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On the open TS/ST correspondence

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Abstract: The topological string/spectral theory correspondence establishes a precise, non-perturbative duality between topological strings on local Calabi-Yau threefolds and the spectral theory of quantized mirror curves. While this duality has been rigorously formulated for the closed topological string sector, the open string sector remains less understood. Building on the results of [1–3], we make further progress in this direction by constructing entire, off-shell eigenfunctions for the quantized mirror curve from open topological string partition functions. We focus on local $\mathbb{F}_0$, whose mirror curve corresponds to the Baxter equation of the two-particle, relativistic Toda lattice. We then study the standard and dual four-dimensional limits, where the quantum mirror curve for local $\mathbb{F}_0$ degenerates into the modified Mathieu and McCoy-Tracy-Wu operators, respectively. In these limits, our framework provides a way to construct entire, off-shell eigenfunctions for the difference equations associated with these operators. Furthermore, we find a simple relation between the on-shell eigenfunctions of the modified Mathieu and McCoy-Tracy-Wu operators, leading to a functional relation between the operators themselves.

Contents

1 Introduction

The topological string/spectral theory (TS/ST) correspondence [1, 2, 4, 5] establishes a precise non-perturbative relation between the partition functions of topological string theory on local Calabi-Yau (CY) threefolds and the spectral properties of certain quantum mechanical operators on the real line. These quantum operators are obtained through the quantization of mirror curves [6, 7] and correspond to Baxter equations for a class of relativistic integrable systems, such as cluster integrable systems [8, 9] or elliptic Ruijsenaars-Schneider (RS) systems [10].

The TS/ST correspondence itself is structured in two parts: one relating closed strings to the spectrum, and the other relating open strings to the eigenfunctions. A central feature of this duality is the relationship between the string coupling g_s and the Planck constant $\hbar$ given by

$$g_s = \frac{4\pi^2}{\hbar} \,. \tag{1.1}$$

This implies that string perturbation theory naturally encodes non-perturbative effects on the spectral theory side. Conversely, the usual WKB expansion in spectral theory gives us the non-perturbative effects on the topological string side. Thus, the TS/ST correspondence bridges perturbative expansions in one theory with non-perturbative phenomena in its dual counterpart. This allows for the derivation of exact, closed-form expressions for many quantities on both sides of the correspondence. Another important consequence of (1.1) is the existence of the so-called self-dual, or maximally symmetric point [11], given by

$$\hbar = 2\pi = g_s \,. \tag{1.2}$$

At this special point, the TS/ST correspondence predicts remarkable simplifications, not only on the string theory side but also in operator theory, see e.g. [1, 2, 4, 5, 12, 13].

In this work, we focus on the example where the underlying CY geometry is local $\mathbb{F}_0$. In the closed string sector, one important statement of the TS/ST correspondence is [4]

$$\det(1+\kappa\rho) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{J}(\mu+\mathrm{i}2\pi k,\xi,\hbar)} \,, \qquad \kappa = \mathrm{e}^{\mu} \,, \tag{1.3}$$

where the operator $\rho : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the inverse of the quantum mirror curve to local $\mathbb{F}_0$, that is $\rho = \mathrm{O}^{-1}$ with

$$\mathrm{O} = \mathrm{e}^{\hat{y}} + \mathrm{e}^{-\hat{y}} + \mathrm{e}^{2\xi}\left(\mathrm{e}^{\hat{x}} + \mathrm{e}^{-\hat{x}}\right) , \qquad [\hat{x},\hat{y}] = \mathrm{i}\hbar \,, \qquad \xi \in \mathbb{R} \,, \, \hbar \in \mathbb{R}_{>0} \,. \tag{1.4}$$

On the r.h.s of (1.3), κ parametrizes the closed string moduli space, while ξ is the so-called "mass" parameter, associated with the residue of the CY one-form at infinity. The function $\mathrm{J}(\mu,\xi,\hbar)$ represents the topological string grand potential, incorporating both perturbative and non-perturbative contributions in $g_s = 4\pi^2/\hbar$, see (3.9). From the perspective of the moduli space, $\mathrm{J}(\mu,\xi,\hbar)$ is defined around the large radius point, and its perturbative part in g_s is given by the standard Gopakumar-Vafa (GV) free energy, see (3.17) and (3.19). The summation over k in (1.3) effectively smooths all the singularities across the closed string moduli space, parametrized by κ, allowing one to move away from the large radius point. Indeed, we can prove that ρ is a trace class operator on $L^2(\mathbb{R})$ [14, 15], and therefore its spectral determinant on the l.h.s. of (1.3) is an entire function of κ. We can expand it around $\kappa \to \infty$, making contact with the large radius point, or around $\kappa = 0$, making contact with the orbifold point [4, 5, 16]. Let us also note that, the summation over k in (1.3) is also essential for ensuring good modular properties of the determinant [11, 17] and for connecting it with q-isomonodromic τ-functions [18–28].

The main motivation of this work is to extend the relation in (1.3) to the open string sector. To this end, the first step is to define the open string grand potential $\mathrm{J}(x, \mu, \xi, \hbar)$, where x is the open string modulus. This was done in [1, 2] by focusing on the case where the brane is inserted in the outer leg of the toric diagram, the explicit form is given in (3.22), (3.31). Analogous to the closed string sector, $\mathrm{J}(x, \mu, \xi, \hbar)$ incorporates both perturbative and non-perturbative contributions in $g_s = 4\pi^2/\hbar$ and is defined in the large radius frame. One limitation of $\mathrm{J}(x, \mu, \xi, \hbar)$ however, is that it is not an entire function of the open string modulus x, which is a desirable property for a background-independent, non-perturbative formulation of open strings [29]. In [1, 2], it was further suggested that achieving this property requires considering a combination of the form

$$\psi(x, \kappa) = \sum_{k \in \mathbb{Z}} \sum_{\sigma} \mathrm{e}^{\mathrm{J}_\sigma(x, \mu + \mathrm{i}2\pi k, \xi, \hbar)}\,, \qquad\qquad \kappa = \mathrm{e}^{\mu}\,, \tag{1.5}$$

where the summation over σ is expected to play a role analogous to the sum over k in (1.3), but for the open string modulus x. In particular, just as the sum over n smooths out all singularities in the closed string moduli, the sum over σ should similarly ensure that (1.5) becomes an entire function of x.

In this paper, building on insights from [1–3], we make this expectation precise by providing an explicit form for this summation over σ in the case of local $\mathbb{F}_0$, at generic values of $\hbar$ and the complex moduli μ and ξ. Specifically, we find that the precise combination to consider is

$$\psi(x, \kappa) = \sum_{k \in \mathbb{Z}} \left(\mathrm{e}^{\mathrm{J}(x, \mu + \mathrm{i}2\pi k, \xi, \hbar)} + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2} + \frac{\pi x}{\hbar} + \mathrm{J}(-x - \mathrm{i}\pi, \mu + \mathrm{i}\pi + \mathrm{i}2\pi k, \xi, \hbar)} \right). \tag{1.6}$$

From the point of view of spectral theory, the combination (1.6) is a solution of the functional difference equation corresponding to the quantized mirror curve (1.4),

$$\psi(x + \mathrm{i}\hbar, \kappa) + \psi(x - \mathrm{i}\hbar, \kappa) + 2\mathrm{e}^{2\xi} \cosh(x) \psi(x, \kappa) + \kappa \psi(x, \kappa) = 0\,. \tag{1.7}$$

Difference equations of the type (1.7) admit many solutions. The proposal (1.6) stands out in three ways. Firstly, (1.6) is always a well-defined function of x, κ, ξ and $\hbar$ solving (1.7), as opposed to just a formal solution[1]. Secondly, (1.6) becomes a proper eigenfunction of the operator (1.4) when κ is a root of the spectral determinant (1.3). Thirdly, the eigenfunctions (1.6) are entire in x for all $\kappa \in \mathbb{C}$. We do not have a complete, mathematical proof for these statements, but we performed many analytic and numerical tests that support them.

[1]By "formal solution", we mean an object that solves (1.7) but that is not a well-defined function. For example, objects defined via divergent series expansions or those with a dense set of poles in their domain of definitions (e.g. when $\hbar \in \mathbb{R}_{>0}$).

By performing a canonical transformation one can further express such eigenfunctions via O(2) matrix models [1]. More specifically we have

$$\psi(x,\kappa) = \int_{\mathbb{R}} \mathrm{d}q\, U(x,q)\Xi(q,\kappa)\,, \tag{1.8}$$

where $U(x,q)$ is the kernel of a unitary transformation, see (2.40), and we have

$$\Xi(q,\kappa) = \exp\left(\frac{\pi}{\sqrt{2}}\frac{q}{\hbar}\right)\mathtt{f}(q)\sum_{N=0}^{\infty}\kappa^N\Psi_N(q)\,, \tag{1.9}$$

where $\mathtt{f}(q)$ is given in (2.11) and $\Psi_N(q)$ is defined by the following unnormalized expectation value within the O(2) matrix model (2.19)

$$\Psi_N(q) = \left\langle \prod_{k=1}^{N}\tanh\left(\frac{\pi}{\sqrt{2}}\frac{q-q_k}{\hbar}\right)\right\rangle. \tag{1.10}$$

The expression (1.9) is particularly interesting because it allows one to easily connect with the conifold frame as we discuss later.

This paper is organized as follows. In section 2, we analyse the matrix model (1.10) in some detail and show how its canonical transformation (1.8) naturally leads to the symmetric structure of the two contributions in (1.6). In section 3, we translate this into a conjecture for the eigenfunctions in terms of open topological strings, as given in (1.6), and we perform several detailed tests of the proposal.

In section 4, we examine two specific limits of our construction: the standard four-dimensional limit [30, 31], defined in (4.1), and the dual four-dimensional limit [18, 32], defined in (4.33). In the standard four-dimensional limit, the difference equation (1.7) reduces to the Fourier-transformed Mathieu operator whose eigenvalue equation reads

$$\sqrt{t}(\phi(x+\mathrm{i}\epsilon,E)+\phi(x-\mathrm{i}\epsilon,E))+x^2\phi(x,E)-E\phi(x,E)=0\,, \tag{1.11}$$

where t, E and ϵ correspond to the 4D limits of ξ, κ and $\hbar$ in (1.7), respectively. As with (1.7), this equation has many formal solutions. However, our construction identifies a special class of eigenfunctions that are entire, even off-shell. These are given by

$$\phi(x,E) = \phi_1\left(\frac{x}{\epsilon},\frac{\sigma}{\epsilon},\frac{t}{\epsilon^4}\right)+\phi_2\left(\frac{x}{\epsilon},\frac{\sigma}{\epsilon},\frac{t}{\epsilon^4}\right) \tag{1.12}$$

with

$$\phi_2(x,\sigma,t) = \phi_1(-x,\sigma,t)\left[\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\partial_\sigma F_{\mathrm{NS}}^{4\mathrm{d}}(\sigma,t)}(\mathrm{e}^{2\pi x}-\mathrm{e}^{2\pi\sigma})-\mathrm{e}^{\frac{\mathrm{i}}{2}\partial_\sigma F_{\mathrm{NS}}^{4\mathrm{d}}(\sigma,t)}(\mathrm{e}^{2\pi x}-\mathrm{e}^{-2\pi\sigma})}{\mathrm{e}^{2\pi\sigma}-\mathrm{e}^{-2\pi\sigma}}\right] \tag{1.13}$$

where $\phi_1(x,\sigma,t)$ is defined in (4.13) with σ and E being related by (4.10). The factor in square brackets in (1.13) is crucial to ensuring that the off-shell function (1.12) is entire. When evaluated on-shell, this factor is ± 1 depending on the parity of the eigenfunction, and our result reproduces the well-known expression for the on-shell eigenfunction in terms of 2d/4d surface defects in the Nekrasov–Shatashvili (NS) phase of the Ω-background [33–41]. On the other hand, in the dual four-dimensional limit (4.33), the operator (1.4) leads to the McCoy-Tracy-Wu operator [18]

$$\mathrm{e}^{4t^{1/4}\cosh\hat{x}}\cosh\left(\frac{\hat{y}}{2}\right)\mathrm{e}^{4t^{1/4}\cosh\hat{x}}\,, \tag{1.14}$$

whose eigenfunctions are computed by 2d/4d surface defects in the GV (or self-dual) phase of the Ω-background [3]. Even though the off-shell eigenfunctions of (1.11) and (1.14) are quite different, when evaluated on-shell, they are related in a remarkably simple way. This in turn provides a clear functional relation between the modified Mathieu (1.11) and McCoy-Tracy-Wu (1.14) operators, see subsection 4.3 and equation (4.49).

In section 5, we conclude and outline some open problems. We also have four appendices that provide technical details and definitions necessary for understanding the results presented in the main text.

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2 Eigenfunctions and matrix models

2.1 The spectral problem

The mirror curve to local $\mathbb{F}_0$ reads [30, 42]

$$\mathrm{e}^{y}+\mathrm{e}^{-y}+\mathrm{e}^{2\xi}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)+\kappa=0\,, \tag{2.1}$$

where κ, ξ are the complex moduli of local $\mathbb{F}_0$. The eigenvalue equation corresponding to the quantization of the mirror curve (2.1) is [6, 7]

$$\left(\mathrm{e}^{\hat{y}}+\mathrm{e}^{-\hat{y}}+\mathrm{e}^{2\xi}\left(\mathrm{e}^{\hat{x}}+\mathrm{e}^{-\hat{x}}\right)\right)\psi(x,\kappa)+\kappa\psi(x,\kappa)=0\,,\qquad [\hat{x},\hat{y}]=\mathrm{i}\hbar\,, \tag{2.2}$$

leading to the following difference equation

$$\psi(x+\mathrm{i}\hbar,\kappa)+\psi(x-\mathrm{i}\hbar,\kappa)+2\,\mathrm{e}^{2\xi}\cosh(x)\psi(x,\kappa)+\kappa\psi(x,\kappa)=0\,, \tag{2.3}$$

which also corresponds to the Baxter equation for the two-particle, relativistic, quantum Toda lattice [43]. In this paper, we always take

$$\xi\in\mathbb{R}\,,\qquad \hbar\in\mathbb{R}_{>0}\,. \tag{2.4}$$

The domain of definition of the multiplication operator $\left(\mathrm{e}^{\hat{x}}+\mathrm{e}^{-\hat{x}}\right)$ consists of functions ψ for which $\mathrm{e}^{\pm x}\psi(x)\in L^2(\mathbb{R})$. Similarly, the difference operator $\left(\mathrm{e}^{\hat{y}}+\mathrm{e}^{-\hat{y}}\right)$ is defined on functions $\psi(x)$ that admit an analytic continuation in the strip

$$\{x\in\mathbb{C}\mid |\mathrm{Im}(x)|<\hbar\}\,, \tag{2.5}$$

such that $\psi(x)$ is square-integrable for all constant $|\mathrm{Im}(x)|<\hbar$, and for which the limits

$$\lim_{\epsilon\to 0^+}\psi(x-\mathrm{i}\hbar+\mathrm{i}\epsilon,\kappa)\,,\qquad\qquad \lim_{\epsilon\to 0^+}\psi(x+\mathrm{i}\hbar-\mathrm{i}\epsilon,\kappa)\,, \tag{2.6}$$

exist in the sense of convergence in $L^2(\mathbb{R})$. The quantized mirror curve is then a symmetric, strictly positive operator defined on the intersection of the domains of the multiplication and difference operators. Hence, one can define its Friedrichs extension, see [14]. It is this self-adjoint extension we will consider in everything that follows, and we refer to it as the quantum mirror curve. This also leads to a purely discrete spectrum $\{E_n\}_{n\in\mathbb{N}}$ for the quantum mirror curve, see [4, 14, 15]. Our conventions for the spectrum are

$$\left(\mathrm{e}^{\hat{y}}+\mathrm{e}^{-\hat{y}}+\mathrm{e}^{2\xi}\left(\mathrm{e}^{\hat{x}}+\mathrm{e}^{-\hat{x}}\right)\right)\psi(x,-\mathrm{e}^{E_n})=\mathrm{e}^{E_n}\psi(x,-\mathrm{e}^{E_n})\,. \tag{2.7}$$

We refer to the variables x,y, and the corresponding operators $\hat{x},\hat{y}$ in (2.1) as outer topological string coordinates for reasons that will become clear later. Following [44, 45] we introduce the matrix model coordinates q,p

$$\begin{aligned} x&=\frac{1}{\sqrt{2}}(q+p)+\xi\,,\\ y&=\frac{1}{\sqrt{2}}(-q+p)+\xi\,. \end{aligned} \tag{2.8}$$

After some creative algebra, one can show that the eigenvalue equation in the q,p coordinate can be written as [45, sec. 2.1]

$$\mathrm{O}\,\Xi(q,\kappa)=-\kappa\,\Xi(q,\kappa)\,, \tag{2.9}$$

$$\mathrm{O}=\frac{\sqrt{2}}{\hbar}(\mathtt{f}^*(\hat{q}))^{-1}\cosh\left(\frac{\hat{p}}{\sqrt{2}}\right)(\mathtt{f}(\hat{q}))^{-1}\,,\qquad\qquad [\hat{q},\hat{p}]=\mathrm{i}\hbar\,, \tag{2.10}$$

where we used

$$\begin{aligned} \mathtt{f}(q)&=\frac{2^{1/4}}{\sqrt{2\pi b}}\exp\left(-\frac{\xi}{2}\right)\exp\left(\frac{q}{2\sqrt{2}}\right)\frac{\Phi_b\left(\frac{q}{\sqrt{2}\pi b}-\frac{\xi}{\pi b}+\mathrm{i}\frac{b}{4}\right)}{\Phi_b\left(\frac{q}{\sqrt{2}\pi b}+\frac{\xi}{\pi b}-\mathrm{i}\frac{b}{4}\right)}\,,\qquad\qquad \hbar=\pi b^2\,,\\ \mathtt{f}^*(q)&=\frac{2^{1/4}}{\sqrt{2\pi b}}\exp\left(-\frac{\xi}{2}\right)\exp\left(\frac{q}{2\sqrt{2}}\right)\frac{\Phi_b\left(\frac{q}{\sqrt{2}\pi b}+\frac{\xi}{\pi b}+\mathrm{i}\frac{b}{4}\right)}{\Phi_b\left(\frac{q}{\sqrt{2}\pi b}-\frac{\xi}{\pi b}-\mathrm{i}\frac{b}{4}\right)}\,,\qquad\qquad \hbar=\pi b^2\,, \end{aligned} \tag{2.11}$$

with Φ_b Faddeev's non-compact quantum dilogarithm, see appendix A. Note that $\mathtt{f}^*(q)=\overline{\mathtt{f}(q)}$ only if $q,\xi,b\in\mathbb{R}$. It is convenient to introduce the inverse operator $\rho=\mathrm{O}^{-1}$ whose integral kernel is

$$\rho(q_1,q_2)=\frac{\mathtt{f}(q_1)\mathtt{f}^*(q_2)}{2\cosh\left(\frac{\pi}{\sqrt{2}}\frac{q_1-q_2}{\hbar}\right)}\,. \tag{2.12}$$

One important property is that ρ is of trace class and its Fredholm determinant admits and expansion in terms of an O(2) matrix model. More precisely

$$\det(1+\kappa\rho)=\prod_{n=0}^{\infty}(1+\kappa \mathrm{e}^{-E_n})=\sum_{N=0}^{\infty}\kappa^N Z(N,\hbar) \tag{2.13}$$

where E_n is determined as discussed around (2.7) and

$$Z(N,\hbar)=\frac{1}{N!}\sum_{s\in S_N}(-)^{\mathrm{sgn}(s)}\int_{\mathbb{R}}\mathrm{d}^N x\prod_{k=1}^{N}\rho\big(x_k,x_{s(k)}\big)\,, \tag{2.14}$$

where S_N is the permutation group of N elements. By applying the Cauchy identity one can further write $Z(N,\hbar)$ as [45]

$$Z(N,\hbar)=\frac{1}{2^N N!}\int_{\mathbb{R}^N}\mathrm{d}^N q\prod_{k=1}^{N}\mathtt{v}(q_k)\prod_{\ell=k+1}^{N}\tanh^2\left(\frac{q_k-q_\ell}{\sqrt{2}b^2}\right), \tag{2.15}$$

$$\mathtt{v}(q)=\mathtt{f}(q)\mathtt{f}^*(q)=\frac{\mathrm{e}^{-\xi}}{\sqrt{2}\pi b^2}\exp\left(\frac{q}{\sqrt{2}}\right)\frac{\Phi_b\left(\frac{q}{\sqrt{2}\pi b}-\frac{\xi}{\pi b}+\mathrm{i}\frac{b}{4}\right)\Phi_b\left(\frac{q}{\sqrt{2}\pi b}+\frac{\xi}{\pi b}+\mathrm{i}\frac{b}{4}\right)}{\Phi_b\left(\frac{q}{\sqrt{2}\pi b}+\frac{\xi}{\pi b}-\mathrm{i}\frac{b}{4}\right)\Phi_b\left(\frac{q}{\sqrt{2}\pi b}-\frac{\xi}{\pi b}-\mathrm{i}\frac{b}{4}\right)}, \tag{2.16}$$

where again $\hbar=\pi b^2$. It is important to stress that (2.13) in entire in κ, in particular the sum on the r.h.s. has an infinite radius of convergence. This is a standard result in Fredholm theory and follows from the trace class property of ρ, see [4, 14, 15].

2.2 Integrating quasi-periodic functions

In the following sections, we frequently encounter integrals involving Faddeev's quantum dilogarithm Φ_b. To compute these integrals, we will make extensive use of Lemma 2.1 from [46], which we briefly review for future reference. Let $f:\mathcal{U}\to\mathbb{C}$ be an analytic function with $\mathcal{U}\subseteq\mathbb{C}$ open, $\mathcal{C}\subseteq\mathcal{U}$ an oriented path, and $a\in\mathbb{C}\setminus\{0\}$ a constant such that the following properties hold:

1. $\mathcal{U}=a+\mathcal{U}$,
2. $f(z)(f(z+a)-f(z))\neq 0$ for all $z\in\mathcal{C}$,
3. $f(z+a)f(z-a)=f^2(z)$ for all $z\in\mathcal{U}$,

then the following equality holds

$$\int_{\mathcal{C}}f(z)\mathrm{d}z=\left(\int_{\mathcal{C}}-\int_{a+\mathcal{C}}\right)\frac{f(z)}{1-f(z+a)/f(z)}\mathrm{d}z\,. \tag{2.17}$$

If we can close the contour on the right-hand side, then the integral reduces to a sum over residues. This will be the case for the integrals of interest to us.

2.3 The eigenfunctions in matrix models coordinates

2.3.1 The general construction

Off-shell eigenfunctions with respect to the matrix model coordinates q, p were found in [1, Sec. 2], following [47]. Let us define

$$\Xi^{\pm}(q;\kappa) = E^{\pm}(q) \sum_{N=0}^{\infty} (\pm\kappa)^N \Psi_N(q)\,, \qquad E^{\pm}(q) = \exp\left(\pm \frac{q}{\sqrt{2}b^2}\right) \mathtt{f}(q)\,, \tag{2.18}$$

with $\mathtt{f}$ given in (2.11) and $\Psi_N(q)$ is defined by the following unnormalized expectation value

$$\Psi_N(q) = \frac{1}{2^N N!} \int_{\mathbb{R}^N} \mathrm{d}^N q \prod_{k=1}^{N} \tanh\left(\frac{q-q_k}{\sqrt{2}b^2}\right) \mathtt{v}(q_k) \prod_{\ell=k+1}^{N} \tanh^2\left(\frac{q_k - q_\ell}{\sqrt{2}b^2}\right), \tag{2.19}$$

where $\mathtt{v}$ is defined in (2.16). Notice that also (2.18) is entire in κ, in parallel with (2.13). As before this follows from the trace class property of ρ.

Using (2.18), we can write the eigenvalue equation (2.9) as

$$\Omega^{\pm}\left(q + \mathrm{i}\frac{\hbar}{\sqrt{2}}\right) + \Omega^{\pm}\left(q - \mathrm{i}\frac{\hbar}{\sqrt{2}}\right) = -\sqrt{2}\hbar\kappa \mathtt{v}(q) \Omega^{\pm}(q,\kappa)\,, \tag{2.20}$$

$$\Omega^{\pm}(q,\kappa) = \exp\left(\pm\frac{q}{\sqrt{2}b^2}\right) \sum_{N=0}^{+\infty} \Psi_N(q) (\pm\kappa)^N\,, \tag{2.21}$$

and (2.20) reads at the level of the Ψ_N

$$\Psi_N\left(q + \frac{\mathrm{i}\pi b^2}{\sqrt{2}}\right) - \Psi_N\left(q - \frac{\mathrm{i}\pi b^2}{\sqrt{2}}\right) = \mathrm{i}\sqrt{2}\pi b^2 \mathtt{v}(q) \Psi_{N-1}(q)\,. \tag{2.22}$$

Therefore the spectral problem in the matrix model coordinates q, p can be formulated as follows. We look for solutions of (2.20) which are analytic in the strip $|\mathrm{Im}(q)| < \pi b^2/\sqrt{2}$ and which belong to $L^2(\mathbb{R})$. In the off-shell eigenfunctions (2.21), the first requirement is already implemented because of the specific form of $\Psi_N(q)$ given in (2.19), as we will verify in the case $N = 1$ below. As for the $L^2(\mathbb{R})$ requirement we have

$$\Omega^{\pm}(q,\kappa) \simeq \begin{cases} \det(1 \pm \kappa\rho) \mathrm{e}^{\pm\frac{q}{\sqrt{2}b^2}} & q \to +\infty \\ \det(1 \mp \kappa\rho) \mathrm{e}^{\pm\frac{q}{\sqrt{2}b^2}} & q \to -\infty \end{cases} \tag{2.23}$$

leading to the quantization condition $\det(1 + \kappa\rho) = 0$ as expected and in agreement with the discussion in subsection 2.1. Indeed the results of [1, 47] is that $\Xi_{\pm}(q;\kappa)$ become true square integrable eigenfunctions of (2.9), and equal to each other up to a factor $(-)^n$, when $\kappa = -\exp(E_n) < 0$ is on-shell. Their canonical transformation will then produce the eigenfunctions $\psi\big(x, -\mathrm{e}^{E_n}\big)$ of (2.7) in the topological string (x, y)-coordinates, as we discuss in subsection 2.4.

2.3.2 The $N = 1$ example for $\hbar \in \pi\mathbb{Q}_{>0}$

In this section, we make use of subsection 2.2 to compute explicitly $\Psi_1(q)$ and test its analytic properties. We have from (2.19)

$$\Psi_1(q) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}p\, \tanh\left(\frac{q-p}{\sqrt{2}b^2}\right)\mathtt{v}(p)\,, \qquad \hbar = \pi b^2\,, \tag{2.24}$$

where $\mathtt{v}$ is given in (2.16). The function $\mathtt{v}$ inherits some quasi-periodicity from the quantum dilogarithms (A.6), namely

$$\begin{aligned}\frac{\mathtt{v}\big(p + \mathrm{i}\sqrt{2}\pi b^2 k\big)}{\mathtt{v}(p)} &= \mathrm{e}^{\mathrm{i}\pi b^2 k} \\ &\prod_{n=0}^{|k|-1}\left\{\frac{\left(1+\mathrm{e}^{s_k \mathrm{i}\pi b^2\left(2n+\frac{1}{2}\right)}\mathrm{e}^{\sqrt{2}p-2\xi}\right)\left(1+\mathrm{e}^{s_k \mathrm{i}\pi b^2\left(2n+\frac{1}{2}\right)}\mathrm{e}^{\sqrt{2}p+2\xi}\right)}{\left(1+\mathrm{e}^{s_k\mathrm{i}\pi b^2}\mathrm{e}^{s_k \mathrm{i}\pi b^2\left(2n+\frac{1}{2}\right)}\mathrm{e}^{\sqrt{2}p-2\xi}\right)\left(1+\mathrm{e}^{s_k\mathrm{i}\pi b^2}\mathrm{e}^{s_k \mathrm{i}\pi b^2\left(2n+\frac{1}{2}\right)}\mathrm{e}^{\sqrt{2}p+2\xi}\right)}\right\}\end{aligned} \tag{2.25}$$

for $k \in \mathbb{Z}$ and $s_k = \mathrm{sgn}(k)$. From these identities one can see that the integrand of (2.24) is quasi-periodic in the sense of the Lemma in subsection 2.2 when $b^2 \in \mathbb{Q}_{>0}$: we have quasi-periodicity under shifts by $\mathrm{i}\sqrt{2}\pi b^2 m = \mathrm{i}\sqrt{2}\pi n$ when $b^2 = n/m$ with n, m positive coprime integers. However, the quasi-periodic shift is trivial when $n \in 2\mathbb{N}_{>0}$, so we can only use the lemma for $b^2 = (2n+1)/m$. In that case, one gets

$$\Psi_1(q) = \frac{1}{2}\left(\int_{\mathbb{R}+\mathrm{i}0} - \int_{\mathbb{R}+\mathrm{i}\sqrt{2}\pi(2n+1)+\mathrm{i}0}\right)\mathrm{d}p\, \tanh\left(\frac{q-p}{\sqrt{2}b^2}\right)\frac{\mathtt{v}(p)}{1-\frac{\mathtt{v}\left(p+\mathrm{i}\sqrt{2}\pi(2n+1)\right)}{\mathtt{v}(p)}}\,, \tag{2.26}$$

$$\frac{1}{1-\frac{\mathtt{v}\left(p+\mathrm{i}\sqrt{2}\pi(2n+1)\right)}{\mathtt{v}(p)}} = -\,\mathrm{i}\frac{\left(1-\mathrm{i}(-)^{n+m}\mathrm{e}^{-m\left(\sqrt{2}p-2\xi\right)}\right)\left(1+\mathrm{i}(-)^{n+m}\mathrm{e}^{m\left(\sqrt{2}p+2\xi\right)}\right)}{(-)^{n+m}4\mathrm{e}^{2m\xi}\sinh\left(\sqrt{2}mp\right)}\,. \tag{2.27}$$

The integrand has an essential singularity at complex infinity, but it doesn't contribute to the integral since $\mathtt{v}(p) \propto \exp\left(\mp p/\sqrt{2}\right)$ when $\mathrm{Re}(p) \to \pm\infty$ with $\mathrm{Im}(p)$ constant. Hence the integration over p reduces to the sum over the residues of the integrand in (2.26) with $\mathrm{Im}(p) \in \left]0, \sqrt{2}\pi(2n+1)\right]$.

Note that $\mathtt{v}(p)$ has poles at

$$\mathtt{v}(p):\qquad \frac{p}{\sqrt{2}} = s\xi \pm \mathrm{i}\pi\left[b^2\left(k+\frac{1}{4}\right)+\left(\ell+\frac{1}{2}\right)\right] \qquad \text{poles} \qquad k,\ell \in \mathbb{N}\,, \tag{2.28}$$

where $s \in \{\pm 1\}$ and the upper sign is coming from the numerator and the lower sign from the denominator of $\mathtt{v}$. Note that all the poles inside the integration contour are simple as a direct consequence of the observation around equation (A.10). Likewise one finds

$$\frac{1}{1-\frac{\mathtt{v}\left(p+\mathrm{i}\sqrt{2}\pi(2n+1)\right)}{\mathtt{v}(p)}}:\qquad \begin{cases}\frac{p}{\sqrt{2}} = \pm\xi + \mathrm{i}\frac{\pi}{m}\left[k + (-)^{n+m}\frac{1}{4}\right] & \text{roots} \\ \frac{p}{\sqrt{2}} = \mathrm{i}\frac{\pi}{2}\frac{k}{m} & \text{poles}\end{cases} \qquad k \in \mathbb{Z}\,. \tag{2.29}$$

One can check that all the poles of $\mathtt{v}$ with positive imaginary part coincide with roots of the denominator, and hence are not realized as poles of the integrand when they are inside the integration contour, where they are simple. However, we do have m poles from the tanh and $2(2n+1)m$ poles from the denominator at

$$p = q + \mathrm{i}\frac{\pi}{\sqrt{2}}b^2(2k+1)\,, \qquad k \in \{0, \cdots, m-1\}\,, \tag{2.30}$$

$$p = \mathrm{i}\frac{\pi}{\sqrt{2}}\frac{\ell}{m}\,, \qquad \ell \in \{1, \cdots, 2(2n+1)m\}\,, \tag{2.31}$$

respectively, which have residues

$$-\sqrt{2}b^2\,, \qquad \text{and} \qquad -\mathrm{i}(-)^{n+m+\ell}\frac{\cosh(2m\xi)}{2\sqrt{2}m}\,. \tag{2.32}$$

This gives finally the following expression for Ψ_1

$$\boxed{\Psi_1(q) = \Psi_1^{(1)}(q) + \Psi_1^{(2)}(q)} \tag{2.33}$$

where

$$\boxed{\begin{aligned} \Psi_1^{(1)}(q) &= -\mathrm{i}\frac{\pi}{\sqrt{2}}b^2\left[1 + \mathrm{i}(-)^{n+m}\cosh(2m\xi)\operatorname{csch}\left(\sqrt{2}mq\right)\right] \\ &\qquad\qquad \sum_{k=0}^{m-1}\mathtt{v}\left(q + \mathrm{i}\frac{\pi}{\sqrt{2}}b^2(2k+1)\right). \\ \Psi_1^{(2)}(q) &= (-)^{n+m+1}\frac{\sqrt{2}\pi}{4m}\cosh(2m\xi)\sum_{\ell=-2n}^{2n+1}(-)^{\ell}\coth\left(\frac{q}{\sqrt{2}b^2} - \mathrm{i}\frac{\pi}{2}\frac{\ell}{2n+1}\right) \\ &\qquad\qquad \sum_{k=0}^{m-1}\mathtt{v}\left(\mathrm{i}\frac{\pi}{\sqrt{2}}\left(\frac{\ell}{m} + b^2(2k+1)\right)\right). \end{aligned}} \tag{2.34}$$

and we remind the reader that we took $\hbar/\pi = b^2 = (2n+1)/m$ with $2n+1$ and m coprime. It is also noteworthy that these functions are real along the real line.

Let us look at the analytic properties of Ψ_1. We can then make the following considerations.

1. Let us consider $\Psi_1^{(1)}(q)$. The simple poles of $\mathtt{v}\Big(q + \mathrm{i}\frac{\pi}{\sqrt{2}}b^2(2k+1)\Big)$ with $|\mathrm{Im}(q)| \leqslant \pi b^2/\sqrt{2}$ coincide with the simple roots of the factor in square brackets and are hence not realized.

2. In addition, $\Psi_1^{(1)}(q)$ has simple poles originating from $\operatorname{csch}\big(\sqrt{2}mq\big)$ at

$$q = \mathrm{i}\frac{\pi}{\sqrt{2}}\frac{r}{m} \qquad r \in \{-(2n+1), \cdots, +(2n+1)\}\,. \tag{2.35}$$

where the upper and lower bound on r come from the requirement of being inside the strip, i.e. $|\mathrm{Im}(q)| \leqslant \pi b^2/\sqrt{2}$.

We now argue that these poles cancel against those of $\Psi_1^{(2)}$. It is then clear that $\Psi_1^{(2)}(q)$ has simple poles at (2.35) originating from the term $\ell = r$ in (2.34). In addition, we find the following residue for $\Psi_1^{(1)}(q)$ at (2.35)

$$(-)^{r+n+m}\frac{\pi}{2}\frac{b^2}{m}\cosh(2m\xi)\sum_{\ell=0}^{m-1}\mathtt{v}\left(\mathrm{i}\frac{\pi}{\sqrt{2}}\left(\frac{r}{m}+b^2(2k+1)\right)\right). \tag{2.36}$$

Likewise $\Psi_1^{(2)}$ has the same residue with the opposite overall sign.

Hence we can conclude that $\Psi_1(q)$ is analytic on the strip $-\pi b^2/\sqrt{2} \leqslant \mathrm{Im}(q) \leqslant \pi b^2/\sqrt{2}$ as expected. Note however that outside the strip there are higher order poles in $\Psi_1^{(1)}$, coming from $\mathtt{v}$, which do not get cancelled by the simple roots of the factor in square brackets or by any poles coming from the periodic part. Hence, Ψ_1 is analytic on the strip, but not entire. Moreover, from our solution (2.33) we get

$$\Psi_1\left(q+\mathrm{i}\frac{\sqrt{2}\pi b^2}{2}\right)-\Psi_1\left(q-\mathrm{i}\frac{\sqrt{2}\pi b^2}{2}\right)=\mathrm{i}\sqrt{2}\pi b^2\mathtt{v}(q) \tag{2.37}$$

which is the difference equation we expect for Ψ_1 from (2.22).

Since $\Psi_1(q)$ reduces to the first spectral trace $Z(1,\hbar)$ in the $q \to +\infty$ limit one gets from (2.33)

$$\boxed{Z\left(1,\frac{(2n+1)\pi}{m}\right)=(-)^{n+m}\frac{\sqrt{2}\pi}{4m}\cosh(2m\xi)\sum_{\ell=1}^{2(2n+1)m}(-)^{\ell}\mathtt{v}\left(\mathrm{i}\frac{\pi}{\sqrt{2}}\frac{\ell}{m}\right)} \tag{2.38}$$

where $\mathtt{v}$ is defined in (2.16) and $2n+1$, m are coprime. This is of course exactly the result one gets when applying the residue technique above directly to the integral defining $Z(1,\hbar)$ in (2.15). The expression (2.38) is, in some sense, complementary to that of [48, eq. (3.55)], which holds for $\mathrm{Im}(b^2) > 0$.

2.4 The eigenfunctions in outer topological string coordinates

2.4.1 The general construction and symmetric structures

One motivation for considering the outer topological string x, y-coordinates is that they establish a direct connection with the open topological string in the presence of a D-brane on the external leg of the toric diagram, as we will discuss in section 3. Consequently, in these coordinates, we obtain a particularly explicit framework for computing these eigenfunctions using topological string partition functions[2].

[2]The matrix model coordinates seem more natural to describe a brane in the internal leg of the toric diagram [49, 50].

The eigenfunctions $\psi(x,\kappa)$ in the topological string (x,y)-coordinates and the eigenfunctions $\Xi(q,\kappa)$ in the matrix model (q,p)-coordinates are then related by a canonical transformation. More precisely

$$\psi^{\pm}(x,\kappa) = \int_{\mathbb{R}} \mathrm{d}q\, U(x,q)\Xi^{\pm}(q,\kappa)\,, \tag{2.39}$$

where from [1]

$$U(x,q) = \frac{2^{1/4}}{\sqrt{2\pi\hbar}} \exp\left(\frac{\mathrm{i}}{\hbar}\left(\frac{x^2}{2} - \sqrt{2}(x-\xi)q + \frac{q^2}{2}\right)\right). \tag{2.40}$$

Hence, if we take the eigenfunctions (2.18), the corresponding eigenfunctions in topological string coordinates are

$$\psi^{\pm}(x,\kappa) = \sum_{N\geqslant 0} (\pm\kappa)^N \psi^{\pm}_N(x) \tag{2.41}$$

where

$$\psi^{\pm}_N(x) = \int_{\mathbb{R}} \mathrm{d}q\, U(x,q) E^{\pm}(q)\Psi_N(q)\,. \tag{2.42}$$

Note that this integral is only well-defined when $E^{\pm}(q)\Psi_N(q)$ is integrable. However, we have

$$\begin{aligned} E^{+}(q)\Psi_N(q) &\simeq \begin{cases} \exp\left[\left(\frac{b^2-2}{2\sqrt{2b^2}}\right)(-q)\right] \exp\left(-\mathrm{i}\frac{2\sqrt{2}}{\pi}\frac{\xi q}{b^2}\right) & q\to+\infty \\ \exp\left[\left(\frac{b^2+2}{2\sqrt{2b^2}}\right)q\right] & q\to-\infty \end{cases} \\ E^{-}(q)\Psi_N(q) &\simeq \begin{cases} \exp\left[\left(\frac{b^2+2}{2\sqrt{2b^2}}\right)(-q)\right] \exp\left(-\mathrm{i}\frac{2\sqrt{2}}{\pi}\frac{\xi q}{b^2}\right) & q\to+\infty \\ \exp\left[\left(\frac{b^2-2}{2\sqrt{2b^2}}\right)q\right] & q\to-\infty \end{cases} \end{aligned}\,. \tag{2.43}$$

Hence, the unitary transformation (2.39) is well-defined only when $b^2 > 2$, as also noted in [1]. This raises the question of how to make sense of the canonical transformation[3] (2.42) when $\hbar = b^2\pi \leqslant 2\pi$. Our strategy to address this issue is the following. In this region, we will momentarily set aside convergence issues and directly apply the Lemma from subsection 2.2. This approach gives a finite result for any value of b^2, even if the starting point was problematic for $b^2 \leqslant 2$. The case $\xi = 0$, $b^2 = 2$ was analysed in [1, p. 15] using the same strategy.

We are now going to study (2.39) and (2.42) in detail. First, it was conjectured in [1, 2] that this integral can be written as the sum of two contributions, which are in a one-to-one correspondence with the two saddles of the integrand on the r.h.s. of (2.42). Second, in [3], a special scaling limit of (2.42) was analysed in detail, and it was found that the two saddles are related in a very simple way. By combining these two observations, we

[3]Curiously, if it were not for the q^2 term in the canonical transformation, the $N=0$ version of this transform would be very close to the "integral analogue of the ${}_1\psi_1$-summation formula of Ramanujan" as given in [51, eq. (51)], which can be explicitly computed in closed form using the method of residues.

can propose the following Ansatz for the expression of the eigenfunctions in the topological string coordinates :

$$\psi^{\pm}(x,\kappa)=\omega^{\pm}(x,\kappa)+\exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}\pm\frac{\pi x}{\hbar}\right)\omega^{\pm}(-x\mp\mathrm{i}\pi,-\kappa)\,, \tag{2.44}$$

for some functions $\omega^{\pm}(x,\kappa)$. By using (2.18) and (2.42), one can easily see that $\omega_{-}(x,\kappa)=\omega_{+}(-x,\kappa)$. Hence, for simplicity of notation, we will often omit the $\pm$ sign and write

$$\psi(x,\kappa)\equiv\psi^{+}(x,\kappa)\ . \tag{2.45}$$

Then our result reads

$$\boxed{\psi(x,\kappa)=\omega(x,\kappa)+\exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}\right)\omega(-x-\mathrm{i}\pi,-\kappa)\,,} \tag{2.46}$$

for some function $\omega(x,\kappa)$.

At the level of components in the κ expansion (2.18), equation (2.44) becomes

$$\psi_N^{\pm}(x)=\omega_N^{\pm}(x)+(-)^N\exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}\pm\frac{\pi x}{\hbar}\right)\omega_N^{\pm}(-x\mp\mathrm{i}\pi)\,, \tag{2.47}$$

for some functions $\omega_N^{\pm}(x)$ where $\omega_N^{-}(x)=(-)^N\omega_N^{+}(-x)$. As before we will often omit the $\pm$ superscript and simply note

$$\psi_N(x)\equiv\psi_N^{+}(x)\ . \tag{2.48}$$

Then (2.47) reads

$$\boxed{\psi_N(x)=\omega_N(x)+(-)^N\exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}\right)\omega_N(-x-\mathrm{i}\pi)}\,, \tag{2.49}$$

for some function $\omega_N(x)$.

We will test this proposal in several ways and, we also give explicit expressions for ω by using topological string theory, see subsection 3.3. As we discuss in appendix C, the case $\xi=0$, $\hbar=2\pi$ analysed in [1, 2] is special and the above structure is hidden.

As a first consistency check of (2.46), let us study the behaviour of these functions under a parity transformation. Using the parity properties of the quantum dilogarithm (A.5) and the expression (2.19), we have

$$\begin{aligned}E^{\pm}(-q)\Psi_N(-q)&=\exp\left(\mp\frac{\sqrt{2}\pi}{\hbar}q\right)\exp\left(\frac{\mathrm{i}}{\hbar}2\sqrt{2}\xi q\right)E^{\pm}(q)(-)^N\Psi_N(q)\\&=\exp\left(\frac{\mathrm{i}}{\hbar}2\sqrt{2}\xi q\right)E^{\mp}(q)(-)^N\Psi_N(q)\,,\\\psi_N^{\pm}(-x)&=\exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}\right)\exp\left(\mp\frac{\pi x}{\hbar}\right)(-)^N\psi_N^{\pm}(x\mp\mathrm{i}\pi)\,,\\&=(-)^N\psi_N^{\mp}(x)\,.\end{aligned} \tag{2.50}$$

The eigenfunctions behave then as

$$\begin{aligned}\Xi^{\pm}(-q,\kappa) &= \exp\left(\mp\frac{\sqrt{2}\pi q}{\hbar}\right)\exp\left(\mathrm{i}\frac{2\sqrt{2}}{\hbar}\xi q\right)\Xi^{\pm}(q,-\kappa)\\ &= \exp\left(\mathrm{i}\frac{2\sqrt{2}}{\hbar}\xi q\right)\Xi^{\mp}(q,\kappa)\\ \psi^{\pm}(-x,\kappa) &= \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}\right)\exp\left(\mp\frac{\pi x}{\hbar}\right)\psi^{\pm}(x\mp\mathrm{i}\pi,-\kappa)\\ &= \psi^{\mp}(x,\kappa)\,,\end{aligned} \tag{2.51}$$

and we see in particular that the eigenfunctions in the x, y-coordinates have a well-defined parity while there is a local phase for generic ξ in the q, p-coordinates. It is important to note that the parity relation for $\psi^{\pm}(x,\kappa)$ in (2.51) is the same as the one relating the two terms in (2.46), ensuring the self-consistency of our proposal (2.46).

2.4.2 The $N = 0$ example for $\hbar \in \pi\mathbb{Q}_{>0}$

In this subsection, we compute[4] $\psi_N = \psi_N^+$ for $N = 0$ and $\hbar \in \pi\mathbb{Q}_{>0}$, which serves two goals. Firstly, it will give some evidence that we could expect the off-shell eigenfunctions of the quantum mirror curve to be entire[5] in x, and secondly, we will use it as an analytical check of our conjecture (2.49). We are interested in

$$\psi_0(x) = \int_{\mathbb{R}} \mathrm{d}q\, U(x,q)E(q)\,, \qquad E(q) = E^+(q)\,. \tag{2.52}$$

It is important to note that the integrand in (2.52) is a meromorphic function of q, which inherits some quasi-periodicity from the quantum dilogarithms (A.7),

$$\frac{U\big(x,q+\mathrm{i}\sqrt{2}\pi k b^2\big)E\big(q+\mathrm{i}\sqrt{2}\pi k b^2\big)}{U(x,q)E(q)} = \mathrm{e}^{-\mathrm{i}\pi k^2 b^2}\mathrm{e}^{-2k\xi}\mathrm{e}^{2kx}\mathrm{e}^{-\sqrt{2}kq}\mathrm{e}^{\mathrm{i}\pi k}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}kb^2} \prod_{\ell=0}^{|k|-1}\left(\frac{1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mathrm{sgn}(k)b^2(4\ell+1)}\mathrm{e}^{\mathrm{sgn}(k)2\xi}\mathrm{e}^{\sqrt{2}q}}{1+\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mathrm{sgn}(k)b^2(4\ell+3)}\mathrm{e}^{-\mathrm{sgn}(k)2\xi}\mathrm{e}^{\sqrt{2}q}}\right) \tag{2.53}$$

where $\hbar = \pi b^2$ and $k \in \mathbb{Z}$. When we take

$$b^2 = \frac{n}{m} \in \mathbb{Q}_{>0}\,, \qquad n, m \in \mathbb{N}_{>0} \text{ and coprime}\,, \qquad \text{and } k = \pm m\,, \tag{2.54}$$

then this simplifies further to

$$\frac{U\big(x,q\pm\mathrm{i}\sqrt{2}\pi n\big)E\big(q\pm\mathrm{i}\sqrt{2}\pi n\big)}{U(x,q)E(q)} = (-)^{(n+1)m}\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}n}\mathrm{e}^{\mp 2m\xi}\mathrm{e}^{\pm 2mx}\mathrm{e}^{\mp\sqrt{2}mq}\left(\frac{1-(-)^m\mathrm{e}^{\mathrm{i}\frac{\pi}{2}n}\mathrm{e}^{2m\xi}\mathrm{e}^{\sqrt{2}mq}}{1-(-)^m\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}n}\mathrm{e}^{-2m\xi}\mathrm{e}^{\sqrt{2}mq}}\right)^{\pm}. \tag{2.55}$$

[4] As explained below (2.47) the result for ψ_N^- follows immediately. Hence we adopt the notation (2.48).

[5] This point was also made in [1], based on a computation of ψ_N for several $N \in \mathbb{N}$ for $\xi = 0$ and $\hbar = 2\pi$ [1, eqs. (2.95), (2.96)]. See also [2, eqs. (4.20), (4.21)] for the case $\xi = 0$, $\hbar = 4\pi$ and $\hbar = 2\pi/3$.

Hence we find that the integrand defining ψ_0 in (2.52) is quasi-periodic in the sense of the quasi-periodic integrand lemma of subsection 2.2. Following the lemma we can rewrite (2.52) as

$$\psi_0(x) = \left(\int_{\mathbb{R}+\mathrm{i}0} - \int_{\mathbb{R}+\mathrm{i}\sqrt{2}\pi n+\mathrm{i}0}\right)\mathrm{d}q \\ \frac{U(x,q)E(q)}{1-(-)^{(n+1)m}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}n}\mathrm{e}^{-2m\xi}\mathrm{e}^{2mx}\mathrm{e}^{-\sqrt{2}mq}\left(\frac{1-(-)^m\mathrm{e}^{\mathrm{i}\frac{\pi}{2}n}\mathrm{e}^{2m\xi}\mathrm{e}^{\sqrt{2}mq}}{1-(-)^m\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}n}\mathrm{e}^{-2m\xi}\mathrm{e}^{\sqrt{2}mq}}\right)}\,. \tag{2.56}$$

The integrand above is a meromorphic function of q with a finite number of poles inside the integration contour and an essential singularity at infinity. Note furthermore from (2.43) that the integrand decays exponentially inside the whole contour for $\mathrm{Re}(q) \to \pm\infty$. Hence we can close the contour at complex infinity and the integral reduces to a sum over the residues of the poles.

To simplify the discussion we will assume that either[6] $\xi \neq 0$ or $n \in (2\mathbb{N}+1)$. There are potential poles of the integrand coming from E at (A.2)

$$q = \pm\sqrt{2}\xi \pm \mathrm{i}\sqrt{2}\pi\left(\left(k+\frac{1}{2}\right)+\frac{n}{m}\left(\ell+\frac{1}{4}\right)\right), \qquad k,\ell\in\mathbb{N}\,, \tag{2.57}$$

and all the poles inside the integration contour in (2.56) are simple as a direct consequence of the observation made around (A.10). However, for each such pole of E there is a coinciding simple pole for the denominator in (2.56), and hence the integrand in (2.56) is analytic around these points. The only poles inside the integration contour are hence coming from the roots of the denominator and are located at

$$\sqrt{2}mq_{\pm,k}(x) = \ln\Bigg[(-)^m\mathrm{e}^{\mathrm{i}\frac{\pi}{2}n}\left(\frac{\mathrm{e}^{2m\xi}}{2}\right)\mathrm{e}^{mx}\Bigg((-)^{n(m+1)}\mathrm{e}^{mx}+\mathrm{e}^{-mx} \\ \pm\,\mathrm{sgn}\left(\arg\left(x+\mathrm{i}\frac{\pi}{2}\right)\right)\sqrt{\left((-)^{n(m+1)}\mathrm{e}^{mx}+\mathrm{e}^{-mx}\right)^2-(-)^{nm}4\mathrm{e}^{-4m\xi}}\Bigg)\Bigg]+\mathrm{i}2\pi k\,, \tag{2.58}$$

where $k\in\mathbb{Z}$ should be such that $0<\mathrm{Im}(q_{\pm,k}(x))\leqslant\sqrt{2}\pi n$ and we use the convention $\mathrm{sgn}(0)=-1$. It is important for later to note that these points are by construction a solution to

$$\begin{aligned} U(x,q_{\pm,k}(x))E(q_{\pm,k}(x)) &= U\left(x,q_{\pm,k}(x)+\mathrm{i}\sqrt{2}\pi n\right)E\left(q_{\pm,k}(x)+\mathrm{i}\sqrt{2}\pi n\right) \\ &= U(x,q_{\pm,k+nm}(x))E(q_{\pm,k+nm}(x))\,. \end{aligned} \tag{2.59}$$

[6]The derivation for $\xi=0$ and $n\in 2\mathbb{N}_{>0}$ is done analogously, yielding the same result as obtained by taking the $\xi\to 0$ limit for $n\in 2\mathbb{N}_{>0}$ on the solution (2.61). This case agrees with the results of [1, 2], see also appendix C.

The residues corresponding with (2.58) and coming from the denominator are given by

$$\mathrm{Res}_\pm(x) = \frac{1}{2\sqrt{2}m} \\ \pm \frac{\mathrm{sgn}\big(\arg(x+\mathrm{i}\frac{\pi}{2})\big)}{2\sqrt{2}m}\left(\frac{(-)^{n(m+1)}\mathrm{e}^{mx}-\mathrm{e}^{-mx}}{\sqrt{\left((-)^{n(m+1)}\mathrm{e}^{mx}+\mathrm{e}^{-mx}\right)^2-(-)^{nm}4\mathrm{e}^{-4m\xi}}}\right). \tag{2.60}$$

It should be noted that the particular choice of branches in (2.58) and (2.60) is to some extent purely conventional. However, we will relate $q_\pm$ and $q_\mp$ in (2.64), and for this purpose it is important to use the particular choice made in (2.58) and (2.60), or something equivalent. In the end, we find that (2.52) is given by

$$\psi_0(x) = \omega_{0,+}(x) + \omega_{0,-}(x), \tag{2.61}$$

$$\omega_{0,\pm}(x) = \mathrm{i}2\pi\mathrm{Res}_\pm(x)\sum_{k=0}^{nm-1} U(x, q_{\pm,k}(x))E(q_{\pm,k}(x))\,, \tag{2.62}$$

where $\hbar = \pi b^2 = \pi n/m$ with $n, m \in \mathbb{N}_{>0}$ and coprime. It is noteworthy that $\omega_{0,\pm}$ and hence ψ_0 can be expressed entirely in terms of elementary functions and the classical dilogarithm Li_2 by using (A.8). See figure 1 for some plots of the functions defined above.

Let us look at the analytic properties of $\omega_{0,\pm}$ and ψ_0. Note first that $\omega_{0,\pm}$ and ψ_0 are analytic along the logarithmic branch cut of $q_{\pm,k}$, since crossing the branch simply amounts to shifting the range of k in the sum in (2.61), which doesn't affect $\omega_{0,\pm}$ or ψ_0 by (2.59). Note furthermore that ψ_0 is analytic along the square root branches of $q_{\pm,k}$ and $\mathrm{Res}_\pm$ as well, since crossing the branches simply interchanges the $\pm$-signs. The only remaining potential singularities are the branch points of the logarithm in $q_{\pm,k}$ and the square roots, and the poles of $E(q_{\pm,k}(x))$. The logarithmic branch point of $q_{\pm,k}$ is never realized, and the square root branch points in $\mathrm{Res}_\pm(x)$ cancel between the $\omega_{0,\pm}$. Furthermore, the poles of E are never reached by $q_{\pm,k}$ when either $\xi \neq 0$ or $n \in (2\mathbb{N}+1)$ as we assumed[7]. Hence we conclude that $\psi_0(x)$ is in fact an entire function of x, even though neither $\omega_{0,+}$ nor $\omega_{0,-}$ is entire, see figure 1. The same structure will reappear when we express the eigenfunctions in terms of the grand potential of topological strings on local $\mathbb{F}_0$ in (3.35).

One can furthermore check that ψ_0 solves the difference equation associated with the quantized mirror curve at $\kappa = 0$, that is

$$\psi_0(x+\mathrm{i}\hbar) + \psi_0(x-\mathrm{i}\hbar) + 2\mathrm{e}^{2\xi}\cosh(x)\psi_0(x) = 0\,. \tag{2.63}$$

However, ψ_0 is not an eigenfunction of the quantized mirror curve, since it is not square integrable in the strip (2.5).

Let us end with the observation that (2.61) is a well-defined, entire function of x that solves (2.63) for all $b^2 = n/m \in \mathbb{Q}_{>0}$, also $b^2 \leqslant 2$, even though the original integral transform as given in (2.52) is only well-defined for $b^2 > 2$.

[7]Though the $\xi \to 0$ limit for even n behaves well, see footnote 6.

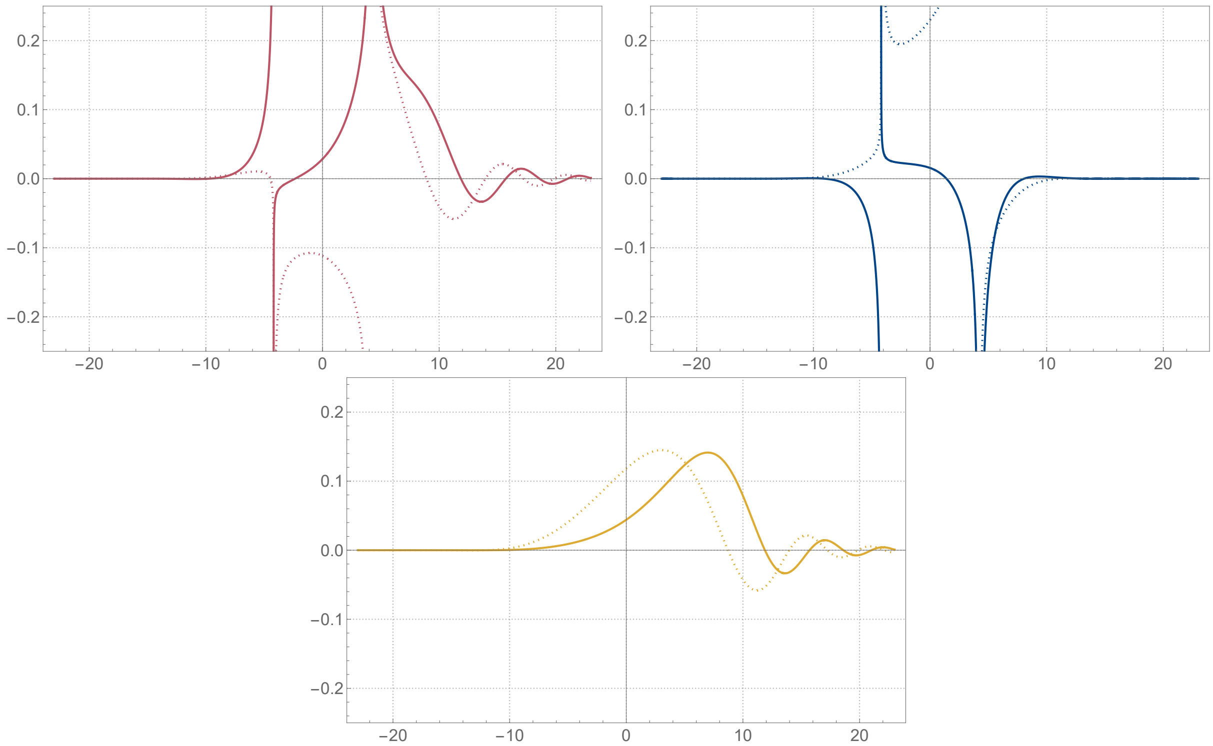

Figure 1. From the upper left in clockwise sense: the first term $\omega_{0,+}$ in (2.61), the second term $\omega_{0,-}$, and their sum ψ_0 for $\hbar = 4\pi$ and $\xi = -7/4$. The real part is the solid line and the imaginary part is the dotted line.

2.4.3 Relating the two terms for $N = 0$ and $\hbar \in \pi\mathbb{Q}_{>0}$

We now want to show that (2.61) can be written as in (2.49) for $N = 0$. For all[8] $x \in \mathbb{C}$, $\xi \in \mathbb{R}$, coprime $n, m \in \mathbb{N}_{>0}$, and $k \in \mathbb{Z}$ on finds the following relation for (2.58) and (2.60)

$$q_{\pm,k}(x) = -q_{\mp,-k-\ell}(-x - \mathrm{i}\pi)\,, \qquad \mathrm{Res}_{\pm}(x) = \mathrm{Res}_{\mp}(-x - \mathrm{i}\pi)\,, \tag{2.64}$$

where $\ell \in \{-1, 0, +1\}$ should be chosen appropriately according to the branches. It should be noted that this symmetry is only visible upon the specific choice of the branch structure we made in (2.58) and (2.60). One can use the parity structure in q in (2.50) to rewrite

$$\begin{aligned} U(x, q_{\pm,k}(x))E(q_{\pm,k}(x)) &= U(x, -q_{\mp,-k-\ell}(-x - \mathrm{i}\pi))E(-q_{\mp,-k-\ell}(-x - \mathrm{i}\pi)) \\ &= \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2} + \frac{\pi x}{\hbar}\right) U(-x - \mathrm{i}\pi, q_{\mp,-k-\ell}(-x - \mathrm{i}\pi))E(q_{\mp,-k-\ell}(-x - \mathrm{i}\pi))\,. \end{aligned} \tag{2.65}$$

[8] The branches in (2.58) have to be chosen differently when $x = -\mathrm{i}\pi/2$.

Note that at the level of $\omega_{0,\mp}$ we can replace $q_{\mp,-k-\ell}$ by $q_{\mp,k}$ as a consequence of (2.59). Hence, we find that (2.61) can be written as

$$\begin{aligned}\psi_0(x) &= \omega_{0,+}(x) + \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2} + \frac{\pi x}{\hbar}\right)\omega_{0,+}(-x-\mathrm{i}\pi) \\ &= \exp\left(\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2} + \frac{\pi x}{\hbar}\right)\omega_{0,-}(-x-\mathrm{i}\pi) + \omega_{0,-}(x)\end{aligned} \tag{2.66}$$

which is exactly the conjectured structure in (2.49), here for $N = 0$ and $\hbar \in \pi\mathbb{Q}_{>0}$.

2.5 The 't Hooft expansion

An important way to test the conjectured structure in (2.49) is by analysing $\psi_N^{\pm}(x)$ (2.42) in a 't Hooft limit. This limit was studied in detail in [1, sec. 3] for the case $\xi = 0$, see also [52]. We will closely follow their approach and use it to argue for the structure in (2.49). The 't Hooft limit is defined by taking

$$\hbar, N, \xi, |q|, |x| \to +\infty\,, \tag{2.67}$$

while keeping the following ratio's constant,

$$\lambda = \frac{N}{\hbar}\,, \qquad \xi_D = \frac{2\pi}{\hbar}\xi\,, \qquad q_D = \frac{2\pi}{\hbar}q\,, \qquad x_D = \frac{2\pi}{\hbar}x\ . \tag{2.68}$$

2.5.1 Preparation

Later on, we will need the solutions of the classical mirror curve (2.1) in the matrix model coordinates q, p (2.8) and the topological string coordinates x, y, which read

$$\begin{aligned}\mathrm{e}^{\frac{1}{\sqrt{2}}p_\sigma(q,\xi,\kappa)} &= P_\sigma\left(\mathrm{e}^{q/\sqrt{2}}, \xi, \kappa\right), \\ \mathrm{e}^{y_\sigma(x,\xi,\kappa)} &= Y_\sigma(\mathrm{e}^x, \xi, \kappa)\,,\end{aligned} \tag{2.69}$$

where $\sigma = \pm 1$ and

$$\begin{aligned}P_\pm(Q,\xi,\kappa) &= \frac{-\mathrm{e}^{-\xi}\kappa \pm \sqrt{(\mathrm{e}^{-\xi}\kappa)^2 - 4(\mathrm{e}^{\xi}Q + \mathrm{e}^{-\xi}Q^{-1})(\mathrm{e}^{-\xi}Q + \mathrm{e}^{\xi}Q^{-1})}}{2\mathrm{e}^{\xi}(\mathrm{e}^{\xi}Q + \mathrm{e}^{-\xi}Q^{-1})}\,, \\ Y_\pm(X,\xi,\kappa) &= -\left(\frac{\mathrm{e}^{2\xi}}{2}\left(X + \frac{1}{X}\right) + \frac{\kappa}{2}\right) \pm \sqrt{\left(\frac{\mathrm{e}^{2\xi}}{2}\left(X + \frac{1}{X}\right) + \frac{\kappa}{2}\right)^2 - 1}\,.\end{aligned} \tag{2.70}$$

Let us also introduce the convenient shorthand notation

$$p_\pm(q_D) = p_\pm(q_D, \xi_D, \kappa_D)\,, \qquad y_\pm(x_D) = y_\pm(x_D, \xi_D, \kappa_D)\,. \tag{2.71}$$

As we discuss below (2.75), there is an implicit definition of κ_D in terms of λ and ξ_D.

2.5.2 In matrix model coordinates

Let us now consider 't Hooft limit as defined in (2.67) and (2.68) on the matrix model

$$E(q)\Psi_N(q)/Z(N,\hbar)\ , \tag{2.72}$$

where the relevant functions are defined in equations (2.11), (2.15), and (2.18) and below. One finds the asymptotic expansion [53]

$$\begin{aligned} E(q) &\simeq \exp\left(\frac{\mathrm{i}}{g_s}\mathcal{E}_0(q_D)+\mathcal{E}_1(q_D)+\mathcal{O}(g_s)\right), \\ \frac{E(q)\Psi_N(q)}{Z(N)} &\simeq \exp\left(\frac{\mathrm{i}}{g_s}\mathcal{T}_0(q_D)+\mathcal{T}_1(q_D)+\mathcal{O}(g_s)\right) \end{aligned} \tag{2.73}$$

The precise form of the functions $\mathcal{T}_1$ and $\mathcal{E}_1$ is not important for our discussion. Thus, we focus on the leading-order terms. Using the quasi-classical expansion of the quantum dilogarithm (A.13) gives

$$\mathcal{E}_0(q_D) = \mathrm{i}\pi\xi_D - \mathrm{i}\pi\frac{q_D}{\sqrt{2}} - 2\mathrm{Li}_2\left(-\mathrm{i}\mathrm{e}^{q_D/\sqrt{2}}\mathrm{e}^{-\xi_D}\right) + 2\mathrm{Li}_2\left(\mathrm{i}\mathrm{e}^{q_D/\sqrt{2}}\mathrm{e}^{\xi_D}\right). \tag{2.74}$$

The computation of $\mathcal{T}_0$ is more involved and requires various matrix model techniques developed in [1, sec. 3.2] and [52, 53]. Following [1, sec. 3.2] we get

$$\mathcal{T}_0(q_D) = \int^{q_D} p_\sigma(q_D')\mathrm{d}q_D' + \mathrm{i}\pi\xi_D \tag{2.75}$$

where p_σ is defined through (2.69) and (2.71). The correct sign $\sigma=\sigma(q_D)\in\{\pm\}$ depends on the region of the complex q_D-plane, however, we will not need it for what follows. In (2.75), we implicitly use the matrix model relation between the 't Hooft coupling λ and κ_D. This relation is obtained as follows:

- There is an explicit relation between the 't Hooft coupling λ and the endpoints of the eigenvalue density, denoted by $a^\pm$. For our matrix model, this relation is given in [53, eqs. (2.76)–(2.80)].

- The endpoints of the cuts, $a^\pm$, are related to the mirror curve parameters ξ_D and κ_D as

$$a^{\pm 2} = a^{\pm 2}(\xi_D,\kappa_D) = \frac{\mathrm{e}^{-2\xi_D}\kappa_D^2}{8} - \cosh(2\xi_D) \pm \sqrt{\left(\frac{\mathrm{e}^{-2\xi_D}\kappa_D^2}{8} - \cosh(2\xi_D)\right)^2 - 1}\,. \tag{2.76}$$

  These correspond to the branch points of the mirror curve in the (p,q) coordinates.

By combining the above points, we obtain an explicit relation between κ_D, ξ_D, and λ, leading to (2.75). We refer to [53] for more details.

2.5.3 In outer topological string coordinates

We are interested in the 't Hooft limit of (2.42). By recalling the conventions (2.48) we have

$$\frac{\psi_N(x)}{Z(N)} = \int_{\mathbb{R}} \mathrm{d}q\, U(x,q) \frac{E(q)\Psi_N(q)}{Z(N)} = \int_{\mathbb{R}} \mathrm{d}q_D\, \frac{U^D(x_D,q_D)}{\sqrt{2\pi g_s}} \frac{E^D(q_D)\Psi_N^D(q_D)}{Z(N)}\,, \tag{2.77}$$

where we defined for future convenience

$$E^D(q_D) = E\left(\frac{2\pi}{g_s} q_D\right), \qquad \Psi_N^D(q_D) = \Psi_N\left(\frac{2\pi}{g_s} q_D\right), \tag{2.78}$$

$$U^D(x_D,q_D) = \sqrt{\frac{(2\pi)^3}{g_s}}\, U\left(\frac{2\pi}{g_s} x_D, \frac{2\pi}{g_s} q_D\right). \tag{2.79}$$

The 't Hooft limit of (2.77) becomes then a simple application of the stationary phase method. The essential ingredient is of course the saddle point equation

$$\partial_{q_D} F(x_D,q_D) + \mathcal{T}_0'(q_D) = -\sqrt{2}(x_D - \xi_D) + q_D + p_\sigma(q_D) = 0\,, \tag{2.80}$$

where we defined from (2.40)

$$F(x_D,q_D) = \frac{x_D^2}{2} - \sqrt{2}(x_D - \xi_D) q_D + \frac{q_D^2}{2}\,. \tag{2.81}$$

This should then be solved to find $q^D(x_D)$. Note that the saddle point equation is exactly the transformation of x in the canonical transformation (2.8), which is a direct consequence of the construction of U (2.40) [1, sec. 2.5] and the form of $\mathcal{T}_0$ in (2.75). Combining this with the transformation of y in (2.8) and using the fact that p_σ and y_σ in (2.69) are both solutions of the classical mirror curve (2.1) yields[9],

$$q_\pm^D(x_D) = \frac{\sqrt{2}}{2}(x_D - y_\pm(x_D))\,, \qquad |\mathrm{Im}(x_D)| < 2\pi\,,\ \xi_D \in \mathbb{R}\,, \tag{2.82}$$

where $y_\pm$ is given in (2.69) and (2.71). Note that the restriction on x_D is the usual domain restriction of the eigenfunctions (2.5) in the dual variables. Observe that $q_\pm^D$ has a simple parity symmetry,

$$q_\pm^D(x_D) = -q_\mp^D(-x_D)\,, \tag{2.83}$$

which will be important in a moment.

As a result of the stationary phase method one finds [54, ch. 5]

$$\frac{\psi_N(x)}{Z(N)} \simeq \sum_{\sigma\in\{\pm\}} \left(\frac{U^D\big(x_D, q_\sigma^D(x_D)\big)}{\sqrt{-\mathrm{i}(\mathcal{T}_0 + F)''(q_\sigma^D(x_D))}} \frac{E^D(q_\sigma^D(x_D))\Psi_N^D(q_\sigma^D(x_D))}{Z(N)} + \mathcal{O}(g_s) \right). \tag{2.84}$$

It should be noted that we didn't make the 't Hooft limit expansion in g_s explicit, and the functions involved are still complicated functions of g_s. However, the form given above is

[9] Up to an integer multiple of $\mathrm{i}\sqrt{2}\pi$ if we are outside the strip $|\mathrm{Im}(x_D)| < 2\pi$.

the most convenient one to understand the relation between the saddles[10]. We found in (2.83) that $q_\pm^D$ has a simple parity symmetry and one can see that the denominator has similarly

$$(\mathcal{T}_0 + F)''(q_\pm^D(x_D)) = p_\pm^{D\prime}(q_\pm^D(x_D)) + 1 = \frac{\sqrt{2}}{q_\pm^{D\prime}(x_D)} = (\mathcal{T}_0 + F)''(q_\mp^D(-x_D))\,, \tag{2.85}$$

where the second equality follows from the x_D derivative of the saddle point equation (2.80). We can then use the parity symmetries of the functions involved to get rid of the minus sign in front of the $q_\mp^D(-x_D)$ in (2.83) to find that the saddles are related by

$$\begin{aligned}\frac{U^D\big(x_D, q_\pm^D(x_D)\big)}{\sqrt{-\mathrm{i}(\mathcal{T}_0+F)''(q_\pm^D(x_D))}}\,\frac{E^D(q_\pm^D(x_D))\Psi_N^D(q_\pm^D(x_D))}{Z(N)} &= \\ (-)^N \mathrm{e}^{\mathrm{i}\frac{g_s}{8}}\mathrm{e}^{\frac{x_D}{2}}\frac{U^D\big(-x_D - \mathrm{i}\frac{g_s}{2}, q_\mp^D(-x_D)\big)}{\sqrt{-\mathrm{i}(\mathcal{T}_0+F)''(q_\mp^D(-x_D))}}&\,\frac{E^D(q_\mp^D(-x_D))\Psi_N^D(q_\mp^D(-x_D))}{Z(N)}\,. \end{aligned}\tag{2.86}$$

It is then again a consequence of the saddle point equation (2.80) that we can shift the argument of $q_\mp(-x_D)$ as well to get

$$\begin{aligned}\frac{U^D\big(x_D, q_\pm^D(x_D)\big)}{\sqrt{-\mathrm{i}(\mathcal{T}_0+F)''(q_\pm^D(x_D))}}\,\frac{E^D(q_\pm^D(x_D))\Psi_N^D(q_\pm^D(x_D))}{Z(N)} = (-)^N \mathrm{e}^{\mathrm{i}\frac{g_s}{8}}\mathrm{e}^{\frac{x_D}{2}} \\ \frac{U^D\big(-x_D - \mathrm{i}\frac{g_s}{2}, q_\mp^D(-x_D - \mathrm{i}\frac{g_s}{2})\big)}{\sqrt{-\mathrm{i}(\mathcal{T}_0+F)''\big(q_\mp^D(-x_D - \mathrm{i}\frac{g_s}{2})\big)}}\,\frac{E^D(q_\mp^D(-x_D - \mathrm{i}\frac{g_s}{2}))\Psi_N^D(q_\mp^D(-x_D - \mathrm{i}\frac{g_s}{2}))}{Z(N)} + \mathcal{O}(g_s)\,,\end{aligned}\tag{2.87}$$

which is exactly the proposed structure (2.49) in the dual variables, up to potential corrections of $\mathcal{O}(g_s)$. Note that factors like the $\exp(\mathrm{i}g_s/8)$ or the shifts of the argument of $q_\pm^D$ are not visible at this level, but are part of the $\mathcal{O}(g_s)$ corrections.

3 The TS/ST correspondence for local $\mathbb{F}_0$

In this section, we first review some aspects of the TS/ST correspondence for the closed string sector and then discuss the generalization to the open string sector. We focus on the particular case where the toric CY threefold is local $\mathbb{F}_0$.

3.1 The quantum mirror map and Wilson loop

Mirror symmetry plays an important role in the TS/ST correspondence, with one of its key components being the quantum mirror map. Originally introduced from a geometrical perspective, this map was defined as the quantization of the A-period in the mirror curve [7, 55]. Subsequently, it was understood that, from a geometric engineering perspective,

[10] See [1, sec 3.3, app. A] for an explicit construction of the subleading order in the 't Hooft limit.

this map could be identified with a Wilson loop in a corresponding five-dimensional gauge theory, see [41, 56–58].

For local $\mathbb{F}_0$ the quantum mirror map is given in [7, sec. 7.2] or [59, eq. (3.58)]. The first few orders in a large μ expansion are

$$t_B(\hbar) = 2\mu - 2\big(\mathrm{e}^{4\xi}+1\big)\mathrm{e}^{-2\mu} - \big(3\mathrm{e}^{8\xi} + 2\big(\mathrm{e}^{\mathrm{i}\hbar}+4+\mathrm{e}^{-\mathrm{i}\hbar}\big)\mathrm{e}^{4\xi}+3\big)\mathrm{e}^{-4\mu} + \mathcal{O}\big(\mathrm{e}^{-6\mu}\big) \tag{3.1}$$

where $\kappa = \exp(\mu)$ and ξ are the complex moduli of the mirror curve (2.1). We will also use

$$t_F(\hbar) = t_B(\hbar) - 4\xi\,. \tag{3.2}$$

If we sent $\hbar \to 0$ we recover the classical mirror map relating the Kähler parameters of local $\mathbb{F}_0$, to the complex moduli μ and ξ. For example, if $\xi = 0$ the classical mirror map is simply given by

$$t_B(0) = 2\mu - 4\,\mathrm{e}^{-2\mu}\;{}_4F_3\left[\begin{matrix}1,\,1,\,\frac{3}{2},\,\frac{3}{2}\\ 2,\,2,\,2\end{matrix}\,\middle|\,16\,\mathrm{e}^{-2\mu}\right], \tag{3.3}$$

where ${}_4F_3$ is the generalized hypergeometric function.

From the perspective of geometric engineering, topological string theory on local $\mathbb{F}_0$ engineers a five-dimensional, $\mathcal{N}=1$, SU(2) SYM theory in the Ω-background [30, 31]. In this gauge theory context, the quantum mirror map corresponds to the inverse of the Wilson loop in the fundamental representation. The latter can be computed via supersymmetric localization [56, 57] and the first few terms read

$$\begin{aligned} W(t_F,t_B,\hbar) &= \mathrm{e}^{t_F/2}+\mathrm{e}^{-t_F/2} + \left[\frac{\big(\mathrm{e}^{t_F/2}+\mathrm{e}^{-t_F/2}\big)}{(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F})(1-\mathrm{e}^{-\mathrm{i}\hbar}\mathrm{e}^{-t_F})}\right]\mathrm{e}^{-t_B} + \\ &\left[\frac{\mathrm{e}^{-2t_F}\big(\mathrm{e}^{t_F/2}+\mathrm{e}^{-t_F/2}\big)}{(1-\mathrm{e}^{\mathrm{i}2\hbar}\mathrm{e}^{-t_F})(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F})^3(1-\mathrm{e}^{-\mathrm{i}\hbar}\mathrm{e}^{-t_F})^3(1-\mathrm{e}^{-\mathrm{i}2\hbar}\mathrm{e}^{-t_F})}\Big(-\big(3\mathrm{e}^{\mathrm{i}\hbar}+4+3\mathrm{e}^{-\mathrm{i}\hbar}\big)\right. \\ &\left. + \big(\mathrm{e}^{\mathrm{i}2\hbar}+\mathrm{e}^{\mathrm{i}\hbar}+1+\mathrm{e}^{-\mathrm{i}\hbar}+\mathrm{e}^{-\mathrm{i}2\hbar}\big)\big(\mathrm{e}^{t_F}+\mathrm{e}^{-t_F}\big)\Big)\right]\mathrm{e}^{-2t_B} + \mathcal{O}\big(\mathrm{e}^{-3t_B}\big) \end{aligned} \tag{3.4}$$

where $t_{B,F}$ are the Kähler parameters. We refer to [60, eq. (3.22)] for the full definition in this specific example. It is easily verified that setting

$$t_B = t_B(\hbar)\,, \qquad t_F = t_F(\hbar)\,, \tag{3.5}$$

in equation (3.4) gives

$$W(t_F(\hbar), t_B(\hbar), \hbar) = \mathrm{e}^{-2\xi}\kappa\,. \tag{3.6}$$

3.2 The closed string sector and the spectrum

Let us first review some important elements of the TS/ST correspondence for the closed string sector [4, 5], see [61, 62] for a review. Note that we focus on the specific case where the CY threefold is local $\mathbb{F}_0$.

One feature of the TS/ST correspondence is that, on the topological string side, the relevant quantities involve a special combination of refined topological string partition functions in the GV ($-\epsilon_1 = \epsilon_2 = g_s$) and NS limit ($\epsilon_1 \to 0, \epsilon_2 = \hbar$) respectively with the relation

$$g_s = \frac{4\pi^2}{\hbar} \ . \tag{3.7}$$

The arguments of these two sets of special functions are typically rescaled with respect to each other. Hence it is convenient to define

$$\alpha^D = \left(\frac{2\pi}{\hbar}\right)\alpha \ . \tag{3.8}$$

The self-dual, or maximally supersymmetric point, is defined at $\hbar = g_s = 2\pi$ [4, 11].

The main quantity is the closed topological string grand potential $\mathrm{J}(\mu, \xi, \hbar)$. This quantity encapsulates both perturbative and non-perturbative contributions to the closed topological string free energy near the large radius point [4, 59, 63]. More precisely we have

$$\mathrm{J}(\mu, \xi, \hbar) = \mathrm{A}(\xi, \hbar) + \mathrm{J}_{\mathrm{p}}(\mu, \xi, \hbar) + \mathrm{J}_{\text{1-loop}}(\mu, \xi, \hbar) + \mathrm{J}_{\mathrm{inst}}(\mu, \xi, \hbar) \, . \tag{3.9}$$

Let us define these quantities.

- We denote with $\mathrm{A}(\xi, \hbar)$ the constant maps contribution whose closed-form reads [45, 64]

$$\mathrm{A}(\xi, \hbar) = \frac{4\xi^3}{3\pi\hbar} + \frac{\hbar\xi}{12\pi} + A_c\left(\frac{\hbar}{\pi}\right) - F_{\mathrm{CS}}(\xi, \hbar) \, , \tag{3.10}$$

$$A_c(k) = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \left(\frac{k}{\pi}\right)^2 \int_0^{+\infty} \mathrm{d}x \, \frac{x}{\mathrm{e}^{kx} - 1} \ln\left(1 - \mathrm{e}^{-2x}\right) , \tag{3.11}$$

$$\begin{aligned} F_{\mathrm{CS}}(\xi, g) = \frac{\hbar^2}{8\pi^4}\Big[\mathrm{Li}_3\Big(-\mathrm{e}^{2\left(\frac{2\pi}{\hbar}\right)\xi}\Big) + \mathrm{Li}_3\Big(-\mathrm{e}^{-2\left(\frac{2\pi}{\hbar}\right)\xi}\Big) - 2\zeta(3)\Big] \\ + \int_0^{+\infty} \mathrm{d}x \, \frac{x}{\mathrm{e}^{2\pi x} - 1} \ln\left[\frac{\sinh^2\left(\frac{\pi^2 x}{\hbar}\right)}{\sinh^2\left(\frac{\pi^2 x}{\hbar}\right) + \cosh^2\left(\left(\frac{2\pi}{\hbar}\right)\xi\right)}\right] , \end{aligned} \tag{3.12}$$

  where Li_3 is the polylogarithm of order 3 and ζ is the Riemann zeta function.

- The polynomial part J_{p} is given by

$$\mathrm{J}_{\mathrm{p}}(\mu, \xi, \hbar) = \frac{t_B^3(\hbar)}{12\pi\hbar} - \frac{\xi t_B^2(\hbar)}{2\pi\hbar} + \left(\frac{\pi}{6\hbar} - \frac{\hbar}{24\pi}\right) t_B(\hbar) - \frac{\pi\xi}{3\hbar} \, , \tag{3.13}$$

  where the quantum mirror map $t_B(\hbar)$ is given in (3.1).

- The one-loop part consists of two contributions: one coming from the one-loop part of the free energy in the NS phase and one being the one-loop part of the free energy in the GV phase of the Ω-background. The sum of these two contributions reads then

$$\mathrm{J}_{\text{1-loop}}(\mu,\xi,\hbar) = \sum_{k=1}^{+\infty}\left[\frac{1}{2\pi k^2}\cot\left(\hbar\frac{k}{2}\right)(1+kt_F(\hbar)) + \frac{\hbar}{4\pi k}\csc^2\left(\hbar\frac{k}{2}\right)\right]\mathrm{e}^{-kt_F(\hbar)}$$
$$- \sum_{k=1}^{+\infty}\frac{1}{2k}\csc^2\left(\left(\frac{4\pi^2}{\hbar}\right)\frac{k}{2}\right)\mathrm{e}^{-k\left(\frac{2\pi}{\hbar}\right)t_F(\hbar)}, \quad (3.14)$$

where $t_F(\hbar)$ is related to the quantum mirror map $t_B(\hbar)$ (3.1) via (3.2). This can be written in closed form in $t_F(\hbar)$ when $\hbar = 2\pi(n/m)$ with $n, m \in \mathbb{N}_{>0}$ coprime,

$$\mathrm{J}_{\text{1-loop}}(\mu,\xi,\hbar) =$$
$$\left(\frac{2\pi^2(m^2-n^2)+3m^2t_F^2(\hbar)}{12\pi^2 nm^2}\right)\ln\left(1-\mathrm{e}^{-mt_F(\hbar)}\right) - \frac{t_F(\hbar)\,\mathrm{Li}_2\left(\mathrm{e}^{-mt_F(\hbar)}\right)}{2\pi^2 nm} - \frac{\mathrm{Li}_3\left(\mathrm{e}^{-mt_F(\hbar)}\right)}{2\pi^2 nm^2}$$
$$+\sum_{k=1}^{m-1}\frac{\csc^2\left(\hbar\frac{k}{2}\right)}{4\pi k}\mathrm{e}^{-kt_F(\hbar)}\left\{\left[\hbar+\left(\frac{1}{k}+t_F(\hbar)\right)\sin(\hbar k)\right]{}_3F_2\left[\begin{matrix}1,\ \frac{k}{m},\ \frac{k}{m}\\ 1+\frac{k}{m},\ 1+\frac{k}{m}\end{matrix}\middle|\,\mathrm{e}^{-mt_F(\hbar)}\right]\right.$$
$$\left.+\left(\frac{k/m}{(1+k/m)^2}\right)\mathrm{e}^{-mt_F(\hbar)}[\hbar+t_F(\hbar)\sin(\hbar k)]\,{}_3F_2\left[\begin{matrix}2,\ 1+\frac{k}{m},\ 1+\frac{k}{m}\\ 2+\frac{k}{m},\ 2+\frac{k}{m}\end{matrix}\middle|\,\mathrm{e}^{-mt_F(\hbar)}\right]\right\}$$
$$-\sum_{k=1}^{n-1}\frac{\csc^2\left(\left(\frac{4\pi^2}{\hbar}\right)\frac{k}{2}\right)}{2k}\mathrm{e}^{-k\left(\frac{2\pi}{\hbar}\right)t_F(\hbar)}\,{}_2F_1\left[\begin{matrix}1,\ \frac{k}{n}\\ 1+\frac{k}{n}\end{matrix}\middle|\,\mathrm{e}^{-n\left(\frac{2\pi}{\hbar}\right)t_F(\hbar)}\right], \quad (3.15)$$

where Li_q is the polylogarithm of order q, and ${}_pF_q$ is the generalized hypergeometric function. It is interesting to note that (3.14) admits also the following integral representation [18, eq. (3.9)]

$$\mathrm{J}_{\text{1-loop}}(\mu,\xi,\hbar) = -\frac{\hbar^2}{8\pi^4}\,\mathrm{Li}_3\left(\mathrm{e}^{-\left(\frac{2\pi}{\hbar}\right)t_F(\hbar)}\right)+$$
$$2\,\mathrm{Re}\int_0^{\infty\mathrm{e}^{\mathrm{i}0}}\mathrm{d}x\ \frac{x}{\mathrm{e}^{2\pi x}-1}\ln\left(1-2\cosh\left(\left(\frac{4\pi^2}{\hbar}\right)x\right)\mathrm{e}^{-\left(\frac{2\pi}{\hbar}\right)t_F(\hbar)}+\mathrm{e}^{-2\left(\frac{2\pi}{\hbar}\right)t_F(\hbar)}\right). \quad (3.16)$$

- The instanton part of the grand potential also consists of two parts: one coming from the instanton part of the NS free energy, and one being the instanton part of the GV free energy. Together they read

$$\mathrm{J}_{\text{inst}}(\mu,\xi,\hbar) = F_{\text{inst}}^{\text{GV}}\left(\left(\frac{2\pi}{\hbar}\right)t_F(\hbar), \left(\frac{2\pi}{\hbar}\right)t_B(\hbar), \frac{4\pi^2}{\hbar}\right)$$
$$+\left(-\frac{1}{2\pi}+\frac{t_F(\hbar)}{2\pi}\partial_{t_F}+\frac{t_B(\hbar)}{2\pi}\partial_{t_B}+\frac{\hbar}{2\pi}\partial_\hbar\right)F_{\text{inst}}^{\text{NS}}(t_F(\hbar),t_B(\hbar),\hbar) \quad (3.17)$$

where $F^{\rm NS}_{\rm inst}$ is the instanton part of the 5d Nekrasov free energy in the NS limit, and $F^{\rm GV}_{\rm inst}$ is similarly the instanton part of the 5d Nekrasov free energy in the GV limit. The leading order reads

$$F^{\rm NS}_{\rm inst}(t_F, t_B, \hbar) = \left[\frac{\mathrm{i}\left(1+\mathrm{e}^{\mathrm{i}\hbar}\right)}{(1-\mathrm{e}^{\mathrm{i}\hbar})(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F})(1-\mathrm{e}^{-\mathrm{i}\hbar}\mathrm{e}^{-t_F})}\right]\mathrm{e}^{-t_B} + \mathcal{O}\left(\mathrm{e}^{-2t_B}\right), \tag{3.18}$$

$$F^{\rm GV}_{\rm inst}(t_F, t_B, g_s) = \left[\frac{2\mathrm{e}^{\mathrm{i}g_s}}{\left(1-\mathrm{e}^{\mathrm{i}g_s}\right)^2\left(1-\mathrm{e}^{-t_F}\right)^2}\right]\mathrm{e}^{-t_B} + \mathcal{O}\left(\mathrm{e}^{-2t_B}\right), \tag{3.19}$$

and the all order constructions can be found in appendix B, equations (B.12) and (B.15) respectively.

Recall that the NS and GV contributions in $\mathrm{J}_{\text{1-loop}}$ (3.14) and $\mathrm{J}_{\rm inst}$ (3.17), when considered separately, have a dense set of pole on the real $\hbar$ or g_s axis, which makes these functions ill-defined. However, in the sum these poles cancel and the resulting function is perfectly well-defined for any value of g_s or $\hbar$. There are, of course, infinitely many ways to cancel these poles, see for instance [65] for a discussion on other proposals. However, the approach here is uniquely suited to yield the correct spectrum of the relativistic quantum integrable system. This mechanism was also used in ABJM theory [59, 66], where it successfully captures non-perturbative effects in its corresponding string dual [67]. Note also that the NS contributions in $\mathrm{J}_{\text{1-loop}}$ (3.14) and $\mathrm{J}_{\rm inst}$ (3.17) are perturbative in $\hbar$ for fixed $t_{B,F}$, but non-perturbative in $g_s = 4\pi^2/\hbar$ for fixed $t^D_{B,F} = (2\pi/\hbar)t_{B,F}$. Vice versa, the GV contributions in $\mathrm{J}_{\text{1-loop}}$ (3.14) and $\mathrm{J}_{\rm inst}$ (3.17) are perturbative in g_s for fixed $t^D_{B,F}$, but non-perturbative in $\hbar = 4\pi^2/g_s$ for fixed $t_{B,F} = (2\pi/g_s)t^D_{B,F}$.

From the point of view of spectral theory, it is then natural to consider the canonical ensemble. More precisely, one of the key statements of the TS/ST correspondence is that

$$\det(1+\kappa\rho) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{J}(\mu+\mathrm{i}2\pi k,\xi,\hbar)}, \qquad \kappa = \exp(\mu), \tag{3.20}$$

where ρ is the operator in (2.12) and J is the grand potential of (3.9). An important point is that (3.20) is entire in the full κ plane, therefore all the singularities in the closed string moduli space are smoothed out and the full quantity is background independent. To extract the partition function around specific points in the moduli space, we need to expand (3.20) accordingly. For instance, expanding around $\kappa = \infty$ leads to the large radius expansion, while expanding around $\kappa = 0$ gives

$$\det(1+\kappa\rho) = \sum_{N=0}^{\infty} \kappa^N Z(N,\hbar), \tag{3.21}$$

where $Z(N,\hbar)$, defined in (2.15), corresponds to the non-perturbative topological string partition function in the conifold frame [4, 16, 45].

3.3 The open string sector and the eigenfunctions

Let us now turn to the open sector of the topological string. As before, we focus on the example of local $\mathbb{F}_0$. The grand potential for the open topological string in the presence of a toric D-brane on an external leg of the toric diagram was introduced in [1, 2]. Here, we follow the formulation in [3, app. A], where the resummation in the open string modulus x is also performed.

The open string grand potential is

$$\mathrm{J}^{\mathrm{open}}(x,\mu,\xi,\hbar) = \mathrm{J}^{\mathrm{open}}_{\mathrm{p}}(x,\xi,\hbar) + \mathrm{J}^{\mathrm{open}}_{\text{1-loop}}(x,\mu,\xi,\hbar) + \mathrm{J}^{\mathrm{open}}_{\mathrm{inst}}(x,\mu,\xi,\hbar)\,, \tag{3.22}$$

where μ is the closed string modulus, ξ is the mass parameter, and x is the open string modulus. The functions appearing on the r.h.s. of (3.22) are defined as follows.

- The polynomial part in x is

$$\mathrm{J}^{\mathrm{open}}_{\mathrm{p}}(x,\xi,\hbar) = -\frac{\mathrm{i}}{\hbar}2\xi x - \frac{\mathrm{i}}{\hbar}\frac{x^2}{2} + \frac{1}{2}\left(\frac{2\pi}{\hbar} - 1\right)x\,. \tag{3.23}$$

- The one loop part is given by [3, eq. (A.24)]

$$\mathrm{J}^{\mathrm{open}}_{\text{1-loop}}(x,\mu,\xi,\hbar) = \ln\Phi_\beta\left(\frac{1}{2\pi\beta}\left(-x - \frac{t_F(\hbar)}{2}\right) + \mathrm{i}\frac{\beta}{2}\right) + \ln\Phi_\beta\left(\frac{1}{2\pi\beta}\left(-x + \frac{t_F(\hbar)}{2}\right) + \mathrm{i}\frac{\beta}{2}\right) \tag{3.24}$$

  where $\hbar = 2\pi\beta^2$ and $\ln\Phi_\beta$ is the logarithm of Faddeev's non-compact quantum dilogarithm, see appendix A. One can express (3.24) in closed form in terms of elementary functions and the classical dilogarithm Li_2 if $\hbar \in 2\pi\mathbb{Q}_{>0}$, by using (A.8). This 1-loop part of the grand potential consists again of contributions from both the NS and GV free energies, just as in the closed sector. See [3, app. A] for details.

- The instanton part $\mathrm{J}^{\mathrm{open}}_{\mathrm{inst}}$ consists also of a part coming from the NS free energy, and a part coming from the GV free energy

$$\mathrm{J}^{\mathrm{open}}_{\mathrm{inst}}(x,\mu,\xi,\hbar) = F^{\mathrm{open}}_{\mathrm{NS,inst}}(x,t_F(\hbar),t_B(\hbar),\hbar) + F^{\mathrm{open}}_{\mathrm{GV,inst}}\left(\left(\frac{2\pi}{\hbar}\right)x, \left(\frac{2\pi}{\hbar}\right)t_F(\hbar), \left(\frac{2\pi}{\hbar}\right)t_B(\hbar), \frac{4\pi^2}{\hbar}\right). \tag{3.25}$$

  Here, $F^{\mathrm{open}}_{\mathrm{NS,inst}}$ represents the NS limit of the refined open topological string free energy associated with a brane inserted on the outer leg of the toric diagram:

$$F^{\mathrm{open}}_{\mathrm{NS,inst}}(x,t_F,t_B,\hbar) = \frac{\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{\frac{t_F}{2}-x}\left(1+\mathrm{e}^{-t_F}+\mathrm{e}^{\mathrm{i}\hbar}(1+\mathrm{e}^{\mathrm{i}\hbar})\mathrm{e}^{-\frac{t_F}{2}-x}\right)}{(1-\mathrm{e}^{\mathrm{i}\hbar})(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F})(1-\mathrm{e}^{-\mathrm{i}\hbar}\mathrm{e}^{-t_F})\left(1+\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{\frac{t_F}{2}-x}\right)\left(1+\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-\frac{t_F}{2}-x}\right)}\mathrm{e}^{-t_B} + \mathcal{O}\left(\mathrm{e}^{-2t_B}\right), \tag{3.26}$$

and the all order definition is given in appendix B, equation (B.12). Similarly, $F^{\text{open}}_{\text{GV,inst}}$ is the open topological string free energy corresponding to a brane inserted in the outer leg of the toric diagram:

$$F^{\text{open}}_{\text{GV,inst}}(x,t_F,t_B,g_s) = \\ -\frac{\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{\frac{t_F}{2}-x}\left(1+\mathrm{e}^{-t_F}-2\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{-\frac{t_F}{2}-x}\right)}{(1-\mathrm{e}^{\mathrm{i}g_s})(1-\mathrm{e}^{-t_F})^2\left(1-\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{\frac{t_F}{2}-x}\right)\left(1-\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{-\frac{t_F}{2}-x}\right)}\mathrm{e}^{-t_B}+\mathcal{O}\left(\mathrm{e}^{-2t_B}\right), \quad (3.27)$$

and the all order expression for (3.27) can be found in appendix B, equation (B.15).

Similar to what was observed in the closed string sector, the role of the NS partition function in (3.22) in the open sector is also purely non-perturbative in g_s and it plays a crucial role in cancelling the poles in g_s of the GV part and making the full expression well defined. Hence (3.22) provide a well-defined non-perturbative completion for the open topological string partition function around the large radius frame. On the other side, from the spectral theory point of view, the perturbative contributions in $\hbar = 4\pi^2/g_s$ are captured by the NS partition function in (3.22), whereas the GV part remains purely non-perturbative in $\hbar$. We refer to [1, 2] for further details.

Note that $\exp\left(J^{\text{open}}_{\text{1-loop}}\right)$ has poles at

$$x = \pm\frac{t_F(\hbar)}{2} - \mathrm{i}2\pi\left(n-\frac{1}{2}\right) - \mathrm{i}\hbar m\,, \qquad n\in\mathbb{N}_{>0},\ m\in\mathbb{N}\,, \quad (3.28)$$

and similarly, $\exp(J^{\text{open}}_{\text{inst}})$ has poles coming from $F^{\text{open}}_{\text{NS,inst}}$ and $F^{\text{open}}_{\text{GV,inst}}$ respectively when

$$\begin{aligned} x &= \pm\frac{t_F(\hbar)}{2} + \mathrm{i}2\pi\left(m-\frac{1}{2}\right) + \mathrm{i}\hbar n\,, \qquad n\in\mathbb{N}_{>0},\ m\in\mathbb{Z}\,,\\ x &= \pm\frac{t_F(\hbar)}{2} + \mathrm{i}2\pi\left(n-\frac{1}{2}\right) + \mathrm{i}\hbar m\,, \qquad n\in\mathbb{N}_{>0},\ m\in\mathbb{Z}\,. \end{aligned} \quad (3.29)$$

Note that only the poles with $n\leqslant N$ occur at order $\exp(-N\,t_B)$ in $F^{\text{open}}_{\text{NS,inst}}$ or $F^{\text{open}}_{\text{GV,inst}}$. These poles should be related to the transition from the external to the internal leg of the toric diagram and they do not disappear in the open string grand potential $\mathrm{J}^{\text{open}}(x,\mu,\xi,\hbar)$. They only disappear in the final combination (3.35). This means that, although the starting point is a brane on the outer leg of the toric diagram, once we consider the combination in (3.35), it no longer matters that the brane was initially placed on the outer leg.

Broadly speaking, the open TS/ST correspondence connects (3.22) to the eigenfunctions of the quantum mirror curve (2.7). It is important to emphasize that there are numerous ways to construct formal solutions to (2.3). For instance, consider

$$\exp\left[\mathrm{J}^{\text{open}}_{\text{p}}(x,\xi,\hbar)+\mathrm{J}^{\text{open}}_{\text{1-loop}}(x,\mu,\xi,\hbar)+F^{\text{open}}_{\text{NS,inst}}(x,t_F(\hbar),t_B(\hbar),\hbar)\right] \quad (3.30)$$

While this expression formally satisfies (2.3), it is not a well-defined function for $\hbar\in\mathbb{R}_{>0}$ due to the dense set of poles at $\hbar\in\pi\mathbb{Q}$, and it fails to satisfy the analytic properties required by a proper eigenfunction as discussed below (2.3). The exponential of (3.22) on the other hand, as well as each individual term in (3.35), would give a well-defined solution to the difference equation for all $\hbar \in \mathbb{R}_{>0}$. However, it does not have the correct analytical properties or asymptotic behaviour to be in the domain of the quantized mirror curve. Hence it does not qualify as an eigenfunction either.

To construct proper eigenfunctions, it is useful to introduce the full grand potential, which is defined as:

$$\mathrm{J}(x,\mu,\xi,\hbar) = \mathrm{J}(\mu,\xi,\hbar) + \mathrm{J}^{\mathrm{open}}(x,\mu,\xi,\hbar) \; . \tag{3.31}$$

It was further conjectured in [1, 2] that one of the terms in the sum (1.5) should be given by:

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} \; . \tag{3.32}$$

We identify this term to be $\omega(x,\kappa)$ in (2.46). In addition, building on the structure found in (2.46) for the second term, we can now express the full eigenfunctions in a compact form using (3.31). Specifically, we have:

$$\psi(x,\kappa) = \sum_{k\in\mathbb{Z}} \sum_{\sigma\in\{1,2\}} \mathrm{e}^{\mathrm{J}_\sigma(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} \, , \qquad\qquad \kappa = \mathrm{e}^{\mu} \, , \tag{3.33}$$

where:

$$\begin{aligned} \mathrm{J}_1(x,\mu,\xi,\hbar) &= \mathrm{J}(x,\mu,\xi,\hbar), \\ \mathrm{J}_2(x,\mu,\xi,\hbar) &= \frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2} + \frac{\pi x}{\hbar} + \mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi,\xi,\hbar). \end{aligned} \tag{3.34}$$

Therefore, starting from the grand potential, we can construct a class of well-defined, entire solutions to the quantum mirror curve (2.2) as

$$\boxed{\psi(x,\kappa) = \sum_{k\in\mathbb{Z}} \left(\mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2} + \frac{\pi x}{\hbar} + \mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi+\mathrm{i}2\pi k,\xi,\hbar)} \right)} \; . \tag{3.35}$$

Let us make some comments on this result

- The grand potential $\mathrm{J}(x,\mu,\xi,\hbar)$ has poles at finite values of x, which can be interpreted as a consequence of the fact that its construction starts from a brane placed on the outer leg of the toric diagram. In contrast, the combination in (3.35) is entire in x, which suggests that, although the starting point involves a brane on the outer leg, this is not important in the final expression.

- As we reviewed in the introduction, the the summation over k in (3.20) smooths all the singularities in the closed string moduli space, parametrized by κ, leading to an entire function in κ. The sum over the two terms in (3.33) and (3.35) play the same role, but for the open string modulus x. Indeed each of the two terms in (3.35) individually has singularities at finite value of x which are smooth out once we add the two terms (and we sum over k). This holds even off-shell, i.e. for $\kappa \neq -\mathrm{e}^{E_n}$.

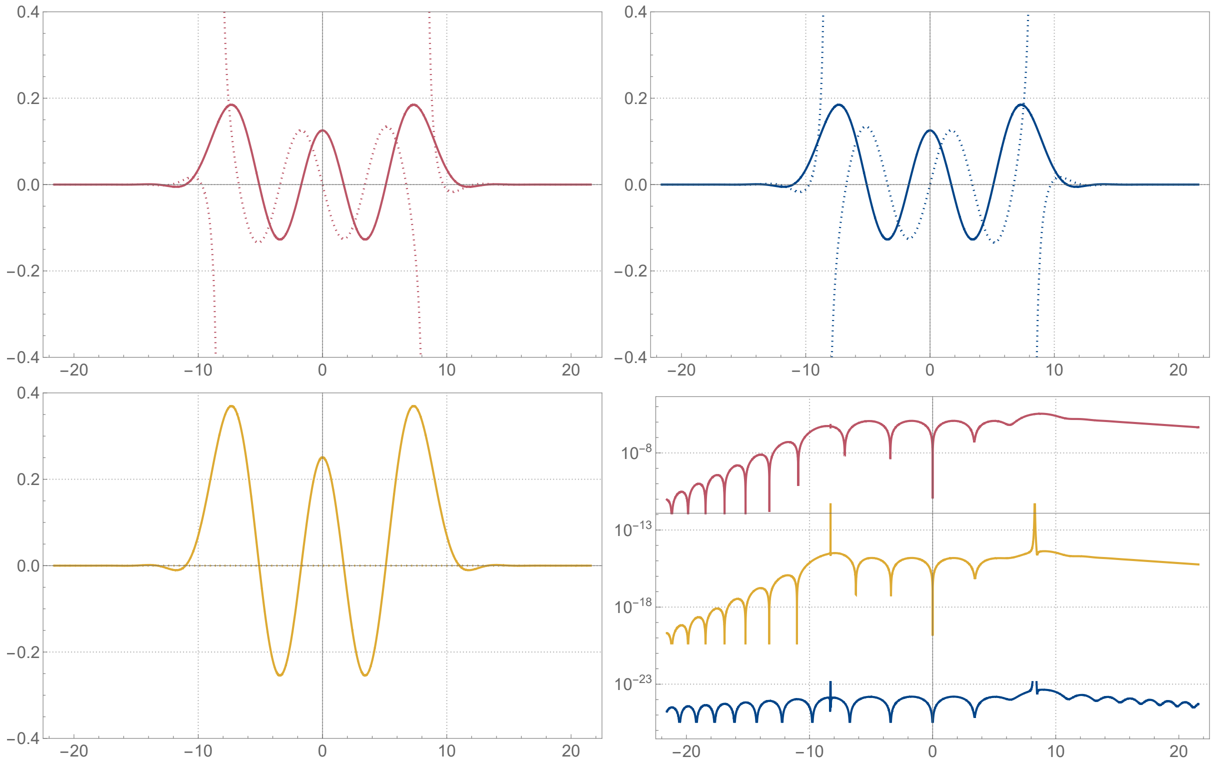

Figure 2. The on-shell eigenfunction from topological strings for $\hbar = 8\pi/3$ and $\xi = -16/53$ at energy $E = E_4$ corresponding to the 4th excited state. From top left to bottom right, the first term in (3.35), the second in (3.35), the complete eigenfunction and the difference with the numerical eigenfunction for 0, 2 and 4 terms in the large t_B expansion of $J^{\text{open}}_{\text{inst}}$. The real part is the solid line and the imaginary part is the dotted line.

- As can be seen in figure 2 and in appendix D, when evaluated on-shell, the two terms in (3.35) have the same real part (which is pole-free) while having opposite imaginary parts (with poles that cancel in the sum).
- For some values of the parameters, the sum in (3.35) is only pointwise convergent. As a result, testing (3.35) for these values becomes more challenging. This is the case for instance if we take $\text{Im}(x) > (2\pi + \hbar)/2$.

Some numerical tests of the above properties are presented in figure 2, figure 3, and appendix D.

4 Four-dimensional limits

It is well-known that topological string theory on non-local CY manifolds can be used to geometrically engineer four-dimensional $\mathcal{N} = 2$ theories [30, 31]. In the case of a local $\mathbb{F}_0$ geometry, the corresponding four-dimensional gauge theory is $\mathcal{N} = 2$, SU(2) SYM. In this section, we study the four-dimensional limit of the open TS/ST correspondence. There are two distinct four-dimensional limits one can implement: the standard limit, discussed in subsection 4.1, and the dual limit, presented in subsection 4.2.

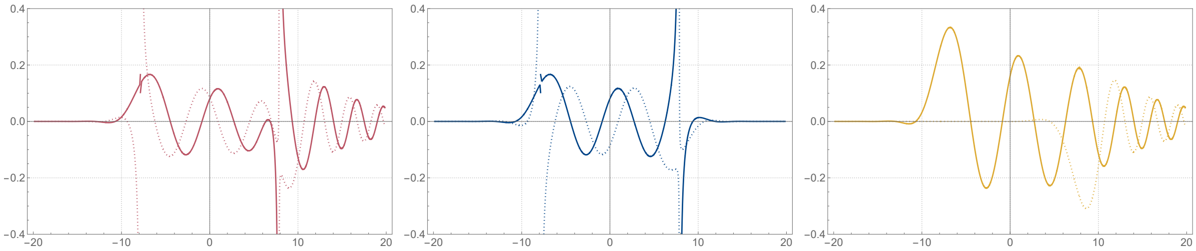

Figure 3. The off-shell eigenfunction from topological strings for $\hbar = 8\pi/3$ and $\xi = -16/53$ at energy $E \approx 7.26$, which is halfway between the 3th and 4th excited state. From top left to bottom, the first term in (3.35), the second term in (3.35) and the complete off-shell eigenfunction. The real part is the solid line and the imaginary part is the dotted line.

4.1 The standard four-dimensional limit

Let us consider the mirror curve for local $\mathbb{F}_0$ (2.1). In the standard 4d limit the parameters of the curve are scaled as [30]

$$x = R x_{4\mathrm{d}}\,, \qquad \mathrm{e}^{2\xi} = \frac{1}{\sqrt{t}R^2}\,, \qquad \kappa = \frac{1}{\sqrt{t}R^2}\big(-2 - R^2 E\big)\,, \qquad \hbar = R\epsilon\,, \tag{4.1}$$

and the limit is taken as $R \to 0$. In this limit (2.2) becomes the Fourier transformed modified Mathieu operator

$$\mathrm{O}_{\mathrm{FMa}} = \sqrt{t}\big(\mathrm{e}^{\hat{y}} + \mathrm{e}^{-\hat{y}}\big) + \hat{x}^2\,, \qquad [\hat{x}, \hat{y}] = \mathrm{i}\epsilon\,, \qquad t, \epsilon > 0\,, \tag{4.2}$$

where we omit the subscripts 4d in the variable x for the sake of notation. The corresponding eigenvalue equation reads

$$\sqrt{t}(\phi(x - \mathrm{i}\epsilon, E) + \phi(x + \mathrm{i}\epsilon, E)) + x^2\phi(x, E) - E\phi(x, E) = 0\,. \tag{4.3}$$

If we perform a Fourier transform on (4.2), i.e. we exchange position and momentum operator, we obtain the modified Mathieu operator in the standard form

$$\Big(\sqrt{t}\big(\mathrm{e}^{q} + \mathrm{e}^{-q}\big) - \epsilon^2\partial_q^2 - E\Big)\widehat{\phi}(q, E) = 0\,. \tag{4.4}$$

The eigenfunctions of (4.3) and (4.4) are related by a Fourier transform,

$$\widehat{\phi}(q, E) = \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx/\epsilon}\, \phi(x, E)\,. \tag{4.5}$$

The quantization condition for the spectrum of (4.2) is derived analogously to the one for (2.2), i.e. by imposing analytic continuation within the strip (2.5), and requiring square integrability of $\phi(x, E)$. On the other hand, for the Fourier transformed operator (4.4), this amounts to ask for square integrability of $\widehat{\phi}(q, E)$. The energy spectrum $\{E_n\}_{n\geqslant 0}$ is the same for both operators.

For the sake of notation, we work at $\epsilon = 1$. The ϵ-dependence can be reinstated by appropriately shifting the variable as follows:

- For the modified Mathieu operator (4.4), we can work at $\epsilon = 1$ and then re-install the ϵ dependence by shifting

$$t \to t/\epsilon^4 , \qquad E \to E/\epsilon^2 . \tag{4.6}$$

- For the Fourier transformed modified Mathieu (4.2) (4.3), we can work at $\epsilon = 1$ and then re-install the ϵ dependence by shifting

$$x \to x/\epsilon , \qquad t \to t/\epsilon^4 , \qquad E \to E/\epsilon^2 . \tag{4.7}$$

4.1.1 Result

The standard four-dimensional limit (4.1) was examined in the context of the closed TS/ST correspondence in [68–70]. In [69] it was shown that in this limit (3.20) gives

$$\det\left(1 + \frac{E}{\mathrm{O}_{\mathrm{FMa}}}\right) = A(t)\left(\frac{\sinh\left(\frac{\mathrm{i}}{2}\partial_\sigma F^{\mathrm{4d}}_{\mathrm{NS}}(\sigma,t)\right)}{\mathrm{i}\sinh\left(2\pi\sigma\right)}\right), \tag{4.8}$$

where $A(t)$ is a normalization constant independent of E, chosen such that the left-hand side, when evaluated at $E = 0$, equals 1. In (4.8), we note by $F^{\mathrm{4d}}_{\mathrm{NS}}$ is the full, four-dimensional NS free energy associated with $\mathcal{N} = 2$, SU(2) SYM

$$\begin{aligned} F^{\mathrm{4d}}_{\mathrm{NS}}(\sigma,t) = &-\psi^{(-2)}(1-\mathrm{i}2\sigma) - \psi^{(-2)}(1+\mathrm{i}2\sigma) - \sigma^2\ln(t) \\ &- \left(\frac{2}{4\sigma^2+1}\right)t - \left(\frac{20\sigma^2-7}{4(4\sigma^2+1)^3(\sigma^2+1)}\right)t^2 + \mathcal{O}\left(t^3\right), \end{aligned} \tag{4.9}$$

where $\psi^{(-2)}$ is the polygamma function of order -2. Higher orders in the t expansion can be found in (B.30) and according to the all order expression (B.28). The variable σ and the energy E are related via the quantum Matone relation[11] [71, 72],

$$E = -t\partial_t F^{\mathrm{4d}}_{\mathrm{NS}}(\sigma,t) , \qquad 2\sigma \notin \mathrm{i}\mathbb{Z} . \tag{4.10}$$

The quantization condition for the operator spectrum, determined by the vanishing of the determinant (4.8), exactly reproduces the NS quantization condition [73]:

$$\partial_\sigma F^{\mathrm{4d}}_{\mathrm{NS}}(\sigma,t) = 2\pi(n+1) , \qquad n \in \mathbb{N} , \tag{4.11}$$

see also [74, 75] and reference therein. For a fixed value of t, we denote by $\{\sigma_n\}_{n\geqslant 0}$ the solution to (4.11). These give the energy spectrum of the operator (4.3) via the quantum Matone relation (4.10).

By implementing the four dimensional limit (4.1) on the eigenfunctions expression appearing on the r.h.s. of (3.35) we find (see subsubsection 4.1.2)

$$\sum_{k\in\mathbb{Z}}\left(\mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}2\pi k,\xi,\hbar)} + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}+\mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi+\mathrm{i}2\pi k,\xi,\hbar)}\right) \to \phi_1(x,\sigma,t) + \phi_2(x,\sigma,t) \tag{4.12}$$

[11] This relation is simply the 4d limit of the Wilson loop (3.4).

where[12]

$$\boxed{\phi_1(x,\sigma,t) = -\mathrm{i}\exp\left(\frac{\mathrm{i}}{4}\partial_\sigma F^{4\mathrm{d}}_{\mathrm{NS}}(\sigma,t)\right)t^{-\mathrm{i}\frac{x}{2}}\Gamma(\mathrm{i}(x+\sigma))\Gamma(\mathrm{i}(x-\sigma))Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}(-x,\sigma,t)} \tag{4.13}$$

$$\boxed{\phi_2(x,\sigma,t) = \phi_1(-x,\sigma,t)\left[\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\partial_\sigma F^{4\mathrm{d}}_{\mathrm{NS}}(\sigma,t)}(\mathrm{e}^{2\pi x}-\mathrm{e}^{2\pi\sigma}) - \mathrm{e}^{\frac{\mathrm{i}}{2}\partial_\sigma F^{4\mathrm{d}}_{\mathrm{NS}}(\sigma,t)}(\mathrm{e}^{2\pi x}-\mathrm{e}^{-2\pi\sigma})}{\mathrm{e}^{2\pi\sigma}-\mathrm{e}^{-2\pi\sigma}}\right]} \tag{4.14}$$

with $Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}$ denoting the instanton part of the NS function in the presence of a surface defect

$$Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}(x,\sigma,t) = 1 + \left[\frac{1+2(\mathrm{i}x+1)}{(1+4\sigma^2)((\mathrm{i}x+1)^2+\sigma^2)}\right]t + \mathcal{O}(t^2)\ . \tag{4.15}$$

Higher-order terms are provided in (B.31), while the full all-order definition is given in (B.29). We obtain then the following eigenfunction of (4.3)

$$\boxed{\phi(x,E,t) = \phi_1(x,\sigma,t) + \phi_2(x,\sigma,t)} \tag{4.16}$$

where E and σ are related as in (4.10). Let us make some comments on the above result.

- The finite difference equation (4.3) has extensive families of formal solutions. For instance, both functions (4.14) and (4.13) are solutions to the Fourier transform Mathieu equation (4.3). Each of these functions is meromorphic, with poles located at $x = \pm\sigma + \mathrm{i}\ell$, where $\ell \in \mathbb{Z}$. However, what makes the solution (4.16) special is that in the summation, all poles cancel, yielding a final expression that is entire in x, even when evaluated at generic values of the energy. For this to happen the factor in the square brackets in (4.14) is crucial.

- Although the symmetric structure of the two contributions from (2.46) is lost in the 4d limit, the key feature that remains is that only the sum (4.16) of these two contributions is entire in x.

- When we evaluate the above eigenfunctions (4.16), (4.14), (4.13) on-shell, i.e. on the locus (4.11), we obtain

$$\begin{aligned}\phi(x,E_n,t) = (-)^n \mathrm{e}^{\mathrm{i}\frac{\pi}{2}n} t^{\frac{\mathrm{i}x}{2}}\Gamma(\mathrm{i}(-x+\sigma_n))\Gamma(\mathrm{i}(-x-\sigma_n))Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}(x,\sigma_n,t) \\ + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}n} t^{-\frac{\mathrm{i}x}{2}}\Gamma(\mathrm{i}(x+\sigma_n))\Gamma(\mathrm{i}(x-\sigma_n))Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}(-x,\sigma_n,t)\,,\end{aligned} \tag{4.17}$$

  which is the well-known form of the eigenfunctions obtained in [33, 35, 37, 38, 41]. Notice that the relation between the two terms in (4.17) once again exhibits the same symmetry structure as in (2.46).

- Let us now consider the asymptotic behaviour of (4.14) and (4.13) for $\mathrm{Re}(x) \to \pm\infty$ with constant $\mathrm{Im}(x)$. The non-trivial asymptotics is determined by the part involving

[12] The overall normalization $-\mathrm{i}\exp\left(\frac{\mathrm{i}}{4}\partial_\sigma F_{\mathrm{NS}}(\sigma,t)\right)$ was added so that the on-shell eigenfunctions are real.

Γ functions as well as the term inside the square brackets in (4.14). For $\phi^{(1)}$ we get an exponential decay in both directions

$$\left|\phi^{(1)}(x+\mathrm{i}y,\sigma,t)\right| \propto \mathrm{e}^{\mp\pi x-(1+2y)\ln(\pm x)}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right), \qquad x\to\pm\infty\,, \tag{4.18}$$

for any constant $y\in\mathbb{R}$. For $\phi^{(2)}$ instead we have

$$\left|\phi^{(2)}(x+\mathrm{i}y,\sigma,t)\right| \propto \begin{cases} \left(\frac{\sinh\left(\frac{\mathrm{i}}{2}\partial_\sigma F^{4\mathrm{d}}_{\mathrm{NS}}(\sigma,t)\right)}{\mathrm{i}\sinh(2\pi\sigma)}\right)\mathrm{e}^{+\pi x-(1-2y)\ln(x)}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right), & x\to+\infty \\ \left(\frac{\sinh\left(\frac{\mathrm{i}}{2}\partial_\sigma F^{4\mathrm{d}}_{\mathrm{NS}}(\sigma,t)-2\pi\sigma\right)}{\mathrm{i}\sinh(2\pi\sigma)}\right)\mathrm{e}^{+\pi x-(1-2y)\ln(-x)}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right), & x\to-\infty \end{cases} \tag{4.19}$$

for any constant $y\in\mathbb{R}$. The overall trigonometric terms come from the factor in square brackets in (4.14). Hence the complete eigenfunction is square-integrable[13] if and only if

$$\frac{\sinh\left(\frac{\mathrm{i}}{2}\partial_\sigma F^{4\mathrm{d}}_{\mathrm{NS}}(\sigma,t)\right)}{\mathrm{i}\sinh(2\pi\sigma)}=0\,, \tag{4.20}$$

which is precisely the vanishing of the Fredholm determinant (4.8).

4.1.2 Derivation

Here we sketch the derivation of the result (4.12). We are interested in the standard 4d limit (4.1) of the eigenfunctions as they appear in terms of the gauge theory grand potential in (3.35). The scaling of the complex structure parameters as given in (4.1) corresponds to the following scaling of the Kähler parameters

$$t_B(\hbar)=-\ln\left(t\hbar^4\right)+2\pi\sigma\hbar\,, \qquad t_F(\hbar)=2\sigma\hbar\,, \tag{4.21}$$

and in addition, the negative sign of κ in (4.1) is reflected in a shift

$$t_{B,F}(\hbar)\to t_{B,F}(\hbar)-\mathrm{i}2\pi\,. \tag{4.22}$$

The standard 4d limit is then taking $\hbar\to 0$ from above while keeping t, σ and $x_{4\mathrm{d}}$ fixed with $\mathrm{Re}(\sigma),t>0$.

We will not keep track of the overall normalization of the 5d eigenfunctions, since we didn't fix it in the first place. Hence, we can freely choose an overall normalization, and it turns out that normalizing the eigenfunctions (3.35) as

$$\mathrm{e}^{-\mathrm{J}(\mu,\xi,\hbar)}\sum_{k\in\mathbb{Z}}\left(\mathrm{e}^{\mathrm{J}(x,\mu+\mathrm{i}\pi(2k-1),\xi,\hbar)}+\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi x}{\hbar}+\mathrm{J}(-x-\mathrm{i}\pi,\mu+\mathrm{i}\pi(2k),\xi,\hbar)}\right), \tag{4.23}$$

will be a convenient choice[14].

[13] Note that each of the two terms themselves are never square-integrable for all $y\in\mathbb{R}$ due to the simple poles at $x+\mathrm{i}y=\pm\sigma+\mathrm{i}k$ for $k\in\mathbb{Z}$.

[14] Interchanging the sum over k with any limit is quite subtle, since the sum is not always uniformly converging in x. However, we will not go into this issue and simply note that interchanging the sum and limit in this case gives a perfectly well-behaved answer in line with our expectations.

Let us first look at the closed sector, closely following [68, 69]. After applying the normalization one gets for the polynomial part of the closed grand potential (3.13)

$$\begin{aligned}&\mathrm{J}_{\mathrm{p}}(\mu+\mathrm{i}\pi\ell,\xi,\hbar)-\mathrm{J}_{\mathrm{p}}(\mu,\xi,\hbar)=\\&\qquad\frac{\pi\ell^2\ln\left(t\hbar^4\right)}{2\hbar}-\mathrm{i}\frac{\pi^2\ell(2\ell^2-1)}{3\hbar}-\mathrm{i}\ell\sigma\ln\left(t\hbar^4\right)-2\pi\ell^2\sigma+\mathcal{O}(\hbar)\,,\end{aligned}\tag{4.24}$$

where $\ell=2k-1$ for the first term and $\ell=2k$ for the second term. Hence one can see that the dominating contributions in the sum over the shifts $k\in\mathbb{Z}$ are given by $k=0,1$ for the first term and by $k=0$ term for the second term. One can already note that this gives a trivial closed sector contribution for the second term, due to our normalization. The 4d limit for the 1-loop and instantons part of the closed grand potential can be dealt with as done in [68, 69] and one finds

$$\begin{aligned}&\mathrm{J}(\mu\pm\mathrm{i}\pi,\xi,\hbar)-\mathrm{J}(\mu,\xi,\hbar)=\\&\qquad\frac{\pi}{2}\frac{\ln(t\hbar^4)}{\hbar}\mp\mathrm{i}\frac{\pi}{2}\pm\frac{\mathrm{i}}{2}\partial_\sigma F_{\mathrm{NS}}^{\mathrm{4d}}(t,\sigma)-\ln\left(\mathrm{e}^{2\pi\sigma}-\mathrm{e}^{-2\pi\sigma}\right)+\mathcal{O}(\hbar)\,.\end{aligned}\tag{4.25}$$

where the 4d NS free energy is given in (4.9).

Let us now turn our attention to the open sector. The polynomial part of the open grand potential (3.23) can be dealt with straightforwardly. To take the standard 4d limit on the 1-loop part (3.24) it is useful to first use the quasi-periodicity of quantum dilogarithm (A.7) to write[15]

$$\begin{aligned}&\mathrm{J}^{\mathrm{open}}_{\text{1-loop}}(x,\mu\pm\mathrm{i}\pi,\xi,\hbar)=-2\ln(\hbar)-\ln\left(x_{\mathrm{4d}}^2-\sigma^2\right)-2\pi x_{\mathrm{4d}}+\ln\left(\mathrm{e}^{2\pi x_{\mathrm{4d}}}-\mathrm{e}^{\mp2\pi\sigma}\right)\\&+\ln\Phi_\beta\left[\beta(\sigma-x_{\mathrm{4d}})+\frac{\mathrm{i}}{2}\left(\beta^{-1}-\beta\right)\right]+\ln\Phi_\beta\left[\beta(-\sigma-x_{\mathrm{4d}})+\frac{\mathrm{i}}{2}\left(\beta^{-1}-\beta\right)\right]+\mathcal{O}(\hbar)\end{aligned}\tag{4.26}$$

$$\begin{aligned}&\mathrm{J}^{\mathrm{open}}_{\text{1-loop}}(-x-\mathrm{i}\pi,\mu,\xi,\hbar)=-2\ln(\hbar)-\ln\left(x_{\mathrm{4d}}^2-\sigma^2\right)\\&+\ln\Phi_\beta\left[\beta(\sigma+x_{\mathrm{4d}})+\frac{\mathrm{i}}{2}\left(\beta^{-1}-\beta\right)\right]+\ln\Phi_\beta\left[\beta(-\sigma+x_{\mathrm{4d}})+\frac{\mathrm{i}}{2}\left(\beta^{-1}-\beta\right)\right]+\mathcal{O}(\hbar)\end{aligned}\tag{4.27}$$

with $\beta=\sqrt{\hbar/2\pi}$. In the end, we are interested in the exponential of the grand potential so all equalities are modulo integer multiples of $\mathrm{i}2\pi$. Let us introduce a variable z which is

$$z=\frac{1}{2}\pm\mathrm{i}(x_{\mathrm{4d}}\pm\sigma)\,,\tag{4.28}$$

with the signs chosen independently and $2|\mathrm{Im}(x\pm\sigma)|<1$ or $0<\mathrm{Re}(z)<1$. To compute the expansion of the quantum dilogarithms in (4.26) and (4.27), we use the integral

[15] When implementing the limit we assume for simplicity $2|\mathrm{Im}(x\pm\sigma)|<1$. However, the final result extends to all values of x in the complex plane.

representations in (A.11) and (A.12). We find that

$$\ln\Phi_\beta\left(\frac{\mathrm{i}}{2}\beta^{-1}-\mathrm{i}\beta z+\mathcal{O}(\beta^3)\right)=$$
$$-\mathrm{i}\frac{\pi}{12\beta^2}+z\ln\left(2\pi\beta^2\right)-\frac{\ln(2\pi)}{2}+\mathrm{i}\frac{\pi}{2}z+\ln\Gamma\left(z+\frac{1}{2}\right)+\mathcal{O}(\beta^2) \quad (4.29)$$

Combining the polynomial and 1-loop parts of the grand potential one finds

$$\mathrm{J}_{\mathrm{p}}^{\mathrm{open}}(x,\xi,\hbar)+\mathrm{J}_{\text{1-loop}}^{\mathrm{open}}(x,\mu\pm\mathrm{i}\pi,\xi,\hbar)=-\frac{\mathrm{i}\pi^2}{3\hbar}-\ln(2\pi\hbar)-\mathrm{i}\frac{\pi}{2}$$
$$+\ln\Gamma(\mathrm{i}(-x_{4\mathrm{d}}+\sigma))+\ln\Gamma(\mathrm{i}(-x_{4\mathrm{d}}-\sigma))+\mathrm{i}\frac{x_{4\mathrm{d}}}{2}\ln(t)+\ln\left(\mathrm{e}^{2\pi x_{4\mathrm{d}}}-\mathrm{e}^{\mp2\pi\sigma}\right)+\mathcal{O}(\hbar) \quad (4.30)$$

$$\frac{\mathrm{i}}{\hbar}\frac{\pi^2}{2}+\frac{\pi}{\hbar}x+\mathrm{J}_{\mathrm{p}}^{\mathrm{open}}(-x-\mathrm{i}\pi,\xi,\hbar)+\mathrm{J}_{\text{1-loop}}^{\mathrm{open}}(-x-\mathrm{i}\pi,\mu,\xi,\hbar)=\frac{\pi}{2}\frac{\ln(t\hbar^4)}{\hbar}-\frac{\mathrm{i}\pi^2}{3\hbar}-\ln(2\pi\hbar)$$
$$+\ln\Gamma(\mathrm{i}(x_{4\mathrm{d}}+\sigma))+\ln\Gamma(\mathrm{i}(x_{4\mathrm{d}}-\sigma))-\mathrm{i}\frac{x}{2}\ln(t)+\mathcal{O}(\hbar)\ . \quad (4.31)$$

Regarding the instanton part for the open sector, one can see that the NS part is a rational function of $\exp((x)-t_F/2)$, $\exp(-t_F)$ and $\exp(-t_B)$. Hence the shifts of $t_{B,F}$ by $\mathrm{i}4\pi k$ act trivially and the only difference between the first and second term is a change in the sign of x. The GV part on the other hand vanishes in the standard 4d limit. The resulting defect instanton partition function can be found in (B.34).

Hence putting all parts of the grand potential for both the closed and open sectors together we get an overall divergent factor

$$\frac{\pi}{2}\frac{\ln(t\hbar^4)}{\hbar}-\frac{\mathrm{i}\pi^2}{3\hbar}-\ln(2\pi\hbar)\,, \quad (4.32)$$

so that after appropriate normalization one finds that (4.23) reduces exactly to (4.14), (4.13) and (4.16) in the standard 4d limit (4.21).

4.2 The dual four-dimensional limit

A few years ago, [18, 32] showed that, starting from the quantum mirror curve (2.7), one can implement another scaling limit to connect with the four-dimensional gauge theory. In this limit, we take

$$4\xi=4\pi\mathrm{i}\sigma-\frac{2\pi}{R}\log\left(R^4t\right)\,,\qquad \mathrm{e}^{2\xi}\kappa=2\cos(2\pi\sigma)\,,\qquad \hbar=\frac{4\pi^2}{R}\,, \quad (4.33)$$

and send $R\to0$. One of the key differences between the limits (4.1) and (4.33) is that in (4.1), we take $\hbar\to0$, whereas in (4.33), we take $\hbar\to\infty$. Hence, we refer to (4.33) as the "dual" four-dimensional limit.

Applying the scaling (4.33) and taking the limit $R \to 0$ on the operator kernel (2.12) yields the integral operator $\rho_{\mathrm{GV}}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with kernel [18]

$$\rho_{\mathrm{GV}}(q,p) = \frac{\mathrm{e}^{-4t^{1/4}\cosh q}\mathrm{e}^{-4t^{1/4}\cosh p}}{\cosh\left(\frac{q-p}{2}\right)}\,, \tag{4.34}$$

and we look for square-integrable eigenfunctions

$$\int_{\mathbb{R}} \mathrm{d}p\ \rho_{\mathrm{GV}}(q,p)\widehat{\varphi}_n(p,\widehat{E}_n,t) = \widehat{E}_n\widehat{\varphi}_n(q,\widehat{E}_n,t)\,. \tag{4.35}$$

Interestingly, (4.34) first appeared in the literature in the 1970s in studies of the 2d Ising model and the theory of Painlevé equations, see [76–79]. We will refer to it as McCoy-Tracy-Wu operator. The connection between the quantum mirror curve (2.7) and the Painlevé kernel (4.34) made it possible to prove the TS/ST correspondence for local $\mathbb{F}_0$ in this particular dual 4d limit [18]. See also [21] for the generalization to all $Y^{N,0}$ geometries.

The spectral problem (4.35) was solved in [3], where it was shown that the eigenfunctions can be explicitly computed using the partition function of four-dimensional, $\mathcal{N}=2$, SU(2) supersymmetric Yang-Mills theory in the GV (or self-dual) phase of the Ω-background ($-\epsilon_1 = \epsilon_2 = 1$), with the inclusion of a surface defect. More precisely we have [3]

$$\begin{aligned}\widehat{\varphi}(q,\widehat{E}_n,t) = \int_{\mathbb{R}} \mathrm{d}x\ \mathrm{e}^{\mathrm{i}2qx} \sum_{k\in\mathbb{Z}} \Bigg(& Z_{\mathrm{GV}}^{2\mathrm{d}/4\mathrm{d}}\left(x, k+\frac{1}{2}+\mathrm{i}\widehat{\sigma}_n, t\right) Z_{\mathrm{GV}}^{4\mathrm{d}}\left(k+\frac{1}{2}+\mathrm{i}\widehat{\sigma}_n, t\right) \\ & + Z_{\mathrm{GV}}^{2\mathrm{d}/4\mathrm{d}}\left(-x-\frac{\mathrm{i}}{2}, k+\mathrm{i}\widehat{\sigma}_n, t\right) Z_{\mathrm{GV}}^{4\mathrm{d}}(k+\mathrm{i}\widehat{\sigma}_n, t)\Bigg)\,,\end{aligned} \tag{4.36}$$

$$\widehat{E}_n = 2\pi\,\mathrm{sech}(2\pi\widehat{\sigma}_n)\,, \tag{4.37}$$

where the gauge theory partition function of the defect is given by

$$Z_{\mathrm{GV}}^{2\mathrm{d}/4\mathrm{d}}(x,\sigma,t) = t^{\mathrm{i}\frac{x}{2}}\Gamma\left(-\mathrm{i}x-\sigma+\frac{1}{2}\right)\Gamma\left(-\mathrm{i}x+\sigma+\frac{1}{2}\right)Z_{\mathrm{GV,inst}}^{2\mathrm{d}/4\mathrm{d}}(x,\sigma,t)\,, \tag{4.38}$$

$$\begin{aligned}Z_{\mathrm{GV,inst}}^{2\mathrm{d}/4\mathrm{d}}(x,\sigma,t) = 1 - & \left[\frac{\widetilde{x}}{2\sigma^2(\widetilde{x}^2-\sigma^2)}\right]t \\ & + \left[\frac{\widetilde{x}(\widetilde{x}+1)^2 - \widetilde{x}(10\widetilde{x}^2+19\widetilde{x}+10)\sigma^2 + (8\widetilde{x}^2+30\widetilde{x}+9)\sigma^4}{4\sigma^4(4\sigma^2-1)^2(\widetilde{x}^2-\sigma^2)\left((\widetilde{x}+1)^2-\sigma^2\right)}\right]t^2 + \mathcal{O}(t^3)\,,\end{aligned} \tag{4.39}$$

with $\widetilde{x} = \mathrm{i}x + 1/2$ and higher orders in the t expansion can be found from the definition (B.32). The $\widehat{\sigma}_n \in \mathbb{R}\setminus\{0\}$ are solutions to

$$\sum_{k\in\mathbb{Z}} Z_{\mathrm{GV}}^{4\mathrm{d}}\left(k+\frac{1}{2}+\mathrm{i}\widehat{\sigma}_n, t\right) = 0\,, \tag{4.40}$$

and $Z^{\rm 4d}_{\rm GV}(\sigma,t)$ is the Nekrasov function in the GV-phase of the Ω-background:

$$Z^{\rm 4d}_{\rm GV}(\sigma,t)=\frac{t^{\sigma^2}}{G(1-2\sigma)G(1+2\sigma)}\left(1+\frac{t}{2\sigma^2}+\frac{(8\sigma^2+1)t^2}{4\sigma^2(4\sigma^2-1)^2}+\mathcal{O}(t^3)\right), \tag{4.41}$$

with G the Barnes G-function, and higher orders in the instanton expansion can be found in (B.33) and according to (B.32). The Fourier transform in (4.36) can be interpreted exactly as in (4.5). Indeed, we can represent the kernel (4.34) in operator form as

$$\mathrm{e}^{-4t^{1/4}\cosh\hat{q}}\frac{1}{\cosh\left(\frac{\hat{p}}{2}\right)}\mathrm{e}^{-4t^{1/4}\cosh\hat{q}}\,, \qquad [\hat{q},\hat{p}]=\mathrm{i}2\pi\,. \tag{4.42}$$

By exchanging momentum and position operator according to the following unitary transformation

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{4\pi}\\ -4\pi & 0\end{pmatrix}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} \tag{4.43}$$

we obtain the Fourier transform operator whose eigenfunctions are given by

$$\begin{aligned}\varphi(x,\widehat{E}_n,t)=\sum_{k\in\mathbb{Z}}\Bigg(&Z^{\rm 2d/4d}_{\rm GV}\left(x,k+\frac{1}{2}+\mathrm{i}\widehat{\sigma}_n,t\right)Z^{\rm 4d}_{\rm GV}\left(k+\frac{1}{2}+\mathrm{i}\widehat{\sigma}_n,t\right)\\ &+Z^{\rm 2d/4d}_{\rm GV}\left(-x-\frac{\mathrm{i}}{2},k+\mathrm{i}\widehat{\sigma}_n,t\right)Z^{\rm 4d}_{\rm GV}(k+\mathrm{i}\widehat{\sigma}_n,t)\Bigg)\,.\end{aligned} \tag{4.44}$$

4.3 Relating modified Mathieu to McCoy-Tracy-Wu

From the preceding discussion, we see that:

1. The eigenfunctions and spectrum of the (Fourier transformed) modified Mathieu operator (4.4) are determined by gauge theory partition functions in the NS phase of the Ω background.

2. The eigenfunctions and spectrum of the (Fourier transformed) McCoy–Tracy–Wu operator (4.34) are determined by gauge theory partition functions in the GV phase of the Ω background.

It was first shown in [80], based on [81],[16] that these two phases of the Ω background can be related by using the Nakajima-Yoshioka blowup equations [83]. This relation was extended to partition functions in the presence of surface defects in [84, 85]. It is therefore natural to ask whether we can relate the eigenfunctions and the spectra of (4.4) and (4.34).

4.3.1 The spectrum

For the modified Mathieu equation the spectrum E_n is given in terms of σ_n (4.10), which is a solution of (4.11) [73], while for the McCoy-Tracy-Wu operator the spectrum $\widehat{E}_n$ is given in terms of $\widehat{\sigma}_n$ (4.37), which is a solution of (4.40) [18]. Using the NS limit of blowup

[16] In [81], the relationship between GV invariants and the NS phase of the Ω-background was used to express the quantization condition of [4] in a more symmetric form [82].

equations without defects, it follows that solutions of (4.11) are mapped to solutions of (4.40), that is [80, 86]

$$\sigma_n = \widehat{\sigma}_n \,. \tag{4.45}$$

This gives a direct, but non-trivial relation between the spectra of the two operators above, namely

$$\boxed{\begin{array}{ll} \text{modified Mathieu:} & E_n = -t\partial_t F^{\text{4d}}_{\text{NS}}(\sigma_n, t) \\ \text{McCoy-Tracy-Wu:} & \widehat{E}_n = 2\pi\,\text{sech}(2\pi\sigma_n) \end{array}} \,. \tag{4.46}$$

Note that the relation between the energy and σ is much simpler in the McCoy-Tracy-Wu case.

4.3.2 The eigenfunctions

Let E_n and $\widehat{E}_n$ be related as in (4.46). Numerically we observe that[17]

$$\boxed{\frac{\phi(x, E_n, t)}{\phi(x_0, E_n, t)} = \frac{\varphi(x, \widehat{E}_n, t)}{\varphi(x_0, \widehat{E}_n, t)}} \tag{4.47}$$

where $\varphi(x, \widehat{E}_n, t)$ are the Fourier transformed eigenfunctions of the McCoy-Tracy-Wu operator in (4.44), $\phi(x, E_n, t)$ are the Fourier transformed eigenfunctions of the modified Mathieu operator (4.16) and x_0 is some arbitrary point which is not a zero of the eigenfunctions. The equality (4.47) should follow from taking the NS limit of the blowup equations in the presence of surface defect [84]. A detailed study will appear elsewhere. Note that, for (4.47) to hold is important that we evaluate both sides on-shell, i.e. at (4.46), (4.11). For generic values of E, $\hat{E}$ (4.47) does not hold.

4.3.3 The operators

Let us now use the relation between the spectrum and the eigenfunctions of the modified Mathieu operator (4.4) and McCoy-Tracy-Wu operator (4.34) to find a relation between the operators themselves. To avoid subtleties related to the domain of definition, it is convenient to work with bounded operators. Since the modified Mathieu operator is strictly positive, it has a bounded inverse. We note by $\rho_{\text{NS}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the inverse of[18]

$$-\frac{1}{4}\partial_q^2 + \sqrt{t}(\mathrm{e}^{2q} + \mathrm{e}^{-2q}) \,. \tag{4.48}$$

Moreover, both ρ_{NS} and ρ_{GV} are positive, self-adjoint, and of trace class, and hence there exists an orthonormal basis of eigenfunctions by the Hilbert-Schmidt theorem. We discussed in the previous section that numerical evidence suggests that these eigenfunctions are the same, and we also know the relation between their spectra as given in (4.46). It follows immediately that

$$\boxed{\rho_{\text{NS}} = \mathfrak{F}(\rho_{\text{GV}}) \,,} \tag{4.49}$$

[17] One could reach this conclusion by studying the commutation relation between the operators themself. However, one must pay special attention to the domain of definition.

[18] Note that there is a factor of 2 multiplying q compared to (4.4), which arises from the factor of 2 in the unitary transformation of the eigenfunctions in (4.36).

$$\boxed{\mathfrak{F} \;:\; \Sigma(\rho_{\rm GV}) \to \Sigma(\rho_{\rm NS}) \;:\; E \mapsto \left(-t\partial_t F_{\rm NS}^{\rm 4d}\left(\frac{1}{2\pi}\text{arcsech}\left(\frac{E}{2\pi}\right), t\right)\right)^{-1},} \tag{4.50}$$

where $F_{\rm NS}^{\rm 4d}$ is defined by (4.9) and (B.28), $\Sigma(\rho) \subset \mathbb{R}_{\geqslant 0}$ is the spectrum of ρ, and $\mathfrak{F}(\rho)$ should be understood in the functional calculus sense [87, def. 7.13]. Note that $\mathfrak{F}$ is strictly increasing on $\Sigma(\rho_{\rm GV})$, hence $\mathfrak{F}^{-1}$ is also well-defined on $\Sigma(\rho_{\rm NS})$.

5 Conclusion and outlook

In this paper, we formulated the open topological string/spectral theory correspondence for local $\mathbb{F}_0$, by generalizing [1, 2] away from the self-dual point. Focusing on local $\mathbb{F}_0$, our main result is encapsulated in (3.35). From the perspective of topological string theory, the right-hand side provides a non-perturbative, background-independent formulation of the open topological string partition function, which is entire in both the closed string modulus κ and the open string modulus x. From the viewpoint of the quantum mirror curve, what makes (3.35) particularly significant is that it provides a solution to the corresponding difference equation (1.7) which is entire, even off-shell. When evaluated on-shell, this solution gives the eigenfunctions of the relativistic two-particle Toda lattice.

We explored the implications of our construction (3.35) in both the standard [30] and dual [18] four-dimensional limits, where the quantum mirror curves reduce to the (Fourier-transformed) Mathieu operator (4.2) and the McCoy-Tracy-Wu operator (4.34), respectively. In the standard 4d limit, our construction provides entire off-shell eigenfunctions of the Fourier-transformed Mathieu operator (1.7), expressed as special combinations of NS functions in the presence of 2d/4d surface defects, see (4.16). When evaluated on-shell, these solutions reproduce the known results [33–41]. In the dual 4d limit, (3.35) reproduces the results of [3], where the eigenfunctions of the McCoy-Tracy-Wu operator are obtained through a special combination of 2d/4d surface defects in the GV phase of the background Ω. Notably, we find that the eigenfunctions of the Mathieu and the McCoy-Tracy-Wu operators, when evaluated on-shell, can be related in a simple and direct manner. This gives an explicit functional relation between the two operators, see (4.49).

Many open questions remain; we summarize some of them below.

- It would be important to understand the geometric meaning of the second term in (3.35). This insight would enable a straightforward generalization to all other toric CY threefolds.

- Non-perturbative effects in the context of the closed TS/ST correspondence have been analyzed from a resurgence perspective in [48, 65, 88–93], see [62] for a review and a more exhaustive list of references. It would be interesting to explore the open version of the TS/ST correspondence through the lens of resurgence, particularly the role that the special combinations (3.35) and (4.16) may play in the context of exact WKB [91, 92, 94–96], as well as the connection with quantum modularity [97].

- In subsection 4.3, we numerically demonstrated an explicit relation between the on-shell eigenfunctions of the modified Mathieu operator and the McCoy-Tracy-Wu operator. It should be possible to derive this relation analytically using blowup equations in the presence of surface defects. The proof will appear elsewhere.

- Over the years, many formal solutions to the functional difference equation (2.3) have been constructed using topological string/gauge theory partition functions. However, most of these proposals are not well-defined for $\hbar \in \mathbb{R}_{>0}$ and, moreover, they do not satisfy the analytic properties required for the eigenfunctions of the relativistic Toda lattice, discussed below (2.3). To our knowledge, the only exceptions are [1, 2, 41].

  The constructions in [1, 2] are specific to the self-dual point $\hbar = 2\pi$ with $\xi = 0$. Our proposal naturally reduces to theirs when these parameter values are imposed.

  On the other hand, the connection to [41] is less straightforward. A key distinction between (3.35) and the eigenfunctions in [41] is that our functions in (3.35) remain entire even off-shell, whereas those in [41] exhibit poles at specific values of x. It would be interesting to understand this better, e.g. via blowup equations.

- In the closed version of the TS/ST correspondence, the sum over integers on the right-hand side of (1.3) has a direct interpretation in the context of q-isomonodromic tau functions [20]. It would be interesting to explore whether the special combinations of the two terms in (3.35) carry any particular meaning from the perspective of q-isomonodromic deformations.

- Another point for future investigation is the relation to fiber-base (FB) duality, i.e., invariance under exchange of t_B and t_F. It is straightforward to verify (at least numerically) that the functions F^{GV} and F^{NS}, as given in equations (3.18) and (3.19), are not invariant under FB duality.[19] The same holds for the full grand potential J.

  However, the Fredholm determinant (1.3) is invariant under FB duality. It would be interesting to explore how this duality manifests at the level of the special eigenfunctions (3.35) we constructed.

- Last but not least, it would be important to establish a rigorous analytic proof of our results, such as demonstrating that (3.35) is entire in x for generic values of $\kappa \in \mathbb{C}$, $\xi \in \mathbb{R}$, and $\hbar \in \mathbb{R}_{>0}$.

We hope to report on some of these topics in the future.

A Faddeev's non-compact quantum dilogarithm

Good summaries of the properties of Faddeev's non-compact quantum dilogarithm $\Phi_b(z)$ can be found in the appendices of [15, 46] and the more comprehensive [51, app. A].

[19] We recall that we are using the Nekrasov representation for these functions, in which they are exact in t_F and expressed as a series in e^{-t_B}. If one further expands in e^{-t_F}, one obtains the GV representation, which is a double series in e^{-t_F} and e^{-t_B}. While this GV representation appears to be invariant under FB duality, it suffers from convergence issues [59], making this apparent invariance not very meaningful.

A.1 Definitions and general properties

The defining representation of the quantum dilogarithm is often taken to be [51, eq. (42)]

$$\Phi_b(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-\mathrm{i}2zu}}{\sinh(bu)\sinh(b^{-1}u)}\frac{\mathrm{d}u}{u}\right), \qquad 2|\mathrm{Im}(z)| < \left|\mathrm{Re}\big(b+b^{-1}\big)\right|. \quad \text{(A.1)}$$

It can be analytically continued to a meromorphic function of z on the whole complex plane with poles and roots at [51, eq. (45)]

$$z = \begin{cases} +\mathrm{i}\big[\big(k+\frac{1}{2}\big)b + \big(\ell+\frac{1}{2}\big)b^{-1}\big] & \text{poles} \\ -\mathrm{i}\big[\big(k+\frac{1}{2}\big)b + \big(\ell+\frac{1}{2}\big)b^{-1}\big] & \text{roots} \end{cases}, \qquad k,\ell \in \mathbb{N}, \quad \text{(A.2)}$$

and with an essential singularity at complex infinity [51, p. 34]. One can determine the order of the poles and roots when $b^2 \in \mathbb{Q}_{>0}$, based on [46, eq. (21)], as we do around (A.10). The parameter b is in general such that $b^2 \in \mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$, but we are mostly interested in $b^2 > 0$, since this corresponds to $\hbar, g_s > 0$. The asymptotic behaviour of the quantum dilogarithm is given by [51, eq. (46)]

$$\Phi_b(z) \simeq \begin{cases} \Phi_b^2(0)\exp(\mathrm{i}\pi z^2) & \mathrm{Re}(z) \gg 1 \\ 1 & \mathrm{Re}(z) \ll -1 \end{cases}, \qquad \mathrm{Re}(b) > 0\,, \quad \text{(A.3)}$$

and the asymptotic behaviour elsewhere in the complex z-plane can be found in [51, eq. (46)].

The non-compact quantum dilogarithm has some important symmetries. There is an inversion and parity symmetry in b [51, p. 34],

$$\Phi_b(z) = \Phi_{b^{-1}}(z) = \Phi_{-b}(z) \quad \text{(A.4)}$$

and the following behaviour under a parity transformation for z [51, eq. (47)] [46, p. 16],

$$\Phi_b(z)\Phi_b(-z) = \Phi_b^2(0)\exp\big(\mathrm{i}\pi z^2\big)\,, \qquad \Phi_b(0) = \exp\Big(\mathrm{i}\frac{\pi}{24}\big(b^2+b^{-2}\big)\Big)\,. \quad \text{(A.5)}$$

One has furthermore that $\overline{\Phi_b(z)} = 1/\Phi_b(\overline{z})$ whenever $b \in \mathbb{R}\setminus\{0\}$ or $|b| = 1$. Another important property for us is the quasi-periodicity in z [51, eq. (48)] [46, eq. (77)],

$$\Phi_b(z + s\mathrm{i}b^{\pm}) = \Big(1 + \mathrm{e}^{s\mathrm{i}\pi b^{\pm 2}}\mathrm{e}^{2\pi b^{\pm}z}\Big)^{-s}\Phi_b(z)\,, \quad \text{(A.6)}$$

where $s \in \{\pm 1\}$. One has similarly

$$\Phi_b(z + \mathrm{i}kb^{\pm}) = \left\{\prod_{n=0}^{|k|-1}\Big(1 + \mathrm{e}^{\mathrm{sgn}(k)\mathrm{i}\pi(2n+1)b^{\pm 2}}\mathrm{e}^{2\pi b^{\pm}z}\Big)^{-\mathrm{sgn}(k)}\right\}\Phi_b(z)\,, \qquad k \in \mathbb{Z}\,, \quad \text{(A.7)}$$

by applying $|k|$ times (A.6).

A.2 Representations and expansions of Faddeev's quantum dilogarithm

One can express the non-compact quantum dilogarithm in terms of elementary functions and the classical dilogarithm when $b^2 \in \mathbb{Q}_{>0}$ [46, eqs. 9, 11, 21],

$$\Phi_b(z) = \frac{\exp\left[\frac{\mathrm{i}}{2\pi nm}\mathrm{Li}_2(\exp(\widetilde{z})) + \left(1+\frac{\mathrm{i}}{2\pi nm}\widetilde{z}\right)\ln(1-\exp(\widetilde{z}))\right]}{D_m\left(\exp\left(\frac{\widetilde{z}}{m}\right);\exp\left(\mathrm{i}2\pi\frac{n}{m}\right)\right)D_n\left(\exp\left(\frac{\widetilde{z}}{n}\right);\exp\left(\mathrm{i}2\pi\frac{m}{n}\right)\right)}\,, \tag{A.8}$$

$$D_k(X;q) = \prod_{\ell=1}^{k-1}\left(1-q^\ell X\right)^{\ell/k}\,, \qquad \widetilde{z} = 2\pi\sqrt{nm}z + \mathrm{i}\pi(n+m)\,, \qquad b^2 = \frac{n}{m}\,, \tag{A.9}$$

where $n, m \in \mathbb{N}_{>0}$ are coprime. It should be noted that this is a representation of the meromorphic quantum dilogarithm in terms of multivalued functions. One can use (A.8) to check that the poles and roots of the quantum dilogarithm (A.2) located at

$$z = \pm\mathrm{i}\left[\left(km + r + \frac{1}{2}\right)b + \left(\ell n + s + \frac{1}{2}\right)b^{-1}\right]\,, \qquad b^2 = \frac{n}{m}\,, \tag{A.10}$$

are of order $1 + k + \ell$, where $k, \ell \in \mathbb{N}$ and $r \in \mathbb{N}_{<m}$, $s \in \mathbb{N}_{<n}$, and the upper, plus sign is for the poles and the lower, minus sign for the roots.

When taking the 4d limits the following representation of Faddeev's quantum dilogarithm comes in useful [65, eqs. (3.2)-(3.8)][20]

$$\ln\Phi_b(z) = -\frac{\mathrm{i}}{2\pi b^2}\mathrm{Li}_2\left(-\mathrm{e}^{2\pi bz}\right) - \mathrm{i}\int_0^{+\infty}\frac{\mathrm{d}t}{1+\mathrm{e}^{2\pi t}}\ln\left(\frac{1+\mathrm{e}^{2\pi bz-2\pi b^2 t}}{1+\mathrm{e}^{2\pi bz+2\pi b^2 t}}\right)\,, \tag{A.11}$$

under the condition[21] that $2|\mathrm{Im}(z)| < b^{-1}$ when $b > 0$. In taking the standard or dual 4d limit on the integral representation above, one may use the following integral representation of the log gamma function [98, p. 8],

$$\ln\Gamma\left(z+\frac{1}{2}\right) = \frac{\ln(2\pi)}{2} - z + z\ln(z) - 2\int_0^{+\infty}\frac{\mathrm{d}t}{1+\mathrm{e}^{2\pi t}}\arctan\left(\frac{t}{z}\right)\,, \qquad \mathrm{Re}(z) > 0\,, \tag{A.12}$$

which can be obtained from Binet's second formula for the log gamma function.

From [51, eqs. 65, 67] one also has the following asymptotic expansion

$$\ln\Phi_b\left(\frac{z}{2\pi b}\right) \simeq \sum_{n=0}^{\infty}\left(\mathrm{i}2\pi b^2\right)^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\partial_z^{2n}\mathrm{Li}_2(-\mathrm{e}^z) = -\frac{\mathrm{i}}{2\pi b^2}\mathrm{Li}_2(-\mathrm{e}^z) + \mathcal{O}\left(b^2\right)\,, \tag{A.13}$$

when $b \to 0$ and where $B_n(1/2)$ are the Bernoulli polynomials evaluated at $1/2$.

[20] There seems to be a constant term missing in [65, eqs. (3.2)-(3.8)].

[21] One can note from the behaviour under shifts of z by ib^{-1} that the representation above has a limited domain of validity: the quantum dilogarithm is quasi-periodic under such shifts while the same shifts act trivially on the right-hand side above. Some numerical checks seem to suggest that the representation above is only valid for $2|\mathrm{Im}(z)| < b^{-1}$ when $\mathrm{Re}(z) \geqslant 0$.

B Special functions from topological strings

In this section we summarize the all order construction of the instanton partition function or instanton free energy for the relevant gauge theories.

One can go from the instanton partition function to the free energy by

$$Z(Q) = \sum_{n\in\mathbb{N}} \frac{Z_n}{n!} Q^n \qquad\qquad F(Q) = \ln(Z(Q)) = \sum_{n\in\mathbb{N}_{>0}} \frac{F_n}{n!} Q^n \tag{B.1}$$

where F_n is given by Faà di Bruno's formula

$$F_n = \sum_{m=1}^{n} (-)^{m-1}\ (m-1)!\ B_{n,m}(Z_1, \cdots, Z_{n-m+1})\,, \tag{B.2}$$

and the $B_{n,m}$ are the partial or incomplete, exponential Bell polynomials, BellY in *Wolfram Mathematica*.

B.1 Five-dimensional, $\mathcal{N}=1$, SU(2) SYM / topological vertex for local $\mathbb{F}_0$

We summarize the refined open topological vertex for local $\mathbb{F}_0$ as presented in [3, app. A.1], which follows [99]. See [100, 101] for earlier works.

A Young diagram, or partition, μ is given by

$$\mu = \{\mu_1, \mu_2, \mu_3, \cdots \mid \forall\ k, \ell \in \mathbb{N}_{>0} : (\mu_k \in \mathbb{N}) \wedge (k \leqslant \ell \Rightarrow \mu_k \geqslant \mu_\ell)\}\,. \tag{B.3}$$

We denote by μ^t the transposed Young diagram

$$\mu^t = \left\{\mu^t_1, \mu^t_2, \mu^t_3, \cdots \mid \forall\ k \in \mathbb{N}_{>0} : \mu^t_k = |\{\ell \in \mathbb{N}_{>0} \mid \mu_\ell \geqslant k\}|\right\}. \tag{B.4}$$

For any pair $k, \ell \in \mathbb{N}_{>0}$ we say that $(k,\ell) \in \mu$ if $1 \leqslant \ell \leqslant \mu_k$ and we define

$$|\mu| = \sum_{k=1}^{+\infty} \mu_k\,, \qquad\qquad ||\mu||^2 = \sum_{k=1}^{+\infty} \mu_k^2\,. \tag{B.5}$$

Let μ, ν be two Young diagrams and define

$$Z_\mu(r_1, r_2) = \prod_{(k,\ell)\in\mu} \left(1 - r_2^{\mu_k-\ell} r_1^{\mu^t_\ell - k+1}\right)^{-1}, \qquad ||Z_\mu(r_1,r_2)||^2 = Z_{\mu^t}(r_1,r_2) Z_\mu(r_2,r_1)\,, \tag{B.6}$$

where $r_1 = \exp(\mathrm{i}\epsilon_1)$ and $r_2 = \exp(-\mathrm{i}\epsilon_2)$, with $\epsilon_{1,2}$ the Ω-background parameters. Note the different sign in the definition of $r_{1,2}$. Define then the Nekrasov factors as [99, eq. (A.7)]

$$N_{\mu,\nu}(Q; r_1, r_2) = \prod_{(k,\ell)\in\nu} \left(1 - Q\ r_2^{\nu_k-\ell} r_1^{\mu^t_\ell-k+1}\right) \prod_{(k,\ell)\in\mu} \left(1 - Q\ r_2^{-\mu_k+\ell-1} r_1^{-\nu^t_\ell+k}\right), \tag{B.7}$$

as well as the combination

$$\begin{aligned} C_{\mu,\nu}(Q_X, Q_F, r_1, r_2) = &\left(N_{\mu^t,\nu}(Q_F; r_1^{-1}, r_2^{-1}) N_{\mu^t,\nu}\left(Q_F \frac{r_1}{r_2}; r_1^{-1}, r_2^{-1}\right)\right)^{-1} \\ &\frac{N_{\emptyset,\mu^t}\left(Q_X \frac{r_1^2}{r_2}; r_1^{-1}, r_2^{-1}\right) N_{\emptyset,\nu}\left(Q_X Q_F \frac{r_1^2}{r_2}; r_1^{-1}, r_2^{-1}\right)}{N_{\emptyset,\mu^t}\left(Q_X \frac{r_1}{r_2}; r_1^{-1}, r_2^{-1}\right) N_{\emptyset,\nu}\left(Q_X Q_F \frac{r_1}{r_2}; r_1^{-1}, r_2^{-1}\right)}\,, \end{aligned} \tag{B.8}$$

where $\emptyset$ is the empty partition and $Q_{X,F}$ is related to the Kähler parameter of the brane and the fibre respectively. The "*open-closed t-brane partition function*" for local $\mathbb{F}_0$ is then given by [99, p. 50, eq. (5.4)]

$$Z_{\text{inst}}^{\text{open-closed}}(Q_X, Q_F, Q_B, r_1, r_2) = \sum_{n\in\mathbb{N}} Q_B^n \sum_{m=0}^{n} \sum_{\substack{|\mu|=m \\ |\nu|=n-m}} r_1^{||\nu^t||^2} r_2^{||\mu^t||^2} ||Z_\mu(r_1,r_2)||^2 ||Z_\nu(r_2,r_1)||^2 C_{\mu,\nu}(Q_X,Q_F,r_1,r_2)\,, \quad \text{(B.9)}$$

where Q_B is related to the Kähler parameter of the base. The closed partition function is then obtained by setting $Q_X = 0$, that is

$$Z_{\text{inst}}(Q_F, Q_B, r_1, r_2) = Z_{\text{inst}}^{\text{open-closed}}(0, Q_F, Q_B, r_1, r_2)\,. \quad \text{(B.10)}$$

Finally we define the t-brane instanton partition function for local $\mathbb{F}_0$ by [99, p. 50, eq. (5.7)]

$$Z_{\text{inst}}^{\text{open}}(Q_X, Q_F, Q_B, r_1, r_2) = \frac{Z_{\text{inst}}^{\text{open-closed}}(Q_X, Q_F, Q_B, r_1, r_2)}{Z_{\text{inst}}(Q_F, Q_B, r_1, r_2)}\,. \quad \text{(B.11)}$$

Two particular phases of the refined topological vertex are of interest to us: the self-dual or Gopakumar-Vafa (GV) phase $\epsilon_1 + \epsilon_2 = 0$ or $r_1 = r_2$, and the Nekrasov-Shatashvili (NS) phase $\epsilon_1 \to 0$ or $r_1 \to 1$. For the NS phase we have

$$\begin{aligned} F_{\text{inst}}^{\text{NS}}(t_F, t_B, \hbar) &= \lim_{\epsilon_1\to 0}(-\epsilon_1) F_{\text{inst}}\left(\mathrm{e}^{-t_F}, \mathrm{e}^{-t_B}, \mathrm{e}^{\mathrm{i}\epsilon_1}, \mathrm{e}^{-\mathrm{i}\hbar}\right), \\ F_{\text{NS,inst}}^{\text{open}}(x, t_F, t_B, \hbar) &= \lim_{\epsilon_1\to 0} F_{\text{inst}}^{\text{open}}\left(-\mathrm{e}^{-\mathrm{i}\epsilon_1/2}\mathrm{e}^{t_F/2}\mathrm{e}^{-x}, \mathrm{e}^{-t_F}, \mathrm{e}^{-t_B}, \mathrm{e}^{\mathrm{i}\epsilon_1}, \mathrm{e}^{-\mathrm{i}\hbar}\right). \end{aligned} \quad \text{(B.12)}$$

Numerically, one can see that the series expansions in t_B of the above quantities are convergent[22], but, to our knowledge, at present there is not a rigorous mathematical proof. At leading order we have

$$F_{\text{inst}}^{\text{NS}}(t_F, t_B, \hbar) = \left[\frac{\mathrm{i}\left(1+\mathrm{e}^{\mathrm{i}\hbar}\right)}{(1-\mathrm{e}^{\mathrm{i}\hbar})(1-\mathrm{e}^{-\mathrm{i}\hbar}\mathrm{e}^{-t_F})(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F})}\right]\mathrm{e}^{-t_B} + \mathcal{O}\left(\mathrm{e}^{-2t_B}\right), \quad \text{(B.13)}$$

$$F_{\text{NS,inst}}^{\text{open}}(x, t_F, t_B, \hbar) = \left[\frac{\mathrm{e}^{\mathrm{i}2\hbar}\mathrm{e}^{\frac{t_F}{2}-x}\left(1+\mathrm{e}^{-t_F}+\mathrm{e}^{\mathrm{i}\hbar}\left(1+\mathrm{e}^{\mathrm{i}\hbar}\right)\mathrm{e}^{-t_F}\mathrm{e}^{\frac{t_F}{2}-x}\right)}{(1-\mathrm{e}^{\mathrm{i}\hbar})(1-\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-t_F})(\mathrm{e}^{\mathrm{i}\hbar}-\mathrm{e}^{-t_F})\left(1+\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{-\frac{t_F}{2}-x}\right)\left(1+\mathrm{e}^{\mathrm{i}\hbar}\mathrm{e}^{\frac{t_F}{2}-x}\right)}\right]\mathrm{e}^{-t_B} + \mathcal{O}\left(\mathrm{e}^{-2t_B}\right) \quad \text{(B.14)}$$

For the GV phase we have similarly

$$\begin{aligned} F_{\text{inst}}^{\text{GV}}(t_F, t_B, g_s) &= F_{\text{inst}}\left(\mathrm{e}^{-t_F}, \mathrm{e}^{-t_B}, \mathrm{e}^{-\mathrm{i}g_s}, \mathrm{e}^{-\mathrm{i}g_s}\right) \\ F_{\text{GV,inst}}^{\text{open}}(x, t_F, t_B, g_s) &= F_{\text{inst}}^{\text{open}}\left(\mathrm{e}^{\mathrm{i}g_s/2}\mathrm{e}^{t_F/2}\mathrm{e}^{-x}, \mathrm{e}^{-t_F}, \mathrm{e}^{-t_B}, \mathrm{e}^{-\mathrm{i}g_s}, \mathrm{e}^{-\mathrm{i}g_s}\right). \end{aligned} \quad \text{(B.15)}$$

[22] There are some issues when $\hbar$ is real, but these are resolved once F_{NS} and F_{GV} are combined in the definition of topological string grand potential J, as discussed in the main text.

The series expansion in t_B of the above quantities is convergent. In the case of $F_{\text{inst}}^{\text{GV}}$, this was rigorously demonstrated in [19][23]. The proof for $F_{\text{GV,inst}}^{\text{open}}$ follows analogously. The leading order reads then

$$F_{\text{inst}}^{\text{GV}}(t_B, t_F, g_s) = \left[\frac{2\mathrm{e}^{\mathrm{i}g_s}}{(1-\mathrm{e}^{\mathrm{i}g_s})^2(1-\mathrm{e}^{-t_F})^2}\right]\mathrm{e}^{-t_B} + \mathcal{O}\big(\mathrm{e}^{-2\,t_B}\big)\,, \tag{B.16}$$

$$F_{\text{GV,inst}}^{\text{open}}(x, t_F, t_B, g_s) = \\ \left[\frac{\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{\frac{t_F}{2}-x}\left(2\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{-\frac{t_F}{2}-x}-1-\mathrm{e}^{-t_F}\right)}{(1-\mathrm{e}^{\mathrm{i}g_s})(1-\mathrm{e}^{-t_F})^2\left(1-\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{\frac{t_F}{2}-x}\right)\left(1-\mathrm{e}^{\mathrm{i}\frac{g_s}{2}}\mathrm{e}^{-\frac{t_F}{2}-x}\right)}\right]\mathrm{e}^{-t_B} + \mathcal{O}\big(\mathrm{e}^{-2\,t_B}\big)\,, \tag{B.17}$$

for the closed and open parts respectively.

B.2 Four-dimensional, $\mathcal{N}=2$, SU(2) SYM

Consider the following four-dimensional limit for the variables in the previous section [30, 102]

$$Q_X = \exp\big(-R\big(z-a+\mathrm{i}\epsilon_1^{4\text{d}}\big)\big) \qquad Q_F = \exp(-2Ra) \qquad Q_B = R^4 t \tag{B.18}$$

$$\epsilon_{1,2} \to R\epsilon_{1,2}\,, \qquad R \to 0\,. \tag{B.19}$$

This is the same scaling as for the standard variables in the standard 4d limit, (4.1), (4.21), and (4.22), and the same scaling as for the dual variables in the dual 4d limit, (4.33). This gives for (B.6)

$$Z_\mu^{4\text{d}}(\epsilon_1,\epsilon_2) = \prod_{(k,\ell)\in\mu}\big((\mu_k-\ell)\epsilon_2 - \big(\mu_\ell^t - k+1\big)\epsilon_1\big)^{-1}\,, \tag{B.20}$$

$$\big|\big|Z_\mu^{4\text{d}}(\epsilon_1,\epsilon_2)\big|\big|^2 = Z_{\mu^t}^{4\text{d}}(\epsilon_1,\epsilon_2)Z_\mu^{4\text{d}}(\epsilon_2,\epsilon_1)\,, \tag{B.21}$$

while the Nekrasov factors (B.7) become

$$N_{\mu,\nu}^{4\text{d}}(\alpha;\epsilon_1,\epsilon_2) = \prod_{(k,\ell)\in\nu}\big((\nu_k-\ell)\epsilon_2 - \big(\mu_\ell^t-k+1\big)\epsilon_1 + \mathrm{i}\alpha\big) \\ \prod_{(k,\ell)\in\mu}\big((-\mu_k+\ell-1)\epsilon_2 - \big(-\nu_\ell^t+k\big)\epsilon_1 + \mathrm{i}\alpha\big)\,, \tag{B.22}$$

and similarly

$$C_{\mu,\nu}^{4\text{d}}(a,\epsilon_1,\epsilon_2) = \big(N_{\mu^t,\nu}^{4\text{d}}(2a;\epsilon_1,\epsilon_2)N_{\mu^t,\nu}^{4\text{d}}(2a-\mathrm{i}(\epsilon_1+\epsilon_2);\epsilon_1,\epsilon_2)\big)^{-1}\,, \tag{B.23}$$

[23] There are some issues when g_s is real, but these are resolved once F_{NS} and F_{GV} are combined in the definition of topological string grand potential J, as discussed in the main text.

$$
\begin{aligned}
C^{2\mathrm{d}/4\mathrm{d}}_{\mu,\nu}(z,a,\epsilon_1,\epsilon_2) = C^{4\mathrm{d}}_{\mu,\nu}(a,\epsilon_1,\epsilon_2)& \\
\frac{N^{4\mathrm{d}}_{\emptyset,\mu^t}(z-a-\mathrm{i}(\epsilon_1+\epsilon_2);\epsilon_1,\epsilon_2)}{N^{4\mathrm{d}}_{\emptyset,\mu^t}(z-a-\mathrm{i}\epsilon_2;\epsilon_1,\epsilon_2)} &\frac{N^{4\mathrm{d}}_{\emptyset,\nu}(z+a-\mathrm{i}(\epsilon_1+\epsilon_2);\epsilon_1,\epsilon_2)}{N^{4\mathrm{d}}_{\emptyset,\nu}(z+a-\mathrm{i}\epsilon_2;\epsilon_1,\epsilon_2)}\,. \qquad \text{(B.24)}
\end{aligned}
$$

Hence we find for the complete 2d/4d defect instanton partition function in a generic phase of the Ω-background

$$
Z^{4\mathrm{d}}_{\mathrm{inst}}(a,t,\epsilon_1,\epsilon_2) = \sum_{n\in\mathbb{N}} t^n \sum_{m=0}^{n} \sum_{\substack{|\mu|=m\\|\nu|=n-m}} (-)^n ||Z^{4\mathrm{d}}_{\mu}(\epsilon_1,\epsilon_2)||^2 ||Z^{4\mathrm{d}}_{\nu}(\epsilon_2,\epsilon_1)||^2 C^{4\mathrm{d}}_{\mu,\nu}(a,\epsilon_1,\epsilon_2)\,, \tag{B.25}
$$

$$
\begin{aligned}
&Z^{\mathrm{D}}_{\mathrm{inst}}(z,a,t,\epsilon_1,\epsilon_2) = \\
&\qquad \sum_{n\in\mathbb{N}} t^n \sum_{m=0}^{n} \sum_{\substack{|\mu|=m\\|\nu|=n-m}} (-)^n ||Z^{4\mathrm{d}}_{\mu}(\epsilon_1,\epsilon_2)||^2 ||Z^{4\mathrm{d}}_{\nu}(\epsilon_2,\epsilon_1)||^2 C^{2\mathrm{d}/4\mathrm{d}}_{\mu,\nu}(z,a,\epsilon_1,\epsilon_2)\,,
\end{aligned} \tag{B.26}
$$

$$
Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{inst}}(z,a,t,\epsilon_1,\epsilon_2) = \frac{Z^{\mathrm{D}}_{\mathrm{inst}}(z,a,t,\epsilon_1,\epsilon_2)}{Z^{4\mathrm{d}}_{\mathrm{inst}}(a,t,\epsilon_1,\epsilon_2)}\,. \tag{B.27}
$$

The NS limit as used in subsection 4.1 is then given by

$$
F^{4\mathrm{d}}_{\mathrm{NS,inst}}(\sigma,t) = \lim_{\epsilon_1\to 0}(-\epsilon_1) F^{4\mathrm{d}}_{\mathrm{inst}}(\sigma,t,\epsilon_1,1)\,, \tag{B.28}
$$

$$
Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}(x,\sigma,t) = \lim_{\epsilon_1\to 0} Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{inst}}\Big(x-\mathrm{i}\frac{\epsilon_1}{2},\sigma,t,\epsilon_1,1\Big)\,. \tag{B.29}
$$

The convergence of (B.28) as series in t for $2\sigma \neq \epsilon_1\mathbb{Z}$ follows from [103]. Similar arguments should hold for (B.29) as well. The first few terms of the NS limit are

$$
\begin{aligned}
F^{4\mathrm{d}}_{\mathrm{NS,inst}}(\sigma,t) = -\left[\frac{2}{4\sigma^2+1}\right]t - \left[\frac{20\sigma^2-7}{4(4\sigma^2+1)^3(\sigma^2+1)}\right]t^2& \\
-\left[\frac{4(144\sigma^4-232\sigma^2+29)}{3(4\sigma^2+1)^5(\sigma^2+1)(4\sigma^2+9)}\right]t^3 + \mathcal{O}\big(t^4\big)\,,&
\end{aligned} \tag{B.30}
$$

$$
\begin{aligned}
&Z^{2\mathrm{d}/4\mathrm{d}}_{\mathrm{NS,inst}}(x,\sigma,t) = 1 + \left[\frac{1-2\widetilde{x}}{(1+4\sigma^2)(\widetilde{x}^2+\sigma^2)}\right]t + \\
&\left[\frac{5+9\widetilde{x}-27\widetilde{x}^2+14\widetilde{x}^3+(91-10\widetilde{x}(17+2\widetilde{x}(-7+2\widetilde{x})))\sigma^2+4(-7+2\widetilde{x})(-5+4\widetilde{x})\sigma^4}{4(1+4\sigma^2)^3(1+\sigma^2)(\widetilde{x}^2+\sigma^2)((\widetilde{x}-1)^2+\sigma^2)}\right]t^2 \\
&\qquad + \mathcal{O}\big(t^3\big)
\end{aligned} \tag{B.31}
$$

where $\widetilde{x} = -\mathrm{i}x - 1$.

For the GV phase we define

$$\begin{aligned} Z^{2d/4d}_{GV,inst}(x,\sigma,t) =& Z^{2d/4d}_{inst}\left(x+\frac{i}{2}, i\sigma, t, -1, 1\right), \\ Z^{4d}_{GV,inst} =& Z^{4d}_{inst}(i\sigma, t, -1, 1) \end{aligned} \tag{B.32}$$

The convergence of (B.32) as a series in t for $2i\sigma \notin \mathbb{Z}$ follows from [104]. The first few orders of the GV phase (B.32) are given by

$$\begin{aligned} Z^{4d}_{GV,inst}(\sigma,t) = 1 + \left[\frac{1}{2\sigma^2}\right]t + \left[\frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)^2}\right]t^2 + \left[\frac{8\sigma^4-5\sigma^2+3}{24\sigma^2(4\sigma^2-1)^2(\sigma^2-1)^2}\right]t^3 \\ + \left[\frac{256\sigma^8-832\sigma^6+972\sigma^4-177\sigma^2+81}{384\sigma^4(4\sigma^2-1)^2(\sigma^2-1)^2(4\sigma^2-9)^2}\right]t^4 + \mathcal{O}(t^5), \end{aligned} \tag{B.33}$$

$$\begin{aligned} Z^{2d/4d}_{GV,inst}(x,\sigma,t) = 1 - \left[\frac{\widetilde{x}}{2\sigma^2(\widetilde{x}^2-\sigma^2)}\right]t \\ + \left[\frac{\widetilde{x}(\widetilde{x}+1)^2-\widetilde{x}(10\widetilde{x}^2+19\widetilde{x}+10)\sigma^2+(8\widetilde{x}^2+30\widetilde{x}+9)\sigma^4}{4\sigma^4(4\sigma^2-1)^2(\widetilde{x}^2-\sigma^2)\left((\widetilde{x}+1)^2-\sigma^2\right)}\right]t^2 + \mathcal{O}(t^3), \end{aligned} \tag{B.34}$$

where we used $\widetilde{x} = ix + 1/2$ for convenience.

C Comment on the $\xi = 0$, $\hbar = 2\pi$ case

The eigenfunctions of (2.7) for $\xi = 0$ and $\hbar = 2\pi$ were studied extensively in [1, 2]. The explicit expression for $\psi_0(x)$ can be found in [1, eqs. (2.95), (2.96)]. However, when written in the form given in [1, eqs. (2.95), (2.96)], the structure presented in (2.49) is not immediately apparent. In this appendix, we clarify the reasons for this.

Let us consider the $\xi \to 0$ limit when $\hbar = 2\pi$ of (2.61). In this case one gets

$$\sqrt{2}q_{\pm,k}(x) = \ln\left[e^x\left(\frac{e^x+e^{-x}\pm\operatorname{sgn}\left(\arg\left(x+i\frac{\pi}{2}\right)\right)\sqrt{(e^x-e^{-x})^2}}{2}\right)\right] + i2\pi k, \tag{C.1}$$

and it should be noted that the important symmetry $q_{\pm,k}(x) = -q_{\mp,-k}(-x-i\pi) \bmod i2\pi$ is still present. Let us now write the two terms (2.61) explicitly for $x \in \mathbb{R}$ and $\hbar = 2\pi$. We find that

$$\begin{aligned} \omega_{0,+}(x) &= i2\pi \lim_{\xi\to 0}\left(\mathrm{Res}_+(x)\sum_{k=0}^{1} U(x,q_{+,k}(x))E(q_{+,k}(x))\right) \\ &= \begin{cases} -\frac{1}{\sqrt{\pi}}e^{ix^2/4\pi}\left(\frac{e^x}{e^{2x}-1}\right) & \text{if } x<0, \\ \frac{e^{i\pi/4}}{\sqrt{2\pi}}e^{-ix^2/4\pi}\left(\frac{e^x(e^x-i)}{e^{2x}-1}\right) & \text{if } x>0, \end{cases} \end{aligned} \tag{C.2}$$

$$
\begin{aligned}
\omega_{0,-}(x) &= \mathrm{i}2\pi \lim_{\xi\to 0}\left(\mathrm{Res}_{-}(x)\sum_{k=0}^{1} U(x, q_{-,k}(x))E(q_{-,k}(x))\right) \\
&= \begin{cases} \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}x^2/4\pi}\left(\frac{\mathrm{e}^{x}(\mathrm{e}^{x}-\mathrm{i})}{\mathrm{e}^{2x}-1}\right) & \text{if } x<0\,, \\ -\frac{1}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}x^2/4\pi}\left(\frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}-1}\right) & \text{if } x>0\,, \end{cases}
\end{aligned} \tag{C.3}
$$

and $\omega_{0,\pm}(x)$ diverges when $x = 0$ but the sum $\psi_0(x)$ is still well-defined. One finds for all $x \in \mathbb{R}$ that

$$
\psi_0(x) = \omega_{0,+}(x) + \omega_{0,-}(x) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}x^2/4\pi}\left(\frac{\mathrm{e}^{x}(\mathrm{e}^{x}-\mathrm{i})}{\mathrm{e}^{2x}-1}\right) - \frac{1}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}x^2/4\pi}\left(\frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}-1}\right), \tag{C.4}
$$

which is indeed the same expression as in [1, eq. (2.95)-(2.96)]. However, if we were to start from the final expression on the right-hand side of (C.4), we would not see the general structure (2.49) which is instead manifest when $\xi \neq 0$.

D Selected plots of the eigenfunctions at various ξ, $\hbar$ and energies

Below we plot (3.35) for various values of the parameters. For the on-shell eigenfunctions, we have always from top left to bottom right, the first term in (3.35), the second term in (3.35), the complete eigenfunction, and the difference with the numerical eigenfunction for 0, 2 and 4 terms in the large t_B expansion of $J^{\text{open}}_{\text{inst}}$. The numerical eigenfunctions are obtained by diagonalizing in the basis of the harmonic oscillator, as in [105]. For the off-shell eigenfunctions, we have similarly from top left to bottom right, the first term in (3.35), the second term in (3.35). The real part is the solid line and the imaginary part is the dotted line.

D.1 The case $\hbar = 4\sqrt{2}$ and $\xi = 2/3$

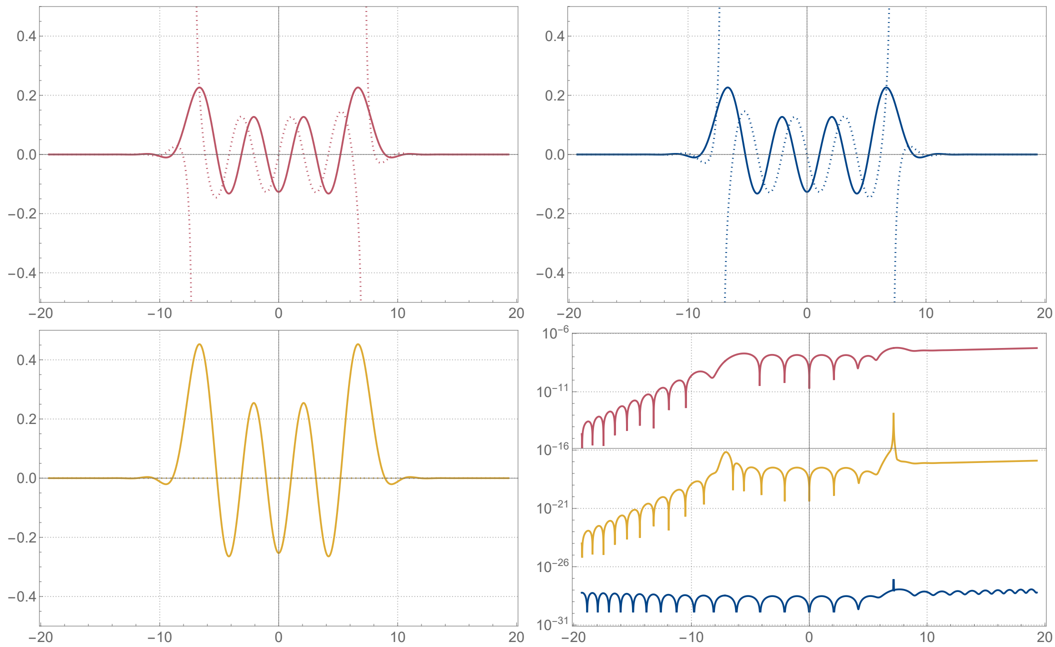

Figure 4. The on-shell eigenfunction from topological strings for $\hbar = 4\sqrt{2}$ and $\xi = 2/3$ at energy $E = E_6 \approx 8.49$. See the explanation at the beginning of appendix D.

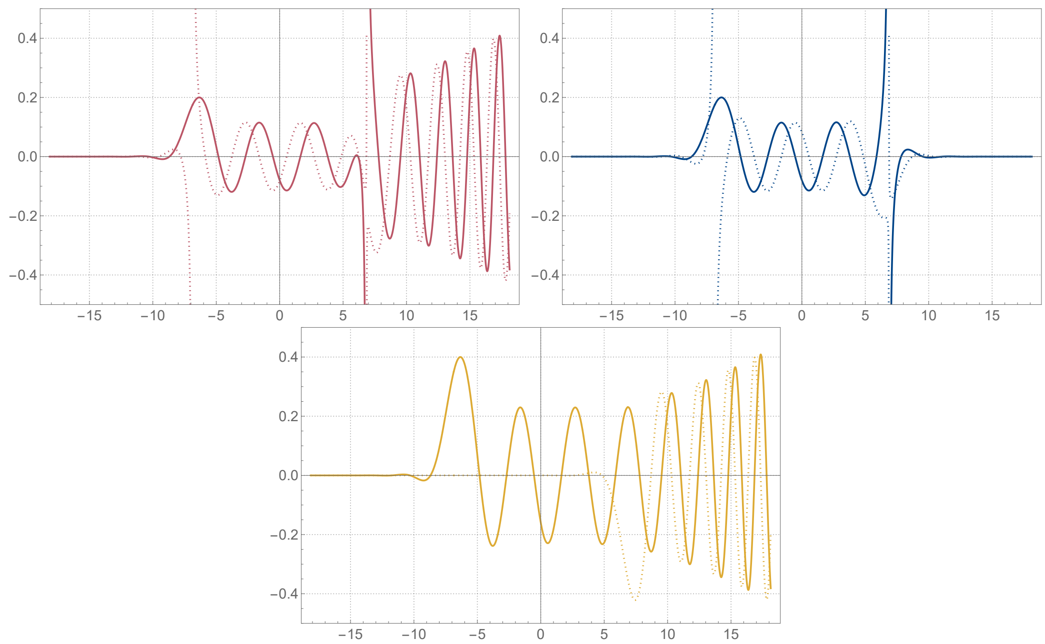

Figure 5. The off-shell eigenfunction from topological strings for $\hbar = 4\sqrt{2}$ and $\xi = 2/3$ at energy $E \approx 8.19$, which is halfway between the 5th and 6th excited states. See the explanation at the beginning of appendix D.

D.2 The case $\hbar = 2\pi/3$ and $\xi = -\ln(3)/3$

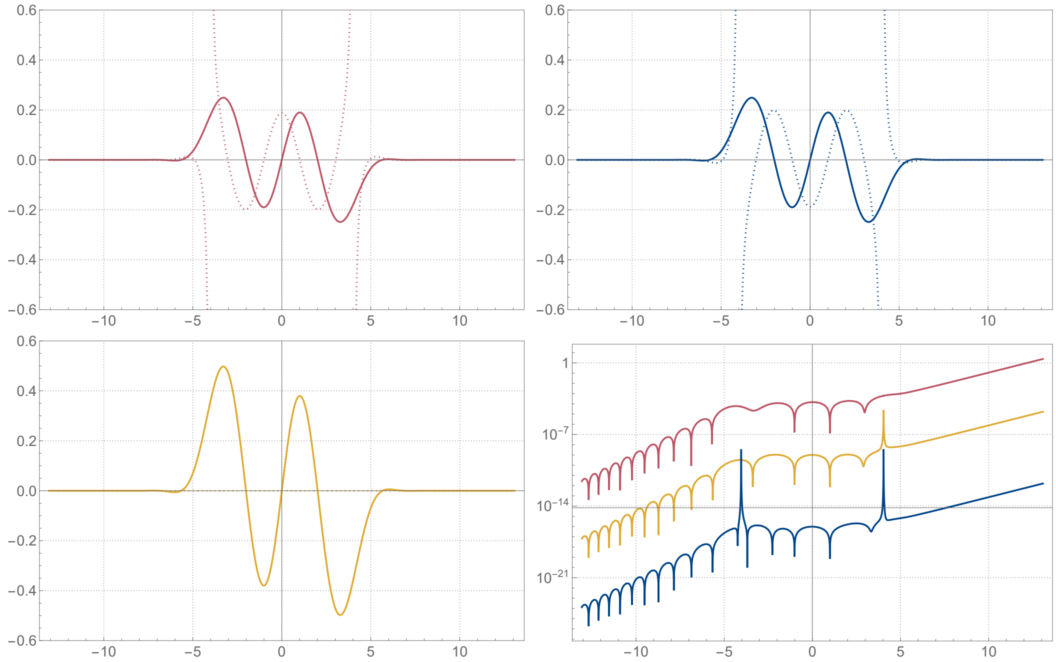

Figure 6. The on-shell eigenfunction from topological strings for $\hbar = 2\pi/3$ and $\xi = -\ln(3)/3$ at energy $E = E_3 \approx 3.3063$. See the explanation at the beginning of appendix D.

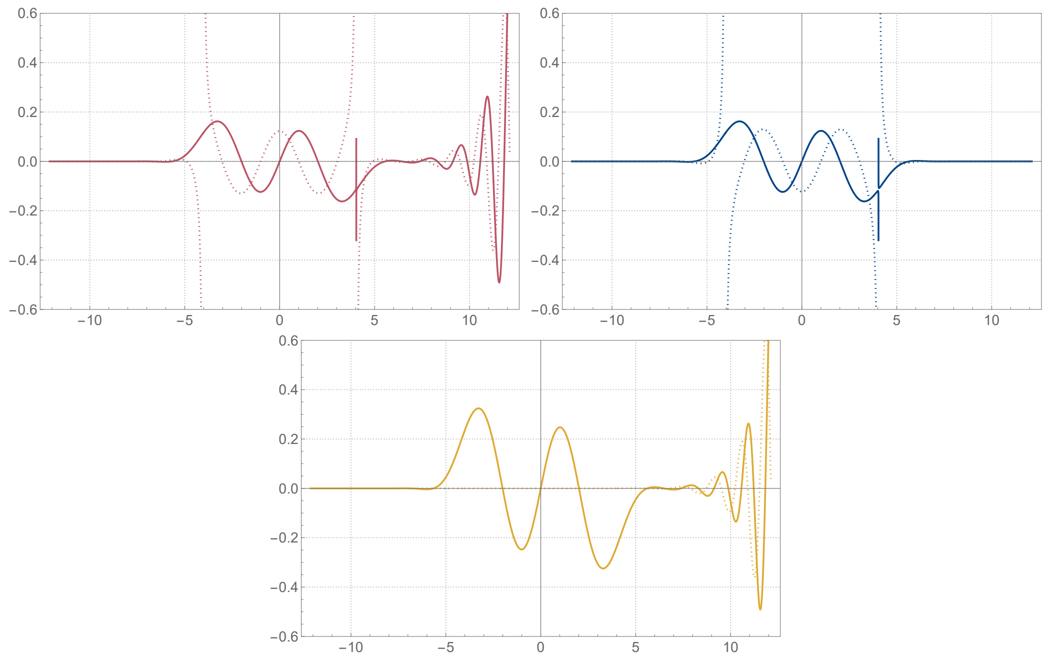

Figure 7. The off-shell eigenfunction from topological strings for $\hbar = 2\pi/3$ and $\xi = -\ln(3)/3$ at energy $E \approx 3.3062$, which is just below the third excited state. See the explanation at the beginning of appendix D.

D.3 The case $\hbar = 3\pi$ and $\xi = \sqrt{7}/4$

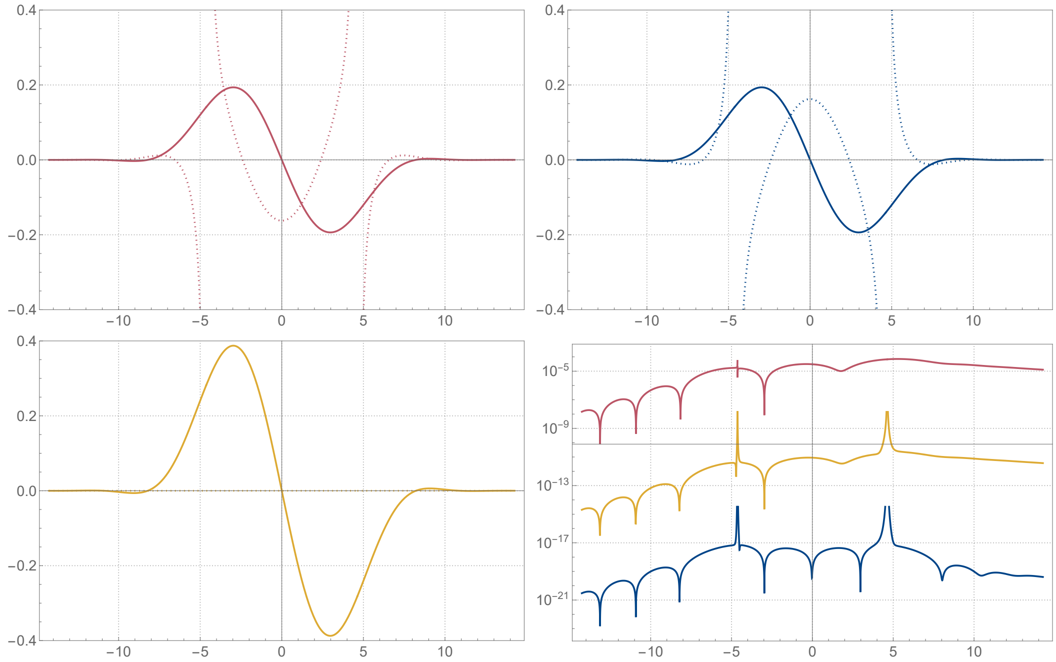

Figure 8. The on-shell eigenfunction from topological strings for $\hbar = 3\pi$ and $\xi = \sqrt{7}/4$ at energy $E = E_1 \approx 5.95$. See the explanation at the beginning of appendix D.

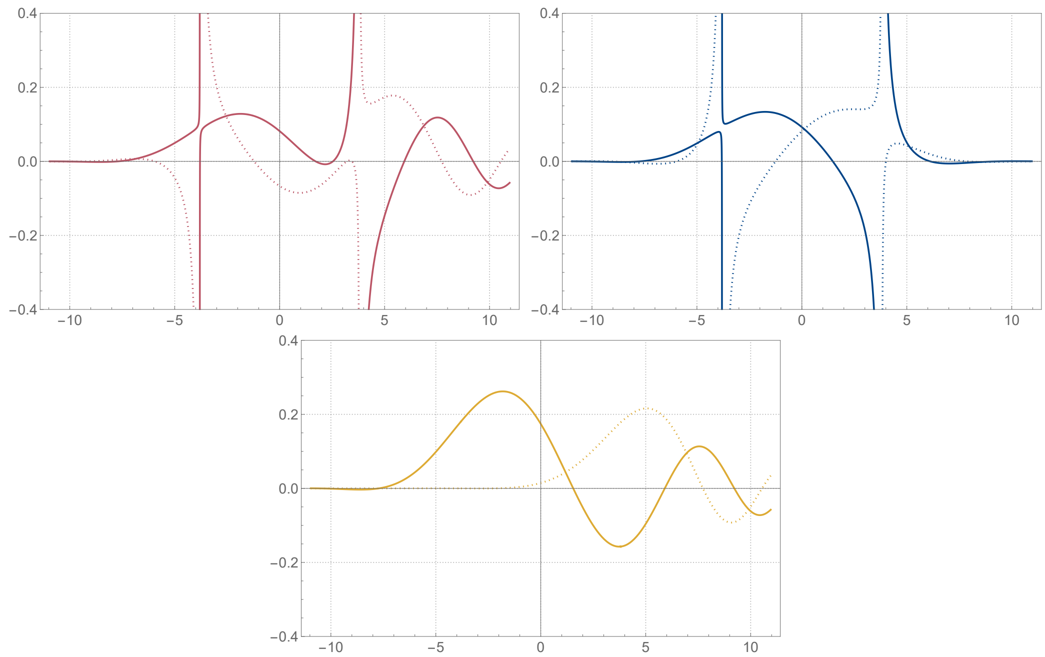

Figure 9. The off-shell eigenfunction from topological strings for $\hbar = 3\pi$ and $\xi = \sqrt{7}/4$ at energy $E \approx 5.13$, which is halfway between the ground state and the first excited state. See the explanation at the beginning of appendix D.

D.4 The case $\hbar = 5\pi/2$ and $\xi = 1/7$

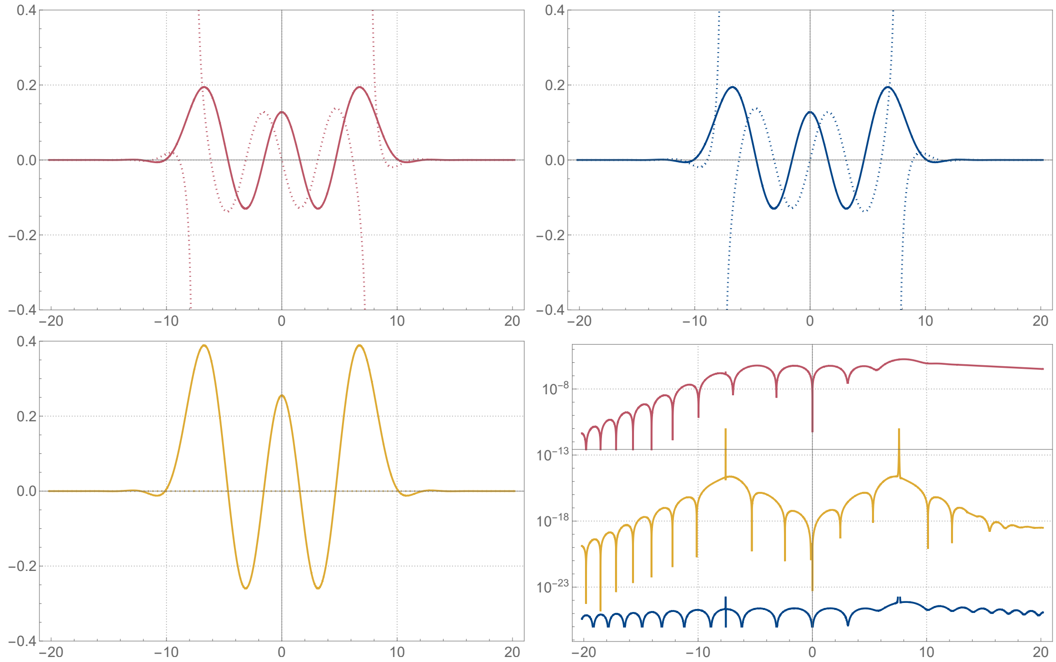

Figure 10. The on-shell eigenfunction from topological strings for $\hbar = 5\pi/2$ and $\xi = 1/7$ at energy $E = E_4 \approx 7.87$. See the explanation at the beginning of appendix D.

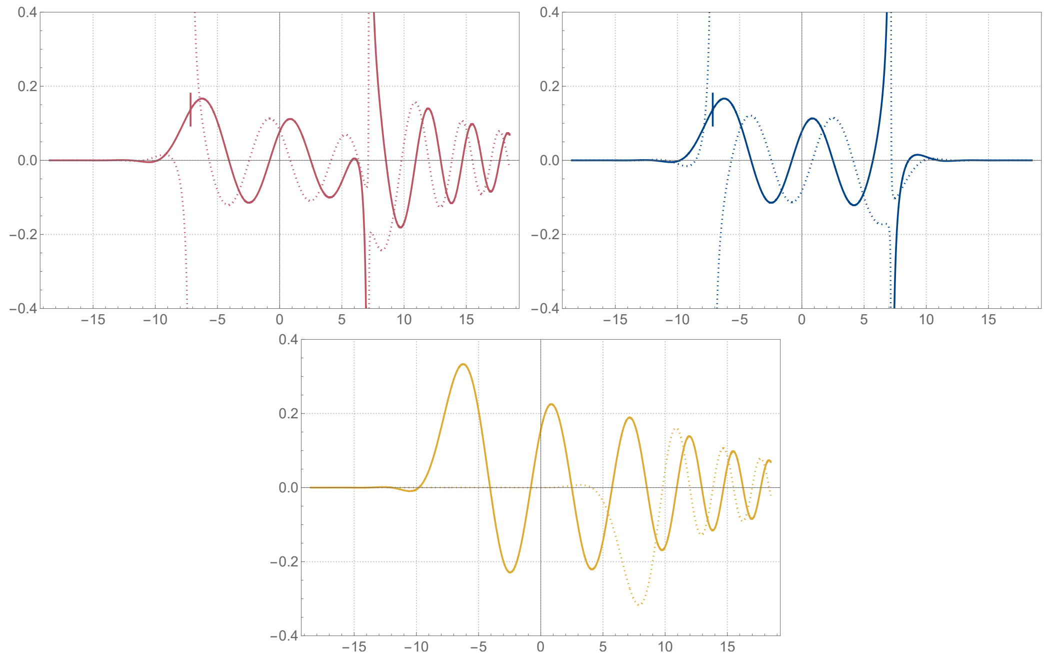

Figure 11. The off-shell eigenfunction from topological strings for $\hbar = 5\pi/2$ and $\xi = 1/7$ at energy $E \approx 7.45$, which is halfway between the 3th and 4th excited state. See the explanation at the beginning of appendix D.

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