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Notes on statistics in High Energy Physics: Discovery and exclusion.

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In the quest of searching for new Physics one may expect that, in an event counting experiment, a certain number of events due to a physical process that is induced by new phenomena. This total number of events, n, can be modeled as a Poisson distribution with mean $s+b$, where s stands for the expected number of signal events, coming from new Physics, and b is the expected number of background events. In order to test statistically if a sign of new Physics is present in the number of events collected by the experiment one can use different methods. One of the most used statistically tests in order to claim discovery or impose upper limits are the so-called q_0 and q_μ respectively, that could be defined as,

$$q_0 = \begin{cases} -2\ln\lambda(0) & \text{if} \quad \hat{\mu} \geq 0, \\ 0 & \text{if} \quad \hat{\mu} < 0, \end{cases} \tag{1}$$

$$q_\mu = \begin{cases} -2\ln\lambda(\mu) & \text{if} \quad \hat{\mu} \geq \mu, \\ 0 & \text{if} \quad \hat{\mu} < \mu, \end{cases} \tag{2}$$

where $\lambda(0) = L(0)/L(\hat{\mu})$ and $\lambda(\mu) = L(\mu)/L(\hat{\mu})$. Here L is the likelihood function and the $\hat{\mu}$ is the value of the parameter μ that maximises the likelihood function. From these tests it is possible to define the corresponding p-values as follows,

$$p_0 = \int_{q_0^{\text{obs}}}^{\infty} f(q_0|0)dq_0, \tag{3}$$

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where the function $f(q_0|0)$ is the probability density function (pdf) of q_0 under the hypothesis of only background and

$$p_\mu = \int_{q_\mu^{\text{obs}}}^{\infty} f(q_\mu|\mu) dq_\mu, \tag{4}$$

with $f(q_\mu|\mu)$ the pdf of q_μ assuming the μ hypothesis.

From the p-values one can obtain a significance using the relation

$$Z = \Phi^{-1}(1-p), \tag{5}$$

where Φ^{-1} is the inverse of the standar cumulative Gaussian distribution. Using this relation a significance of $Z = 5$ (5σ) that corresponds to $p = 2.87 \times 10^{-7}$ is the level at which a discovery is considered, whereas a significance of $Z = 1.64$ that corresponds to $p = 0.05$ is the level at which a signal hypothesis can be excluded with a 95% confidence level (C.L.)

In an experiment one may define the expected significance that would allow to reject different values of the signal hypothesis, μ. In the case of discovery it is relevant to know the median that under the hypothesis of a nominal signal ($\mu = 1$) one can reject the only-background hypothesis ($\mu = 0$). In the case of exclusion one may look for the median that, under the only-background hypothesis is able to reject a non-zero value of μ. The expressions of the medians for the discovery and exclusion cases are

$$\text{med}[Z_0|\mu'] = \sqrt{q_{0,A}}, \tag{6}$$
$$\text{med}[Z_\mu|0] = \sqrt{q_{\mu,A}}. \tag{7}$$

Here the subindex A indicates that the q-tests must be evaluated in the data sets known as Asimov data sets. The Asimov data set is defined as the set that when used to evaluate the estimators of all the parameters, one obtains the true values of the parameters. Considering a bin counting experiment and assuming Poisson distributions and we fix the value of μ to be μ', then the expected number of events in each bin is $\nu_i = \mu' s_i + b_i$. So the data set that is the set of the number of events in each bin, n_i, that gives us the parameter μ' when maximising the likelihood is defined as

$$n_{i,A} = E[n_i] = \nu_i = \mu' s_i + b_i, \tag{8}$$

that basically defines the Asimov data set. This is the set that we will use to evaluate the statistical tests $\text{med}[Z_0|\mu']$ and $\text{med}[Z_\mu|0]$ from Eqs. (6)-(7).

1 Evidence and/or discovery without uncertanties.

Let us now obtain the significance which allows to quantify the sensitivity of evidence or discovery. If we know the number of background events, b, and we parameterise the number of signal events as μs, the likelihood can be written as

$$L(\mu) = \frac{(\mu s + b)^n}{n!} e^{-(\mu s + b)}. \tag{9}$$

From Eq. (1) we must compute the quantity $q_0 = -2\ln(L(0)/L(\hat{\mu}))$, so

$$L(0) = \frac{b^n}{n!}e^{-b}, \tag{10}$$

for $L(0)$ and for the case of $L(\mu)$

$$L(\hat{\mu}) = \frac{(\hat{\mu}s + b)^n}{n!}e^{-(\hat{\mu}s+b)}. \tag{11}$$

The condition that maximises the likelihood in this case is $\hat{\mu} = (n-b)/s$, so we have $\hat{\mu}s+b = n$ and then

$$L(\hat{\mu}) = \frac{n^n}{n!}e^{-n}. \tag{12}$$

If we compute the ratio of likelihoods that is defined as $\lambda(0)$ we obtain

$$\lambda(0) = \frac{L(0)}{L(\hat{\mu})} = \frac{b^n}{n^n}e^{-b+n}. \tag{13}$$

Then

$$\begin{aligned} q_0 =& -2\ln\lambda(0) = -2\ln\left(\frac{L(0)}{L(\hat{\mu})}\right) = -2\ln\left(\frac{b^n}{n^n}e^{-b+n}\right) = \\ =& -2\left(n\ln n - n\ln n + (n-b)\right) = 2\left(n\ln\frac{n}{b} + b - n\right). \end{aligned} \tag{14}$$

If we evaluate the Asimov set for the nominal value $\mu' = 1$ that according to Eq. (8) basically means $n = s + b$, we obtain

$$\begin{aligned} \mathrm{med}[Z_0|1] =& \sqrt{q_{0,A}} = \sqrt{2\left(n\ln\frac{n}{b} + b - n\right)} = \sqrt{2\left((s+b)\ln\frac{s+b}{b} + b - (s+b)\right)} = \\ =& \sqrt{2\left[(s+b)\ln\left(\frac{s+b}{b}\right) - s\right]}. \end{aligned} \tag{15}$$

The median obtained for the discovery sensitivity without uncertainties is

$$\boxed{\mathrm{med}[Z_0|1] = \sqrt{2\left[(s+b)\ln\left(\frac{s+b}{b}\right) - s\right]}.} \tag{16}$$

Sometimes the expected number of background events is much larger than the ones expected for the signal, $s \ll b$. In that case one can take the limit of the expression of Eq. (16). For that purpose one can write Eq. (16) as

$$\mathrm{med}[Z_0|1] = \sqrt{2\left((s+b)\ln\left(1 + \frac{s}{b}\right) - s\right)}. \tag{17}$$

One can easily expand the logarithm at second order in s/b as

$$\ln\left(1+\frac{s}{b}\right) \approx \frac{s}{b} - \frac{s^2}{2b^2} + \mathcal{O}((s/b)^3). \tag{18}$$

Substituting in the expression of the significance

$$\begin{aligned}\mathrm{med}[Z_0|1] \approx & \sqrt{2\left((s+b)\left(\frac{s}{b} - \frac{s^2}{2b^2} + \mathcal{O}((s/b)^3)\right) - s\right)} = \\ = & \sqrt{2\left(\frac{s^2}{b} - \frac{s^3}{2b^2} + s - \frac{s^2}{2b} - s\right)} = \\ = & \sqrt{2\left(\frac{s^2}{2b} + \mathcal{O}((s/b)^2)\right)} = \frac{s}{\sqrt{b}}.\end{aligned} \tag{19}$$

This expression is commonly used in the literature to make naive estimates of the evidence or discovery of a given process in the context of new Physics. In some cases the estimate could be a really good approach since some of the scenarios fulfill the limit $s/b \ll 1$.

2 Exclusion limits without uncertainties.

Let us now compute the expression for the exclusion limits. Given the likelihood,

$$L(\mu) = \frac{(\mu s + b)^n}{n!} e^{-(\mu s + b)}, \tag{20}$$

and taking $L(\hat{\mu})$ from Eq. (12) we are interested in calculating $\lambda(\mu)$,

$$\lambda(\mu) = \frac{L(\mu)}{L(\hat{\mu})} = \frac{(\mu s + b)^n}{n^n} e^{-(\mu s + b) + n}. \tag{21}$$

Then

$$\begin{aligned}q_\mu = & -2\ln\lambda(\mu) = -2\left[n\ln(\mu s + b) - (\mu s + b) + n - n\ln n\right] = \\ = & 2\left[\mu s + b - n + n\ln n - n\ln(\mu s + b)\right]\end{aligned} \tag{22}$$

We take the value $\mu' = 0$ as we are dealing with a null hypothesis, so according to Eq. (8) $n = b$ and renaming $\mu s \to s$[1], then,

$$\mathrm{med}[Z_\mu|0] = \sqrt{q_{\mu,A}} = \sqrt{2\left[s + b\ln b - b\ln(s+b)\right]} = \sqrt{2\left[b\ln\left(\frac{b}{s+b}\right) + s\right]}. \tag{23}$$

So the median obtained for the exclusion sensitivity without uncertainties is

$$\boxed{\mathrm{med}[Z_\mu|0] = \sqrt{q_{\mu,A}} = \sqrt{2\left[b\ln\left(\frac{b}{s+b}\right) + s\right]}.} \tag{24}$$

[1] This redefinition holds since μs is the number of signal events.

The equation that we have derived above is the one that should be used to compute the exclusion limits for a given luminosity in an experiment. Usually, assuming that the limit $b \gg s$ holds the exclusion limits are computed using the same simplified expression for the discovery significance of Eq. (19), $s/\sqrt{b}$. This approximation is valid because in the case where $b \gg s$, we can expand the logarithm

$$\mathrm{med}[Z_\mu|0] \approx \sqrt{2\left(s - b\left(\frac{s}{b} - \frac{s^2}{2b^2}\right)\right)} = \frac{s}{\sqrt{b}}. \tag{25}$$

However, if we want to set exclusion limits of a process of new Physics the total number of events include both the signal and background events so the correct limit to be taken is $s + b \gg s$. In that limit the logarithm that appears in Eq. (24) can be expanded as

$$\begin{aligned} \ln\left(\frac{b}{s+b}\right) &= \ln\left(\frac{b+s-s}{s+b}\right) = \ln\left(1 - \frac{s}{s+b}\right) \approx \\ &\approx -\frac{s}{s+b} - \frac{1}{2}\left(\frac{s}{s+b}\right)^2 + \mathcal{O}\left((s/(s+b))^3\right). \end{aligned} \tag{26}$$

If we substitute this expansion in the expression of the median of Eq. (24) we obtain,

$$\begin{aligned} \mathrm{med}[Z_\mu|0] &\approx \sqrt{2\left[b\left(-\frac{s}{s+b} - \frac{1}{2}\left(\frac{s}{s+b}\right)^2 + s\right)\right]} = \sqrt{2\left[s - \frac{bs}{s+b} - \frac{bs^2}{2(s+b)^2}\right]} = \\ &= \sqrt{2\left[\frac{s(s+b) - bs}{s+b} - \frac{bs^2}{2(s+b)^2}\right]} = \sqrt{2\left[\frac{s^2}{s+b} - \frac{b}{2}\frac{s^2}{(s+b)^2}\right]} = \\ &= \sqrt{2\frac{s^2}{(s+b)}\left[1 - \frac{b}{2(s+b)}\right]} = \sqrt{2\frac{s^2}{(s+b)}\left[1 - \frac{b+s-s}{2(s+b)}\right]} = \\ &= \sqrt{2\frac{s^2}{(s+b)}\left[\frac{1}{2} + \frac{s}{2(s+b)}\right]} = \sqrt{\frac{s^2}{(s+b)}\left[1 + \frac{s}{s+b}\right]} = \\ &= \sqrt{\frac{s^2}{(s+b)} + \mathcal{O}(\left(\frac{s}{(s+b)}\right)^2)} = \frac{s}{\sqrt{s+b}}. \end{aligned} \tag{27}$$

In order to stablish exclusion limits in a naive approach assuming that the total number of background events is greater than the signal the relation that should be used is

$$\mathrm{med}[Z_\mu|0] \approx \frac{s}{\sqrt{s+b}}. \tag{28}$$

3 Evidence and/or discovery with uncertanties.

$$L(s,b) = \frac{(s+b)^n}{n!} e^{-(s+b)} \frac{(\tau b)^m}{m!} e^{-\tau b} \tag{29}$$

$$\hat{s} = n - m/\tau, \quad \hat{b} = m/\tau, \tag{30}$$

$$\hat{\hat{b}}(s) = \frac{n + m - (1+\tau)s + \sqrt{[n + m - (1+\tau)s]^2 + 4(1+\tau)sm}}{2(1+\tau)} \tag{31}$$

$$\hat{\hat{b}}(0) = \frac{n+m}{1+\tau} \tag{32}$$

$$L(0, \hat{\hat{b}}) = \frac{(\hat{\hat{b}}(0))^n}{n!} e^{-\hat{\hat{b}}(0)} \frac{(\tau\hat{\hat{b}}(0))^m}{m!} e^{-\tau\hat{\hat{b}}(0)} = \frac{\left(\frac{n+m}{1+\tau}\right)^n}{n!} e^{-\left(\frac{n+m}{1+\tau}\right)} \frac{\left(\tau\left(\frac{n+m}{1+\tau}\right)\right)^m}{m!} e^{-\tau\left(\frac{n+m}{1+\tau}\right)} \tag{33}$$

$$L(\hat{s}, \hat{b}) = \frac{(\hat{s}+\hat{b})^n}{n!} e^{-(\hat{s}+\hat{b})} \frac{(\tau\hat{b})^m}{m!} e^{-\tau\hat{b}} = \frac{n^n}{n!} e^{-n} \frac{m^m}{m!} e^{-m} \tag{34}$$

$$\begin{aligned}\lambda(0) =& \frac{L(0,\hat{\hat{b}})}{L(\hat{s},\hat{b})} = \left(\frac{n+m}{n(1+\tau)}\right)^n \left(\frac{\tau(n+m)}{m(1+\tau)}\right)^m \exp\left\{-\frac{\tau(n+m)+n+m}{1+\tau} + n + m\right\} = \\ =& \left(\frac{n+m}{n(1+\tau)}\right)^n \left(\frac{\tau(n+m)}{m(1+\tau)}\right)^m\end{aligned} \tag{35}$$

$$\begin{aligned}q_0 =& -2\ln\lambda(0) = -2\ln\left[\left(\frac{n+m}{n(1+\tau)}\right)^n \left(\frac{\tau(n+m)}{m(1+\tau)}\right)^m\right] = \\ =& -2\left[n\ln\left(\frac{n+m}{(1+\tau)n}\right) + m\ln\left(\frac{\tau(n+m)}{m(1+\tau)}\right)\right]\end{aligned} \tag{36}$$

Assuming $n \to s+b$ and $m \to \tau b$

$$\begin{aligned}\mathrm{med}[Z_0|1] =& \sqrt{q_0} = \left[-2\left((s+b)\ln\left[\frac{(s+b)+\tau b}{(1+\tau)(s+b)}\right] + \tau b \ln\left[\frac{s+(1+\tau)b}{b(1+\tau)}\right]\right)\right]^{1/2} = \\ =& \left[-2\left((s+b)\ln\left[\frac{s+b(1+\tau)}{(1+\tau)(s+b)}\right] + \tau b\ln\left[\frac{s}{b(1+\tau)} + 1\right]\right)\right]^{1/2}\end{aligned} \tag{37}$$

We take $\sigma_b^2 = b/\tau$

$$\begin{aligned}\mathrm{med}[Z_0|1] =& \left[-2\left((s+b)\ln\left[\frac{s+b(1+b/\sigma_b^2)}{(1+b/\sigma_b^2)(s+b)}\right] + \frac{b^2}{\sigma_b^2}\ln\left[\frac{s}{b(1+b/\sigma_b^2)} + 1\right]\right)\right]^{1/2} = \\ =& \left[-2\left((s+b)\ln\left[\frac{b^2+\sigma_b^2(s+b)}{(s+b)(\sigma_b^2+b)}\right] + \frac{b^2}{\sigma_b^2}\ln\left[1 + \frac{s\sigma_b^2}{b(b+\sigma_b^2)}\right]\right)\right]^{1/2} = \\ =& \left[2\left((s+b)\ln\left[\frac{(s+b)(\sigma_b^2+b)}{b^2+\sigma_b^2(s+b)}\right] - \frac{b^2}{\sigma_b^2}\ln\left[1 + \frac{\sigma_b^2 s}{b(b+\sigma_b^2)}\right]\right)\right]^{1/2}.\end{aligned} \tag{38}$$

So the median obtained for the exclusion sensitivity with uncertainties is

$$\boxed{\text{med}[Z_0|1] = \left[2\left((s+b)\ln\left[\frac{(s+b)(\sigma_b^2+b)}{b^2+\sigma_b^2(s+b)}\right] - \frac{b^2}{\sigma_b^2}\ln\left[1+\frac{\sigma_b^2 s}{b(b+\sigma_b^2)}\right]\right)\right]^{1/2}.} \tag{39}$$

In the limit where $\sigma_b^2/b \ll 1$ we take the first line of Eq. (38) for simplicity,

$$\text{med}[Z_0|1] = \left[-2\left((s+b)\ln\left[\frac{s+b(1+b/\sigma_b^2)}{(s+b)(1+b/\sigma_b^2)}\right] + \frac{b^2}{\sigma_b^2}\ln\left[1+\frac{s}{b(1+b/\sigma_b^2)}\right]\right)\right]^{1/2}. \tag{40}$$

We can see the first term in the limit $\sigma_b^2/b \ll 1$,

$$\ln\left[\frac{s+b(1+b/\sigma_b^2)}{(s+b)(1+b/\sigma_b^2)}\right] = \ln\left[\frac{(s+b)\frac{\sigma_b^2}{b}+b}{(s+b)(1+\sigma_b^2/b)}\right] \overset{\sigma_b^2/b\ll 1}{\approx} \ln\left[\frac{b}{s+b}\right]. \tag{41}$$

And the second term,

$$\frac{b^2}{\sigma_b^2}\ln\left[1+\frac{s}{b/(1+b/\sigma_b^2)}\right] = \frac{b^2}{\sigma_b^2}\ln\left[1+\frac{s}{b}\frac{\sigma_b^2}{(\sigma_b^2+b)}\right] \overset{\sigma_b^2/b\ll 1}{\approx}$$
$$\frac{b^2}{\sigma_b^2}\ln\left[1+\frac{s}{b}\frac{\sigma_b^2}{b}\right] \approx \frac{b^2}{\sigma_b^2}\left[\frac{s}{b}\frac{\sigma_b^2}{b}\right] = s, \tag{42}$$

where we have used the expansion of the logarithm when $x \ll 1$ is $\ln(1+x) \approx x + \mathcal{O}(x^2)$. So if we write them into the definition of $\text{med}[Z_0|1]$ we have,

$$\text{med}[Z_0|1] = \left[-2\left((s+b)\ln\left[\frac{s+b(1+b/\sigma_b^2)}{(s+b)(1+b/\sigma_b^2)}\right] + \frac{b^2}{\sigma_b^2}\ln\left[1+\frac{s}{b(1+b/\sigma_b^2)}\right]\right)\right]^{1/2} \approx$$
$$\approx \left[-2\left((s+b)\ln\left[\frac{b}{s+b}\right]+s\right)\right]^{1/2} = \left[2\left((s+b)\ln\left[\frac{s+b}{b}\right]-s\right)\right]^{1/2}, \tag{43}$$

that is exactly what we obtained for discovery in the absence of uncertainties in Eq. (16).

We can compute the limit of Eq. (39) where $s/b \ll 1$, calculating every term separately. For the sake of simplicity we define $a = s/b$, so Eq. (39) can be written as,

$$\text{med}[Z_0|1] = \left[2\left((s+b)\ln\left[\frac{(s+b)(\sigma_b^2+b)}{b^2+\sigma_b^2(s+b)}\right] - \frac{b^2}{\sigma_b^2}\ln\left[1+\frac{\sigma_b^2 s}{b(b+\sigma_b^2)}\right]\right)\right]^{1/2} =$$
$$= \left[2\left(b(1+a)\ln\left[\frac{(1+a)(\sigma_b^2+b)}{b+\sigma_b^2(1+a)}\right] - \frac{b^2}{\sigma_b^2}\ln\left[1+a\frac{\sigma_b^2}{b+\sigma_b^2}\right]\right)\right]^{1/2}. \tag{44}$$

The first term in that limit ($s/b = a \ll 1$),

$$\ln\left[\frac{(1+a)(\sigma_b^2+b)}{b+\sigma_b^2(1+a)}\right] = \ln\left[\frac{\sigma_b^2+b+\sigma_b^2 a+ab}{b+\sigma_b^2+\sigma_b^2 a}\right] = \ln\left[1+\frac{ab}{b+\sigma_b^2(1+a)}\right]. \tag{45}$$

In the limit where $a \ll 1$ the logarithm can be written as

$$\ln\left[1+\frac{ab}{b+\sigma_b^2(1+a)}\right] \approx \frac{b}{b+\sigma_b^2}a + \frac{1}{2}\frac{(-b^2-2b\sigma_b^2)}{(b+\sigma_b^2)^2}a^2 + \mathcal{O}(a^3). \tag{46}$$

In the same limit the second term is written as

$$\ln\left[1+a\frac{\sigma_b^2}{b+\sigma_b^2}\right] \approx \frac{\sigma_b^2}{b+\sigma_b^2}a - \frac{1}{2}\frac{\sigma_b^4}{(b+\sigma_b^2)^2}a^2 + \mathcal{O}(a^3). \tag{47}$$

4 Exclusion limits with uncertanties.

$$\lambda(\mu) = \frac{L(s, \hat{\hat{b}})}{L(\hat{s}, \hat{b})} \tag{48}$$

$$L(\hat{s}, \hat{b}) = \frac{(\hat{s}+\hat{b})^n}{n!} e^{-(\hat{s}+\hat{b})} \frac{(\tau\hat{b})}{m!} e^{-\tau\hat{b}}, \tag{49}$$

$$L(s, \hat{\hat{b}}) = \frac{(s+\hat{\hat{b}})^n}{n!} e^{-(s+\hat{\hat{b}})} \frac{(\tau\hat{\hat{b}})}{m!} e^{-\tau\hat{\hat{b}}}, \tag{50}$$

Taking the definitions of $\hat{s}$, $\hat{b}$ and $\hat{\hat{b}}$ from Eqs.(30) and (31). Then we have,

$$L(\hat{s}, \hat{b}) = \frac{(\hat{s}+\hat{b})^n}{n!} e^{-(\hat{s}+\hat{b})} \frac{(\tau\hat{b})^m}{m!} e^{-\tau\hat{b}} = \frac{n^n}{n!} e^{-n} \frac{m^m}{m!} e^{-m}. \tag{51}$$

$$\begin{aligned}
\lambda(\mu) =& \frac{(s+\hat{\hat{b}})^n}{n^n} \frac{e^{-(s+\hat{\hat{b}})}}{e^{-n}} \frac{(\tau\hat{\hat{b}})^m}{m^m} \frac{e^{-\tau\hat{\hat{b}}}}{e^{-m}} = \frac{(s+\hat{\hat{b}})^n}{n^n} \frac{(\tau\hat{\hat{b}})^m}{m^m} \exp\left\{-(s+\hat{\hat{b}}) - \tau\hat{\hat{b}} + n + m\right\} = \\
=& \frac{(s+\hat{\hat{b}})^n}{n^n} \frac{(\tau\hat{\hat{b}})^m}{m^m} \exp\left\{n + m - s - (1+\tau)\hat{\hat{b}}\right\}.
\end{aligned} \tag{52}$$

$$\begin{aligned}
q_0 =& -2\ln\lambda(\mu) = -2\ln\left[\frac{(s+\hat{\hat{b}})^n}{n^n} \frac{(\tau\hat{\hat{b}})^m}{m^m} \exp\left\{n + m - s - (1+\tau)\hat{\hat{b}}\right\}\right] = \\
=& -2\left[n\ln\left(\frac{s+\hat{\hat{b}}}{n}\right) + m\ln\left(\frac{\tau\hat{\hat{b}}}{m}\right) + n + m - s - (1+\tau)\hat{\hat{b}}\right]
\end{aligned} \tag{53}$$

For the computation of the exclusion limit we are calculating med$[Z_\mu|0]$ so $\mu' = 0$ and hence Eq. (8) becomes $n = b$ and as in the same case as in the discovery with uncertainties $m = \tau b$.

$$\begin{aligned}
q_0 =& -2\ln\lambda(\mu) = -2\left[b\ln\left(\frac{s+\hat{\hat{b}}}{b}\right) + \tau b\left(\frac{\hat{\hat{b}}}{b}\right) + b + \tau b - s - (1+b)\hat{\hat{b}}\right] = \\
=& -2\left[b\ln\left(\frac{s+\hat{\hat{b}}}{b}\right) + \tau b\ln\left(\frac{\hat{\hat{b}}}{b}\right) + (b-\hat{\hat{b}})(1+\tau) - s\right] = \\
=& -2\left[b\ln\left(\frac{s+\hat{\hat{b}}}{b}\right) + \frac{b^2}{\sigma_b^2}\ln\left(\frac{\hat{\hat{b}}}{b}\right) + (b-\hat{\hat{b}})\left(1+\frac{b}{\sigma_b^2}\right) - s\right] = \\
=& 2\left[b\ln\left(\frac{b}{s+\hat{\hat{b}}}\right) - \frac{b^2}{\sigma_b^2}\ln\left(\frac{\hat{\hat{b}}}{b}\right) - (b-\hat{\hat{b}})\left(1+\frac{b}{\sigma_b^2}\right) + s\right].
\end{aligned} \tag{54}$$

Where in the last step we made use of the fact $\tau = b/\sigma_b^2$. Let us now follow with the expression of $\hat{\hat{b}}$.

$$\begin{aligned}
\hat{\hat{b}}(s) =& \frac{n+m-(1+\tau)s+\sqrt{[n+m-(1+\tau)s]^2+4(1+\tau)sm}}{2(1+\tau)} = \\
=& \frac{b+\tau b-(1+\tau)s+\sqrt{[b+\tau b-(1+\tau)s]^2+4(1+\tau)s\tau b}}{2(1+\tau)} = \\
=& \frac{(1+\tau)(b-s)+\sqrt{(1+\tau)^2(b-s)^2+4(1+\tau)s\tau b}}{2(1+\tau)} = \\
=& \frac{(b-s)+\sqrt{(b-s)^2+\dfrac{4s\tau b}{(1+\tau)}}}{2} = \frac{(b-s)+\sqrt{(b+s)^2-\dfrac{4bs}{(1+\tau)}}}{2} = \\
=& \frac{(b-s)+\sqrt{(b+s)^2-\dfrac{4bs}{(1+b/\sigma_b^2)}}}{2} = \frac{(b-s)+\sqrt{(b+s)^2-\dfrac{4bs\sigma_b^2}{(\sigma_b^2+b)}}}{2}.
\end{aligned} \tag{55}$$

$$\boxed{\hat{\hat{b}}(s) = \frac{(b-s)+\sqrt{(b+s)^2-\dfrac{4bs\sigma_b^2}{(\sigma_b^2+b)}}}{2}.} \tag{56}$$

$$\boxed{\mathrm{med}[Z_0|0] = \left[2\left(b\ln\left(\frac{b}{s+\hat{\hat{b}}}\right)-\frac{b^2}{\sigma_b^2}\ln\left(\frac{\hat{\hat{b}}}{b}\right)+(\hat{\hat{b}}-b)\left(1+\frac{b}{\sigma_b^2}\right)+s\right)\right]^{1/2}} \tag{57}$$

If we take the limit where $b/\sigma_b^2 \gg 1$, that is $\tau \gg 1$ then

$$\hat{\hat{b}}(s) = \frac{(b-s)+\sqrt{(b+s)^2-\dfrac{4bs}{(1+\tau)}}}{2} \xrightarrow{\tau\gg 1} b \tag{58}$$

Making the substitution in Eq. (57) and taking the limit $b/\sigma_b^2 \gg 1$,

$$\begin{aligned}
\mathrm{med}[Z_0|0] \xrightarrow{\tau\gg 1}& \left[2\left(b\ln\left(\frac{b}{s+b}\right)-\frac{b^2}{\sigma_b^2}\ln\left(\frac{b}{b}\right)+(b-b)\left(1+\frac{b}{\sigma_b^2}\right)+s\right)\right]^{1/2} = \\
=& \left[2\left(b\ln\left(\frac{b}{s+b}\right)+s\right)\right]^{1/2}.
\end{aligned} \tag{59}$$

That is basically the same result as we have in Eq. (24) for the exlusion without uncertainties.