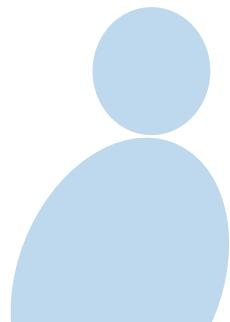


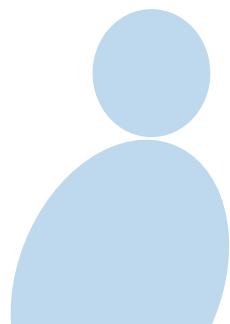
# eMasters in Communication Systems

**Prof. Aditya  
Jagannatham**



**Elective Module:**

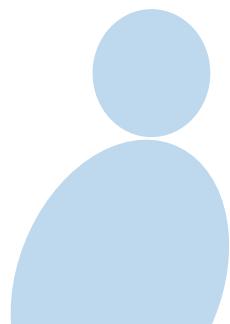
**Detection for Wireless  
Communication**



# Chapter 8

## Detection of Random Signals

$\bar{s}_1, \bar{s}_0, \bar{s}_1$   
Deterministic.  
 $\bar{s}$  Random signal ?



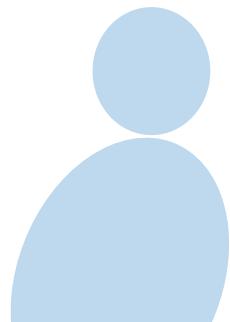
# Detection of Random Signals

- Consider a **random signal detection problem**
- Under  $\mathcal{H}_0$

Signal is absent.

$$\begin{aligned} y(1) &= v(1) \\ y(2) &= v(2) \\ &\vdots \\ y(N) &= v(N) \end{aligned}$$

iid Gaussian  
noise samples  
mean = 0  
var =  $\sigma^2$



# Detection of Random Signals

- Consider a **random signal detection** problem
- Under  $\mathcal{H}_0$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$
$$\bar{y} = \bar{V}$$
$$\mathcal{H}_0$$

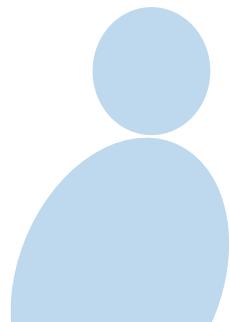
# Detection of Random Signals

- This can be written in the **vector form**.
- Under  $\mathcal{H}_0$

$$\begin{bmatrix} y(1) \\ y(z) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(z) \\ \vdots \\ v(N) \end{bmatrix}$$

*$N \times 1$  noise vector*

$$\bar{y} = \bar{v}$$

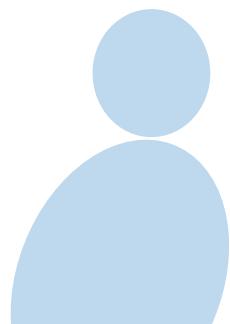


# Detection of Random Signals

- This can be written in the **vector form**.
- Under  $\mathcal{H}_0$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$\tilde{\mathbf{y}}$                        $\bar{\mathbf{v}}$



# Detection of Random Signals

- Under  $\mathcal{H}_1$

$$y(1) = s(1) + v(1)$$

$$y(z) = s(z) + v(z)$$

$$\vdots$$
$$y(N) = s(N) + v(N)$$

$$s(1), s(z), \dots, s(N)$$

iid Gaussian samples

Signal Present  
Random

iid Gaussian

mean = 0  
var =  $\sigma_s^2$

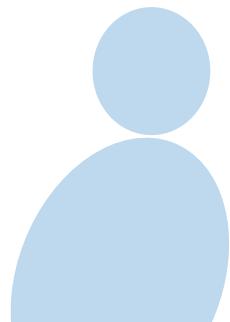
$s(i)$  independent  
of  $v(j)$

$E\{s(i)v(j)\} = 0$   
for all  $i, j$

# Detection of Random Signals

- Under  $\mathcal{H}_1$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$
$$\bar{y} = \bar{s} + \bar{v}$$



# Detection of Random Signals

- This can be written in the **vector form**.

- Under  $\mathcal{H}_1$

$$\begin{bmatrix} y(1) \\ y(z) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(z) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(z) \\ \vdots \\ v(N) \end{bmatrix}$$

$\bar{y} = \bar{s} + \bar{v}$

Gaussian signal vector

Gaussian noise vector

# Detection of Random Signals

- This can be written in the **vector form**.
- Under  $\mathcal{H}_1$

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

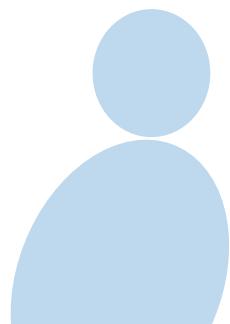
$$\bar{y} = \bar{s} + \bar{v}$$

$$E\{\bar{v}\} = 0$$
$$E\{\bar{v}\bar{v}^T\} = \sigma_v^2 I$$

*Noise covariance*

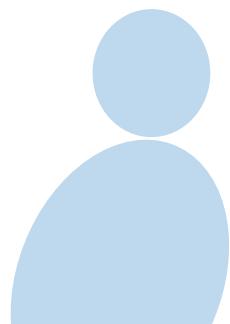
$$E\{\bar{s}\} = 0$$
$$E\{\bar{s}\bar{s}^T\} = \sigma_s^2 I$$

*signal covariance matrix*



# Detection of Random Signals

- $s(i)$  are i.i.d. **Gaussian** mean 0 and variance  $\sigma_s^2$
- $v(i)$  are i.i.d. **Gaussian** mean 0 and variance  $\sigma^2$



# Detection of Random Signals

- The likelihoods are as follows
- Under  $\mathcal{H}_0$  we have  $y(i) = v(i)$

likelihood under  
 $\mathcal{H}_0$ .  
only Gaussian  
noise present.

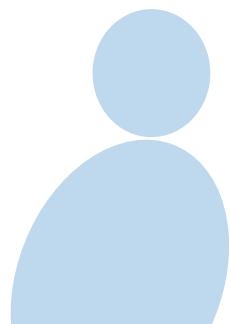
$$p(\bar{y}; \mathcal{H}_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2(1)/2\sigma^2}{2\sigma^2}} \times \dots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2(N)/2\sigma^2}{2\sigma^2}}$$
$$= \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{N}{2}} e^{-\frac{\|\bar{y}\|^2}{2\sigma^2}}$$

likelihood under  $\mathcal{H}_0$   
NULL hypothesis -

## Detection of Random Signals

- The likelihoods are as follows
- Under  $\mathcal{H}_0$  we have  $y(i) = v(i)$

$$\begin{aligned} p(\bar{\mathbf{y}}; \mathcal{H}_0) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2(1)}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2(N)}{2\sigma^2}} \\ &= \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2} \|\bar{\mathbf{y}}\|^2} \end{aligned}$$



# Detection of Random Signals

$y(i) = \text{sum of Gaussian} \Rightarrow y(i) = \text{Gaussian}$

- Under  $\mathcal{H}_1$  we have  $y(i) = s(i) + v(i)$   
 $E\{y(i)\} = E\{s(i)\} + E\{v(i)\} = 0$
- $y(i)$  are i.i.d. Gaussian mean 0, variance  $\sigma^2 + \sigma_s^2$   $y(i) \sim \mathcal{N}(0, \sigma_s^2 + \sigma^2)$ .

$$\begin{aligned} E\{y^2(i)\} &= E\{(s(i) + v(i))^2\} \\ &= E\{s^2(i)\} + E\{v^2(i)\} + 2E\{s(i)v(i)\} \\ &= \sigma_s^2 + \sigma^2 \end{aligned}$$

# Detection of Random Signals

- Under  $\mathcal{H}_1$

$$p(\bar{y}; \mathcal{H}_1) = \frac{1}{\sqrt{2\pi(\sigma_s^2 + \sigma^2)}} e^{-\frac{\bar{y}^2(1)}{2(\sigma_s^2 + \sigma^2)}} \times \dots \times \frac{1}{\sqrt{2\pi(\sigma_s^2 + \sigma^2)}} e^{-\frac{\bar{y}^2(N)}{2(\sigma_s^2 + \sigma^2)}}$$

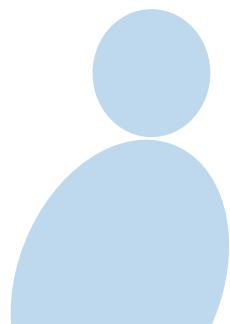
$$= \left( \frac{1}{2\pi(\sigma^2 + \sigma_s^2)} \right)^{\frac{N}{2}} e^{-\frac{1}{2} \frac{\|\bar{y}\|^2}{\sigma^2 + \sigma_s^2}}$$

$\left. \begin{array}{l} p(\bar{y}; \mathcal{H}_1) \\ \text{likelihood under } \mathcal{H}_1. \end{array} \right\}$

## Detection of Random Signals

- Under  $\mathcal{H}_1$

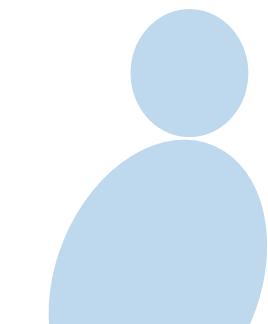
$$\begin{aligned} p(\bar{\mathbf{y}}; \mathcal{H}_1) &= \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_s^2)}} e^{-\frac{y^2(1)}{2(\sigma^2 + \sigma_s^2)}} \times \cdots \times \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_s^2)}} e^{-\frac{y^2(N)}{2(\sigma^2 + \sigma_s^2)}} \\ &= \left( \frac{1}{2\pi(\sigma^2 + \sigma_s^2)} \right)^{\frac{N}{2}} e^{-\frac{1}{2(\sigma^2 + \sigma_s^2)} \|\bar{\mathbf{y}}\|^2} \end{aligned}$$



# Detection of Random Signals

- Choose  $\mathcal{H}_0$  if LRT Likelihood Ratio Test

$$\frac{\frac{P(\bar{y}; \mathcal{H}_0)}{P(\bar{y}; \mathcal{H}_1)}}{\left( \frac{1}{2\pi\sigma^2} \right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2} \cdot \frac{\|\bar{y}\|^2}{\sigma^2}}} \geq \tilde{\gamma}$$
$$\frac{\left( \frac{1}{2\pi(\sigma^2 + \sigma_s^2)} \right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2} \frac{\|\bar{y}\|^2}{\sigma^2 + \sigma_s^2}}}{\left( \frac{1}{2\pi\sigma^2} \right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2} \frac{\|\bar{y}\|^2}{\sigma^2}}} \geq \tilde{\gamma}$$



# Detection of Random Signals

Take natural log on both sides when choose  $f_0$  when

$$\frac{N}{2} \ln \left( \frac{\sigma_s^2 + \sigma^2}{\sigma^2} \right) + \| \bar{y} \|^2 \left( \frac{1}{2} \cdot \frac{1}{\sigma^2 + \sigma_s^2} - \frac{1}{2\sigma^2} \right) \geq \ln \tilde{\gamma}$$

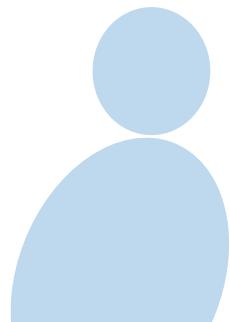
$$\frac{N}{2} \ln \left( \frac{\sigma^2 + \sigma_s^2}{\sigma^2} \right) - \ln \tilde{\gamma} \geq \| \bar{y} \|^2 \left( \frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2 + \sigma_s^2)} \right)$$

$$\Rightarrow \| \bar{y} \|^2 \leq \frac{\frac{N}{2} \ln \left( \frac{\sigma_s^2 + \sigma^2}{\sigma^2} \right) - \ln \tilde{\gamma}}{\left( \frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2 + \sigma_s^2)} \right)} = \sigma$$

# Detection of Random Signals

choose  $H_0$

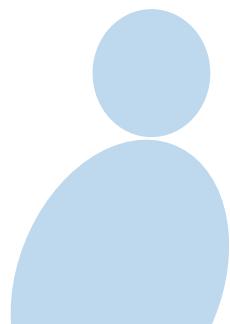
$$\|\bar{y}\|^2 \leq \gamma$$



# Detection of Random Signals

- Choose  $\mathcal{H}_0$  if

$$\frac{\frac{p(\bar{\mathbf{y}}; \mathcal{H}_0)}{p(\bar{\mathbf{y}}; \mathcal{H}_1)}}{\left( \frac{1}{2\pi\sigma^2} \right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2} \|\bar{\mathbf{y}}\|^2}} \geq \tilde{\gamma}$$
$$\left( \frac{1}{2\pi(\sigma^2 + \sigma_s^2)} \right)^{\frac{N}{2}} e^{-\frac{1}{2(\sigma^2 + \sigma_s^2)} \|\bar{\mathbf{y}}\|^2} \geq \tilde{\gamma}$$

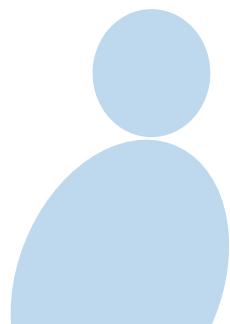


# Detection of Random Signals

- Choose  $\mathcal{H}_0$  if

$$\frac{N}{2} \ln \left( \frac{\sigma^2 + \sigma_s^2}{\sigma^2} \right) + \|\bar{\mathbf{y}}\|^2 \left( -\frac{1}{2\sigma^2} + \frac{1}{2(\sigma^2 + \sigma_s^2)} \right) \geq \ln \tilde{\gamma}$$

$$\underline{\|\bar{\mathbf{y}}\|^2} \leq \frac{\frac{N}{2} \ln \left( \frac{\sigma^2 + \sigma_s^2}{\sigma^2} \right) - \ln \tilde{\gamma}}{\left( \frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2 + \sigma_s^2)} \right)} = \underline{\gamma}$$



## Detection of Random Signals

*energy of signal.*

- Choose  $\mathcal{H}_0$  if  $\|\bar{y}\|^2 = |y(1)|^2 + \dots + |y(N)|^2$

$$\|\bar{y}\|^2 \leq \tau$$

- Choose  $\mathcal{H}_1$  if

$$\|\bar{y}\|^2 > \tau$$

# Detection of Random Signals

- Choose  $\mathcal{H}_0$  if

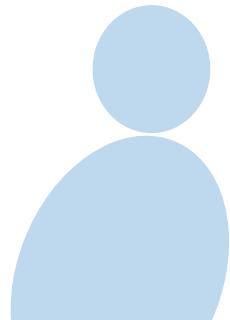
$$\|\bar{\mathbf{y}}\|^2 \leq \gamma$$

- Choose  $\mathcal{H}_1$  if

$$\|\bar{\mathbf{y}}\|^2 > \gamma$$

ENERGY  
DETECTOR

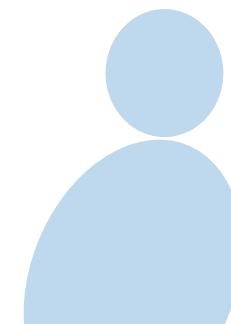
Very Simple  
Low complexity  
Efficient



# Detection of Random Signals

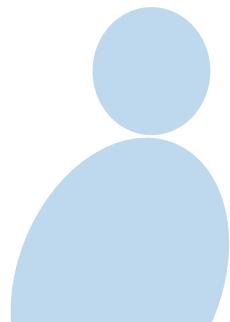
- This is termed as the ENERGY DETECTOR.

Non coherent  
detector



# **Detection of Random Signals**

- This is termed as the **Energy Detector**.



# Detection of Random Signals

- The  $P_{FA}$  is given as follows

$$\Pr(\|\bar{\mathbf{y}}\|^2 \geq \gamma; \mathcal{H}_0)$$

$$\Pr(|y(1)|^2 + \dots + |y(N)|^2 \geq \gamma; \mathcal{H}_0)$$

# Detection of Random Signals under $H_0$

$$y(i) = v(i) \sim \mathcal{N}(0, \sigma^2)$$

$$\Rightarrow y(1) + \dots + y(N) \geq T$$

$$\Rightarrow \left(\frac{y(1)}{\sigma}\right)^2 + \dots + \left(\frac{y(N)}{\sigma}\right)^2 \geq \frac{T}{\sigma^2}$$

$$\frac{y(i)}{\sigma} = \frac{v(i)}{\sigma} \sim \mathcal{N}(0, 1)$$

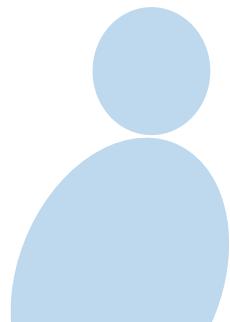
Standard Gaussian  
RV mean = 0  
var = 1.

# Detection of Random Signals

- The  $P_{FA}$  is given as follows

$$\Pr(\|\bar{\mathbf{y}}\|^2 \geq \gamma; \mathcal{H}_0)$$

$$y^2(1) + \cdots + y^2(N) \geq \gamma$$



# Detection of Random Signals

$$\left(\frac{y(1)}{\sigma}\right)^2 + \cdots + \left(\frac{y(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$
$$\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$

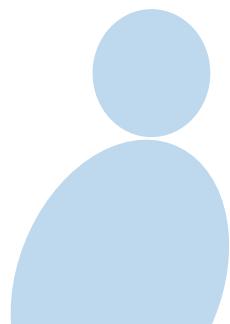
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Sum of Squares of  $N$  standard  
zero mean unit variance Gaussian RVs.

This is  $\chi_N^2$  # number of  
degrees of freedom

# Detection of Random Signals

$$\frac{v(i)}{\sigma} \sim \mathcal{N}(0, 1)$$

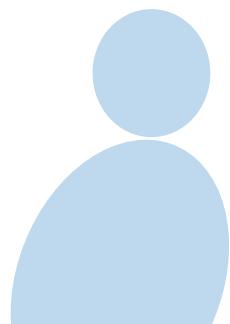


# Detection of Random Signals

$$\frac{v(i)}{\sigma} \sim \mathcal{N}(0,1)$$
$$\left(\frac{v(1)}{\sigma}\right)^2 + \dots + \left(\frac{v(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$

Central Chi Squared  
RV with  $N$  degrees  
of Freedom

Central  $\chi^2_N$ .

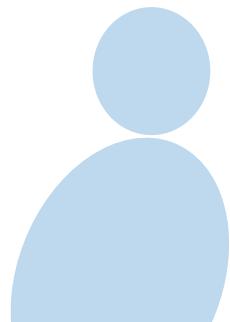


# Detection of Random Signals

$$\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2$$

- Sum of squares of  $N$  iid zero mean Gaussian RVs of Unit Variance .

- This is  $\chi^2_N$  · RV.



# Detection of Random Signals

$$\left(\frac{v(1)}{\sigma}\right)^2 + \cdots + \left(\frac{v(N)}{\sigma}\right)^2$$

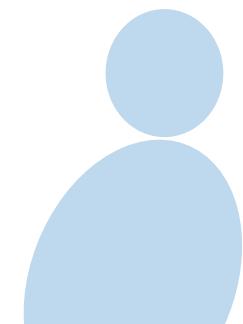
- Sum of squares of  $N$  iid zero-mean Gaussian RVs.

- This is  $\chi_N^2$  RV N = # degrees of freedom

## Detection of Random Signals

- $\chi_N^2$ -Central chi-squared RV  
with  $N$  degrees of freedom

Zero mean.  
Each Gaussian has zero mean.



# Detection of Random Signals

- PDF is given as

$$f_X(x) = \frac{1}{2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} x^{\frac{N}{2}-1} e^{-\frac{1}{2}x}, \quad x \geq 0$$

$$\Gamma\left(\frac{N}{2}\right) = \int_0^\infty t^{\frac{N}{2}-1} e^{-t} dt$$

$$\Gamma m = (m-1)!$$

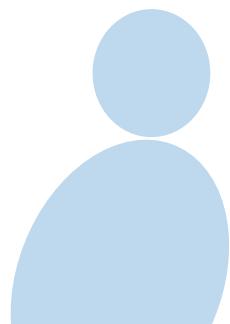
$\chi_N^2$ : Central Chi Squared RV  
 $N$  degrees of freedom

$$\frac{N}{2}-1$$

$$e^{-\frac{1}{2}x}$$

$$x \geq 0$$

+ ve number m.

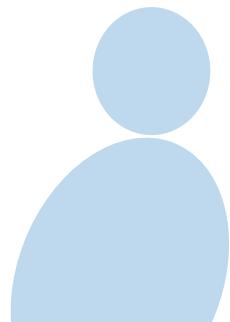


# Detection of Random Signals

- PDF is given as

$$f_X(x) = \frac{1}{2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} x^{\frac{N}{2}-1} e^{-\frac{1}{2}x}, x \geq 0$$

$$\Gamma\left(\frac{N}{2}\right) = \int_0^{\infty} t^{\frac{N}{2}-1} e^{-t} dt$$



# Detection of Random Signals

Under  $H_0$ , Probability decision =  $H_1$ .

- $P_{FA}$  is given as

$$\Pr\left(\left(\frac{V(1)}{\sigma}\right)^2 + \left(\frac{V(2)}{\sigma}\right)^2 + \cdots + \left(\frac{V(N)}{\sigma}\right)^2 > \frac{T}{\sigma^2}\right)$$

$$= Q_{X_N^2} \left( \frac{T}{\sigma^2} \right)$$

CCDF of Chi squared RV with N degrees of freedom.

# Detection of Random Signals

- $P_{FA}$  is given as

$$\Pr\left(\frac{v^2(1)}{\sigma^2} + \frac{v^2(2)}{\sigma^2} + \cdots + \frac{v^2(N)}{\sigma^2} \geq \frac{\gamma}{\sigma^2}\right)$$

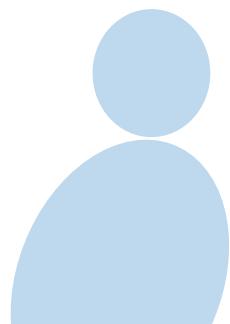
$$P_{FA} = Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2}\right)$$

Probability of False Alarm

# Detection of Random Signals

- $Q_{\chi_N^2}(\cdot)$ : CCDF of the  $\chi_N^2$  RV with  $N$   
degrees of freedom

(Complementary cumulative  
Distribution Function)



# Detection of Random Signals

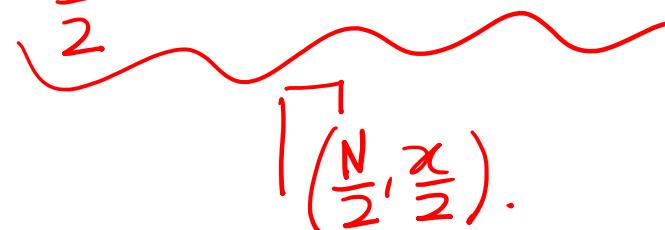
- The CCDF of the  $\chi_N^2$  RV with  $N$  degrees of freedom is

$$Q_{\chi_N^2}(x) = \int_{\frac{x}{2}}^{\infty} \frac{1}{2^{\frac{N}{2}} \sqrt{\left(\frac{N}{2}\right)}} e^{-\frac{1}{2}t} t^{\frac{N}{2}-1} dt$$
$$t = 2u \Rightarrow dt = 2du$$
$$= \frac{1}{2^{\frac{N}{2}} \sqrt{\left(\frac{N}{2}\right)}} \int_{\frac{x}{2}}^{\infty} e^{-u} (2u)^{\frac{N}{2}-1} 2 du$$

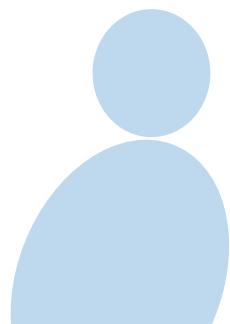
# Detection of Random Signals

$$Q_{X_N^2}(x) = \frac{1}{\Gamma\left(\frac{N}{2}\right)} \int_x^{\infty} e^{-u} u^{\frac{N}{2}-1} du$$

Upper incomplete Gamma function



$$Q_{X_N^2}(x) = \frac{\Gamma\left(\frac{N}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{N}{2}\right)}$$



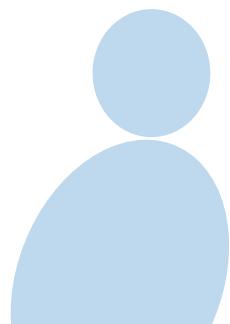
# Detection of Random Signals

$$F(m, x) = \int_x^{\infty} e^{-t} t^{m-1} dt \quad \left. \right\} \text{Upper incomplete Gamma function}$$

# Detection of Random Signals

- The **CCDF** of the  $\chi_N^2$  RV with  $N$  degrees of freedom is

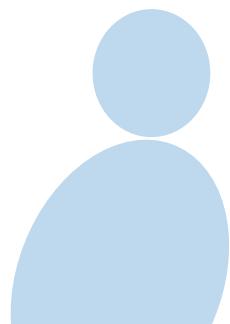
$$Q_{\chi_N^2}(x) = \frac{1}{2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} \int_x^{\infty} t^{\frac{N}{2}-1} e^{-\frac{1}{2}t} dx$$



# Detection of Random Signals

- The CCDF of the  $\chi_N^2$  RV with  $N$  degrees of freedom is

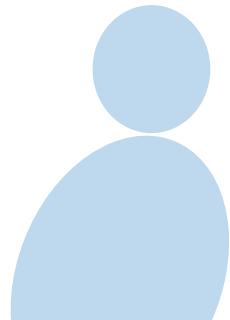
$$\begin{aligned} Q_{\chi_N^2}(x) &= \frac{1}{2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} \int_{\frac{x}{2}}^{\infty} (2u)^{\frac{N}{2}-1} e^{-\frac{1}{2}u^2} 2du \\ &= \frac{1}{\Gamma\left(\frac{N}{2}\right)} \underbrace{\int_{\frac{x}{2}}^{\infty} u^{\frac{N}{2}-1} e^{-u} du}_{\text{Upper Incomplete Gamma Function}} = \frac{\Gamma\left(\frac{N}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \end{aligned}$$



# Detection of Random Signals

$$P_{FA} = Q_{\chi_N^2} \left( \frac{\bar{x}}{\sigma^2} \right)$$
$$P_{FA} = \frac{\Gamma \left( \frac{N}{2}, \frac{\bar{x}}{2\sigma^2} \right)}{\Gamma \left( \frac{N}{2} \right)}$$

Probability of False Alarm



## Detection of Random Signals

- The  $P_D$  is given as follows Probability Signal is detected & under  $H_1$ .

$$\Pr(\|\bar{y}\|^2 \geq \gamma; \mathcal{H}_1)$$

$$= \Pr(|y(1)|^2 + \dots + |y(N)|^2 \geq \gamma; \mathcal{H}_1)$$

# Detection of Random Signals

$$|y(1)|^2 + |y(2)|^2 + \dots + |y(N)|^2 \geq \tau \text{ under } H_1$$

$$y(i) = s(i) + v(i) \sim \mathcal{N}(0, \sigma^2 + \sigma_s^2)$$

$$\left| \frac{y(1)}{\sqrt{\sigma^2 + \sigma_s^2}} \right|^2 + \dots + \left| \frac{y(N)}{\sqrt{\sigma^2 + \sigma_s^2}} \right|^2 \geq \frac{\tau}{\sigma^2 + \sigma_s^2}$$

*sum of squares of N zero mean  
iid Gaussian RVs of unit variance*

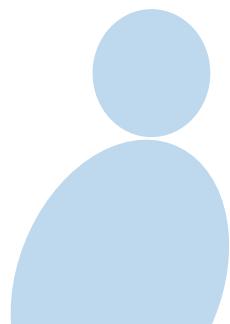
$$\Rightarrow \chi_N^2 \text{ RV}$$

# Detection of Random Signals

- The  $P_D$  is given as follows

$$\Pr(\|\bar{\mathbf{y}}\|^2 \geq \gamma; \mathcal{H}_1)$$

$$y^2(1) + \cdots + y^2(N) \geq \gamma$$

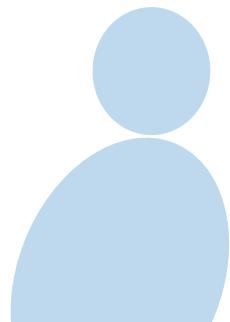


# Detection of Random Signals

$$\left( \frac{y(1)}{\sqrt{\sigma^2 + \sigma_s^2}} \right)^2 + \dots + \left( \frac{y(N)}{\sqrt{\sigma^2 + \sigma_s^2}} \right)^2 \geq \frac{\gamma}{\sigma^2 + \sigma_s^2}$$

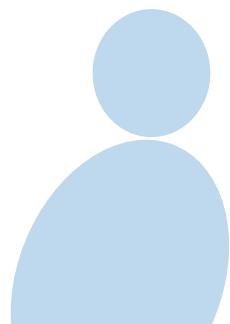
$$\geq \frac{\gamma}{\sigma^2 + \sigma_s^2}$$

$\chi_N^2$  Random variable .



# Detection of Random Signals

$$\frac{y(i)}{\sqrt{\sigma^2 + \sigma_s^2}} \sim \mathcal{N}(0, 1) \quad \text{Standard Gaussian RV.}$$



# Detection of Random Signals

$$\frac{y(i)}{\sqrt{\sigma^2 + \sigma_s^2}} \sim \mathcal{N}(0,1)$$

Sum of squares  
of N Standard  
Normal. RV's.

$$\left( \frac{y(1)}{\sqrt{\sigma^2 + \sigma_s^2}} \right)^2 + \dots + \left( \frac{y(N)}{\sqrt{\sigma^2 + \sigma_s^2}} \right)^2 \geq \frac{\gamma}{\sigma^2 + \sigma_s^2}$$

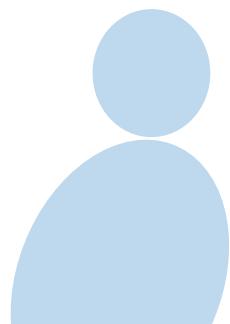
$\chi^2_N$ .

# Detection of Random Signals

- $P_D$  is given as

$$\Pr \left( \left| \frac{y(1)}{\sqrt{\sigma^2 + \sigma_s^2}} \right|^2 + \cdots + \left| \frac{y(N)}{\sqrt{\sigma^2 + \sigma_s^2}} \right|^2 \geq \frac{T}{\sigma^2 + \sigma_s^2} \right)$$

$$Q_{X_N^2} \left( \frac{T}{\sigma^2 + \sigma_s^2} \right)$$

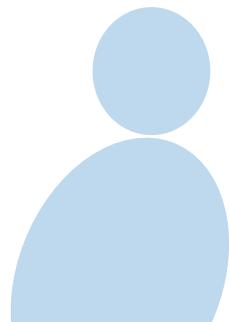


# Detection of Random Signals

- $P_D$  is given as

$$\Pr \left( \left( \frac{y(1)}{\sqrt{\sigma^2 + \sigma_s^2}} \right)^2 + \cdots + \left( \frac{y(N)}{\sqrt{\sigma^2 + \sigma_s^2}} \right)^2 \geq \frac{\gamma}{\sigma^2 + \sigma_s^2} \right)$$
$$P_D = Q_{\chi_N^2} \left( \frac{\gamma}{\sigma^2 + \sigma_s^2} \right) = \frac{\Gamma \left( \frac{N}{2}, \frac{\gamma}{2(\sigma^2 + \sigma_s^2)} \right)}{\Gamma \left( \frac{N}{2} \right)}.$$

Probability of detection



# Detection of Random Signals

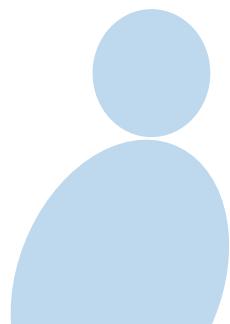
$$P_{FA} = Q_{X_N^2} \left( \frac{\gamma}{\sigma^2} \right)$$

$$P_D = Q_{X_N^2} \left( \frac{\gamma}{\sigma^2 + \sigma_s^2} \right)$$

$$ROC = ? \quad \gamma = \sigma^2 Q_{X_N^2}^{-1}(P_{FA})$$

$$P_D = Q_{X_N^2} \left( \frac{\sigma^2 Q_{X_N^2}^{-1}(P_{FA})}{\sigma^2 + \sigma_s^2} \right)$$

ROC  
Receiver operating characteristic



# Detection of Random Signals

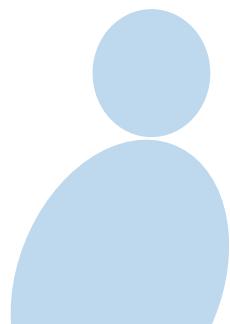
- PDF for  $N = 2$  is given as

$$f_X(x) = \frac{1}{2^{\frac{N}{2}} \Gamma_{\frac{N}{2}}} \cdot e^{-\frac{1}{2}x} x^{\frac{N}{2}-1} = \frac{1}{2^1 \cdot \Gamma_1} e^{-\frac{1}{2}x} x^0 = \frac{1}{2} e^{-\frac{1}{2}x}$$

$$f_X(x) = \frac{1}{2} e^{-\frac{1}{2}x}$$

$$\Gamma(1) = (1-1)! = 0! = 1$$

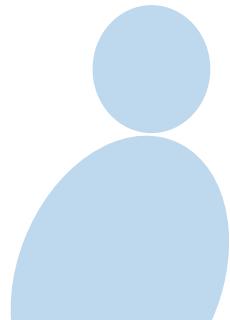
$\chi^2$  ~~Central Chi squared  
RV with 2 degrees  
of Freedom~~



# Detection of Random Signals

- PDF for  $N = 2$  is given as

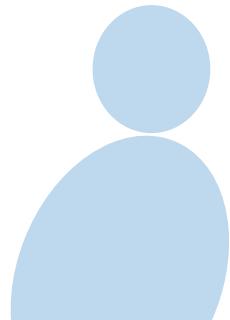
$$\begin{aligned}f_X(x) &= \frac{1}{2^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} x^{\frac{N}{2}-1} e^{-\frac{1}{2}x} \\&= \frac{1}{2\Gamma(1)} e^{-\frac{1}{2}x} = \frac{1}{2} e^{-\frac{1}{2}x} x \geq 0 \\ \Gamma(1) &= \int_0^\infty e^{-t} dt = 1\end{aligned}$$



# Detection of Random Signals

- CCDF for  $N = 2$  is given as

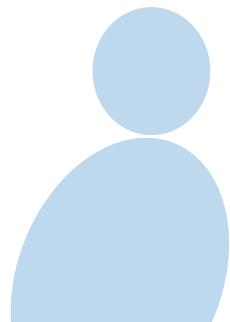
$$\begin{aligned} Q_{X_2}(x) &= \int_x^{\infty} \frac{1}{2} e^{-\frac{1}{2}t} dt \\ &= -e^{-\frac{1}{2}t} \Big|_x^{\infty} \\ &= e^{-\frac{1}{2}x} \end{aligned}$$



# Detection of Random Signals

- CCDF for  $N = 2$  is given as

$$Q_{\chi^2_2}(x) = \int_x^\infty \frac{1}{2} e^{-\frac{1}{2}t} = e^{-\frac{1}{2}x}$$



# Detection of Random Signals

- Therefore  $P_D, P_{FA}$  are

Receiver operating  
characteristic  
 $N = 2$

$$P_D = Q_{\chi^2} \left( \frac{\gamma}{\sigma^2 + \sigma_s^2} \right) = e^{-\frac{\gamma}{2(\sigma^2 + \sigma_s^2)}}$$

$$P_{FA} = Q_{\chi^2} \left( \frac{\gamma}{\sigma^2} \right) = e^{-\frac{\gamma}{2\sigma^2}} \Rightarrow \gamma = -2\sigma^2 \ln P_{FA}$$

$$\begin{aligned} \Rightarrow e^{-\frac{1}{2(\sigma^2 + \sigma_s^2)} \cdot (-2\sigma^2 \ln P_{FA})} &= e^{\frac{\sigma^2}{\sigma^2 + \sigma_s^2} \ln P_{FA}} \\ &= e^{\ln P_{FA} \frac{\sigma^2}{\sigma^2 + \sigma_s^2}} \end{aligned}$$

$$P_{FA}^{\frac{\sigma^2}{\sigma^2 + \sigma_s^2}} = P_D$$

# Detection of Random Signals

- Therefore  $P_D, P_{FA}$  are

$$P_D = Q_{\chi^2_2} \left( \frac{\gamma}{\sigma^2 + \sigma_s^2} \right) = e^{-\frac{1}{2}\frac{\gamma}{\sigma^2 + \sigma_s^2}}$$

$$P_{FA} = Q_{\chi^2_2} \left( \frac{\gamma}{\sigma^2} \right) = e^{-\frac{1}{2}\frac{\gamma}{\sigma^2}}$$

Probability of  
False Alarm ·  $P_{FA}$  .

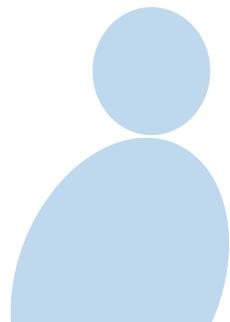
Probability of  
detection

$P_D$  .

# Detection of Random Signals

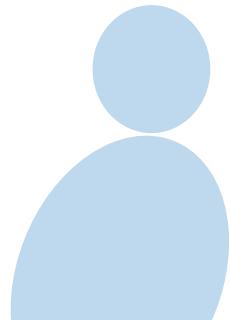
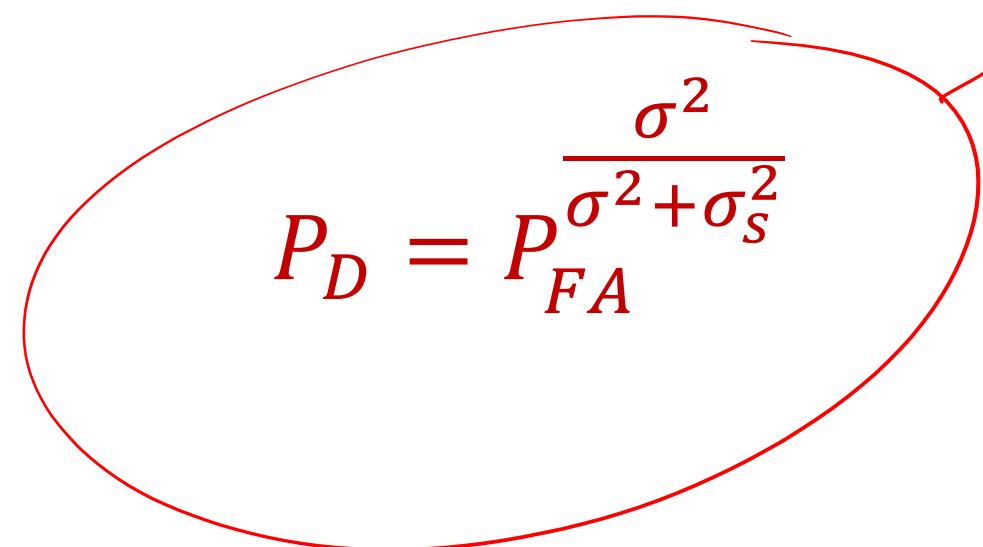
- ROC can be evaluated as

$$\gamma = -2\sigma^2 \ln P_{FA}$$



# Detection of Random Signals

$$P_D = e^{-\frac{1}{2} \frac{\gamma}{2\sigma^2 + \sigma_s^2}} = e^{-\frac{1}{2} \times \frac{-2\sigma^2 \ln P_{FA}}{\sigma^2 + \sigma_s^2}}$$
$$= e^{\frac{\sigma^2}{\sigma^2 + \sigma_s^2} \ln P_{FA}} = e^{\ln P_{FA}^{\frac{\sigma^2}{\sigma^2 + \sigma_s^2}}} \quad ROC$$



# Chi-Squared Approximation

$\chi^2_N$

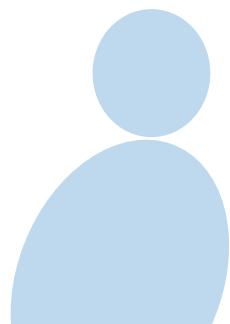
- Chi-squared RV with  $N$  degrees of freedom

Central Chi Squared  
RV

$$U = U_1^2 + U_2^2 + \cdots + U_N^2$$

$$U_i \sim N(0, 1)$$

$U_i$  are iid Standard Normal RVs.

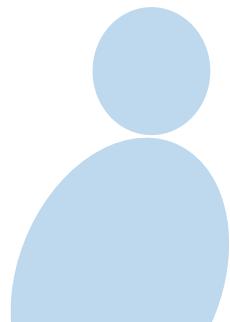


## Chi-Squared Approximation

- Chi-squared RV with  $N$  degrees of freedom

$$u = u_1^2 + u_2^2 + \cdots + u_N^2$$

$$u_i \sim N(0,1).$$



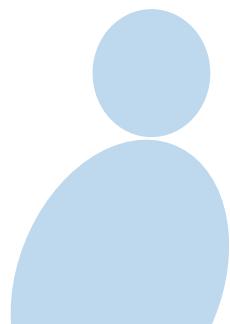
# Chi-Squared Approximation

$$u = u_1^2 + u_2^2 + \cdots + u_N^2$$

- What happens when  $N \rightarrow \infty$ ?

$\xrightarrow{u \rightarrow \text{Gaussian RV}}$   
Central Limit  
Theorem

$u_i^2 \underset{iid}{\sim} \text{RVs}$ .  
Sum of  $N$  iid



## Chi-Squared Approximation

$$u = u_1^2 + u_2^2 + \cdots + u_N^2$$

$$\begin{aligned} E\{u\} &= ? \\ E\{(u - \bar{u})^2\} &= ? \end{aligned}$$

$$\rightarrow \mathcal{N}(?, ?)$$

- What happens when  $N \rightarrow \infty$ ?
  - Becomes **Gaussian** because of **Central Limit Theorem.**

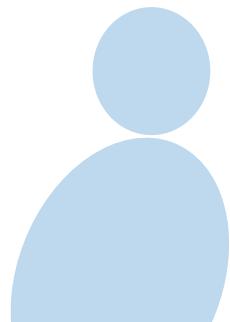
# Chi-Squared Approximation

- Mean is

$$\begin{aligned} E\{u\} &= E\left\{u_1^2 + u_2^2 + \cdots + u_N^2\right\} \\ &= \sum_{i=1}^N E\{u_i^2\} = \sum_{i=1}^N 1 = N \end{aligned}$$

$u_i \sim \mathcal{N}(0, 1)$   
 $E\{u_i^2\} = 1$

Mean =  $N$

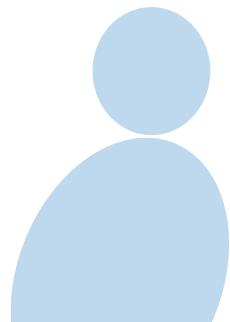


# Chi-Squared Approximation

- Mean is

$$E\{u_i^2\} = 1$$

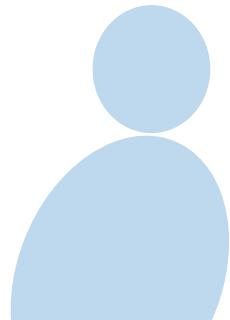
$$\begin{aligned} E\{u\} &= E\{u_1^2 + u_2^2 + \cdots + u_N^2\} \\ &= N \end{aligned}$$



# Chi-Squared Approximation

- **Variance** can be found as follows

$$\begin{aligned}\text{Variance} &= E\{(u - E\{u\})^2\} \\ &= E\{u^2\} - \underline{(E\{u\})^2} \\ &= E\{u^2\} - N^2\end{aligned}$$

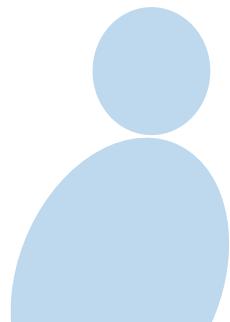


# Chi-Squared Approximation

- **Variance** can be found as follows

$$\text{Variance} = E\{u^2\} - E^2\{u\}$$

$$= \cancel{E\{u^2\}} + N^2$$



## Chi-Squared Approximation

$$\begin{aligned} E\{u^2\} &= E\left\{\left(\sum_{i=1}^N u_i^2\right)^2\right\} \\ &= E\left\{\sum_{i=1}^N u_i^4 + 2 \sum_{i=1}^N \sum_{j=1}^{i-1} u_i^2 u_j^2\right\} \\ &= \sum_{i=1}^N \left( E\{u_i^4\} \right) + 2 \sum_{i=1}^N \sum_{j=1}^{i-1} E\{u_i^2 u_j^2\} \end{aligned}$$

$$\begin{aligned} E\{u_i^4\} &= 3\sigma^4 = 3 \quad \text{Since } u_i, u_j \text{ independent} \\ E\{u_i^2 u_j^2\} &= E\{u_i^2\} E\{u_j^2\} \\ &= 1 \times 1 = 1 \end{aligned}$$

# Chi-Squared Approximation

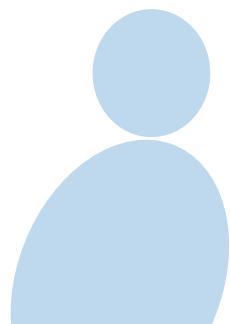
$$E\{U^2\} = \sum_{i=1}^N 3 + 2 \left( \sum_{i=1}^N \sum_{j=1}^{i-1} 1 \right)$$

Total number of combinations  
 $i \neq j = Nc_2$   
 $= \frac{N(N-1)}{2}$

$$= 3N + 2 \cdot \frac{N(N-1)}{2}$$

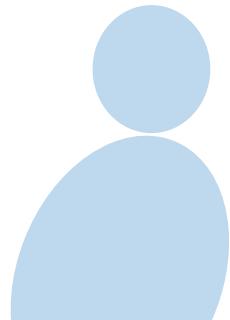
$$= 3N + (N^2 - N)$$

$$= N^2 + 2N$$



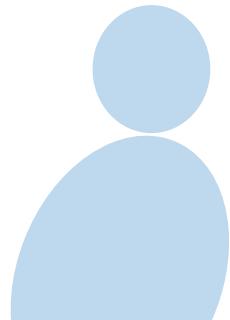
# Chi-Squared Approximation

$$\begin{aligned} E\{u^2\} &= E \left\{ \left( \sum_{i=1}^N u_i^2 \right)^2 \right\} \\ &= E \left\{ \sum_{i=1}^N u_i^4 + \sum_{i=1}^N \sum_{j=1}^{i-1} 2u_i^2 u_j^2 \right\} \end{aligned}$$



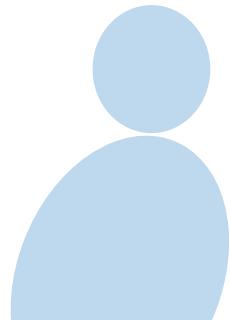
# Chi-Squared Approximation

$$\begin{aligned} &= \sum_{i=1}^N E\{u_i^4\} + \sum_{i=1}^N \sum_{j=1}^{i-1} 2E\{u_i^2 u_j^2\} \\ &= \sum_{i=1}^N E\{u_i^4\} + \sum_{i=1}^N \sum_{j=1}^{i-1} 2E\{u_i^2 u_j^2\} \end{aligned}$$



# Chi-Squared Approximation

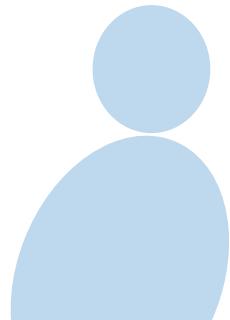
$$\begin{aligned} &= \sum_{i=1}^N 3 + \sum_{i=1}^N \sum_{j=1}^{i-1} 2 = 3N + 2 \times {}^N C_2 \\ &= 3N + N(N - 1) = N^2 + 2N \end{aligned}$$



# Chi-Squared Approximation

- **Variance** can be found as follows

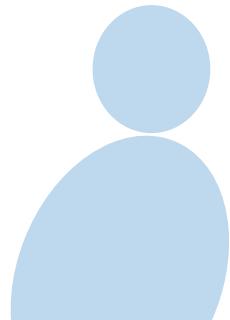
$$\begin{aligned}\text{Variance} &= E\{U^2\} - E\{U\}^2 \\ &= N^2 + 2N - N^2 \\ &= 2N = \underbrace{\text{Variance of } U}.\end{aligned}$$



# Chi-Squared Approximation

- **Variance** can be found as follows

$$\begin{aligned}\text{Variance} &= E\{u^2\} - E^2\{u\} \\ &= N^2 + 2N - N^2 = 2N\end{aligned}$$



# Chi-Squared Approximation

- Therefore,

$$\chi_N^2 \approx \mathcal{N}(N, 2N)$$

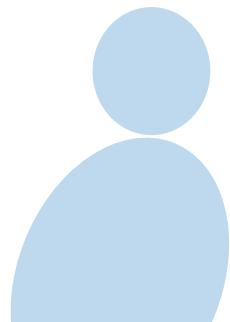
Normal Distribution  
mean  
variance

$N \rightarrow \infty$

# Chi-Squared Approximation

- Therefore,

$$\chi_N^2 \approx \mathcal{N}(N, 2N)$$

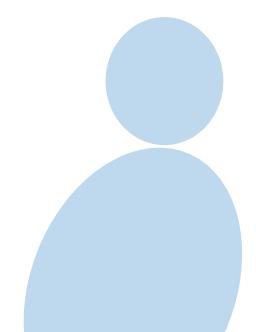


# Chi-Squared Approximation

- The  $P_D$  can be found as

$$\begin{aligned} P_D &= Q_{\chi^2_N} \left( \frac{\tilde{I}}{\sigma_s^2 + \sigma^2} \right) = \Pr \left( \chi_N^2 \geq \frac{\tilde{I}}{\sigma_s^2 + \sigma^2} \right) \\ &= \Pr \left( \mathcal{N}(N, 2N) \geq \frac{\tilde{I}}{\sigma_s^2 + \sigma^2} \right) \\ &= Q \left( \frac{\frac{\tilde{I}}{\sigma_s^2 + \sigma^2} - N}{\sqrt{2N}} \right) \end{aligned}$$

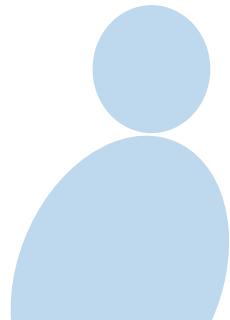
*Q function  
of standard:  
Gaussian RV*



# Chi-Squared Approximation

- The  $P_D$  can be found as

$$P_D = Q_{\chi_N^2} \left( \frac{\tilde{\gamma}}{\sigma^2 + \sigma_s^2} \right)$$
$$= Q \left( \frac{\frac{\tilde{\gamma}}{\sigma^2 + \sigma_s^2} - N}{\sqrt{2N}} \right)$$



# Chi-Squared Approximation

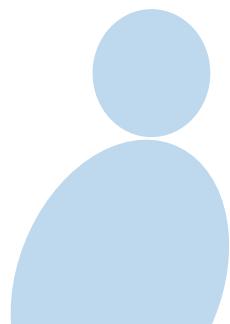
- The  $P_{FA}$  can be found as

$$\begin{aligned} P_{FA} &= Q_{\chi_N^2}\left(\frac{\tilde{\sigma}}{\sigma^2}\right) = \Pr\left(\chi_N^2 \geq \frac{\tilde{\sigma}}{\sigma^2}\right) \\ &= \Pr\left(\mathcal{N}(N, 2N) \geq \frac{\tilde{\sigma}}{\sigma^2}\right) \quad \text{For large } N. \\ &= Q\left(\frac{\frac{\tilde{\sigma}}{\sigma^2} - N}{\sqrt{2N}}\right) = P_{FA} \end{aligned}$$

# Chi-Squared Approximation

- The  $P_{FA}$  can be found as

$$P_{FA} = Q_{\chi_N^2} \left( \frac{\tilde{\gamma}}{\sigma^2} \right) = Q \left( \frac{\frac{\tilde{\gamma}}{\sigma^2} - N}{\sqrt{2N}} \right)$$



# Chi-Squared Approximation

- ROC found as

$$P_{FA} = Q\left(\frac{\tilde{\sigma}/\sigma^2 - N}{\sqrt{2N}}\right)$$
$$\Rightarrow \tilde{\sigma} = \left(\sqrt{2N} Q^{-1}(P_{FA}) + N\right) \sigma^2$$

# Chi-Squared Approximation

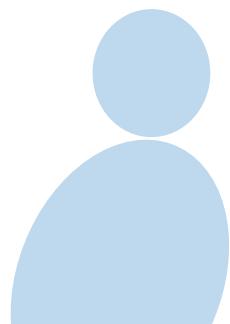
- ROC found as

$$P_D = Q\left(\frac{\frac{\tilde{x}}{\sigma^2 + \sigma_s^2} - N}{\sqrt{2N}}\right) = Q\left(\frac{\frac{\sigma^2}{\sigma_s^2 + \sigma^2} (\sqrt{2N} \Phi^{-1}(P_{FA}) + N) - N}{\sqrt{2N}}\right)$$

$$P_D = Q\left(\frac{\sigma^2}{\sigma_s^2 + \sigma^2} \Phi^{-1}(P_{FA}) - \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} \cdot \sqrt{\frac{N}{2}}\right)$$

For large  $N$   
 $\chi_N^2 \rightarrow N(N, 2N)$

Receiver Operating characteristic  
ROC

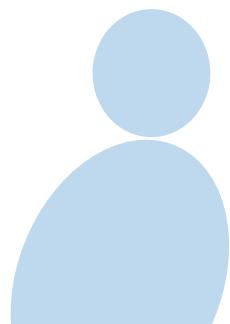


# Chi-Squared Approximation

- ROC found as

$$P_D = Q\left(\frac{\frac{1}{\sigma^2 + \sigma_s^2} \left( \sigma^2 (\sqrt{2N} Q^{-1}(P_{FA}) + N) \right) - N}{\sqrt{2N}}\right)$$

$$P_D = Q\left(\frac{\sigma^2}{\sigma^2 + \sigma_s^2} Q^{-1}(P_{FA}) - \frac{\sigma_s^2}{\sigma^2 + \sigma_s^2} \sqrt{\frac{N}{2}}\right)$$



Instructors may use this white area (14.5 cm / 25.4 cm) for the text.  
Three options provided below for the font size.

**Font: Avenir (Book), Size: 32, Colour: Dark Grey**

**Font: Avenir (Book), Size: 28, Colour: Dark Grey**

**Font: Avenir (Book), Size: 24, Colour: Dark Grey**

**Do not use the space below.**

