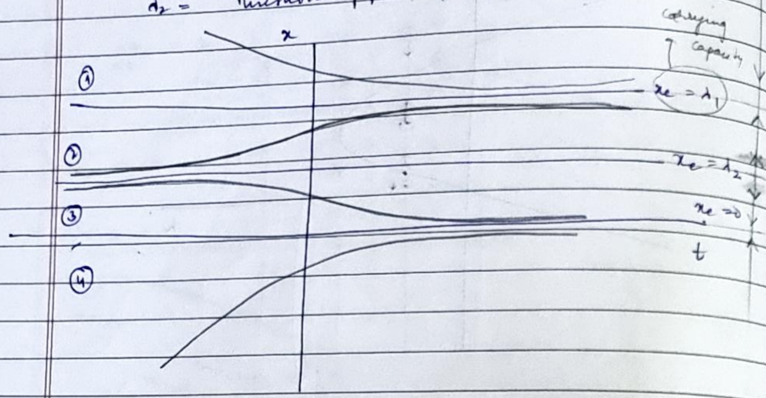
2) $\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$

$0 < \lambda_2 < \lambda_1$

λ_1 = carrying capacity

λ_2 = threshold population (min population)



① → asymptotes → $\lambda_1, \lambda_2, 0$

(Extinction when the initial population is less than the threshold population)

Point of intersection of stable & unstable curves is bifurcation pt

→ $\frac{dx}{dt} = ax - ax^2$

$\frac{dx}{dt} = a - x^2$

$\frac{dx}{dt} = ax - x^2$

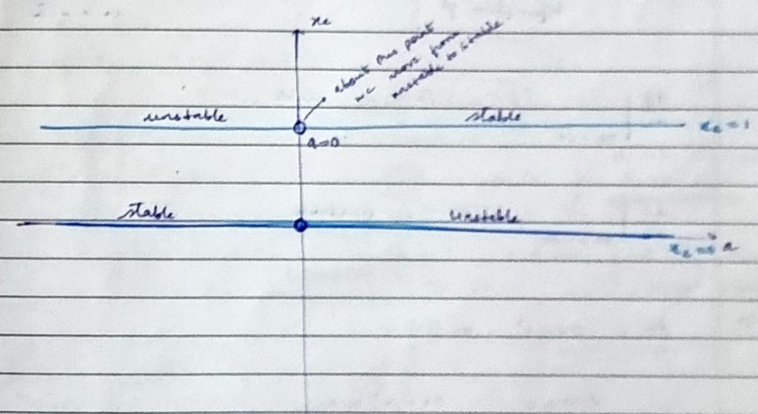
$\frac{dx}{dt} = ax - x^3$

① $\frac{dx}{dt} = ax(1-x) = f(x)$

• Bifurcation diagram

$\frac{df}{dx} = a - 2ax$ → nature of x_e will depend on a → stable or unstable

$f(x) = 0 \rightarrow x_e = 0, \quad x_e = 1$



The system has bifurcation at $a = 0$.

$\frac{df}{dx}$ → has no meaning until and unless calculated at x_e

$\left.\frac{df}{dx}\right|_{x_e=0} = a \begin{cases} a>0 \rightarrow \text{unstable} \\ a<0 \rightarrow \text{stable} \end{cases}$

$\frac{df}{dx} < 0 \rightarrow$ stable

$\frac{df}{dx} > 0 \rightarrow$ unstable

$\left.\frac{df}{dx}\right|_{x_e=1} = -a \begin{cases} a<0 \rightarrow \text{unstable} \\ a>0 \text{ stable} \end{cases}$

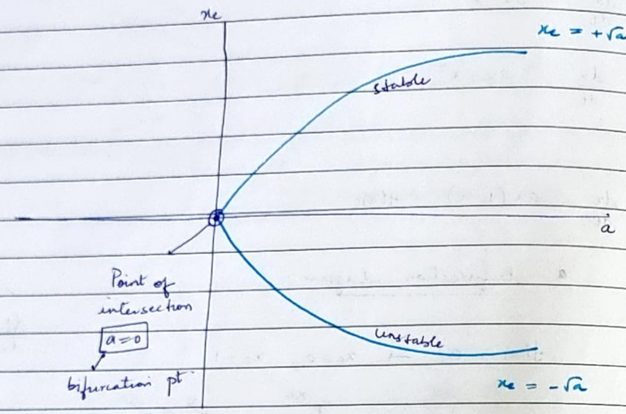
(2) $\frac{dx}{dt} = a - x^2 = f(x) = 0$

$x^2 = a$

$\boxed{x_e = \pm\sqrt{a}}$ → equilibrium soln.

↳ $a > 0$

$\frac{df}{dx} = -2x$



$\left.\frac{df}{dx}\right|_{x_e=\sqrt{a}} = -2\sqrt{a}$ → stable

$\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = 2\sqrt{a}$ → unstable.

(3) $\frac{dx}{dt} = ax - x^2 = f(x)$

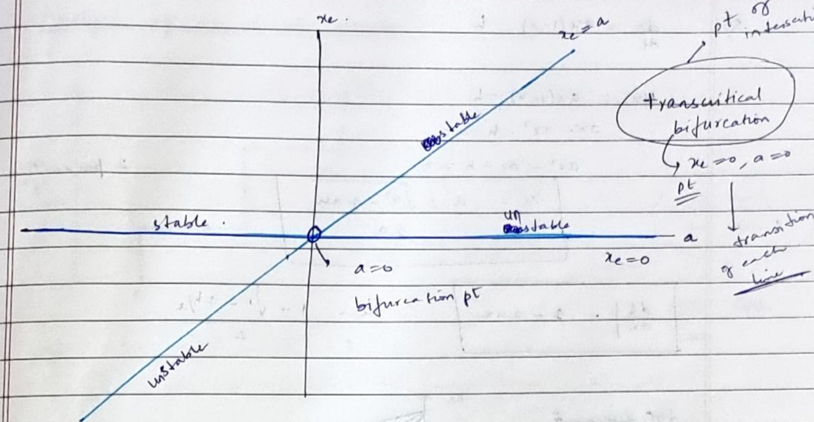
$f(x) = 0 \qquad ax - x^2 = 0$

$\boxed{x_e = 0} \quad \boxed{x_e = a}$

$\frac{df}{dx} = a - 2x.$

$\left.\frac{df}{dx}\right|_{x_e=0} = a$ { $a>0$ unstable, $a<0$ stable

$\left.\frac{df}{dx}\right|_{x_e=a} = -a$ { $a>0$ stable, $a<0$ unstable.



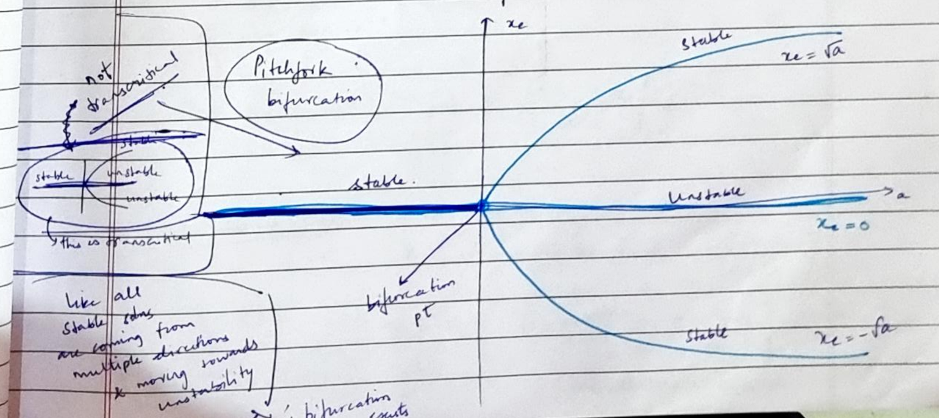
(4) $\frac{dx}{dt} = ax - x^3 = f(x)$

$f(x) = 0 \qquad ax - x^3 = 0$

$x_e = 0, \quad x_e = \pm\sqrt{a} \quad a>0$

$\frac{df}{dx} = a - 3x^2$ $\qquad \left.\frac{df}{dx}\right|_{x_e=0} = a$ { $a>0$ unstable, $a<0$ stable

$\left.\frac{df}{dx}\right|_{x_e=-\sqrt{a}} = -2a$ { $a>0$ stable $\qquad \left.\frac{df}{dx}\right|_{x_e=\sqrt{a}} = -2a$ { $a>0$ stable



$1.5 \geq 0.5\sqrt{1-\frac{4h}{a}}$

$1 - \frac{4h}{a} > 0$

$\frac{h}{a} < \frac{1}{4}$

$h < \frac{a}{4}$

Date
Page

a) $\frac{dx}{dt} = ax(1-x) - h$

$f(x) = ax(1-x) - h$

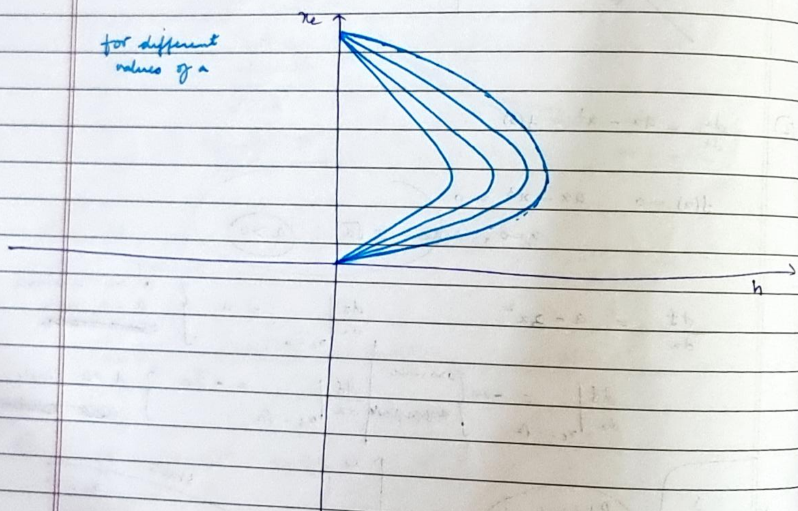
$ax - ax^2 - h = 0.$

$ax^2 - ax + h = 0$ 2 parameters

$$\boxed{x_e = \frac{a \pm \sqrt{a^2 - 4ah}}{2a}}$$

$$\boxed{\left.\frac{df}{dx}\right| = a - 2ax.} \qquad x_e = \frac{1 \pm \sqrt{1 - 4h/a}}{2}$$

for different values of a



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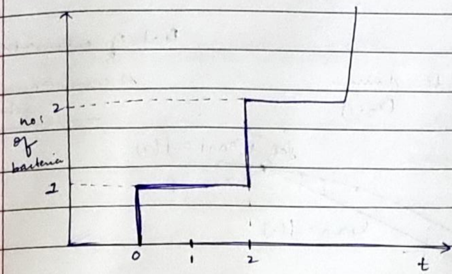
Analysis ~~of~~ in discrete domain



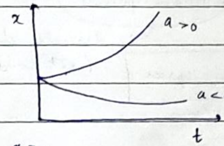
x, t both are discrete

at t = 0 no. of bacteria = 1

t = 1 the no. of bacteria will remain same = 1

→ The no. of bacteria is very large → ∴ taking continuous domain works

$\frac{dx}{dt} = ax$



$$\frac{x_{n+1} - x_n}{\Delta t} = a x_n$$

$$x_{n+1} = (1 + a\Delta t)\, x_n$$

⟹ d) $\frac{dx}{dt} = ax(1-x)$

$$x_{n+1} = (1 + a\Delta t)\, x_n \left[1 - \left(\frac{a\Delta t}{1 + a\Delta t}\right) x_n\right]$$

⟹ Equilibrium soln :- $x_{n+1} = x_n$

$$x_n = (1 + a\Delta t)\, x_n \left[1 - \frac{a\Delta t}{1 + a\Delta t} x_n\right]$$

$$x_n \left[1 - (1 + a\Delta t)\left(1 - \frac{a\Delta t\, x_n}{1 + a\Delta t}\right)\right] = 0.$$

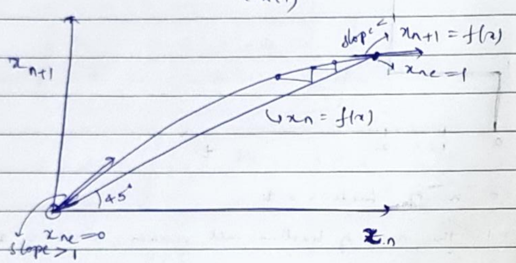
$x_{ne} = 0$ $\quad$ $x_{ne} = 1$

→ Graphical method to get equilibrium soln

Plot $f(x) = x$

Plot $g(x)$ = discrete domain (x_{n+1})

Point of intersection is equilibrium soln



① discretize the system

② Find equilibrium soln by plotting $f(x) = x$, $g(x) = x_{n+1}$ — Fixed points → discrete domain (continuous domain → equilibrium soln)

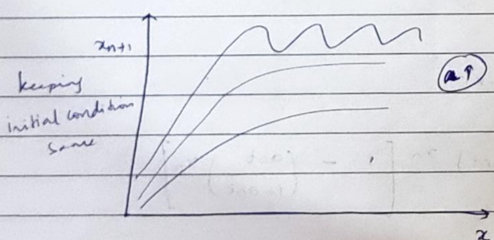
③ slope > 1 → unstable

slope < 1 → stable

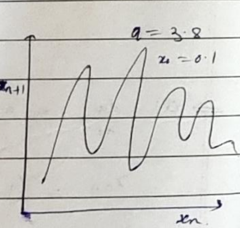
→ All the pts lie in metric spaces ↳ discrete domain

Q) $x_{n+1} = a(x_n)(1-x_n)$

as you change a it starts oscillating



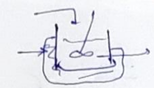
a↑



→ change initial condition & check ✓ $a = 3.8$, $x_0 = 0.11$

→ future behavior has changed completely

→ chaotic system → future changes drastically depending on initial condition



metric spaces → have measurement or units defined.

Reactor stability Analysis

Transient operation of a jacketed CSTR

$$\frac{dc}{dt} = \frac{F}{V}(c_f - c) - r$$ → makes the system non-linear

$$\frac{dT}{dt} = \frac{F}{V}(T_f - T) + \left(\frac{-\Delta H}{\rho c_p}\right) r - \frac{UA}{V\rho c_p}(T - T_j)$$

→ There can be multiple ss solutions. We need to identify which solution is stable. If the conditions change (stability), nature of solutions will change.

① Find steady state solution. Newton raphson → to find ss

Q) $\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$

$\frac{dx_2}{dt} = 2x_2 = f_2$

} No bifurcation plot

SS: $x_1^2 - x_2^2 - 1 = 0$ $\quad x_1^2 = 1$ $\quad x_{1e} = \pm 1$

$2x_2 = 0$ $\quad \boxed{x_{2e} = 0}$

(dynamical variable is a variable)

$$\underbrace{\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$ ↳ equilibrium soln

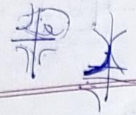
→ (determine jacobian) $J = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$

$\frac{\partial f_1}{\partial x_1}$ $\frac{\partial f_1}{\partial x_2}$ $\frac{\partial f_2}{\partial x_1}$ $\frac{\partial f_2}{\partial x_2}$

→ Determine the jacobian at equilibrium soln.

$J|_{e_1} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ ↳ $\lambda|_{e_1} = \pm 2$ → saddle.

$J|_{e_2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ → source soln. ↳ $\lambda|_{e_2} = 2, 2$

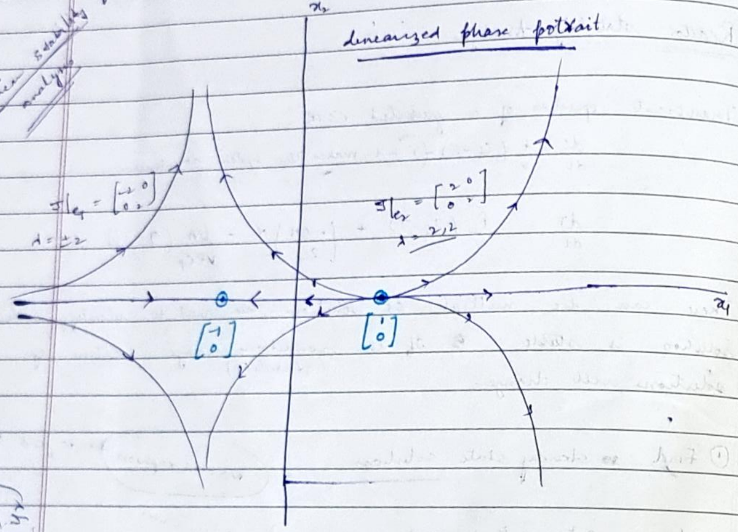


linear

$\lambda_1 = \lambda_2$ not parabolic

→ find eigen values.

linear stability analysis



$$ce^{t}\underline{v} + te^{t}$$

$x - x_e$

$$y = f(x)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{bmatrix}$$

→ $y = f(x)$

→ $y = y_e + \frac{dy}{dx}\Big|_e (x - x_e) + O\left(\frac{d^2y}{dx^2}\right)$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_{e_1} + J\Big|_{e_1} \begin{bmatrix} x_1 - x_{e_1} \\ x_2 - x_{2e_2} \end{bmatrix}$$

$$\begin{pmatrix} f_1 - f_1|_{e_1} \\ f_2 - f_{2e_1} \end{pmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - x_{1e_1} \\ x_2 - x_{2e_2} \end{bmatrix}$$

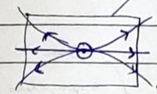
$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 - x_{1e_1} \\ x_2 - x_{2e_2} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Hartman - Grobman theorem

Orbit structure → all phase lines

topologically → qualitatively



hyperbolic equilibrium point :- pt which doesn't correspond to eigenvalue 0

(even if one of the eigen value is 0 → don't linearize) → linearized phase potrait will not work

$$\frac{F}{V} C_f - \frac{F}{V} C_s - k e^{-E/RT_s} C_s = 0$$

$$C_s = \frac{\left(\frac{F C_f}{V}\right)}{\left(\frac{F}{V}\right) + k e^{-E/RT_s}}$$

$$\frac{F T_f}{V} - \left(\frac{F}{V}\right) T_s + \left(\frac{-\Delta H}{\rho C_p}\right)\left(k e^{-E/RT_s}\right) C_s - \frac{UA}{V\rho C_p} T_s + \frac{UA T_j}{V\rho C_p} = 0$$

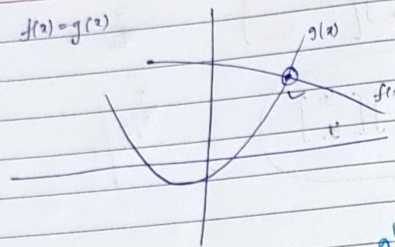
heat of reaction

$$\underbrace{\frac{\left[-\frac{(\Delta H)\, k\, (F/V)}{\rho C_p} C_f\right] e^{-E/RT_s}}{\left(\frac{F}{V}\right) + k e^{-E/RT_s}}}_{\text{nonlinear}} = \underbrace{\left(\frac{UA}{V\rho C_p} + \frac{F}{V}\right) T_s - \left(\frac{UA T_j}{V\rho C_p} + \frac{F T_f}{V}\right)}_{\text{linear}}$$

heat added to the system or taken out

convective heat transfer due to reactor fluid and jacket fluid

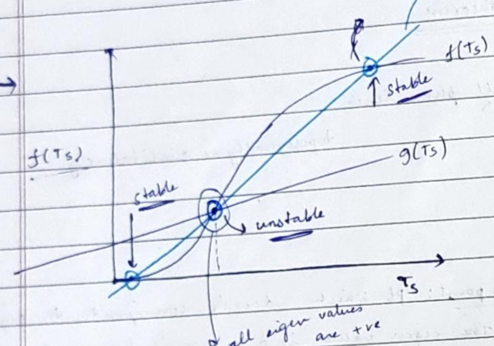
$f(x) = g(x)$



(plot to find the point of intersection)

g(Ts) (3 ss solutions)

↳ all solutions are physically realisable

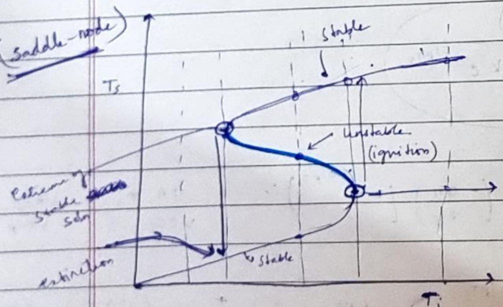


stable
eigenvalues = −ve (or)
Re(eigenvalues) = −ve
might have complex eigen values.

all eigen values are +ve

will vary Tj to maintain desired stable soln

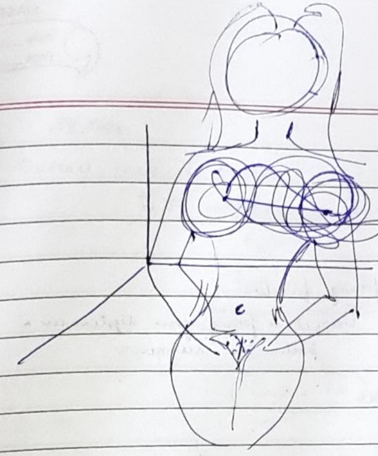
(saddle-node)



(bifurcation diagram)

when there is a small change in temp, the ss temp shoots up (ignition)

(lorenz attractor)



Chaotic (lorenz attractor)

Why do we use transform domain analysis?

① disturbance

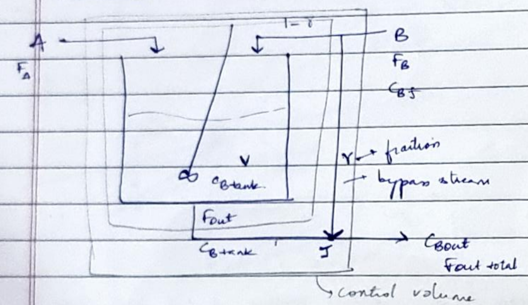
② response

③ steady-state solution

$$\frac{dx}{dt} = ax + \underbrace{b(t)}_{\text{input / forcing function}}$$

because it is forcing your system in a particular manner

TRANSFORM DOMAIN ANALYSIS



→ control volume

No reaction

(all densities are similar) → assumption

~~$\frac{d}{dt}$ …~~

~~…~~

$$(1-r)F_B C_{Bf} - C_{Btank} F_{out} = \frac{d(C_{Btank} V)}{dt}$$

Forcing function

$$\boxed{\frac{dC_{Btank}}{dt} + \frac{\overset{F_A}{F_{out}}}{V} C_{Btank} = \frac{(1-r)F_B C_{Bf}}{V}} \rightarrow ①$$

dynamical system

C_{Btank} → dynamical variable

Model eqn is given in terms of a particular variable → that is dynamical variable

$$\frac{dx}{dt} + f(x) = g(t)$$

(x is dynamical variable) — $g(t)$: Not dynamical variable

---

steady state Vs equilibrium

⇒ Overall material balance

$$\cancel{\rho_A} F_A + \cancel{\rho_B} F_B = F_{out\,total}\,\cancel{\rho_{mix}}$$

$F_A >> F_B$

densities are similar

$$\boxed{F_{out\,total} = F_A}$$

$$\cancel{\rho_A} F_A + \cancel{\rho_B}(1-r)F_B = \cancel{\rho_{mix}} F_{out}$$

$$F_{out} = F_A \checkmark$$

→ at junction J (volume of junction = 0)

$$\left(F_{out}(C_{Btank}) + r F_B C_{Bf} = C_{Bout} \times F_{out\,total}\right)$$

$$\boxed{F_A C_{Btank} + r F_B C_{Bf} = F_A C_{Bout}}$$

C_{Btank} → dynamical variable

C_{Bout} is not DV

$$\boxed{C_{Bout} = C_{Btank} + \left(\frac{rF_B}{F_A}\right) C_{Bf}} \rightarrow ②$$

⇒ $$\frac{dC_{Btank}}{dt} + \frac{F_A}{V} C_{Btank} = \frac{(1-r)F_B C_{Bf}}{V}$$

make the coeff 1

$$\frac{V}{F_A}\frac{dC_{Btank}}{dt} + C_{Btank} = \frac{(1-r)F_B C_{Bf}}{F_A}$$

$$u = \frac{F_B C_{Bf}}{F_A}$$

$$\boxed{C_{Bout} = C_{Btank} + r u}$$

$x = C_{Btank}$

$y = C_{Bout}$

⇒ $$\frac{V}{F_A}\frac{dx}{dt} + x = (1-r)u$$

$$y = x + ru$$

dependence of diff system variables on u, y?

$x(0) - ky$

$$\underline{A(\tau s+1) + Bs = \chi s+1}$$

take laplace transform

$$\left[\left(\frac{V}{F_A}\right)s + 1\right]\bar{x}(s) = (1-r)\,\bar{u}(s)$$

deviation variable form

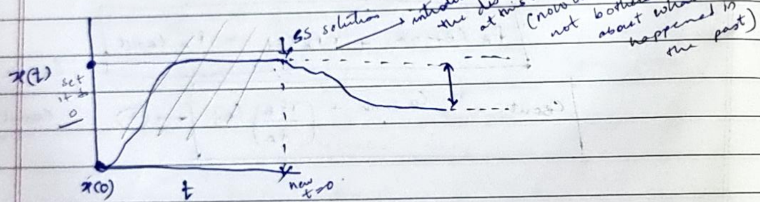
$$\frac{\bar{x}(s)}{\bar{u}(s)} = \frac{1-r}{\left(\frac{V}{F_A}\right)s+1}$$

transform domain is between 2 steady states

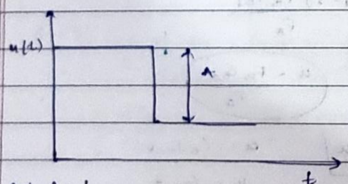
* ~~transform~~

$$\boxed{\bar{y}(s) = \bar{x}(s) + r\,\bar{u}(s)}$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{\left(\frac{Vr}{F_A}\right)s+1}{\left(\frac{V}{F_A}\right)s+1}$$ → effect of input variable on output variable



$$\boxed{\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{\chi s+1}{\tau s+1}}$$



$\bar{u}(s) = \frac{A}{s}$

(step input)

let A = 1

$$\bar{y}(s) = \frac{1}{s}\left(\frac{\chi s+1}{\tau s+1}\right) = \frac{\chi}{\tau s+1} + \frac{1}{s(\tau s+1)}$$

$$\frac{A}{s} + \frac{B}{\tau s+1} = \frac{\chi}{\tau s+1} + \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{\chi-\tau}{\tau s+1} + \frac{1}{s}$$

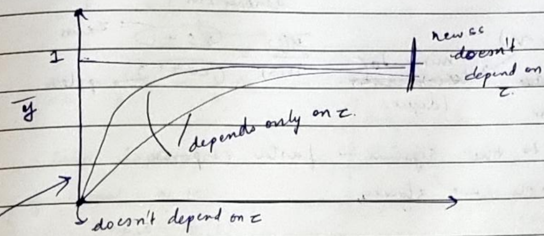
$$\boxed{C(\tau s+1) + Bs = \chi s+1}$$

NPTEL

~~$\bar{y}(t) = 1 - \tau$~~

$$\boxed{\bar{y}(t) = 1 - \left(1 - \frac{\chi}{\tau}\right)e^{-t/\tau}} \checkmark$$

χ, τ → system variables
V, F_A, r



We want to know how the system behaves when we introduce the disturbance

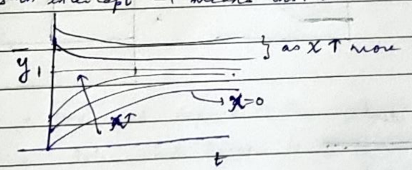
$\chi = 0$, $\tau = 1$ new SS is 1 bcoz we have given unit disturbance

τ = time const → lag ~~rate~~ indicates the rate of dynamics.

smaller τ → faster dynamics

→ ~~monotone~~ monotonous.

→ χ → ~~rate~~ lead time const → $\chi\uparrow$, faster response.
↳ results in intercept → means instantaneous response.



⇒ $a_1 \frac{dy}{dt} + y = b\,f(t)$ → deviation variable form.

$$\boxed{\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{k}{\tau s+1}}$$

k = gain, τ = time const

↳ degree of the polynomial in the denominator will tell you about the order of the system

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{xs+1}{\tau s+1} \Bigg\} \rightarrow \text{order of the system} = (1,1)$$

~~denominator~~ denominator | ~~denominator~~ numerator

order :- (p, q)

~~numerator~~ (degree) denominator

numerator (degree)

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{xs+1}{\tau s+1}$$

$xs+1 \rightarrow$ zeros

$\tau s+1 \rightarrow$ poles

→ add zero to the system → faster response

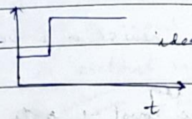
→ addn of pole → slower

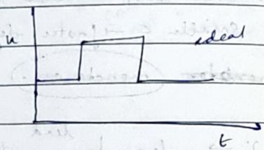
⇒ ideal step function

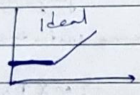
why ideal?

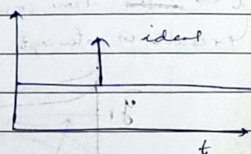
↳ No fluctuations

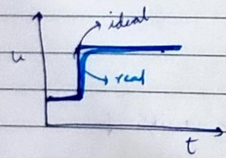
response of the system that is generating the input (this will take some time)











→ Equilibrium soln and state-space domain, transform domain S.S solution are same

↳ due to physics.

→ tolerance = $x_{i+1} - x_i \rightarrow 0$.

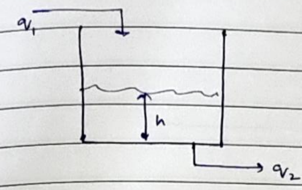
if we give the initial guess $= 0$ → you are already at the fixed pt.

initial guess $\neq 0, 1$

(usually when we normalize, 0, 1 are equilibrium soln)

↳ the iteration will give 0 and you will never get the desired values.

→



$$\frac{dh}{dt} = \frac{1}{A} q_1 - \frac{1}{A} q_2$$

take laplace

if output $= q_2(t)$

Input $= q_1(t)$

↳ In this case you get (1, 1) order system.

↳ here h is dynamical ~~var~~ variable not the output

$$\left[\begin{array}{l} \text{Output} = h \qquad q_2 = \alpha h. \\ \text{Input} = q_1(t) \\ \quad ↳ \text{take laplace} \\ \qquad ↳ \text{order} = 1 \end{array}\right]$$

$$\Rightarrow g(s) = \frac{K(xs+1)}{(\tau_1 s+1)(\tau_2 s+1)} \longrightarrow y(t) = AK\left(1 - \left(\frac{\tau_1 - x}{\tau_1 - \tau_2}\right)e^{t/\tau_1} - \left(\frac{\tau_2 - x}{\tau_2 - \tau_1}\right)e^{t/\tau_2}\right)$$

↳ for step response.

poles $= -1/\tau_1, -1/\tau_2$

Zero $= -1/x$

→ If i add a zero what happens to the system?

If i add a pole what happens to the system?

→ $\tau_1 = 1.15$

$\tau_2 = 0$

$x = 0$

K = 1

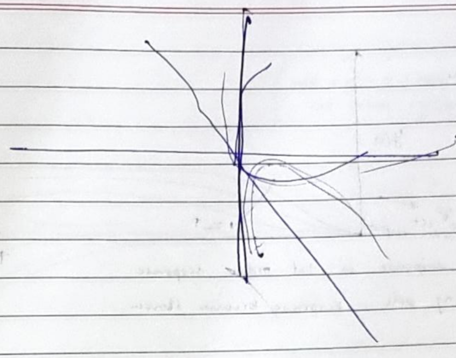
A = 1 → step unity

$\tau_1 \downarrow$, dynamics is faster

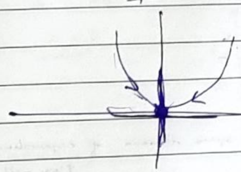
↓

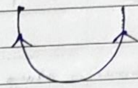
reaches quickly to 1

$|\lambda_1| > |\lambda_2|$



$e^{-2t}\begin{bmatrix}1\\0\end{bmatrix}$ $\boxed{\begin{bmatrix}0\\1\end{bmatrix}}$ e^{-3t}

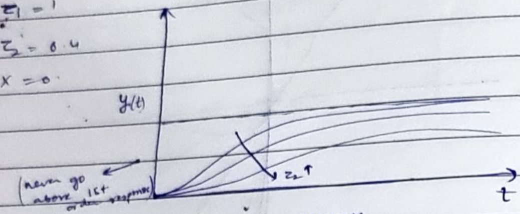




→ $\tau_1 = 1$

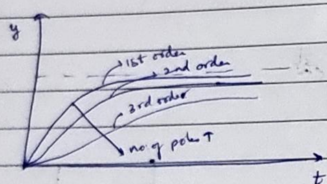
$\tau_2 = 0.4$

$x = 0.$



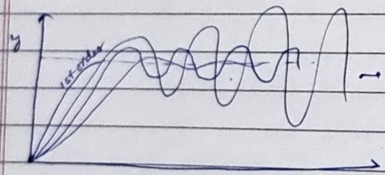
fastest response is 1st order response

by adding pole, response becomes slower.



→ (Behaviour is always monotonous)

↳ (In state space domain if eigenvalues are real & you will get monotonous)



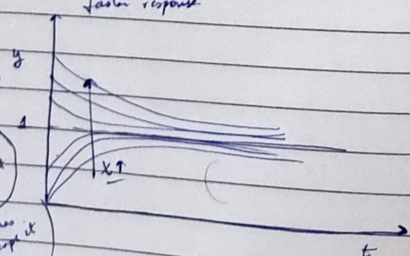
→ non monotonous. (2nd order)

→ real poles → monotonous response.

→ $\tau_1 = 0.25$

$x \neq 0$

faster response



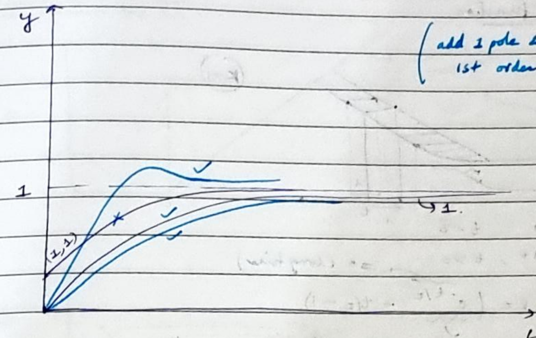
(adding of zero pushes it up)

(adding of poles ~~keeps~~ keeps it to 0)

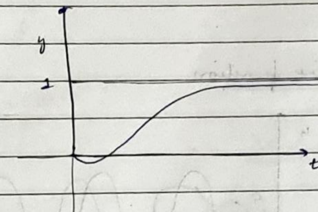
(add 1 pole & 1 zero to 1st order systems)

it will hang onto 0

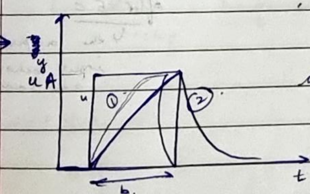
$\tau_1 \neq 0$, $\tau_2 \neq 0$, $x \neq 0$ } (2 poles, 1 zero)



→ $\tau_1 \neq 0$, $\tau_2 \neq 0$ } $\tau_1, \tau_2 > 0$, $x < 0$

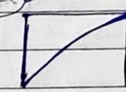


(Inverse response)

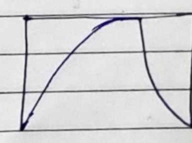


depends on τ whether the response will reach A

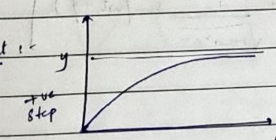
if $b = 4\tau$

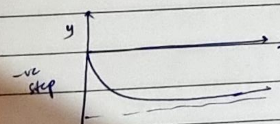


→ if $b >> 4\tau$

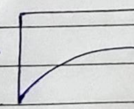


→ step input :



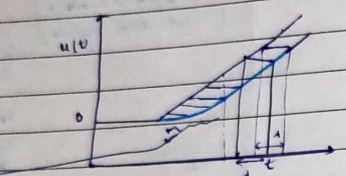


→ if $\tau >> b$



Ideal ramp function

(k = 1)



distance between input & output keeps existing

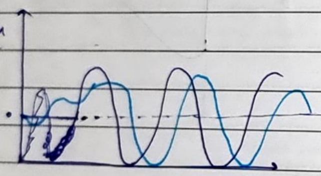
$$u(t) = \begin{cases} 0 & t<0 \\ At, & t>0 \end{cases}$$

$$y(t) = Ak\tau\left(\underbrace{e^{-t/\tau}}_{=0 \text{ (long time)}} + t/\tau - 1\right)$$

this will matter in the initial stages.

(after 4τ the distance between input & output will not change)

ideal sine function

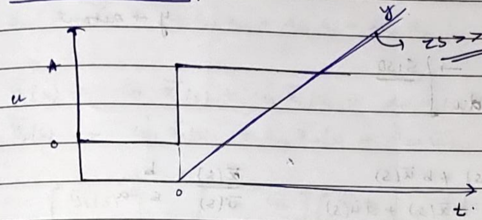


after 5τ
↳ the output and input behaviours are same.

$$u(t) = \begin{cases} 0 & t<0 \\ A\sin\omega t & t>0 \end{cases}$$

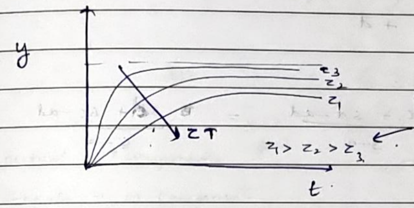
$$y(t) = AK\left[\frac{\omega\tau}{1+\omega^2\tau^2}\, e^{-t/\tau} + \frac{1}{\sqrt{1+\omega^2\tau^2}}\sin(\omega t+\phi)\right]$$

Ideal step function



$y(t) = Akt$

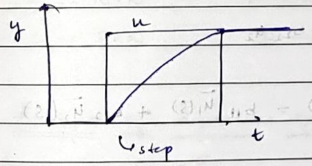
unstable system



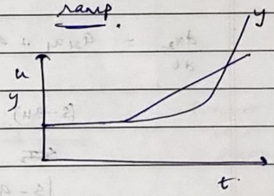
$\tau_1 > \tau_2 > \tau_3$

$$g(s) = \frac{k}{\tau s+1}$$

$\tau s >> 1$

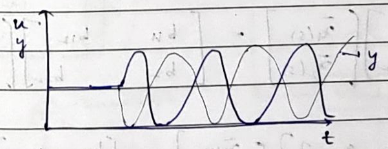


ramp



(Transfer function indicates stability)

sine function



MIMO system

$x_i \to$ input variables

$y \to$ output

$$\left.\begin{aligned}\frac{dx}{dt} &= ax + bu \\ y &= cx + du\end{aligned}\right\} \text{SISO}$$

$$s\bar{x}(s) = a\bar{x}(s) + b\bar{u}(s) \qquad \frac{\bar{x}(s)}{\bar{u}(s)} = \frac{b}{s-a}$$

$$\bar{y}(s) = c\bar{x}(s) + d\bar{u}(s)$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{cb}{s-a} + d$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{bc + sd - ad}{s-a} = \frac{sd + bc - ad}{s-a}$$

order $\to (1,1)$

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2$$

$$(s-a_{11})\bar{x}_1(s) - a_{12}\bar{x}_2(s) = b_{11}\bar{u}_1(s) + b_{12}\bar{u}_2(s)$$

$$(s-a_{22})\bar{x}_2(s) - a_{21}\bar{x}_1(s) = b_{21}\bar{u}_1(s) + b_{22}\bar{u}_2(s)$$

$$\begin{bmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22}\end{bmatrix}\begin{bmatrix}\bar{x}_1(s) \\ \bar{x}_2(s)\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}\bar{u}_1(s) \\ \bar{u}_2(s)\end{bmatrix}$$

$$\left\{ s\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\right\}\begin{bmatrix}\bar{x}_1(s) \\ \bar{x}_2(s)\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}\bar{u}_1(s) \\ \bar{u}_2(s)\end{bmatrix}$$

$$\boxed{(s\underline{I} - \underline{A})\,\bar{\underline{x}}(s) = \underline{B}\,\bar{\underline{u}}(s)}$$

$$y_1 = c_{11}x_1 + c_{12}x_2 + d_{11}u_1 + d_{12}u_2$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + d_{21}u_1 + d_{22}u_2$$

Output $\to y_i$

Input $\to u_i$

$$\bar{y}_1(s) = c_{11}\bar{x}_1(s) + c_{12}\bar{x}_2(s) + d_{11}\bar{u}_1(s) + d_{12}\bar{u}_2(s)$$

$$\bar{y}_2(s) = c_{21}\bar{x}_1(s) + c_{22}\bar{x}_2(s) + d_{21}\bar{u}_1(s) + d_{22}\bar{u}_2(s)$$

$$\begin{bmatrix}\bar{y}_1(s) \\ \bar{y}_2(s)\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{bmatrix}\begin{bmatrix}\bar{x}_1(s) \\ \bar{x}_2(s)\end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{bmatrix}\begin{bmatrix}\bar{u}_1(s) \\ \bar{u}_2(s)\end{bmatrix}$$

$$\bar{y}(s) = \underline{C}\,\bar{\underline{x}}(s) + \underline{D}\,\bar{\underline{u}}(s)$$

$$\underset{p\times 1}{\bar{y}(s)} = \Big\{\underbrace{\underline{C}\big[(s\underline{I}-\underline{A})^{-1}\underline{B}\big] + \underline{D}}_{\text{Transfer function}}\Big\}_{P\times M}\,\bar{u}(s)$$

output $\to$ P components

Inputs $\to$ m components

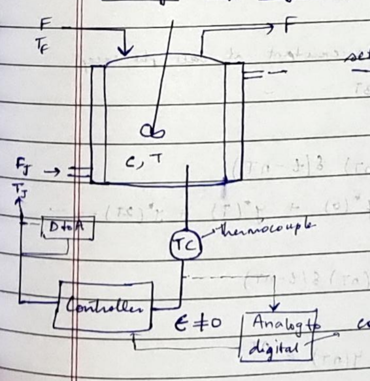
$x \to N\times 1$ $\qquad$ $y \to P\times 1$

$\underline{A} \to N\times N$ $\qquad$ $\underline{B} \to N\times M$

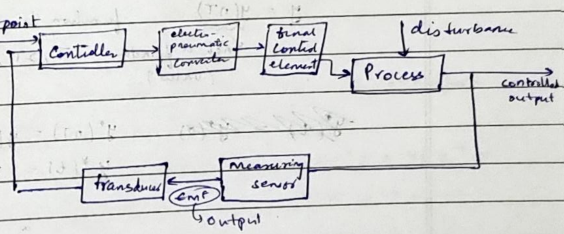
$\underline{C} \to P\times N$ $\qquad$ $\underline{D} \to P\times M$

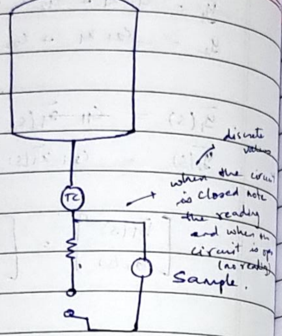
Analysis of dynamics of discrete-time systems

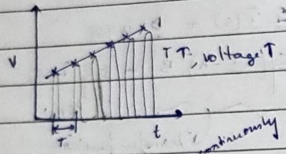
Process = maintain const temp & conc



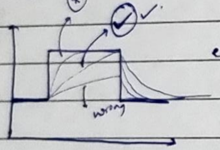
continuous signal to discrete (to feed into the controller)







Sampler — signal is changing continuously



electro mechanical switch.

If you close the switch for longer time, original signal will change

$y = y(t)$

$y = \underset{\sim}{y(nT)}$ function is non-existent at all pts except $T, 2T, 3T$ - - -

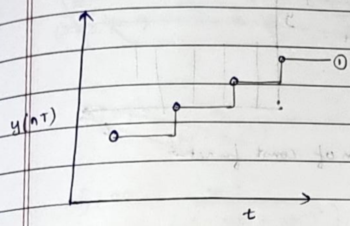
✓not a function (value)

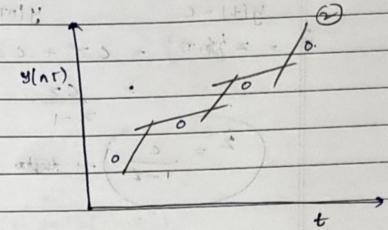
$y^*(nT) = y(nT)\ \delta(t - nT)$

$y^*(t) = y^*(0) + y^*(T) + y^*(2T)$ - - -

$$y^*(t) = \sum_{n=0}^{\infty} y(nT)\ \delta(t - nT)$$

$$y^*(s) = \sum_{n=0}^{\infty} y(nT)\ e^{-nTs}$$





$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT}(t - nT) + \frac{1}{2!}\left(\frac{d^2m}{dt^2}\right)_{nT}(t - nT)^2 + \cdots$$

write taylor series expansion to convert this discrete system to continuous

sampler is holding the signal

→ zero order hold

$m(t) = m(nT) \longrightarrow$ ① → continuous signal but not the best one

$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT}(t - nT) \longrightarrow$ ②

→ $y(t) = c + a(t - b)$

1st order hold.

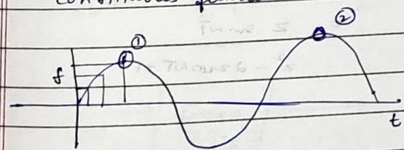
→ continuous domain → laplace transform → variable changes ↳ output that we get is in s.

→ discrete domain → z - transform

Z-transform

$$z\{y(0), y(T), y(2T) \ldots\} = \sum_{n=0}^{\infty} y(nT)\ z^{-n}$$

continuous function ⟶ discrete function ⟶ Z-transform



0.3
0.45
0.65
0.9
1

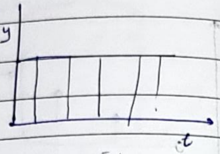
$z\{y(nT)\} = 0.3z^0 + 0.45z^{-1} + 0.65z^{-2} + $ —

(If we have ①, ② only, we will not know whether the function is sine or const)

$\frac{a}{1-r} \to a P$

$y(t) = c$

$y(nT) = \{c, c, c \ldots\}$



$\hat{z} = z\{y(nT)\} = c z^{0} + c z^{-1} + c z^{-2} + \ldots$

$= \frac{cz}{z-1}$

$\hat{z} = \frac{c}{1-z^{-1}} \to$ z transform of const function

$\Rightarrow y(t) = e^{-at}$

$y(nT) = \{1, e^{-aT}, e^{-2aT}, \ldots\}$

$\hat{z} = 1 z^{0} + (e^{-aT}) z^{-1} + e^{-2aT}(z^{-2}) + \ldots$

$\hat{z} = \frac{1}{1 - e^{-aT} z^{-1}}$

$\to y(t) = \sin \omega t$

$y(nT) = \{0, \sin \omega T, \sin 2\omega T, \ldots\}$

$\hat{z} = \sin \omega T z^{-1} + \sin 2\omega T z^{-2} + \ldots$

$\frac{e^{i\omega T} - e^{-i\omega T}}{2}$

$= \frac{z \sin \omega T}{z^{2} - 2z\cos \omega T + 1}$

Q) Invert the z-transform

$\hat{y}(z) = \frac{z}{z^{2} - 4z + 3}$

$\frac{z}{(z-3)(z-1)}$

$\cancel{\frac{z}{(z-3)(z-1)} \Rightarrow \frac{a}{z-3} + \frac{b}{z-1}}$

$\cancel{\hat{y}(z) = \frac{1}{(1-3z^{-1})}}$

$\hat{y}(z) = \frac{z^{-1}}{(1 - 4z^{-1} + 3(z^{-1})^{2})} = \frac{z^{-1}}{(1-3z^{-1})(1-z^{-1})}$

$\frac{1}{2} \frac{3z^{-1} - z^{-1}}{(1-3z^{-1})(1-z^{-1})}$

$\frac{1}{2} \frac{3z^{-1} - 1 - z^{-1} + 1}{(1-3z^{-1})(1-z^{-1})}$

$\Rightarrow \frac{1}{2}\left[\frac{-1}{1-z^{-1}} + \frac{1}{1-3z^{-1}} \right]$

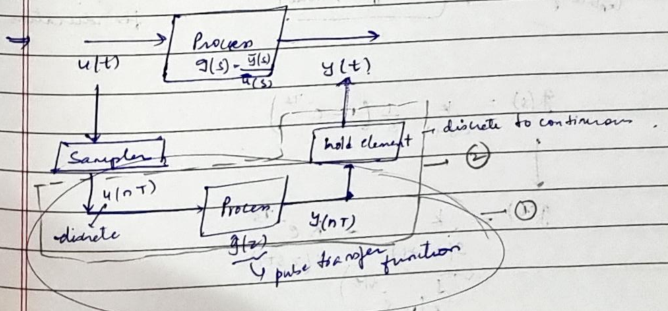
$\hat{y}(z) = \frac{1}{2} \frac{1}{1-3z^{-1}} - \frac{1/2}{1-z^{-1}}$

$\frac{1}{1 - e^{-aT} z^{-1}}$

$e^{-aT} = 3$ $\quad aT = -\ln 3 \quad a = \frac{-\ln 3}{T}$

$y(t) = \frac{1}{2}(e^{-at}) - \frac{1}{2} \quad a = \frac{-\ln 3}{T}$

$\Rightarrow$



→ "No hold" pulse transfer function ①

$$g(s) = \frac{k}{\tau s+1}$$

invert TF ↓

$$g(t) = \frac{k}{\tau}\left\{g(s)\right\}^{-1}\left(\frac{1}{s+1/\tau}\right)$$

$$g(t) = \frac{k}{\tau} e^{-t/\tau}$$

$$g(nT) = \left\{\frac{k}{\tau}, \frac{k}{\tau}e^{-T/\tau}, \frac{k}{\tau}e^{-2T/\tau} \ldots\right\}$$

$$\hat{g}(z) = \frac{k}{\tau}\left(\frac{1}{1-e^{-T/\tau}z^{-1}}\right)$$

→ Pulse TF of a first order process with zero order ②

$$g_p(s) = \frac{k}{\tau s+1}$$

$$g_p(t) = \frac{k}{\tau} e^{-t/\tau}$$ — process

$$g_h(s) = \frac{1}{s}(1-e^{-Ts})$$

→ shift operator

step input (holds) laplace of input

(holds for T time)

hold element maintains signal at const value for duration T

$$g(s) = \frac{k}{\tau s+1}\,\frac{1}{s}\left(1-e^{-Ts}\right)$$

↓

$$g(t) = \frac{k}{s(\tau s+1)} - \frac{k e^{-Ts}}{s(\tau s+1)}$$

$$g(t) = \begin{cases} k(1-e^{-t/\tau}) & t<T \\ k(1-e^{-t/\tau}) - k(1-e^{-(t-T)/\tau}) & t>T \end{cases}$$

→ shift operator for z transform ⇒ z^{-1}

$$\hat{g}(z) = \frac{k(1-e^{-T/\tau})z^{-1}}{1-e^{-T/\tau}z^{-1}}$$ → TF — utility is same as laplace transform

→ step response of a discrete time first order system with zero-order hold element

$$g(s) = \frac{k}{\tau s+1}$$

$$\bar{u}(s) = \frac{A}{s}$$

$$\bar{y}(s) = \frac{Ak}{s(\tau s+1)}$$

$$\hat{u}(z) = \frac{A}{1-z^{-1}}$$

$$\hat{y}(z) = \frac{k(1-e^{-T/\tau})z^{-1}}{1-e^{-T/\tau}z^{-1}} \cdot \frac{A}{1-z^{-1}}$$

→ inversion of this will give you response of discrete time