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Packaged Quantum States and Symmetry: A Group-Theoretic Framework for Gauge-Invariant Packaged Entanglements

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Abstract

We present a group-theoretic framework showing that any nontrivial representation of a finite or compact group inherently induces packaged entanglement (in which every internal quantum numbers are inseparably entangled) in multiparticle quantum systems. In this framework, each single-particle excitation carries an inseparable block of internal quantum numbers dictated by its irreducible representation. The local gauge constraints or superselection rules forbid any partial fractionation of charges or quantum labels. We illustrate this principle through gauge symmetries ($U(1)$, $SU(2)$, and $SU(3)$), discrete symmetries (charge conjugation C, parity P, time reversal T, and their combinations), and p-form symmetries. In each case, physical requirements (e.g., gauge invariance or superselection) ensure that the resulting states cannot be factorized. These naturally explain phenomena like Bell-type structures, color confinement, and hybrid gauge-invariant configurations. The packaging principle bridges concepts in gauge theory, quantum information, and topological classifications. Our results may be useful for applications in exotic hadron spectroscopy, the exploration of extended symmetries in field theory, and quantum technologies.

Contents

1 Introduction

Recently, we proposed **packaged entangled states** in which the **internal quantum numbers (IQNs)** (color, flavor, electric charge, or other discrete labels) must inseparably entangled.[1, 2] When tracing back to quantum field excitation, we find that the packaged entanglement is caused by local gauge invariance [3, 4, 5, 6] and superselection rules [7, 8, 9, 10]. Even in single-particle level, whenever a particle (antiparticle) is created, its IQNs are locked in indivisible, **irreducible-representation (irrep)** blocks [11, 12]. For example, in QCD color degrees of freedom among quarks are locked together into color-singlet bound states [13, 14, 15], ensuring that no partial color can be factorized from a hadron's wavefunction. When extending this to multi-particle systems, the single-particle irrep blocks can form packaged entangled states in a single charge sector without breaking the superselection rules. Analogous constraints arise when $\mathbb{Z}_2$ symmetries [16, 17], $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, or $\hat{C}\hat{P}\hat{T}$ operations [18] act on multiparticle systems. Superselection rules [7, 8, 9] fix the allowed sectors and enforces Bell-like [19] or GHZ-like [20, 21, 22, 23] superpositions in which internal charges or quantum labels must remain inseparably paired.

On the other hand, symmetry governs conservation laws [24] and dictates which interactions and states are possible.[4, 12, 25, 26, 27, 28, 29] Therefore, symmetry is very important in modern theoretical physics. For example, in quantum field theory (QFT) local gauge invariance enforces that physical states lie in specific charge or color sectors.[13, 14, 15, 30, 31] This confines quarks into color-neutral hadrons or organizes excitations by total electric charge. Discrete symmetries [18] such as $\hat{C}$ (charge conjugation), $\hat{P}$ (parity), and $\hat{T}$ (time reversal) further constrain how external degrees of freedom (e.g., spin, momentum) couple to internal quantum labels (e.g., color, flavor). This often yields states that exhibit robust forms of entanglement.

In this work, we present a unified group-theoretic framework demonstrating that any nontrivial representation [12, 17] (i.e., dimension > 1) of a finite or compact group G inevitably induces packaged entanglement. Each single-particle excitation carries an irreducible block of internal quantum numbers, forbidding partial fractionation of charges. Combining multiple excitations then forces the total state to reside in a single net-charge (or net-color) sector, leading to nonfactorizable, gauge-invariant entangled states.

We illustrate this principle through a range of examples: (1) Discrete $\mathbb{Z}_2$ Symmetries [16, 17]: Charge conjugation pairs particles and antiparticles, yielding Bell-type superpositions, while parity or time reversal flips momentum or spin, creating inseparable bipartite states. (2) Non-Abelian Symmetries [4] (e.g., $SU(2), SU(3)$): Confinement in QCD [13, 14, 15] enforces that quarks in $\mathbf{3}$ must combine to form color-singlet hadrons ($\mathbf{1}$), entangling their color degrees of freedom. Likewise, flavor or isospin irreps in $SU(2)$ or $SU(3)$ remain locked into irreducible multiplets. (3) Extended or Higher-Form Symmetries [32, 33]: In confining lattice gauge theories, line or surface operators carrying discrete or continuous flux also become irreducible objects that cannot be fractionated, generalizing the packaging principle to vortex lines and domain walls.[34, 35, 36, 37, 38]

Using a representation-theoretic framework [16, 17], we show that once a system is restricted to fixed irreps (whether by superselection or local gauge constraints), the resulting wavefunctions necessarily exhibit entanglement patterns that are packaged into irreducible blocks of quantum numbers. Using the ideas of packaging and group theory, therefore, we can explain a variety of fundamental phenomena, such as color confinement [30, 31], discrete symmetry transformations, and topological excitations. It points toward potential applications in quantum information and advanced gauge-invariant technologies.

2 Packaging Principle and Group-Theoretic Foundations

It is important to understand the transformation properties of single-particle creation operators under a symmetry group, and therefore understand why internal quantum numbers (IQNs) appear as inseparable, packaged units. In this section, we explain single-particle states and then multi-particle systems by appealing to the irreducibility ensured by Schur's lemma [39, 40] and the Clebsch-Gordan decomposition [41, 42].

2.1 Single-Particle Irreps and the Packaging Principle

Let G be a compact Lie group representing a local gauge symmetry [43, 44]. In quantum field theory, each single-particle excitation (e.g. a quark or a gluon in QCD) is created by an operator

$$\hat{a}_\alpha^\dagger(\mathbf{p}),$$

where the index α labels the basis states of an irreducible representation (irrep) $\mathbf{R}_\alpha$ of G. For instance, in QCD with $G = \mathrm{SU}(3)$ quarks transform in the fundamental $\mathbf{3}$, antiquarks in the conjugate $\overline{\mathbf{3}}$, and gluons in the adjoint $\mathbf{8}$.

Under a gauge transformation $g \in G$, the creation operator transforms as

$$U(g)\,\hat{a}_i^\dagger\,U(g)^{-1} = \sum_{j=1}^{d_\alpha} \Big[D^{(\alpha)}(g)\Big]_{ij}\,\hat{a}_j^\dagger,$$

where $d_\alpha = \dim(\mathbf{R}_\alpha)$, $D^{(\alpha)}(g)$ is the $d_\alpha \times d_\alpha$ matrix of the group element g in the irrep $\mathbf{R}_\alpha$, and indices i, j run over the internal basis states of $\mathbf{R}_\alpha$. This ensures the operator $\hat{a}^\dagger$ carries a definite gauge-charge structure (color, flavor, etc.) consistent with the representation.

Because $\mathbf{R}_\alpha$ is irreducible, no proper, nonzero invariant subspace exists inside a single-particle operator's gauge-space. In other words, each creation operator transforms in an irreducible representation. By Schur's lemma, the IQNs of a single-particle operator cannot be decomposed into independent parts. They are "packaged" as a single, inseparable block. For example, a quark in the $\mathbf{3}$ cannot be split into sub-components of color, and the same holds for any other gauge charge.

Example 1 (Single-Particle Packaging under $\mathrm{SU}(N)$)**.** *For $G = \mathrm{SU}(N)$ a single-particle field operator belongs to an irrep labeled by a Young diagram. For example:*

- *In $\mathrm{SU}(3)$ color: a quark is in the $\mathbf{3}$, an antiquark in $\overline{\mathbf{3}}$, and a gluon in the $\mathbf{8}$.*
- *In $\mathrm{SU}(2)$ isospin: an isospin-$\frac{1}{2}$ particle is in the doublet $\mathbf{2}$, while isospin-1 states form the triplet $\mathbf{3}$.*

In each case, the representation is irreducible and the associated quantum numbers appear as a complete unit.

2.2 Multi-Particle States and Clebsch-Gordan Decomposition

Once the single-particle irreps $\mathbf{R}_\alpha$ are defined, we can form a multi-particle state by taking the tensor product of those irreps

$$\mathbf{R}_{\alpha_1} \otimes \mathbf{R}_{\alpha_2} \otimes \cdots \otimes \mathbf{R}_{\alpha_n}$$

The theory of representation then breaks this product into a direct sum of irreps

$$\mathbf{R}_{\alpha_1} \otimes \mathbf{R}_{\alpha_2} \otimes \cdots \otimes \mathbf{R}_{\alpha_n} = \bigoplus_\gamma \mathbf{R}_\gamma,$$

where the direct sum runs over all irreps $\mathbf{R}_\gamma$ that appear. This process is called the Clebsch-Gordan method [41, 42]. The physical consequence is that the IQNs become entangled. Only specific combinations (the singlet in a confining theory) can exist as free states.

In a confining theory such as QCD, only the singlet subrepresentation $\mathbf{1}$ appears as a free state. Non-singlet parts (like $\mathbf{8}$ in SU(3)) cannot exist as free states because of confinement, but they can appear in short-range interactions. In this way, packaged entangled states stay in a well-defined total representation $\mathbf{R}_\gamma$. In QCD color, we usually restrict to the color-singlet piece.

Example 2 (Decompositions of Color Structure)**.** *In QCD the color structure of hadrons is determined by these decompositions:*

- *Two Quarks:* $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$*. Here, the symmetric* $\mathbf{6}$ *and the antisymmetric* $\overline{\mathbf{3}}$ *represent the only possible ways to combine two* $\mathbf{3}$*'s.*
- *Quark-Antiquark:* $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$*. The singlet* $\mathbf{1}$ *is color-neutral and corresponds to the observed mesons, while the octet* $\mathbf{8}$ *remains confined.*
- *Larger Products: For a baryon (qqq), one further combines the diquark (typically in* $\overline{\mathbf{3}}$*) with another quark* $\mathbf{3}$ *to yield a singlet.*

For the convenience of later discussion, we now outline the Clebsch-Gordan procedure as follows:

1. Identify the highest weight (or use Young diagrams) for the irreps.
2. Combine the diagrams according to symmetry rules (symmetric or antisymmetric).
3. Use dimension formulas (e.g. the hook-length formula for SU(3)) to check consistency.
4. Decompose the tensor product into a direct sum, identifying each sub-irrep.

This decomposition shows that, due to the irreducibility of the single-particle states, any multi-particle wavefunction is necessarily entangled in its internal (gauge) degrees of freedom. For instance, the color wavefunction of a meson is not a simple product but an entangled state that projects onto the color-singlet subspace.

2.3 Theorem: Existence of Group Associated Packaged States

We now state a theorem about the existence of packaged states associate to a finite or compact gauge group G. We show that for any finite or compact group G, one may construct a Hilbert space of multi-particle excitations in which the internal quantum numbers are packaged into irreducible blocks.

Theorem 1 (Existence of Group Associated Packaged States)**.** *Let G be any finite or compact group (Abelian or non-Abelian) acting as a symmetry of a quantum physical system. Then there exists a Hilbert space $\mathcal{H}$ carrying a (unitary or projective) representation*

$$U : G \to \mathcal{U}(\mathcal{H})$$

with the following properties:

1. ***Existence of Single-Particle Packaged States:*** *Every single-particle creation operator $\hat{a}_\alpha^\dagger$ transforms irreducibly under G. Hence, its internal quantum numbers (e.g., electric charge, color, flavor) are locked together as a single, inseparable unit.*

2. ***Existence of Multi-Particle Packaged Product States:*** *The full Hilbert space (e.g., constructed as a Fock space) decomposes into charge sectors*

$$\mathcal{H} \cong \bigoplus_{Q \in \hat{G}} \mathcal{H}_Q, \tag{1}$$

   where each subspace $\mathcal{H}_Q$ comprises packaged states with fixed net charge (or total representation) Q and transforms according to a representation of G.

3. ***Existence of an Orthonormal Basis of Packaged Entangled States:*** *For any charge sector $\mathcal{H}_Q$ with $\dim \mathcal{H}_Q > 1$, there exists an orthonormal basis*

$$\mathcal{B}_Q = \Big\{ |\Psi_\alpha^Q\rangle \in \mathcal{H}_Q : \alpha = 1, 2, \ldots, \dim \mathcal{H}_Q, \quad \langle \Psi_\alpha^Q | \Psi_\beta^Q \rangle = \delta_{\alpha\beta} \Big\}, \tag{2}$$

   where each $|\Psi_\alpha^Q\rangle$ is a packaged entangled state in the sense that its internal quantum numbers appear as inseparably entangled. In particular, these states cannot be expressed as a simple tensor product in which the IQNs are partially factorized.

4. ***Module Isomorphism:*** *The action of G on each charge sector $\mathcal{H}_Q$ is isomorphic (as a G-module) to the unique abstract irreducible representation space V_Q associated with Q. That is, there exists an isomorphism*

$$\Phi_Q : \mathcal{H}_Q \to V_Q,$$

   satisfying

$$\Phi_Q\Big(U(g)\psi\Big) = \sigma_Q(g)\, \Phi_Q(\psi) \tag{3}$$

   for all $g \in G$ and $\psi \in \mathcal{H}_Q$, where $\sigma_Q : G \to \mathrm{GL}(V_Q)$ is the abstract irreducible representation corresponding to Q.

Proof. We prove the theorem in four steps.

1. **Existence of Single-Particle Packaged States:** By Wigner's theorem [42], any symmetry is represented by a (unitary or anti-unitary) operator. For finite or compact groups G, one may choose a unitary representation

$$U : G \to \mathcal{U}(\mathcal{H}).$$

Every finite-dimensional unitary representation of G is completely reducible. Hence, the single-particle Hilbert space $\mathcal{H}_1$ decomposes as

$$\mathcal{H}_1 \cong \bigoplus_{R \in \hat{G}} n_R \, V_R,$$

where each V_R is the carrier space for the irrep R and n_R is its multiplicity. Associating each creation operator $\hat{a}^\dagger_{R,i}$ (with label i) with the corresponding subspace V_R ensures that its IQNs appear as a single, inseparable block. Therefore, the single-particle decomposition and packaging confirm the existence of G-associated single-particle packaged states.

2. **Existence of Multi-Particle Packaged Product States:** Multi-particle states are constructed as tensor products of single-particle states. If the individual states transform in the irreps $R_1, R_2, \ldots, R_n$, then the combined n-particle state transforms according to the tensor product representation

$$R_1 \otimes R_2 \otimes \cdots \otimes R_n.$$

By applying the Clebsch-Gordan decomposition, this tensor product decomposes into a direct sum of irreps:

$$R_1 \otimes R_2 \otimes \cdots \otimes R_n \cong \bigoplus_{Q \in \hat{G}} N_Q \, \sigma_Q,$$

where N_Q is the multiplicity of the irrep σ_Q and Q labels the net charge. So the full Hilbert space decomposes as in Eq. (1), with each $\mathcal{H}_Q$ consisting of packaged states that transform irreducibly under G. Therefore, the Fock space decomposition and charge sectors confirm the existence of G-associated multi-particle packaged product states.

3. **Existence of an Orthonormal Basis of Packaged Entangled States:** A charge sector $\mathcal{H}_Q$ is indeed a Clebsch-Gordan subspace. Since $\mathcal{H}_Q$ is a finite-dimensional Hilbert space, it has a basis of linear independent packaged states, The basis includes both product states and entangled states. We want to convert the packaged product states into packaged entangled states to obtain a basis of all packaged entangled states for $\mathcal{H}_Q$.

   In local gauge theories, superselection rules forbid coherent superpositions of packaged states from different charge sectors $\mathcal{H}_Q$, but allow coherent superpositions of packaged states from the same charge sectors $\mathcal{H}_Q$. Therefore, we can safely use the packaged product states to construct packaged entangled states.

   Let us now takes a linear combination of tensor products in the charge sectors $\mathcal{H}_Q$:

$$|\Psi^Q_\alpha\rangle = \sum_{i_1,\ldots,i_n} c^{(\alpha)}_{i_1,\ldots,i_n} \, |\phi_{i_1}\rangle \otimes |\phi_{i_2}\rangle \otimes \cdots \otimes |\phi_{i_n}\rangle,$$

where each $|\phi_{i_k}\rangle$ is a single-particle state carrying a full, irreducible set of IQNs, and each $|\phi_{i_1}\rangle \otimes |\phi_{i_2}\rangle \otimes \cdots \otimes |\phi_{i_n}\rangle$ is a packaged product states. We need to prove that $|\Psi_\alpha^Q\rangle$ must be packaged entangled state.

Now, suppose that $|\Psi_\alpha^Q\rangle$ is a packaged product state (factorizable), i.e.,

$$|\Psi_\alpha^Q\rangle = |\chi_1\rangle \otimes |\chi_2\rangle \otimes \cdots \otimes |\chi_n\rangle,$$

with each factor $|\chi_k\rangle$ describing a subset of the IQNs that, when combined, yield the net charge Q. However, by linear algebra, this is possible only if all the $|\phi_{i_1}\rangle$ differ only by an overall scalar, $\cdots$, and all the $|\phi_{i_n}\rangle$ differ only by an overall scalar. In this sense, the packaged product states in $\mathcal{H}_Q$ are not distinct packaged product states, but the same packaged product states up to a multiplicative constant. Hence we get a contradiction whenever the packaged product states in $\mathcal{H}_Q$ are truly distinct states (linearly independent). Thus, every superposition $|\Psi_\alpha^Q\rangle$ is necessarily a packaged entangled state.

After obtaining all packaged entangled states, we may apply the Gram-Schmidt procedure [45] to construct a orthonormal basis for $\mathcal{H}_Q$:

$$\mathcal{B}_Q = \{\, |\Psi_1^Q\rangle, |\Psi_2^Q\rangle, \ldots, |\Psi_{\dim \mathcal{H}_Q}^Q\rangle, \quad \langle \Psi_\alpha^Q | \Psi_\beta^Q \rangle = \delta_{\alpha\beta} \}.$$

where each $|\Psi_\alpha^Q\rangle$ is a packaged entangled state. This proves the existence of G-associated multi-particle packaged entangled states.

(4) Module Isomorphism. For each $g \in G$ and $|\Psi_\alpha^Q\rangle \in \mathcal{B}_Q$, the transformation law is given by

$$U(g)|\Psi_\alpha^Q\rangle = \sum_{\beta=1}^{\dim \mathcal{H}_Q} D_{\beta\alpha}^{(Q)}(g)\, |\Psi_\beta^Q\rangle,$$

where $D^{(Q)}(g)$ is the matrix representing g in the abstract irrep σ_Q on V_Q. Define the mapping

$$\Phi_Q : \mathcal{H}_Q \to V_Q,$$

by sending the basis $\{|\Psi_\alpha^Q\rangle\}$ to the standard basis of V_Q. Then, for all $g \in G$ and $\psi \in \mathcal{H}_Q$,

$$\Phi_Q\Big(U(g)\psi\Big) = \sigma_Q(g)\, \Phi_Q(\psi),$$

which proves the isomorphism of the G-module $\mathcal{H}_Q$ with V_Q.

Collecting these steps, we conclude that: internal quantum numbers are packaged at the single-particle level, and the full representation of G on the Hilbert space decomposes into charge sectors. Within each fixed sector $\mathcal{H}_Q$, there exist an orthonormal basis of packaged entangled states, and orthonormal basis is isomorphic as a G-module to the abstract irreducible representation corresponding to Q. $\square$

This theorem formalizes the idea that gauge invariance enforces a decomposition of the Hilbert space into charge sectors wherein the internal quantum numbers are inseparable. In confining theories such as QCD, only the singlet sector is physical, while in other contexts, different sectors correspond to distinct observable multiplets.

Remark 1 (Scope of the theorem)**.** *The construction is general for any finite or compact group G acting unitarily on the Hilbert space. However, we also note the following important caveats and refinements:*

Anti-Unitary Symmetries: If G contains anti-unitary elements (e.g. time-reversal or certain charge-conjugation operations), the representation must be handled using appropriate complex conjugation conventions. In our theorem we assume a unitary G-action. Extensions to anti-unitary or projective cases can be made but require modified technical treatments.

Infinite-Dimensional Representations: For compact groups, all irreps are finite-dimensional, so the construction directly applies. For non-compact groups, the irreps may be infinite-dimensional, ao additional technical conditions are required. Here, we restrict our discussion to finite-dimensional irreps.

Superselection (Gauge) vs. Global Symmetry: In local gauge theories, superselection rules enforce that states from different charge sectors cannot coherently superpose. This reinforces the packaging principle. For global symmetries, although cross-sector superpositions are allowed in principle, one typically focuses on a definite Q sector. This is because the total charge is conserved and the Hamiltonian is block-diagonalized accordingly.

For finite groups, the group algebra decomposes into a finite direct sum of irreps. For compact Lie groups, although infinitely many irreps exist, each physically relevant excitation carries a finite-dimensional irrep. Consequently, multi-particle states exhibit packaged entanglement that reflects the irreducible structure of G. In local gauge theories, superselection rules restrict physical states to a fixed charge sector, reinforcing that internal quantum numbers are packaged as complete, inseparable units.

2.4 Projection Operators

To classify packaged states by their irreps, we will construct projection operators [46, 47, 48] that extract the subspace $\mathcal{H}_R$ associated with a given irreducible representation R of G.

(a) Finite Gauge Groups

For a finite group G, let χ_R denote the character of an irreducible representation R, which satisfy the orthogonality relation

$$\frac{1}{|G|}\sum_{g\in G}\chi_R(g)\,\chi^*_{R'}(g)=\delta_{RR'}.$$

Then, given a representation $U: G\to\mathcal{U}(\mathcal{H})$, the projection operator onto the subspace transforming as R is defined by

$$P_R=\frac{d_R}{|G|}\sum_{g\in G}\chi^*_R(g)\,U(g), \tag{4}$$

where $d_R=\dim V_R$ and $|G|$ is the order of the group and $U(g)$ is the unitary (or projective) representation of $g\in G$ on $\mathcal{H}$.

One may verify that P_R is idempotent and self-adjoint:

$$P_R^2=P_R,\quad P_R^\dagger=P_R,$$

and P_R also commutes with all $U(g)$

$$P_R\, U(g) = U(g)\, P_R \quad \forall\, g \in G.$$

Thus, P_R is an orthogonal projector that can project the Hilbert space $\mathcal{H}$ onto the subspace $\mathcal{H}_R$ transforming as V_R.

(b) Compact Lie Groups

For a compact Lie group G, let χ_R and d_R be as before. We replace the orthogonality relation by a normalized Haar measure $d\mu(g)$ such that

$$\int_G d\mu(g) = 1.$$

In Eq.(4), we replace the sum by an integral with respect to the Haar measure $d\mu(g)$. Then the projection operator P_R becomes

$$P_R = d_R \int_G \chi_R^*(g)\, U(g)\, d\mu(g). \tag{5}$$

Similarly, we verify that P_R is satisfies

$$P_R^2 = P_R \quad \text{and} \quad P_R^\dagger = P_R,$$

This projection operator extracts the subspace $\mathcal{H}_R$ that transforms as the irreducible representation R of G.

These projection operators are mathematically natural: they are idempotent, self-adjoint, and commute with the group action. They also carry important physical meaning. For instance, in QCD the projection onto the color-singlet subspace enforces that observable hadrons are confined, as only states with net color zero (i.e., invariant under G) are free to propagate. Superselection rules then prohibit coherent superpositions of states from different charge sectors.

3 Classification of Packaged States Based on Gauge Groups

In gauge theories, the internal quantum numbers (IQNs) of single-particle excitations are packaged as inseparable units by virtue of their irreducible transformation properties. As a result, multi-particle states inherit a structure that reflects the underlying representation theory of the gauge group G.[5, 49] In this section, we briefly review the main results of the representation-theoretic decomposition and explain how projection operators play a leading role in identifying the physically relevant (e.g. color-singlet) subspaces that underlie phenomena such as confinement and superselection.

3.1 Classification Based on Finite Gauge Groups

Although many gauge theories have continuous symmetry groups (*e.g.* $U(1)$ or $SU(N)$), finite gauge groups [39, 50, 51, 52] can also serve as local symmetries. In fact, for a finite group G, the representation theory is fully reducible, which simplifies defining distinct charge sectors. In particular, for cyclic groups $\mathbb{Z}_N$, most irreps are one-dimensional. These give rise to discrete superselection rules that further refine the packaging principle.

Finite local gauge groups arise in various ways. For example, remnants of broken continuous symmetries in high-energy physics, where a continuous symmetry is reduced to a discrete subgroup. And symmetry-protected topological (SPT) phases in condensed matter, where a finite group acts as an on-site symmetry and can be gauged to study topological orders. We now outline how packaged states emerge in a finite gauge theory.

General Finite Gauge Group G. Let G be a finite gauge group whose representations are completely reducible. In our framework, every single-particle excitation carries internal quantum numbers as an inseparable block by transforming in an irreducible representation (irrep) of G. If a single-particle operator carries a local gauge charge Q, its state transforms under G according to an irrep labeled by a discrete index d. In other words, a packaged state is uniquely characterized by the combined label (Q, d).

To formalize this, we define the unitary representation on the Hilbert space $\mathcal{H}$:

$$U : G \to \mathrm{U}(\mathcal{H}),$$

Since G is finite, we can decompose $\mathcal{H}$ into the direct sum of subspaces corresponding to the irreps $\{V_d\}$,

$$\mathcal{H} \cong \bigoplus_d \mathcal{H}_{Q,d}.$$

where each subspace $\mathcal{H}_{Q,d}$ carries a particular "local gauge charge" Q (or net representation) and also transforms under a discrete label d from one of the irreps of G. In many finite groups (especially cyclic ones like $\mathbb{Z}_N$), these irreps can be one-dimensional, so each sector $\mathcal{H}_{Q,d}$ is in effect a 1D representation that cannot be further fractionated.

We can use a projection operator defined in Eq. (4) onto the discrete sector labeled by d. Given the character χ_d of the irrep V_d, we have

$$P_d = \frac{1}{|G|} \sum_{g \in G} \chi_d^*(g)\, U(g).$$

Applying P_d to any state $|\Psi\rangle$ that carries the local gauge charge Q produces the component

$$|\Psi_d\rangle \;=\; P_d\,|\Psi\rangle \;\in\; \mathcal{H}_{Q,d}.$$

which lies entirely in the sector with discrete quantum number d.

Hence, each state is labeled (Q, d) with no partial splitting of the discrete label d. This is an immediate manifestation of the packaging principle: once a local gauge charge Q is assigned, the discrete label d cannot be partially changed.

$\mathbb{Z}_N$ as a Gauge Group. Let us now consider the cyclic group $\mathbb{Z}_N$ as a local gauge group. The transformation varies from point to point and a gauge field U_{xy} is introduced to ensure local invariance. The physical Hilbert space is restricted to gauge-invariant states

Let $g(x) \in \mathbb{Z}_N$ be a local gauge transformation, where for each spacetime point x,

$$g(x) \;\in\; \{\,\omega^{k(x)} : k(x) \in \{0, \ldots, N-1\}\}, \quad \omega = e^{2\pi i/N}.$$

A matter field $\psi(x)$ carrying discrete charge $q \in \{0, 1, \ldots, N-1\}$ transforms by

$$\psi(x) \;\mapsto\; \omega^{\,q}\,\psi(x).$$

To maintain local gauge invariance, a gauge field U_{xy} is introduced on links between points x and y so that the covariant derivative transforms homogeneously. Its transformation law is:

$$U_{xy} \mapsto \Big[\omega^{q(x)} - \omega^{q(y)}\Big]\, U_{xy}.$$

Physical states $|\text{phys}\rangle$ must be invariant under local gauge transformations (must satisfy Gauss's law constraints). For instance, if $G(x)$ is the generator of local gauge transformations at site x, then

$$G(x)\,|\text{phys}\rangle = |\text{phys}\rangle \quad \forall x.$$

Usually, in confining theories, only the gauge singlet (net charge zero) sector is truly gauge-invariant.

The irreducible representations of $\mathbb{Z}_N$ (which, in the 1D case, are given by $\rho_d(g) = \exp(2\pi i\, d/N)$) label the local gauge charge at each point. However, due to the local nature, the fields must be dressed by gauge fields and the physical Hilbert space is projected onto gauge invariant states (e.g. net $\mathbb{Z}_N$ charge zero). In this way, the internal quantum numbers are packaged together, but the constraint is enforced locally via Gauss's law.

$\mathbb{Z}_2$ Gauge Group. In the special case of the finite group $\mathbb{Z}_2$ (with elements $\{e, \hat{O}\}$ and $\hat{O}^2 = I$),[50] one typically represents the nontrivial element by, say, $U_o = \sigma_x$ in a two-dimensional Hilbert space with basis $\{|+\rangle, |-\rangle\}$. The corresponding projectors

$$P_+ = \frac{1}{2}\Big(I + U_o\Big), \quad P_- = \frac{1}{2}\Big(I - U_o\Big)$$

then partition the Hilbert space into two superselection sectors. Consequently, a packaged state carrying a local gauge charge Q is assigned the full label $(Q,+)$ or $(Q,-)$, ensuring that its discrete internal quantum numbers remain completely inseparable.

Therefore, when G is finite, its complete reducibility and often 1D irreps guarantee that discrete labels cannot be partially changed within a single local gauge charge sector. The packaging principle thus remains in force: each single-particle or multi-particle wavefunction belongs to a definite irreps block $\mathcal{H}_{Q,d}$ with no partial factorization allowed. Hence, packaged quantum states in finite gauge theories are classified by (Q, d), combining the usual gauge charge Q with the discrete label d.

3.2 Classification Based on Compact Lie Groups

Compact Lie groups, such as $U(1)$ and $SU(N)$, have a richer structure.[4, 30, 53, 54] We treat Abelian and non-Abelian cases separately.

3.2.1 The Abelian Case: $U(1)$

For the Abelian gauge group $U(1)$, every irreducible representation (irrep) is one-dimensional and can be labeled by a continuous or discrete charge q (integer in many physical settings).[53] Specifically, one has

$$\rho_q\Big(e^{i\theta}\Big) \;=\; e^{\,i\,q\,\theta}.$$

In standard QED, for instance, q takes integer values (in units of the elementary charge). By the Peter-Weyl theorem, the full Hilbert space decomposes as

$$\mathcal{H} \;\cong\; \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q,$$

where $\mathcal{H}_q$ is the subspace of states carrying net $U(1)$ charge q.

(1) Single-Particle Representations. Suppose a single-particle field operator $\hat{\psi}(x)$ has charge q. Under a local gauge transformation $g(x) = e^{i\,\alpha(x)}$, it transforms by

$$\hat{\psi}(x) \;\mapsto\; e^{\,i\,q\,\alpha(x)}\,\hat{\psi}(x).$$

Because each irrep of $U(1)$ is one-dimensional, the phase $e^{\,i\,q\,\alpha(x)}$ is an inseparable block. One cannot factor out or split the charge q among multiple sub-pieces within the same operator – *e.g.*, you cannot split an electron's charge -1 into fractional parts. Formally, for the single-particle creation operator $\hat{a}_q^\dagger$, one writes:

$$U\Big(e^{\,i\theta}\Big)\,\hat{a}_q^\dagger\,U\Big(e^{\,i\theta}\Big)^{-1} \;=\; e^{\,i\,q\,\theta}\,\hat{a}_q^\dagger,$$

illustrating that $\hat{a}_q^\dagger$ itself furnishes a 1D representation with charge q.

(2) Multi-Particle States and Additive Charges. For multi-particle excitations, consider the product of field operators:

$$\hat{\psi}_{q_1}^\dagger(x_1)\,\hat{\psi}_{q_2}^\dagger(x_2)\;\cdots\;\hat{\psi}_{q_n}^\dagger(x_n).$$

each operator transforms as

$$\hat{\psi}_{q_i}^\dagger(x_i) \mapsto e^{iq_i\alpha(x_i)}\,\hat{\psi}_{q_i}^\dagger(x_i).$$

Each $\hat{\psi}_{q_i}^\dagger(x_i)$ picks up a phase $e^{\,i\,q_i\,\alpha(x_i)}$ under $g(x) = e^{\,i\,\alpha(x)}$. Since $U(1)$ is Abelian, the total gauge transformation is a simple product of phases:

$$\prod_{i=1}^{n} e^{\,i\,q_i\,\alpha(x_i)} \;=\; e^{\,i\,\Big(q_1+q_2+\cdots+q_n\Big)\,\alpha},$$

so the net charge of the multi-particle state is

$$Q_{\text{tot}} = q_1 + q_2 + \cdots + q_n.$$

Hence, in a non-confining theory such as QED, states with nonzero net charge can appear as free asymptotic states. Nevertheless, the superselection rule states that different net charges $\mathcal{H}_q$ do not mix:

$$\mathcal{H} \;=\; \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q.$$

(3) Projecting onto a Definite Charge Sector. Because $U(1)$ is a continuous group, one defines the projector P_q using the normalized Haar measure on $U(1)$. Concretely,

$$P_q \;=\; \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-i\,q\,\theta}\, U\Big(e^{i\,\theta}\Big),$$

where $U\Big(e^{i\,\theta}\Big)$ is the unitary operator implementing the global gauge transformation $e^{i\,\theta} \in U(1)$. By construction, $P_q^2 = P_q$ and

$$P_q\,\mathcal{H} \;=\; \mathcal{H}_q.$$

Hence, for any state $|\psi\rangle \in \mathcal{H}$, the component $|\psi_q\rangle = P_q\,|\psi\rangle$ lies entirely in the charge-q sector. This superselection structure ensures that a physical state must carry a definite net charge; cross-sector interference is disallowed.

Therefore, for an Abelian gauge group $U(1)$, each single-particle creation operator $\hat{a}_q^\dagger$ transforms irreducibly with a single packaged charge q. Multi-particle states acquire net charge $Q_{\text{tot}} = q_1 + \cdots + q_n$ in an additive fashion. The superselection rule, enforced via the projection operator P_q, forbids coherent superpositions of different net charges $q \neq q'$. All internal quantum numbers relevant to $U(1)$ remain packaged and cannot be split among multiple excitations. This is precisely the packaging principle as applied to an Abelian local gauge group.

(4) Comparison: Abelian $U(1)$ vs. Discrete $\mathbb{Z}_N$. Both these gauge groups exhibit packaged irreps at the single-particle level: For $U(1)$, each irreps is labeled by an integer q. For $\mathbb{Z}_N$, each irreps is labeled by a residue $a \in \{0, \ldots, N-1\}$.

In both cases, irreps are 1-dimensional, so the group action is a pure phase. The packaging arises from the impossibility of dividing that label (integer q or residue a) among separate operators. A single electron operator, for instance, cannot be split in half to share partial electric charge. Similarly, a single $\mathbb{Z}_N$-charged excitation with label $a \neq 0$ remains an indivisible discrete block.

For non-confining $U(1)$ (like standard electromagnetism), net-charged states appear freely in nature (electrons, protons, etc.). For a local $\mathbb{Z}_N$ gauge theory with confinement, only net zero flux (mod N) emerges as a free state. Nonzero flux might remain bound or exist only as part of composite excitations that collectively sum to zero mod N.

Therefore, in Abelian $U(1)$ symmetry, the classification of packaged states is simple. Each single-particle excitation carries a definite integer charge, and multi-particle states have additive charges. States belonging to different charge sectors are physically distinct due to superselection rules.

3.2.2 Non-Abelian Gauge Groups SU(N)

In non-Abelian gauge theories [5, 49, 51], the packaging principle implies that each single-particle operator carries an indivisible set of gauge quantum numbers. For SU(N), single-particle operators transform in irreducible representations (irreps) labeled by Young diagrams (with up to $N-1$ rows). Multi-particle states are constructed as tensor products of these irreps and decompose into a direct sum of irreps via Clebsch-Gordan rules. In a local gauge theory (e.g. QCD), only the gauge-singlet subspace (the $\mathbf{1}$) is physically observed, whereas in global symmetries (such as isospin or flavor), non-singlet multiplets may appear externally. We now describe the general structure before specializing to SU(2) and SU(3).

(1) Single-Particle Irreps. Let $\hat{a}_i^\dagger$ denote a creation operator that transforms in an irrep $\mathbf{R}$ of $\mathrm{SU}(N)$. Its transformation under $g \in \mathrm{SU}(N)$ is given by

$$U(g)\,\hat{a}_i^\dagger\,U(g)^{-1} = \sum_{j=1}^{d_\mathbf{R}} D_{ji}^{(\mathbf{R})}(g)\,\hat{a}_j^\dagger,$$

where $D^{(\mathbf{R})}(g)$ is a $d_\mathbf{R} \times d_\mathbf{R}$ matrix. No proper invariant subspace exists within an irreducible representation. Thus, the internal quantum numbers (e.g. color or flavor) are packaged as a single block.

(2) Multi-Particle Tensor Products and Decomposition. For a system of n excitations with each particle transforming in an irrep $\mathbf{R}_{\alpha_i}$, the multi-particle state belongs to

$$\mathbf{R}_{\alpha_1} \otimes \mathbf{R}_{\alpha_2} \otimes \cdots \otimes \mathbf{R}_{\alpha_n}.$$

Using Clebsch-Gordan (or Young diagram) methods, this product decomposes into a direct sum of irreps:

$$\mathbf{R}_{\alpha_1} \otimes \mathbf{R}_{\alpha_2} \otimes \cdots \otimes \mathbf{R}_{\alpha_n} \cong \bigoplus_\beta N_\beta\,\mathbf{R}_\beta.$$

For local gauge groups (e.g. color SU(3)), physical states are restricted to the singlet (or net-neutral) subspace. For global symmetries (e.g. isospin), non-singlet multiplets may be observed as free states.

(3) Module Isomorphism. Once the Hilbert space is decomposed as

$$\mathcal{H} \cong \bigoplus_\beta \mathcal{H}_\beta,$$

the theorem guarantees an isomorphism of $\mathrm{SU}(N)$-modules:

$$\Phi_\beta : \mathcal{H}_\beta \to V_\beta,$$

where V_β is the abstract representation space for the irrep $\mathbf{R}_\beta$. This means that for all $g \in \mathrm{SU}(N)$ and $\psi \in \mathcal{H}_\beta$,

$$\Phi_\beta\Big(U(g)\psi\Big) = \rho_\beta(g)\,\Phi_\beta(\psi),$$

with $\rho_\beta(g)$ the matrix representation of g in V_β.

3.2.3 SU(2) Gauge Group

To classify packaged states with SU(2) gauge group, we consider a hypothetical gauge theory with gauge group SU(2) (often referred to as two-color QCD), where the internal gauge charge is represented by SU(2).[4] In this theory, quarks transform in the fundamental (doublet) representation of SU(2).

Let $\hat{q}_i^\dagger$ be a quark creation operator, where the index $i = 1, 2$ denotes the two internal gauge components. Under a gauge transformation $g \in \mathrm{SU}(2)$, the operator transforms as

$$U(g)\,\hat{q}_i^\dagger\,U(g)^{-1} = \sum_{j=1}^{2} D_{ji}^{(1/2)}(g)\,\hat{q}_j^\dagger,$$

with $D^{(1/2)}(g)$ being the 2×2 matrix of the fundamental representation. The fundamental doublet is irreducible and hence the internal gauge charge is fully packaged into a single block. There is no way to split the charge q into independent pieces.

Now consider constructing a two-quark state. Since each quark transforms as a doublet $\mathbf{2}$, the tensor product of two doublets decomposes as

$$\mathbf{2} \otimes \mathbf{2} \cong \mathbf{3} \oplus \mathbf{1}.$$

A standard derivation using Young diagrams or Clebsch-Gordan coefficients shows that the symmetric combination forms the triplet ($\mathbf{3}$) and the antisymmetric combination forms the singlet ($\mathbf{1}$). For instance, an explicit expression for the singlet state is

$$|\Psi_{\mathbf{1}}\rangle = \frac{1}{\sqrt{2}}\Big(|q,1;\, q,2\rangle - |q,2;\, q,1\rangle\Big).$$

This state is invariant under SU(2) gauge transformations and is a packaged entangled state, meaning that the internal gauge indices are inseparably combined into a single, gauge-invariant unit.

Remark 2 (Isomorphism and Classification)**.** *By applying the projection operator method (as described in the previous subsection on projection operators) to the two-color theory, one projects the full Hilbert space $\mathcal{H}$ onto the singlet subspace $\mathcal{H}_{\mathbf{1}}$. In this subspace, the gauge transformation acts trivially:*

$$U(g)\,|\Psi_{\mathbf{1}}\rangle = |\Psi_{\mathbf{1}}\rangle,$$

for all $g \in \mathrm{SU}(2)$. Thus, we have an isomorphism

$$\Phi_{\mathbf{1}} : \mathcal{H}_{\mathbf{1}} \to V_{\mathbf{1}},$$

where $V_{\mathbf{1}}$ is the one-dimensional abstract representation space corresponding to the singlet. This example illustrates the general principle that each single-particle operator carries an indivisible gauge charge and that multi-particle states can be projected onto invariant, packaged subspaces.

3.2.4 SU(3) Gauge Group

For SU(3), the irreps are typically labeled by two nonnegative integers (p, q) (or equivalently by Young diagrams).[17, 30, 54] The fundamental representation is $(1, 0)$ (denoted by $\mathbf{3}$) and its conjugate is $(0, 1)$ (denoted by $\bar{\mathbf{3}}$). Other common irreps include the symmetric product of two fundamentals, $(2, 0)$ (denoted by $\mathbf{6}$), the adjoint representation $(1, 1)$ (denoted by $\mathbf{8}$), and so on.

To classify packaged states with SU(3) gauge group, we focus on quantum chromodynamics (QCD), where the internal gauge charge is color. The relevant irreps are: quarks transform in the fundamental representation $\mathbf{3}$, antiquarks transform in the conjugate representation $\overline{\mathbf{3}}$, and gluons transform in the adjoint representation $\mathbf{8}$.

A quark creation operator $\hat{q}_c^\dagger$ (with $c = 1, 2, 3$) transforms as

$$U(g)\,\hat{q}_c^\dagger\,U(g)^{-1} = \sum_{d=1}^{3} D^{(\mathbf{3})}_{dc}(g)\,\hat{q}_d^\dagger,$$

where $D^{(\mathbf{3})}(g)$ is the matrix representation of g in the fundamental. By the packaging principle, A quark state is a packaged state, where the full color degree of freedom is carried as a single, inseparable unit.

A multi-quark state is usually obtained by the tensor-product of individual quark states. We need to decompose these tensor products so that we can reduce symmetry and identify states invariant under G (correspond to physical observables) and reveal superselection rules. These are important on classifying the packaged states.

There are a number of decompositions. Here we only consider two of them that are directly applied to explain observable phenomena:

1. **Decomposition $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$.**

   The left side represents the tensor-product of a quark and an antiquark (a meson state). After decomposition, the singlet ($\mathbf{1}$) corresponds to color-neutral mesons, while the octet ($\mathbf{8}$) represents confined color-octet states that never appear as free particles.

   The only packaged entangled state is the color-singlet (meson) given by

   $$|M\rangle = \frac{1}{\sqrt{3}} \sum_{c=1}^{3} |q, c;\, \bar{q}, c\rangle.$$

   This state is invariant under SU(3) gauge transformations, i.e., in which the color indices are completely contracted.

2. **Decomposition $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$.**

   The left side represents the tensor-product of three quarks (a baryon state). After decomposition, The color singlet is the only observable state.

   The only packaged entangled state is the color-singlet (baryon) given by

   $$|B\rangle = \frac{1}{\sqrt{6}} \epsilon_{abc} \, |q, a;\, q, b;\, q, c\rangle,$$

   where ϵ_{abc} is the totally antisymmetric tensor. This construction guarantees that the three color indices are fully entangled into an indivisible color-neutral package.

More complicated multi-quark systems (e.g., tetraquarks or pentaquarks) are constructed by forming tensor products of quarks and antiquarks, and then projecting onto the color-singlet subspace. The projection is achieved via the corresponding Clebsch-Gordan coefficients or Young diagram methods. In local gauge theories such as QCD, only the color-singlet (or net-neutral) states are physically observable due to confinement.

In both the meson and baryon cases, the invariant subspace (e.g., $\mathcal{H}_{\mathbf{1}}$) is isomorphic to the abstract one-dimensional representation $V_{\mathbf{1}}$ of SU(3). That is, there exists an intertwiner

$$\Phi_{\mathbf{1}} : \mathcal{H}_{\mathbf{1}} \to V_{\mathbf{1}},$$

satisfying

$$\Phi_{\mathbf{1}}\Big(U(g)|\Psi\rangle\Big) = \rho_{\mathbf{1}}(g)\, \Phi_{\mathbf{1}}(|\Psi\rangle)$$

for all $g \in \mathrm{SU}(3)$ and $|\Psi\rangle \in \mathcal{H}_{\mathbf{1}}$. This formalizes the idea that the packaged entangled color singlet is an irreducible gauge-invariant object.

Remark 3 (Local Versus Global Symmetries)**.** *For local gauge symmetries (e.g. QCD color* $\mathrm{SU}(3)$*), only the gauge-singlet (*$\mathbf{1}$*) states are free due to confinement. Non-singlet states remain internal. For global symmetries such as isospin or flavor, non-singlet multiplets (e.g. the baryon octet or decuplet in* $\mathrm{SU}(3)_{\textit{flavor}}$*) can appear as observable free states. In both cases, however, the packaging principle holds: each single-particle operator carries an irreducible set of quantum numbers that cannot be partially factored.*

Thus, the packaged states in non-Abelian gauge theories are classified by the irreducible subrepresentations obtained from the tensor product of single-particle blocks. This classification explains physical phenomena such as color confinement [30, 31] and superselection, highlighting that the internal degrees of freedom are intrinsically entangled and cannot be partially separated. In QCD, only packaged entangled states like color-singlet are observable. While in global symmetries non-singlet multiplets may appear.

4 Classification of Packaged States Based on Discrete Groups

In many physical systems, in addition to continuous gauge symmetries, discrete symmetries play an important role.[29, 42] When these discrete symmetries are present, they further constrain or partition the internal (packaged) degrees of freedom. In other words, the packaging principle (the internal quantum numbers (IQNs) are inseparable) is refined by imposing that states also lie in definite discrete symmetry sub-sectors. In this section, we discuss how discrete symmetries refine the classification of packaged states and how projection operators and superselection rules ensure that the packaged (internal) charges remain intact.

4.1 Pure Global Discrete Symmetry Groups and Their Role in Packaging

A **discrete symmetry** is one whose symmetry group D is finite. We have developed the theoretic framework of cyclic group $\mathbb{Z}_N$ as a local gauge group in Section 3.1. We can use the similar results for discrete symmetry groups. Let us now consider $\mathbb{Z}_N$ as a pure global discrete symmetry.[55, 56]

When $\mathbb{Z}_N$ is treated as a global symmetry, the transformation is the same across entire spacetime and there is no accompanying gauge field. For a fixed $g \in \mathbb{Z}_N$, the transformation is applied uniformly:

$$\psi(x) \to \psi'(x) = \exp\!\left(\frac{2\pi i\, q}{N}\right) \psi(x),$$

where q is a fixed charge for the field $\psi(x)$, independent of x.

There is no link variable or local Gauss's law constraint. The entire Hilbert space is partitioned into global sectors labeled by the eigenvalue of the global operator $U(g)$.

States in different global sectors (i.e. different eigenvalues of $U(g)$) cannot be coherently superposed. However, unlike in the local gauge case, nontrivial sectors can be physically realized. For example, a system might have two distinct vacua labeled by different $\mathbb{Z}_N$ charges.

The internal quantum number (or discrete charge) is still packaged, meaning that a single operator carries a full phase $\exp(2\pi i\, q/N)$. The full label of a state is then given by its gauge charge (if any) and the global discrete charge. In this case, the Hilbert space decomposes as

$$\mathcal{H} = \bigoplus_{q=0}^{N-1} \mathcal{H}_q,$$

where no additional local constraint forces, for instance, a net zero charge. The projector onto a discrete sector is given by

$$P_q = \frac{1}{N} \sum_{k=0}^{N-1} \exp\biggl(-\frac{2\pi i\, qk}{N}\biggr) U_k,$$

with

$$U_k = \exp\biggl(\frac{2\pi i\, \hat{q}\, k}{N}\biggr).$$

Comparison $\mathbb{Z}_N$ as a Local Gauge Group vs. as a Global Discrete Symmetry. We have treated cyclic group $\mathbb{Z}_N$ as a local gauge group (see Section 3.1) and as a global discrete symmetry group. In both cases, the internal discrete charge is packaged as an inseparable unit. In the gauge case, additional local constraints (Gauss's law) enforce that the overall state must be gauge invariant (often forcing net charge zero), while in the global case the discrete label q simply labels different superselection sectors. We now include a comparison table to summarizes these differences:

Table 1: Comparison $\mathbb{Z}_N$ as a Gauge Group & as a Global Discrete Symmetry

Feature	$\mathbb{Z}_N$ as Gauge Group	$\mathbb{Z}_N$ as Global Discrete Symmetry
Transformation	$g(x)$ depends on x; local gauge transformation	g is uniform over spacetime
Gauge Field	Yes; link variables U_{xy} exist	No gauge field
Constraint	Gauss's law forces physical states to be gauge invariant (often net zero charge)	No local constraint; states in any sector can be physical
Projector	$P_q = \frac{1}{N}\sum_{k=0}^{N-1} e^{-2\pi i\, qk/N}\, U_k(x)$ with local $U_k(x)$	$P_q = \frac{1}{N}\sum_{k=0}^{N-1} e^{-2\pi i\, qk/N}\, U_k$ with $U_k = \exp\Bigl(\frac{2\pi i\, \hat{q}\, k}{N}\Bigr)$
Physical Superselection	Only gauge-invariant (typically trivial) sectors appear	Different sectors (labeled by q) are distinct, and no mixing occurs

In the following discussions, we will focus on the simplest $\mathbb{Z}_2$ symmetry group.

4.2 Corollary: Packaged Entanglements Based on $\mathbb{Z}_2$ Symmetry Operators

Discrete symmetry operators can also induce a characteristic form of entanglement. We now consider the simplest discrete group $\mathbb{Z}_2$.[57, 58] Referring to Theorem 1, we obtain the following corollary:

Corollary 1 (Bell-like Entanglements under $\mathbb{Z}_2$ Symmetries)**.** *Consider a system that consists of two identical subsystems, labeled A and B, each described by a set of quantum numbers $Q = (Q_1, Q_2, \ldots, Q_n)$. If there is a $\mathbb{Z}_2$ symmetry operator $\hat{O}$ such that $\hat{O}\,|Q\rangle = |{-Q}\rangle$, $\hat{O}\,|{-Q}\rangle = |Q\rangle$, then in the two-level Hilbert space:*

1. *Mathematically, one can construct four Bell-like entangled states:*
$$\begin{aligned} |\Psi^{\pm}\rangle_{AB} &= \frac{1}{\sqrt{2}}\Big(|Q\rangle_A\,|{-Q}\rangle_B \;\pm\; |{-Q}\rangle_A\,|Q\rangle_B\Big), \\ |\Phi^{\pm}\rangle_{AB} &= \frac{1}{\sqrt{2}}\Big(|Q\rangle_A\,|Q\rangle_B \;\pm\; |{-Q}\rangle_A\,|{-Q}\rangle_B\Big). \end{aligned} \tag{6}$$
Each of these states is an eigenstate of $\hat{O} \otimes \hat{O}$ with eigenvalue ± 1.

2. *The physical realization of the states in Eq. (6) are subjected to superselection rules, i.e., if $\hat{O}$ is the charge conjugation operator $\hat{C}$, then states $|\Phi^{\pm}\rangle_{AB}$ are not allowed.*

3. *Applying the tensor product of the $\mathbb{Z}_2$ groups, we naturally obtain the GHZ states, W states, and Dicke states.*

Proof. By assumption, $\hat{O}$ swaps $|Q\rangle \leftrightarrow |{-Q}\rangle$.

1. On the two-subsystem Hilbert space, we have:
$$(\hat{O}_A\otimes\hat{O}_B)\Big(|Q\rangle_A\,|{-Q}\rangle_B\Big) = |{-Q}\rangle_A\,|Q\rangle_B, \quad (\hat{O}_A\otimes\hat{O}_B)\Big(|{-Q}\rangle_A\,|Q\rangle_B\Big) = |Q\rangle_A\,|{-Q}\rangle_B.$$
and the linear combinations in Eq. (6) transform into each other up to a sign:
$$(\hat{O}_A \otimes \hat{O}_B)\,|\Psi^{\pm}\rangle_{AB} = \pm\,|\Psi^{\pm}\rangle_{AB}, \quad (\hat{O}_A \otimes \hat{O}_B)\,|\Phi^{\pm}\rangle_{AB} = \pm\,|\Phi^{\pm}\rangle_{AB}.$$
Therefore, each $|\Psi^{\pm}\rangle$ and $|\Phi^{\pm}\rangle$ is an eigenstate with eigenvalue ± 1.

2. In case of charge conjugation $\hat{C}$ (for ± 1 electric charges), superselection may forbid mixing $|{+Q}\rangle$ and $|{-Q}\rangle$ unless the total charge of the bipartite system is zero. States $|\Phi^{\pm}\rangle_{AB}$ try to superpose states with different charge, which is not allowed. Physically, one must ensure the allowed total charge sector remains consistent with gauge invariance.

3. By applying the tensor product of the $\mathbb{Z}_2$ groups, we naturally obtain the GHZ states, W states, and Dicke states.

□

4.3 Special $\mathbb{Z}_2$ Symmetry Operators and Packaged Entanglements

The group $\mathbb{Z}_2$ is the simplest example of a discrete symmetry. It has two elements, $\{e, \hat{O}\}$, with the property that $\hat{O}^2 = e$. In this case, the operator $\hat{O}$ may represent charge conjugation ($\hat{C}$), parity ($\hat{P}$), time reversal ($\hat{T}$), or combinations such as $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$, and $\hat{C}\hat{P}\hat{T}$.

According to Corollary 1, a $\mathbb{Z}_2$ symmetry gives rise to $\mathbb{Z}_2$-type entanglements. Because $\mathbb{Z}_2$ group is not a gauge group, it can result in both internal entanglement (packaged entanglement) and external entanglement (spin, momentum, etc.). This depends on the specific operation represented by the $\mathbb{Z}_2$ group.

4.3.1 Internal: Charge Conjugation $\hat{C}$

In QFT, charge conjugation $\hat{C}$ is an operator invented to perform a reversal operation that transforms each particle $|P\rangle$ into its antiparticle $|\bar{P}\rangle$, and vice versa, $|P\rangle \leftrightarrow |\bar{P}\rangle$.[7, 59] Although $\hat{C}$ can be defined as either unitary or anti-unitary in different conventions, here we treat it as a linear (unitary) operator.

(1) Single-Particle. At single-particle level, the charge conjugation $\hat{C}$ is a unitary operator. It transforms each particle $|P\rangle$ into its antiparticle $|\bar{P}\rangle$ and vice versa:

$$\hat{C}\,|P\rangle \;=\; |\bar{P}\rangle, \quad \hat{C}\,|\bar{P}\rangle \;=\; |P\rangle,$$

where $|P\rangle$ has net charge $+Q$ and $|\bar{P}\rangle$ has net charge $-Q$. This indicates that $\hat{C}$ swaps charges, i.e., $\hat{C} : |{+Q}\rangle \leftrightarrow |{-Q}\rangle$.

In fact, $\hat{C}$ reverse all IQNs (electric charge, flavor, color, baryon number, etc.),

$$\hat{C} : \text{IQNs} \leftrightarrow -\text{IQNs}.$$

But $\hat{C}$ does not affect external degree of freedoms (spin, momentum, position, etc.) unless combined with other discrete symmetries like $\hat{P}$ or $\hat{T}$.

Under charge conjugation $\hat{C}$, we can write the basis of a single-particle space as

$$\{|P\rangle,\, |\bar{P}\rangle\}.$$

where $\hat{C}$ swaps these states.

(2) Particle-Antiparticle Pair and Packaged Bell States. For a particle-antiparticle system, the product state basis is:

$$\{|P\rangle_A\,|P\rangle_B, |\bar{P}\rangle_A\,|\bar{P}\rangle_B, |P\rangle_A\,|\bar{P}\rangle_B, |\bar{P}\rangle_A\,|P\rangle_B\}.$$

The corresponding net charges of the product states are:

$$\{+2Q, -2Q, 0, 0\}$$

According to Corollary 1, superselection forbids mixing different total charges $+2Q$ with $-2Q$. Only the net-zero charge sector is allowed in a particle-antiparticle pair. More specifically:

1. Forbidden superposition ($+2Q$ vs. $-2Q$): Superposing $|P\rangle_A|P\rangle_B$ with $|\bar{P}\rangle_A|\bar{P}\rangle_B$ would mix total charge $+2Q$ and $-2Q$, which is blocked by the charge superselection rule. Thus, it is not physically realizable.

2. Physically allowed superposition (0 vs. 0): Superposing $|P\rangle_A\,|\bar{P}\rangle_B$ with $|\bar{P}\rangle_A\,|P\rangle_B$ would mix total charge 0 and 0, which obeys the charge superselection rule. Thus, it is physically realizable. We have packaged entangled states:

$$\left|\Psi_P^{\pm}\right\rangle_{AB} \;=\; \frac{1}{\sqrt{2}}\Big(|P\rangle_A\,|\bar{P}\rangle_B \;\pm\; |\bar{P}\rangle_A\,|P\rangle_B\Big). \tag{7}$$

Under tensor product $\hat{C}\otimes\hat{C}$, we have:

$$\hat{C}\otimes\hat{C} \;:\; |P\rangle_A\,|\bar{P}\rangle_B \;\leftrightarrow\; |\bar{P}\rangle_A\,|P\rangle_B,$$

and

$$\hat{C} \otimes \hat{C} \left|\Psi_P^{\pm}\right\rangle_{AB} = \pm\left|\Psi_P^{\pm}\right\rangle_{AB}.$$

We see that packaged entangled states $\left|\Psi_P^{\pm}\right\rangle_{AB}$ become the eigenstates of $\hat{C} \otimes \hat{C}$ with eigenvalues ± 1.

Thus, charge conjugation yields these internal, packaged entangled states within the net-zero charge sector. They show how gauge or superselection constraints guide us to physically meaningful superpositions.

4.3.2 External: Parity $\hat{P}$ and Time-Reversal $\hat{T}$

Contrast to charge conjugation $\hat{C}$, both parity $\hat{P}$ and time-reversal $\hat{T}$ flip external two-level, but leave the IQNs unchanged.

(1) Parity $\hat{P}$. In QFT, parity $\hat{P}$ is an operator invented to perform a spatial mirror operation, i.e., inverts space: $\mathbf{x} \leftrightarrow -\mathbf{x}$.[60, 61]

At single-particle level, the parity $\hat{P}$ is a pure unitary operator. In addition to the spatial coordinates $\mathbf{x}$, $\hat{P}$ also flips momentum $\mathbf{p} \leftrightarrow -\mathbf{p}$ because momentum is related to spatial coordinates. Consider a single-particle with two-level momenta. Let us label right-moving vs. left-moving single-particle states as $|+\rangle$, $|-\rangle$, we have

$$\hat{P}\,|+\rangle = |-\rangle, \quad \hat{P}\,|-\rangle = |+\rangle.$$

Because spin is similar to orbital angular momentum, $\hat{P}$ also flips (or preserves) spin states (depending on spin's vector or pseudovector nature). In many formulations (especially in the nonrelativistic limit), parity leaves the spin unchanged (up to a phase). But here we are considering a context where parity acts non-trivially on spin (a flip operation on an external two-level system). Let us consider a single-particle with spin-$\frac{1}{2}$ and label the up vs. down states as $|\uparrow\rangle$, $|\downarrow\rangle$, we have

$$\hat{P}\,|\uparrow\rangle = |\downarrow\rangle, \quad \hat{P}\,|\downarrow\rangle = |\uparrow\rangle.$$

But $\hat{P}$ leaves IQNs (like electric charge or baryon number) unchanged, as they do not transform under spatial inversion.

(2) Time-Reversal $\hat{T}$. In QFT, time-reversal $\hat{T}$ is an operator invented to perform a time-reversal operation, i.e., inverts time: $t \leftrightarrow -t$.[62]

In addition to the direction (flow) of time t, $\hat{T}$ flips momentum $\mathbf{p} \leftrightarrow -\mathbf{p}$ because momentum is related to time t.

At single-particle level, the time-reversal $\hat{T}$ is an anti-unitary operator. This means that $\hat{T}$ includes both a unitary spin/coordinate transformation and a complex-conjugation operation. There is subtlety for single spin-$\frac{1}{2}$. On a single spin-$\frac{1}{2}$ space, $\hat{T}$ satisfies $\hat{T}^2 = -\mathbf{I}$, which implies that we cannot have a single spin-$\frac{1}{2}$ state that is strictly an eigenstate of $\hat{T}$ with a real eigenvalue. This is the essence of Kramers degeneracy: every level is at least twofold degenerate. However, we can define states that are ± 1 eigenstates under $\hat{T}_A \otimes \hat{T}_B$ (see two-particle systems). This yields the same Bell structure as above, which can exhibit $\hat{T}^2 = +\mathbf{I}$ on certain subspaces. In this case, it is consistent to say certain bipartite states have ± 1 eigenvalues under $\hat{T}$. In this sense, we say that $\hat{T}$ also flips spin/angular momentum $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$.

But $\hat{T}$ leaves internal quantum numbers (electric charge Q, baryon number B, etc.) invariant, since these do not directly depend on the sign of time.

(3) Two-Particle Momentum/Spin Bell States. Let us now move to the two-particle systems, where we need to consider the tensor product $\hat{P}_A \otimes \hat{P}_B$ and $\hat{T}_A \otimes \hat{T}_B$.

Tensor product $\hat{P}_A \otimes \hat{P}_B$ acts like a reflection on each local subsystem (in spin or momentum space). This produces Bell-type entangled states with well-defined symmetry under spatial inversion, which are ± 1 eigenstates under $\hat{P}_A \otimes \hat{P}_B$. Similarly, we can also define Bell-type entangled states that are ± 1 eigenstates under tensor product $\hat{T}_A \otimes \hat{T}_B$.

- **Momentum:** For a two-particle system of two-level momentum basis, the product state basis is:

$$\{|+\rangle_A\,|+\rangle_B,\ |-\rangle_A\,|-\rangle_B,\ |+\rangle_A\,|-\rangle_B,\ |-\rangle_A\,|+\rangle_B\}.$$

  Both parity $\hat{P}$ and time reversal $\hat{T}$ flip momentum. Applying Corollary 1, we again obtain four Bell states of momentum in a two-level momentum basis:

$$\begin{aligned} |\Psi_m^\pm\rangle_{AB} &= \tfrac{1}{\sqrt{2}}\Big(|+\rangle_A\,|-\rangle_B \pm |-\rangle_A\,|+\rangle_B\Big), \\ |\Phi_m^\pm\rangle_{AB} &= \tfrac{1}{\sqrt{2}}\Big(|+\rangle_A\,|+\rangle_B \pm |-\rangle_A\,|-\rangle_B\Big), \end{aligned} \tag{8}$$

  which satisfy

$$\hat{P}_A \otimes \hat{P}_B\,|\Psi_m^\pm\rangle_{AB} = \pm|\Psi_m^\pm\rangle_{AB}, \quad \hat{P}_A \otimes \hat{P}_B\,|\Phi_m^\pm\rangle_{AB} = \pm|\Phi_m^\pm\rangle_{AB}.$$

  and

$$\hat{T}_A \otimes \hat{T}_B\,|\Psi_m^\pm\rangle_{AB} = \pm|\Psi_m^\pm\rangle_{AB}, \quad \hat{T}_A \otimes \hat{T}_B\,|\Phi_m^\pm\rangle_{AB} = \pm|\Phi_m^\pm\rangle_{AB}.$$

- **Spin:** For a two-particle system of spin-$\frac{1}{2}$, the product state basis for spin is:

$$\{|\uparrow\rangle_A\,|\uparrow\rangle_B,\ |\downarrow\rangle_A\,|\downarrow\rangle_B,\ |\uparrow\rangle_A\,|\downarrow\rangle_B,\ |\downarrow\rangle_A\,|\uparrow\rangle_B\}.$$

  Both parity $\hat{P}$ and time reversal $\hat{T}$ flip spin. Applying Corollary 1, we again obtain four Bell states of spin:

$$\begin{aligned} |\Psi_s^\pm\rangle_{AB} &= \tfrac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A\,|\downarrow\rangle_B \pm |\downarrow\rangle_A\,|\uparrow\rangle_B\Big), \\ |\Phi_s^\pm\rangle_{AB} &= \tfrac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A\,|\uparrow\rangle_B \pm |\downarrow\rangle_A\,|\downarrow\rangle_B\Big), \end{aligned} \tag{9}$$

  which satisfy

$$\hat{P}_A \otimes \hat{P}_B\,|\Psi_s^\pm\rangle_{AB} = \pm|\Psi_s^\pm\rangle_{AB}, \quad \hat{P}_A \otimes \hat{P}_B\,|\Phi_s^\pm\rangle_{AB} = \pm|\Phi_s^\pm\rangle_{AB}.$$

  and

$$\hat{T}_A \otimes \hat{T}_B\,|\Psi_s^\pm\rangle_{AB} = \pm|\Psi_s^\pm\rangle_{AB}, \quad \hat{T}_A \otimes \hat{T}_B\,|\Phi_s^\pm\rangle_{AB} = \pm|\Phi_s^\pm\rangle_{AB},$$

From above discussion, we see that entangled states

$$|\Psi_m^\pm\rangle_{AB},\ |\Phi_m^\pm\rangle_{AB},\ |\Psi_s^\pm\rangle_{AB},\ |\Phi_s^\pm\rangle_{AB}$$

are the ± 1 eigenstates of both parity $\hat{P}$ and time reversal $\hat{T}$.

Especially, in either momentum or spin space, time reversal $\hat{T}$ classifies certain bipartite states into ± 1 symmetry sectors and therefore show a neat link between $\hat{T}$ invariance and Bell-type entanglement. Even though $\hat{T}$ is anti-unitary, for these two-subsystem states (with integer total spin or balanced momentum pairs), it is meaningful to speak of $\hat{T}$-eigenstates.

Finally, we conclude that $\mathbb{Z}_2$ symmetry operators exemplify the simplest instance of Theorem 1: whenever a flip transformation has a nontrivial irrep (dimension 2), it forces a 1D submodule to vanish and yields entangled states in that subspace. If the transformation flips internal quantum numbers (charge), then the resulting states are necessarily Bell-like packaged entangled states.

4.3.3 Combined $\hat{P}\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, and $\hat{C}\hat{P}\hat{T}$

We have seen how the discrete symmetries $\hat{C}$ (charge conjugation), $\hat{P}$ (parity), and $\hat{T}$ (time reversal) each generates entangled states (Bell-like states) in the bipartite systems. Now we would like to go a step further to combine these operators into products such as $\hat{P}\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, and $\hat{C}\hat{P}\hat{T}$.[18, 63, 64, 65] The combined operators $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, and $\hat{C}\hat{P}\hat{T}$ act on both external (spin, momentum) and internal (charge) quantum numbers simultaneously. We then obtains hybrid packaged entangled states that cannot be factorized into a pure spin part times a pure charge part.

The hybrid packaged entangled states are especially relevant for analyzing CP or CPT violation in meson systems (e.g. kaons $|K^0\rangle, |\bar{K}^0\rangle$), where one forms CP eigenstates with definite entanglement structure. For example, we can use the hybrid packaged entangled states to classify states by ± 1 (or sometimes complex phases) under each discrete transformation, analyzing discrete-symmetry tests or violations (e.g. CP violation in neutral mesons), and enforcing superselection rules (e.g. total charge) while still forming valid entangled superpositions.

Below we will discuss each of these combined symmetries and show how they produce hybridized entangled states.

(1) $\hat{P}\hat{T}$ Symmetry. From earlier discussions, we know that $\hat{P}$ inverts space ($\mathbf{x} \leftrightarrow -\mathbf{x}$, $\mathbf{p} \leftrightarrow -\mathbf{p}$), and $\hat{T}$ reverses time ($t \leftrightarrow -t$, again $\mathbf{p} \leftrightarrow -\mathbf{p}$, flips spin, etc.). Acting twice on external two-level sometimes leads to a trivial net action on certain subspaces (e.g. $\hat{P}\hat{T} \approx \hat{I}$ on spin-0 or momentum-symmetric states). Symbolically:

$$\hat{P}\,\hat{T}\,|\Psi_s^{\pm}\rangle_{AB} \;=\; \pm\,\hat{P}\,|\Psi_s^{\pm}\rangle_{AB} \;=\; |\Psi_s^{\pm}\rangle_{AB}, \quad \hat{P}\,\hat{T}\,|\Phi_s^{\pm}\rangle_{AB} \;=\; \pm\,\hat{P}\,|\Phi_s^{\pm}\rangle_{AB} \;=\; |\Phi_s^{\pm}\rangle_{AB},$$

and similarly for momentum Bell states $|\Psi_m^{\pm}\rangle_{AB}, |\Phi_m^{\pm}\rangle_{AB}$. Thus, certain two-particle states become invariant (up to signs) under $\hat{P}\hat{T}$. In some frameworks (particularly integer-spin subspaces), $\hat{P}\hat{T}$ can act as the identity.

(2) $\hat{C}\hat{P}$ and $\hat{C}\hat{T}$ Symmetry. $\hat{C}\hat{P}$ simultaneously flips internal charges and inverts spatial coordinates (momenta/spins). Similarly, $\hat{C}\hat{T}$ flips charge and reverses spatial coordinates (momenta/spins). These operations mix internal and external two-level, allowing one to define states with definite eigenvalues under $\hat{C}\hat{P}$ or $\hat{C}\hat{T}$. Below we focus on $\hat{C}\hat{P}$ for concreteness. The same reasoning applies to $\hat{C}\hat{T}$, with time reversal $\hat{T}$ replacing $\hat{P}$.

For a two-particle system, if we consider internal IQNs $|P\rangle, |\bar{P}\rangle$, external spin-$\frac{1}{2}$ states $|\uparrow\rangle, |\downarrow\rangle$, and external momentum states $|+\rangle, |-\rangle$, then we obtain hybrid packaged entangled states:

$$\begin{aligned} |\Psi_h^{\pm}\rangle_{AB} &= \frac{1}{\sqrt{2}}\Big(|P,\uparrow,+\rangle_A\,|\bar{P},\downarrow,-\rangle_B \;\pm\; |\bar{P},\downarrow,-\rangle_A\,|P,\uparrow,+\rangle_B\Big), \\ |\Phi_h^{\pm}\rangle_{AB} &= \frac{1}{\sqrt{2}}\Big(|P,\uparrow,+\rangle_A\,|\bar{P},\uparrow,+\rangle_B \;\pm\; |\bar{P},\downarrow,-\rangle_A\,|P,\downarrow,-\rangle_B\Big). \end{aligned} \tag{10}$$

These are the eigenstates of $(\hat{C}\hat{P})_A \otimes (\hat{C}\hat{P})_B$ and $(\hat{C}\hat{T})_A \otimes (\hat{C}\hat{T})_B$ with eigenvalue ± 1, i.e.,

$$(\hat{C}\hat{P})_A \otimes (\hat{C}\hat{P})_B\,|\Psi_h^{\pm}\rangle_{AB} \;=\; \pm\,|\Psi_h^{\pm}\rangle_{AB}, \quad (\hat{C}\hat{P})_A \otimes (\hat{C}\hat{P})_B\,|\Phi_h^{\pm}\rangle_{AB} \;=\; \pm\,|\Phi_h^{\pm}\rangle_{AB}$$

and

$$(\hat{C}\hat{T})_A \otimes (\hat{C}\hat{T})_B\,|\Psi_h^{\pm}\rangle_{AB} \;=\; \pm\,|\Psi_h^{\pm}\rangle_{AB}, \quad (\hat{C}\hat{T})_A \otimes (\hat{C}\hat{T})_B\,|\Phi_h^{\pm}\rangle_{AB} \;=\; \pm\,|\Phi_h^{\pm}\rangle_{AB}.$$

These show that we mixed internal charges with external spin and momentum in hybridized packaged entangled states.

Example 3 (Neutral-Meson Systems (K^0, B^0, etc.))**.** *An important application arises in neutral mesons, such as kaons $|K^0,\uparrow\rangle$ vs. $|\bar{K}^0,\downarrow\rangle$. In analyzing CP violation or correlated decays, we often looks at eigenstates of $\hat{C}\hat{P}$.*

1. *For a single meson:*

$$\hat{C}\hat{P}\,|K^0,\uparrow\rangle = e^{i\alpha}\,|\bar{K}^0,\downarrow\rangle, \quad \hat{C}\hat{P}\,|\bar{K}^0,\downarrow\rangle = e^{-i\alpha}\,|K^0,\uparrow\rangle,$$

   This leads to the well-known CP-eigenstates

$$|K_{1,2}\rangle = \tfrac{1}{\sqrt{2}}\Big(|K^0,\uparrow\rangle \pm |\bar{K}^0,\downarrow\rangle\Big).$$

2. *For two mesons (e.g. in $\phi \to K^0\bar{K}^0$ decays), we can build correlated states that are eigenstates of $(\hat{C}\hat{P})_A \otimes (\hat{C}\hat{P})_B$. In that sense, they become packaged or hybrid entangled states in the internal (strangeness) degree of freedom and can exhibit correlated decays that test CP symmetry.*

(3) $\hat{C}\hat{P}\hat{T}$ Symmetry. In quantum field theory, CPT is a fundamental combined symmetry guaranteed by the CPT theorem: any Lorentz-invariant local QFT must be invariant under $\hat{C}\hat{P}\hat{T}$. In this case, one indeed consider the particles and antiparticles as an integrated system, which is invariant under the flip of charge $\hat{C} : Q \mapsto -Q$, spatial coordinates $\hat{P} : \mathbf{x} \mapsto -\mathbf{x}$, and time $\hat{T} : t \mapsto -t$.

However, here we are dealing with packaged entangled states, $\hat{P}\hat{T}$ may act trivially (or as an overall reflection) on certain subspaces. Consequently, $\hat{C}\hat{P}\hat{T}$ effectively reduces to $\hat{C}$. Regardless, for analyzing discrete-symmetry properties in meson-antimeson pairs, or more generally for exploring whether a process respects or breaks CPT, one can construct states with well-defined $\hat{C}\hat{P}\hat{T}$ transformation properties. If time reversal $\hat{T}$ is anti-unitary, extra care in defining eigenstates is needed, but in multi-particle systems it can still be consistent to speak of ± 1 CPT eigenvalues in certain subspaces.

Thus, these combined symmetries ($\hat{P}\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{C}\hat{P}\hat{T}$) yield a rich structure of hybrid entangled states, which can be systematically classified by how each operator acts on the internal and external DOFs. This classification underpins many discrete-symmetry tests in particle physics, such as detecting CP violation in neutral-meson decays, or examining whether CPT invariance might be violated in exotic scenarios.

4.4 Combination of Discrete and Gauge Symmetries: $G \times D$

(1) Combined Symmetry Group. In many theories, the symmetry group comprises both a continuous local gauge symmetry G and an additional discrete symmetry D.[66, 67, 68] For example, in quantum chromodynamics (QCD) the gauge group G might be SU(3) (local color symmetry), while an additional discrete symmetry D (such as charge conjugation C or baryon parity) is present. In such cases, the full classification of packaged states is refined by a combined label. The total symmetry group is given by the direct product

$$\mathcal{G} = G \times D.$$

The Hilbert space $\mathcal{H}$ of the theory then naturally decomposes into sectors characterized by both the net gauge (local) charge and the discrete charge. That is, if $\hat{Q}$ is the self-adjoint operator for the gauge charge and d labels the eigenvalues of the discrete symmetry, then one may write

$$\mathcal{H} = \bigoplus_{Q \in \sigma(\hat{Q})} \mathcal{H}_Q \quad \text{with} \quad \mathcal{H}_Q = \bigoplus_{d \in D} \mathcal{H}_{Q,d}\,.$$

Here, $\mathcal{H}_{Q,d}$ is the subspace of states with net gauge charge Q and discrete label d. For example, if baryon number is broken down to $\mathbb{Z}_N$, then each state carries a combined label (Q, d). In the case of a local gauge symmetry, superselection rules ensure that states belonging to different $\mathcal{H}_Q$ cannot interfere. In contrast, since D is usually a global symmetry, it often allows additional structure within a given gauge sector; that is, the discrete symmetry further partitions $\mathcal{H}_Q$ into finer sub-sectors and packages the wavefunction by imposing additional superselection rules.

(2) Irreps of the Combined Group $G \times D$. When a field transforms under both G and D, a single-particle creation operator $\hat{a}^\dagger$ carries a composite index:

$$\hat{a}^\dagger_{\alpha,d}(\mathbf{p}),$$

where α labels the irreducible representation (irrep) of the continuous gauge group G (for example, $\mathbf{3}$ for quarks in SU(3)) and d labels the discrete charge under D (for example, the eigenvalue ± 1 under a $\mathbb{Z}_2$ symmetry). Under a combined transformation $(g, \delta) \in G \times D$, the operator transforms as

$$U_{(g,\delta)}\, \hat{a}^\dagger_{\alpha,d}\, U^{-1}_{(g,\delta)} = \sum_{\alpha',d'} \Big[D^{(\alpha,d)}(g,\delta) \Big]_{\alpha',\, d';\, \alpha,\, d} \hat{a}^\dagger_{\alpha',d'}\,.$$

If the discrete group D is Abelian (e.g. $\mathbb{Z}_N$), the discrete part of the representation is typically one-dimensional (a phase factor, e.g. ω^k with $\omega = e^{2\pi i/N}$). Thus, the total transformation is a tensor product of the continuous and discrete parts, and the state's full charge is given by the combined label (Q, d).

(3) Physical Interpretation. While the local gauge symmetry G imposes a strict superselection rule (so that no state may be a superposition of different net gauge charges), the discrete symmetry D - being a global symmetry - allows an additional refinement within each gauge-charge sector. For instance, even if a state has net gauge charge Q, it can still carry a discrete label d from D. The decomposition

$$\mathcal{H}_Q = \bigoplus_{d \in D} \mathcal{H}_{Q,d}$$

means that states with different discrete labels cannot interfere with one another. In this sense, the discrete symmetry further packages the internal degrees of freedom, refining the overall structure to a combined charge (Q, d).

Example 4 ($SU(2) \times C$ (Charge Conjugation))**.** *For $SU(2)$ gauge symmetry, the fundamental representation is 2-dimensional. Let C denote charge conjugation, which, in this context, interchanges particles and antiparticles (in SU(2), a quark and an antiquark both transform as doublets, but with conjugate labels).*

- *A single-particle operator might be written as $\hat{a}^\dagger_{\alpha,c}(\mathbf{p})$, where $\alpha \in \{1, 2\}$ labels the $SU(2)$ index and $c \in \{+, -\}$ denotes the eigenvalue of the charge conjugation operator C (i.e. whether it is a particle or an antiparticle).*
- *Under a combined transformation $(g, C) \in SU(2) \times \mathbb{Z}_2$, the operator transforms as*

$$U(g, C)\, \hat{a}^\dagger_{\alpha,c}\, U(g, C)^{-1} \;=\; \sum_\beta D^{(\mathbf{2})}_{\beta\alpha}(g)\, \chi_c(C)\, \hat{a}^\dagger_{\beta,c'}\,,$$

  where $\chi_c(C)$ is the one-dimensional character of $\mathbb{Z}_2$ (for instance, $\chi_+(C) = +1$ and $\chi_-(C) = -1$).
- *The full Hilbert space then decomposes into sectors labeled by the net $SU(2)$ charge (which may be chosen by the rules of confinement) and a discrete label from C.*

Example 5 ($SU(3) \times \mathbb{Z}_N$ (Baryon Parity))**.** *In QCD, the local gauge group is $SU(3)$ (color). Suppose, in addition, there is an exact discrete symmetry $\mathbb{Z}_N$ associated with baryon parity (a remnant of a broken $U(1)$).*

- *Each quark creation operator is in the **3** of $SU(3)$ and carries a definite baryon number (say, $-1/3$) along with a discrete $\mathbb{Z}_N$ label. For simplicity, denote the discrete charge by $d \in \{0, 1, \ldots, N - 1\}$.*
- *A meson state, for example, is built from one quark and one antiquark:*

$$|M\rangle \;=\; \frac{1}{\sqrt{3}} \sum_{c=1}^{3} |q_c, d\rangle \otimes |\bar{q}_c, \bar{d}\rangle \;,$$

  where the color indices are summed so that the state is a color singlet, and the discrete charges satisfy $d + \bar{d} \equiv 0 \pmod{N}$.
- *The Hilbert space is then decomposed as*

$$\mathcal{H}_0 = \bigoplus_{d \in \mathbb{Z}_N} \mathcal{H}_{0,d}\,,$$

  where $\mathcal{H}_{0,d}$ is the subspace with net color singlet and discrete label d.
- *The discrete symmetry further refines the state classification: states with different $\mathbb{Z}_N$ labels belong to distinct superselection sectors, so that no physical observable can create interference between, say, $\mathcal{H}_{0,0}$ and $\mathcal{H}_{0,1}$.*

5 Classification of Packaged States Based on Differential Forms

In common gauge theories, we usually work with pointlike particles or local fields. We call the symmetries related to these objects 0-form symmetries. In modern physics, however, p-form symmetries are also important.[33, 69, 70, 71, 72, 73] In a p-form symmetry, the symmetry acts on objects that have codimension p, which can also be in packaged states.

In this section, we will introduce p-form symmetries with a focus on 1-form or 2-form in confining gauge theories. We will generalize the packaging principle (no partial factorization) to extended flux lines and surfaces. In other words, the flux carried by a Wilson loop or the flux on a membrane must appear as a complete, inseparable unit. We will also discuss mixed packaged objects and symmetries that combine 0-form charges and 1-form or 2-form fluxes. Finally, we will relate winding (topological) superselection on lattices to 1-form (or higher-form) charges. These topological sectors remain superselected because local operators cannot alter global flux quantum numbers.

5.1 Extended Packaged Objects and p-Form Symmetries

We now describe extended packaged objects in detail and show how it naturally leads to the notion of p-form symmetries acting on these extended packaged operators.

(1) Extended Packaged Objects. Consider an extended operator $O^{(p)}$ defined on a p-dimensional submanifold $\mathcal{M}_p$ of spacetime.[34, 35, 36] Such an operator might represent a flux tube (if $p = 1$) or a surface defect (if $p = 2$). Suppose $O^{(p)}$ is labeled by a charge (or flux) γ taking values in a group G (or in one of its representations). This charge γ is analogous to the electric charge or color charge in pointlike particles. It is fully packaged and cannot be subdivided. In analogy with the pointlike case, the packaging principle asserts that the operator $O^{(p)}$ transforms irreducibly under G; that is, the full flux label γ appears as an inseparable unit.

Assume that under a group transformation $g \in G$, the extended operator transforms as

$$U(g)\, O^{(p)}\, U(g)^{-1} = \rho(g)\, O^{(p)},$$

where ρ is a representation of G on the space of such operators. Since $\rho(g)$ is irreducible, there is no invariant subspace of $O^{(p)}$; hence, the entire flux label γ remains intact. If one attempted to split the operator into two parts,

$$O^{(p)} \stackrel{?}{=} O_1^{(p)} \otimes O_2^{(p)},$$

with individual labels γ_1 and γ_2 such that $\gamma_1 \cdot \gamma_2 = \gamma$, then by the linearity of the representation one would have

$$U(g)\, (O_1^{(p)} \otimes O_2^{(p)})\, U(g)^{-1} = \rho_1(g)\, O_1^{(p)} \otimes \rho_2(g)\, O_2^{(p)}.$$

If the original $O^{(p)}$ transforms irreducibly, no nontrivial decomposition into invariant subspaces exists. Therefore, the entire extended object must be treated as a single packaged block, and its flux cannot be fractionated. This is the extended analogue of the statement that, for pointlike excitations, the full internal charge is inseparable.

Example 6 (Flux Lines in Confining Theories)**.** *In a confining* SU(N) *gauge theory, a flux line (or Wilson loop) carries a center charge (e.g. in* $\mathbb{Z}_N$*). Under a center transformation,*

$$U(g)\,W(C)\,U(g)^{-1} = e^{2\pi ik/N}\,W(C),$$

where k is the charge of the loop. Since $W(C)$ is an irreducible operator, it is impossible to split the loop into parts that would carry fractional center charge. The entire flux line is a packaged object: its center charge is inseparable, similar to how a quark's color charge cannot be split.

In lattice formulations or theories with periodic boundary conditions, extended objects (flux lines or surfaces) may wrap around non-contractible cycles. This leads to topologically protected superselection sectors where the total flux is conserved.

(2) p-Form Symmetries. When dealing with extended operators, it is natural to introduce the concept of a p-form symmetry [32, 33, 37, 38]. In d-dimensional spacetime, a p-form global symmetry is characterized by conserved charges measured on $(d-p)$-dimensional manifolds. The charged operators under a p-form symmetry are supported on p-dimensional submanifolds.

Definition 1 (p-Form Symmetries)**.** *A p-form symmetry in d dimensions is defined by the existence of topological operators $U_g(\Sigma_{d-p})$ for each $g \in G^{(p)}$, where Σ_{d-p} is a closed $(d-p)$-dimensional manifold. These operators act on charged p-dimensional objects (such as Wilson loops for $p = 1$ or surface operators for $p = 2$) by*

$$U_g(\Sigma_{d-p})\,O^{(p)}\,U_g(\Sigma_{d-p})^{-1} = \chi(g)\,O^{(p)},$$

where $\chi(g)$ is a character (or, more generally, a matrix) of $G^{(p)}$.

In our context, a p-form symmetry ensures that the extended operator (which may represent a flux or topological defect) is an inseparable entity. It carries its full charge as one packaged unit. For example, in an SU(N) gauge theory the center symmetry $\mathbb{Z}_N$ acts as a 1-form symmetry on Wilson loops. Under a center transformation, a Wilson loop in the fundamental representation picks up a phase $e^{2\pi i/N}$, confirming that it carries an indivisible 1-form charge.

Example 7 (SU(N) Center as a 1-Form Symmetry)**.** *In an* SU(N) *gauge theory with center* $\mathbb{Z}_N$*, a Wilson loop $W(C)$ transforms as:*

$$U(g)\,W(C)\,U(g)^{-1} = e^{2\pi ik/N}\,W(C),$$

where k is the 1-form charge. The irreducibility of this transformation prevents one from splitting $W(C)$ into separate components with fractional charges.

5.2 Corollary: Existence of p-Form Symmetry Associated Packaged States

We now apply Theorem 1 to the case of p-form symmetries by replacing pointlike charged excitations with extended operators.

Corollary 2 (Existence of p-Form Symmetry Associated Packaged States)**.** *Consider a p-form symmetry group $G^{(p)}$ (finite or compact, including groups such as $U(1)$) in $(d+1)$-dimensional spacetime. This symmetry acts on operators supported on p-dimensional submanifolds (for example, line operators when $p=1$ or surface operators when $p=2$). Each extended operator is an irreducible block and is labeled by a complete p-form charge that cannot be split. Then the Hilbert space $\mathcal{H}$ of states built from such extended operators decomposes into charge sectors*

$$\mathcal{H} \cong \bigoplus_{Q \in \hat{G}^{(p)}} \mathcal{H}_Q,$$

where each sector $\mathcal{H}_Q$ consists of states with a definite total p-form charge Q. Specifically:

1. ***Symmetry Action on Extended Operators:*** *Let $O^{(p)}$ denote a p-dimensional charged operator. Then for any $g \in G^{(p)}$ and for any closed $(d-p)$-dimensional manifold Σ_{d-p} that links with the support of $O^{(p)}$, the topological operator $U_g(\Sigma_{d-p})$ acts as*
$$U_g(\Sigma_{d-p})\, O^{(p)}\, U_g(\Sigma_{d-p})^{-1} = \chi(g)\, O^{(p)},$$
where $\chi(g)$ is the character (or matrix) specifying the irreducible transformation. In this way, $O^{(p)}$ is seen to be a packaged object.

2. ***Charge Sectors and Irreducible Representations:*** *The full Hilbert space decomposes into sectors $\mathcal{H}_Q$ with fixed total p-form charge Q. In a given sector, every state $|\Psi_Q\rangle \in \mathcal{H}_Q$ transforms under an irreducible representation ρ_Q of $G^{(p)}$; that is, for all $g \in G^{(p)}$,*
$$U(g)\,|\Psi_Q\rangle = \rho_Q(g)\,|\Psi_Q\rangle.$$
For Abelian p-form symmetries, $\rho_Q(g)$ is a one-dimensional phase factor, while for non-Abelian symmetries, ρ_Q may be higher-dimensional.

3. ***Packaged Entangled States:*** *Within each fixed sector $\mathcal{H}_Q$, one can construct an orthonormal basis of packaged entangled states. In these states, the charge degrees of freedom of the individual extended operators are locked together in a way that yields a definite total charge Q. This is achieved by taking appropriate linear combinations (a projection) of product states of extended operators, analogous to the 0-form case discussed in Theorem 1.*

4. ***Module Isomorphism to the Abstract Representation:*** *Let V_Q denote the abstract vector space carrying the irreducible representation ρ_Q of $G^{(p)}$. Then there exists an isomorphism of $G^{(p)}$-modules*
$$\Phi_Q : \mathcal{H}_Q \to V_Q,$$
such that for every $g \in G^{(p)}$ and every basis state $|\Psi_Q^{(i)}\rangle \in \mathcal{H}_Q$,
$$\Phi_Q\Big(U(g)\,|\Psi_Q^{(i)}\rangle\Big) = \rho_Q(g)\,\Phi_Q\Big(|\Psi_Q^{(i)}\rangle\Big).$$
This mapping guarantees that the space of packaged entangled states in the charge sector is precisely isomorphic to the representation space for the irreducible representation associated with the p-form charge Q. In particular, when extended operators carrying charges Q_1 and Q_2 are combined, the resulting composite state carries the charge given by the group product $Q_1 \cdot Q_2$ (or, in the Abelian case, the sum $Q_1 + Q_2$ modulo any constraints), with the corresponding representation obtained via the tensor product and its Clebsch-Gordan decomposition.

Thus, any collection of p-form charged operators can be regarded as a single composite object transforming in an irreducible representation of $G^{(p)}$. This corollary extends the packaging principle of Theorem 1 to higher-form symmetries, demonstrating that the multi-object state space breaks into irreducible charge sectors that transform under $G^{(p)}$ exactly as dictated by the group structure.

5.3 Classification Based on 1-Form Symmetry

A 1-form symmetry acts on line operators (such as Wilson loops or vortex lines) rather than on local (pointlike) fields. In lattice gauge theory [14, 74, 75, 76], there are plenty of these examples where flux loops or vortex lines carry definite 1-form charges.

(1) Flux-Loop States in Lattice Gauge Theory. A flux-loop state [75] in a lattice gauge theory can be written as

$$|\Psi_{\text{flux}}\rangle \;=\; \sum_{\{r_\ell\}} \beta\Big(\{r_\ell\}\Big) \;\bigotimes_{\ell} |r_\ell\rangle,$$

where each state $|r_\ell\rangle$ is a basis state on link ℓ (carrying electric flux or a gauge element) and the coefficients $\beta(\{r_\ell\})$ enforce Gauss's law at every vertex. In particular, nonzero amplitudes β select a specific winding sector, so this wavefunction describes a packaged flux configuration.

A 1-form packaged state, such as a flux line, carries its entire flux as a single, inseparable unit. Any attempt to divide the flux among parts would violate the irreducibility (packaging) condition.

(2) Field-Theoretic Description of 1-Form Charges. At the field-theory level, a 1-form symmetry may be represented by a topological current $J^{\mu_1\cdots\mu_{d-1}}$ or a two-form gauge field $B_{\mu\nu}$. In either case, the 1-form charge is measured on a codimension-1 manifold and captures the total flux (or vortex linking) through that manifold. Practically, we often labels how a Wilson loop picks up a phase upon linking with a topological defect insertion.

Because local operators only act on small, localized regions, they cannot globally reconfigure flux lines that wrap non-contractible cycles. We cannot locally cut or rearrange entire flux loops, which is why winding numbers or global flux sectors exist.

(3) Global Flux and Winding. In 2D or 3D lattice gauge theories with periodic boundary conditions (like a torus), we can defines integer fluxes such as

$$W_x \;=\; \sum_{\ell\in\text{loop}_x} E_\ell, \quad W_y \;=\; \sum_{\ell\in\text{loop}_y} E_\ell, \quad \ldots,$$

where loop_x or loop_y traverse non-contractible cycles, and E_ℓ is an integer electric flux on link ℓ. Each set $\{W_x, W_y, \ldots\}$ labels a distinct global flux (winding) sector. A state $|W_x, W_y, \ldots\rangle$ is topologically different from another state with different flux numbers. Local operators cannot change the net flux around a cycle, so these winding sectors do not mix.

Such invariance shows how a 1-form symmetry controls flux lines' behave. The winding number acts like the charge for this 1-form symmetry. States like $|W\rangle$ and $|W'\rangle$ represent different 1-form irreps: discrete flux ($\mathbb{Z}$ or $\mathbb{Z}_N$), or continuous real-valued flux ($U(1)$).

(4) Winding (Topological) Superselection. On a torus or similar periodic manifold, changing W_x or W_y requires nonlocal flux manipulation. This results in winding-number superselection. A local Hamiltonian cannot change the net flux that goes around a cycle. By Gauss's law, flux lines must either close on themselves or terminate on charges. Hence, any loop that circles the system remains stable unless a large (instanton-like) process intervenes.[69, 70] Thus, states $|W = 0\rangle$ and $|W = 1\rangle$ (or any integer label) form distinct superselection sectors. Local measurements cannot show interference between them.

(5) Instantons / Large Gauge Transformations. In certain special cases (e.g., lower dimensionality or special boundary conditions), large gauge transformations or instanton events can connect states with different winding [**?**], creating a finite amplitude for transitions between different sectors $|W\rangle \leftrightarrow |W'\rangle$. In many cases (for example, in large volumes or strong coupling) such tunneling is negligible. Thus, the winding sectors remain fixed.

(6) 1-Form Packaged Entanglement. Line operators with 1-form charges obey the above rules. Their irreducible representation prevents any partial splitting of the charge.

Example 8 (Discrete 1-Form Example: $\mathbb{Z}_N$ Flux)**.** *Consider a 1-form symmetry group $\mathbb{Z}_N$. Each line operator transforms in a 1D irrep labeled by $k \in \{0, \ldots, N-1\}$. Denote the line operators by*

$$\hat{\mathcal{W}}^{\dagger}_{(k)}(\text{loop}).$$

It picks up a phase ω^k, with $\omega = e^{2\pi i/N}$, under the 1-form transformation.

Suppose we have a two-line system:

$$\hat{\mathcal{W}}^{\dagger}_{(k_1)}(\text{loop}_1)\, \hat{\mathcal{W}}^{\dagger}_{(k_2)}(\text{loop}_2)\, |0\rangle \;\in\; \mathbf{R}_{k_1} \;\otimes\; \mathbf{R}_{k_2}.$$

Because each $\mathbf{R}_k$ is 1D, a multi-line state dimension is $\dim(\mathbf{R}_{k_1} \otimes \mathbf{R}_{k_2}) = 1$. But the total 1-form charge is $(k_1 + k_2) \bmod N$.

If physical boundary conditions require net zero flux (like no external source), we project onto $k_1 + k_2 \equiv 0 \mod N$. Mathematically:

$$P_{\text{net}=0} \;=\; \frac{1}{N} \sum_{j=0}^{N-1} \Big(U_j\Big),$$

where U_j are the 1-form transformations shifting each line operator's label. Only states with $k_1 + k_2 = 0 \mod N$ remain.

Such projection can entangle line operators if they are localizable or partially distinct in the wavefunction. We cannot have half a line with charge k_1 and half with (k_2) inside the same line. Instead, we either have two separate lines or a single line in irreps $k_1 + k_2$.

Even though each line operator is one-dimensional, the full label $k_1 + k_2 \mod N$ remains intact, meaning the flux is packaged as a whole.

Example 9 (Continuous 1-Form Example: $U(1)$ Flux)**.** *For a continuous 1-form group $U(1)$, a line operator may pick up a phase $e^{i\theta\,\Phi}$, where $\Phi \in \mathbb{R}$ is the flux. If two flux lines have fluxes Φ_1 and Φ_2, the the total flux is $\Phi_1 + \Phi_2$. If physical states require net flux zero, then $\Phi_1 + \Phi_2 = 0$. This condition inevitably correlates the lines in a non-separate way.*

In both cases, packaged entanglement comes from the requirement that line operators must form a complete net representation (often net zero flux).

5.4 Classification Based on 2-Form Symmetry

A 2-form symmetry in 3+1D (i.e. $d = 4$) acts on codimension-2 surfaces, i.e., 2D surfaces in spacetime.[32, 33, 34, 35, 36, 37, 38] Examples include Wilson surfaces in certain topological field theories or membrane excitations in higher-rank gauge theories.

(1) Basic Ideas. A 2-form symmetry can be envisioned as a topological current $J^{\mu_1\mu_2}$ (a 2-form current) whose charge is measured on codimension-2 surfaces. In 3+1D, a closed 2D surface Σ can detect the 2-form charge by

$$Q(\Sigma) \;=\; \int_{\Sigma} \star\, J.$$

Under a 2-form symmetry transformation, a 2D surface operator $U(\Sigma)$ may be multiplied by a phase or a more general group element. In discrete cases (for example, $\mathbb{Z}_N$), each surface operator is in a 1D irreps of $\mathbb{Z}_N$. If that irreps label is nontrivial, one cannot split the surface into parts with fractional labels. This mirrors the packaging principle for 0-form or 1-form charges.

Example 10 (2-Form Center Symmetry)**.** *In some 4D gauge theories, there is a 2-form center symmetry acting on surfaces that measure the flux of a higher-rank gauge field* $B_{\mu\nu}$*. This symmetry fixes the net sheet flux so that local operators cannot change part of the flux on a surface.*

(2) Surfaces as 2-Form Charges. In 3+1D, a surface operator $\hat{\mathcal{S}}^{\dagger}(\Sigma)$ creates a 2D surface Σ. It transforms in an irrep of the 2-form group. If its irreps label is ℓ, then one cannot break the surface into parts with different label. By irreps indivisibility, a 2D surface is packaged with its full 2-form charge: For discrete $\mathbb{Z}_N$ case, the surface operator might pick up a phase $\exp\!\big(2\pi i\, k/N\big)$ upon linking with certain defects. The label $k \mod N$ cannot be split between partial surfaces. For continuous $U(1)$ case, the operator might accumulate a continuous phase $\exp\!\big(i\,\alpha\, Q(\Sigma)\big)$. In both cases, the whole surface is a single unit. Just as a quark's color charge is packaged, a surface operator's flux label is packaged as an inseparable unit.

Similar to that of 1-form flux line charges, we cannot split 2-form charges, which is the direct analogue of the packaging principle for extended objects.

Example 11 (3+1D Membrane/Surface Excitations)**.** *A topological membrane is a closed 2D surface carrying discrete/continuous flux of a 2-form field* $B_{\mu\nu}$*. A Wilson surface generalizes a Wilson loop to a 2D operator with* $\exp(\int B)$ *for a 2-form field.*

If the theory confines such surfaces (energy grows with area), then they cannot appear as free excitations. This leads to superselection for net 2-form flux much like flux tubes in 1-form symmetry.

(3) 2-Form Surface Operators in Detail. Consider a surface operator $\hat{\mathcal{S}}^{\dagger}(\Sigma)$ that inserts a 2D surface Σ. It transforms in some irreps of the 2-form group:

1. Discrete $\mathbb{Z}_N$: The operator may pick up a phrase $\exp(2\pi i\, k/N)$ if it links with a topological defect carrying $\mathbb{Z}_N$ 2-form charge. The label $k \mod N$ is fixed for the entire surface.

2. Continuous $U(1)$: The operator may accumulate a continuous phase $\exp\Big(i\alpha\, Q(\Sigma)\Big)$ and the entire surface is a single block.

This is similar to the Clebsch-Gordan approach for color or 1-form flux lines. The 2D surfaces combine additively. The irreducibility means that one cannot split a surface into pieces with different topological labels.

(4) Topological Sectors and 2-Form Superselection. If the 2-form symmetry is confining (energy scaling with surface area), then a single nontrivial surface is not a free particle. We usually need boundary excitations or additional surfaces to cancel out the net 2-form flux. Thus, different total 2-form charges on large surfaces can form distinct superselection sectors.

- 1-Form vs. 2-Form: In 1-form symmetry, flux tubes [71, 72] exhibit an area or perimeter law. In 2-form symmetry, membranes can exhibit an area law that ensures they close on themselves or attach to lower-dimensional excitations.
- Local Operators: They cannot globally re-assign the 2-form flux carried by a large surface. Only a globally nontrivial transformation can change it. Locally, the 2-form charge is protected and results in superselection among 2D defect sectors.

Like 1-form winding around a torus, a 2-form symmetry in 3+1D can define flux $\int_{T^2} B_{\mu\nu}$ on a 2D cycle. The fluxes in different flux sectors are not connectable and therefore are superselected.

(5) Mathematical Derivations. For discrete $\mathbb{Z}_N$ 2-form symmetry in 3+1D, we often introduces a 2-form gauge field $B_{\mu\nu} \in \mathbb{Z}_N$.

Then the action can be written as:

$$S \;\approx\; \frac{N}{2\pi} \sum_{\text{plaquettes } p} B_p \wedge dB_p \quad \text{(schematic)}.$$

If a flux encloses a hole or boundary, the gauge transformations only shift $B_{\mu\nu}$ but cannot break the flux.

We can define a surface operator:

$$U(\Sigma) \;=\; \exp\Big(i \sum_{p\subset\Sigma} B_p\Big),$$

to label the discrete flux crossing Σ. If $U(\Sigma)$ belongs to irreps $\ell \mod N$, then it picks up $\exp\Big(2\pi i\, \ell/N\Big)$ that links a topological defect.

The entire surface Σ carries the same label ℓ. Splitting it to $\ell' \neq \ell$ and $\ell - \ell'$ would yield two distinct surface operators, not a single irreps. This is forbidden on 2-form surfaces, which is similar to no partial color or no partial 1-form flux.

(6) 2-Form Packaged Entanglement. Surface operators with 2-form charges are packaged as a single block. When several surfaces are present, the net 2-form flux is the sum (modulo the group law). If physical conditions require net zero flux, then the surfaces must combine into an entangled state that cannot be factored into independent parts.

Example 12 (Surfaces in Discrete 2-Form Groups)**.** *Take $\mathbb{Z}_N$ again, but now as a 2-form group. Each 2D surface operator $\hat{\mathcal{S}}^\dagger_{(m)}(\Sigma)$ transforms with a phase ω^m. Suppose we have two surfaces with charges m_1 and m_2. The total system is:*

$$\hat{\mathcal{S}}^\dagger_{(m_1)}(\Sigma_1)\,\hat{\mathcal{S}}^\dagger_{(m_2)}(\Sigma_2)\,|0\rangle \;\in\; \mathbf{R}_{m_1} \otimes \mathbf{R}_{m_2}.$$

Again, each factor is 1D, but the net 2-form flux is $(m_1 + m_2) \bmod N$. If the theory confines 2-form flux (like an area law for membranes), only total flux 0 might appear as a stable or gauge-allowed boundary condition. Therefore, we project onto $m_1 + m_2 = 0 \bmod N$. If you had one big surface carrying label m, you cannot fractionate it into partial sub-surfaces with $m_1 \neq m_2$ inside the same operator; that would break the irreps structure.

Example 13 (Surfaces in Continuous 2-Form)**.** *One can also have a continuous 2-form gauge field $B_{\mu\nu} \in U(1)$. A surface operator obtains a phase $\exp\Big(i\,\alpha \cdot \Phi_\Sigma\Big)$, where Φ_Σ is the total flux through Σ. Combining surfaces to form net flux zero can again entangle them if the boundary conditions or global constraints require neutrality.*

Thus, the entire 2D surface remains a single block in a 2-form irreps label, and partial fractionation is disallowed. If you have multiple surfaces, net flux neutrality can tie them together in a single wavefunction.

5.5 Mixed Objects and Symmetries

In realistic gauge theories, multiple forms of charges (0-form, 1-form, 2-form, etc.) often coexist.[77] For instance, pointlike excitations with 0-form charges (quarks or electrons), line operators with 1-form flux (Wilson loops or vortex lines), and surface operators with 2-form charges (membranes or domain walls in 3+1D).

A natural question is how these distinct charges combine into a single physical state and why cannot be partially separated.

(1) Constructing a Mixed State. Suppose we have a local operator $\hat{a}^\dagger_\alpha$ (0-form), a line operator $\hat{\mathcal{W}}^\dagger_\gamma$ (1-form), and a surface operator $\hat{\mathcal{S}}^\dagger_\delta$ (2-form). A general state may be written as:

$$\hat{a}^\dagger_{\alpha_1}(\mathbf{x}_1)\,\hat{a}^\dagger_{\alpha_2}(\mathbf{x}_2)\,\hat{\mathcal{W}}^\dagger_{\gamma_1}(\text{loop}_1)\,\hat{\mathcal{S}}^\dagger_{\delta_1}(\text{surface}_1)\,\Big|0\Big\rangle,$$

where each $\hat{a}^\dagger_{\alpha_i}$ carries a 0-form irreps $\mathbf{R}_{\alpha_i}$ (for example, color $\mathbf{3}$ or electric charge ± 1), $\hat{\mathcal{W}}^\dagger_\gamma$ is a line operator in a 1-form irrep $\mathbf{R}_\gamma$ (for example, center $\mathbb{Z}_N$), and $\hat{\mathcal{S}}^\dagger_\delta$ is a surface operator in a 2-form irrep $\mathbf{R}_\delta$.

This state may represent a hadron that encloses a flux loop or domain wall. Each part is a complete irreducible block: the local charges (0-form quarks) must neutralize color, the line operator (1-form flux) might close into a loop or attach to sources, the surface operator (2-form) might represent a discrete membrane or domain.

(2) Product of Irreps. Let $\mathcal{G}_0$ be the 0-form gauge group (for example, SU(3) color), $\mathcal{G}_1$ be the 1-form group (for example, center symmetry $\mathbb{Z}_N$), and $\mathcal{G}_2$ be the 2-form group (if applicable). The full symmetry group may be written as

$$\mathcal{G} \;=\; \mathcal{G}_0 \;\times\; \mathcal{G}_1 \;\times\; \mathcal{G}_2,$$

If you pick four excitations labeled $\alpha, \beta, \gamma, \delta$, then the combined color/flavor/flux wavefunction transforms in the tensor product

$$\mathbf{R}_\alpha \otimes \mathbf{R}_\beta \otimes \mathbf{R}_\gamma \otimes \mathbf{R}_\delta.$$

Physical states must either project onto the neutral representation in each sector (for example, a color singlet or net zero flux) or meet boundary conditions that allow a nonzero net charge. This projection step creates an entangled wavefunction across local and extended parts.

(3) No Partial Factorization. The packaging principle means that one cannot split a single irreps of the 1-form or 2-form group. For example, a line operator $\hat{\mathcal{W}}_\gamma^\dagger$ with label γ cannot be divided into two parts with different labels unless they are separate operators. The same is true for a surface operator.

This is the extended version of color quarks (0-form): the full color irreps $\mathbf{3}$ is locked into one creation operator. Similarly, a flux line with irreps γ or a membrane with irreps δ is locked as a single block.

Example 14. *Mixed Objects:*

1. *Hadron + Flux Tube: Imagine two quarks in color* $\mathbf{3}\times\mathbf{3}$ *together with a line operator carrying a center flux from a* $\mathbf{Z}_N$ *group. If the system must be a color singlet* $\mathbf{1}$ *and has net zero flux, the the wavefunction will mix the color and flux DOFs in an entangled manner. No partial splitting of color or flux is allowed.*

2. *Vortex Lines in 2D: In 2D, a vortex line is pointlike, but the same rule applies: one cannot split a vortex if it forms an irreducible representation of a discrete 1-form.*

3. *Membrane in 3+1D: A 2-form surface operator with a discrete label* $\ell \in \{0, \ldots, N-1\}$ *must appear as one complete block. When combined with local excitations, the total state must have net zero 2-form charge.*

(4) Superselection Sectors of Mixed Objects. A superselection sector of mixed objects is a subspace of the Hilbert space labeled by a charge (0-form, 1-form, or 2-form) that cannot be changed by local operators.

If a gauge theory confines color (0-form) or flux (1-form, 2-form), then non-singlet (or non-neutral) states exist only as internal excitations or short-range bound states. They do not appear as free states. In toroidal lattices or nontrivial manifolds, the net flux/winding on non-contractible loops or surfaces is discrete. Local moves cannot reassign that global flux, so each sector is superselected.

Thus, states with different net 1-form or 2-form charges are disjoint, much as color singlet and nonsinglet sectors are for 0-form charges. This extends to all forms simultaneously: each sector for 0-form, 1-form, 2-form, etc. must satisfy the neutrality or boundary condition.

(5) Mathematical Illustrations.

1. Tensor Product of Multiple Sectors: Let $\mathbf{R}_0$ be the 0-form color irreps, $\mathbf{R}_1$ be the 1-form flux irreps, and $\mathbf{R}_2$ be the 2-form surface irreps, etc. A combined multi-particle state belongs to
$$\mathbf{R}_0 \otimes \mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \cdots .$$

If the theory requires net neutrality in each sector, we project onto the trivial subrep in each sector. This step usually produces an entangled wavefunction across color, flux lines, and surfaces.

2. Projector onto Neutral/Trivial Representations: In each sector, one defines

$$P_{\text{trivial}} \;=\; \frac{1}{|G_{\text{form}}|} \sum_{g \in G_{\text{form}}} U_g,$$

where U_g implements the gauge transformation or global symmetry transformation in that sector. Only states with net charge 0 survive. Attempting to remain in a single subrep also enforces packaging: no partial irreps mixing.

3. Local versus Global Operations: Local operators act on small regions and cannot change the global charge or flux that runs around noncontractible cycles. Only large gauge transformations can change the sector.

Thus, the extended packaging principle emerges mathematically: the irreps structure in each form group forbids partial fractionation, and the superselection or confinement conditions enforce that physically observed states are net neutral across all forms.

(6) Packaged Entanglement of Mixed Objects. Because each local or extended operator is an irreps block, the full wavefunction must lie in a single overall representation. This requirement produces nontrivial packaged entanglement among 0-form charges, 1-form flux, or 2-form flux. For example, a hadron may contain both color charges (quarks) and a flux loop or domain wall. The flux and local charges become interwoven, and the state cannot be factored into separate parts. In this way, line operators can bind with local excitations to form a gauge-invariant whole. Such mixed states show that the packaging principle applies to both local and extended charges.

Now consider a gauge theory with 0-form (color/electric) charges, plus 1-form flux lines, plus 2-form surfaces in 3+1D. A general multi-object state may be written as:

$$\Bigl(\hat{a}^\dagger_{\alpha_1}\hat{a}^\dagger_{\alpha_2}\cdots\Bigr)_{\text{0-form}} \times \Bigl(\hat{\mathcal{W}}^\dagger_{\gamma_1}\hat{\mathcal{W}}^\dagger_{\gamma_2}\cdots\Bigr)_{\text{1-form}} \times \Bigl(\hat{\mathcal{S}}^\dagger_{\delta_1}\hat{\mathcal{S}}^\dagger_{\delta_2}\cdots\Bigr)_{\text{2-form}} |0\rangle,$$

where each local or extended operator is an irreps block under its respective group. The total state then belongs to

$$\mathbf{R}_{\alpha_1} \otimes \cdots \otimes \mathbf{R}_{\gamma_1} \otimes \cdots \otimes \mathbf{R}_{\delta_1} \otimes \cdots .$$

Physical states often require net neutrality in each form:

- 0-form: color singlet $\mathbf{1}$, or electric net zero charge.
- 1-form: total net zero charge or consistent with boundary conditions.
- 2-form: net surface net zero charge or allowed by boundary data.

Mathematically, one introduces projectors onto the trivial representation in each sector. For example, one can applies a projector

$$P^{(\text{0-form})}_{\text{singlet}} \;\wedge\; P^{(\text{1-form})}_{\text{net}=0} \;\wedge\; P^{(\text{2-form})}_{\text{net}=0}$$

to the entire multi-object Hilbert space. Only states that pass all three projectors remain as physically valid states.

Because these conditions tie together the local charges (0-form) and the extended fluxes (1-form or 2-form), the resulting state can be entangled across different forms of charge. For example, if a quark system tries to neutralize color while a line operator tries to neutralize flux, the constraints can force an overall superposition that correlates quark color states with flux line states.

Example 15 (Example: Baryon + Flux Tube + Membrane)**.** *Suppose we have 3 quarks in color* $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$*. A line operator with center flux label* γ *and a surface operator with some 2-form label* δ*. The entire system is in*

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \ \otimes \ \mathbf{R}_\gamma \ \otimes \ \mathbf{R}_\delta.$$

We require color neutrality $\mathbf{1} \subset \mathbf{3}^{\otimes 3}$*, plus 1-form net zero charge, plus 2-form net zero charge. Applying projectors, we obtain a physical subspace that is usually a small dimensional space of superpositions. We correlate color states with the line and surface flux states to ensure net* $(0,0,0)$ *in each form. Finally, we obtain a packaged entangled state that cannot be factored into separate "color* $\times$ *flux* $\times$ *surface" parts.*

5.6 Extended Packaging Principle

Based on the original packaging principle defined for pointlike charges (0-form) [2], we now extend it to higher-form charges:

Definition 2 (Extended Packaging Principle)**.** *Let a gauge theory include higher-form symmetries (1-form, 2-form, or in general p-form). Under these symmetries, extended objects (lines, surfaces, etc.) carry flux or charge labels in irreducible representations of the p-form group. Also, superselection rules prevent coherent mixing of different net flux (or winding) sectors. Then:*

1. ***No Partial Factorization of*** p***-Form Charges:*** *Each creation or annihilation operator for an extended excitation (for example, a line with 1-form charge* γ*) must transform as a complete irreducible representation of the p-form group. One cannot split or partly factor the flux label* γ*. In other words, no single extended object can be divided into fractional flux parts.*

2. ***Single Net-Flux Superselection Sector:*** *Physical states cannot mix different net fluxes or winding numbers. All superpositions must lie in a single net-flux sector. The total flux around noncontractible loops or surfaces is fixed. This prevents interference between different sectors.*

3. ***Flux Packaged Entanglement:*** *Within a fixed flux sector, multiple extended excitations (lines, surfaces) can form a nonseparable superposition. We call this flux packaged entanglement because it uses irreducible representations of the p-form group in a way that forbids partial flux splitting. One cannot entangle part of the flux while leaving the rest separate, just as color or electric charge is locked together in the 0-form case.*

4. ***Hybridization with 0-Form or External DOFs:*** *In theories that also have ordinary gauge charges (0-form) or external degrees of freedom (such as spin or*

momentum), one can form hybrid states that mix local particles with extended excitations. Measuring an external or 0-form degree of freedom can collapse the entire extended flux wavefunction while preserving the net flux. Similarly, internal 0-form charges can be entangled with 1-form or 2-form fluxes, each subject to its own superselection rule.

The extended packaging principle asserts that all charges and fluxes remain as inseparable, irreducible blocks. It doesn't matter whether they are 0-form (local) charges like a color charge in **3**, 1-form charges such as a flux line with label γ, or even higher-form charges. In each sector, the corresponding charge is fully packaged and cannot be broken into smaller pieces. Moreover, net flux or winding rules impose superselection so that states in different topological sectors do not mix. When these sectors combine, the full state is given by the tensor product of these packaged irreducible blocks. This ensures that one cannot peel off a fraction of a charge from any single sector.

6 Representation-Theoretic Invariants and Entanglement Measures

In previous sections, we classified packaged states by decomposing the Hilbert space into charge sectors corresponding to irreducible representations of the gauge group G.[8, 9, 10] This classification reveals the structure of internal quantum numbers (IQNs) and how they are inseparably packaged into each single-particle operator. However, this classification alone does not quantify the degree of entanglement among these internal degrees of freedom. In gauge theories, additional constraints arise from superselection rules and the requirement that physical observables be gauge invariant. As a result, standard entanglement measures (such as concurrence or negativity) must be adapted to this context.[78, 79, 80, 81]

In this section, we develop representation-theoretic invariants that serve as entanglement measures tailored for these packaged states.[82, 83, 84, 85] These invariants are constructed so that they remain unchanged under the action of the gauge group G and respect the superselection rules. In particular, we focus on invariants obtained by projecting onto singlet subspaces or by constructing gauge-invariant combinations of the state coefficients. This analysis complements the earlier classification by providing quantitative tools to assess entanglement in the presence of gauge constraints.

6.1 Classification of Packaged Entanglements Using Group Invariants

Let $|\Psi\rangle \in \mathcal{H}_Q$ be a state in a fixed charge (or color) sector. In gauge theories, only states within a single irreducible charge sector are physically allowed, and all observables must commute with the gauge transformations. Consequently, any entanglement measure must be constructed from gauge-invariant quantities. Here we describe two complementary approaches.

(1) Invariants from Projection onto Singlet Subspaces. Suppose that the representation space V in which $|\Psi\rangle$ lives can be decomposed as

$$V \cong V_{\mathbf{1}} \oplus V_{\text{rest}},$$

where $V_{\mathbf{1}}$ is the singlet subspace. Then one may define the gauge-invariant quantity

$$P_{\mathbf{1}} = \langle \Psi | \hat{\mathcal{P}}_{\mathbf{1}} | \Psi \rangle,$$

where $\hat{\mathcal{P}}_{\mathbf{1}}$ is the projection operator onto $V_{\mathbf{1}}$. By construction, $P_{\mathbf{1}}$ is invariant under G and measures the singlet (i.e. fully packaged) component of $|\Psi\rangle$. In confining theories, only states with $P_{\mathbf{1}} = 1$ may appear as free particles.

(2) Gauge-Invariant Partial Transpose and Concurrence. In conventional quantum information, the partial transpose of a density matrix (or the concurrence derived from it) is a common measure of entanglement. However, when the Hilbert space carries gauge indices, a naive partial transpose may break gauge invariance. One strategy is to first project the state onto a gauge-invariant subspace and then perform the partial transpose on the remaining (external) degrees of freedom. Specifically, let

$$\rho = |\Psi\rangle\langle\Psi|$$

be the density matrix of the state, and let $P_{\mathbf{1}}$ be the projection operator onto the singlet subspace. Then define

$$\tilde{\rho} = P_{\mathbf{1}}\, \rho\, P_{\mathbf{1}}.$$

The partial transpose is taken on the degrees of freedom that are not associated with the gauge symmetry. This ensures that the resulting negativity (or other measure) is gauge invariant.

Detailed Derivation: For a state $|\Psi\rangle$ in $\mathcal{H}_Q$, assume we have expanded it in a basis $\{|\Psi^{(i)}\rangle\}$ that is adapted to the decomposition

$$\mathcal{H}_Q \cong V_{\mathbf{1}} \oplus V_{\text{rest}}.$$

The projection operator $\hat{\mathcal{P}}_{\mathbf{1}}$ is constructed from the invariant tensors (or by the projection methods discussed earlier) so that for any $g \in G$,

$$U(g)\, \hat{\mathcal{P}}_{\mathbf{1}}\, U(g)^{-1} = \hat{\mathcal{P}}_{\mathbf{1}}.$$

Then the invariant quantity

$$P_{\mathbf{1}} = \langle \Psi | \hat{\mathcal{P}}_{\mathbf{1}} | \Psi \rangle$$

remains constant under the gauge transformation, i.e.,

$$P_{\mathbf{1}} = \langle \Psi | U(g)^{-1}\, \hat{\mathcal{P}}_{\mathbf{1}}\, U(g) | \Psi \rangle.$$

This is a clear, representation-theoretic invariant that quantifies the packaged (singlet) content of $|\Psi\rangle$.

6.2 Relating Superselection Rules to the Center of the Group

For non-Abelian gauge groups, the center $Z(G)$ plays a crucial role in enforcing superselection rules. For example, in $\mathrm{SU}(N)$ the center is $\mathbb{Z}_N$. States that transform nontrivially under $Z(G)$ cannot be superposed with those that are invariant under $Z(G)$.

(1) Center Charges and Superselection. If a state $|\Psi\rangle$ satisfies

$$U(z)|\Psi\rangle = \chi(z)|\Psi\rangle \quad \forall z \in Z(G),$$

then the character $\chi(z)$ labels a superselection sector. In a confining theory, only states with trivial center charge (e.g., the color singlet in QCD) are observed as free states.

(2) Invariant Quantities from Center Charges. One can define a gauge-invariant fidelity

$$F = \langle\Psi|\hat{\mathcal{P}}_{Z=1}|\Psi\rangle,$$

where $\hat{\mathcal{P}}_{Z=1}$ projects onto the sector with trivial center charge. Such invariants ensure that any entanglement measure is computed within a single superselection sector, preserving gauge invariance.

6.3 Examples of Representation-Theoretic Entanglement Measures

Example 16 (Color Singlet Overlap in QCD)**.** *Consider a meson state formed by a quark* ($\mathbf{3}$) *and an antiquark* ($\overline{\mathbf{3}}$) *in QCD:*

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}.$$

A general state can be written as

$$|\Psi_M\rangle = \alpha\,|\mathbf{1}\rangle + \sum_{a=1}^{8} \beta_a\,|\mathbf{8}, a\rangle.$$

The gauge-invariant singlet overlap is defined by

$$P_{\mathbf{1}} = \langle\Psi_M|\hat{\mathcal{P}}_{\mathbf{1}}|\Psi_M\rangle = |\alpha|^2.$$

If $P_{\mathbf{1}} = 1$*, the state is a pure meson (color singlet). If* $P_{\mathbf{1}} = 0$*, it lies entirely in the octet and is not observable as a free particle.*

Example 17 (Gauge-Invariant Negativity in a Bipartite System)**.** *Consider a bipartite system* $\mathcal{H}_A \otimes \mathcal{H}_B$ *where each subsystem carries both gauge and external degrees of freedom. A naive partial transpose on the full Hilbert space may break gauge invariance. Instead, one first projects onto the gauge-invariant subspace:*

$$\tilde{\rho} = P_{\mathbf{1}}\,\rho\,P_{\mathbf{1}},$$

where $\rho = |\Psi\rangle\langle\Psi|$*. One then performs the partial transpose on the external degrees only. The resulting negativity is gauge invariant and quantifies the entanglement among the measurable degrees of freedom.*

7 Extending to Full Spacetime Symmetry: $G\times$Poincaré

Thus far, we have classified packaged quantum states by decomposing the Hilbert space into sectors corresponding to irreducible representations (irreps) of a gauge group G (finite or compact). In a full quantum field theory, however, the symmetry is not only given by G but by the direct product $G \times$ Poincaré.

In 1939, Eugene Wigner's [12] laid the groundwork for understanding spacetime symmetries in quantum mechanics by classifying the unitary irreducible representations of the Poincaré group. Later Lochlainn O'Raifeartaigh [86] demonstrated the incompatibility of nontrivial mixing between internal and spacetime symmetries in the context of Lie algebras. In 1967, Coleman and Mandula (Coleman-Mandula theorem) [87] proved that the symmetry group of a relativistic QFT with a mass gap and analytic S-matrix must factorize as $G \times$ Poincaré. This rules out nontrivial mixing of spacetime and internal symmetries in the context of Lie algebras. In 1975, Haag et al. [88] extended the Coleman-Mandula theorem by introducing supersymmetry (fermionic generators). This originated the graded Lie algebras (super-Poincaré algebra) to unify spacetime and internal symmetries.

In this section, we extend packaged states to incorporate spacetime symmetry and show how the packaging principle persists when combining gauge and Lorentz/Poincaré representations. In particular, we address the following questions:

- How does one construct single-particle states that are simultaneously in an irreducible representation of both G and the Poincaré group?
- How are the internal (gauge) degrees of freedom packaged together with spacetime properties (such as momentum and spin/helicity) in a single creation operator?
- How does one consistently combine these to form multi-particle states that respect both gauge invariance and Poincaré invariance?

We now present the detailed derivations.

7.1 Combining Gauge and Lorentz Representations

(1) Single-Particle States as Irreps of $G \times$ Poincaré. In a relativistic quantum field theory, elementary particles are classified by irreps of the full symmetry group

$$\mathcal{G} = G \times \text{Poincaré}.$$

According to Wigner's classification, an elementary particle state is labeled by its four-momentum p^μ (with $p^2 = m^2$) and by its internal quantum numbers (such as spin or helicity) arising from the little group (a subgroup of the Lorentz group). When gauge degrees of freedom are present, each field operator $\hat{\psi}(x)$ transforms in a tensor product representation:

$$\hat{\psi}(x) \quad \longrightarrow \quad D^{(G)}(g) \otimes D^{(\text{Lorentz})}(\Lambda),$$

where $D^{(G)}(g)$ is an irreducible representation of the gauge group G (e.g. $\mathbf{3}$ for quarks) and $D^{(\text{Lorentz})}(\Lambda)$ is the representation of the Lorentz transformation Λ (or its induced representation of the Poincaré group).

For example, for a massive particle the state may be labeled as

$$|p, \sigma\rangle_\alpha,$$

with p^μ on-shell, σ denoting the spin (or helicity) degrees of freedom, and α labeling the gauge index (or the gauge irreducible block). The creation operator is then written as

$$\hat{a}^\dagger_{\alpha,\sigma}(\mathbf{p}),$$

and under a transformation $g \in G$ and a Lorentz transformation Λ,

$$U(g,\Lambda)\,\hat{a}^\dagger_{\alpha,\sigma}(\mathbf{p})\,U(g,\Lambda)^{-1} = \sum_{\beta,\sigma'} \left[D^{(G)}_{\beta\alpha}(g) \otimes D^{(\text{Lorentz})}_{\sigma'\sigma}(\Lambda)\right]\,\hat{a}^\dagger_{\beta,\sigma'}(\Lambda\mathbf{p}).$$

Because the representation is a tensor product of two irreps, the entire creation operator is a packaged state of both gauge and spacetime (Lorentz) quantum numbers. The packaging principle implies that within a single creation operator the gauge quantum numbers cannot be split off from the Lorentz ones.

(2) Mathematical Derivation of the Combined Representation. Let V_G be the representation space for the gauge group G and V_L be the representation space for the Lorentz (or little) group. Then the full single-particle Hilbert space is

$$\mathcal{H}_1 \cong V_G \otimes V_L,$$

which is an irrep of $G\times$Poincaré (or, more precisely, a direct sum of such irreps if multiple species exist). By complete reducibility, every single-particle state is an inseparable tensor product of its gauge part and its Lorentz part. In other words, there is no meaningful way to factor out a subset of the gauge quantum numbers from a state without affecting its Lorentz transformation properties.

(3) Consequences for Superselection and Partial Factorization. Since the internal gauge charge and the Lorentz quantum numbers are combined into a single irreducible block, we have:

1. **No Partial Factorization:** A single creation operator $\hat{a}^\dagger_{\alpha,\sigma}(\mathbf{p})$ is irreducible with respect to the full group. One cannot, for example, separate half of the color charge from the momentum or spin part.

2. **Superselection Rules:** Since the total state lies in a definite irreducible sector of $G \times$ Poincaré, it automatically obeys superselection rules. In a local gauge theory, this implies that the net gauge charge (e.g. color) is fixed, and states in different sectors (e.g. color singlet vs. non-singlet) cannot mix.

7.2 Discrete Spacetime Symmetries and Their Impact

Beyond the continuous symmetries of $G \times$ Poincaré, discrete symmetries such as charge conjugation (C), parity (P), and time reversal (T) further constrain the structure of packaged states. Although these discrete transformations act on the full state, they also respect the packaging principle.

For a given state $|p,\sigma\rangle_\alpha$, discrete transformations act as follows:

- **Charge Conjugation (C):** This operation exchanges particles with antiparticles and, correspondingly, interchanges a representation V_G with its conjugate V_G^*. Thus, if $\hat{a}^\dagger_{\alpha,\sigma}(\mathbf{p})$ creates a particle in V_G, then under C it transforms into an operator that creates the antiparticle in V_G^*.

- **Parity (P):** Parity reverses the spatial coordinates, $\mathbf{x} \to -\mathbf{x}$, and acts non-trivially on the Lorentz part V_L (for example, by flipping helicity for massless states or by changing orbital angular momentum). The gauge part V_G remains unaffected.

- **Time Reversal (T):** Time reversal reverses momenta and spin directions and typically involves complex conjugation. Again, the gauge sector remains packaged as a whole.

Since these discrete operations either leave the gauge part invariant or exchange it with its conjugate in a well-defined manner, the overall structure of the packaged state remains intact. In particular, a state that is a gauge singlet (or belongs to a specific irreducible G-sector) will remain so under discrete transformations.

7.3 Multi-Particle States and Total Quantum Numbers

In constructing multi-particle states, one must combine the single-particle representations of $G \times$ Poincaré using the standard tensor product rules. Suppose we have n particles:

$$\mathcal{H}^{(n)} \cong \bigotimes_{i=1}^{n} \Big(V_G^{(i)} \otimes V_L^{(i)} \Big).$$

This tensor product can be rearranged as

$$\mathcal{H}^{(n)} \cong \left(\bigotimes_{i=1}^{n} V_G^{(i)} \right) \otimes \left(\bigotimes_{i=1}^{n} V_L^{(i)} \right).$$

(1) Decomposition in the Gauge Sector: The product

$$\bigotimes_{i=1}^{n} V_G^{(i)}$$

decomposes into a direct sum of irreps of G:

$$\bigotimes_{i=1}^{n} V_G^{(i)} \cong \bigoplus_{Q \in \hat{G}} N_Q \, V_Q,$$

where N_Q denotes the multiplicity of the irrep V_Q. In confining theories, the physical requirement is that only the gauge singlet $V_{\mathbf{1}}$ appears in the asymptotic spectrum.

(2) Decomposition in the Lorentz Sector: Simultaneously, the Lorentz (or Poincaré) parts combine to yield states of definite total momentum, spin, and possibly orbital angular momentum. For massive particles, one obtains states with a well-defined spin J. For massless particles, one obtains helicity eigenstates.

(3) Overall Isomorphism: Thus, the full multi-particle Hilbert space decomposes as

$$\mathcal{H}^{(n)} \cong \bigoplus_{Q \in \hat{G}} \bigoplus_{\Sigma} \mathcal{H}_{Q,\Sigma},$$

where Σ collectively denotes the Lorentz quantum numbers (momentum, spin, etc.) of the composite state. In each sector, the state transforms as an irreducible representation of $G \times$ Poincaré and remains a packaged state. The internal gauge quantum numbers remain inseparable from the overall quantum state.

Example 18 (A Hybrid Meson in QCD)**.** *Consider a quark-antiquark pair in QCD. Each quark carries a color index (in $V_{\mathbf{3}}$ or $V_{\overline{\mathbf{3}}}$) and spin degrees of freedom. Their single-particle state is*

$$|p, \sigma\rangle_\alpha \quad \textit{with } \alpha \in \{1, 2, 3\}.$$

Under the full symmetry, the quark transforms as

$$U(g,\Lambda)\,\hat{a}^{\dagger}_{\alpha,\sigma}(\mathbf{p})\,U(g,\Lambda)^{-1} = \sum_{\beta,\sigma'} D^{(\mathbf{3})}_{\beta\alpha}(g)\,D^{(L)}_{\sigma'\sigma}(\Lambda)\,\hat{a}^{\dagger}_{\beta,\sigma'}(\Lambda\mathbf{p}).$$

For the quark-antiquark pair, the tensor product of the gauge parts is

$$\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{1}\oplus\mathbf{8},$$

and the physical meson state is projected onto the color-singlet subspace. Simultaneously, the Lorentz parts (spin and momentum) combine to yield a state of definite total spin J (and parity P and charge conjugation C if defined). The final state is a packaged quantum state of both gauge and spacetime degrees of freedom:

$$|M(J^{PC}),p\rangle = |\mathbf{1}_{color}, J^{PC}, p\rangle.$$

The construction ensures that neither the gauge charge nor the Lorentz indices can be factored out separately. They are completely entangled within a single irreducible block.

Physically, this extension underlines that observables must be invariant under both gauge and Poincaré transformations. In confining theories, for example, the physical states (such as hadrons) appear as color-singlets that are simultaneously characterized by well-defined momentum and spin. The interplay between the internal packaging and the external symmetry assures that the robust entanglement among IQNs is maintained even when one considers the full, relativistic context.

8 Cohomological and Topological Approaches to Packaged States

In previous sections, we classified packaged quantum states based solely on the representation theory of gauge groups (finite or compact) and their tensor product decompositions. In many modern quantum field theories and strongly correlated systems, however, additional topological [52, 89, 90, 91, 92, 93, 94] and cohomological [52, 95, 96] data emerge. These data are especially relevant in the presence of symmetry-protected topological (SPT) phases or discrete gauge theories.

Incorporating cohomological data refines the notion of packaging by encoding additional topological or anomaly-related information into the internal quantum numbers. Whereas the standard irreducible representation captures the inseparability of IQNs, the projective representations arising from nontrivial cocycles indicate that even these packaged units may carry an extra phase ambiguity. This additional structure can affect how excitations combine and interact. For example, when two projectively represented excitations are fused, the associated cocycles multiply. This potentially leading to modified fusion rules or braiding statistics. Such phenomena are important in the study of fractional quantum Hall systems and topological orders.

In this section, we extend our classification of packaged states by incorporating group cohomology and topological invariants. In doing so, we capture how extra cocycle phases and topological terms further refine the notion of a packaged state. Furthermore, these methods focus on the internal (gauge) degrees of freedom (our packaged states) that remain inseparable and how they may be twisted by nontrivial cohomology classes.

8.1 Group Cohomology Classification

Group cohomology has emerged as a powerful tool in the classification of topological phases and symmetry-protected topological (SPT) orders.[95, 96] In the context of packaged quantum states, the idea is to refine the labeling of internal charges by associating them with (possibly twisted) irreducible representations. In many cases, the extra data are captured by a 2-cocycle $\omega \in Z^2(G, U(1))$ (or higher cocycles in higher dimensions), which leads to projective representations.

(1) Basics of Group Cohomology. For a group G and an abelian coefficient group A (usually $U(1)$), the second cohomology group $H^2(G, A)$ classifies equivalence classes of 2-cocycles. A 2-cocycle is a function

$$\omega : G \times G \to U(1)$$

satisfying the cocycle condition:

$$\omega(g_2, g_3)\, \omega(g_1, g_2 g_3) = \omega(g_1, g_2)\, \omega(g_1 g_2, g_3) \quad \forall\, g_1, g_2, g_3 \in G.$$

Two 2-cocycles ω and ω' are considered equivalent if there exists a function $f : G \to U(1)$ such that

$$\omega'(g_1, g_2) = \frac{f(g_1) f(g_2)}{f(g_1 g_2)}\, \omega(g_1, g_2).$$

The group $H^2(G, U(1))$ then classifies the different projective classes of representations of G.

(2) Projective Representations and Twisted (Packaged) States. A projective representation of G on a vector space V is a map

$$\tilde{\rho} : G \to \mathrm{GL}(V)$$

such that

$$\tilde{\rho}(g_1)\, \tilde{\rho}(g_2) = \omega(g_1, g_2)\, \tilde{\rho}(g_1 g_2),$$

where $\omega(g_1, g_2) \in U(1)$ is a 2-cocycle. In the context of packaged quantum states, a single-particle operator $\hat{a}^\dagger$ may transform not as a strict (linear) irrep, but as a projective representation with an extra phase given by ω. In other words, packaged states can be labeled by twisted irreps $\mathbf{R}^\omega$. The presence of a nontrivial cocycle enriches the structure: the way internal quantum numbers add when combining excitations is modified by the cocycle.

(3) Twisted Tensor Products. Suppose we have two single-particle operators, $\hat{a}^\dagger$ and $\hat{b}^\dagger$, that transform under projective representations $\tilde{\rho}_1$ and $\tilde{\rho}_2$ with the same 2-cocycle ω. Their tensor product transforms as:

$$\tilde{\rho}_1(g) \otimes \tilde{\rho}_2(g) \;:\; (\hat{a}^\dagger \otimes \hat{b}^\dagger) \mapsto \omega(g_1, g_2) \left[\tilde{\rho}_1(g) \otimes \tilde{\rho}_2(g)\right] (\hat{a}^\dagger \otimes \hat{b}^\dagger).$$

Because the cocycle is multiplicative, the full multi-particle state remains in a fixed ω-twisted sector. Thus, the packaged state is now labeled by both the irrep $\mathbf{R}$ and the cocycle ω, and no partial splitting of the gauge (or internal) charge is possible unless the cocycle is trivial.

(4) Higher Cohomology and Topological Phases. In higher dimensions, one may encounter nontrivial 3-cocycles or 4-cocycles (i.e., elements of $H^3(G, U(1))$ or $H^4(G, U(1))$) that play a role in the classification of topological orders and SPT phases. For instance, in 2+1D topological orders, excitations may acquire fractional statistics, which can be understood via a 3-cocycle in $H^3(G, U(1))$. The packaging principle extends naturally: each local excitation is associated with a fixed cohomology class ω, and when excitations combine, the topological phase (or braiding phase) remains inseparable.

While the cohomological approach provides a richer classification scheme, it is not without limitations. One limitation is computational complexity: determining the relevant cocycles and analyzing their consequences for composite states can be challenging, especially for non-Abelian groups or in higher dimensions. Moreover, although these cohomological invariants capture subtle topological features and anomalies, their physical interpretation may not always be as transparent as that of standard representation theory. Nonetheless, the novel aspect of this approach is its ability to predict new, robust forms of packaged entanglement that are protected by topological invariants. In particular, if a packaged state transforms projectively, its internal structure is not only inseparable in the usual sense but also carries a nontrivial topological twist, which may have observable consequences in processes sensitive to anomalies or topological phases.

8.2 Anomalies and Topological Terms

Anomalies and topological terms provide another viewpoint on the classification of packaged states, especially when global symmetries are involved or when the theory has nontrivial boundary modes.

(1) 't Hooft Anomalies and Wess-Zumino Terms. A global symmetry anomaly (or 't Hooft anomaly) [97] implies that a symmetry cannot be realized by an on-site operator in a strictly local Hilbert space. Instead, such anomalies manifest as additional topological terms in the effective action, such as the Wess-Zumino (WZ) term [90]. Mathematically, the WZ term is associated with a nontrivial element in $H^{d+1}(G, U(1))$ for a d-dimensional system. For example, in 1+1D, the WZ term can enforce that the edge modes transform projectively, leading to a nontrivial 2-cocycle in $H^2(G, U(1))$.

(2) Implications for Packaged States. In the presence of an anomaly, the internal degrees of freedom of a packaged state may acquire extra phase factors, so that the state transforms as a projective representation rather than a linear one. The packaging principle remains intact: the full set of internal quantum numbers, including the anomalous phase, is still inseparable. In other words, even though the anomaly modifies the transformation law to

$$\tilde{\rho}(g_1)\tilde{\rho}(g_2) = \omega(g_1, g_2)\, \tilde{\rho}(g_1 g_2),$$

the internal charge remains a single unit labeled by ω. Consequently, superselection rules enforce that states with different anomaly (cocycle) classes cannot interfere.

(3) Derivation: Anomalous Edge Modes. Consider a 2+1D system with chiral fermions at its edge. The effective theory of the edge modes may exhibit a chiral anomaly, captured by a nontrivial 2-cocycle ω. Let $|\Psi\rangle$ be an edge state that transforms projectively:

$$U(g)|\Psi\rangle = \tilde{\rho}(g)|\Psi\rangle,$$

with

$$\tilde{\rho}(g_1)\tilde{\rho}(g_2) = \omega(g_1, g_2)\,\tilde{\rho}(g_1 g_2).$$

Even though the anomaly modifies the phase acquired under composition of group elements, the state $|\Psi\rangle$ is still a packaged state: its internal quantum numbers (including the anomalous phase) are fully contained within the single irreducible projective representation. This ensures that any entanglement measure must respect the cocycle structure.

(4) Topological Terms and Braiding Phases. In higher dimensions (e.g., 3+1D), topological terms associated with nontrivial cocycles (such as 3-cocycles in $H^3(G, U(1))$) can lead to fractional statistics and braiding phases among excitations. In such cases, when multiple packaged states are combined, the overall wavefunction acquires a topological phase determined by the product of cocycles. Despite the nontrivial braiding, the internal topological charge remains inseparable and is characterized by a fixed cohomology class.

9 Hybrid Systems: External $\otimes$ Internal $\otimes$ (Global) Symmetries

In realistic quantum field theories (e.g. the Standard Model), elementary particles are characterized not only by a local gauge symmetry G_{local} (e.g. color SU(3) or electroweak SU(2)×U(1)) [4, 6, 30, 98] but also by a global symmetry G_{global} (e.g. flavor or isospin) [54, 99] and by the spacetime (Lorentz or Poincaré) symmetry [12, 87]. In our context, a packaged quantum state is one in which the internal gauge quantum numbers are inseparably packaged into a single irreducible block. In this section we show how to combine the local (internal) and global symmetry sectors with the Lorentz symmetry in forming full physical states. Our aim is to preserve the packaging principle for the local gauge charges while allowing for free mixing in the global (flavor) sector and accounting for the external degrees of freedom.

9.1 Extended Hybridization of Internal and External Degrees of Freedom with Global Symmetries

(1) The Combined Group Structure. Consider a theory with symmetry group [87, 88]

$$\mathcal{G} = G_{\text{local}} \times G_{\text{global}} \times \text{Lorentz},$$

where G_{local} is a local gauge group (e.g. SU(3) for QCD, U(1) for electromagnetism), G_{global} is a global symmetry (e.g. flavor SU(N_f), baryon number, isospin), for which mixing of different states is allowed, and the Lorentz group (or Poincaré group) provides external degrees of freedom such as momentum and spin (or helicity). By the packaging principle, the internal gauge degrees of freedom are bundled into irreducible representations (irreps) that cannot be partially factored.

A single-particle creation operator in such a theory can be labeled by three indices:

$$\hat{a}^{\dagger}_{f,\alpha,\sigma}(\mathbf{p}),$$

where f denotes the global (flavor) index and transforms under G_{global}, α denotes the local gauge index and transforms under an irreducible representation V_G of G_{local}, σ denotes the spin (or helicity) index, and $\mathbf{p}$ is the three-momentum. Together these transform under the Lorentz group. The packaging principle requires that α is inseparable. For example, a quark is always in a full $\mathbf{3}$ of color.

(2) Transformation Properties. Under a transformation $(g, h, \Lambda) \in G_{\text{local}} \times G_{\text{global}} \times$ Lorentz, the creation operator transforms as

$$U(g, h, \Lambda)\, \hat{a}^{\dagger}_{f,\alpha,\sigma}(\mathbf{p})\, U(g, h, \Lambda)^{-1} = \sum_{\beta,\sigma',f'} D^{(G)}_{\beta\alpha}(g)\, D^{(L)}_{\sigma'\sigma}(\Lambda)\, \hat{a}^{\dagger}_{f',\beta,\sigma'}(\Lambda\mathbf{p})\, D^{(global)}_{f'f}(h),$$

where $D^{(G)}(g)$ is the matrix representation of g on the gauge (internal) space V_G, $D^{(L)}(\Lambda)$ represents the Lorentz (or little group) transformation on the spin degrees of freedom, and $D^{(global)}(h)$ represents the transformation on the global (flavor) indices.

By the packaging principle, $D^{(G)}(g)$ is irreducible and cannot be decomposed into smaller parts. The local gauge part remains locked in its full irreducible representation. Thus, the internal gauge charge is packaged. While the Lorentz and global parts may allow for superpositions and mixing (for example, linear combinations of different flavor states).

(3) Consequences for Multi-Particle States. For a multi-particle state, one takes the tensor product of individual single-particle spaces:

$$\mathcal{H}^{(n)} \cong \bigotimes_{i=1}^{n} \Big(V_G^{(i)} \otimes V_{global}^{(i)} \otimes V_L^{(i)} \Big).$$

This tensor product can be rearranged as

$$\mathcal{H}^{(n)} \cong \left(\bigotimes_{i=1}^{n} V_G^{(i)} \right) \otimes \left(\bigotimes_{i=1}^{n} V_{global}^{(i)} \right) \otimes \left(\bigotimes_{i=1}^{n} V_L^{(i)} \right).$$

Therefore, we have:

1. In the gauge sector $\bigotimes_{i=1}^{n} V_G^{(i)}$, the packaging principle forces a decomposition into irreps such that the local gauge charge remains inseparable. In confining theories, one projects onto the gauge-singlet subspace:

$$\bigotimes_{i=1}^{n} V_G^{(i)} \cong \bigoplus_{Q \in \hat{G}_{\text{local}}} N_Q\, V_Q,$$

   where, for example, $V_{\mathbf{1}}$ is the color singlet in QCD.

2. In the global sector $\bigotimes_{i=1}^{n} V_{global}^{(i)}$, no such constraint exists, so linear combinations are allowed. This leads to observable flavor multiplets.

3. The Lorentz sector combines in the usual way to yield total momentum and spin.

Thus, a full multi-particle state is expressed as a direct sum over sectors:

$$\mathcal{H}^{(n)} \cong \bigoplus_{Q,\Sigma} \mathcal{H}_{Q,\Sigma},$$

where Q labels the irreps of the local gauge group (subject to superselection), and Σ denotes the set of Lorentz quantum numbers (momentum, spin, etc.). The flavor part, meanwhile, may mix freely.

Example 19 (QCD with Flavor)**.** *Consider quantum chromodynamics (QCD) with:*

- *Local gauge group:* $\mathrm{SU}(3)_{\text{color}}$*. Each quark is a full* $\mathbf{3}$ *(or* $\overline{\mathbf{3}}$*) and its color charge is packaged.*
- *Global symmetry: an approximate* $\mathrm{SU}(3)_{\text{flavor}}$ *that rotates up, down, and strange quarks.*
- *Lorentz symmetry: providing momentum and spin.*

A quark creation operator is labeled as $\hat{q}^\dagger_{f,c,\sigma}(\mathbf{p})$*, where* $c = 1, 2, 3$ *(color),* $f = u, d, s$ *(flavor), and* σ *denotes the spin. Physical hadrons must be color singlets. For example, a meson is formed from a quark-antiquark pair:*

$$|M\rangle = \frac{1}{\sqrt{3}} \sum_{c=1}^{3} |q, c; \bar{q}, c\rangle \otimes |\mathit{flavor}\rangle \otimes |\mathit{Lorentz}\rangle.$$

Here, the color part is strictly packaged into a singlet by contracting the indices, while the flavor part can be a nontrivial linear combination (e.g. the π^0 *state might be* $(u\bar{u} - d\bar{d})/\sqrt{2}$*). The overall state is then a tensor product of a gauge-invariant (and hence packaged) color singlet, a freely mixed flavor multiplet, and the external Lorentz part.*

(4) Packaging for Extended Hybridization of Symmetries. A single-particle state in our hybrid system is given by

$$|p, \sigma\rangle_{f,\alpha} \in V_G \otimes V_{global} \otimes V_L,$$

where the local gauge part V_G is an irreducible (packaged) representation that cannot be decomposed. Under a transformation (g, h, Λ),

$$U(g, h, \Lambda)\, |p, \sigma\rangle_{f,\alpha} = \sum_{\beta,\sigma',f'} D^{(G)}_{\beta\alpha}(g)\, D^{(L)}_{\sigma'\sigma}(\Lambda)\, |\Lambda p, \sigma'\rangle_{f',\beta}\, D^{(global)}_{f'f}(h).$$

For multi-particle states, the decomposition of the gauge sector enforces superselection rules (e.g., only color singlets are physical in confining theories), while the global part can mix and the Lorentz part is combined in the usual manner.

The packaging principle applies to the local gauge sector: every single-particle creation operator carries its full local gauge charge as an indivisible unit. When combined with global symmetries and Lorentz symmetry, the full physical state is represented as a tensor product:

$$\mathcal{H} \cong \left(\bigoplus_{Q \in \hat{G}_{\text{local}}} \mathcal{H}_Q \right) \otimes V_{global} \otimes V_L.$$

Thus, while the flavor (global) and Lorentz (external) degrees of freedom allow for superpositions and interference, the local gauge part remains rigidly packaged. This structure explains why in QCD, for example, no free state with nonzero color exists, even though the flavor and Lorentz parts can vary widely.

Finally we see that, in a hybrid system with symmetry

$$G_{\text{local}} \times G_{\text{global}} \times \text{Lorentz},$$

the physical state of a particle (or composite system) is given by a tensor product

$$V_G \otimes V_{global} \otimes V_L,$$

where the local gauge part V_G is an inseparable (packaged) block by virtue of its irreducibility, while the global and Lorentz parts may be superposed. This construction is critical for understanding phenomena such as color confinement (where only the packaged singlet state is observed) and flavor mixing (where global symmetries allow nontrivial multiplets), and it provides a complete description of the observable quantum numbers in a fully relativistic setting.

10 Practical Computations and Examples

In this section we illustrate how the packaging principle is implemented in explicit computations. Unlike traditional state classifications, here our focus is on packaged quantum states in which internal gauge degrees of freedom (such as electric charge or color) are irreducibly bundled into single operators that cannot be partially factorized. We provide examples from QED and QCD, along with derivations that demonstrate how the net quantum numbers are computed and how projection onto invariant (singlet) subspaces is achieved.

10.1 QED: Net Electric Charge Sectors

In quantum electrodynamics (QED), the gauge group is $U(1)$. Every electron or positron creation operator carries a fixed electric charge ($-e$ for electrons, $+e$ for positrons) as an inseparable unit. Let us first derive the charge sectors.

Let $\hat{a}^\dagger_{e^-}$ and $\hat{b}^\dagger_{e^+}$ denote electron and positron creation operators, respectively. Under a $U(1)$ gauge transformation, represented by

$$U(e^{i\theta}) = \exp\Big(i\theta\hat{Q}\Big),$$

these operators transform as

$$U(e^{i\theta})\,\hat{a}^\dagger_{e^-}\,U(e^{i\theta})^{-1} = e^{-i\theta}\,\hat{a}^\dagger_{e^-}, \qquad U(e^{i\theta})\,\hat{b}^\dagger_{e^+}\,U(e^{i\theta})^{-1} = e^{i\theta}\,\hat{b}^\dagger_{e^+}.$$

Thus, a single electron state is an irreducible (1D) packaged state with charge $-e$.

For a multi-particle state, suppose we have n creation operators with charges q_i (each q_i being $\pm e$). The combined state transforms as

$$U(e^{i\theta}) \prod_{i=1}^{n} \hat{a}^\dagger_i |0\rangle = \exp\Big(i\theta \sum_{i=1}^{n} q_i/e\Big) \prod_{i=1}^{n} \hat{a}^\dagger_i |0\rangle.$$

Hence, the total electric charge is

$$Q_{\text{tot}} = \sum_{i=1}^{n} q_i,$$

and the Hilbert space decomposes as

$$\mathcal{H} \cong \bigoplus_{Q \in e\mathbb{Z}} \mathcal{H}_Q.$$

Here, each $\mathcal{H}_Q$ is one-dimensional with respect to the charge label, though it may be tensored with additional degrees of freedom (momentum, spin, etc.).

Finally, we list representative states (net-charge sectors in QED) in the following table:

Table 2: Example:

Sector $\mathcal{H}_Q$	Examples
$Q = 0$	$\lvert 0\rangle$, $\lvert e^- e^+\rangle$, multi-photon states
$Q = -e$	$\lvert e^-\rangle$, $\lvert e^- e^- e^+\rangle$
$Q = +e$	$\lvert e^+\rangle$, $\lvert e^+ e^+ e^-\rangle$
$Q = -2e$	$\lvert e^- e^-\rangle$
$\cdots$	

The fact that each electron or positron operator carries its full charge as a single block exemplifies the packaging principle in an Abelian context.

10.2 QCD: Color Representations and Invariant (Singlet) States

Quantum chromodynamics (QCD) features a non-Abelian gauge group SU(3). Here the internal quantum numbers (color) are packaged in a nontrivial way, and only color-singlet combinations are physically observable.

(1) Single-Particle Packaging. A quark creation operator $\hat{q}_c^\dagger$ carries color index $c \in \{1, 2, 3\}$ and transforms in the fundamental representation $\mathbf{3}$ of SU(3):

$$U(g)\, \hat{q}_c^\dagger\, U(g)^{-1} = \sum_{d=1}^{3} D_{dc}^{(\mathbf{3})}(g)\, \hat{q}_d^\dagger.$$

By the packaging principle, the color degree of freedom is inseparable - a quark always appears as a complete $\mathbf{3}$.

(2) Multi-Particle Decomposition. For multi-particle states, one must take tensor products of the single-particle representations. For example, a meson is formed by a quark-antiquark pair:

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}.$$

The color singlet $\mathbf{1}$ is the physically observable state. An explicit meson wavefunction is given by

$$|M\rangle = \frac{1}{\sqrt{3}} \sum_{c=1}^{3} |q, c;\, \bar{q}, c\rangle.$$

This state is gauge invariant, as can be seen by applying $U(g)$ and using the unitarity of $D^{(\mathbf{3})}(g)$. The antisymmetrization (or contraction of indices) ensures that the full color charge is packaged into a single, inseparable unit.

Similarly, a baryon state is formed by combining three quark operators:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \cdots ,$$

with the color singlet given by

$$|B\rangle = \frac{1}{\sqrt{6}}\,\epsilon_{abc}\,|q,a;\;q,b;\;q,c\rangle.$$

Here ϵ_{abc} is the totally antisymmetric tensor. This guarantees that the three color indices are fully entangled, and the quarks are inseparably packaged into a color-neutral state.

(3) Exotic States and Projection Operators. More exotic states (tetraquarks, pentaquarks) involve higher tensor products, such as

$$\mathbf{3}^{\otimes k}\otimes\overline{\mathbf{3}}^{\otimes m}.$$

A physical state must lie in the color-singlet subspace. One can systematically perform Clebsch-Gordan decompositions (or use Young tableau techniques) to isolate the singlet channel. The projection operator onto the color singlet for SU(3) can be constructed analogously to the finite group case, but with integration over the Haar measure:

$$P_{\mathbf{1}} = d_{\mathbf{1}}\int_{\mathrm{SU}(3)} d\mu(g)\,\chi_{\mathbf{1}}^*(g)\,U(g).$$

Since $\chi_{\mathbf{1}}(g) = 1$ and $d_{\mathbf{1}} = 1$, this operator projects out the invariant (singlet) component.

(4) Practical Classification Table for QCD. For QCD, only color-singlet combinations are observed. A simplified table of multi-quark packaged states is: These clas-

Table 3: Example: Color-Singlet States in QCD

State	Color Composition
Meson	$q\bar{q}$ with $\mathbf{3}\otimes\overline{\mathbf{3}}\to\mathbf{1}$
Baryon	qqq with $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\to\mathbf{1}$
Tetraquark	$qq\bar{q}\bar{q}$ with proper contraction to yield $\mathbf{1}$
Pentaquark	$qqqq\bar{q}$ with proper contraction to yield $\mathbf{1}$

sification tables arise naturally by enforcing the packaging principle: each quark is an inseparable block in the $\mathbf{3}$ (or $\overline{\mathbf{3}}$), and only the full contraction yielding the singlet $\mathbf{1}$ is physical.

10.3 Bound-State Wavefunctions and Quantum Information Models

(1) Bethe-Salpeter Equation. Relativistic bound states are commonly computed using the Bethe-Salpeter equation:

$$\Gamma(p,P) \;=\; \int\frac{d^4k}{(2\pi)^4}\,K(p,k,P)\,S\Big(k+\frac{P}{2}\Big)\,\Gamma(k,P)\,S\Big(k-\frac{P}{2}\Big),$$

where $\Gamma(p,P)$ is the vertex function, $K(p,k,P)$ is the interaction kernel (e.g., one-gluon exchange in QCD), and S are the propagators. In a confining theory like QCD, the solution $\Gamma(p,P)$ must be projected onto the color-singlet channel. One typically expands:

$$\Gamma_{\alpha\beta}(p,P) \;=\; \frac{\delta_{\alpha\beta}}{\sqrt{3}}\,\Gamma_{\text{singlet}}(p,P)\;+\;\Gamma_{\text{octet}}(p,P),$$

and only Γ_{singlet} corresponds to a physical meson state. This explicit projection embodies the packaging principle at the level of bound-state wavefunctions.

(2) Potential Models. In potential models (such as the Cornell potential, $V(r) \sim -\alpha/r + \sigma r$), hadron wavefunctions are often factorized as

$$\Psi_{\alpha\beta}(\mathbf{r}) = \varphi_{\text{color}}(\alpha, \beta)\, \Phi_{\text{spin,space}}(\mathbf{r}).$$

For mesons, the color part is fixed by the packaged state:

$$\varphi_{\text{color}}(\alpha, \beta) = \frac{\delta_{\alpha\beta}}{\sqrt{3}},$$

which ensures that the quark and antiquark form an inseparable color singlet. The external (spin and spatial) part $\Phi_{\text{spin,space}}(\mathbf{r})$ is then solved for using the potential model. This separation demonstrates how the packaging principle constrains the internal gauge part, while the external part retains the usual quantum mechanical degrees of freedom.

(3) Quantum Information Toy Models. To illustrate packaged states in a controlled setting, one may construct quantum information toy models on small lattices. For example, consider a lattice model where each site has a local Hilbert space

$$\mathcal{H}_{\text{site}} \cong \mathbb{C}^d \otimes \mathbb{C}^{d'},$$

where $\mathbb{C}^d$ represents the local gauge (color) degree of freedom and $\mathbb{C}^{d'}$ represents an external degree (e.g., spin or a global symmetry label). Gauge invariance forces one to project onto subspaces where the local gauge indices combine to yield a singlet. For instance, in an SU(2) gauge model, if each site carries a two-dimensional gauge degree of freedom, physical states on links or plaquettes must be in the gauge singlet (or invariant) sector. Explicit computations using projection operators (as described in earlier sections) then yield the packaged entangled states in the model.

In practical computations:

- For QED, the packaging is straightforward: each operator carries a fixed electric charge, and the net charge is the sum of the individual charges.
- For QCD, the non-Abelian nature of SU(3) requires the decomposition of tensor products (via Clebsch-Gordan coefficients or Young diagram methods) to isolate color-singlet (packaged) states.
- Bound-state wavefunctions (via the Bethe-Salpeter equation or potential models) incorporate the packaging principle by projecting onto gauge-invariant subspaces.
- Quantum information toy models provide a finite-dimensional setting where gauge invariance and packaging can be explicitly computed.

These computations underscore that in all cases the internal gauge charges remain as **inseparable blocks**, ensuring that packaged quantum states are fundamentally distinct from traditional quantum states where such constraints may be absent.

11 Discussion

Our analysis reveals that the inseparability of internal quantum numbers (IQNs) is not an incidental property but rather a direct consequence of the group-theoretic structure underpinning gauge symmetries. In our framework, each single-particle creation operator transforms under an irreducible representation (irrep) of the symmetry group G. No proper invariant subspace exists within these irreps, so any multiparticle state constructed via tensor products of such operators necessarily carries packaged internal charges.

We illustrated the principle through various examples. In the case of discrete $\mathbb{Z}_2$ symmetries, we showed how charge conjugation $\hat{C}$, parity $\hat{P}$, and time-reversal $\hat{T}$ yield Bell-like entangled states that are rigorously split by superselection rules. For continuous groups such as $SU(2)$ and $SU(3)$, our discussion of Clebsch-Gordan decompositions highlights how color confinement in QCD or isospin structure in nuclear physics naturally emerge from the requirement that physical states belong to a net singlet subspace.

Furthermore, we extended the packaging idea from pointlike particles to flux lines, membranes, or domain walls in theories with higher-form symmetries and even hybrid configurations involving local (gauge), global, and spacetime (Lorentz/Poincaré) symmetries. In doing so, we connected our results with practical approaches such as Bethe-Salpeter equations, lattice gauge theory simulations, and quantum-information models, illustrating that the packaging principle has broad relevance from particle physics to quantum computing.

These insights not only deepen our understanding of gauge-invariant entanglement but also suggest new applications. For example, the robust entangled subspaces predicted by our theory could serve as protected resources for quantum error correction. Likewise, the group-theoretic techniques presented here may offer new strategies for designing resource-efficient simulations of lattice gauge theories.

12 Conclusion

We have shown that whenever a finite or compact group G has a nontrivial representation acting on a quantum system, it necessarily enforces packaged entanglement: internal quantum numbers must appear as inseparable blocks within single superselection sectors. This packaging principle provides a unified explanation for phenomena ranging from the Bell-like or GHZ-like states that emerge under discrete $\mathbb{Z}_2$ operations (e.g., charge conjugation, parity, time reversal) to the color-singlet confinement of quarks observed in non-Abelian gauge theories such as QCD. Beyond standard gauge theories, the same reasoning applies to extended or higher-form symmetries, such as flux tubes and topological defects that cannot be fractionated. By integrating irreducible representations with local gauge invariance and superselection rules, this packaging framework illuminates why certain entangled structures arise and remain robust across high-energy physics and quantum information contexts. The irreps-based entanglement constraints can guide the construction of gauge-invariant quantum states that are naturally resistant to local noise and offers potential for fault-tolerant quantum simulation and computation. Thus, discrete symmetry tests, lattice gauge theory simulations, and the study of exotic beyond-Standard-Model sectors may all benefit from these insights into how symmetry underlies and safeguards irreducible forms of quantum entanglement.

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