# eMasters – Communication Systems
# E920 Wireless Communications

## Notes, References, Questions, Problems, and Solutions

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# Key Concepts in Signal Processing

- Power Signal
  - A signal is said to be power signal if its average power is finite i.e. $0 < P < \infty$
  - Total energy is $\infty$
  - Periodic signals are examples of power signals
  - The average power of a signal is defined as the mean power dissipated by the signal such as voltage or current in a unit resistance over a period
- Energy Signal
  - A signal is said to be energy signal if and only if its total energy E is finite i.e. $0 < E < \infty$
  - Average power 0
  - Non-periodic signals are examples of energy signals
- Taking the reference of electric circuits/signals
  - The instantaneous power is $p(t) = v(t).i(t)$
  - Applying Ohm's law ($v(t) = i(t)R$)
    - $p(t) = \frac{v^2(t)}{R}\ or\ i^2(t)R$
  - When the value of resistance is 1Ω, the power dissipated in it is known as **normalized power**
    $\Rightarrow Normalized\ power\ -\ p(t) = v^2(t) = i^2(t)$
- If $v(t)\ or\ i(t)$ is denoted by a continuous time signal $x(t)$
  - The instantaneous power is equals to the square of the amplitude of the signal
  - $p(t) = |x(t)|^2$

# Key Concepts in Signal Processing

- Power and energy - Continuous Time Case
  - **Average power** or normalized power of continuous time signal $x(t)$ is given by
    - $P = \lim_{T\to\infty} \frac{1}{T} \int_{-\left(\frac{T}{2}\right)}^{\left(\frac{T}{2}\right)} |x(t)|^2 dt$ **Watts**
  - **Total energy** or normalized energy of a continuous time signal is defined as
    - $E = \lim_{T\to\infty} \int_{-\left(\frac{T}{2}\right)}^{\left(\frac{T}{2}\right)} |x(t)|^2 dt$ **Joules**
- Power and energy - Discrete Time Case
  - For the discrete time signal $x(n)$, the integrals above are replaced by summations
  - Hence the **total energy** or normalized energy of $x(n)$ is
    - $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$
  - The **average power** or normalized power of $x(n)$ is given by
    - $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$
- Important Points
  - Power and energy signals are mutually exclusive – no signal can be both power signal and energy signal
  - A signal is neither energy nor power signal if both energy and power of the signal are equal to infinity
  - All practical signals have finite energy; thus they are energy signals
  - All finite duration and finite amplitude signals are energy signals
  - A signal whose amplitude is constant over infinite duration is a power signal
  - The energy of a signal is not affected by the time shifting and time inversion. It is only affected by the time scaling

# Key Concepts in Signal Processing

- Average power
  - The average power of a signal is defined as the mean power dissipated by the signal such as voltage or current in a unit resistance over a period
  - $P = \lim_{T\to\infty} \frac{1}{T}\int_{-(\frac{T}{2})}^{(\frac{T}{2})} |x(t)|^2 dt$
- Parseval's Power Theorem
  - The power of a signal is equal to the sum of square of the magnitudes of various harmonic components present in the discrete spectrum
  - $P = \sum_{n=-\infty}^{\infty} |C_n|^2$
- Energy
  - Energy $E_s$ of a continuous time signal $x(t)$ is defined as the area under the curve of square of magnitude of that signal
  - $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$
  - The energy signal exists only of the energy (E) of the signal is finite, i.e., only if $0 < E < \infty$
- Rayleigh's Energy Theorem
  - Energy of a function i.e. the integral of the square of magnitude of a function is equal to the integral of the square of magnitude of its Fourier transform
  - $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\sum_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

# Energy Spectral Density and Autocorrelation

- Energy Spectral Density or Energy Density or Energy Density Spectrum - ψ(ω)
  - Distribution of energy of a signal in the frequency domain
  - $\psi(\omega) = |X(\omega)|^2$
- Autocorrelation
  - Gives the measure of degree of similarity between a signal (time series) and its time delayed version
  - The autocorrelation function of an energy signal x(t) is given by
    - $R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$
    - $\tau$ is called the delayed parameter
- Relationship between ESD and autocorrelation
  - The autocorrelation function $R(\tau)$ and the energy spectral density (ESD) function ψ(ω) form a Fourier transform pair
    - $R(\tau) \overset{FT}{\leftrightarrow} \psi(\omega)$

# Power Spectral Density and Autocorrelation

- Autocorrelation
  - Gives the measure of similarity between a signal and its time-delayed version expressed as
    - $R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)^*\}$
      - $X(t_1)\ -\ value\ of\ X\ at\ instant\ t_1$
      - $X(t_2)^*\ -complex\ conjugate\ value\ of\ X\ at\ instant\ t_2$
  - The autocorrelation function of a power (or periodic) signal $x(t)$ with any time period T is given by
    - $R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-(\frac{T}{2})}^{(\frac{T}{2})} x(t)x^*(t-\tau)dt$
    - $\tau$ is called the delayed parameter
  - For wide sense stationary process (WSS), the auto-correlation function) is:
    - $R_X(\tau) = E\{X(t)X(t+\tau)\}$
- PSD or Power Density or Power Density Spectrum
  - The distribution of average power of a signal in the frequency domain and denoted by $S(\omega)$
    - $S(\omega) = \lim_{\tau\to\infty} \frac{|X(\omega)|^2}{\tau}$
  - Power spectral density, PSD ($S_x(f)$) – of $x(t)$ is the Fourier transform ($\mathcal{F}$) of autocorrelation function $R_x(\tau)$ of $x(t)$
    - $S_x(f) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_x(\tau)e^{-2j\pi f_\tau}d\tau$
- Relationship between PSD and autocorrelation function
  - The power spectral density function $S(\omega)$ and the autocorrelation function $R(\tau)$ of a power signal form a Fourier transform
    - $R(\tau) \overset{FT}{\leftrightarrow} S(\omega)$

# Modern Communication Technologies and Systems

- Cutting edge wireless technologies
  - Multiple Antennae Systems
    - Multiple Input and Multiple Output - MIMO Technology
  - OFDM – Orthogonal Frequency Division Multiplexing
    - Large bandwidth divided into several sub-bands and multiple uses sub-carriers
  - CDMA
    - Spreading code
- Modern cellular and wifi systems built on cutting edge wireless technologies
  - LTE
  - 5G – NR
  - 802.11 AC, 802.11 AX

# Principles and Models of Modern Wireless Systems

- Large Scale Fading
  - Due to path loss of signal as a function of distance and shadowing by large structures – buildings and hills
  - Occurs as mobile moves through a distance of the order of the cell size
  - Frequency independent
- Small Scale Fading
  - Due to constructive and destructive interference of the multiple signal paths between the transmitter and receiver
  - Occurs at the spatial scale of the order of the carrier wavelength
  - Frequency dependent

# Modern Wireline Digital Communication System

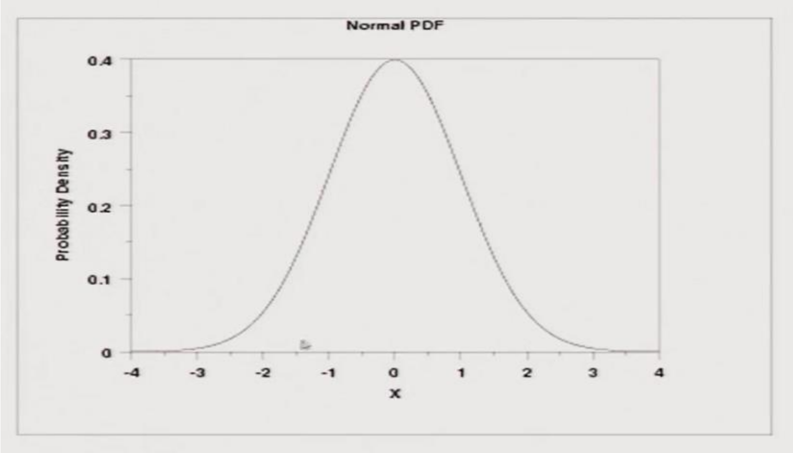
- Channel is fixed
- Signal to Noise Power Ratio
- Simple model of wireline communication system
  - Four components
    - Received signal *y*
    - Transmitted signal *x*
    - Noise *n*
    - Channel
  - SNR – Signal to Noise Power Ratio
  - Simplified model
    - $y = x + n$
    - *n* – Additive noise
- Signal Energy
  - The energy of a continuous-time signal *x(t)* is defined as:
    - $E_x = \int_{-\infty}^{\infty} |x(t)^2| dt$
  - The energy of a discrete-time signal x[n] is
    - $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
  - Physical interpretation
    - Energy above does not refer to a specific physical property
    - Instead, it describes the size of the signal
    - The energy above, however, can be related to electrical energy
    - If *x(t)* is the voltage signal across a load of resistance R, then the energy supplied to that load is $\frac{E_x}{R}$

# Modern Wireline Digital Communication System

- Signal Power or Average power
  - The power of a continuous-time signal *x(t)*
    - $P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$
    - If the signal is periodic => $P_x = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$
  - The power of discrete-time signal
    - $P_x = \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2$
    - When the signal is periodic
      - $\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$
      - N is the period of periodic signal
  - Physical interpretation
    - Power above does not refer to a specific physical property
    - Instead, it describes the size of the periodic signal
    - The power above, however, can be related to electrical power
    - If *x(t)* is the voltage signal across a load of resistance R
      - Then the instantaneous power supplied to that load is $\frac{x^2(t)}{R}$
      - Based on $P = \frac{V^2}{R} = V \times I = V\ -\ Voltage, I\ -Current, R\ -Resistance$
  - Expected average value of the power of a signal $x$ - $P = E\{|x|^2\}$
  - $E\{*\}\ -Expected\ value\ or\ average$

# White Gaussian Noise

- Most common model for noise is Gaussian - the noise samples follow the Gaussian density function
- In other words noise PDF is Gaussian in nature – probability distribution/density function
- Gaussian variables/systems are represented by the Gaussian PDF function
  - $f_G(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(n-\mu)^2}{2\sigma^2}}$
  - Peak at the centre is the mode
  - Peak coincides with the mean (a measure of centrality)
  - Unimodal distribution
  - Width/spread of PDF is the variance




- Typically, noise has a zero mean ($\mu = 0$) value, then
  - Noise PDF - $f_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}}$
- In the context of zero mean noise, variance $\sigma^2$ becomes power
- $\sigma^2 = Variance = \ Power\ of\ zero\ mean\ noise = \frac{N_0}{2} = E\{|N|^2\}$
  - Measure of spread
  - Larger the spread, larger the noise power

# Additive White Gaussian Noise

- Where does $\frac{\mathrm{N}_0}{2}$ come from?
- White noise
  - To say that $f_N(n)$ is a white noise means merely the successive samples are uncorrelated
  - $E\{f_N(n).\,f_N(n+m)\} = \begin{cases} \sigma_{f_N}^2, m = 0 \\ 0\,, m \neq 0 \end{cases} \triangleq \sigma_{f_N}^2 \delta(m)$
  - Where $E\{f_N(n).\,f_N(n+m)\}$ denotes the expected value random variables of $f_N$
  - In other words, the autocorrelation function of white noise is an impulse at lag 0
  - Since power spectral density is the Fourier transform of the autocorrelation function, the PSD of white noise is constant
  - Therefore white noise power spectral density is flat or constant across the frequency spectrum
  - Power spectral density of white noise is given as
    - $PSD \;-\; S_{nn}(\Omega) = \frac{N_0}{2} \;-\; constant\ and\ here\ \Omega\ is\ frequency$
  - Similar to white light – contains all frequency components
- White Gaussian noise is additive in nature

# Random Process Characterization - Power Spectral Density

- For deterministic signal
  - a Fourier transform gives the spectrum of the signal (distribution of power across different frequencies in the spectrum)
- For random process or signal
  - Power spectral density gives the distribution of power across different frequencies in the spectrum
- Random process is a random variable at every instant of time
- One of the important tools to characterize random process is the ***power spectral density***
- Power Spectral Density
  - Measure of the signal's power
  - The power spectrum $S_{xx}(f)$ of a time series $x(t)$ describes the distribution of power into frequency components composing that signal
  - According to Fourier Analysis, any physical signal can be decomposed into a number of discrete frequencies or spectrum of frequencies over a continuous range
  - The statistical average of a certain signal as analysed in terms of its frequency content is called Spectrum
  - PSD refers to the spectral energy distribution that is depicted per unit time
  - Summation or integration of the spectral components yields the total power for a physical processor or variance in a statistical process which is identical to what would be obtained by integrating $x^2(t)$ over the time domain as per Parseval's theorem (Rayleigh's Energy Theorem or Rayleigh's Identity)
- Power spectral density of a random process or variable is derived from the auto-correlation function
- PSD of a random process or signal is the Fourier transform of auto-correlation function

# Stationary Process

- Strict/strong stationary process is a stochastic process whose unconditional joint probability distribution does not change when shifted in time
- Consequently, parameters such as mean and variance also do not change over time
- Definitions
  - Joint probability
    - Given two random variables that are defined in the same probability space, the joint probability distribution is the corresponding probability distribution on all possible pairs of outputs
  - Probability space
    - Also called a probability triplet comprising of
      - **Sample space** of all possible outcomes - $\Omega$
      - **Event space** set of events/outcomes in the sample space - $\mathcal{F}$
      - **Probability function** $P$ Event probability in the event space – between 0 and 1
    - Represented by $\{\Omega, \mathcal{F}, P\}$
- Strict sense stationarity - SSS
  - Let $\{X_t\}$ be a stochastic process
  - Let $F_X\left(x_{t_{1+\tau}}, x_{t_{2+\tau}}, \ldots, x_{t_{n+\tau}}\right)$ represent cumulative distribution function of the unconditional joint distribution of $\{X_t\}$ at times $t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau$
  - Then $\{X_t\}$ is said to be strictly stationary or strict sense stationary if:
    - $F_X\left(x_{t_{1+\tau}}, x_{t_{2+\tau}}, \ldots, x_{t_{n+\tau}}\right) = F_X\left(x_{t_1}, x_{t_2}, \ldots, x_{t_n}\right)$ for all $\tau, t_1, t_2, \ldots, t_n \in \mathbb{R}$ and for all $n \in \mathbb{N} > 0$
    - Since $\tau$ does not affect $F_X(.)$, $F_X$ is not a function of time
  - **All statistical properties** of $X_t$ - all orders of moments – are time invariant or invariant under time translation
- Wide sense stationarity – WSS
  - A random process is said to be WSS if only its first (mean) and second moments (autocorrelation) are invariant under time translation and higher-order moments may vary with time

# Auto-correlation, Cross Correlation and Power Spectral Density

- Correlation
  - If X,Y are two complex-valued random variables, the correlation between these two random variables is defined as $E\{XY^*\}$
  - A higher correlation between X,Y indicate a higher degree of similarity between the values assumed by these random variables
  - Degree/strength of the correlation can be measured by correlation coefficient $-1 \leq R \leq 1$
  - A positive correlation coefficient
    - As one variable increases, the other variable also tends to increase
  - A negative correlation coefficient
    - As one variable increases, the other variable tends decreases
  - Zero correlation coefficient
    - No relationship between the variables
- Auto-correlation or serial correlation
  - Degree to which a time series is correlated with itself over time
  - Measure of how the values of a variable at different time points are related to each other
  - Autocorrelation is commonly used in time series analysis to detect patterns and trends in data
- Cross Correlation
  - Degree of similarity between two time series or between two signals at different lags or time intervals
- Power spectral density
  - PSD is calculated using the Fourier transform of a signal or time series
  - PSD is usually plotted with frequency on x-axis and power or energy on y-axis

# PSD of Random Process and White Noise

- One of the important tools to characterize a random process is the power spectral density
- PSD of a random process depicts the distribution of power/energy across different frequencies in the spectrum
- PSD of a random process is derived from the auto-correlation function
- Auto-correlation of Random Process
  - Two samples of noise at time k is $n(k)$, and time (n+l) is $n(k+l)$ a lag of l from k
  - The correlation between these two samples i.e. the expected value of $n(k) * n(k+l)$ – $E\{n(k) and\ n(k+l)\}$
  - If the correlation depends only on lag l and does not depend on the time instant k, such a random process is known as *Wide Sense Stationary Random Process*
- White Gaussian Noise
  - Typically the *white noise* is a wide sense stationary random process
  - Particularly for white noise, **the auto-correlation is simply an impulse** - $\frac{N_0}{2}\delta(l)$
    - $R_{nn}(l) = E\{n(k) * n(k+l)\} = \frac{N_0}{2}\delta(l)$
  - Any two samples of white noise are un-correlated and coupled with the fact these are Gaussian which means these are independent noise samples- IID – independent identically distributed samples
  - When we take the Fourier transform of white noise auto-correlation function we get the PSD
  - The PSD of white noise i.e. Fourier transform of an impulse is flat across entire frequency spectrum - $\frac{N_0}{2}$

# Notes – Power of a Signal and PSD of White Noise

- Power of a signal
  - Measure of the amount of energy in the signal
  - Commonly computed as mean squared value of the signal over a given interval
  - The power P of continuous signa x(t) over the interval [t1, t2]
    - $P = \frac{1}{(t_2 - t_1)} * \int_{t_1}^{t_2} |x(t)|^2 dt \; - \; where \; |x(t)|^2 \; represents \; the \; magnitude \; squared \; at \; of \; the \; signal \; at \; time \; t$
  - The power of a signal is closely related to its variance which is how much the signal values deviate from their mean value
  - The variance of a continuous signal x(t) over an interval [t1,t2] is defined as
    - $Var = \sigma^2 = \frac{1}{t_1 - t_2} * \int_{t_1}^{t_2} (x(t) - \mu)^2 dt \; - \mu \; is \; the \; mean \; value \; of \; the \; signal \; over \; t_1 and \; t_2 \; interval$
  - Why the power of a signal is equal to its variance, let's use the following property of variance
    - $Var = E\{x^2\} - E\{x\}^2 \; - E\{.\} \; is \; expectation \; operator$
    - If the signal x(t) has zero mean i.e. $E\{x\} = 0$, then $Var = \sigma^2 = E\{x^2\} - 0 = E\{x^2\}$
  - In other words, the variance of a zero mean signal is equal to the expected value of the signal's magnitude squared
  - For a signal with non-zero mean, the power can be obtained by subtracting the mean value of the signal before computing the mean squared value

# Wireline SNR – Signal to Noise Power Ratio

- $y = x + n$
- $\mathrm{SNR} = \frac{E|x|^2}{E\{|n|^2\}} = \frac{P}{\frac{N_0}{2}} = \frac{2P}{\mathrm{N}_0} \sim \frac{P}{N_0} \sim \frac{P}{\sigma^2}$ - P is the power of the signal
- Approximately constant because channel is fixed
- Hence no variations or fluctuations in SNR and hence performance is fixed

# Performance of Communication System

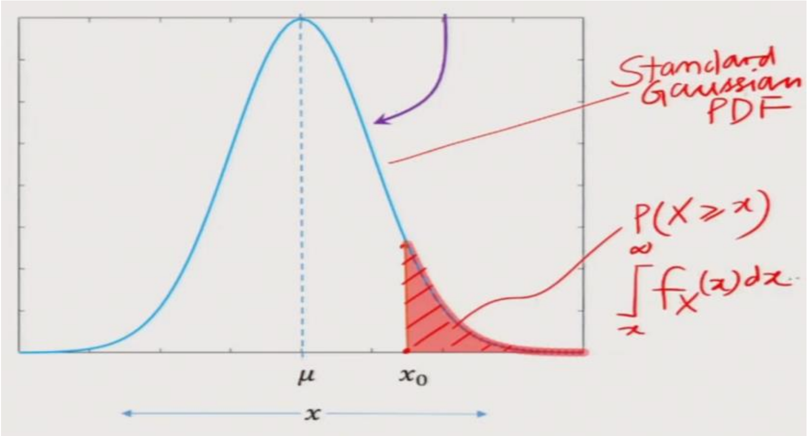
- BER is the probability that a single received bit is in error
- Bit Error Rate is an important metric for communication system performance

# Digital Modulation - BPSK

- Mapping of information bits to signals that can be transmitted over the channel
- There are various formats for digital modulation
  - BPSK, QPSK, QAM
- BPSK – Binary Phase Shift Keying
  - $x \in \{+A, -A\}$ :
  - Two Phases - $0^o, 180^o$ are employed to indicate the information
  - Signal Constellation : $\{+A, -A\}$ - 2 points/symbols
  - A is amplitude-voltage
  - $0 \rightarrow +A$
  - $1 \rightarrow -A$
  - If there M points/symbols in the constellation, the number of bits per symbol will be
    - $\log_2 m$
  - Number of bits per symbol in BPSK - $\log_2 2 = 1$
  - Consider signal power P $\Rightarrow A = \sqrt{P} \Rightarrow x \in \{\sqrt{P}, -\sqrt{P}\}$
  - Expected value - $E\{|x|^2\} = P$
- Communication system model for BPSK
  - $y = x + n$

# BPSK Wireline Performance

- Signal power – P
- Noise power - $\frac{N_0}{2}$
- SNR is $\frac{P}{(\frac{N_0}{2})} = \frac{2P}{N_0}$
- BER for BPSK over wireline channel
  - $BER = Q\left(\sqrt{\frac{2P}{N_0}}\right) = Q(\sqrt{SNR})$
- Standard Gaussian RV
  - $Mean\ \mu = 0, Variance\ \sigma^2 = 1$
  - PDF - $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)}$
- Gaussian Q function
  - Q function is the CCDF – Complementary CDF of standard Gaussian RV $-f_X(x)$ - $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)} dx$
  - The CDF is the probability that the value of the random variable is equal to or less than a certain value
  - Hence the **Complementary CDF** is the probability that the value of the random variable exceeds certain value
  - CDF - $\mathbb{P}(X \leq x) = F_X(x)$
  - CCDF - $\mathbb{P}(X > x) = \bar{F}_X(x) = 1 - CDF$
  - Hence $CCDF(x) = Q(x) = \mathbb{P}(X > x_0) = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)} dx \leq \frac{1}{2} e^{-\left(\frac{1}{2}x^2\right)}$
  - $Q(x)$ is bounded by $\mathbf{Q(x)} \leq \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{e}^{-\left(\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x}^{\mathbf{2}}\right)}$
- PDF can be obtained from CDF $F_X(x)$
  - PDF - $f_X(x) = \frac{dF_X(x)}{dx}$
- $SNR_{dB} = 10 \log_{10} SNR \Rightarrow SNR = 10^{\frac{SNR_{dB}}{10}}$




**CCDF aka Q Function of Gaussian RV**

# QPSK

- Quadrature Phase Shift Keying
  - Quadrature - $90^o$
- Quadrature Carrier Multiplexing
  - Cos and Sine waves - $90^o$ to each other – Orthogonal
- Constellation is given as
  - $x_I + jx_Q$
    - $x_I \; - In\,phase\,component \; - \cos(2\pi f_c t)$
    - $x_Q \; - Quadrature\,component \; - \sin(2\pi f_c t)$
    - Above two are orthogonal carriers
  - Any communication signal can be expressed as a combination of two signals
    - $x_I(t) + jx_Q(t) \; - complex\,representation\,of\,passband\,signal$
  - 2 times the rate bandwidth
  - $x_I \in \{+A, -A\}, x_Q \in \{+A, -A\}$
  - QPSK constellation $x_I + jx_Q\; is \;\{A + jA, A - jA, -A + jA, -A - jA\}\; where\, symbols\, or\, points\; M = 4$
  - QPSK bits per symbol = $\log_2 4 = 2\; bits$
  - If the power is P= $2A^2 \Rightarrow Ampliture\; A = \sqrt{\frac{P}{2}}$
  - Phases of the symbols are - $45^o, 135^o, 225^0, 315^o$
  - Phase differences between any successive points/symbols are $90^o$

# QPSK

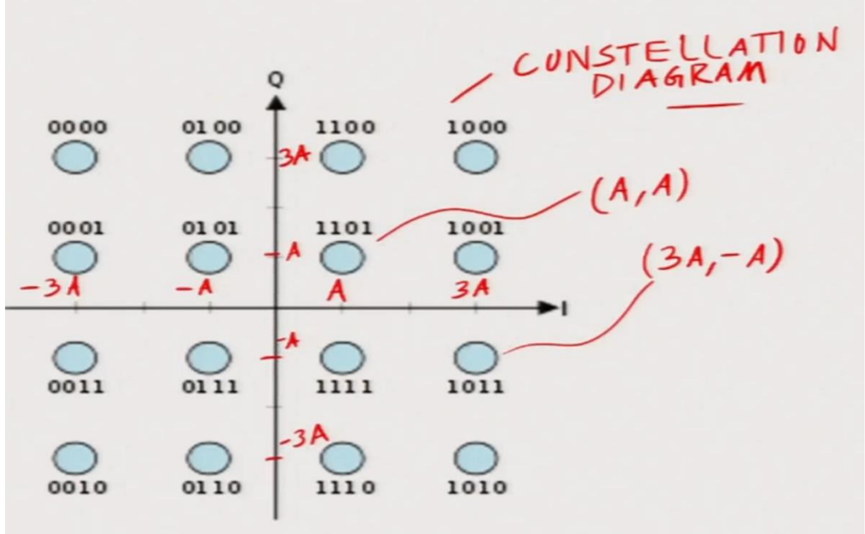
- Mapping of the Symbols can be:
  - $(A, A) \rightarrow 00$
  - $(A, -A) \rightarrow 01$
  - $(-A, A) \rightarrow 10$
  - $(-A, -A) \rightarrow 11$
- Communication system model
  - $\underbrace{(y_I + jy_Q)}_{y} = \underbrace{(x_I + jx_Q)}_{x} + \underbrace{(n_I + jn_Q)}_{n}$
  - $y, x, n$ - $are\ complex\ baseband\ representations\ of\ received\ signal, transmitted\ signal, noise, respectively$ and are complex quantities
- Signal power P
  - For power $A = \frac{\sqrt{P}}{2}$
  - In phase and quadrature components each will have half the power => P/2
  - Signal amplitude A depends on the power and cannot be chosen arbitrarily
- Noise power
  - Noise will have real and complex parts each of which will have a power of $\frac{N_0}{2}$ and hence total noise power will be $\frac{N_0}{2} + \frac{N_0}{2} = N_0$
  - $n_I, n_Q$ are Gaussian with $\frac{N_0}{2} \Rightarrow Total\ Noise\ power = N_0$
- SNR for this system is $SNR_{QPSK} = \frac{P}{N_0}$ - P is the total power of both the streams (In phase and quadrature phase)
- QPSK can be represented as two parallel streams of BPSK
  - $y_I = x_I + n_I$
  - $y_Q = x_Q + n_Q$
- BER of each BPSK stream (In phase or quadrature) is $Q\left(\sqrt{SNR}\right) = Q(\sqrt{\frac{P}{N_0}})$

# QPSK

- QPSK symbol is in error when either of the bits (in phase and quadrature) is in error
- Symbol Error Rate of QPSK
  - $SER_{QPSK} = 1 - \left(1 - Q\left(\sqrt{SNR_{QPSK}}\right)\right)^2$
  - $SER_{QPSK} \approx 2 * BER_{BPSK} = 2Q\left(\sqrt{SNR}\right) = 2Q\left(\sqrt{\frac{P}{N_0}}\right)$

# QAM – Quadrature Amplitude Modulation

- HOM – Higher Order Modulation
- Most important constellations
- Used in 4G,LTE, 5G-NR etc.
- Generalization of QPSK
  - QPSK is 4 QAM
- Also called as $2^{2n}\ QAM$
- QAM is known as M-QAM – M is the number of symbols
- QCM – Quadrature Carrier Multiplexing
- Number of bits per symbols - $\log_2 M$
- Square constellation
- 16 QAM
  - $x_I \in \{-3A, -A, A, 3A\}$
  - $x_Q \in \{-3A, -A, A, 3A\}$
  - $x_I + jx_Q = -3A - j3A, -3A - jA, -3A + jA \dots\}$
  - A is amplitude and depends on the power and cannot be chosen arbitrarily
- QAM allows to transmit at very high bit rates
  - 1024 QAM has $\log_2 1024 = \log_2 2^{10} = 10\ bits\ per\ symbol$



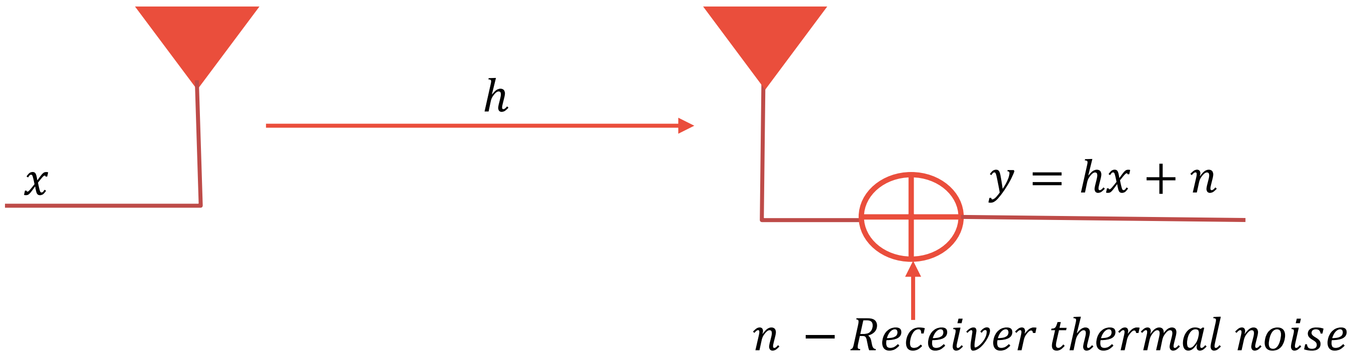
# QAM

- AMC – Adaptive modulation and coding – is employed by mobile communication systems for effective communication
- $Signal\ power\ -\ P = E\{|x|^2\} = E_S,\ Noise\ power\ -\ N_0 = E\{|n|\}^2$
- Symbol Error Rate for m-ary QAM is:
  - $\mathbf{SER} \approx \mathbf{4}\left(\mathbf{1} - \frac{\mathbf{1}}{\sqrt{\boldsymbol{M}}}\right)\boldsymbol{Q}\left(\sqrt{\frac{\mathbf{3}\boldsymbol{P}}{\boldsymbol{N}_{\mathbf{0}}(\boldsymbol{M}-\mathbf{1})}}\right) - \boldsymbol{M} - \boldsymbol{modulation\ order}$

# Wireless Channel and Performance

- Multiple propagation paths – LOS, and several NLOS – multipath propagation due to scatterers
- Multipaths exists due to large objects or scatters
- Multipath propagation leads to multiple copies of the signal at the receiver
- Multiple signal copies causes superposition of the signals at the receiver resulting interference – constructive and destructive
- Because of interference, the SNR varies or fluctuates
- Due to interference, received signal power fluctuates or varies and this phenomenon is called fading
- Wireless channel is also called fading channel - where the received power dips significantly is termed as deep fade

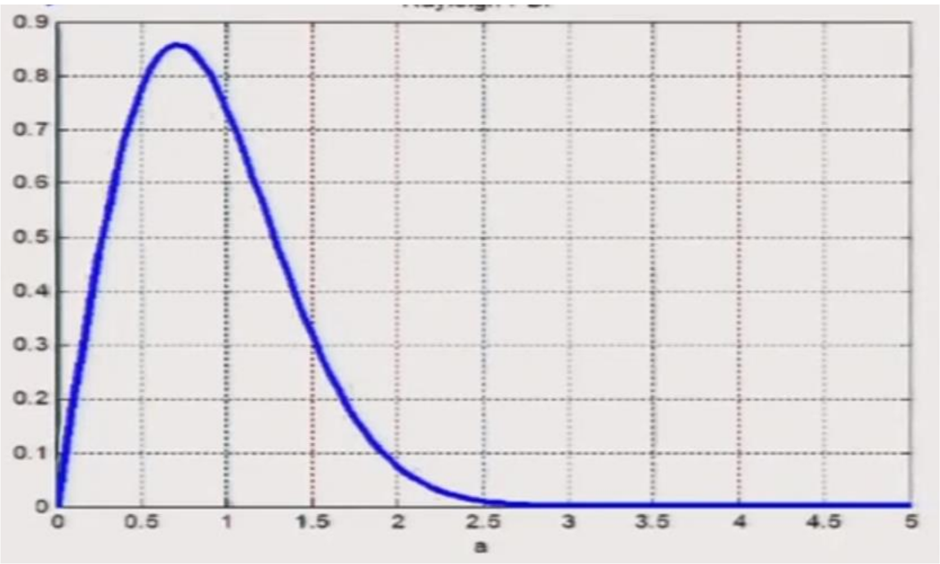
# Wireless Channel Model




- Wireline channel model
  - $y = x + n$
- Wireless channel model
  - $\boldsymbol{y = hx + n}$
  - **h** fading channel coefficient and is complex quantity
  - **h** has multiplicative effect on the signal
  - **h** is represented as
    - $\boldsymbol{h_I + jh_Q}$ - In phase and quadrature entities
    - Alternatively as $\boldsymbol{u + jv}$
  - **h** determines the output power – large if |h| is large, small if |h| is small
- Fading channel coefficient
  - Random in nature and modelled as $\mathrm{h} = u + jv$
  - $u$ real part and $v$ is imaginary part and $-$ $u$ and $v$ are independent Gaussian RVs
  - Their mean is zero i.e. $E\{u\} = E\{v\} = 0$ and variance is $\frac{1}{2}$ i.e. $E\{u^2\} = E\{v^2\} = \frac{1}{2}$
  - **h** is a symmetric complex Gaussian RV
  - Mean - $E\{h\} = E\{u\} + jE\{v\} = 0$ and Variance - $E\{|h|^2\} = E\{u^2 + v^2\} = E\{u^2\} + E\{v^2\} = \frac{1}{2} + \frac{1}{2} = 1$
  - **h** is a complex Gaussian variable with mean zero and variance unity – unit variance complex Gaussian
  - Polar form of h is : $h = ae^{j\phi}$
    - $a\ is\ magnitude = |h| = \sqrt{u^2 + v^2}$
    - $\phi\ is\ phase = \angle h$

# Fading Channel Coefficient

- Amplitude
  - The channel coefficient **h** in the polar form is $ae^{j\phi} \ - a \ is \ amplitude \ and \ \phi \ is \ phase$
  - The amplitude *a* follows the **Rayleigh PDF**
    - $f_A(a) = \begin{cases} 2ae^{-a^2}, a \geq 0 \\ 0, a < 0 \end{cases}$
  - Rayleigh fading channel




- Phase
  - $\phi$ phase is uniformly distributed across $-\pi, \pi \ or \ 0, \pi$
  - PDF of phase = $f_\Phi(\phi) = \begin{cases} \frac{1}{2\pi}, -\pi < \phi \leq \pi \\ 0, \quad Otherwise \end{cases}$

# Symbol Detection – Equalization and Estimation

- Channel estimation
  - Technique used for determining the value of **h** to be accounted for at the receiver
  - Performed by measuring the transmitted pilot stream of symbols – sequence of known symbols – at the receiver
- Equalization
  - Account or cancel or invert the effect of the channel coefficient *h* is known as ***equalization***
  - Dividing the received signal by a channel coefficient – a single number – is called as **Single Tap Equalizer**
    - $z = \frac{1}{h} y = \frac{1}{h}(hx + n) = x + \frac{n}{h}$
- It's not always possible to simply divide by a coefficient to recover the original signal – especially when there is Inter-Symbol-Interference
- Equalization process
  - Example : $x \in \{A, -A\}$
  - Simple signal detection (z) at the receiver can be carried out as follows:
    - $z = \begin{cases} \geq 0 \Rightarrow \hat{x} \sim +A \\ < 0 \Rightarrow \hat{x} \sim -A \end{cases} - \hat{x}\ is\ an\ estimate\ or\ detected\ symbol\ or\ hard\ decision\ or\ slicing$
    - This is termed as a **Threshold Detector** and threshold is zero
- Mathematically rigorous framework that can guarantee signal detection is Maximum Likelihood (ML) Detector

# Wireless Channel Output SNR

- Wireless system $y = hx + n$
- The output power of the channel is = $|h|^2 * E\{|x|\}^2 = |h|^2 * P = \mathrm{a}^2 P$
  - **a** is amplitude and real value whereas **h** is complex quantity
- The channel output SNR, represented as $SNR_o$ is
  - $SNR_o = \frac{Signal\ Power}{Noise\ Power} = \frac{|h|^2 P}{\frac{N_0}{2}} = |h|^2 * SNR_{Transmitter} = a^2 SNR_{Transmitter}$

# Wireless Channel Performance - BER

- Instantaneous BER of BPSK = $Q\left(\sqrt{SNR_0}\right) = Q(\sqrt{a^2 * SNR_{Transmitter}}$
  - $SNR_0$ - The output SNR of the wireless channel
- Instantaneous BER depends on amplitude ***a* (|h|)** which is a random quantity – Rayleigh random variable
- Hence BER is an average with respect to the PDF of ***a*** which is follows Rayleigh PDF from the channel model of Rayleigh Fading channel - $f_A(a) = 2ae^{-a^2}$
- The average of a function $g(a)$ whose PDF is $f_A(a)$ is $\int_{-\infty}^{\infty} g(a).\, f_A(a).\, da$
  - The lower limit $-\infty$ is actually zero
- BER of Wireless BPSK
  - $\int_0^{\infty} Q(\sqrt{a^2.SNR}).\, 2ae^{-a^2}.\, da$
  - $= \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{2+SNR}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{2P/N_0}{2+2P/N_0}}\right)$
- BER of Wireless QPSK
  - $\frac{1}{2}\left(1 - \sqrt{\frac{P/N_0}{2+P/N_0}}\right)$

# BER of Rayleigh Fading Channel

- Wireline
  - The Q function – CCDF - the tail probability of Gaussian RV, x is $\int_x^{\infty} \frac{1}{\sqrt{2}\,\pi} \times e^{-\frac{t^2}{2}} \times dt$
  - The Q function can be approximately upper bounded by
    - $\boldsymbol{Q(x) \leq \frac{1}{2} \times e^{-\frac{1}{2} \times x^2}}$
    - $\Rightarrow Q\left(\sqrt{SNR}\right) \leq \frac{1}{2} \times e^{-\frac{1}{2} \times \left(\sqrt{SNR}\right)^2} \leq \frac{1}{2} \times e^{-\frac{1}{2} \times SNR}$
  - **Wireline BER decreases exponentially with an increase in SNR**
- Wireless
  - $\frac{1}{2} \times \left(1 - \sqrt{\frac{SNR}{2+SNR}}\right) = \frac{1}{2} \times \left(1 - \sqrt{\frac{1}{\frac{2}{SNR}+1}}\right) = \frac{1}{2} \times \left(1 - \left(\frac{2}{SNR} + 1\right)^{-\frac{1}{2}}\right)$
    - At higher SNR, $\frac{2}{SNR}$ will be very small
    - Applying Taylor series $(1 + x)^{-\frac{1}{2}}$, where x is very small $(1 + x)^{-\frac{1}{2}} \approx (1 - \frac{1}{2} \times x)$ is first order Taylor series approximation
  - Applying first order Taylor series approximation to BER
    - $\frac{1}{2} \times \left(1 - \left(1 - \frac{1}{2} \times \frac{2}{SNR}\right)\right) = \frac{1}{2} \times \frac{1}{SNR}$
  - **Wireless BER only decreases as $\boldsymbol{\frac{1}{SNR}}$**
- The wireless channel requires an order of $10^4$ times more transmit signal power to maintain the same BER
- This is due to deep fade or wireless channel being deep fading channel

# Deep Fade

- Output signal power $\ll$ noise power
- For deep fade $|h|^2 P < N_0 \Rightarrow |h|^2 = a^2 < \frac{N_0}{P} = \frac{1}{SNR} \Rightarrow \boldsymbol{a^2 < \frac{1}{SNR}}$
- Probability of Deep Fade
  - $P_{DF} = \Pr\left(a^2 < \frac{1}{SNR}\right) \Rightarrow \Pr\left(a < \frac{1}{\sqrt{SNR}}\right)$
    - $a$ $-$ *Rayleigh random variable* whose $PDF$ *is* $f_A(a) = 2ae^{-a^2}$
  - $\int_0^{1/\sqrt{SNR}} f_A(a)\, da = \int_0^{1/\sqrt{SNR}} 2ae^{-a^2}\, da = -e^{a^2}\Big|_0^{\frac{1}{\sqrt{SNR}}} = 1 - e^{-\frac{1}{SNR}}$
  - $\boldsymbol{P_{DF} = \left(1 - e^{-\frac{1}{SNR}}\right)}$
  - As per Taylor series, $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$
  - When x is small, $e^{-x} = 1 - x$
  - Simplifying further, $P_{DF} = \left(1 - \left(1 - \frac{1}{SNR}\right)\right) = \frac{1}{SNR}$
  - $\boldsymbol{BER = \frac{1}{2SNR} = \frac{1}{2} P_{DF} \Rightarrow BER \propto \frac{1}{2} P_{DF}}$
- BER is highly correlated with deep fade

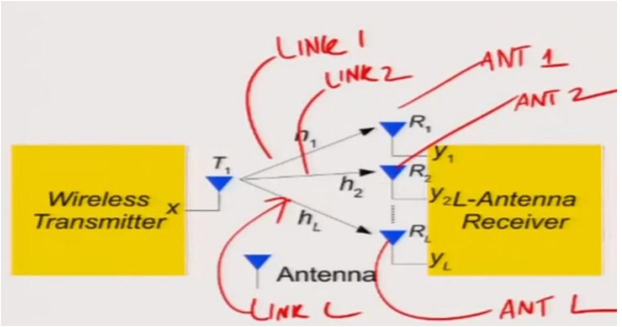
# BER and SER of QPSK and QAM

- QPSK
  - SNR of BPSK $= \frac{P}{\left(\frac{N_0}{2}\right)}$
  - SNR of QPSK (2 BPKS streams) $= \frac{P}{\left(\frac{N_0}{2}+\frac{N_0}{2}\right)} = \frac{P}{N_0}$
  - BER of each BPSK stream is $\frac{1}{2} \times \frac{1}{SNR} = \frac{1}{2} \times \frac{1}{\left(\frac{P}{N_0}\right)} = \frac{1}{2} \times \frac{1}{SNR}$
  - Overall SER - $2 \times BER = 2 \times \frac{1}{2} \times \frac{1}{SNR} = \frac{1}{SNR}$
- SER of M-ary QAM
  - $4 \times \left(1 - \frac{1}{\sqrt{M}}\right) \times \frac{1}{2} \times \frac{1}{\frac{3P}{N_0(M-1)}} = 4 \times \left(1 - \frac{1}{\sqrt{M}}\right) \times \frac{1}{2} \times \frac{(M-1)}{3 \times SNR} \propto \boldsymbol{\frac{1}{SNR}}$
- Thus BER and SER are proportional to $\frac{1}{SNR}$
- Thus the wireless channel performance cannot be increased simply by changing the modulation
- The solution to improve the wireless performance is diversity – multiple antennae

# Multiple Antenna Diversity

# Multiple Antennas and Diversity

- Diversity is achieved through multiple links between transmitter and receiver
- One simple technique for diversity is to use multiple antennas
- Mathematical model
  - $y_i = h_i \times x + n_i$
    - $h_i$ - channel coefficient between the transmit antenna and $i^{th}$ receive antenna
    - $x$ – Input symbol vector
    - $y_i$ - Output vector on $l^{th}$ receive antenna
    - $n_i$ - Noise vector
- For L antenna system
  - $y_1 = h_1 \times x + n_1$
  - $y_2 = h_2 \times x + n_2$
  - $y_3 = h_3 \times x + n_3$
  - …
  - $y_L = h_L \times x + n_L$




SIMO Diversity

# SIMO System – Weighted Linear Combining

- Vector form
  - $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ y_L \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ . \\ . \\ h_L \end{bmatrix} [x] + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ . \\ . \\ n_L \end{bmatrix}$$
- $$\underbrace{\bar{y}}_{output\ vector} = \underbrace{\bar{h}}_{channel\ vector} x + \underbrace{\bar{n}}_{noise\ vector}$$
- $y_1, y_2, y_3, \ldots, y_L$ are the output symbols on the antennas at the receiver
- How to process these output samples from receive antennas?
- **Weighted linear combining** - $w_1^* y_1 + w_2^* y_2 + w_3^* y_3 + \cdots + w_L^* y_L$ - filtering operation
- Assuming $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ . \\ . \\ w_L \end{bmatrix} \Rightarrow \bar{w}^H = \underbrace{[w_1^* \ \ w_2^* \ \ w_3^* \ldots . w_L^*]}$
- $\bar{w}^H$ - Hermitian (transpose and complex conjugate) of $\bar{w}$ and H is the Hermitian operator
- $\Rightarrow \underbrace{[w_1^* \ \ w_2^* \ \ w_3^* \ldots . w_L^*]}_{\bar{w}^H} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ y_L \end{bmatrix}}_{\bar{y}}$
- **$\bar{w}$ is known as beamformer or beamforming vector**

# SIMO - Beamformer

- How to choose $\overline{w}$?
  - Choose beamformer to maximize output SNR
- Output of beamformer
  - $\underbrace{\overline{w}^H\overline{y}}_{output\ of\ the\ beamformer} = \overline{w}^H\left(\overline{h}x+\overline{n}\right) = \underbrace{\overline{w}^H\overline{h}x}_{signal} + \underbrace{\overline{w}^H\overline{n}}_{noise}$
  - Signal power $= \left|\overline{w}^H\overline{h}\right|^2 \times P$
  - Noise power $= E\left\{\overline{|w^H n|^2}\right\} = E\left\{\left|\left[w_1^*\ w_2^*\ w_3^*\ w_4^* \ldots w_L^*\right]\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ . \\ . \\ n_L \end{bmatrix}\right|\right\} = E\{|w_1^* n_1 + w_2^* n_2 + \cdots + w_L^* n_L|^2\}$
  - Noise samples $n_1, n_2, \ldots, n_L$ across antennas are i.i.d – independent and identically distributed
    - $\Rightarrow E\{n_1 n_2^*\} = \begin{cases} if\ i \neq j \Rightarrow E\{n_i\} \times E\{n_j^*\} = 0 \times 0\ , as\ these\ are\ zero\ mean\ noise\ samples \\ if\ i = j, E\{n_i n_j^*\} = E\{|n_i|^2\} = N_0 \end{cases}$
  - The output noise power at the output of the beamformer is: $E\{|\overline{w}^H\overline{n}|^2\} = N_0\left|\left|\overline{w}\right|\right|^2$
    - $\left|\left|\overline{w}\right|\right| = \sqrt{|w_1|^2 + |w_2^2| + |w_3|^2 + \cdots |w_L|^2}$
    - $\left|\left|\overline{w}\right|\right|^2 = (|w_1|^2 + |w_2^2| + |w_3|^2 + \cdots |w_L|^2)$
- Hence SNR at the output of beamformer:
  - $SNR_0 = \frac{\left|\overline{w}^H\overline{h}\right|^2 E\{|x|^2\}}{N_0\left|\left|w\right|\right|^2} = \frac{\left|\overline{w}^H\overline{h}\right|^2 P}{N_0\left|\left|w\right|\right|^2}$

# SIMO - Beamformer

- SNR at the output of beamformer:
  - $SNR_0 = \frac{\left|\overline{w}^H \overline{h}\right|^2 E\{|x|^2\}}{N_0 ||w||^2} = \frac{\left|\overline{w}^H \overline{h}\right|^2 P}{N_0 ||w||^2}$
- Remembering Cauchy-Schwartz inequality
  - $\left|\overline{w}\overline{h}\right|^2 \leq \left||\overline{w}|\right|^2 \left||\overline{h}|\right|^2 \Rightarrow \left|\overline{\boldsymbol{w}}^{\boldsymbol{H}} \overline{\boldsymbol{h}}\right|^{\mathbf{2}} \leq \left||\overline{\boldsymbol{w}}|\right|^{\mathbf{2}} \left||\overline{\boldsymbol{h}}|\right|^{\mathbf{2}}$ – Remember $w^H is\ a\ transposed\ complex\ conjugate\ of\ w$
- Using Cauchy-Schwartz inequality for $SNR_0$
  - $\frac{\left|\overline{w}^H \overline{h}\right|^2 P}{N_0 ||w||^2} \leq \frac{\cancel{||w||^2} ||h||^2 P}{N_0 \cancel{||w||^2}} = ||h||^2 \times \frac{P}{N_0}$ - Maximum output SNR
  - $Max(SNR_0) = ||h||^2 \times \frac{P}{N_0}$ - this is for QPSK
  - For BPSK, $Max(SNR_0) = ||h||^2 \times \frac{P}{\frac{N_0}{2}} = ||h||^2 \times \frac{2P}{N_0}$
  - Hence maximum output SNR occurs when we choose $\overline{w} \propto \overline{h}\ or\ \overline{w} = k\overline{h}$ - that means $\overline{w}$ should be aligned with channel – matched filter
  - $\Rightarrow \overline{\boldsymbol{w}} = \frac{\overline{\boldsymbol{h}}}{||\boldsymbol{h}||} = \boldsymbol{Maximal\ Ratio\ Combiner}$
    - $\left||\overline{h}|\right| = Norm\ of\ \overline{h} = \sqrt{|h_1|^2 + |h_2|^2 + \cdots |h_L|^2}$
    - $||\overline{w}|| = 1$ – Unit-norm beamformer
  - Since $\frac{\overline{h}}{||\overline{h}||}$ is the combiner that maximizes the noise power ratio, it is know as **MRC – Maximal Ratio Combiner**
  - **Output SNR of *Maximal Ratio Combiner* is**: $\boldsymbol{SNR_0} = \frac{\boldsymbol{P}}{\boldsymbol{N_0}} \times \left||\overline{\boldsymbol{h}}|\right|^{\mathbf{2}}$
- Uses of Beamforming
  - Maximizes the signal ratio – SNR
  - Supress the interference from un-desired use or jammers

# BERs for Multiple Antenna System

- BER for BPSK
  - $^{(2L-1)}C_{(L-1)} \times \frac{1}{2^L} \times \frac{1}{SNR^L}$
- BER of Multi-antenna system $\propto \frac{1}{SNR^L}$
- L is called diversity order
- If the BER decreases as $\frac{1}{SNR^d}$, then the diversity order is $d$

# Deep Fade in Multi-Antenna System

- Deep fade occurs when the signal is buried in noise i.e. $SNR_0 \leq 1$
- $\Rightarrow \frac{P}{N_0}\left||h|\right|^2 \leq 1\ where\ \left||h|\right|^2 (norm) = |h_1|^2 + |h_2|^2 + \cdots + |h_L|^2$
  - $|h_1|^2, |h_2|^2, \ldots |h_L|^2\ -\ magnitudes\ of\ each\ component\ of\ \bar{h}$
- $\Rightarrow SNR_0\left||h|\right|^2 \leq 1 \Rightarrow \left||h|\right|^2 \leq \frac{1}{SNR_0}$
- $\left||h|\right|^2$ is known as ***chi-squared random variable*** and let's say $g = \left||h|\right|^2$
- The PDF of chi-squared $\chi^2$ variable $f_G(g) = \frac{g^{L-1}e^{-g}}{(L-1)!}$
- So then the $P\,r\left(g \leq \frac{1}{SNR_0}\right) = \boldsymbol{P_{DF}} = \int_0^{\frac{1}{SNR_0}} f_G(g)dg = \int_0^{\frac{1}{SNR_0}} \frac{g^{L-1}e^{-g}}{(L-1)!}dg = \frac{1}{L!} \times 1/SNR^L \propto \boldsymbol{\frac{1}{SNR^L}}$
- $\boldsymbol{BER} \propto \boldsymbol{P_{DF}} \propto \boldsymbol{\frac{1}{SNR^L}}$

# Multi-Antenna System - Deep Fade Justification

- Say $E_i$ is the event that $i^{th}$ link between receiver and transmitter of SIMO is in deep fade – assumption is each $E_i$ $(E_1, E_2, \ldots, E_L)$ is an independent event
- Then the probability that L-antenna SIMO system has to be in deep fade for a given receiver
  - $P_{DF} = \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_L) = \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \cdots \times \mathbb{P}(E_L)$
- Since $E_1, E_2, E_3, \ldots E_L$ are independent, links between the receiver and transmitter antennae are independently fading
- This link independence is obtained when the antenna spacing is large
- Rule of thumb for antenna spacing $\Rightarrow$ $Antenna\ Spacing \geq \frac{\lambda}{2} = \frac{c}{2 \times f_c} = \frac{3 \times 10^8 m/s}{2 \times f_c}$
  - $f_c$ - Carrier frequency

# MIMO System

- Multiple-Input Multiple-Output Antenna System
- Key technology used in
  - 4G LTE
  - 5G-NR
  - WLAN – 802.11n, AC, AX
- MIMO can lead to a significant increase in data rates via parallel transmissions – Same time, same power, and same bandwidth
- This phenomenon – multiplexing several information streams in the spatial domain - is called **Spatial Multiplexing**
- MIMO system model
  - ***r*** is the receive antennae and ***t*** is the transmit antennae
  - It'll be called as ***rxt*** MIMO system
  - Mathematically represented as
  - $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ y_r \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1t} \\ h_{21} & h_{22} & h_{23} & \dots & h_{2t} \\ h_{31} & h_{32} & h_{33} & \dots & h_{3t} \\ . & . & . & . & . \\ . & . & . & . & . \\ h_{r1} & h_{r2} & h_{r3} & \dots & h_{rt} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_t \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ . \\ . \\ n_r \end{bmatrix}$$
  - $\bar{y} = H\bar{x} + \bar{n}$
    - $\bar{y}\ - output\ vector\ - rx1$
    - $H\ - channel\ vector\ - rxt$
    - $\bar{x}\ - Input\ vector\ - tx1$
    - $\bar{n}\ - noise\ vector\ - rx1$
- $h_{ij}$ is the channel coefficient between $i^{th}$ receive antenna and $j^{th}$ transmit antenna

# MIMO Receiver

- Given $\bar{y}$, how to determine $\bar{x}$?
- We have $r$ equations and $t$ unknowns for $\bar{y} = H\bar{x}$
- $\bar{x} = H^{-1}\bar{y}$
- If H is non-singular (determinant ≠0, i.e. invertible i.e. $H^{-1}$ exists)
  - $\bar{y} = H\bar{x}$ has a unique solution ⇒ $\boldsymbol{\hat{x}} = \boldsymbol{H^{-1}\bar{y}}$
  - $\hat{x}$ is called estimated vector
- What happens when $r > t$
  - More equations – y (r) than unknowns – x (t) in MIMO system model
  - That means more rows than columns ⇒ tall matrix – NOT a square matrix
  - **H is not invertible**
    - Inverse exists only for square matrices when the determinant is NOT zero (can be positive or negative)
  - $\bar{y} = H\bar{x}\ has$ no solution for $\hat{x}$
- How to determine $\hat{x}$?
- Approximate solution - Find $\hat{x}$ such that $\bar{e}$ is minimum - $\bar{y} - H\hat{x} = \bar{e}$
- Minimize the error $\min||e||^2 = \min||\bar{y} - H\hat{x}||^2$ - Minimize the square of norm of the error
- This is known as **Least Squares Problem – LS problems**
- Solution to LS problem is $\boldsymbol{\hat{x} = \left(H^H H\right)^{-1} H^H \bar{y}}$ **- Zero Forcing - ZF - Receiver**
- $(H^H H)^{-1} H^H$ is termed as *pseudo-inverse* of H – Tall matrix formula
  - Because $(H^H H)^{-1} H^H . H = \mathrm{I}$

# MIMO LMMSE Receiver

- Another popular MIMO receiver is LMMSE receiver
- Linear Minimum Mean Square Error Receiver
  - Estimate $\hat{x} = C^H y$ - Linear transformation for the estimate
  - Minimize $E\left\{\left||C^H y - \hat{x}|\right|^2\right\}$
- To derive the solution for LMMSE receiver, we need
  - Covariance matrix of $\bar{x}$ $- R_{xx} = E\{\bar{x}\bar{x}^H\}$
  - Covariance matrix of $\bar{y}$ $- R_{yy} = E\{\bar{y}\bar{y}^H\}$
  - Cross-covariance matrix of $-R_{xy} = E\{\bar{x}\bar{y}^H\}$
- General expression of LMMSE receiver is
  - LMMSE estimate $\hat{x} = R_{xy}R_{yy}^{-1}\bar{y}$
- LMMSE receiver derivation
  - The transmit symbols to be IID – Independent and identically distributed
  - $R_{xx} = E\{\bar{x}\,\bar{x}^H\} = PI_{t\times t}\begin{cases} 0, if\ i \neq j \\ E\{|x_i|^2\}, = PI\ if\ i = j \end{cases}$
  - $R_{xy} = E\{\bar{x}\bar{y}^H\} = P.I.H^H = P.H^H$
  - $R_{yy} = E\{\bar{y}\bar{y}^H\} = \mathrm{P.HH^H} + \mathrm{N_0.I}$
- $\hat{x} = PH^H(P.HH^H + N_0 I)^{-1}\bar{y}$ - rxr
- $\hat{x} = P(P.H^H H + N_0 I)^{-1}H^H\bar{y}$
- Alternate LMMSE expression
  - $\hat{x} = \left(H^H H + \frac{N_0}{P} I\right)^{-1} H^H\bar{y} = \left(H^H H + \frac{1}{SNR}\right)^{-1} H^H\bar{y}$
- At high SNR ($SNR\ \rightarrow \infty$) $\Rightarrow \frac{1}{SNR} = 0 \Rightarrow LMMSE\ \hat{x} = (H^H H)^{-1}H^H\bar{y}$ - tens to be zero forcing receiver

# MIMO LMMSE Receiver Derivation

- $\hat{\boldsymbol{x}} = \boldsymbol{R}_{\boldsymbol{xy}}\boldsymbol{R}_{\boldsymbol{yy}}^{-1}\overline{\boldsymbol{y}}$
- The transmit symbols are IID – Independent and identically distributed
- $R_{xx} = E\{\bar{x}\,\bar{x}^H\}$
  - $\Rightarrow R_{xx} = PI_{t\times t}\begin{cases} 0, if\ i \neq j \\ E\{|x_i|^2\}, = PI\ if\ i = j \end{cases}$
- $R_{yy} = E\{\bar{y}\bar{y}^H\}$
  - $\Rightarrow R_{yy} = E\{\bar{y}\bar{y}^H\} = E\{(H\bar{x}+\bar{n})(H\bar{x}+\bar{n})^H\} = E\{(H\bar{x}+\bar{n})(\bar{x}^H H^H + \bar{n}^H)\}$
  - $\Rightarrow E\{H\bar{x}\bar{x}^H H^H + H\bar{x}\bar{n}^H + \bar{n}\bar{x}^H H^H + \bar{n}\bar{n}^H\} =$
    - $H \underbrace{E\{\bar{x}\bar{x}^H\}}_{Co-variance\ of\ \bar{x}=P} H^H + H \underbrace{E\{\bar{x}\bar{n}^H\}}_{Cross\ co-variance\ between\ transmit\ and\ noise\ vectors=0} + \underbrace{E\{\bar{n}\bar{x}^H\}}_{Cross\ co-variance\ between\ noise\ andtransmitvectors=0} H^H + \underbrace{E\{\bar{n}\bar{n}^H\}}_{Noise\ co-variance\ N_0 I}$
  - $R_{yy} = P.HH^H + N_0$
- $R_{xy} = E\{\bar{x}\bar{y}^H\}$
  - $Deriving\ similarly\ like\ above$:
  - $R_{xy} = P.I.H^H = P.H^H$
- So $\hat{\boldsymbol{x}} = \boldsymbol{R}_{\boldsymbol{xy}}\boldsymbol{R}_{\boldsymbol{yy}}^{-1}\overline{\boldsymbol{y}} = \boldsymbol{PH}^{\boldsymbol{H}}\underbrace{\left(\boldsymbol{PHH}^{\boldsymbol{H}} + \boldsymbol{N}_{\boldsymbol{0}}\boldsymbol{I}\right)^{-1}}_{\boldsymbol{r}\times\boldsymbol{r}}\overline{\boldsymbol{y}}$
- Simplifying further, $\hat{\boldsymbol{x}} = \left(\boldsymbol{HH}^{\boldsymbol{H}} + \frac{\boldsymbol{N}_{\boldsymbol{0}}}{\boldsymbol{P}}\boldsymbol{I}\right)^{-1}\boldsymbol{H}^{\boldsymbol{H}}\overline{\boldsymbol{y}} = \underbrace{\left(\boldsymbol{HH}^{\boldsymbol{H}} + \frac{\boldsymbol{1}}{\boldsymbol{SNR}}\boldsymbol{I}\right)^{-1}}_{\boldsymbol{t}\times\boldsymbol{t}}\overline{\boldsymbol{y}}$
- Form One - $\hat{\boldsymbol{x}}_{\boldsymbol{LMMSE}} = \boldsymbol{P}.\boldsymbol{H}^{\boldsymbol{H}}\left(\boldsymbol{HH}^{\boldsymbol{H}} + \boldsymbol{N}_{\boldsymbol{0}}\boldsymbol{I}\right)^{-1}\overline{\boldsymbol{y}}$ - inverse to be taken for $r \times r$
- Form Two - $\hat{\boldsymbol{x}}_{\boldsymbol{LMMSE}} = \left(\boldsymbol{HH}^{\boldsymbol{H}} + \frac{\boldsymbol{1}}{\boldsymbol{SNR}_{\boldsymbol{0}}}\boldsymbol{I}\right)^{-1}\boldsymbol{H}^{\boldsymbol{H}}\overline{\boldsymbol{y}}$ - inverse to be taken for $t \times t$ – lower computational complexity as $t \ll r$
- At high SNR ($SNR\ \rightarrow \infty$) $\Rightarrow \frac{1}{SNR} = 0 \Rightarrow LMMSE\ \hat{x} = (H^H H)^{-1}H^H\bar{y}$ - tens to be **zero forcing receiver**

# SVD – Singular Value Decomposition

- Mathematical technique for decomposing a matrix into 3 components
  - A left singular matrix
  - A diagonal singular values matrix
  - A right singular matrix
- Represented by $H = U_{\mathrm{r}\times r}\Sigma_{\mathrm{r}\times t}V_{t\times t}^{H}$
- Provides a way to analyze the properties of a matrix – rank, eigenvalues, eigenvectors, singular values etc. which are related to the matrix norm and condition number
- One of the most important techniques for MIMO processing
- Used for decomposing the channel between transmitter and receiver into set of parallel sub-channels, each having different gains and phases
- Used to find the optimal beamforming vectors that maximize the SNR for each sub-channel – a technique known as **Singular Value Beamforming**
- Also used for transmit diversity – usage of multiple antennas at the transmitter to improve the reliability of transmission
- By decomposing the channel matrix using SVD
  - A transmitter can send different signals on each of the sub-channels with different powers based on their corresponding singular values
  - This technique is known as STBC – Space-time Block Coding
  - STBC improves the error performance of wireless systems

# SVD – Singular Value Decomposition

- Given $r \geq t$, SVD is defined as
  - $H = U_{rxr}\Sigma_{rxt}V_{txt}^{H}$
- $\Sigma$ has the structure
  - $\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  - Diagonal matrix
  - $t$ diagonal singular values
  - $(r-t) \times t$ zero matrix
  - $\sigma_i \geq 0$ and arranged in decreasing order $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \sigma_t \geq 0$ and are singular values - real, non-negative
    - Singular values of any matrix are non-negative real number numbers
    - Whereas eigenvalues can be complex, negative, or positive
  - Rank of H is number of non-zero singular values
- U,V satisfy the properties
  - Ortho-normal matrices - Columns are orthogonal and norm $||column||^2$ is 1
    - $U = [\bar{u}_1\ \bar{u}_2\ \bar{u}_3\ \ldots \bar{u}_r] \Rightarrow \bar{u}_i.\bar{u}_j = 0\ and\ ||\bar{u}_i||^2 = 1$
    - $V = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3\ \ldots \bar{v}_r] \Rightarrow \bar{v}_i.\bar{v}_j = 0\ and||\bar{v}_i||^2 = 1$
  - Unitary matrices
    - $U^H U = UU^H = I$
    - $V^H V = VV^H = I$
  - $U$ is known as left singular matrix and $\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots \bar{u}_r$ left singular vectors and $\bar{u}_1$ is called **dominant left singular vector**
  - $V$ is known as right singular matrix and $\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots \bar{v}_r$ left singular vectors and $\bar{v}_1$ is called **dominant right singular vector**

# SVD Relation to Eigenvalue Decomposition

- $$HH^H = U \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & \sigma_t^2 & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{r \times t} U^H$$
  - $\sigma_1^2, \sigma_2^2, \sigma_3^2 \dots \sigma_t^2$ are non-zero eigenvalues and rest $(r-t)$ values are zero of $HH^H$
  - U contains eigenvectors $\bar{u}_1, \bar{u}_2, \bar{u}_3, \dots \bar{u}_r$ (the left singular vectors) of $HH^H$

- $$H^H H = V \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_t^2 \end{bmatrix}_{t \times t} V^H$$
  - $\sigma_1^2, \sigma_2^2, \sigma_3^2 \dots \sigma_t^2$ - are $t$ non-zero eigenvalues of $H^H H$
  - $V$ contains eigenvectors $\bar{v}_1, \bar{v}_2, \bar{v}_3, \dots \bar{v}_t$ (the right singular vectors) of $H^H H$

# SVD in MIMO Processing

- MIMO channel model $\bar{y} = H\bar{x} + \bar{n}$
- Substitute $H = U\Sigma V^H$
- At the receiver without pre-coding at the transmitter
  - $\bar{y} = U\Sigma V^H \bar{x} + \bar{n}$
  - Process using $U^H \Rightarrow \tilde{y} = U^H \bar{y}$
    - $\Rightarrow \tilde{y} = U^H(H\bar{x} + \bar{n}) = U^H H\bar{x} + U^H\bar{n} = U^H U\Sigma V^H \bar{x} + U^H\bar{n} = \boldsymbol{\Sigma V^H \bar{x}} + \boldsymbol{\tilde{n}}$ $(\tilde{n} = U^H\bar{n})$
  - $U^H$ is the combiner or receive beam former or receive filter
- At the transmitter with pre-coding
  - Pre-process/multiply prior to transmission using $V: \bar{x} = V\tilde{x}$
    - $\tilde{\mathrm{x}}$ $- \mathit{Original\ symbol\ vector}$
    - $\bar{x} - \mathit{Transmitted\ vector}$
    - $V - \mathit{Precoder\ or\ Transmit\ beamforming\ matrix}$
- At the receiver with transmitter pre-coding
  - $\bar{y} = U\Sigma V^H(V\tilde{x}) + \bar{n}$
  - Process using $U^H \Rightarrow \tilde{y} = U^H \bar{y}$
    - $\Rightarrow \tilde{y} = U^H(H\bar{x} + \bar{n}) = U^H HV\tilde{x} + U^H\bar{n} = U^H U\Sigma V^H V\tilde{x} + U^H\bar{n} = \boldsymbol{\Sigma\tilde{x}} + \boldsymbol{\tilde{n}}$ $(\tilde{n} = U^H\bar{n})$
  - $U^H$ is the combiner or receive beam former or receive filter

# SVD in MIMO Processing

- SVD-based transmitted pre-coded signals at the receive antennas output
  - $\tilde{y} = \boldsymbol{\Sigma}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{n}}\ (\tilde{n} = U^H \bar{n})$

- $\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ . \\ . \\ \tilde{y}_r \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ . \\ . \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ . \\ . \\ \tilde{n}_r \end{bmatrix}$
- $\tilde{y}$ will have more than t components as $r \gg t$. $(t+1), (t+2), \ldots r$ can be ignored as these are noise
- Hence the system of MIMO equations with SVD are:
  - $\tilde{y}_1 = \sigma_1 \tilde{x}_1 + \tilde{n}_1$
  - $\tilde{y}_2 = \sigma_2 \tilde{x}_2 + \tilde{n}_2$
  - $\tilde{y}_3 = \sigma_3 \tilde{x}_3 + \tilde{n}_3$
  - …
  - $\tilde{y}_t = \sigma_t \tilde{x}_t + \tilde{n}_t$
- Decoupled channels – spatial multiplexing
- Consider $i^{th}$channel $\Rightarrow \tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{n}_i$
  - Signal power = $E\{|\tilde{x}|^2\} = P_i$
  - Noise power = $N_0$
- $SNR_o^i = \sigma^2 . \frac{P_i}{N_0}$

# Capacity of MIMO Wireless Systems

- Shannon's channel capacity – maximum data rate for error free transmission
  - $\log_2(1 + SNR)$
- Maximum data rate for $i^{th}$ channel in MIMO is $\log_2(1 + \sigma^2 . \frac{P_i}{N_0})$
- So, the total data transmission capacity of the MIMO channel
  - $\sum_{i=1}^{t} R_i = \sum_{i=1}^{t} \log_2\left(1 + SNR_o^i\right) = \sum_{i=1}^{t} \log_2\left(1 + \sigma^2 . \frac{P_i}{N_0}\right)$
- Let's call the maximum transmit power at the transmitter – total permissible transmit power – as $P_0$
- Therefore sum of all MIMO streams/links/modes $P_0 = \sum_{i=1}^{t} P_i$
- What is the possible maximum error-free transmission rate of MIMO channel per a unit bandwidth – bits/second/hertz
- Solution is as per below optimization problem
  - $Maximize \ \sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{N_0}\right)$
    - $subject\ to\ the\ constraint\ \sum_{i=1}^{t} P_i = P_0$
- Solved using Lagrange multiplier $-f(x) + \lambda g(x) - KKT\ framework$
- The optimal power solution for maximum error-free transmission rate is
  - $P_j = \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_j^2}\right)^+ + indicates\ that\ if\ the\ value\ < 0, then\ the\ value\ is\ 0$
  - $P_j\ is \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_j^2}\right) if\ P_j \geq 0, else\ 0$
  - $P_j = \begin{cases} \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_j^2}\right), if\ \frac{1}{\lambda} \geq \frac{N_0}{\sigma_j^2} \\ 0, if\ \frac{1}{\lambda} < \frac{N_0}{\sigma_j^2} \end{cases}$

# MIMO Capacity

- Optimal power allocation to various MIMO streams/modes - Water-filling/pouring power allocation
- $If\ \sigma_j\ is\ small \Rightarrow \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_j^2}\right)\ is\ small \Rightarrow$ Weaker channels are allocated with lower power ( based on this water-filling power allocation)
- Corollary of the above, stronger channels will be allocated more power

# Alamouti Code

- The main idea behind Alamouti coding is to transmit the same data over multiple antennas in a way that maximizes diversity and protects against error – error protection coding
  - If one of the antennas experiences fading or interference, the data can still be recovered from the other antenna
- This is achieved by transmitting the data in a space-time block code, where the data is transmitted over multiple time slots and multiple antennas improving the reliability and capacity of transmission
- Particularly useful when the transmitted signal is subjected to fading due to multipath propagation
- It is a 2x1 Orthogonal STBC – used for 2 transmit antennae and 1 receive antenna system (a 1x2 (rxt) diverse system – MISO system)
- It encodes two complex symbols onto two transmit antennae and transmits over two consecutive time slots providing diversity and improved SNR at the receiver
  - In the first time slot, the two complex symbols are transmitted on the two antennas
  - In the second time slot, the same two symbols are transmitted again, but with the conjugate of one symbol transmitted on the second antenna
- Achieves diversity order 2 without the knowledge of the channel state information (CSI) for beamforming (channel coefficients being sent by the receiver) at the transmitter

# Alamouti Code

- MISO channel is represented by $[h_1\ h_2] = \bar{h}^T$
  - $h_1\ - channel\ coefficient\ between\ transmit\ antenna\ 1\ and\ receive\ antenna\ 1$
  - $h_2 - channel\ coefficient\ between\ transmit\ antenna\ 2\ and\ receive\ antenna\ 1$
- The first transmit vector $t_1$ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
  - $x_1 - transmit\ antenna\ 1$
  - $x_2 - transmit\ antenna\ 2$
- **The second transmit vector $\boldsymbol{t_2}$ is $\begin{bmatrix} -\boldsymbol{x_2^*} \\ \boldsymbol{x_1^*} \end{bmatrix}$**
- Hence $y_1 = [h1\ h2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_1 = h_1x_1 + h_2x_2 + n_1$
- $y_2 = [h_1\ h_2]\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + n_2 = -h_1x_2^* + h_2x_1^* + n_2 \Rightarrow [h_2 - h_1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_2$
- $\Rightarrow y_2^* = -h_1^*x_2 + h_2^*x_1 + n_2^*$
- $y_2^* = [h_2^*\ - h_1^*]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_2^*$
- Hence output system of equations
  - $\begin{bmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2^*} \end{bmatrix} = \begin{bmatrix} \boldsymbol{h_1} & \boldsymbol{h_2} \\ \boldsymbol{h_2^*} & -\boldsymbol{h_1^*} \end{bmatrix}\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} + \begin{bmatrix} \boldsymbol{n_1} \\ \boldsymbol{n_2^*} \end{bmatrix}$

# Alamouti Code

- $$\underbrace{\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix}}_{Output\ Vector - \bar{y}} = \underbrace{\begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}}_{Channel\ Vector\ -H} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{Transmit\ Vector - \bar{x}} + \underbrace{\begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}}_{Noise\ Vector\ -\bar{n}}$$
- **Columns of channel matrix of Alamouti system are orthogonal vectors**
- **Hence Alamouti code is known as Orthogonal Space-time Block Code – OSTBC**
- Can be extended to arbitrary number of receive antennae
- Since the matrix is orthogonal, decoding can be simply performed by multiplying by inverse
- Inverse of Alamouti Matrix - $H^{-1} = \frac{1}{||h||^2} H^H = \frac{1}{||\mathrm{h}||^2} \begin{bmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{bmatrix}$
  - $\left||h|\right|^2 = |h_1|^2 + |h_2|^2$ - Norm square of $1 \times 2$ channel vector $=$ $|[h_1 \ \ h_2]|^2 = \left(\sqrt{|h_1|^2 + |h_2|^2}\right)^2 = (h_1^2 + h_2^2)$
- **Decoder o/p =** $\boldsymbol{H^{-1}\bar{y} = \frac{1}{\left||h|\right|^2} \times H^H \times \bar{y}, where\ \bar{y} = \begin{bmatrix} y_1 \\ y_2^* \end{bmatrix}}$
- Alamouti code transmits two symbols per two time instants with a net of one symbol per a time instant i.e. *R=1* – such a code is termed a *full rate code*

# SNR and BER of Alamouti Code

- SNR
  - The available power P at the transmitter is split equally across two transmit streams/antennae
  - $\Rightarrow E\{|x_1|^2\} = E\{|x_2|^2\} = \frac{P}{2}$
  - **Channel output SNR of each stream is -** $\boldsymbol{SNR_0 = \left|\left|h\right|\right|^2 \times \frac{\frac{P}{2}}{N_0}}$
  - $\Rightarrow \frac{1}{2} \times \left|\left|h\right|\right|^2 \times \frac{P}{N_0} = \frac{1}{2} \times \left|\left|h\right|\right|^2 \times SNR_{tx} = \frac{1}{2} \times (|h_1|^2 + |h_2|^2) \times SNR_{tx}$
    - Where for BPSK $SNR = \frac{P}{N_0}$ and QPSK $SNR = \frac{2P}{N_0}$
  - $\boldsymbol{SNR_o = \frac{1}{2} \times \left|\left|h\right|\right|^2 \times \frac{\frac{P}{2}}{N_0} = \frac{1}{2} \times \left|\left|h\right|\right|^2 \times SNR \ where \left|\left|h\right|\right|^2 = (|h_1|^2 + |h_2|^2)}$
  - Alternatively, $\mathrm{SNR}_0 = \frac{1}{2} \times \frac{\left|\left|h\right|\right|^2}{\sigma^2} P$
- BER
  - **The output BER of Alamouti Scheme for BPSK and QPSK** $= \boldsymbol{\frac{3}{SNR^2}}$

# OFDM Technology

# Single and Multi-carrier Modulation

- Single Carrier System
  - For a bandwidth B/2, the symbol time is $\frac{1}{B}$
    - Example : For a single carrier system of 10 MHz bandwidth, the symbol time is $\frac{1}{10\ MHz} = 0.1\ \mu s$
  - As the bandwidth increases, symbol time decreases or shrinks
  - Due to multi-path in wireless channel, different copies of the transmitted signal arrive with different delays - $\tau_0, \tau_1, \tau_2, \dots, \tau_{l-1}$
  - As a result, multi-path components are spread over time which is called as **Delay Spread**
  - **For large bandwidth**
    - **Delay Spread >> Symbol Time/Duration** $\Rightarrow \boldsymbol{t_{ds}} \gg \boldsymbol{t_s}$
  - At higher bandwidths, the multi-path signal copies with different delays superimpose at the receiver causing **ISI – Inter-symbol Interference increasing BER**
- How to eliminate ISI?
- ISI can be eliminated using Multi-carrier Modulation (MCM) using OFDM

# What is OFDM

- OFDM is widely employed in most of the modern cellular and wifi systems
- MIMO-OFDM is one of the most widely used technologies today
  - 4G LTE, 5G-NR
  - 802.11n, 802.11ac, 802.11ax
- Due its transmission over very large bandwidth, it enables ultra-high data rates
- OFDM exploits frequency dimension and MIMO exploits space dimension
  - Space + Frequency Multiplexing => Extra-ordinarily high data rates

# OFDM - Multi-carrier Modulation

- Instead of using one carrier, use N carriers/sub-carriers
- Divides the bandwidth into N sub-bands/carriers
- Each sub-band bandwidth is $\frac{B}{N}$ across N bands, B is total bandwidth $\left[-\frac{B}{2}, +\frac{B}{2}\right]$
- The sub-carrier bandwidth is also called as sub-carrier spacing, sub-carrier bandwidth etc. – $\frac{B}{N}$
- Sub-carriers are placed at $\ldots. -\frac{2B}{N}, -\frac{B}{N}, 0, \frac{B}{N}, \frac{2B}{N}, \frac{3B}{N}, \ldots$
- **If $\boldsymbol{\frac{B}{N}}$ is called as $\boldsymbol{f_0 \Rightarrow N\ sub - carrier}$ placement is** $\ldots, \boldsymbol{-2f_0, -f_0, 0, f_0, 2f_0}, \ldots.$
- As per signals and systems, the mathematical representation of a generic carrier of frequency is
  - $f_c = e^{j2\pi f_c t}$
- $k^{th}$ sub-carrier is placed at $kf_0$, where $\mathrm{f}_0 = \frac{\mathrm{B}}{\mathrm{N}}$
  - $\boldsymbol{f_k = kf_0 = e^{j2\pi k f_0 t}}$
- Modulate each sub-carrier with a symbol
  - $k^{th}$ modulated sub-carrier $x_k(t)$ = $\boldsymbol{X_k} \times \boldsymbol{e^{j2\pi k f_0 t}}$
- Hence the aggregate transmit signal
  - Becomes the sum of all modulated sub-carrier signals
  - $x(t) = \sum_k X_k e^{j2\pi k f_0 t}$
  - Also termed as symbol loaded on sub-carrier $kf_0$

# Fourier Series

- Fourier series exists for a continuous periodic signal
- For **continuous periodic signals**, **Fourier series** or **complex exponential Fourier series** or **trigonometric Fourier** series will be used
- For **continuous aperiodic** signals, **Fourier transform** is used
- For **discrete aperiodic** signals, **Discrete-Time Fourier transform** is used
- For **discrete periodic** signals, **Discrete Fourier Series** is used
- For **discrete limited** signals, **FFT or DFT** is used

# OFDM - Demodulation

- Considering a noiseless and non-fading channel/scenario
  - $y(t) = x(t) = \sum_k X_k e^{j2\pi k f_0 t}$
  - $\sum_k X_k e^{j2\pi k f_0 t}$ is a Fourier series or complex exponential Fourier series of $x(t)$
    - Fundamental frequency $f_0 = \frac{B}{N}$ and various $X_k$ representing the Fourier coefficients $X_l$
- All frequencies $kf_0$ are multiples of the fundamental frequency $f_0 = \frac{1}{T_0} = \frac{B}{N}$
- Therefore to extract $X_l$ coefficient of a Fourier coefficient corresponding to the frequency $f_l$
  - Multiply the input signal $y(t)$ with $e^{-j2\pi l f_0 t}$ (complex conjugate of the input), and $f_0$ $or$ $\frac{1}{T}$, and integrate over a limit of $\left[0, \frac{1}{f_0}\right]$ $or$ $\left[-\frac{1}{2f_0}, \frac{1}{2f_0}\right]$ $or$ $\left[0, \frac{1}{f_0}\right]$ $or$ $\left[-\frac{T}{2}, \frac{T}{2}\right]$
  - $\Rightarrow X_l = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} y(t) \times \left(e^{j2\pi l f_0 t}\right)^* dt = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_k X_k e^{j2\pi k f_0 t} \times e^{-j2\pi l f_0 t}\, dt$
- **$\frac{1}{f_0} = T$ is the OFDM symbol duration $\Rightarrow T = \frac{1}{f_0} = \frac{1}{\frac{B}{N}} = \frac{N}{B} = N \times \left(\frac{1}{B}\right) \Rightarrow \frac{N}{B} \gg \frac{1}{B} \gg$ Delay Spread**
- **Thus OFDM eliminates ISI**

# OFDM - Demodulation

- **Extracting $l^{th}$ symbol**
  - $f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} y(t) \times (e^{j2\pi l f_0 t})^* dt = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_k X_k e^{j2\pi k f_0 t} \times e^{-j2\pi l f_0 t}\, dt = f_0 \int_0^{\frac{1}{f_0}} \sum_k X_k e^{j2\pi k f_0 t} \times e^{-j2\pi l f_0 t}\, dt$
- Also called as coherent demodulation
- Rewriting the above equation:
  - $X_l = \sum_k X_k (f_0 \int_0^{\frac{1}{f_0}} e^{j2\pi(k-l) f_0 t} dt)$
    - $f_0 \int_0^{\frac{1}{f_0}} e^{j2\pi(k-l) f_0 t} dt \begin{cases} 0, if\ k \neq l \\ 1, if\ k = l \end{cases}$
    - The sub-carriers $e^{j2\pi k}, e^{j2\pi l}$ are orthogonal and hence their product when they are not equal $k \neq l$ is zero
  - $X_l = \sum_k X_k\, \delta(k-l)\ - discrete\ delta\ function$
- The sub-carrier orthogonality can leveraged to extract the symbol
  - $\Rightarrow$ inner product between two sub-carriers that are orthogonal is zero
- In summary
  - Orthogonal – sub-carriers are orthogonal
  - Frequency Division – Dividing bandwidth into multiple sub-carriers
  - Multiplexing – Parallel transmission of multiple symbols over multiple sub-carriers in the same band – Wide Bandwidth Channel
  - Enables transmission over large bandwidth without ISI – extremely high data rates

# OFDM Generation – IDFT ⇒ IFFT

- OFDM comprises of 1000s of sub-carriers ⇒ difficult to generate!
- Is there a simple technique to generate OFDM?
- Yes – using sampling!
- Considering signal BW is limited to $\frac{B}{2}$ $(i.e.\ f_{max})$, the sampling frequency is $f_s = 2 \times \frac{B}{2} = B$ (Nyquist Criterion)
- Sampling duration $t_s = \frac{1}{B} = \frac{1}{f_s}$
- The $l^{th}$ sample will be at t $= l \times t_s = l \times \frac{1}{B} = \frac{l}{B}$
- $\Rightarrow x(l) = \sum_k X_k e^{j2\pi k f_0 t} = \sum_k X_k e^{j2\pi k \frac{B\,l}{NB}} = \sum_k X_k e^{\frac{j2\pi kl}{N}}$
- **Scaling $x(l)$ by $\frac{1}{N}$ factor ⇒ $\frac{1}{N}\sum_k X_k e^{j2\pi\frac{kl}{N}}$**
- The expression $\frac{1}{N}\sum_k X_k e^{j2\pi\frac{kl}{N}} dt$ is representation of IDFT – **Inverse Discrete Fourier Transform**
- IDFT can be efficiently implemented using IFFT – **Inverse Fast Fourier Transform**
- IFFT algo is widely implemented in DSP chipsets – **computational complexity is $Nlog_2N$**

$$\underbrace{X_0, X_1, X_2, \dots X_{n-1}}_{Frquency\ Domain}$$

Loaded Sub-carriers

⇩

Sampling - $f_s = 1/B$

⇩

IDFT

⇩

IFFT

$$x(l) = \frac{1}{N}\sum_{k=0}^{(N-1)} X_k e^{\frac{j2\pi kl}{N}}$$

⇩

$x(0), x(1), x(N-1)$

Time domain samples

Tx

IFFT

Load/Modulate

$$x(l) = \frac{1}{N}\sum_{k=0}^{(N-1)} X_k e^{j2\pi\frac{kl}{N}}$$

**(Time Domain Samples)**

Rx

$(IFFT)^{-1} \Rightarrow FFT$

Demodulation

$$H_k = \sum_{l=0}^{(L-1)} h(l) e^{-j2\pi\frac{kl}{N}}$$

Zero padded FFT of the channel taps

(N-L) zeros

**Sub-carrier channel coefficients**

$$Y_k = \sum_{l=0}^{(N-1)} y(l) e^{-j2\pi\frac{kl}{N}}$$

**Frequency Domain Samples**

# ISI Channel Model

- Fading channel model for a symbol $k$, $y(k) = hx(k) + n(k)$
- ISI channel model with interference from other/previous symbols in the band, for symbol $k$:
  - $y(k) = h(0)x(k) + h(1)x(k-1) + h(2)x(k-2) + \cdots + h(L-1)x\big(k-(L-1)\big) + v(k)$
    - $\{v(k)\ is\ noise\}$
- Simplifying
  - $y(k) = \sum_{l=0}^{l=(L-1)} h(l)x(k-l) + v(k)\ -convolution\ operation$
  - $h(0), h(1), h(2), \dots h(L-1)$ are called ***channel taps – channel filters***
- $y(k)$ can be further simplified as $y(k) = h * x + v(k)$
- So, channel performs ***linear convolution***

# Convolution

- Convolution is a mathematical operation on two functions (*f* and *g)* that produce a third function *(f*g)* that expresses how the shape of one is modified by the other
- The term *convolution* refers to both the resultant function and the process of computing it
- The resultant function is defined as the integral of the product of the two functions after one is reflected on y-axis and shifted
  - $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$
- Some features of convolution are similar to cross-correlation

# Cyclic Prefix – CP and Demodulation

- $\underbrace{x(N-\tilde{L}),\ldots x(N-2),x(N-1)}_{Cyclic\ Prefix},\quad \underbrace{x(0),x(1),x(2),\ldots x(N-1)}_{Original\ samples\ generated\ by\ IFFT}$
- The cyclic prefix are $\tilde{L}$ samples from the tail of OFDM generated by IFFT
- Prior to transmission, the above cyclic prefix is prefixed to the IFFT output forming OFDM block
- So the total number of samples in OFDM block are $(N+\tilde{L}), \tilde{L} \ll N$
- $\tilde{L}$ is usually a maximum of 25%*N
- What is the effect of CP?
- Because of the addition of CP as prefix to the output of IFFT of OFDM, the linear convolution becomes circular convolution
  - $y(k) = h * x + v(k) \Rightarrow Prefix\ with\ CP \Rightarrow y(k) = h \circledast x + v$
- **The output of OFDM channel is $\boldsymbol{y(k) = h \circledast x + v}$**
- The output of the sub-carrier k - $y(k) = h \circledast x + v \Rightarrow FFT \Rightarrow Y_K = H_k \times X_k + V_k$
- When an FFT is taken for a circular convoluted time domain signal, it becomes the product between corresponding coefficients

# OFDM System Model

- $k$=0,1,2,…N-1 – sub-carriers
- The OFDM System Model is
  - $Y_0 = H_0 \times X_0 + V_0$
  - $Y_1 = H_1 \times X_1 + V_1$
  - $Y_2 = H_2 \times X_2 + V_2$
  - …
  - $Y_{N-1} = H_{N-1} \times X_{N-1} + V_{N-1}$
- Channel coefficient $H_k$ are given as
  - $H_k = \sum_0^{L-1} h(l) e^{-j2\pi\frac{kl}{N}} \Rightarrow h(0), h(1), h(2), \dots h(L-1), 0,0,0, \dots 0\ (N - L\ zeros)$
  - N Point Zero Padded FFT $\Rightarrow \mathrm{H_0, H_1, H_2, \dots H_{N-1}}\ - sub - carrier\ channel\ coefficients$
- CP removal
  - OFDM block = $\underbrace{x\left(N-\tilde{L}\right), \dots x(N-2), x(N-1)}, \underbrace{x(0), x(1), x(2), \dots x(N-1)}$
  - The prefixed CP is removed by the receiver and receiver processes the output from $x(0)\ or\ y(0)$
    - $y(0) = h(0)x(0) + h(1)x(N-1) + \cdots + h(L-1)x(N-L+1) + \cdots + 0 \times x(2) + 0 \times x(1) + v(0)$
  - This process is known as CP removal
- What is OFDM achieving?
  - Converting a time-domain ISI channel into N parallel ISI-free (flat fading) sub-carrier channels!
    - ISI – Frequency selective, dispersive channel
    - Flat fading – Frequency flat channel, ISI free

# OFDM Examples

- The expressions $e^{j\theta}$ and $e^{-j\theta}$ can be simplified using Euler's formula
  - $e^{j\theta} = cos(\theta) + jsin(\theta)$
  - $e^{-j\theta} = cos(-\theta) + jsin(-\theta) = cos(\theta) - jsin(\theta)$
    - $cos(-\theta)$= $cos(\theta)$ - Since cosine is even function
    - $sin(-\theta)$= $-sin(\theta)$ - Since sine is odd function
- $H_3 = h(0) + h(1)e^{-\frac{j3\pi}{2}}$
  - $= h(0) + h(1)\left(\cos\left(-\frac{3\pi}{2}\right) - jsin\left(\frac{3\pi}{2}\right)\right)$
  - $= h(0) + h(1)(0 -$
- Effective bit rate $= Number\ of\ bits\ per\ symbol\ * \frac{N}{T_{symbols}+T_{CP}}$

# BER Performance of OFDM Systems

- Consider channel taps - $h(0), h(1), h(2), h(3), \ldots, h(L-1)$
- Assume they are Rayleigh fading with unit power $E\{|h(l)|^2\} = 1$
- Noise samples $v(l)$ are i.i.d with power $N_0 = E\{|v(l)|^2\}$
- Symbols loaded on sub-carriers have power P
- Effective SNR for QPSK as
  - $\boldsymbol{SNR_{OFDM-QPSK} = \rho_{eff} = \frac{L}{N} \times \frac{P}{N_0} = \frac{L}{N} SNR}$
    - L – number of channel taps
    - N – number of sub-carriers
- BER of OFDM for QPSK
  - $\boldsymbol{BER_{OFDM-QPSK}} = \frac{1}{2} \times \left(1 - \sqrt{\frac{\rho_{eff}}{2+\rho_{eff}}}\right) \approx \frac{1}{2} \times \frac{1}{\rho_{eff}} \approx \frac{1}{2} \times \frac{1}{\frac{L}{N} SNR}$
  - $\boldsymbol{BER_{OFDM-QPSK} \approx \frac{1}{2} \times \frac{N}{L} \times \frac{1}{SNR}}$

# MIMO-OFDM Channel Coefficients

- MIMO channel coefficients in the frequency domain corresponding to sub-carrier $k$ between $i^{th}$ receive antenna and $j^{th}$ transmit antenna $H_{ij}(k)$
- $h_{ij}(0), h_{ij}(1), h_{ij}(2), \dots h_{ij}(L-1), \underbrace{0,0,0,0 \dots 0}_{(N-L\ zeros)} \Rightarrow FFT(N-pt) \Rightarrow$

  $H_{ij}(0), H_{ij}(1), \dots, H_{ij}(N-1)$
- $H_{ij}(k)$ are given by the FFT of the channel taps and are frequency domain coefficients
- $Y(k) = H(k) \times X(k) + W(k)$
  - $H(k) - Flat\ MIMO\ channel$
  - There are $k = 0,1,2, \dots (N-1) - n\ flat\ parallel\ MIMO\ channels$
- Converted a frequency selective MIMO channel to a flat fading MIMO channel using OFDM

# MIMO-OFDM

- Exploits **spatial multiplexing** of MIMO and **frequency multiplexing** of OFDM leading to ultra-high data rates
- MIMO-OFDM channel model
  - $r$ – number ***receive*** antennas, $t$ – number of transmit antennas
  - Channel taps between receive antenna *i* and transmit antenna *j*
  - $h_{ij}(0), h_{ij}(1), h_{ij}(2), \ldots, h_{ij}(L-1), h_{ij}(l) - l^{th}\ channel\ tap\ between\ i^{th} receive\ antenna\ and\ j^{th} transmit\ antenna\ with\ l\ taps$
  - Total number of taps are $r \times t \times l$
- Transmission
  - On each transmit antenna $j$ load the sub-carriers as below – i.e. perform IFFT
    - $X_j(0), X_j(1), X_j(2), \ldots X_j(N-1) \Rightarrow X_j(k) - Symbol\ loaded\ on\ sub-carrier\ k\ at\ transmit\ antenna\ j$ – Frequency domain samples
  - Total number of symbols = $n \times t$
  - IFFT can be performed as shown below
    - $X_j(0), X_j(1), X_j(2), \ldots X_j(N-1) \Rightarrow \mathrm{IFFT} \Rightarrow \mathrm{x_j(0), x_j(1), x_j(2), \ldots, x_j(N-1)}$ – time domain samples on transmit antenna $j$
  - One IFFT per each transmit antenna – total $t$ IFFT blocks at the transmitter
  - Add CP on each transmit antenna
    - $\underbrace{x_j\left(N-\tilde{L}\right), \ldots x_j(N-2), x_j(N-1),}_{CP\ -\tilde{L}\ samples}\ \underbrace{x_j(0), x_j(1), x_j(2), \ldots, x_j(N-1)}_{original\ samples\ after\ IFFT\ -N\ samples} \Rightarrow \left(N+\tilde{L}\right) samples$
  - What is the size of total transmission block – one OFDM block?
    - $\left(N+\tilde{L}\right) \times t \ = \ t \times (N + N_{CP}) \times t$
- Channel
  - Because of addition of CP, the linear convolution becomes circular convolution
  - $y_i(k) = \sum_{j=1}^{t} h_{ij} \circledast x_j + w_j(k)$
- Receiver
  - FFT is performed at each receive antenna after removing CP
  - $y_i(0), y_i(1), y_i(2), y_i(N-1)(CP\ removed) \Rightarrow FFT \Rightarrow Y_i(0), Y_i(1), Y_i(2), \ldots, Y_i(N-1)$
  - $y_i(k)$ is output on sub-carrier $k$
  - One FFT at each receive antenna $\Rightarrow r$ FFTs

# Net MIMO-OFDM System Model

- $$\underbrace{\begin{bmatrix} Y_1(k) \\ Y_2(k) \\ Y_3(k) \\ . \\ . \\ . \\ Y_r(k) \end{bmatrix}}_{Y(k)} = \underbrace{\begin{bmatrix} H_{11}(k) & \cdots & H_{1t}(k) \\ \vdots & \ddots & \vdots \\ H_{r1}(k) & \cdots & H_{rt}(k) \end{bmatrix}}_{H(k)} \times \underbrace{\begin{bmatrix} X_1(k) \\ X_2(k) \\ . \\ . \\ X_t(k) \end{bmatrix}}_{X(k)} + \underbrace{\begin{bmatrix} W_1(k) \\ W_2(k) \\ . \\ . \\ W_r(k) \end{bmatrix}}_{W(k)}$$
- $Y(k)$ – $r \times 1$ vector of outputs for all Rx antennas on sub-carrier $k$ – output symbol vector
- $H(k)$ - $r \times t$ MIMO channel matrix for sub-carrier $k$ – Channel matrix
- $X(k)$ - $t \times 1$ vector of symbols loaded on sub-carrier $k$ for all Tx antennas – transmit symbol vector
- $W(k)$ - $r \times 1$ Noise vector for sub-carrier $k$ – noise vector

# MIMO-OFDM

- MIMO-OFDM model for sub-carrier $k$ is
  - $Y(k) = H(k) \times X(k) + W(k)\ - Frequency\ domain\ model$
- How many such parallel MIMO systems are there?
  - One for each sub-carrier → N parallel MIMO channels
  - $Y(0) = H(0)X(0) + W(0)$
  - $Y(1) = H(1)X(1) + W(1)$
  - $Y(2) = H(2)X(2) + W(2)$
  - ….
  - $Y(N-1) = H(N-1)X(N-1) + W(N-1)$
- How to recover $X(k)$?
  - Any MIMO receiver – ZF or LMMSE receiver
- ZF receiver
  - ZF receiver : $\hat{X}(k) = H^{\dagger}(k)Y(k) \Rightarrow \underbrace{\left(H^H(k)H(k)\right)^{-1}}_{Pseudo\ Inverse\ of\ H(k)} H^H(k)Y(k)$
- LMMSE receiver
  - $\hat{X}(k) = \left(H^H(k)H(k) + \frac{1}{SNR}I\right)^{-1} H^H(k)Y(k)$

# Characterization of Wireless Channel

# Wireless Channel Model

- Wireless channel model
  - $h = \sum_0^{L-1} a_i \delta(t - \tau)$
    - $L$ – number of multi-path components/channels
    - $a_i$ - Channel attenuation
    - $\tau_i$ - delay
- An important characteristic of a multi-path channel is the time delay spread it causes to the received signal
- This delay spread equals the time delay between
  - The arrival of the first received signal component (LOS or multi-path)
  - Last received signal component associated with a single transmitted pulse
- If the delay spread is
  - Small compared to the inverse of the signal bandwidth, then there is little time spreading in the received signal
  - Relatively large compared to the inverse of the signal bandwidth, then there is significant time spreading of the received signal which leads to substantial signal distortion
- Maximum delay spread
  - $T_d = \tau_{L-1} - \tau_0$
    - $\tau_{L-1}$ - Last multi-path component received
    - $\tau_0$ - First multi-path component received

# Wireless Channel Model – Delay Spread

- Another metric for the delay spread is RMS delay spread
- Let $g_i = |a_i|^2$
- Mean delay (Weighted average of delays) $\bar{\tau} = \frac{\sum_i g_i \tau_i}{\sum_i g_i}$
- RMS Delay Spread - $T_{d,rms} = \sqrt{\frac{\sum_i g_i (\tau_i - \bar{\tau})^2}{\sum_i g_i}}$
- Typical delay spread in outdoor distances – 2-3 $\mu$s
- When does ISI occur?
  - $T_d (Delay\ Spread) \geq \frac{1}{2} T_s\ (Symbol\ Time)$ - ISI
- On the other hand
  - $T_d (Delay\ Spread) < \frac{1}{2} T_s\ (Symbol\ Time)$ – No ISI

# ISI Channel Model

- Non ISI channel model
  - $y(k) = hx(k) + n(k)$
  - $y(k)$ depends only on $x(k)$
- ISI channel model
  - $y(k) = h(0)x(k) + h(1)x(k-1) + h(2)x(k-2) + \cdots + h(L-1)x(k-L+1) + n(k)$
  - $y(k) = h * x + n - *$ - linear convolution
    - $h(0), h(1), \ldots h(L-1)$ – channel taps or taps of the channel filter or coefficients of channel filter – FIR – Finite Impulse Response
  - $y(k)$ depends on $x(k), x(k-1), \ldots.$
- Delay spread
  - $T_d = 2\mu s$
  - $T_s \leq 4\mu s = 2T_d \Rightarrow \frac{1}{B} = 4\mu s$
  - $\Rightarrow B \geq \frac{1}{2T_d} = \frac{1}{4\mu s} = 250\ kHz$
  - So, ISI occurs when the bandwidth is greater than 250 kHz – this bandwidth is known as Coherence bandwidth and is represented by $B_c$ of the channel
- The relationship between $T_d$ and $B_c$ is $\boldsymbol{B_c = \frac{1}{2T_d}}$ **-** $T_d$ - channel delay spread
- Hence ISI occurs if:
  - $T_d \geq \frac{1}{2} T_s$
  - $B \geq B_c \geq \frac{1}{2T_d}$
- So, $B_c \propto \frac{1}{T_d}$ - coherence bandwidth is inversely proportional to delay spread

# Delay Spread

- Say delay spread $T_d = 0$
  - Channel model $h(t) = \delta(t)$
  - Fourier transform $|H(f)| = 1$
  - BW=$B_C = \infty$
- Frequency domain interpretation of $B_c$
  - When signal bandwidth $B_s/B \leq B_c$, the output spectrum is undistorted
    - Channel response is flat over signal bandwidth
    - Such a channel is termed as **flat fading channel**
    - No ISI and no distortion
  - When signal bandwidth $B_s/B \geq B_c$
    - Fading is frequency selective
    - Channel response is NOT flat and varies over signal bandwidth
    - Signal spectrum is distorted frequency domain and is ISI in time domain

# Doppler Shift

- What happens when a mobile moves away or towards base station/transmitter
  - Angle $\theta$ and velocity $v$
  - Change in the observed/received frequency by the receiver
  - This is termed as doppler shift
- Doppler shift $f_D = \frac{v \times cos\theta}{c} \times f_c$
  - $v$ – velocity of the mobile
  - $c$ – speed of light
  - $f_c$ - carrier frequency
  - $\theta$ – angle between velocity vector and line joining mobile to base station
- Impact of doppler shift on the channel
  - Time varying delay $\Rightarrow$ Time Varying Channel
  - Such a channel is called Time Selective Channel
- How fast is channel varying?
  - Let $T_c$ denote the time over which channel is constant
  - This $T_c$ is termed **Coherence Time**
  - $T_c = \frac{1}{4f_D} \propto \frac{1}{f_D}$
- Higher velocity => coherence time is small
- Doppler bandwidth = $2f_D = B_D$

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