ONE STEP AT A TIME

Running probably looks like the simple act of placing one foot in front of the other at a speed faster than walking. Though your feet and legs do most of the work, your entire body is vigorously awakened. Most people have an inexpressible fascination for running, it probably stems from an unconscious throwback to a time when our ancestors travelled only by foot. If you already have made or want to make running a part of your daily or weekly routine, remember that like all forms of exercise it calls for a mental discipline that enhances who you are as a person. It improves your body and mind.

RUNNING TAKES YOU PLACES!

BIG FIVE MARATHON

In the Big Five Marathon, participants get to run in South African terrain alongside zebras, giraffes, and lions.

GREAT WALL MARATHON

Participants run along one of mankind's most wondrous long-winding wonders of the world. The Great Wall Marathon is carried out in three different formats: a marathon (42.19 km), a half marathon (21.1 km) and an 8.5 km run.

BAGAN TEMPLE MARATHON

Bagan, in Myanmar, is home to a plethora of temples that date back to hundreds of years. It is considered one of the richest archaeological sites in Asia. Participants of the Bagan Temple Marathon run amidst this treasure trove of history.

PETRA DESERT MARATHON

The Petra Desert Marathon has its participants running through Petra, an ancient city where antiquated Eastern traditions combine with Hellenistic architecture. The arid Jordanian landscape contains caves, monasteries, and tombs carved into rock and is the land of the Bedouins—the nomads of the Middle East.

POLAR CIRCLE MARATHON

Imagine running a marathon in one of the coldest regions on the planet! The Polar Circle Marathon has its participants make their way through the ice sheet and the gravel road that leads to Kangerlussuaq, a town just north of the Polar Circle.

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PDC

1. Bequette, process dynamics: Modeling, analysis and simulation
2. Ogunnaike and Ray, Process dynamics, modeling and control
3. stephanopoulos, chemical process control

→ facweb.iitkgp.ac.in/~parag
  >courses >process - - . } ~~ slides

Why do you need process control?

(1) Process demand

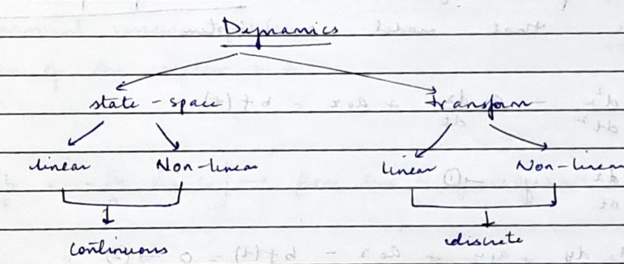
(2)

Process dynamics? → If you leave the system to settle down to a specific variable → dynamics will tell you what will happen.

Without process control also we can maintain the system ~~to~~ at a specific condition.

→ Transform domain analysis → take laplace and then invert it (analysis → going from one state to the other as a response.)

→ state - space analysis



Dynamics

→ Deals with motion of bodies under the action of forces

→ During motion, the coordinates of the system relative to the frame of reference $\boxed{\text{change with time}}$

→ Transient behaviour during staged-operations:

$$h_n \frac{dx_n(i,t)}{dt} = L_{n-1} x_{n-1}(i,t) + V_{n+1}(t)\, y_{n+1}(i,t) - V_n \ldots$$

changing with time.

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Dynamical system

Atleast one variable is a function of time

→ order of the system → order of ODE that models the system → works well for 1 variable

→ $\frac{dc}{dt} = \frac{F}{V}(c_f - c) - r$

Jacketed CSTR: $\frac{dT}{dt} = \frac{F}{V}(T_f - T) + \frac{(-\Delta H)}{\rho c_p} r - \frac{UA}{V \rho c_p}(T - T_j)$

} If we have 2 ODE, what is the order?

→ Order of the system is the "number of first order" ODE's that model the system.

$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b f(t)$

$\frac{dx}{dt} = y$ → ①

$a_2 \frac{dy}{dt} + a_1 y + a_0 x - b f(t) = 0$ → ②

→ n th order ODE can be converted to system of n first order ODE.

Linear system

A system is said to be a linear system if its governing dynamical equations are linear

$\hat{L}(\underline{u} + \underline{v}) = \hat{L}(\underline{u}) + \hat{L}(\underline{v})$
$\hat{L}(\alpha \underline{u}) = \alpha \hat{L}(\underline{u})$
} for operator $(\hat{L})$

→ A system which does not follow these rules

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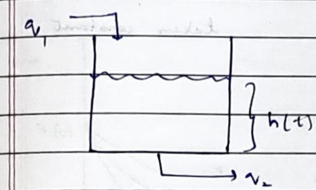
$\frac{dx}{dt} = \lambda x$
$\hat{L} = \frac{d}{dt} - \lambda$
} linear

$\frac{dx}{dt} = \lambda x^2$
$\hat{L} = \frac{d(\cdot)}{dt} - (\cdot)^2$
} non-linear

Q)



$\frac{dh(t)}{dt} = \frac{1}{A}(q_1 - q_2)$ ⟶ Cannot say whether it is linear or not (depends on q_1, q_2)

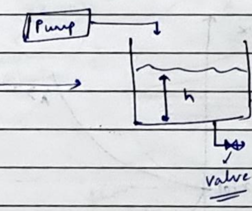
Dynamical variable = h(t)

Order of the system = 1.

→ $\left[\frac{dh}{dt} = -\frac{\sqrt{h}}{A}\sqrt{2g}\, C_d\right]$ → flow due to gravity → depends on valve.

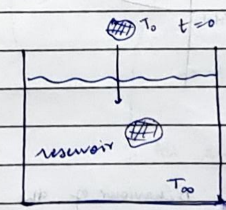
→ $\frac{dh}{dt} = \frac{h}{A}$ → ✓

→ $\frac{dh}{dt} = \frac{1}{A} f(t) - \frac{1}{A} g(h)$ ⟶



$\frac{dh}{dt} + \frac{1}{A} g(h) = \frac{1}{A} f(t)$

Q)



$\frac{dT}{dt} = \frac{-hA_s}{\rho V C}(T - T_\infty)$

↳ all the parameters are const (h, A_s, ρ, V, C)

↳ then it is linear

$$\frac{dT}{dt} = -\frac{hA_s}{\rho V C}(T - T_\infty)$$

$t = 0, \quad T = T_0$

Let $T^* = T - T_\infty$

$$-\frac{hA_s}{\rho V C} = \lambda$$

for small temp range $\rho, V, c, \ldots$ can be taken constant.

$$\frac{dT^*}{dt} = \lambda T^*$$

$$T^*(t) = T^*(0)\, e^{\lambda t}$$

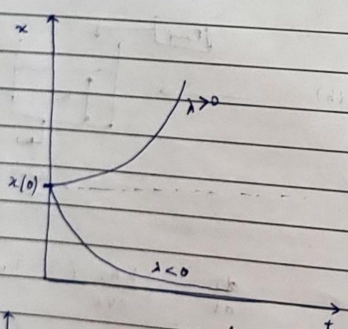
$$\boxed{T = T_\infty + (T_0 - T_\infty)e^{\lambda t}}$$

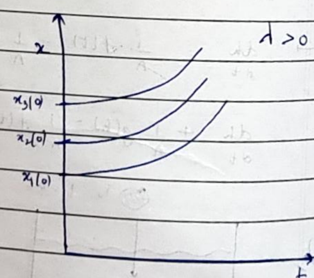
⇒ $\frac{dx}{dt} = \lambda x$ → Linear ODE if λ = const or only $f(t)$

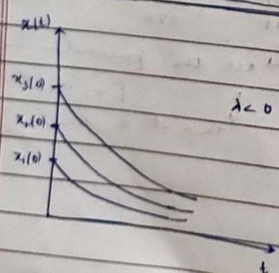
→ linear, 1st order, autonomous system

$$\boxed{x(t) = x(0)\, e^{\lambda t}}$$

$\left.\begin{array}{l}\lambda x = 0 \\ x_e = 0\end{array}\right\}$ equilibrium





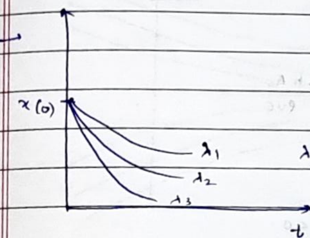


Behaviour of the system changes because of λ (parameter)

What Dynamics tells u?

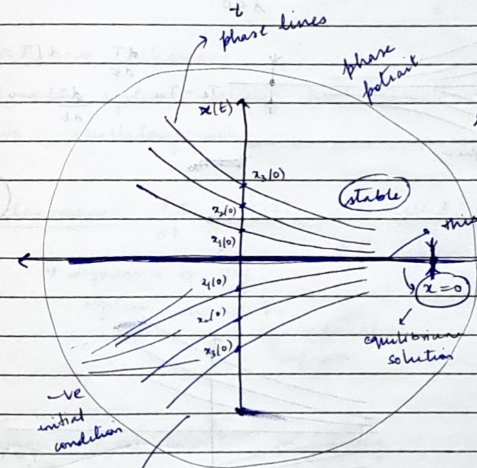
→ ① Want to know whether change is taking place fast or slow (time)

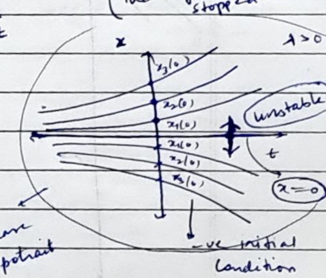
② fate of the system depending upon the parameter



$\lambda_3 > \lambda_2 > \lambda_1$

Long time behaviour does not depend on the magnitude of λ.





for $T(t)$ T cannot be -ve & t cannot be -ve

but this is just for a general case

→ Once you achieve '0', it will stay there forever.

All the non-equilibrium solution converge to equilibrium as we move ahead in time

phase potrait

not a soln

line to indicate stability

$\lambda < 0$ → equilibrium solution is stable equilibrium soln

$\lambda > 0$ → unstable

⇒ Unstable: move forward in time solutions diverge

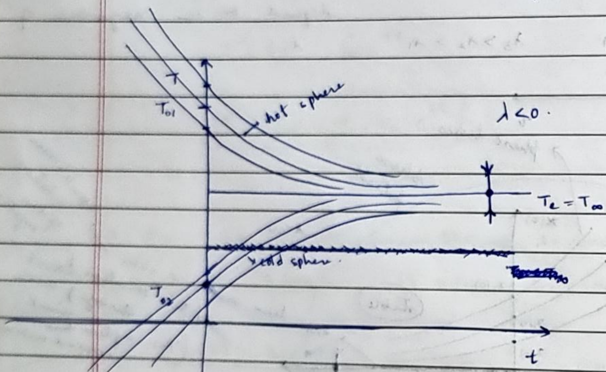
move backward in time solutions will converge.

bifurcation of system → fate of the system depends on λ

$\frac{dx}{dt} = \lambda x$ → the system shows bifurcation at $\boxed{\lambda = 0}$

$\lambda = 0^+, \lambda = 0^-$ → have different behaviour.

⇒ $T(t) = T_\infty + (T_0 - T_\infty)e^{\lambda t}$ $\lambda = -\frac{hA_s}{\rho V c}$

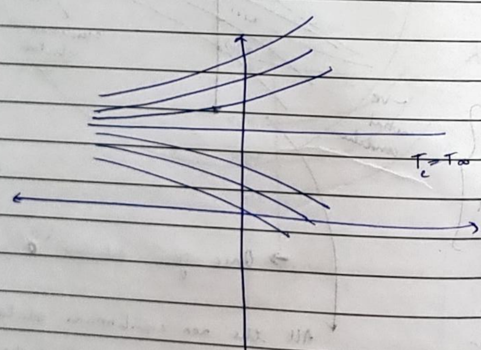


$\lambda < 0$.

$\frac{dT}{dt} = \lambda(T - T_\infty)$

$\frac{dT}{dt} = 0$ → at equilibrium soln

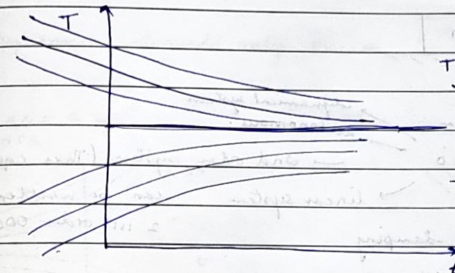
$T_e = T_\infty$



$\lambda > 0$ (not possible)

Draw the phase potrait without solving the equation

$\frac{dT}{dt} = \lambda(T - T_\infty)$



$T_e = T_\infty$

$T > T_\infty$ → $\boxed{\frac{dT}{dt} < 0}$ → always.

$T < T_\infty$ → $\frac{dT}{dt} > 0$ → always.

→ Monotonous behaviour

→ 1st order systems always have monotonous behaviour → can never have oscillatory behaviour

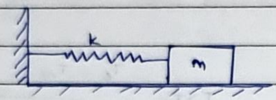
→ Autonomous: $\frac{dx}{dt} = f(x)$ are called autonomous equations & the system is called autonomous system

dynamics of the system cannot be changed by external influence

⇒ Desmos.com / calculator → for plotting.

Q) k = spring constant

m = mass



Free undamped system : — dynamical system, autonomous.

$$m\frac{d^2x}{dt^2} + kx = 0$$ → 2nd order system (This equation can be written into 2 1st order ODE) → linear system

Free vibration with damping,

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

→ Derivative on LHS and all the other things on RHS.

$$\frac{d^n x}{dt^n} = f(x) \rightarrow \text{autonomous.}$$

— dynamical variable :- x, y → for higher order system dynamical variable is a vector

→ $m\frac{d^2x}{dt^2} + kx = 0$

$$\boxed{\frac{dx}{dt} = y} \rightarrow (1)$$

$$\boxed{\frac{dy}{dt} = -\frac{k}{m}x} \rightarrow (2)$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{autonomous system}$$

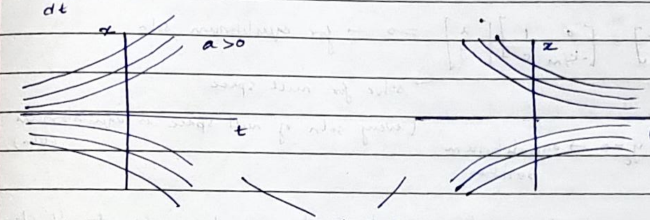
$\underline{x}$ → dynamical variable

$$\boxed{\frac{d}{dt}\underline{x} = \underline{\underline{A}}\,\underline{x}} \rightarrow \text{autonomous}$$

→ linear 1st order ODE $\frac{dx}{dt} = ax$, $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ → autonomous system

→ Dynamics depends only on a

$\frac{dx}{dt} = ax$



bifurcation

bifurcation parameter = a

bifurcation point = a = 0

— a = eigen value of the system

⇒ The solution of $\frac{dx}{dt} = \underline{A}\,\underline{x}$ is

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v_i}$$ → corresponding eigen vector

Mathematica

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda_1 = -i\sqrt{\frac{k}{m}}$ $\lambda_2 = i\sqrt{\frac{k}{m}}$

$v_1 = \begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-iat} \begin{bmatrix} i/a \\ 1 \end{bmatrix} + c_2 e^{iat} \begin{bmatrix} -i/a \\ 1 \end{bmatrix}$$

$a = \sqrt{\frac{k}{m}}$

↓ convert this imaginary eqn ~~[illegible]~~ to real

⇒ $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ → for equilibrium soln

→ solve for null space

(every soln of null space is equilibrium soln)

$x_e = 0,\ y_e = 0$ → equilibrium position

Once we reach equilibrium soln — dynamical variable doesn't change.

→ $\underline{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

⇒ Convert the dynamical equations ~~to~~ into matrix equations & analyze

1. the equilibrium soln
2. the phase potrait
3. the stability of the system
4. The effect of different parameters on the dynamical behaviour of the system.

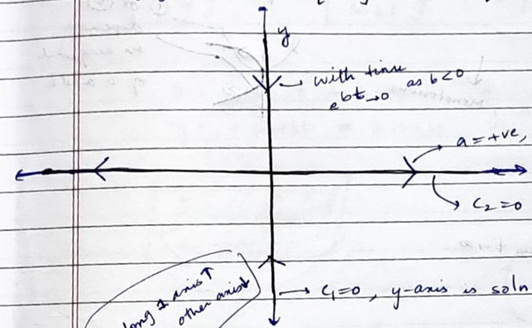
⇒ phase potrait — various components of the system change with time (variables)

Case-1

⇒ $\underline{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $\left.\begin{matrix} \lambda_1 = a \\ \lambda_2 = b \end{matrix}\right\}$ purely real.

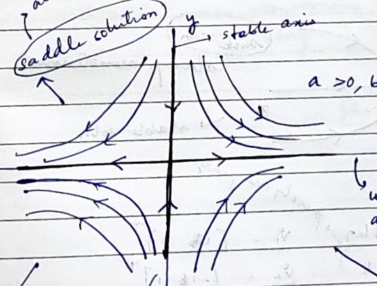
$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{at}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a > 0$
$b < 0$.



along one axis ↑ along other axis ↓

saddle solution



we are plotting for -ve time also

for $a<0, b>0$ ↳ arrows will be reversed

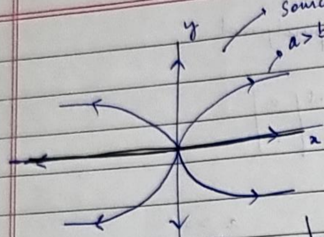
$x = c_1 e^{at}$
$y = c_2 e^{bt}$

draw this on desmos

monotonous.

phase potrait

⇒ eigen values will govern the fate of the system



Source solution

$a > b$

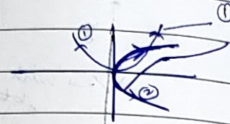
$a > 0$

$b > 0$

(unstable always)

$$\boxed{\frac{dy}{dx} = \frac{c_2 b e^{(b-a)t}}{c_1 a}}$$

monotonous.

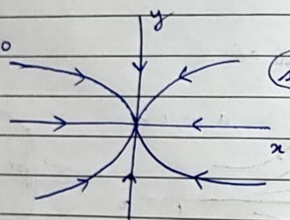


① or ② depends on magnitude of a and b

if $a > b$ $\frac{dy}{dx} \to 0$ as $t \to \infty$

if $b > a$ $\frac{dy}{dx} \to \infty$, as $t \to \infty$.

→ $a < 0$, $b < 0$



sink soln

always monotonous

→ stable soln.

Case-2 → $\underline{A} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ $\quad \left.\begin{matrix}\lambda_1 = -ib \\ \lambda_2 = ib\end{matrix}\right\}$ purely imaginary $\quad \underline{v_1} = [i, 1]^T$, $\underline{v_2} = [-i, 1]^T$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-ibt} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{ibt} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1(\cos bt - i\sin bt) \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2(\cos bt + i\sin bt) \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = (c_1+c_2) \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} + i \begin{bmatrix} (c_1-c_2)\cos bt \\ -(c_1-c_2)\sin bt \end{bmatrix}$$

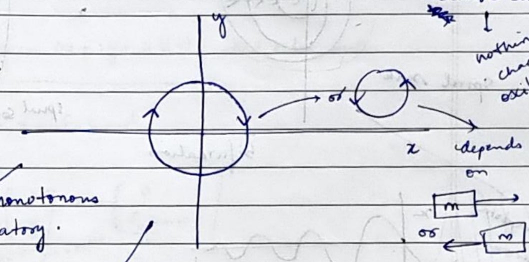
Bifurcation :- Saddle to source
source to saddle
Source to sink

⇒ $z_1 + iz_2$ → is solution then z_1 & z_2 both will be soln.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} + \beta \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix}$$ → linear combination of z_1 & z_2

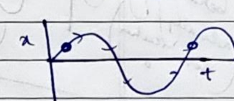
$\left.\begin{matrix} x = \alpha \sin bt + \beta \cos bt \\ y = \alpha \cos bt - \beta \sin bt \end{matrix}\right\}$ → $x^2 + y^2 = \alpha^2 + \beta^2$ → circle → centre solution

nothing changes oscillates



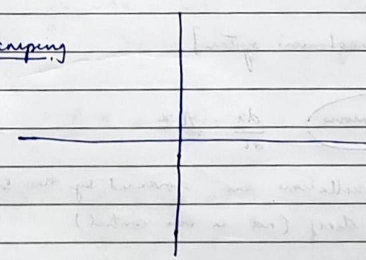
non-monotonous oscillatory.

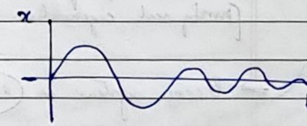
phase potrait



oscillating

→ Damping





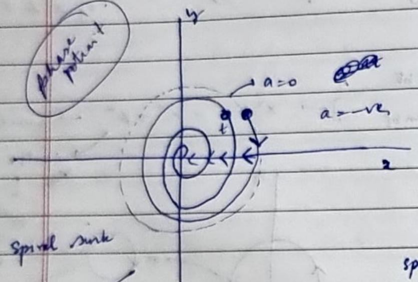
Case-3 $\underline{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $\quad \lambda_1 = a - ib$, $\lambda_2 = a + ib$ ↳ contribution of both real & imaginary.

$\underline{v_1} = [i, 1]^T$

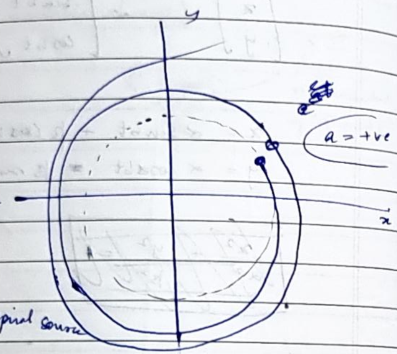
$\underline{v_2} = [-i, 1]^T$

$$x = e^{at}(c_1 \sin bt + c_2 \cos bt)$$
$$y = e^{at}(c_1 \cos bt - c_2 \sin bt)$$

phase portrait

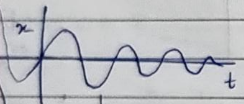


spiral sink



spiral source

bifurcation



asymptotic behaviours actually never reaches 0

→ decay depends on m, c, k
(delaying the decay by maintaining the same system is not possible)

→ Spring with no oscillations? (displacement without oscillation)
① when damping is very high
② brakes of cycle
[purely real eigenvalues → monotonous system]

→ Becoz system is autonomous $\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x}$
→ Oscillations are governed by the system itself (not in our control)

→ to change the system dynamics → non-autonomous → impose

2) $M\frac{d^2x}{dt^2} + kx = \boxed{F_0 \sin \omega t}$ → forcing functions
→ like superimposing non-autonomous on autonomous
→ forced.

Forced vibration without damping

→ Forced vibrations with damping:-

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \sin \omega t$$

⇒ $y = \frac{dx}{dt}$ → $\frac{dx}{dt} = 0x + 1y + 0f(t)$ 2nd order system

$$M\frac{dy}{dt} + cy + kx = F_0 \sin \omega t$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector}} + \underbrace{\begin{bmatrix} 0 \\ \frac{f_0 \sin \omega t}{m} \end{bmatrix}}_{\text{vector}}$$

$$\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{b}(t)$$

→ $g(t)$
→ making the system non-autonomous

→ $\frac{dx}{dt} = ax + b(t)$

$\frac{dx}{dt} - ax = b(t)$
→ use integrating factor to solve

$$\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{b}(t)$$
$$\frac{dx}{dt} - \underline{\underline{A}}\,\underline{x} = \underline{b}(t)$$

This method can be used to solve autonomous & non-autonomous systems.

pre-Multiply both the sides by $\underline{\underline{P}}^{-1}$

$$\frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) - \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{P}}^{-1}\underline{b}(t)$$

$\underline{P^{-1} x} = \underline{y}$, $P^{-1} b(t) = g(t)$; $P^{-1}AP = \Lambda$ → lambda

$\frac{d}{dt}(y) - \Lambda y = g(t)$ By doing all this, identity of eigen values is preserved.

$\frac{d}{dt}(x) - A x = b(t)$

⇒ If P is a non-singular matrix $P^{-1}AP = B$ then A and B are similar matrices.

→ The operation $P^{-1}AP = B$ is called similarity of transformation
→ Similar matrices have same eigen values.

→ $P = [v_1 \mid v_2]$

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

→ If x is an eigenvector of A with an eigenvalue λ the $P^{-1}x$ will be the eigenvector of B with same eigenvalue λ

$e^{-\Lambda t} \frac{dy}{dt} - e^{-\Lambda t} \Lambda y = e^{-\Lambda t} g(t)$

$e^x = 1 + x + \frac{x^2}{2!} \ldots$

$e^{\Lambda t} = I + \Lambda t + \frac{1}{2}\Lambda t^2 \ldots$

Integration of matrix – integrate all elements
differentiation of matrix → different " "

$\int \frac{d}{dt}[y e^{-\Lambda t}] = \int e^{-\Lambda t} g(t)\, dt$

Q) Solve autonomous problem using similarity solution.

→ eigen vectors of A → v_1, v_2, v_3 ...

$P = [v_1 \mid v_2 \mid \ldots]$

made up of eigen vectors.

→ solve for y → then multiply by P

$\frac{dx}{dt} = A x$

$\frac{d}{dt}(P^{-1}x) = P^{-1}A x$

$\frac{d}{dt}(P^{-1}x) = (P^{-1}AP)(P^{-1}x)$

$\frac{dy}{dt} = \Lambda y$

$e^{-\Lambda t} \frac{dy}{dt} - \Lambda e^{-\Lambda t} y = e^{-\Lambda t} g(t)$

$y e^{-\Lambda t} = \int \underbrace{e^{-\Lambda t} g(t)}_{\text{vector}} dt$

integrate individual elements.

Q) $m \frac{d^2x}{dt^2} + kx = 0$

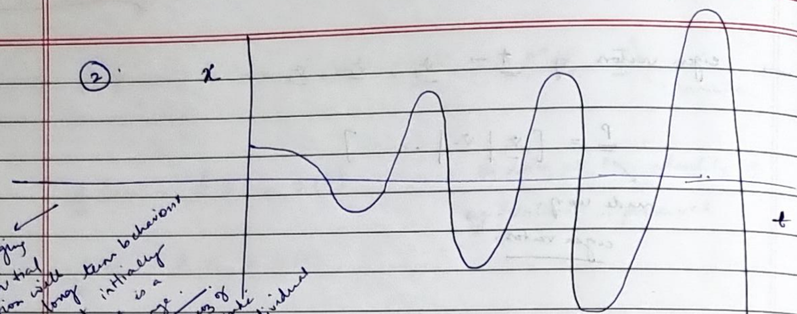
$\frac{d^2x}{dt^2} + x = 0$ → ① $x(t) = c_1 \sin(t) + c_2 \cos(t)$

$\frac{d^2x}{dt^2} + x = \sin t$ → ② $x(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2} t \cos t$

no damping → unstable

because of t the oscillatory behaviour will ↑ with time

②



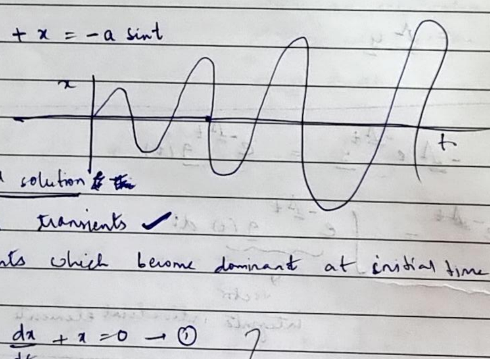
Changing the initial condition will not change long term behaviour but initially there is a change ↓ because of magnitude of individual terms)

→ No we have a method using which we can control the rate of growth of the function

damping → $e^{at}(c_1 \sin t + c_2 \cos t)$ → grows faster

no damping → eqn ② → ~~rate of gro~~ growth rate is slower

a) $\frac{d^2x}{dt^2} + x = -a \sin t$



Analytical solution ~~to the~~

→ Initial transients ✓

→ Components which become dominant at initial time & long time

Q) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$ → ①

$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = \sin t$ → ② } damping

Oscillations can be sustained by using some external force

① $x(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right)$

② $x(t) = c_1 e^{-t/2} \sin \frac{\sqrt{3}}{2}t + c_2 \cdot e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) - \cos t$

long term

→ 1st order system – always monotonous.

→ 2nd order may have non-monotonous behaviour.

In 2nd order non-damped

with external force ↓ sustained oscillation

with external force ↓ can increase continuously

} All of these depends on eigen values

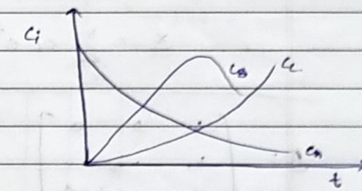
Q) $A \to B \to C$

Initial conc: C_{A0}, C_{B0}, C_{C0}.

$\frac{dC_A}{dt} = -k_1 C_A$

$\frac{dC_B}{dt} = k_1 C_A - k_2 C_B$

$\frac{dC_C}{dt} = k_2 C_B$



3rd order → coz 3 ODE

$\boxed{C_A(t) = C_{A0} e^{-k_1 t}}$

$\frac{d}{dt}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$

$\frac{dC_B}{dt} = k_1 C_A - k_2 C_B$

$\frac{dC_B}{dt} + k_2 C_B = k_1 C_{A0} e^{-k_1 t}$

$e^{k_2 t}\frac{dC_B}{dt} + e^{k_2 t} k_2 C_B = k_1 C_{A0} e^{(k_2-k_1)t}$

$\int \frac{d}{dt}\left(C_B e^{k_2 t}\right) dt = \int k_1 C_{A0} e^{(k_2-k_1)t} dt$

$C_B e^{k_2 t} = \frac{k_1}{k_2-k_1} C_{A0} e^{(k_2-k_1)t} + \alpha$

at $t=0$: $C_{B0} = \frac{k_1}{k_2-k_1} C_{A0} + \alpha$

$\boxed{\alpha = C_{B0} - \frac{k_1}{k_2-k_1} C_{A0}}$

~~$C_B = \left(\frac{k_1}{k_2-k_1}\right) C_{A0} e^{-k_1 t} + \left(C_{B0} - \frac{k_1}{k_2-k_1} C_{A0}\right)$~~

$\boxed{C_B = C_{B0} e^{-k_2 t} + \frac{C_{A0} k_1}{k_2-k_1}\left(e^{-k_1 t} - e^{-k_2 t}\right)}$

$\frac{dC_C}{dt} = k_2 C_B$

~~$\frac{d}{dt}$~~ dynamical variable :- $\underline{x} = [C_A \quad C_B \quad C_C]^T$

dynamical eqn :- $\frac{d\underline{x}}{dt} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \underline{x}$

→ eqn is linear

k_i = const

assume :- $[C_{A1} \quad C_{B1} \quad C_{C1}]^T$ + $[C_{A2} \quad C_{B2} \quad C_{C2}]^T$ ✓

$\alpha [C_{A1} \quad C_{B1} \quad C_{C1}]$

to prove linearity

order = 3.

$\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x}$ → autonomous system

$\underline{x} = \sum_{i=1}^{3} c_i e^{+\lambda_i t} \underline{v_i}$

$\underline{A} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix}$

$\lambda_1 = 0$ // $v_1 = (0, 0, 1)$

$\lambda_2 = -k_1$ $v_2 = \left(\frac{-k_1+k_2}{k_2}, \frac{-k_1}{k_2}, 1\right)$

$\lambda_3 = -k_2$ $v_3 = (0, -1, 1)$

$\frac{d}{dt}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

equilibrium soln

$\begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -k_1 & 0 & 0 & 0 \\ k_1 & -k_2 & 0 & 0 \\ 0 & k_2 & 0 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} -k_1 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & k_2 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -k_1 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$-k_1 C_{Ae} = 0$

$-k_2 C_{Be} = 0$

$\boxed{C_{Ae} = 0 \quad C_{Be} = 0}$

$\begin{bmatrix} C_{Ae} \\ C_{Be} \\ C_{Ce} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}$

C_{Ce} → arbitrary.

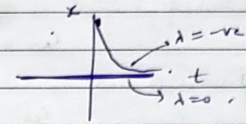
C_{Ce} → depends on C_{A0}, C_{B0}

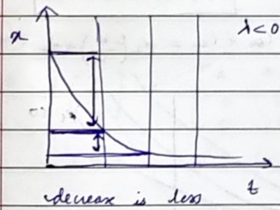
~~Both~~ λ_2, λ_3 → both are −ve ∴ system is stable

$\frac{dx}{dt} = \lambda x$

$\boxed{x(t) = x(0)\, e^{\lambda t}}$

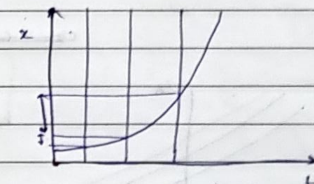
$\frac{dx}{dt} = 0$ $\boxed{x = \text{const}}$





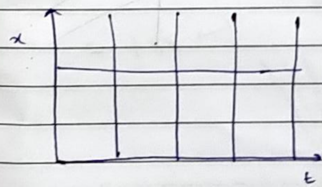
decrease is less as we go with time

$t \to \infty, \; x \to 0$



as $t \to \infty$, vertical gain → ∞

→ gain ↑ with time



λ=0 → something in the system is absolute const (not changing with time) (after a long time)

C_C at $t \to \infty$ — $C_{Ce} = C_{A0} + C_{B0} + C_{C0}$

$$\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 e^{-k_1 t} \begin{bmatrix} \frac{k_2 - k_1}{k_2} \\ -k_1/k_2 \\ 1 \end{bmatrix} + \alpha_3 e^{-k_2 t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{C_A = \alpha_2 e^{-k_1 t} \frac{k_2 - k_1}{k_2}}$$

$$C_{A0} = \alpha_2 \frac{k_2 - k_1}{k_2}$$

$$\alpha_2 = \frac{C_{A0} k_2}{k_2 - k_1}$$

$$C_B = \alpha_2 e^{-k_1 t} \left(\frac{-k_1}{k_2}\right) - \alpha_3 e^{-k_2 t}$$

$$C_C = \alpha_1 + \alpha_2 e^{-k_1 t} + \alpha_3 e^{-k_2 t}$$

$$C_{B0} + C_{C0} = \frac{C_{A0} k_2}{k_2 - k_1}\left(\frac{-k_1}{k_2}\right) + \frac{C_{A0} k_2}{k_2 - k_1} + \alpha_1$$

$$C_{B0} = \alpha_2 \frac{-k_1}{k_2} - \alpha_3$$

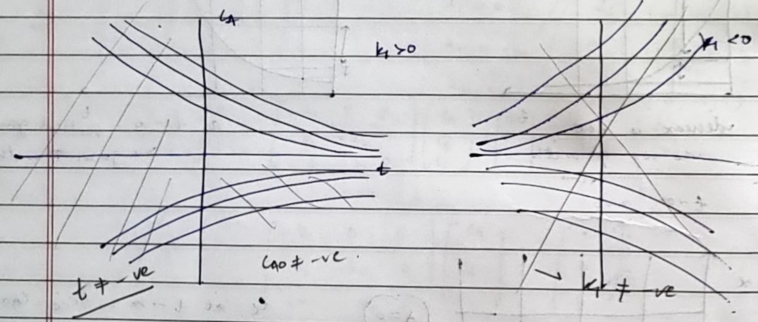
$$\boxed{C_A = C_{A0} e^{-k_1 t}}$$

$$C_{C0} = \alpha_1 + \alpha_2 + \alpha_3$$

$$\boxed{\alpha_1 = C_{B0} + C_{C0} - C_{A0}}$$

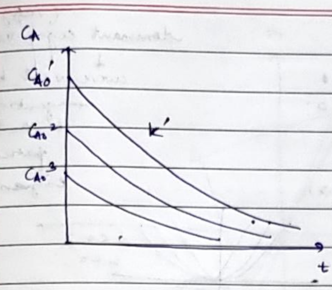
~~$C_{B0} + C_{A0}$~~

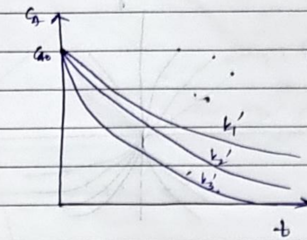
$$C_{B0} + C_{C0} = \alpha_2\left(\frac{-k_1}{k_2}\right) + \alpha_2 + \alpha_1$$



$$\alpha_2\left(\frac{k_2 - k_1}{k_2}\right) + \alpha_1$$

$$C_{A0} + \alpha_1$$





$k_1' < k_2' < k_3'$

$$C_B(t) = \frac{k_1 C_{A0}}{k_2 - k_1}\left(e^{-k_1 t} - e^{-k_2 t}\right) + C_{B0} e^{-k_2 t}$$

$\frac{k_1}{k_2}$ → ratio matters [B is desired product]

$\frac{C_{A0}}{C_{B0}}$, also matters

rate of reaction $= k_1 C_A^n$

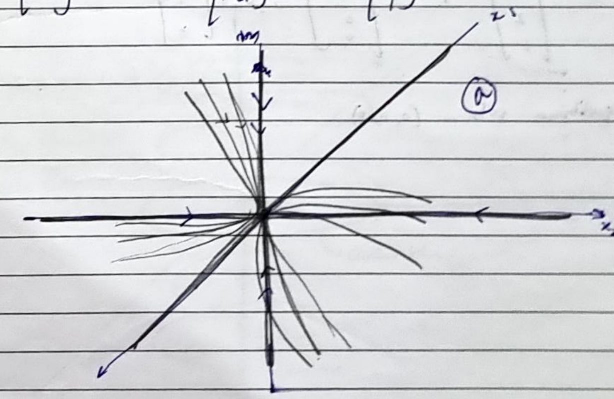
depends on both k_1, C_A

Q) →

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = c$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

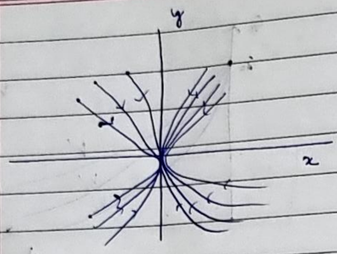


$a < 0$
$b < 0$ } stable system
$c < 0$

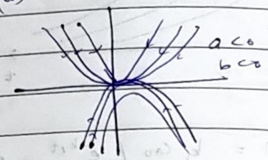
(0,0,0) equilibrium soln

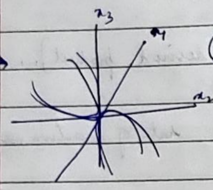
(magnitude) dominant eigen value
↓
curve will be tangential to that particular eigen vector

$a<0$
$b<0$.

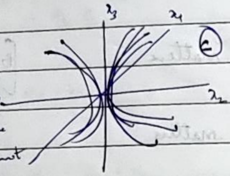


(or)

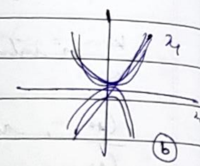




(a)
↓
eigen value dominant



(c)



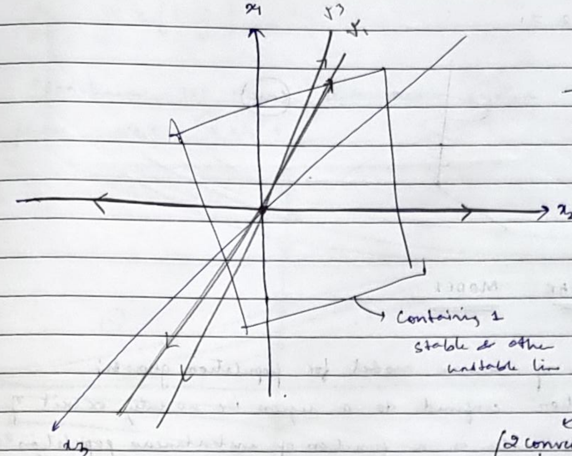
(b)

Q) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1$

$\underline{v_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \underline{v_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

equilibrium soln → (0, 0, 0)



→ only 1 line is stable
→ the plane is unstable.

join (0,0,0)
(3,2,1)

Containing 1 stable & other unstable line

Consider a plane containing stable line and unstable line

stable subspace

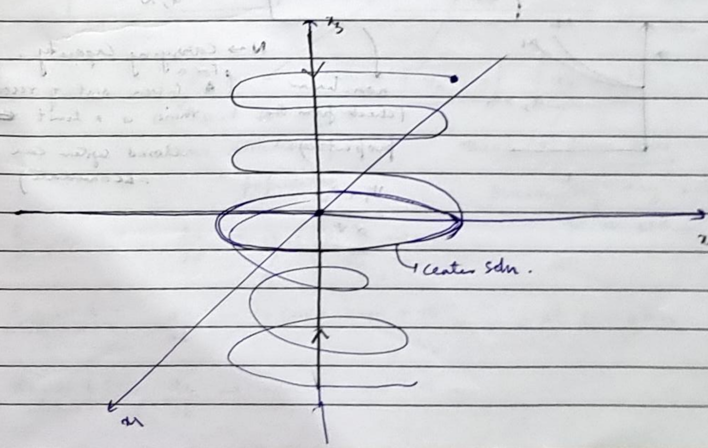
(2 converging lines) unstable subspace

Q) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

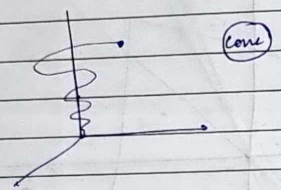
$\lambda_1 = i \qquad \lambda_2 = -i \qquad \lambda_3 = -1$

$\underline{v_1} = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} \qquad \underline{v_2} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \qquad \underline{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(sustained oscillations)
↓
bcoz $\lambda = \pm i$



if $\lambda = 2 \pm i$



NON-LINEAR MODEL

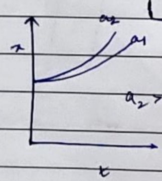
Assumptions of linear model for population growth:

1. Population confined to a region i.e. no entry or exit of members
2. growth rate is a function of instantaneous population
3. No death, only from the birth, no explicit birth rate term

$$\frac{dx}{dt} = ax \rightarrow \text{only 1 parameter}$$

4. growth rate proportional to the instantaneous population only for small population.
5. Negative growth rate at large population so as to limit the population

$$\boxed{\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)} \rightarrow 2 \text{ parameters } a, N$$



non-linear (check from the properties)

$v_1 + v_2$

αv_1

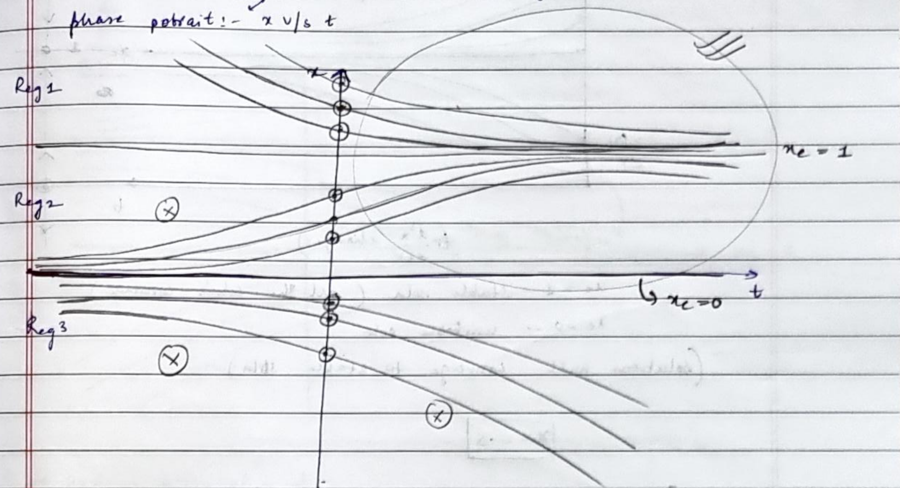
$N \rightarrow$ Carrying Capacity. (For a Given amt of resources, there is a limit a closed system can accomodate)

(For higher order system, draw the phase potrait considering all the dynamical variable)

$\rightarrow \frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$

Non-linear, 1st order, autonomous system

phase potrait :- x v/s t



$y = \frac{x}{N}$

$$N \frac{dy}{dt} = a\, yN(1-y)$$

$$\boxed{\frac{dy}{dt} = a\, y(1-y)} \implies \frac{dx}{dt} = a\, x(1-x)$$

use this as the model with max carrying capacity as 1

$a > 0$

we want to know evolution of population

$\boxed{x_e = 0}$ $\boxed{x_e = 1}$

these are phase lines

Eventually any solution will go to either $x_e = 0$ or $x_e = 1$

Region 1 :- $\frac{dx}{dt} \rightarrow \underset{a}{(+)}\underset{x}{(+)}\underset{1-x}{(-)} \rightarrow -ve$

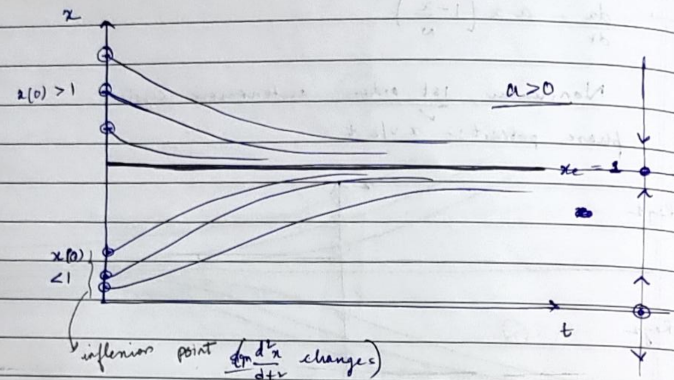
Region 2 : $\frac{dx}{dt} \rightarrow (+)(+)(+) \rightarrow +ve$

at equilibrium soln $\frac{dx}{dt} \rightarrow 0$

Region 3 :- $\frac{dx}{dt} \rightarrow (+)(-)(+) \rightarrow -ve$

$$a(ax)(1-x) - 2ax(ax)(1-x) = 0$$

$$x = 0, 1, \tfrac{1}{2}$$



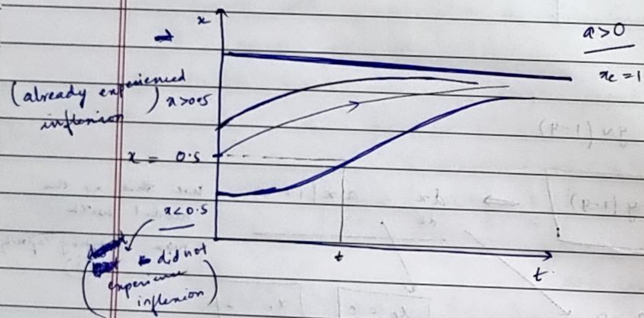
inflexion point ($\frac{d^2x}{dt^2}$ changes)

$x_e = 1$ → stable soln (all the soln's converge)

$x_e = 0$ → unstable soln

(Solutions will converge to stable soln)

$$\boxed{x = 0.5}$$



$$\frac{dx}{dt} = ax - ax^2 = f(x)$$

$$\left.\frac{df}{dx}\right|_{x_e} = \left. a - 2ax \right|_{x_e}$$

$$\left.\frac{df}{dx}\right|_{x_e} = a - 2a = -a$$

$$\left.\frac{df}{dx}\right|_{0} = a$$

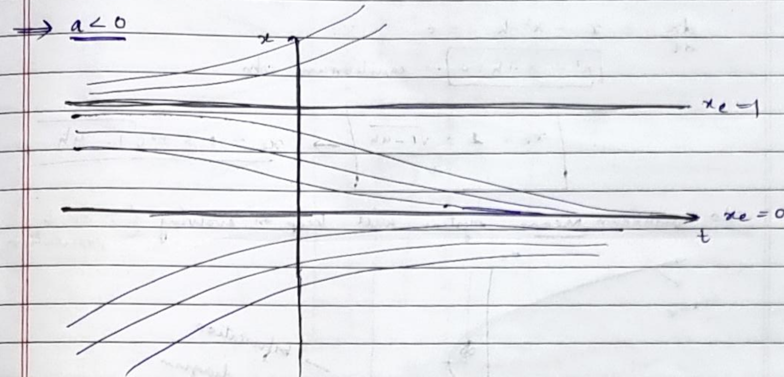
$$x = \frac{1}{ce^{-at}+1}$$

$$x(0) = \frac{1}{c+1} \qquad c+1 = \frac{1}{x(0)} \qquad c = \frac{1}{x(0)} - 1$$

$\frac{df}{dx} = +ve$ → at equilibrium soln → unstable soln

$\frac{df}{dx} = -ve$ → at equilibrium soln → stable soln

⇒ $a < 0$



⇒ $a = 0$ → bifurcation pt

Q) $$\frac{dx}{dt} = ax(1-x)$$

$$\frac{dx}{x(1-x)} = a\,dt$$

$$\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int a\,dt$$

$$\ln\left(\frac{x}{1-x}\right) = at + c$$

$$\frac{x}{1-x} = Ce^{at}$$

$$x = Ce^{at}(1-x)$$

$$\boxed{x = \frac{Ce^{at}}{1+Ce^{at}}} \qquad x = \frac{x(0)\,e^{at}}{1 - x(0) + x(0)\,e^{at}}$$

Q) $\frac{dx}{dt} = x(1-x) - h$ h = exit of members at a const rate

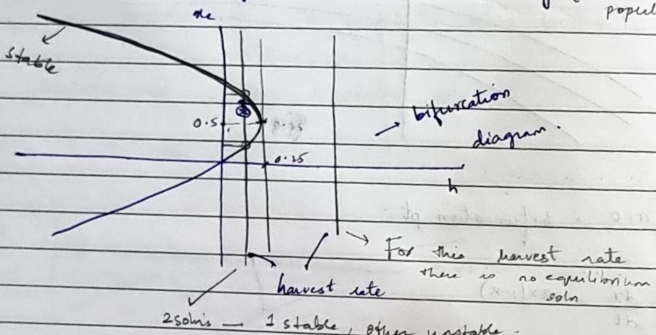
x = population of fish in the pound.

$\frac{dx}{dt} = x - x^2 - h = 0$

$\boxed{x^2 - x + h = 0}$ → equilibrium soln

$\boxed{x_e = \frac{1 \pm \sqrt{1-4h}}{2}}$ → $x_e = 0.5 \pm 0.5\sqrt{1-4h}$

No equilibrium Means :- system will keep on evolving (↑ or ↓ population)



$f = x(1-x) - h$ (h – const)

$\frac{df}{dx} = 1 - 2x$

$x > 0.5$ → stable

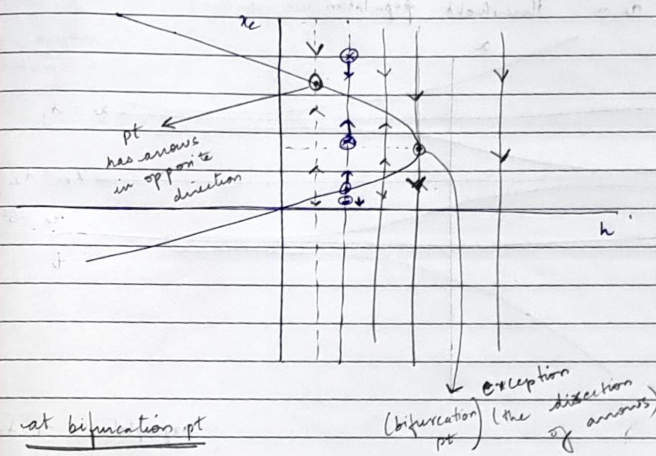
$x < 0.5$ → unstable.

h → bifurcation parameter

$h < 0.25$ → u get equilibrium

$h > 0.25$ → No equilibrium

bifurcation diagram → variation of equilibrium dynamical variable with bifurcation parameter.



at bifurcation pt

1 side u

→ $\frac{dx}{dt} = f(x) = ax$

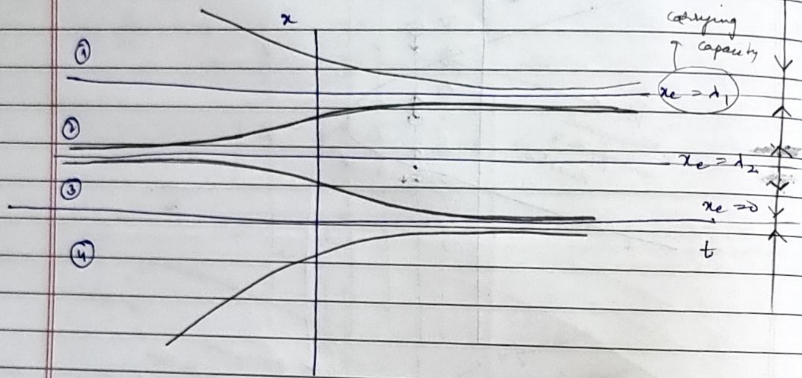
$\frac{df}{dx} = a = 0$ } $\frac{df}{dx}\Big|_{x_e} > 0$ Unstable

$\frac{df}{dx}\Big|_{x_e} < 0$ → stable.

2) $\frac{dx}{dt} = -ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right)$

λ_1 = carrying capacity $\qquad 0 < \lambda_2 < \lambda_1$

λ_2 = threshold population (min population)



① →

asymptotes → $\lambda_1, \lambda_2, 0$

(Extinction when the initial population is less than the threshold population)