denotes that probability that X takes value less than x and will be a function of x.

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Cumulative Distribution Function (CDF)

The CDF of a random variable X is defined as the probability that X takes value less than x

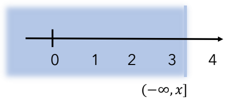
$$F_X(x) = \mathbb{P}(\{\omega: X(\omega) \le x\})$$

$$= \mathbb{P}(\{\omega: X(\omega) \in (-\infty, x]\})$$

$$= \mathbb{P}_X\left((-\infty, x]\right)$$

It is nothing but the probability Law $\mathbb{P}_X(B_x)$ of a random variable X for $B_x = (-\infty, x]$.

CDF at x can be seen as the probability mass of the interval $(-\infty, x]$.



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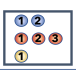
Example: Pick a Ball



- $X(\omega)$ = the number of the ball in ω.

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(E_B)$ $\quad E_B = \{\omega: X(\omega) \in B\}$

$B = (-\infty, x]$.

$x < 1$	$\mathbb{P}_X(B) = 0$
$x = 1$	$\mathbb{P}_X(B) = 1/2$
$1 < x < 2$	$\mathbb{P}_X(B) = 1/2$
$x = 2$	$\mathbb{P}_X(B) = 5/6$
$2 < x < 3$	$\mathbb{P}_X(B) = 5/6$
$x = 3$	$\mathbb{P}_X(B) = 1$
$3 < x$	$\mathbb{P}_X(B) = 1$

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 1 \\ 1/2 & x = 1 \\ 1/2 & 1 < x < 2 \\ 5/6 & x = 2 \\ 5/6 & 2 < x < 3 \\ 1 & x = 3 \\ 1 & 3 < x \end{cases}$$

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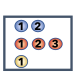
Example: Pick a Ball

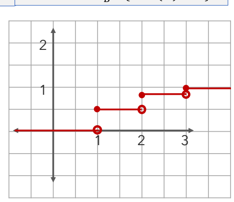


- $X(\omega)$ = the number of the ball in ω.

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(E_B)$ $\quad E_B = \{\omega: X(\omega) \in B\}$

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 1 \\ 1/2 & x = 1 \\ 1/2 & 1 < x < 2 \\ 5/6 & x = 2 \\ 5/6 & 2 < x < 3 \\ 1 & x = 3 \\ 1 & 3 < x \end{cases}$$



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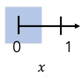
Example: Pick a Number

Pick a number in (0,1) Probability space $(\mathbb{R}, \mathcal{B}(0,1), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.



$B = (-\infty, x]$.

x	$E_B = \{\omega : X(\omega) \in B\}$	$\mathbb{P}_X(B)$
$x < 0$	ϕ	0
$x = 0$	ϕ	0
$0 < x < 4$	$\left(0, \frac{x}{4}\right)$	$\frac{x}{4}$
$x = 4$	$(0,1)$	1
$x > 4$	$(0,1)$	1

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Example: Pick a Number

Pick a number in (0,1) Probability space $(\mathbb{R}, \mathcal{B}(0,1), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.

x	$\mathbb{P}_X(B)$
$x < 0$	0
$x = 0$	0
$0 < x < 4$	$\frac{x}{4}$
$x = 4$	1
$x > 4$	1

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ \frac{x}{4} & 0 < x < 4 \\ 1 & x = 4 \\ 1 & x > 4 \end{cases}$$

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Example: Pick a Number

Pick a number in (0,1) Probability space $(\mathbb{R}, \mathcal{B}(0,1), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

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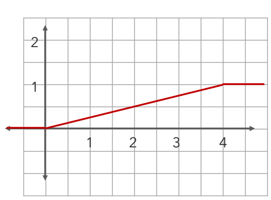

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Properties of CDF

1. $$F_X(\infty) = 1$$

Proof:

Recall $F_X(x) = \mathbb{P}(E_x)$ where $E_x = \{\omega: X(\omega) \leq x\}$

$$E_\infty = \{\omega: X(\omega) \leq \infty\}$$

Since for every outcome, $X(\omega) < \infty$, therefore,

$$E_\infty = \Omega$$

$$F_X(\infty) = \mathbb{P}(\Omega) = 1$$

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Properties of CDF

2. $$F_X(-\infty) = 0$$

Proof:

Recall $F_X(x) = \mathbb{P}(E_x)$ where $E_x = \{\omega: X(\omega) \leq x\}$

$$E_{-\infty} = \{\omega: X(\omega) \leq -\infty\}$$

Since for every outcome, $X(\omega) > -\infty$, therefore,

$$E_{-\infty} = \phi$$

$$F_X(\infty) = \mathbb{P}(\phi) = 0$$

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Properties of CDF

3. $$\mathbb{P}(X > x) = 1 - F_X(x)$$

Proof:

Note that $\{X \leq x\}$ is a short form of saying $\{\omega: X(\omega) \leq x\}$. Therefore $\{X \leq x\}$ is an event.

Now, the event $\{X \leq x\}$ and the event $\{X > x\}$ are disjoint. Their union is Ω.

Therefore, from finite additivity property of probability

$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$ for disjoint events A_1 and A_2

$$\mathbb{P}(\{X \leq x\}) + \mathbb{P}(\{X > x\}) = \mathbb{P}(\Omega) = 1$$
$$\mathbb{P}(\{X > x\}) = 1 - \mathbb{P}(\{X \leq x\})$$

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Properties of CDF

4. $F_X(x)$ is monotonically increasing. i.e.
If $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$

Proof:

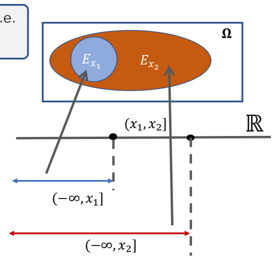
Recall $F_X(x_1) = \mathbb{P}(E_{x_1})$

where $E_{x_1} = \{\omega: X(\omega) \le x_1\} = \{X \le x_1\}$

Recall $F_X(x_2) = \mathbb{P}(E_{x_2})$

where $E_{x_2} = \{X \le x_2\}$

Now, E_{x_2} is a subset of E_{x_1}.



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Properties of CDF

$F_X(x)$ is monotonically increasing. i.e.
If $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$

Proof:

Recall $F_X(x_1) = \mathbb{P}(E_{x_1})$

where $E_{x_1} = \{\omega: X(\omega) \le x_1\} = \{X \le x_1\}$

Recall $F_X(x_2) = \mathbb{P}(E_{x_2})$

where $E_{x_2} = \{X \le x_2\}$

Now, E_{x_2} is a subset of E_{x_1}.

Property of Probability:
Monotonicity
If A ⊂ B, then $\mathbb{P}$(A) ≤ $\mathbb{P}$(B)

$\mathbb{P}(E_{x_1}) \le \mathbb{P}(E_{x_2})$

$F_X(x_1) \le F_X(x_2)$

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Properties of CDF

5. If $x_1 < x_2$, then
$\mathbb{P}((x_1, x_2]) = F_X(x_2) - F_X(x_1)$

Proof:

Recall $F_X(x_1) = \mathbb{P}(E_{x_1})$

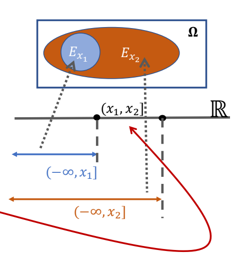
where $E_{x_1} = \{\omega: X(\omega) \le x_1\} = \{X \le x_1\}$

Recall $F_X(x_2) = \mathbb{P}(E_{x_2})$

where $E_{x_2} = \{X \le x_2\}$

The event $\{x_1 < X \le x_2\}$ is equivalent to $E_{x_2} \setminus E_{x_1}$

i.e. $\{X \le x_2\}$ is the union of $\{X \le x_1\}$ and the event $\{x_1 < X \le x_2\}$.



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Properties of CDF

5. If $x_1 < x_2$, then
$\mathbb{P}((x_1, x_2]) = F_X(x_2) - F_X(x_1)$

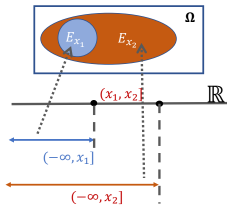
Proof:

Now, $\{X \leq x_2\}$ is the union of $\{X \leq x_1\}$ and the event $\{x_1 < X \leq x_2\}$.

Therefore, from finite additivity property of probability

$\mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_1 \cup A_2)$
for disjoint events A_1 and A_2

$\mathbb{P}(\{X \leq x_1\}) + \mathbb{P}(\{x_1 < X \leq x_2\}) = \mathbb{P}(\{X \leq x_2\})$
$F_X(x_1) + \mathbb{P}(\{x_1 < X \leq x_2\}) = F_X(x_2)$



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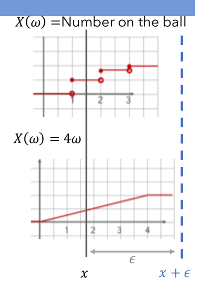
Properties of CDF

5. $F_X(x)$ is right continuous
$\lim_{\epsilon \to 0} F_X(x + \epsilon) = F_X(x)$



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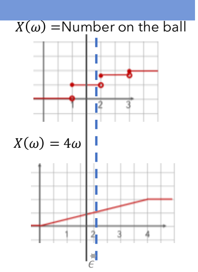

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Properties of CDF

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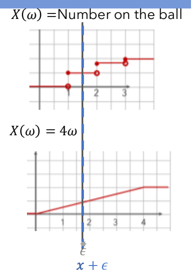
$\lim_{\epsilon \to 0} F_X(x+\epsilon) = F_X(x)$



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Properties of CDF

5. $F_X(x)$ is right continuous
$\lim_{\epsilon \to 0} F_X(x+\epsilon) = F_X(x)$

Proof:

$$F_X(x) = \mathbb{P}(\{\omega : X(\omega) \le x\})$$

Similarly,

$$F_X(x+\epsilon) = \mathbb{P}(\{\omega : X(\omega) \le x+\epsilon\})$$

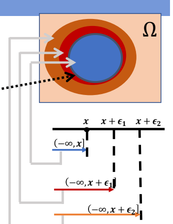
$F_X(x+\epsilon)$ denotes the probability of outcomes that map to a number up to x (including x) along with some additional numbers from the range $(x, x+\epsilon]$.

As $\epsilon \to 0$, the additional range shrinks and eventually no elements are left in it.

This means that as $\epsilon \to 0$,

$F_X(x+\epsilon)$ will denote the probability of outcomes that map to a number up to x (including x) only. = the same as $F_X(x)$

Boundary denotes the event $\{X = x\}$



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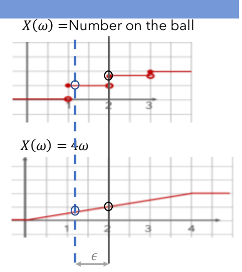
Properties of CDF

6. $F_X(x)$ may not be left continuous
$\lim_{\epsilon \to 0} F_X(x-\epsilon) = F_X(x^-) \neq F_X(x)$



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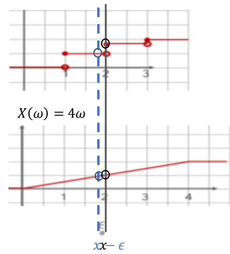
Properties of CDF

6. $F_X(x)$ may not be left continuous

$\lim_{\epsilon \to 0} F_X(x-\epsilon) = F_X(x^-) \neq F_X(x)$

$F_X(x^-) \leq F_X(x)$

$F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\})$



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Properties of CDF

6. $F_X(x)$ may not be left continuous

$\lim_{\epsilon \to 0} F_X(x-\epsilon) = F_X(x^-) \neq F_X(x)$

Proof:

$$F_X(x) = \mathbb{P}(\{\omega: X(\omega) \leq x\})$$
$$F_X(x-\epsilon) = \mathbb{P}(\{\omega: X(\omega) \leq x-\epsilon\})$$

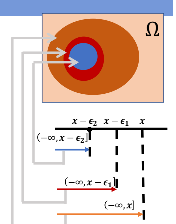
$F_X(x-\epsilon)$ denotes the probability of outcomes that map to a number up to $x-\epsilon$

As $\epsilon \to 0$, the upper limit comes closer to x, however, it will never include outcomes that map to x.

This means that as $\epsilon \to 0$,

$F_X(x-\epsilon)$ will denote the probability of outcomes that map to a number up to x (excluding x).

It is not the same as $F_X(x)$, if there are some outcomes with non-zero probability that map to $\{x\}$



$\boxed{F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\})}$

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