EE908 Assignment-7 Solution

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eMasters in Communication Systems, IITK
EE908: Optimization in SPCOM
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Q1. Show that the following two problems are duals of each other.

$$p^* = \min \max_i (P^T u)_i$$
$$s.t.\ \ u \geq 0, \sum_{j=1}^{n} v_j = 1$$

And

$$d^* = \max \min_j (Pv)_j$$
$$s.t\ \ v \geq 0, \sum_{j=1}^{n} v_j = 1$$

Does it hold that $p^* = d^*$?
This result is the famous minimax theorem of two-person zero-sum games, first provided in Von Neumann's 1928 paper titled Zur Theorie der Gesellschaftsspiele.
Solution:
Reformulating the primal problem,

$$p* = \min \lambda\ \ s.t.* \ (P^T u)_i \leq \lambda\ \forall i, u \geq 0$$

p^* is the smallest λ such that all components of $P^T u$ are less than or equal to λ

Reformulating the dual problem,

$$d^* = \max \mu\ \ \ s.t. (Pv)_j \geq \mu\ \forall j, v \geq 0, \Sigma_{j=1}^{n} v_j = 1$$

d^* is the largest μ such that all components of Pv are greater than or equal to μ

Primal problem's constraint $(P^T u)_i \leq \lambda$ can be rewritten as
$(P^T u)_i \leq \lambda 1$
Taking transpose,
$u^T P \leq \lambda 1^T$

Dual problem's constraint $(Pv)_j \geq \mu$ can be rewritten as
$Pv \geq \mu$
As per strong duality, if there is an optimal solution to both the primal and dual problems - p^* and $d^*, p^* = d^*$

Per minimax theorem,
$\min_{u \geq 0} \max_i (P^T u)_i = \max_{v \geq 0, \Sigma_j v_j = 1} \min_j (Pv)_j$
Thus $p^* = d^*$ holds.

Q2. Find the dual of the penalty function approximation

$$\min \sum_{i=1}^{m} \phi(r_i)$$
$$s.t.\ \ r = Ax - b$$

Where ϕ is the deadzone linear penalty function

$$\phi(u) = \begin{cases} 0, & |u| \le 1 \\ |u| - 1, & |u| > 1 \end{cases}$$

Solution:

Lagrangian $L(x, r, \lambda) = \sum_{i=1}^{m} \phi(r_i) + \nu^T(Ax - b - r)$

The minimum is bounded iff $A^t\nu = 0$,

$$\therefore\ g(\nu) = \begin{cases} -b^T\nu + \sum_{i=1}^{m}\left(\min_{r_i}(\phi(r_i) - \nu_i r_i)\right), & A^T\nu = 0 \\ -\infty, & otherwiese \end{cases}$$

$\min_{r_i}(\phi(r_i) - \nu_i r_i) = -\max_{r_i}\left(\nu_i r_i - \phi(r_i)\right) = -\phi^*(\nu_i)$

So, the general dual can be expressed as,

$\max\left(-b^T\nu - \sum_{i=1}^{m}\phi^*(\nu_i)\right)\ s.t. A^T\nu = 0$

The dual of the dead-zone linear function approximation problem is:

$\max -b^T\nu - \|\nu\|_1\ s.t\ A^T\nu = 0,\ \ \|\nu\|_\infty \le 1$

Q3. Consider the following non-convex problem

$$p^* = \min x^T A x$$
$$s.t.\ \ x_i \in \{-1, 1\}$$

Where $A \in \mathbb{S}^{n\times n}$.

Show that

$$n\lambda_{min}(A) \le p^* \le \sum_{i,j} A_{ij}$$

Hint: Express the constraint as $x_i^2 = 1$ and use weak duality

Solution:

$x_i \in \{-1, 1\} \Rightarrow x_i^2 = 1 \Rightarrow p^* = \min_{x\in\{-1,1\}^n} x^T A x$

Upper bound on p^* (when $x = 1$)$x^T A x = \sum_{i,j} A_{ij} x_i x_j = \sum_{i,j} A_{ij}$

Thus $p^* \le \sum_{i,j} A_{ij}$

Lower bound:

A is symmetric matrix $\Rightarrow$ Rayleigh quotient $\frac{x^T A x}{x^T x} \ge \lambda_{\min}(A)\ (Eigen\ value)$

$x^T x = n$

$\therefore \dfrac{x^T A x}{n} \ge \lambda_{min}(A) \Rightarrow x^T A x \ge n\lambda_{min}(A)$

The lower bound $p^* \ge n\lambda_{min}(A)$

Combing both the bounds,

$n\lambda_{min}(A) \le p^* \le \sum_{i,j} A_{ij}$

QED

Q4. Find the dual of the convex piece-wise linear minimization problem:

$$\min \max_{i=1,\ldots m} (a_i^T x + b_i)$$

Solution:

Say $t \ge a_i^T x + b_i\ \ \forall i = 1, \ldots m$

Primal problem $\Rightarrow \min t\ \ s.t. t \ge a_i^T x + b_i\ \ \forall i = 1, \ldots m$

$\Rightarrow L(x, t, \lambda) = t + \sum_{i=1}^{m}\left(\lambda_i(a_i^T x + b_i - t)\right)$

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"...dare to dream; care to win..."

Dual function $g(x,t,\lambda) = \min_t L(x,t,\lambda) = \min_t \left(t + \Sigma_{i=1}^m \left(\lambda_i(a_i^T x + b_i - t)\right)\right)$
$= \min_t\left(t(1 - \Sigma_{i=1}^m \lambda_i)\right) + \Sigma_{i=1}^m \left(\lambda_i(a_i^T x + b_i)\right)$
For the dual to be bounded below, t coefficient must be zero
$1 - \Sigma_{i=1}^m \lambda_i = 0$
$\Sigma_{i=1}^m \lambda_i = 1$
The dual function becomes:
$g(\lambda) = \Sigma_{i=1}^m \lambda_i(a_i^T x + b_i)$

Minimizing w.r.t x:
$g(\lambda) = \min_x((\Sigma_{i=1}^m \lambda_i a_i)^T x + \Sigma_{i=1}^m \lambda_i b_i)$
$(\Sigma_{i=1}^m \lambda_i a_i)^T x = 0$ for the dual function to be bounded.
$\Sigma_{i=1}^m \lambda_i a_i = 0$
$\therefore\ g(\lambda) = \Sigma_{i=1}^m \lambda_i b_i$

Dual problem is to maximize the dual function $g(\lambda)$ subject to dual constraints
$\max_\lambda \Sigma_{i=1}^m (\lambda_i b_i)$
$s.t. \Sigma_{i=1}^m \lambda_i = 1$
$\Sigma_{i=1}^m \lambda_i a_i = 0$
$\lambda_i \geq 0, i = 1 \dots m$

Q5. Consider the following convex optimization problem:

$$\min \sum_{i=1}^{m} exp(x_i - 1) + y$$
$$s.t. Ax - b + y1 \geq 0$$

Use appropriate change of variables and elimination to show that it can equivalently be written as:

$$\min \log\left(\sum_{i=1}^{m} e^{u_i}\right)$$
$$s.t.\ \ Au - b \geq 0$$

If it holds that A1=1
Solution:
Rewriting the original objective function,
$\Sigma_{i=1}^m e^{(x_i-1)} + \Sigma_{i=1}^m y = \Sigma_{i=1}^m e^{(x_i-1)} + my$

Rewriting the original constraint,
$Ax - b + y1 \geq 0 \Rightarrow Ax - b + y.1 \geq 0 \Rightarrow Ax - b + y \geq 0$

Changing the variable,
Say $u_i = x_i - 1 \Rightarrow x_i = u_i + 1$ and substituting into the constraints,
$A(u+1) - b + y \geq 0 \Rightarrow A(u+1) - b + y \geq 0$
Since $A1 = 1$
$A(u+1) = Au + A1 = Au + 1$
Thus, the constraints become,
$Au + 1 - b + y \geq 0 \Rightarrow Ay - b + (1+y) \geq 0$

Defining a new variable $z = 1 + y$ and rewriting the constraints,
$Au - b + z \geq 0 \Rightarrow z \geq b - Au$

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Substituting $x_i = u_i + 1$ into the objective function,
$\Sigma_{i=1}^{m} e^{x_i - 1} + y = \Sigma_{i=1}^{m} e^{u_i} + my$
$z = 1 + y \Rightarrow y = z - 1$
∴ Objective function becomes,
$\Sigma_{i=1}^{m} e^{u_i} + m(z-1)$

Simplifying,
$\Sigma_{i=1}^{m} e^{u_i} + mz - m$

Dropping the constant term $-m$
$\Sigma_{i=1}^{m} e^{u_i} + mz \; s.t. z \geq b - Au$

z acts as a slack variable and its minimum feasible value is determined by the constraint
$z = \max(b - Au)$

Hence minimizing $\Sigma_{i=1}^{m} e^{u_i} + mz$ is equivalent to minimizing the log-sum-exp term
$\min \log(\Sigma_{i=1}^{m} e^{u_i}) \; s.t. Au - b \geq 0$

QED

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