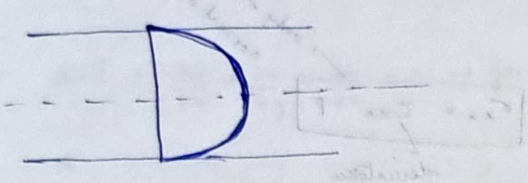
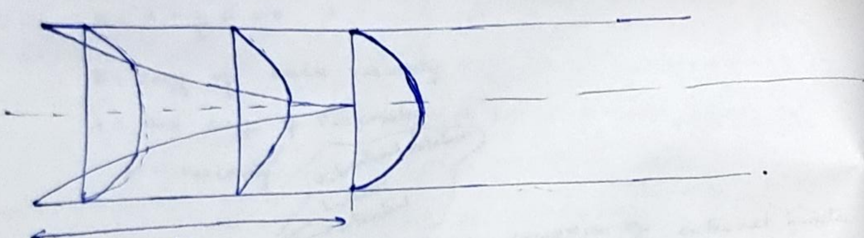
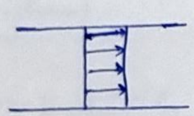
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] + g_x$$

→ 

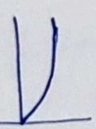
L_T = length over which the velocity profile develops fully



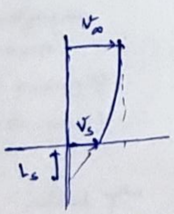
L_T

→ 

→ inviscid flow (or) if surface has infinite slip

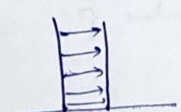
→ 

Boundary layer is formed

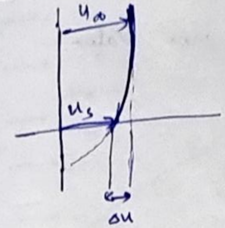


v_s → slip velocity
L_s → slip length
finite slip

$v_\infty > v_s$
free stream velocity
BL will form

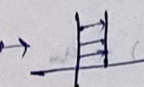


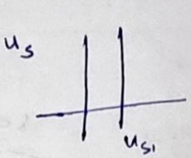
↳ slip length is infinite
NO BL
$v_s \geq v_\infty$

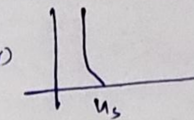


Suppose u_∞ starts to reduce

→ profile qualitatively remains same but Δu reduces

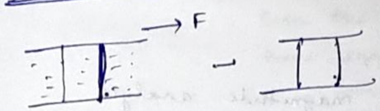
→ when $u_\infty = u_s$ → $\Delta u = 0$ → 

→ $u_\infty < u_s$ 

(or) 

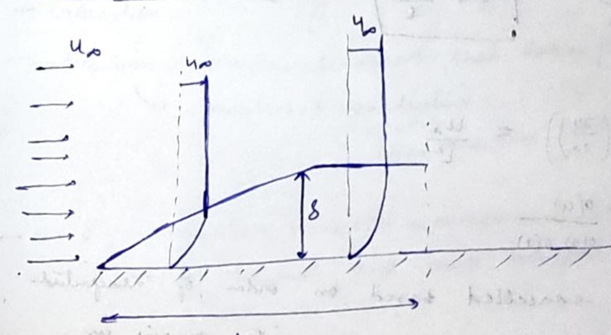
$u_{s_1} < u_s$

→ Couette flow — if the top surface has infinite slip there will be no deformation



BOUNDARY LAYER

→ Flow over a flate plate (semi infinite pool of liquid)



steady state
incompressible
2-D flow
Newtonian

Continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x-component :- $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

→ convection

→ molecular level transport

→ whole transport is due to advection

y-component :- $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g$

Order of magnitude analysis :- looking at a max value a particular variable can attain

order — $O(u) \approx u_\infty$

$O(v) \rightarrow$ don't know

$O(x) \rightarrow L$ (large)

$O(y) \rightarrow \delta(x)$

$\delta << L$

$\delta <<< L$ ⊗

$\frac{\delta}{L} \approx \epsilon$ (<<1 very small) → from the geometry

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

don't consider sign in order of magnitude analysis

$$O\left(\frac{\partial u}{\partial x}\right) \approx O\left(\frac{\partial v}{\partial y}\right)$$

$\frac{O(u)}{O(x)}$ $\quad \frac{u_\infty}{L} \approx \frac{v}{\delta}$ $\quad \boxed{v \approx \frac{u_\infty \delta}{L}}$ → $O(v) << O(u)$

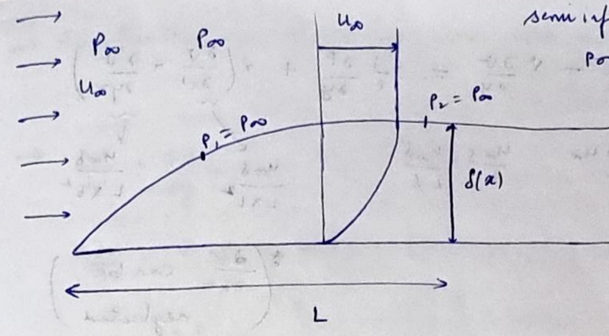
$v \neq 0$

$$\rightarrow O\left(\frac{\partial^2 u}{\partial x^2}\right) \approx O\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right) \approx \frac{u_\infty}{L^2}$$

$\hookrightarrow \frac{O(u)}{O(x) \cdot O(x)}$

→ No term can be cancelled based on order of magnitude analysis of an equation which has its genesis in mass balance.

→ Inside the BL → 2D flow, outside BL → 1D flow



semi infinite Pool

$O(u) \approx u_\infty$

$O(x) \approx L$

$O(y) \approx \delta$

$O(v) \approx \frac{\delta}{L} u_\infty \approx \epsilon u_\infty$

x-component :- (2D, Newtonian flow)

$$\underbrace{u\frac{\partial u}{\partial x}}_{u_\infty \cdot \frac{u_\infty}{L}} + \underbrace{v\frac{\partial u}{\partial y}}_{\frac{u_\infty u_\infty \delta}{L \times \delta}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\frac{u_\infty}{L^2}} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\frac{u_\infty}{\delta^2}}\right)$$

Both the terms have same order ∴ cannot neglect

$\frac{\partial^2 u}{\partial x^2}$ can be neglected

$O\left(\frac{\partial^2 u}{\partial x^2}\right) << O\left(\frac{\partial^2 u}{\partial y^2}\right)$ ✓

$$\boxed{\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{inertia}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\nu\left(\frac{\partial^2 u}{\partial y^2}\right)}_{\text{viscous/stress}}}$$

inertia $\frac{u_\infty^2}{L}$ $\quad$ pressure $\frac{1}{\rho} O\left(\frac{\partial p}{L}\right)$ $\quad$ viscous/stress $\nu \frac{u_\infty}{\delta^2}$

→ convection

→ advection = entire transport that takes place in the flow field.

$\hookrightarrow$ convective + conductive (stress term in NVSE)

→ v = 0, momentum transport → no movement at all → no diffusive transport also but in Thermal and mass transport → diffusive terms will be there.

$$P = p + \rho g y$$

$$\frac{\partial P}{\partial y} = \frac{\partial p}{\partial y} + \rho g$$

$$\frac{\partial P}{\partial x} = \frac{\partial p}{\partial x}$$

y-component :- $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

$\frac{u_\infty \times \delta u_\infty}{L \cdot L} \qquad \frac{u_\infty \delta \times u_\infty \delta}{L \cdot L\delta} \qquad \frac{u_\infty \delta}{L \times L^2} \qquad \frac{u_\infty \delta}{L \times \delta^2}$

$\left(\frac{\partial^2 v}{\partial x^2}\right.$ can be neglected $\left.\right)$

→ $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \rho g + \nu\left(\frac{\partial^2 v}{\partial y^2}\right)$

$P = f(x, y)$

$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$

$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\left(\frac{dy}{dx}\right)$

→ x-component :- $O\left(\frac{u_\infty^2}{L}\right) \approx \frac{O(P)}{L} \approx O\left(\frac{\nu u_\infty}{\delta^2}\right)$ → assume

– If we consider $O\left(u\frac{\partial u}{\partial x} + \dots\right) \approx O\left(\frac{\partial P}{\partial x}\right)$ then in y component

balance consider $O\left(u\frac{\partial v}{\partial x} + \dots\right) \approx O\left(\frac{\partial P}{\partial y}\right)$

– $O\left(\frac{1}{\rho}\frac{\partial P}{\partial x}\right) \approx \nu\frac{u_\infty}{\delta^2}$ $\qquad O\left(\frac{\partial P}{\partial x}\right) \approx \mu\frac{u_\infty}{\delta^2}$

$O\left(\frac{1}{\rho}\frac{\partial P}{\partial y}\right) \approx \nu\frac{u_\infty}{\delta^2}\frac{\delta}{L}$ $\qquad O\left(\frac{\partial P}{\partial y}\right) \approx \mu\frac{u_\infty}{\delta L}$

→ $\frac{\left(\frac{dP}{dx}\right)}{\left(\frac{\partial P}{\partial x}\right)} = 1 + \frac{\left(\frac{\partial P}{\partial y}\right)\frac{dy}{dx}}{\left(\frac{\partial P}{\partial x}\right)} = 1 + \frac{\mu\frac{u_\infty}{\delta L} \times \frac{\delta}{L}}{\mu\frac{u_\infty}{\delta^2}} = 1 + \left(\frac{\delta}{L}\right)^2$

ε^2

$\varepsilon^2 << 1$

Based on the order. $\boxed{\frac{dP}{dx} = \frac{\partial P}{\partial x}}$ → $P \neq f(y)$

$\frac{\partial P}{\partial y} = 0$

→ Eventually, → $\frac{\partial P}{\partial x} = 0$ → more like an assumption.

→ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial P}{\partial y} = 0$

B.C :-

① At $y = 0$, $u = 0$, $v = 0$

② at $y = \delta(x)$, $u = u_\infty$, $v = 0$.

Balance between inertia and friction

$\frac{u_\infty^2}{L} \approx \nu\frac{u_\infty}{\delta^2}$

$\delta^2 = \frac{\nu L}{u_\infty}$

$\delta^2 = \nu\frac{1}{L u_\infty}L^2$

$\boxed{\delta = \left(\frac{\mu}{\rho}\frac{1}{u_\infty L}\right)^{1/2} L}$ $\qquad \delta = \left(\frac{\nu}{u_\infty}\right)^{1/2} L^{1/2}$ $\qquad \delta \propto L^{1/2}$

$Re_L = \frac{u_\infty L \rho}{\mu}$ $\qquad \boxed{Re_x = \frac{u_\infty x \rho}{\mu}}$ → $f(x)$

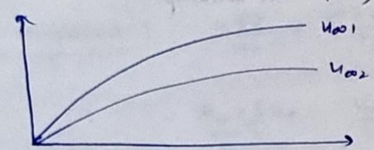
external flow.

$\delta \propto x^{1/2}$ Based on order of magnitude analysis

$\boxed{\delta = \left(\frac{\nu}{u_\infty}\right)^{1/2} x^{1/2}}$ $\qquad$ $u_\infty \uparrow \rightarrow \delta \downarrow$

$\boxed{\delta = \left(\frac{1}{Re_L}\right)^{1/2} L}$ $\qquad \boxed{\frac{\delta}{L} = \left(\frac{1}{Re_L}\right)^{1/2}}$

$u_{\infty 1}, u_{\infty 2}$ | $u_{\infty 2} > u_{\infty 1}$



$$\frac{\delta}{L} = (Re_{,L})^{-1/2}$$

$$\boxed{Re_{,L} = \frac{L^2}{\delta^2}}$$

for a BL $Re_{,L}$ is a geometric parameter ***

$\frac{\rho v \ell}{\mu}$ → little diff → Reynolds no.

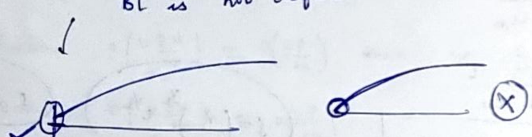
significance :- geometric parameter

$$\delta = \left(\frac{\nu}{u_\infty}\right)^{1/2} x^{1/2}$$

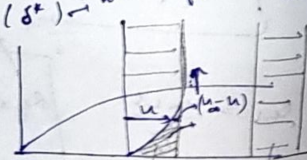
$$\boxed{\frac{d\delta}{dx} = \frac{1}{2}\left(\frac{\nu}{u_\infty}\right)^{1/2} \frac{1}{x^{1/2}}}$$

BL :- locus of end pt of thickness

$\left.\frac{d\delta}{dx}\right|_{x=0} = \infty$ → at the tip of the edge the shape of BL is not defined



Boundary layer displacement thickness (δ^*) → how much mass is not flowing



$$u_\infty \delta^* = \int_0^\infty (u_\infty - u)\, dy$$

$$\boxed{\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy}$$

to the length where there is no flow.

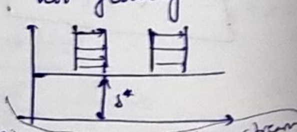
i.e. $\int_0^\infty (u_\infty - u)\, dy$

this mass is generic of v

mass that is not flowing

As it is a semi infinite pool, u_∞ doesn't change

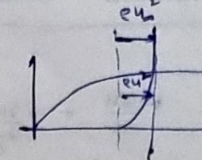
mass is flowing out of the BL and mixed with free stream velocity

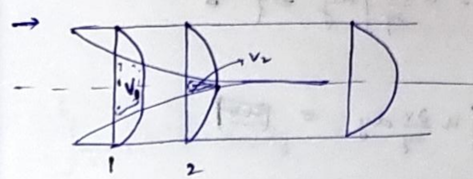


Boundary layer momentum thickness (δ^{**} or θ)

$$\delta^{**} = \theta = \int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy$$

$$\rho u_\infty^2 \theta = \int_0^\infty \rho u (u_\infty - u)\, dy$$





$V_2 > V_1$

→ It is defined as a vertical distance dy which the external potential flow is displaced outward due to the decrease in velocity in the boundary layer.

Momentum integral method for BL

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

Integrate over BL thickness

$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy = \int_0^\delta \nu\left(\frac{\partial^2 u}{\partial y^2}\right)dy$$

$$\int_0^\delta \left(u\frac{\partial u}{\partial x}\right)dy + \int_0^\delta \left(v\frac{\partial u}{\partial y}\right)dy = \int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy$$

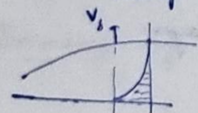
$$\int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta\left[\frac{\partial}{\partial y}(uv) - u\frac{\partial v}{\partial y}\right]dy$$

$$= [uv]_0^\delta - \int u\frac{\partial v}{\partial y}dy$$

$$= uv\big|_\delta - \int u\frac{\partial v}{\partial y}dy$$

$$= u_\infty v_\delta - \int u\frac{\partial v}{\partial y}dy$$

$v_\delta \rightarrow$ drainage velocity $\quad v_\delta = \int_0^\delta \frac{\partial v}{\partial y}dy$



$$\int_0^\delta v\frac{\partial u}{\partial y}dy = u_\infty\int_0^\delta \frac{\partial v}{\partial y}dy - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$= -u_\infty\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy.$$

→ LHS

$$\int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy - u_\infty\int_0^\infty \frac{\partial u}{\partial x}dy$$

$$= \int_0^\delta \frac{\partial}{\partial x}\left(u^2 - uu_\infty\right)dy.$$

$$2\int_0^\delta u\frac{\partial u}{\partial x}dy - u_\infty\int_0^\delta\left(\frac{\partial u}{\partial x}\right)dy = \int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy$$

$$= \int_0^\delta \nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)dy$$

$$= \left[\frac{\mu}{\rho}\frac{\partial u}{\partial y}\right]_{y=0}^{y=\delta} \qquad \because \left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$$

$$= -\frac{\tau_w}{\rho} \rightarrow \text{wall shear stress}$$

$$\int_0^\delta \frac{\partial}{\partial x}\left(u^2 - uu_\infty\right)dy = \frac{-\tau_w}{\rho}$$

$$u_\infty^2\int_0^\delta \frac{d}{dx}\frac{u}{u_\infty}\left(\frac{u_\infty - u}{u_\infty}\right)dy = \frac{\tau_w}{\rho}$$

$$u_\infty^2\int_0^\delta \frac{d}{dx}\left(\frac{u}{u_\infty}\right)\left(1 - \frac{u}{u_\infty}\right)dy = \frac{\tau_w}{\rho}$$

$$\boxed{u_\infty^2\frac{d\theta}{dx} = \frac{\tau_w}{\rho}}$$

→ Karman - Pohlhausen approximate method

→ assume

$$\frac{u}{u_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 \quad \text{where } \eta = \frac{y}{\delta}$$

BC

① at $y=0$, $u=0$ → at $\eta=0$, $\frac{u}{u_\infty}=0$

② At $y=\delta$, $u=u_\infty$ → at $\eta=1$, $\frac{u}{u_\infty}=1$

(3) at $y=0$, $\frac{\partial^2 u}{\partial y^2}=0$ (as wall shear stress is const → $\frac{\partial u}{\partial y}$ = const)

at $\eta=0$, $\frac{\partial^2}{\partial \eta^2}\left(\frac{u}{u_\infty}\right)=0$

(4) at $y=\delta$, $\frac{\partial u}{\partial y}=0$ → at $\eta=1$, $\frac{\partial}{\partial \eta}\left(\frac{u}{u_\infty}\right)=0$

$\boxed{a_0=0}$

$\boxed{1 = a_1 + a_2 + a_3}$

$\frac{\partial}{\partial \eta}\left(\frac{u}{u_\infty}\right) = a_1 + 2a_2\eta + 3a_3\eta^2 \longrightarrow \boxed{a_1+2a_2+3a_3=0}$

$\frac{\partial^2}{\partial \eta^2}\left(\frac{u}{u_\infty}\right) = 2a_2 + 6a_3\eta \longrightarrow \boxed{a_2=0}$

$a_1+a_3=1$

$a_1+3a_3=0$

$-3a_3+a_3=1$ $\boxed{a_3=-1/2}$ $\boxed{a_1=3/2}$

$$\boxed{\frac{u}{u_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3}$$

$$\theta = \int_0^\infty \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy$$

$$\theta = \delta\int_0^\infty \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1-\frac{3}{2}\eta+\frac{1}{2}\eta^3\right)d\eta$$

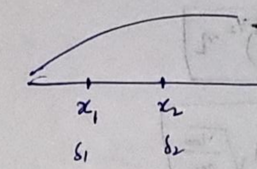
$$\theta = \delta\left(\frac{39}{280}\right)$$

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = -$$

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho u_\infty^2}$$

$$\frac{39}{280}\frac{d\delta}{dx} = \frac{\tau_w}{\rho u_\infty^2}$$

$$\frac{39}{280}\frac{d\delta}{dx} = \frac{3\mu u_\infty}{2\delta\rho u_\infty^2}$$



$$\frac{\delta^2}{2} = \frac{140\nu x}{13 u_\infty}$$

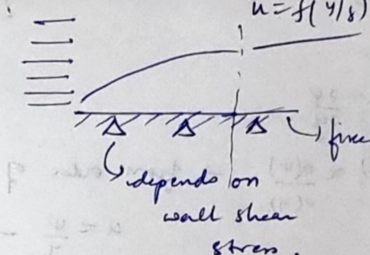
at $x=0$, $\delta=0$

$\boxed{\delta = \frac{4.64\,x}{\sqrt{Re_x}}}$ → $\delta=f(x)$ is known → now at any particular x velocity profile is also known

$u=f(y/\delta)$.

→ utility of BL analysis

To get the wall shear stress (construction)



→ $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$ **Blasius solution**

PDE → ODE

$u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial \psi}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial \psi}{\partial y}\right)$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\left(\frac{\partial^2 \psi}{\partial y^2}\right) = \nu\frac{\partial^3 \psi}{\partial y^3}$$

at $y=0$, $u=0$ → $\frac{\partial \psi}{\partial y}=0$ → $\psi \neq f(y)$

$v=0$ → $\frac{\partial \psi}{\partial x}=0$ → $\psi \neq f(x)$

ψ = const → $\boxed{\psi=0}$

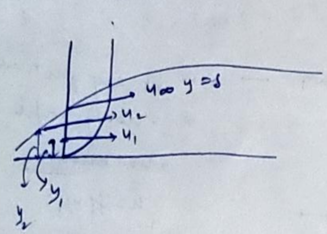
$$\frac{\delta}{x} \approx (Re_x) \qquad \delta = \left(\frac{\nu x}{u_\infty}\right)$$

$$\frac{y}{\delta} = \eta$$

$$\boxed{y = \left(\frac{\nu x}{u_\infty}\right)^{1/2} \eta}$$

$$\boxed{\eta = y\sqrt{\frac{u_\infty}{\nu x}}}$$

→ similarity parameter



$u_2 > u_1$

$u = \bar{f}(y)$

$\boxed{u = \bar{f}(\eta)}$

$$\boxed{\frac{u}{u_\infty} = \bar{f}(\eta)}$$

$$u = \frac{\partial \psi}{\partial y}$$

$$o(u) \approx \frac{o(\psi)}{o(y)}$$ → from order of magnitude analysis

$$u \approx \frac{\psi}{y} \rightarrow \frac{u}{u_\infty} \approx \frac{\psi}{y u_\infty}$$

$y = \delta\eta$

$$\frac{u}{u_\infty} \approx \frac{\psi}{\left(\frac{\nu x}{u_\infty}\right)^{1/2} u_\infty \eta}$$

$$\frac{u}{u_\infty} = \frac{\psi}{\sqrt{\nu x u_\infty}\,\eta}$$

$$\boxed{\frac{\psi}{\sqrt{\nu x u_\infty}} \approx \eta \cdot \frac{u}{u_\infty}}$$

$$\approx \eta \cdot \bar{f}(\eta)$$

η → similarity parameter

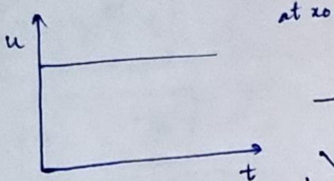
$$\boxed{\frac{\psi}{\sqrt{\nu x u_\infty}} \approx f(\eta)}$$

$$\boxed{\psi = \sqrt{\nu x u_\infty}\, f(\eta)}$$

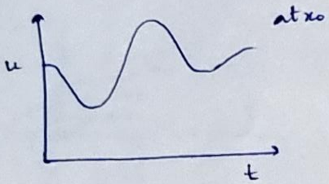
TURBULANCE

→ Some sort of motion with fluctuation component.

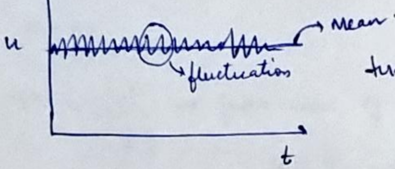
→ mean motion + fluctuation = turbulance.



→ velocity at a particular pt as a function of t
→ Eulerian coordinate
→ steady flow.

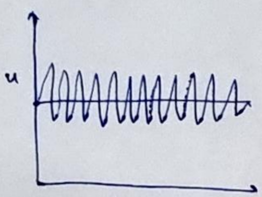


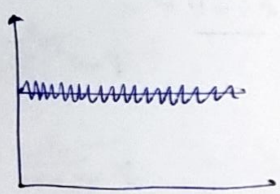
→ unsteady flow



turbulance.

(In true sense, steady turbulent flow is not possible)





fluctuation amplitude is high

turbulance

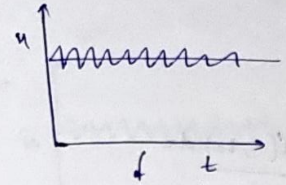
$\boxed{Re > 4000}$ → turbulance

→ Description of turbulent flow consists of a superimposed streaming & fluctuating motion as per reynold's decomposition.

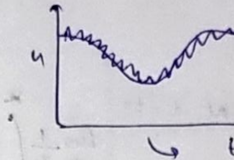
→ Reynold's decomposition of Turbulence

$$u(y,t) = \bar{u}(y) + u'(\vec{r},t)$$

$\vec{r}$



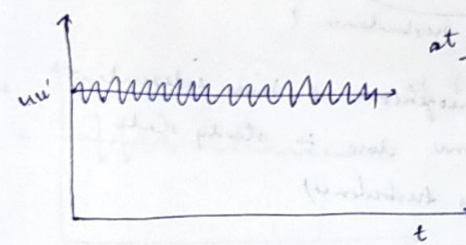
Mean velocity = const



mean velocity = f(t)

$$\underbrace{u(y,t)}_{\text{instantaneous velocity}} = \underbrace{\bar{u}(y)}_{\text{mean velocity}} + \underbrace{u'(\vec{r},t)}_{\text{fluctuation component}}$$

Despite considering 1D motion, fluctuation happens in all possible planes. (which $\vec{r}$ takes care of)



$\bar{u} = \text{const}$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u'(x_0,t)\,dt = 0$$

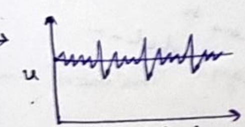
stationary turbulence

fluctuation amplitude remains unaltered with time

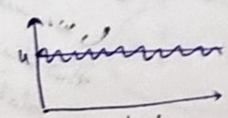
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u'(x_0,t)\,dt \neq 0 \rightarrow \text{non-stationary turbulence}$$

$\bar{u}$ → need not be const necessarily

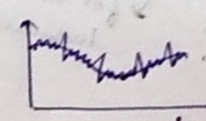
→



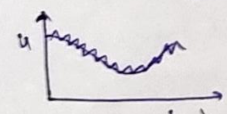
non-stationary



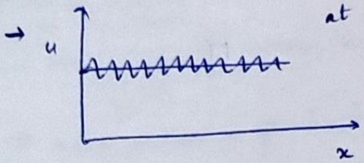
stationary



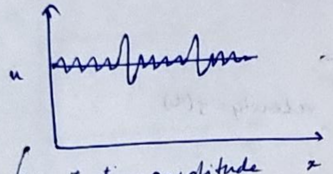
non-stationary



??

→ 

at $t = t_0$



at $t = t_0$

fluctuation amplitude changes along the flow field

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x \underbrace{u'(x, t_0)}\, dx$$

at a particular instant of time t_0 how fluctuation varies with $\underline{x}$

→ If $\lim_{x \to \infty} \frac{1}{x} \int_0^x u'(x, t_0)\, dx = 0$ → fluctuation amplitude remains same throughout the flow field

Homogeneous turbulence

→ Can we have steady state turbulence ?

If we have stationary, homogeneous & time independent mean velocity, we can come close to steady state turbulence (steady uniform turbulence)

→ $\boxed{\overline{u'} = \frac{1}{t} \int_0^t u' dt}$

time average of fluctuation

for stationary turbulence $= 0$

[whenever we use mean ($\overline{u'}$) it will be time average]

We will limit all the discussions to stationary turbulence

→ $\frac{1}{t} \int_0^t u'^2 dt \quad \int_0^t u'v' dt$

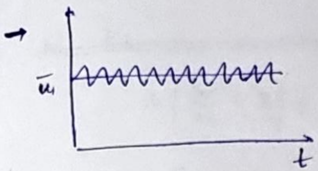
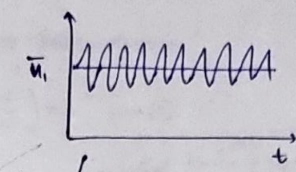
non zero.

→ u, $\bar{u}$, u', $\overline{u'}$ → mean fluctuation (time average)

u → instantaneous velocity

$\bar{u}$ → mean velocity (time average)

u' → fluctuation

→  

stationary turbulence

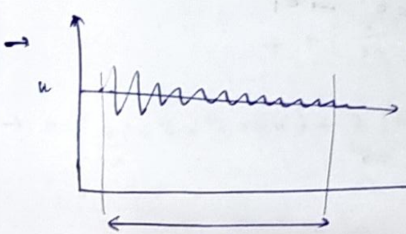
Intensity of fluctuation is higher

distinguish these 2 plots by

$$I = \frac{\sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})}}{\bar{u}}$$

→ $\boxed{\overline{u'^2} = \overline{v'^2} = \overline{w'^2}}$ → fluctuation in all the 3 directions are same

[isotropic turbulence]

→ 

at $t = t_0$

→ fluctuation amplitude is reducing

l (correlation length) → over length l physical nature of turbulence changes

→ Turbulent reynolds number $Re_l' = \frac{u' l}{\nu}$

→ **Reynold's decomposition**

$$u(y,t) = \bar{u}(y) + u'(r^c, t)$$

$\overline{u'} = 0$
$\overline{v'} = 0$
$\overline{w'} = 0$
$\overline{u'v'} \neq 0$

$$\overline{f+g} = \bar{f} + \bar{g}$$

$$\overline{f \cdot g} = \bar{f} \cdot \bar{g}$$

$$\bar{\bar{f}} = \bar{f}$$

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ → incompressible fluid

Substitute the reynold's decomposed form & take the time average

$$\frac{\partial}{\partial x}(\bar{u}+u') + \frac{\partial}{\partial y}(\bar{v}+v') + \frac{\partial}{\partial z}(\bar{w}+w') = 0$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \rightarrow E_1$$

take time average

$$\overline{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}} = 0$$

$$\overline{\frac{\partial \bar{u}}{\partial x}} + \overline{\frac{\partial \bar{v}}{\partial y}} + \overline{\frac{\partial \bar{w}}{\partial z}} + \overline{\frac{\partial u'}{\partial x}} + \overline{\frac{\partial v'}{\partial y}} + \overline{\frac{\partial w'}{\partial z}} = 0$$

$$\frac{\partial \bar{\bar{u}}}{\partial x} + \frac{\partial \bar{\bar{v}}}{\partial y} + \frac{\partial \bar{\bar{w}}}{\partial z} + \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} + \frac{\partial \overline{w'}}{\partial z} = 0$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \cancel{\frac{\partial \overline{u'}}{\partial x}} + \cancel{\frac{\partial \overline{v'}}{\partial y}} + \cancel{\frac{\partial \overline{w'}}{\partial z}} = 0$$

0 0 0 → stationary turbulence

$$\Rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \rightarrow E_2$$

from E_1, E_2 → $$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Continuity is valid for ~~streaming~~ mean velocity & fluctuation velocity

x component N-S equation

$$\rho\left[\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \rightarrow (1)$$

0 assume

~~We take the conservative form of eqn~~

$$u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \rightarrow (2)$$

(1) + (2)

inertial terms → & terms (convective terms)

$$\rho\left[\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

conservative form ★★★

Now we substitute the reynold's decomposition:-

$$\rho\left[\frac{\partial}{\partial x}(\bar{u}+u')(\bar{u}+u') + \frac{\partial}{\partial y}(\bar{u}+u')(\bar{v}+v') + \frac{\partial}{\partial z}(\bar{u}+u')(\bar{w}+w')\right] = \frac{-\partial(\bar{p}+p')}{\partial x}$$
$$+ \mu\left[\frac{\partial^2}{\partial x^2}(\bar{u}+u') + \frac{\partial^2}{\partial y^2}(\bar{u}+u') + \frac{\partial^2}{\partial z^2}(\bar{u}+u')\right]$$

$$\rightarrow \rho\left[\frac{\partial}{\partial x}(\bar{u}^2 + u'^2 + 2\bar{u}u') + \frac{\partial}{\partial y}(\bar{u}\bar{v} + \bar{u}v' + u'\bar{v} + u'v') + \frac{\partial}{\partial z}(\bar{u}\bar{w} + \bar{u}w' + u'\bar{w} + u'w')\right]$$
$$= \frac{-\partial}{\partial x}(\bar{p}+p') + \mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{\partial^2 u'}{\partial z^2}\right]$$

take time average.

$$\overline{\bar{u}\bar{v} + u'v' + \bar{u}v' + u'\bar{v}} = \overline{\bar{u}\bar{v}} + \overline{u'v'} + \bar{u}\cancel{\overline{v'}} + \cancel{\overline{u'}}\bar{v}$$
$$= \bar{u}\bar{v} + \overline{u'v'}$$

$\frac{\partial \overline{p'}}{\partial x} = 0$

$$\frac{\partial}{\partial x}(\bar{u}^2 + \overline{u'^2}) + \frac{\partial}{\partial y}(\bar{u}\bar{v} + \overline{u'v'}) + \frac{\partial}{\partial z}(\bar{u}\bar{w} + \overline{u'w'}) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} +$$

$$\frac{\mu}{\rho}\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right]$$

→ $\left(\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} + \frac{\partial}{\partial x}\overline{u'^2} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right]$

→ $\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \bar{u}\bar{w}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right] - \left[\frac{\partial}{\partial x}(\overline{u'^2}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'})\right]$

→ $\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right) = -\frac{\partial \bar{p}}{\partial x} + \underbrace{\mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right]}_{\text{viscous stress/resistance}}$

$\underbrace{-\rho\left[\underbrace{\frac{\partial}{\partial x}\overline{u'^2}}_{+ve} + \underbrace{\frac{\partial}{\partial y}\overline{u'v'}}_{-ve} + \frac{\partial}{\partial z}\overline{u'w'}\right]}_{\text{turbulent stress/resistance}}$ (−ve)

fluid particles face shear stress (resistance)

lamina flow field (pathline)

Laminar flow: limiting case of fluctuation where fluctuations are 0.

Turbulent :- all the particles have random fluctuation

– pathline (turbulent)

due to fluctuation

← fluctuations cause some additional to fluid flow. (analogous to some additional stress)

[resistance is due to shear stress & fluctuations in turbulent flow]

$+\underbrace{\mu_T}_{\text{flow parameter}}\underbrace{\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)}_{\text{turbulent stress}}$

$= -\frac{\partial \bar{p}}{\partial x} + \mu_E\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)$

total resistance; effective viscosity ($\mu_E = \mu + \mu_T$)

$= -\frac{\partial \bar{p}}{\partial x} + \left[\frac{\partial \tau_{xx_E}}{\partial x} + \frac{\partial \tau_{xy_E}}{\partial y} + \frac{\partial \tau_{xz_E}}{\partial z}\right]$ (effective)

where $\tau_{xy_E} = \tau_{xy_V} + \tau_{xy_T}$

$\boxed{\tau_{xy_V} = \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)}$

$\cancel{\tau_{xy_T} = \mu_T}$ $\boxed{\tau_{xy_T} = -\rho\,\overline{u'v'}} = \mu_T\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)$

$\frac{\mu_T}{\mu}$ = relative strength of the resistances.

$\mu_T > \mu$ → opposition to flow due to turbulence is high

$\mu > \mu_T$ → " " " " " viscosity "

→ $\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu_E}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)$

$\frac{\mu_E}{\rho} = \frac{\mu}{\rho} + \frac{\mu_T}{\rho} = \nu + \epsilon$

eddy diffusivity

In pipe flow → turbulent flow is not favoured.

Mixing is the only operation where turbulence is favoured

→ eddy diffusivity is flow parameter not a physical parameter

→ mixing faster ↑ eddy diffusivity ↑

→ 1D flow

$\tau_{xy_V} = \mu\frac{\partial \bar{u}}{\partial y}$

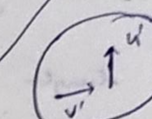
$\tau_{xy_T} = \mu_T\frac{\partial \bar{u}}{\partial y}$

$\mu_T\frac{\partial \bar{u}}{\partial y} = -\rho\,\overline{u'v'}$ ⇒ $\boxed{\epsilon = -\frac{\overline{u'v'}}{\frac{\partial \bar{u}}{\partial y}}}$

$\boxed{\mu_T = \frac{-\rho\overline{u'v'}}{\frac{\partial \bar{u}}{\partial y}}}$

−ve sign so $\mu_T > 0$ or depends on $\frac{\partial \bar{u}}{\partial y}$

$\boxed{\overline{u'v'} = -ve}$



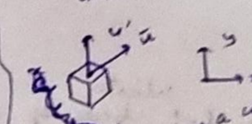
like a vector

out = +ve

In = −ve

$-\rho\overline{u'v'} > 0$

$\overline{u'v'} = -ve$.

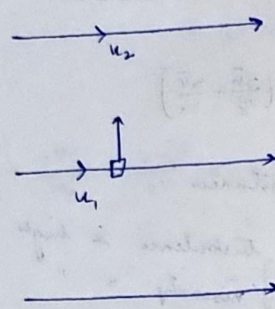


fluid is a continuum

if one particle comes out (v') one other particle should come in

Prandtl mixing length (l)

dynamics → Cannot solve N-S eqn
CFD → can solve N S eqn.



Laminar to turbulent → no: of variables increased but no: of equations are same (4 eq)

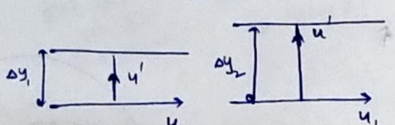
Turbulent
u, v, w, u', v', w', p'

Laminar
u, v, w, p

beco3 of some fluctuation some fluid goes out from level j to level $j+1$. As mass at level j has decreased, u_1 decreases. and u_2 has increased. Now there is a velocity gradient beco3 of fluctuation.

$\boxed{\left(\frac{\partial \bar{u}}{\partial y}\right) \propto u'} \Rightarrow u' \propto \frac{\partial \bar{u}}{\partial y}$

$u' = l \frac{\partial \bar{u}}{\partial y}$ → indicative of how far the particle is moving



strength of the fluctuation is an indication of how far it will go from base pathline

$$l = \frac{u'}{\left(\frac{\partial \bar{u}}{\partial y}\right)}$$

Flow 1 Flow 2
→ velocity gradient is same in both cases
→ high u', high l

→ l → indicates beco3 of fluctuation how much a fluid particle deviates from base pathline.

utility of l :- gives an expression of u'

assumption :- $|u'| = |v'|$ → for isotropic ~~assumption~~ turbulence.

$u'v' = -l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$ $\boxed{u' = -v'}$

$$\boxed{-\rho\overline{u'v'} = \rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2}$$

$\mu_T \longrightarrow -\rho\overline{u'v'} = \mu_T \frac{\partial \bar{u}}{\partial y} = \rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2$

$$\boxed{\mu_T = \rho l^2 \frac{\partial \bar{u}}{\partial y}}$$

$\boxed{l = ky}$ → for a flat plate ↑ y

k → Von karman constant.

→ higher l → more effective mixing