eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 3

(CRB)

Cramer-Rao Bound

Cramer-Rao Bound CRB

- This is a fundamental **lower bound** on MSE of the parameter estimate.

Mean Square Error

$$E\left\{(\hat{h}-h)^2\right\}$$

Cramer-Rao Bound

Indian

- The result is named in honor of **Harald Cramér** and **C. R. Rao**

Path Breaking Principle.

Harald Cramér

- **Harald Cramér** was a Swedish mathematician
  - specializing in ***mathematical statistics***

C. R. Rao

- **Calyampudi Radhakrishna Rao**, known as **C. R. Rao** is an Indian-American mathematician and statistician
- He has been described as "**a living legend**"

C. R. Rao

- His work has greatly influenced **statistics**,
  - and various other fields such as *economics*, *genetics*, *anthropology*, *geology*, *medicine* etc

- Has also been described as one of the **top 10 Indian scientists** of all time

Cramer-Rao Bound

- Consider the observation vector $\bar{\mathbf{y}}$
- The likelihood is

Likelihood of h

$$p(\bar{\mathbf{y}}; h)$$

Cramer-Rao Bound

- The log-likelihood is

Natural log of Likelihood function

$$\ln p(\bar{y}; h)$$

Cramer-Rao Bound

- The log-likelihood is

semicolon

$$\ln p(\bar{\mathbf{y}}; h)$$

Cramer-Rao Bound

- Let $\hat{h}$ be any unbiased estimator of h
- i.e.

$$\underbrace{E\{\hat{h}\}}_{\text{Mean of Estimate.}} = h$$

True value of parameter

Cramer-Rao Bound

- Let $\hat{h}$ be any unbiased estimator of h
- i.e.

$$E\{\hat{h}\} = h$$

Cramer-Rao Bound

Measure of information embodied by parameter

- The lower bound on MSE of $\hat{h}$ is

Fisher information parameter

$$\underbrace{E\left\{\left(\hat{h}-h\right)^2\right\}}^{\text{MSE}} \geq \frac{1}{I(h)}$$

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{y};h)\right)^2\right\}$$

Cramer-Rao Bound

unbiased
Estimator

- The lower bound on MSE of $\hat{h}$ is

MSE of UE $\geq$ Reciprocal of FI

$$E\left\{\left(\hat{h}-h\right)^2\right\} \geq \frac{1}{I(h)}$$

Fisher
information

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}, h)\right)^2\right\}$$

Cramer-Rao Bound

- $I(h)$ is termed the Fisher Information

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}, h)\right)^2\right\}$$

Example

- Let us explore this for our first model
- Noisy measurements

Noisy observations

Unknown parameter

iid Gaussian noise samples
mean = 0
var = σ^2

$$y(1) = h + v(1)$$
$$y(2) = h + v(2)$$
$$\vdots$$
$$y(N) = h + v(N)$$

Example

- Recall

$$p(\bar{\mathbf{y}}; h) = \overbrace{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-h\right)^2}}^{\text{Likelihood of } h}$$

Example

- Recall

Take ln to obtain Log Likelihood.

$$p(\bar{\mathbf{y}}; h) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k)-h)^2}$$

Example

- Therefore,

$$\ln p(\bar{\mathbf{y}}; h) =$$

$$\frac{N}{2} \ln \cdot \frac{1}{2\pi\sigma^2} - \frac{1}{2\sigma^2} \sum_{k=1}^{N} \left(y(k) - h\right)^2$$

Example

- Therefore,

Log Likelihood
for Noisy measurement
problem

$$\ln p(\bar{\mathbf{y}}; h)$$

$$= \frac{N}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k) - h)^2$$

Example

- It follows that

$$\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}};h) =$$

Partial Derivative of Log Likelihood wrto. unknown parameter h.

$$\frac{\partial}{\partial h}\left\{ \underbrace{\cancel{\frac{N}{2}\ln\frac{1}{2\pi\sigma^2}}}_{\text{constant.}} - \frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k)-h)^2 \right\}$$

$$-\frac{1}{2\sigma^2}\cdot\sum_{k=1}^{N}2(y(k)-h)(-1)$$

Example

$$y(k) = h + v(k)$$
$$\Rightarrow \underline{y(k) - h} = v(k).$$

$$\frac{\partial}{\partial h} \ln p(\bar{y}; h) = \frac{1}{\sigma^2} \sum_{k=1}^{N} \left(y(k) - h \right).$$

$$= \underbrace{\frac{1}{\sigma^2} \sum_{k=1}^{N} v(k)}$$

Example

- It follows that

$$\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}};h)$$

$$=\frac{\partial}{\partial h}\left(\frac{N}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right)-\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k)-h)^2\right)$$

$$=\frac{1}{\sigma^2}\sum_{k=1}^{N}(y(k)-h)=\frac{1}{\sigma^2}\sum_{k=1}^{N}v(k)$$

Example-Fisher Information

- The Fisher Information is

Fisher information

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}, h)\right)^2\right\}$$

Example-Fisher Information

$$E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{y};h)\right)^2\right\}$$

$$= E\left\{\left(\frac{1}{\sigma^2}\sum_{k=1}^{N} v(k)\right)^2\right\}$$

$$= \frac{1}{\sigma^4}\cdot E\left\{\left(\sum_{k=1}^{N} v(k)\right)^2\right\}$$

$$= \frac{1}{\sigma^4} E\left\{\left(\sum_{k=1}^{N} v(k)\right)\cdot\left(\sum_{l=1}^{N} v(l)\right)\right\}$$

Example-Fisher Information

$$= \frac{1}{\sigma^4} \cdot E\left\{ \sum_{k=1}^{N} \sum_{l=1}^{N} v(k)v(l) \right\}$$

iid. Gaussian noise samples
mean = 0
var = σ^2

$$= \frac{1}{\sigma^4} \sum_{k=1}^{N} \sum_{l=1}^{N} E\{v(k)v(l)\}$$

$$E\{v(k)v(l)\} \to \sigma^2 \delta(k-l)$$

$$= \begin{cases} \sigma^2 & k = l \\ 0 & k \neq l \end{cases}$$

$$= \frac{1}{\sigma^4} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma^2 \delta(k-l)$$

$$= \frac{1}{\sigma^4} \cdot \sum_{k=1}^{N} \sigma^2 = \frac{N\sigma^2}{\sigma^4} = \frac{N}{\sigma^2}$$

$I(h)$
Fisher information

Example-Fisher Information

- The Fisher Information is

$$
\begin{aligned}
I(h) &= E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}; h)\right)^2\right\} \\
&= E\left\{\left(\frac{1}{\sigma^2}\sum_{k=1}^{N} v(k)\right)^2\right\}
\end{aligned}
$$

Example-Fisher Information

$$I(h) = \frac{1}{\sigma^4} E\left\{ \left(\sum_{k=1}^{N} v(k) \right)^2 \right\}$$

$$= \frac{1}{\sigma^4} E\left\{ \left(\sum_{k=1}^{N} v(k) \right) \left(\sum_{l=1}^{N} v(l) \right) \right\}$$

Example-Fisher Information

$$I(h) = \frac{1}{\sigma^4} E\left\{ \sum_{k=1}^{N} \sum_{l=1}^{N} v(l)\, v(k) \right\}$$

$$= \frac{1}{\sigma^4} \sum_{k=1}^{N} \sum_{l=1}^{N} E\{v(l)v(k)\} = \frac{1}{\sigma^4} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma^2 \delta(k-l)$$

$$= \frac{1}{\sigma^4} \sum_{k=1}^{N} \sigma^2 = \frac{N\sigma^2}{\sigma^4} = \frac{N}{\sigma^2}$$

Example – Cramer Rao Bound (CRB)

- The CRB for the problem is

$$E\{(\hat{\mu}-\mu)^2\} \geq \frac{1}{I(\mu)}$$

$$= \frac{1}{N/\sigma^2} = \frac{\sigma^2}{N}$$

Cramer Rao Bound.

Example – Cramer Rao Bound (CRB)

- The CRB for the problem is

$$\frac{1}{I(h)} = \frac{\sigma^2}{N}$$

Example – Cramer Rao Bound (CRB)

- Therefore, it follows that MSE of any unbiased estimator is

Mean squared Error

$$E\left\{\left(\hat{h}-h\right)^2\right\} \geq \frac{\sigma^2}{N}$$

Example – Cramer Rao Bound (CRB)

- Recall that the MSE of the MLE is exactly

$$E\left\{\left(\hat{h}_{ML} - h\right)^2\right\} = \frac{\sigma^2}{N}$$

Any other estimator cannot have lower MSE!!

Maximum Likelihood Estimate

achieves the CRB!!

Example – Cramer Rao Bound (CRB)

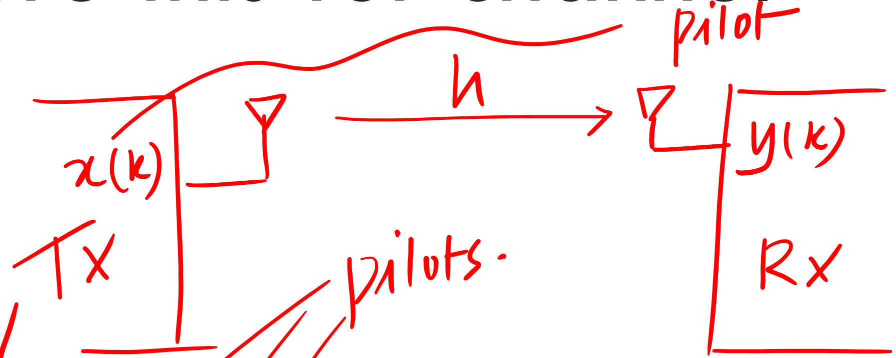
- Therefore, MLE achieves the CRB

  'Efficient Estimator'

- It is termed an **efficient estimator.**

Example- Channel Estimation

- Let us now explore this for channel Estimation



Fading channel coeff.

Outputs:

$$\begin{cases} y(1) = h\,x(1) + v(1) \\ y(2) = h\,x(2) + v(2) \\ \vdots \\ y(N) = h\,x(N) + v(N) \end{cases}$$

iid Gaussian noise samples
mean $= 0$
var $= \sigma^2$

$y(k)$ Gaussian
Mean $= h\,x(k)$
var $= \sigma^2$

Example- Channel Estimation

- Recall

$$p(\bar{\mathbf{y}}; h) = \underbrace{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k) - h\,x(k)\right)^2}}_{\text{Likelihood of } h}$$

channel coefficient

Example- Channel Estimation

- Recall

$$p(\bar{\mathbf{y}};h) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2}$$

Example- Channel Estimation

- Therefore,

log likelihood.

$$\ln p(\bar{\mathbf{y}}; h) =$$

$$\frac{N}{2} \ln \frac{1}{2\pi\sigma^2} - \frac{1}{2\sigma^2} \sum_{k=1}^{N} \left(y(k) - h\, x(k) \right)^2$$

Example- Channel Estimation

- Therefore,

$$\ln p(\bar{\mathbf{y}}; h)$$
$$= \frac{N}{2} \ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2} \sum_{k=1}^{N} \big(y(k) - hx(k)\big)^2$$

Example- Channel Estimation

- It follows that

$$\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}};h) =$$

Partial Derivative of Log likelihood w.r.t. unknown parameter h

constant $\Rightarrow \frac{\partial}{\partial h} = 0$.

$$\frac{\partial}{\partial h}\left\{ \cancel{\frac{N}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right)} - \frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2 \right\}$$

$$\cancel{-}\frac{1}{\cancel{2}\sigma^2}\sum_{k=1}^{N}\cancel{2}\left(y(k)-hx(k)\right)\left(\cancel{-}x(k)\right)$$

Example- Channel Estimation

$$\frac{\partial}{\partial h} \ln p(\bar{y}; h) = \frac{1}{\sigma^2} \sum_{k=1}^{N} \left(y(k) - h x(k) \right) x(k).$$

$$y(k) = h x(k) + v(k)$$

$$\Rightarrow y(k) - h x(k) = v(k)$$

$$= \frac{1}{\sigma^2} \sum_{k=1}^{N} v(k) x(k).$$

Example- Channel Estimation

- It follows that

$$\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}};h)$$

$$=\frac{\partial}{\partial h}\left(\frac{N}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right)-\frac{1}{2\sigma^2}\sum_{k=1}^{N}\left(y(k)-hx(k)\right)^2\right)$$

$$=\frac{1}{\sigma^2}\sum_{k=1}^{N}x(k)(y(k)-h)=\frac{1}{\sigma^2}\sum_{k=1}^{N}x(k)v(k)$$

Example- Channel Estimation

- The Fisher Information is

Fisher information

$= E\{$ square of $\{$ partial Derivative of $\{$ Log likelihood wrto h $\}\}\}$,

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}, h)\right)^2\right\}$$

Example- Channel Estimation

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{y};h)\right)^2\right\}$$

$$= E\left\{\left(\frac{1}{\sigma^2}\cdot\sum_{k=1}^{N} x(k)v(k)\right)^2\right\}$$

$$= \frac{1}{\sigma^4}\cdot E\left\{\left(\sum_{k=1}^{N} x(k)v(k)\right)^2\right\}$$

$$= \frac{1}{\sigma^4}\cdot E\left\{\left(\sum_{k=1}^{N} x(k)v(k)\right)\left(\sum_{l=1}^{N} x(l)v(l)\right)\right\}$$

$$= \frac{1}{\sigma^4} E\left\{\sum_{k=1}^{N}\sum_{l=1}^{N} x(k)x(l)v(k)v(l)\right\}$$

Example- Channel Estimation

$\delta(u)$ = Discrete Delta Function.

$$= \frac{1}{\sigma^4} \sum_{k=1}^{N} \sum_{l=1}^{N} x(k)\, x(l)\, E\{v(k)\, v(l)\}$$

iid zero mean Noise samples

$$E\{v(k)\, v(l)\} = \sigma^2 \delta(k-l).$$

considering Real quantities.

$$= \frac{1}{\sigma^4} \cdot \sum_{k=1}^{N} \sum_{l=1}^{N} x(k)\, x(l)\, \sigma^2 \delta(k-l)$$

$$= \frac{1}{\sigma^4} \cdot \sum_{k=1}^{N} x^2(k)\, \sigma^2 = \underbrace{\frac{1}{\sigma^2} \cdot \|\bar{x}\|^2}_{\text{Fisher information.}}$$

Example- Channel Estimation

- The Fisher Information is

$$I(h) = E\left\{\left(\frac{\partial}{\partial h}\ln p(\bar{\mathbf{y}}, h)\right)^2\right\}$$

$$= E\left\{\left(\frac{1}{\sigma^2}\sum_{k=1}^{N} x(k)v(k)\right)^2\right\} = \frac{\|\bar{\mathbf{x}}\|^2}{\sigma^2}$$

Example- Channel Estimation

- The CRB for the problem is

Cramer Rao bound.

$$E\left\{\left(\hat{h}-h\right)^2\right\} \geq \frac{1}{I(h)}$$

$$= \frac{1}{\|\bar{x}\|^2/\sigma^2} = \frac{\sigma^2}{\|\bar{x}\|^2}$$

Fundamental.
lower bound on
MSE of any unbiased
Estimator.

Example- Channel Estimation

- The CRB for the problem is

$$\frac{1}{I(h)} = \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

Example- Channel Estimation

- Therefore, it follows that MSE of any unbiased estimator is

MSE of any unbiased Estimator

$$E\left\{\left(\hat{h}-h\right)^2\right\} \geq \frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

Example- Channel Estimation

- Recall that the MSE of the MLE is exactly

$$E\left\{\left(\hat{h}_{ML}-h\right)^2\right\}=\frac{\sigma^2}{\|\bar{\mathbf{x}}\|^2}$$

Maximum Likelihood Estimate.

Example- Channel Estimation

- Therefore, MLE achieves the CRB

There cannot be any better unbiased Estimator!!

- It is termed an **efficient estimator**.

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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