Solutions of Tutorial-1

Problem set 1.1

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v}+\boldsymbol{w}=(2,3)$ and $\boldsymbol{v}-\boldsymbol{w}=(6,-1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0,0)$.

17 All vectors $c\boldsymbol{v}+c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v}+\boldsymbol{w}$ and back beyond $(0,0)$. With $c\geq 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with $0\leq c\leq 1$ and $0\leq d\leq 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$ then $c\boldsymbol{v}+d\boldsymbol{w}$ fills the unit square. But when $\boldsymbol{v}=(a,0)$ and $\boldsymbol{w}=(b,0)$ these combinations only fill a segment of a line.

Problem set 1.2

2 $\|\boldsymbol{u}\|=1$ and $\|\boldsymbol{v}\|=5$ and $\|\boldsymbol{w}\|=\sqrt{5}$. Then $|\boldsymbol{u}\cdot\boldsymbol{v}|=0<(1)(5)$ and $|\boldsymbol{v}\cdot\boldsymbol{w}|=10<5\sqrt{5}$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\|=(\frac{4}{5},\frac{3}{5})=(0.8,0.6)$. The vectors $\boldsymbol{w},(2,-1)$, and $-\boldsymbol{w}$ make $0°,90°,180°$ angles with $\boldsymbol{w}$ and $\boldsymbol{w}/\|\boldsymbol{w}\|=(1/\sqrt{5},2/\sqrt{5})$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}=10/5\sqrt{5}$.

9 If $v_2w_2/v_1w_1=-1$ then $v_2w_2=-v_1w_1$ or $v_1w_1+v_2w_2=\boldsymbol{v}\cdot\boldsymbol{w}=0$: perpendicular!

The vectors $(1,4)$ and $(1,-\frac{1}{4})$ are perpendicular.

22 $v_1^2w_1^2+2v_1w_1v_2w_2+v_2^2w_2^2\leq v_1^2w_1^2+v_1^2w_2^2+v_2^2w_1^2+v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2+v_2^2w_1^2-2v_1w_1v_2w_2$ which is $(v_1w_2-v_2w_1)^2\geq 0$.

Problem set 1.3

1 $2\boldsymbol{s}_1+3\boldsymbol{s}_2+4\boldsymbol{s}_3=(2,5,9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x}=(2,3,4)$:

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix}=\begin{bmatrix}(\text{row }1)\cdot\boldsymbol{x}\\(\text{row }2)\cdot\boldsymbol{x}\\(\text{row }2)\cdot\boldsymbol{x}\end{bmatrix}=\begin{bmatrix}2\\5\\9\end{bmatrix}.$$

2 The solutions are $y_1=1,y_2=0,y_3=0$ (right side $=$ column 1) and $y_1=1,y_2=3,y_3=5$. That second example illustrates that the first n odd numbers add to n^2.

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

8
$$\begin{aligned} x_1 - 0 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \\ x_4 - x_3 &= b_4 \end{aligned} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned} \quad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\boldsymbol{b}$$

Problem set 2.1

1 The row picture for $A = I$ has 3 perpendicular planes $x = 2$ and $y = 3$ and $z = 4$. Those are perpendicular to the x and y and z axes: $z = 4$ is a horizontal plane at height 4.

The column vectors are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$. Then $\boldsymbol{b} = (2, 3, 4)$ is the linear combination $2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

4 If $z = 2$ then $x + y = 0$ and $x - y = 2$ give the point $(x, y, z) = (1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

9 (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x} = (18, 5, 0)$ and $(3, 4, 5, 5)$. By rows or by columns: **9** separate multiplications when A is 3 by 3.

17 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces $\begin{bmatrix} y \\ z \\ x \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Q is the *inverse* of P. Later we write $QP = I$ and $Q = P^{-1}$.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.