Fourier series of odd & even functions.

A. Odd function. $[-l, l]$

Let $f(x)$ be an odd function defined in $[-l, l]$.

i.e $f(-x) = -f(x) \quad \forall x \in [-l, l]$.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = 0.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos\frac{n\pi x}{l}}_{\text{even}}}_{\text{odd}}\,dx = 0.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\sin\frac{n\pi x}{l}}_{\text{odd}}}_{\text{even}}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$f(x) = \frac{a_0}{2}^{=0} + \sum_{n=1}^{\infty} a_n^{=0} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}; \quad -l \le x \le l.$$

B. Even funct^{n}:

Let $f(x)$ be an even funct^{n} defined in $[-l, l]$.

So, $f(-x) = f(x) \quad \forall x \in [-l, l]$.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\cos\frac{n\pi x}{l}}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx .$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin\frac{n\pi x}{l}}_{\text{odd}}}_{\text{odd}}\,dx = 0 .$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \overbrace{b_n}^{0} \sin\frac{n\pi x}{l} .$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad ; \quad -l \le x \le l .$$

Find Fourier series of $f(x)$;

Ex-1. $$f(x) = \begin{cases} -\frac{(\pi + x)}{2}, & -\pi \le x < 0 . \\ \frac{\pi - x}{2}, & 0 \le x < \pi . \end{cases}$$

$f(x) = \frac{\pi - x}{2}, \quad 0 \le x \le \pi$ According to the given form of $f(x)$

$f(-x) = \frac{\pi + x}{2}$. So, $f(-x) = -f(x)$.

$\therefore$ given $f(x)$ is an odd funct^{n}.

$\therefore$ In the fourier series expansion of $f(x)$,

$a_0 = 0, \quad a_n = 0 .$

$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$.

$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{\pi - x}{2}\sin nx\, dx$.

$= \frac{1}{\pi}\left[(\pi - x)\frac{\cos nx}{n}\Big|_{\pi}^{0} - \int_0^{\pi}\frac{\cos nx}{n}\, dx \right]$.

$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\sin nx}{n^2}\Big|_0^{\pi} \right] = \frac{1}{n}$.

$\therefore\ f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$.

$= \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots$.

Ex 2. Find F. S. of $f(x) = x^2$ in $(-\pi, \pi)$.

Note that $f(x)$ is an even funct. $\therefore$ its F. S. expansion will contain the constant term & cosine terms.

i.e it is of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2\pi^2}{3}$; $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$.

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \quad // \quad -l \le x \le l.$$

Half range fourier series in $[0, l]$

To get either a fourier cosine series of $f(x)$ in $[0, l]$

or, to " a " sine series of $f(x)$ in $[0, l]$.

∮ Half-range fourier sine series of $f(x)$ in $[0, l]$

Define a new function $\phi(x)$ in $[-l, l]$ such that

$$\phi(x) = \begin{cases} f(x)\ ; & \text{in } [0, l] \\ -f(-x). & \text{in } [-l, 0]. \end{cases}$$

$\therefore$ $\phi(x)$ is an odd function in $[-l, l]$.

$$\therefore \phi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad ; -l \le x \le l.$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad ; \quad 0 \le x \le l.$$

because $f(x) = \phi(x)$ in $0 \le x \le l$.

Half-range Fourier cosine series of $f(x)$ in $[0, l]$

Define $\phi(x)$ in $[-l, l]$ as

$$\phi(x) = \begin{cases} f(x) & \text{in } [0, l] \\ f(-x) & \text{in } [-l, 0] \end{cases}$$

$\therefore$ $\phi(x)$ is an even func. in $[-l, l]$.

$$\therefore \phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad ; -l \le x \le l.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad ; \quad 0 \le x \le l.$$

Ex1. Find a ~~series of cosine~~ fourier cosine series for the funct. $f(x) = x$ on the interval $0 \le x \le \pi$.

Solution. Define $\phi(x)$ such that

$$\phi(x) = \begin{cases} f(x) & \text{in } 0 \le x \le \pi. \\ f(-x) & \text{in } -\pi \le x \le 0. \end{cases}$$

Then,

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \;, \quad -\pi \le x \le \pi.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \;; \quad 0 \le x \le \pi.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \phi(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx.$$

$$= \frac{\cancel{2}}{\pi} \cdot \frac{x^2}{\cancel{2}}\Big|_0^{\pi} = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \phi(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x \cdot \frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{2}{\pi} \cdot \frac{1}{n^2}\Big[\cos nx\Big]_0^{\pi} = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n - 1\right\}\cos nx \;; \quad 0 \le x \le \pi.$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}.$$

2. Find fourier sine series for $f(x) = x^2$ in $[0, \pi]$.

Sol. Define $\phi(x)$ in $[-\pi, \pi]$ such that

$$\phi(x) = \begin{cases} f(x) & ; x \in [0, \pi] \\ -f(-x) & ; x \in [-\pi, 0] \end{cases}$$

$\therefore \phi(x)$ is an odd funct^n. of x in $[-\pi, \pi]$.

$$\therefore \phi(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad , \quad x \in [-\pi, \pi].$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad , \quad x \in [0, \pi].$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx \, dx.$$

$$= \frac{2}{\pi} \left[x^2 \frac{\cos nx}{n} \Big|_{\pi}^{0} + \int_0^{\pi} 2x \frac{\cos nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi^2}{n} (-1)^n + 2 \left\{ x \frac{\sin nx}{n^2} \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n^2} dx \right\} \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi^2}{n} (-1)^n + \frac{2}{n^2} \cdot \frac{\cos nx}{n} \Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{n} (-1)^{n+1} + \frac{2}{n^3} \left\{ (-1)^n - 1 \right\} \right]$$

$$= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left\{(-1)^n - 1\right\}.$$

$$\therefore x = \sum_{n=1}^{\infty} b_n \sin nx \; ; \qquad 0 \le x \le \pi.$$

$$b_n = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left\{(-1)^n - 1\right\} \; //.$$

$$T_n(x) = \cos(n \cos^{-1} x)$$

1. Represent
$$f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

in (a) a Fourier sine series.

(b) a Fourier cosine series.

(c) Fourier series with period 1.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi}{b-a}\right)x + b_n \sin\left(\frac{2n\pi}{b-a}\right)x \right]$$

Here take $a=0,\ b=1$. $\quad a \le x \le b$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{2n\pi}{b-a}\right)x\,dx.$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin\left(\frac{2n\pi}{b-a}\right)x\,dx.$$

(a) Define $\phi(x)$: $\quad \phi(x) = \begin{cases} f(x); & 0 < x < 1 \\ -f(-x); & -1 < x < 0. \end{cases}$

(b) Define $\phi(x)$: $\quad \phi(x) = \begin{cases} f(x); & 0 < x < 1 \\ f(-x); & -1 < x < 0. \end{cases}$

Behaviour of ~~F.S.~~ at the end points or at interior points of $f(x)$ in $[-l, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \quad [-l, l]$$

The series converges to ~~f(x) behaves~~ $f(x_0)$ for $-l < x_0 < l$, if x_0 is not a discontinuity of $f(x)$.

The series converges to $\dfrac{f(x_0-0) + f(x_0+0)}{2}$ for $-l<x_0<l$ if x_0 is an ordinary discontinuity of $f(x)$

At an interior pt. x_0 $(-l < x_0 < l)$, series converges to

$$\frac{f(x_0-0) + f(x_0+0)}{2}$$

At the end points (at $x = \pm l$) $f(x)$ converges to

$$\frac{1}{2}\left[f(-l+0) + f(l-0)\right].$$

Behaviours of half-range Fourier series at intermediate points & at end points

A. Half-range Fourier cosine series in $[0,l]$

$$\phi(x) = \begin{cases} f(x), & 0 \le x \le l \\ f(-x), & -l \le x \le 0 \end{cases}$$

At an interior pt. F.S. converges to

$$\frac{\phi(x_0+0)+\phi(x_0-0)}{2}, \; -l < x_0 < l.$$

$\therefore$ At a pt. x_0, $[x_0 \in (0,l)]$, F.S. converges to

$$\frac{\phi(x_0+0)+\phi(x_0-0)}{2}, \quad 0 < x_0 < l.$$

$\therefore$ At a pt. x_0, $[x_0 \in (0,l)]$ F.S. converges to,

$$\frac{f(x_0+0)+f(x_0-0)}{2}, \quad 0 < x_0, l).$$

Behaviour of the half-range cosine series at $x = 0$.

At $x=0$, series converges to $\dfrac{\phi(0+0)+\phi(0-0)}{2}$.

$\therefore$ At $x=0$, series " " $\dfrac{f(0+0)+f(0+0)}{2}$

$$= f(0+0).$$

$\therefore$ At $x = l$, series converges to $\dfrac{\phi(l-0) + \phi(-l+0)}{2}$

$\therefore$ At $x = l$, half range cosine " " " $\dfrac{f(l-0) + f(l-0)}{2}$

$= f(l-0)$.

For a <u>half range F. C. series</u>,

the series converges to $\frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$; $x_0 \in (0,l)$.

At $x = 0$ it " " $f(0+0)$.

At $x = l$ " " " $f(l-0)$.

B. Half-range Fourier sine series in $[0,l]$.

$$\phi(x) = \begin{cases} f(x), & 0 \le x \le l \\ -f(-x); & -l \le x \le 0. \end{cases}$$

For a ~~The~~ <u>Half-range F. S. series</u>,

the series converges to $\frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$; $x_0 \in (0,l)$.

At $x = 0$ & at $x = l$ it converges to 0.

Finding sum of infinite series of constants using Fourier series.

Parseval's theorem.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Pf. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \rightarrow (1).$ $-\pi \le x \le \pi.$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Multiply both sides of (1) by $f(x)$ & integrate between $-\pi$ and π

$$\int_{-\pi}^{\pi}\{f(x)\}^2\,dx = \frac{a_0}{2}\underline{\int_{-\pi}^{\pi} f(x)\,dx} + \sum_{n=1}^{\infty} a_n \underline{\int_{-\pi}^{\pi}\cos nx\, f(x)\,dx} + \sum_{n=1}^{\infty} b_n \underline{\int_{-\pi}^{\pi}\sin nx\, f(x)\,dx}.$$

$$= \frac{a_0}{2}\times \pi a_0 + \sum_{n=1}^{\infty} a_n \times \pi a_n + \sum_{n=1}^{\infty} b_n \times \pi b_n.$$

or, $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

1. From F.S. expansion of $f(x) = x^2$ in $-\pi < x < \pi$, prove that

1. $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

or, $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$x^2 = f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

1. $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(x)\}^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2). \longrightarrow (1).$$

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}; \quad b_n = 0.$$

$\therefore$ subst. in (1),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2 dx = \frac{\cancel{4}}{9}\cdot\frac{\pi^4}{\cancel{4}} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16(-1)^{2n}}{n^4}.$$

or $$\frac{1}{\cancel{2}\pi}\times\cancel{2}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

or, $$\frac{1}{\cancel{\pi}}\cdot\frac{\pi^{\cancel{5}\,4}}{5} - \frac{\pi^4}{9} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

or, $$\frac{4\cdot\pi^4}{45}\times\frac{1}{\cancel{8}\,2} = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$