eMasters in Communication Systems

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Elective Module:

Detection for Wireless Communication

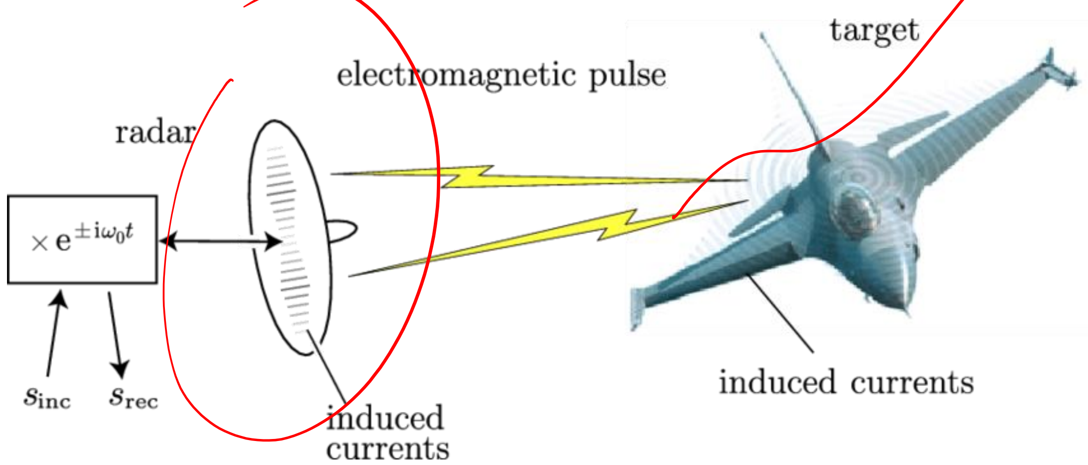
Chapter 1

Likelihood Ratio Test (LRT) for Detection

Detection

- Detect the ***presence or absence*** of a phenomenon.
- RADAR: To detect the presence or absence of an ***aircraft***.

REFLECTION



Detection

- **Communication systems:**
- Detect transmitted symbol belonging to a ***digital constellation***

Quadrature Phase Shift Keying

$\log_2 4 = 2$ bits/sym.

$$\underbrace{\{A + jA, A - jA, -A + jA, -A - jA\}}_{\text{QPSK Constellation}}$$

"DECISION RULE".

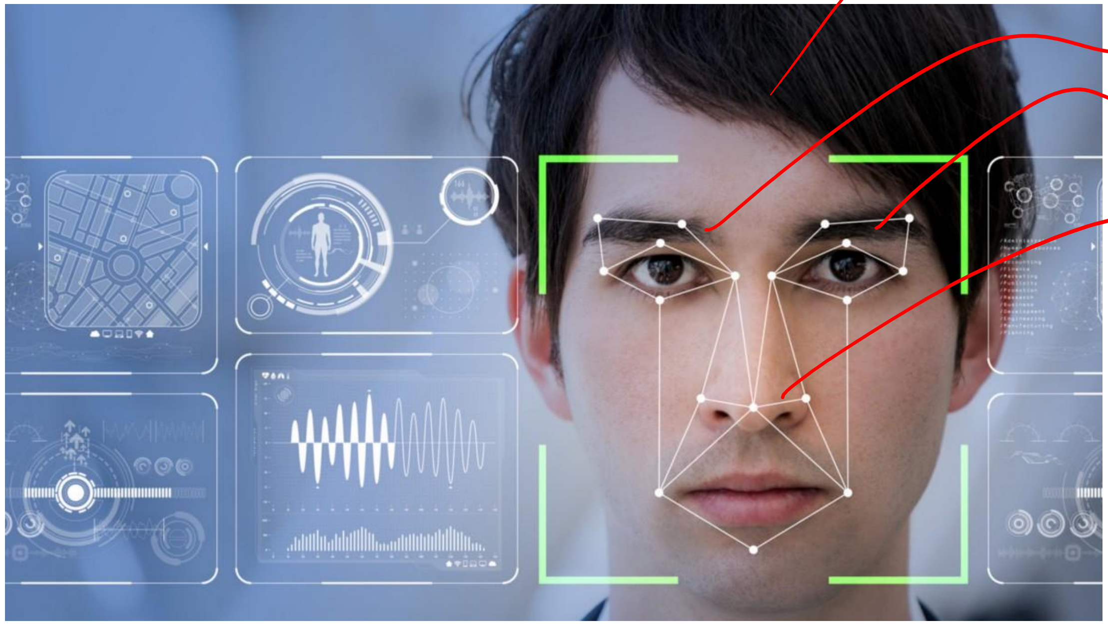
Detection

- **Machine Learning:** Given images, to detect a *face*.

'FACE RECOGNITION'

EYES

NOSE.



Detection

Two hypotheses: $\mathcal{H}_0, \mathcal{H}_1$

- Canonical detection problem: ***Binary Hypothesis testing problem.***
- Two hypotheses:
- $\mathcal{H}_0$: NULL hypothesis
- $\mathcal{H}_1$: ALTERNATIVE hypothesis.

Detection

- Canonical detection problem: ***Binary Hypothesis testing problem*.**
- Two hypotheses:
- $\mathcal{H}_0$: **NULL Hypothesis**
- $\mathcal{H}_1$: **Alternative Hypothesis**

choosing or deciding in favor of one of these hypotheses.

BINARY. HYPOTHESIS.

Detection

NULL hypothesis.

- **Mathematical Model**

Observed Samples.

Signal is absent.

$\mathcal{H}_0$:

$y(1) = v(1)$

$y(2) = v(2)$

$\vdots$

$y(N) = v(N)$

iid Gaussian Noise samples

Mean $= 0$

Var $= \sigma^2$

$\mathcal{N}(0, \sigma^2)$

Output = Noise.

Detection

- **Mathematical Model**

$\mathcal{H}_0$:

$$y(1) = v(1)$$
$$y(2) = v(2)$$
$$\vdots$$
$$y(N) = v(N)$$

Detection

- **Mathematical Model**

$$\mathcal{H}_0: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$\bar{y}$ $\bar{v}$

$$\bar{\mathbf{y}} = \bar{v}$$

$\bar{y} = \bar{v}$

$N \times 1$ Observation vector

$N \times 1$ noise vector

$$E\{\bar{v}\bar{v}^T\} = \sigma^2 I$$

white Noise.

NOISE covariance matrix.

$$E\{\bar{V}\bar{V}^T\} = E\left\{\begin{bmatrix} V(1)\\ V(2)\\ \vdots \\ V(N)\end{bmatrix}\begin{bmatrix} V(1) & V(2) & \cdots & V(N)\end{bmatrix}\right\}$$

$$= E\left\{\begin{bmatrix} V^2(1) & V(1)V(2) & \cdots \\ V(2)V(1) & V^2(2) & \\ \vdots & & \ddots \end{bmatrix}\right\} = \begin{bmatrix} \sigma^2 & 0 & \cdots \\ 0 & \sigma^2 & \\ \vdots & & \ddots \end{bmatrix}$$

$$= \sigma^2 I_{N\times N}.$$

$$E\{V(1)V(2)\}$$

$$= \underbrace{\cancel{E\{V(1)\}}}_{0}\cdot\underbrace{\cancel{E\{V(2)\}}}_{0}$$

$$= 0$$

Detection

- **Mathematical Model**

$$\mathcal{H}_0: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\bar{\mathbf{y}} = \bar{\mathbf{v}}$$

$\mathcal{H}_0 : \bar{y} = \bar{v}$

$\bar{y} = 0 + \bar{v}$

NULL Signal.

Detection

- **Mathematical Model**

Alternative hypothesis.

$\mathcal{H}_1$:

Signal samples.

$y(1) = s(1) + v(1)$

Output = Signal + Noise.

$y(2) = s(2) + v(2)$

$\vdots$

$y(N) = s(N) + v(N)$

Detection

- **Mathematical Model**

$\mathcal{H}_1$:

$$
\begin{aligned}
y(1) &= s(1) + v(1) \\
y(2) &= s(2) + v(2) \\
&\vdots \\
y(N) &= s(N) + v(N)
\end{aligned}
$$

Detection

- **Mathematical Model**

$$\mathcal{H}_1: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \underbrace{\begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix}}_{\bar{s}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\bar{v}}$$

$\bar{s}$ ~ Signal.

$\bar{v}$ ~ Noise, iid Gaussian.

$$E\{\bar{v}\} = 0$$

$$E\{\bar{v}\bar{v}^T\} = \sigma^2 I$$

$$\bar{\mathbf{y}} = \bar{s} + \bar{v}$$

Detection

- **Mathematical Model**

$$\mathcal{H}_1: \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\underset{N\times 1}{\bar{\mathbf{y}}} = \underset{N\times 1}{\bar{\mathbf{s}}} + \underset{N\times 1}{\bar{\mathbf{v}}}$$

Detection

$v(1), v(2), \ldots, v(N)$.

- Noise samples in $\bar{\mathbf{v}}$ are i.i.d. **Gaussian**, mean 0 variance is σ^2.
- $\bar{\mathbf{s}}$ is a ***known signal***

CONSTANT KNOWN SIGNAL

Detection

- Write it in the compact form

$\mathcal{H}_0 \colon \bar{\mathbf{y}} = \bar{\mathbf{v}}, \bar{\mathbf{y}} \sim \mathcal{N}(0, \sigma^2 I)$

N Dimensional.
Multidimensional.
Gaussian.

$\mathcal{H}_1 \colon \bar{\mathbf{y}} = \bar{\mathbf{s}} + \bar{\mathbf{v}}, \bar{\mathbf{y}} \sim \mathcal{N}(\bar{s}, \sigma^2 I)$.

$$E\{\bar{y}\} = E\{\bar{s} + \bar{v}\}$$
$$= \bar{s} + \cancel{E\{\bar{v}\}}\ 0$$
$$= \bar{s}$$

Detection

TEST: Given $\bar{y}$ how to choose between $\mathcal{H}_0, \mathcal{H}_1$?

- Write it in the compact form

$$\mathcal{H}_0 : \bar{\mathbf{y}} = \bar{\mathbf{v}}, \underline{\bar{\mathbf{y}} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})}$$

$$\mathcal{H}_1 : \bar{\mathbf{y}} = \bar{\mathbf{s}} + \bar{\mathbf{v}}, \underline{\bar{\mathbf{y}} \sim \mathcal{N}(\bar{\mathbf{s}}, \sigma^2 \mathbf{I})}$$

Recall

What is joint PDF?

$V(1), V(2), \ldots, V(N)$

$V(i) \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{V^2(i)}{2\sigma^2}}$

- **Gaussian random vector $\bar{\mathbf{v}}$**
- **i.i.d. samples-mean 0, variance σ^2**

$E\{V(i)\} = 0$

$E\{V^2(i)\} = \sigma^2$

$E\{V(i)V(j)\} = 0$
$i \neq j$

- Joint PDF?

Recall

iid $\Rightarrow$ joint PDF = Product of marginal PDFs.

$$f_{\bar{\mathbf{V}}}(\bar{\mathbf{v}}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v^2(1)}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2(N)/2\sigma^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N} v^2(i)}{2\sigma^2}} = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\|\bar{v}\|^2}{2\sigma^2}}$$

joint PDF Noise samples.

MultiVariate Gaussian PDF iid Gaussian RVs.

mean: 0

cov $= \sigma^2 I$.

$$\|\bar{v}\|^2 = v_1^2 + v_2^2 + \cdots + v_N^2$$

Recall

NOISE VECTOR.

$$f_{\bar{\mathbf{v}}}(\bar{\mathbf{v}}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v^2(1)}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v^2(N)}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N} v^2(i)}{2\sigma^2}} = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\|\bar{\mathbf{v}}\|^2}{2\sigma^2}}$$

Likelihood — PDF

$y(i) = v(i)$ — $\mathcal{N}(0,\sigma^2)$, $\mathcal{N}(0,\sigma^2)$

- **Likelihood of hypothesis $\mathcal{H}_0$?**

$$p(\bar{\mathbf{y}};\mathcal{H}_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}$$

Likelihood of $\mathcal{H}_0$.

Likelihood

- **Likelihood of hypothesis $\mathcal{H}_0$?**

How likely is $\mathcal{H}_0$?
$\propto$ PDF of $\bar{y}$.

$$p(\bar{\mathbf{y}};\mathcal{H}_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}$$

Likelihood

- **Likelihood of hypothesis $\mathcal{H}_1$?**

$\mathcal{N}(s(i), \sigma^2)$

$y(i) = s(i) + v(i)$

shifting mean to $s(i)$

$$\underbrace{p(\bar{\mathbf{y}}; \mathcal{H}_1) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N} (y(i)-s(i))^2}{2\sigma^2}}}_{\text{Likelihood of } \mathcal{H}_1}$$

Likelihood

- **Likelihood** of hypothesis $\mathcal{H}_1$?

$$p(\bar{\mathbf{y}}; \mathcal{H}_1) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}{2\sigma^2}}$$

Likelihood Ratio Test

- **Which hypothesis to choose?**

ALMOST

LOGIC: choose hypothesis with higher likelihood!

Likelihood Ratio Test (LRT).

- Choose $\mathcal{H}_0$ if

$$p(\bar{\mathbf{y}}; \mathcal{H}_0) \geq \tilde{\gamma} \times p(\bar{y}; \mathcal{H}_1)$$

Likelihood of $\mathcal{H}_0$

Likelihood of $\mathcal{H}_1$

Arbitrary Threshold.

Likelihood Ratio

$$\Rightarrow \quad \frac{p(\bar{y}; \mathcal{H}_0)}{p(\bar{y}; \mathcal{H}_1)} \geq \tilde{\gamma}$$

Likelihood Ratio Test

Fundamental Principles in Detection.

(LRT)

- Choose $\mathcal{H}_0$ if

$$p(\bar{\mathbf{y}};\mathcal{H}_0) \geq \tilde{\gamma} \times p(\bar{\mathbf{y}};\mathcal{H}_1)$$

Likelihood Ratio $\geq \tilde{\gamma}$

$$\Rightarrow \underbrace{\frac{p(\bar{\mathbf{y}};\mathcal{H}_0)}{p(\bar{\mathbf{y}};\mathcal{H}_1)} \geq \tilde{\gamma}}_{\text{Likelihood Ratio Test (LRT)}}$$

ML

LRT is General. Setting $\tilde{\gamma} = 1$ in LRT gives ML.

ML is special case of LRT. Maximum Likelihood (ML) Decision rule

- If $\tilde{\gamma} = 1$ it becomes ML detection
- Choose $\mathcal{H}_0$ if

$$p(\bar{y};\mathcal{H}_0) \geq p(\bar{y};\mathcal{H}_1)$$

$$\Rightarrow \frac{p(\bar{y};\mathcal{H}_0)}{p(\bar{y};\mathcal{H}_1)} \geq 1 \quad \tilde{\gamma}$$

ML

- If $\tilde{\gamma} = 1$ it becomes ML detection
- Choose $\mathcal{H}_0$ if

$$p(\bar{\mathbf{y}}; \mathcal{H}_0) \geq p(\bar{\mathbf{y}}; \mathcal{H}_1)$$

$\tilde{\gamma}$

$$\underbrace{\Rightarrow \frac{p(\bar{\mathbf{y}}; \mathcal{H}_0)}{p(\bar{\mathbf{y}}; \mathcal{H}_1)} \geq 1}_{\text{Maximum Likelihood}}$$

Likelihood

- $\tilde{\gamma}$ is an ***arbitrary threshold***

LRT

Simplify LRT

- Choose $\mathcal{H}_0$ if

$$\frac{p(\bar{\mathbf{y}};\mathcal{H}_0)}{p(\bar{\mathbf{y}};\mathcal{H}_1)} \geq \tilde{\gamma} \Rightarrow \frac{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{N/2}} e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}}{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{N/2}} e^{-\frac{\sum_{i=1}^{N} (y(i)-s(i))^2}{2\sigma^2}}} \geq \tilde{\gamma}$$

Take ln on both sides

$$\Rightarrow \sum_{i=1}^{N} (y(i)-s(i))^2 - \sum_{i=1}^{N} y^2(i) \geq 2\sigma^2 \ln \tilde{\gamma}$$

LRT

- Choose $\mathcal{H}_0$ if

constant

$$\Rightarrow \sum_{i=1}^{N} \cancel{y^2(i)} + s^2(i) - 2y(i)s(i) - \sum_{i=1}^{N} \cancel{y^2(i)} \geq 2\sigma^2 \ln \tilde{\gamma}$$

$$\Rightarrow \sum_{i=1}^{N} y(i)s(i) \leq \underbrace{\frac{\sum_{i=1}^{N} s^2(i) - 2\sigma^2 \ln \tilde{\gamma}}{2}}_{\gamma}$$

LRT

$$\bar{s} = \begin{bmatrix} s(1) \\ \vdots \\ s(N) \end{bmatrix} \quad \bar{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}$$

- Choose $\mathcal{H}_0$ if

$$\sum_{i=1}^{N} y(i)\,s(i) \leq \gamma$$

$$\gamma = \frac{\overbrace{\sum_{i=1}^{N} s^2(i)}^{\|\bar{s}\|^2} - 2\sigma^2 \ln \tilde{\gamma}}{2} = \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2}$$

$F(\tilde{\gamma})$.

MATCHED FILTER

$$\Rightarrow \bar{s}^T \bar{y} \leq \gamma$$

LRT

- Choose $\mathcal{H}_0$ if

LRT

- Choose $\mathcal{H}_0$ if

$$\frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}}{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}{2\sigma^2}}} \geq \tilde{\gamma}$$

$$\Rightarrow \frac{e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}}{e^{-\frac{\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}{2\sigma^2}}} \geq \tilde{\gamma}$$

LRT

- Choose $\mathcal{H}_0$ if

$$\ln \frac{e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}}{e^{-\frac{\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}{2\sigma^2}}} \geq \ln \tilde{\gamma}$$

$$\Rightarrow \frac{\sum_{i=1}^{N}\left(y(i)-s(i)\right)^2}{2\sigma^2} - \frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2} \geq \ln \tilde{\gamma}$$

LRT

- This can be simplified as follows

$$\Rightarrow \frac{\sum_{i=1}^{N} y^2(i) + s^2(i) - 2y(i)s(i)}{2\sigma^2} - \frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}$$

$$\geq \ln \tilde{\gamma}$$

LRT

- Choose $\mathcal{H}_0$ if

$$\frac{\sum_{i=1}^{N} s^2(i) - 2y(i)s(i)}{2\sigma^2} \geq \ln \tilde{\gamma}$$

$$\sum_{i=1}^{N} y(i)s(i) \leq \underbrace{\frac{\sum_{i=1}^{N} s^2(i) - 2\sigma^2 \ln \tilde{\gamma}}{2}}_{\gamma}$$

LRT

- Choose $\mathcal{H}_0$ if

$$\sum_{i=1}^{N} y(i)s(i) \leq \gamma$$

$$\Rightarrow \bar{s}^T \bar{y} \leq \gamma$$

LRT

- Choose $\mathcal{H}_0$ if

$$\sum_{i=1}^{N} y(i)s(i) \le \frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \gamma$$

$$\bar{\mathbf{s}}^T \bar{\mathbf{y}} \le \gamma$$

LRT

- Choose $\mathcal{H}_1$ if

$$\sum_{i=1}^{N} y(i)\,s(i) > \gamma$$

$$\Rightarrow \bar{s}^T \bar{y} > \gamma$$

LRT

- Choose $\mathcal{H}_1$ if

$$\sum_{i=1}^{N} y(i)s(i) > \gamma$$

$$\bar{\mathbf{s}}^T\bar{\mathbf{y}} > \gamma$$

LRT

- The final **LRT** is given as
- Choose $\mathcal{H}_0$ if $\bar{\mathbf{s}}^T \bar{\mathbf{y}} \le \gamma$
- Choose $\mathcal{H}_1$ if $\bar{\mathbf{s}}^T \bar{\mathbf{y}} > \gamma$

Signal is absent

Likelihood Ratio Test

$\sum_{i=1}^{N} s(i)y(i) \le \gamma$

Test Statistic.

Threshold.

$\sum_{i=1}^{N} s(i)y(i) > \gamma$

LRT

- $\sum_{i=1}^{N} y(i)s(i)$ is the ***test statistic***.
- $\bar{\mathbf{s}}^T \bar{\mathbf{y}}$ $= \sum_{i=1}^{N} y(i)s(i)$.
- ***Matched filter***

Maximizes SNR in White Gaussian noise.

Optimal for Signal detection in AWGN.

ML Detector

$\tilde{\gamma} = 1$ Maximum Likelihood is a Special case of LRT

- $\tilde{\gamma} = 1$

$\ln 1 = 0$

$$\gamma = \frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \frac{\|\bar{s}\|^2}{2}$$

$$\left.\gamma = \frac{\|\bar{s}\|^2}{2}\right\} \text{ML}$$

ML Detector

- $\tilde{\gamma} = 1$

$$\gamma = \overbrace{\frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \frac{\|\bar{\mathbf{s}}\|^2}{2}}^{ML}$$

Performance

Probability of incorrect decision

$\Rightarrow$ Prob of error!

- *What is the performance of the detector?*

Performance

FALSE ALARM:

SIGNAL ABSENT BUT DECISION IS SIGNAL PRESENT

- **Probability of false alarm: P_{FA}**
- Probability that the test ***falsely detects the presence*** of signal under $\mathcal{H}_0$.

Given $\mathcal{H}_0$, what is prob. decision is $\mathcal{H}_1$?

Lower P_{FA} $\Rightarrow$ Better test!

Performance

When does FA occur?

Signal ABSENT

- This happens when under $\mathcal{H}_0$

under H_0

$y(i) = v(i)$

$$\sum_{i=1}^{N} y(i)s(i) > \gamma$$

$$\bar{s}^T \bar{y} > \gamma$$

$$\Rightarrow \sum_{i=1}^{N} v(i)s(i) > \gamma$$

Performance

FALSE ALARM.

- This happens when under $\mathcal{H}_0$

$$\underbrace{\sum_{i=1}^{N} y(i)s(i)}_{\bar{s}^T\bar{y}\,\sim\,\mathcal{N}(0,\sigma^2\|\bar{s}\|^2)} > \gamma \qquad \bar{s}^T\bar{y} > \gamma$$

under $\mathcal{H}_0 \Rightarrow \bar{s}^T\bar{v} > \gamma$

Performance

- Note under $\mathcal{H}_0$

NULL Hypothesis -

$\Rightarrow$ NO SIGNAL

Output = Noise.

$$y(i) = v(i) \sim \mathcal{N}(0, \sigma^2)$$

$var = \sigma^2$

Gaussian mean = 0

Performance

- Note under $\mathcal{H}_0$

$$y(i) = v(i) \sim \mathcal{N}(0, \sigma^2)$$

Performance

Performance

TEST STATISTIC $= \sum_{i=1}^{N} y(i)s(i)$

- This implies that

$$\underbrace{E\left\{\sum_{i=1}^{N} y(i)s(i)\right\}}_{0} = E\left\{\sum_{i=1}^{N} v(i)s(i)\right\}$$

under $\mathcal{H}_0 : y(i) = v(i)$

0

$$= \sum_{i=1}^{N} s(i)\,\cancel{E\{v(i)\}}$$

Performance

- This implies that

$$E\left\{\sum_{i=1}^{N} y(i)s(i)\right\} = \frac{1}{N}\sum_{i=1}^{N} s(i)E\{v(i)\}$$
$$= 0$$

Performance

$$\sum_{i=1}^{N} y(i)s(i) \text{ under } H_0 .$$

- The variance is

$$E\left\{\left(\sum_{i=1}^{N} y(i)s(i)\right)^2\right\} = E\left\{\left(\sum_{i=1}^{N} v(i)s(i)\right)^2\right\}$$

$$= E\left\{ \sum_{i=1}^{N} s^2(i)\, v^2(i) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} v(i)v(j)s(i)s(j) \right\}$$

Performance

Because noise samples are independent!

0

$$= \sum_{i=1}^{N} E\{v^2(i)\} s^2(i) + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} \cancel{E\{v(i)v(j)\}} s(i)s(j)$$

$$= \underbrace{\cancel{E\{v(i)\}}}_{0} \underbrace{\cancel{E\{v(j)\}}}_{0}$$

$$= \sum_{i=1}^{N} \sigma^2 s^2(i) = \sigma^2 \sum_{i=1}^{N} s^2(i) = \sigma^2 \|s\|^2$$

Performance

Variance of Test statistic under $\mathcal{H}_0$.

- The variance is

$$E\left\{\left(\sum_{i=1}^{N} y(i)s(i)\right)^2\right\} = E\left\{\left(\sum_{i=1}^{N} v(i)s(i)\right)^2\right\}$$

$$= \sum_{i=1}^{N} s^2(i)E\{v^2(i)\} = \sum_{i=1}^{N} s^2(i)\sigma^2 = \sigma^2\|\bar{\mathbf{s}}\|^2$$

Performance

- Therefore

$$\sum_{i=1}^{N} y(i)s(i) \sim \mathcal{N}(0, \sigma^2 \|\bar{\mathbf{s}}\|^2)$$

under H_0
Gaussian
Mean = 0
Var = $\sigma^2 \|\bar{s}\|^2$

$\mathcal{N}(0, \sigma^2 \|\bar{s}\|^2)$

Performance

- Therefore, P_{FA} is given as

$$\Pr\left(\sum_{i=1}^{N} y(i)s(i) > \gamma\right)$$

What Probability?

Gaussian RV
mean = 0
var = $\sigma^2\|s\|^2$

What is probability that Gaussian RV with mean = 0 var = $\sigma^2\|s\|^2$ is $> \gamma$?

$$= \Pr\left(\mathcal{N}(0, \sigma^2\|s\|^2) > \gamma\right)$$

Performance

Var = $\sigma^2 \|\bar{s}\|^2$

std.dev = $\sqrt{\text{var}}$

= $\sigma \|\bar{s}\|$.

Gaussian mean = 0

var = 1

- Therefore, P_{FA} is given as

$$\Pr\left(\sum_{i=1}^{N} y(i)s(i) > \gamma\right)$$

$$= \Pr(\mathcal{N}(0, \sigma^2 \|\bar{\mathbf{s}}\|^2) > \gamma)$$

$$= \Pr\left(\frac{\mathcal{N}(0, \sigma^2\|\bar{s}\|^2)}{\sigma\|\bar{s}\|} > \frac{\gamma}{\sigma\|\bar{s}\|}\right)$$

Performance

- Therefore, P_{FA} is given as

$\mathcal{N}(0,1)$

$$= \Pr\left(\frac{\mathcal{N}\left(0, \sigma^2 \|\bar{\mathbf{s}}\|^2\right)}{\sigma \|\bar{\mathbf{s}}\|} > \frac{\gamma}{\sigma \|\bar{\mathbf{s}}\|} \right)$$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{\gamma}{\sigma \|\bar{s}\|} \right) = Q\left(\frac{\gamma}{\sigma \|\bar{s}\|} \right) = P_{FA}$$

Performance

- We use the result

mean

var $= \sigma^2$

std.dev $= \sigma$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$\Rightarrow \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

Standard Gaussian RV

mean $= 0$

var $= 1$.

Performance

- Therefore, P_{FA} is given as

$$= \Pr\left(\frac{\mathcal{N}(0,\sigma^2\|\bar{\mathbf{s}}\|^2)}{\sigma\|\bar{\mathbf{s}}\|} > \frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = \underline{Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)}$$

P_{FA}.

Gaussian Q-function

- What is $Q(\cdot)$?
- This is the **Gaussian Q function**

Gaussian
Q function.

- $Q(\cdot)$ is the **Complementary Cumulative Distribution Function** (CCDF) of the standard Gaussian RV $\mathcal{N}(0,1)$

CDF = cumulative distribution function = $Pr(X \leq x)$

CCDF = $Pr(X > x) = 1 - Pr(X \leq x)$.

- **Standard Gaussian RV**: Mean $\mu = 0$, Variance $\sigma^2 = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

PDF of standard Gaussian RV.
$\mu = 0$
$\sigma^2 = 1$.

- $Q(\cdot)$ is the **Complementary Cumulative Distribution Function** (CCDF) of the standard Gaussian RV

- **Standard Gaussian RV**: Mean $\mu = 0$, Variance $\sigma^2 = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

- $Q(x)$ is defined as

CCDF of Standard Gaussian RV $\mathcal{N}(0,1)$.

Complementary cumulative distribution function.

$$P(X \geq x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = Q(x)$$

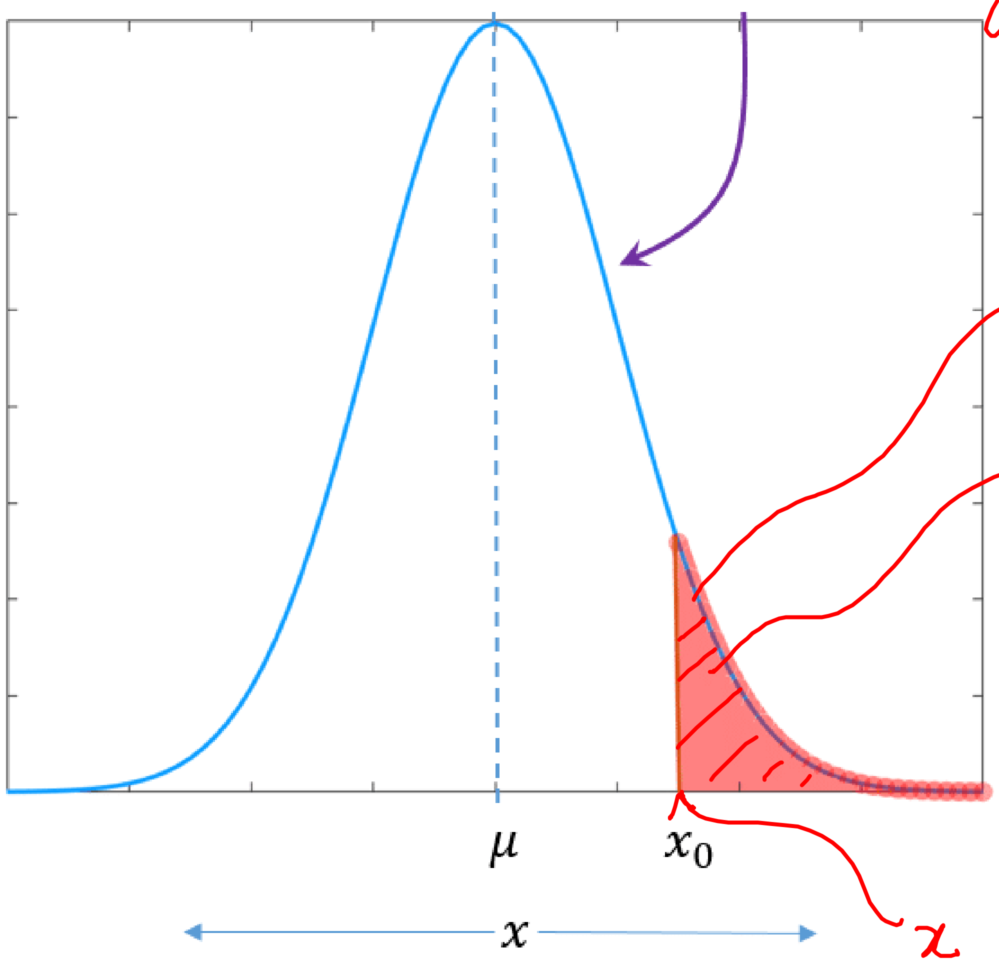
- $Q(x)$ is defined as

$$\Pr\left(\mathcal{N}(0,1) \geq \frac{\gamma}{\sigma\|S\|}\right)$$

$$P(X \geq x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$Q\left(\frac{\gamma}{\sigma\|S\|}\right) = \int_{\frac{\gamma}{\sigma\|S\|}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$P_{FA} = Q\left(\frac{\gamma}{\sigma \|s\|}\right).$$



Gaussian Q function

Tail Probability of standard Gaussian RV.

Area under PDF of standard Gaussian RV

x

Performance

Given $\mathcal{H}_1$ ie Signal is Present

Decision is Signal Present

- **Probability of Detection: P_D**
- Probability that the test ***correctly detects the presence*** of signal under $\mathcal{H}_1$.

Under $\mathcal{H}_1$, decision is $\mathcal{H}_1$, what is probability?

Performance

under $\mathcal{H}$,

$y(i) = s(i) + v(i)$

- This happens when under $\mathcal{H}_1$

$\bar{s}^T \bar{y} > \gamma$

$$\sum_{i=1}^{N} y(i)s(i) > \gamma$$

what Prob?

$\Rightarrow \bar{s}^T(\bar{s} + \bar{v}) > \gamma$

ideally P_D very high ≈ 1

Performance

- This happens when under $\mathcal{H}_1$

$$\overbrace{\sum_{i=1}^{N} y(i)s(i) > \gamma}^{\text{under } \mathcal{H}_1.}$$

Performance

- Note under $\mathcal{H}_1$

$\sim \mathcal{N}(s(i),\sigma^2)$ $\qquad$ $\sim \mathcal{N}(0,\sigma^2)$

$y(i) = s(i) + v(i)$

shifts mean

Gaussian

$$y(i) \sim \mathcal{N}(s(i), \sigma^2)$$

mean = $s(i)$

var = σ^2

Performance

- This implies that

$$E\left\{\sum_{i=1}^{N} y(i)s(i)\right\} = E\left\{\sum_{i=1}^{N}\left(v(i)+s(i)\right)s(i)\right\}$$

Linear combination of Gaussian RVs. $y(i)$
$= \sum y(i)s(i)$ also Gaussian.
mean $= \|\bar{s}\|^2$

$$= \sum_{i=1}^{N} s^2(i) + s(i)\cancel{E\{v(i)\}}^{0} = \sum_{i=1}^{N} s^2(i) = \|\bar{s}\|^2$$

Performance

- This implies that

$$E\left\{\sum_{i=1}^{N} y(i)s(i)\right\} = \sum_{i=1}^{N} s(i)E\{s(i) + v(i)\}$$

$$= \sum_{i=1}^{N} s^2(i) = \|\bar{\mathbf{s}}\|^2$$

Performance

$E\{(X-\mu)^2\}$.

- The variance

mean

$$E\left\{\left(\sum_{i=1}^{N} y(i)s(i) - \sum_{i=1}^{N} s^2(i)\right)^2\right\} = E\left\{\left(\sum_{i=1}^{N} s(i)\left(y(i)-s(i)\right)\right)^2\right\}$$

$$= E\left\{\left(\sum_{i=1}^{N} y(i)v(i)\right)^2\right\} = \sigma^2\|s\|^2$$

under H_1, $\underline{\text{var} = \sigma^2\|s\|^2}$.

Performance

- This implies that

$$E\left\{\left(\sum_{i=1}^{N} s(i)\big(y(i)-s(i)\big)\right)^2\right\} = E\left\{\left(\sum_{i=1}^{N} s(i)v(i)\right)^2\right\}$$

$$= \sum_{i=1}^{N} s^2(i)E\{v^2(i)\} = \sigma^2\|\bar{\mathbf{s}}\|^2$$

Performance

- Therefore

under $\mathcal{H}_1$

std. dev. $= \sigma \|\bar{s}\|$.

Gaussian mean $= \|\bar{s}\|^2$

var $= \sigma^2 \|\bar{s}\|^2$

$$\sum_{i=1}^{N} s(i)y(i) \sim \mathcal{N}(\|\bar{\mathbf{s}}\|^2, \sigma^2 \|\bar{\mathbf{s}}\|^2)$$

Performance

- Therefore, P_D is given as

$$\Pr\left(\sum_{i=1}^{N} s(i)y(i) > \gamma\right)$$

$$= \Pr\left(\mathcal{N}\left(\|s\|^2, \sigma^2\|s\|^2\right) > \gamma\right)$$

Performance

$$\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

- Therefore, P_D is given as

$$\Pr\left(\sum_{i=1}^{N} s(i)y(i) > \gamma\right)$$

$$= \Pr\left(\mathcal{N}\left(\|\bar{\mathbf{s}}\|^2, \sigma^2\|\bar{\mathbf{s}}\|^2\right) > \gamma\right)$$

$$= \Pr\left(\underbrace{\frac{\mathcal{N}\left(\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2\right) - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} > \frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

Performance

- Therefore, P_D is given as

$$= \Pr\left(\frac{\mathcal{N}(\|\bar{\mathbf{s}}\|^2, \sigma^2\|\bar{\mathbf{s}}\|^2) - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|} > \frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$\mathcal{N}(0,1)$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) = Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

Performance

Probability of detection.

- Therefore, P_D is given as

$$= \Pr\left(\frac{\mathcal{N}(||\bar{\mathbf{s}}||^2, \sigma^2||\bar{\mathbf{s}}||^2) - ||\bar{\mathbf{s}}||^2}{\sigma||\bar{\mathbf{s}}||} > \frac{\gamma - ||\bar{\mathbf{s}}||^2}{\sigma||\bar{\mathbf{s}}||}\right)$$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{\gamma - ||\bar{\mathbf{s}}||^2}{\sigma||\bar{\mathbf{s}}||}\right) \underbrace{= Q\left(\frac{\gamma - ||\bar{\mathbf{s}}||^2}{\sigma||\bar{\mathbf{s}}||}\right)}_{P_D.}$$

Performance

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{s}\|}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

Performance

$Pr(\text{decision} = \mathcal{H}_1 \mid \mathcal{H}_0)$

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$Pr(\text{decision} = \mathcal{H}_1 \mid \mathcal{H}_1)$

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

characterize performance of Test!

Performance

- What is role of γ?

Threshold.

What is the role played by γ

- Closer Examination:

mean $= ||\bar{s}||^2$

$$\mathcal{H}_1 : \bar{\mathbf{y}}^T\bar{\mathbf{s}} \sim \mathcal{N}(||\bar{\mathbf{s}}||^2, \sigma^2||\bar{\mathbf{s}}||^2)$$
$$\mathcal{H}_0 : \bar{\mathbf{y}}^T\bar{\mathbf{s}} \sim \mathcal{N}(0, \sigma^2||\bar{\mathbf{s}}||^2)$$

mean $= 0$

Observe only means are different!

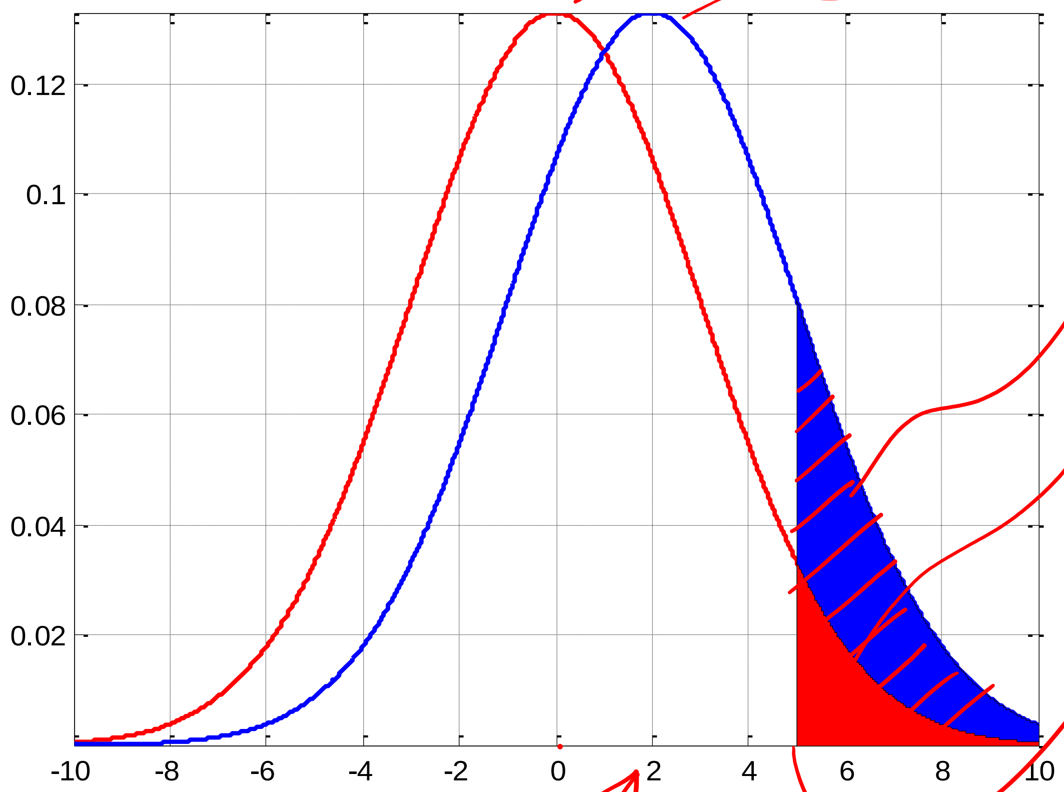
Performance

Two Gaussians shifted means!

$\mathcal{H}_0$ $\mathcal{H}_1$

Purple = P_D
Area $\mathcal{H}_1$.

Red Area = P_{FA}
$\mathcal{H}_0$



γ

$\|s\|^2$

$\gamma \to -\infty$

Both areas $\to 1$

$\Rightarrow P_D \to 1$

$P_{FA} \to 1$.

As $\gamma \to \infty$

Both areas $\to 0$

$\Rightarrow P_D \to 0$

$P_{FA} \to 0$.

Performance

- What is role of γ?
- Case 1: $\gamma = \infty$
- $P_D = 0$
- $P_{FA} = 0$

$P_D = 0$

$P_{FA} = 0$

P_D very poor!

P_{FA} very good!

Performance

- What is role of γ?
- Case 1: $\gamma = -\infty$
- $P_D = 1$

Very good!

- $P_{FA} = 1$

very poor!

Performance

- What is role of γ?
- γ helps tradeoff P_D versus P_{FA}

γ helps tradeoff P_D vs P_{FA}.

γ high $\Rightarrow$ P_D low, P_{FA} low.

γ low $\Rightarrow$ P_D high, P_{FA} high.

Receiver Operating Characteristic

ROC

- The receiver operating characteristic (ROC) is P_D as a function of P_{FA} — Eliminate γ

Receiver Operating Characteristic

- Given P_{FA}

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\mathbf{s}\|}\right)$$

$$\Rightarrow \gamma = \sigma\|\bar{s}\|\, Q^{-1}(P_{FA})$$

Receiver Operating Characteristic

- Given P_{FA}

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$\Rightarrow \gamma = \sigma\|\bar{\mathbf{s}}\|Q^{-1}(P_{FA})$$

Receiver Operating Characteristic

- Therefore

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$\Rightarrow P_D = Q\left(\frac{\sigma\|\bar{s}\| Q^{-1}(P_{FA}) - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

Receiver Operating Characteristic

$\frac{\|\bar{s}\|^2}{\sigma^2} = SNR$

- Therefore

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\|\bar{s}\|^2}{\sigma^2}}\right)$$

$$\underbrace{Q\left(Q^{-1}(P_{FA}) - \sqrt{SNR}\right)}_{ROC.}$$

$$\Rightarrow P_D = Q\left(\frac{\sigma\|\bar{\mathbf{s}}\|Q^{-1}(P_{FA}) - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) - \frac{\|\bar{s}\|}{\sigma}\right)$$

Receiver Operating Characteristic

- Therefore

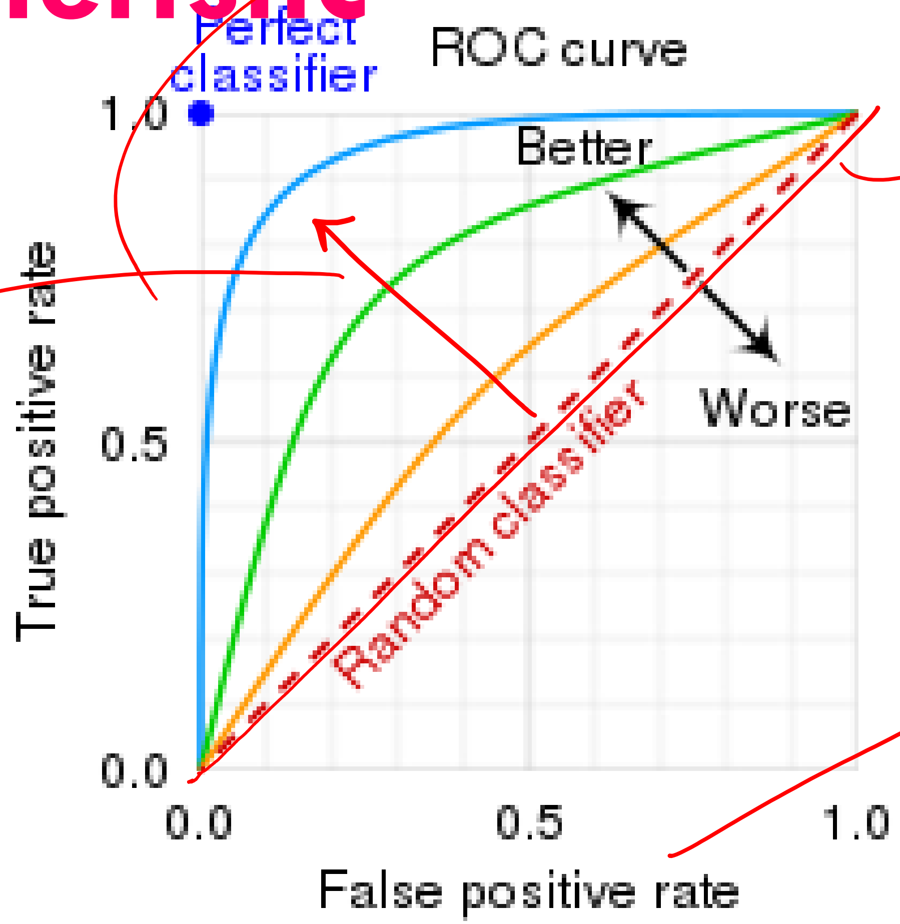
$$\Rightarrow P_D = Q\left(Q^{-1}(P_{FA}) - \frac{\|\bar{\mathbf{s}}\|}{\sigma}\right)$$

ROC = Receiver operating characteristic

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\|\bar{\mathbf{s}}\|^2}{\sigma^2}}\right) = Q\left(Q^{-1}(P_{FA}) - \sqrt{SNR}\right)$$

$$SNR = \frac{\|\bar{s}\|^2}{\sigma^2}$$ } Signal to noise power ratio.

Receiver Operating Characteristic



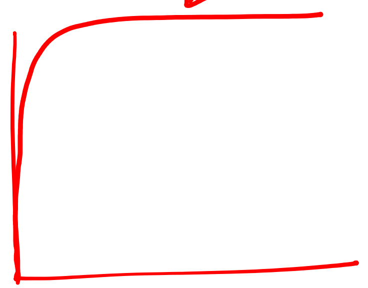
P_D

Given P_{FA} we want maximum P_D.

45°.

$P_{FA} = P_D$.

ideal ROC.



Better Performance.

P_{FA}

Performance ML

Maximum Likelihood.

choose $\mathcal{H}_0$ if $\dfrac{p(\bar{y};\mathcal{H}_0)}{p(\bar{y};\mathcal{H}_1)} \geq 1$ $\tilde{\gamma}=1$

- Note that for ML $\tilde{\gamma} = 1$

$\gamma = ?$

$$\gamma = \frac{\|\bar{\mathbf{s}}\|^2 - 2\sigma^2 \underbrace{\ln \tilde{\gamma}}_{=0}}{2} = \frac{\|\bar{\mathbf{s}}\|^2}{2} = \gamma$$

For $\gamma = \dfrac{\|\bar{s}\|^2}{2}$

LRT reduces to ML

Performance ML

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\|\bar{s}\|^2/2}{\sigma\|\bar{s}\|}\right) = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)$$

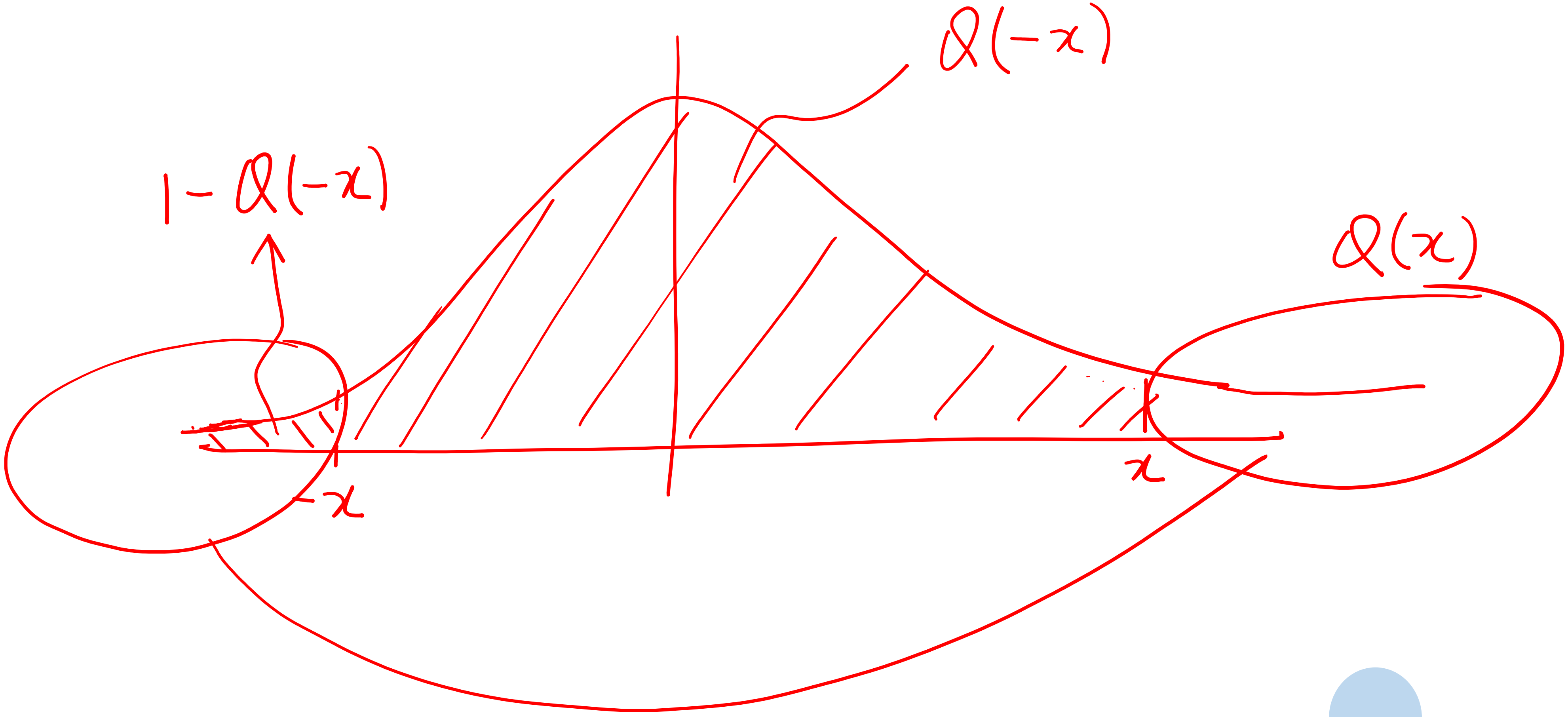
$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\|\bar{s}\|^2/2 - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) = Q\left(-\frac{\|\bar{s}\|}{2\sigma}\right)$$

$$P_{MD} = 1 - P_D = 1 - Q\left(-\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)$$

$$1 - Q(-x) = Q(x)$$

Performance ML

$$\underbrace{P(X \leq -x)}_{1-Q(-x)} = \underbrace{P(X \geq x)}_{Q(x)}$$



Performance ML

- Therefore, P_{FA}, P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\|\bar{\mathbf{s}}\|^2/2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right)$$

$$= Q\left(\sqrt{\frac{\|\bar{s}\|^2}{4\sigma^2}}\right)$$

$$= Q\left(\frac{\sqrt{SNR}}{2}\right).$$

$$P_D = Q\left(\frac{\gamma - \|\bar{\mathbf{s}}\|^2}{\sigma\|\bar{\mathbf{s}}\|}\right) = Q\left(-\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right)$$

$$P_{MD} = 1 - P_D = Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right)$$

Performance ML

Both hypotheses.
EQUIPROBABLE.

- Consider

$$\Pr(\mathcal{H}_0) = \Pr(\mathcal{H}_1) = \frac{1}{2}$$

Performance ML

Probability of error

- Therefore, P_e is

$$\Pr(\mathcal{H}_0)\overbrace{\Pr(\mathcal{H}_1|\mathcal{H}_0)}^{P_{FA}}$$
$$+\Pr(\mathcal{H}_1)\overbrace{\Pr(\mathcal{H}_0|\mathcal{H}_1)}^{P_{MD}}$$

$$= \Pr(H_0)P_{FA} + \Pr(H_1)P_{MD} = \frac{1}{2}P_{FA} + \frac{1}{2}P_{MD}.$$

Performance ML

- Therefore, P_e *is*

$$\Pr(\mathcal{H}_0)\Pr(\mathcal{H}_1|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\Pr(\mathcal{H}_0|\mathcal{H}_1)$$

Probability of error

Overall prob of erroneous decision.

$$\frac{1}{2}P_{FA} + \frac{1}{2}P_{MD} = Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right)$$

Example

ASK. #symbols = 2 $\{0, A\}$ #bits/sym $= \log_2 2 = 1$.

- Consider Amplitude shift keying
- $s = A$ N = 1.

$$\mathcal{H}_0 : y = v$$

$$\mathcal{H}_1 : y = A + v$$

Example

- Therefore, P_e *is*

In communication

$$\sigma^2 = \frac{N_0}{2}$$

$$Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right) = Q\left(\frac{A}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{A^2}{2N_0}}\right)$$

Example

- Therefore, P_e *is*

$$Q\left(\frac{\|\bar{\mathbf{s}}\|}{2\sigma}\right) = Q\left(\frac{A}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{A^2}{2N_0}}\right)$$

Example

Average energy/bit

E_b.

- Each symbol carries one bit
- Let E_b be the energy per bit
- Let each symbol be equiprobable

$\Pr(\mathcal{H}_0)$ $\Pr(\mathcal{H}_1)$

$$\Pr(0) = \Pr(A) = \frac{1}{2}$$

Example

Average energy

energy for $\mathcal{H}_0$

Energy $\mathcal{H}_1$

$$E_b = \Pr(\mathcal{H}_0) \times 0 + \Pr(\mathcal{H}_1) \times A^2$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} A^2 = \frac{1}{2} A^2$$

$$\Rightarrow A^2 = 2 E_b$$

$$\Rightarrow A = \sqrt{2 E_b}.$$

Example

$$E_b = \frac{1}{2} \times 0 + \frac{1}{2} \times A^2$$

$$\Rightarrow A^2 = 2E_b$$

Example

- Therefore, P_e *is*

$$Q\left(\sqrt{\frac{A^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Bit Error Rate for ASK.

- This is termed as bit error rate

Example

BER: Very important for comm.

- Therefore, P_e *is*

$$Q\left(\sqrt{\frac{A^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- This is termed as bit error rate (BER)

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Font: Avenir (Book), Size: 28, Colour: Dark Grey

Font: Avenir (Book), Size: 24, Colour: Dark Grey

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