EE908 Assignment-5 Solution

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eMasters in Communication Systems, IITK
EE908: Optimization in SPCOM
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Q1. Consider the following linear program $\min c^T x \; s.t. Ax \leq b$ where A is square and full rank

(a) When is the problem infeasible

Solution:

When the set of constraints represented by $Ax \leq b$ cannot be satisfied by any vector x.

Few scenarios this can happen are:

- If there is no x such that $x \leq A^{-1}b$ (A is square and full rank ⇒ invertible) – Empty feasible region or b is NOT a combination of columns of A i.e. $b \notin \mathcal{R}(A) \Rightarrow$ The optimal value is ∞
- If A, x, c, b have incompatible dimensions - $x, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
- When the solution is infeasible, the solution to the problem, is said to have a value of ∞

(b) When is the problem unbounded below

Solution:

- When a feasible solution exists to this problem and the objective function $c^T x < 0$ is such that it decreases indefinitely in the feasible region.
- This condition occurs when $x < 0 \; and \; c > 0 \; or \; x > 0 \; and \; c < 0$ and the constraint is satisfied i.e for any vector $x \; s.t. Ax \leq b$.
- When the solution is unbounded below, the problem is said to have a solution value of $-\infty$

(c) When does the problem have a finite solution, and what is it?

Solution:

- The LP will have a finite solution when the objective function $c^T x$ has a definitive minimum value achievable/solvable within the feasible and bounded region/solutions for $any \; x \; s.t. Ax \leq b$.
- The finite solution for this problem is an optimal value:
  $x^* = argmin \{c^T x \mid Ax \leq b\}$ or $x^* = \min_x c^T x \;\; s.t. Ax \leq b$

Q2. Show that any linear programming problem can be expressed as $\min c^T x \;\; s.t. Ax = b, x_i \geq 0, i = 1,2, \ldots n$

Solution:

The general form of optimization is:

$x^* = \min_x c^T x$

$s.t. Wx \leq c, i = 1,2, \ldots m$ – inequality constraint (less than)

$Dx = h$ – equality constraints (x can be +ve or -ve)

To convert above general form to LP form, we need to use two tricks

1. Use ***slack variable s*** to convert the inequality to equality
   $Wx \leq c \Rightarrow Wx + s = c \; s.t. \;\; s \geq 0$
   $[W \quad I] \begin{bmatrix} x \\ s \end{bmatrix} = c \Rightarrow Ay = c \; , A = [W \; I], y = \begin{bmatrix} x \\ s \end{bmatrix}$
2. To bring x into the positive domain select $y = \begin{bmatrix} x \\ s \end{bmatrix}, x \geq 0, s > 0,$

$y = u - v\ select\ u, v\ s.t\ \ u \geq 0, v \geq 0, u \geq v$

Hence,

$\min c^T y$

$s.t. Ay = b, y_i \geq 0, i = 1,2, \ldots n$

QED

Q3. Conser the following linear program $\min c^T x\ s.t. Ax = b$

(a) When is the problem infeasible?

Solution:

The solution is infeasible when the given constraint is not met:

- When $\{x \mid Ax \neq b\}$ i.e. there is no vector x that satisfies all constraints $Ax = b$ – Unmet or inconsistent constraints – $b \notin dom\ Ax$
- If some of the constraints of $Ax = b$, $A_{12}x_1 = b_1, A_{22}x_2 = b_2$ etc. are met but some are not met $A_{11}x_1 \neq b_1$ - Contradictory constraints
- When the solution is infeasible, the solution to the given minimization problem is said to have a value of ∞

(b) When is the problem unbounded below

Solution:

- When the constraints $Ax = b$ are met (feasible solution exists) and the objective function $c^T x < 0$ is such that it decreases indefinitely in the feasible region.
- This condition occurs when $x < 0\ and\ c > 0\ or\ x > 0\ and\ c < 0$ and the constraint $Ax = b$ is satisfied for any vector x i.e. $\{b \in Ax, b \in \mathbb{R}^m, Ax \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n\}$.
- When the solution is unbounded below, the problem is said to have a solution value of $-\infty$

(c) When does the problem have a finite solution, and what is it?

Solution:

- The linear program $\min c^T x$ has a finite solution when a feasible solution exists and bounded (not un-bounded below) i.e. $\exists x\ s.t. Ax = b$ and the objective function $c^T x$ has a lower bound within the feasible region - cannot decrease indefinitely without violating the given constraint
- The finite solution exists when $x > 0\ and\ c > 0$, the constraint $Ax = b$ is satisfied for any vector x $\{b \in Ax, b \in \mathbb{R}^m, Ax \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n\}$ and the minimum value of the objective function for this condition is **0**.

Q4. Conser the following linear program $\min c^T x\ s.t. a^T x \leq b$

(a) When is the probe infeasible?

Solution:

The solution is infeasible when the given constraint is not met:

- When $\{x \mid a^T x > b\}$ i.e. there are no vectors $x\ and\ a$ that satisfy the constraint $a^T x \leq b$ – Unmet constraint
- When the solution is infeasible, the solution to the given minimization problem is said to have a value of ∞

(b) When is the problem unbounded below

Solution:

- When the constraints $a^x \leq b$ is met (feasible solution exists) and the objective function $c^T x < 0$ is such that it decreases indefinitely in the feasible region.

- This condition occurs when $x < 0 \; and \; c > 0 \; or \; x > 0 \; and \; c < 0$ and the constraint $a^T x \leq b$ is satisfied.
- When the solution is unbounded below, the problem is said to have a solution value of $-\infty$

(c) When does the problem have a finite solution, and what is it?
Solution:

- The linear program $\min c^T x$ has a finite solution when a feasible solution exists and bounded (not un-bounded below) i.e. $\exists x \; s.t. \, a^T x \leq b$ and the objective function $c^T x$ has a lower bound within the feasible region - cannot decrease indefinitely without violating the given constraint
- The finite solution exists when $x > 0 \; and \; c > 0$, the constraint $a^T x \leq b$ is satisfied for any vector x and the minimum value of the objective function for this condition is **0**.

Q5. Solve the following optimization problem for $A \succ 0, \min x^T A x \; s.t. \|x\|_2^2 = 1$
Hint: Given that the eigenvalue decomposition $A = U\Sigma U^T$, use the change of variable $y = U^T x$
Solution:

- The minimization objective program is a quadratic program
- However, the equality constraint $\|x\|_2^2$ – level-2 norm - is a non-linear constraint and let's use change of variable as suggested in the hint

Say $y = U^T x \Rightarrow x = Uy$
The objective function $x^T A x = x^T U \Sigma U^T x = (U^T x)^T \Sigma (U^T x) = y^T \Sigma y$
Constraint $\|x\| = x^T x = (Uy)^T (Uy) = y^T U^T U y = y^T y = \|y\|$ $(U \; is \; Orthogonal \Rightarrow U^T U = I)$
$\Rightarrow \|x\|_2^2 = \|y\|_2^2$
Now the optimization problem becomes - $\min\limits_{y} y^T \Sigma y \;\; s.t. \|y\|_2^2 = 1$
Since Σ is a diagonal matrix has positive eigenvalues $(A \succ 0)$ in its diagonal, the minimum of $y^T \Sigma y$ occurs when y corresponds to the eigenvector associated with the smallest eigenvalue of Σ.

Let $\lambda_{min}(A)$ be the smallest eigenvalue of Σ and u_{mim} be the corresponding eigenvector.
Then the solution to minimization problem is $y = u_{min}$
$x = Uy$ and $y = u_{min}$, the solution is $x = Uu_{min}$

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