Quantisations of exactly solvable ghostly models

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ABSTRACT: We investigate an exactly solvable two-dimensional Lorentzian coupled quantum system that in a certain parameter regime can be transformed to a higher time derivative theory (HTDT) with preserved symplectic structure. By transforming the system's Lagrangian, we explicitly map it onto the standard Pais-Uhlenbeck formulation, revealing a direct correspondence in their dynamical and Poisson bracket structures. We quantise the model in two alternative ways. First we derive the eigensystem of the Hamiltonian by solving the Schrödinger equation through an Ansatz that leads to a set of coupled three-term recurrence relations, that we solve exactly, identifying normalisable wavefunctions and their associated energy spectra. We compare our results with a Fock space construction, finding exact agreement. On the basis of the exact solutions we report several specific physical properties of the ghost model investigated with a focus on the localisation properties of the system.

1. Introduction

Ghost-ridden quantum systems manifest in various forms. Typically, this means that parts of their spectra are not bounded from below or that they contain non-normalisable states, leading to a violation of unitarity. They may have linear momentum terms in their Hamiltonians, as for instance found in the dissipative Bateman oscillator [2,3] and Pais-Uhlenbeck oscillators [4] in the Ostrogradsky scheme [1], or exhibit non-positive definite kinetic energy terms of Lorentzian type [5]. In this work, we investigate a model of the latter type.

In many cases the relation to the most prominent ghost ridden systems, higher time derivative theories (HTDT), is well known and often exploited in their analysis. These systems have undergone extensive investigations for more than half a century due their extremely attractive property of being renormalizable [4,6–10]. In particular, they have been considered as potential candidates for quantising gravity [11] and have also been proposed as models in various fields such as cosmology [12–15], finite temperature physics [16], black hole physics [17], in a massless particle descriptions of bosons and fermions [18,19] and in supersymmetric theories [20,21].

The formulation of consistent quantum versions of ghost ridden systems remains a challenge as they usually contain non-normalisable states and/or posses unbounded spectra. Several proposals have been made to cure these deficiencies [22–26]. Various schemes to quantise finite particle theories have been explored, such as BRST quantisation [27–29], Dirac quantisation [30–34], deformation quantisation [35], polymer quantisation [36] or imaginary scaling quantisation [25,37]. Schemes extending to field theories have also been explored, e.g. for scalar field theories [38–40].

In this work, we quantise a ghostly model in form of a two-dimensional Lorentzian coupled quantum system in two alternative ways. First, we directly solve the Schrödinger equation by making a suitable Ansatz that leads to a set of coupled three term recurrence relations. These relations are akin to the central set of equation occurring in quasi-exactly solvable models [41–46] and by appealing to techniques developed in that context we systematically solve them. Alternatively we use the fact that in a certain parameter regime the model can be symplectically mapped to the Pais-Uhlenbeck oscillator so that we can use its classical solutions as the basis for a Fock space construction. In the regime where the two systems coincide, we find perfect agreement between these two approaches.

Our manuscript is organised as follows: In section 2 we present the precise symplectic transformation from our ghost model to the Pais-Uhlenbeck oscillator. In section 3 and 4 we elaborate on the two alternative ways to quantise and solve the ghost system. In section 5 we study various physical properties of our model such as probability densities and uncertainty relations by comparing them also to the classical solutions. Our conclusions are presented in section 6.

2. A ghostly model related to the Pais-Uhlenbeck oscillator

We consider a two-dimensional Hamiltonian system with Hamiltonian given by

$$\mathcal{H} = \left(p_x^2 - p_y^2\right) + \nu^2 x^2 + \Omega y^2 + gxy, \qquad \nu, \Omega, g \in \mathbb{R}. \tag{2.1}$$

We note that the model is similar to a model recently proposed in [5] when the function in there is chosen to be quadratic. Since the kinetic term contains a negative term, we expect the model to be "ghostly", in the sense that its spectrum might not be bounded from below or that unitarity is violated in its evolution. This is a feature shared with HTDTs, and in fact we argue that $\mathcal{H}$ can be viewed as a HTDT in disguise.

To make this connection explicit, we first note that the Legendre transform

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{y}} \dot{y} - \mathcal{L}. \tag{2.2}$$

of the Lagrangian

$$\mathcal{L} = \frac{1}{4} \left(\dot{x}^2 - \dot{y}^2\right) - \nu^2 x^2 - \Omega y^2 - gxy, \tag{2.3}$$

leads to the Hamiltonian in (2.1). The classical dynamics for this model is described by the two coupled second order Euler-Lagrange equations resulting from (2.3)

$$\kappa_1 \left(\frac{1}{2}\ddot{x} + gy + 2\nu^2 x\right) = 0, \qquad \kappa_2 \left(\frac{1}{2}\ddot{y} - gx - 2\Omega y\right) = 0. \tag{2.4}$$

We have included here explicitly two constant overall factors κ_1 and κ_2 for later purpose. In contrast, the Euler-Lagrange equations directly derived from the higher time-derivative PU Lagrangian in standard form

$$\mathcal{L}_{\text{PU}} = \frac{1}{2}\ddot{q}^2 - \frac{\zeta}{2}\dot{q}^2 + \frac{\xi}{2}q^2, \quad \zeta, \xi \in \mathbb{R}^+ \tag{2.5}$$

is of fourth order

$$\ddddot{q} + \zeta\ddot{q} + \xi q = 0, \tag{2.6}$$

The standard parametrisation in terms of the frequencies ω_1 and ω_2 are $\zeta = \omega_1^2 + \omega_2^2$ and $\xi = \omega_1^2\omega_2^2$. There are multiple options to obtain (2.6) from (2.4) and thus relate the corresponding dynamics in the respective phase spaces of the two models. Here, having the quantisation in mind, we choose an option that also preserves the symplectic structure, i.e. loosely speaking the Poisson bracket structure. Using the transformation map

$$\Gamma: \quad x \mapsto \mu_0 q + \mu_2\ddot{q}, \qquad y \mapsto \nu_0 q + \nu_2\ddot{q}, \tag{2.7}$$

with constants

$$\mu_0 = \frac{\sqrt{2}(g - 2\Omega)}{\sqrt{\nu^2 + \Omega - g}}, \qquad \nu_0 = \frac{\sqrt{2}(g - 2\nu^2)}{\sqrt{\nu^2 + \Omega - g}}, \qquad \mu_2 = -\nu_2 = \frac{1}{\sqrt{2}\sqrt{\nu^2 + \Omega - g}}, \tag{2.8}$$

$$\zeta = 4\left(\nu^2 - \Omega\right), \qquad \xi = 4\left(g^2 - 4\nu^2\Omega\right), \qquad \kappa_1 = \kappa_2 = 2\sqrt{2}\sqrt{\nu^2 + \Omega - g}. \tag{2.9}$$

converts both equations (2.4) individually into the PU equation of motion (2.6). Ensuring the reality of the coordinates restricts the parameter regime to $\nu^2 + \Omega > g$. At the same time this transformation preserves the Poisson bracket structure for the phase space variables $\{x_1 = x, x_2 = y, p_1 = p_x = \dot{x}/2, p_2 = p_y = -\dot{y}/2\}$ compared to those for $\{q, \dot{q}, \ddot{q}, \dddot{q}\}$, see [47] for details,

$$\{x_i, p_j\} = \delta_{ij}, \qquad \Leftrightarrow \qquad \{\dot{q}, \ddot{q}\}_1 = 1, \quad \{\dddot{q}, q\}_1 = 1, \quad \{\ddot{q}, \dddot{q}\}_1 = \zeta. \tag{2.10}$$

When directly transforming $\mathcal{L}$ in (2.3) by means of (2.7) we obtain $\mathcal{L} = \mathcal{L}_{\text{PU}} - \ddot{q}^2 - \dot{q}\dddot{q}$. Thus, the two Lagrangians differ only by surface terms, which vanish upon integration, i.e. we may use $\int \ddot{q}^2 dt = -\int \dot{q}\dddot{q} dt$.

It is well known that the PU oscillator exhibits a qualitatively different behaviour at its points of degeneracy, which is therefore a feature we also anticipate to find in our model. Comparing (2.6) with the standard formulation of the PU equation in terms of the natural frequencies ω_1 and ω_2 allows to identify the degeneracy condition $\omega_1 = \omega_2$ within the parameter space of our model (2.1)

$$\dddot{x} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\omega_2^2 x = 0, \quad \Rightarrow \quad \omega_1^2 = \omega_2^2, \quad \Leftrightarrow \quad \zeta^2 = 4\xi \quad \Leftrightarrow \quad g^2 = \left(\nu^2 + \Omega\right)^2. \tag{2.11}$$

We note that the point of degeneracy also marks the boundary of the validity for the transformation (2.7)-(2.9) between our model in (2.1) and the PU-oscillator. The point of degeneracy will play a critical role as a boundary value for the model domain in parameter space in shaping the spectral and dynamical properties of the system.

3. Eigensystem from coupled three term recurrence relations

We shall now solve the Schrödinger equation $H\psi_N(x,y) = E_N\psi_N(x,y)$, $N = 0, 1, 2, \ldots$ by making an Ansatz of Gaussian form dressed with polynomials in x, y for the wavefunctions

$$\psi_N(x,y) = P_N(x,y)e^{-\frac{\alpha x^2}{2} - \frac{\beta y^2}{2} + \gamma xy}, \qquad P_N(x,y) := \sum_{i=0}^{N} \sum_{j=0}^{\lfloor \frac{N-i}{2} \rfloor} \sigma_i^{N-2j} x^i y^{N-i-2j}. \quad (3.1)$$

Here $\lfloor z \rfloor := \max(n \in \mathbb{Z} | n \le z)$ is the floor function that gives the greatest integer less than or equal to z. The Ansatz is similar to one previously used in [48] to solve a coupled harmonic oscillator model. The procedure arising from this consists of solving coupled three term recurrence relations that arise from reading off the powers in $x^i y^j$ upon the substitution of (3.1) into the Schrödinger equation.

3.1 The "ground state"

We start with the "ground state", or more precisely the state with $N = 0$, and an overall normalisation factor $P_0 = \sigma_0^0$, which is in fact not always the state of lowest energy. A solution with energy $E_0 = \alpha - \beta$ is obtained when the model parameters ν, Ω, g are related to the parameters entering via the wavefunction Ansatz, α, β, γ, as

$$\alpha^2 - \gamma^2 = \nu^2, \qquad \gamma^2 - \beta^2 = \Omega, \qquad g + 2(\alpha - \beta)\gamma = 0, \quad (3.2)$$

corresponding to the coefficients of $x^2\psi_0$, $y^2\psi_0$, and $xy\psi_0$, respectively.

3.1.1 The non-degenerate case

Since α, β, γ are just auxiliary variables we need to replace them by the actual model parameters ν, Ω, g. When solving the equations in (3.2) we obtain four different branches of the theory

$$\alpha_\epsilon^\eta = \frac{2\nu^2 + \sigma_\epsilon}{\Sigma_\epsilon^\eta}, \quad \beta_\epsilon^\eta = \frac{2\Omega - \sigma_\epsilon}{\Sigma_\epsilon^\eta}, \quad \gamma_\epsilon^\eta = \frac{-g}{\Sigma_\epsilon^\eta}, \ \Sigma_\epsilon^\eta = 2\eta\sqrt{\nu^2 - \Omega + \sigma_\epsilon}, \ \sigma_\epsilon = \epsilon\sqrt{g^2 - 4\nu^2\Omega}, \quad (3.3)$$

labelled by all combinations of $\epsilon = \pm 1$, $\eta = \pm 1$. We treat these solutions as different superselections sectors of the theory that are unrelated. Notice that for the degeneracy condition in (2.11) we have $\sigma_{-1} = -(\nu^2 - \Omega)$, so that $\Sigma_{-1}^\eta = 0$ and the solutions in (3.3) are not defined. This indicates the common feature that the degenerate case needs to be treated separately. Requiring further that $\psi_0(x,y) \in L^2(\mathbb{R}^2)$ is square integrable on $\mathbb{R}^2$ with $\alpha, \beta, \gamma \in \mathbb{R}$ leads to the constraints $\alpha > 0$, $\beta > 0$ and $\gamma^2 < \alpha\beta$[1]. This allows us to exclude the two cases with $\epsilon = 1$, since then the inequalities can not be satisfied in the

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[1]This is easily seen: The integral

$$\int_{\mathbb{R}^2} \left| e^{-\frac{\alpha x^2}{2} - \frac{\beta y^2}{2} + \gamma xy} \right|^2 dx dy = \int_{\mathbb{R}^2} e^{-(x,y)M(x,y)^\intercal} dx dy, \qquad M = \begin{pmatrix} \alpha & -\gamma \\ -\gamma & \beta \end{pmatrix} \quad (3.4)$$

is finite when M is positive definite, i.e. when $\alpha > 0$, $\beta > 0$ and $\det M > 0 \Leftrightarrow \gamma^2 < \alpha\beta$.

parameter space, leaving the two options with $\epsilon = -1$. In addition, we demand the energy E_0 to be real so that $\Sigma_\epsilon^\eta, \sigma_\epsilon \in \mathbb{R}$. Combining these requirement restricts the parameter space domain in the two remaining cases to

$$\epsilon = -1, \eta = -1 : \quad |g| < -\nu^2 - \Omega, \tag{3.5}$$

$$\epsilon = -1, \eta = 1 : \quad |g| < \nu^2 + \Omega \;\wedge\; |g| < 2\nu^2 \;\wedge\; g^2 > 4\nu^2 \Omega. \tag{3.6}$$

In both of these cases we can find regions in parameter space that satisfy these inequalities as depicted in figure 1.
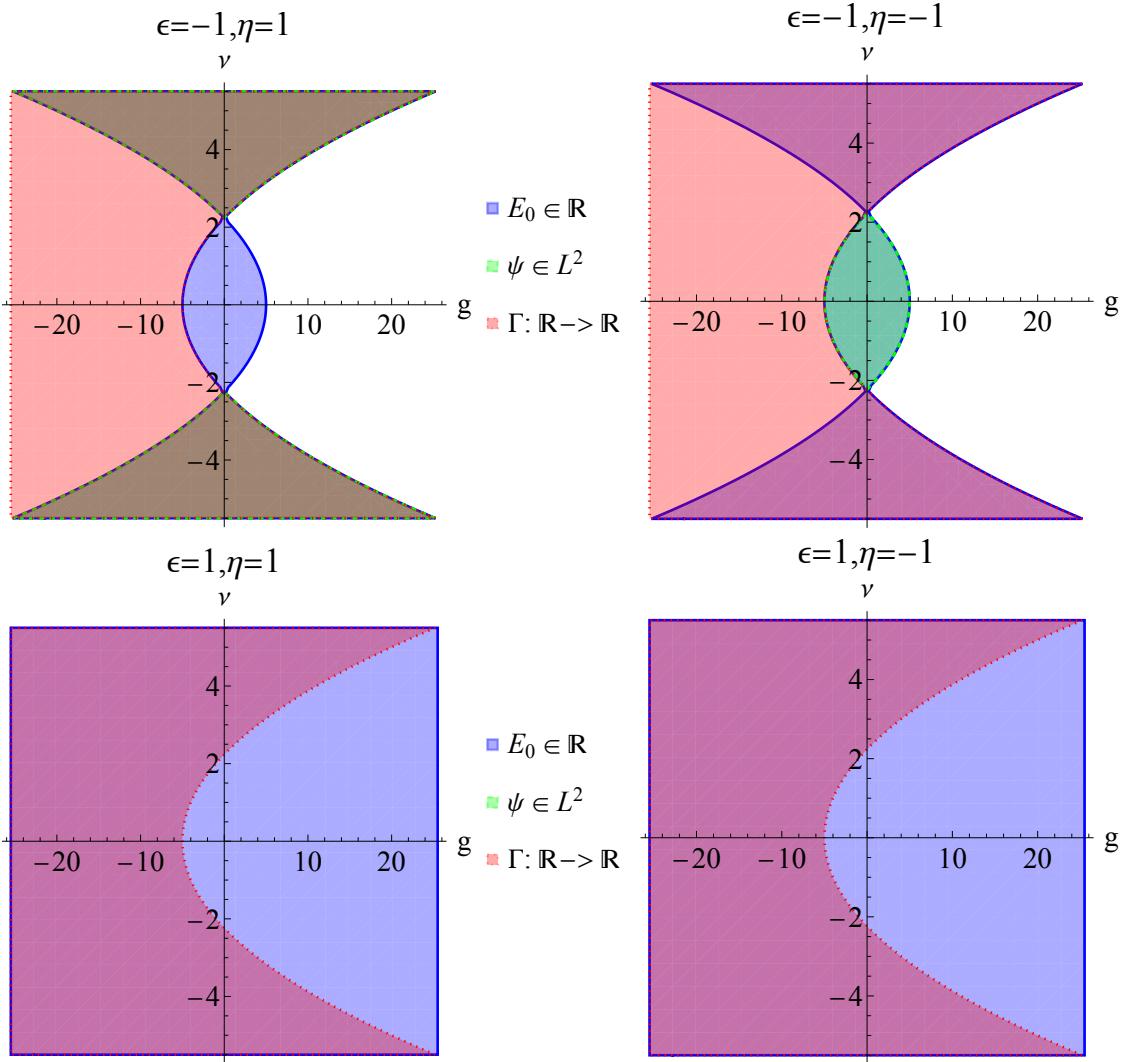

Figure 1: Parameter space domain for the three constraints real ground state energies, square integrability and real symplectic map to the PU oscillator, with $\Omega = -5$.

We notice that in both cases the strong coupling regime, i.e. large g, does not admit normalisable solutions with real "ground state" energy. Furthermore, we observe that when enforcing the map Γ in (2.7) to be real and well-defined, only leaves the brown region for the

case $\epsilon = -\eta = 1$, whereas for the case $\epsilon = \eta = -1$ we have normalisabilty and real ground state energies, but can not make contact to the PU-oscillator in any region of the parameter space in a well-defined manner. For both cases with $\epsilon = 1$ the reality of the ground state energy together with the validity of the map Γ can be guaranteed in large parts of the parameter regime, but the wavefunctions are never normalisable. We summarize these features in table 1, indicating by a $\checkmark$-sign that regions in parameter space can be found in which the three conditions on the left hold and a $\times$-sign when the conditions lead to an empty set in parameter space.

	$\epsilon = -\eta = -1$	$\epsilon = \eta = -1$	$\epsilon = \eta = 1$	$\epsilon = -\eta = 1$
$E_0 \in \mathbb{R},\ \psi_0 \in L^2,\ \Gamma : \mathbb{R} \mapsto \mathbb{R}$	$\checkmark$	$\times$	$\times$	$\times$
$E_0 \in \mathbb{R},\ \psi_0 \in L^2$	$\checkmark$	$\checkmark$	$\times$	$\times$
$\psi_0 \in L^2,\ \Gamma : \mathbb{R} \mapsto \mathbb{R}$	$\checkmark$	$\checkmark$	$\times$	$\times$
$E_0 \in \mathbb{R},\ \Gamma : \mathbb{R} \mapsto \mathbb{R}$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$

Table 1: Non-empty ($\checkmark$) and empty ($\times$) domains in parameter space when imposing combinations of the constraints related to the reality of the ground state energy, square integrability or a well-defined map from the ghost model to the PU-oscillator.

It is clear from the Ansatz for ψ_N, that the constraints resulting from the normalisability requirement also hold for $N > 0$, as in that case the wavefunction only differs by an overall polynomial factor that does not affect the convergence behaviour. Below we see that also the constraints (3.2) remain valid, so that (3.5) holds for all wavefunctions, but additional constraint may arise to ensure the reality of the spectra.

3.1.2 The degenerate case

Solving (3.2) for α, β, γ in the frequency degenerate case when $g = \eta(\nu^2 + \Omega)$ yields the four solutions

$$\alpha_\epsilon = \epsilon \frac{\Omega - 3\nu^2}{2\sqrt{2}\sqrt{\nu^2 - \Omega}}, \quad \beta_\epsilon = \epsilon \frac{\nu^2 - 3\Omega}{2\sqrt{2}\sqrt{\nu^2 - \Omega}}, \quad \gamma_\epsilon^\eta = \eta\epsilon \frac{\nu^2 + \Omega}{2\sqrt{2}\sqrt{\nu^2 - \Omega}}, \qquad (3.7)$$

In this case the constraint $\alpha > 0$, $\beta > 0$ and $\gamma^2 < \alpha\beta$ can not be satisfied for *any* of the four solutions, which in turn implies that the wavefunctions can not be normalised in the degenerate case. The reality of the energy E_0 is guaranteed in this case with the simple constraint $\nu^2 > \Omega$. The map Γ is also not defined in this case.

3.2 The three term recurrence relations

Upon the validity of (3.2) and fixing N, we obtain a three term recurrence relation in n for the coefficients σ_n^N and $\lfloor N/2 \rfloor$ three term recurrence relations with a non-homogenous term for the coefficients σ_n^{N-2k} with $k = 1, \ldots$. The relations read

$$a_n^{N,k} \sigma_{n-1}^{N-2k} + b_n^{N,k} \sigma_n^{N-2k} + c_n \sigma_{n+1}^{N-2k} = f_n^{N,k} \sigma_n^{N-2k+2} + g_n \sigma_{n+2}^{N-2k+2}, \qquad (3.8)$$

where the coefficients are

$$a_n^{N,k} := 2(2k + n - N - 1)\gamma, \quad b_n^{N,k} := E - (2n+1)\alpha + (2N - 4k - 2n + 1)\beta, \qquad (3.9)$$

$$c_n := 2(n+1)\gamma, \ f_n^{N,k} := (N - 2k - n + 1)(N - 2k - n + 2), \ g_n := -(n+1)(n+2),$$

with $\sigma_n^m = 0$ for $n = 0, \ldots, N$, $k = 0, \ldots, \lfloor(N-n)/2\rfloor$ and $N < n < 0$, $\lfloor(N-n)/2\rfloor < m < 0$. We stress that when reading off powers in x and y there will be additional terms besides those reported in (3.8), which, however, all cancel out when imposing the constraints (3.2).

Defining the $(N+1)$-dimensional vector $\Sigma_N^k = (\sigma_0^{N-2k}, \sigma_1^{N-2k}, \ldots, \sigma_{N-1}^{N-2k}, \sigma_N^{N-2k})$ and the matrices

$$
M_N^k = \begin{pmatrix}
b_0^{N,k} & c_0 & 0 & \cdots & 0 & 0 \\
a_1^{N,k} & b_1^{N,k} & c_1 & \cdots & 0 & 0 \\
0 & a_2^{N,k} & b_2^{N,k} & \ddots & 0 & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & a_{N-1}^{N,k} & b_{N-1}^{N,k} & c_{N-1} \\
0 & 0 & \cdots & 0 & a_N^{N,k} & b_N^{N,k}
\end{pmatrix}, \quad
B_N^k = \begin{pmatrix}
f_0^{N,k} & 0 & g_0 & 0 & \cdots & 0 & 0 \\
0 & f_1^{N,k} & 0 & g_1 & \cdots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & g_{N-2} \\
0 & 0 & \ddots & 0 & 0 & f_{N-1}^{N,k} & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & f_N^{N,k}
\end{pmatrix},
$$
(3.10)

it is convenient to cast the recurrence relations (3.8) into matrix form

$$
M_N^k \left(\Sigma_N^k\right)^{\mathsf{T}} = B_N^k \left(\Sigma_N^{k-1}\right)^{\mathsf{T}}, \qquad k = 1, \ldots, \left\lfloor \frac{N-i}{2} \right\rfloor.
$$
(3.11)

These equations are easily solved to

$$
\left(\Sigma_N^k\right)^{\mathsf{T}} = \prod_{\ell=1}^{k} \left[\left(M_N^\ell\right)^{-1} B_N^\ell \right] (\Sigma_N^0)^{\mathsf{T}}.
$$
(3.12)

Next we discuss how to find the initial set of coefficients Σ_N^0 and elaborate on how to compute the inverse of M_N^ℓ efficiently by exploiting the fact that they are tridiagonal matrices.

3.2.1 Energy quantisation from the $k = 0$ recurrence relation

In the first relation of (3.8) for $k = 0$ the term on the right hand side is vanishing so that the relation reduces to a genuine three term relation in n

$$
-2(N+1-n)\gamma\sigma_{n-1}^N + [E - (2n+1)\alpha + (2N - 2n + 1)\beta]\sigma_n^N + 2(n+1)\gamma\sigma_{n+1}^N = 0. \quad (3.13)
$$

These relations are the key characteristic for quasi-exactly solvable models [41–46], allowing us to exploit that structure to solve for all coefficients σ_n^N. Typically the three-term relation reduces to a two-term relation at certain values of n, in our case when $n = N + 1$, which enforces a quantisation condition for the energy E on which we elaborate at first. For the matrix relation $M_N^0 \left(\Sigma_N^0\right)^{\mathsf{T}} = 0$ to have non-trivial solutions we require that the determinant $|M_N^0|$ is vanishing. Noting that M_N^0 is a tridiagonal matrix, we can compute its determinant recursively from

$$
R_{t+1}^{N,0} = b_{t-1}^{N,0} R_t^{N,0} - a_{t-1}^{N,0} c_{t-2}^{N,0} R_{t-1}^{N,0},
$$
(3.14)

with $R_{n\leq 0}^{N,0} = 0$, $R_1^{N,0} = 1$ for $t = 1, \ldots, N + 1$ where R_t^N denotes the determinant of the $(t-1) \times (t-1)$ leading principal submatrix in the top left corner. Hence $|M_N^0| = R_{N+2}^N$,

see e.g. [49]. Solving this equation we obtain the conveniently factorised expression for the determinant

$$|M_N^0| = g\left(E, N, \frac{N}{2} + 1\right)^{\frac{(N+1 \bmod 2)}{2}} \prod_{k=n}^{\lfloor \frac{N+1}{2} \rfloor} g(E, N, n), \qquad (3.15)$$

where the function $g(E, n, m)$ is defined as

$$g(E, n, m) = [E+(2m-2n-3)\alpha+(2m-1)\beta][E+(1-2m)\alpha+(3-2m+2n)\beta]+4\gamma^2(2-2m+n)^2, \qquad (3.16)$$

with $(N + 1 \bmod 2)$ equal 1 or 0 for N even or odd, respectively and $\lfloor x \rfloor$ denoting the floor function defined above. Thus the determinant of M_N^0 is vanishing whenever the quadratic equations is solved in E, i.e. $g(E, n, m) = 0$, namely for the energies

$$E_{Nn}^{\pm} = (N + 1)(\alpha - \beta) \pm (2 - 2n + N)\sqrt{(\alpha + \beta)^2 - 4\gamma^2}, \quad n = 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor, \quad (3.17)$$

$$E_{NN} = (1 + N)(\alpha - \beta), \qquad (N + 1 \bmod 2). \qquad (3.18)$$

We depict a few sample spectra in figure 2, where we distinguish between the four branches. First of all we notice that the spectra for the two normalisable solutions are not bounded from below, whereas one of those corresponding to the non-normalisable cases is bounded from below and the other from above.

Having fixed all parameters of the model but one, i.e. the coupling strength g between the two oscillators, we observe the typical avoided level crossing [50] for fixed values of N. However, the level may cross for different values of N, which indicates that they do not couple because they belong to different symmetry classes. These will be identified below.

At the boundary of the domain the normalisable and non-normalisable solutions exhibit qualitatively distinct behaviour as is seen from the limit to the boundary value

$$\lim_{g \to \nu^2 + \Omega} E_{N,n}^{\pm} = (N + 1)\eta\epsilon\sqrt{\nu^2 - \Omega} + \epsilon|\nu^2 - \Omega| \pm (1 - \epsilon)\left(1 - n + \frac{N}{2}\right)\sqrt{2(\nu^2 - \Omega)}, (3.19)$$

$$\lim_{g \to \nu^2 + \Omega} E_{N,N}^{\pm} = (N + 1)\eta\epsilon\sqrt{\nu^2 - \Omega} + \epsilon|\nu^2 - \Omega|. \qquad (3.20)$$

We observe that for $\epsilon = -1$ the values for the energies $E_{N,n}^+$ and $E_{N,n}^-$ are distinct, whereas for $\epsilon = -1$ the spectra develop an exceptional point [51–53] at the boundary with both values becoming identical to $E_{N,N}$ and complex conjugate pairs thereafter. This is the well known behaviour for $\mathcal{PT}$-symmetric quantum mechanical systems where the transition from the $\mathcal{PT}$-symmetric to the spontaneously broken regime occurs at the exceptional point [54], albeit here for a real Hermitian Hamiltonian. The Hamiltonian is evidently $\mathcal{PT}$-symmetric under $\mathcal{PT} : p_x \to p_x, p_y \to p_y, x \to -x$ and $y \to -y$ and the condition that the eigenstates of $\mathcal{H}$ are also eigenstates of the $\mathcal{PT}$-operator no longer holds in the broken regime when the parameters α, β, γ become complex.

Notice that in the frequency degenerate case (3.7) we always have $(\alpha + \beta)^2 - 4\gamma^2 = 0$ throughout the entire parameter regime, so that also the energies are indeed degenerate.
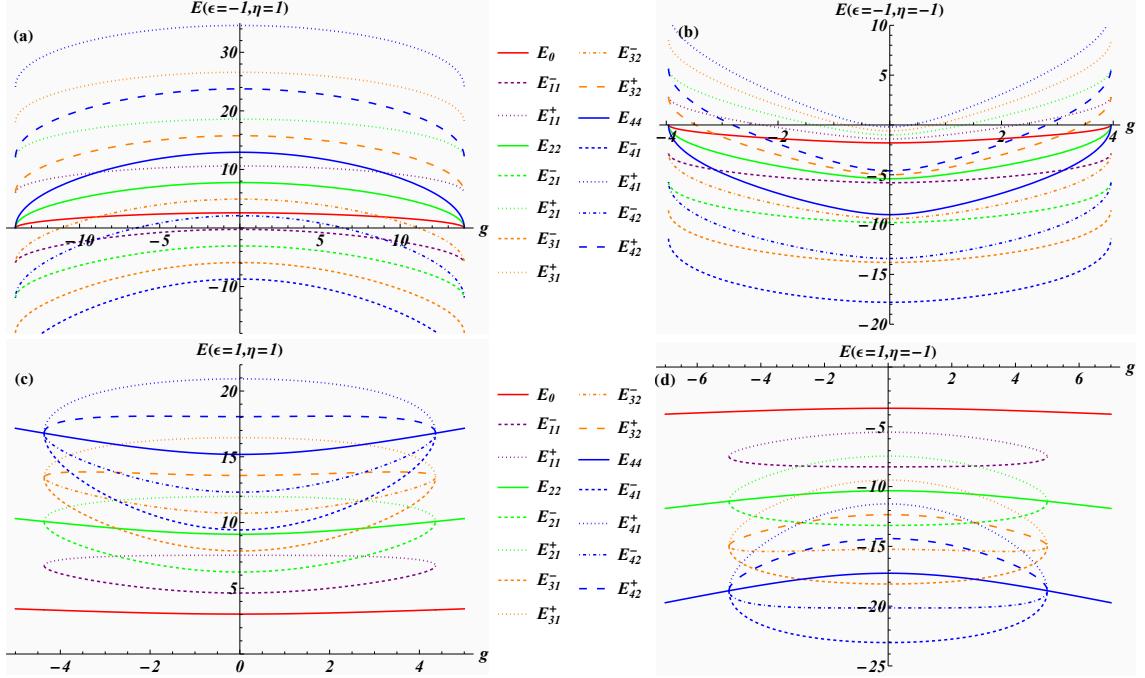
Figure 2: Real energy spectra for normalisable eigenstates in panels (a) and (b) with $\nu = 4$, $\Omega = -2$, $|g| < 14$ and $\nu = 0.2$, $\Omega = -4$, $|g| < 3.96$ and for non-normalisable eigenstates in panels (c) and (d) with $\nu = 0.8$, $\Omega = -5$ and $\nu = 1$, $\Omega = -6$, respectively.

3.2.2 Solution of the $k = 0$ three term recurrence relation

Next we provide a closed solution for the coefficients σ_n^N following [45, 55] in solving the recurrence relations. For this purpose we first transform (3.8) to the canonical form for three-term recurrence relation

$$\tau_{n+1} = \tau_n + \lambda_n \tau_{n-1}, \tag{3.21}$$

with

$$\sigma_n^N := \kappa \tau_n \prod_{k=0}^{n-1} r_k = \kappa \tau_n \frac{[\alpha - (2N+1)\beta - E]}{2n!(\alpha + \beta)} \frac{\Gamma\left[n - \frac{E - \alpha + (2N+1)\beta}{2(\alpha + \beta)}\right]}{\Gamma\left[\frac{3\alpha - (2N-1)\beta - E}{2(\alpha + \beta)}\right]} \left(\frac{\alpha + \beta}{\gamma}\right)^n \tag{3.22}$$

where κ is an overall constant and

$$r_n := -\frac{b_n^{N,0}}{c_n^{N,0}} \qquad s_n := -\frac{a_n^{N,0}}{c_n^{N,0}}, \qquad \lambda_n := \frac{s_n}{r_{n-1} r_n}. \tag{3.23}$$

The recurrence relation (3.21) is solved by

$$\tau_n = 1 + \sum_{p=1}^{\lfloor n/2 \rfloor} S(n-1, p), \tag{3.24}$$

where $\lfloor x \rfloor$ is defined as above and $S(n-1, p)$ as the p-fold sum

$$S(n, p) := \sum_{k_1=1}^{n+2(1-p)} \lambda_{k_1} \sum_{k_2=2+k_1}^{n+2(2-p)} \lambda_{k_2} \cdots \sum_{k_{p-1}=2+k_{p-2}}^{n-2} \lambda_{k_{p-1}} \sum_{k_p=2+k_{p-1}}^{n} \lambda_{k_p}. \tag{3.25}$$

The first expressions are computed to $\tau_0 = 1$, $\tau_1 = 1$, $\tau_2 = 1 + \lambda_1$, $\tau_3 = 1 + \lambda_1 + \lambda_2$, $\tau_4 = 1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_1 \lambda_3$, $\tau_5 = 1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_4$ etc. Thus we compute the first coefficients to

$$\sigma_0^N = \kappa, \tag{3.26}$$
$$\sigma_1^N = \kappa r_0,$$
$$\sigma_2^N = \kappa \left(s_1 + r_0 r_1 \right),$$
$$\sigma_3^N = \kappa \left(s_1 r_2 + s_2 r_0 + r_0 r_1 r_2 \right),$$
$$\sigma_4^N = \kappa \left(s_1 s_3 + s_2 r_0 r_3 + s_1 r_2 r_3 + r_0 r_1 r_3 + r_0 r_1 r_2 r_3 \right),$$
$$\sigma_5^N = \kappa \left(s_1 s_3 r_4 + s_1 s_4 r_2 + s_2 s_4 r_0 + s_1 r_2 r_3 r_4 + s_2 r_0 r_3 r_4 + s_3 r_0 r_1 r_4 + s_4 r_0 r_1 r_2 + r_0 r_1 r_2 r_3 r_4 \right).$$

We have now fully determined the vector Σ_N^0.

3.2.3 Solutions of the non-homogeneous three term recurrence relations, $k \neq 0$

Closed formal solutions to the non-homogeneous three term recurrence relations (3.8) with $k = 1, 2, \ldots$ were found in (3.12). The expressions still involve the inverse of the matrix M_N^ℓ, which can be computed efficiently by exploiting the fact that it is a tridiagonal matrix admitting the closed formula

$$\left(M_N^\ell \right)_{ij}^{-1} = \begin{cases} (-1)^{i+j} \dfrac{R_i^{N,\ell} T_{j+1}^{N,\ell}}{|M_N^0|} \displaystyle\prod_{k=i}^{j-1} c_{k-1}^{N,\ell}, & \text{if } i < j, \\[2ex] \dfrac{R_i^{N,\ell} T_{i+1}^{N,\ell}}{|M_N^0|}, & \text{if } i = j, \\[2ex] (-1)^{i+j} \dfrac{R_j^{N,\ell} T_{i+1}^{N,\ell}}{|M_N^0|} \displaystyle\prod_{k=j}^{i-1} a_k^{N,\ell}, & \text{if } i > j. \end{cases} \tag{3.27}$$

Here the $R_t^{N,\ell}$ are the determinants of the $(t-1) \times (t-1)$ leading principal submatrix in the top left corner of M_N^ℓ computed recursively from

$$R_{t+1}^{N,\ell} = b_{t-1}^{N,\ell} R_t^{N,\ell} - a_{t-1}^{N,\ell} c_{t-2}^{N,\ell} R_{t-1}^{N,\ell}, \tag{3.28}$$

with $R_{n\leq 0}^{N,\ell} = 0$, $R_1^{N,\ell} = 1$ for $t = 1, \ldots, N+1$. The $T_t^{N,\ell}$ are determinants of the $(t-1) \times (t-1)$ trailing principal submatrix in the lower right corner of M_N^ℓ starting at row t, which can be computed from

$$T_t^{N,\ell} = b_{t-1}^{N,\ell} T_{t+1}^{N,\ell} - a_t^{N,\ell} c_{t-1}^{N,\ell} T_{t+2}^{N,\ell}, \tag{3.29}$$

with $T_{n>N+2}^{N,\ell} = 0$, $T_{N+2}^{N,\ell} = 1$ for $t = N, N-1 \ldots, 1$.

Note that the quantisation condition has selected the energy E such that $|M_N^0| = 0$, which in turn ensures that $|M_N^\ell| \neq 0$ for $\ell = 1, 2, \ldots$ so that all relevant inverse matrices in the procedure exist.

3.3 First explicit examples

Having found the entire eigensystem it is useful to report some explicit examples. The first solutions for the polynomials P_N are easily found. Up to an overall multiplicative constant c_0^N they read

$$P_0 = 1, \tag{3.30}$$

$$P_1 = y - \frac{E - \alpha + 3\beta}{2\gamma}x,$$

$$P_2 = x^2 + y^2 + \frac{(E - 3\alpha + 3\beta)(E - \alpha + 5\beta)\left[x^2(E - \alpha + \beta) - 2\right]}{8\gamma^2(E - \alpha + \beta)} - \frac{(E - \alpha + 5\beta)}{2\gamma}xy. \tag{3.31}$$

Together with the energies from the quantisation condition (3.17) and (3.18), the lowest level eigensolutions are

$$N = 0 : P_0 = 1, \quad E_0 = \alpha - \beta, \tag{3.32}$$

$$N = 1 : P_1^1 = y - \frac{\alpha + \beta - \kappa}{2\gamma}x, \quad E_1^1 = 2(\alpha - \beta) - \kappa, \tag{3.33}$$

$$P_1^2 = y - \frac{\alpha + \beta + \kappa}{2\gamma}x, \quad E_1^2 = 2(\alpha - \beta) + \kappa,$$

$$N = 2 : P_2^1 = x^2 + y^2 - \frac{\alpha + \beta}{\gamma}xy, \quad E_2^1 = 3(\alpha - \beta), \tag{3.34}$$

$$P_2^2 = \frac{\kappa^2 + (\alpha + \beta)\kappa}{2\gamma^2(\beta - \alpha + \kappa)} + x^2\left[\frac{(\alpha + \beta)(\alpha + \beta - \kappa)}{2\gamma^2} - 1\right] - \frac{(\alpha + \beta - \kappa)}{\gamma}xy + y^2,$$

$$P_2^3 = \frac{\kappa^2 - (\alpha + \beta)\kappa}{2\gamma^2(\beta - \alpha - \kappa)} + x^2\left[\frac{(\alpha + \beta)(\alpha + \beta + \kappa)}{2\gamma^2} - 1\right] - \frac{(\alpha + \beta + \kappa)}{\gamma}xy + y^2,$$

$$E_2^2 = 3(\alpha - \beta) - 2\kappa, \qquad E_2^3 = 3(\alpha - \beta) + 2\kappa,$$

with $\kappa := \sqrt{(\alpha + \beta)^2 - 4\gamma^2}$.

4. Bosonic Fock space construction

Next we second quantise the model and compare with the results obtained in the previous section. Our starting point is the well-known mode expansion solution for the PU-equation [4] in the non-degenerate case

$$q(t) = ae^{-it\omega_1} + a^\star e^{it\omega_1} + be^{-it\omega_2} + b^\star e^{it\omega_2}, \qquad a, a^\star, b, b^\star \in \mathbb{C}. \tag{4.1}$$

Imposing the frequencies to be real implies that the parametrisation (2.11) limits the regimes to $0 < \zeta$, $0 < \xi < \zeta^2/4$. In model parameter space these values correspond to the restrictions $\nu^2 > \Omega$, $g^2 > 4\nu^2\Omega$, $g^2 > (\nu^2 + \Omega)^2$, which we already encountered previously. Expressing the frequencies in terms of the model parameters gives rise to eight different regimes depending on their respective ordering and signs. We obtain

$$\omega_i^{\epsilon\eta} = \frac{\epsilon}{\sqrt{2}}\sqrt{\zeta + \eta\sqrt{\zeta^2 - 4\xi}}, \tag{4.2}$$

with ζ and ξ unambiguously defined in (2.9), so that we have the regimes

$$\text{I}: \omega_2^{++} > \omega_1^{+-} > 0, \ \text{II}: \omega_1^{++} > \omega_2^{+-} > 0, \quad \text{III}: \omega_2^{++} > -\omega_1^{--} > 0, \ \text{IV}: -\omega_1^{-+} > \omega_2^{+-} > 0,$$
$$\text{V}: 0 > \omega_2^{--} > \omega_1^{-+}, \ \text{VI}: 0 > \omega_1^{--} > \omega_2^{-+}, \quad \text{VII}: 0 > \omega_2^{--} > -\omega_1^{++}, \ \text{VIII}: 0 > -\omega_1^{+-} > \omega_2^{-+}.$$

Using the solution (4.1) the transformation (2.7) then yields by direct calculation

$$x = \bar{\mu}_1 \left(a e^{-it\omega_1} + a^\star e^{it\omega_1} \right) + \bar{\mu}_2 \left(b e^{-it\omega_2} + b^\star e^{it\omega_2} \right), \tag{4.3}$$

$$y = \bar{\nu}_1 \left(a e^{-it\omega_1} + a^\star e^{it\omega_1} \right) + \bar{\nu}_2 \left(b e^{-it\omega_2} + b^\star e^{it\omega_2} \right), \tag{4.4}$$

$$p_x = \frac{i}{2} \omega_1 \bar{\mu}_1 \left(a^* e^{it\omega_1} - a e^{-it\omega_1} \right) + \frac{i}{2} \omega_2 \bar{\mu}_2 \left(b^* e^{it\omega_2} - b e^{-it\omega_2} \right), \tag{4.5}$$

$$p_y = \frac{i}{2} \omega_1 \bar{\nu}_1 \left(a e^{-it\omega_1} - a^* e^{it\omega_1} \right) + \frac{i}{2} \omega_2 \bar{\nu}_2 \left(b e^{-it\omega_2} - b^* e^{it\omega_2} \right), \tag{4.6}$$

with $\bar{\mu}_i := \mu_0 - \mu_2 \omega_i^2$, $\bar{\nu}_i := \nu_0 - \nu_2 \omega_i^2$. When solved for $a, a^\star, b, b^\star$ this gives

$$a = \left[\omega_1 \bar{\mu}_2 y - \omega_1 \bar{\nu}_2 x - 2i(\bar{\nu}_2 p_x + \bar{\mu}_2 p_y) \right] \frac{e^{it\omega_1}}{2\omega_1 \left(\omega_1^2 - \omega_2^2 \right) (\mu_2 \nu_0 - \mu_0 \nu_2)}, \tag{4.7}$$

$$b = \left[\omega_2 \bar{\nu}_1 x - \omega_2 \bar{\mu}_1 y + 2i(\bar{\nu}_1 p_x + \bar{\mu}_1 p_y) \right] \frac{e^{it\omega_2}}{2\omega_2 \left(\omega_1^2 - \omega_2^2 \right) (\mu_2 \nu_0 - \mu_0 \nu_2)}, \tag{4.8}$$

and $a^\star, b^\star$ simply being the complex conjugates of a, b. Next we promote x, y, p_x, p_y and $a, a^\star, b, b^\star$ to operators $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ and $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$, respectively. The Poisson brackets (2.10) become equal time commutators $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$ provided that

$$\left[\hat{a}, \hat{a}^\dagger \right] = \frac{\nu_0^2 - \mu_0^2 + 2(\mu_0 \mu_2 - \nu_0 \nu_2)\omega_2^2 + (\nu_2^2 - \mu_2^2)\omega_2^4}{\omega_1 \left(\omega_1^2 - \omega_2^2 \right)^2 (\mu_2 \nu_0 - \mu_0 \nu_2)^2} = \frac{1}{2\omega_1(\omega_1^2 - \omega_2^2)}, \tag{4.9}$$

$$\left[\hat{b}, \hat{b}^\dagger \right] = \frac{\nu_0^2 - \mu_0^2 + 2(\mu_0 \mu_2 - \nu_0 \nu_2)\omega_1^2 + (\nu_2^2 - \mu_2^2)\omega_1^4}{\omega_2 \left(\omega_1^2 - \omega_2^2 \right)^2 (\mu_2 \nu_0 - \mu_0 \nu_2)^2} = \frac{1}{2\omega_2(\omega_2^2 - \omega_1^2)}. \tag{4.10}$$

In the last step we used (2.9) and (2.11) to replace

$$\Omega \to \frac{1}{8} \left(\rho - \omega_1^2 - \omega_2^2 \right), \ \ \nu^2 \to \frac{1}{8} \left(\rho + \omega_1^2 + \omega_2^2 \right), \tag{4.11}$$

where $\rho = \sqrt{16g^2 + \left(\omega_1^2 - \omega_2^2 \right)^2}$. We have excluded here the choice with a minus sign in front of the square root, as in that case Γ always maps to complex solutions. Using these creation and annihilation operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$, where we still need to identify which one is which, the second quantised version of our Hamiltonian (2.1) becomes

$$\hat{\mathcal{H}} = 2 \left(\omega_1^2 - \omega_2^2 \right) \left(\omega_1^2 \hat{a}^\dagger \hat{a} - \omega_2^2 \hat{b}^\dagger \hat{b} \right) + \frac{1}{2}(\omega_1 + \omega_2). \tag{4.12}$$

In this formulation the observed level crossing in the energy spectrum is evident as the model is simply build from two commuting copies of $SU(1,1)$ algebras. Next we construct the bosonic Fock space for this Hamiltonian.

4.1 The ground state

As candidates for the ground state we take again $\psi_0^{\epsilon,\eta}$ from (3.1), with ϵ, η labelling the four solution (3.3) for α, β, γ. Translating from the variables Ω, ν to ω_1, ω_2 these solutions become

$$\alpha_\epsilon^\eta = \frac{\rho + \Omega_{12}^\epsilon}{4\eta\sqrt{\Omega_{12}^\epsilon}}, \qquad \beta_\epsilon^\eta = \frac{\rho - \Omega_{12}^\epsilon}{4\eta\sqrt{\Omega_{12}^\epsilon}}, \qquad \gamma_\epsilon^\eta = -\frac{g}{\eta\sqrt{\Omega_{12}^\epsilon}}, \tag{4.13}$$

with $\Omega_{12}^\epsilon = \omega_1^2 + \omega_2^2 + 2\epsilon\sqrt{\omega_1^2\omega_2^2}$. Next we identify which operator annihilated the state $\psi_0^{\epsilon,\eta}$ depending on the choices for ϵ, η and the regimes. We find

$$\hat{a}\psi_0^{-+} = 0, \qquad \hat{a}^\dagger\psi_0^{--} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{II}, \text{III}, \text{VI}, \text{VII}\}, \tag{4.14}$$

$$\hat{a}^\dagger\psi_0^{-+} = 0, \qquad \hat{a}\psi_0^{--} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{I}, \text{IV}, \text{V}, \text{VIII}\}, \tag{4.15}$$

$$\hat{b}\psi_0^{-+} = 0, \qquad \hat{b}^\dagger\psi_0^{--} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{I}, \text{III}, \text{V}, \text{VII}\}, \tag{4.16}$$

$$\hat{b}^\dagger\psi_0^{-+} = 0, \qquad \hat{b}\psi_0^{--} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{II}, \text{IV}, \text{VI}, \text{VIII}\}, \tag{4.17}$$

and

$$\hat{a}\psi_0^{++} = 0, \qquad \hat{a}^\dagger\psi_0^{+-} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{I}, \text{II}, \text{VII}, \text{VIII}\}, \tag{4.18}$$

$$\hat{a}^\dagger\psi_0^{++} = 0, \qquad \hat{a}\psi_0^{+-} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{III}, \text{IV}, \text{V}, \text{VI}\}, \tag{4.19}$$

$$\hat{b}\psi_0^{++} = 0, \qquad \hat{b}^\dagger\psi_0^{+-} = 0, \qquad \text{for } \omega_1, \omega_2 \in \{\text{I}, \text{II}, \text{III}, \text{IV}\}, \tag{4.20}$$

$$\hat{b}^\dagger\psi_0^{++} = 0, \qquad \hat{b}\psi_0^{+-} = 0 \qquad \text{for } \omega_1, \omega_2 \in \{\text{V}, \text{VI}, \text{VII}, \text{VIII}\}. \tag{4.21}$$

Notice that according to our table 1 only ψ_0^{-+} are candidates posses non-empty regimes where all three desirable properties are satisfied.

In order to identify the ground state energy it is most convenient to bring all the annihilation operators to the right in the expression for the Hamiltonian. Thus it is useful to re-write the Hamiltonian in some slightly alternative forms by using the commutation relations (4.7) and (4.8). We easily obtain

$$\hat{\mathcal{H}} = 2\left(\omega_1^2 - \omega_2^2\right)\left(\omega_1^2\hat{a}\hat{a}^\dagger - \omega_2^2\hat{b}^\dagger\hat{b}\right) + \frac{1}{2}(\omega_2 - \omega_1), \tag{4.22}$$

$$= 2\left(\omega_1^2 - \omega_2^2\right)\left(\omega_1^2\hat{a}^\dagger\hat{a} - \omega_2^2\hat{b}\hat{b}^\dagger\right) + \frac{1}{2}(\omega_1 - \omega_2), \tag{4.23}$$

$$= 2\left(\omega_1^2 - \omega_2^2\right)\left(\omega_1^2\hat{a}\hat{a}^\dagger - \omega_2^2\hat{b}\hat{b}^\dagger\right) - \frac{1}{2}(\omega_1 + \omega_2). \tag{4.24}$$

Using the relations (4.14)-(4.17) to identify the version with all annihilation operators to the right, we can directly read off the ground state energies in the different regimes as

$$\text{III}, \text{VII}: \quad \hat{\mathcal{H}}\psi_0^{-+} = \frac{1}{2}(\omega_1 + \omega_2)\psi_0^{-+}, \quad \hat{\mathcal{H}}\psi_0^{--} = -\frac{1}{2}(\omega_1 + \omega_2)\psi_0^{--}, \tag{4.25}$$

$$\text{I}, \text{V}: \quad \hat{\mathcal{H}}\psi_0^{-+} = \frac{1}{2}(\omega_2 - \omega_1)\psi_0^{-+}, \quad \hat{\mathcal{H}}\psi_0^{--} = \frac{1}{2}(\omega_1 - \omega_2)\psi_0^{--}, \tag{4.26}$$

$$\text{II}, \text{VI}: \quad \hat{\mathcal{H}}\psi_0^{-+} = \frac{1}{2}(\omega_1 - \omega_2)\psi_0^{-+}, \quad \hat{\mathcal{H}}\psi_0^{--} = \frac{1}{2}(\omega_2 - \omega_1)\psi_0^{--}, \tag{4.27}$$

$$\text{IV}, \text{VIII}: \quad \hat{\mathcal{H}}\psi_0^{-+} = -\frac{1}{2}(\omega_1 + \omega_2)\psi_0^{-+}, \quad \hat{\mathcal{H}}\psi_0^{--} = \frac{1}{2}(\omega_1 + \omega_2)\psi_0^{--}. \tag{4.28}$$

For the ground state ψ_0^{-+} we used version (4.12) in the regions III, VII, version (4.22) in the regions I V, (4.23) in the regions II, VI and (4.24) in the regions IV, VIII. For ψ_0^{--} we used version (4.12) in the regions IV, VIII, version (4.22) in the regions II VI, (4.23) in the regions I, V and (4.24) in the regions III, VII. When translating back to α, β, γ we find that in all cases we obtain $E_0 = \alpha - \beta$, i.e. we have $\hat{\mathcal{H}}\psi_0 = (\alpha - \beta)\psi_0$ in all cases in the respective regimes.

4.2 The energy spectrum and the excited states

The eigensystem can be defined in the usual way by building up a Fock space from our sets of creation and annihilation operators. Since we observed in the previous section that we obtain identical expressions when we convert the frequency representation back to the original model parameters, it suffices to consider one regime. For definiteness we take here the regime I. The exited states are defined by applying successive powers of the two creation operators, $\hat{a}$ and $\hat{b}^\dagger$ in this case, on the ground state. The normalised eigenfunctions of the exited states are then

$$\bar{\psi}_{n,m} := \frac{1}{\sqrt{N_{n,m}}}\hat{a}^n \left(\hat{b}^\dagger\right)^m \psi_0^{-+}, \quad N_{n,m} = \frac{2\pi}{\sqrt{n!m!\omega_1\omega_2}}\frac{1}{2^{n+m}\omega_2^m\omega_1^n(\omega_1^2 - \omega_2^2)^{n+m}}. \quad (4.29)$$

The corresponding eigenenergy spectrum is computed to

$$\bar{E}_{n,m} = \left(\frac{1}{2} - n\right)\omega_1 + \left(m - \frac{1}{2}\right)\omega_2, \qquad n, m \in \mathbb{N}_0. \quad (4.30)$$

We observe the typical feature in HTDT of the spectrum being unbounded from below for normalisable states, as in the regime I both frequencies are positive. For the non normalisable states we may, however, construct a bounded system. For instance, in the regime I we find

$$\hat{\mathcal{H}}\psi_{n,m}^{++} = \left[\frac{1}{2}(n + 1)\omega_1 + \frac{1}{2}(m + 1)\omega_2,\right]\psi_{n,m}^{++}, \qquad n, m \in \mathbb{N}_0, \quad (4.31)$$

which clearly corresponds to a spectrum bounded from below since both frequencies are positive in the regime I. Evidently, both cases suffer from undesirable features, but given the choice between normalisable wavefunction and unbounded spectra from below versus non-normalisable wavefunctions and bounded spectra, the latter option seems to be less attractive.

Finally, we directly compare the eigensystem obtained here with the solution found from the recurrence relations in the previous section by transforming the α, β, γ dependence into an ω_1, ω_2 dependence. Starting with the energy spectrum from (3.17), (3.18) we obtain in the regime I

$$E_{Nn}^+ = \left(\frac{1}{2} - n\right)\omega_1 + \left(N - n + \frac{3}{2}\right)\omega_2 = \bar{E}_{n-1,1+N-n}, \ n = 1, \ldots, \left\lfloor\frac{N}{2}\right\rfloor, \quad (4.32)$$

$$E_{Nn}^- = \left(n - N - \frac{3}{2}\right)\omega_1 + \left(n - \frac{1}{2}\right)\omega_2 = \bar{E}_{1+N-n,n-1}, \quad (4.33)$$

$$E_{NN} = -\left(\frac{N}{2} + \frac{1}{2}\right)\omega_1 + \left(\frac{N}{2} + \frac{1}{2}\right)\omega_2 = \bar{E}_{N/2,N/2}, \quad (N + 1 \bmod 2). \quad (4.34)$$

The same identification for the quantum numbers then holds for the wavefunctions. Ignoring overall factors as the wavefunctions for the solutions of the recurrence are not normalised yet, we identify

$$\psi^+_{Nn} \propto \bar{\psi}_{n-1,1+N-n}, \qquad \psi^-_{Nn} \propto \bar{\psi}_{1+N-n,n-1}, \qquad \psi_{NN} \propto \bar{\psi}_{N/2,N/2}. \tag{4.35}$$

Thus, both approaches are consistent with each other and lead to the same result.

5. Model properties

Having explicitly solved the quantum system, we are now in a position to calculate all relevant physical properties of our ghost model. We begin by computing the probability distribution $|\psi(x,y)|^2$ in $\mathbb{R}^2$. Our results for the normalisable wavefunctions are shown in figure 3.
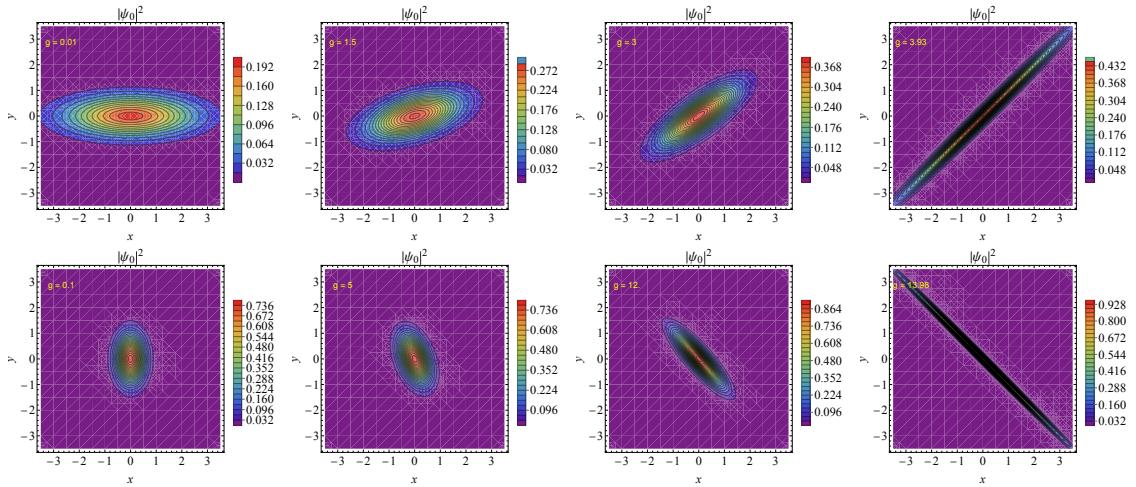
Figure 3: Probability densities for the "ground state" for different coupling constants g for the $\epsilon = \eta = -1$ branch with $\nu = 4$, $\Omega = -2$ in the upper row and $\epsilon = -\eta = -1$ branch with $\nu = 0.2$, $\Omega = -4$ in the lower row.

For a small coupling constant g, we observe that the ground state is most localized at the origin, with the probability density falling off symmetrically in the negative and positive x and y directions, depending on the strength of the potential in each direction. As the coupling constant increases, the horizontal axis of the elliptical region in which the particle is localized begins to tilt. The ellipse becomes increasingly squeezed until its axis aligns with the $y = x$ or $y = -x$ diagonal in the $\epsilon = -\eta = -1$ or $\epsilon = \eta = -1$ case, respectively. Eventually the wavefunction collapses into a line when g reaches the boundary of the defining domain in parameter space.

For the exited states the behaviour is similar for each of the nodes of the wavefunctions as seen in figure 4 for an example.

Another interesting quantity to compute is the uncertainty relation

$$(\Delta x \Delta p_x)_\psi = \sqrt{\langle x^2 \rangle_\psi - \langle x \rangle^2_\psi} \sqrt{\langle p_x^2 \rangle_\psi - \langle p_x \rangle^2_\psi} = \sqrt{\langle x^2 \rangle_\psi} \sqrt{\langle p_x^2 \rangle_\psi}. \tag{5.1}$$
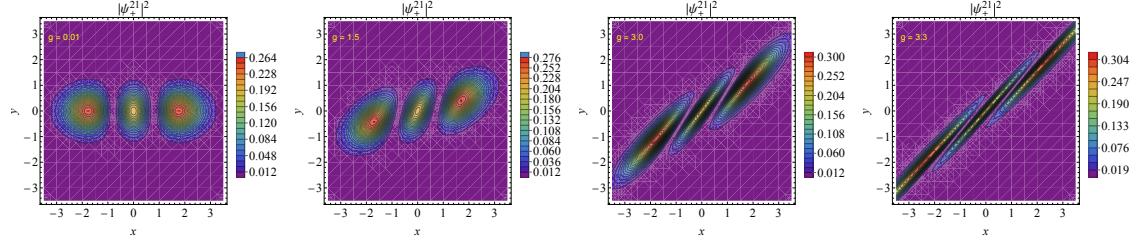
Figure 4: Probability densities for the ψ_{21}^+ state for different coupling constants g for the $\epsilon = \eta = -1$ branch with $\nu = 4$, $\Omega = -2$.

Using the explicit solution we find for the first examples

$$(\Delta x \Delta p_x)_{\psi_{11}} = \frac{1}{2} \sqrt{\frac{\alpha\beta}{\alpha\beta - \gamma^2}}, \tag{5.2}$$

$$(\Delta x \Delta p_x)_{\psi_{22}} = \frac{3}{2} \sqrt{\frac{\alpha\beta}{\alpha\beta - \gamma^2}}, \tag{5.3}$$

$$(\Delta x \Delta p_x)_{\psi_{11}^\pm} = \frac{1}{2} \sqrt{5 \pm \frac{4(\alpha+\beta)}{\sqrt{(\alpha+\beta)^2 - 4\gamma^2}} + \gamma^2 \left(\frac{4}{(\alpha+\beta)^2 - 4\gamma^2} + \frac{3}{\alpha\beta - \gamma^2} \right)}, \tag{5.4}$$

$$(\Delta x \Delta p_x)_{\psi_{21}^\pm} = \frac{1}{2} \sqrt{13 \pm \frac{12(\alpha+\beta)}{\sqrt{(\alpha+\beta)^2 - 4\gamma^2}} + \gamma^2 \left(\frac{16}{(\alpha+\beta)^2 - 4\gamma^2} + \frac{5}{\alpha\beta - \gamma^2} \right)}. \tag{5.5}$$

Similar expressions are obtained for the uncertainties in the y direction. We depict these expressions in figure 5.
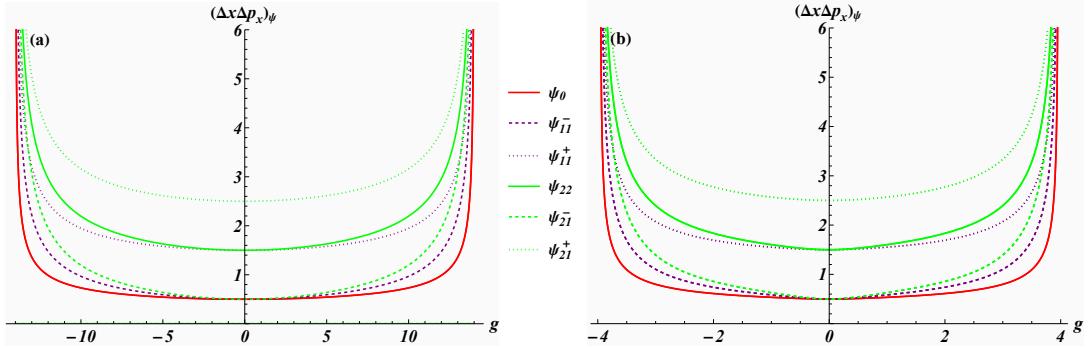

Figure 5: Uncertainty relations (5.2)-(5.5) for varying coupling constant g for the $\epsilon = \eta = -1$ branch with $\nu = 4$, $\Omega = -2$ in panel (a) and $\epsilon = -\eta = -1$ branch with $\nu = 0.2$, $\Omega = -4$ in panel (b).

Reassuringly, we note first that the fundamental bound of $1/2$ is respected by all solutions and even saturated for small g for ψ_0, ψ_{11}^- and ψ_{22}^-. Furthermore, we observe that when the coupling constant approaches the boundary of the domain in the parameter regime the uncertainties tend to infinity. This behaviour is qualitatively reproduced in the classical phase space. In figure 6 we plot the classical solutions (4.3)-(4.6) with constants $a = b = 0.5$ from $t = 0$ to some arbitrary large time $t = 150$ for the regime I. We observe

that for small values of the coupling constant the solutions are well localised in the (x, p_x)-phase space. This localisation becomes more and more fuzzy when the coupling constant increases and eventually a whole region is filled out when the boundary of the defining domain is approached. When the cut-off time becomes very large the phase space is also filled for small values of g, but it takes considerable longer to achieve that.
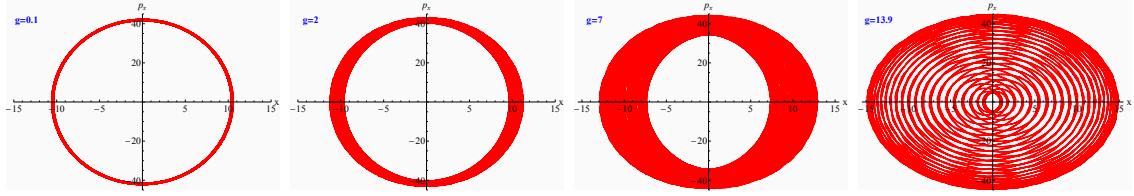

Figure 6: Classical trajectories from (4.3)-(4.6) in the (x, p_x) phase space in regime I with $\Omega = -1$, $\nu = 4$.

6. Conclusions

In this work, we investigated an exactly solvable ghostly quantum system and explored its quantisation through two distinct approaches. By explicitly mapping our system onto the Pais-Uhlenbeck oscillator via a symplectic transformation, we established a direct correspondence between their dynamical structures in certain parameter regimes, thereby demonstrating that our model can be interpreted as a higher time-derivative theory (HTDT) in disguise. This correspondence allowed us to quantise the system in two alternative ways where the transformation map is well-defined.

We obtained exact solutions to the Schrödinger equation using an Ansatz that led to theories defined in four different superselection sectors. In each of them we were led to a set of coupled three-term recurrence relations, which we solved systematically to construct normalisable wavefunctions and their associated energy spectra. Our findings were further validated by a second-quantisation approach, where we built a bosonic Fock space representation of the model. Both methods yielded consistently compatible results, confirming the reliability of our analytical solutions.

One of the insights from our study is the intricate balance between normalisability and spectral boundedness in ghostly quantum systems similar to the features found in HTDT. While the sectors with normalisable solutions exhibit unbounded spectra, the two sectors with non-normalisable wavefunctions can lead to bounded spectra either from above or below. The $\mathcal{PT}$-symmetry of the system is most relevant in the two latter cases in which it explains the occurrence of exceptional points and a spontaneously broken regime with pairs of complex conjugate eigenvalues.

The model properties in the low coupling regime revealed an interesting behaviour with the coupling constant approaching the boundary of the defining parameter regime. Our analysis of probability densities and uncertainty relations illustrated how the coupling parameter influences the localisation properties of the system, with increasing interactions leading to pronounced quantum fluctuations and loss of phase space localisation.

At this stage we did not provide any analysis about the model's behaviour at the point in parameter space that is equivalent to the degenerate point in the PU-model, which is known to be problematic already at the classical level with its well-known divergent solutions and so far no widely accepted quantised version. Similarly it is unclear whether our ghostly model possess consistent solutions in the strong coupling regime.

Our results contribute to the broader understanding of higher-derivative quantum models and ghostly excitations in quantum mechanics, providing a novel exactly solvable framework for further explorations. Future directions include extending this approach to field-theoretic models, investigating interactions with external potentials, and long term exploring potential applications to quantum gravity scenarios where higher-derivative corrections play a crucial role.

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