Integer Programming dual

We may not dualize all the constraints

Integer Program :

$$\min_x c^T x \qquad x \in \mathbb{R}^n$$
$$Ax \le b$$
$$x_i \in \{0,1\} \leftarrow \text{not dualize}$$

NP-hard

time $= O(e^n)$

define $f(x) = \{ c^T x \mid x_i \in \{0,1\} \}$

$\operatorname{dom} f = \{0,1\}^n$

$$\min_x f(x) \qquad \text{(non-convex obj)}$$
$$Ax \le b \;\ldots\; \lambda \in \mathbb{R}^m$$

$$L(x,\lambda) = c^T x + \lambda^T (Ax - b)$$

$$\operatorname{dom} L = \underbrace{\{0,1\}^n}_{x} \times \underbrace{\mathbb{R}^m}_{\lambda}$$

$$\mathbb{B}^n \times \mathbb{R}^m$$

$$g(\lambda) = \min_{x \in \mathbb{B}^n} L(x,\lambda) = \min_{x \in \mathbb{B}^n} c^T x + \lambda^T (Ax - b)$$

$$= \min_{x \in \mathbb{B}^n} (c + A^T\lambda)^T x - b^T\lambda$$

$$= \min_{x_i \in \{0,1\}} \sum_{i=1}^{n} [c + A^T\lambda]_i x_i - b^T\lambda$$

$$= \sum_{i=1}^{n} \min_{x_i \in \{0,1\}} [c + A^T\lambda]_i x_i - b^T\lambda$$

<u>Aside</u>

$$\min_{x_i \in \{0,1\}} h_i x_i$$

pick $x_i^* = \begin{cases} 0 & h_i \geq 0 \\ 1 & h_i \leq 0 \end{cases}$

so $h_i x_i = \begin{cases} 0 & h_i \geq 0 \\ h_i & h_i \leq 0 \end{cases} = \min\{h_i, 0\}$

so $$g(\lambda) = \sum_{i=1}^{m} \min\left([c + A^T\lambda]_i, 0 \right) - b^T\lambda$$

Dual problem $$\max_{\lambda \geq 0} g(\lambda)$$

$$= \max_{t, \lambda} \sum_{i=1}^{m} t_i - b^T\lambda$$

$[c + A^T\lambda]_i \geq t_i$

$0 \geq t_i$

$\min\left([c + A^T\lambda]_i, 0 \right) \geq t_i \quad i = 1 \ldots m$

Dual of IP : $D = \max_{t,\lambda} \ \mathbb{1}^T t - b^T \lambda$ (L.P)

$$c + A^T \lambda \geq t$$
$$t \leq 0$$

from weak duality : $P \geq D$ ← can be easily found

practically used a lot for solving IP

Takeaways :
- some constraints may be included in dom
- need not dualize them
- get different dual problems