EE910: Digital Communication Systems-I

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

April 11, 2022

Lecture #2C: Signal space representation of waveforms

Signal space concepts

- The *inner product* of two generally complex valued signals $x_1(t)$ and $x_2(t)$ is defined as

$$\begin{aligned}\langle x_1(t), x_2(t)\rangle &\triangleq \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt \\ \langle x_1(t), x_2(t)\rangle &= 0 \qquad \textbf{(orthogonality)}\end{aligned}$$

- The *norm of a signal*

$$\|x(t)\| = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{\mathcal{E}_x}$$

where, $\mathcal{E}_x$ is the energy in $x(t)$.

Signal space concepts

- A set of m signals is *orthonormal* if they are
  - Orthogonal;
  - Unit norm.

Signal space concepts

- A set of m signals is *linearly independent* if no signal can be represented as a linear combination of the remaining signals.
-

$$\begin{aligned} \|x_1(t)+x_2(t)\| &\leq \|x_1(t)\|+\|x_2(t)\| \quad \text{(Triangle inequality)} \\ |\langle x_1(t), x_2(t)\rangle| &\leq \|x_1(t)\|\cdot\|x_2(t)\| \;\; \text{(Cauchy- Schwartz inequality)} \\ &= \sqrt{\mathcal{E}_{x_1}\mathcal{E}_{x_2}} \end{aligned}$$

equivalently

$$\left|\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt\right| \leq \left|\int_{-\infty}^{\infty}|x_1(t)|^2dt\right|^{1/2}\left|\int_{-\infty}^{\infty}|x_2(t)|^2dt\right|^{1/2}$$

with equality when $x_2(t) = \alpha x_1(t)$ for some complex number α.

Orthogonal expansions of signals

- A set of orthonormal functions $\{\phi_n(t), \;\; n = 1, 2, \cdots, K\}$

$$\langle \phi_n(t), \phi_m(t)\rangle = \int_{-\infty}^{\infty}\phi_n(t)\phi_m^*(t)dt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

- Approximation of signal $s(t)$ by $\hat{s}(t)$ is

$$\hat{s}(t) = \sum_{k=1}^{K} s_k\phi_k(t)$$

- Approximation error

$$e(t) = s(t) - \hat{s}(t)$$

Orthogonal expansions of signals

- Energy in the error signal

$$\begin{aligned}\mathcal{E}_e &= \int_{-\infty}^{\infty} |s(t) - \hat{s}(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \left| s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right|^2 dt\end{aligned}$$

the coefficients $\{s_k\}$ are selected such that the error energy $\mathcal{E}_e$ is minimized (in mean square error sense) and are given by

$$s_n = \langle s(t), \phi_n(t) \rangle = \int_{-\infty}^{\infty} s(t)\phi_n^*(t)dt, \qquad n = 1, 2, \cdots, K$$

Gram Schmidt procedure

- A set of orthogonal signals from the set of finite energy waveforms $\{s_m(t), \quad m = 1, 2, \cdots, M\}$ is constructed as follows. choose a signal waveform randomly from the set $\{s_m(t), \quad m = 1, 2, \cdots, M\}$, $s_1(t)$

$$\begin{aligned}\phi_k(t) &= \frac{s_k(t)}{\sqrt{\int_{-\infty}^{\infty} |s_k(t)|^2 dt}} = \frac{s_k(t)}{\sqrt{\mathcal{E}_k}}, \qquad \text{For k=1} \\ \gamma_k(t) &= s_k(t) - \sum_{i=1}^{k-1} c_{ki}\phi_i(t) \\ \phi_k(t) &= \frac{\gamma_k(t)}{\sqrt{\mathcal{E}_k}} \qquad \text{For } k > 1\end{aligned}$$

Gram Schmidt procedure

where,

$$c_{ki} = \langle s_k(t), \phi_i(t) \rangle = \int_{-\infty}^{\infty} s_k(t)\phi_i^*(t)dt$$
$$\mathcal{E}_k = \int_{-\infty}^{\infty} \gamma_k^2(t)dt$$

- A signal $s_m(t)$ can be written in the term of set of orthonormal waveforms $\phi_n(t)$ as

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\phi_n(t) \qquad \text{for } m = 1, 2, \cdots, M$$

Gram Schmidt procedure

- Series expansion of the signal represents orthogonal projection of $s_i(t)$ onto the space spanned by the N basis function.
- Expansion coefficient s_{ik} can be interpreted as the projection of the ith signal onto the kth basis function.
- Each signal is represented as a point in N-dimensional signal space.
- The basis set for the signal set are the basis functions.

Gram Schmidt procedure

- For a fixed set of basis orthonormal waveforms $\phi_n(t)$, signals $\{s_m(t)\}$ can be written equivalently as vectors

$$\mathbf{s}_m = \begin{bmatrix} s_{m1} & s_{m2} & \cdots & s_{mN} \end{bmatrix}^T \qquad \text{for } m = 1, 2, \cdots, M$$

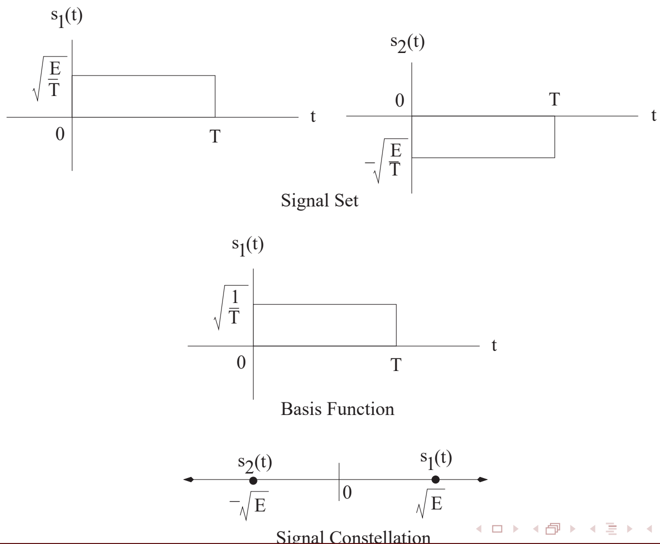
and by orthogonality of the basis

$$\langle s_k(t), s_l(t) \rangle = \langle \mathbf{s}_k, \mathbf{s}_l \rangle$$

Gram Schmidt procedure

- Note that the functions $\{\phi_n(t)\}$ obtained from the Gram Schmidt procedure are not unique.
- For the different order of orthogonalization process of $\{s_m(t)\}$, the orthonormal waveforms $\{\phi_n(t)\}$ will be different and the corresponding vector representation of the signal $s_m(t)$, $\mathbf{s}_m$ will be different.
- The dimensionality of the signal space N will not change, and the vectors $\{\mathbf{s}_m\}$ will retain their geometric configuration, i.e. their lengths and their inner products will be invariant to the choice of the orthonormal functions $\{\phi_n(t)\}$.

Orthonormal Basis Sets



Gram Schmidt Procedure

- Let the set of M signals be denoted by $s_1(t), s_2(t), \cdots, s_M(t)$, defined over the interval $[0, T]$.
- First basis function is defined by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

where E_1 is the energy of the signal $s_1(t)$ chosen arbitrarily from the set.
- $s_1(t)$ can then be represented as

$$\begin{aligned} s_1(t) &= \sqrt{E_1}\phi_1(t) \\ &= s_{11}\phi_1(t) \end{aligned}$$

where the coefficient $s_{11} = \sqrt{E_1}$ and $\phi_1(t)$ has unit energy.

Gram Schmidt Procedure

- Next using $s_2(t)$ we define

$$s_{21}(t) = \int_0^T s_2(t)\phi_1(t)dt$$

- Let $g_2(t)$, a function orthogonal to $\phi_1(t)$ over the interval $[0, T]$ be defined as

$$g_2(t) = s_2(t) - s_{21}(t)\phi_1(t)$$

- Second basis function can be defined as

$$\begin{aligned}\phi_2(t) &= \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} \\ &= \frac{s_2(t) - s_{21}(t)\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}\end{aligned}$$

where E_2 is the energy of the signal $s_2(t)$.

Gram Schmidt Procedure

- Continuing in this fashion we can define

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}(t)\phi_j(t)$$

where the coefficients s_{ij} are defined as

$$s_{ij}(t) = \int_0^T s_i(t)\phi_j(t)dt \quad , j = 1, 2, \cdots, i-1$$

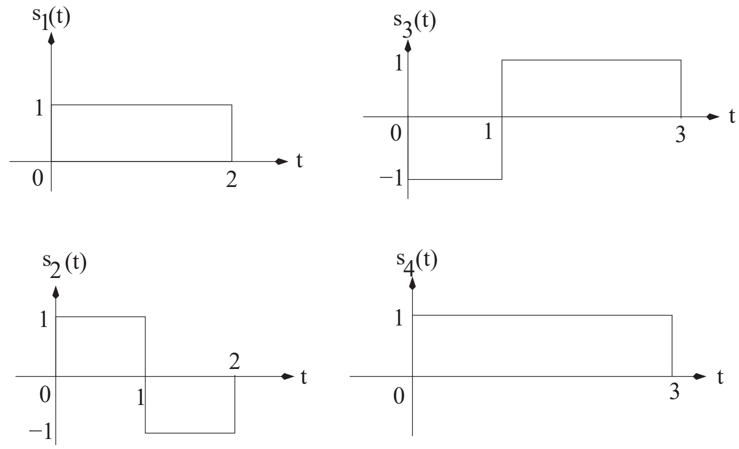
- Set of basis functions that form the orthonormal set can be defined as

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}} \quad , i = 1, 2, \cdots, N$$

- If the signals $s_1(t), s_2(t) \cdots, s_M(t)$ forms a linearly independent set, then $N = M$, otherwise $N < M$

Gram Schmidt Procedure

- Apply Gram-Schmidt procedure to the signals given below



Gram Schmidt Procedure

- Signal $s_1(t)$ has energy 2, so $\phi_1(t) = s_1(t)/\sqrt{2}$.
- $\phi_1(t)$ and $s_2(t)$ are orthogonal, so $\phi_2(t) = s_2(t)/\sqrt{2}$, where $E_2 = 2$.
- $g_3(t) = s_3(t) + \sqrt{2}\phi_2(t)$.
- $g_3(t)$ has unit energy, so $\phi_3(t) = g_3(t)$.
- $g_4(t) = s_4(t) - \sqrt{2}\phi_1(t) - \phi_3(t) = 0$.
- Thus $s_4(t)$ is linear combination of $\phi_1(t)$ and $\phi_3(t)$.
- The dimensionality of the signal set is $N = 3$.

Gram Schmidt Procedure

Solution: