Regular Schwarzschild Black Hole in Modified Gravity

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Classical general relativity predicts a singularity at the center of a black hole, where known laws of physics break down. This suggests the existence of deeper, yet unknown principles of Nature. Among various theoretical possibilities, one of the most promising proposals is a transition to a de Sitter phase at high curvatures near the black hole center. This transition, originally proposed by Gliner and Sakharov, ensures the regularity of metric coefficients and avoids the singularity. In search for such a regular black hole solution with finite curvature scalar, we propose a metric function g_{rr} that exhibits a de Sitter-like core in the central region. An appealing feature of this metric is the existence of a *single* event horizon resembling the Schwarzschild black hole. Furthermore, the entire spacetime geometry is determined by the black hole mass alone, in agreement with the Isarel-Carter *no-hair theorem* for a charge-less, non-rotating black hole. To determine the gravitational action consistent with such a solution, we consider a general Lagrangian density $f(\mathcal{R})$ in place of the Einstein-Hilbert action. By numerically solving the resulting field equation, we find that, in addition to the Einstein-Hilbert term, a Padé approximant in the Ricci scalar $\mathcal{R}$ can produce such regular black hole solutions. To assess the physical viability of these black hole solutions, we verify that the proposed metric satisfies the principal energy conditions, namely, the dominant energy condition, weak energy condition, and null energy condition, throughout the entire spacetime. Furthermore, in agreement with Zaslavskii's regularity criterion, the metric satisfies the strong energy condition in the range $r \geq r_h/2$, where r_h is the event horizon.

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I. INTRODUCTION

Black holes (BHs) are among the many important and beautiful outcomes of the general theory of relativity. It also serves as a theoretical laboratory to inspect the fundamental behaviour of the Nature. The first BH solution, the Schwarzschild solution [1], came

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as a unique static vacuum solution to Einstein's field equations, $G_{ab} = 8\pi T_{ab} = 0$. This solution gives the $4D$ metric $ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2$, with $F(r) = (1 - 2m/r)$, for a black hole of ADM mass m, and $d\Omega^2$ is the line element on a 2-sphere of unit radius. Although this metric is the first solution of the Einstein field equations describing the spacetime due to a massive gravitating body, it also gives rise to the fundamental question of what happens near the central singularity, as the Kretschmann scalar $R_{abcd}R^{abcd}$ diverges in an unbounded manner like r^{-6} as $r \to 0$.

The divergence of the curvature scalar and the consequent breakdown of general relativity indicates some more rudimentary properties of a BH. One simple proposition is the transition, at some scale, of the spacetime itself to a de Sitter type geometry, with the hope that this core might help in regulating the divergence of the curvature scalar by giving some finite upper bound.

This possibility required replacing the metric coefficient g_{rr} of a BH with a de Sitter core at the centre with an eqaution of state $\rho = -p$, as first suggested by Gliner and Sakharov [2, 3]. Black holes with unique properties, such as, non-divergent metric coefficients and regular Kretschmann scalar as $r \to 0$, are generally known as regular black holes.

In this context, Wenda and Zhu [4] proposed that the exterior geometry of a Schwarzschild BH can be joined with a singularity-free de Sitter interior at the event horizon. But this direct matching of a de Sitter core with a exterior vacuum can never be carried out exactly at the horizon, as this will violate the continuity of pressure condition [5], $T^{ab}N_b = 0$, with N_b the normal to the boundary. However, although matching is not allowed at the horizon, but in principle this is allowed elsewhere [6].

Since the known laws of fundamental physics break down near the central singularity, it gives a possible indication that quantum effects might start dominating in the region near $r \approx 0$ resulting in a singularity-free spacetime geometry. Quantum fluctuations might induce nonlinear curvature $\mathcal{R}$ terms in the effective gravitational Lagrangian, that might remove the singularity. In addition to having a de Sitter type core in the centre, we hypothesize that modification of the Einstein-Hilbert action by replacing the Ricci scalar $\mathcal{R}$ with a functional $f(\mathcal{R})$, can result in a regular BH. In this study we adopt these scenario in search for a regularized black hole solution in $f(\mathcal{R})$ gravity.

The rest of the paper is organized as follows. In Section II, a general background of regular BHs is given, that also includes the required regularity conditions on the black hole metric. Section III gives the proposed model for regular BHs and obtains its equivalent $f(\mathcal{R})$ gravity model. Furthermore, the energy conditions of the present model are discussed in Section IV. Finally, Section V concludes the paper with a discussion.

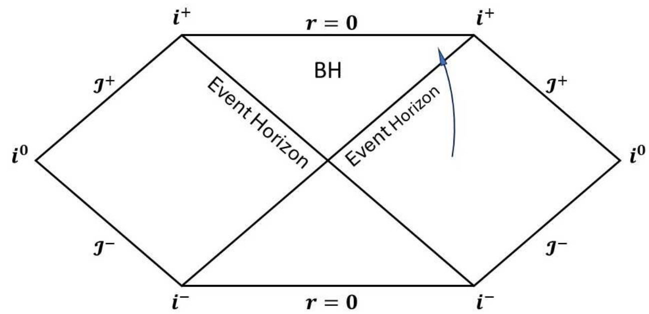

FIG. 1. Penrose diagram with a single event horizon representing the regularized Schwarzschild BH without a singularity at $r = 0$. The curve with an arrow indicates the motion of a timelike particle.

II. BACKGROUND

A. Regular Black Holes

The first regular BH solution without any central singularity was proposed by Bardeen [7], with redshift function

$$F(r) = 1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} \ , \tag{1}$$

where q is a magnetic charge and m the ADM mass of the BH.

Another regular BH solution of high importance was introduced by Hayward [8] by considering the Einstein tensor to be proportional to the square of the curvature. A detailed discussion on regular BH can be found in [9].

Bardeen's regular solutions can be found as a solution of Einstein gravity coupled with nonlinear electrodynamics (NLED) in the presence of an electromagnetic charge q [10, 11], where the central singularity is avoided by the self gravitating magnetic field.

A general procedure of constructing a regular BH solution, in the presence of a magnetic or electric charge, by considering a coupling of the gravitational field with NLED has been offered in [12–14]. In these analyses, a two-parameter family of BH solutions were obtained, and by choosing a particular region of the parameter space, the singularity at the center was removed. This method for obtaining regular BHs can be generalized to include the cosmological constant and asymptotic AdS geometry.

The above formalism can be extended further in the context of modified theories of gravity and NLED. Without specifying any definitive form for $f(\mathcal{R})$ and the NLED Lagrangian, regular solutions were obtained by choosing an appropriate form for the mass function $m(r)$, as was shown in [15]. Furthermore, such BHs have two horizons, namely an event horizon and a Cauchy horizon. Interestingly all energy conditions are satisfied throughout the spacetime, except for the strong energy condition, which is violated near the Cauchy horizon. Fortunately, it has been shown recently that, in general, the spacetime structure of a static, spherically symmetrical, regular BH violates the strong energy condition in any region inside the event horizon, in such a way that the Tolman mass becomes negative [16].

Some of the other static regular BH solutions with electric and magnetic charge can be found in [17–27], and with rotation in [28–33] and some on alternative theories of gravity in [34–40].

B. Conditions on the redshift function

We shall consider a static, spherically symmetric black hole spacetime with the metric given by

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad (2)$$

where $F(r)$ is designated as the redshift function. A surface of area $4\pi r^2$ is trapped if $F(r) < 0$ and untrapped if $F(r) > 0$, and the horizon of the BH is defined as the root of $F(r = r_h) = 0$. The trapping horizons, in this case also the killing horizons, are located at $r = r_h$.

In order that the spacetime is asymptotically Minkowskian, we require

$$F(r) \to 1 \quad \text{as} \quad r \to \infty \ . \quad (3)$$

Since we are interested in regularising a Schwarzschild BH, we further require a Schwarzschild-like spacetime for $r \gg r_h$, that is

$$F(r) \sim 1 - \frac{2m}{r} \quad \text{for} \quad r \gg r_h, \quad (4)$$

where m is the ADM mass of the BH.

Furthermore, following Gliner, Sakharov, Bardeen, and Hayward [2, 3, 7, 8], we regularize the BH interior with a de Sitter core, requiring

$$F(r) \to 1 - \frac{1}{3}\Lambda r^2 \quad \text{as} \quad r \to 0 \ , \quad (5)$$

giving a non-divergent curvature at the centre of the BH.

A red shift function $F(r)$, satisfying the conditions 3, 4, and 5, is expected to give a regular Schwarzschild-like BH spacetime with ADM mass m.

III. MODEL FOR A REGULAR SCHWARZSCHILD BLACK HOLE

A. Regularisation of the singularity

We regularize the BH spacetime by considering a simple model for the redshift function $F(r)$, written as

$$F(r) = 1 - \frac{2mr^2}{(r+l)^3} \ . \quad (6)$$

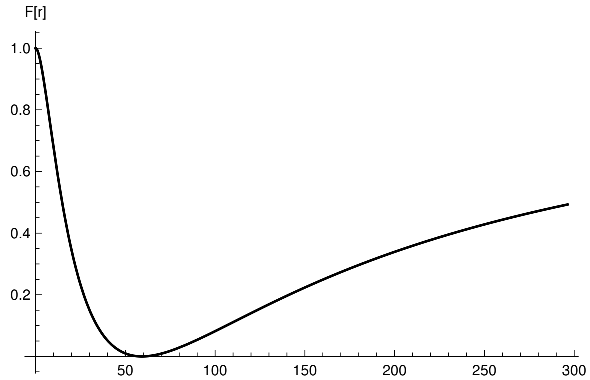

FIG. 2. The red shift function $F(r)$ given by equation 6 for $m = 100$, showing that $F(r) = 0$ at the horizon $r = 2l = \frac{16}{27}m = 59.2593$ of the regularised Schwarzschild BH. All numerical values are in Planck units, here and henceforth.

Clearly, this function satisfies all three conditions 3, 4, and 5, in particular

$$\begin{aligned} F(r) &\sim 1 - \frac{2m}{r} \quad \text{for} \quad r \gg r_h, \\ &\sim 1 - \frac{1}{3}\Lambda r^2 \quad \text{for} \quad r \ll l, \end{aligned} \quad (7)$$

where $\Lambda = 6m/l^3$.

The horizons are given by $F(r_h) = 0$, giving three roots. Upon requiring all three roots to be real, it turns out that one root is negative and other two roots are equal and positive for

$$l = \frac{8}{27}m \ , \quad (8)$$

defining a *single* event horizon at radius

$$r_h = 2l, \quad (9)$$

the *single* event horizon being a desired property of the spacetime that can mimic the *Schwarzschild black hole*.

Moreover, the metric parameter l being related with the mass m via 8, the entire spacetime geometry is determined by the mass of the black hole alone. This is in agreement with the Isarel-Carter *no-hair theorem* [41–43] for a charge-less, non-rotating black hole, such as the Schwarzschild black hole.

Figure 1 displays the Penrose diagram for such a spacetime with a single event horizon representing the regularised Schwarzschild BH without a singularity at $r = 0$.

Figure 2 displays the profile of the redshift function $F(r)$, where the zero of the function defines the black hole horizon.

The metric 2, with the red shift function 6, yields the Ricci scalar $\mathcal{R}$ as

$$\mathcal{R} = \frac{24l^2m}{(r+l)^5} \quad (10)$$

and the Kretschmann scalar

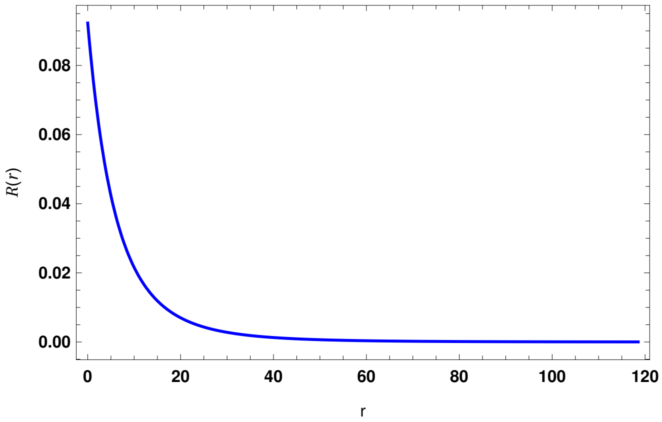

FIG. 3. Radial profile of the Ricci scalar $\mathcal{R}(r)$ given by equation 10 for $m = 100$. At the origin $r = 0$, the Ricci scalar has a finite value $\frac{24m}{l^3} = 24 \times (\frac{27}{8})^3 \frac{1}{m^2} = 0.09226$, indicating a regular spacetime inside ($r < 2l = 59.2593$) of the regularised Schwarzschild BH.

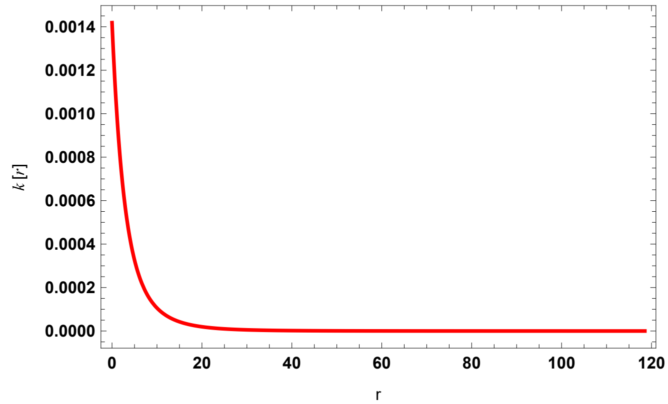

FIG. 4. Radial profile of the Kretschmann scalar $\mathcal{K}(r)$ given by equation 11 for $m = 100$. At the origin $r = 0$, the Kretschmann scalar has a finite value $\frac{96m^2}{l^6} = 96 \times (\frac{27}{8})^6 \frac{1}{m^4} = 0.001419$, indicating a regular spacetime inside ($r < 2l = 59.2593$) of the regularised Schwarzschild BH.

$$\mathcal{K} = R_{abcd}R^{abcd} = 48m^2 \frac{(2l^4 + 7l^2r^2 - 2lr^3 + r^4)}{(r+l)^{10}} \, . \tag{11}$$

Thus the Ricci scalar $\mathcal{R}$ as well as the Kretschmann scalar $\mathcal{K}$ are finite and regular upon approaching the center, $r \to 0$. These features indicate regularity of the BH spacetime at the center. Moreover, we have $\mathcal{R} \to 0$ and $\mathcal{K} \to 0$ as $r \to \infty$, showing asymptomatic flatness of the spacetime. Figures 3 and 4 illustrate these features clearly.

The metric 2, with the red shift function 6, yields the components of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$, as

$$\begin{aligned} G^t_t &= G^r_r = -\frac{6lm}{(r+l)^4} \, , \\ G^\theta_\theta &= G^\phi_\phi = \frac{6lm(r-l)}{(r+l)^5}, \end{aligned} \tag{12}$$

showing that all the coefficients falls off very rapidly, $\mathcal{O}(r^{-4})$ for $r \gg l$.

These components of the Einstein tensor can be used to relate with the components of the effective energy-momentum tensor Θ_{ab}, defined by $G_{ab} = \Theta_{ab}$. Thus, the energy density $-\Theta^t_t$ is given by

$$-G^t_t = -\Theta^t_t = \frac{6l}{m^{1/3}} \left(\frac{E(r)^{\frac{1}{3}}}{r} \right)^4 \tag{13}$$

and

$$g^{rr} = 1 - \frac{2E(r)}{r} \, , \tag{14}$$

where the energy $E(r)$ is defined by

$$E(r) = \frac{m}{(1 + l/r)^3} \, . \tag{15}$$

Equations 10, 13 and 15 suggest that $G^t_t \sim \mathcal{R}^{4/5}$. Thus assuming $G^{ab} = \Theta^{ab} \sim \mathcal{R}^{4/5}$, it can be shown that the component $dE/dr = -(1/2)r^2\Theta^t_t$ of the Einstein equations yields a simple singularity-free solution for the metric coefficient g^{rr} in the form 6 with appropriate boundary conditions.

B. The equivalent $f(\mathcal{R})$ gravity model

In this work, we shall be interested in constructing an equivalent $f(\mathcal{R})$ theory of gravity in the Jordan frame that mimics the above regularized spacetime geometry.

Consequently, we begin by considering the general action

$$\mathcal{A} = \frac{1}{2}M_p^2 \int d^4x\sqrt{-g}f(\mathcal{R}), \tag{16}$$

where $g = \det[g_{ab}]$, g_{ab} being the metric of the spacetime, given by equation 2. We shall take the regularised red shift function $F(r)$ defined in equation 6.

Extremum of the action 16 yields the modified field equation as

$$f'(\mathcal{R})R_{ab} - \frac{1}{2}f(\mathcal{R})g_{ab} = \left[\nabla_a\nabla_b - g_{ab}\Box\right] f'(\mathcal{R}), \tag{17}$$

where $f'(\mathcal{R}) = \frac{df}{d\mathcal{R}}$.

The Ricci tensor R_{ab} can be obtained from equation 17 in the form

$$R_{ab} = \frac{1}{f'(\mathcal{R})}\left[\frac{1}{2}f(\mathcal{R})g_{ab} + \left[\nabla_a\nabla_b - g_{ab}\Box\right] f'(\mathcal{R})\right] \, , \tag{18}$$

which is the Ricci scalar in vacuum in the $f(\mathcal{R})$ model.

On the other hand, when we regularised the BH metric in the previous section, we obtained the field equation as $R_{ab} - \frac{1}{2}\mathcal{R}g_{ab} = \Theta_{ab}$, or equivalently

$$R_{ab} = \Theta_{ab} - \frac{1}{2}\Theta g_{ab} \, . \tag{19}$$

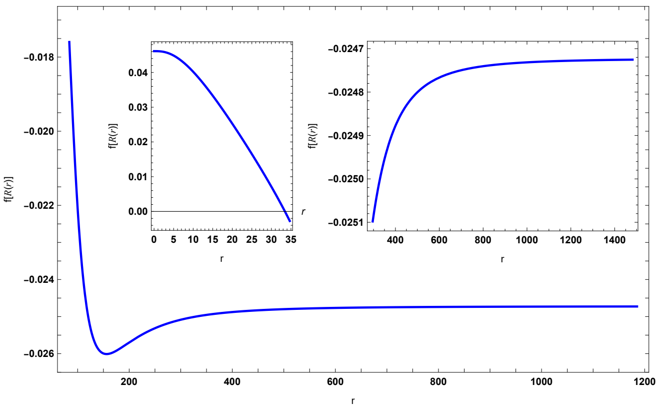

FIG. 5. Radial profile of the Lagrangian $f[\mathcal{R}(r)]$ in the modified gravity action 16, obtained by numerical integration of the differential equation 24 with initial conditions $f[\mathcal{R}(r=0)] = \frac{12m}{l^3} = 12 \times (\frac{27}{8})^3 \frac{1}{m^2} = 0.046132$ for $m = 100$ and $f'[\mathcal{R}(r=0)] = 0$. The first inset shows its behaviour for small radial coordinates where it crosses over to negative values at $r \approx 33.5$, which is slightly higher than $l = \frac{8}{27}m = 29.6296$. The second inset shows that it approaches zero for large radial coordinates.

Equation 18 represents the Ricci tensor of the vacuum spacetime in $f(\mathcal{R})$ gravity. Consequently, in order to find an equivalent $f(\mathcal{R})$ representation for the regularised BH, we obtain from equation 19,

$$\frac{1}{f'(\mathcal{R})}\left[\frac{1}{2}f(\mathcal{R})g_{ab}+\left[\nabla_a\nabla_b - g_{ab}\Box\right]f'(\mathcal{R})\right] = \Theta_{ab}-\frac{1}{2}\Theta g_{ab} \,. \tag{20}$$

Taking trace on both sides, and using the fact that $\mathcal{R} = -\Theta$, we have

$$\Box f'(\mathcal{R}) = \frac{1}{3}\Big[2f(\mathcal{R}) - \mathcal{R}f'(\mathcal{R})\Big] \,. \tag{21}$$

Substitution of equation 21 in equation 17 gives

$$f'(\mathcal{R})R_{ab} = \frac{1}{3}g_{ab}\Big[\mathcal{R}f'(\mathcal{R})-\frac{1}{2}f(\mathcal{R})\Big]+\nabla_a\nabla_b f'(\mathcal{R}), \tag{22}$$

Equation 10 expresses the Ricci scalar as a function of the radial coordinate, $\mathcal{R}(r)$. We can thus express the derivative as

$$f'[\mathcal{R}(r)] = \frac{df}{d\mathcal{R}} = -\frac{(r+l)^6}{120l^2m}\,\frac{df}{dr} \,. \tag{23}$$

Consequently, the "$\theta\theta$" or "$\phi\phi$" component of equation 22 leads to

$$A(r)\frac{d^2 f}{dr^2} + B(r)\frac{df}{dr} + C(r)f = 0, \tag{24}$$

with

$$\begin{aligned} A(r) &= (r+l)^6 F(r) \,, \\ B(r) &= 2lmr(r+l)(3r-l) + 6(r+l)^5 F(r) \,, \\ C(r) &= -20l^2mr \,, \end{aligned} \tag{25}$$

where $F(r)$ is the regularised red shift function given by equation 6.

C. Numerical Integration

We numerically integrate the differential equation 24 with initial conditions $f[\mathcal{R}(r=0)] = \frac{12m}{l^3}$ and $f'[\mathcal{R}(r=0)] = 0$ as appropriate for our model in order to obtain $f[\mathcal{R}(r)]$ as a function of the radial coordinate r in the range $r \in [0,\infty)$.

Figure 5 shows the radial profile of $f[\mathcal{R}(r)]$. It is clear from the plot that in the vicinity of the origin, $r \approx 0$, the behaviour of $f(\mathcal{R})$ differs significantly from Einstein gravity $f(\mathcal{R}) = \mathcal{R}$, and it asymptotically approaches zero at long distances. Therefore, our choice of the redshift function $F(r)$ given by 6 modifies the internal structure of the BH appropriately, by replacing the central singularity at $r = 0$ with a de Sitter core while keeping the long distance ($r \gg 2l$) behaviour the same as that of a Schwarzschild BH with ADM mass m.

To find the dependency of $f[\mathcal{R}]$, shown in figure 5, as a function of the Ricci scalar $\mathcal{R}$, we obtain the best fit by using the Padé approximant, given by

$$f[\mathcal{R}] = a_0\mathcal{R} + \frac{a_0 - a_1 r + a_2 r^2 + a_3 r^3 - a_4 r^4}{1 - b_1 r + b_2 r^2 - b_3 r^3 + b_4 r^4} \,, \tag{26}$$

where the radial coordinate is expressed as a function of $\mathcal{R}$ by using equation 10 as

$$r = \left(\frac{24l^2m}{\mathcal{R}}\right)^{1/5} - l \,, \tag{27}$$

with $a_0 \gg a_1 \gg \cdots \gg a_4$, and $b_1 \gg b_2 \gg \cdots \gg b_4$, and $a_i, b_i > 0$. The first term on the RHS in equation 26 implies the Einstein-Hilbert Lagrangian density with a constant prefactor.

D. Solution near the origin

Analytical solution of equation 24 near the origin $r = 0$ can be found by expanding its coefficients given by equations 25 in powers of r/l. Upon expanding up to order $\mathcal{O}(r/l)$, we obtain the solution as

$$f(x) = \frac{12m}{C_3}\exp\left[-x\left(6 - \frac{10m}{bl} + bx\right)\right]\left(C_1\mathbf{H}_\lambda\left[\frac{c+b^2lx}{b^{3/2}l}\right] + C_2\mathbf{F}_1^1\left[-\frac{\lambda}{2},\frac{1}{2},\frac{(c+b^2lx)^2}{b^3l^2}\right]\right) \tag{28}$$

where $x = r/l$, with $\mathbf{H}_\lambda(z)$ the Hermite polynomial of order λ and $\mathbf{F}^1_1(i;j;z)$ is the Kummer's function of the first kind. The constants $\lambda, C_1, C_2, C_3, b$ and c are given by

$$
\begin{aligned}
&\lambda = \frac{a}{b^2 l^2} \ , \\
&C_1 = b^{3/2}\left[\kappa \mathbf{F}^1_1\left[1-\frac{\lambda}{2};\frac{3}{2};\frac{c^2}{b^3l^2}\right] - b^3l^2(3bl-5m)\mathbf{F}^1_1\left[-\frac{\lambda}{2};\frac{1}{2};\frac{c^2}{b^3l^2}\right]\right] , \\
&C_2 = l\left[b^8\left(b^3l^2+30blm-50m^2\right)\mathbf{H}_{\lambda-1}\left[\frac{c}{b^{3/2}l}\right] + b^{19/2}l\left(3bl-5m\right)\mathbf{H}_\lambda\left[\frac{c}{b^{3/2}l}\right]\right], \\
&C_3 = l^3\Big(b^3l^2+30blm-50m^2\Big)\left[b^{13/2}(3bl-10m)\mathbf{H}_\lambda\left[\frac{c}{b^{3/2}l}\right]\mathbf{F}^1_1\left[1-\frac{\lambda}{2};\frac{3}{2};\frac{c^2}{b^3l^2}\right] + b^8 l\mathbf{H}_{\lambda-1}\left[\frac{c}{b^{3/2}l}\right]\mathbf{F}^1_1\left[-\frac{\lambda}{2};\frac{1}{2};\frac{c^2}{b^3l^2}\right]\right], \\
&b = m-3, \\
&c = -9l-10m+3lm,
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
a &= 27l^2+90lm-27l^2m+50m^2-30lm^2+9l^2m^2-l^2m^3, \\
\kappa &= 3l^3b^4-10l^2(m-12)b^2m-450lbm^2+500m^3 \ .
\end{aligned}
\tag{30}
$$

It can be verified from 28 that $f(x=0) = \frac{12m}{l^3}$ and $f'(x=0) = 0$, consistent with our initial conditions.

IV. ENERGY CONDITIONS

We further investigate if the metric 2 with the red-shift function 6 satisfies the four energy conditions: the Weak Energy Condition (WEC), Null Energy Condition (NEC), Strong Energy Condition (SEC) and Dominant Energy Condition (DEC) [44–46].

Consider a timelike four velocity t^μ and null-like four velocity n^μ. For a stationary observer in the comoving frame $t^\mu = [1/\sqrt{F}, 0, 0, 0]$ and for a null radial trajectory $n^\mu = [1, F, 0, 0]$, satisfying $t^\mu t_\mu = -1$ and $n^\mu n_\mu = 0$. Then using the methodology shown in [47], we have for SEC

$$
R_{\mu\nu}t^\mu t^\nu = \frac{6lm(r-l)}{(r+l)^5} \ \geq 0, \quad \forall \ r \in [l,\infty) \tag{31}
$$

which through the Raychaudhuri equation suggests that the proposed gravity model is attractive. Clearly the SEC is satisfied only in the range $r \in [l,\infty)$. However, the SEC is violated in the range $r \in [0,l]$, which is consistent with the findings of [16], which states that for any static, spherically symmetric regular BH, the SEC is violated for any region inside the event horizon.

On the other hand, the WEC is satisfied in the entire region of the spacetime,

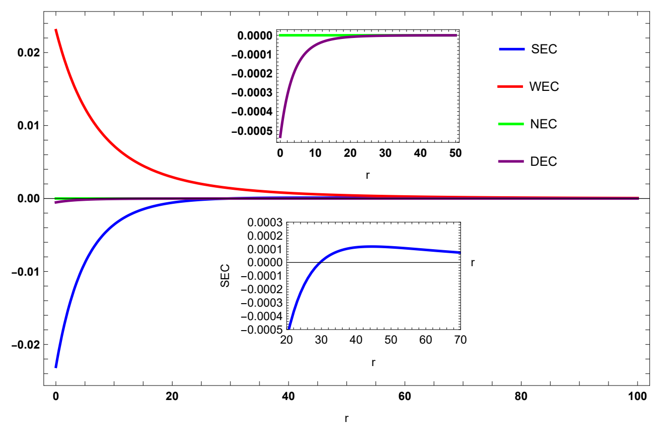

FIG. 6. Radial profiles of SEC, WEC, DEC, NEC given by equations 31, 32, 33, 34 with the regularised red shift function 6 for $m = 100$.

$$
\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]t^\mu t^\nu = \frac{6lm}{(r+l)^4} \ \geq 0, \quad \forall \ r \in [0,\infty) \ , \tag{32}
$$

indicating that the energy density measured by any observer is always non-negative, in agreement with Hawking and Ellis [48].

The DEC is also satisfied in the entire region of the spacetime,defining a single event

$$
\begin{aligned}
&\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\left[R_{\lambda\rho} - \frac{1}{2}Rg_{\lambda\rho}\right]g^{\nu\rho}t^\mu t^\lambda \\
&\qquad = -\left[\frac{6lm}{(r+l)^4}\right] \leq 0 \ , \quad \forall \ r \in [0,\infty) \ ,
\end{aligned}
\tag{33}
$$

which implies non-negative energy density in addition of having a local energy flow vector which is non-spacelike.

Furthermore, the NEC is satisfied throughout the spacetime,

$$\Big[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big]n^{\mu}n^{\nu} = 0 \ , \ \forall \ r \in [0, \infty) \ , \tag{34}$$

having its fundamental importance in signifying boundedness from below of the corresponding Hamiltonian.

Figure 6 illustrates radial profiles of the above energy conditions 31, 32, 33, and 34 derived from the red shift function $F(r)$ given by equation 6. The figure clearly shows that WEC, NEC and DEC are satisfied throughout the spacetime, $r \in [0, \infty)$. On the other hand, SEC holds only in the range $r \in [l, \infty)$, whereas its violation in the range $r \in [0, l]$ is acceptable as it occurs inside the event horizon [16].

V. DISCUSSION AND CONCLUSION

In this work, we explored a regular black hole metric characterized by a non-diverging metric coefficients at $r = 0$, as described by Equation 6. This specific metric structure exhibits a de Sitter-like core at small radial distances, with an effective cosmological constant given by $\Lambda = 6m/l^3$. At larger distances, the metric smoothly transitions into a Schwarzschild-like geometry, ensuring compatibility with well-established results in the weak-field limit.

An attractive feature of this metric model is the existence of a *single* event horizon resembling the Schwarzschild black hole. Furthermore, the metric parameter l is found to be related with the mass m of the black hole, via 8. This is in agreement with the Isarel-Carter *no-hair theorem* for a charge-less, non-rotating black hole.

One of the crucial aspects of constructing a physically meaningful regular black hole solution is ensuring that all curvature invariants remain finite throughout the spacetime. In our model, we verified that both the Ricci scalar $\mathcal{R}$ and the Kretschmann scalar $R_{abcd}R^{abcd}$ remain well-behaved and finite for all $r \in [0, \infty)$. This confirms that the proposed metric successfully removes the central singularity, replacing it with a regular core consistent with fundamental principles of physics.

To determine the underlying gravitational action that gives rise to such a regular black hole solution, we adopted a general $f(\mathcal{R})$ gravity framework, in place of the Einstein-Hilbert action. By demanding that the gravitational field equations arising from this action leads to our regular black hole metric in vacuum, we obtained a differential equation (24) that the function $f(\mathcal{R})$ must satisfy to generate such a regular black hole solution.

A significant outcome of this analysis is the specific form of $f(\mathcal{R})$, expressed as a Padé approximant in the Ricci scalar. Our numerical results, presented in Figure 5, confirm that the function $f(\mathcal{R})$ starts from a de Sitter core at small r and smoothly transitions to an asymptotically flat spacetime. A best-fit analysis reveals that the leading-order term in $f(\mathcal{R})$ corresponds to the Einstein-Hilbert Lagrangian density $\mathcal{R}$, consistent with the fact that general relativity remains valid asymptotically in the large distance limit.

Apart from constructing a regular black hole solution, it is essential to verify the energy conditions to assess physical viability of such black holes. We therefore confirmed that the proposed regular metric satisfies the principal energy conditions. Our numerical analysis, illustrated in Figure 6, shows that the weak energy condition (WEC), the null energy condition (NEC), the dominant energy condition (DEC) hold throughout the entire spacetime. The strong energy condition (SEC) is satisfied in the range $r \geq r_h/2$, which is consistent with Zaslavskii's criterion, that confirms that SEC can be violated anywhere inside the event horizon for regular black holes [16]. These energy conditions further support the physical credibility of our regularized metric model.

Overall, our findings demonstrate that it is possible to construct a physically viable regular black hole solution with a *single* event horizon resembling the Schwarzschild BH with a de Sitter core in the $f[\mathcal{R}]$ gravity framework. This approach not only eliminates the central singularity but also ensures consistency with energy conditions, as well as agreement with the Isarel-Carter *no-hair theorem*.

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