EE901
Probability and Random Processes

Module 10
Random Processes

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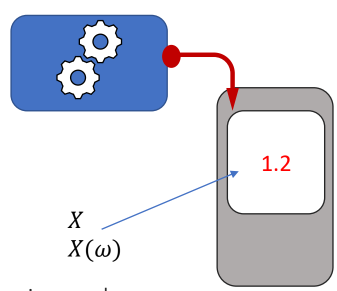
Random Processes

- A random variable maps each outcome to a number.
- In other words, whenever we do the experiment, we observe a value of random variable (corresponding to the outcome)



- A random process maps each outcome to a waveform instead.

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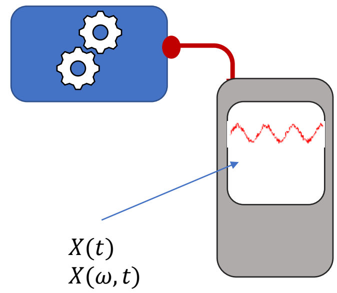
Random Processes

- A random process maps each outcome to a waveform instead.
- Every time we observe this random experiment, we see a waveform (instead of a single value as seen in random variable case)

Example: Measuring noise in a device.

Think of an underlying sample space.

Instead of observing the actual ω, we observe an electrical signal which uniquely depends on the outcome.



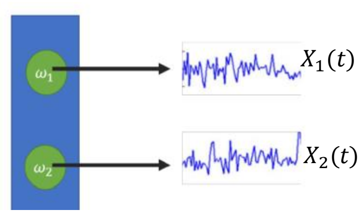
Random Processes

- Example: Let (Ω, F, P) be the probability space.
- Ω has only two outcomes ω_1 and ω_2.
- Let $X_1(t)$ and $X_2(t)$ be two functions.
- Using the map

$$\omega_1 \rightarrow X_1(t), \omega_2 \rightarrow X_2(t)$$

- Define

$$X(\omega, t) = \begin{cases} X_1(t) & \text{if } \omega = \omega_1 \\ X_2(t) & \text{if } \omega = \omega_2 \end{cases}$$

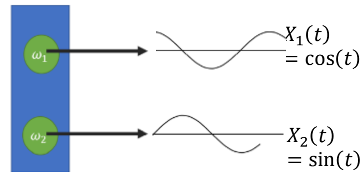


- $X(\omega, t)$ represents a random process.

Random Processes

- Define a random process $X(\omega, t)$

$$X(\omega, t) = \begin{cases} X_1(t) & \text{if } \omega = \omega_1 \\ X_2(t) & \text{if } \omega = \omega_2 \end{cases}$$



- What is the probability that we see a sin curve?
- What is the probability that $X(t) = 0$ at $t = \pi/2$?

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Random Process Interpretations

- Now, if we fix ω, say $\omega = \omega_1$

  $X(\omega, t) = X(\omega_1, t)$ = Deterministic Function

  Assigning these deterministic functions to each outcome results in a RP.

- If we fix t, say $t = t_0$, then

  $X(\omega, t) = X(\omega, t_0)$ = Random Variable

- Collection of $(X(\omega, t))$ i.e. random variables for a set of multiple time values (either discrete or continuous) results in a random process.

- If we fix both $\omega = \omega_1$ and $t = t_0$, then

  $X(\omega, t) = X(\omega_1, t_0)$ = Deterministic Number

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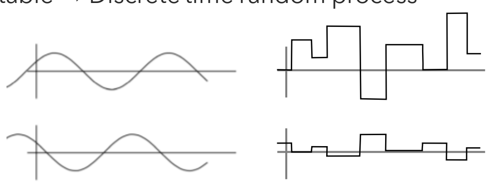
Continuous and Discrete Time RP

- For each outcome ω, the associated function X(ω, t) is called a sample path, realization, or trajectory.

- Let T = the range of t
- If T is uncountable → Continuous time random process
- If T is countable → Discrete time random process



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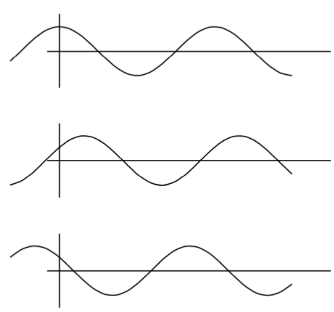
Example: Signal with Random Phase

- Let a random process
  $X(\omega, t) \;=\; A\cos(2\pi f t + \Phi(\omega))$
- where $\Phi(\omega)$ is a RV with ~ Unif$[0, 2\pi]$
- If we change $\omega, \Phi(\omega)$ will change,
- For each ω, we will get cos trajectory with different initial phase.



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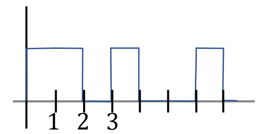
Example: Bernoulli Process

A Bernoulli process is a finite or infinite sequence of independent random variables $X_1, X_2, X_3 \cdots$ such that

For each i, the value of X_i is either 0 or 1

For all values of i, the probability that $X_i = 1$ is p.



Here, $T = \{1, 2, 3, \dots \dots, n\}$.

$$X(\omega, t) = \{X(\omega, 1), X(\omega, 2), X(\omega, 3), \cdot \cdot \cdot, X(\omega, n)\}$$

where each of $X(\omega, k)$ is a Bernoulli RV with probability p.

All X_i's independent and identically distributed.

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Examples: Brownian motion

- Motion of a solid molecule in a liquid.
- Let $X(\omega, t)$ denotes the location of a molecule at time t.

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Brownian Process

A Brownian Process W(t) is such that

$W(0) = 0$

It has Independent increments

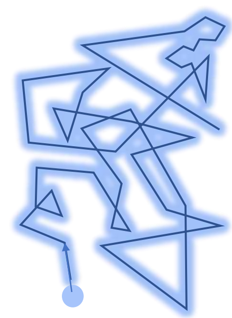
i.e., $let\ t_1 \leq t_2 \leq t_3$ then,

$W(t_2) - W(t_1)$ is independent of $W(t_3) - W(t_2)$

Let (s, t) be a time interval then ,

W(t) – W(s) is a Gaussian with Mean 0 and Variance $\sigma^2(t-s)$.

It has continuous sample paths



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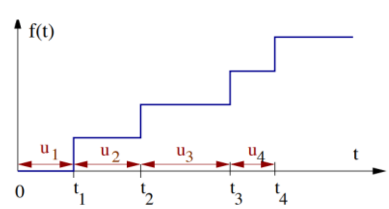
Counting Process

- Each waveform consists of a random set of increment points
- At every increment point, the process value increases by a fixed amount.



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