Eigenvalues and Eigenvectors of Symmetric Matrices

Rohit Budhiraja, IITK

Applied Linear Algebra for Wireless Communications

Recap and agenda for today's class

- Discussed eigenvalues and eigenvectors for a generic matrix A
- Discuss these concepts for symmetric matrices
    - Chapter 6.4 of the book

Eigenvalues, Eigenvectors, Diagonalization (recap)

- Almost all vectors change direction when they are multiple by A i.e., $A\mathbf{x} = \mathbf{y}$
- Certain exceptional vectors $\mathbf{x}$ are in the same direction as Ax

$$A\mathbf{x} = \lambda\mathbf{x}$$

- $\mathbf{x}$ is an eigenvector and λ is eigenvalue
  - λ tell us whether the eigenvector is shrunk or stretched or left unchanged
  - λ can be complex also
- Suppose $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$
  - We have $AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$

Eigenvalues, Eigenvectors of Symmetric Matrix

- What is special about $S\mathbf{x} = \lambda\mathbf{x}$ when S is real and symmetric i.e., $S = S^T$
  - Symmetric matrix has real eigenvalues
  - Eigenvectors are chosen orthonormal when they correspond to different eigenvalues
- Why do we use the word "choose"?
  - Because eigenvectors need not be unit vectors, we can decide their lengths
  - We will choose eigenvectors of length one, which are orthonormal
- Suppose $S\mathbf{x} = \lambda\mathbf{x}$, $\lambda = a + ib$ might be complex, with conjugate $\bar{\lambda} = a - ib$
- Components of $\mathbf{x}$ may be complex
  - We change signs of their imaginary parts and denote it as $\bar{\mathbf{x}}$
- Note that $\bar{\lambda}\bar{\mathbf{x}} = \overline{\lambda\mathbf{x}}$ and $\overline{S\mathbf{x}} = \overline{S}\bar{\mathbf{x}} = S\bar{\mathbf{x}}$

Eigenvalues of symmetric matrix are real

- We can take conjugate of $S\mathbf{x} = \lambda\mathbf{x}$, with S being real

$$\overline{S\mathbf{x}} = \overline{\lambda\mathbf{x}} \text{ leads to } S\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}\text{. Transposes to } \bar{\mathbf{x}}^T S = \bar{\mathbf{x}}^T \bar{\lambda}$$

- Take inner product of the first equation with $\bar{\mathbf{x}}$ and last equation with $\mathbf{x}$

$$\bar{\mathbf{x}}^T S \bar{\mathbf{x}} = \bar{\mathbf{x}}^T \lambda \bar{\mathbf{x}} = \lambda ||\bar{\mathbf{x}}||^2$$
$$\bar{\mathbf{x}}^T S \bar{\mathbf{x}} = \bar{\mathbf{x}}^T \bar{\lambda} \bar{\mathbf{x}} = \bar{\lambda} ||\bar{\mathbf{x}}||^2$$

- This implies that $\lambda = \bar{\lambda}$, which implies that λ is real
- Eigenvector come from solving $(S - \lambda I)\mathbf{x} = \mathbf{0}$, they are also real
- Important fact is that they are also perpendicular which we prove now

Eigenvectors are chosen orthogonal

- Suppose $S\mathbf{x} = \lambda_1 \mathbf{x}$ and $S\mathbf{y} = \lambda_2 \mathbf{y}$, and we have
$$(\lambda_1 \mathbf{x})^T \mathbf{y} = (S\mathbf{x})^T \mathbf{y} = \mathbf{x}^T S^T \mathbf{y} = \mathbf{x}^T S \mathbf{y} = \mathbf{x}^T \lambda_2 \mathbf{y} = \lambda_2 \mathbf{x}^T \mathbf{y} \tag{1}$$
- We also have
$$(\lambda_1 \mathbf{x})^T \mathbf{y} = \mathbf{x}^T \lambda_1 \mathbf{y} = \lambda_1 \mathbf{x}^T \mathbf{y} \tag{2}$$
- LHS of Eq. (1)= LHS of Eq. (2). RHS of Eq. (1)= RHS of Eq. (2):
$$\lambda_2 \mathbf{x}^T \mathbf{y} = \lambda_1 \mathbf{x}^T \mathbf{y} \Rightarrow (\lambda_2 - \lambda_1)\mathbf{x}^T \mathbf{y} = 0$$
- Since $\lambda_1 \neq \lambda_2$, we have $\mathbf{x}^T \mathbf{y} = 0$
- $\mathbf{x}$ and $\mathbf{y}$ may not have unit length but we can choose them to have so
$$S\mathbf{x} = \lambda_1 \mathbf{x} \Rightarrow S \frac{\mathbf{x}}{||\mathbf{x}||} = \lambda_1 \frac{\mathbf{x}}{||\mathbf{x}||}$$

Eigenvectors are chosen orthogonal

- Symmetric matrix S has orthogonal eigenvector matrices Q
- Every 2 by 2 symmetric matrix S can be decomposed as

$$\begin{aligned} S &= Q\Lambda Q^{-1} = Q\Lambda Q^T = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} \\ &= \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T \end{aligned}$$

- Every n by n symmetric matrix S can be similarly decomposed as

$$S = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \cdots + \lambda_n \mathbf{x}_n \mathbf{x}_n^T$$

- Spectral Theorem: Every symmetric matrix can be factorized as $S = Q\Lambda Q^T$ with real eigenvalues in λ and orthonormal eigenvectors in the columns of Q

Complex Eigenvalues of Real Matrices

- For a real non-symmetric matrix, complex eigenvalues and eigenvectors occur in pair

$$\text{If } A\mathbf{x} = \lambda\mathbf{x} \text{ then } A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$$

- For $A = \begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}$, $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$
- Eigenvectors must be $\mathbf{x}$ and $\bar{\mathbf{x}}$ as A is real

$$A\mathbf{x} = \begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (cos\,\theta + i\ sin\,\theta) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$A\bar{\mathbf{x}} = \begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (cos\,\theta - i\ sin\,\theta) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

All Symmetric Matrices are Diagonalizable

- When no eigenvalues of A are repeated, eigenvectors are independent
  - A can be diagonalized
- But a repeated eigenvalue can produce a shortage of eigenvectors
  - This sometimes happens for nonsymmetric matrices
- It never happens for symmetric matrices
  - There are always enough eigenvectors to diagonalize