EE901
Probability and Random Processes

Module 6
Multiple Random Variables

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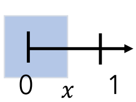
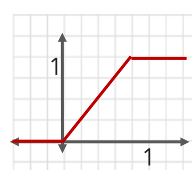
Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$X(\omega) = \omega$ for each $\omega \in \Omega$.



CDF is $F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$



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Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$Y(\omega) = 1 - \omega$ for each $\omega \in \Omega$.

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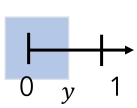
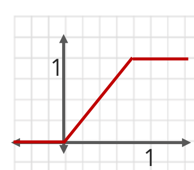


CDF is $F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$



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Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$Y(\omega) = 1 - \omega$ for each $\omega \in \Omega$.

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

$Y(\omega) = 1 - X(\omega)\ \forall\, \omega$

$Y = X$

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Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$Z(\omega) = \omega^2$ for each $\omega \in \Omega$.

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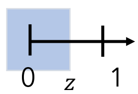
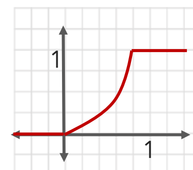


CDF is $F_Z(x) = \begin{cases} 0 & z \le 0 \\ \sqrt{z} & 0 < z < 1 \\ 1 & z \ge 1 \end{cases}$



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Example: Pick a Number

Pick a number in (0,1) Probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$Z(\omega) = \omega^2$ for each $\omega \in \Omega$.

$Z(\omega) = X(\omega)^2$ for each $\omega \in \Omega$. $Z = X^2$

$X(\omega) = \sqrt{Z(\omega)}$, for each $\omega \in \Omega$. $X = \sqrt{Z}$

$Y = 1 - \sqrt{Z}$

$Z = (1 - Y)^2$

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Multiple RVs on the same Probability Space

- There can be many RVs defined on the same probability space.

$$X(\omega) = \omega, \qquad Y(\omega) = 1 - \omega, \qquad Z(\omega) = \omega^2$$

- In some cases, we can express one random variable using other random variables.

$$Y = 1 - X, \qquad Y = 1 - \sqrt{Z}$$
$$X = \sqrt{Z}, \qquad X = 1 - Y$$

- In some cases, it may not be possible to determine the value of one RV from other.
  - Consider the experiment to pick a number in the range $(-1,\ 1)$

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Multiple RVs on the same Probability Space

- Consider following RVs defined on the same probability space.

$$X(\omega) = \omega, \qquad Y(\omega) = 1 - \omega, \qquad Z(\omega) = \omega$$

- They all the same CDF, however,

$$X = Z, \qquad Y \neq Z \ \text{ instead } Y = 1 - Z$$

- How do we distinguish the two RVs?
- If we don't have knowledge about the definitions of two random variables, just by looking at their CDF we cannot infer the relation between them.
- We need to look at what values they take simultaneously. Some kind of joint distribution.

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Joint Distribution

- In addition, there are cases where there is no direct relation

Dice roll Probability space $(\Omega, 2^{\Omega}, \mathbb{P})$

$X(\omega) = 1$ if ω is even, otherwise 0

$Y(\omega) = 1$ if ω is less than 4, otherwise 0

Notice further that both X and Y have the same distribution, but they are not equal.

In fact there is no relation between the two.
You can not derive the exact value of Y from X. At least no always.

We can talk about the set of pair of values X and Y can take together and their corresponding probabilities.

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Joint Distribution

Dice roll. Probability space $(\Omega, 2^{\Omega}, \mathbb{P})$ $X(\omega) = 1$ if ω is even otherwise 0 $Y(\omega) = 1$ if $\omega < 4$ otherwise 0

Pair of values that X and Y can take together and their corresponding probabilities.

$X = 1, Y = 1.$ ω is even and less than 4 which means ω is 2. Probability = 1/6.

$X = 1, Y = 0.$ ω is even and not less than 4 which means ω is 4 or 6. Probability = 1/3.

$X = 0, Y = 1.$ ω is odd and less than 4 which means ω is 1 and 3. Probability = 1/3.

$X = 0, Y = 0.$ ω is odd and not less than 4 which means ω is 5. Probability = 1/6.

Not all values are equi-probable!!

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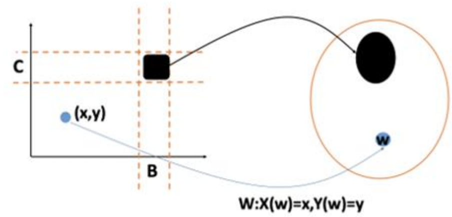
Joint Distribution of RVs

- Let X and Y be two random variables on the same probability space.

- The joint probability law is defined as



$$\mathbb{P}_{X,Y}(A) = \mathbb{P}(\{\omega : (X(\omega), Y(\omega)) \in A\})$$

if $A = B \times C$,

$$\mathbb{P}_{X,Y}(B \times C) = \mathbb{P}(\{\omega : X(\omega) \in B, Y(\omega) \in C\})$$

where B and C are some borel sets.

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Joint CDF

- The joint CDF is given by

$$\mathrm{F}_{XY}(x, y) = \mathbb{P}(\{\omega : \mathrm{X}(\omega) \leq x \text{ and } \mathrm{Y}(\omega) \leq y\})$$

- In other words, if $E_{x,y} = \{\omega : X(\omega) \leq x, Y(\omega) \leq y\}$
  then $F_{X,Y}(x, y) = \mathbb{P}(E_{x,y})$

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Example: Coin Toss

 

$$\Omega = \{HH, TH, HT, TT\}$$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = HH \text{ or } HT \\ 0 & \text{if } \omega = TH \text{ or } TT \end{cases}, \quad Y(\omega) = \begin{cases} 1 & \text{if } \omega = HH \text{ or } TH \\ 0 & \text{if } \omega = HT \text{ or } TT \end{cases}$$

ω	$X(\omega)$	$Y(\omega)$
HH	1	1
HT	1	0
TH	0	1
TT	0	0

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Example: Coin Toss

$$E_{x,y} = \{\omega : X(\omega) \le x, Y(\omega) \le y\}$$

ω	$X(\omega)$	$Y(\omega)$
HH	1	1
HT	1	0
TH	0	1
TT	0	0

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Example: Coin Toss

$$E_{x,y} = \{\omega : X(\omega) \le x, Y(\omega) \le y\}$$

ω	$X(\omega)$	$Y(\omega)$
HH	1	1
HT	1	0
TH	0	1
TT	0	0

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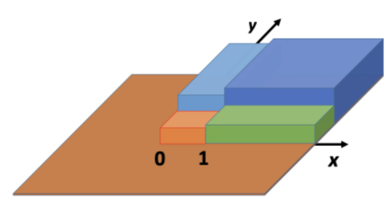
Example: Coin Toss

$$\mathbb{P}(\{HH\}) = p_1, \mathbb{P}(\{HT\}) = p_2$$

$$\mathbb{P}(\{TH\}) = p_3, \mathbb{P}(\{TT\}) = p_4$$

Then

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } , y < 0 \\ p_4 & 0 \le x < 1, 0 \le y < 1 \\ p_2 + p_4 & 1 \le x, 0 \le y < 1 \\ p_3 + p_4 & 0 \le x < 1, 1 \le y \\ 1 & 1 \le x, 1 \le y \end{cases}$$



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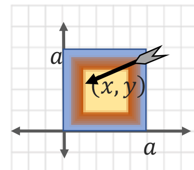
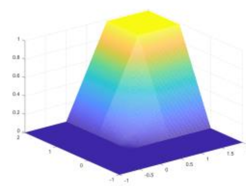
Example: Dart Throw

- Consider a random experiment where a dart is thrown on a board B. The outcome is the location where the dart hits the board. Board area is 1.
- $\Omega = B$. Each outcome ω is a 2D coordinate (x,y). Assume a uniform probability measure which means
  - $\mathbb{P}(A) = |A|$ for any set A on the board.
  - Let $X(\omega)$ and $Y(\omega)$ denote the x and y coordinate of the outcome.



- Joint CDF of X and Y can be computed as

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y) = $$
$$= \begin{cases} 0 & \text{if } x \text{ OR } y < 0 \\ \min(x,a)\min(y,a) & \text{if } a > x, y > 0 \\ 1 & \text{if } x, y > a \end{cases}$$



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Properties of Joint CDF

$$E_{x,y} = \{\omega : X(\omega) \le x, Y(\omega) \le y\} \qquad F_{X,Y}(x,y) = \mathbb{P}(E_{x,y})$$

$$\lim_{x\to\infty, y\to\infty} F_{X,Y}(x,y) = 1$$

$$\lim_{x\to-\infty, y\to-\infty} F_{X,Y}(x,y) = 0$$

$$\lim_{x\to\infty, y\to-\infty} F_{X,Y}(x,y) = 0$$

$$\lim_{x\to-\infty, y\to\infty} F_{X,Y}(x,y) = 0$$

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Properties of Joint CDF

If $x < x'$ and $y < y'$, then $F(x, y) \leq F(x', y')$.

$\lim_{x \to \infty} F_{X,Y}(x, y) = F_Y(y)$

$\lim_{y \to \infty} F_{X,Y}(x, y) = F_X(x)$

Joint CDF is right continuous.

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Joint PMF of DRVs

- Let X and Y be two DRVs on the same probability space.
- Then the joint PMF is defined as

$$\begin{aligned} p_{X,Y}(x, y) &= \mathbb{P}(X = x, Y = y) \\ &= \mathbb{P}(\{\omega : X(\omega) = x, Y(\omega) = y\}) \end{aligned}$$

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Example: Coin toss


$$\Omega = \{HH, TH, HT, TT\}$$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = HH \text{ or } HT \\ 0 & \text{if } \omega = TH \text{ or } TT \end{cases}, \quad Y(\omega) = \begin{cases} 1 & \text{if } \omega = HH \text{ or } TH \\ 0 & \text{if } \omega = HT \text{ or } TT \end{cases}$$

ω	$X(\omega)$	$Y(\omega)$
HH	1	1
HT	1	0
TH	0	1
TT	0	0

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Example: Coin toss

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

ω	$X(\omega)$	$Y(\omega)$
HH	1	1
HT	1	0
TH	0	1
TT	0	0

x	y
1	1
1	0
0	1
0	0

$E = \{X = x, Y = y\}$
{HH}
{HT}
{TH}
{TT}

$\mathbb{P}(\{X = x, Y = y\})$

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Example: Coin Toss

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

$$\mathbb{P}(\{HH\}) = p_1, \mathbb{P}(\{HT\}) = p_2$$
$$\mathbb{P}(\{TH\}) = p_3, \mathbb{P}(\{TT\}) = p_4$$

x	y	$E = \{X = x, Y = y\}$	$p_{X,Y}(x,y)$
1	1	{HH}	p_1
1	0	{HT}	p_2
0	1	{TH}	p_3
0	0	{TT}	p_4

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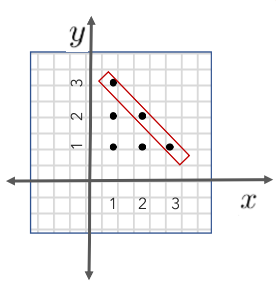
Probability Law in terms of PMF

- For any 2D set A, the probability that X, Y takes value in A is given as

$$\mathbb{P}_{X,Y}(A) = \sum_{(x,y)\in\mathcal{R}(X,Y)} p_{X,Y}(x,y)\mathbb{1}((x,y) \in A)$$



Consider the PMF. Each point is equi-probable.
Compute the probability that X and Y are equal.

Here, $A = \{(x,y)\colon x = y\}$ which is a line.

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