EE910: Digital Communication Systems-I

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Lecture #7B: Error probability of orthogonal signaling with noncoherent detection

Error probability of orthogonal signaling with noncoherent detection

- Let us assume M equiprobable, equal-energy, carrier modulated orthogonal signals are transmitted over an AWGN channel.
- These signals are noncoherently demodulated at the receiver and and then optimally detected.
- The lowpass equivalent of the signals can be written as M N-dimensional vectors ($N = M$)

$$\begin{aligned} s_1 &= (\sqrt{2\mathcal{E}_s}, 0, 0, ..., 0) \\ s_2 &= (0, \sqrt{2\mathcal{E}_s}, 0, ..., 0) \\ &\vdots \\ s_N &= (0, 0, ..., 0, \sqrt{2\mathcal{E}_s}) \end{aligned} \tag{1}$$

Error probability of orthogonal signaling with noncoherent detection

- Without loss of generality we can assume that s_{1l} is transmitted.
- Therefore the received vector will be

$$\mathbf{r}_l = e^{j\phi}\mathbf{s}_{1l} + \mathbf{n}_l \tag{2}$$

where $\mathbf{n}_l$ is a complex circular zero-mean Gaussian random vector with variance of each complex component equal to $2N_0$.

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- The optimal receiver computes and compares $|\mathbf{r}_l.\mathbf{s}_{ml}|$, for all $1 \leq m \leq M$. This results in

$$\begin{aligned} |\mathbf{r}_l.\mathbf{s}_{1l}| &= |2\mathcal{E}_s e^{j\phi} + \mathbf{n}_l.\mathbf{s}_{1l}| \\ |\mathbf{r}_l.\mathbf{s}_{1l}| &= |\mathbf{n}_l.\mathbf{s}_{ml}| \quad 2 \leq m \leq M \end{aligned} \tag{3}$$

Error probability of orthogonal signaling with noncoherent detection

- For $1 \leq m \leq M$, $\mathbf{n}_l.\mathbf{s}_{ml}$ is a circular zero-mean complex Gaussian random variable with variance $4\mathcal{E}_s N_0$ ($2\mathcal{E}_s N_0$ per real and imaginary parts).
- From equation (3) we have

$$\begin{aligned} Re[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(2\mathcal{E}_s cos\phi, 2\mathcal{E}_s N_0) \\ Im[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(2\mathcal{E}_s sin\phi, 2\mathcal{E}_s N_0) \\ Re[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(0, 2\mathcal{E}_s N_0) \quad 2 \leq m \leq M \\ Im[\mathbf{r}_l.\mathbf{s}_{1l}] &\sim N(0, 2\mathcal{E}_s N_0) \quad 2 \leq m \leq M \end{aligned} \tag{4}$$

Error probability of orthogonal signaling with noncoherent detection

- From the definition of Rayleigh and Ricean random variables, we conclude that random variables $R_m,\ 1 \leq m \leq M$ defined as

$$R_m = |\mathbf{r}_l.\mathbf{s}_{ml}| \quad 1 \leq m \leq M \tag{5}$$

  are independent random variables.
- R_1 has a Ricean distribution with parameters $s = 2\mathcal{E}_s$ and $\sigma^2 = 2\mathcal{E}_s N_0$, and
- $R_m, 2 \leq m \leq M$, are Rayleigh random variables with parameter $\sigma^2 = 2\mathcal{E}_s N_0$.

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- In other words,

$$p_{R_1}(r_1) = \begin{cases} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{r_1^2+s^2}{2\sigma^2}}, & r_1 > 0 \\ 0, & otherwise \end{cases} \tag{6}$$

  and

$$p_{R_m}(r_m) = \begin{cases} \frac{r_m}{\sigma^2} e^{-\frac{r_m^2}{2\sigma^2}}, & r_m > 0 \\ 0, & otherwise \end{cases} \tag{7}$$

  for $2 \leq m \leq M$.

Error probability of orthogonal signaling with noncoherent detection

- Since by assumption $\mathbf{s}_{1l}$ is transmitted, a correct decision is made at the receiver if $R_1 > R_m$ for $2 \leq m \leq M$.
- Although random variables R_m for $1 \leq m \leq M$ are statistically independent, the events $R_1 > R_2, R_1 > R_3, ..., R_1 > R_M$ are not independent due to the existence of the common R_1
- To make them independent, we need to condition on $R_1 = r_1$ and then average over all values of r_1.

Error probability of orthogonal signaling with noncoherent detection

- Therefore

$$\begin{aligned} P_c &= P[R_2 < R_1, R_3 < R_1, ..., R_M < R_1] \\ &= \int_0^\infty P[R_2 < r_1, R_3 < r_1, ..., R_M < r_1 | R_1 = r_1] p_{R_1}(r_1) dr_1 \\ &= \int_0^\infty (P[R_2 < r_1])^{M-1} p_{R_1}(r_1) dr_1 \end{aligned} \tag{8}$$

- But

$$\begin{aligned} P[R_2 < r_1] &= \int_0^{r_1} p_{R_2}(r_2) dr_2 \\ &= 1 - e^{-\frac{r_1^2}{2\sigma^2}} \end{aligned} \tag{9}$$

Error probability of orthogonal signaling with noncoherent detection

- Using the binomial expansion, we have

$$\left(1 - e^{-\frac{r_1^2}{2\sigma^2}}\right)^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-\frac{nr_1^2}{2\sigma^2}} \tag{10}$$

- Substituting into equation (8), we get

$$\begin{aligned} P_c &= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^\infty e^{-\frac{nr_1^2}{2\sigma^2}} \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{r_1^2+s^2}{2\sigma^2}} dr_1 \\ &= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^\infty \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{(n+1)r_1^2+s^2}{2\sigma^2}} dr_1 \\ &= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-\frac{ns^2}{2(n+1)\sigma^2}} \int_0^\infty \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{(n+1)r_1^2+\frac{s^2}{n+1}}{2\sigma^2}} dr_1 \end{aligned} \tag{11}$$

Error probability of orthogonal signaling with noncoherent detection

- By introducing a change of variables

$$s' = \frac{s}{\sqrt{n+1}} \tag{12}$$

$$r' = r_1\sqrt{n+1} \tag{13}$$

the integral in equation (11) becomes

$$\begin{aligned} \int_0^\infty \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) e^{-\frac{(n+1)r_1^2+\frac{s^2}{n+1}}{2\sigma^2}} dr_1 &= \frac{1}{n+1} \int_0^\infty \frac{r'}{\sigma^2} I_0(\frac{r's'}{\sigma^2}) e^{-\frac{s'^2+r'^2}{2\sigma^2}} dr' \\ &= \frac{1}{n+1} \end{aligned} \tag{14}$$

where we have used the fact that the area under a Ricean pdf is equal to 1.

Error probability of orthogonal signaling with noncoherent detection

- Substituting Equation (14) into Equation (11) and noting that $\frac{s^2}{2\sigma^2} = \frac{4\mathcal{E}^2}{4\mathcal{E}_s N_0} = \frac{\mathcal{E}_s}{N_0}$ we obtain

$$P_c = \sum_{n=0}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} e^{-\frac{n}{n+1}\frac{\mathcal{E}_s}{N_0}} \tag{15}$$

- Then the probability of a symbol error becomes

$$P_e = \sum_{n=0}^{M-1} \frac{(-1)^{n+1}}{n+1} \binom{M-1}{n} e^{-\frac{n log_2 M}{n+1}\frac{\mathcal{E}_b}{N_0}} \tag{16}$$

Error probability of orthogonal signaling with noncoherent detection

- For binary orthogonal signalling, including binary orthogonal FSK with noncoherent detection, Equation (16) simplifies to

$$P_b = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}} \tag{17}$$

- Comparing this result with coherent detection of binary orthogonal signals for which the error probability is given by

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) \tag{18}$$

- Using the inequality $Q(x) \leq \frac{1}{2}e^{-x^2/2}$, we conclude that $P_b(\text{noncoherent}) \geq P_b(\text{coherent})$, as expected.

Error probability of orthogonal signaling with noncoherent detection

- For error probabilities less than 10^{-4}, the difference between the performance of coherent and noncoherent detection of binary orthogonal is less than 0.8 dB
- For $M > 2$, we may compute the probability of a bit error by making use of the relationship

$$P_b = \frac{2^{k-1}}{2^k - 1} P_e \tag{19}$$

Error probability of orthogonal signaling with noncoherent detection