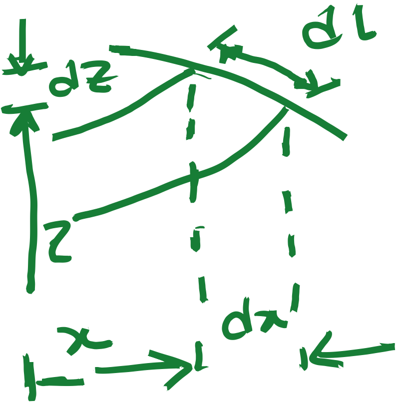
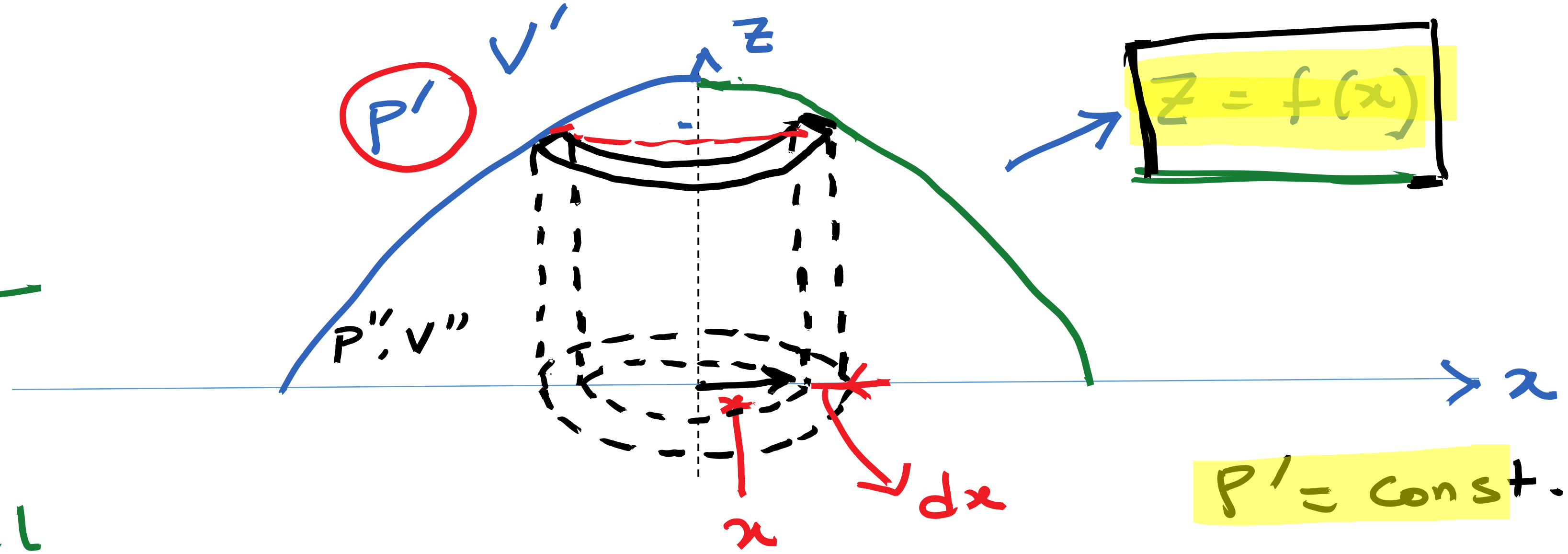
Generalized Expression of Young Laplace Equation for an axi-symmetric Surface

21.03.2022

Lecture: 28





$\boxed{z = f(x)}$

$P'' \rightarrow Pr$, $V'' \rightarrow Vol$] Inside

P', V'] $\rightarrow$ outside

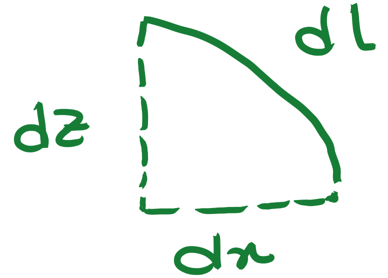
$V = V' + V''$

$P' = const.$

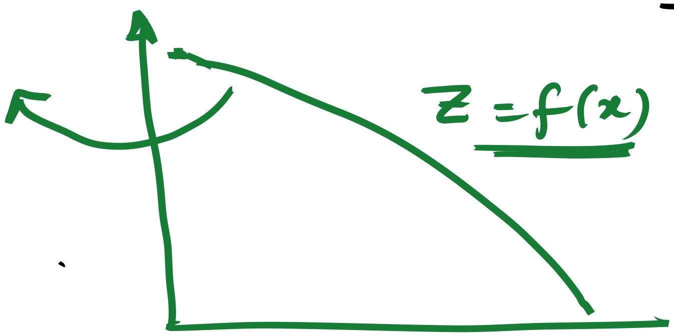
$V = const$

$T = const$

$V'' = const$



$V'' = constant$



(1)

$$F = \gamma A + V''(\Delta P) + C$$

Plugging in Expression of A and V.

$$F = \gamma \int_0^R 2\pi x \left(1 + Z_x^2\right)^{1/2} dx + (\Delta P) \int_0^R 2\pi x Z \cdot dx + C$$

(2) We would like to evaluate F as a function of shape, so that we can find out the equilibrium shape → which corresponds to minimize F. ⇒ FIND out the shape for which F is minimum.

$Z = f(x)$

Necessary Cond.

$\frac{dZ}{dx} = 0$,

Sufficiency Cond

$\frac{d^2Z}{dx^2} = +ve$

The general form of the eqn. which needs to be minimized

$$F = \int_0^R f\left(x, Z, \frac{dZ}{dx}\right) dx$$

$$F = \int_0^R f(x, z, z_x)\,dx$$

Problem related to integration of a differential.

Functional * NEW

Calculus of Variations

Necessary condition for Optimization of a Functional is. NEW

$$\boxed{\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z_x}\right) = 0}$$

$$F = \gamma \int_0^R 2\pi x (1 + z_x^2)^{1/2}\,dx + (\Delta P) \int_0^R 2\pi x z \cdot dx + C$$

Some expression without the integral

For our eqn. $\boxed{f = \gamma \cdot 2\pi x (1 + z_x^2)^{1/2} + \Delta P \cdot 2\pi x z} + C$

$$F = \gamma \int_0^R 2\pi x \cdot \left(1 + Z_x^2\right)^{1/2} dx + \int_0^R \underline{(\Delta P)} \, (2\pi x \, Z) \, dx .$$

$$F = \int_0^R f\left(x, Z, \frac{dZ}{dx}\right) dx \longrightarrow \underline{\text{Functional}}$$

Calculus of Variations.

We are looking at the necessery condition for optimization of a Functional.

* Please check diff between F and f

$$\frac{\partial f}{\partial Z} - \frac{d}{dx}\left(\frac{df}{dZ_x}\right) = 0$$

Does not involve F
But it involves "f"

$$f = \gamma \cdot 2\pi x \left(1 + Z_x^2\right)^{1/2} + \Delta \cdot 2\pi x \cdot Z$$

$$\frac{\partial f}{\partial Z} - \frac{d}{dx}\left(\frac{\partial f}{\partial Z_x}\right) = 0$$

$$f = \gamma \cdot 2\pi x \left(1 + Z_x^2\right)^{1/2} + \Delta p \cdot 2\pi x \cdot Z \cdot$$

Please treat 'Z' and "Z_x"s "$\frac{dz}{dx}$" as independent variables, x also

$$\frac{\partial f}{\partial Z} = \Delta p \cdot 2\pi x$$

$$\frac{\partial f}{\partial Z_x} = \gamma \cdot 2\pi x \frac{1}{2}\left(1 + Z_x^2\right)^{-1/2} \cdot 2 Z_x = \gamma \cdot 2\pi x \cdot Z_x \left(1 + Z_x^2\right)^{-1/2} \cdot$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial Z_x}\right) = 2\pi\gamma \Big[Z_x \left(1 + Z_x^2\right)^{-1/2} + x Z_{xx} \left(1 + Z_x^2\right)^{-1/2} - x Z_x \frac{1}{2}\left(1 + Z_x^2\right)^{-3/2} \cdot 2 Z_x \cdot Z_{xx} \Big]$$

We take the Second and the third term:$\rightarrow$

$$x Z_{xx}(1+Z_x^2)^{-1/2} - x Z_x^2 \cdot Z_{xx}(1+Z_x^2)^{-3/2}$$

$$= x Z_{xx}(1+Z_x^2)^{-1/2}\left[1 - \frac{Z_x^2}{1+Z_x^2}\right]$$

$$= x Z_{xx}(1+Z_x^2)^{-1/2}\left[\frac{1+\cancel{Z_x^2}-\cancel{Z_x^2}}{1+Z_x^2}\right]$$

$$= x Z_{xx}(1+Z_x^2)^{-3/2}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial Z_x}\right) = 2\pi\gamma\left[Z_x(1+Z_x^2)^{-1/2} + x\, Z_{xx}(1+Z_x^2)^{-3/2}\right]$$

$$\frac{\partial f}{\partial Z} = \Delta p \cdot 2\pi x.$$

Necessary condition is $\frac{\partial f}{\partial Z} - \frac{d}{dx}\left(\frac{\partial f}{\partial Z_x}\right) = 0$

$$\Delta p \cdot 2\pi x = 2\pi\gamma\left[\frac{Z_x}{x(1+Z_x^2)^{1/2}} + \frac{Z_{xx}}{(1+Z_x^2)^{3/2}}\right]$$

$$\Rightarrow \Delta p = \gamma\left[\frac{Z_x}{x(1+Z_x^2)^{1/2}} + \frac{Z_{xx}}{(1+Z_x^2)^{3/2}}\right]$$

Generalized Expression of Young - Laplace Eqn for an axi-symmetric Surface:

$$\Delta p = \gamma \left[\frac{Z_x}{x(1+Z_x^2)^{1/2}} + \frac{Z_{xx}}{(1+Z_x^2)^{3/2}} \right]$$

$1/R_2 \rightarrow$ out of Plane curvature

$1/R_1 \rightarrow$ In Plane curvature

$$\Delta p = \gamma \left[\frac{1}{R_2} + \frac{1}{R_1} \right]$$

At the point How to measure R_1 and R_2?

For an axi-symmetric surface, if $Z = f(x)$ is known. $\rightarrow$ Then at any point Z_x and Z_{xx} can be determined

$$\Delta p = \gamma \left[\frac{Z_x}{x(1+Z_x^2)^{1/2}} + \frac{Z_{xx}}{(1+Z_x^2)^{3/2}} \right]$$

$1/R_2 \rightarrow$ $\quad$ $1/R_1 \rightarrow$

$$\Pi = -\phi^{LW}$$

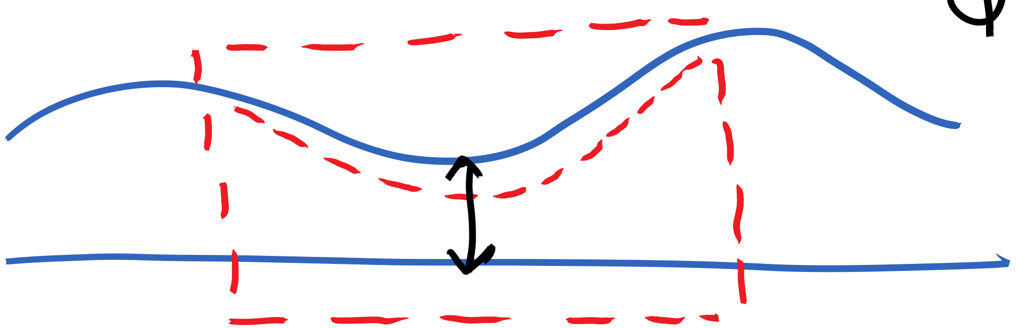
Disjoining Pressure

Neg. Disjoing Pre

→ Conjoing Pressure

Effective Interfece Potential

Thin Film



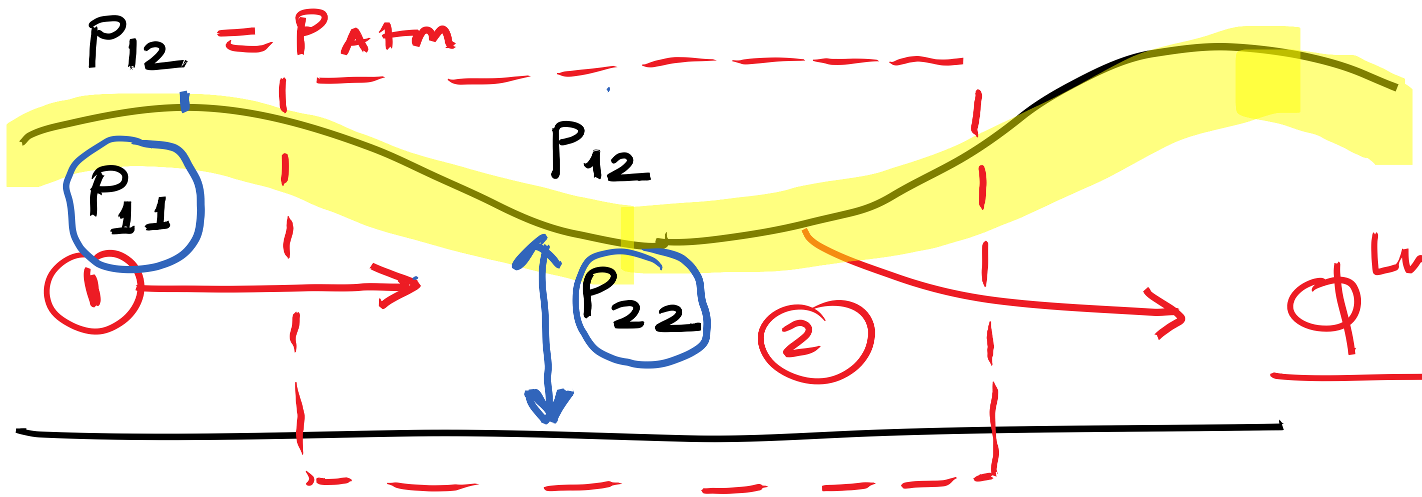
$$\phi^{LW} = \frac{\partial(\Delta G^{Ex})}{\partial h} \rightarrow +ve$$

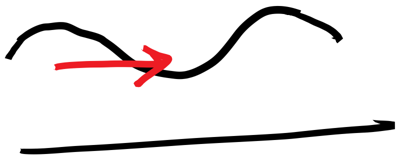
Growth of Fluctuation favored.

< Disjoining Pressure >

For Negative Disjoing Pressure.

Does Negative Disjoing Pressure GUARENTEE Rupture of of The Film





$\phi^{LW} = +ve$ / Π = Negative

$P_{11} > P_{22}$

NOT ONLY DISJOINING PRESSURE.

Now we have Laplece Pressure → $P_{11} > P_{22}$

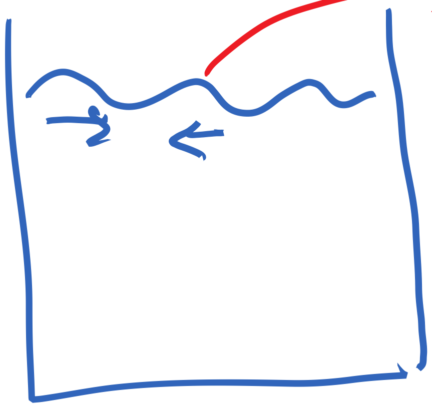
Laplece Pressure will try to trigger flow From Zone 1 to Zone 2 → Try will try to Level the film.

Necessery Condition

(1) Negative Disjoing Pressure

(2) Strength of Disjoing Pressure > Strength of the stabilizing effect of Laplace Pressure

Then only the Film will rupture



Laplace Pressure mediated Flattening

Interfacial Interaction / if attractive (Negative Disjoing Pressure or Positive AE) favors instability

Surface Tension thro' Laplace Pressure Tries to stabilize!