Non-linear logistics growth model.

$$\frac{dx}{dt} = ax\left(1-\frac{x}{N}\right)$$

growth parameter   carrying capacity

max. population that can be supported.

Consider a population $x(t)$ with growth parameter N set to 1.

$$\frac{dx}{dt} = ax(1-x)$$

$$\frac{dx}{x(1-x)} = a\,dt$$

$$\Rightarrow \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = a\,dt$$

$$\Rightarrow \ln\left(\frac{x}{1-x}\right) = at - \ln c$$

$$\Rightarrow \frac{x}{1-x} = ce^{at}$$

$$\Rightarrow \frac{1}{x} - 1 = \frac{1}{ce^{at}}$$

$$\Rightarrow \boxed{x(t) = \frac{ce^{at}}{1+ce^{at}}} \rightarrow (1)$$

Initial conditions: $x(0) = \frac{c}{1+c} \Rightarrow \frac{1}{x_0} = \frac{1}{c} + 1$

$$\Rightarrow c = \frac{x_0}{1-x_0}$$

$$\therefore x(t) = \frac{\left(\frac{x_0}{1-x_0}\right)e^{at}}{1+\left(\frac{x_0}{1-x_0}\right)e^{at}}$$

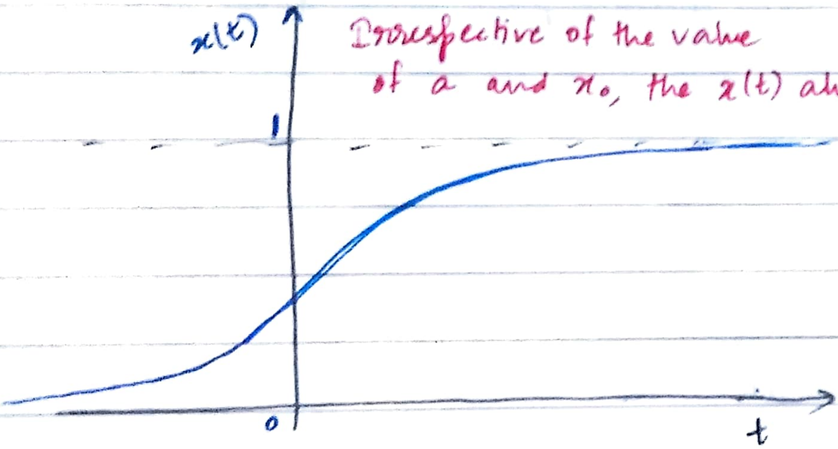
$$\boxed{x(t) = \frac{x_0 e^{at}}{1-x_0+x_0 e^{at}}}$$

↳ From non-linear model.

$$x(t) = x_0 e^{at}$$

↳ From linear model.

Upon plotting on desmos to understand the dynamics,

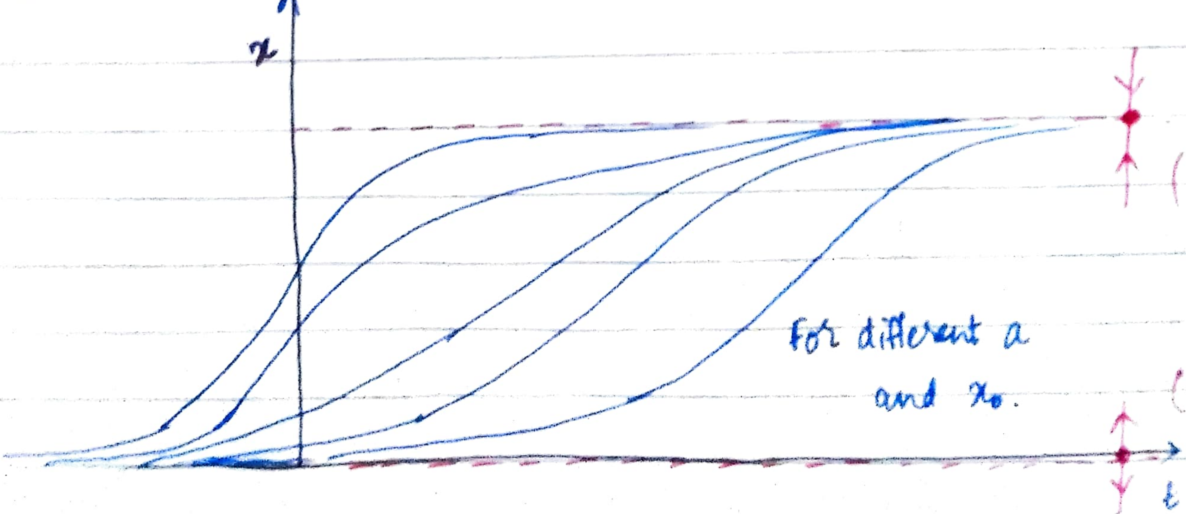


Irrespective of the value of a and x_0, the $x(t)$ always saturates at 1 ($\approx N$ as set).

i.e., this curve always saturates at carrying capacity N.

phase portrait:

Equilibrium solutions: $\frac{dx}{dt} = ax(1-x) = 0 \Rightarrow \left.\begin{array}{l} x=0 \\ x=1 \end{array}\right\}$ equilibrium solutions



$x_e = 1$ (Stable solⁿ) - Sink

For different a and x_0.

(Unstable solⁿ) - Source

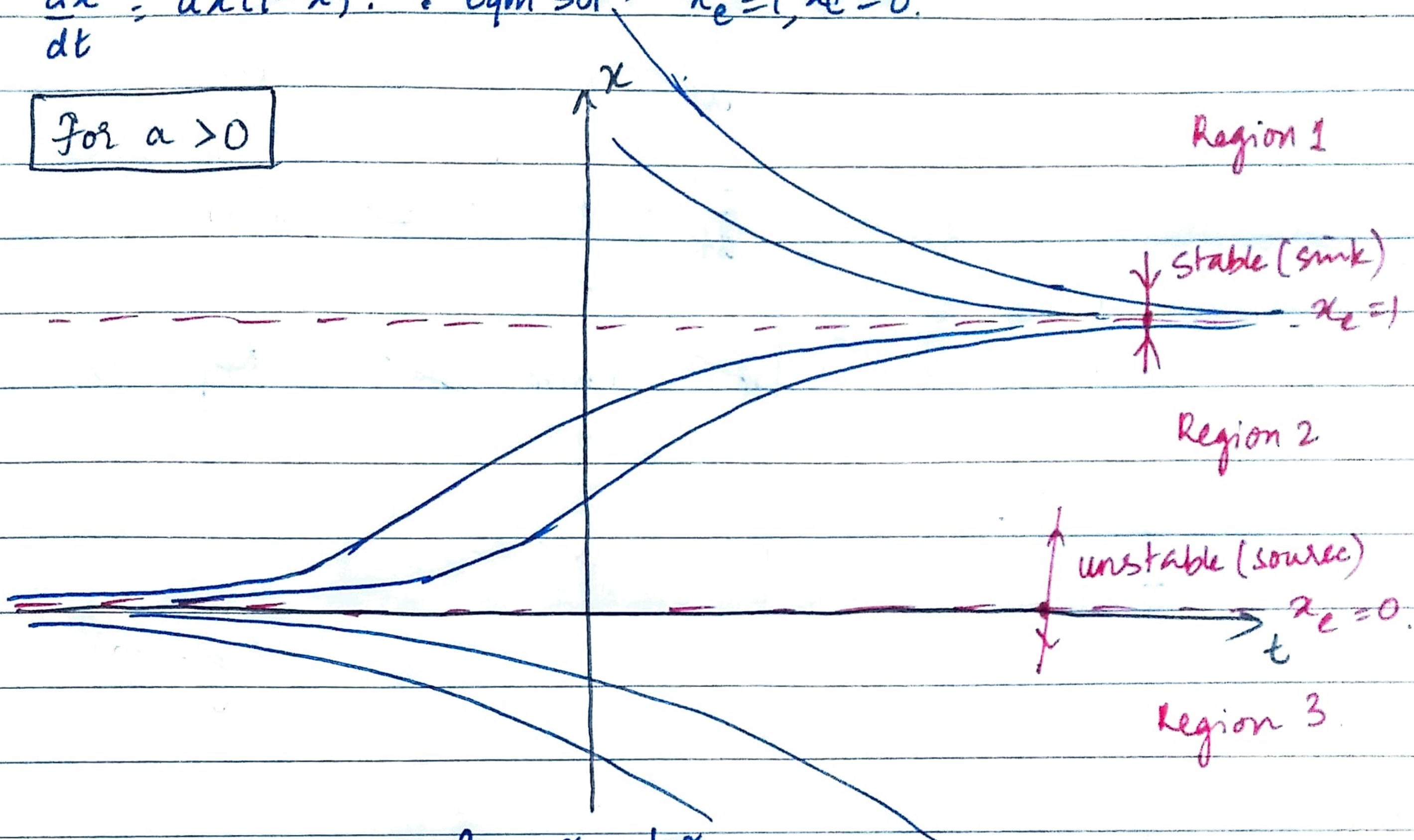
$x_e = 0$

classmate

Making phase portrait without solving the DE.

$\frac{dx}{dt} = ax(1-x)$. ∴ Eqm sol^ns $x_e = 1$, $x_e = 0$.

$\boxed{\text{For } a > 0}$



a x 1−x

Region 1 : $\frac{dx}{dt} \rightarrow$ (+ve)(+ve)(−ve) as $x > 1$

$\Rightarrow \frac{dx}{dt} \rightarrow$ −ve and has asymptote at $x = 1$

Region 2 : $\frac{dx}{dt} \rightarrow$ (+ve)(+ve)(+ve) as $0 < x < 1$

$\Rightarrow \frac{dx}{dt} \rightarrow$ +ve and has asymptotes at $x = 0$ & $x = 1$

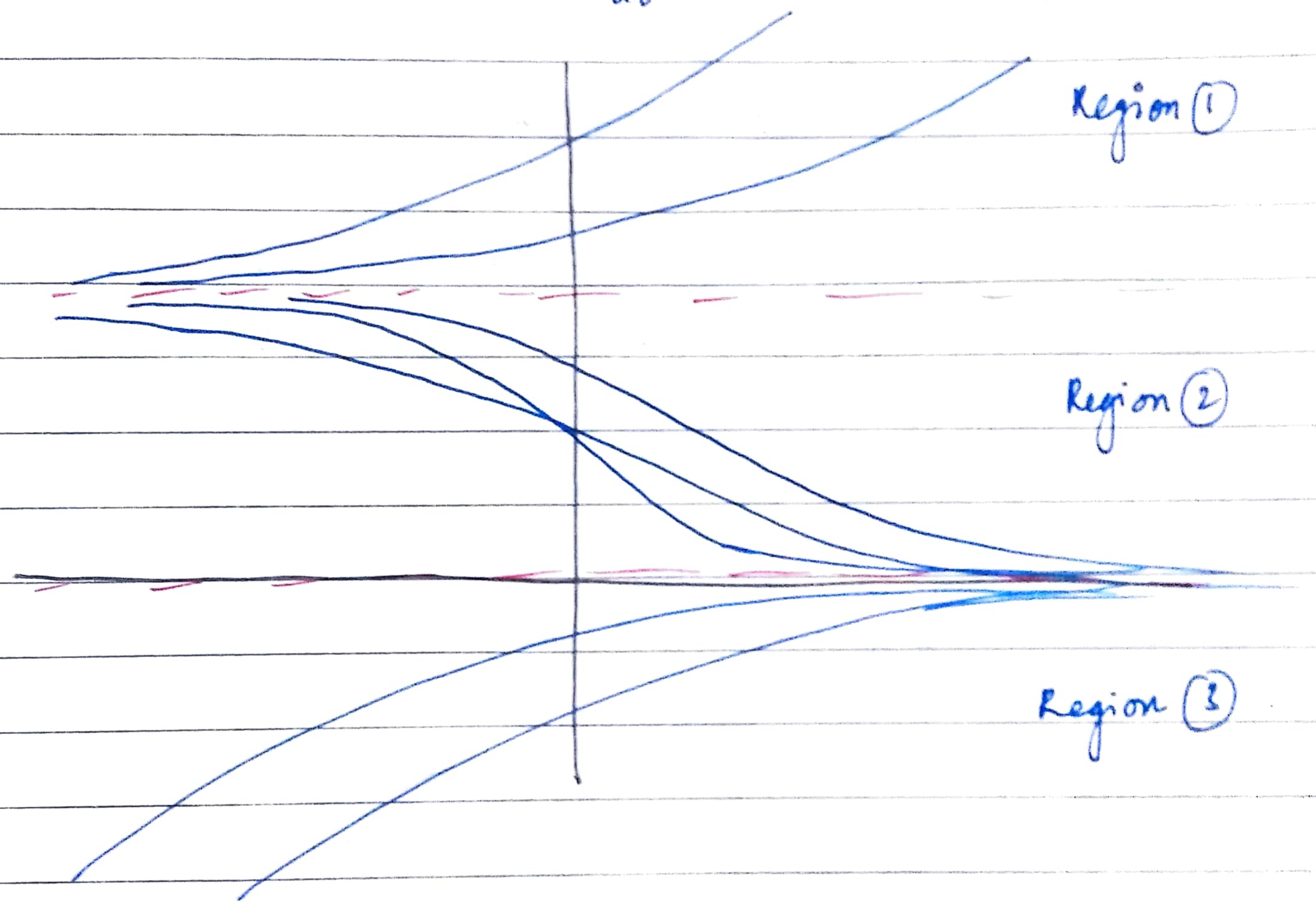
Region 3 : $\frac{dx}{dt} \rightarrow$ (+ve)(−ve)(+ve) as $x < 0$

$\Rightarrow \frac{dx}{dt} \rightarrow$ −ve and asymptote at $x = 0$.

Similarly for $a<0$, we get $\frac{dx}{dt} \rightarrow +ve$ in ① region

$\frac{dx}{dt} \rightarrow -ve$ in ② region

$\frac{dx}{dt} \rightarrow +ve$ in ③ region.



As can be seen, the phase portraits at $a>0$ and $a<0$ are mirror images of each other.

Now, let's take the example of fish harvesting in a pond. In this case at every interval, some amount of fish is taken from the pond and sold. So, there is a constant rate of withdrawal of fish from the pond that also has to be accounted for, which hasn't been done so far.

Fish harvesting in a pond.

Consider a constant harvesting rate of "h".

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h$$

Let's consider $a = N = 1$ as effects of 'h' are of concern.

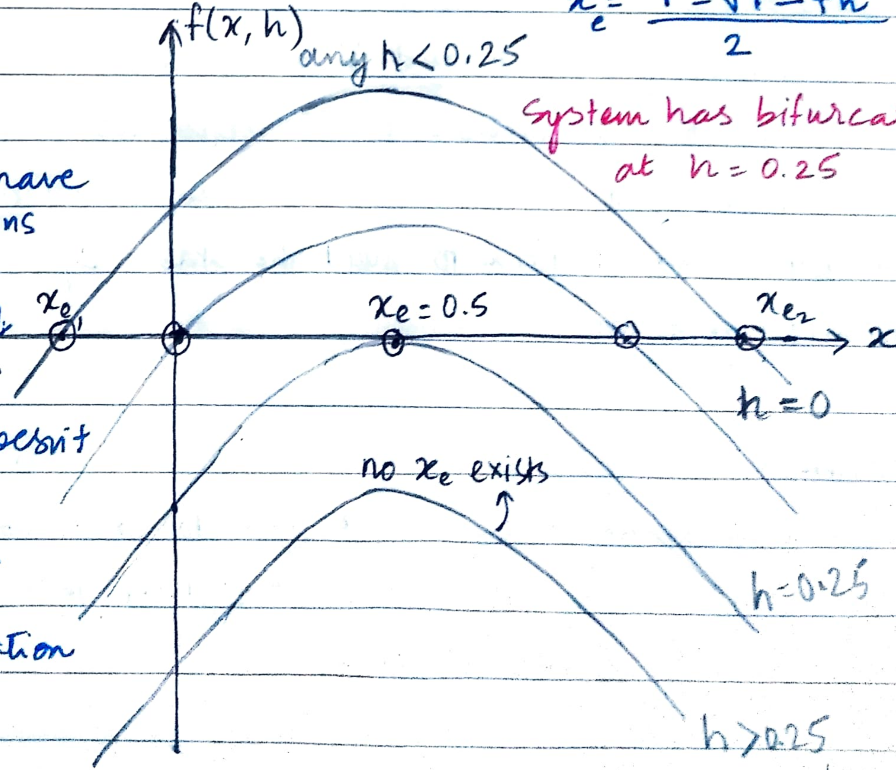
$$f(x,h) = \frac{dx}{dt} = \cancel{ax}\, x(1-x) - h$$

$$\therefore\ f(x,h) = -x^2 + x - h$$ $\longrightarrow$ For equilibrium solution, $f(x,h) = 0$ or $\frac{dx}{dt} = 0$.

i.e.; at equilibrium, $x^2 - x + h = 0$

So, equilibrium solution is a function of h.

$$x_e = \frac{1 \pm \sqrt{1-4h}}{2}$$



System has bifurcation at $h = 0.25$

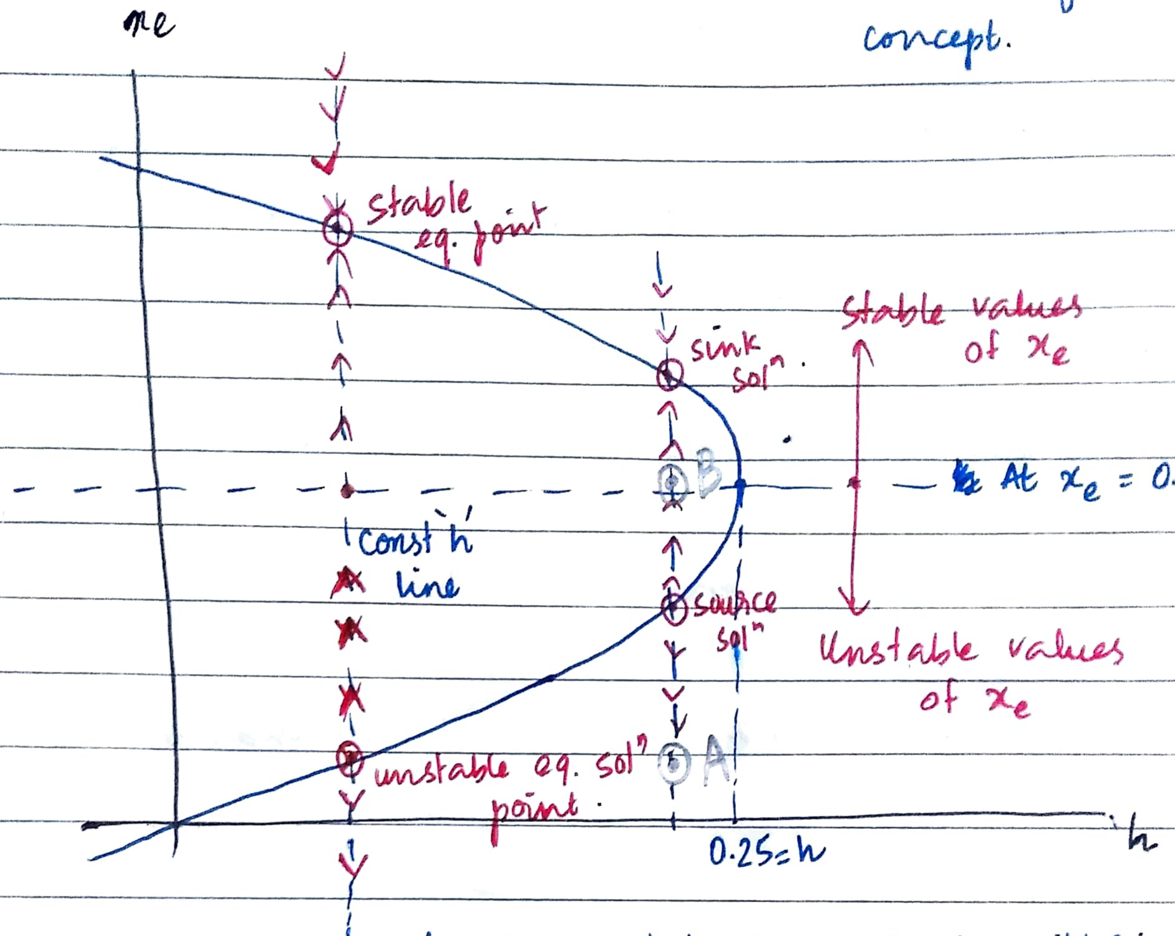
At $h_e < 0.25$, we have 2 equilibrium sol^{ns}

At $h_e = 0.25$, $x_e = 0.5$] single.

At $h_e > 0.25$, x_e doesn't exist

At any value of $h_e > 0.25$ population can't thrive.

See lecture from 1:05:00 for this concept.



↳ at every const h, there are 2 equilibrium solutions. One of it is stable and the other is unstable.

Let's consider points A and B on a constant 'h' line as shown. At point A, the population goes to zero as it is lower than the eq. solution and closer to the source solution. But at B, the population would grow to meet the stable eq. solution. The population doesn't die down.

Problem:

Imagine a population that goes to extinction if the initial population is below a certain "critical threshold". The features of such a population are -

classmate

- upper limit on population given by carrying capacity
- exponential growth during I.C. and saturation later.
- extinction when initial population is below critical threshold.

Such dynamics are given by -

$$\frac{dx}{dt} = -rx\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$ where x is the population

$r \to$ growth parameter

$\lambda_1, \lambda_2 \to$ carrying capacity and critical threshold such that $0 < \lambda_2 < \lambda_1$

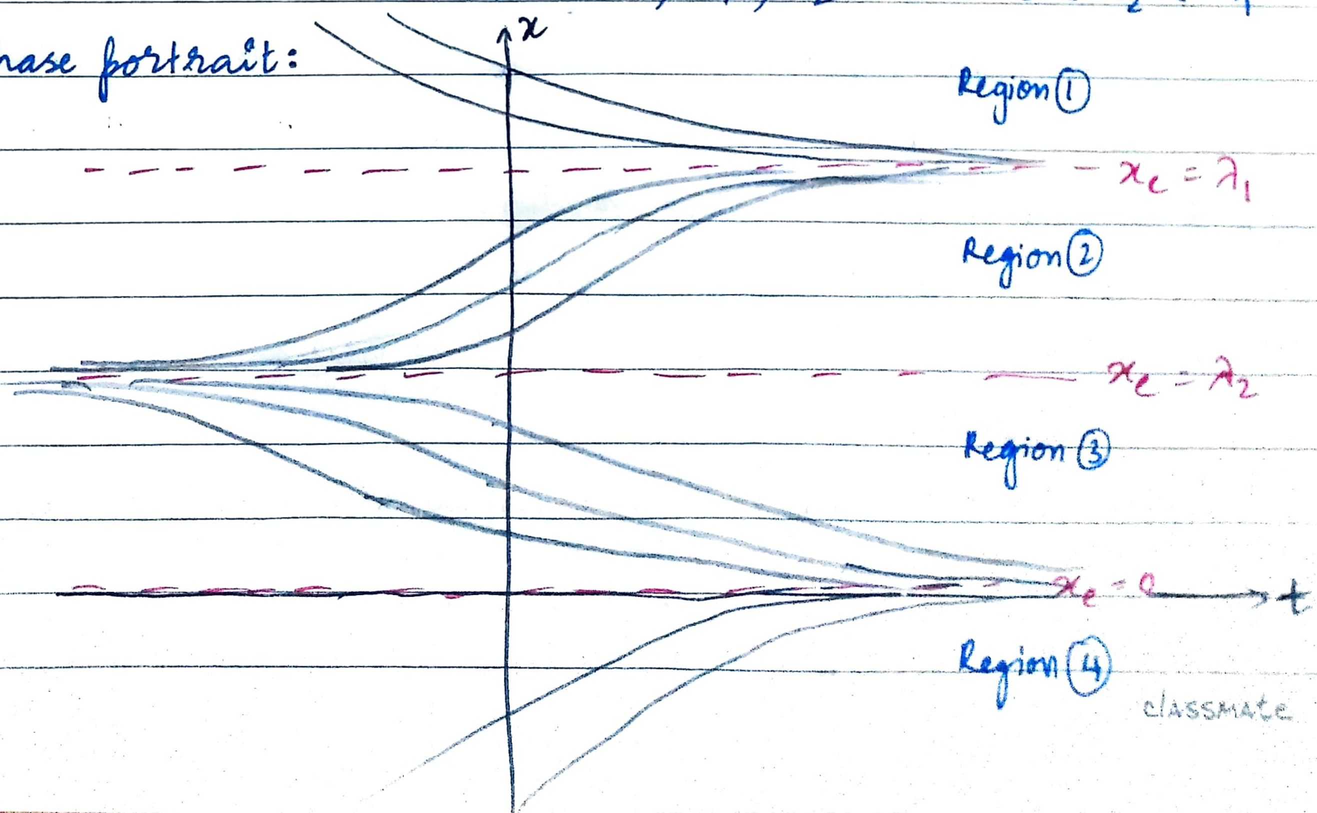
Logistic growth with threshold.

$$\frac{dx}{dt} = -rx\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$

For equilibrium solutions; $-rx\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) = 0$

$\Rightarrow \quad x = 0, \lambda_1, \lambda_2$ with $0 < \lambda_2 < \lambda_1$

Phase portrait:



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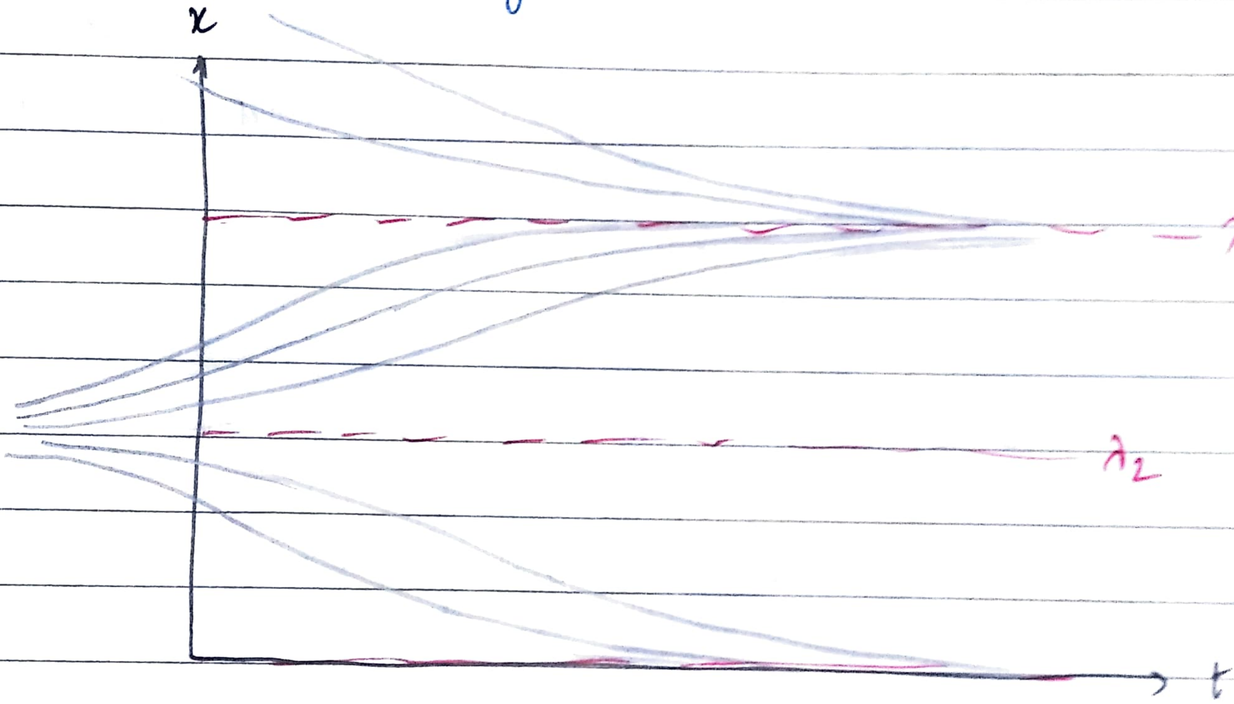
$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$

R① : $x > \lambda_1$ and $x > \lambda_2$

$$\frac{dx}{dt} = (-ve)(-ve)(-ve) = -ve$$

Similarly, doing this for other regions, phase portrait can be generated as seen before.

We can discard the regions ~~we~~ with $t<0$ and $x<0$.



a) What are eqlm populations?

$x_e = 0, \lambda_1, \lambda_2$

b) Solve analytically and compare solutions.

~~Done~~ Do it ~~[illegible]~~ - integrate.

c) Develop phase portraits using analytical solution.

~~Done~~ Do it

d) Develop phase portrait without solving governing equation explicitly.

Done above.

e) Analyse sol^n's and phase portraits for

$x_0 > \lambda_1 \longrightarrow$ converges to $x_e = \lambda_1$

$x_0 < \lambda_1$ and $x_0 > \lambda_2 \longrightarrow$ converges to $x_e = \lambda_1$

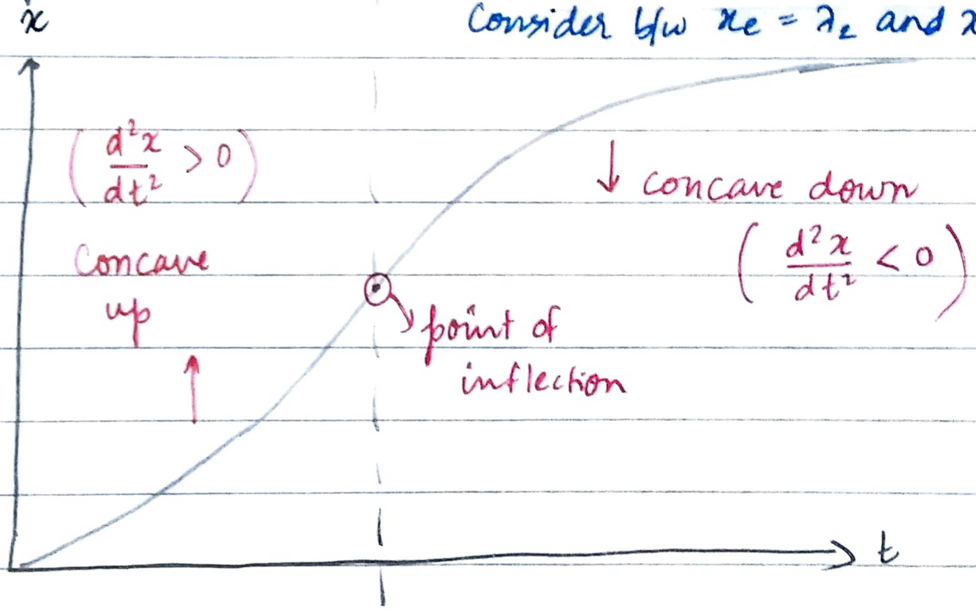
$x_0 < \lambda_2 \longrightarrow$ converges to $x_e = 0 \longrightarrow$ population dies down.

f) Comment upon the bifurcation of the system.

Bifurcation at 'r' $= 0$.

g) Determine the population(s) at which any one solution curve in phase portrait exhibits inflection.

Consider b/w $x_e = \lambda_2$ and $x_e = \lambda_1$



$$\frac{d^2x}{dt^2} = 0 \text{ gives the inflection point}$$

$$\frac{dx}{dt} = -rx\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$

$$0 = \frac{d^2x}{dt^2} = -r\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) - rx\left(\frac{-1}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) - rx\left(1-\frac{x}{\lambda_1}\right)\left(\frac{-1}{\lambda_2}\right)$$

$$0 = -(\lambda_1 - x)(\lambda_2 - x) + x(\lambda_2 - x) + x(\lambda_1 - x)$$

classmate

$$0 = (x-\lambda_1)(x-\lambda_2) + x(x-\lambda_2) + x(x-\lambda_1)$$

$$\Rightarrow (x-\lambda_2)(2x-\lambda_1) + x(x-\lambda_1) = 0$$

$$\Rightarrow 3x^2 - 2x\lambda_2 - x\lambda_1 - x\lambda_1 + \lambda_1\lambda_2 = 0$$

$$\Rightarrow 3x^2 - 2x(\lambda_1+\lambda_2) + \lambda_1\lambda_2 = 0.$$

$$x = \frac{2(\lambda_1+\lambda_2) \pm \sqrt{4(\lambda_1+\lambda_2)^2 - 12\lambda_1\lambda_2}}{6}$$

$$= \frac{(\lambda_1+\lambda_2)}{3} \pm \frac{\sqrt{4(\lambda_1^2+\lambda_2^2) - 4\lambda_1\lambda_2}}{6}$$

$$x = \frac{(\lambda_1+\lambda_2) \pm \sqrt{\lambda_1^2+\lambda_2^2 - \lambda_1\lambda_2}}{3}$$

Lecture 5

06.02.21

(Non-linear) Dynamics in discrete domain

$\frac{dx}{dt} = ax$ ← population growth in initial stages

$x(t) = x(0) \cdot e^{at}$

$t \in \mathbb{R}^{+} \Rightarrow x(t) \in \mathbb{R}^{+}$

There is an inherent flaw in this. Since x can be any real value through this model; it goes against reality as population can never be a non-integer value. How can we have fractional numbes of fish, per say?

Now how to do it in discrete range?

$\frac{dx}{dt} = ax, \quad t = t_0, t_1, t_2 \ldots$

$x = f(t)$ → continuous model

$x_{n+1} = g(x_n)$

$\frac{x_{n+1} - x_n}{\Delta t} = a x_n$

$\Rightarrow x_{n+1} = x_n + a \Delta t \, x_n.$

$\Rightarrow x_{n+1} = b x_n$ where $b = (a\Delta t + 1)$ → discrete model.

Irrespective of the model being discrete or continuous, the solutions should evolve similarly, i.e., need to have the same characteristics. This can be seen upon plotting.

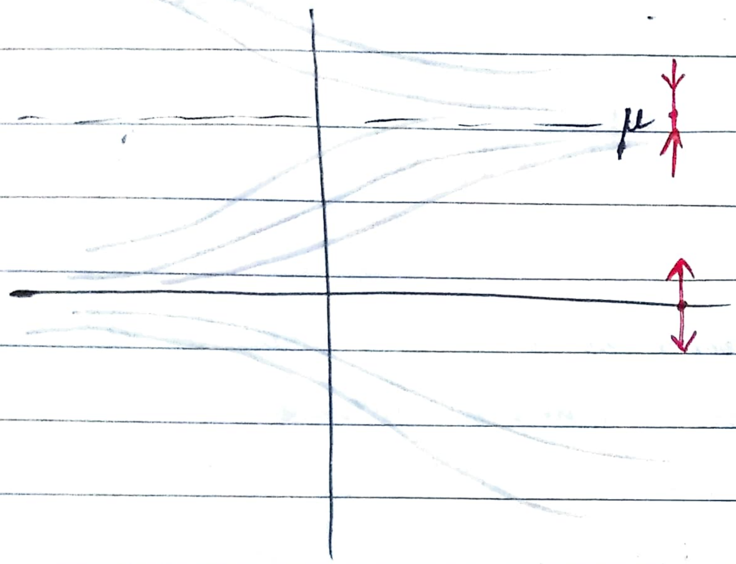
Now, let's consider the logistic model.

$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$

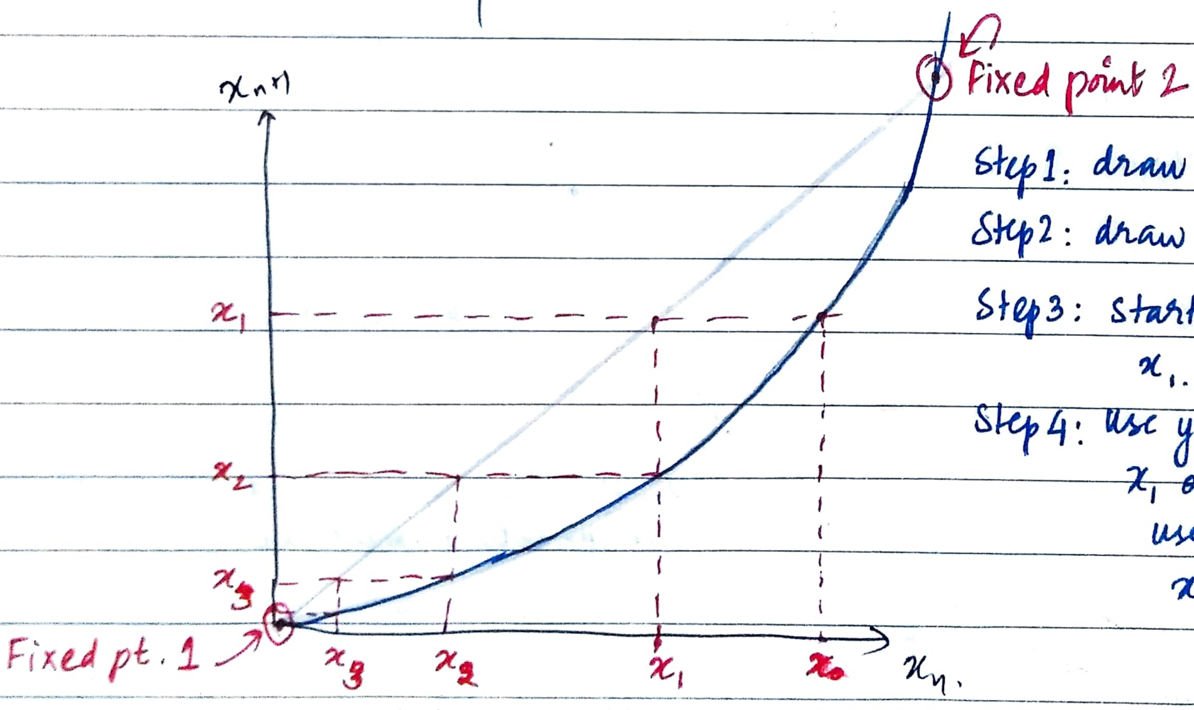
$$\frac{x_{n+1}-x_n}{\Delta t} = a x_n\left(1-\frac{x_n}{N}\right) \Rightarrow x_{n+1} = x_n\left[1+a\Delta t\left(1-\frac{x_n}{N}\right)\right]$$

$$\Rightarrow x_{n+1} = \cancel{}\ \lambda x_n\left(1-\frac{x_n}{\mu}\right) \quad \text{where } \lambda = (1+a\Delta t)$$

$$\mu = \frac{N(1+a\Delta t)}{a\Delta t}$$



in continuous domain for a particular range of 'a', or 'λ'.



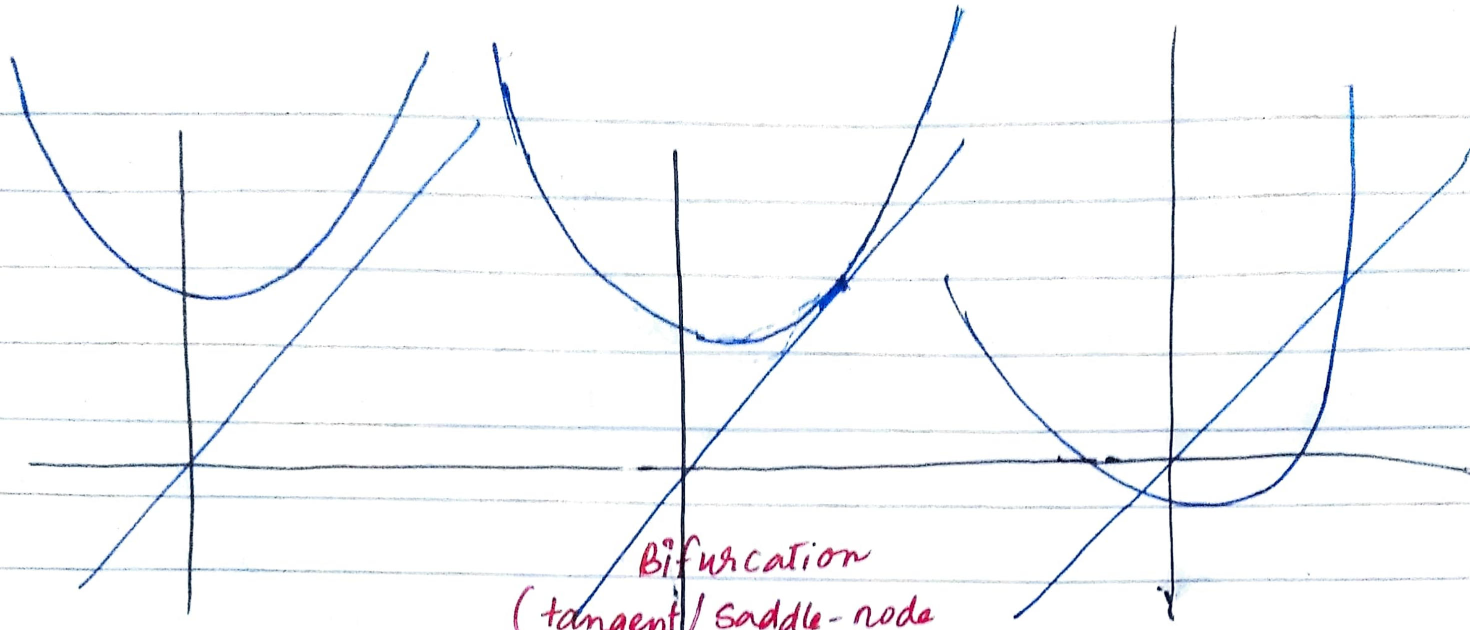
Step 1: draw the curve.

Step 2: draw $y = x$ line.

Step 3: start with x_0 to find x_1.

Step 4: use $y = x$ line to plot x_1 on x axis and use that to find x_2 and so on.

Equilibrium solution in discrete domain is called 'fixed point'. Now, we know that eventually, as can be seen from the plot, the steps converge to meet at $(0,0)$ where $x_n = x_{n+1}$. So, it is a 'fixed point.' Similarly, the other intersection point is the other fixed point. So, there are 2 F.Ps corresponding to 2 equil.^{m} solutions in continuous domain

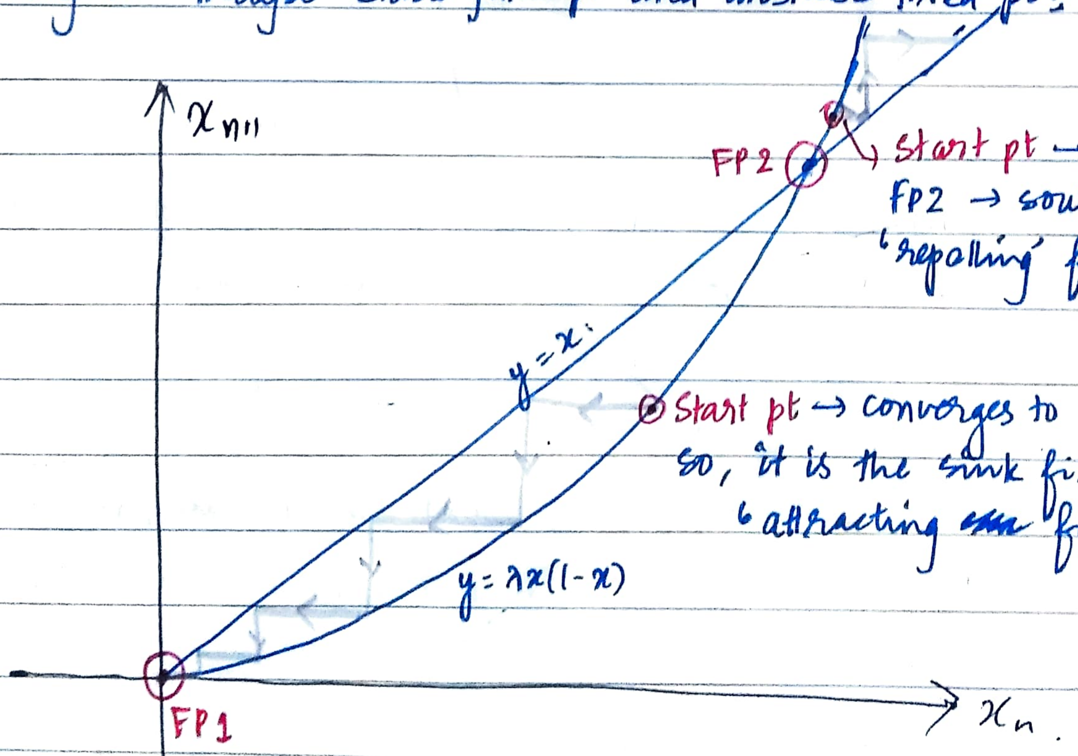


No equilibrium Solution

Bifurcation (tangent / saddle-node bifurcation).
One equilibrium sol^n

2 equilibrium sol^{n}s.

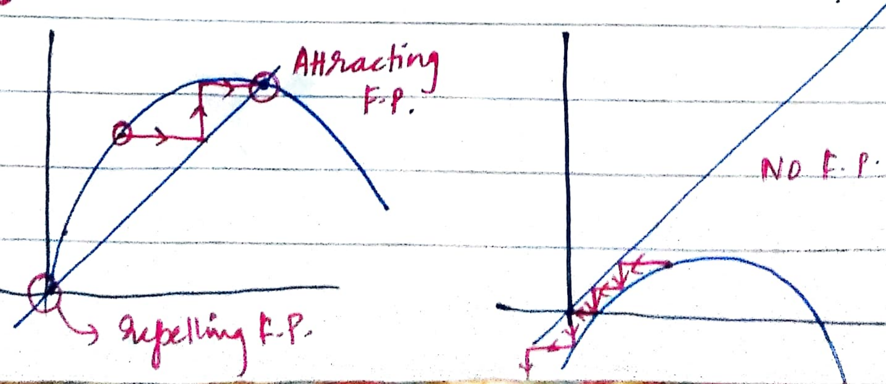
System can have 0, 1 or 2 sol^{n}s based on the value of λ.
So, how to capture the concept of stable & unstable eq^m solutions in continuous domain, into discrete domain? That is, is there something analogous - maybe stable fixed pt and unstable fixed pt?



Start pt → diverges from FP2 → source F.P. or 'repelling' fixed point.

Start pt → converges to (0,0) F.P.
So, it is the sink fixed point or 'attracting ~~as~~ fixed point'

Similarly,



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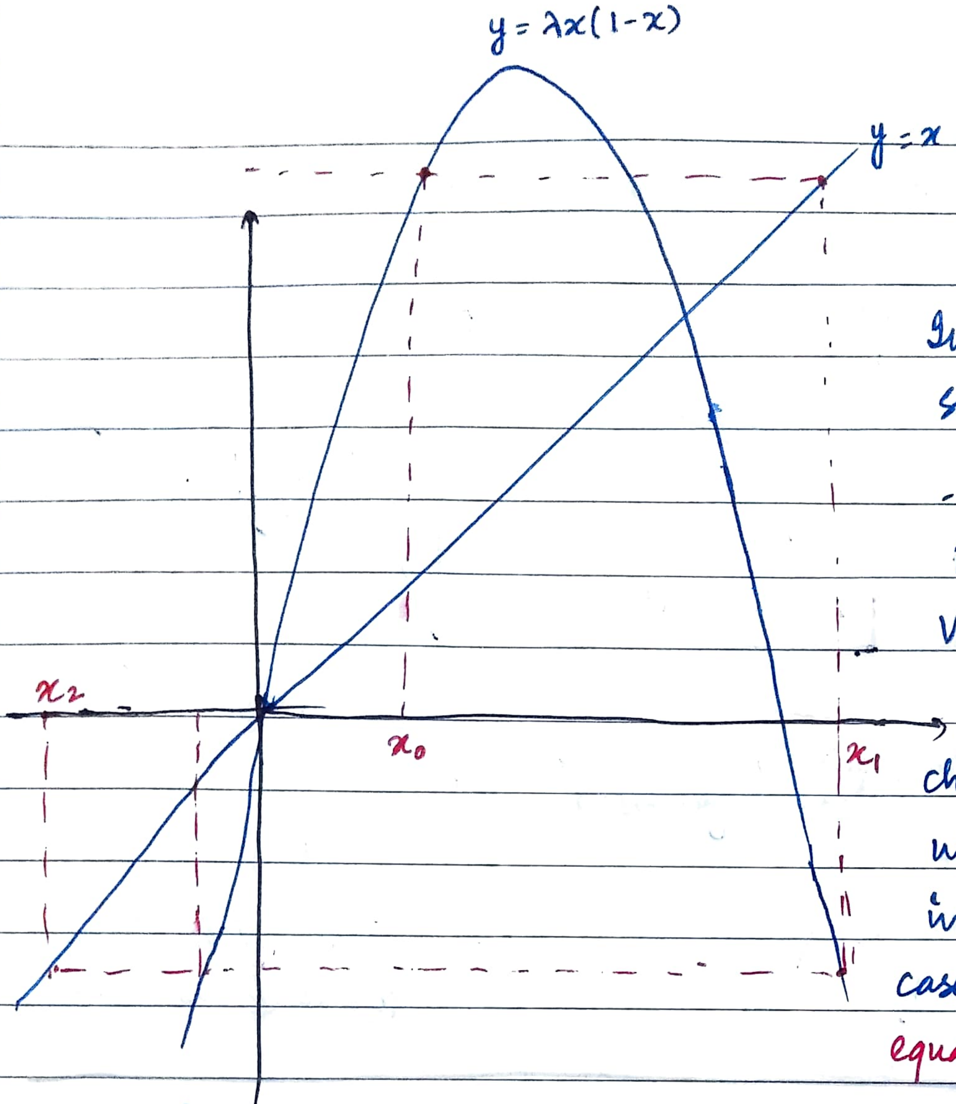


In this case, it is oscillating. Such cases can be interestingly elegant.

This also happens at some value of the parameter λ.

In this case, the solutions change drastically not only with λ, but also based on initial condition x_0. In this case, the eq. is called chaotic equation or situation is called 'chaos'

Chaos – Chaotic eqs. are inherently non-linear but every non-linear eq. is not chaotic. An interesting application of chaotic eqs. is random number generation.

So, how to determine if a F.P. is attracting or repelling?

→ For attracting Fixed point ; $|f'(x)|_{F.P} < 1$

For repelling ; $|f'(x)|_{F.P} > 1$

(If $f' = 1$, tangent bifurcation — attracting on one side, repelling on the other)

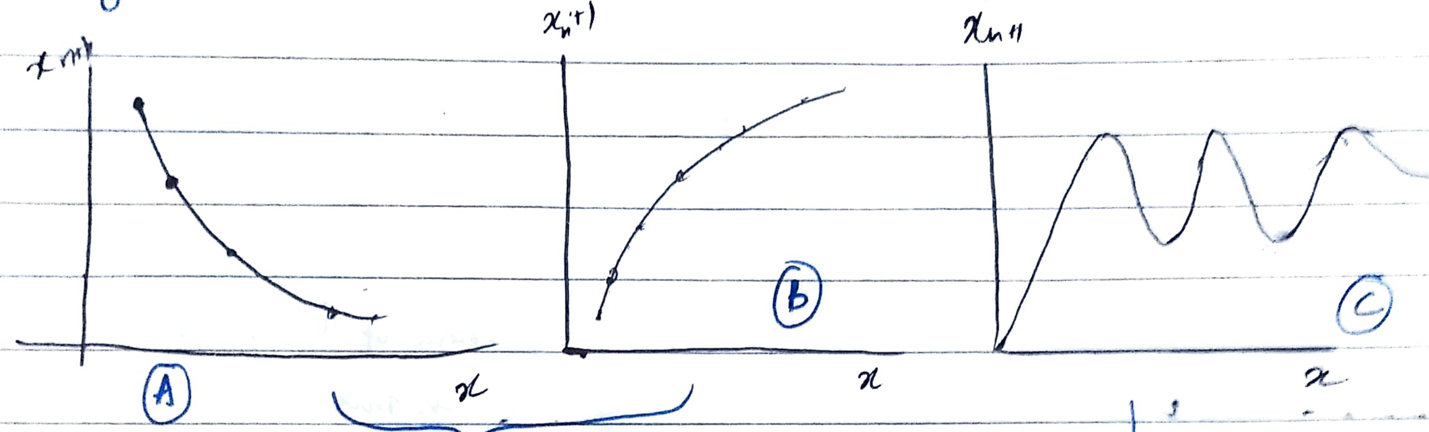
So, below are the concluding steps.

1. Find the deterministic eq. (say, $f(x) = \lambda x(1-x)$)
2. Solve to find F.P → F.P is the intersection of $f(x)$ and $y = x$.

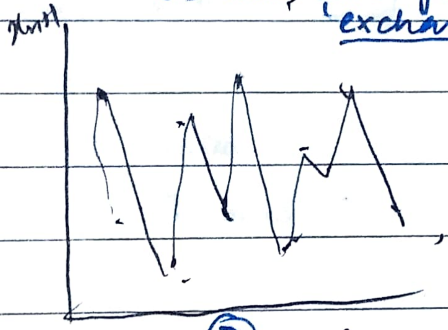
   So, $x = \lambda x(1-x)$

   Solutions give F.Ps.
3. Find $|f'(x)|_{F.P.}$ to determine if each of them are attracting or repelling.

Try plotting $\lambda x(1-x)$ vs x. You'll see that as λ increases, the system changes as -



this transition pt is called <u>'exchange bifurcation'</u>



D

Transition from C to D is called

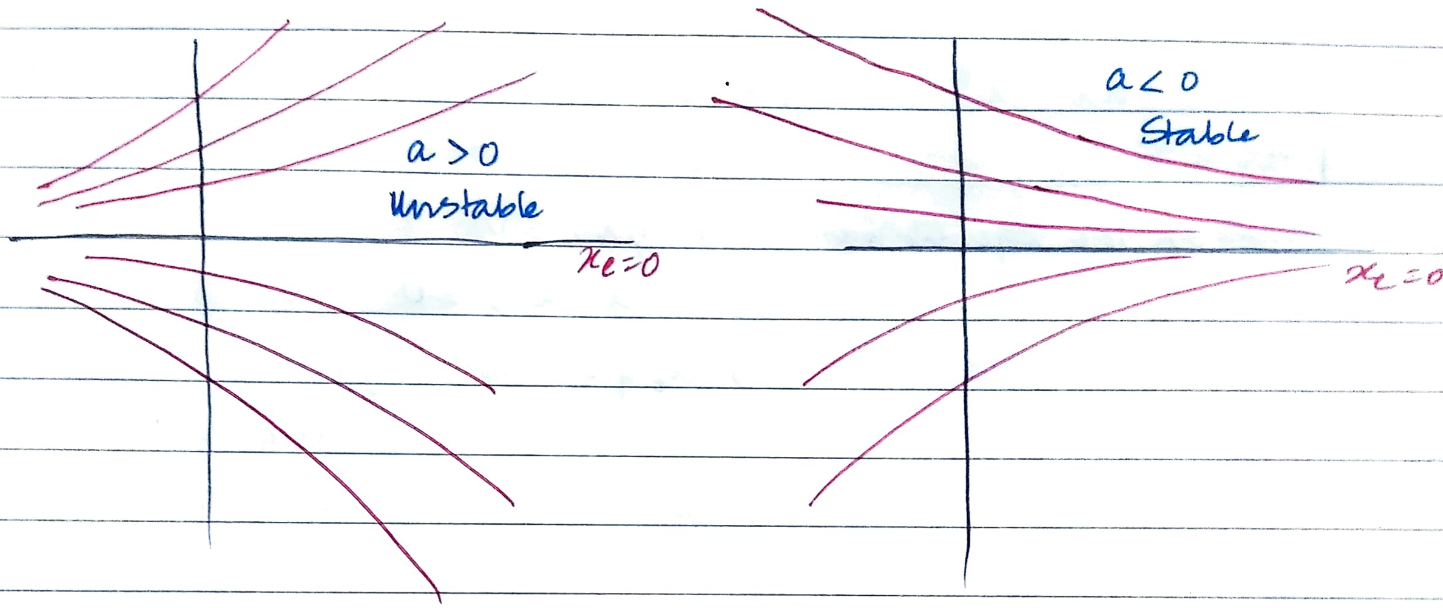
Chaos is very prominent whenever there is a discrete model or discrete domain. We are forced to use discrete models even for a continuous domain situation because today, everything is modeled in computers which are inherently discrete due to their truncation. So, we may have to use discrete model because the system is inherently in discrete domain (say, population dynamics) or because we are forced to use discrete model due to the usage of a computer (due to truncation).

Lecture 6

12.02.21 — Bifurcations in non-linear systems.

Let's take a relook at linear systems again.

$$\frac{dx}{dt} = ax.$$



So, we said that the system has bifurcation at $a = 0$.

But can we determine this without drawing a plot all the time.

Condition for equilibrium : $f(a, x) = 0$

$$\Rightarrow ax = 0$$

$$\Rightarrow x_e = 0.$$

So, what's the condition for bifurcation?

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{x_e} = 0}$$

$$\Rightarrow \left.\frac{\partial}{\partial x}(ax)\right|_{x_e = 0} \Rightarrow a = 0.$$

In $\frac{dx}{dt} = ax$, a is the eigen value.

Condition for stability : a (eigen value λ) < 0

for unstability : a (eigen value λ) > 0

That means, at bifurcation, system goes from stable to unstable or vice-versa. Hence, eigen value (λ) $= 0$.
Now, let's try this for a non-linear system.

$$\boxed{\frac{dx}{dt} = ax - x^3 = f(a, x)}$$

Condition for equilibrium : $f(a,x) = 0$

$\Rightarrow x(a - x^2) = 0$

$\Rightarrow x_e = 0, \; x_e = \pm\sqrt{a}$

$\downarrow$

3 eq. sol^{n}s.

What's the nature of equilibrium solutions — stable or unstable?

$f(a, x) = ax - x^3$

Condition for bifurcation : $\left.\frac{\partial f}{\partial x}\right|_{x_e} = 0.$

$\Rightarrow \left.(a - 3x^2)\right|_{x_e} = 0.$

<u>Case 1</u>: $x_e = 0$; $\lambda = \left.\frac{\partial f}{\partial x}\right|_{x_e = 0} = a$

That means, x_e is stable when $a < 0$ (eigen value < 0)

x_e is unstable when $a > 0$ (eigen value > 0).

<u>Case 2</u>: $x_e = \sqrt{a}$; $\lambda = \left.\frac{\partial f}{\partial x}\right|_{x_e = \sqrt{a}} = -2a$

$\therefore$ $x_e = \sqrt{a}$ is stable when $a > 0$

$x_e = \sqrt{a}$ is unstable when $a < 0$

classmate

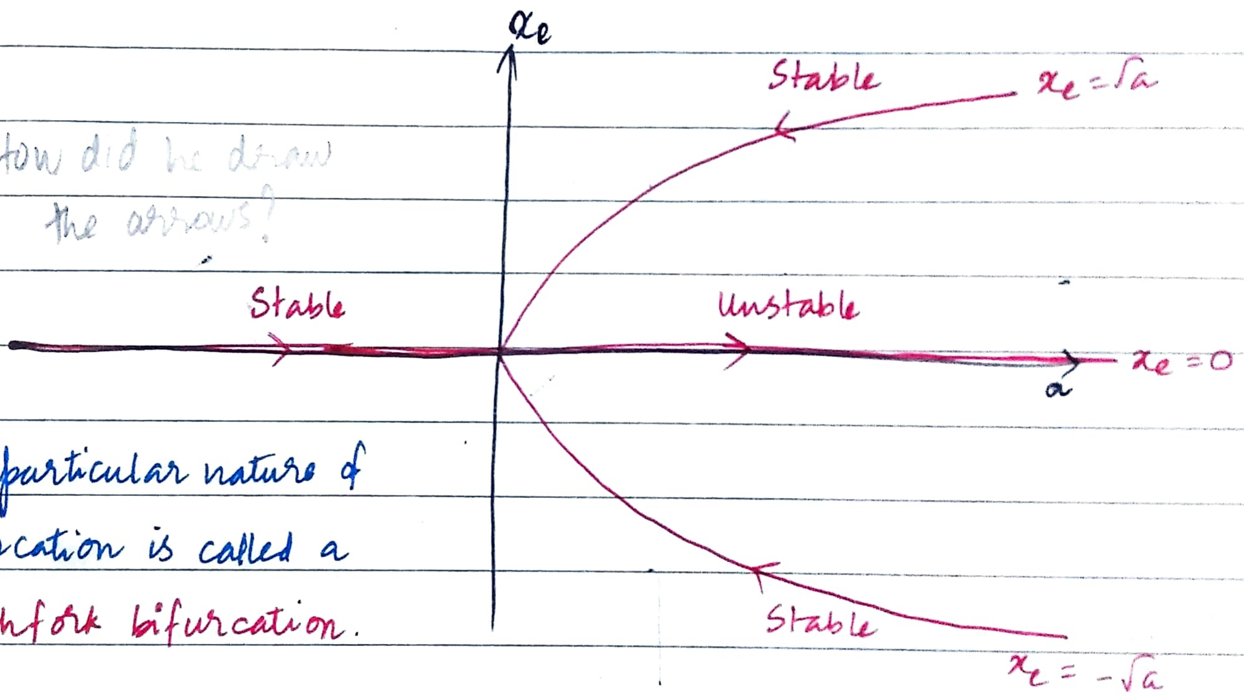
<u>Case 3</u>: $x_e = -\sqrt{a}$; $\lambda = \left.\frac{\partial f}{\partial x}\right|_{x_e = -\sqrt{a}} = -2a$

$\therefore$ $x_e = -\sqrt{a}$ is stable at $a > 0$

$x_e = -\sqrt{a}$ is unstable at $a < 0$

Using this, let's draw a Bifurcation diagram, that gives equilibrium solutions as a function of bifurcation parameter

$x_e = 0$, $x_e = \sqrt{a}$, $x_e = -\sqrt{a}$.



Q: How did he draw the arrows?

This particular nature of bifurcation is called a Pitchfork bifurcation.

Let's take another example: $\boxed{\frac{dx}{dt} = a - x^2 = f(a, x)}$

Condition for equilibrium : $f(a, x) = 0$

$a - x^2 = 0$

$x_e = \sqrt{a}$, $x_e = -\sqrt{a}$.

Condition for bifurcation: $\left.\frac{\partial f}{\partial x}\right|_{x_e} = 0$

Case 1: $\left.\frac{\partial f}{\partial x}\right|_{x_e = \sqrt{a}} = -2x_e = \underbrace{-2\sqrt{a}}$.

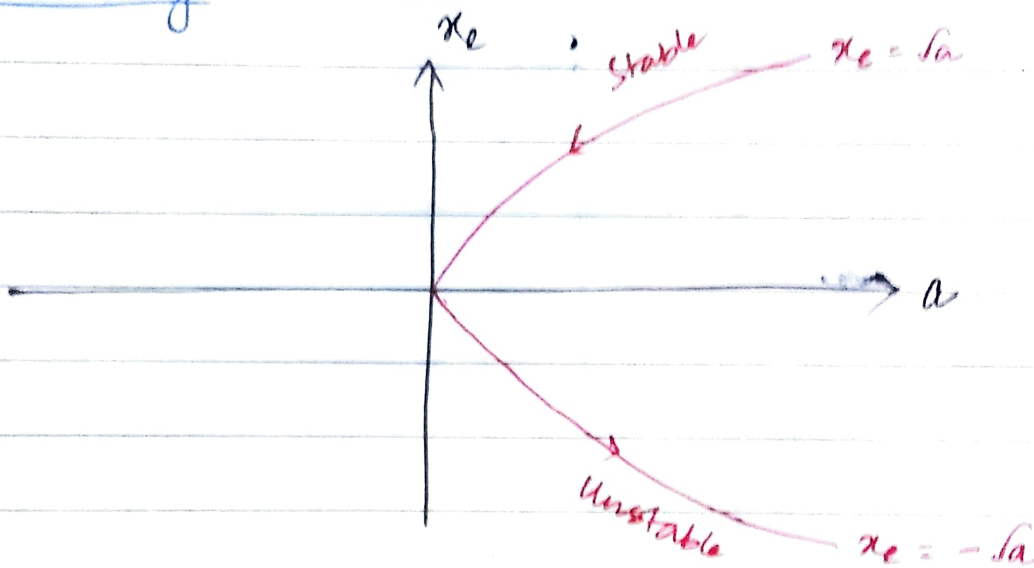
$\sqrt{a}$ is always +ve.

$\therefore x_e = \sqrt{a}$ is always stable.

Case 2: $\left.\frac{\partial f}{\partial x}\right|_{x_e = -\sqrt{a}} = -2x_e = 2\sqrt{a}$

$\therefore x_e = -\sqrt{a}$ is always unstable

Bifurcation diagram:



Let's take another example: $\boxed{\frac{dx}{dt} = ax - x^2} = f(a, x)$

Condition at equilibrium: $f(a, x) = 0$

$$ax - x^2 = 0$$

$$x(a - x) = 0$$

$$x_e = 0, \quad x_e = a$$

Condition for bifurcation: $\left.\frac{\partial f}{\partial x}\right|_{x_e} = 0$

classmate

Case 1: $x_e = 0$; $\lambda = \left.\frac{\partial f}{\partial x}\right|_{x_e=0} = a - 2x_e = a$

$x_e = 0$ is stable when $a < 0$

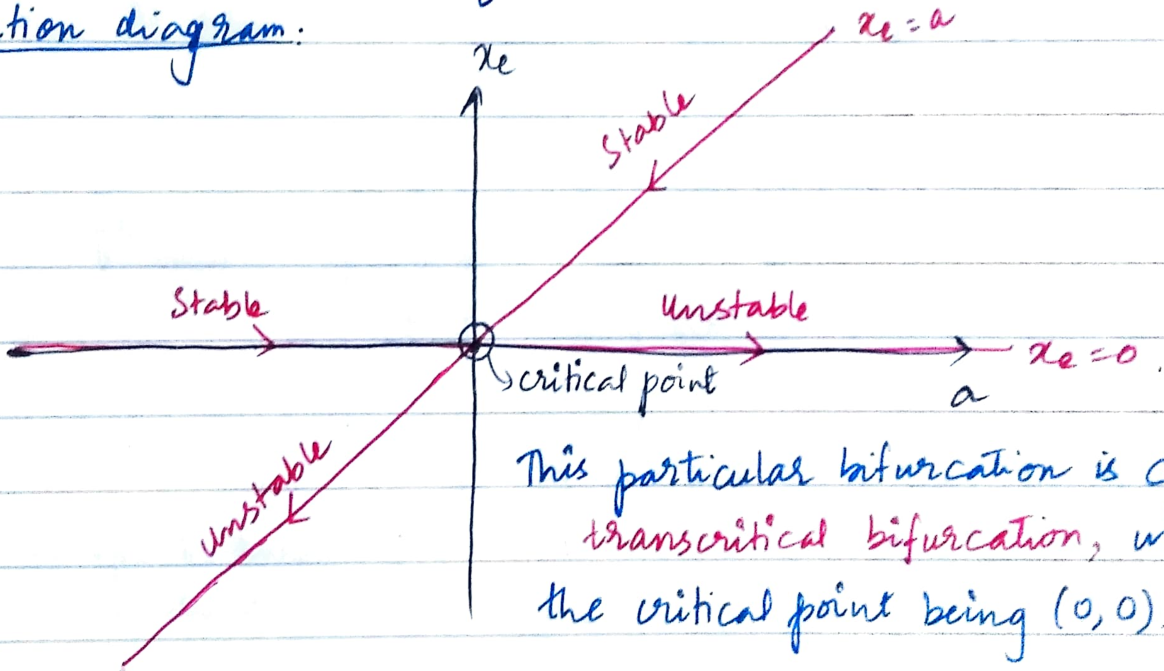
$x_e = 0$ is unstable when $a > 0$

case 2: $x_e = a$; $\lambda = \left.\frac{\partial f}{\partial x}\right|_{x_e=a} = a - 2(a) = -a$

$x_e = a$ is stable for $a > 0$

$x_e = a$ is unstable for $a < 0$.

Bifurcation diagram:



This particular bifurcation is called transcritical bifurcation, with the critical point being $(0,0)$.

Two-state non-linear system:

Given,
$$\boxed{\begin{cases} \frac{dx}{dt} = x^2 - y^2 - 1 = f(x,y) \\ \frac{dy}{dt} = 2y = g(x,y) \end{cases}}$$

Condition for equilibrium :

$f(x,y) = 0$ and $g(x,y) = 0$

$\Rightarrow x_e^2 - y_e^2 - 1 = 0$ $\Rightarrow +2y_e = 0$

$\Rightarrow x_e^2 - 1 = 0$ $\boxed{y_e = 0}$

$\Rightarrow \boxed{x_e = \pm 1}$

$\therefore$ Eq. Solutions : $[1\ 0]^T$, $[-1\ 0]^T$

So, how to determine eigen values for a higher order system?

$f(x,y) = x^2 - y^2 - 1$

$g(x,y) = +2y.$

$\frac{\partial f}{\partial x} = 2x$; $\frac{\partial f}{\partial y} = -2y$; $\frac{\partial g}{\partial x} = 0$; $\frac{\partial g}{\partial y} = 2$

Jacobian matrix $\underline{\underline{J}} = \begin{bmatrix} 2x & -2y \\ 0 & 2 \end{bmatrix}$

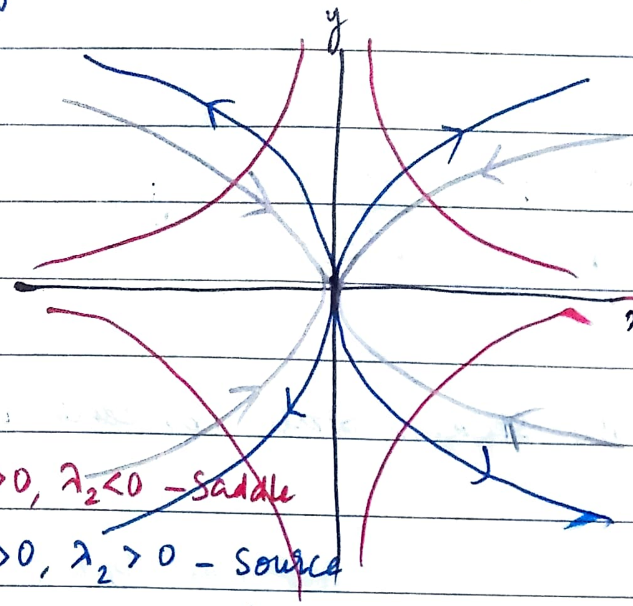
$$\left.\begin{aligned} \underline{\underline{J}}\Big|_{e_1} \text{ or } J\Big|_{[1\ 0]^T} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ \underline{\underline{J}}\Big|_{e_2} \text{ or } J\Big|_{[-1\ 0]^T} &= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}\right\} \text{Jacobians at equilibrium solutions.}$$

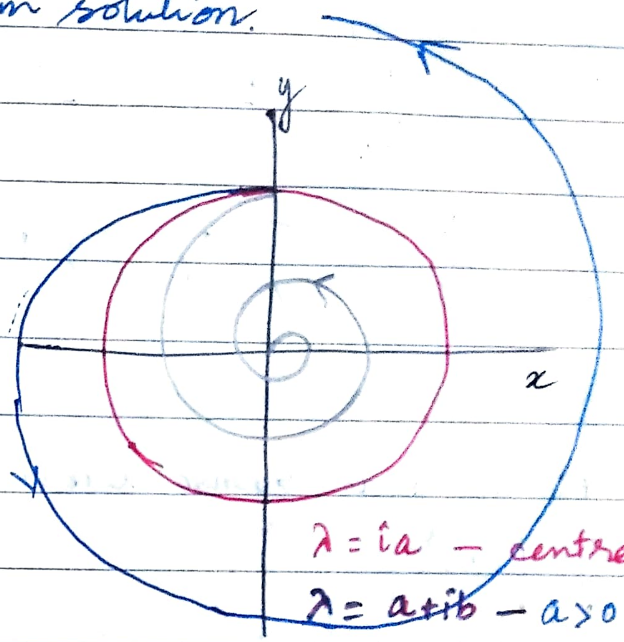
Now, let's look at the eigen values of Jacobians.

$\underline{\underline{J}}\Big|_{e_1}$ has $\lambda_1 = \lambda_2 = 2 \rightarrow$ greater than zero.

$\underline{\underline{J}}\Big|_{e_2}$ has $\underset{\text{negative}}{\underset{\downarrow}{\lambda_1 = -2}}$, $\underset{\text{positive.}}{\underset{\downarrow}{\lambda_2 = 2}}$

Now the most imp. thing to determine is - Behaviour of non-linear system "close" to the equilibrium solution.





$\lambda_1 > 0, \lambda_2 < 0$ – Saddle

$\lambda_1 > 0, \lambda_2 > 0$ – Source

$\lambda_1 < 0, \lambda_2 < 0$ – Sink

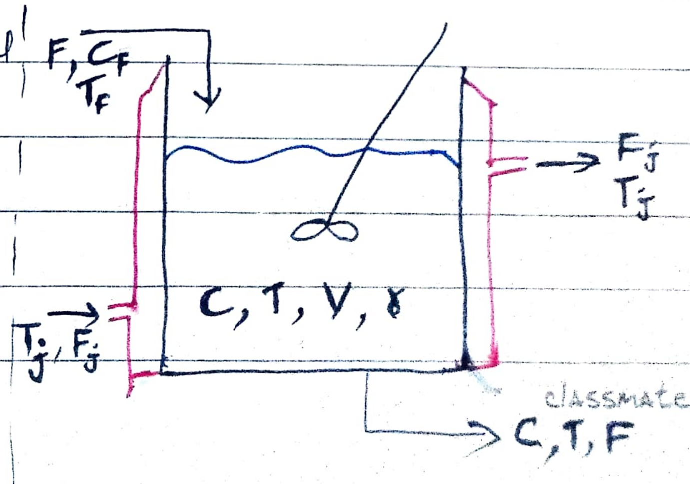
$\lambda = ia$ – centre

$\lambda = a + ib$ – $a > 0$ → spiral source

$\lambda = a + ib$ – $a < 0$ → spiral sink

Problem: Chemical Reactor Analysis

In this problem, we take the case of a jacketted CSTR and understand its dynamical behaviour. CSTR's are meant to be operated at steady state. However, before the steady state is reached, there are transients in the system which must die out with time. Due to the provision of heating / cooling by a fluid in the jacket, this arrangement is also called diabatic or isothermal reactor. We use the following input and dynamical variables: (see the diagram).



Mass and energy balance over the reactor give the following dynamical eqs:

$$\frac{dC}{dt} = \frac{F}{V}(C_f - C) - r$$

$$\frac{dT}{dt} = \underbrace{\frac{F}{V}(T_f - T)}_{\text{due to feed}} + \underbrace{\left(\frac{-\Delta H}{\rho C_p}\right) r}_{\text{reaction}} - \underbrace{\frac{UA}{V\rho C_p}(T - T_j)}_{\text{heat exchange b/w jacket and reactor}}$$

a) For the given system, determine the steady state concentration profile.

We can determine S.S/ eq^n solution from the null space.

$$\left.\begin{aligned} \frac{dC}{dt} &= \frac{F}{V}(C_f - C) - r = 0 \\ \frac{dT}{dt} &= \frac{F}{V}(T_f - T) + \left(\frac{-\Delta H}{\rho C_p}\right) r - \frac{UA}{V\rho C_p}(T - T_j) = 0 \end{aligned}\right\} \text{This set of eqs. have to be solved simultaneously}$$

* Plug the above equations in MATLAB and use "fsolve".

The steady state sol^n obtained can depend highly upon the initial condition given in this case.

Let's consider the possible I.V. ranges.

- Conc. 'C' : Maximum value when no reaction ⇒ $C = C_f$
  : Minimum value when high 'r' ⇒ $C = 0$
- Temp 'T' : Min T : R.T (reactor temp)
  Max T : T_{upper}

Solve for multiple initial guesses. Upon doing so, we'll get 3 equilibrium solutions. Let's understand why and what this means.

At steady state:

$\Rightarrow \frac{F}{V}(C - C_f) - r = 0 \longrightarrow$ (1)

$\Rightarrow \frac{F}{V}(T_f - T) + \left(\frac{-\Delta H}{\rho C_p}\right) r + \frac{UA}{V\rho C_p}(T - T_j) = 0$

$\Rightarrow \left(\frac{-\Delta H}{\rho C_p}\right) r = \frac{F}{V}\left[T - T_F\right] + \frac{UA}{V\rho C_p}(T - T_j)$

$\Rightarrow \underbrace{\frac{-\Delta H k_o C_s e^{-\Delta E/RT_s}}{\rho C_p}}_{\text{reaction rate}} = \underbrace{\left(\frac{F T_{Fs}}{V} - \frac{UA T_{Js}}{V\rho C_p}\right) + \left(\frac{F}{V} + \frac{UA}{V\rho C_p}\right) T_s}_{\text{convective energy transfer}}$ (2)

reaction rate (Energy generated due to the reaction).

convective energy transfer

connection of reaction mix + Jacket heat transfer.

LHS of eq (2): $\frac{-\Delta H k_o C_s e^{-\Delta E/RT_s}}{\rho C_p} \longrightarrow (\alpha e^{\beta/T}) . C_s \longrightarrow$ (3)

constant α , constant β

C_s → S.S. conc. → also depends on temp.

From Eq (1): $\frac{F}{V}(C_s - C_{fs}) - k_o e^{-\Delta E/RT_s} C_s = 0$
at S.S.

$\Rightarrow C_s = \dfrac{\frac{F}{V} C_{Fs}}{\frac{F}{V} + k_o e^{-\Delta E/RT_s}}$

Substituting this in eq. (3), we get,

$(\alpha e^{\beta/T}) . \left(\dfrac{F/V \, C_{Fs}}{F/V + k_o e^{+\beta/T}}\right)$ which can be written as -

LHS $= \dfrac{a e^{-1/T_s}}{b + e^{-1/T_s}}$

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Lecture 7 : 19th Feb

Disease dynamics of infectious diseases and epidemics

Assumption : Total population is constant and well mixed.

Population : N

Problem: If a small no. of infected persons are introduced to this closed population, then how to describe the spread of the infection as a function of time?

The dynamics are similar to a reaction in series:

$A \rightarrow B \rightarrow C$

$$\left.\begin{aligned} \frac{dC_A}{dt} &= -k_1 C_A \\ \frac{dC_B}{dt} &= k_1 C_A - k_2 C_B \end{aligned}\right\} \text{Linear}$$

$$\frac{dC_C}{dt} = k_2 C_B$$

However, in the model for spread of infectious diseases, the system becomes quickly non-linear.

SIR model :

S → Susceptibles

I → Infectious → can spread to others.

R → Recovered / Dead / Quarantined until recovery.

→ essentially ones that can't be infected again

such that, $S \longrightarrow I \longrightarrow R$ (like in reaction in series)

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If the no. of S is more, then higher is the possibility of infections.
Also, more the I, more the rate of conversion of S to I.

$$\therefore \frac{dS}{dt} = -\gamma SI \rightarrow \text{rate} \uparrow \text{ as } S\uparrow \text{ or } I\uparrow \qquad \rightarrow (\mathrm{I})$$

$$\frac{dI}{dt} = \gamma SI - \underbrace{aI}_{\text{rate of recovery}} \qquad \rightarrow (\mathrm{II})$$

$$\frac{dR}{dt} = aI \qquad \rightarrow (\mathrm{III})$$

This is called the SIR model proposed by Kermack - McKendrick model. In this model, $r > 0$ and $a > 0$.
If N is the total population;

$$S + I + R = N$$

At $t = 0$; $S(0) = S_0$, $I(0) = I_0$; $R(0) = 0$

Based on this model, answer the following -
a) Given the initial conditions, will the infection spread?
b) If yes, will it decline (naturally) after some time?

$$\frac{dI}{dt} = rSI - aI$$

If the infection has to spread, $\frac{dI}{dt}$ must be '>0'

At $t = 0$, $\left.\frac{dI}{dt}\right|_0 = rS_0I_0 - aI_0 > 0$ for infection to spread.
< 0 for infection to die away.

$\Rightarrow I_0(rS_0 - a) > 0$

$\Rightarrow S_0 > a/r$ → Susceptible population in the beginning must be $> a/r$ for infection to spread.

$\frac{a}{r}$ is called the relative removal rate

(r/a is called the infection contact rate).

If relative removal rate is smaller, epidemic bound to happen.

$\frac{r S_0}{a} = R_0$ (this is called R_0 → tells how severely the infection would spread)

↓

called the Basic reproduction rate of the infection.

For epidemic $R_0 > 1$

For COVID-19 ;

$R_0 = 1.86$ at the beginning of lockdown (March 2020)

$R_0 = 1.55$ during lockdown (April 2020)

$R_0 \simeq 1.25$ during lockdown (May - June 2020)

$R_0 \simeq 1.05$ in September 2020.

⋮

$R_0 < 1$ until last week (2nd week of Feb).

This gives rise to the concept of "Herd immunity"

WKT, $R_0 = \frac{S_0 r}{a}$ → what if we make S_0 very small?

↓

R_0 would go less than 1.

So, if a large no. of people recover from disease or get vaccinated, then S_0 can become small enough to reduce R_0.

This eventually is bound to happen always according to the SIR model. But, we don't want herd immunity to be attained

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through large no. of deaths.

Now, let's see what is the condition for 'I' curve to reach a maxima?

$$\frac{dI}{dt} = -rSI + aI$$

$$\frac{dS}{dt} = rSI$$

$$\frac{dI}{dS} = \frac{-rSI + aI}{rSI} = -1 + \frac{a}{rS}$$

$$\Rightarrow \frac{dI}{dS} = -1 + \frac{\rho}{S} \quad \text{where } \frac{a}{r} = \rho$$

$$\Rightarrow I = \int \left(-1 + \frac{\rho}{S}\right) dS$$

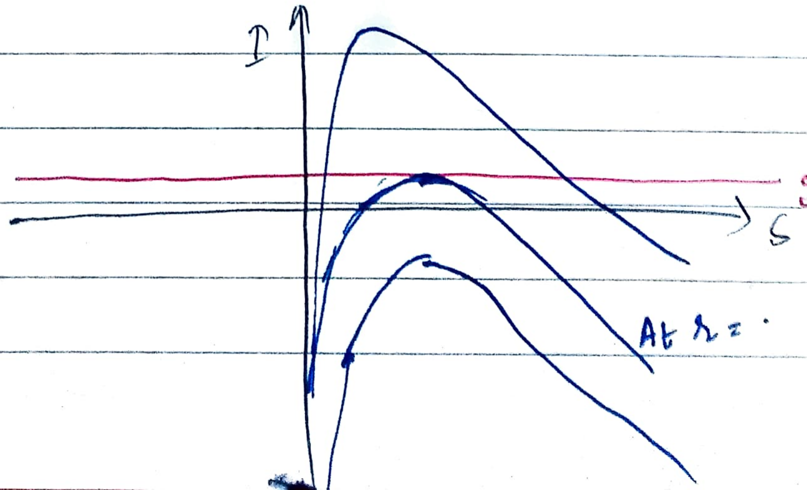
$$\Rightarrow I = -S + \rho \ln S + c.$$

At $t = 0$, $S = S_0$, $I = I_0$ $\Rightarrow$ $I_0 + S_0 - \rho \ln S_0 = c$

Also, WKT, $S + I + R = S_0 + I_0 = N$

$$\therefore \quad I + S - \rho \ln S = N - \rho \ln S_0$$

$$\Rightarrow \boxed{I = N - S + \rho \ln \frac{S}{S_0}}$$ → Plot this on desmos. (I vs. S)



There is always a maxima for I curve

$$\frac{dI}{dt} = -rSI + aI$$

At maxima,

$$\frac{dI}{dt} = 0 \Rightarrow \boxed{S = \frac{a}{r} \text{ at maxima}} \rightarrow r\ S = \rho \text{ at } I = I_{max}$$

$$\therefore \boxed{I_{max} = N - \rho + \rho \ln\left(\frac{\rho}{S_0}\right)} \rightarrow$$ Gives an estimate of no. of people that may be infected.

Let's see $\frac{dR}{dS} = \frac{aI}{-rSI} = \frac{-\rho}{S}$

$$\Rightarrow dR = -\rho \frac{dS}{S}$$

$$\Rightarrow R = -\rho \ln S + c$$

At $t = 0$, $R_0 = 0$, $S = S_0 \Rightarrow 0 = -\ln S_0 + c$

$$\therefore R = -\rho \ln S + \rho \ln S_0$$

$$\Rightarrow \frac{-R}{\rho} = \ln \frac{S}{S_0} \Rightarrow \boxed{S = S_0 e^{-R/\rho}}$$

What do we understand from this?

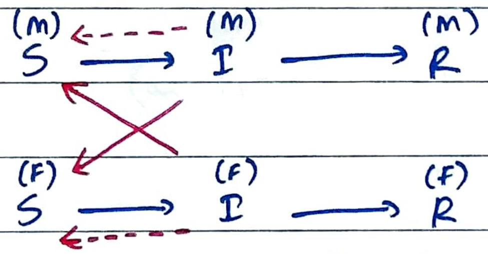
Models for STDs:

For an influenza epidemic model, we can assume that the population is well-mixed. But this is not valid for STDs.

Flu models: $\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$

For STDs : $\boxed{\begin{matrix} S(\text{Male}) \\ S(\text{Female}) \end{matrix}} \longrightarrow \boxed{\begin{matrix} I(\text{Male}) \\ I(\text{Female}) \end{matrix}} \longrightarrow \boxed{\begin{matrix} R(\text{male}) \\ R(\text{female}) \end{matrix}}$

For AIDS; the earliest models (before LGBTQ+) can be represented as –



'⟶' shows spread of infection : Infectious male infects a susceptible female.

'----›' after LGBTQ+ considerations.

For infections like Gonohrrhea; a recovered person immediately joins the susceptible pool, represented by arrows from R(M) to S(M) and R(F) to S(F).

These models are much more complex & non-linear.

Let's look at some realistic models of epidemics where we account for natural deaths, population growths, etc.

P. T. O.

<u>Class Test 2</u> : Syllabus → Non linear dynamics
lectures 4 – 7
Pen-paper test → <u>March 5th</u>
11:30 AM to 12:30 PM

b ↓ recruitment rate

Susceptible $\xrightarrow{r_1}$ Natural death

↓

Infectives $\xrightarrow{r_2}$ Natural Death

↙ ↘

AIDS ⟶ Natural Death.

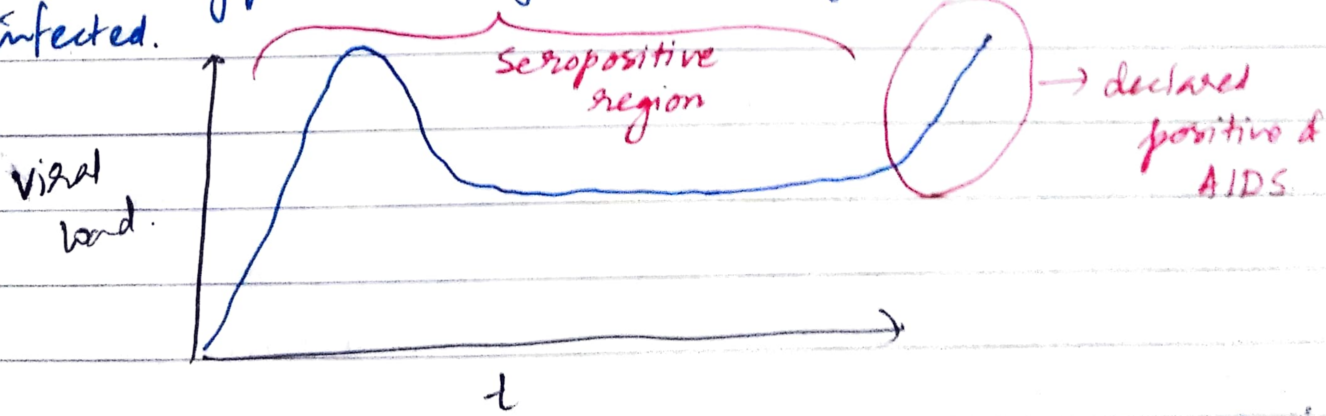
↓

Death due to infection

Seropositive (Non-infectious) ⟶ Natural Death

What's seropositive?

Not every person having virus causing AIDS is not declared infected.



This wave is due to T-cells which fight for a while reducing viral load.