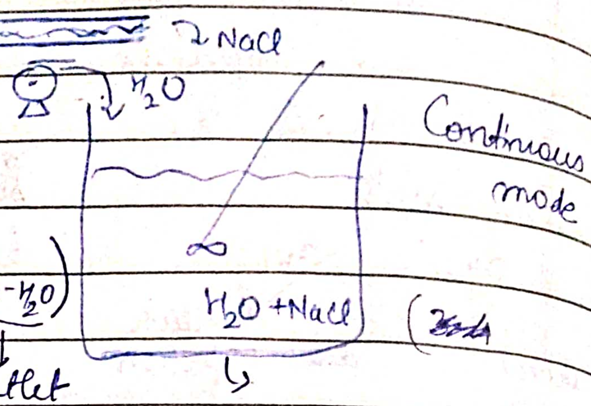
26/2/21

Analysis of Dynamical Systems in Transform Domain

Inpute rate - output rate = assumulation rate

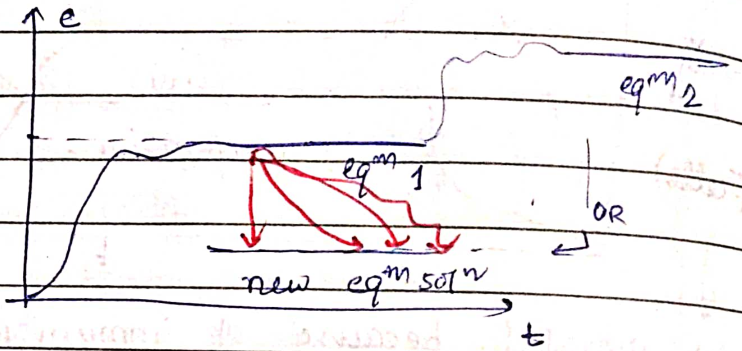


$$\Rightarrow \frac{d}{dt}(C_{NaCl}) = f\left(Q_{H_2O}, \dot{m}_{NaCl}, \underbrace{Q_{NaCl-H_2O}}_{\text{outlet}}\right)$$

Determine eq^m sol^n to the above eq^n - why needed?

Once we reach eq^m,

fluctuations after SS, we will have reach new eq^m sol^n.



Objective: How does the system evolve b/w 2 eq^m solutions.

- Dynamics b/w 2 eq^m sol^n is Transfer Domain modeling:

First order dynamics -

$$a_1 \frac{dy}{dt} + a_0 y = bu(t) \quad \text{---} \quad (1)$$

$$\Rightarrow \frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0}u(t)$$

Standard notation $\Rightarrow \tau \frac{dy}{dt} + y = Ku(t) \quad \text{---} \quad (2)$

output variable (y)

$u(t)$ ← input variable / forcing function

$\tau = \frac{a_1}{a_0}$ and $K = \frac{b}{a_0}$

τ ↑ Time constant

K ↑ Process gain / static gain / gain

Here y: C_{NaCl}

u: $\dot{m}_{NaCl}$ → forcing a change in system ⇒ output variable system at steady state "when a disturbance is introduced" to the system $(t=0)$

this means, the system was at steady state before.

At that time, this eqⁿ is valid:

$$\boxed{t=0} \quad \tau \frac{dy_s}{dt} + y_s = K u_s(t) \qquad \text{--- (3)}$$

Governing eqⁿ is the eqⁿ at 1st steady state.

(2) - (3)

$$\tau \frac{d(y-y_s)}{dt} + (y-y_s) = K(u(t)-u_s(t))$$

Let $y-y_s = y^*$ — deviation variable

$$\boxed{\tau \frac{dy^*}{dt} + y^* = K u^*(t)} \qquad \text{--- (4)}$$

We can conveniently apply Laplace transform to eqⁿ (4)

$$\tau s \bar{y}^*(s) - \underbrace{y^*(0)}_{=0} + \bar{y}^*(s) = K \bar{u}^*(s)$$

=0 (so, the eqⁿ 4 is convenient)

$$(\tau s+1)\bar{y}^*(s) = K\bar{u}^*(s)$$

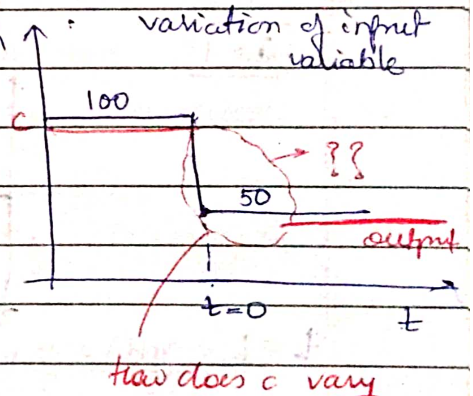
$$\bar{y}^*(s) = \frac{K\,\bar{u}^*(s)}{(\tau s+1)} \Rightarrow \boxed{\frac{\bar{y}^*(s)}{\bar{u}^*(s)} = \frac{K}{(\tau s+1)}} = g(s)$$

Input NaCl $\dot{m}$ = 100 kg/h

suddenly the flow rate drops to 50 kg/h

This type of forcing fⁿ is called an

a) ideal step input



$$u(t) = \begin{cases} 0 & t<0 \\ A & t>0 \end{cases}$$

$$g(s) = \frac{K}{\tau s+1} = \frac{\bar{y}^*(s)}{\bar{u}^*(s)}$$

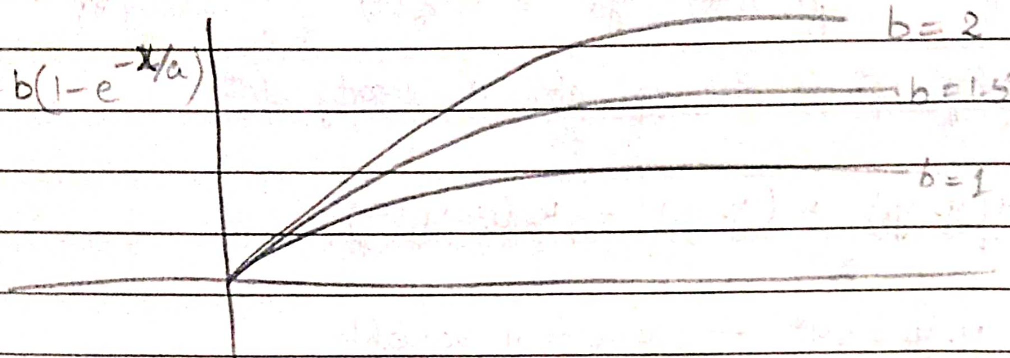
$$\Rightarrow \bar{y}^*(s) = \frac{K}{\tau s+1} \cdot \frac{A}{s} \quad ; \quad \bar{u}^*(s) = \frac{A}{s}$$

$$\Rightarrow \bar{y}^*(s) = AK\left[-\frac{\tau}{\tau s+1} + \frac{1}{s}\right] \leftarrow \text{partial fraction}$$

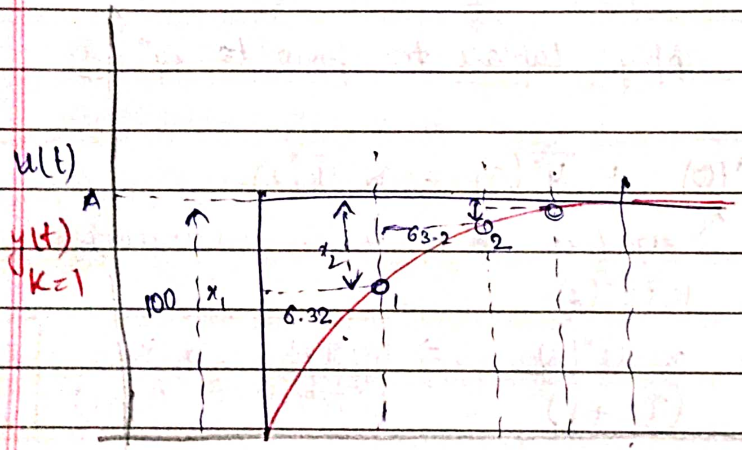
$$\bar{y}^*(s) = AK\left[\frac{1}{s} - \frac{1}{s + 1/\tau}\right]$$

$$y^*(t) = AK\left[1 - e^{-t/\tau}\right]$$

$f(x) = b(1 - e^{-x/a})$



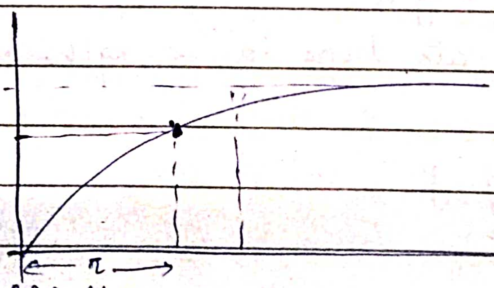
Convention: input in blue



Point 1 is 63.2% of x_1

Point 2 is 63.2% of the remaining (that is x_2) of one 1 time constant before's

$\boxed{a = 1}$



In 4 time constants we reach ~ 98.2% of overall remaining

$t = \tau,\ y(t) = 0.632\, y_{eq}$

In 5 time constants, 99.3%

Q In what time will system reach new SS?

$\tau = a_1/a_0$ from system

$4\tau \rightarrow 98.2\%$

$5\tau \rightarrow 99.3\%$

infinite time $\rightarrow 100\%$

$2\tau - 86.45\%$

$3\tau - 95.01\%$

$4\tau - 98.17\%$

τ: time reqd to ~~can~~ reach 63.% of step forced input f^n

Does not mean τ is specifically linked to step f^ns.

Date
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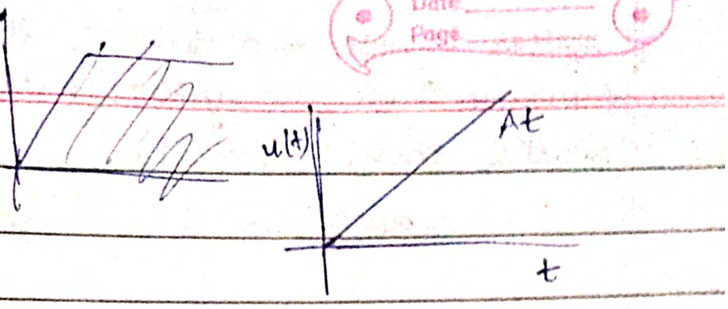
Take another forcing f^n

b) Ideal ramp function:

$$u(t) = \begin{cases} 0, & t<0 \\ At, & t>0 \end{cases}$$



$g(x) = x$

$f(x) = a\left(e^{-x/a} + x/a - 1\right)$

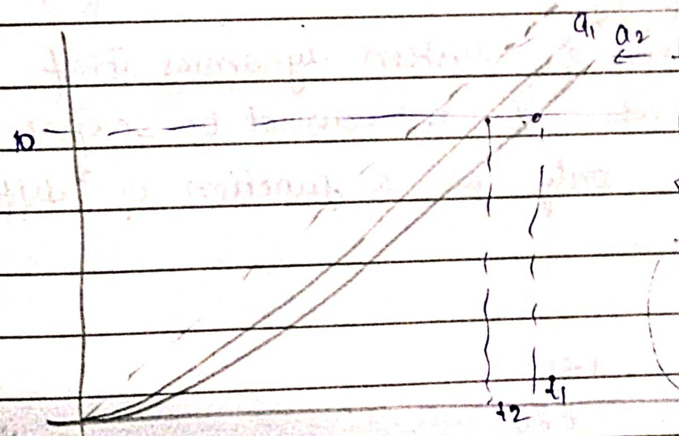
$$\bar{y}^*(s) = \frac{K}{\tau s+1}\,\bar{u}^*(s) \quad ; \quad \bar{u}^*(s) = \frac{A}{s^2}$$

$$\bar{y}^*(s) = KA\left[\frac{1}{\tau s+1}\cdot\frac{1}{s^2}\right]$$

$$= AK\left[\frac{1}{s+1/\tau} - \frac{1}{s} + \frac{1}{\tau s^2}\right]$$

[illegible]

$$\boxed{y(t) = AK\tau\left(e^{-t/\tau} + \frac{t}{\tau} - 1\right)}$$



a_2 ← $t \to \infty$ the f^n never merge

output follows the input but will never be able to catch the input

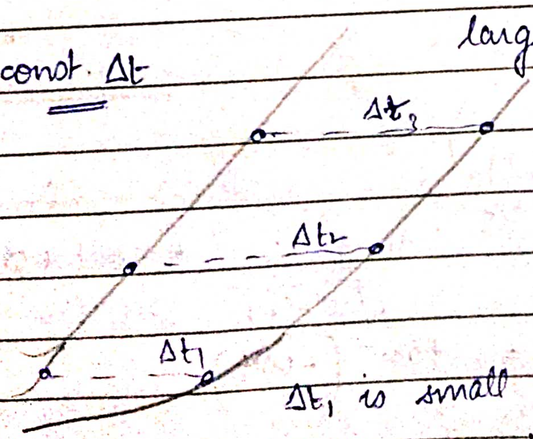
unlike step input.

always a lag in time difference

We wanted "10". Reaches "10" after some time.

large $\tau \Rightarrow$ more lag

Goal: To reach const. Δt



Δt_1 is small

$\Delta t \uparrow$ then becomes constant

When will the Δt be constant? — after 5τ

" τ "

If system adjusts to a new SS, there will be transience in the system, but after some time, no transience occurs, then. $(\sim 5\tau)$

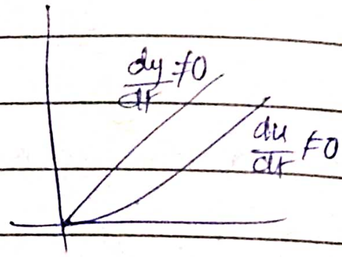
so, concept of τ is there ~~for~~ in other systems also.

- Time invariant : $\frac{dy}{dt} = 0$ & $\frac{du}{dt} = 0$

- Time independent : not a f^n of "t"

$\Rightarrow$ In case of ramp f^n, neither of the input or output are zero.

$\Rightarrow$ Here, the difference b/w u & y in slopes is invariant with time.



$\Rightarrow$ $\frac{d}{dt}(u-y) = 0$

System in absolute sense is never time invariant.

$\Rightarrow$ Step ~~step~~ input -

τ defined in terms of system dynamics itself.

$\Rightarrow$ for ~~th~~ other functions, τ ~~not~~ cannot be expressed so, only as a function of t difference.

c) Sinusoidal input :

$$u(t) = \begin{cases} 0 & t<0 \\ A\sin\omega t & t>0 \end{cases}$$

$$\bar{u}^*(s) = \frac{A\omega}{s^2+\omega^2}$$

$A\omega K\left[\frac{M\omega}{s^2+\omega^2} + \frac{N/\tau}{\tau s+1}\right]$

$M\omega s + M\omega + \frac{N}{\tau}s^2 + \frac{N\omega^2}{\tau}$

$$\bar{y}^*(s) = \frac{A\omega}{s^2+\omega^2}\left(\frac{K}{\tau s+1}\right)$$

$$y(t) = AK\left[\frac{\omega\tau}{1+\tau^2\omega^2}e^{-t/\tau} + \frac{1}{\sqrt{\omega^2\tau^2+1}}\sin(\omega t+\phi)\right]$$

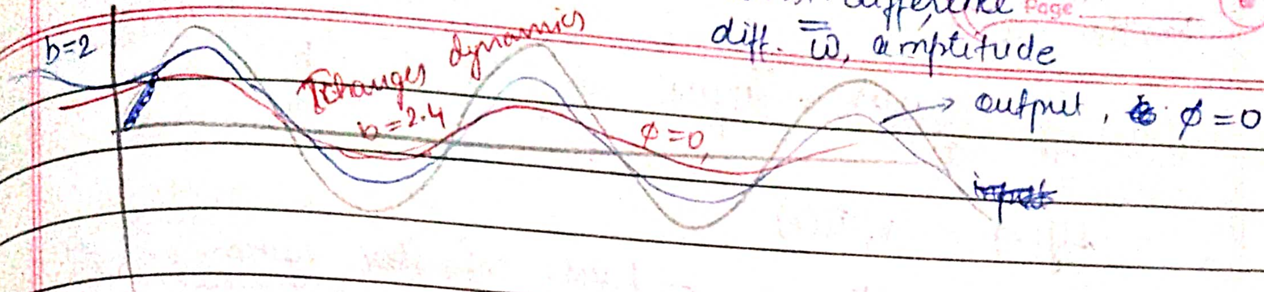
where $\phi = \tan^{-1}(-\omega\tau)$

Heaviside cover-up

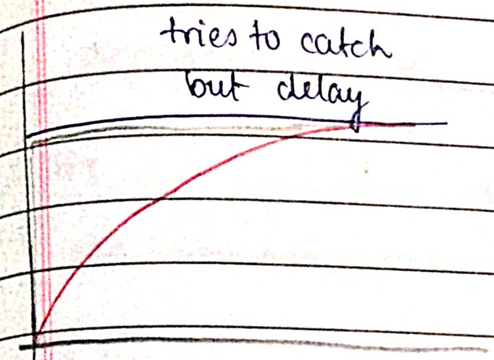
$$A_K = \lim_{s\to r_K}(s-r_K)Y(s)$$

$\phi = 0 \Rightarrow \tau = 0$ (no lag)

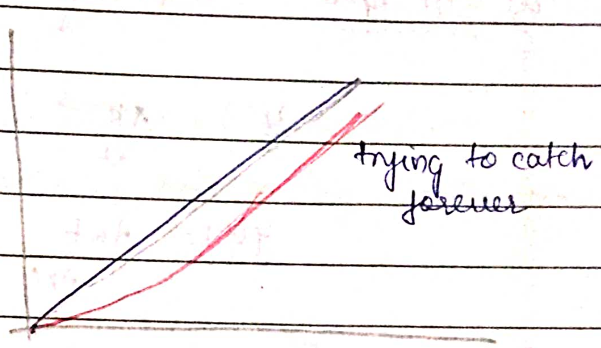
Ultimate response : $\underline{\text{large } t}$
sinusoidal
const. difference
diff. ω, amplitude



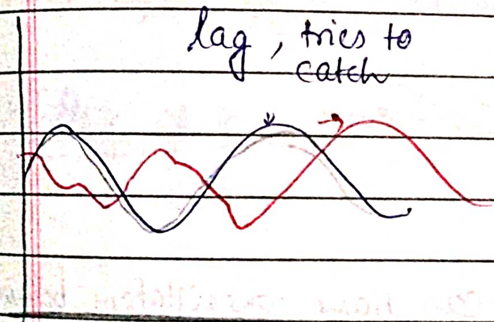
Step



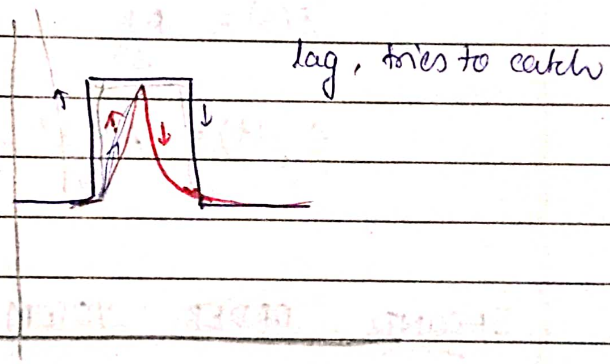
Ramp



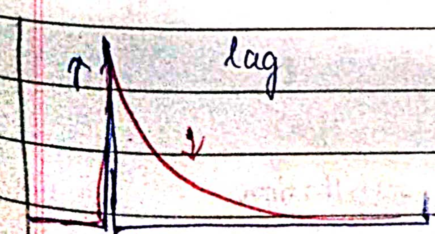
sinusoid



pulse



Impulse



This happens because of non-zero time constant, τ.

If $\tau \to 0$, system respons very quickly
$\Rightarrow$ system will adapt to new state w/o transience.

$$\tau \frac{dy}{dt} + y = Ku(t)$$

Pure gain systems.

$$g(s) = \frac{k}{\tau s + 1}$$

when, $\tau = 0$, $g(s) = K$ $\rightarrow$ only gain in system

& no time const.

$$a_1 \frac{dy}{dt} + a_0 y = b u(t)$$

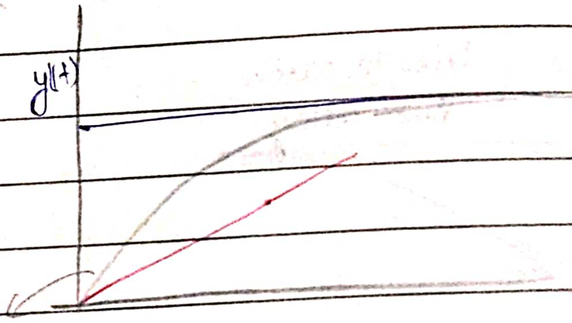
Consider $a_0 = 0$

$$\frac{dy}{dt} = \frac{b}{a_1} u(t) = K_1 u(t) \quad ; \quad K_1 = \frac{b}{a_1}$$

$$s\bar{y}(s) = K_1 \bar{u}(s)$$

$$g(s) = \frac{K_1}{s} \quad \text{— pure capacitor system}$$

a) Step input ; $\bar{u}(s) = \frac{A}{s}$

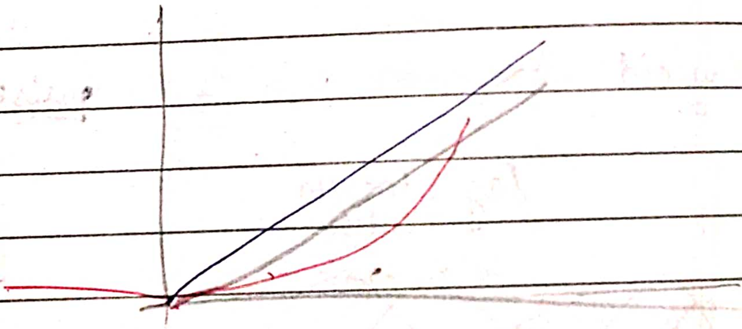


$$\bar{y}(s) = \frac{AK}{s^2}$$

$$y(t) = AKt$$

$AK(1-e^{-t/\tau})$

b) Ramp : $\bar{u}(s) = \frac{A}{s^2}$



$$\bar{y}(s) = \frac{AK}{s^3}$$

$$\Rightarrow y(t) = \frac{AKt^2}{2}$$

SECOND ORDER SYSTEM — can have oscillatory behavior

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b u(t)$$

$$\left(\frac{a_2}{a_0}\right)\frac{d^2y}{dt^2} + \left(\frac{a_1}{a_0}\right)\frac{dy}{dt} + y = \left(\frac{b}{a_0}\right)u(t)$$

$\frac{a_2}{a_0} = \tau^2$, τ = natural period of oscillation

$\frac{a_1}{a_0} = 2\xi\tau$; ξ = damping coefficient — signifies whether the response to new change will be oscillatory ... etc.

$\frac{b}{a_0} = K$; K = gain

$\tau^2 \frac{d^2 \bar{y}(\cancel{s})}{dt^2}$

Deviat^n & LT

$$\tau^2\left(s^2\bar{y}(s) - s y(0) - \cancel{s} y''(0)\right) + 2\tau\xi\left(s\bar{y}(s) - y'(0)\right) = K\bar{u}(s)$$

$$g(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$$ ← sol^n of quadric eq^n

↓

partial fraction.

↓

asymptotic response, oscillatory

from state space analysis

2nd order systems show osc. response ∴ but ~~to~~ they can ~~do~~ show that also b/w 2 SSs.

Problems — convert to standard form (forcing f^n: hT∞

11/3 Analysis of the transfer function :

- State-space domain : $\frac{dx_1}{dt} = f_1(x_1, \ldots)$

$$\frac{dx_2}{dt} = f_2(x_1, \ldots)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, \ldots)$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$ ← Matrix Eq^n

↑ Coeff matrix ↑ dynamical vector

↓

null space → eq^m solutions

eigen values → stability analysis

- Transform domain analysis : $\frac{dy}{dt} = f(y, u)$ ← input variable

↑ output variable

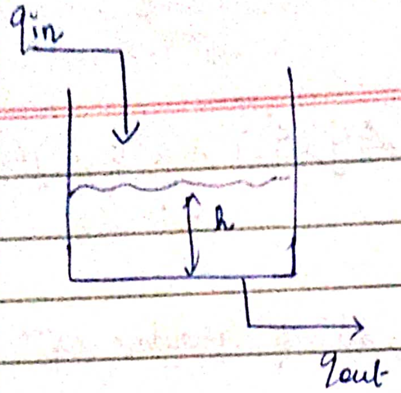
↓

Laplace tranform ($g(s)$ ← transfer f^n)

↑ tells about dynamical features of the system

$$\Rightarrow g(s) = \frac{\bar{y}(s)}{\bar{u}(s)} \quad \Rightarrow y(t) = L^{-1}\left[g(s)\,\bar{u}(s)\right]$$

Order $= 1$



$$A\frac{dh}{dt} = q_{in} - q_{out}$$

$$\frac{dh}{dt} = \frac{q_{in}}{A} - \frac{q_{out}}{A}$$

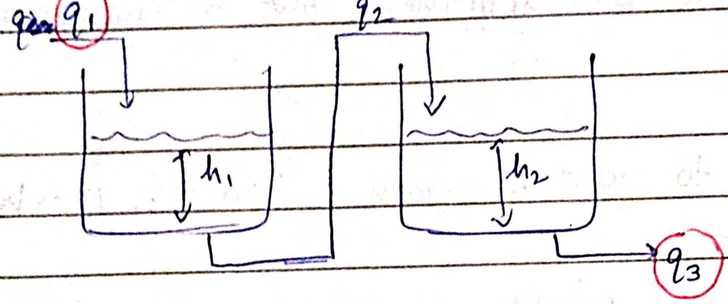
If we want to analyze dynamics of h

h : dynamical variable → output variable

How will h change?

→ Either by changing q_{in} or q_{out}

Now, if we have 2 tanks



$$A_1\frac{dh_1}{dt} = q_1 - q_2 \quad \text{--- (I)}$$

$$A_2\frac{dh_2}{dt} = q_2 - q_3 \quad \text{--- (II)}$$

Order $= 2$

~~Our interest can be~~

- Overall input & overall output of the system -

say, output variable $= q_3$ (it might go to a column)

input variable $= q_1$

Eq^n (1) is a 1st order ODE

$$g_1(s) = \frac{K_1}{\tau_1 s + 1}$$

Eq^n (2) is a 1st order ODE

$$g_2(s) = \frac{K_2}{\tau_2 s + 1}$$

$$\bar{u}(s) \longrightarrow \boxed{g_1(s)} \xrightarrow{\bar{y}_1(s)} \boxed{g_2(s)} \xrightarrow{\bar{y}_2(s)}$$

$$g_1(s) = \frac{\bar{y}_1(s)}{\bar{u}(s)} \qquad g_2(s) = \frac{\bar{y}_2(s)}{\bar{y}_1(s)}$$

$$\Rightarrow g_1(s)\, g_2(s) = \frac{\bar{y}_2(s)}{\bar{u}(s)} \quad \begin{matrix} \leftarrow \text{Final output} \\ \leftarrow \text{first input} \end{matrix}$$

Overall transfer f^n for n^{th} order system:

$$\bar{g}(s) = \prod_{i=1}^{n} g_i(s)$$

For a two-tank system, $g(s) = \dfrac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)}$ → 2^{nd} order

→ Def^n of order in State space domain : no. of 1^{st} order ODEs

→ " " " in Transform domain : degree of polynomial of the denom^r of the transfer f^n.

I) Two tanks in a series:

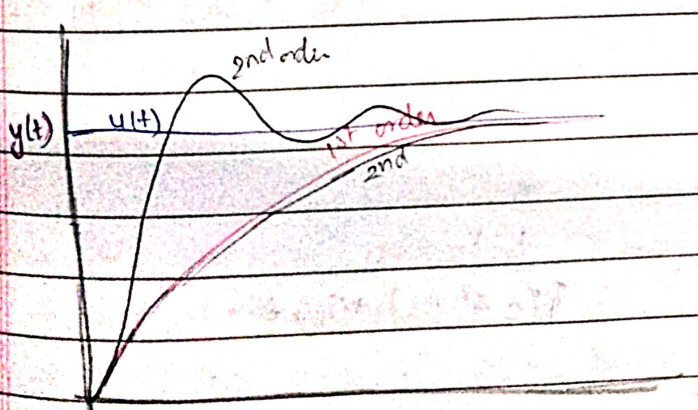
$$g(s) = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

II) spring-mass system:

$$g(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s+1}$$

1^{st} order $g_o(s) = \dfrac{K}{\tau s+1}$ 2^{nd} order: $\dfrac{K}{\tau^2 s^2 + 2\zeta\tau s+1}$

We want to know the difference b/w these two ↑ system subjected to a unit step change.



2nd order: ~~we ha~~ → monotonous
→ oscillatory

We also saw this in S-S domain

λ → pure im → oscillatory
↳ complex → " (- - -
↳ real → monotonous

~~y(t)~~ $= \dfrac{\bar{y}(s)}{\bar{u}(s)} = \dfrac{K}{(\tau_1 s+1)(\tau_2 s+1)}$

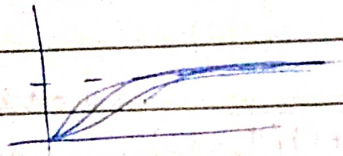
$\bar{u}(s) = \frac{1}{s}$

$$y(t) = L^{-1}\left\{ \frac{K}{s(\tau_1 s+1)(\tau_2 s+1)} \right\}$$

$$y(t) = K\left\{ 1 - \frac{\tau_1}{\tau_1-\tau_2} e^{-t/\tau_1} - \left(\frac{\tau_2}{\tau_2-\tau_1}\right) e^{-t/\tau_2} \right\}$$

Desmos :

whatever the values of τ_1, τ_2 — no oscillations in system



Qualitative analysis :

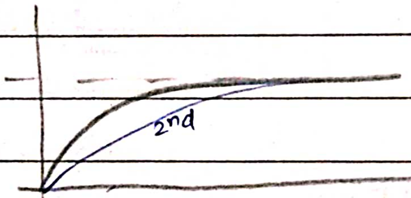
→ 2nd order transfer f^n obtained by multiplication of 1st order transfer f^ns never show oscillations.

* 1st order systems in series

→ How does it compare against 1st order dynamics.

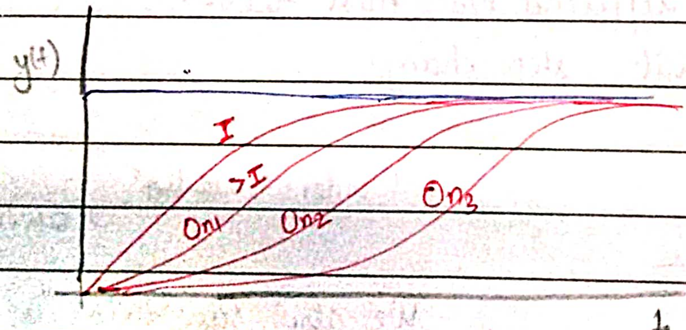
$$g(s) = \frac{K}{\tau s+1} \qquad \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{\tau s+1}$$

$$\Rightarrow y(t) = K\left(1 - e^{-t/\tau}\right)$$



Never does the 2nd order faster than 1st order.

A higher order system is always slower than 1st order sys.



$O_{n_3} > O_{n_2} > O_{n_1} > I$

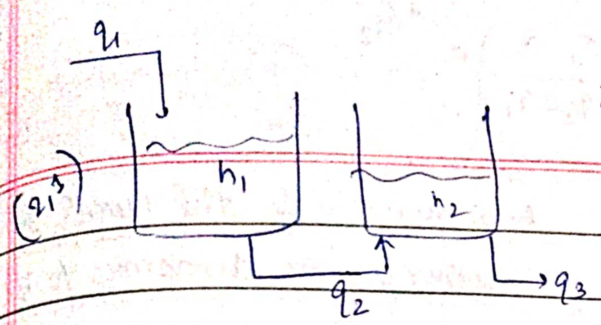
$$g(s) = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{K_1 K_2}{\tau_1 \tau_2 s^2 + (\tau_1+\tau_2) s + 1}$$

~~↓ corresponds to~~

$$\bar{u}(s) \longrightarrow \boxed{g_1(s)} \xrightarrow{\bar{y}_1(s)} \boxed{g_2(s)} \longrightarrow \bar{y}_2(s)$$

This means that the level of liquid in tank 1 is not affected by the level of liquid in tank 2.

Non-interacting systems.



There can be a backflow depending on the levels of h_1 & h_2

$y_1 = f(y_1, y_2, u)$ | $a_1 \dfrac{dy_1}{dt} + a_2 y_1 + a_3 y_2 = a_4 u$

$y_2 = f(y_1, y_2)$ | $b_1 \dfrac{dy_2}{dt} + b_2 y_2 - b_3 y_1 = 0$

$$\Rightarrow \frac{a_1}{a_2}\frac{dy_1}{dt} + y_1 + \frac{a_3}{a_2} y_2 = \frac{a_4}{a_2} u_1 \Rightarrow \left(\frac{a_1}{a_2} s + 1\right) \bar{Y}_1(s) + \frac{a_3}{a_2} \bar{Y}_2(s) = \frac{a_4}{a_2} \bar{U}(s)$$

$$\Rightarrow \frac{b_1}{b_2}\frac{dy_2}{dt} - \frac{b_3}{b_2} y_1 + y_2 = 0 \Rightarrow \left(\frac{b_1}{b_2} s + 1\right) \bar{Y}_2(s) - \frac{b_3}{b_2} \bar{Y}_1(s) = 0$$

Simultaneously solve the algebraic eqⁿs for $\bar{Y}_1, \bar{Y}_2$

dynamics of the 1st tank

$$\frac{\bar{Y}_1(s)}{\bar{U}(s)} = \frac{\left[\dfrac{a_4 b_1 s + b_2 a_4}{a_2 b_2 + a_3 b_3}\right]}{\left(\dfrac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right) s^2 + \left(\dfrac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right) s + 1}$$

output of 1st tank / input

$$\frac{\bar{Y}_2(s)}{\bar{U}(s)} = \frac{\left(\dfrac{a_4 b_3}{a_2 b_2 + a_3 b_3}\right)}{\left(\dfrac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right) s^2 + \left(\dfrac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right) s + 1}$$

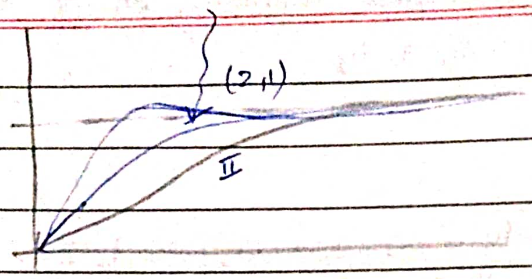
overall dynamics

→ Both are 2nd order dynamics.
→ Not a multiplication of 2 TFs.
→ 1st tank also 2nd order because it takes time to adjust to the disturbance from 2nd tank.
→ 1st tank dynamics, $g(s) = \dfrac{x s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$

Order of the system → (2,1)

$$y(t) = L^{-1}\left\{\frac{x s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}\right\}$$

$$y(t) = 1 - \left(\frac{\tau_1 - x}{\tau_1 - \tau_2}\right)e^{-t/\tau_1} - \left(\frac{\tau_2 - x}{\tau_2 - \tau_1}\right)e^{-t/\tau_2}$$

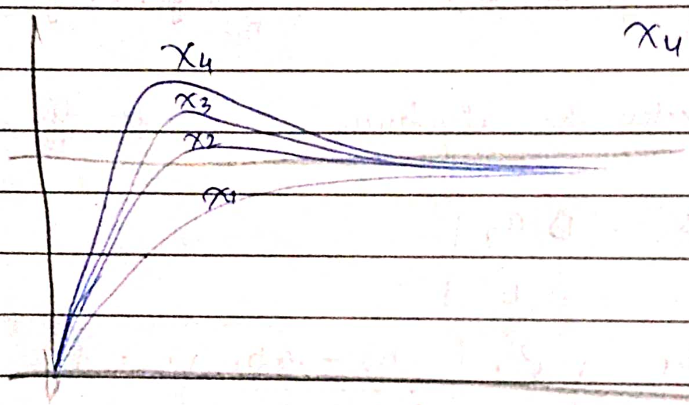


Any quantity in the numr is pushing the dynamics to be faster.

x is making system over-enthusiastic

x is changing dynamics to an extent that $\bar{u}(t)$ is also crossed. but ultimately comes to $u(t)$

Hence, it can overshoot.



$x_4 > x_3 > x_2 > x_1 > x$

$$g(s) = \frac{xs + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

← zero

← poles

More poles – more sluggish

more zeros – faster, enthusiastic

Multiple inputs

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m$$

$\vdots$

$$\frac{dx_n}{dt} = a_{1n}x_1 + a_{2n}x_2 + \dots a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m$$

$$\Rightarrow y_1 = C_1x_1 + C_2x_2 + \dots + C_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \dots + d_{1m}u_m$$

$\vdots$

p no of outputs.

$$y_p = C_{p1}x_1 + C_{p2}x_2 + \dots + C_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \dots + d_{pm}u_m$$

n-dynamical variables
m- input variables
p- output variables

MIMO

Date
Page

$$\frac{d\underline{X}}{dt} = \underline{A}\underline{X} + \underline{B}\underline{U}$$

$$\frac{d\underline{Y}}{dt} = \underline{C}\underline{X} + \underline{D}\underline{U}$$

Transform

$$\begin{cases} s\bar{x}_1 = a_{11}\bar{x}_1 + a_{12}\bar{x}_2 + \dots + a_{1n}\bar{x}_n + b_{11}\bar{u}_1 + b_{12}\bar{u}_2 + \dots + b_{1m}\bar{u}_m \\ \vdots \\ s\bar{x}_n = a_{n1}\bar{x}_1 + a_{n2}\bar{x}_2 + \dots + a_{nn}\bar{x}_n + b_{n1}\bar{u}_1 + b_{n2}\bar{u}_2 + \dots + b_{nm}\bar{u}_m \end{cases}$$

$$\begin{cases} \bar{y}_1 = c_{11}\bar{x}_1 + c_{12}\bar{x}_2 + \dots + c_{1n}\bar{x}_n + d_{11}\bar{u}_1 + d_{12}\bar{u}_2 + \dots + d_{1m}\bar{u}_m \\ \vdots \\ \bar{y}_p = c_{p1}\bar{x}_1 + c_{p2}\bar{x}_2 + \dots + c_{pn}\bar{x}_n + d_{p1}\bar{u}_1 + d_{p2}\bar{u}_2 + \dots + d_{pm}\bar{u}_m \end{cases}$$

n+p equations to be solved simultaneously

$$\left\{ s\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \vdots \\ \vdots & & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \right\} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ \vdots & & & \\ b_{n1} & \dots & & b_{nm} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_m \end{bmatrix}$$

$$\Rightarrow (s\underline{I} - \underline{A})\underline{\bar{X}} = \underline{B}\underline{\bar{U}}$$

$$\underline{\bar{X}} = (s\underline{I} - \underline{A})^{-1}\underline{B}\underline{U}$$ ← Laplace transform of dynamical variable

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ \vdots & & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_m \end{bmatrix}$$

$$\underline{\bar{Y}(s)} = \underline{\underline{C}}\,\bar{X} + \underline{\underline{D}}\,\bar{U}$$

$$\bar{Y}(s) = \underline{\underline{C}}\left\{[sI - A]^{-1} B\bar{U}\right\} + \underline{\underline{D}}\bar{U}$$

Transfer f^n matrix $\boxed{\dfrac{\bar{Y}(s)}{\bar{U}(s)} = \underline{\underline{C}}\{sI - A\}^{-1}B + \underline{\underline{D}}}$

For a SISO system $\rightarrow$ transfer f^n

For a MIMO system $\rightarrow$ transfer f^n matrix

Interconversion of state space $\Leftrightarrow$ Transfer domain

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + bu \\ y &= cx + du \end{aligned}\right\} \text{state-space domain model}$$

$$s\bar{x}(s) = a\bar{x}(s) + b\bar{u}(s)$$

$$\Rightarrow \bar{x}(s) = \frac{b}{(s-a)}\,\bar{u}(s)$$

$$\bar{y}(s) = c\bar{x}(s) + d\bar{u}(s)$$

$$\Rightarrow \bar{y}(s) = \frac{cb}{(s-a)}\bar{u}(s) + d\bar{u}(s)$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{cb}{s-a} + d$$

$$\bar{g}(s) = \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{ds - ad + cb}{s-a}$$

$$\bar{u}(s) \longrightarrow \boxed{g(s)} \longrightarrow \bar{y}(s)$$

block diagram

~~Can~~ Get ~~some~~ ~~form~~ to block diagram:

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{ds + (cb - ad)}{s-a}$$

$$s\bar{y}(s) - a\bar{y}(s) = ds\,\bar{u}(s) + (cb - ad)\,\bar{u}(s)$$

$s\bar{y}(s)$ must come from $\frac{dy}{dt}$

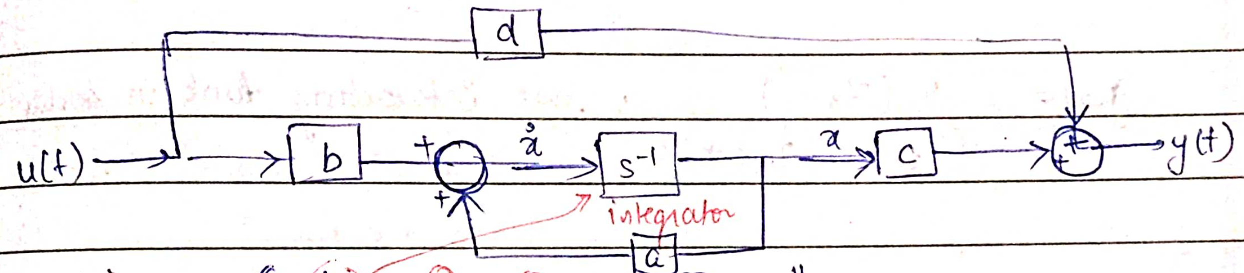
$$\Rightarrow \underbrace{\frac{dy}{dt} - ay}_{\text{output}} = \underbrace{\left(\frac{du}{dt}\right)d}_{\text{input}} + \underbrace{(cb - ad)u}_{\text{state-space domain model.}}$$

We have converted all outputs in terms of inputs & derivatives.

$\Downarrow$

Block diagram

- should have input, output, & those operations



$\frac{d(1)}{dt}$ - derivative

$$\left\{\begin{aligned} \frac{dx}{dt} &= ax + bu \quad \text{—(1)} \\ y &= cx + du \quad \text{—(2)} \end{aligned}\right.$$

Integrator : $\boxed{s^{-1}}$

Relationship b/w a & x is s.t. if we supply $\frac{dx}{dt}$ to with.

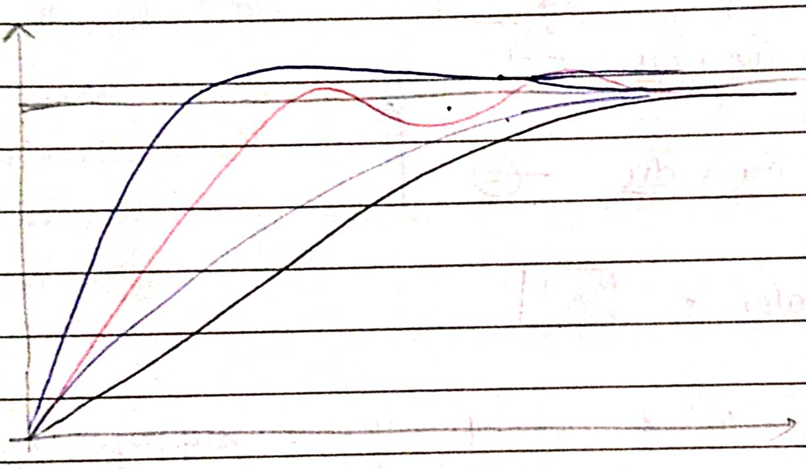
13/3/21 **Analysis of Transfer Functions :**

$$g(s) = \frac{K}{\tau s + 1}$$; liquid level in tank

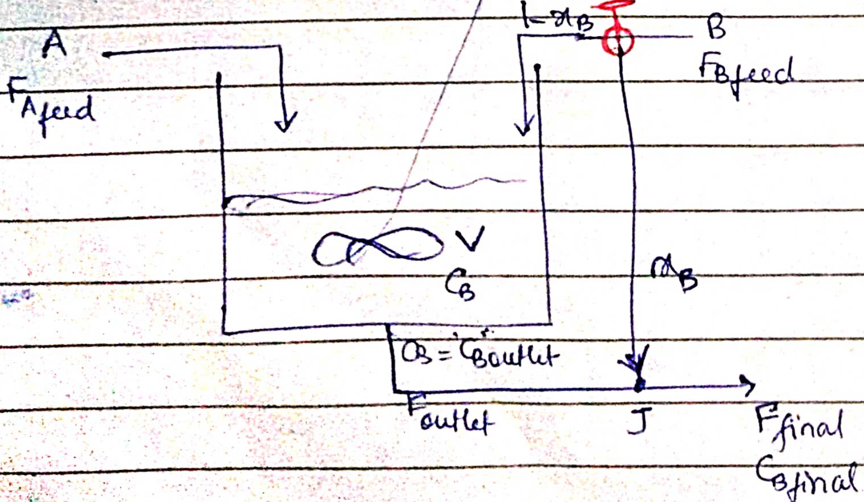
$$g(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$$; spring-mass

$$g(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$$; two tanks (non-interacting) in series

$$g(s) = \frac{K(\chi s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$; two interacting tank in series



Q. Consider a mixing tank in which a small quantity of liquid B is added to a large volumetric flow rate of liquid A. A fract^n α of B is fed to the outlet of tank.



Control value to ~~acco~~ get desired $C_{B final}$

Material balance around the tank:

input − output = accumulation

At steady state,

$$F_{A feed} + F_{B feed} = F_{outlet}$$

Since, $F_{A feed} >> F_{B feed} \Rightarrow F_{outlet} \approx F_{A feed}$

Material balance at the junction J:

$$F_{outlet} + x_B F_{B feed} = F_{final}$$

$$\Rightarrow F_{A feed} + x_B F_{B feed} = F_{final}$$

$\because F_{A feed} >> F_{B feed}$

$$\Rightarrow F_{outlet} \approx F_{final} \approx F_{A feed} \quad \text{—— (1)}$$

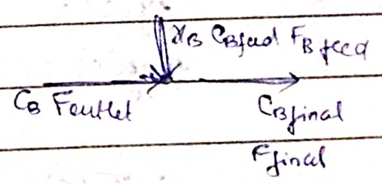
Material balance for B in the tank:

$$0 + (1-x_B) F_{B feed} \cdot C_{B feed} - (F_{outlet})\, C_{B outlet} = \frac{V\,dC_B}{dt}$$

$$(1-x_B) F_{B feed} \cdot C_{B feed} - F_{A feed}\, C_B = \frac{V\,dC_B}{dt}$$

$$\Rightarrow \frac{V}{F_{A feed}} \frac{dC_B}{dt} + C_B = \left(\frac{F_{B feed}\, C_{B feed}}{F_{A feed}}\right)(1-x_B) \quad \text{—— (2)}$$

At junction J:



x_1

$$C_B F_{outlet} + x_B C_{B feed} F_{B feed} = C_{B final} F_{final}$$

$$\Rightarrow C_B F_{A feed} + x_B C_{B feed} F_{B feed} = C_{B final} F_{final}$$

$$\Rightarrow C_B F_{A feed} + x_B C_{B feed} F_{B feed} = C_{B final} F_{A feed}$$

$$\Rightarrow C_{B final} = C_B + \frac{F_{B feed}}{F_{A feed}} x_B C_{B feed} \quad \text{—— (3)}$$

$\frac{d(\;)}{dt}$ dyn. vari

Dynamical eq's:

$C_B \leftarrow$ dynamical variable $= x$

$C_{B\,final} \leftarrow$ output variable $= y$

$$\frac{V}{F_{feed}}\frac{dC_B}{dt} + C_B = \left(\frac{F_{Bfeed}\,C_{Bfeed}}{F_{feed}}\right)(1-x_B) \leftarrow a_1\frac{dx}{dt} + a_0 x = bu$$

$\frac{F_{Bfeed}\,C_{Bfeed}}{F_{feed}} \leftarrow$ input variable $= u$

$\frac{V}{F_{feed}} = \tau$, $K = (1-x_B)$, $a_0 = 1$

$$\Rightarrow \tau\frac{dx}{dt} + x = (1-x_B)\,u \quad \text{—— I}$$

$$\Rightarrow y = x + x_B u \quad \text{—— II}$$

Our domain cast in such a form that we can use transform domain analysis:

$$s\tau\bar{x}(s) + \bar{x}(s) = (1-x_B)\,\bar{u}(s)$$

$$\Rightarrow \bar{x}(s) = \frac{(1-x_B)\,\bar{u}(s)}{\tau s + 1}$$

$$\bar{y}(s) = \bar{x}(s) + x_B\bar{u}(s)$$

$$\Rightarrow \bar{y}(s) = \left[\frac{1-x_B}{\tau s+1} + x_B\right]\bar{u}(s)$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{1-\cancel{x_B} + \tau s x_B + \cancel{x_B}}{\tau s+1}$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{(x_B\tau)s + 1}{(\tau s+1)} = \frac{\lambda s+1}{(\tau s+1)}$$

The transfer f^n of the system has the form:

$$\boxed{g(s) = \frac{\lambda s + 1}{\tau s + 1}}$$

(1,1) order system

$$g(s) = \frac{1}{\tau s+1} + \frac{\chi s}{\tau s+1}$$

This system is similar to having 2 systems in series

Step input : $u(s) = \frac{A}{s} = \frac{1}{s}$

$$y(s) = \frac{1\cdot(\chi s+1)}{s(\tau s+1)}$$

$$\Rightarrow y(t) = L^{-1}\left[\quad\right]$$

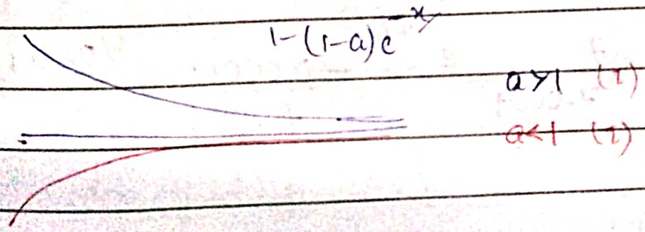
$$y(t) = 1 - \left(1-\frac{\chi}{\tau}\right)e^{-t/\tau}$$

$\frac{\text{lead time const}}{\text{lag time const}}$

Des

Pure capacitive response – quickly follows input

Here, pure capacitive response superimposed on 1st order dynamics



Lead Lag system

$$\boxed{g(s) = \frac{\chi s+1}{\tau s+1}}$$

eg: $g(s) = \frac{1}{5s+1}$

We need to look at what & how the system is composed of –

eg:

$$\longrightarrow \boxed{\frac{-2s+1}{5s+1}} \longrightarrow \boxed{\frac{1}{-2s+1}} \longrightarrow$$

Lead-lag system $g_1(s)$ $g_2(s)$

$\boxed{\text{Pole-zero cancellation}}$

$$g(s) = g_1(s)\, g_2(s) = \frac{-2s+1}{5s+1} \times \frac{1}{-2s+1} = \frac{1}{5s+1}$$

Poles of 1 component canceled zeros of 2nd component
We could've imagine a 1st order dynamics from g(s).
But it is not so!

Q

$(-2s+1) \longrightarrow$ Under what conditions / when do we have poles & zeroes in RHP and LHP?

$\longrightarrow$ Feedback comes from different direction?

Take $g_2(s) = \dfrac{1}{-2.0001 s + 1}$

Overall TF: $g_3(s) = \dfrac{-2s+1}{5s+1} \times \dfrac{1}{-2.0001s+1}$

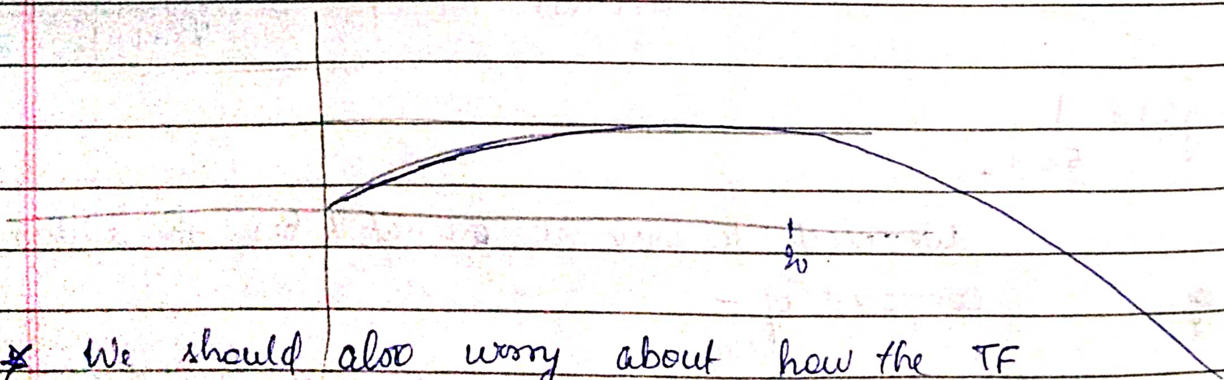
$$g_3(s) = \frac{-2s+1}{-10.0005 s^2 + 2.9999 s + 1}$$

$$g_4(s) = \frac{1}{5s+1}$$

Response to unit step ~~s~~ function:

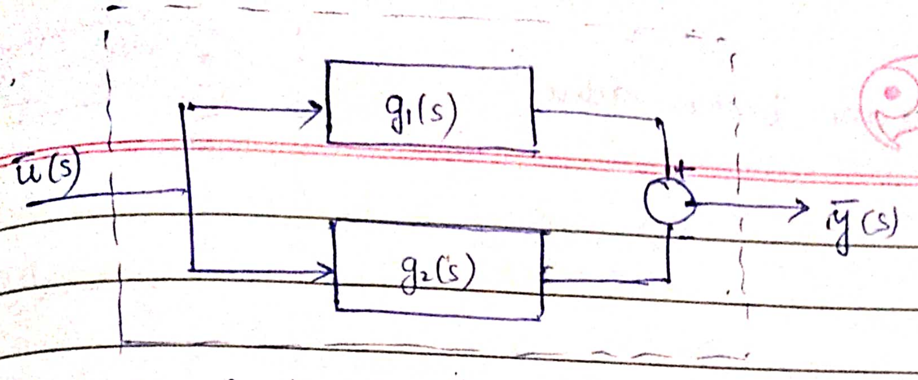
$$y_4(t) = 1 - e^{-t/5}$$

$$y_3(t) = 1 - \frac{7}{7.0001} e^{-t/5} - \frac{0.0001}{7.0001} \cdot e^{t/2.0001}$$



* We should also worry about how the TF has come

Systems were arranged in series $\uparrow$ multiplicat^n of TFs



effect of g_1 is to amplify the signal
g_2 is to dampen " "
(opposes)

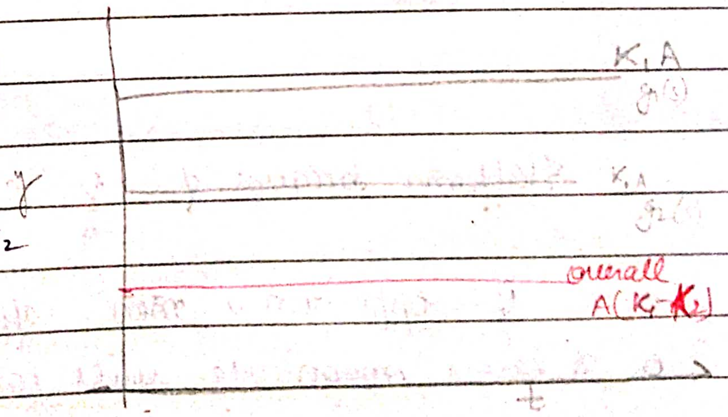
$$g(s) = g_1(s) - g_2(s)$$
$$= \frac{K_1}{\tau_1 s+1} - \frac{K_2}{\tau_2 s+1}$$

↑ main mode ↑ opposition mode

$$\bar{y}(s) = \bar{u}(s)\, g_1(s) - \bar{u}(s)\, g_2(s)$$
$$\Rightarrow \bar{y}(s) = \frac{K_1}{s(\tau_1 s+1)} - \frac{K_2}{s(\tau_2 s+1)}$$
$$y(t) = L^{-1}\left[\frac{K_1}{s(\tau_1 s+1)}\right] - L^{-1}\left[\frac{K_2}{s(\tau_2 s+1)}\right]$$
$$y(t) = AK_1\left(1-e^{-t/\tau_1}\right) - AK_2\left(1-e^{-t/\tau_2}\right)$$

If the transfer f^n is of the above form, then what would be the ultimate response of the system

$$\lim_{t\to\infty} y(t) = A(K_1-K_2)$$

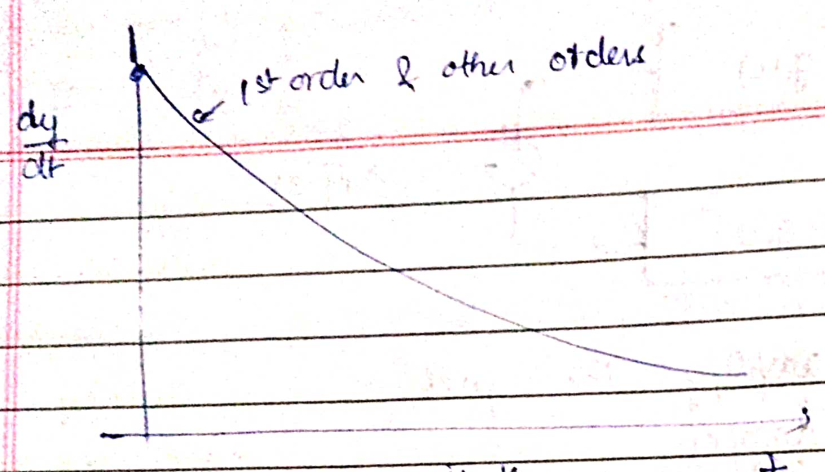


$$\frac{dy}{dt} = \frac{AK_1}{\tau_1} e^{-t/\tau_1} - \frac{AK_2}{\tau_2} e^{-t/\tau_2}$$

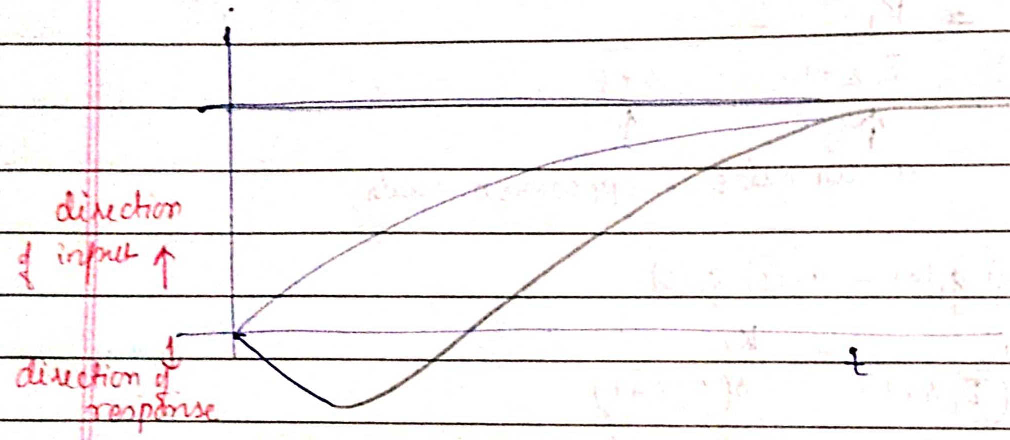
$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{K_1}{\tau_1} - \frac{K_2}{\tau_2}$$

$K_1 > K_2$ ← known

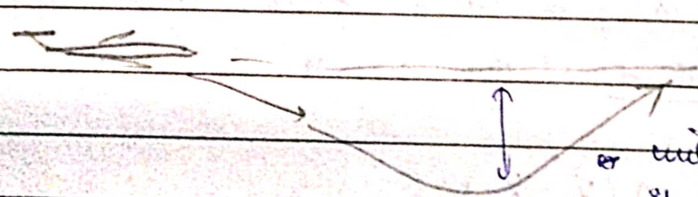
But what about τ_1 & τ_2 ? ← no comment on this



Here, $\frac{dy}{dt} \begin{cases} >0 & \text{if } K_1/\tau_1 > K_2/\tau_2 \\ <0 & \text{if } K_1/\tau_1 < K_2/\tau_2 \end{cases}$



The fate of the system as $t \to \infty$ will always be as known before.
But initially, there is a negative slope. Because it has to reach to a +ve value, there will be inversion.



Happens because of $\frac{K_1}{\tau_1} < \frac{K_2}{\tau_2}$

- If opposition is more, effect of opposition is more.
- A larger magnitude will cause larger delay in response
- In such systems, we may or may not have inverse response

$$g(s) = \frac{K_1}{\tau_1 s+1} - \frac{K_2}{\tau_2 s+1}$$

$$= \frac{K_1(\tau_2 s+1) - K_2(\tau_1 s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$= \frac{(K_1\tau_2 - K_2\tau_1)s + (K_1 - K_2)}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$= \frac{K(\chi s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$\chi = \frac{K_1\tau_2 - K_2\tau_1}{K_1 - K_2}$$

$$K = K_1 - K_2$$

$\chi = f(K_1\tau_2 - K_2\tau_1)$

Depending on K_1, K_2, τ_1, τ_2, χ can be positive or χ can be negative.

$$\eta = \frac{K_2\tau_1 - K_1\tau_2}{K_1 - K_2}$$

$$g(s) = K\left(\frac{-\eta s+1}{(\tau_1 s+1)(\tau_2 s+1)}\right)$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{K(-\eta s+1)}{s(\tau_1 s+1)(\tau_2 s+1)}\right]$$

$$y(t) = AK\left[1 + \frac{(\eta+\tau_1)}{(\tau_2-\tau_1)}e^{-t/\tau_1} + \left(\frac{\eta+\tau_2}{\tau_1-\tau_2}\right)e^{-t/\tau_2}\right]$$

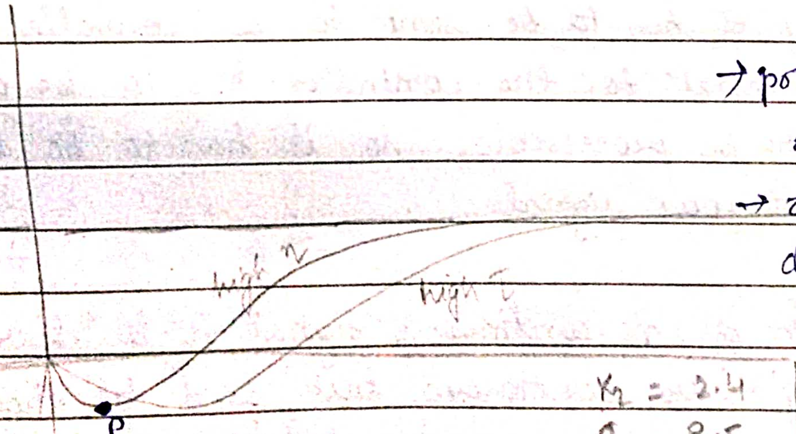
Ultimate response, $t \to \infty$

$$\lim_{t\to\infty} y(t) = AK$$

η corresponds to zero

τ_1, τ_2 ~~Poles~~ " to poles

Dermas



→ poles make dynamics sluggish

→ zeroes make dynamics faster

$K_2 = 2.4$

$\tau_1 = 8.5$

$\tau_2 = 1.1$

$K_1 = 4.9$

$$t = \frac{\tau_1\tau_2}{\tau_1-\tau_2}\ln\left(\frac{K_2\tau_1}{K_1\tau_2}\right)$$

Slope inverts at P, $\frac{dy}{dt} = 0$

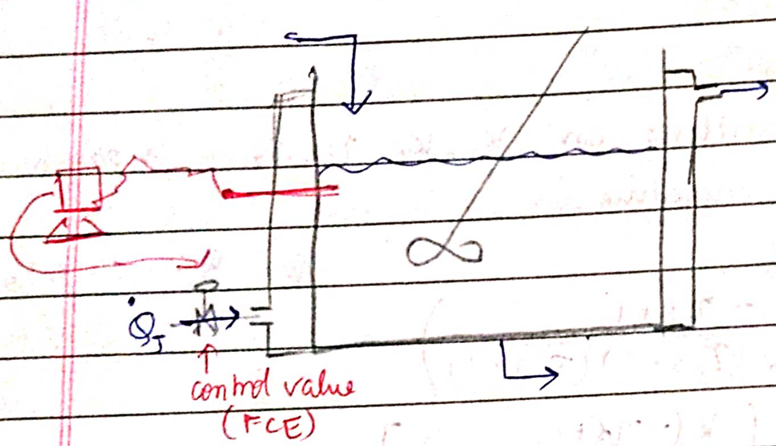
- Pure capacity / pure . — depending on magnitudes.

26/3/21

Dynamics of discrete-time systems

$\bar{u}(s) \longrightarrow \boxed{g(s)} \longrightarrow \bar{y}(s) \rightarrow L^{-1}\{\bar{y}(s)\} = y(t) \rightarrow$ dynamics of output variable

$u(t), x(t), y(t) \rightarrow$ continuous functions (in time)



Jacketed CSTR

Process temp : T_p

Set ~~pt~~ temp : T_s

If $T_p > T_s$, $\dot{Q}_J \uparrow$

$T_p < T_s$, $\dot{Q}_J \downarrow$

→ How $\dot{Q}_J$ will change with T_p or $(T_s - T_p)$ will come from model.

→ Model Eqⁿ's : will be continuous ODEs.

→ Depending on $|T_s - T_p|$, value position will change

$$\epsilon = \cancel{T_s - T_p}\ T_p - T_s$$

→ How will you know ϵ?

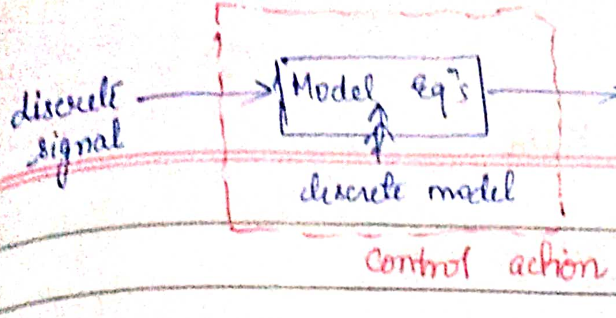
We should have a temp measuring device, say thermocouple

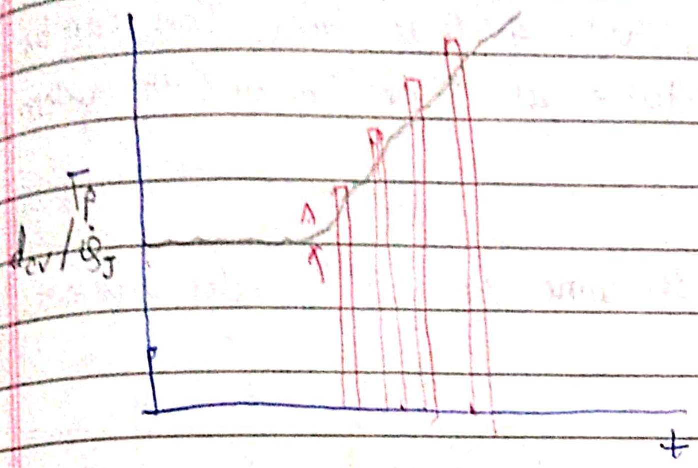
→ The only ϵ has to be sent to a controller - computer.

→ The only input to the controller has to be a digital signal for it to be processible. So, it has to be discrete.

→ T_p: continuous signal

* Conversion of a continuous signal to a discrete time signal.
* Model eqⁿ's are continous but feed to them are discrete. So, they need to be converted to discrete time domain
* Conversion of discrete time output signal to a continuous signal.



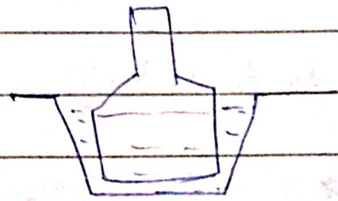


A - Control action invoked

from A, FCE acts and diff values openings as Tp changes.

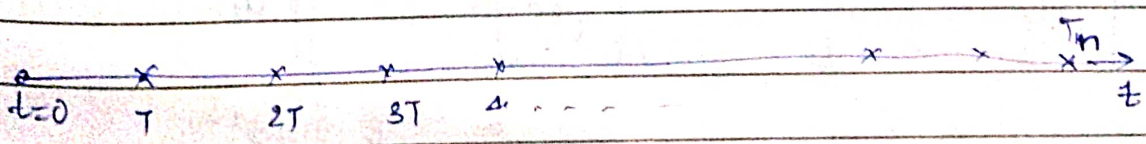
Value open  value close

* We would like to have automatic control



- Value receives continuously increasing Tp as disturbance and is acted upon to regulate.
- But as model is discrete, output is discrete, so act^n is discrete
- But we do not want this, so we should convert output to continuous domain.

Conversion of continuous models to discrete-time models :



$T = \Delta t$

$$\frac{dy}{dt} = \frac{y_{n+1} - y_n}{T}$$

y_n : amplitude/ value of cont. f^n at 'T_n'

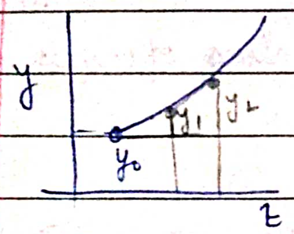
$$\frac{dy}{dt} = f(y,a)$$

$$\tau \frac{dy}{dt} + y = Ku(t)$$

$$\tau \frac{(y_{n+1} - y_n)}{T} + y_n = K u_n$$

$$\Rightarrow \boxed{y_{n+1} = \left(1 - \frac{T}{\tau}\right) y_n + \frac{K T u_n}{\tau}}$$

This means that the amplitude ~~of it~~ at time T_{n+1} can be obtained from the amplitude at time T_n and the system parameters



Do same for 2nd order system.

Multiple input - Multiple output system :

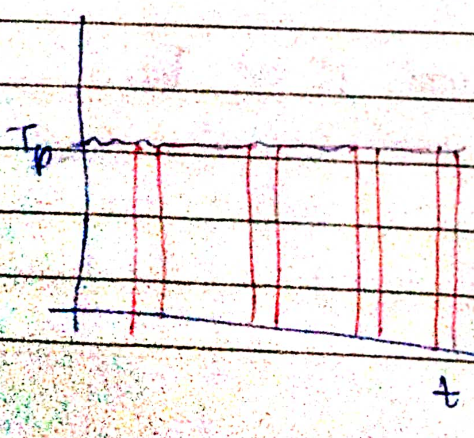
$$\frac{dy_1}{dt} + a_{11} y_1 + a_{12} y_2 = b_{11} u_1 + b_{12} u_2$$

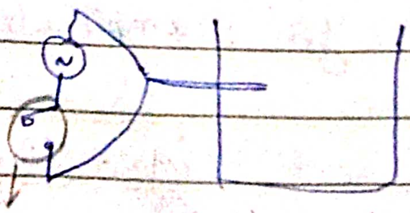
$$\frac{dy_2}{dt} + a_{21} y_1 + a_{22} y_2 = b_{21} u_1 + b_{21} u_2$$

$$\begin{cases} \frac{y_{1,n+1} - y_{1,n}}{T} + a_{11} y_{1n} + a_{12} y_{2n} = b_{11} u_{1n} + b_{12} u_{2n} \\ \frac{y_{2,n+1} - y_{2,n}}{T} + a_{21} y_{1n} + a_{22} y_{2n} = b_{21} u_{1n} + b_{22} u_{2n} \end{cases}$$

$$\begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y_{1n} \\ y_{2n} \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} u_{1n} \\ u_{2n} \end{bmatrix}$$

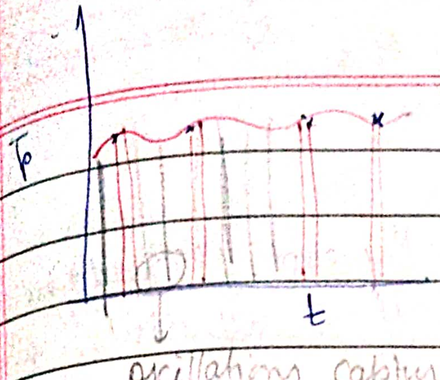
Conversion of a continuous input signal to a discrete time signal





input to it is continuous but I decide how to take output

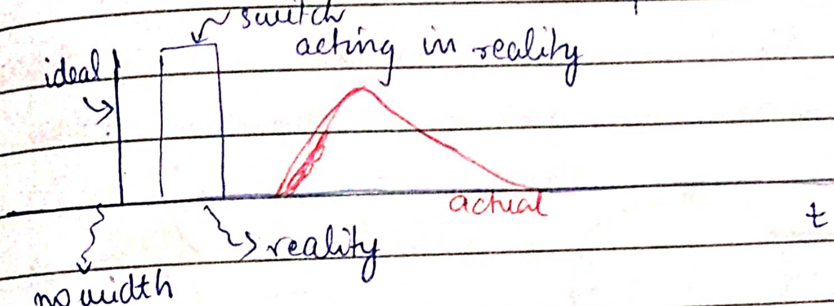
Limitation:



The observation here is const T_p but this may not be the reality.

- ✓ Sampling method
- ✓ System dependent on fineness of discretization.
- ✓ Correct sampling to retain resolut^n.

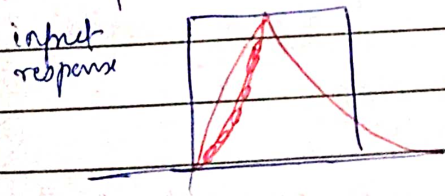
Value opened, stays for some time before ~~car~~ closing.



But the act^n of switch is like a pulse response.

Current has to rise from 0 to some finite value

Dynamics governed by τ, ..



y^* - amplitude ~~ds~~ -

Discrete time

discrete $\quad y^*(t) = y^*(0) + y^*(T) + y^*(2T) + \ldots\ldots$

$\Rightarrow y^*(t) = \underset{\text{cont.}}{y(0)} \; \delta(T) + y(T)\,\delta(t-T) + \ldots\ldots$

$$y^*(t) = \sum_{n=0}^{\infty} \underbrace{y(nT)}_{\text{a number}} \delta(t-nT)$$ ← discrete time signal from a contin. signal

$$\bar{y}^*(s) = \sum_{n=0}^{\infty} y(nT)\, \mathcal{L}\,\delta(t-nT)$$

$$\Rightarrow \bar{y}^*(s) = \sum_{n=0}^{\infty} y(nT)\, e^{-nTs}$$

Conversion of a discrete output signal to a continuous ~~input~~ output signal

$m(T), m(2T), m(3T) \ldots$ — My cont f^n should be s.t. if I substitute $t = T$,

$\downarrow$

continuous $m(t)$

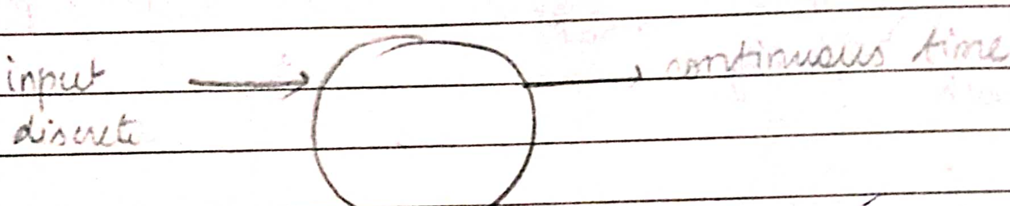
TL expansion about nT

$$m(t) = m(nT) + \left.\frac{dm}{dt}\right|_{nT} (t - nT) + \frac{1}{\lfloor 2} \left.\frac{d^2m}{dt^2}\right|_{nT} (t-nT)^2$$

~~about~~

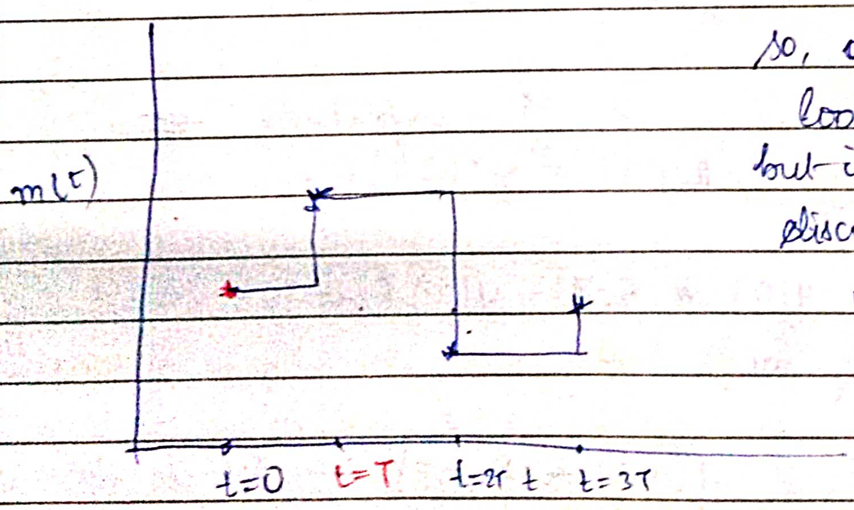
I know this

discrete — discrete — discrete



Problem: be careful with no. of terms in TL series expansion. We retain only the zeroth order derivative. — zeroth order hold

$$m(t) \simeq m(nT)$$



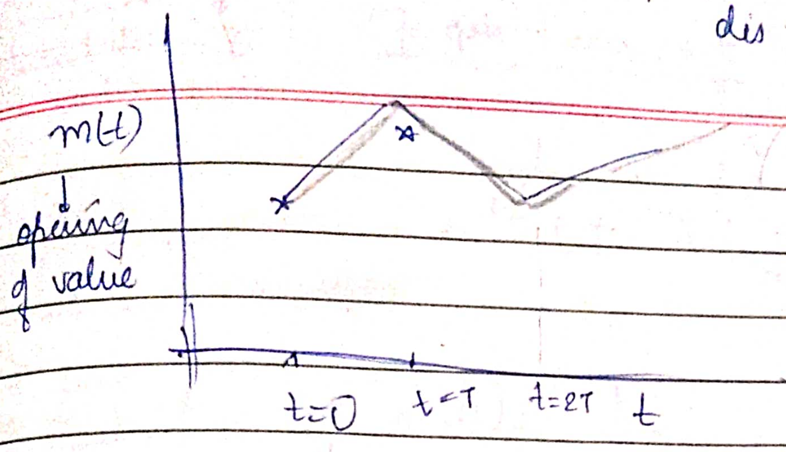
so, our output f^n will look like a step f^n but it looks like a ~~continuous~~ discrete system, so not very useful.

1st order hold: $m(t) = m(nT) + \left.\frac{dm}{dt}\right|_{nT} (t - nT)$

dis — dis — cont.

$\sim \quad y = c + mx \rightarrow$ linear in time

linear f^n's in b/w
dis + effect of gradient



b/w 2 discrete signals, now our value is opened continuously

Z-transforms : Discrete time systems

Laplace transformat^n to continuous functions

$$Z[y(t)] = \bar{y}(z) = \sum_{n=0}^{\infty} y(nT) z^{-n}$$

$$z\{y(0), y(T), y(2T) \dots\} = \sum_{n=0}^{\infty} y(nT) z^{-n}$$

from prev
~~Whe~~ $$y^*(s) = \sum_{n=0}^{\infty} y(nT) e^{-nTs}$$

When $z = e^{Ts}$

$$\bar{y}^*(s) = \sum_{n=0}^{\infty} y(nT) z^{-n} = \bar{z}(s)$$

- All properties of Z transformation are very similar to L. transform

Format : Laplace
Model → input-output → transform f^n → $y(t) \to L^{-1}\{\bar{g}(s)\bar{u}(s)\}$

Format : Z-transform
Model → input-output → transform f^n → $y(t) \to z^{-1}$

Z-transformat^n of a unit step f^n

① Unit step f^n: $\{1, 1, 1, \ldots\}$

$$\bar{Z}(u) = 1.z^0 + 1.z^{-1} + 1.z^{-2} + \ldots$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$\bar{Z}(u) = \frac{1}{1-z^{-1}} \quad \Big| \quad L(u) = \frac{A}{s}$$

② **Ramp f^n** : $u = \{0, aT, 2aT, 3aT, \ldots\}$

$$Z(u) = 0.z^0 + aTz^{-1} + 2aTz^{-2} + 3aTz^{-3} + \ldots$$

$$= aT(1 + 2z^{-1} + 3z^{-2} + \ldots)z^{-1}$$

$$= \frac{aTz^{-1}}{(1-z^{-1})^2}$$

Dynamics of a discrete time first-order system :

$$y_{n+1} = ay_n + bu_n$$

Unit step : $u_n = 1 \quad \forall n$

$$y_{n+1} = ay_n + b$$

$$y_1 = ay_0 + b$$

$$y_2 = ay_1 + b = a(ay_0 + b) + b$$

$$\Rightarrow y_2 = a^2 y_0 + (a+1)b$$

$$\vdots$$

$$\boxed{y_n = a^n y_0 + b(a^{n-1} + a^{n-2} + \ldots + 1)}$$

Deviation variable at $t=0$, $y_0 = 0$

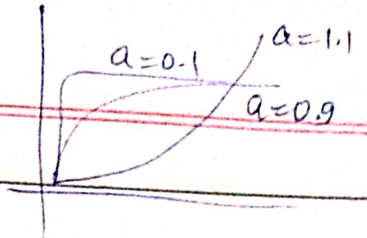
$$y_n = b(a^{n-1} + a^{n-2} + \ldots + 1)$$

$$y_n = b\left[\frac{1-a^n}{1-a}\right]$$

excel

y-n

a	1.1
b	1.1



imp. for: $\boxed{a<1}$ stable

$\boxed{a>1}$ unstable

Do with z-transform

$$g(z) = \frac{\bar{y}(z)}{\bar{u}(z)} \Rightarrow \bar{y}(z) = \bar{u}(z)\,g(z)$$

$$\bar{u}(z) = \frac{1}{1-z^{-1}}$$

g(z) unknown

$$g(s) = \frac{K}{\tau s+1}$$

$$g(t) = \frac{K}{\tau} e^{-t/\tau}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{K}{\tau} e^{-nT/\tau} z^{-n}$$

$$= \underbrace{\frac{K}{\tau}}_{b} \cdot \frac{1}{(1-\underbrace{e^{-nT/\tau}}_{a} z^{-1})}$$

$$g(z) = \frac{b}{1-az^{-1}}$$

$$y(z) = \left(\frac{b}{1-az^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$

Tables

Inverse

$$y(n) = b\left(\frac{1-a^n}{1-a}\right)$$