Lecture - 14
4/9/17

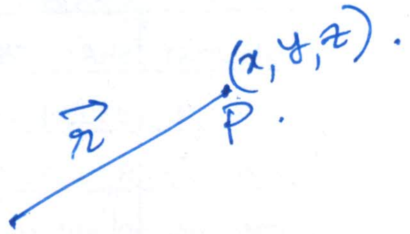
Fourier series.

$f(x+2l) = f(x) \rightarrow f$ is periodic with period $2l$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \; ; \; -l \le x \le l.$$

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + \cdots$$

$$+ b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots$$



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$\hat{i}, \hat{j}, \hat{k} \rightarrow$ unit vectors along x, y, z directions.

$(\hat{i}\cdot\hat{j}) = 0 = (\hat{i}, \hat{k}) = (\hat{j}\cdot\hat{k})$

$\langle \underset{\sim}{a}, \underset{\sim}{b} \rangle$ = inner ~~vector~~ product of $\underset{\sim}{a}, \underset{\sim}{b}$

$\hat{i}\cdot\hat{j} = \langle \hat{i}, \hat{j} \rangle$ = inner product of $\hat{i}, \hat{j}$

= dot " " $\hat{i}, \hat{j}$.

If $\langle \hat{i}, \hat{j} \rangle = 0$ we say vectors $\hat{i}, \hat{j}$ are orthogonal

$\langle \underset{\sim}{a}, \underset{\sim}{b} \rangle = 0$, " " " $\underset{\sim}{a}, \underset{\sim}{b}$ are orthogonal

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\, dx = 0 \; ; \; m \neq n .$$

Inner product for $\{f_n(x)\}$ may be defined as

$$\int_a^b f_n(x)\, f_m(x)\, dx \quad \text{where } a \leq x \leq b .$$

If $\int_a^b f_n(x)\, f_m(x)\, dx = 0$, we say the set $\{f_n(x)\}$ to be orthogonal.

If you add $g(x)$ to the set $\{f_n(x)\}$ & see $\int_a^l g(x)\, f_n(x)\, dx \neq 0$, then $\{f_n(x)\}$ is said to be complete orthogonal set.

Then it is possible to express $F(x)$ as.

$$F(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots$$

$$= \sum_{n=1}^{\infty} a_i f_i(x) .$$

It can be shown that $\left\{ \cos \frac{n\pi x}{l} ;\ n = 0, 1, 2, 3, \dots \right\}$ and $\left\{ \sin \frac{n\pi x}{l} ;\ n = 1, 2, \dots \right\}$ are two complete orthogonal sets.

Also, $\left\{ \cos \frac{n\pi x}{l} \right\}$ & $\left\{ \sin \frac{n\pi x}{l} \right\}$ are mutually orthogonal over the interval $[-l, l]$.

$$\int_{-l}^{l} \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = \begin{cases} 0 & ; m \neq n \\ l & , m = n \neq 0 \\ 2l & , m - \end{cases}$$

$$\int_{-l}^{l} \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \begin{cases} 0 & , m \neq n \\ 0 & , m = n = 0 \\ l & , m = n \neq 0 \end{cases}$$

$$\int_{-l}^{l} \cos^2 \frac{m\pi x}{l}\, dx = \frac{1}{2}\int_{-l}^{l}\left(1 + \cos \frac{2m\pi x}{l}\right) dx = l$$

$$\int_{-l}^{l} \sin^2 \frac{m\pi x}{l}\, dx = \frac{1}{2}\int_{-l}^{l}\left(1 - \cos \frac{2m\pi x}{l}\right) dx$$

$$\int_{-l}^{l} \cos \frac{n\pi x}{l} \cdot 1\, dx = 0 \ ; \ n \neq 0$$

$$\int_{-l}^{l} \cos \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad (1)$$

$-l \le x \le l$

Integrate both sides of (1) w. r. to x between $-l$ and l.

$$\int_{-l}^{l} f(x)\,dx = \frac{a_0}{2}\int_{-l}^{l} dx + \sum_{n=1}^{\infty} a_n \int_{-l}^{l} \cos\frac{n\pi x}{l}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-l}^{l} \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{a_0}{2}\times 2l + 0 + 0 = l a_0.$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

Multiply both sides of (1) with $\cos\frac{m\pi x}{l}$ and integrate between $(-l, l)$. This gives

$$\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx = \frac{a_0}{2}\underbrace{\int_{-l}^{l} \cos\frac{m\pi x}{l}\,dx}_{0}$$

$$+ \sum_{n=1}^{\infty} a_n \int_{-l}^{l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-l}^{l} \cos\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$= 0 + a_m l + 0$$

$$a_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx.$$

To find b_n, multiply (1) by $\sin \frac{m\pi x}{l}$ & integrate between $(-l, l)$.

$$\int_{-l}^{l} f(x) \sin \frac{m\pi x}{l}\, dx = \frac{a_0}{2} \underbrace{\int_{-l}^{l} \sin \frac{m\pi x}{l}\, dx}_{=0}$$

$$+ \sum_{n=1}^{\infty} a_n \underbrace{\int_{-l}^{l} \cos \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx}_{=0} + \sum_{n=1}^{\infty} b_n \int_{-l}^{l} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx.$$

$$= 0 + 0 + b_m \cdot l.$$

$$\therefore\ b_m = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{m\pi x}{l}\, dx.$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right), \quad -l \le x \le l}$$

then, $a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx, \quad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

(1) Find the Fourier series of x in $(-\pi, \pi)$

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x\, dx = 0.$$

$$a_n = \cancel{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx.$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x} \; \underset{\text{even}}{\cos nx}}_{\text{odd}}\, dx = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x} \; \underset{\text{odd}}{\sin nx}}_{\text{even}}\, dx = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx\, dx.$$

$$= \frac{2}{\pi} \left[x \frac{\cos nx}{n} \Big|_{\pi}^{0} + \underbrace{\int_{0}^{\pi} \frac{\cos nx}{n}\, dx}_{=0} \right].$$

$$= \frac{2}{\cancel{\pi}} \times -\frac{\cancel{\pi}}{n} \cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\therefore x = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx.$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

$-\pi \leq x \leq \pi$

Thm. At the end points (a, b). Fourier series of $f(x)$ converges to, $\dfrac{f(a+) + f(b-)}{2}$.

At the end points $(-\pi, \pi)$, Fourier series of x converges to $\dfrac{\pi + (-\pi)}{2} = 0$.

Note 1) The series on the r.h.s converges to $f(x)$ for all x, in $(-\pi, \pi)$, if $f(x)$ is continuous. It converges to $\dfrac{f(x_0-0)+f(x_0+0)}{2}$, if $f(x)$ is discontinuous at $x=x_0$

2) At $x = \pm\pi$, it converges to, $\dfrac{f(-\pi+0) + f(\pi-0)}{2}$.

Ex-2. Find fourier series of $f(x)$,

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi. \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 1.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx.$$

$$= \frac{1}{\pi}\,\frac{\cos nx}{n}\Big|_{\pi}^{0} = \frac{1}{\pi n}\left[1 - \cos n\pi\right].$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & n = \text{even}. \\ \frac{2}{n\pi}, & n = \text{odd} \end{cases}.$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left\{1-(-1)^n\right\}\sin nx.$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

At $(-\pi, \pi)$ it should converge to $\quad -\pi \le x \le \pi$.

$$\frac{f(-\pi+0)+f(\pi+0)}{2},$$

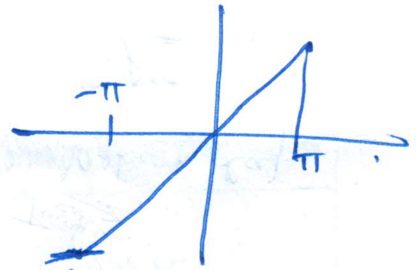
$\underline{\underline{x = 0.}}$

In order that $f(x)$ can be expressed in terms of a Fourier series in (a, b), $f(x)$ must satisfy Dirichlet's conditions.

1). $\int_a^b |f(x)|\, dx$ exists.

2) $f(x)$ has finite no of finite discontinuities in (a, b).

3) $f(x)$ has finite no. of maxima and minima in (a, b).



$$\cancel{f(x)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$$

x.

$x \in R$?

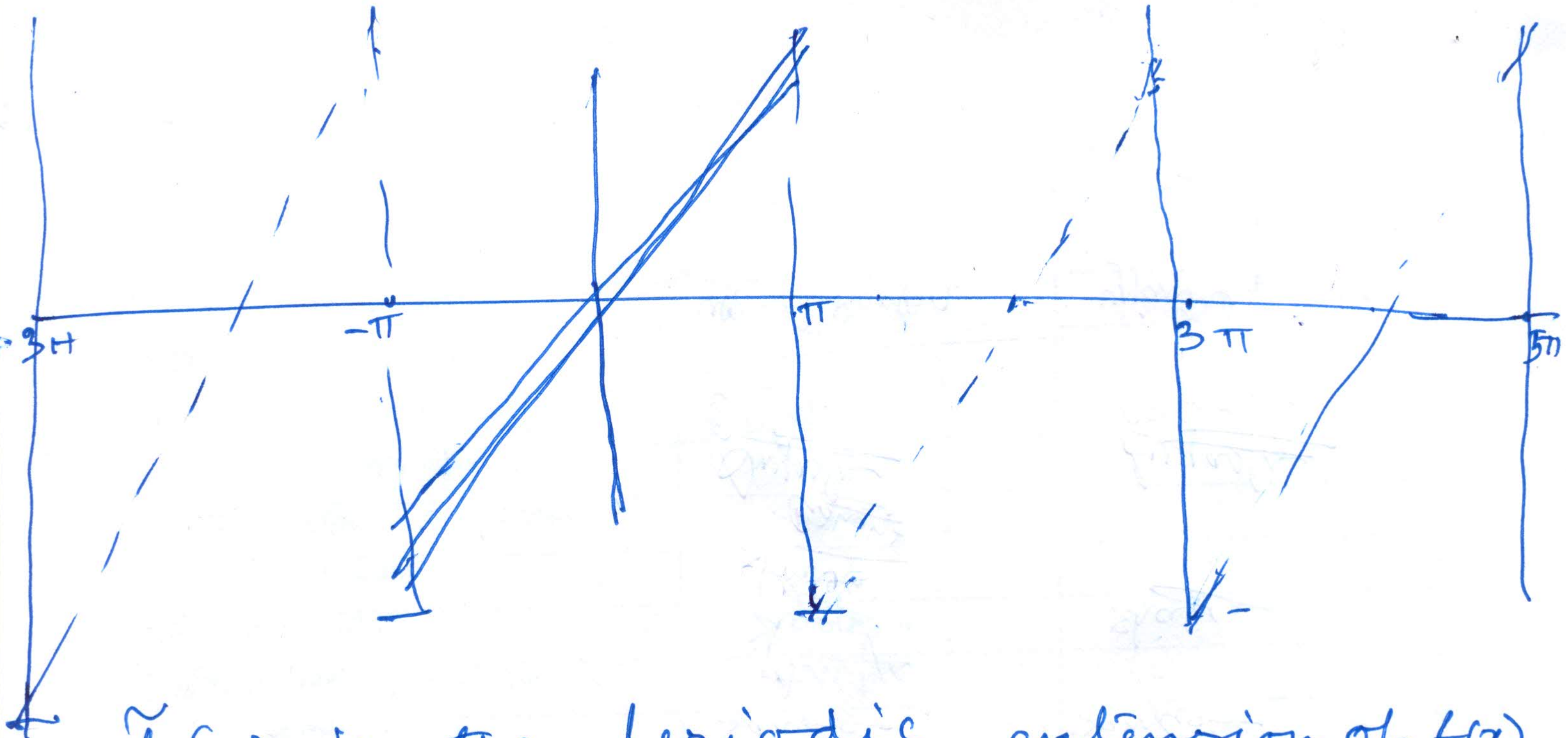
periodic extension $\tilde{f}(x)$ of $f(x)$.

$$\underset{f(x)}{x} = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right].$$

$-\pi \le x \le \pi$

But above series is not valid beyond $[-\pi, \pi]$.

But $2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right]$ is valid for all x & is a period funct.. of period 2π..



$\tilde{f}(x)$ is the periodic extension of $f(x)$ in $(-\pi, \pi)$ such that

$$\tilde{f}(x) = f(x) \quad \text{in } -\pi \le x \le \pi.$$

$$\& \quad \tilde{f}(x) = \tilde{f}(x + 2\pi).$$

Q. Given $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} dx$, $\rightarrow (1')$ $\quad -l \le x \le l$.

with

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \rightarrow (2')$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \rightarrow (3'), \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \rightarrow (4')$$

Find Fourier series of $f(x)$ in $(0, 2\pi)$ & the expressions for the coefficients.

Put $l = \pi$ in (1') – (4')

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$-\pi \le x \le \pi$ $\rightarrow (1)$

with $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \rightarrow (2)$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \rightarrow (3), \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \rightarrow (4).$$

Let

$$y = x + \pi$$

When, $-\pi \le x \le \pi$, then $0 \le y \le 2\pi$.

Then from (1),

$$f(y-\pi) = \phi(y), \text{ say} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n(y-\pi) + \sum_{n=1}^{\infty} b_n \sin n(y-\pi).$$

$$\text{or, } \phi(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left\{ \cos ny \cos n\pi + \underbrace{\cancel{\sin ny \sin n\pi}}_{=0} \right\} + \sum_{n=1}^{\infty} b_n \left\{ \sin ny \cos n\pi - \underbrace{\cancel{\cos ny \sin n\pi}}_{=0} \right\}.$$

$$\phi(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n \cos ny + \sum_{n=1}^{\infty} b_n (-1)^n \sin ny$$

$$\boxed{\phi(y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos ny + \sum_{n=1}^{\infty} B_n \sin ny, \quad 0 \le y \le 2\pi}$$

$$A_n = a_n (-1)^n = (-1)^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

$y = x + \pi$

$$= (-1)^n \times \frac{1}{\pi} \int_0^{2\pi} f(y-\pi) \cos n(y-\pi) \, dy.$$

$$= \underline{(-1)^n} \times \frac{1}{\pi} \int_0^{2\pi} \phi(y) \cos ny \, \underline{(-1)^n} \, dy$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} \phi(y) \cos ny \, dy.$$

Similarly show that,

$$A_0 = \frac{1}{\pi} \int_0^{2\pi} \phi(y) \, dy \quad \& \quad B_n = \frac{1}{\pi} \int_0^{2\pi} \phi(y) \sin ny \, dy$$

Fourier series of $f(x)$ in $(0, 2\pi)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

$$0 \leq x \leq 2\pi.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx.$$

Fourier series of $f(x)$ in (a, b).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{(b-a)} + \sum_{n=1} b_n \sin \frac{2n\pi}{(b-a)} x.$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\, dx.$$

$$a_n = \frac{2}{b-a}\int_a^b f(x) \cos\left(\frac{2n\pi}{b-a}\right) x\, dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x) \sin\left(\frac{2n\pi}{b-a}\right) x\, dx. \quad //.$$