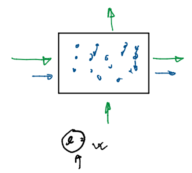
Energy Conservation

Wednesday, January 13, 2021 7:29 PM

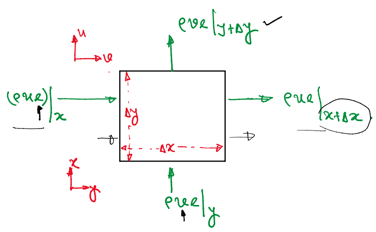
Energy transport equation.

$$\left\{\begin{array}{l}\text{Rate of energy}\\ \text{accumulation}\\ \text{in the CV}\end{array}\right\}_{(1)} = \left\{\begin{array}{l}\text{Net transfer}\\ \text{of energy by}\\ \text{fluid flow}\end{array}\right\}_{(2)} + \left\{\begin{array}{l}\text{Net transfer}\\ \text{of energy by}\\ \text{conduction}\end{array}\right\}_{(3)}$$

$$+ \left\{\begin{array}{l}\text{Rate of internal}\\ \text{heat generation}\end{array}\right\}_{(4)} - \left\{\begin{array}{l}\text{Net work transfer}\\ \text{from the CV to}\\ \text{the surrounding}\end{array}\right\}_{(5)}$$



For (1) and (2)



where

e is the total specific energy of the system

$$e = \hat{u} + \frac{1}{2}v_s^2$$

$\hat{u}$ = Specific internal energy $= \frac{u}{m} = \hat{u}$

$\frac{1}{2}v_s^2$ = Specific kinetic energy $= \frac{\frac{1}{2}mv_s^2}{m} = \frac{1}{2}v_s^2$

(1) Rate of energy accumulation

$$\frac{\partial}{\partial t}(\rho e)\Delta x\Delta y \quad \cdots\cdots (1)$$

(2) Net transfer of energy by fluid flow

i) Rate of energy in

$$\rho u e|_x \Delta y + \rho v e|_y \Delta x$$

ii) Rate of energy out

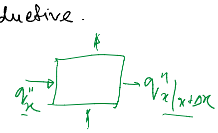
$$\rho u e|_{x+\Delta x}\Delta y + \rho v e|_{y+\Delta y}\Delta x$$

Net [in − out]

$$= -\left[\frac{\partial(\rho u e)}{\partial x} + \frac{\partial(\rho v e)}{\partial y}\right]\Delta x\Delta y \quad \cdots\cdots (2)$$

(3) The transport element here is the conductive heat flux.

$$q'' = \frac{\text{Conductive heat}}{\text{Area} \times \text{time}}.$$



So, the net conductive heat transfer will be.

$$-\left[\frac{\partial}{\partial x}(q''_x) + \frac{\partial}{\partial y}(q''_y)\right]\Delta x\Delta y \quad \cdots\cdots (3)$$

(4) Total internal heat generation

$$q'''\Delta x\Delta y \cdots\cdots (4)$$

$$q''' = \frac{\text{heat}}{\text{Volume} \times \text{time}}$$

= Volumetric heat generation

(5) Net rate of work done by the fluid element against the surrounding

i) Work against the body force.

ii) Work against the surface force.

$F = mg$

$g_x = \frac{F}{m}$, (g_y)

$g_x \times \frac{\Delta x}{\Delta t} \to u$ +

$g_y \times v$.

a) Work against the pressure force.

b) Work against the viscuss force.

i) Rate of work against the body force.

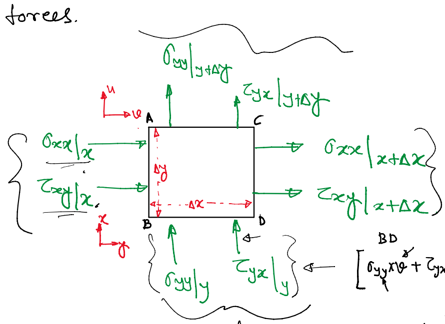
$$-\rho\left[u g_x + v g_y\right]\Delta x\Delta y \cdots\cdots (5) \rightarrow \quad g = \frac{\text{Force}}{\text{mass}}.$$

ii) Rate of work against the surface forces.

a) Pressure.

$$-\left[\frac{\partial}{\partial x}(pu) + \frac{\partial}{\partial y}(pv)\right]\Delta x \Delta y \quad \cdots\cdots (6a)$$

b) Viscous forces.



$\tau = \begin{bmatrix} \sigma_{xx} & \\ & \sigma_{yy} \end{bmatrix}$

$\left[\sigma_{xx}|_x u + \tau_{xy}|_x v\right] \Delta y$

$\tau_{xy} = \tau_{yx} =$

BD $\left[\sigma_{yy} v + \tau_{yx} u\right]$ $\tau_{ji} =$

direction ⊥ to the surface along which the force is acting. {direction of applied forces.}

Net work done by the viscous force.

$$\left[\sigma_{xx}u + \tau_{xy}v\right]_x \Delta y + \left[\sigma_{yy}v + \tau_{yx}u\right]_y \Delta x$$

$$-\left[\sigma_{xx}u + \tau_{xy}v\right]_{x+\Delta x} \Delta y - \left[\sigma_{yy}v + \tau_{yx}u\right]_{y+\Delta y} \Delta x$$

$\nabla.(\tau.\vec{v})$.

↓ After simplification

$$= -\left[\frac{\partial}{\partial x}(\sigma_{xx}u) + \frac{\partial}{\partial x}(\tau_{xy}v) + \frac{\partial}{\partial y}(\sigma_{yy}v) + \frac{\partial}{\partial y}(\tau_{yx}u)\right]\Delta x \Delta y$$

After substituting the constitutive equations and simplifications [Home work]

$$= [\mu \Phi] \Delta x \Delta y \quad \cdots\cdots (6b)$$

Assumptions.

i) Incompressible flow $\nabla.v = 0$

ii) NF

Constitutive equations

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{xx} = -2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu(\nabla.\vec{v})$$

$$\sigma_{yy} = -2\mu\frac{\partial v}{\partial y} + \frac{2}{3}\mu(\nabla.\vec{v})$$

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2$$

So, the net work done by the fluid element on the surrounding

$= 5 + 6a + 6b$

$$= -\left[\rho(ug_x + vg_y) + \frac{\partial}{\partial x}(pu) + \frac{\partial}{\partial y}(pv) - \mu\Phi\right] \cdots\cdots (6)$$

Substituting equation (1) → (6) into the transport equation we get

$$\boxed{\rho\frac{D(e+\psi)}{Dt} = q''' - \nabla.(p\vec{v}) - (\nabla.q'') + \mu\Phi} \quad \cdots\cdots (7)$$

where $g = -\nabla\psi$ ψ = potential energy.

This is the equation of transport of total energy $(e+\psi) = \hat{u} + \frac{1}{2}v_s^2 + \psi$

1st law of thermodynamics.

According to 1st law we have.

$q = \Delta U + W$

According to 1st law we have.

$$Q = \Delta U + W$$

So, the heat supplied to the system is used to increase the internal energy only.

q''

using this condition equation (7) can be written as

$$\rho \frac{De}{Dt} = q''' - (\nabla \cdot q'') - \nabla \cdot (P\vec{v}) + \mu\phi \quad \cdots (8)$$

where $e = \hat{u}$ only

$e \to f(T)$

using thermodynamic relation $\hat{u} \approx e$.

$$H = U + PV$$

We can write the specific enthalpy as $\hat{h}$

$$\frac{H}{m} = h = e + P\left(\frac{1}{\rho}\right) \longrightarrow e = h - P(1/\rho)$$

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} \quad \cdots (9)$$

From Fourier law of heat conduction

$h = f(T)$

$dh = c_p dT$

$$q'' = -k\nabla T$$

Substituting equation (P) & (9) into equation (8) we get

$$\rho\frac{Dh}{Dt} = q''' - \nabla \cdot q'' - P(\nabla \cdot \vec{v}) + \mu\phi + \frac{DP}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt}$$

$$\rho\frac{Dh}{Dt} = q''' + \nabla \cdot (k\nabla T) + \mu\phi - \frac{P}{\rho}\left[\underbrace{\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v})}_{\approx 0 \text{ (Not an assumption)}}\right] + \frac{DP}{Dt}$$

$$\boxed{\rho\frac{Dh}{Dt} = q''' + \nabla \cdot (k\nabla T) + \frac{DP}{Dt} + \mu\phi} \quad \cdots (11)$$

if $dh = c_p dT$ → const Pr. → Closed system. → ideal gas. X

From thermodynamic relation we have.

$$dH = Tds + vdp$$

$s = f(T, P)$

So, $$dh = Tds + \frac{1}{\rho}dp \quad \cdots (12)$$

Note : $dh = c_p dT$ is only valid for ideal gas.

Table 1.1 Summary of thermodynamic relations[a] and models

	Internal Energy $du = T\,ds - P\,dv$	Enthalpy $dh = T\,ds + v\,dP$	Entropy $ds = \frac{1}{T}du + \frac{P}{T}dv$
Pure substance	$du = c_v\,dT$	$dh = c_p\,dT$	$ds = \frac{c_p}{T}dT - \left(\frac{\partial v}{\partial T}\right)_P dP$
	$+\left[T\left(\frac{\partial P}{\partial T}\right)_v - P\right]dv$	$+\left[-T\left(\frac{\partial v}{\partial T}\right)_p + v\right]dP$	$= \frac{c_v}{T}dT + \left(\frac{\partial P}{\partial T}\right)_v dv$
Ideal gas	$du = c_v\,dT$	$dh = c_p\,dT$	$ds = c_p\frac{dT}{T} - R\frac{dP}{P}$
			$= c_v\frac{dT}{T} + R\frac{dv}{v}$
			$= c_v\frac{dP}{P} + c_p\frac{dv}{v}$
Incompressible liquid	$du = c\,dT$	$dh = c\,dT + v\,dP$	$ds = c\frac{dT}{T}$

Now, $s = f(T, P)$

$$ds = \left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP \quad \cdots (13)$$

Using Maxwell's relation we have.

Maxwell relation

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_P$$

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left[\frac{\partial(1/\rho)}{\partial T}\right]_P$$

$$= \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{\beta}{\rho} \quad \cdots \quad \beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$$

Where β is the volumetric expansion coefficient

$$\beta = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_P = \frac{1}{1/\rho}\frac{\partial(1/\rho)}{\partial T} = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$$

We also have.

$$\left(\frac{\partial s}{\partial T}\right)_P = \frac{c_p}{T}$$

So, we got

$$ds = \frac{c_p}{T}dT - \frac{\beta}{\rho}dp \quad \text{[from equation 13]}$$

and

$$dh = Tds + \left(\frac{1}{\rho}\right)dp = c_p dT - \frac{\beta T}{\rho}dp + \frac{1}{\rho}dp$$

$$dh = c_p dT + \frac{1}{\rho}(1 - \beta T)dp$$

$$\rho\frac{Dh}{Dt} = \rho c_p\frac{DT}{Dt} + (1 - \beta T)\frac{DP}{Dt} \quad \cdots (14)$$

← Relation between h & T

$$\boxed{\rho c_p\frac{DT}{Dt} = \nabla \cdot (\nabla kT) + q''' + \beta T\frac{DP}{Dt} - \nabla \cdot (\tau \cdot \vec{v})}$$

No assumption.

Substituting into eq (11)

$$\boxed{\rho c_p\frac{DT}{Dt} = \nabla \cdot (k\nabla T) + q''' + \beta T\frac{DP}{Dt} + \mu\phi} \quad \cdots (15)$$

$$\boxed{\rho c\frac{DT}{Dt} = \nabla \cdot (\nabla kT) + q''' + \mu\Phi}$$

i) NF ii) Ic.

general form with (i) (ii) & (iii) assumptions

Special cases: i) Ideal gas $\left[\beta = \frac{1}{T}\right]$

$$\boxed{\rho C_p \frac{DT}{Dt} = \nabla \cdot (K \nabla T) + q''' + \frac{DP}{Dt} + \mu \bar{\Phi}} \quad \cdots (16)$$

$\tau_{xy} \times v$

$\nabla \cdot [\tau \cdot \vec{v}]$

C_p

C_v

C

2) Incompressible liquid $(\beta = 0)$ [NF]

$$\boxed{\rho C \frac{DT}{Dt} = \nabla \cdot (K \nabla T) + q''' + \mu \bar{\Phi}} \quad \cdots (17)$$

$C_p = C_v \approx C.$

3) Most of the convection problem

a) constant K

b) Zero heat generation

c) negligible viscous dissipation $\phi \approx 0$

$$\boxed{\rho C \frac{DT}{Dt} = K \nabla^2 T} \quad \cdots (18)$$