

Four-twist effects on excitations in symmetric orbifold CFTs

Bin Guo^{†1} and Shaun D. Hampton^{‡2}

[†] School of Physics, Central South University,
Changsha 418003, China

[‡] School of Physics,
Korea Institute for Advanced Study,
Seoul 02455, Korea

Abstract

Symmetric orbifold CFTs contain twist operators that can join and split copies of the CFT. In this paper, we study the effects of four twist-2 operators on two copies of a single free boson. A recent study analyzed their effects on the vacuum, finding a nontrivial left-right mixing that arises from the fact that the covering surface is a torus, while the effects of one or two twist-2 operators do not produce such mixing. Here, we extend this analysis to excited states and find a similar left-right mixing. Furthermore, we explore the continuum, or high-energy, limit and show that the left-right mixing becomes negligible in this limit.

¹guobin@csu.edu.cn

²sdh2023@kias.re.kr

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1 Introduction

Symmetric orbifold CFTs have played a significant role in understanding the $\text{AdS}_3/\text{CFT}_2$ correspondence [1–3], providing examples of CFTs with known holographic duals. The tensionless limit of string theory [4, 5] corresponds to the free point of the symmetric orbifold CFT [3, 6–8], while the supergravity limit corresponds to the strongly coupled regime. A marginal deformation moves the CFT from the free point toward the strongly coupled regime [9–11]. This deformation involves a twist-2 operator. Studying the effects of this twist-2 operator is essential for understanding the interpolation between the tensionless limit and the supergravity limit.

The symmetric orbifold CFT is constructed by taking multiple copies of a seed CFT and orbifolding by the permutation group. Twisted sectors occur when some copies are joined together, while in untwisted sectors, all copies remain unjoined. The twist-2 operator can join and split copies of the CFT. Its effects have been explored in various studies. For studies on a single twist-2 operator, see [12–18], and for studies on two twist-2 operators, see [19–21].

One significant effect of the twist-2 operator is the creation of pairs of excitations from the vacuum. In studies with one or two twist operators, pair creation was found to occur independently in either the left-moving or right-moving sector. This means that only pairs consisting of either two left-moving modes or two right-moving modes can be created. Additionally, left-moving pair creation is a holomorphic function, while right-moving pair creation is an antiholomorphic function. However, recent research has shown that pair creation of four twist-2 operators involves mixing between the left- and right-moving sectors [22]. As a result, pairs can be created with one left-moving mode and one right-moving mode. Moreover, in this case, left-moving pair creation is no longer a holomorphic function, and similarly, right-moving pair creation is no longer an antiholomorphic function.

As shown in [22], this left-right mixing comes from a torus covering map. On the torus, the left- and right-moving sectors of free fields can couple through the nontrivial periodicity. This coupling on the covering space is then transferred to the left-right mixing when four twist operators are present. In the original base space, these nontrivial periodicities appear as monodromy conditions.

In this paper, we extend the results of [22], which studied the effect of four twist-2 operators on the vacuum, to explore their effect on excited states. We follow the setup in [22], where four twist-2 operators act on the untwisted sector of two copies of a free boson and ultimately return to the untwisted sector. Beyond pair creation, the presence of excited initial modes gives rise to two additional effects: Propagation, which describes how an initial mode transforms into a final mode, and Contraction, which accounts for the annihilation of two initial modes. We will show that these effects also exhibit left-right mixing, similar to pair creation. We also examine the continuum limit, where the energy of the excitation mode becomes very high. In this limit, the left-right mixing falls off rapidly, making it negligible at high energies. This behavior is useful for understanding the effects of multiple twist operators in the continuum limit.

The paper is organized as follows: In Section 2, we introduce the symmetric orbifold CFT, discussing the effect of twist operators and reviewing the relevant correlator required for our computations. Sections 3 and 4 provide a detailed derivation of the expressions for propagation and contraction, respectively. In Section 5, we analyze the properties of these effects numerically. Section 6 examines the continuum limit and large separation limits. Finally, in Section 7 we collect our results, and in Section 8 we discuss and conclude.

2 Effect of four twist operators

2.1 Symmetric product orbifold CFT

The symmetric product orbifold CFT is described by the target space

$$M^N/S_N \quad (2.1)$$

where M denotes the target space of the seed CFT, and S_N is the symmetric group acting on N elements. In this paper, we focus on the case where $N = 2$ and $M = \mathbb{R}$, corresponding to two copies of a free boson. These two copies are denoted as $X^{(1)}$ and $X^{(2)}$.

The base space is a cylinder parameterized by a complex coordinate w

$$w = \tau + i\sigma, \quad -\infty < \tau < \infty, \quad 0 \leq \sigma < 2\pi \quad (2.2)$$

We also use the complex plane coordinate z , related to the cylinder coordinate by $z = e^w$.

The action of S_2 permutes the two copies of the field, giving rise to both untwisted and twisted sectors. In the untwisted sector, the fields $X^{(1)}$ and $X^{(2)}$ are periodic

$$X^{(i)}(\tau, \sigma + 2\pi) = X^{(i)}(\tau), \quad i = 1, 2 \quad (2.3)$$

For each copy, we define left and right moving modes at constant τ as follows

$$\begin{aligned} \alpha_n^{(i)} &= \int_{\sigma=0}^{2\pi} \frac{dw}{2\pi} e^{nw} \partial_w X^{(i)} = \oint \frac{dz}{2\pi} z^{nw} \partial_z X^{(i)}, \\ \bar{\alpha}_n^{(i)} &= \int_{\sigma=0}^{2\pi} \frac{d\bar{w}}{2\pi} e^{n\bar{w}} \partial_{\bar{w}} X^{(i)} = \oint \frac{d\bar{z}}{2\pi} \bar{z}^{nw} \partial_{\bar{z}} X^{(i)} \end{aligned} \quad (2.4)$$

where n is an integer. The vacuum state $|0\rangle^{(i)}$ for each copy i is defined by

$$\alpha_n^{(i)} |0\rangle^{(i)} = 0, \quad \bar{\alpha}_n^{(i)} |0\rangle^{(i)} = 0, \quad n \geq 0 \quad (2.5)$$

The commutation relations are given by

$$[\alpha_m^{(i)}, \alpha_n^{(j)}] = m\delta^{ij}\delta_{m+n,0}, \quad [\bar{\alpha}_m^{(i)}, \bar{\alpha}_n^{(j)}] = m\delta^{ij}\delta_{m+n,0}, \quad [\alpha_m^{(i)}, \bar{\alpha}_n^{(j)}] = 0 \quad (2.6)$$

These relations can be derived from the OPEs

$$\begin{aligned}\partial X^{(i)}(z)\partial X^{(j)}(z') &\sim -\frac{\delta^{ij}}{(z-z')^2} + \text{regular terms} \\ \bar{\partial}X^{(i)}(\bar{z})\bar{\partial}X^{(j)}(\bar{z}') &\sim -\frac{\delta^{ij}}{(\bar{z}-\bar{z}')^2} + \text{regular terms}\end{aligned}\quad (2.7)$$

There is also a twisted sector characterized by the boundary conditions

$$X^{(1)}(\tau, \sigma + 2\pi) = X^{(2)}(\tau), \quad X^{(2)}(\tau, \sigma + 2\pi) = X^{(1)}(\tau) \quad (2.8)$$

To describe these two fields, it is convenient to use a single field defined on an extended base space with $0 \leq \sigma < 4\pi$ and periodicity 4π . Since these fields and their mode expansions are not directly used in this paper, we refer the reader to [12, 23, 24] for detailed discussions.

In the z -plane, the ground state of this twisted sector corresponds to a local operator called the twist operator, denoted as σ_2 . This operator has conformal dimensions $h = \bar{h} = 1/16$. As we focus specifically on this twist-2 operator, we omit the subscript 2 for simplicity.

For studies on general twist operators and their correlations, we refer the reader to [25–30]. In the D1D5 CFT, a marginal deformation operator can be constructed using the twist-2 operator. This marginal operator moves the D1D5 CFT from the free point towards the gravity regime and is essential for understanding anomalous dimensions [11, 31–38], thermalization [39, 40], various aspects of black hole microstates [41–43] and scrambling [44], etc.

2.2 Monodromy

Correlation functions involving twist operators exhibit effects arising from monodromy, a phenomenon that occurs when a field is transported around twist operators in a closed loop. These effects were initially developed for \mathbb{Z}_N orbifolds in [45]. Here we provide a brief review of monodromy within the context of our setup.

Let us consider the antisymmetric combination of the two field copies, defined away from the twist operator's location as

$$X \equiv \frac{1}{\sqrt{2}}(X^{(1)} - X^{(2)}) \quad (2.9)$$

This field X is antisymmetric under the interchange of the two copies. Now, consider the z -plane with insertions of the twist operator σ . When X traces a loop around any twist operator, the boundary conditions (2.8) cause $X^{(1)}$ and $X^{(2)}$ to exchange. Consequently, X obtains a minus sign upon a 2π rotation around a twist operator. Therefore, X is periodic only under a 4π rotation around the twist operator.

The twist operators create branch cuts of order 2 on the z -plane, with $X^{(1)}$ and $X^{(2)}$ representing fields defined on the two distinct Riemann sheets. The field X becomes

periodic when encircling a closed loop on the Riemann surface. More generally, X satisfies a periodicity condition along C on the Riemann surface associated with the branched z -plane [45]

$$0 = \Delta_C X = \oint_C dz \partial X + \oint_C d\bar{z} \bar{\partial} X \quad (2.10)$$

These conditions impose nontrivial constraints that can couple the left and right moving sectors. Such constraints arise from loops on the covering surface that cannot be contracted to a point. In the case of two twist operators the covering surface is a sphere which has no nontrivial loops. Therefore, there are no nontrivial monodromy constraints in this case. The first instance of nontrivial monodromy effects occur with four twist operators, where the covering surface becomes a torus. In this paper, we focus on this scenario, particularly the coupling between left and right movers, which was recently first explored in [22].

2.3 The effect of four twist operators

In this section, we describe various effects produced by four twist operators. In this paper, we focus on the simplest case where only two copies of the seed CFT are present, such that $N = 2$. The ground state in the untwisted sector is given by

$$|0\rangle \equiv |0\rangle^{(1)} |0\rangle^{(2)} |\bar{0}\rangle^{(1)} |\bar{0}\rangle^{(2)} \quad (2.11)$$

where $|0\rangle^{(i)} |\bar{0}\rangle^{(i)}$ denotes the vacuum state for each copy i . Consider an initial state in the untwisted sector containing some left and right moving excitations

$$\alpha_{-n_1} \alpha_{-n_2} \dots \bar{\alpha}_{-m_1} \bar{\alpha}_{-m_2} \dots |0\rangle \quad (2.12)$$

where $n_k, m_k > 0$. The excitations here are antisymmetric combinations of modes between the two copies

$$\alpha_p \equiv \frac{1}{\sqrt{2}} (\alpha_p^{(1)} - \alpha_p^{(2)}) \quad \bar{\alpha}_p \equiv \frac{1}{\sqrt{2}} (\bar{\alpha}_p^{(1)} - \bar{\alpha}_p^{(2)}) \quad (2.13)$$

The symmetric combinations of the modes remain invariant under the orbifolding procedure and thus are not affected by the action of the twist operators: they simply pass through the twist operators [12, 18].

Let us consider the effect of applying four twist operators on this initial state. The first twist operator joins the two untwisted copies into a twist-2 copy, while the second twist operator splits it back into two untwisted copies; similarly, the third and fourth twist operators repeat this process. After applying all four twist operators, we return to a state in the untwisted sector. We define this final state as ϕ

$$|\phi\rangle = \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-n_1}^{(i_1)} \alpha_{-n_2}^{(i_2)} \dots \bar{\alpha}_{-m_1}^{(j_1)} \bar{\alpha}_{-m_2}^{(j_2)} \dots |0\rangle \quad (2.14)$$

To determine the state ϕ , we use the covering map. On the covering surface, the fields are free and follow Wick contraction rules, leading to three basic rules for computing the state ϕ :

(i) Contraction: Two modes in the initial state can ‘contract’, producing a constant. Four such contraction terms arise from different combinations of left and right moving modes, defined as follows

$$C^{\alpha\alpha}[m, n] \quad C^{\alpha\bar{\alpha}}[m, n] \quad C^{\bar{\alpha}\bar{\alpha}}[m, n] \quad (2.15)$$

For instance, a contraction between two left moving modes $\alpha_{-m}\alpha_{-n}$ produces $C^{\alpha\alpha}[m, n]$, while a contraction between a left moving mode and a right moving mode $\alpha_{-m}\bar{\alpha}_{-n}$ yields $C^{\alpha\bar{\alpha}}[m, n]$. In the contraction process, all possible pairs of modes are considered. Each mode either contracts with another mode, producing the constants above, or passes through the twist, as described in step (ii) below.

(ii) Propagation: Any modes that remain after the contraction pass through the twist operator and become new modes under the action of the twist operator, as follows

$$\begin{aligned} \alpha_{-n} &\rightarrow \sum_{p>0} f_{n,p}^{\alpha\alpha} \alpha_{-p} + \sum_{p>0} f_{n,p}^{\alpha\bar{\alpha}} \bar{\alpha}_{-p} \\ \bar{\alpha}_{-n} &\rightarrow \sum_{p>0} f_{n,p}^{\bar{\alpha}\alpha} \alpha_{-p} + \sum_{p>0} f_{n,p}^{\bar{\alpha}\bar{\alpha}} \bar{\alpha}_{-p} \end{aligned} \quad (2.16)$$

(iii) Pair creation: After the previous two steps, the modes in the initial state have either been contracted or propagated through the twist. What remains is the twist operator acting on the untwisted vacuum, which gives the state

$$\begin{aligned} |\chi\rangle &\equiv \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) |0\rangle \\ &= \mathcal{C} \exp \left(\sum_{m,n>0} \gamma_{m,n}(w_i, \bar{w}_i) \alpha_{-m} \alpha_{-n} + \sum_{m,n>0} \beta_{m,n}(w_i, \bar{w}_i) \alpha_{-m} \bar{\alpha}_{-n} \right. \\ &\quad \left. + \sum_{m,n>0} \bar{\gamma}_{m,n}(w_i, \bar{w}_i) \bar{\alpha}_{-m} \bar{\alpha}_{-n} \right) |0\rangle \end{aligned} \quad (2.17)$$

where the coefficient \mathcal{C} is

$$\mathcal{C} \equiv \langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) |0\rangle \quad (2.18)$$

which has been computed in previous works [25, 45]. To better understand these rules, consider an example with a single initial mode

$$\prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-n} |0\rangle = \left(\sum_{p>0} f_{n,p}^{\alpha\alpha} \alpha_{-p} + \sum_{p>0} f_{n,p}^{\alpha\bar{\alpha}} \bar{\alpha}_{-p} \right) |\chi\rangle \quad (2.19)$$

We first apply the propagation rule (2.23) to propagate the initial mode through the twist. Then the twist operator acts on the untwisted vacuum to produce pairs according to the pair creation rule (2.17). Now consider an example with two initial left moving modes

$$\begin{aligned} & \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-m} \alpha_{-n} |0\rangle \\ &= \left(\left(\sum_{p>0} f_{m,p}^{\alpha\alpha} \alpha_{-p} + \sum_{p>0} f_{m,p}^{\alpha\bar{\alpha}} \bar{\alpha}_{-p} \right) \left(\sum_{q>0} f_{n,q}^{\alpha\alpha} \alpha_{-q} + \sum_{q>0} f_{n,q}^{\alpha\bar{\alpha}} \bar{\alpha}_{-q} \right) + C^{\alpha\alpha}[m,n] \right) |\chi\rangle \quad (2.20) \end{aligned}$$

The first term in parentheses arises from the propagation of the two initial modes, while the second term comes from the contraction. The state χ , which captures pair creation, is given by (2.17). Similarly, for an initial state consisting of one left moving mode and one right moving mode

$$\begin{aligned} & \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-m} \bar{\alpha}_{-n} |0\rangle \\ &= \left(\left(\sum_{p>0} f_{m,p}^{\alpha\alpha} \alpha_{-p} + \sum_{p>0} f_{m,p}^{\alpha\bar{\alpha}} \bar{\alpha}_{-p} \right) \left(\sum_{q>0} f_{n,q}^{\bar{\alpha}\alpha} \alpha_{-q} + \sum_{q>0} f_{n,q}^{\bar{\alpha}\bar{\alpha}} \bar{\alpha}_{-q} \right) + C^{\alpha\bar{\alpha}}[m,n] \right) |\chi\rangle \quad (2.21) \end{aligned}$$

In the appendix of [22], it was shown that the above rule arises from Wick contraction on the covering space. We also note that the coefficients γ and β , which characterize pair creation processes, were computed explicitly in [22]. These results are summarized in section 7, where we have collected all of our results together. Furthermore as we will show, the coefficients of the contractions can be seen morally as the conjugates of the pair creation coefficients. In this work, we will compute both the propagation and the contraction, thereby completing the full set of effects produced by four twist operators.

2.4 Propagation

Using (2.19), which contains a single left moving mode in the initial state, α_{-n} , we can apply a left moving mode in the final state, α_p , which yields

$$f_{n,p}^{\alpha\alpha} = \frac{1}{p} \frac{\langle 0 | \alpha_p \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) | 0 \rangle} = \frac{1}{p} \frac{\langle 0 | \alpha_p \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle} \quad (2.22)$$

or a right moving mode $\bar{\alpha}_p$, which yields

$$f_{n,p}^{\alpha\bar{\alpha}} = \frac{1}{p} \frac{\langle 0 | \bar{\alpha}_p \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) | 0 \rangle} = \frac{1}{p} \frac{\langle 0 | \bar{\alpha}_p \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle} \quad (2.23)$$

where we have mapped the amplitudes to the z -plane using $z = e^w$. Note that the Jacobian factors arising from this mapping of the twist operators cancel out between the numerator

and denominator. The mode before the twist, α_{-n} , is defined by the contour at $\tau < \tau_i$, corresponding to $|z| < |z_i|$, while the modes after the twist, α_p and $\bar{\alpha}_p$, are defined by the contour at $\tau > \tau_i$, corresponding to $|z| > |z_i|$. By inserting the modes (2.4), (2.13) mapped to the appropriate z -plane locations, into the above correlation functions (2.22) and (2.23), we obtain

$$f_{n,p}^{\alpha\alpha} = -\frac{1}{p} \oint_{|z|>|z_i|} \frac{dz}{2\pi} z^p \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} g(z, z'; z_i, \bar{z}_i) \quad (2.24a)$$

$$f_{n,p}^{\alpha\bar{\alpha}} = -\frac{1}{p} \oint_{|\bar{z}|>|\bar{z}_i|} \frac{d\bar{z}}{2\pi} \bar{z}^p \int_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} b(\bar{z}, z'; z_i, \bar{z}_i) \quad (2.24b)$$

where the correlation functions g and b are defined as

$$g(z, z'; z_i, \bar{z}_i) \equiv -\frac{\langle 0 | \partial_z X \partial_{z'} X \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle} \quad (2.25a)$$

$$b(\bar{z}, z'; z_i, \bar{z}_i) \equiv -\frac{\langle 0 | \partial_{\bar{z}} X \partial_{z'} X \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle} \quad (2.25b)$$

We note that b measures correlations between left and right moving fields, an effect that doesn't arise in the two twist case. The expressions for $f_{n,p}^{\bar{\alpha}\alpha}$ and $f_{n,p}^{\bar{\alpha}\bar{\alpha}}$ can be obtained by simply switching the barred with unbarred coordinates in the expressions above.

2.5 Contraction

To derive the analogous expressions for the contractions, we can act on the states (2.20) and (2.21) with just the vacuum $\langle 0 |$ which gives the following expressions

$$\begin{aligned} C^{\alpha\alpha}[m, n] &= \frac{\langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-m} \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) | 0 \rangle} = \frac{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) \alpha_{-m} \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle} \\ C^{\alpha\bar{\alpha}}[m, n] &= \frac{\langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) \alpha_{-m} \bar{\alpha}_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(w_i, \bar{w}_i) | 0 \rangle} = \frac{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) \alpha_{-m} \bar{\alpha}_{-n} | 0 \rangle}{\langle 0 | \prod_{i=1}^4 \sigma(z_i, \bar{z}_i) | 0 \rangle} \end{aligned} \quad (2.26)$$

where we have mapped them to the z -plane using the transformation $z = e^w$. The modes are placed before the twist operators, defined by a contour at $\tau < \tau_i$, which corresponds to $|z| < |z_i|$. Again substituting the mode definitions from (2.4) and (2.13) at the appropriate z -plane locations into the above expressions yields the contractions

$$\begin{aligned} C^{\alpha\alpha}[m, n] &= - \oint_{|z|<|z_i|} \frac{dz}{2\pi} z^{-m} \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} g(z, z'; z_i, \bar{z}_i) \\ C^{\alpha\bar{\alpha}}[m, n] &= - \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-m} \oint_{|\bar{z}|<|\bar{z}_i|} \frac{d\bar{z}}{2\pi} \bar{z}^{-n} b(\bar{z}, z'; z_i, \bar{z}_i) \end{aligned} \quad (2.27)$$

where $C^{\bar{\alpha}\bar{\alpha}}[m, n]$ is simply the barred version of $C^{\alpha\alpha}[m, n]$ where one can simply replace the barred quantities with the unbarred quantities.

2.6 Review of the correlators

The effects discussed above are governed by two correlation functions: $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$. The general expressions for these correlators at arbitrary twist locations z_i, \bar{z}_i are provided in [22], building upon the specific cases discussed in [45]. In this section, we summarize the results and outline the method used to derive them, while referring the reader to [22] for further technical details.

The correlators g and b are given by

$$\begin{aligned}
& g(z, z'; z_i, \bar{z}_i) \\
&= \frac{1}{2} \frac{(z - z_1)^{\frac{1}{2}}(z - z_3)^{\frac{1}{2}}(z' - z_2)^{\frac{1}{2}}(z' - z_4)^{\frac{1}{2}}}{(z - z_2)^{\frac{1}{2}}(z - z_4)^{\frac{1}{2}}(z' - z_1)^{\frac{1}{2}}(z' - z_3)^{\frac{1}{2}}} \frac{1}{(z - z')^2} \\
&\quad + \frac{1}{2} \frac{(z - z_2)^{\frac{1}{2}}(z - z_4)^{\frac{1}{2}}(z' - z_1)^{\frac{1}{2}}(z' - z_3)^{\frac{1}{2}}}{(z - z_1)^{\frac{1}{2}}(z - z_3)^{\frac{1}{2}}(z' - z_2)^{\frac{1}{2}}(z' - z_4)^{\frac{1}{2}}} \frac{1}{(z - z')^2} \\
&\quad + \frac{A(z_i, \bar{z}_i)}{(z - z_1)^{\frac{1}{2}}(z - z_2)^{\frac{1}{2}}(z - z_3)^{\frac{1}{2}}(z - z_4)^{\frac{1}{2}}(z' - z_1)^{\frac{1}{2}}(z' - z_2)^{\frac{1}{2}}(z' - z_3)^{\frac{1}{2}}(z' - z_4)^{\frac{1}{2}}} \\
& b(\bar{z}, z'; z_i, \bar{z}_i) \\
&= \frac{B(z_i, \bar{z}_i)}{(\bar{z} - \bar{z}_1)^{\frac{1}{2}}(\bar{z} - \bar{z}_2)^{\frac{1}{2}}(\bar{z} - \bar{z}_3)^{\frac{1}{2}}(\bar{z} - \bar{z}_4)^{\frac{1}{2}}(z' - z_1)^{\frac{1}{2}}(z' - z_2)^{\frac{1}{2}}(z' - z_3)^{\frac{1}{2}}(z' - z_4)^{\frac{1}{2}}} \tag{2.28}
\end{aligned}$$

The functions $A(z_i, \bar{z}_i)$ and $B(z_i, \bar{z}_i)$ are defined as

$$\begin{aligned}
A(z_i, \bar{z}_i) &\equiv -A(x, \bar{x})(z_1 - z_3)(z_2 - z_4) \\
B(z_i, \bar{z}_i) &\equiv B(x, \bar{x})|z_1 - z_3||z_2 - z_4| \tag{2.29}
\end{aligned}$$

where the cross ratio x is given by

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{2.30}$$

The functions $A(x, \bar{x})$ and $B(x, \bar{x})$ are expressed as

$$\begin{aligned}
A(x, \bar{x}) &= x(1 - x) \frac{d}{dx} \ln(F(x)\bar{F}(1 - \bar{x}) + F(1 - x)\bar{F}(\bar{x})) \\
B(x, \bar{x}) &= \frac{1}{\pi(F(x)\bar{F}(1 - \bar{x}) + F(1 - x)\bar{F}(\bar{x}))} \tag{2.31}
\end{aligned}$$

where $F(x)$ is the hypergeometric function

$$F(x) \equiv {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{2.32}$$

The functions $A(x, \bar{x})$ and $B(x, \bar{x})$ can also be written as

$$\begin{aligned} A(x, \bar{x}) &= \frac{(E(x) - (1-x)K(x))\bar{K}(1-\bar{x}) - (E(1-x) - xK(1-x))\bar{K}(\bar{x})}{2(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \\ B(x, \bar{x}) &= \frac{\pi}{4(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \end{aligned} \quad (2.33)$$

where $K(y)$ and $E(y)$ denote the complete elliptic integrals of the first and second kinds. This form will be used for explicit computations in later sections.

In the following, we outline the steps to compute the correlation functions $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$.

1. Map to the covering space. First consider the twist operator locations at the special positions $z_1 = 0, z_2 = x, z_3 = 1, z_4 = \infty$. In this setup, the covering map can be found explicitly and is given by

$$z = \frac{\mathcal{P}(t) - e_1}{e_2 - e_1} \quad (2.34)$$

where \mathcal{P} is the Weierstrass \mathcal{P} function, and e_i 's are its values at the half-periods of the torus. This maps the base space, which is a sphere, into a torus. The torus modulus parameter τ is then related to the cross ratio x .

2. Two point function on the torus. Mapping to the covering space resolves the branch points associated with the twist operators. As a result, the correlation functions $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$, which involve two free fields and four twist operators, transform into two-point functions of free fields on the torus. This two point function is well-known in the literature.

To reproduce the double pole when the two points are close on the torus, the two-point functions are uniquely determined by the Weierstrass \mathcal{P} function up to two additional constants. These constants are fixed by imposing the condition that the fields are periodic on the torus along both periods $(1, \tau)$. On the base space, these periodicity conditions correspond to the monodromy conditions (2.10).

3, Correlation function on the base space. Thanks to the form of the covering map, the two point function, which depends on the torus coordinate t and the torus modulus parameter τ , can be expressed in terms of the original base space coordinate z and the cross ratio x .

4. Correlation function at arbitrary twist locations. Once the correlation functions at specific twist locations are obtained, an $SL(2, \mathbb{C})$ transformation can be applied to move the twist operators to arbitrary locations z_i . This yields the correlation functions $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$.

2.7 Properties of the correlators

There are two important properties of the correlation functions $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$ that will be useful. For further technical details, see [22].

Twist interchange symmetries Since the four twist operators are identical, the correlation functions $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$ are invariant under the interchange of twist locations, i.e., $z_i \leftrightarrow z_j$. For an explicit verification of this symmetry, see section 3.5 in [22]. Consequently, the contraction, propagation and pair creation coefficients are completely symmetric among the four twist operator locations.

Reduction to two twist operators The correlation functions $g(z, z'; z_i, \bar{z}_i)$ and $b(\bar{z}, z'; z_i, \bar{z}_i)$, which involve four twist operators, can be reduced to a correlation function with two twist operators when any two of the four twist operators are brought together. For example, when $z_3 \rightarrow z_4$, the term $\prod_{i=1}^4 \sigma(z_i, \bar{z}_i)$ in (2.25a) and (2.25b) is replaced by $\sigma(z_2, \bar{z}_2)\sigma(z_1, \bar{z}_1)$. For an explicit verification of this reduction, see section 3.6 in [22]. As a result, the contraction, propagation and pair creation coefficients for four twist operators reduce to their corresponding values for two twist operators.

3 Propagation for four twists

In this section, we derive the propagation coefficients for four twist operators, using the correlators reviewed in the previous section.

3.1 Left moving propagation

We begin by recalling the left moving propagation coefficients given in (2.24a)

$$f_{n,p}^{\alpha\alpha} = -\frac{1}{p} \oint_{|z|>|z_i|} \frac{dz}{2\pi} z^p \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} g(z, z'; z_i, \bar{z}_i) \quad (3.1)$$

Inserting the expression for the amplitude g from (2.28) and separating it into three terms, we obtain

$$f_{n,p}^{\alpha\alpha} \equiv f_I^{\alpha\alpha} + f_{II}^{\alpha\alpha} + f_{III}^{\alpha\alpha} \quad (3.2)$$

where the three terms are defined as follows

$$f_I^{\alpha\alpha} = -\frac{1}{2p} \oint_{|z|>|z_i|} \frac{dz}{2\pi} z^p \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} \frac{(z-z_1)^{\frac{1}{2}}(z-z_3)^{\frac{1}{2}}(z'-z_2)^{\frac{1}{2}}(z'-z_4)^{\frac{1}{2}}}{(z-z_2)^{\frac{1}{2}}(z-z_4)^{\frac{1}{2}}(z'-z_1)^{\frac{1}{2}}(z'-z_3)^{\frac{1}{2}}} \frac{1}{(z-z')^2}$$

$$\begin{aligned}
f_{\text{II}}^{\alpha\alpha} &= -\frac{1}{2p} \oint_{|z|>|z_i|} \frac{dz}{2\pi} z^p \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} \frac{(z-z_2)^{\frac{1}{2}}(z-z_4)^{\frac{1}{2}}(z'-z_1)^{\frac{1}{2}}(z'-z_3)^{\frac{1}{2}}}{(z-z_1)^{\frac{1}{2}}(z-z_3)^{\frac{1}{2}}(z'-z_2)^{\frac{1}{2}}(z'-z_4)^{\frac{1}{2}}} \frac{1}{(z-z')^2} \\
f_{\text{III}}^{\alpha\alpha} &= \frac{1}{p} A(x, \bar{x})(z_1 - z_3)(z_2 - z_4) \oint_{|z|>|z_i|} \frac{dz}{2\pi} z^p \oint_{|z'|<|z_i|} \frac{dz'}{2\pi} z'^{-n} \\
&\quad \times \frac{1}{(z-z_1)^{\frac{1}{2}}(z-z_2)^{\frac{1}{2}}(z-z_3)^{\frac{1}{2}}(z-z_4)^{\frac{1}{2}}(z'-z_1)^{\frac{1}{2}}(z'-z_2)^{\frac{1}{2}}(z'-z_3)^{\frac{1}{2}}(z'-z_4)^{\frac{1}{2}}} \tag{3.3}
\end{aligned}$$

Note that the first and second terms are related by a simple interchange of the twist locations

$$f_{\text{II}}^{\alpha\alpha} = f_{\text{I}}^{\alpha\alpha}(z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4) \tag{3.4}$$

In the forthcoming computations, we evaluate the contours for $|z| > |z_i|$ at $z = \infty$ and for $|z| < |z_i|$ at $z = 0$, as these contours can be freely deformed to these respective locations. For $z \rightarrow \infty$, where $|z| \gg |z_i|$, we expand the integrand factors as

$$(z - z_i)^{\pm\frac{1}{2}} = z^{\pm\frac{1}{2}}(1 - z_i z^{-1})^{\pm\frac{1}{2}} = \sum_{k \geq 0}^{\pm\frac{1}{2}} C_k (-1)^k z_i^k z^{-k \pm \frac{1}{2}} \tag{3.5}$$

where ${}^p C_q$ is the binomial coefficient

$${}^p C_q = \frac{p!}{q!(p-q)!} \tag{3.6}$$

Similarly, for $z \rightarrow 0$, where $|z| \ll |z_i|$, we expand the integrand as

$$(z - z_i)^{\pm\frac{1}{2}} = (-z_i)^{\pm\frac{1}{2}}(1 - z_i^{-1} z)^{\pm\frac{1}{2}} = (-1)^{\pm\frac{1}{2}} \sum_{k \geq 0}^{\pm\frac{1}{2}} C_k (-1)^k z_i^{-k \pm \frac{1}{2}} z^k \tag{3.7}$$

Utilizing these expansions, next we compute each term in the propagation.

Term I and II To compute the first term in the propagation $f_{\text{I}}^{\alpha\alpha}$, we substitute the expansions (3.5) and (3.7) into (3.3)

$$\begin{aligned}
f_{\text{I}}^{\alpha\alpha} &= -\frac{1}{2p} \left(\frac{z_2 z_4}{z_1 z_3} \right)^{\frac{1}{2}} \sum_{k_i, k'_i \geq 0}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} \\
&\quad (-1)^{\sum k_i + \sum k'_i} z_1^{k_1 - k'_1} z_2^{k_2 - k'_2} z_3^{k_3 - k'_3} z_4^{k_4 - k'_4} \oint_{z=\infty} \frac{dz}{2\pi} z^{p - \sum k_i} \oint_{z'=0} \frac{dz'}{2\pi} z'^{\sum k'_i - n} \frac{1}{(z-z')^2} \tag{3.8}
\end{aligned}$$

Focusing on the integrations in the last line, we first perform the contour integral around the point $z' = 0$, which yields

$$(n - \sum k'_i) i \oint_{z=\infty} \frac{dz}{2\pi} z^{p-n-\sum k_i + \sum k'_i - 1} \Big|_{n-\sum k'_i > 0} \quad (3.9)$$

To compute the contour integral around $z = \infty$, we make the coordinate transformation $z = \frac{1}{u}$, which maps the pole at $z = \infty$ to a pole at $u = 0$. This yields the expression

$$(n - \sum k'_i) i \oint_{u=0} \frac{du}{2\pi} u^{-p+n+\sum k_i - \sum k'_i - 1} \Big|_{n-\sum k'_i > 0} = -(n - \sum k'_i) \delta_{p-n-\sum k_i + \sum k'_i, 0} \Big|_{n-\sum k'_i > 0} \quad (3.10)$$

Substituting this result back, we find

$$f_I^{\alpha\alpha} = \frac{1}{2p} \left(\frac{z_2 z_4}{z_1 z_3} \right)^{\frac{1}{2}} \sum_{k_i, k'_i \in D}^{} \frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} (-1)^{\sum k_i + \sum k'_i} z_1^{k_1-k'_1} z_2^{k_2-k'_2} z_3^{k_3-k'_3} z_4^{k_4-k'_4} (n - \sum k'_i) \quad (3.11)$$

where the sum is taken over the region satisfying

$$D = \{k_i, k'_i \geq 0, n - \sum k'_i > 0, p - n - \sum k_i + \sum k'_i = 0\} \quad (3.12)$$

From the third condition above, we choose

$$k_4 = p - n - (k_1 + k_2 + k_3) + k'_1 + k'_2 + k'_3 + k'_4 \quad (3.13)$$

Imposing the first and second conditions, we then deduce the following constraints on the remaining indices

$$\begin{aligned} k_3 &\leq p - n - k_1 - k_2 + k'_1 + k'_2 + k'_3 + k'_4 \\ k_2 &\leq p - n - k_1 + k'_1 + k'_2 + k'_3 + k'_4 \\ k_1 &\leq p - n + k'_1 + k'_2 + k'_3 + k'_4 \\ k'_4 &\leq n - (k'_1 + k'_2 + k'_3 + 1) \\ k'_3 &\leq n - (k'_1 + k'_2 + 1) \\ k'_2 &\leq n - (k'_1 + 1) \\ k'_1 &\leq n - 1 \end{aligned} \quad (3.14)$$

with the additional constraint that

$$k'_4 \geq n - p - (k'_1 + k'_2 + k'_3) \quad (3.15)$$

which yields

$$\begin{aligned}
f_I^{\alpha\alpha} = & \frac{1}{2p} \left(\frac{z_2 z_4}{z_1 z_3} \right)^{\frac{1}{2}} (-1)^{p+n} \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \sum_{k'_4=\max[0, n-p-k'_1-k'_2-k'_3]}^{n-k'_1-k'_2-k'_3-1} \\
& \sum_{k_1=0}^{p-n+k'_1+k'_2+k'_3+k'_4} \sum_{k_2=0}^{p-n-k_1+k'_1+k'_2+k'_3+k'_4} \sum_{k_3=0}^{p-n-k_1-k_2+k'_1+k'_2+k'_3+k'_4} (n - k'_1 - k'_2 - k'_3 - k'_4) \\
& -\frac{1}{2} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3+k'_4} \\
& z_1^{k_1-k'_1} z_2^{k_2-k'_2} z_3^{k_3-k'_3} z_4^{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3} \quad (3.16)
\end{aligned}$$

For the second term of propagation, $f_{II}^{\alpha\alpha}$, we can simply make the switches $z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4$ as indicated in (3.4).

Term III Let's now compute the third term of propagation, $f_{III}^{\alpha\alpha}$. Inserting the expansions (3.5) and (3.7) into the third term in (3.3), we obtain

$$\begin{aligned}
f_{III}^{\alpha\alpha} = & \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{\sqrt{z_1 z_2 z_3 z_4}} \\
& \times \left(\frac{1}{p} \sum_{k_i \geq 0}^{-\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} (-1)^{\sum k_i} z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{k_4} \oint_{z=\infty} \frac{dz}{2\pi} z^{p-\sum k_i-2} \right) \\
& \times \left(\sum_{k'_i \geq 0}^{-\frac{1}{2}} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} (-1)^{\sum k'_i} z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-k'_4} \oint_{z'=0} \frac{dz'}{2\pi} z'^{\sum k'_i-n} \right) \quad (3.17)
\end{aligned}$$

where the contour integrals can be evaluated as follows

$$\begin{aligned}
\oint_{z=\infty} \frac{dz}{2\pi} z^{p-\sum k_i-2} &= \oint_{u=0} \frac{du}{2\pi} u^{-p+\sum k_i} = i\delta_{p-\sum k_i-1,0} \\
\oint_{z'=0} \frac{dz'}{2\pi} z'^{\sum k'_i-n} &= i\delta_{n-\sum k'_i-1,0} \quad (3.18)
\end{aligned}$$

This gives us

$$\begin{aligned}
f_{III}^{\alpha\alpha} = & -\frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{\sqrt{z_1 z_2 z_3 z_4}} \\
& \times \left(\frac{1}{p} \sum_{\substack{k_i \geq 0 \\ p-\sum k_i-1=0}}^{-\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} (-1)^{\sum k_i} z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{k_4} \right) \\
& \times \left(\sum_{\substack{k'_i \geq 0 \\ n-\sum k'_i-1=0}}^{-\frac{1}{2}} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} (-1)^{\sum k'_i} z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-k'_4} \right) \quad (3.19)
\end{aligned}$$

The summation region can be rewritten as

$$\begin{aligned}
p - k_1 - k_2 - k_3 - k_4 - 1 = 0 &\implies k_4 = p - k_1 - k_2 - k_3 - 1 \\
k_4 \geq 0 &\implies p - k_1 - k_2 - 1 \geq k_3 \\
k_3 \geq 0 &\implies p - k_1 - 1 \geq k_2 \\
k_2 \geq 0 &\implies p - 1 \geq k_1
\end{aligned}$$

$$\begin{aligned}
n - k'_1 - k'_2 - k'_3 - k'_4 - 1 = 0 &\implies k'_4 = n - k'_1 - k'_2 - k'_3 - 1 \\
k'_4 \geq 0 &\implies n - k'_1 - k'_2 - 1 \geq k'_3 \\
k'_3 \geq 0 &\implies n - k'_1 - 1 \geq k'_2 \\
k'_2 \geq 0 &\implies n - 1 \geq k'_1
\end{aligned} \tag{3.20}$$

which gives

$$\begin{aligned}
f_{\text{III}}^{\alpha\alpha} = & - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{\sqrt{z_1 z_2 z_3 z_4}} \\
& \times \left(\frac{(-1)^p}{p} \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-k_1-1} \sum_{k_3=0}^{p-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-k_1-k_2-k_3-1} \right. \\
& \quad \left. z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{p-k_1-k_2-k_3-1} \right) \\
& \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\
& \quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right)
\end{aligned} \tag{3.21}$$

Final expression for the left moving propagation Inserting the three terms (3.16), (3.4) and (3.21) into (3.2), we obtain

$$\begin{aligned}
f_{n,p}^{\alpha\alpha} = & \frac{1}{2p} (-1)^{p+n} \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \sum_{k'_4=\max[0, n-p-k'_1-k'_2-k'_3]}^{n-k'_1-k'_2-k'_3-1} \\
& \sum_{k_1=0}^{p-n+k'_1+k'_2+k'_3+k'_4} \sum_{k_2=0}^{p-n-k_1+k'_1+k'_2+k'_3+k'_4} \sum_{k_3=0}^{p-n-k_1-k_2+k'_1+k'_2+k'_3+k'_4} (n - k'_1 - k'_2 - k'_3 - k'_4) \\
& -\frac{1}{2} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3+k'_4} \\
& \times \left(\left(\frac{z_2 z_4}{z_1 z_3} \right)^{\frac{1}{2}} z_1^{k_1-k'_1} z_2^{k_2-k'_2} z_3^{k_3-k'_3} z_4^{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3+k'_4} \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{z_1 z_3}{z_2 z_4} \right)^{\frac{1}{2}} z_2^{k_1 - k'_1} z_1^{k_2 - k'_2} z_4^{k_3 - k'_3} z_3^{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3} \Big) \\
& - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{\sqrt{z_1 z_2 z_3 z_4}} \\
& \times \left(\frac{(-1)^p}{p} \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-k_1-1} \sum_{k_3=0}^{p-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-k_1-k_2-k_3-1} \right. \\
& \quad \left. z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{p-k_1-k_2-k_3-1} \right) \\
& \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\
& \quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \tag{3.22}
\end{aligned}$$

The right moving propagation follows a similar derivation and can be obtained by taking the complex conjugate of the above expression

$$f_{n,p}^{\bar{\alpha}\bar{\alpha}} = (f_{n,p}^{\alpha\alpha})^* \tag{3.23}$$

This means replacing barred quantities with unbarred ones and vice versa, $z_i \leftrightarrow \bar{z}_i$. The final result is summarized in section 7.

3.2 left-right moving propagation

In this section, we compute the propagation coefficients $f_{n,p}^{\alpha\bar{\alpha}}$, which describe a left moving excitation being converted into a right moving excitation. The computation closely parallels the derivation of $f_{\text{III}}^{\alpha\alpha}$ in the previous section. We start with (2.24b)

$$f_{n,p}^{\alpha\bar{\alpha}} = -\frac{1}{p} \oint_{\bar{z}=\infty} \frac{d\bar{z}}{2\pi} \bar{z}^p \oint_{z'=0} \frac{dz'}{2\pi} z'^{-n} b(\bar{z}, z'; z_i, \bar{z}_i) \tag{3.24}$$

Inserting the correlation function (2.28) yields

$$\begin{aligned}
f_{n,p}^{\alpha\bar{\alpha}} = & -\frac{B(z_i, \bar{z}_i)}{p} \left(\oint_{\bar{z}=\infty} \frac{d\bar{z}}{2\pi} \bar{z}^p \frac{1}{(\bar{z} - \bar{z}_1)^{\frac{1}{2}} (\bar{z} - \bar{z}_2)^{\frac{1}{2}} (\bar{z} - \bar{z}_3)^{\frac{1}{2}} (\bar{z} - \bar{z}_4)^{\frac{1}{2}}} \right. \\
& \times \left. \left(\oint_{z'=0} \frac{dz'}{2\pi} z'^{-n} \frac{1}{(z' - z_1)^{\frac{1}{2}} (z' - z_2)^{\frac{1}{2}} (z' - z_3)^{\frac{1}{2}} (z' - z_4)^{\frac{1}{2}}} \right) \right) \tag{3.25}
\end{aligned}$$

The contour integration and the summation index constraints follow in a similar manner as for $f_{\text{III}}^{\alpha\alpha}$, and we thus obtain

$$f_{n,p}^{\alpha\bar{\alpha}} = \frac{B(x, \bar{x}) |z_1 - z_3| |z_2 - z_4|}{\sqrt{z_1 z_2 z_3 z_4}}$$

$$\begin{aligned}
& \times \left(\frac{(-1)^p}{p} \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-k_1-1} \sum_{k_3=0}^{p-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-k_1-k_2-k_3-1} \right. \\
& \quad \left. \bar{z}_1^{k_1} \bar{z}_2^{k_2} \bar{z}_3^{k_3} \bar{z}_4^{p-k_1-k_2-k_3-1} \right) \\
& \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\
& \quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \tag{3.26}
\end{aligned}$$

The propagation $f_{n,p}^{\bar{\alpha}\alpha}$ follows a similar derivation and can be obtained by taking the complex conjugate of the above expression

$$f_{n,p}^{\bar{\alpha}\alpha} = (f_{n,p}^{\alpha\bar{\alpha}})^* \tag{3.27}$$

This means that we replace barred quantities with unbarred ones and vice versa, $z_i \leftrightarrow \bar{z}_i$. The final result is summarized in section 7.

4 Contraction for four twists

In this section, we derive the contraction coefficients for four twist operators, using the correlators (2.28).

4.1 Left moving contraction

We begin by recalling the left moving contraction coefficients as given in (2.27)

$$C^{\alpha\alpha}[m, n] = - \oint_{z=0, |z|>|z'|} \frac{dz}{2\pi} z^{-m} \oint_{z'=0} \frac{dz'}{2\pi} z'^{-n} g(z, z'; z_i, \bar{z}_i) \tag{4.1}$$

where the poles encircled by both contours are at 0, and we have chosen a particular ordering, $|z| > |z'|$, to perform the computation though the result does not depend on this specific choice. Inserting the expression for the correlation function g , recorded in (2.28), yields an expression with three terms

$$C^{\alpha\alpha}[m, n] = C_I^{\alpha\alpha} + C_{II}^{\alpha\alpha} + C_{III}^{\alpha\alpha} \tag{4.2}$$

which are defined as

$$C_I^{\alpha\alpha} \equiv -\frac{1}{2} \oint_{z=0, |z|>|z'|} \frac{dz}{2\pi} z^{-m} \oint_{z'=0} \frac{dz'}{2\pi} z'^{-n} \frac{(z-z_1)^{\frac{1}{2}}(z-z_3)^{\frac{1}{2}}(z'-z_2)^{\frac{1}{2}}(z'-z_4)^{\frac{1}{2}}}{(z-z_2)^{\frac{1}{2}}(z-z_4)^{\frac{1}{2}}(z'-z_1)^{\frac{1}{2}}(z'-z_3)^{\frac{1}{2}}} \frac{1}{(z-z')^2}$$

$$\begin{aligned}
C_{\text{II}}^{\alpha\alpha} &\equiv -\frac{1}{2} \oint_{z=0, |z|>|z'|} \frac{dz}{2\pi} z^{-m} \oint_{z'=0} \frac{dz'}{2\pi} z'^{-n} \frac{(z-z_2)^{\frac{1}{2}}(z-z_4)^{\frac{1}{2}}(z'-z_1)^{\frac{1}{2}}(z'-z_3)^{\frac{1}{2}}}{(z-z_1)^{\frac{1}{2}}(z-z_3)^{\frac{1}{2}}(z'-z_2)^{\frac{1}{2}}(z'-z_4)^{\frac{1}{2}}} \frac{1}{(z-z')^2} \\
C_{\text{III}}^{\alpha\alpha} &\equiv A(x, \bar{x})(z_1-z_3)(z_2-z_4) \left(\oint_{z=0} \frac{dz}{2\pi} z^{-m} \frac{1}{(z-z_1)^{\frac{1}{2}}(z-z_2)^{\frac{1}{2}}(z-z_3)^{\frac{1}{2}}(z-z_4)^{\frac{1}{2}}} \right) \\
&\times \left(\oint_{z'=0} \frac{dz'}{2\pi} z'^{-n} \frac{1}{(z'-z_1)^{\frac{1}{2}}(z'-z_2)^{\frac{1}{2}}(z'-z_3)^{\frac{1}{2}}(z'-z_4)^{\frac{1}{2}}} \right) \tag{4.3}
\end{aligned}$$

We observe a simple relation between terms I and II through pairwise interchanges of twist locations

$$C_{\text{II}}^{\alpha\alpha} = C_{\text{I}}^{\alpha\alpha}(z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4) \tag{4.4}$$

Term I and II Let's begin by computing $C_{\text{I}}^{\alpha\alpha}$. Inserting the expansions around $z=0$ (3.7) into the expression for $C_{\text{I}}^{\alpha\alpha}$ in (4.3), we obtain the following expression

$$\begin{aligned}
C_{\text{I}}^{\alpha\alpha} &= -\frac{1}{2} \sum_{k_i, k'_i \geq 0}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} \\
&\quad (-1)^{\sum k_i + \sum k'_i} z_1^{-(k_1+k'_1)} z_2^{-(k_2+k'_2)} z_3^{-(k_3+k'_3)} z_4^{-(k_4+k'_4)} \\
&\quad \oint_{z=0, |z|>|z'|} \frac{dz}{2\pi} z^{-m+\sum k_i} \oint_{z'=0} \frac{dz'}{2\pi} z'^{-n+\sum k'_i} \frac{1}{(z-z')^2} \tag{4.5}
\end{aligned}$$

Focusing on the last line, we first perform the contour integral over z' , and then over z

$$\begin{aligned}
&(n-k'_1-k'_2-k'_3-k'_4)i \oint_{z=0} \frac{dz}{2\pi} z^{-m-n+k_1+k_2+k_3+k_4+k'_1+k'_2+k'_3+k'_4-1} \Big|_{n-k'_1-k'_2-k'_3-k'_4>0} \\
&= -(n-k'_1-k'_2-k'_3-k'_4)\delta_{m+n-k_1-k_2-k_3-k_4-k'_1-k'_2-k'_3-k'_4,0} \Big|_{n-k'_1-k'_2-k'_3-k'_4>0} \tag{4.6}
\end{aligned}$$

This gives

$$\begin{aligned}
C_{\text{I}}^{\alpha\alpha} &= \frac{1}{2} \sum_{k_i, k'_i \in D}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} \\
&\quad (-1)^{\sum k_i + \sum k'_i} z_1^{-(k_1+k'_1)} z_2^{-(k_2+k'_2)} z_3^{-(k_3+k'_3)} z_4^{-(k_4+k'_4)} (n-k'_1-k'_2-k'_3-k'_4) \tag{4.7}
\end{aligned}$$

with the sum taken over the region

$$D = \{ k_i, k'_i \geq 0, m+n-\sum k_i - \sum k'_i = 0, n-\sum k'_i > 0 \} \tag{4.8}$$

Following a similar procedure as before, this region can be written as

$$k_3 \leq m+n-(k'_1+k'_2+k'_3+k'_4+k_1+k_2)$$

$$\begin{aligned}
k_2 &\leq m + n - (k'_1 + k'_2 + k'_3 + k'_4 + k_1) \\
k_1 &\leq m + n - (k'_1 + k'_2 + k'_3 + k'_4) \\
k'_4 &\leq n - (k'_1 + k'_2 + k'_3 + 1) \\
k'_3 &\leq n - (k'_1 + k'_2 + 1) \\
k'_2 &\leq n - (k'_1 + 1) \\
k'_1 &\leq n - 1
\end{aligned} \tag{4.9}$$

which gives

$$\begin{aligned}
C_I^{\alpha\alpha} = & \frac{1}{2}(-1)^{m+n} \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \sum_{k'_4=0}^{n-k'_1-k'_2-k'_3-1} \sum_{k_1=0}^{m+n-k'_1-k'_2-k'_3-k'_4} \\
& \sum_{k_2=0}^{m+n-k'_1-k'_2-k'_3-k'_4-k_1} \sum_{k_3=0}^{m+n-k'_1-k'_2-k'_3-k'_4-k_1-k_2} (n - k'_1 - k'_2 - k'_3 - k'_4) \\
& z_1^{-k_1-k'_1} z_2^{-k_2-k'_2} z_3^{-k_3-k'_3} z_4^{-m-n+k_1+k_2+k_3+k'_1+k'_2+k'_3}
\end{aligned} \tag{4.10}$$

For the second term in the contraction, $C_{II}^{\alpha\alpha}$, we can simply make the switches $z_1 \leftrightarrow z_2$ and $z_3 \leftrightarrow z_4$ in the above expression, as indicated in (4.4).

Term III Now, let's compute the third term in the contraction, $C_{III}^{\alpha\alpha}$. The derivation is similar to that of $f_{III}^{\alpha\alpha}$ and $f_{n,p}^{\alpha\bar{\alpha}}$, but with a different pole structure. Inserting (3.7) into the expression for $C_{III}^{\alpha\alpha}$ (4.3) yields

$$\begin{aligned}
C_{III}^{\alpha\alpha} = & \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{z_1 z_2 z_3 z_4} \\
& \times \left(\sum_{k_i \geq 0} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} (-1)^{\sum k_i} z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-k_4} \oint_{z=0} \frac{dz}{2\pi} z^{-m+\sum k_i} \right) \\
& \times \left(\sum_{k'_i \geq 0} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} (-1)^{\sum k'_i} z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-k'_4} \oint_{z'=0} \frac{dz'}{2\pi} z'^{-n+\sum k'_i} \right)
\end{aligned} \tag{4.11}$$

Performing the contour integrals of z and z' yields

$$\begin{aligned}
C_{III}^{\alpha\alpha} = & - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{z_1 z_2 z_3 z_4} \\
& \times \left(\sum_{\substack{k_i \geq 0 \\ m-\sum k_i-1=0}} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{k_4}^{-\frac{1}{2}} (-1)^{\sum k_i} z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-k_4} \right)
\end{aligned}$$

$$\times \left(\sum_{\substack{k'_i \geq 0 \\ n - \sum k'_i - 1 = 0}}^{-\frac{1}{2}} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{k'_4}^{-\frac{1}{2}} (-1)^{\sum k'_i} z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-k'_4} \right) \quad (4.12)$$

The region of the summation can be rewritten as follows

$$\begin{aligned} m - k_1 - k_2 - k_3 - k_4 - 1 &= 0 \implies k_4 = m - k_1 - k_2 - k_3 - 1 \\ k_4 &\geq 0 \implies m - k_1 - k_2 - 1 \geq k_3 \\ k_3 &\geq 0 \implies m - k_1 - 1 \geq k_2 \\ k_2 &\geq 0 \implies m - 1 \geq k_1 \end{aligned}$$

$$\begin{aligned} n - k'_1 - k'_2 - k'_3 - k'_4 - 1 &= 0 \implies k'_4 = n - k'_1 - k'_2 - k'_3 - 1 \\ k'_4 &\geq 0 \implies n - k'_1 - k'_2 - 1 \geq k'_3 \\ k'_3 &\geq 0 \implies n - k'_1 - 1 \geq k'_2 \\ k'_2 &\geq 0 \implies n - 1 \geq k'_1 \end{aligned} \quad (4.13)$$

which gives

$$\begin{aligned} C_{\text{III}}^{\alpha\alpha} &= - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{z_1 z_2 z_3 z_4} \\ &\times \left((-1)^m \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{m-k_1-1} \sum_{k_3=0}^{m-k_1-k_2-1}^{-\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{m-k_1-k_2-k_3-1} \right. \\ &\quad \left. z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-m+k_1+k_2+k_3+1} \right) \\ &\times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1}^{-\frac{1}{2}} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\ &\quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \end{aligned} \quad (4.14)$$

Final expression for the left moving contraction By inserting the expressions (4.10), (4.4) and (4.14) into (4.2), we obtain

$$\begin{aligned} C^{\alpha\alpha}[m, n] &= \frac{1}{2} (-1)^{m+n} \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \sum_{k'_4=0}^{n-k'_1-k'_2-k'_3-1} \sum_{k_1=0}^{m+n-k'_1-k'_2-k'_3-k'_4} \\ &\quad \sum_{k_2=0}^{m+n-k'_1-k'_2-k'_3-k'_4-k_1} \sum_{k_3=0}^{m+n-k'_1-k'_2-k'_3-k'_4-k_1-k_2} (n - k'_1 - k'_2 - k'_3 - k'_4) \\ &\quad -\frac{1}{2} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{m+n-k_1-k_2-k_3-k'_1-k'_2-k'_3-k'_4} \end{aligned}$$

$$\begin{aligned}
& \times \left(z_1^{-k_1-k'_1} z_2^{-k_2-k'_2} z_3^{-k_3-k'_3} z_4^{-m-n+k_1+k_2+k_3+k'_1+k'_2+k'_3} \right. \\
& \quad \left. + z_2^{-k_1-k'_1} z_1^{-k_2-k'_2} z_4^{-k_3-k'_3} z_3^{-m-n+k_1+k_2+k_3+k'_1+k'_2+k'_3} \right) \\
& - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{z_1 z_2 z_3 z_4} \\
& \times \left((-1)^m \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{m-k_1-1} \sum_{k_3=0}^{m-k_1-k_2-1} -\frac{1}{2} C_{k_1} -\frac{1}{2} C_{k_2} -\frac{1}{2} C_{k_3} -\frac{1}{2} C_{m-k_1-k_2-k_3-1} \right. \\
& \quad \left. z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-m+k_1+k_2+k_3+1} \right) \\
& \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1} -\frac{1}{2} C_{k'_2} -\frac{1}{2} C_{k'_3} -\frac{1}{2} C_{n-k'_1-k'_2-k'_3-1} \right. \\
& \quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \tag{4.15}
\end{aligned}$$

The right moving contraction follows a similar derivation and can be obtained by taking the complex conjugate of the above expression

$$C^{\bar{\alpha}\bar{\alpha}}[m, n] = (C^{\alpha\alpha}[m, n])^* \tag{4.16}$$

This means that we replace barred quantities with unbarred ones and vice versa, $z_i \leftrightarrow \bar{z}_i$. The final result is summarized in section 7.

4.2 Left-right moving contraction

For $C^{\alpha\bar{\alpha}}[m, n]$, the computation is very similar to that of $C_{\text{III}}^{\alpha\alpha}$ which is the third term of the left moving contraction. First recalling the expression for $C^{\alpha\bar{\alpha}}[m, n]$ recorded in (2.27), we have

$$C^{\alpha\bar{\alpha}}[m, n] = - \oint_{z'=0} \frac{dz'}{2\pi} z'^{-m} \oint_{\bar{z}=0} \frac{d\bar{z}}{2\pi} \bar{z}^{-n} b(\bar{z}, z'; z_i, \bar{z}_i) \tag{4.17}$$

Inserting the correlation function for b recorded in (2.28) into the above expression, we obtain

$$\begin{aligned}
& C^{\alpha\bar{\alpha}}[m, n] \\
& = - B(x, \bar{x}) |z_1 - z_3| |z_2 - z_4| \left(\oint_{z'=0} \frac{dz'}{2\pi} z'^{-m} \frac{1}{(z' - z_1)^{\frac{1}{2}} (z' - z_2)^{\frac{1}{2}} (z' - z_3)^{\frac{1}{2}} (z' - z_4)^{\frac{1}{2}}} \right) \\
& \times \left(\oint_{\bar{z}=0} \frac{d\bar{z}}{2\pi} \bar{z}^{-n} \frac{1}{(\bar{z} - \bar{z}_1)^{\frac{1}{2}} (\bar{z} - \bar{z}_2)^{\frac{1}{2}} (\bar{z} - \bar{z}_3)^{\frac{1}{2}} (\bar{z} - \bar{z}_4)^{\frac{1}{2}}} \right) \tag{4.18}
\end{aligned}$$

Notice that the above contour integral is similar to the integral in $C_{\text{III}}^{\alpha\alpha}$, as given in (4.3). We therefore obtain a final expression similar to (4.14)

$$\begin{aligned}
C^{\alpha\bar{\alpha}}[m, n] &= \frac{B(x, \bar{x})|z_1 - z_3||z_2 - z_4|}{|z_1 z_2 z_3 z_4|} \\
&\times \left((-1)^m \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{m-k_1-1} \sum_{k_3=0}^{m-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{m-k_1-k_2-k_3-1} \right. \\
&\quad \left. z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-m+k_1+k_2+k_3+1} \right) \\
&\times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\
&\quad \left. \bar{z}_1^{-k'_1} \bar{z}_2^{-k'_2} \bar{z}_3^{-k'_3} \bar{z}_4^{-n+k'_1+k'_2+k'_3+1} \right) \tag{4.19}
\end{aligned}$$

4.3 Relating contraction and pair creation

Since both the initial and final states belong to the untwisted sector, a time-reversal symmetry exists that relates contraction and pair creation. In this section, we derive these relations and establish a similar relation for the propagation coefficients. These relations can serve as a cross-check for our results collected in section 7.

The contraction coefficients are given by (2.26)

$$\begin{aligned}
C_{m,n}^{\alpha\alpha}(z_i, \bar{z}_i) &= \frac{\langle 0 | \prod_i \sigma(z_i, \bar{z}_i) \alpha_{-m} \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_i \sigma(z_i, \bar{z}_i) | 0 \rangle} \\
C_{m,n}^{\alpha\bar{\alpha}}(z_i, \bar{z}_i) &= \frac{\langle 0 | \prod_i \sigma(z_i, \bar{z}_i) \alpha_{-m} \bar{\alpha}_{-n} | 0 \rangle}{\langle 0 | \prod_i \sigma(z_i, \bar{z}_i) | 0 \rangle} \tag{4.20}
\end{aligned}$$

Here, we keep the discussion general, allowing the number and order of twist operators to be arbitrary, rather than restricting to the specific case of four twist-2 operators considered in this paper. To relate these coefficients to pair creation, we use the Hermitian conjugate of the twist operator

$$(\sigma(z, \bar{z}))^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \sigma(1/\bar{z}, 1/z) \tag{4.21}$$

where h and \bar{h} are the left and right moving dimensions of σ , respectively. By taking the Hermitian conjugate of the contraction coefficients in (4.20), we obtain

$$\begin{aligned}
(C_{m,n}^{\alpha\alpha}(z_i, \bar{z}_i))^* &= \frac{\langle 0 | \alpha_n \alpha_m \prod_i \sigma(1/\bar{z}_i, 1/z_i) | 0 \rangle}{\langle 0 | \prod_i \sigma(1/\bar{z}_i, 1/z_i) | 0 \rangle} = 2mn \gamma_{m,n}(1/\bar{z}_i, 1/z_i) \\
(C_{m,n}^{\alpha\bar{\alpha}}(z_i, \bar{z}_i))^* &= \frac{\langle 0 | \bar{\alpha}_n \alpha_m \prod_i \sigma(1/\bar{z}_i, 1/z_i) | 0 \rangle}{\langle 0 | \prod_i \sigma(1/\bar{z}_i, 1/z_i) | 0 \rangle} = mn \beta_{m,n}(1/\bar{z}_i, 1/z_i) \tag{4.22}
\end{aligned}$$

Notice that the prefactor in (4.21) cancels between the numerator and denominator. Taking the complex conjugate of these expressions, we find the relation

$$C_{m,n}^{\alpha\alpha}(z_i, \bar{z}_i) = 2mn \gamma_{m,n}(1/z_i, 1/\bar{z}_i), \quad C_{m,n}^{\alpha\bar{\alpha}}(z_i, \bar{z}_i) = mn \beta_{m,n}(1/z_i, 1/\bar{z}_i) \quad (4.23)$$

A similar relation can be derived for the propagation coefficients. Recall that the propagation coefficients are given by

$$\begin{aligned} f_{n,p}^{\alpha\alpha}(z_i, \bar{z}_i) &= \frac{1}{p} \frac{\langle 0 | \alpha_p \prod_i \sigma(z_i, \bar{z}_i) \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_i \sigma(z_i, \bar{z}_i) | 0 \rangle} \\ f_{n,p}^{\alpha\bar{\alpha}}(z_i, \bar{z}_i) &= \frac{1}{p} \frac{\langle 0 | \bar{\alpha}_p \prod_i \sigma(z_i, \bar{z}_i) \alpha_{-n} | 0 \rangle}{\langle 0 | \prod_i \sigma(z_i, \bar{z}_i) | 0 \rangle} \end{aligned} \quad (4.24)$$

Applying Hermitian conjugation to these expressions gives

$$\begin{aligned} (f_{n,p}^{\alpha\alpha}(z_i, \bar{z}_i))^* &= \frac{1}{p} \frac{\langle 0 | \alpha_n \prod_i \sigma(1/\bar{z}_i, 1/z_i) \alpha_{-p} | 0 \rangle}{\langle 0 | \prod_i \sigma(1/\bar{z}_i, 1/z_i) | 0 \rangle} = \frac{n}{p} f_{p,n}^{\alpha\alpha}(1/\bar{z}_i, 1/z_i) \\ (f_{n,p}^{\alpha\bar{\alpha}}(z_i, \bar{z}_i))^* &= \frac{1}{p} \frac{\langle 0 | \bar{\alpha}_n \prod_i \sigma(1/\bar{z}_i, 1/z_i) \alpha_{-p} | 0 \rangle}{\langle 0 | \prod_i \sigma(1/\bar{z}_i, 1/z_i) | 0 \rangle} = \frac{n}{p} f_{p,n}^{\bar{\alpha}\alpha}(1/\bar{z}_i, 1/z_i) \end{aligned} \quad (4.25)$$

Taking the complex conjugate of these expressions yields

$$f_{n,p}^{\alpha\alpha}(z_i, \bar{z}_i) = \frac{n}{p} f_{p,n}^{\alpha\alpha}(1/z_i, 1/\bar{z}_i), \quad f_{n,p}^{\alpha\bar{\alpha}}(z_i, \bar{z}_i) = \frac{n}{p} f_{p,n}^{\bar{\alpha}\alpha}(1/z_i, 1/\bar{z}_i) \quad (4.26)$$

5 Numerical analysis

In this section, we analyze the propagation and contraction coefficients numerically. We will work in the Lorentzian signature on the cylinder coordinate ($t = -i\tau, \sigma$), as defined in (2.2). Notice that σ here is the circular coordinate of the base space and not the twist operator.

First, we note that the propagation and contraction coefficients are invariant under any permutation of the twist locations $z_i \leftrightarrow z_j$ ($w_i \leftrightarrow w_j$) for $i, j = 1, 2, 3, 4$. This symmetry arises because the correlation functions from which they are derived, g and b , are invariant under such twist interchanges, as reviewed in subsection (2.7). We have also numerically verified this symmetry.

By direct inspection of the expressions collected in section 7, as well as through numerical testing, we find that the propagation coefficients exhibit the following anti-periodicity under shifts by 2π in any one of the twist locations

$$f_{n,p}^{\alpha\alpha}(t_i + 2\pi) = -f_{n,p}^{\alpha\alpha}(t_i), \quad f_{n,p}^{\alpha\bar{\alpha}}(t_i + 2\pi) = -f_{n,p}^{\alpha\bar{\alpha}}(t_i) \quad (5.1)$$

In contrast, the contraction coefficients are periodic in 2π , similar to the pair creation coefficients studied in [22]

$$C^{\alpha\alpha}[n, p](t_i + 2\pi) = C^{\alpha\alpha}[n, p](t_i), \quad C^{\alpha\bar{\alpha}}[n, p](t_i + 2\pi) = C^{\alpha\bar{\alpha}}[n, p](t_i) \quad (5.2)$$

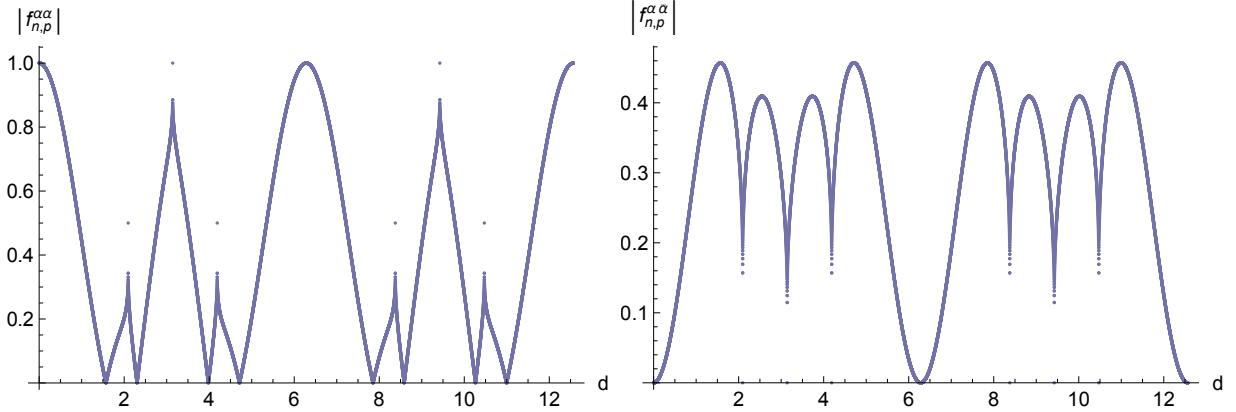


Figure 1: $|f_{n,p}^{\alpha\alpha}|$ (left) and $|f_{n,p}^{\alpha\bar{\alpha}}|$ (right) vs. distance d between equally spaced twist operators in time, with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = d$, $t_3 = 2d$, $t_4 = 3d$ with a range of d from 0 to 4π for $n = p = 1$.

We note that in our analysis we fix the twist operators to be located at the same point along the circle which we fix to be $\sigma_i = 0$, and we study separations in the time direction. The reason that there is a shift by a minus sign for propagation is because the modes utilized in our analysis, we recall, are written as an antisymmetric combination of copy 1 and copy 2 modes since we've taken the number of copies of the CFT to be $N = 2$. Therefore a 2π shift in t_i for any one twist operator maps copy 1 into copy 2 and copy 2 into copy 1. Since propagation, $f^{\alpha\alpha}, f^{\alpha\bar{\alpha}}$, multiplies a linear combination of modes in the copy 1 - copy 2 basis this gives an overall minus sign unlike the case for pair creation, γ, β , which multiplies a bilinear combination of modes in the copy 1 - copy 2 basis and contraction, $C^{\alpha\alpha}, C^{\alpha\bar{\alpha}}$ which doesn't multiply any modes at all.

In the following, we focus on the propagation effects, as the contraction effects are very similar to pair creation, a relationship we proved explicitly in (4.23). Let us consider a configuration in which all twist operators are equally spaced in time, with the separation denoted by d , and positioned at the same spatial location, $\sigma_i = 0$. Due to the twist interchange symmetry and the periodicity, the plot exhibits reflective symmetry about $d = \pi$ and periodicity of 2π , as shown in fig.1.

Let us note several other features of propagation from fig.1. When all four twist operators coincide, they can be replaced by the identity operator since there is no effect at all. In this case the initial mode simply propagates unaltered on the base space as time evolves. Therefore, propagation, characterizing the mode transition $\alpha \rightarrow \alpha$, is given by $|f_{n,n}^{\alpha\alpha}| = 1$ and $|f_{n,p}^{\alpha\alpha}| = 0$ for $p \neq n$ with the transition $\alpha \rightarrow \bar{\alpha}$ yielding $|f_{n,p}^{\alpha\bar{\alpha}}| = 0$, as expected, a feature which is also explicitly highlighted in fig.3 where the left plot is for equal energy between initial and final modes, $n = p = 1$, and the right plot is for differing energies between the initial and final modes, $n = 1, p = 3$. As the twist operators are separated in time, the probability to transition into modes with different energies and modes in different sectors becomes nonzero. At $d = \frac{2\pi}{3}$ the first and fourth twist operators

are separated by 2π which, because of periodicity conditions (5.1), is equivalent to the first and fourth twist operators coinciding, which yields the two twist configuration. At this separation mixed propagation vanishes and standard propagation reduces to the two twist result. Increasing the separation to $d = \pi$ corresponds to the first and third twist operators being separated by 2π and the second and fourth twist operators being separated by 2π . Which, again through (5.1), is equivalent to the first and third twist operators coinciding and the second and fourth twist operators coinciding with the two pairs being separated by a distance of π . For this configuration propagation for $\alpha \rightarrow \alpha$ again returns to a value of $|f_{n,n}^{\alpha\alpha}| = 1$, $|f_{n,p}^{\alpha\alpha}| = 0$ for $p \neq n$ and for $\alpha \rightarrow \bar{\alpha}$ to a value of $|f_{n,p}^{\alpha\bar{\alpha}}| = 0$. These features are demonstrated numerically in fig.1. Fig.2 shows the behavior of the left-right mixed propagation $|f_{n,p}^{\alpha\bar{\alpha}}|$ as a function of the equal time separation, d , between consecutive twist operators. It takes the largest value at $n = p = 1$ and $d = \frac{\pi}{2}$, which is the widest separation among the four twist operators within a periodic 2π interval. This aligns with the role of twist operators, which join and split copies of the CFTs. In time order, the first and third twist operators join two untwisted CFT copies into a twisted copy, while the second and fourth split it back into two untwisted copies. As a result, consecutive twist operators tend to cancel each other's effects when they are close. Therefore, the maximum effect arises at $d = \frac{\pi}{2}$, where the twist operators are most widely separated. For higher energy modes (shorter wavelengths), smaller separations dominate, resulting in stronger cancellations and smaller propagation coefficients. These behaviors are similar to the left-right mixed pair creation studied in [22].

As can be seen in fig.4, for small time separations between twist operators we see that propagation $|f_{n,p}^{\alpha\bar{\alpha}}|$ is highly peaked when the final energy is equal to the initial energy which in this case is, $n = p = 1$, and sharply decays for $p \neq n$. For the same separation the value of mixed propagation, $|f_{n,p}^{\alpha\bar{\alpha}}|$, is small for any value of p with a decaying envelope starting from $p = 1$. This behavior is expected, again demonstrating the cancellation effects by twist operators which act close together.

Fig.5 shows values of $|f_{n,p}^{\alpha\bar{\alpha}}|$ and $|f_{n,p}^{\alpha\alpha}|$ vs. n, p for maximum equal separation of consecutive twist operators in time, $d = \frac{\pi}{2}$. We see that $|f_{n,p}^{\alpha\alpha}|$ strongly oscillates for various values of n and p , giving zero of $n = p$ which is consistent with expected behavior. For mixed propagation $|f_{n,p}^{\alpha\bar{\alpha}}|$ we see that it peaks at $n = p = 1$ and yields decaying local maxima for larger values of n with $p = 1$. While the lowest mode for both n, p yields the highest transition probability, for higher energy initial left moving modes, $n > 1$, for which $|f_{n,p}^{\alpha\bar{\alpha}}|$ is nonzero, for each fixed n , the largest transition is to a right moving mode with the lowest energy, $p = 1$. This nicely captures the behavior of nontrivial monodromy as a low energy effect.

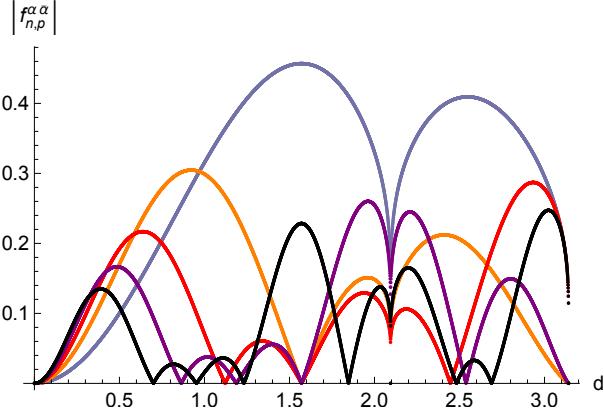


Figure 2: $|f_{n,p}^{\alpha\bar{\alpha}}|$ vs. distance d between equally spaced twist operators in time for initial mode $n = 1$ and final modes $p = 1$ (blue), 2 (orange), 3 (red), 4 (purple), 5 (black) for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = d$, $t_3 = 2d$, $t_4 = 3d$ for $0 \leq d \leq \pi$.

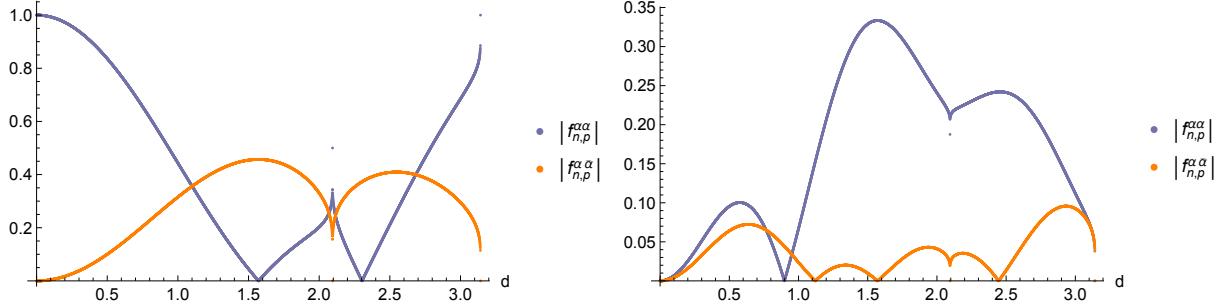


Figure 3: $|f_{n,p}^{\alpha\alpha}|$ (blue) and $|f_{n,p}^{\alpha\bar{\alpha}}|$ (orange) vs. distance d between equally spaced twist operators in time for initial mode $n = 1$ and final mode $p = 1$ (left) and $p = 3$ (right), for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = d$, $t_3 = 2d$, $t_4 = 3d$ for $0 \leq d \leq \pi$.

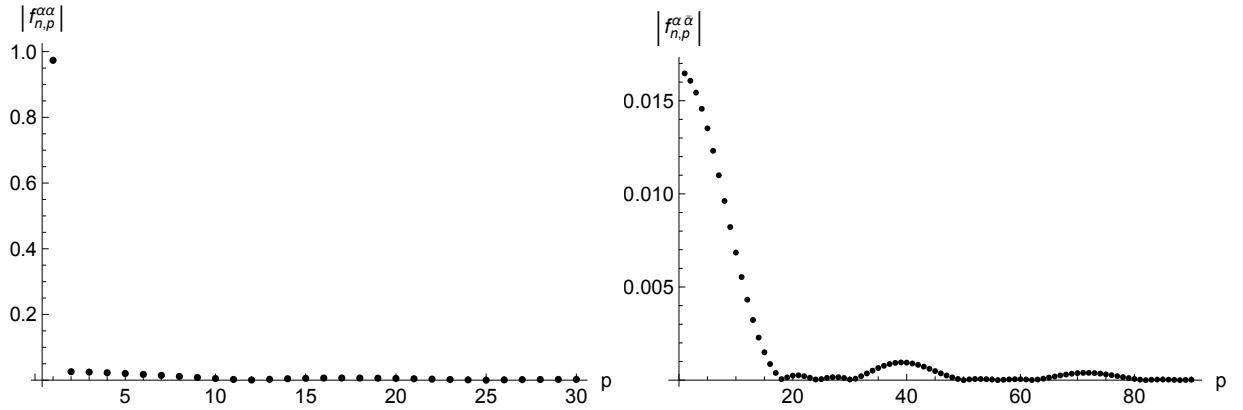


Figure 4: $|f_{n,p}^{\alpha\alpha}|$ (left) and $|f_{n,p}^{\alpha\bar{\alpha}}|$ (right) vs. p , the final mode number, with fixed initial mode number $n = 1$ for equally spaced twist operators for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = \frac{\pi}{16}$, $t_3 = \frac{\pi}{8}$, $t_4 = \frac{3\pi}{16}$.

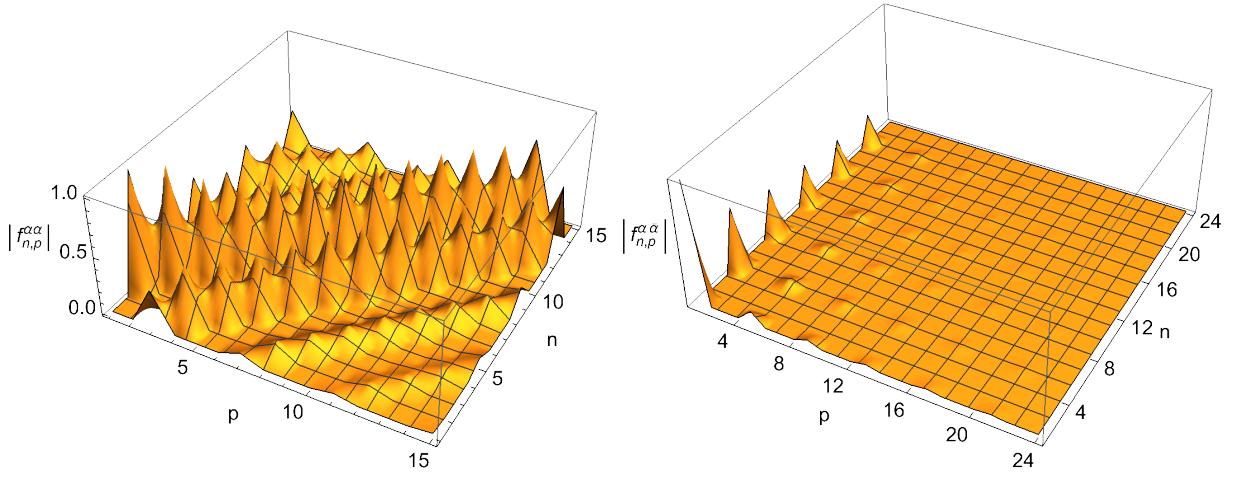


Figure 5: $|f_{n,p}^{\alpha\alpha}|$ (left), $|f_{n,p}^{\alpha\bar{\alpha}}|$ (right) vs. n, p for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = \frac{\pi}{2}$, $t_3 = \pi$, $t_4 = \frac{3\pi}{2}$. Although n, p are integers, smooth interpolation is used for better visualization.

6 Some limits

The following question is crucial for understanding the effect of multiple twist operators: Can we reproduce the effect of multiple twist operators by summing over intermediate states and using the effects of one twist or fewer twists? If so, it would simplify the computation of multiple twist effects. In this section, we will investigate this question by analyzing the effect of four-twist operators in two different limits.

The first is the continuum limit, where the energy corresponding to mode numbers becomes very large. In [16], it is assumed that in this limit, the effects of multiple twists can be reproduced by combining the effects of single twists. We will find that our results strongly support this assumption. The second limit is the large separation limit, where the distance between twists becomes very large while the mode numbers remain relatively low. In this case, we find that nontrivial monodromy effects still persist, even though the twist operators are well-separated.

6.1 Continuum Limit

In [20, 21, 23], the large-energy (or continuum) limit was taken to obtain approximate expressions for one- and two-twist effects, including pair creation and propagation. This involved considering mode numbers which were $\gg 1$. For the single-twist case, the continuum limit is more straightforward to apply, as the expressions can be obtained in closed analytic form. For the propagation of a single twist, we have

$$f_{np}^{B(1)} \approx \begin{cases} \frac{i}{(n-p)\pi} \sqrt{\frac{n}{p}} e^{w_0(p-n)} & n \neq p \\ \frac{1}{2} & n = p \end{cases} \quad (6.1)$$

where $n, p \gg 1$. In this process, two untwisted copies join into a twist-2 copy. Here (1) labels an initial copy, and B denotes the bosonic field. For more details, see [16, 17]. The parameter w_0 is the location of the twist operator on the cylinder. For pair creation, the continuum limit is given by

$$\gamma_{mn}^B \approx \frac{1}{2\pi} \frac{1}{\sqrt{mn}} \frac{1}{m+n} e^{w_0(m+n)} \quad (6.2)$$

For the two-twist case, however, the situation is a bit more complicated. The exact expressions are not simple analytic functions as in the single twist case but are instead given by finite sums involving binomial coefficients and powers of twist locations. Furthermore, since the presence of two or more twist operators introduces twist separations, the expressions for the corresponding twist effects have an oscillatory behavior governed by this separation. The separation between the two twists in Lorentzian signature is denoted by $\Delta w = i(\Delta t + \Delta\sigma)$. The continuum limit for propagation in the two-twist case is given by

$$f_{np}^{B,(1)(1')} \approx \begin{cases} \frac{1}{(n-p)\pi} \sqrt{\frac{n}{p}} \sin\left((n-p)\frac{\Delta w}{2\pi i}\right) \text{sgn}\left(\frac{\Delta w}{2\pi i} - 1\right), & n \neq p \\ \left|\frac{\Delta w}{2\pi i} - 1\right|, & n = p \end{cases} \quad (6.3)$$

where (1) denotes initial copy 1 and (1') denotes final copy 1. For more details, see [21]. We observe that for $n \neq p$, the amplitude $\frac{1}{(n-p)\pi} \sqrt{\frac{n}{p}}$ of the oscillatory function matches that of the single-twist case in (6.1).

For pair creation, the oscillatory function has not yet been determined explicitly but its amplitude has and matches that of the single-twist case [20]

$$\gamma_{mn}^{B,(1)(1)} \sim \frac{1}{\sqrt{mn}(m+n)} \quad (6.4)$$

where both modes correspond to copy-1 excitations in the final state. Since the amplitude of the oscillatory function for both pair creation and propagation remains the same in the one- and two-twist cases, it has been conjectured that for an arbitrary number of twist operators they should similarly display this same amplitude [21]. The oscillating factor however is expected to become increasingly more complicated as the number of twist operators increases.

Turning to the four-twist results computed here and in [22], we note that the pair creation and propagation expressions in (7.5) and (7.8) share similarities with the two-twist case, involving finite sums but with an additional nontrivial monodromy term. The oscillatory behavior of the four-twist expressions is more complicated due to the increased number of possible combinations of twist separations. To simplify, we consider a configuration where all four twist operators are at the same spatial location, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and are equally spaced in time at $t_1 = 0$, $t_2 = d$, $t_3 = 2d$ and $t_4 = 3d$. We will study how the oscillatory behavior depends on d .

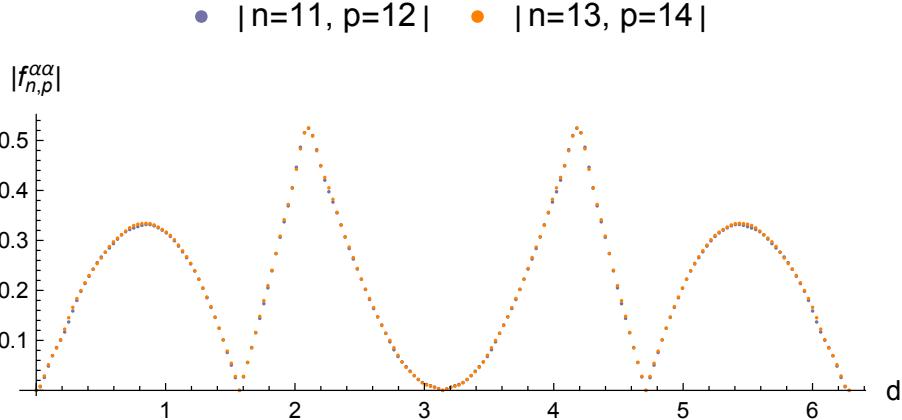


Figure 6: The plot of $|f_{n,p}^{aa}|$ vs. d for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0, t_2 = d, t_3 = 2d, t_4 = 3d$, over the range $0 \leq d \leq 2\pi$ for $n = 11, p = 12$ (blue) and $n = 13, p = 14$ (orange).

For propagation, we plot the energy differences for $p - n = 1$ in fig.6 and for $p - n = 2$ in fig.7. Unlike the one- and two-twist continuum limits, which exhibit an analytic form that clearly separates the amplitude from the oscillatory function as illustrated in (6.1) and (6.3), the four-twist case does not display such a clean separation. Nevertheless, we observe that in the continuum limit ($n, p \gg 1$) with fixed $n - p$, the propagation approaches a fixed profile, as seen in fig.6 and fig.7, where the two curves match precisely. This behavior is also seen in the one- and two-twist cases discussed earlier. Similarly, for pair creation in the continuum limit ($m, n \gg 1$) with fixed $m + n$, the profile approaches a fixed form, as shown in fig.6.

Returning to the question of whether the effect of four twist operators can be produced by combining the effects of two pairs of two twists. The four-twist effects include nontrivial monodromy terms, such as those with coefficients $A(x, \bar{x})$ and $B(x, \bar{x})$ in the expressions summarized in section 7. In the continuum limit, these nontrivial monodromy terms decay much more rapidly than the local terms. This rapid falloff at higher energies indicates that these terms are important only at low energies, as they arise from nontrivial cycles on the torus covering surface, with their size determined by the separation of the twist operators. We demonstrate this behavior by comparing the full result, the local term and the monodromy term both for pair creation, $\gamma_{m,n}$ for energies $m = 8, n = 9$, shown in fig.10, and for propagation, $f_{n,p}^{aa}$ for energies $n = 11, p = 12$ shown in fig.8, where each effect is plotted as function of equal time twist separation d . We note that when the twists are sufficiently separated, the monodromy terms decay quickly. For separations in which the twist operators are close together, either directly or through (anti)periodic shifts of 2π , the monodromy terms have nontrivial behavior and may have values comparable to those of the local terms. This is because short separations correspond to energies comparable to those in the high energy limit which, for these separations, is no longer reliable. The negligibility of the monodromy terms at higher energies strongly supports the assumption

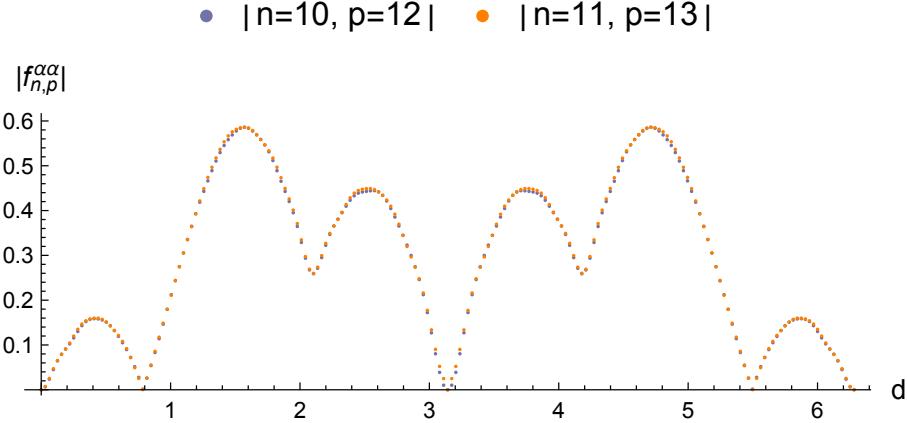


Figure 7: The plot of $|f_{n,p}^{\alpha\alpha}|$ vs. d for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0, t_2 = d, t_3 = 2d, t_4 = 3d$, over the range $0 \leq d \leq 2\pi$ for $n = 10, p = 12$ (blue) and $n = 11, p = 13$ (orange).

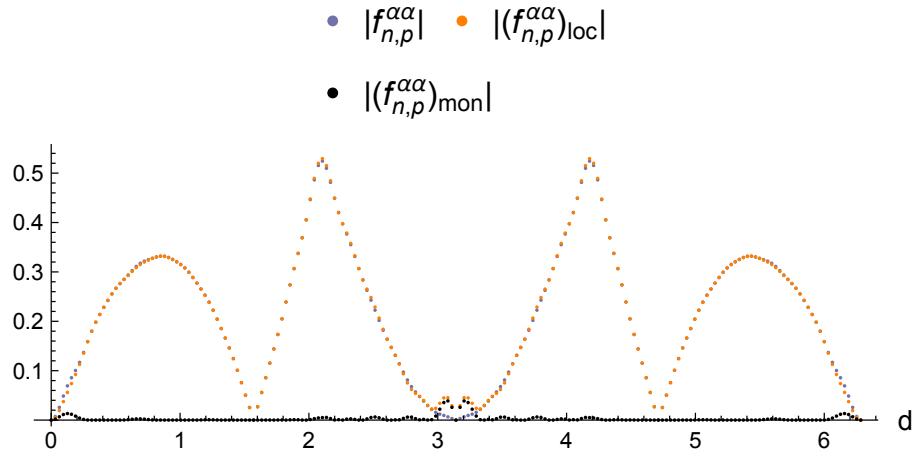


Figure 8: The plot of $|f_{n,p}^{\alpha\alpha}|$ (blue), $|(f_{n,p}^{\alpha\alpha})_{\text{local}}|$ (orange), $|(f_{n,p}^{\alpha\alpha})_{\text{mono}}|$ (black) vs. d for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0, t_2 = d, t_3 = 2d, t_4 = 3d$, over the range $0 \leq d \leq 2\pi$ for $n = 11, p = 12$.

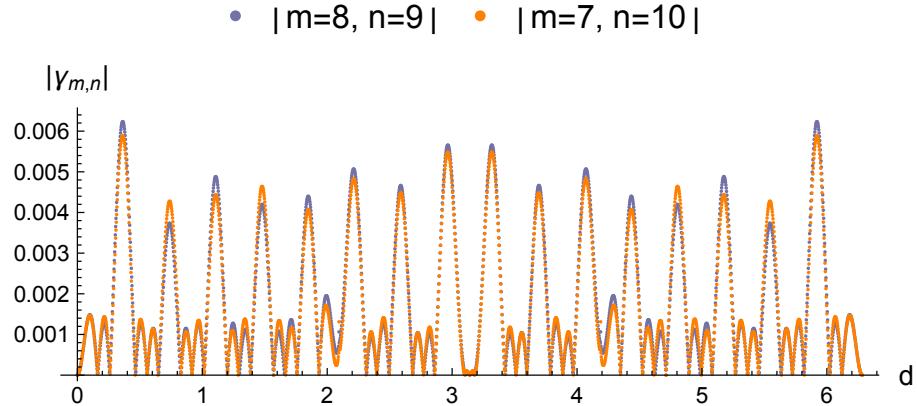


Figure 9: The plot of $|\gamma_{m,n}|$ vs. d for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = d$, $t_3 = 2d$, $t_4 = 3d$, over the range $0 \leq d \leq 2\pi$ for $m = 8, n = 9$ (blue) and $m = 7, n = 10$ (orange).

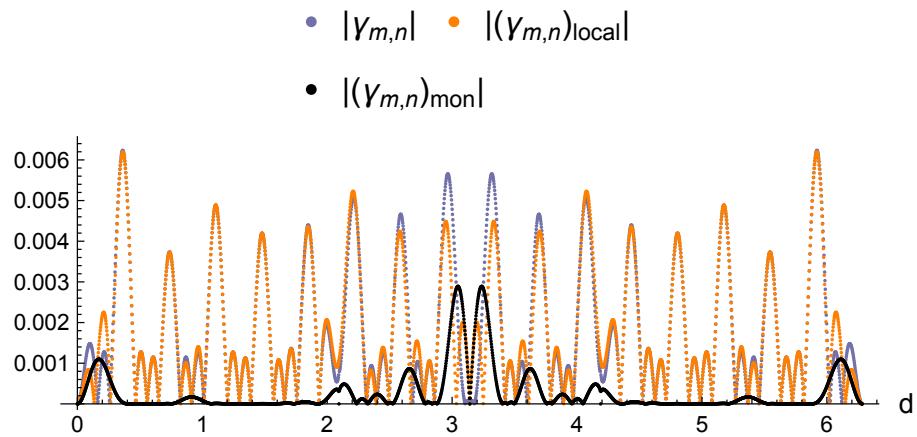


Figure 10: The plot of $|\gamma_{m,n}|$ (blue), $|(\gamma_{m,n})_{\text{local}}|$ (blue) and $|(\gamma_{m,n})_{\text{mon}}|$ vs. d for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, and $t_1 = 0$, $t_2 = d$, $t_3 = 2d$, $t_4 = 3d$, over the range $0 \leq d \leq 2\pi$ for $m = 8, n = 9$.

that the effect of multiple twist operators at high energy can be reproduced by combining the effects of fewer twist operators.

6.2 Small and large separation limit

We now consider the small and large separation limits of twist effects for four twist operators. To do this, we make the following transformations

$$c_1 \equiv z_1 - z_2, \quad d \equiv z_2 - z_3, \quad c_2 \equiv z_3 - z_4, \quad s = \frac{1}{4}(z_1 + z_2 + z_3 + z_4) \quad (6.5)$$

We will focus on the case where c_1 , c_2 and s are held fixed, while considering the limits $d \rightarrow 0$ or $d \rightarrow \infty$.

Small separation limit $d = z_2 - z_3 \rightarrow 0$

Let us consider the limit where $z_2 \rightarrow z_3$ or $d \rightarrow 0$ with c_1, c_2 and s fixed. First we analyze the monodromy functions $A(z_i, \bar{z}_i)$ and $B(z_i, \bar{z}_i)$, from (7.1)

$$\begin{aligned} A(z_i, \bar{z}_i) &= -(z_1 - z_3)(z_2 - z_4)A(x, \bar{x}) \\ &= -(z_1 - z_3)(z_2 - z_4) \\ &\quad \times \frac{(E(x) - (1-x)K(x))\bar{K}(1-\bar{x}) - (E(1-x) - xK(1-x))\bar{K}(\bar{x})}{2(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \end{aligned} \quad (6.6)$$

and

$$\begin{aligned} B(z_i, \bar{z}_i) &= |z_1 - z_3||z_2 - z_4|B(x, \bar{x}) \\ &= \frac{|z_1 - z_3||z_2 - z_4|\pi}{4(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \end{aligned} \quad (6.7)$$

By inserting (6.5) into (6.6) and (6.7), and recalling the relation

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \quad (6.8)$$

we can expand around $d \sim 0$, keeping only the leading order terms. The resulting behavior as $d \rightarrow 0$ is

$$\begin{aligned} A(z_i, \bar{z}_i) &\approx \frac{c_1 c_2}{2 \log |d|} \rightarrow 0 \\ B(z_i, \bar{z}_i) &\approx -\frac{|c_1 c_2|}{2 \log |d|} \rightarrow 0 \end{aligned} \quad (6.9)$$

This is consistent with the expectation that the monodromy terms vanish as two twist operators are brought together. In this limit, the four twist configuration reduces to the

two twist configuration, which does not contain nontrivial monodromy terms. For pair creation (7.5), (7.6) and propagation (7.8), (7.10) for the lowest mode numbers, we find the following limits as $d \rightarrow 0$

$$\begin{aligned}\gamma_{1,1} &\approx \frac{1}{16}(c_1 + c_2)^2 + \frac{1}{4} \frac{c_1 c_2}{\log |d|} \rightarrow \frac{1}{16}(c_1 + c_2)^2 = \frac{1}{16}(z_1 - z_2 + z_3 - z_4)^2 \\ \beta_{1,1} &\approx -\frac{|c_1 c_2|}{2 \log |d|} \rightarrow 0 \\ f_{1,1}^{\alpha\alpha} &\approx \frac{1}{2} \left(\sqrt{\frac{3c_1 + c_2 + 4s}{-c_1 - 3c_2 + 4s}} + \sqrt{\frac{-c_1 - 3c_2 + 4s}{3c_1 + c_2 + 4s}} \right) = \frac{1}{2} \left(\sqrt{\frac{z_1}{z_4}} + \sqrt{\frac{z_4}{z_1}} \right) \\ f_{1,1}^{\alpha\bar{\alpha}} &\approx -\frac{8|c_1 c_2|}{\sqrt{(-c_1 - 3c_2 + 4s)(3c_1 + c_2 + 4s)(-c_1 + c_2 + 4s)^2} \log |d|} \rightarrow 0\end{aligned}\tag{6.10}$$

In the limit where two of the four twists come together, the left-moving effects remain, approaching the two-twist result [20, 21]. The left-right mixed effects vanish because there is no nontrivial monodromy in the two-twist case. These results have also been proven more generally in [22].

Large separation limit $d = z_2 - z_3 \rightarrow \infty$

Let us now consider the limit $d = z_2 - z_3 \rightarrow \infty$ again with c_1, c_2, s held fixed. Expanding (6.6) and (6.7), around $d \sim \infty$ and explicitly writing the leading order terms, we find the limits for the two monodromy functions to be

$$\begin{aligned}A(z_i, \bar{z}_i) &\approx \frac{d^2}{4 \log |d|} \rightarrow \infty, \\ B(z_i, \bar{z}_i) &\approx \frac{|d|^2}{4 \log |d|} \rightarrow \infty\end{aligned}\tag{6.11}$$

Considering the lowest mode number contributions for pair creation (7.5), (7.6) and propagation (7.8), (7.10), in the limit that $d \rightarrow \infty$, we find that

$$\begin{aligned}\gamma_{1,1} &\approx \frac{d^2}{8 \log |d|} \rightarrow \infty \\ \beta_{1,1} &\approx \frac{|d|^2}{4 \log |d|} \rightarrow \infty \\ f_{1,1}^{\alpha\alpha} &\approx 1 \\ f_{1,1}^{\alpha\bar{\alpha}} &\approx \frac{\bar{d}}{d \log \bar{d}} \rightarrow 0\end{aligned}\tag{6.12}$$

For $\gamma_{1,1}$, in the large d limit, the dominant contribution is naively expected to come from the vacuum state in the intermediate states between the second and third twists, scaling

proportionally to d^2 . The additional factor of $\frac{1}{\log|d|}$ arises from the nontrivial monodromy effects, which cannot be simply reproduced by summing over the excited intermediate states, as they contribute only lower powers of d . In contrast, for $f_{1,1}^{\alpha\alpha}$, the dominant contribution comes from the first excited states α_{-1} between the second and third twists (since the vacuum contribution vanishes, as the initial state contains only one boson, which cannot turn into a vacuum by the first two twists). This reproduces the same scaling as in the above result. There is no nontrivial monodromy effect at leading order in the large d limit. The discrepancy in the leading behavior of γ and f may arise because, in the case of pair creation, the vacuum contributes, while in propagation, it does not. We conjecture that the following is generally true: for pair creation (and contraction), there is a nontrivial monodromy effect at leading order in the large d limit, whereas there is none in propagation. It will be interesting to explore this further.

7 Collecting Results

Here we collect the expressions for all the effects of the twist operators: pair creation, as computed in [22], along with propagation and contraction, which are computed in this paper.

In the following the absolute value is given by $|z| = \sqrt{z\bar{z}}$, and the functions A and B and are defined as

$$\begin{aligned} A(x, \bar{x}) &= \frac{(E(x) - (1-x)K(x))\bar{K}(1-\bar{x}) - (E(1-x) - xK(1-x))\bar{K}(\bar{x})}{2(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \\ B(x, \bar{x}) &= (B(x, \bar{x}))^* = \bar{B}(\bar{x}, x) = \frac{\pi}{4(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \\ \bar{A}(\bar{x}, x) &= (A(x, \bar{x}))^* \\ &= \frac{(\bar{E}(\bar{x}) - (1-\bar{x})\bar{K}(\bar{x}))K(1-x) - (\bar{E}(1-\bar{x}) - \bar{x}\bar{K}(1-\bar{x}))K(x)}{2(K(x)\bar{K}(1-\bar{x}) + K(1-x)\bar{K}(\bar{x}))} \end{aligned} \quad (7.1)$$

where $K(y)$ and $E(y)$ are the complete elliptic integrals of the first and second kinds respectively. The cross ratio is given by (7.1).

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad \bar{x} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)}, \quad z_j = e^{w_j}, \quad \bar{z}_j = e^{\bar{w}_j} \quad (7.2)$$

where $j = 1, 2, 3, 4$. In the following, we will perform complex conjugation. In practice, this means replacing all z_i with \bar{z}_i and vice versa, as well as using the complex conjugates of $A(x, \bar{x})$ and $B(x, \bar{x})$ from (7.1).

In Euclidean signature, the cylinder coordinates w_j are given by

$$w_j = \tau_j + i\sigma_j, \quad \bar{w}_j = \tau_j - i\sigma_j \quad (7.3)$$

while in Lorentzian signature

$$w_j = i(t_j + \sigma_j), \quad \bar{w}_j = i(t_j - \sigma_j) \quad (7.4)$$

Pair creation

The left-moving pair creation coefficient is given by

$$\begin{aligned} \gamma_{m,n} = & \frac{1}{4mn} (-1)^{m+n} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-(k_1+1)} \sum_{k_3=0}^{n-(k_1+k_2+1)} \sum_{k_4=0}^{n-(k_1+k_2+k_3+1)} \sum_{k'_1=0}^{m+n-(k_1+k_2+k_3+k_4)} \\ & \sum_{k'_2=0}^{m+n-(k_1+k_2+k_3+k_4+k'_1)} \sum_{k'_3=0}^{m+n-(k_1+k_2+k_3+k_4+k'_1+k'_2)} (n - (k_1 + k_2 + k_3 + k_4)) \\ & \times \left(z_1^{k_1+k'_1} z_2^{k_2+k'_2} z_3^{k_3+k'_3} z_4^{m+n-(k_1+k'_1+k_2+k'_2+k_3+k'_3)} \right. \\ & \quad \left. + z_2^{k_1+k'_1} z_1^{k_2+k'_2} z_4^{k_3+k'_3} z_3^{m+n-(k_1+k'_1+k_2+k'_2+k_3+k'_3)} \right) \\ & - A(x, \bar{x})(z_1 - z_3)(z_2 - z_4) \\ & \times \left(\frac{1}{\sqrt{2}m} (-1)^m \sum_{k'_1=0}^{m-1} \sum_{k'_2=0}^{m-(k'_1+1)} \sum_{k'_3=0}^{m-(k'_1+k'_2+1)} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{m-(k'_1+k'_2+k'_3+1)} \right. \\ & \quad \left. z_1^{k'_1} z_2^{k'_2} z_3^{k'_3} z_4^{m-(k'_1+k'_2+k'_3+1)} \right) \\ & \times \left(\frac{1}{\sqrt{2}n} (-1)^n \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-(k_1+1)} \sum_{k_3=0}^{n-(k_1+k_2+1)} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{n-(k_1+k_2+k_3+1)} \right. \\ & \quad \left. z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{n-(k_1+k_2+k_3+1)} \right) \end{aligned} \quad (7.5)$$

The left-right mixed pair creation coefficient is

$$\begin{aligned} \beta_{m,n} = & B(x, \bar{x}) |z_1 - z_3| |z_2 - z_4| \\ & \times \left(\frac{1}{m} (-1)^m \sum_{k'_1=0}^{m-1} \sum_{k'_2=0}^{m-(k'_1+1)} \sum_{k'_3=0}^{m-(k'_1+k'_2+1)} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C'_{k'_3}^{-\frac{1}{2}} C_{m-(k'_1+k'_2+k'_3+1)} \right. \\ & \quad \left. z_1^{k'_1} z_2^{k'_2} z_3^{k'_3} z_4^{m-(k'_1+k'_2+k'_3+1)} \right) \\ & \times \left(\frac{1}{n} (-1)^n \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-(k_1+1)} \sum_{k_3=0}^{n-(k_1+k_2+1)} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{n-(k_1+k_2+k_3+1)} \right. \\ & \quad \left. z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{n-(k_1+k_2+k_3+1)} \right) \end{aligned}$$

$$\bar{z}_1^{k_1} \bar{z}_2^{k_2} \bar{z}_3^{k_3} \bar{z}_4^{n-(k_1+k_2+k_3+1)} \Big) \quad (7.6)$$

The right-moving pair creation coefficient is the complex conjugate of the left-moving pair creation coefficient

$$\bar{\gamma}_{m,n} = (\gamma_{m,n})^* \quad (7.7)$$

Propagation

The left-moving propagation coefficient is given by

$$\begin{aligned} f_{n,p}^{\alpha\alpha} = & \frac{1}{2p} (-1)^{p+n} \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \sum_{k'_4=\max[0,n-p-k'_1-k'_2-k'_3]}^{n-k'_1-k'_2-k'_3-1} \\ & \sum_{k_1=0}^{p-n+k'_1+k'_2+k'_3+k'_4} \sum_{k_2=0}^{p-n-k_1+k'_1+k'_2+k'_3+k'_4} \sum_{k_3=0}^{p-n-k_1-k_2+k'_1+k'_2+k'_3+k'_4} (n - k'_1 - k'_2 - k'_3 - k'_4) \\ & - \frac{1}{2} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3+k'_4} \\ & \times \left(\left(\frac{z_2 z_4}{z_1 z_3} \right)^{\frac{1}{2}} z_1^{k_1-k'_1} z_2^{k_2-k'_2} z_3^{k_3-k'_3} z_4^{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3} \right. \\ & \quad \left. + \left(\frac{z_1 z_3}{z_2 z_4} \right)^{\frac{1}{2}} z_2^{k_1-k'_1} z_1^{k_2-k'_2} z_4^{k_3-k'_3} z_3^{p-n-k_1-k_2-k_3+k'_1+k'_2+k'_3} \right) \\ & - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{\sqrt{z_1 z_2 z_3 z_4}} \\ & \times \left(\frac{(-1)^p}{p} \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-k_1-1} \sum_{k_3=0}^{p-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-k_1-k_2-k_3-1} \right. \\ & \quad \left. z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{p-k_1-k_2-k_3-1} \right) \\ & \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\ & \quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \end{aligned} \quad (7.8)$$

The right-moving propagation is given by the complex conjugation

$$f_{n,p}^{\bar{\alpha}\bar{\alpha}} = (f_{n,p}^{\alpha\alpha})^* \quad (7.9)$$

The left-right mixed propagation coefficient is given by

$$f_{n,p}^{\alpha\bar{\alpha}} = \frac{B(x, \bar{x}) |z_1 - z_3| |z_2 - z_4|}{\sqrt{z_1 z_2 z_3 z_4}}$$

$$\begin{aligned}
& \times \left(\frac{(-1)^p}{p} \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-k_1-1} \sum_{k_3=0}^{p-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{p-k_1-k_2-k_3-1} \right. \\
& \quad \left. \bar{z}_1^{k_1} \bar{z}_2^{k_2} \bar{z}_3^{k_3} \bar{z}_4^{p-k_1-k_2-k_3-1} \right) \\
& \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\
& \quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \tag{7.10}
\end{aligned}$$

and

$$f_{n,p}^{\bar{\alpha}\alpha} = (f_{n,p}^{\alpha\bar{\alpha}})^* \tag{7.11}$$

Contraction

The left-moving contraction is given by

$$\begin{aligned}
C^{\alpha\alpha}[m, n] &= \frac{1}{2} (-1)^{m+n} \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \sum_{k'_4=0}^{n-k'_1-k'_2-k'_3-1} \sum_{k_1=0}^{m+n-k'_1-k'_2-k'_3-k'_4} \\
&\quad \sum_{k_2=0}^{m+n-k'_1-k'_2-k'_3-k'_4-k_1} \sum_{k_3=0}^{m+n-k'_1-k'_2-k'_3-k'_4-k_1-k_2} (n - k'_1 - k'_2 - k'_3 - k'_4) \\
&\quad -\frac{1}{2} C_{k'_1}^{\frac{1}{2}} C_{k'_2}^{\frac{1}{2}} C_{k'_3}^{\frac{1}{2}} C_{k'_4}^{\frac{1}{2}} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{m+n-k_1-k_2-k_3-k'_1-k'_2-k'_3-k'_4} \\
&\quad \times \left(z_1^{-k_1-k'_1} z_2^{-k_2-k'_2} z_3^{-k_3-k'_3} z_4^{-m-n+k_1+k_2+k_3+k'_1+k'_2+k'_3} \right. \\
&\quad \left. + z_2^{-k_1-k'_1} z_1^{-k_2-k'_2} z_4^{-k_3-k'_3} z_3^{-m-n+k_1+k_2+k_3+k'_1+k'_2+k'_3} \right) \\
&\quad - \frac{A(x, \bar{x})(z_1 - z_3)(z_2 - z_4)}{z_1 z_2 z_3 z_4} \\
&\quad \times \left((-1)^m \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{m-k_1-1} \sum_{k_3=0}^{m-k_1-k_2-1} -\frac{1}{2} C_{k_1}^{-\frac{1}{2}} C_{k_2}^{-\frac{1}{2}} C_{k_3}^{-\frac{1}{2}} C_{m-k_1-k_2-k_3-1} \right. \\
&\quad \left. z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-m+k_1+k_2+k_3+1} \right) \\
&\quad \times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} -\frac{1}{2} C_{k'_1}^{-\frac{1}{2}} C_{k'_2}^{-\frac{1}{2}} C_{k'_3}^{-\frac{1}{2}} C_{n-k'_1-k'_2-k'_3-1} \right. \\
&\quad \left. z_1^{-k'_1} z_2^{-k'_2} z_3^{-k'_3} z_4^{-n+k'_1+k'_2+k'_3+1} \right) \tag{7.12}
\end{aligned}$$

The right-moving contraction is given by the complex conjugation

$$C^{\bar{\alpha}\bar{\alpha}}[m, n] = (C^{\alpha\alpha}[m, n])^* \quad (7.13)$$

The left-right mixed contraction is

$$\begin{aligned} C^{\alpha\bar{\alpha}}[m, n] &= \frac{B(x, \bar{x})|z_1 - z_3||z_2 - z_4|}{|z_1 z_2 z_3 z_4|} \\ &\times \left((-1)^m \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{m-k_1-1} \sum_{k_3=0}^{m-k_1-k_2-1} z_1^{-k_1} z_2^{-k_2} z_3^{-k_3} z_4^{-m+k_1+k_2+k_3+1} \right) \\ &\times \left((-1)^n \sum_{k'_1=0}^{n-1} \sum_{k'_2=0}^{n-k'_1-1} \sum_{k'_3=0}^{n-k'_1-k'_2-1} \bar{z}_1^{-k'_1} \bar{z}_2^{-k'_2} \bar{z}_3^{-k'_3} \bar{z}_4^{-n+k'_1+k'_2+k'_3+1} \right) \end{aligned} \quad (7.14)$$

and

$$C^{\bar{\alpha}\alpha}[m, n] = (C^{\alpha\bar{\alpha}}[m, n])^* = C^{\alpha\bar{\alpha}}[n, m] \quad (7.15)$$

8 Discussion and Conclusion

In this paper, we investigate the effect of four twist-2 operators on excited states, extending the results of [22], which focused on their effect on the vacuum. We consider a simple setup where these operators act on the untwisted sector of two copies of a free boson. As shown in [22], the effect on the vacuum is captured by a set of coefficients called pair creation, which describes the production of two modes in the final state. In this paper, we find that the effect on excited states requires two additional sets of coefficients: Propagation, which describes how one mode transforms into another, and Contraction, which describes the annihilation of two initial modes. A novel feature discovered in [22] is that pair creation involves mixing between the left- and right-moving sectors. We show that this mixing also appears in the effects on excited states, meaning that propagation and contraction also exhibit left-right mixing.

To compute these effects, a six-point function involving two bosonic fields and four twist operators is required. The results are provided in [45] for specific locations of the twist operators and in [22] for general locations. These results can be obtained using a covering map, which maps the six-point function into a two-point function of the free boson on a torus. This torus two-point function naturally exhibits left-right mixing, which becomes

the mixing in the effect of the four twist-2 operators. For higher twists and more twist operators, this mixing is generally present whenever the covering map is not a sphere.

The ultimate goal is to understand the effect of multiple twist operators. To do this, we examine two limits of the effect of four twist operators to see if it can be simplified and reproduced by combining the effects of fewer twist operators. First, we look at the continuum limit, where the energy of the modes is large. In this case, we find that the nontrivial monodromy terms decay much faster compared to the other terms, and the mixing between left- and right-moving sectors becomes negligible. This supports the idea that, at high energies, the effect of multiple twist operators can be reproduced by combining the effects of a single twist operator [19]. This is because the effect of one twist operator does not involve left-right mixing, and combining them in this way will not introduce such mixing. It would be valuable to explore this further and compare it with the work in [19] to better understand the effect of multiple twist operators in the continuum limit.

The second limit we consider is where the first and second twists are close together, the third and fourth twists are also close together, with a large distance between the second and third twists. Additionally, the mode numbers involved are relatively low. Naively, in the limit where the separation between the second and third twists is large, only the lowest possible energy state between them contributes. Since this state is generated by the first two twists, there is no left-right mixing. Similarly, because the final state is produced from this intermediate state through two more twists, no such mixing should occur. Therefore, at leading order in the large-distance limit, we would expect no left-right mixing. However, our analysis shows that such mixing does occur at leading order in this limit. This suggests that the nontrivial monodromy effect, and thus the left-right mixing, is a low-energy phenomenon that cannot be ignored even when the twists are far apart.

There are several directions for further exploration. First, investigating the continuum limit and comparing it with the findings in [19], to gain a better understanding of the effects of multiple twist operators in the continuum limit. Second, extending the computation to supersymmetric cases, particularly in the D1D5 CFT, by incorporating fermions. Third, in the D1D5 system, the twist-2 operator is part of the marginal deformation that moves the theory away from the free point towards the supergravity regime in moduli space. Understanding the role of this left-right mixing in this process would be interesting.

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A Bogoliubov transformation with left-right mixing

The Bogoliubov transformation is a linear transformation that mixes creation and annihilation operators. For a simple example, see section 4 of [18]. In this appendix, we extend this example to include left-right mixing to help readers better understand the new effects explored in this paper.

To mimic the behavior of left- and right-moving bosons, we introduce two free bosons a and \bar{a} , each has its own set of creation and annihilation operators. These operators satisfy the following commutation relations

$$\begin{aligned} [a, a^\dagger] &= 1, \quad [\bar{a}, \bar{a}^\dagger] = 1, \\ [a, a] &= [a, \bar{a}] = [\bar{a}, \bar{a}] = [a, \bar{a}^\dagger] = 0 \end{aligned} \tag{A.1}$$

The corresponding A-vacuum is defined by

$$a|0\rangle_A = 0, \quad \bar{a}|0\rangle_A = 0 \tag{A.2}$$

Now, consider a linear transformation that mixes the creation and annihilation operators

$$\begin{aligned} a &= \alpha_1 b + \alpha_2 b^\dagger + \alpha_3 \bar{b} + \alpha_4 \bar{b}^\dagger \\ a^\dagger &= \alpha_1^* b^\dagger + \alpha_2^* b + \alpha_3^* \bar{b}^\dagger + \alpha_4^* \bar{b} \\ \bar{a} &= \bar{\alpha}_1 b + \bar{\alpha}_2 b^\dagger + \bar{\alpha}_3 \bar{b} + \bar{\alpha}_4 \bar{b}^\dagger \\ \bar{a}^\dagger &= \bar{\alpha}_1^* b^\dagger + \bar{\alpha}_2^* b + \bar{\alpha}_3^* \bar{b}^\dagger + \bar{\alpha}_4^* \bar{b} \end{aligned} \tag{A.3}$$

The new set of creation and annihilation operators satisfies the following commutation relations

$$\begin{aligned} [b, b^\dagger] &= 1, \quad [\bar{b}, \bar{b}^\dagger] = 1, \\ [b, b] &= [b, \bar{b}] = [\bar{b}, \bar{b}] = [b, \bar{b}^\dagger] = 0 \end{aligned} \tag{A.4}$$

These impose constraints on the transformation parameters given by

$$\begin{aligned} |\alpha_1|^2 - |\alpha_2|^2 + |\alpha_3|^2 - |\alpha_4|^2 &= 1 \\ |\bar{\alpha}_1|^2 - |\bar{\alpha}_2|^2 + |\bar{\alpha}_3|^2 - |\bar{\alpha}_4|^2 &= 1 \\ \alpha_1 \bar{\alpha}_2 - \alpha_2 \bar{\alpha}_1 + \alpha_3 \bar{\alpha}_4 - \alpha_4 \bar{\alpha}_3 &= 0 \\ \alpha_1 \bar{\alpha}_1^* - \alpha_2 \bar{\alpha}_2^* + \alpha_3 \bar{\alpha}_3^* - \alpha_4 \bar{\alpha}_4^* &= 0 \end{aligned} \tag{A.5}$$

The corresponding B-vacuum is defined by

$$b|0\rangle_B = 0, \quad \bar{b}|0\rangle_B = 0 \tag{A.6}$$

This B-vacuum differs from the A-vacuum. To find the relation between them, we apply the linear transformations (A.3) to the A-vacuum condition (A.2), which gives

$$\begin{aligned} (\alpha_1 b + \alpha_2 b^\dagger + \alpha_3 \bar{b} + \alpha_4 \bar{b}^\dagger) |0\rangle_A &= 0 \\ (\bar{\alpha}_1 b + \bar{\alpha}_2 b^\dagger + \bar{\alpha}_3 \bar{b} + \bar{\alpha}_4 \bar{b}^\dagger) |0\rangle_A &= 0 \end{aligned} \quad (\text{A.7})$$

The solution to these equations is

$$|0\rangle_A = e^{\frac{\gamma}{2}b^\dagger b^\dagger + \beta b^\dagger \bar{b}^\dagger + \frac{\delta}{2}\bar{b}^\dagger \bar{b}^\dagger} |0\rangle_B \quad (\text{A.8})$$

where the coefficients satisfy

$$\begin{aligned} \alpha_1 \gamma + \alpha_2 + \alpha_3 \beta &= 0, & \alpha_1 \beta + \alpha_3 \delta + \alpha_4 &= 0 \\ \bar{\alpha}_1 \gamma + \bar{\alpha}_2 + \bar{\alpha}_3 \beta &= 0, & \bar{\alpha}_1 \beta + \bar{\alpha}_3 \delta + \bar{\alpha}_4 &= 0 \end{aligned} \quad (\text{A.9})$$

This result illustrates that the A-vacuum, defined with respect to a, \bar{a} , is not the B-vacuum, defined with respect to b, \bar{b} .

Instead, the A-vacuum appears as an excited state containing pairs of b and \bar{b} bosons from the perspective of the B-vacuum. This phenomenon is analogous to the pair creation effect associated with twist operators. In this analogy, the B-vacuum represents the state before the application of the twist operator, while the A-vacuum corresponds to the state after its application. As evident from (A.8), the linear transformation that mixes the left and right moving bosons also induces pair creation involving these modes. This is the left-right mixing in the ansatz (2.17). In the following, we will demonstrate that this mixing also affects contractions and propagations.

Let's now consider an excitation above the vacuum. Using the commutation relations (A.4), we obtain

$$\begin{aligned} a^\dagger |0\rangle_A &= [(\alpha_2^* \gamma + \alpha_4^* \beta) b^\dagger + (\alpha_2^* \beta + \alpha_4^* \delta) \bar{b}^\dagger] |0\rangle_A \\ \bar{a}^\dagger |0\rangle_A &= [(\bar{\alpha}_2^* \gamma + \bar{\alpha}_4^* \beta) b^\dagger + (\bar{\alpha}_2^* \beta + \bar{\alpha}_4^* \delta) \bar{b}^\dagger] |0\rangle_A \end{aligned} \quad (\text{A.10})$$

This result shows that an a -excitation (left moving) can give rise to both a b -excitation (left moving) and a \bar{b} -excitation (right moving), and similarly for a \bar{a} -excitation (right moving). These propagation coefficients can be identified as

$$\begin{aligned} f^{a^\dagger b^\dagger} &\equiv \alpha_2^* \gamma + \alpha_4^* \beta, & f^{a^\dagger \bar{b}^\dagger} &\equiv \alpha_2^* \beta + \alpha_4^* \delta \\ f^{\bar{a}^\dagger b^\dagger} &\equiv \bar{\alpha}_2^* \gamma + \bar{\alpha}_4^* \beta, & f^{\bar{a}^\dagger \bar{b}^\dagger} &\equiv \bar{\alpha}_2^* \beta + \bar{\alpha}_4^* \delta \end{aligned} \quad (\text{A.11})$$

which allows us to write

$$\begin{aligned} a^\dagger |0\rangle_A &= (f^{a^\dagger b^\dagger} b^\dagger + f^{a^\dagger \bar{b}^\dagger} \bar{b}^\dagger) |0\rangle_A \\ \bar{a}^\dagger |0\rangle_A &= (f^{\bar{a}^\dagger b^\dagger} b^\dagger + f^{\bar{a}^\dagger \bar{b}^\dagger} \bar{b}^\dagger) |0\rangle_A \end{aligned} \quad (\text{A.12})$$

Next, consider a state with two excitations

$$\begin{aligned}
a^\dagger \bar{a}^\dagger |0\rangle_A &= (\alpha_1^* b^\dagger + \alpha_2^* b + \alpha_3^* \bar{b}^\dagger + \alpha_4^* \bar{b})(\bar{\alpha}_1^* b^\dagger + \bar{\alpha}_2^* b + \bar{\alpha}_3^* \bar{b}^\dagger + \bar{\alpha}_4^* \bar{b})|0\rangle_A \\
&= (\alpha_1^* b^\dagger + \alpha_2^* b + \alpha_3^* \bar{b}^\dagger + \alpha_4^* \bar{b})(f^{\bar{a}^\dagger b^\dagger} b^\dagger + f^{\bar{a}^\dagger \bar{b}^\dagger} \bar{b}^\dagger)|0\rangle_A \\
&= ((f^{a^\dagger b^\dagger} b^\dagger + f^{a^\dagger \bar{b}^\dagger} \bar{b}^\dagger)(f^{\bar{a}^\dagger b^\dagger} b^\dagger + f^{\bar{a}^\dagger \bar{b}^\dagger} \bar{b}^\dagger) + C^{a^\dagger \bar{a}^\dagger})|0\rangle_A
\end{aligned} \tag{A.13}$$

Here, in addition to the propagation terms from (A.12), there is an extra term $C^{a^\dagger \bar{a}^\dagger}$, which accounts for the contraction of an a -excitation (left moving) and a \bar{a} -excitation (right moving), leading to their annihilation. Similar contractions can occur between two a -excitations or two \bar{a} -excitations. These contraction coefficients are defined as

$$\begin{aligned}
C^{a^\dagger \bar{a}^\dagger} &\equiv \alpha_2^* f^{\bar{a}^\dagger b^\dagger} + \alpha_4^* f^{\bar{a}^\dagger \bar{b}^\dagger} \\
C^{a^\dagger a^\dagger} &\equiv \alpha_2^* f^{a^\dagger b^\dagger} + \alpha_4^* f^{a^\dagger \bar{b}^\dagger} \\
C^{\bar{a}^\dagger \bar{a}^\dagger} &\equiv \bar{\alpha}_2^* f^{\bar{a}^\dagger b^\dagger} + \bar{\alpha}_4^* f^{\bar{a}^\dagger \bar{b}^\dagger}
\end{aligned} \tag{A.14}$$

For more excitations, the pattern follows similarly, using the three kinds of coefficients, contraction, propagation, and pair creation as described in section 7, where these coefficients are promoted to matrices with indices corresponding to the mode numbers.

In the model discussed in section 4 of [18], there is no left-right mixing, which is the case for one and two twist operators. This corresponds to taking $\alpha_3 = \alpha_4 = \bar{\alpha}_3 = \bar{\alpha}_4 = \beta = 0$ in the above example and for simplicity, only focusing on the unbarred bosons.

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