Assignment 5 Solution

Digital Communication System-I

May 2023

1. (a) The probability of receiving bit-1 and bit-0 after the first transition can be given by

$$Pr(y=1) = Pr(y=1|x=0) * Pr(x=0) + Pr(y=1|x=1) * Pr(x=1) = 0.62$$
$$Pr(y=0) = Pr(y=0|x=0) * Pr(x=0) + Pr(y=0|x=1) * Pr(x=1) = 0.38$$

Hence, the probability of receiving bit-1 at node B is

$$Pr(z=1) = Pr(z=1|y=0)*Pr(y=0)+Pr(z=1|y=1)*Pr(y=1) = 0.572,$$

and similarly, $Pr(z=0) = 0.428$.

2. (b) MAP and ML decision rules are same when all the messages are equiprobable, *i.e.*, $Pr(x=0) = Pr(x=1) = 0.5$ The MAP decision rule is given as,

$$\hat{m} = \arg \max_{1 \le m \le M} \frac{P_m p\left(r \mid s_m\right)}{p(r)}.$$

Since p(r) is independent of m, then MAP decision rule can be written as, $\hat{m} = \arg\max_{1 \le m \le M} P_m p\left(r \mid s_m\right)$.

If the messages are equiprobable, the optimal decision rule reduces to, $\hat{m} = \arg\max_{1 \le m \le M} p\left(r \mid s_m\right)$, which is same as the ML decision rule.

Therefore, the statement: MAP decision rule is same as the ML decision rule if $Pr(y=0|x=1) = Pr(y=1|x=0) = 0.5$, is false.

3. (c) We need optimum detection rule to minimize the probability of error. The receiver makes an optimal decision to minimize the probability of disagreement between the transmitted message m and the detected message $\hat{m}$ given by $P_e = P[\hat{m} \neq m]$.

4. (c) The bit error probability for BPSK modulated signal with coherent detection over an AWGN channel with the power spectral density of the channel noise is $N_0/2$ is

$$Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = 0.1586.$$

Therefore, $\frac{E_b}{N_0} = 0.5 \times Q^{-1}(0.1586) = 0.5 =$ -3 dB.

5. (a) The probability of error for equiprobable binary signaling scheme over AWGN channel is given as $P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$.

   Now, for orthogonal signaling scheme $d_{12} = \sqrt{4\epsilon_b}$.

   $\therefore P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2\epsilon_b}{N_0}}\right)$.

6. (a) Probability of error depends on Euclidean distances among signal vectors in signal space diagram.

7. (d) $r_{th} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln\left(\frac{1-p}{p}\right) = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln\left(\frac{0.8}{0.2}\right) = \frac{0.35N_0}{\sqrt{\mathcal{E}_b}}$.

8. (b) From the definition, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du$. Hence, we can write $Q(-x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^\infty \exp\left(-\frac{u^2}{2}\right) du$, where 1 can be represented as $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{u^2}{2}\right) du$. Therefore, $Q(x) = 1 - Q(-x)$. Hence, $Q(x) \neq Q(-x)$, and thus Q function is not an even function.

   And, $Q'(x) = \frac{\partial Q(x)}{\partial x} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, where $Q'(x) > 0$ for $x > 0$. Hence, $Q(x)$ is an increasing function.

9. (a) From MAP decision rule,

$$\hat{m} = \arg \max_{1 \le m \le M} \left[\frac{N_0}{2} \log\left(P_m\right) - \frac{1}{2}\mathcal{E}_m + r^T S_m\right]$$

   If the signals are equiprobable, then $P_m = \frac{1}{M}$ and the ML decision rule is given as

$$\hat{m} = \arg \max_{1 \le m \le M} \left[\frac{N_0}{2} \log\left(P_m\right) - \frac{1}{2} \left\|r - S_m\right\|^2\right] = \arg \min_{1 \le m \le M} \left\|r - S_m\right\|$$

   Also from $\arg\min_{1 \le m \le M} \left\|r - S_m\right\|^2 = \arg\min_{1 \le m \le M} \left(\|r\|^2 - 2r^T S_m + \left\|S_m\right\|^2\right) = \arg\max_{1 \le m \le M} \left(\int_0^T r(t) S_m(t) dt - \frac{1}{2} \int_0^T \left|S_m(t)\right|^2 dt\right)$ $\therefore$ The correct option is only (a).

10. (d) Impulse response of matched filter is given as h(t) = s(T-t)

    $\therefore h(t) = \frac{A}{T} \sin 2\pi f_c (T - t)$

    $\therefore$ Output of matched filter at t = T is derived as

    $g(t = T) = h(T) * s(T) = \int_0^T \big(s(\tau) h(T - \tau)\big) d\tau$

    $g(t = T) = \int_0^T \left(\frac{A}{T} \sin\ 2\pi f_c \tau\right)\left(\frac{A}{T} \sin\ 2\pi f_c (T - (T - \tau))\right)$

    $g(t = T) = \frac{A^2}{2T} - \frac{A^2}{8\pi f_c T^2} \sin(4\pi f_c T)$