EE901
Probability and Random Processes

Module 9
Limit Theorems

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1

Deviation and Limit Theorems

Suppose a system designed by us works when the input voltage is between 220 V± 5V i.e. between 215V and 225V.

Suppose the input voltage is a random variable with mean 220V.

What is the probability that system works?

- If the distribution is known, the exact deviation probability can be computed.
- When the distribution is not known, some bounds may be useful depending on the known information.

2

Deviation and Limit Theorems

- Suppose we want to find the value of a parameter X from an experiment.

- Suppose the value obtained is X_1, which has some noise in the observation.
- X_1 tends to be around the true value of the parameter but it is not exactly equal. It may be very far away for the particular instance of experiment.

- We try to do the same experiment multiple times, and get values $X_1, X_2, \dots, X_n$
- We take the average of all these values, known as *sample average*.
- We hope that the average of these values will give the value of true parameter

- Questions: how many trials do we need? What is the probabilistic guarantee?

3

Deviations of Random Variables

4

Markov's Inequality

- Let X be a positive RV, then the probability that X is more than t, is upper bounded as

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}[X]}{t}$$

Example:

- Let, $\mathbb{E}[X] = 3$, the $\mathbb{P}[X > 9] \leq \frac{1}{3}$

- This means that X can be more than 9 at maximum one third of the time.

5

Chebyshev's Inequality

- Let X be a RV with mean μ and variance σ^2.
- The probability that X deviates from the mean more than t, is upper bounded as

$$\mathbb{P}(|X - \mu| > t) \leq \frac{\sigma^2}{t^2}$$

6

Limits for n –Sample Average

7

Mean and Variance of Sum of RVs

- If there are n different independent RVs X_i 's.

$$Z = \sum_{i=1}^{n} X_i$$

$$\mathbb{E}[Z] = \sum \mathbb{E}[X_i] \qquad \text{Var}(Z) = \sum \text{Var}(X_i)$$

- n –sample average or sample mean

$$S_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \mathbb{E}[S_n] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[X_i]$$

$$\text{Var}[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}[X_i]$$

8

Mean and Variance of Average of RVs

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \mathbb{E}[S_n] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] \qquad \text{Var}[S_n] = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}[X_i]$$

- If all of them have the same mean m and variance σ^2 then

$$\mathbb{E}[S_n] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}nm = m$$

$$\text{Var}[S_n] = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}[X_i] = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}$$

$$\lim_{n\to\infty}\text{Var}[S_n] = 0$$

- When the variance of any RV is 0, it implies that RV is constant and equal to its mean.

$$S_n \to m$$

Law of Large numbers

Weak Law of Large Numbers

If X_i's are n independent and identically distributed RVs then,

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}[X]$$

in *probability* convergence.

This means that

$$\mathbb{P}(|S_n - \mathbb{E}[X]| > \epsilon) \to 0$$

The probability that the $n-$ sample average is more than ϵ away than the mean

Central Limit Theorem

Let $T_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mathbb{E}[X]) = \sqrt{n}(S_n - \mathbb{E}[X])$

$$\mathbb{E}[T_n] = \sqrt{n}(\mathbb{E}[S_n] - \mathbb{E}[X]) = 0$$

$$\text{Var}[T_n] = \frac{1}{n}\sum_{i=1}^{n}\text{Var}[X_i] = \frac{1}{n}n\text{Var}[X] = \text{Var}[X]$$

Central Limit Theorem states that

as $n \to \infty$, T_n's distribution converges to the Gaussian distribution.

$$T_n = \sqrt{n}(S_n - \mathbb{E}[X]) \xrightarrow{d} \underline{\mathcal{N}(0, \text{Var}[X])}$$

Central Limit Theorem

$T_n = \sqrt{n}(S_n - \mathbb{E}[X]) \xrightarrow{d} \mathcal{N}(0, \text{Var}[X])$ Let the variance of $X = \sigma^2$

$T_n \to \mathcal{N}(0, \sigma^2)$

$\frac{T_n}{\sigma} \to \mathcal{N}(0,1)$ Normal RV.

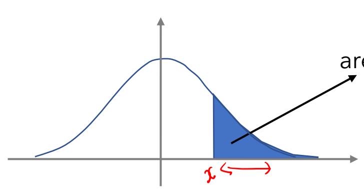
Further

$$Z_n = \frac{T_n}{\sqrt{\text{Var}[X]}} \xrightarrow{d} \mathcal{N}(0,1)$$

σ

$\mathbb{P}(T_n > c) = \mathbb{P}\left(Z_n > \frac{c}{\sigma}\right) = \phi\left(\frac{c}{\sigma}\right)$ where ϕ is the CCDF of the standard Gaussian RV.

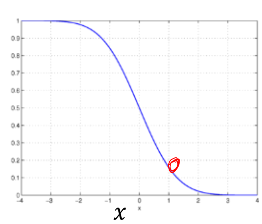
CCDF of a



area= $\phi(x)$

$$= \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}\exp\left(-\frac{u^2}{2}\right)du.$$

$\phi(x)$



Central Limit Theorem

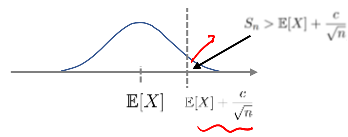
$$\mathbb{P}\left(T_n > c\right) = \mathbb{P}\left(Z_n > \frac{c}{\sigma}\right) = \phi\left(\frac{c}{\sigma}\right)$$

Recall

$$T_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mathbb{E}[X]) = \sqrt{n}(S_n - \mathbb{E}[X])$$

Hence,

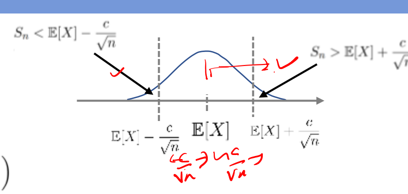
$$\mathbb{P}\left[S_n > \mathbb{E}[X] + \frac{c}{\sqrt{n}}\right] = \phi\left(\frac{c}{\sigma}\right)$$



13

Deviation from the Mean

$$\mathbb{P}\left[S_n > \mathbb{E}[X] + \frac{c}{\sqrt{n}}\right] = \phi\left(\frac{c}{\sigma}\right)$$



$$\mathbb{P}\left[S_n < \mathbb{E}[X] - \frac{c}{\sqrt{n}} \text{ OR } S_n > \mathbb{E}[X] + \frac{c}{\sqrt{n}}\right] = 2\phi\left(\frac{c}{\sigma}\right)$$

The probability that the n−sample average is more than $\frac{c}{\sqrt{n}}$ away from the mean

$$\mathbb{P}\left[|S_n - \mathbb{E}[X]| > \frac{c}{\sqrt{n}}\right] = 2\phi\left(\frac{c}{\sigma}\right)$$

The probability that the n−sample average is not more than $\frac{c}{\sqrt{n}}$ away from the mean

$$\mathbb{P}\left[|S_n - \mathbb{E}[X]| < \frac{c}{\sqrt{n}}\right] = 1 - 2\phi\left(\frac{c}{\sigma}\right)$$

14

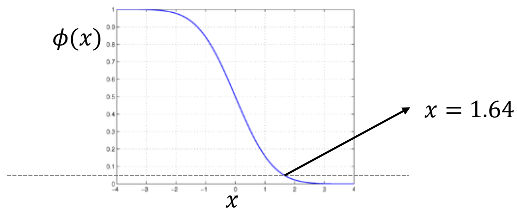
Deviation from the Mean

- For any n, find a_n such that the n–average is within a_n limit from the mean with 90% probability

$$\mathbb{P}\left[|S_n - \mathbb{E}(X)| < a_n\right] \geq 0.9 \qquad \mathbb{P}\left[|S_n - \mathbb{E}[X]| < \frac{c}{\sqrt{n}}\right] = 1 - 2\phi\left(\frac{c}{\sigma}\right)$$

- Find c such that $1 - 2\varphi\left(\frac{c}{\sigma}\right) = 0.9.$

$$c = \sigma\,\varphi^{-1}(0.05) \quad = 1.64\,\sigma$$



15

Deviation from the Mean

- For any n, find a_n such that the n–average is within a_n limit from the mean with 90% probability

$$\mathbb{P}\left[|S_n - \mathbb{E}(X)| < a_n\right] \geq 0.9 \qquad \mathbb{P}\left[|S_n - \mathbb{E}[X]| < \frac{c}{\sqrt{n}}\right] = 1 - 2\phi\left(\frac{c}{\sigma}\right)$$

- Find c such that $1 - 2\varphi\left(\frac{c}{\sigma}\right) = 0.9.$

$$c = \sigma\,\varphi^{-1}(0.05) \quad = 1.64\,\sigma$$

- c is independent of n
- The confidence interval

$$(\mu - a_n, \mu + a_n) = \left(\mu - \sigma\frac{\varphi^{-1}(0.05)}{\sqrt{n}}, \mu + \sigma\frac{\varphi^{-1}(0.05)}{\sqrt{n}}\right)$$

- The interval gets smaller as $n \to \infty$

16