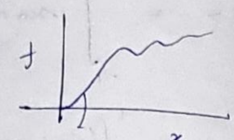
$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin n\pi x + \sum_{n=1}^{\infty} b_n \cos n\pi x$ → fourier series

→ why is fourier series used?
(or) any other expansion



you have to write the function in terms of cos, sin

a_n = weight function

~~Adj~~

Adjoint operator

$$\underline{A} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$\langle \underline{A}\underline{u}, \underline{v} \rangle = 14$

$\langle \underline{u}, \underline{A}\underline{v} \rangle = -2$

→ $\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\langle \underline{A}\underline{u}, \underline{v} \rangle = \left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= a_{11}u_1v_1 + a_{12}u_2v_1 + a_{21}u_1v_1 + a_{22}u_2v_2$$

$$= u_1(a_{11}v_1 + a_{21}v_2) + (u_2(a_{12}v_1 + a_{22}v_2))$$

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11}v_1 + a_{21}v_2 \\ a_{12}v_1 + a_{22}v_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= \langle \underline{u}, \underline{A}^* \underline{v} \rangle$$

$$\langle \underline{A}\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{A}^* \underline{v} \rangle$$

$\langle u, \underline{A}^* v \rangle = 14$

$\underline{A}^*$ is adjoint operator of $\underline{A}$

Every matrix can be an operator

→ $\circledast : \hat{I} \times V^{2\times1} \longrightarrow V^{2\times1}$ → generate new vectors within the same vector space

(operator maps the vector) vector

$\circledast : \hat{I} \times V^{m\times n} \rightarrow V^{n\times p}$

$2\times3 \quad 3\times1$ → 2×1 original vector space

Q) $\underline{A} = \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix}$

$\underline{u} = [u_1 \ u_2]^T$

$\underline{v} = [v_1 \ v_2]^T$

$$\langle \underline{A}\underline{u}, \underline{v} \rangle = \left\langle \begin{bmatrix} 1 & -2i \\ 3 & i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} u_1 - 2iu_2 \\ 3u_1 + iu_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= (u_1 - 2iu_2)v_1 + (3u_1 + iu_2)v_2$$

$$= u_1(v_1 + 3v_2) + u_2(-2iv_1 + iv_2)$$

$= \left< \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 + 3v_2 \\ -2iv_1 + iv_2 \end{bmatrix} \right>$

X

$= \left< \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -2i & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right>$

$\underline{\underline{A}}^* = \begin{bmatrix} 1 & 3 \\ +2i & -i \end{bmatrix}$

complex number
↓
Inner product is different
take complex conjugate

$\underline{\underline{A}}^* = \bar{A}^T$ → need not derive this

a) $\underline{\underline{A}} = \begin{bmatrix} 3 & 2+i & 2 \\ 2-i & 2 & -i \\ 2 & i & 1 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 3 & 2-i & 2 \\ 2+i & 2 & i \\ 2 & -i & 1 \end{bmatrix}$

$\underline{\underline{A}}^* = \bar{\underline{\underline{A}}}^T = \begin{bmatrix} 3 & 2+i & 2 \\ 2-i & 2 & -i \\ 2 & i & 1 \end{bmatrix}$

Hermitian matrix
(Hermitian conjugate)

$\underline{\underline{A}} = \bar{\underline{\underline{A}}}^T = \underline{\underline{A}}^*$

$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{\underline{A}}^* \underline{v} \rangle$

if $\underline{\underline{A}}^* = \underline{\underline{A}}$

$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{\underline{A}}\,\underline{v} \rangle$ → self adjoint operator

d) The eigen values of a self adjoint operator are <u>real</u>

$\langle \underline{\underline{A}}\,\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{\underline{A}}\,\underline{v} \rangle$

let $\underline{u}$ be an eigen vector of $\underline{A}$ with the corresponding eigen value as λ

find eigen vector

$\boxed{\underline{A}\,\underline{u} = \lambda \underline{u}}$ → scaling

$\langle \underline{A}\,\underline{u}, \underline{u} \rangle = \langle \lambda \underline{u}, \underline{u} \rangle = \lambda \langle \underline{u}, \underline{u} \rangle \rightarrow (1)$

$\langle \underline{u}, \underline{A}\,\underline{u} \rangle = \langle \underline{u}, \lambda \underline{u} \rangle = \bar{\lambda} \langle \underline{u}, \underline{u} \rangle \rightarrow (2)$

(1) = (2)
→ for self adjoint

$\lambda = \bar{\lambda}$

$\Rightarrow \lambda \in \mathbb{R}$

→ The eigen vectors of a self adjoint operator corresponding to distinct eigen values are always orthogonal.

$\underline{A}\,\underline{u} = \lambda_u \underline{u}$

$\underline{A}\,\underline{v} = \lambda_v \underline{v}$

$\langle \underline{A}\,\underline{u}, \underline{v} \rangle = \langle \lambda_u \underline{u}, \underline{v} \rangle = \lambda_u \langle \underline{u}, \underline{v} \rangle \rightarrow (1)$

$\langle \underline{u}, \underline{A}\,\underline{v} \rangle = \lambda_v \langle \underline{u}, \underline{v} \rangle \rightarrow (2)$

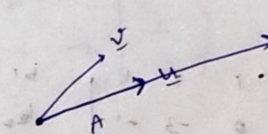
(1) = (2)

→ $\boxed{\langle \underline{u}, \underline{v} \rangle = 0}$

iff $\langle \underline{u}, \underline{v} \rangle = 0$

⇒ $A = \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\underline{u} = \begin{bmatrix} 9 \\ 369 \end{bmatrix}$ $\underline{A}\,\underline{u} = \underline{v}$



if $\underline{u}$ is eigen vector → $\underline{A}\,\underline{u} = \lambda \underline{u}$

eigen vector
↓
special vector such that direction is same never changes

eigenvector ✓ eigen vector ⓧ eigenvalue

a) $\frac{d^2 f}{dx^2} + \alpha f = 0$

$\hat{L} = \frac{d^2}{dx^2} + \alpha$ — it is a self adjoint operator

$\langle \hat{L}f, g \rangle = \langle f, \hat{L}g \rangle$

$\int_0^1 \left(\frac{d^2 f}{dx^2} + \alpha f\right) g \, dx$

$= g \frac{df}{dx}\Big|_0^1 - f \underbrace{\frac{dg}{dx}\Big|_0^1}_{\text{force } g' = 0} + \dots$

~~$g \frac{df}{dx} + \alpha f$~~

if $f'(0) = f'(1) = 0$

$g'(0) = g'(1) = 0$ — neumann

a) Identify the solvability condition & determine the range space of the following equation

$x_1 + x_2 + x_3 = b_1$

$2x_1 - x_2 + x_3 = b_2$

$x_1 - 2x_2 = b_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & -1 & 1 & b_2 \\ 1 & -2 & 0 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -3 & -1 & b_2 - 2b_1 \\ 0 & -3 & -1 & b_3 - b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - (b_2 - 2b_1) \end{array}\right]$$

$b_3 - b_1 - b_2 + 2b_1 = 0$

$b_3 - b_2 + b_1 = 0$

$b_3 = b_2 - b_1$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_2 - b_1 \end{bmatrix} \Rightarrow \boxed{DOF = 2}$$

$b_1 = \alpha \quad b_2 = \beta$

dimension = 2

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

basis $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Fredholm's alternative method (range space)

$\underline{\underline{A}}\, \underline{x} = \underline{b}$ is solvable if $\langle \underline{b}, \underline{y} \rangle = 0 \;\forall\; \underline{y} : \underline{\underline{A}}^{*} \underline{y} = 0$ — null space of adjoint operator

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\underline{\underline{A}}^{*} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\underline{\underline{A}}^{*} \underline{y} = 0 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-3y_2 - 3y_3 = 0$$

$$y_1 + 2y_2 + y_3 = 0$$

$$\boxed{y_3 = -y_2}$$

$$y_1 + 2y_2 - y_2 = 0$$

$$\boxed{y_1 = -y_2} \qquad y_2 = \alpha$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ -\alpha \end{bmatrix} \qquad \text{DOF} = 1$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\langle \underline{b}, \underline{y} \rangle = \left\langle \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \alpha \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\rangle = 0$$

$$-b_1 + b_2 - b_3 = 0 \qquad \boxed{\text{DOF} = 2}$$

$$b_3 = b_2 - b_1$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvalue Problems

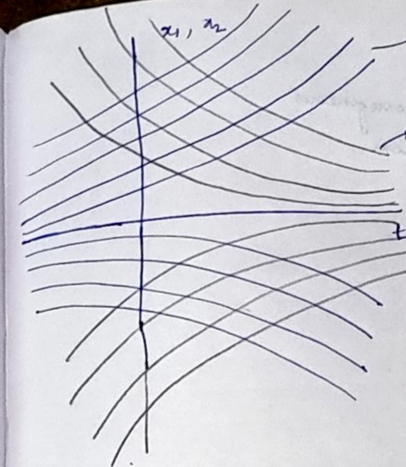
$$\frac{dx}{dt} = 2x \Rightarrow x = e^{2t} \times c_1$$

$$\frac{dx}{dt} = -3x \Rightarrow x = e^{-3t} \times c_2$$

$$\rightarrow \left.\begin{aligned} \frac{dx_1}{dt} &= 2x_1 \\ \frac{dx_2}{dt} &= -3x_2 \end{aligned}\right\} \text{ simultaneous equations}$$

$$\{x_1, x_2\} = \{c_1 e^{2t}, c_2 e^{-3t}\}$$

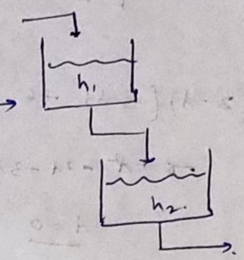
$$\hookrightarrow \underline{x}$$



points of intersection are the solution of the simultaneous equations (This will make a linear vector space)

$$\Rightarrow \left.\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + 3x_2 \\ \frac{dx_2}{dt} &= -2x_1 - 3x_2 \end{aligned}\right\}$$

Ex →

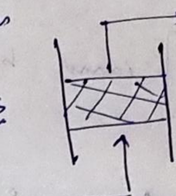


$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

→ linear, autonomous

no $f(t)$ on RHS

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$



x_i, P, T

gaseous stream in general

Packed bed.

$$\frac{dx}{dt} = ax \quad \text{eigenvalue} \rightarrow L\underline{x} = a\underline{x}$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \qquad \Big\} \; a \rightarrow \underline{\underline{A}} \text{ (similar equations)}$$

$$x = ce^{at}$$

$$\underline{x} = \underline{c}\, e^{\lambda t} \underline{v}$$

→ eigenvector

→ eigen value.

$$\underline{x} = e^{\lambda t} \underline{v}$$

$\ne f(t)$

→ corresponding eigen vector

$x = f(x, P, T)$

$P(x, T)$

$T(x, P)$

$$\frac{d}{dt}(\underline{x}) = \frac{d}{dt}(e^{\lambda t}\underline{v}) = \lambda e^{\lambda t}\underline{v}$$

→ const

$$\boxed{\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}}$$

$$\underline{\underline{A}}\,\underline{x} = (e^{\lambda t}\underline{v})\underline{\underline{A}} = e^{\lambda t}\underline{\underline{A}}\,\underline{v} = \lambda e^{\lambda t}\underline{v}$$

$$\frac{dx}{dt} = \underline{\underline{A}}\,x$$

$$\boxed{x = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i}$$ — only for homogeneous problem.

→ $\underline{\underline{A}} = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ -2 & -3-\lambda \end{vmatrix}$$

$$(2-\lambda)(-3-\lambda) + 6 = 0$$

$$-6 + \lambda^2 - 2\lambda + 3\lambda + 6 = 0$$

$$\lambda = 0 \qquad \lambda = -1$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ cannot be an eigenvector. (eigenvector only gives the direction)

$$\underline{A}\,\underline{v_1} = 0\,\underline{v_1}$$

$$\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2v_1 + 3v_2 = 0$$

$$v_2 = \alpha$$

$$v_1 = -\frac{3\alpha}{2}$$

$$\underline{V_1} = \alpha\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} = \alpha'\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\underline{A}\,\underline{V_2} = (-1)\,\underline{V_2}$$

$$\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} v \\ u \end{bmatrix} = -\begin{bmatrix} v \\ u \end{bmatrix}$$

$\cancel{2v_1 + 3v_2 + v_1 = 0}$ $\quad 2v + 3u = -v$

$$-2v - 3u = -u.$$

$$3v + 3u = 0 \qquad \boxed{v + u = 0}$$

$$-2v - 2u = 0$$

$$v + u = 0$$

$$\underline{V_2} = \beta\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{0t}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} ✓$$

$$x_1 = -3c_1 - c_2 e^{-t}$$
$$x_2 = 2c_1 + c_2 e^{-t}$$

all the solutions lie in the vector space.

1st order ODE ↳ always IVP

Q) $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 5x = 0$

$x(0) = c_1,\ \left.\dfrac{dx}{dt}\right|_0 = c_2$ → IVP

$x(0) = c_1,\ x(1) = c_2$ → BVP

$$D^2 + 6D + 5 = 0$$

$$(D = -1, -5)$$

$$x(t) = c_1 e^{-t} + c_2 e^{-5t}$$

(Method)

let $\frac{dx}{dt} = y$

$$\frac{dy}{dt} + 6y + 5x = 0$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 0-\lambda & 1 \\ -5 & -6-\lambda \end{vmatrix} = 0$$

$$+\lambda(6+\lambda) + 5 = 0$$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = -1, -5$$

$$\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_2 = -v_1$$

$$-5v_1 - 6v_2 = -v_2$$

$$v_1 + v_2 = 0$$

$$\underline{V_1} = \alpha\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_2 = -5v_1$$

$$-5v_1 - 6v_2 = -5v_2$$

$$v_2 + 5v_1 = 0$$

$$\underline{V_2} = \beta\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Advantages → 2nd method why?

Find derivative directly

nth order equation

$\Rightarrow$ Q) $\frac{d^2y}{dx^2} - y = 0 \Rightarrow y(x) = c_1 e^{x} + c_2 e^{-x}$

$\frac{d^2y}{dx^2} + y = 0 \Rightarrow y(x) = c_1 \cos x + c_2 \sin x$

Q) $\frac{dy}{dx} = 0y + 1z$

$\frac{dz}{dx} = y + 0z.$

$$\frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix}$$

$$\lambda = 1, -1$$

$$\underline{A}\,\underline{v_1} = 1\,\underline{v_1}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 = v_2$$

$\rightarrow$ $\underline{A}\,\underline{v_2} = \underline{v_2}$

$\lambda_1 = 1 \rightarrow \underline{v_1} = [1 \; 1]^T$

$\lambda_2 = -1 \rightarrow \underline{v_2} = [1 \; -1]^T$

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 e^{x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-x} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Q) $\frac{dy}{dx} = z$

$\frac{dz}{dx} = -1y$

$$\frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda_1 = i$

$\lambda_2 = i$

$$\begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} = \begin{bmatrix} iv_1 \\ iv_2 \end{bmatrix}$$

$\boxed{v_2 = iv_1}$ $\quad -v_1 = iv_2$

$$\underline{v_1} = \alpha \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -i\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} = \begin{bmatrix} -iv_1 \\ -iv_2 \end{bmatrix}$$

$v_2 = -iv_1$ $\quad \underline{v_2} = \beta \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 e^{ix}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-ix}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$y = c_1' \cos x + i c_2' \sin x$

→ $y = z_1 + i z_2$

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$a_2 \frac{d^2}{dx^2}(z_1 + iz_2) + a_1 \frac{d}{dx}(z_1 + iz_2) + a_0(z_1 + iz_2) = 0.$$

$$\underbrace{a_2 \frac{d^2 z_1}{dx^2} + a_1 \frac{dz_1}{dx} + a_0 z_1}_{=0} + i\underbrace{\left(a_2 \frac{d^2 z_2}{dx^2} + a_1 \frac{dz_2}{dx} + a_0 z_2\right)}_{=0} = 0.$$

so both z_1 & z_2 are original solution.

→ Re $= c_1' \cos x$

Im $= c_2' \sin x$

$$\boxed{y(x) = d_1 \cos x + d_2 \sin x}$$

⇒ Frobenius method

$$\frac{d^2y}{dx^2} - y = 0$$

$$\frac{d^2y}{dx^2} + y = 0$$

assume

$$y = \sum_{i=0}^{\infty} c_i x^{i+k}$$

$$\frac{dy}{dx} = \sum_{i=0}^{\infty} c_i (i+k) x^{i+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k-2} \rightarrow \text{substitute this in ODE}$$

$$\sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k-2} - \sum_{i=0}^{\infty} c_i x^{i+k} = 0$$

$$c_0 k(k-1) x^{k-2} + c_1 (k+1) k x^{k-1} + c_2 (2+k)(k+1) x^k +$$

$$c_3 (k+3)(k+2) x^{k+1} + c_4 (k+4)(k+3) x^{k+2} + \dots$$

$$- c_0 x^k - c_1 x^{k+1} - c_2 x^{k+2} \dots = 0$$

$\therefore$ $c_0 k(k-1) = 0$ & $c_1 (k+1)(k) = 0$ → bcoz on the RHS no terms are there for power $k-2, k-1$

$\Rightarrow k = 0$

$$\rightarrow x^k (c_2 (2+k)(k+1) - c_0) + x^{k+1} (c_3 (k+3)(k+2) - c_1) +$$

$$x^{k+2} (c_4 (k+4)(k+3) - c_2) \dots = 0$$

→ Individual coeff $= 0$

$$c_2 (2+k)(k+1) - c_0 = 0$$

$$c_3 (k+3)(k+2) - c_1 = 0$$

$$c_4 (k+4)(k+3) - c_2 = 0$$

$$\downarrow$$

$$c_4 = \frac{1}{4 \cdot 3} \frac{1}{2 \cdot 1} c_0$$

$$\boxed{n(n-1) c_n - c_{n-2} = 0}$$

$c_n = \frac{c_0}{n!}$ → n = even

n = odd

$$c_n = \frac{c_1}{n!}$$

$$c_2 = \frac{1}{2!} c_0 \qquad c_3 = \frac{1}{3!} c_1$$

$$c_4 = \frac{1}{4!} c_0 \qquad c_5 = \frac{1}{5!} c_1$$

$$c_6 = \frac{1}{6!} c_0 \qquad c_7 = \frac{1}{7!} c_1$$

$$\rightarrow y = \sum_{i=0}^{\infty} c_i x^i$$

$$y = c_0 + c_1 x + \frac{1}{2!} c_0 x^2 + \frac{1}{3!} c_1 x^3 + \dots$$

$$y = c_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + c_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

$$y = c_0 \cosh x + c_1 \sinh x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$y = \left(\frac{c_0 + c_1}{2}\right) e^x + \left(\frac{c_0 - c_1}{2}\right) e^{-x} \qquad \boxed{y = d_1 e^x + d_2 e^{-x}}$$

basis & orthogonal

$y = c_1 \sin x + c_2 \cos x$

basis & orthogonal

why this method :-

$$\frac{d^2y}{dx^2} + (x) y = 0$$

$$\frac{d^2y}{dx^2} - (x) y = 0$$

won't be able to solve using other methods

$\therefore$ use frobenius method

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$a_1, a_2, a_0 \in C$

$a_i \in f(x)$ → use frobenius method can be used.

$a_i \in f(y)$ → non linear eqn

STURM - LIOUVILLE THEORY

$\frac{d^2y}{dx^2} + y = 0 \rightarrow$ solved using 3 methods — ①

① $aD^2 + bD + c = 0$

② Converting this to simultaneous DE then into matrix eqn (eigenvalues & then eigenvectors)

③ Frobenius method.

$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \rightarrow \lambda$ is a parameter

$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \rightarrow \lambda = \lambda(t) \rightarrow \lambda$ is a parameter not a variable (as it is not a function of x, y)

$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \rightarrow \lambda = \lambda(x) \rightarrow \lambda$ is variable so use frobenius method.

if this is dependent variable then the equation is non linear

Q1) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad n \geq 0$ — ② (2nd order ODE)

Q2) $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \quad -1 < x < 1$

all the terms are multiplied by independent variable) → all are linear equation

(linear or non-linear systems are defined by model equation not solution)

$(eg: y = \sin x / \cos x,\ y = e^x / e^{-x})$

take $y = y_1 + y_2$ → DE will satisfy the ODE then take $y = \alpha y_1$

$y = mx + c$ → only form of linear function

Q) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ → Bessel equation

$y = \sum_{i=0}^{\infty} c_i x^{i+k}, \quad \frac{dy}{dx} = \sum_{i=0}^{\infty} c_i (i+k) x^{i+k-1},$

$\frac{d^2y}{dx^2} = \sum_{i=0}^{\infty} c_i (i+k)(i+k-1) x^{i+k-2}$

$\rightarrow \sum_{i=0}^{\infty} c_i (i+k)(i+k-1)(x^{i+k}) + \sum_{i=0}^{\infty} c_i (i+k) x^{i+k} - n^2 \sum_{i=0}^{\infty} c_i x^{i+k} + \sum_{i=0}^{\infty} c_i x^{i+k+2} = 0$

$c_0(k)(k-1) + c_0(k) - n^2 c_0 = 0$ → coeff of x^k.

$c_0 k^2 - n^2 c_0 = 0$ → coeff of x^k.

$c_0(k^2 - n^2) = 0.$

$c_0 = 0$ (or) $\boxed{k = \pm n}$ → k is known

Two solutions → one involving n & one involving −n.

(a) $\boxed{k = +n}$

$\sum_{i=0}^{\infty} c_i (i+n)(i+n-1) x^{i+n} + \sum_{i=0}^{\infty} c_i (i+n) x^{i+n} - n^2 \sum_{i=0}^{\infty} c_i x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$

coeff x^{n+1}

$c_1(1+n)(n) + c_1(1+n) - c_1 n^2 = 0$

$c_1((n+n^2) + 1 + n - n^2) = 0$

$\boxed{c_1 = 0}$ (or) $n + 1 + n = 0$

$n = -1/2$ ⊗ not possible.

∴ c_1 has to be 0.

$\sum_{i=0}^{\infty} c_i (i^2 + in - i + in + n^2 - n) x^{i+n} + \sum_{i=0}^{\infty} c_i (i+n) x^{i+n} - n^2 \sum_{i=0}^{\infty} c_i x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$

$\sum_{i=0}^{\infty} c_i (i^2 + 2in - i - \cancel{n} + \cancel{n^2} + (i + \cancel{n}) - \cancel{n^2}) x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0.$

$$\boxed{\sum_{i=0}^{\infty} c_i (i^2 + 2in) x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0}$$

coeff of x^{n+2}.

$c_2(4 + 4n) + c_0 = 0.$

$c_2 = \frac{-c_0}{4(n+1)}$

Coeff of x^{n+3}

$c_3(9 + 6n)x^{n+3} + c_1 x^{n+3}$

$\boxed{c_3(9 + 6n) + c_1 = 0}$

$c_3 = \frac{-c_1}{3(2n+3)}$

$c_1 = 0 \Rightarrow \boxed{c_3 = 0}$

$C_1 = C_3 = C_5 = C_7 = \dots = 0$

$\to C_4(16+8n)\,x^{n+4} + C_2 x^{n+4}$

$$\boxed{C_4 = \frac{-C_2}{8(n+2)}}$$

$$C_4 = \frac{+C_0}{4(n+1)(8)(n+2)}$$

$$C_4 = \frac{C_0}{32(n+1)(n+2)}$$

$$C_6 = f(C_0)$$

$$y(x) = (C_0 + C_2 x^2 + C_4 x^4 \dots)\,x^n$$

$$\boxed{y(x) = C_0 x^n\left(1 + \alpha_1 x^2 + \alpha_2 x^4 + \dots\right)}$$

oscillatory nature

one bessel function

C_0 is unknown.

Bessel function for specific value of C_0

$k=-n$ other bessel function

(8) A metallic rod of infinite length at temp T_0 is dipped in an infinite reservoir at temperature T_∞. The radius of the rod is R. Determine the spatio-temporal evolution of the temperature of the rod if the governing equation is given by

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

neglected (∞ length) (large)

range will be limited to T_0, T_∞

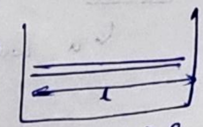
$T_\infty \to$ cold

$T_0 \to$ rod is hot

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\cancel{\frac{\partial^2 T}{\partial \theta^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right)$$

$=0$ cylindrically symmetric

0



$L >>> R$

T_∞ to $T_0 \to L$ order of magnitude analysis Compared to radial.

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)$$

$$T = X(t)\,Y(r)$$

$$Y\frac{dX}{dt} = \alpha\left(X\frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr}X\right)$$

$$\frac{1}{X}\frac{dX}{dt} = \alpha\left(\frac{1}{Y}\frac{d^2Y}{dr^2} + \frac{1}{Yr}\frac{dY}{dr}\right) = \text{const}$$

$$\frac{\alpha}{Y}\left(\frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr}\right) = \beta$$

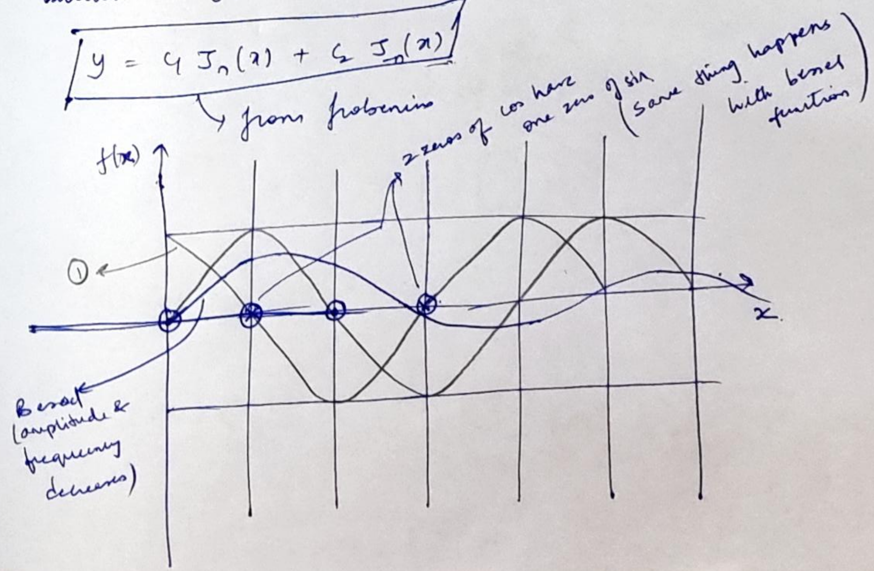
$$\frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr} - \frac{\beta}{\alpha}Y = 0$$

$$\boxed{r^2\frac{d^2Y}{dr^2} + r\frac{dY}{dr} - \left(\frac{\beta r^2}{\alpha}\right)Y = 0}$$

solution to this equation is bessel function.

$$\boxed{y = C_1 J_n(x) + C_2 J_{-n}(x)}$$

from frobenius



→ sphere $T = f(\theta, \phi)$

↳ u will get $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \quad -1 < x < 1$

↳ legendre ~~polynomial~~ equation

solution - legendre polynomial

Instead of infinite series we get polynomial bcoz series get truncated

$\boxed{c_{n+2} = (\ldots) c_n}$ → recursion relation

self adjoint operator
∴ we get orthogonal polynomials (eigen values are real)

⇒ Any equation of the above kind can be written

as $\hat{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)$ ⟶ sturm liouville operator.

$\hat{L}y + \lambda q(x) y = 0$ ⟶ sturm liouville problem

→ 4 independent variables in any problem - x, y, z, t
(other things are parameters)