eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 10

Online/ Sequential Estimation

Online Estimation

- Consider the **SISO channel estimation** problem.

$$y = hx + n$$

SISO channel.

Wireless System Model

- The corresponding model is

$$y(1) = hx(1) + v(1)$$

$$y(2) = hx(2) + v(2)$$

$$\vdots$$

$$y(N) = hx(N) + v(N).$$

Wireless System Model

- The corresponding model is

$$
\begin{gathered}
y(1) = hx(1) + v(1) \\
y(2) = hx(2) + v(2) \\
\vdots \\
y(N) = hx(N) + v(N)
\end{gathered}
$$

Wireless System Model

- The channel estimate is

$$\hat{h} = \frac{\bar{x}^T \bar{y}}{\bar{x}^T \bar{x}} = \frac{\sum_{k=1}^{N} x(k) y(k)}{\sum_{k=1}^{N} x^2(k)}.$$

$$\bar{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}.$$

Wireless System Model

- The channel estimate is ML Estimate.

$$\hat{h} = \frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}}$$

Wireless System Model

- The quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \qquad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Wireless System Model

- The quantities are

$$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad \bar{\mathbf{y}} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Wireless System Model

- This can also be termed as estimate at time N

Estimate at time N.

Estimate is updated with time.

$$\hat{h}(N) = \frac{\sum_{k=1}^{N} x(k)y(k)}{\sum_{k=1}^{N} x^2(k)} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

$$\underbrace{\hat{h}(1), \hat{h}(2), \ldots, \hat{h}(N)}_{\text{Sequence of Estimates.}}$$

Wireless System Model

time N+1

- Consider now the next output

$$y(N+1) = hx(N+1) + v(N+1)$$

$y(N+1)$: output
$x(N+1)$: input/pilot.
} New set of output/input.

- Do we need to repeat estimation?
  - Can we simply **update** the previous estimate? $\hat{h}(N) \xrightarrow{\text{update}} \hat{h}(N+1)$?

Sequential Estimation

- This update process is termed **sequential estimation.**
  - Since estimation is carried out **sequentially** as outputs arrive…

$y(N+1), y(N+2), y(N+3), \ldots$

$$\underbrace{\hat{h}(N), \hat{h}(N+1), \hat{h}(N+2), \ldots}_{\text{SEQUENCE.}}$$

Online Estimation

- This is also termed **online estimation**.
  - Since estimation is being carried out **continuously**... Estimator is ONLINE!
  - And never stops

Online Estimation

- This can be achieved as follows.
- Note estimate at time $N + 1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N+1} x(k)\,y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

Channel Estimate
Time N+1.

Online Estimation

- This can be achieved as follows.
- Note estimate at time $N+1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N+1} x(k)y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

Online Estimation

- MSE at time N is

$p(N)$

MSE at time N.

$$p(N) = \frac{\sigma^2}{\|\bar{x}\|^2}$$

$$\Rightarrow \|\bar{\mathbf{x}}\|^2 = \frac{\sigma^2}{p(N)}.$$

Online Estimation

- MSE at time N is

$$p(N) = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2} \Rightarrow \|\bar{\mathbf{x}}\|^2 = \frac{\sigma^2}{p(N)}$$

Wireless System Model

- Therefore, it follows that

Estimate at time N.

$$\hat{h}(N) = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

$$\Rightarrow \bar{x}^T \bar{y} = \hat{h}(N) \cdot \|\bar{x}\|^2$$

$$= \frac{\sigma^2}{p(N)} \cdot \hat{h}(N).$$

Wireless System Model

- Therefore, it follows that

$$\hat{h}(N) = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\bar{\mathbf{x}}^T \bar{\mathbf{x}}} = \frac{\bar{\mathbf{x}}^T \bar{\mathbf{y}}}{\|\bar{\mathbf{x}}\|^2}$$

MSE at time N.

$$\Rightarrow \bar{\mathbf{x}}^T \bar{\mathbf{y}} = \hat{h}(N) \frac{\sigma^2}{p(N)}$$

Estimate at Time N

Online Estimation

- Estimate at time $N+1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N+1} x(k)y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sum_{k=1}^{N} x(k)y(k) + x(N+1)y(N+1)}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}.$$

Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N+1} x(k)y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sum_{k=1}^{N} x(k)y(k) + x(N+1)y(N+1)}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}$$

Online Estimation

- Estimate at time $N + 1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N} x(k)y(k) + x(N+1)y(N+1)}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}$$

$$= \frac{\bar{x}^T \bar{y} + x(N+1)\,y(N+1)}{\|\bar{x}\|^2 + x^2(N+1)}.$$

$$= \frac{\frac{\sigma^2}{p(N)} \cdot \hat{h}(N) + x(N+1)\,y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}.$$

Online Estimation

- Estimate at time $N+1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N} x(k)y(k) + x(N+1)y(N+1)}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}$$

$$= \frac{\frac{\sigma^2}{p(N)}\hat{h}(N) + x(N+1)y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

Online Estimation

- Estimate at time $N+1$ is

$$\hat{h}(N+1) = \frac{\frac{\sigma^2}{p(N)}\hat{h}(N) + x(N+1)y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\hat{h}(N)\left\{\frac{\sigma^2}{p(N)} + x^2(N+1) - x^2(N+1)\right\} + x(N+1)y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

Online Estimation

- Estimate at time $N+1$ is update.

$$\hat{h}(N+1) = \hat{h}(N) + \frac{x(N+1)y(N+1) - \hat{h}(N)x^2(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \hat{h}(N) + \frac{p(N)x(N+1)}{\underbrace{\sigma^2 + p(N)x^2(N+1)}_{k(N+1)}} \overbrace{\left\{ y(N+1) - \hat{h}(N)x(N+1) \right\}}^{e(N+1)}$$

Online Estimation

- Estimate at time $N+1$ is

$$\hat{h}(N+1) =$$

Online Estimation

Estimate at time N+1

- Estimate at time $N + 1$ is

Gain

Prediction error

$$\hat{h}(N+1) = \underbrace{\hat{h}(N) + k(N+1)\, e(N+1)}_{\text{Update Rule.}}$$

Innovation.

Estimate at time N.

Online Estimation

- Estimate at time $N+1$ is

$$\hat{h}(N+1) = \frac{\frac{\sigma^2}{p(N)}\hat{h}(N) + x(N+1)y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\hat{h}(N)\left(\frac{\sigma^2}{p(N)} + x^2(N+1) - x^2(N+1)\right) + x(N+1)y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

Online Estimation

- Estimate at time $N+1$ is

$$\hat{h}(N+1) = \hat{h}(N) + \frac{x(N+1)y(N+1) - \hat{h}(N)x^2(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \hat{h}(N) + \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}\Big(y(N+1) - \hat{h}(N)x(N+1)\Big)$$

Online Estimation

- This can be expressed as

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)$$

$$k(N+1) = \left.\frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}\right\} \text{Gain.}$$

$$e(N+1) = y(N+1) - \underline{\hat{h}(N)x(N+1)} \Big\} \text{Error.}$$

Online Estimation

MSE at time N ·

Gain

error

- This can be expressed as

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)$$

$$k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}$$

$$e(N+1) = \left(y(N+1) - \hat{h}(N)x(N+1)\right)$$

Online Estimation

- Thus we have derived the online estimator

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)$$

MSE Update

MSE at time N+1.

- The MSE can be updated as follows

$$p(N+1) = \frac{\sigma^2}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sigma^2}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \frac{\sigma^2}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sigma^2}{\underbrace{\sum_{k=1}^{N} x^2(k)}_{\|\bar{x}\|^2} + x^2(N+1)} = \frac{\sigma^2}{\frac{\sigma^2}{p(N)} + x^2(N+1)}.$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \frac{\sigma^2}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\sigma^2 p(N)}{\sigma^2 + p(N)x^2(N+1)}$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \left(\frac{\sigma^2}{\sigma^2 + p(N)x^2(N+1)} \right) p(N).$$

$$= \left(1 - \frac{p(N)x^2(N+1)}{\sigma^2 + p(N)x^2(N+1)} \right) p(N).$$

$$= \left(1 - \underbrace{\frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N\ 1)}}_{k(N+1).} \cdot x(N+1) \right) p(N).$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \left(1 - k(N+1)x(N+1)\right)p(N).$$

MSE update.

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \frac{\sigma^2}{\dfrac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\sigma^2 p(N)}{\sigma^2 + p(N)x^2(N+1)}$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \frac{\sigma^2 p(N)}{\sigma^2 + p(N)x^2(N+1)}$$

$$= \left(\frac{\sigma^2}{\sigma^2 + p(N)x^2(N+1)}\right)p(N)$$

$$= \left(1 - \frac{p(N)x^2(N+1)}{\sigma^2 + p(N)x^2(N+1)}\right)p(N)$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \left(1 - \frac{p(N)x^2(N+1)}{\sigma^2 + p(N)x^2(N+1)}\right)p(N)$$

$$= \left(1 - \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}x(N+1)\right)p(N)$$

$$= \big(1 - k(N+1)x(N+1)\big)p(N)$$

MSE Update

- The MSE can be updated as follows

$$p(N+1) = \big(1 - k(N+1)x(N+1)\big)p(N)$$

MSE update Rule.

MSE at time N+1.

MSE at time N

Online Estimation Vector Parameter

- Consider now the MISO channel estimation problem

$$\text{N outputs.}\left\{\begin{aligned} y(1) &= \bar{\mathbf{x}}^T(1)\bar{\mathbf{h}} + v(1) \\ y(2) &= \bar{x}^T(2)\bar{h} + v(2) \\ &\vdots \\ y(N) &= \bar{x}^T(N)\bar{h} + v(N) \end{aligned}\right.$$

Online Estimation Vector Parameter

- Consider now the MISO channel estimation problem

$$y(1) = \bar{\mathbf{x}}^T(1)\bar{\mathbf{h}} + v(1)$$
$$y(2) = \bar{\mathbf{x}}^T(2)\bar{\mathbf{h}} + v(2)$$
$$\vdots$$
$$y(N) = \bar{\mathbf{x}}^T(N)\bar{\mathbf{h}} + v(N)$$

MISO Channel Model

- This can be written in the matrix form vectors

$$\bar{y} = X\bar{h} + \bar{v}$$

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{y}} = \underbrace{\begin{bmatrix} \bar{x}^T(1) \\ \bar{x}^T(2) \\ \vdots \\ \bar{x}^T(N) \end{bmatrix}}_{X} \bar{h} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{v}}$$

MISO Channel Model

N×M
Pilot matrix

- This can be written in the matrix form vectors

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{\mathbf{y}}} = \underbrace{\begin{bmatrix} \bar{\mathbf{x}}^T(1) \\ \bar{\mathbf{x}}^T(2) \\ \vdots \\ \bar{\mathbf{x}}^T(N) \end{bmatrix}}_{\mathbf{X}} \bar{\mathbf{h}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{\mathbf{v}}}$$

ML Estimate

Estimate at time N.

- The LS estimate at time N is

$$\hat{\mathbf{h}}(N) = (X^T X)^{-1} X^T \bar{y}$$

ML Estimate

- The LS estimate at time N is

$$\hat{\mathbf{h}}(N) = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

Online Estimation Vector Parameter

- Consider now a new output

Time N+1.

$$y(N+1) = \bar{x}^T(N+1)\bar{h} + v(N+1)$$

Pilot vector time N+1.

Output at time N+1

Online Estimation Vector Parameter

- Consider now a new output
  $$y(N+1) = \bar{\mathbf{x}}^T(N+1)\bar{\mathbf{h}} + v(N+1)$$

Online Estimation Vector Parameter

- How to update $\hat{\mathbf{h}}(N)$?

To obtain $\hat{h}(N+1)$?

Try to deduce from scalar model.

Online Estimation

- The scalar model can be modified as

$$\underbrace{\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)}_{scalar\ parameter}$$

Gain vector

M X 1 vector

scalar

$$\hat{h}(N+1) = \underbrace{\hat{h}(N) + \bar{k}(N+1)\, e(N+1)}_{\text{Vector parameter.}}$$

Online Estimation

- The scalar model can be modified as

$$\underbrace{\hat{h}(N+1) = \hat{h}(N) + k(N+1)e(N+1)}_{scalar\ parameter}$$

$$\underbrace{\hat{\mathbf{h}}(N+1) = \hat{\mathbf{h}}(N) + \bar{\mathbf{k}}(N+1)e(N+1)}_{vector\ parameter}$$

Online Estimation

- The scalar model can be modified as

$$\underbrace{k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}}_{scalar\ parameter}$$

$$\underbrace{\bar{k}(N+1) = \frac{P(N)\bar{x}(N+1)}{\sigma^2 + \bar{x}^T(N+1)P(N)\bar{x}(N+1)}}_{\text{Vector Parameter.}}$$

Online Estimation

- The scalar model can be modified as

$$\underbrace{k(N+1) = \frac{p(N)x(N+1)}{\sigma^2 + p(N)x^2(N+1)}}_{scalar\ parameter}$$

$$\underbrace{\bar{\mathbf{k}}(N+1) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}}_{vector\ parameter}$$

Online Estimation

- The scalar model can be modified as

$$\underbrace{e(N+1) = \left(y(N+1) - x(N+1)\hat{h}(N)\right)}_{scalar\ parameter}$$

$$\underbrace{e(N+1) = y(N+1) - \bar{x}^T(N+1)\,\hat{h}(N).}_{\text{Vector parameter.}}$$

Online Estimation

- The scalar model can be modified as

$$\underbrace{e(N+1) = \left(y(N+1) - x(N+1)\hat{h}(N)\right)}_{scalar\ parameter}$$

$$\underbrace{e(N+1) = \left(y(N+1) - \bar{\mathbf{x}}^{T}(N+1)\hat{\mathbf{h}}(N)\right)}_{vector\ parameter}$$

Online Estimation Vector Parameter

- Therefore, the net model is

$$\hat{\mathbf{h}}(N+1) = \hat{\mathbf{h}}(N) + \bar{\mathbf{k}}(N+1)e(N+1)$$

$$\bar{\mathbf{k}}(N+1) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}$$

$$e(N+1) = \left(y(N+1) - \bar{\mathbf{x}}^T(N+1)\hat{\mathbf{h}}(N)\right)$$

Error Covariance Update

- The error covariance at time N is

$$\mathbf{P}(N) = \sigma^2 (X^T X)^{-1}$$

Error Covariance Update

- The error covariance at time N is

$$\mathbf{P}(N) = \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$$

Error Covariance Update

- Inspired by the scalar case, this can be updated for time N as

$$\underbrace{p(N+1) = \big(1 - k(N+1)x(N+1)\big)p(N)}_{scalar\ parameter}$$

error covariance at time N.

$$\underbrace{P(N) = \Big(I - \bar{k}(N+1)\,\bar{x}^T(N+1)\Big)P(N).}_{\text{vector parameter.}}$$

Error Covariance Update

- Inspired by the scalar case, this can be updated for time $N+1$ as

$$\mathbf{P}(N+1) = \left(\mathbf{I} - \bar{\mathbf{k}}(N+1)\bar{\mathbf{x}}^T(N+1)\right)\mathbf{P}(N)$$

Online Example

- Consider the problem

Output vector

N=4.

Pilot matrix

$$\bar{\mathbf{y}} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

Online Example

- The ML estimate is

$$X^T X = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} = 4I$$

$$\hat{\mathbf{h}}(\overset{\hat{h}(4)}{N}) = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \frac{1}{4} \cdot I \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$$

Online Example

- The ML estimate is

$\hat{h}(4)$

Estimate at time $N=4$.

$$\hat{\mathbf{h}}(N) = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$$

Online Example

- Let $\sigma^2 = 4$. Error covariance is

Error covariance matrix $N = 4$.

$$\mathbf{P}(N) = \sigma^2 (X^T X)^{-1} = 4 \cdot \frac{1}{4} I$$

$$P(4) = I$$

Online Example

- Let $\sigma^2 = 4$. Error covariance is

$$\mathbf{P}(N) = \sigma^2(\mathbf{X}^T\mathbf{X})^{-1} = \mathbf{I}$$

Online Example

- Consider now a new input-output

$N+1=5$

$$y(5) = -2, \bar{\mathbf{x}}(5) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$y(N+1) = -2 \quad \bar{x}(N+1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\bar{\mathbf{k}}(5) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}$$

$$= \frac{I \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix}}{4 + \begin{bmatrix} -2 & 2 \end{bmatrix}^T \cdot I \begin{bmatrix} -2 \\ 2 \end{bmatrix}} = \frac{1}{12}\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

scalar

$$\bar{\mathbf{k}}(5) = \frac{\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}{\sigma^2 + \bar{\mathbf{x}}^T(N+1)\mathbf{P}(N)\bar{\mathbf{x}}(N+1)}$$

No matrix inversion!

$$= \frac{\mathbf{I} \times \begin{bmatrix} -2 \\ 2 \end{bmatrix}}{4 + \begin{bmatrix} -2 & 2 \end{bmatrix} \mathbf{I} \times \begin{bmatrix} -2 \\ 2 \end{bmatrix}} = \frac{1}{12} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$e(5) = \left(y(5) - \bar{\mathbf{x}}^T(5)\hat{\mathbf{h}}(4)\right)$$

$$= \quad -2 - \begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} = -2 - (2-1)$$

$$= -2 - 1 = -3 = e(5)$$

Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$e(5) = \left(y(5) - \bar{\mathbf{x}}^T(5)\hat{\mathbf{h}}(4)\right)$$

$$= -2 - \begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = -3$$

Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\hat{\mathbf{h}}(5) = \hat{\mathbf{h}}(4) + \bar{\mathbf{k}}(5)e(5)$$

$\hat{h}(5)$.

$$= \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -2 \\ 2 \end{bmatrix}(-3) = \begin{bmatrix} -1 + \frac{1}{2} \\ -\frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}$$

Online Example

- The estimate $\hat{\mathbf{h}}(5)$ can be evaluated as follows

$$\hat{\mathbf{h}}(5) = \hat{\mathbf{h}}(4) + \bar{\mathbf{k}}(5)e(5)$$

update procedure has very low complexity!

$$= \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -2 \\ 2 \end{bmatrix}(-3) = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}$$

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