Linear independence and different subspaces

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Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

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  - Systematically calculate complete solution of $A\mathbf{x} = \mathbf{b}$

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- Discuss the linear independence, column space and row space today
  - Chapter 3.4 and 3.5 of the book

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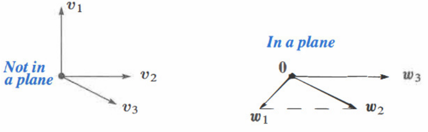

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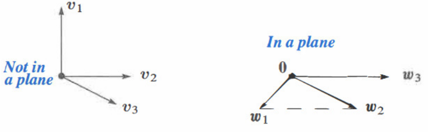

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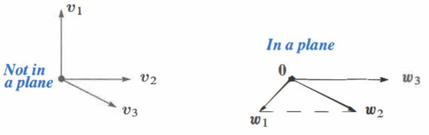

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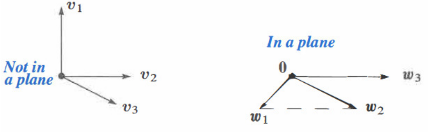

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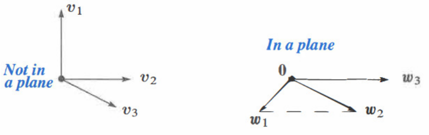

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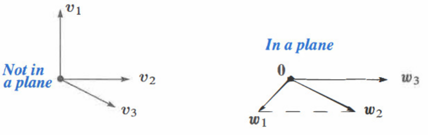

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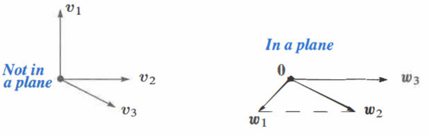

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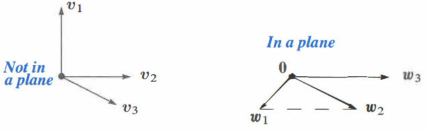

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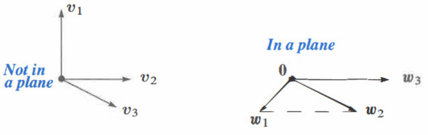

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  - Any set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n > m$

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  - There are n pivots and no free variables, only $\mathbf{x} = 0$ is in the $N(A)$

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- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ span only a line in $\mathbf{R}^2$

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- Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis for $\mathbf{R}^n$ when they are the columns of an $n \times n$ invertible matrix

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  - Also, $C(A^T)$ is the same as $C(R^T)$

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- Pivot columns of A are a basis for its column space
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Four Subspaces of example R matrix (1)

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$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ with } m = 3,\ n = 5 \text{ and } r = 2$$

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