eMasters in Communication Systems

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Elective Module:

Estimation for Wireless Communication

Chapter 8

Minimum Mean Square Error

MMSE Estimation

MMSE Estimation

- MMSE = Minimum Mean Squared Error.

ML = Maximum Likelihood.

MMSE Estimation

- Consider **observation vector** $\bar{\mathbf{y}}$ $N \times 1$
- **Unknown parameter** vector $\bar{\mathbf{h}}$ $M \times 1$

$$\bar{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} \qquad \bar{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}$$

MMSE Estimation

BAYESIAN PHILOSOPHY.

- However, in this, $\bar{\mathbf{y}}, \bar{\mathbf{h}}$ are **random** in nature

ML : $\bar{h}$ Deterministic unknown

RANDOM.

Fundamental.

MMSE Estimation

minimize.

- The cost-function for the **MMSE estimate** of $\bar{\mathbf{h}}$ is given as

$$\text{ML}: \underbrace{\min \left\| \bar{y} - X\bar{h} \right\|^2}_{\text{LEAST SQUARES.}}$$

minimum Mean Square Error

$$\text{MMSE}: \min \underbrace{E\left\{ \underbrace{\left\| \underbrace{\hat{h} - \bar{h}}_{\text{Error}} \right\|^2}_{\text{Square Error}} \right\}}_{\text{MEAN SQUARE Error}}$$

MMSE Estimation

- The cost-function for the **MMSE estimate** of $\bar{\mathbf{h}}$ is given as

$$\min \overbrace{E\left\{\left\|\hat{\mathbf{h}}-\bar{\mathbf{h}}\right\|^2\right\}}^{\text{MMSE cost function}}$$

MMSE Estimation

- The expression for the **MMSE estimate** is

Simple But challenging in practice!

$$\hat{h} = E\{\bar{h} \mid \bar{y}\}$$

in general fairly difficult to evaluate!

General Expression for MMSE Estimate

conditional expectation of $\bar{h}$ given $\bar{y}$.

MMSE Estimation

- The expression for the **MMSE estimate** is

$$\hat{\mathbf{h}} = E\left\{\bar{\mathbf{h}} \,|\, \bar{\mathbf{y}}\right\}$$

General. Expression

MMSE Estimation

zero mean
Gaussian.
Random vectors.

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, this can be simplified as

zero mean.

$E\{\bar{h}\} = 0 \quad E\{\bar{y}\} = 0$

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = R_{hy} R_{yy}^{-1} \bar{y}$$

$E\{\bar{h}|\bar{y}\}$.

Note: MMSE only when $\bar{h}, \bar{y}$ are jointly Gaussian

MMSE Estimation

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, this can be simplified as

MMSE Estimate

$\bar{h}, \bar{y}$ are zeromean. Gaussian.

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

MMSE Estimation

- $\mathbf{R}_{yy}$ is covariance matrix of $\bar{y}$

$$\mathbf{R}_{yy} = E\{\bar{y}\,\bar{y}^T\} \quad N \times N$$

$$E\left\{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}\begin{bmatrix} y(1) & y(2) & \cdots & y(N) \end{bmatrix}\right\}$$

MMSE Estimation

- $\mathbf{R}_{yy}$ is

covariance matrix
$\bar{y}$

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

MMSE Estimation

- $\mathbf{R}_{hy}$ is

cross covariance of $\bar{h}, \bar{y}$

zero mean

$M \times N$

$$\mathbf{R}_{hy} = E\{\bar{h}\,\bar{y}^T\}$$

$$= E\left\{ \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} \begin{bmatrix} y(1) & y(2) & \cdots & y(N) \end{bmatrix} \right\}$$

MMSE Estimation

- $\mathbf{R}_{hy}$ is

$$\mathbf{R}_{hy} = E\{\bar{\mathbf{h}}\bar{\mathbf{y}}^T\}$$

$$\underline{\hat{h} = R_{hy} R_{yy}^{-1} \bar{y}}\,.$$

MMSE Estimation

- Consider now the MISO channel estimation model

multiple input single output.

$$\overset{N\times 1}{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}} = \overbrace{\begin{bmatrix} \bar{x}^T(1) \\ \bar{x}^T(2) \\ \vdots \\ \bar{x}^T(N) \end{bmatrix}}^{X \sim N \times M} \overset{M\times 1}{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}} + \overset{N\times 1}{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}$$

$$\bar{y} = X\bar{h} + \bar{v}$$

MMSE Estimation

- Consider now the MISO channel estimation model

Gaussian.

Gaussian

Gaussian

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\bar{\mathbf{y}}} = \underbrace{\begin{bmatrix} \bar{\mathbf{x}}^T(1) \\ \bar{\mathbf{x}}^T(2) \\ \vdots \\ \bar{\mathbf{x}}^T(N) \end{bmatrix}}_{\mathbf{X}} \bar{\mathbf{h}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{\mathbf{v}}}$$

$$\bar{y} = X\bar{h} + \bar{v}$$

MMSE Estimation

Gaussian

Gaussian

Linear Estimation model : $\bar{y} = \underbrace{X \bar{h} + \bar{v}}_{\text{Linear combination of Gaussian Random Vectors.}}$: Linear Gaussian model.

Unknown.

$\Rightarrow \bar{y}$ is also Gaussian.

$\Rightarrow \bar{y}, \bar{h}$ Gaussian.

MMSE Estimation

- We have

unknown parameter vector

$$\bar{\mathbf{h}} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}$$

MMSE Estimation

- We have

$$\bar{\mathbf{h}} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}$$

MMSE Estimation

independent
identically
distributed

- Let the elements h_i be zero-mean i.i.d. Gaussian of variance σ_h^2

variance

$$E\{h_i\} = 0$$

$$E\{h_i^2\} = \sigma_h^2$$

$$i \neq j \quad E\{h_i h_j\} = E\{h_i\} \cdot E\{h_j\} = 0$$

MMSE Estimation

- Then,

covariance matrix of $\bar{h}$?

$M \times M$

$$\mathbf{R}_{hh} = E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\} = E\left\{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}\begin{bmatrix} h_1 & h_2 & \cdots & h_M \end{bmatrix}\right\}$$

$$= E\left\{\begin{bmatrix} h_1^2 & h_1 h_2 & & \\ h_2 h_1 & h_2^2 & & \\ \vdots & & \ddots & \\ & & & h_M^2 \end{bmatrix}\right\}$$

MMSE Estimation

$$\mathbf{R}_{hh} = \begin{bmatrix} \sigma_h^2 & 0 & \cdots & 0 \\ 0 & \sigma_h^2 & & \\ \vdots & & \ddots & \\ 0 & & & \sigma_h^2 \end{bmatrix}$$

covariance matrix of $\bar{h}$

$$R_{hh} = \sigma_h^2 \, I$$

$M \times M$

MMSE Estimation

- Then,

$$\mathbf{R}_{hh} = E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\}$$
$$= E\left\{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix}\begin{bmatrix} h_1 & h_2 & \dots & h_M \end{bmatrix}\right\}$$

MMSE Estimation

Diagonal elements h_i^2

$E\{h_i^2\} = \sigma_h^2$

- Then,

$$\mathbf{R}_{hh} = E\left\{\begin{bmatrix} h_1^2 & h_1 h_2 \\ h_2 h_1 & h_2^2 \end{bmatrix}\right\}$$

Off diagonal elements $h_i h_j$

$E\{h_i h_j\} = 0.$

MMSE Estimation

- Then,

$$\mathbf{R}_{hh} = \begin{bmatrix} \sigma_h^2 & 0 & & \\ 0 & \sigma_h^2 & & \\ & & \ddots & \\ & & & \end{bmatrix} = \sigma_h^2 \mathbf{I}$$

MMSE Estimation

- Similarly, let the elements $v(i)$ be zero-mean i.i.d. Gaussian of variance σ^2

$$\bar{v} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

iid zeromean
Gaussian
Variance σ^2

MMSE Estimation

- Then,

$$\mathbf{R}_{vv} = E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\} = \sigma^2 I$$

NxN matrix

MMSE Estimation

- Then,

covariance matrix of $\bar{v}$

$$\mathbf{R}_{vv} = E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^H\} = \sigma^2\mathbf{I}$$

MMSE Estimation

- Consider now the MISO channel estimation model

We need: R_{yy}, R_{hy}?

How to evaluate?

$$\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}$$

MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

$$= E\{(X\bar{h}+\bar{v})(X\bar{h}+\bar{v})^T\}$$

$$= E\{X\bar{h}\bar{h}^T X^T + \bar{v}\bar{h}^T X^T + X\bar{h}\bar{v}^T + \bar{v}\bar{v}^T\}$$

$$= X E\{\bar{h}\bar{h}^T\} X^T + \cancel{E\{\bar{v}\bar{h}^T\}}^{\,0} X^T$$

$$+ X \cancel{E\{\bar{h}\bar{v}^T\}}^{\,0} + E\{\bar{v}\bar{v}^T\}$$

MMSE Estimation

$$E\{\bar{h}\bar{v}^T\} = ? \quad E\{\bar{v}\bar{h}^T\} = ?$$

$\bar{v}$: Thermal noise @ Receiver

$\bar{h}$: Random channel because of scattering environment

$\Rightarrow$ $\bar{h}, \bar{v}$ are independent

$$\Rightarrow \quad E\{\bar{h}\bar{v}^T\} = 0$$

$$E\{\bar{v}\bar{h}^T\} = 0.$$

MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$

$$= X E\{\bar{h}\bar{h}^T\} X^T + E\{\bar{v}\bar{v}^T\}$$

$$= X \sigma_h^2 I X^T + \sigma^2 I$$

$$= \sigma_h^2 X X^T + \sigma^2 I$$

MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = E\{\overline{\mathbf{y}}\,\overline{\mathbf{y}}^T\}$$

$$=$$

MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = E\{\bar{\mathbf{y}}\bar{\mathbf{y}}^T\}$$
$$= \mathrm{E}\left\{(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}})(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}})^T\right\}$$
$$= E\left\{\mathbf{X}\bar{\mathbf{h}}\bar{\mathbf{h}}^T\mathbf{X}^T\right\} + E\left\{\bar{\mathbf{v}}\bar{\mathbf{h}}^T\mathbf{X}^T\right\}$$
$$+ E\left\{\mathbf{X}\bar{\mathbf{h}}\bar{\mathbf{v}}^T\right\} + E\left\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\right\}$$

MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = \mathbf{X}E\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\}\mathbf{X}^T + \cancelto{0}{E\{\bar{\mathbf{v}}\bar{\mathbf{h}}^T\}\mathbf{X}^T}$$
$$+\cancelto{0}{\mathbf{X}E\{\bar{\mathbf{h}}\bar{\mathbf{v}}^T\}} + E\{\bar{\mathbf{v}}\bar{\mathbf{v}}^T\}$$

MMSE Estimation

- It follows that

$$\mathbf{R}_{yy} = \sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}$$

covariance matrix of $\bar{y}$

N X N matrix.

MMSE Estimation

cross covariance matrix $\bar{h}, \bar{y}$

- Also

$$\mathbf{R}_{hy} = E\{\bar{h}\bar{y}^T\}$$

$$= E\{\bar{h}(X\bar{h}+\bar{v})^T\}$$

$$= E\{\bar{h}(\bar{h}^T X^T + \bar{v}^T)\}$$

$$= E\{\bar{h}\bar{h}^T\}X^T + \overset{0}{\cancel{E\{\bar{h}\bar{v}^T\}}}$$

$$= \sigma_h^2 X^T.$$

MMSE Estimation

- Also

M X N . Wide matrix

$$\mathbf{R}_{hy} = \sigma_h^2 X^T$$

MMSE Estimation

- Also

$$\mathbf{R}_{hy} = E\left\{\bar{\mathbf{h}}\bar{\mathbf{y}}^T\right\}$$

$$= \mathrm{E}\left\{\bar{\mathbf{h}}\left(\mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{v}}\right)^T\right\}$$

$$= E\left\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\mathbf{X}^T\right\} + \left\{\bar{\mathbf{h}}\bar{\mathbf{v}}^T\right\}$$

MMSE Estimation

- Also

$$\mathbf{R}_{hy} = E\left\{\bar{\mathbf{h}}\bar{\mathbf{h}}^T\right\}\mathbf{X}^T + \left\{\bar{\mathbf{h}}\bar{\mathbf{v}}^T\right\}$$
$$= \sigma_h^2\mathbf{X}^T$$

MMSE Estimation

- Therefore, the MMSE estimate is given as

MMSE Estimate.

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}}|\bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$= \sigma_h^2 X^T \left(\sigma_h^2 X X^T + \sigma^2 I\right)^{-1} \bar{y}$$

MMSE Estimate.

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = E\{\bar{\mathbf{h}} | \bar{\mathbf{y}}\} = \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\bar{\mathbf{y}}$$

$$= \sigma_h^2 \overbrace{\mathbf{X}^T\left(\sigma_h^2\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{I}\right)^{-1}}^{Q_2}\bar{\mathbf{y}}$$

MMSE Estimation

- Note

$$\sigma_h^2\mathbf{X}^T\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{X}^T = \sigma_h^2\mathbf{X}^T\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{X}^T$$

$$\Rightarrow X^T(\sigma_h^2 X X^T + \sigma^2 I) = (\sigma_h^2 X^T X + \sigma^2 I) X^T$$

$$\Rightarrow \underbrace{(\sigma_h^2 X^T X + \sigma^2 I)^{-1} X^T}_{Q_1} = \underbrace{X^T(\sigma_h^2 X X^T + \sigma^2 I)^{-1}}_{Q_2}.$$

MMSE Estimation

- Note

$$\overbrace{\left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1} X^T}^{Q_1}$$

$$= \underbrace{X^T \left(\sigma_h^2 X X^T + \sigma^2 I\right)^{-1}}_{Q_2}$$

MMSE Estimation

- Note

$$\sigma_h^2 \mathbf{X}^T \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{X}^T = \sigma_h^2 \mathbf{X}^T \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{X}^T$$

$$\mathbf{X}^T \left(\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I} \right) = \left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} \right) \mathbf{X}^T$$

MMSE Estimation

- Note

$$\left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I}\right)^{-1} \mathbf{X}^T = \mathbf{X}^T \left(\sigma_h^2 \mathbf{X} \mathbf{X}^T + \sigma^2 \mathbf{I}\right)^{-1}$$

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \overbrace{\sigma_h^2 \mathbf{X}^T \big(\overbrace{\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}}^{N \times N} \big)^{-1} \bar{\mathbf{y}}}^{\text{Form 1}}$$

$$= \underbrace{\sigma_h^2 \big(\underbrace{\sigma_h^2 X^T X + \sigma^2 I}_{M \times M \text{ matrix}} \big)^{-1} X^T \bar{y}}_{\text{Form 2}}$$

Typically $M << N$

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} =$$

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \sigma_h^2 \left(\sigma_h^2 X^T X + \sigma^2 I \right)^{-1} X^T \bar{y}$$

$$SNR = \frac{\sigma_h^2}{\sigma^2}$$

$$= \left(X^T X + \frac{\sigma^2}{\sigma_h^2} I \right)^{-1} X^T \bar{y}$$

$$\hat{h} = \left(X^T X + \frac{1}{SNR} I \right)^{-1} X^T \bar{y}$$

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \sigma_h^2 \mathbf{X}^T \left(\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I} \right)^{-1} \bar{\mathbf{y}}$$

$$= \sigma_h^2 \left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{X}^T \bar{\mathbf{y}}$$

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \left(\mathbf{X}^T\mathbf{X} + \frac{\sigma^2}{\sigma_h^2}\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

MMSE Estimation

- Therefore, the MMSE estimate is given as

$$\hat{\mathbf{h}} = \sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$\underset{M\times 1}{\hat{h}} = \underbrace{\left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\underset{M\times M}{\mathbf{I}}\right)^{-1}\mathbf{X}^T\underset{N\times 1}{\bar{\mathbf{y}}}}_{\text{MMSE Estimate.}}$$

MMSE Estimation- High SNR

- At high $SNR \to \infty \Rightarrow \frac{1}{SNR} \to 0$

$$\hat{h} = \left(X^T X + \frac{1}{SNR} I\right)^{-1} X^T \bar{y}$$

$$\to \left(X^T X\right)^{-1} X^T \bar{y}$$

ML Estimate

Maximum Likelihood Estimate !

MMSE Estimation- High SNR

- At high $SNR \to \infty \Rightarrow \frac{1}{SNR} \to 0$

MMSE

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} \mathbf{X}^T\bar{\mathbf{y}}$$

$$\to (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

ML

MMSE Estimation- High SNR

- Therefore **MMSE estimate** approaches the ML Estimate.

at high SNR !!

MMSE Estimation- High SNR

- Therefore **MMSE estimate** approaches the **ML Estimate**.

at high SNR!!

MMSE Estimation Example

Example:

Consider

4x1
N=4

M=2
4X2

$$\bar{\mathbf{y}} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

What is $\hat{\mathbf{h}}$ when $SNR = -6dB = \frac{1}{4}$

MMSE Estimation Example

- The **MMSE estimate** can be calculated as follows

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$$

MMSE Estimation Example

- The **MMSE estimate** can be calculated as follows

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

MMSE Estimation Example

- $SNR = -6\,dB \approx \frac{1}{4}$

$$\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I} = 4I + \frac{1}{1/4}I$$

$$= 4I + 4I$$

$$= 8I$$

MMSE Estimation Example

- $SNR = -6\,dB \approx \frac{1}{4}$

$$\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8I$$

MMSE Estimation Example

- $SNR = -6\ dB \approx \frac{1}{4}$

$$\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

MMSE Estimation Example

$$\frac{1}{8}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I} = 8I$$

$$\left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} = (8I)^{-1} = \frac{1}{8}I$$

MMSE Estimation Example

$$\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\left(\mathbf{X}^T\mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

MMSE Estimation Example

$$\left(\mathbf{X}^T\mathbf{X} + \frac{1}{\mathrm{SNR}}\mathbf{I}\right)^{-1}\mathbf{X}^T =$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

MMSE Estimation Example

$$\left(\mathbf{X}^T\mathbf{X} + \frac{1}{\mathrm{SNR}}\mathbf{I}\right)^{-1}\mathbf{X}^T$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

MMSE Estimation Example

MMSE Estimate of $\bar{h}$

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{\text{SNR}}\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$\underbrace{\hat{h}}$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$

MMSE Estimation Example

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X} + \frac{1}{\mathrm{SNR}}\mathbf{I}\right)^{-1}\mathbf{X}^T\bar{\mathbf{y}}$$

$$= \frac{1}{8}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}-2\\ -3\\ 1\\ -2\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-4\\ -2\end{bmatrix} = \overbrace{\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{4}\end{bmatrix}}^{\hat{h}}$$

MMSE Covariance

$\hat{h} - \bar{h} = \bar{e}$

$E\{\bar{e}\bar{e}^T\}$

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, the error covariance is

covariance matrix of error

Error $\underline{\text{Covariance}}$

$$E\left\{\left(\hat{\mathbf{h}} - \bar{\mathbf{h}}\right)\left(\hat{\mathbf{h}} - \bar{\mathbf{h}}\right)^T\right\} =$$

$$= R_{hh} - R_{hy} R_{yy}^{-1} R_{yh}.$$

MMSE Covariance

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, the error covariance is

$$E\left\{\left(\hat{\mathbf{h}}-\bar{\mathbf{h}}\right)\left(\hat{\mathbf{h}}-\bar{\mathbf{h}}\right)^{T}\right\}$$

$$=\mathbf{R}_{hh}-\mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

MMSE Covariance

- For $\bar{\mathbf{h}}, \bar{\mathbf{y}}$, jointly Gaussian, zero-mean, the error covariance is

$$E\left\{\left(\hat{\mathbf{h}}-\bar{\mathbf{h}}\right)\left(\hat{\mathbf{h}}-\bar{\mathbf{h}}\right)^{T}\right\}$$
$$=\mathbf{R}_{hh}-\mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

MMSE Error Covariance

- Note that

$$\mathbf{R}_{yh} = \mathbf{R}_{hy}^T$$

$$= R_{yh} = E\{\bar{y}\bar{h}^T\} = E\{(\bar{h}\bar{y}^T)^T\}$$

$$= \left(E\{\bar{h}\bar{y}^T\}\right)^T = R_{hy}^T$$

$$= \left(\sigma_h^2 X^T\right)^T = \sigma_h^2 X$$

MMSE Error Covariance

- Note that

$$\mathbf{R}_{yh} = \mathbf{R}_{hy}^{T} = \left(\sigma_h^2 \mathbf{X}^{T}\right)^{\boldsymbol{T}} = \sigma_h^2 \mathbf{X}$$

MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\overbrace{\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}}^{\text{Error Covariance}}$$

$$= \sigma_h^2 I - \sigma_h^2 X^T\left(\sigma_h^2 X X^T + \sigma^2 I\right)^{-1}\sigma_h^2 X$$

$$= \sigma_h^2 I - \sigma_h^2\left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1}\sigma_h^2 X^T X$$

MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$
$$= \sigma_h^2\mathbf{I} - \sigma_h^2\mathbf{X}^T\left(\sigma_h^2\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{I}\right)^{-1}\sigma_h^2\mathbf{X}$$

MMSE Error Covariance

- Note

$$\mathbf{X}^T\left(\sigma_h^2\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{I}\right)^{-1}$$

$N \times N$

$$= \left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1} X^T$$

$M \times M$

MMSE Error Covariance

- Note

$$\mathbf{X}^T\left(\sigma_h^2\mathbf{X}\mathbf{X}^T+\sigma^2\mathbf{I}\right)^{-1}$$

$$=\left(\sigma_h^2\mathbf{X}^T\mathbf{X}+\sigma^2\mathbf{I}\right)^{-1}\mathbf{X}^T$$

MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

$$= \sigma_h^2 \mathbf{I} - \sigma_h^2 \mathbf{X}^T \left(\sigma_h^2 \mathbf{X}\mathbf{X}^T + \sigma^2 \mathbf{I}\right)^{-1} \sigma_h^2 \mathbf{X}$$

$$= \sigma_h^2 I - \sigma_h^2 \left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1} \sigma_h^2 X^T X$$

MMSE Error Covariance

- Therefore, one obtains the error covariance

$$\mathbf{R}_{hh} - \mathbf{R}_{hy}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yh}$$

$$= \sigma_h^2\mathbf{I} - \sigma_h^2\mathbf{X}^T\left(\sigma_h^2\mathbf{X}\mathbf{X}^T + \sigma^2\mathbf{I}\right)^{-1}\sigma_h^2\mathbf{X}$$

$$= \sigma_h^2\mathbf{I} - \sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}\sigma_h^2\mathbf{X}^T\mathbf{X}$$

MMSE Error Covariance

- This can be simplified as

$$\sigma_h^2\mathbf{I} - \sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}\sigma_h^2\mathbf{X}^T\mathbf{X}$$

Add & Subtract $\sigma^2 I$

$$= \sigma_h^2 I - \sigma_h^2\left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1}\left(\underbrace{\sigma_h^2 X^T X + \sigma^2 I}_{} - \sigma^2 I\right)$$

$$= \cancel{\sigma_h^2 I} - \cancel{\sigma_h^2 I} + \sigma_h^2\sigma^2\left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1}$$

$$= \sigma_h^2\sigma^2\left(\sigma_h^2 X^T X + \sigma^2 I\right)^{-1}$$

MMSE Error Covariance

- This can be simplified as

$$\mathrm{SNR} = \frac{\sigma_h^2}{\sigma^2}$$

$$= \sigma_h^2 \sigma^2 \left(\sigma_h^2 X^T X + \sigma^2 I \right)^{-1}$$

$$= \sigma^2 \left(X^T X + \frac{\sigma^2}{\sigma_h^2} I \right)^{-1}$$

$$= \underbrace{\sigma^2 \left(X^T X + \frac{1}{\mathrm{SNR}} I \right)}_{\text{Error Covariance.}}$$

MMSE Error Covariance

- This can be simplified as

$$\sigma_h^2\mathbf{I} - \sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}\left(\sigma_h^2\mathbf{X}^T\mathbf{X}+\sigma^2\mathbf{I} - \sigma^2\mathbf{I}\right)$$

$$= \sigma_h^2\mathbf{I} - \sigma_h^2\mathbf{I} + \sigma^2\sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}$$

$$= \sigma^2\sigma_h^2\left(\sigma_h^2\mathbf{X}^T\mathbf{X} + \sigma^2\mathbf{I}\right)^{-1}$$

MMSE Error Covariance

- This can be simplified as

$$\sigma^2 \sigma_h^2 \left(\sigma_h^2 \mathbf{X}^T \mathbf{X} + \sigma^2 \mathbf{I}\right)^{-1}$$

$$= \left(\frac{1}{\sigma^2}\mathbf{X}^T \mathbf{X} + \frac{1}{\sigma_h^2}\mathbf{I}\right)^{-1}$$

$$= \sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR}\mathbf{I}\right)^{-1}$$

MMSE Error Covariance

- MSE is

Minimum Mean Square Error?

$$= E\{\|\hat{h} - \bar{h}\|^2\}$$

$$= \mathrm{Tr}\left\{\sigma^2 \left(X^T X + \frac{1}{SNR} I\right)^{-1}\right\}$$

MMSE Error Covariance

- MSE is

$$Tr\left\{\sigma^2\left(\mathbf{X}^T\mathbf{X}+\frac{1}{SNR}\mathbf{I}\right)^{-1}\right\}$$

MMSE Error Covariance Example

- MSE is

$$Tr\left\{\sigma^2\left(\mathbf{X}^T\mathbf{X}+\frac{1}{SNR}\mathbf{I}\right)^{-1}\right\}$$

MMSE Estimation Example

Example:

$$10\log_{10}\sigma^2 = 3$$
$$\Rightarrow \sigma^2 = 10^{0.3} \approx 2$$

Consider

$$\bar{\mathbf{y}} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$$

What is Error cov and MSE when $SNR = -6dB$ $= \frac{1}{4}$ and $\sigma^2 = 3dB$ $= 2$

MMSE Error Covariance

- This can be simplified as

$$\sigma^2 \left(\mathbf{X}^T \mathbf{X} + \frac{1}{SNR} \mathbf{I} \right)^{-1}$$

$$= 2\left(4I + 4I\right)^{-1} = 2 \cdot \times \frac{1}{8} I = \frac{1}{4} I$$

MMSE Error Covariance

- This can be simplified as

$$\sigma^2\left(\mathbf{X}^T\mathbf{X}+\frac{1}{SNR}\mathbf{I}\right)^{-1}$$

$$=2(4\mathbf{I}+4\mathbf{I})^{-1}=\frac{1}{4}\mathbf{I}$$

MMSE Error Covariance

- MSE is

$$\mathrm{Tr}\left\{\frac{1}{4}\underbrace{I}_{2\times 2}\right\} = \mathrm{Tr}\left\{\begin{bmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{bmatrix}\right\}$$

$$= 2\times\frac{1}{4} = \frac{1}{2}.$$

MMSE Error Covariance

- MSE is

$$Tr\left\{\frac{1}{4}\mathbf{I}\right\} = 2 \times \frac{1}{4} = \frac{1}{2}$$

Minimum mean square error.

Instructors may use this white area (14.5 cm / 25.4 cm) for the text.
Three options provided below for the font size.

Font: Avenir (Book), Size: 32, Colour: Dark Grey

Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

Do not use the space below.