

STAT 302 2023/24 Winter Term 2  
Module 6 worksheet 1

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## New concepts

Continuous RV; uniform distribution; expected values; function of continuous RV

## Thought experiment

Imagine the following setup: you take a perfect string of length  $l$ , put a coordinate system along its length, with the origin at one end. Next, imagine having to cut the string at a *randomly selected* location, in a way that any location is **equally likely** to be chosen. Let the random variable  $X$  represent the location of the cut.

1. What do you think is the probability of the cut falling:

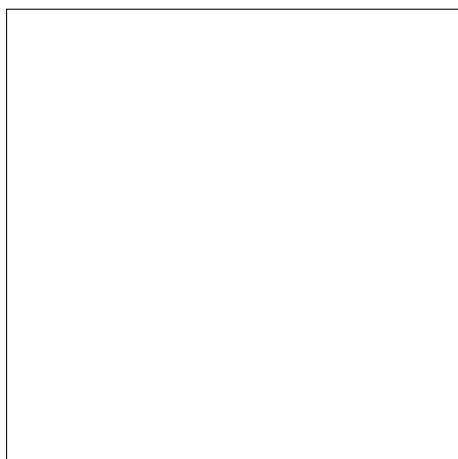
in the first half of the string? \_\_\_\_; in the first quarter of the string? \_\_\_\_;

at a location that has negative value? \_\_\_\_; at a location greater than  $l$ ? \_\_\_\_.

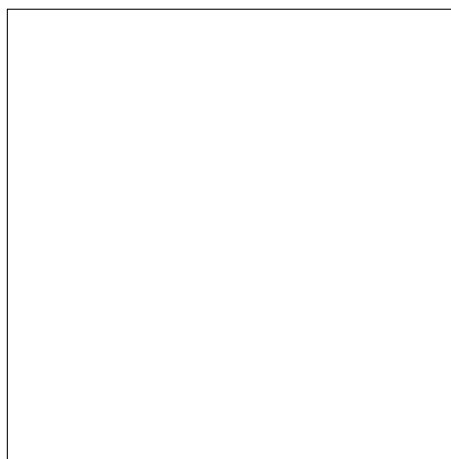
2. What is the general rule for the cut falling within a proportion  $a$  of its length, say within  $[0, al]$ , where  $a \in (0, 1]$ ?  $P(0 \leq X \leq al) = \underline{\hspace{2cm}}$ .

3. In other words, prob. of the cut falling at a location between  $-\infty$  to  $c$  (inclusive) is:

$$F(c) = P(X \leq c) = \begin{cases} \quad, c < 0 \\ \quad, 0 \leq c < l \\ \quad, c \geq l \end{cases}$$



(a) Plot  $F(a)$



(b) Plot  $F'(a)$  where the derivative exists

4. How does  $P(X \in [0, 1/2])$  compare to  $P(X \in [0, 1/2))$ ? \_\_\_\_\_.

## Continuous uniform distribution over $[a, b]$

A RV  $X$  with continuous uniform distribution has density function that is uniform (i.e. non-zero constant) over the interval  $[a, b]$  on the real line and 0 everywhere else. We say  $X \sim \mathcal{U}(a, b)$ .

**Q1.** Find  $f(x)$ , the probability density function (pdf) of a  $\mathcal{U}(a, b)$  RV.

Given a density function, we can find the corresponding CDF because \_\_\_\_\_.

**Q2.** Find  $F(x)$ , the CDF of a  $\mathcal{U}(a, b)$  RV.

**Q3.** Suppose  $X \sim \mathcal{U}(0, 10)$ . Find the following using the CDF

(a)  $P(X > 3)$

(b)  $P(2 \leq X < 12)$

(c)  $P(X > 5 | X > 2)$

## Expected values and probabilities of events

In general, for continuous RVs, calculation of expected values and probabilities of event  $X \in B$ ,  $B$  being some interval on the real line, is similar to the calculation in the discrete case except that we replace summation with integration:

	Discrete	Continuous
$F(x)$	$\sum_{k \leq x} f(k)$	
$E[g(x)]$	$\sum_k g(k) f(k)$	
$P(X \in B)$	$\sum_{k \in B} f(k)$	

**Q4.** For  $X \sim \mathcal{U}(a, b)$ , calculate  $E[X]$

For practice, try to calculate  $E[X^2]$  and  $Var(X)$  for a  $\mathcal{U}(a, b)$  RV at home.

## Transformations of a continuous RV

Let  $X$  be a RV with a known CDF  $F_X(x)$  and  $Y = g(X)$ . We want to find CDF and pdf of  $Y$ .

- **When  $g$  is strictly monotonic**, the inverse function  $g^{-1}$  exists. This makes getting the CDF of  $Y$ ,  $F_Y(y)$  relatively easy.

**step 1.** Identify the range of  $Y$ . This informs us when  $F_Y(y)$  will be 0 and when it will be 1.

**step 2.** Find  $F_Y(y)$  by noting:  
if  $g$  strictly monotonic increasing,

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) =$$

if  $g$  strictly monotonic decreasing,

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) =$$

**step 3.** Find pdf  $f_y(y)$  by noting  $f_y(y) = F'(y)$

- **But what if  $g$  not strictly monotonic?** Then we need to think about the shape of the function, maybe consider separating the domain of  $g(\cdot)$  into regions where the function is strictly monotonic within each region...

**Q5.** Let  $X \sim \mathcal{U}(-1, 1)$  Find the cdf and pdf of  $Y = X^2$