EE901 Probability and Random Processes

Module -3
Distribution of Random Variables

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Probability of a Singleton Set

$$\boxed{F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\})}$$

- If F_X is continuous everywhere, the probability of a singleton set $\{x\}$ is zero.
- This means there is no point x where probability mass is concentrated.
- We say the mass is diffused and a mass density exists.
- If for any x, probability of $\{x\}$ is non-zero then, there is a discontinuity or a jump there.
- At those point, probability mass is concentrated.

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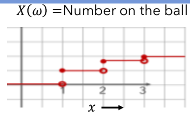
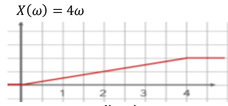
Types of Random Variables

- Depending on how probability mass is distributed, random variables are divided into two types:
  - Discrete
  - Continuous
  - Mixed
- RV is a function, so the set of values it takes is known as its range.

$X(\omega)$ =Number on the ball



$X(\omega) = 4\omega$



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Discrete Random Variable (RV)

- A random variable is called a discrete RV if its range is finite or countable.
- Its CDF will have discontinuities and has discrete levels
- Let us start with an example.

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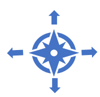
Example: Discrete RV

- Pick a direction with equal probability
  - $\Omega = \{N, W, E, S\}$
  - X is 1 if picked direction is N, otherwise it is 0
  - $X(\omega) = 1(\omega = N)$
  - X takes only two values 0 and 1, hence it is a discrete RV
  - What is its probability law and CDF?



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Example: Pick a Direction

$X(\omega)$ = Indicator that North is picked

$\mathbb{P}_X(B) = \Pr[X \in B] = \mathbb{P}(\mathrm{E}_B)$	$\mathrm{E}_B = \{\omega: X(\omega) \in B\}$

$B = (-\infty, x]$.

$x < 0$	$\mathbb{P}_X(B) = 0$
$x = 0$	$\mathbb{P}_X(B) = 3/4$
$0 < x < 1$	$\mathbb{P}_X(B) = 3/4$
$x = 1$	$\mathbb{P}_X(B) = 1$
$1 < x$	$\mathbb{P}_X(B) = 1$

CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 0 \\ 3/4 & x = 0 \\ 3/4 & 0 < x < 1 \\ 1 & x = 1 \\ 1 & 1 < x \end{cases}$$

- Set B that does not include 1 or 0 in them have no probability mass in them.
- This indicates that all the probability mass is concentrated at these two points only.

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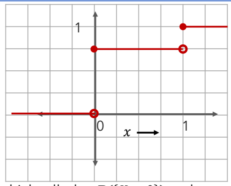
Example: Pick a Direction

$X(\omega)$ = Indicator that North is picked
CDF is $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big)$

$$= \begin{cases} 0 & x < 0 \\ 3/4 & x = 0 \\ 3/4 & 0 < x < 1 \\ 1 & x = 1 \\ 1 & 1 < x \end{cases}$$



- There are two jumps at $X = 0$ and $X = 1$ which tells that $\mathbb{P}(\{X = 0\})$ and $\mathbb{P}(\{X = 1\})$ are non-zero.
- Rest singleton elements have zero probability.
- All the probability mass is concentrated at these two points only with 1 having ¼ mass and 0 having ¾ mass.

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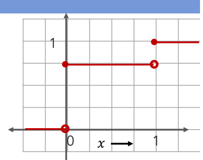
Example: Pick a Direction

Another way to compute CDF

- X takes only two values 0 and 1.
- $X = 0$ is equivalent to the event {W,S,E} which has 3/4 probability *i.e.* $\mathbb{P}[\{X = 0\}] = 3/4$
- $X = 1$ is equivalent to the event {N} which has 1/4 probability *i.e.* $\mathbb{P}[\{X = 1\}] = 1/4$
- Since $F_X(-\infty) = 0$, start with 0 at $-\infty$.
- Reach $x = 0$ and then make a jump of $\mathbb{P}[\{X = 0\}]$ to reach 3/4 and continue until $x = 1$.
- At $x = 1$, make a jump of $\mathbb{P}[\{X = 1\}]$ to reach 3/4+1/4=1.
- Stay at 1 as x approaches infinity.



No of Jumps = No of values X can take

Sum of all jumps = 1

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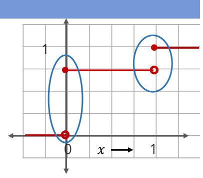
Example: Pick a Direction

- Specifying all jumps (jump points and jump amounts) fixes the distribution including CDF.
- Hence, distribution can be specified by specifying $\mathbb{P}[\{X = x\}]$ for all x.

$$p_X(x) = \mathbb{P}(\{X = x\}) = \begin{cases} 1/4 & \text{if } x = 1 \\ 3/4 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$



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Example: Pick a Direction

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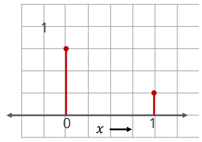

This is known as probability mass function or PMF.

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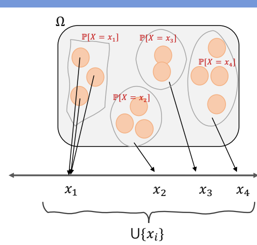
Discrete RV

- Let RV takes only finite number of values x_i's

$$\Omega = \bigcup_i \{X = x_i\}$$

$$\mathbb{P}(\Omega) = \sum_i \mathbb{P}(\{X = x_i\})$$

$$1 = \sum_i \mathbb{P}(\{X = x_i\})$$



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Discrete RV

$$1 = \sum_i \mathbb{P}(\{X = x_i\})$$

This indicates that $\mathbb{P}(\{X = x_i\})$'s can not be all zeros otherwise their sum cannot be non-zero.

The probability mass is concentrated at these points.

The PMF at a value x is defined as the probability that X takes the value x

$$p_X(x) = \mathbb{P}(\{X = x_i\})$$

This is also true for RVs that take countable number of values.

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Discrete RV

- Also note that

$$\{X \in (-\infty, x]\} = \bigcup_{\{x_i \le x\}} \{X = x_i\}$$

$$F_X(x) = \sum_{x_i \le x} p_X(x_i)$$

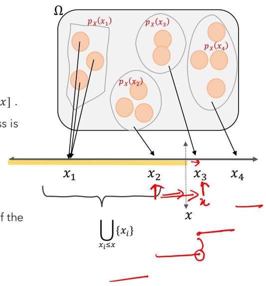
As we increase x, more mass is accumulated in $(-\infty, x]$.

As a new x_i is added when x passes it, a sudden mass is added which results in a jump.

There are jumps in CDF, one corresponding to each value in the range.

$$\boxed{F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\})}$$

- At other places, $F_X(x)$ remains constant. So, plot of the CDF of such a RV looks like stair case. Hence, the name "Discrete" RV.



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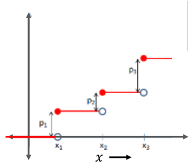
CDF and PMF of a Discrete RV

- The distribution can be specified by listing $\mathbb{P}[X \le x]$ for all x.

$$F_X(x) = \mathbb{P}(\{X \le x\})$$

- This is CDF.

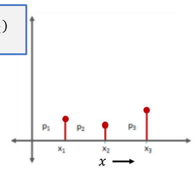


$$\boxed{F_X(x) = \sum_{x_i \le x} p_X(x_i)}$$

- The distribution can be specified by listing $\mathbb{P}[X = x]$ for all x.

$$p_X(x) = \mathbb{P}(\{X = x\})$$

- This is PMF.



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Discrete RV from Countable Ω

- It is not required to have sample space with finite/countable number of outcomes.
- Let us take another example:
- Pick a number between 0 and 1
  - $\Omega = (0,1)$
  - X is 1 if picked number is less than 0.5, otherwise it is 0
  - $X(\omega) = 1(\omega < 0.5)$
  - X takes only two values 0 and 1, hence it is a discrete RV
  - What is its CDF?

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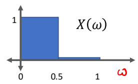
- Pick a number between 0 and 1. $\Omega = (0,1)$. $X(\omega) = 1(\omega < 0.5)$



$P((a,b)) = b-a$

CDF $B = (-\infty, x]$

$x<0$ $(-\infty, x]$ set of negative numbers $\phi \to 0$

$x=0$ $(-\infty, 0] = (.5,1)$ $\to \frac{1}{2}$

$\omega < 0.5$

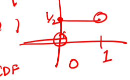
$(0, 0.5) \to 1$

$(0.5, 1) \to 0$

$1 > x > 0$ $(-\infty, x] = (.5,1) \to 1/2$

$x = 1$ $(-\infty, 1] = (0,1) \to 1$

$x > 1$ $\to 1$



Continuous RV

- When RV takes only finite/countable number of values x_i's, we saw that
$$1 = \sum_i \mathbb{P}(\{X = x_i\})$$
- Therefore, $\mathbb{P}(\{X = x_i\})$ can not be zeros otherwise the sum cannot be non-zero.

- Now, let us consider a RV that take uncountable number of values.
- If each $\mathbb{P}(\{X = x\})$ is non-zero, then the summation will be infinite.
- Hence, $\mathbb{P}(\{X = x\})$ should be zero for all x *except few countable number of points.*
- Let us take $\mathbb{P}(\{X = x\}) = 0$ for all x.
- Now, since
$$F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\}),$$
there will not be any jumps or discontinuities.
- Corresponding CDF will be continuous.

Continuous RV

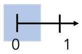
$$\mathbb{P}(\{X = x\}) = 0 \text{ for all } x.$$

- Now, since
$$F_X(x) - F_X(x^-) = \mathbb{P}(\{X = x\}),$$
there will not be any jumps. Corresponding CDF will be continuous.
- Probability mass is not concentrated at any points.

- A RV is a continuous RV if its CDF $F_X(x)$ is continuous with x.

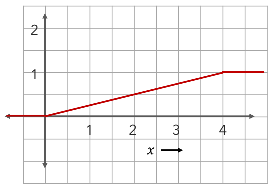
Example: Pick a Number

Pick a number in (0,1)   Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.



CDF is $F_X(x) = \mathbb{P}_X((-\infty, x])$

$$= \begin{cases} 0 & x \leq 0 \\ x/4 & 0 < x < 4 \\ 1 & x \geq 4 \end{cases}$$



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Continuous RV

- A RV is a continuous RV if its CDF $F_X(x)$ is continuous with x.
- As we increase x, more mass is accumulated in $(-\infty, x]$.
- So CDF $F_X(x)$ increases.
- However, probability mass is not concentrated at any points, but it is diffused over an interval.

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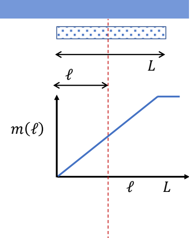
Density of a Measure (Mass)

- Take a uniform rod with length L
- Now, take a segment of length ℓ from one side.
- Its mass is $m(\ell)$ is a function of ℓ. Bigger the length, larger the mass.
- If I take a segment of zero length, it will not have any mass.
- However, combining all these segments, we get the complete rod which has a mass.



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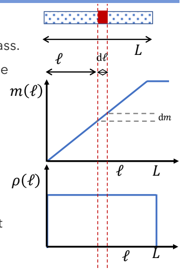
Density of a Measure (Mass)

- Take a uniform rod with length L and mass M.
- If I take a segment of zero length, it will not have any mass.
- If we take a very small segment of length $\mathrm{d}\ell$, what will be its mass $\mathrm{d}m$?
- Let us take its density as $\rho = M/L$.

  $\mathrm{d}m = \rho\ \mathrm{d}\ell$ which will contribute to $m(\ell)$.
- Total mass will be the sum over these segments

$$M = \int \mathrm{d}m = \int_0^L \rho \mathrm{d}\ell \ = \rho L = M$$

For uniformly built rod, density is defined as mass per unit length.



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Density of a Measure (Mass)

- Take a non-uniform rod with length L and mass M.
- Let the density be $\rho(\ell)$.
- If we take a very small segment of length $\mathrm{d}\ell$, its mass is

$$\mathrm{d}m = \rho(\ell)\ \mathrm{d}\ell$$

- Therefore, the total mass

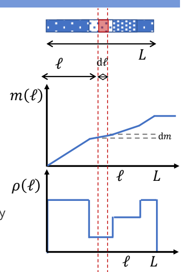
$$M = \int \mathrm{d}m = \int_0^L \rho(\ell) \mathrm{d}\ell$$

- The mass of a segment (a, b)

$$M = \int_a^b \rho(\ell) \mathrm{d}\ell$$

- Denote the mass of a segment of length ℓ from one side by $m(\ell)$. Density is defined as derivative of mass $m(\ell)$ with ℓ.

$$\rho(\ell) = \frac{\mathrm{d}m}{\mathrm{d}\ell}$$



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Continuous RV

- A RV is a continuous RV if its CDF $F_X(x)$ is continuous with x.
- Since the CDF $F_X(x)$ can be seen as the (probability) mass of interval $(-\infty, x]$, a density can be defined here also such that

$$F_X(x) = \mathbb{P}_X\big((-\infty, x]\big) = \int_{-\infty}^{x} f_X(x) \mathrm{d}x$$

- Therefore, the density is

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x)$$

This is known as probability density function or PDF.

- The probability of X taking a value in the interval $[a, b]$

$$\mathbb{P}_X([a, b]) = \int_a^b f_X(x) \mathrm{d}x$$

- The probability of X taking a value in the interval $(a, a+\epsilon)$ where ϵ is very small,

$$\mathbb{P}_X([a, b]) = \int_a^{a+\epsilon} f_X(x) \mathrm{d}x = f_X(x)\epsilon$$

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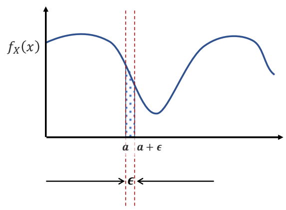
Continuous RV

$\mathbb{P}_X([a,b]) = f_X(x)\,\epsilon$



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Example: Pick a Number

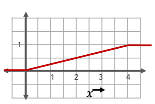
Pick a number in (0,1)   Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ $X(\omega) = 4\omega$ for each $\omega \in \Omega$.

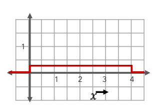
CDF is

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x/4 & 0 < x < 4 \\ 1 & x \ge 4 \end{cases}$$

PDF is

$$f_X(x) = \begin{cases} 0 & x \le 0 \\ 1/4 & 0 < x < 4 \\ 0 & x \ge 4 \end{cases}$$





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Mixed RV

- $\mathbb{P}(\{X = x\})$ should be zero for all x *except few countable number of points.*
- Let us take $\mathbb{P}(\{X = x\}) = 0$ for all x except one point.

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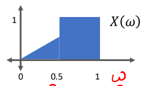
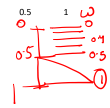
Pick a number between 0 and 1. $\Omega = (0,1)$.

$$X(\omega) = \begin{cases} \omega & \text{if } \omega < 0.5 \\ 1 & \text{if } \omega \geq 0.5 \end{cases}$$



CDF	B_x $(-\infty, x]$	$P_X(B_x)$	$F_X(x)$
$x<0$	B_x	ϕ	0
$x=0$			0
$0<x<0.5$	$(0,x]$	$(0,x]$	x
$1>x\geq 0.5$	$(0,0.5]\cup(0.5,x]$		x
$x=1$	$(0\cdot 0.5)\cup(0.5,1)$		1



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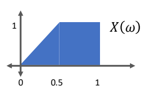
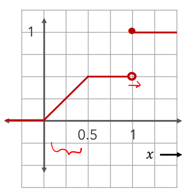
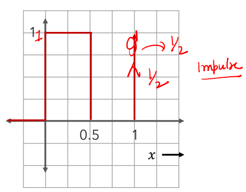
Pick a number between 0 and 1. $\Omega = (0,1)$.

$$X(\omega) = \begin{cases} \omega & \text{if } \omega < 0.5 \\ 1 & \text{if } \omega \geq 0.5 \end{cases}$$







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