RECESS: FROM .... am/pm TO .... am/pm LUNCH: FROM .... am/pm TO .... am/pm

TIME	PERIOD	1	2	3	4	5	6	7	8	9	10
	FROM	am pm	am pm	am pm	am pm	am pm	am pm	am pm	am pm	am pm	am pm
	TO	am pm	am pm	am pm	am pm	am pm	am pm	am pm	am pm	am pm	am pm
MONDAY	SUBJECT										
	TEACHER										
TUESDAY	SUBJECT										
	TEACHER										
WEDNESDAY	SUBJECT										
	TEACHER										
THURSDAY	SUBJECT										
	TEACHER										
FRIDAY	SUBJECT										
	TEACHER										
SATURDAY	SUBJECT										
	TEACHER										

$\cancel{\cos(A+B) = \cos A \cos B -}$

$\cos((w-v)t) =$

$\cos(A+B) = \cos A \cos B -$

$\cos(A-B) \quad \cos A + \sin A$

GNK/ SA0718010

I N D E X

NAME: Madhuri Sarda ROLL NO.: ....

STD.: .... DIV./SEC.: .... SUBJECT: Advance mathematical techniques.

S. No.	Date	Title	Page No.	Teacher's Sign/Remarks
		~~class test → 1/9/22 → SDE~~		
mid sem topics won't be part of end sem		PAD → chap 1-4 of notes ↳ syllabus covered upto 9th sept		mid sem
		16th → voluntary class → PAD		
		20/10/22 → class test (elliptic & hyperbolic PDE)		
		End sem → after mid sem.		

$$-u_{30}(1-x) = \sum_{n=1}^{\infty} C_6 \sin n\pi x \qquad t=0$$

$$-u_{30}\int_0^1 (1-x)\sin n\pi x\, dx = \frac{C_6}{2}$$

CYLINDRICAL COORDINATE SYSTEM (r, θ, z)

2D transient

Problem:- (2 dimensional too long θ - symmetry)

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

at $t=0$, $u = u_0$

at $r=0$, u = finite → physical sc.

$r=1$, $u=0$

$$u = T(t)\, R(r)$$

$$R\frac{dT}{dt} = T\frac{1}{r}\left(\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right)$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \text{Const}$$

Case 1 :- Const $=0$

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

$$r\frac{dR}{dr} = C_1$$

$$\frac{dR}{dr} = \frac{C_1}{r}$$

$$\boxed{R = C_1 \ln r + C_2}$$

$\boxed{C_1 = 0}$ → $r=0$, R - finite

$$\boxed{R = C_2}$$

$r=1$, $R=0$, $\boxed{C_2=0}$ — [Trivial soln] → ∴ const $\neq 0$

Case 2 :- const = +ve.

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \lambda^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda^2 R r = 0$$

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \lambda^2 R r = 0$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 r^2 R = 0$$

consider, $y = \lambda r$

$$\frac{dR}{dr} = \frac{dR}{dy}\cdot\frac{dy}{dr} = \lambda\frac{dR}{dy}$$

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{dy}\left(\lambda\frac{dR}{dy}\right)\cdot\frac{dy}{dr} = \lambda^2\frac{d^2R}{dy^2}$$

$$r^2\lambda^2\frac{d^2R}{dy^2} + r\lambda\frac{dR}{dy} - \lambda^2 r^2 R = 0$$

$$y^2\frac{d^2R}{dy^2} + y\frac{dR}{dy} - y^2 R = 0$$

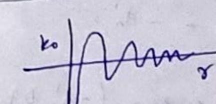
$$\boxed{y\frac{d^2R}{dy^2} + \frac{dR}{dy} - yR = 0.}$$ → Bessel equation of 2nd kind of order '0'.

→ $R = C_1 I_0(y) + C_2 k_0(y)$

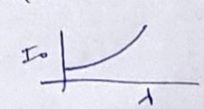
$R(r) = C_1 I_0(\lambda R) + C_2 k_0(\lambda r)$

at $r=0$, R = finite

⇓ $C_2 = 0$.



$$R(r) = C_1 I_0(\lambda r)$$



$r=1$, $R=0$ ⇒ $0 = C_1 I_0(\lambda)$, $I_0(\lambda) \neq 0$

$\boxed{C_1 = 0}$

Const $= \lambda^2$ → not possib

Proof :- Bessel functions are orthogonal function w.r.t r.

$$\iiint y_m y_n \underline{r}\, dr = 0 \quad \forall\, m \neq n$$

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 r R = 0$$

λ_m, λ_n are 2 distinct eigenvalues

$\downarrow \qquad\qquad \searrow$

$J_0(\lambda_m r) \qquad J_0(\lambda_n r)$

① $\frac{d}{dr}\left(r\frac{d}{dr}J_0(\lambda_m r)\right) + \lambda_m^2 r J_0(\lambda_m r) = 0 \quad \times\ r\, J_0(\lambda_n r)$

② $\frac{d}{dr}\left(r\frac{d}{dr}\left(J_0(\lambda_n r)\right)\right) + \lambda_n^2 r J_0(\lambda_n r) = 0 \quad \times\ r\, J_0(\lambda_m r)$

$$J_0(\lambda_m) = J_0(\lambda_n) = 0.$$

$$\int_0^1 J_0(\lambda_n r)\frac{d}{dr}\left(r\frac{d}{dr}J_0(\lambda_m r)\right)dr + \lambda_m^2\int r\, J_0(\lambda_m r)\, J_0(\lambda_n r)\, dr -$$

$$\int_0^1 J_0(\lambda_m r)\frac{d}{dr}\left(r\frac{d}{dr}J_0(\lambda_m r)\right)dr - \lambda_m^2\int r\, J_0(\lambda_m r)\, J_0(\lambda_n r)\, dr$$

$$= 0.$$

$$\rightarrow J(\cancel{\lambda_n r})\, r\frac{d\left(J_0(\lambda_m r)\right)}{dr}\Bigg|_0^1 - \int_0^1 \frac{dJ_0(\lambda_n r)}{dr}\, r\, \frac{dJ_0(\lambda_m r)}{dr}\, dr$$

$r=1$

$$- J_0(\cancel{\lambda_m r})\, r\frac{d\left(J_0(\lambda_n r)\right)}{dr}\Bigg|_0^1 + \int_0^1 \frac{dJ_0(\lambda_n r)}{dr}\, r\, \frac{dJ_0(\lambda_m r)}{dr}\, dr$$

$r=1$

$$= \underset{\neq 0}{(\lambda_n^2 - \lambda_m^2)}\int_0^1 \underbrace{J_0(\lambda_m r)\, J_0(\lambda_n r)\, r\, dr}_{=0}.$$

$u = T(t)\, R(r)\, \Theta(\theta)$

$$R\Theta \frac{dT}{dt} = T\Theta \frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{RT}{r^2}\frac{d^2\Theta}{d\theta^2}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}$$

$$\frac{r^2}{T}\frac{dT}{dt} = \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$$

$$\frac{r^2}{T}\frac{dT}{dt} - \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = +\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \text{const}$$

<u>case 1</u>: const $= 0$

$\cancel{\Theta = c_1\theta}$ $\quad \Theta = c_1\theta + c_2$

$$\Theta|_{\pi} = \Theta|_{-\pi}$$

$$c_1\pi + c_2 = -c_1\pi + c_2$$

$$\boxed{c_1 = 0}$$

$\boxed{\Theta = c_2}$ → eigen function

$$\left.\frac{d\Theta}{d\theta}\right|_{\pi} = \left.\frac{d\Theta}{d\theta}\right|_{-\pi} \rightarrow \text{always valid}$$

0 is the eigen value | Eigen function = const

<u>case - 2</u> :- const = +ve

$$+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \alpha^2$$

$$\Theta = c_1 e^{\alpha\theta} + c_2 e^{-\alpha\theta}$$

$$c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = c_1 e^{-\alpha\pi} + c_2 e^{\alpha\pi}$$

$$\boxed{c_1 = c_2}$$

$$\Theta = c_1\left(e^{\alpha\theta} + e^{-\alpha\theta}\right)$$

$$\left.\frac{d\Theta}{d\theta}\right|_{\pi} = \left.\frac{d\Theta}{d\theta}\right|_{-\pi}$$

$$c_1\left(\alpha e^{\alpha\pi} - \alpha e^{-\alpha\pi}\right) = c_1\left(\alpha e^{-\alpha\pi} - \alpha e^{\alpha\pi}\right)$$

$$\boxed{c_1 = 0}$$

$c_1 = 0, \; c_2 = 0.$

[trivial soln]

Case 3: const $= -\alpha^2$

$$\frac{d^2\theta}{d\theta^2} + \alpha^2\theta = 0$$

$$\theta = c_1\sin\alpha\theta + c_2\cos\alpha\theta$$

$$c_1\sin\alpha\pi + c_2\cos\alpha\pi = c_1\sin\alpha(-\pi) + c_2\cos\alpha(-\pi)$$

$2c_1\sin\alpha\pi = 0$ → $\sin\alpha\pi = 0$ $\quad \alpha\pi = n\pi$

$\boxed{\alpha = n}$ — $n = 1, \ldots$

$$c_1\alpha\cos\alpha\pi - c_2\alpha\sin\alpha\pi = c_1\alpha\cos\alpha(-\pi) - c_2\alpha\sin\alpha(-\pi)$$

$\boxed{c_2\alpha\sin\alpha\pi = 0}$

$\alpha\sin\alpha\pi = 0$.

$\alpha = 0$ & $\alpha\pi = n\pi$

$\alpha = 1, 2, \ldots$

$\boxed{\alpha_n = n}$ → $n = 1, 2, 3 \ldots$

⇒ Eigen values for θ soln $= 0, 1, 2, \ldots$ → combining Case 1 & 3

Eigen function :- $[\Theta(\theta) = c_3\sin n\theta + c_4\cos n\theta]$ ✓

for case 1 & case 3 → $n = 0, 1, 2, \ldots \infty$

→ $$\frac{r^2}{T}\frac{dT}{dt} - \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -n^2$$

$$\frac{1}{T}\frac{dT}{dt} - \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{-n^2}{r^2}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2} = \text{const} = -\lambda^2$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2} = -\lambda^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R + \lambda^2 rR = 0$$

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \frac{n^2}{r}R + \lambda^2 rR = 0.$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R + \lambda^2r^2R = 0.$$

$y = \lambda r$

$$u(x,y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi y$$

$y = 1 \quad u = 1$

$$1 = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi$$

use orthogonality

$$\int_0^1 \sin m\pi x\,dx = \sum_{n=1}^{\infty}\int_0^1 c_n \sin n\pi x \sinh n\pi \sin m\pi x\,dx$$

$$\left[\frac{-\cos m\pi x}{m\pi}\right]_0^1 = \left(c_n\int_0^1 \sin^2 n\pi x\,dx\right)\sinh n\pi$$

$$\frac{1-\cos n\pi}{n\pi} = c_n \sinh n\pi \cdot \frac{1}{2}$$

$$c_n = 2(1-\cos n\pi)$$

$$c_{n1} = \frac{\int_0^1\int_{-\pi}^{\pi} f(r,\theta)\, J_n(\lambda_{mn} r) \ldots}{\int\int r\, J_n^2(\lambda_{mn} r)\sin^2(n\theta)\,dr\,d\theta}$$

Simpsons rule

$$c_{n2} = \frac{\int_0^1\int_{-\pi}^{\pi} r f(r,\theta)\, J_n(\lambda_{mn} r)\cos n\theta\; r\,dr\,d\theta}{\int\int r\, J_n^2(\lambda_{mn} r)\cos^2(n\theta)\,dr\,d\theta}$$

Madhuri Sarda

18CH30030

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

at $\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\} u=0$

$y=0 \longrightarrow u=0$.

$y=1 \rightarrow u=1$

10/10

$$u = X(x)\, Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha_n^2$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha_n^2$$

$$\frac{d^2X}{dx^2} + \alpha_n^2 X = 0$$

C_1

$\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\} \rightarrow u=0 \rightarrow X=0$

$$\alpha_n = n\pi$$

$$\boxed{X_n = C_1 \sin n\pi x}$$

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} = -\alpha_n^2$$

$$\frac{d^2Y}{dy^2} - \alpha_n^2 Y = 0$$

$$Y_n = C_2 e^{\alpha_n y} + C_3 e^{-\alpha_n y}$$

$$y=0 \rightarrow Y_n = 0$$

$$0 = C_2 + C_3$$

$$\boxed{C_2 = -C_3}$$

$$\boxed{Y_n = C_2 \left(e^{\alpha_n y} - e^{-\alpha_n y}\right)}$$

$$Y_n = 2C_2 \left(\frac{e^{\alpha_n y} - e^{-\alpha_n y}}{2}\right)$$

$$\boxed{Y_n = 2C_2 \sinh \alpha_n y}$$

$\iint r J_n(\alpha_{mn} r)$ ~~...~~ $(n\theta)\, r\, dr\, d\theta$

$$u(x,y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh \alpha_n y$$

$y = 1 \to u = 1$

$$1 = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh \alpha_n$$

use orthogonality

$$\int_0^1 \sin m\pi x \, dx = \sum_{n=1}^{\infty} \int_0^1 c_n \sin n\pi x \sinh \alpha_n \sin m\pi x \, dx$$

$$\left[\frac{-\cos m\pi x}{m\pi} \right]_0^1 = \left(c_n \int_0^1 \sin^2 n\pi x \, dx \right) \sinh \alpha_n$$

$$\frac{1 - \cos n\pi}{n\pi} = c_n \sinh \alpha_n \, 1/2$$

$$c_n = 2\left(\frac{1 - \cos n\pi}{n\pi} \right) \frac{1}{\sinh \alpha_n}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{2(1-\cos n\pi)}{n\pi} \frac{1}{\sinh \alpha_n} \sin n\pi x \sinh \alpha_n y}$$

$$v = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2 t)\sin(n\pi r)$$

$$u_0 r = \sum_{n=1}^{\infty} c_n \sin(n\pi r)$$

$$c_n = 2u_0 \int_0^1 r \sin(n\pi r)\, dr$$

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2 t)\frac{\sin(n\pi r)}{r}}$$

<u>2 Dim in space</u>

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$\nabla^2 u = 0$. & ϕ symmetry

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0$$

$r=0$, $u =$ finite

$r=1$, $u = f(\theta)$

$u = R(r)\,T(\theta)$

$$\frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0.$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0.$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{-1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = \lambda \text{ (+ve const)}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \lambda T\sin\theta = 0 \qquad t = \cos\theta$$

$$\frac{dT}{d\theta} = \frac{dT}{dt}\times\frac{dt}{d\theta} = -\sin\theta\frac{dT}{dt}$$

$$\sin\theta\frac{dT}{d\theta} = -\sin^2\theta\frac{dT}{dt} = -(1-t^2)\frac{dT}{dt}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = -\frac{d}{d\theta}\left((1-t^2)\frac{dT}{dt}\right)$$

$$= -\frac{d}{dt}\left((1-t^2)\frac{dT}{dt}\right)\frac{dt}{d\theta}$$

$$= \sin\theta\left(\frac{d}{dt}\left((1-t^2)\frac{dT}{dt}\right)\right)$$

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^n P_n(\cos\theta)$$

at $r=1$, $u = f(\theta)$

$$f(\theta) = \sum_{n=1}^{\infty} C_n \underbrace{P_n(\cos\theta)}$$ legendre polynomials are orthogonal functions w.r.t weight function $\sin\theta$.

$$C_n = \frac{\int_0^{\pi} f(\theta) P_n(\cos\theta)\sin\theta\, d\theta}{\int_0^{\pi} P_n^2(\cos\theta)\sin\theta\, d\theta}$$

Ex-3

$\nabla^2 u = 0$, & no ϕ symmetry.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

$$u = R(r) \times \Theta(\theta) \times \Phi(\phi)$$

$$\frac{\Theta\Phi}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Theta R}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0.$$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

Periodic BCs on Φ :-

physical BC :- $\begin{cases} \Phi|_{\pi} = \Phi|_{-\pi} \\ \frac{d\Phi}{d\phi}\Big|_{\pi} = \frac{d\Phi}{d\phi}\Big|_{-\pi} \end{cases}$

$$\frac{d^2\Phi}{d\phi^2} + \underbrace{\text{const}}_{= m^2}\Phi = 0.$$

$m = n \rightarrow n = 0, 1, 2, \ldots \infty \rightarrow$ eigen values.

$$\boxed{\Phi_m(\phi) = C_1\sin(m\phi) + C_2\cos(m\phi)}$$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{m^2}{\sin^2\theta}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = \lambda$$ for non-trivial soln

→ 3D in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

at $r=0$, u - finite

$r=1$, $u = f(\theta,\phi)$

$\left.\begin{array}{l}\theta=0\\ \theta=\pi\end{array}\right\}$ u = finite

$u|_{\phi=\pi} = u|_{-\pi}$

$\frac{\partial u}{\partial\phi}\Big|_{\pi} = \frac{\partial u}{\partial\phi}\Big|_{-\pi}$

$u = R(r)\,\Theta(\theta)\,\Phi(\phi)$

$$\frac{\Theta\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \text{const } (0 \text{ \& } +ve) = m^2$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$\Phi|_{\pi} = \Phi|_{-\pi}$

$\frac{d\Phi}{d\phi}\Big|_{\pi} = \frac{d\Phi}{d\phi}\Big|_{-\pi}$

$\Phi_m = A\sin(mx) + B\cos mx$

$m = 0, 1, 2, \dots, \infty$

$$\rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = +ve = \lambda$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2\Theta}{\sin^2\theta} + \lambda\Theta = 0$$

$t = \cos\theta \quad -1 \le t \le 1$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dt}\frac{dt}{d\theta} = -\sin\theta\frac{d\Theta}{dt}$$

$$\boxed{\sin\theta\frac{d\Theta}{d\theta} = -(1-t^2)\frac{d\Theta}{dt}}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\frac{d}{d\theta}\left[(1-t^2)\frac{d\Theta}{dt}\right]$$

$$= -\frac{d}{dt}\left((1-t^2)\frac{d\Theta}{dt}\right) \times \frac{dt}{d\theta}.$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] = \frac{d}{dt}\left[(1-t^2)\frac{d\Theta}{dt}\right]$$

$$\frac{d}{dt}\left[(1-t^2)\frac{d\Theta}{dt}\right] - \frac{m^2}{1-t^2}\Theta + \lambda\Theta = 0$$

→ associated legendre equation for $\lambda = n(n+1)$

$$\Theta(t) = c_1 P_n^m(t) + c_2 Q_n^m(t)$$

$$\boxed{\Theta(\cos\theta) = c_1 P_n^m(\cos\theta)}$$

$n \rightarrow$ degree

m = order

→ eigen values → $\lambda = n(n+1)$

$n = m, m+1, \dots, m+2, \dots, \infty$

→ r-direction

$$\frac{1}{R_{mn}}\frac{d}{dr}\left(r^2\frac{dR_{mn}}{dr}\right) = n(n+1)$$

$$r^2\frac{d^2R_{mn}}{dr^2} + 2r\frac{dR_{mn}}{dr} - n(n+1)R_{mn} = 0$$ → euler's equation

Discrete Domain

$$Ax = b$$

$$\boxed{x = A^{-1} b}$$

→ $u = \underset{\sim}{L^{-1}} f$

↳ adjoint operator

$L u = f$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

} → green's function method

at $t=0$, $u = u_0$

$x=0$, $u = u_{01}$

$x=1$, $\frac{\partial u}{\partial x} + \beta u = \alpha$

4 sources of non homogeneity.

Particular location $(x \mid (x_0, t_0))$

(force all BC too)

~~~~ force RH in the governing equation to unit impulse.

↳ force $f(x,t)$ to unit impulse. located at (x_0, t_0)

→ dirac delta function

Definition of green's function (g)

Construction of g (causal green's function)

$$\frac{\partial g(x,t \mid x_0, t_0)}{\partial t} - \frac{\partial^2 g}{\partial x^2}(x,t \mid x_0,t_0) = \delta(x - x_0)\delta(t - t_0)$$

at $t=0$, $g=0$

$x=0$, $g=0$

$x=1$, $\frac{\partial g}{\partial x} + \beta g = 0$.

Adjoint operator

$L^{*} = ?$
~~~~

Governing equation of g^* (adjoint green function)

$$L^* g^* = \delta(x - x_1)\,\delta(t - t_1)$$

$$-\frac{\partial g^*}{\partial t} - \frac{\partial^2 g^*}{\partial x^2} = \delta(x - x_1)\,\delta(t - t_1)$$

$Bg^* = ?$ → Bilinear concomittant term should be set to 0.

$$g^* = f_2(x, t)$$

$$\langle g^*, Lg \rangle = \langle L^* g^*, g \rangle + \underbrace{J(g, g^*)}_{=0 \text{ to get } g^*}$$

step 1 :- Construction of green's functions → check if $L = L^*$ & $B = B^*$ or not

step 2 :- solution of g.

step 3 :- obtain L^*

step 4 :- Construction of adj green's function g^*

step 5 :- solution of $g^*(x,t)$

step 6 :- Connect g^* with u & get soln of $u(x,t)$

if yes: omit 3,4,5 Directly connect u with g

if no: all steps should be done

→ For laplacian operator (L) → it is self adjoint

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$L = L^*, \quad B = B^*$$

Properties of Dirac delta

(i) $\delta(x - x_0) = 1$ at $x = x_0$

$= 0 \ \forall\ x \neq x_0$

(ii) shifting property

$$\int_{x_0 - \epsilon}^{x_0 + \epsilon} f(x)\,\delta(x - x_0)\,dx = f(x_0)$$

→ **Relation between g & g^***

$$Lu = f \quad \text{subject to: } Bu = h$$

Causal green's function :- ① $Lg_{x|x_0} = \delta(x - x_0)$ subject to $Bg = 0$

Adjoint green's function :- ② $L^* g^*_{(x|x_1)} = \delta(x - x_1)$ subject to $Bg^* = 0$

Take inner product of equation ① wrt g^*

Take inner product of equation ② wrt g

and substract.

$$\langle g^*, Lg\rangle - \langle g, L^* g^*\rangle = \langle g^*(x|x_1), \delta(x-x_0)\rangle - \langle g(x|x_0), \delta(x-x_1)\rangle$$

$$\langle u, Lv\rangle = \langle u, L^* v\rangle + J(u,v)$$

$$\langle L^* g^*, g\rangle + \cancel{J(g,g^*)} - \langle g, L^* g^*\rangle = g^*(x_0|x_1) - g(x_1|x_0)$$

$\cancel{J(g,g^*)} = 0$ ∵ all BC are homogeneous.

$$\boxed{g^*(x_0|x_1) = g(x_1|x_0)}$$ → to find g^* → valid for 1D, 2D, 3D...

Example:- $g(x|x_0) = x - x_0 + x x_0$

↓ $x \to x_1$

$g(x_1|x_0) = x_1 - x_0 + x_1 x_0$.

↳ $= g^*(x_0|x_1)$

$g^*(x_0|x_1) = x_1 - x_0 + x_1 x_0$

$$\boxed{g^*(x|x_1) = x_1 - x + x_1 x}$$

Connection between u & g^*

$Lu = f$, $Bu = h$ → ①

~~eqn~~ $L^* g^*(x|x_1) = \delta(x - x_1)$ → ②

$Bg^* = 0$

take inner product of ① with g^*

" " " ② with g

& substract.

$$\langle g^*, Lu\rangle - \langle u, L^* g^*\rangle = \langle f, g^*\rangle - u(x_1)$$

$$\langle L^* g^*, u\rangle + J(g^*, u) - \langle u, L^* g^*\rangle = \langle f g^*\rangle - u(x_1)$$

$$\boxed{u(x_1) = \langle f, g^*\rangle + J(g^*, u)}$$ → $\ne 0$ bcoz BC of u are NH

Ex:- ~~If~~ $f = x$

$g^* = x_1 x$

$$\langle f, g^*\rangle = \int_0^1 x\, x_1\, dx = \frac{x_1}{3}$$

$u(x_1) = p(x_1)$ → purely a function of x_1

$x_1 \to x$.

$$\boxed{u(x) = \frac{x}{3} + \frac{1}{2} + x^2}$$

→ Q) $\frac{d^2u}{dx^2} = x$

$\frac{d^2 g(x|x_0)}{dx^2} = \delta(x - x_0)$ $\quad 0 < x < 1$

for $x < x_0 \Rightarrow \frac{d^2 g_1}{dx^2} = 0$ → lower half solution

$x > x_0 \to \frac{d^2 g_2}{dx^2} = 0$. → upper half solution

$g(x|x_0) = A_1 u_1(x) + B_1 u_2(x)$ for $x < x_0$.

$= A_2 u_1(x) + B_2 u_2(x)$ for $x > x_0$.

(Consts will be diff for upper & lower half)

→ Relationship between g & g^* ⇒ $g(x_1|x_0) = g^*(x_0|x_1)$

solution of g

$$\frac{d^2 g}{dx^2} = \delta(x - x_0)$$

For a typical 2nd order equation:-

$\frac{d^2 g}{dx^2} = 0$ for $x \ne x_0$

$g = c_1 A(x) + c_2 B(x)$ $\forall\ 0 \le x < x_0$ → lower half solution g_1

$= c_3 A(x) + c_4 B(x)$ $\forall\ x_0 < x \le 1$ → upper half solution g_2

at $x=0$, $g=0$

$g_1(0)=0$

$B=0$

$\boxed{g_1 = Ax}$

at $x=1$, $g=0$

$C+D=0$

$\boxed{D=-C}$

$\boxed{g_2 = C(x-1)$}

$g = Ax \quad \forall\ 0 \le x < x_0$

$= C(x-1) \quad \forall\ x_0 < x \le 1$

→ Continuity of green's function

$Ax_0 = C(x_0-1)$

→ Jump discontinuity

$C-A=1$

$\boxed{A = x_0-1}$ $\quad$ $\boxed{C = x_0}$

$$g = \begin{cases} (x_0-1)x & \forall\ 0 \le x < x_0 \\ x_0(x-1) & \forall\ x_0 < x \le 1 \end{cases}$$

$\frac{d^2g}{dx^2} = \delta(x-x_0) \longrightarrow ②$

$\int ① xg - \int ② u$

$$\int_0^1 g\frac{d^2u}{dx^2}dx - \int_0^1 u\frac{d^2g}{dx^2} = \int_0 ugdx - \int_0^1 \delta(x-x_0)\,u\,dx$$

$$g\frac{du}{dx}\Big|_0^1 - u\frac{dg}{dx}\Big|_0^1 = \int xgdx - u(x_0)$$

$$\underbrace{\cancel{g(1)}}_{0}u'(1) - \underbrace{\cancel{g(0)}}_{0}u'(0) - u(1)\frac{dg}{dx}\Big|_1 + u(0)\frac{dg}{dx}\Big|_0 = \int xgdx - u(x_0)$$

$$-2x_0 + 1\cdot(x_0-1) = \int xgdx - u(x_0)$$

$$\boxed{u(x_0) = \underbrace{\int xgdx}_{\text{governing equation NH}} + \overbrace{x_0+1}^{BC}}$$

↳ 3 sources of NH

∴ 3 terms.

$$u(x_0) = \int_0^{x_0} x(x_0-1)x\,dx + \int_{x_0}^{1} x\, x_0(x-1)\,dx + x_0 + 1$$

$$u(x_0) = \left[(x_0-1)\frac{x^3}{3}\right]_0^{x_0} + x_0\left(\left[\frac{x^3}{3}\right]_{x_0}^{1} - \left[\frac{x^2}{2}\right]_{x_0}^{1}\right) + x_0 + 1$$

$$\cancel{u(x_0) = \frac{x_0^3(x_0-1)}{3} + \frac{x_0}{3} - \frac{x_0^4}{3} - \frac{1}{2} + \frac{x_0^2}{2} + x_0 + 1}$$

$$\boxed{u(x_0) = \frac{x_0^3}{6} + \frac{5x_0}{6} + 1}$$

$x_0 \to x$

$$u(x) = \frac{x^3}{6} + \frac{5}{6}x + 1$$

Ex-2 $\frac{d^2u}{dx^2} = x \to (1)$

at $x = 0 \to du/dx = 1$

$x = 1 \to u = 2$

(2) $\frac{d^2g}{dx^2} = \delta(x - x_0)$ subject to $\frac{dg}{dx} = 0$ at $x = 0$

$g = 0$ at $x = 1$

$g = Ax + B$ for $x < x_0$

$= Cx + D$ for $x > x_0$.

$\frac{dg}{dx} = 0 \to A = 0$ $\boxed{g = B} \to x < x_0$

$0 = C + D \to x = 1$

$C = -D$

$\boxed{g = c(x-1)} \to x > x_0$

$g = B \quad x < x_0$

$= c(x-1) \quad x > x_0$

continuity of green's function

$B = c(x_0 - 1)$

Jump discontinuity

$$\left.\frac{dg}{dx}\right|_{x_0^+} - \left.\frac{dg}{dx}\right|_{x_0^-} = 1$$

$c - 0 = 1$

$\boxed{c = 1}$

$\boxed{B = (x_0 - 1)}$

$g = x_0 - 1 \quad x < x_0$

$= x - 1 \quad x > x_0$

$$\int g\frac{d^2u}{dx^2}dx - \int u\frac{d^2g}{dx^2}dx = \int xg\,dx - u(x_0)$$

By parts

$$\cancel{g(1)u'(1)} - g(0)u'(0) - u(1)g'(1) - u(0)\cancel{g'(0)} = \int xg\,dx - u(x_0)$$

$$u(x_0) = \int x\underbrace{g}_{}dx + 1 + x_0$$

$$\int_0^{x_0} \quad \int_{x_0}^{1}$$

$$u(x_0) = \int_0^{x_0} x(x_0-1)\,dx + \int_{x_0}^{1} x(x-1)\,dx + x_0 + 1$$

$$u(x_0) = \left[\frac{x^2}{2}(x_0-1)\right]_0^{x_0} + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x_0}^{1} + x_0 + 1$$

$$u(x_0) = \frac{x_0^3}{2} - \frac{x_0^2}{2} + \frac{1}{3} - \frac{1}{2} - \frac{x_0^3}{3} + \frac{x_0^2}{2} + x_0 + 1$$

$$u(x_0) = \frac{x_0^3}{6} + x_0 + 5/6$$

$$u(x) = \frac{x^3}{6} + x + 5/6$$

Solution of non-homogeneous PDE

Elliptic equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

at $x=0 \rightarrow u=a$ $\quad$ $y=0,\ u=0$

$x=1 \rightarrow u=0 \,|\, \frac{\partial u}{\partial x}=0$ $\quad$ $y=1,\ u=b$

① Construction of Causal green's Function:

$$Lg = \delta(x-x_0)\,\delta(y-y_0)$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0)$$

$L\phi = -\lambda\phi$ type

at $x=0$, $x=1$; $y=0$, $y=1$ $\rightarrow$ 2 independent eigen value problems

$g=0 \quad \frac{\partial g}{\partial x}=0$

$$g(x,y|x_0,y_0) = \sum a_i \phi_i(x,y)$$

ϕ_i = eigen function

(full eigen function expansion method)

corresponding eigen value problem.

$$L\phi + \lambda\phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \lambda\phi = 0$$

eigen function is function of both x,y.

at $x=0,1$; $y=0,1$ $\quad \phi=0 \quad \frac{\partial \phi}{\partial x}=0$

$$\phi(x,y) = X(x)\,Y(y)$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \lambda XY = 0$$

→ formulating eigen value problem in both x,y directions

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda = -\alpha^2$$

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0$$

$x=0$, $x=1$: $X=0 \quad \frac{\partial X}{\partial x}=0$

$$\boxed{X_n = C_1 \sin n\pi x}$$

$\alpha_n = n\pi$

$n = 1, 2, \ldots \infty$

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} - \lambda = -\alpha^2$$

$$\frac{d^2Y}{dy^2} + k^2 Y = 0 \quad \text{at } \left.\begin{matrix} y=0 \\ y=1 \end{matrix}\right\} Y=0$$

$$\lambda_{mn} = (m^2+n^2)\pi$$

$$Y_m = C_2 \sin m\pi y$$

$$k_m = m\pi, \quad m = 1, 2, \dots$$

$$\boxed{\phi_{m,n}(x,y) = C_{mn} \sin n\pi x \sin m\pi y}$$

$$g(x,y) = \sum_m \sum_n a_{mn}\, \phi_{mn}(x,y)$$

norm → length

~~$g(x,y) = \sum_m \sum_n a_{mn} \sin$~~

$$a_{mn} = \frac{\langle g, \phi_m \rangle}{\langle \phi_{mn}, \phi_{mn} \rangle} \rightarrow \text{normalization}$$

$$\langle \phi_{mn}, \phi_{mn} \rangle = 1 \rightarrow$$ **MAKE UR LYF EASY!!!**

$$\iint \phi_{mn}^2 \, dx\, dy = 1$$

$$\int_0^1\int_0^1 C_{mn}^2 \sin^2 n\pi x \sin^2 m\pi y \, dx\, dy = 1$$

$$\boxed{C_{mn} = 2}$$

$$\phi_{mn}(x,y) = 2 \sin n\pi x \sin m\pi y$$

$$a_{mn} = \langle g, \phi_{mn} \rangle$$

Estimation of a_{mn}

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0) \longrightarrow ①$$

$$\frac{\partial^2 \phi_{mn}}{\partial x^2} + \frac{\partial^2 \phi_{mn}}{\partial y^2} + \lambda_{mn}\phi_{mn} = 0 \longrightarrow ②$$

$$\langle 1, \phi_{mn} \rangle - \langle 2, g \rangle$$

$$\iint \phi_{mn} \frac{\partial^2 g}{\partial x^2} dx\, dy + \iint \phi_{mn} \frac{\partial^2 g}{\partial y^2} dx\, dy - \iint g \frac{\partial^2 \phi_{mn}}{\partial x^2} dx\, dy -$$

$$\iint g \frac{\partial^2 \phi_{mn}}{\partial y^2} dx\, dy - \lambda_{mn} \iint \phi_{mn}\, g \, dx\, dy = \phi_{mn}(x_0, y_0)$$

$$\text{LHS} = \int_y \left[\cancel{\phi_{mn} \frac{\partial g}{\partial x}\Big|_0^1} - \int_0^1 \frac{\partial \phi_m}{\partial x}\frac{\partial g}{\partial x} dx \right] dy + \int_x \left[\cancel{\phi_{mn} \frac{\partial g}{\partial y}\Big|_0^1} - \int_0^1 \frac{\partial \phi_m}{\partial y}\frac{\partial g}{\partial y} dx \right]$$

cancel

$$- \int_y \left[\cancel{g \frac{\partial \phi_m}{\partial x}\Big|_0^1} - \int \frac{\partial g}{\partial x}\frac{\partial \phi_{mn}}{\partial x} dx \right] dy$$

$$- \int_x \left[\cancel{g \frac{\partial \phi_{mn}}{\partial y}\Big|_0^1} - \int \frac{\partial g}{\partial y}\frac{\partial \phi_{mn}}{\partial y} dy \right] dx$$

$$0 = \lambda_{mn} \langle \phi_{mn}, g \rangle + \phi_{m,n}(x_0, y_0)$$

$$\langle \phi_{m,n}, g \rangle = -\phi_{m,n}(x_0, y_0)$$

$$a_{mn} = -2\,\frac{\sin(n\pi x_0)\sin(m\pi y_0)}{\pi^2(m^2+n^2)}$$

$$g(x, y | x_0, y_0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-4)\, \frac{\sin(n\pi x_0)\sin(m\pi y_0) \sin n\pi x \sin m\pi y}{\pi^2(m^2+n^2)}$$

→ $g^* = ?$ ~~Col → Self adjoint~~

$\langle v, Lu \rangle$

$u = g, \quad v = g^*$

$$\langle g^*, Lg \rangle = \iint g^* \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) dx\, dy$$

$$= \iint g^* \frac{\partial^2 g}{\partial x^2} dx\, dy + \iint g^* \frac{\partial^2 g}{\partial y^2} dx\, dy$$

$$= \int_y \left[g^* \frac{\partial g}{\partial x}\Big|_0^1 - \int \frac{\partial g^*}{\partial x}\frac{\partial g}{\partial x} dy\, dx \right] + \int_x \left[g^* \frac{\partial g}{\partial y}\Big|_0^1 - \frac{\partial g^*}{\partial y}\frac{\partial g}{\partial y} dy\, dx \right]$$

$$= -\iint \frac{\partial g^*}{\partial x}\frac{\partial g}{\partial x} dx\, dy - \iint \frac{\partial g^*}{\partial y}\frac{\partial g}{\partial y} dy\, dx$$

$$= -\left[\int_y \left[-\frac{\partial g^*}{\partial x} g \right]_0^1 - \int \frac{\partial^2 g^*}{\partial x^2} g \, dx\, dy - \left[\int_x \left[-\frac{\partial g^*}{\partial y} g \right]_0^1 - \int \frac{\partial^2 g^*}{\partial y^2} g \, dx\, dy \right] \right]$$

$$= \iint g \left(\frac{\partial^2 g^*}{\partial x^2} + \frac{\partial^2 g^*}{\partial y^2} \right) dx\, dy = \iint g (L^* g^*)\, dx\, dy$$

$$= \langle g, L^* g^* \rangle$$

$$L^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\underline{L^* = L}$$

$\Rightarrow$ ① $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$

② $\dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0)$

$$\iint g \frac{\partial^2 u}{\partial x^2} dx\, dy + \iint g \frac{\partial^2 u}{\partial y^2} dx\, dy = \iint g f \, dx\, dy - u(x_0, y_0)$$

$$- \iint u \frac{\partial^2 g}{\partial x^2} dx\, dy - \iint u \frac{\partial^2 g}{\partial y^2} dy\, dx$$

LHS :- $\displaystyle\int_y \left[g \frac{\partial u}{\partial x}\Big|_0^1 - \int \cancel{\frac{\partial g}{\partial x} \frac{\partial u}{\partial x} dx} \right] dy$

$\displaystyle+ \int_x \left[g \frac{\partial u}{\partial y}\Big|_0^1 - \int \frac{\partial g}{\partial y} \frac{\partial u}{\partial y} dy \right] dx$

$\displaystyle- \int_y \left[u \frac{\partial g}{\partial x}\Big|_0^1 - \int \cancel{\frac{\partial u}{\partial x} \frac{\partial g}{\partial x} dx} \right] dy$

$\displaystyle- \int_x \left[u \frac{\partial g}{\partial y}\Big|_0^1 - \int \cancel{\frac{\partial u}{\partial y}} \frac{\partial g}{\partial y} dy \right] dx$

$$= \int_y \left[g(1) \frac{\partial u}{\partial x}\Big|_{x=1} - g(0) \frac{\partial u}{\partial x}\Big|_{x=0} \right] dy$$

$$+ \int_x \left[g(y=1) \frac{\partial u}{\partial y}\Big|_{y=1} - g(0) \frac{\partial u}{\partial y}\Big|_{y=0} \right] dx.$$

$$- \int_y \left[u(1) \frac{\partial g}{\partial x}\Big|_1 - u(0) \frac{\partial g}{\partial x}\Big|_{x=0} \right] dy$$

$$- \int_x \left[u(1) \frac{\partial g}{\partial y}\Big|_{y=1} - u(0) \frac{\partial g}{\partial y}\Big|_{x=0} \right] dx$$

LHS :- $\displaystyle = a \int \frac{\partial g}{\partial x}\Big|_{x=0} dy - b \int \frac{\partial g}{\partial y}\Big|_{y=1} dx$

$$a \int_{y=0}^{1} \frac{\partial g}{\partial x}\Big|_{x=0} dy - b \int_{x=0}^{1} \frac{\partial g}{\partial y}\Big|_{y=1} dx = \iint g f \, dx\, dy - u(x_0, y_0)$$

$$u(x_0, y_0) = \iint g f \, dx\, dy + b \int_x \frac{\partial g}{\partial y}\Big|_{y=1} dx - a \int_y \frac{\partial g}{\partial x}\Big|_{x=0} dy.$$

$R_{mn}(r) = c_3 r^n$

$$u(r,\theta,\phi) = \sum_{n=m}^{\infty} \sum_{m=0}^{n} r^n P_n^m(\cos\theta)\left[C_{mn}\sin(m\phi) + D_{mn}\cos(m\phi)\right]$$

Using orthogonal properties of eigen functions.

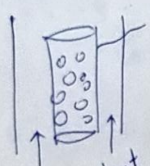
at $r=1$, $u = f(r,\theta)$

$$C_{mn} = \frac{\int_{-\pi}^{\pi}\int_{\theta=0}^{\pi} f(r,\theta)\, P_n^m(\cos\theta)\sin(m\phi)\sin\theta\, d\theta\, d\phi}{\int_{-\pi}^{\pi}\int_{\theta=0}^{\pi} {P_n^m}^2(\cos\theta)\sin^2 m\phi\,\sin\theta\, d\theta\, d\phi}$$

$$D_{mn} = \frac{\int_{-\pi}^{\pi}\int_{0}^{\pi} f(r,\theta)\, P_n^m(\cos\theta)\cos(m\phi)\sin\theta\, d\phi}{\int_{-\pi}^{\pi}\int_{0}^{\pi} {P_n^m}^2(\cos\theta)\cos^2(m\phi)\sin\theta\, d\theta\, d\phi}$$

Solution of non-homogeneous PDEs

Solution of non-homogeneous PDEs



coolant

$T_w(t) < T_{mp}$ of MOC of reactor element.

T_{mp} → melting pt

$$\rho C_p \frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \pm h(r,t)$$

NH governing eqn

$h(r,t)$ → generation term. because of this term, the equation is NH

\+ → source term

− → sink term

→ $\frac{\partial u}{\partial t} = \nabla^2 u + f$ $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}$

$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

$\boxed{Lu = f}$ L → operator

↘ continuous domain

↘ soln $u = L^{-1} f$

Discrete Domain

$Ax = b$

$\boxed{x = A^{-1} b}$

→ $u = L^{-1} f$

↳ adjoint operator

$Lu = f$

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t)$

} → green's function method

at $t=0$, $u = u_0$

$x=0$, $u = u_{01}$

$x=1$, $\frac{\partial u}{\partial x} + \beta u = \alpha$

4 sources of non homogeneity. (force all BC too)

~~forcast~~ force NH in the governing equation to unit impulse.

Particular location $(x \mid (x_0, t_0))$

> force $f(x,t)$ to unit impulse located at (x_0,t_0) → dirac delta function

Definition of green's function (g)

Construction of g (causal green's function)

$$\frac{\partial g(x,t \mid x_0,t_0)}{\partial t} - \frac{\partial^2 g}{\partial x^2}(x,t \mid x_0,t_0) = \delta(x-x_0)\,\delta(t-t_0)$$

at $t=0$, $g=0$

$x=0$, $g=0$

$x=1$, $\frac{\partial g}{\partial x} + \beta g = 0$.

Adjoint operator

$L^* = ?$