Solutions of Tutorial-4

Problem set 3.4

1 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \\ c \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is solved by $\boldsymbol{c} = (1, 1, -4, 1)$. Then $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{v}_4 = \mathbf{0}$ (dependent).

5 (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ columns add to $\mathbf{0}$.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

(changing -1's to 1's for the matrix A in solution **7** above makes A invertible.)

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False*. The zero vector is always in the row space.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

24 (a) *False* $A = [1 \;\; 1]$ has dependent columns, independent row (b) *False* Column space $\neq$ row space for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) *True*: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) *False*, columns may be dependent, in that case not a basis for $\boldsymbol{C}(A)$.

Problem set 3.5

2 A: Row space basis $=$ row $1 = (1,2,4)$; nullspace $(-2,1,0)$ and $(-4,0,1)$; column space basis $=$ column $1 = (1,2)$; left nullspace $(-2,1)$. B: Row space basis $=$ both rows $= (1,2,4)$ and $(2,5,8)$; column space basis $=$ two columns $= (1,2)$ and $(2,5)$; nullspace $(-4,0,1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y} = \mathbf{0}$: the rows of B are independent.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & -3 \\ 3 & -1 \end{bmatrix}$

(e) *Impossible* Row space $=$ column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$. So $\boldsymbol{v} = \mathbf{0}$.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

Problem set 4.1

3 (a) One way is to use these two columns directly: $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$

(b) Impossible because $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal subspaces: $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ is not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular

(d) Rows orthogonal to columns makes A times A = zero matrix ρ. An example is $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(e) $(1,1,1)$ in the nullspace (columns add to the zero vector) and also $(1,1,1)$ is in the row space: no such matrix.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is also in the *nullspace* of A^{T}. Those subspaces are perpendicular. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \mathbf{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column and row spaces are the *same*. The nullspace is always perpendicular to the row space. (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" $\boldsymbol{x}$ and $\boldsymbol{z}$ have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

20 If $\boldsymbol{V}$ is the whole space $\mathbf{R}^4$, then $\boldsymbol{V}^{\perp}$ contains only the *zero vector*. Then $(\boldsymbol{V}^{\perp})^{\perp}$ = all vectors perpendicular to the zero vector = $\mathbf{R}^4 = \boldsymbol{V}$.

25 If the columns of A are unit vectors, all mutually perpendicular, then $A^{\mathrm{T}}A = I$. Simple but important ! We write Q for such a matrix.

Problem set 4.2

1 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 5/3$; $\boldsymbol{p} = 5\boldsymbol{a}/3 = (5/3, 5/3, 5/3)$; $\boldsymbol{e} = (-2, 1, 1)/3$

(b) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = -1$; $\boldsymbol{p} = \boldsymbol{a}$; $\boldsymbol{e} = \mathbf{0}$.

2 (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$
(b) The projection of $\boldsymbol{b} = (1,1)$ onto $\boldsymbol{a} = (1,-1)$ is $\boldsymbol{p} = (0,0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

The picture for part (a) has the vector $\boldsymbol{b}$ at an angle θ with the horizontal $\boldsymbol{a}$. The picture for part (b) has vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ at a $90°$ angle.

23 If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $\boldsymbol{e} = \mathbf{0}$.

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$. But the $(2,2)$ entry of $P^{\mathrm{T}}P$ is the length squared of column 2.