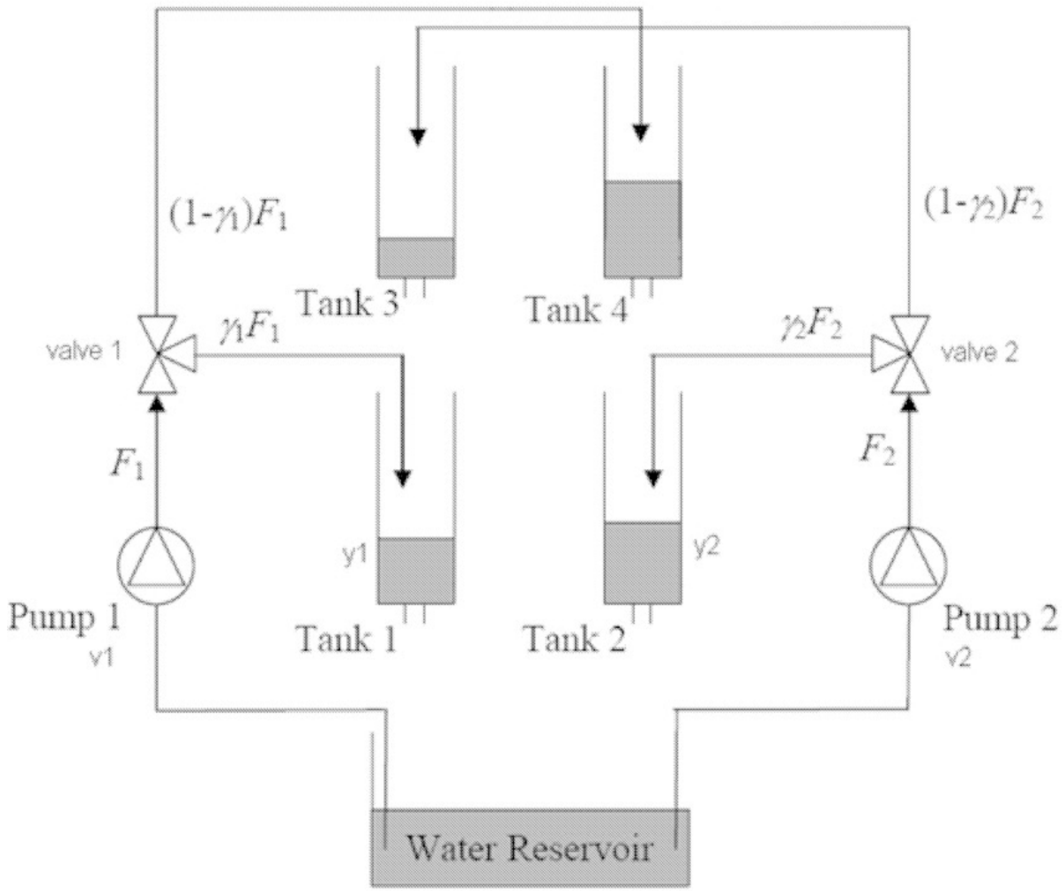
Interaction Analysis
for
Multi-Loop Control

Multivariable Process Control

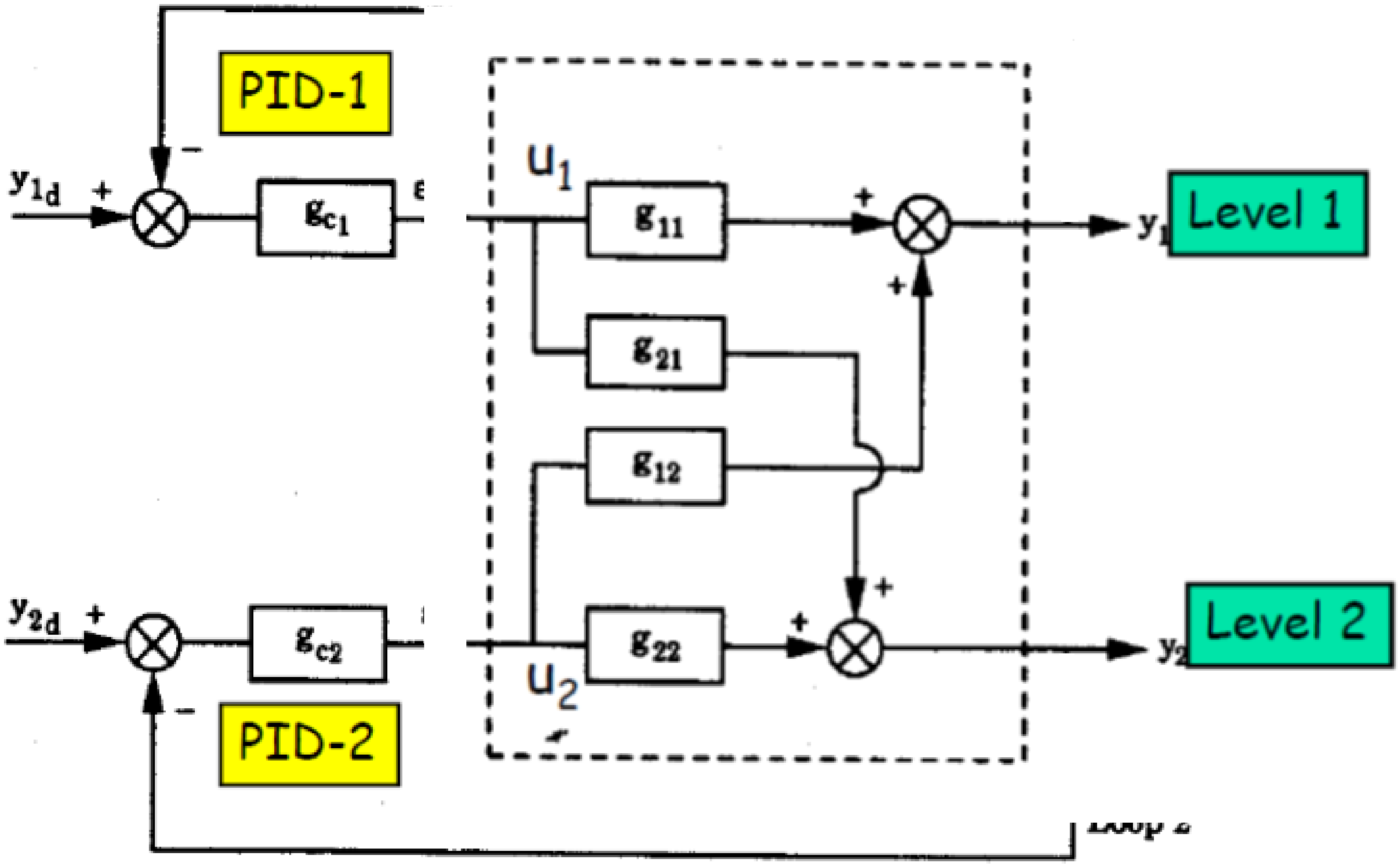
- Industrial Processes
  - multivariable (multiple inputs influence same output) and exhibit strong interaction among the variables
- Conventional Control scheme
  - Multiple *Single Input Single Output* PID controllers used for controlling plant (Multi-Loop Control)
- Consequences
  - Loop Interactions
  - Lack of coordination between different PID loops
  - Neighboring PID loops can co-operate with each other or end up opposing / disturbing each other

Quadruple Tank

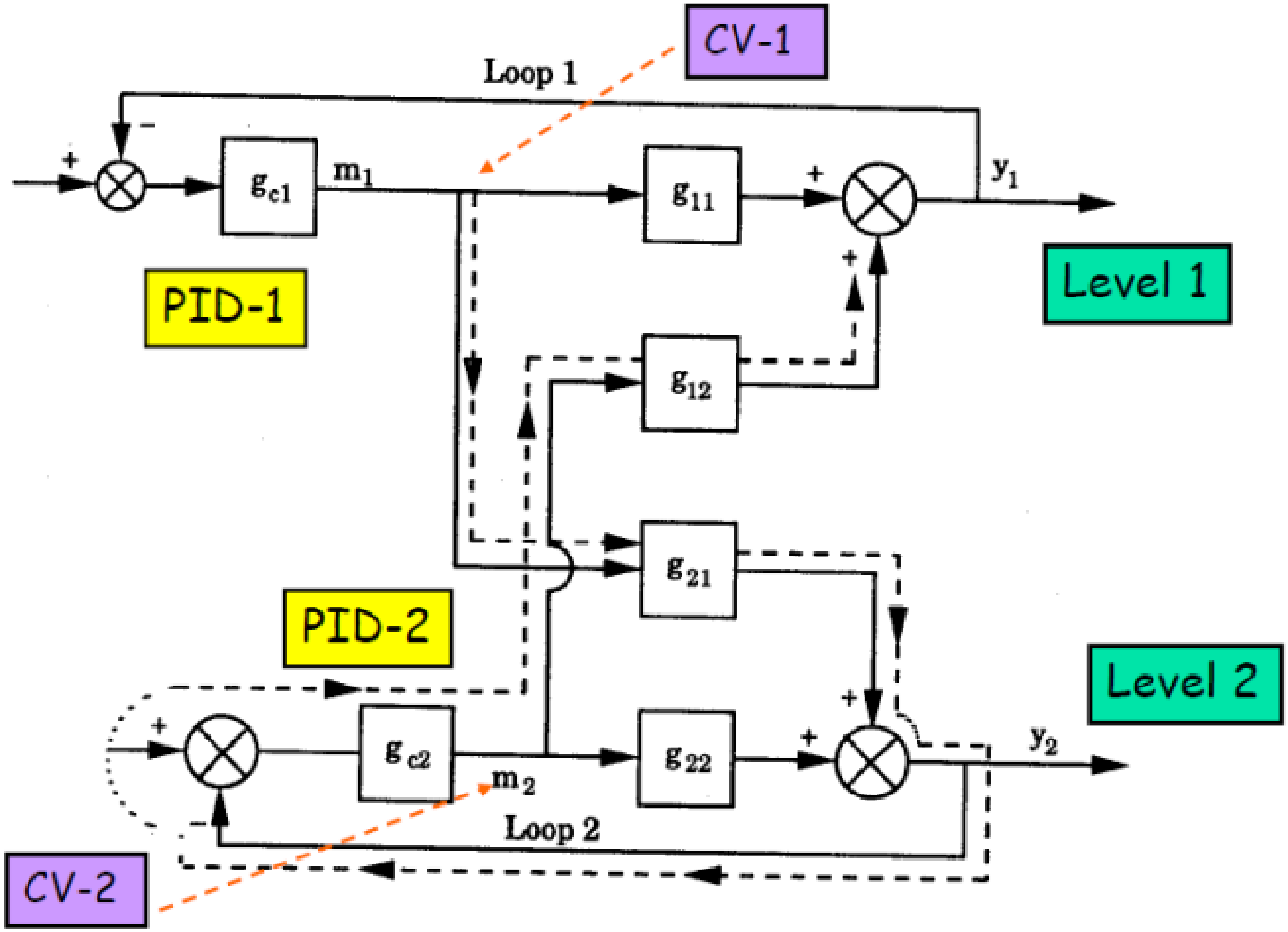


$$y_1(s) = g11(s)u_1(s) + g12(s)u_2(s)$$
$$y_2(s) = g21(s)u_1(s) + g\_22(s)u_2(s)$$

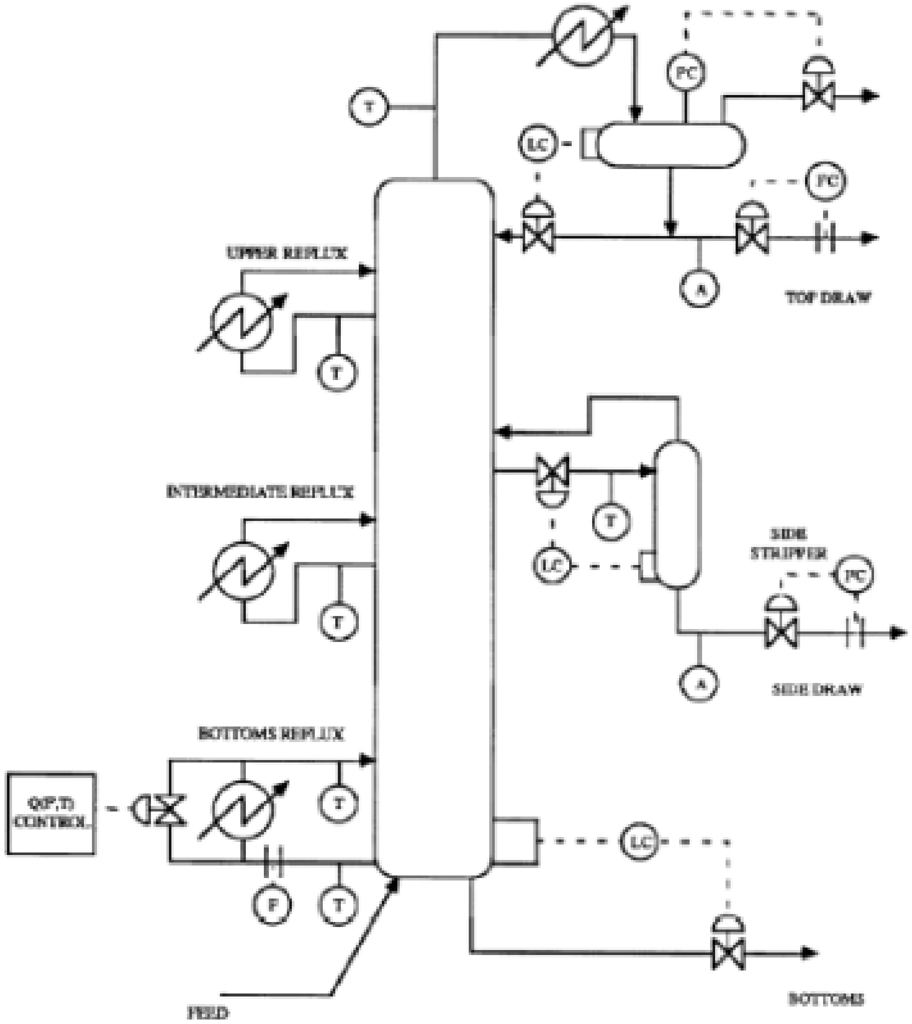
Block Diagram : Quad Tank



Loop Interactions : Quad Tank



Shell Control Problem



Controlled Outputs :

($y1$) Top End Point
($y2$) Side Endpoint
($y3$) Bottom Reflux Temperature

Manipulated Inputs :

($u1$) Top Draw
($u2$) Side Draw
($u3$) Bottom Reflux Duty

Unmeasured Disturbances:

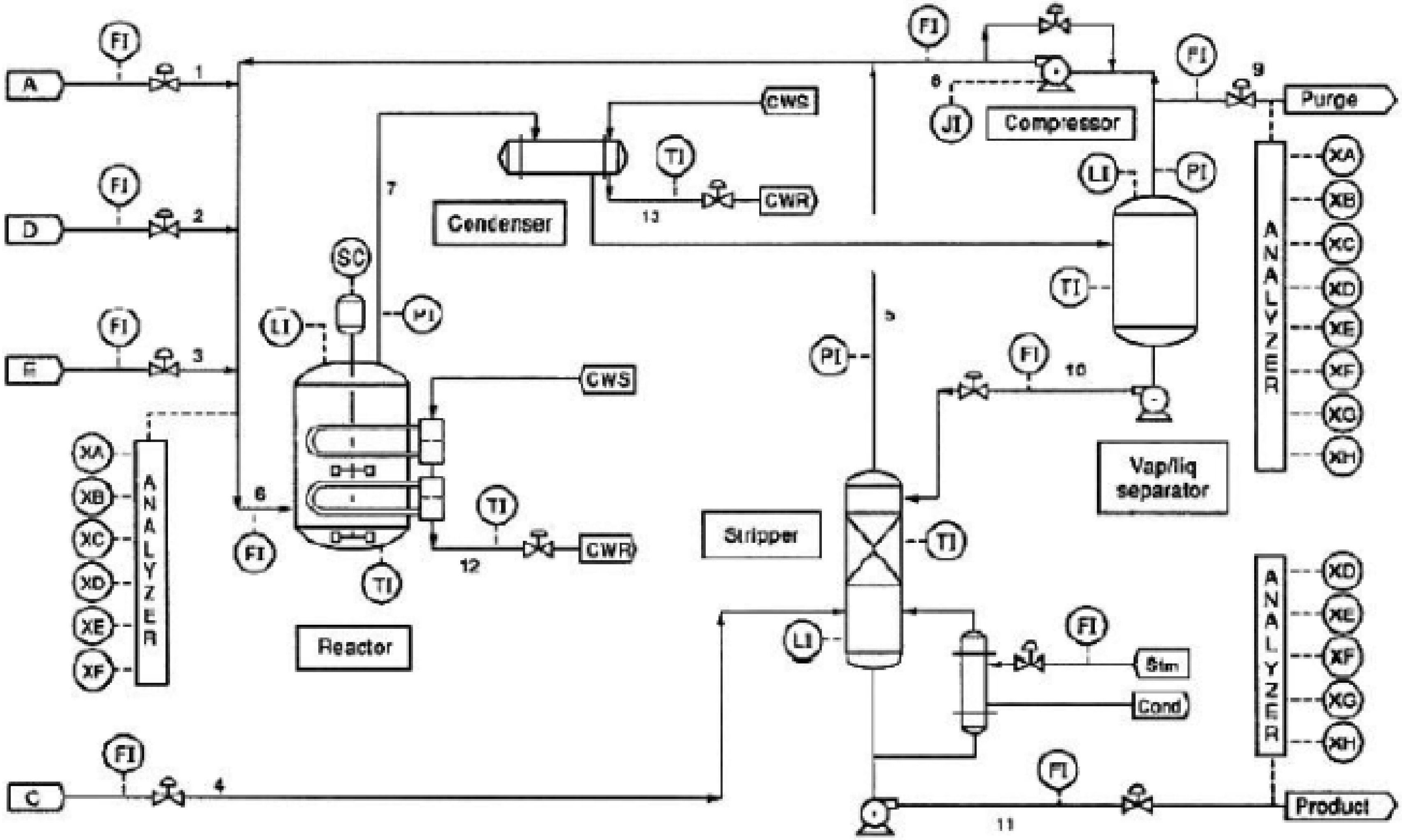
(d_1) Upper reflux
(d_2) Intermediate reflux

Shell Control Problem

- Control Scheme 1
  - PID-1: (y1) Top End Point - (u1) Top Draw
  - PID-2: (y2) Side Endpoint - (u2) Side Draw
  - PID-3: (y3) Bottom Reflux Temperature – (u3) Bottom Reflux Duty
- Control Scheme 2
  - PID-1: (y2) Side End Point - (u1) Top Draw
  - PID-2: (y1) Top Endpoint - (u2) Side Draw
  - PID-3: (y3) Bottom Reflux Temperature –(u3) Bottom Reflux Duty

How to examine above options systematically and reach a decision ?

Tennesse Eastman Problem



Primary Controlled Variable: G conc in Product and Product flow rate

Input - Output Variables

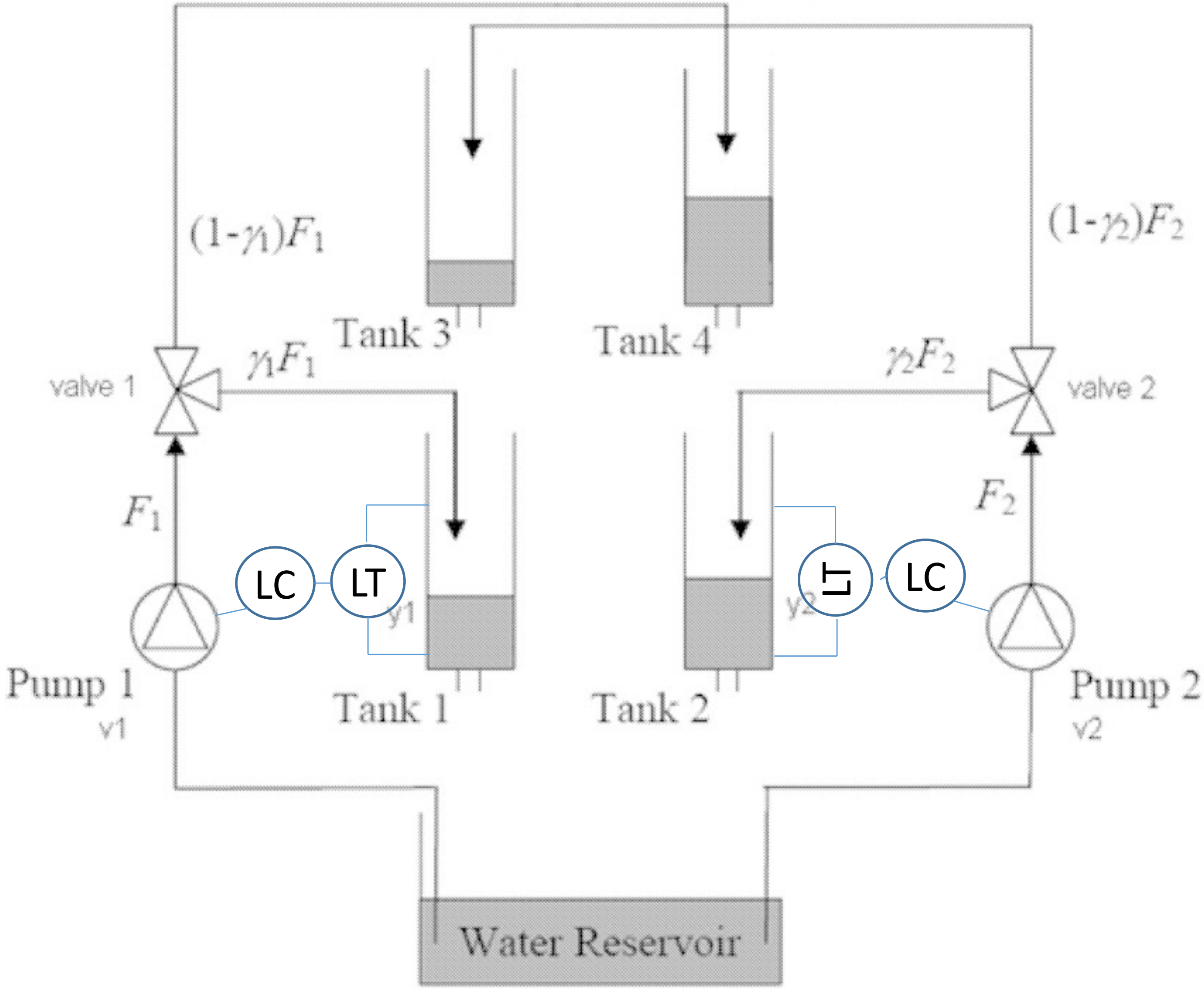
Controlled variable	*Penalty* on error	*Manipulated* Input	*Penalty on* input moves
		Reactor level setpoint	1
		Stripper level setpoint	1
% G in product	5	Separator level setpoint	1
production rate, F11	5	Reactor pressure setpoint	0.2
% B in purge	10	F1	1
% A in feed	2	F2	1
% E in feed	5	F4	1
reactor pressure	5	F8	1
		Reactor temperature	1
		Separator temperature	1

Loop Interactions

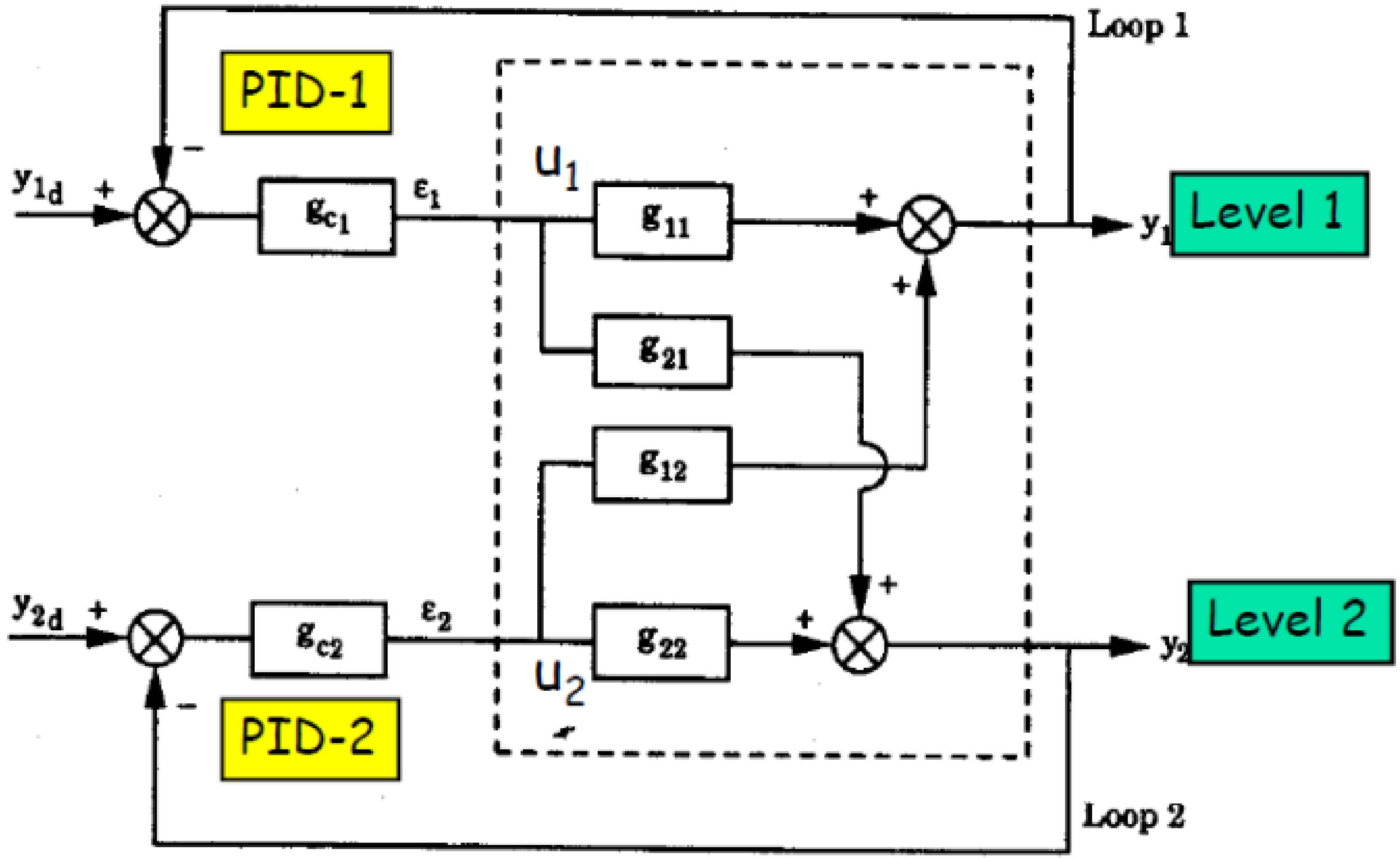
- Large loop interactions :
  - can lead to poor quality of control due to lack of coordination among PID controllers

- Solution strategy:
  - Choose controller pairing with minimal interactions
  - De-tune the controllers to minimize loop interactions
  - Design multi-variable controllers, which simultaneously Change all inputs by considering errors in all the outputs

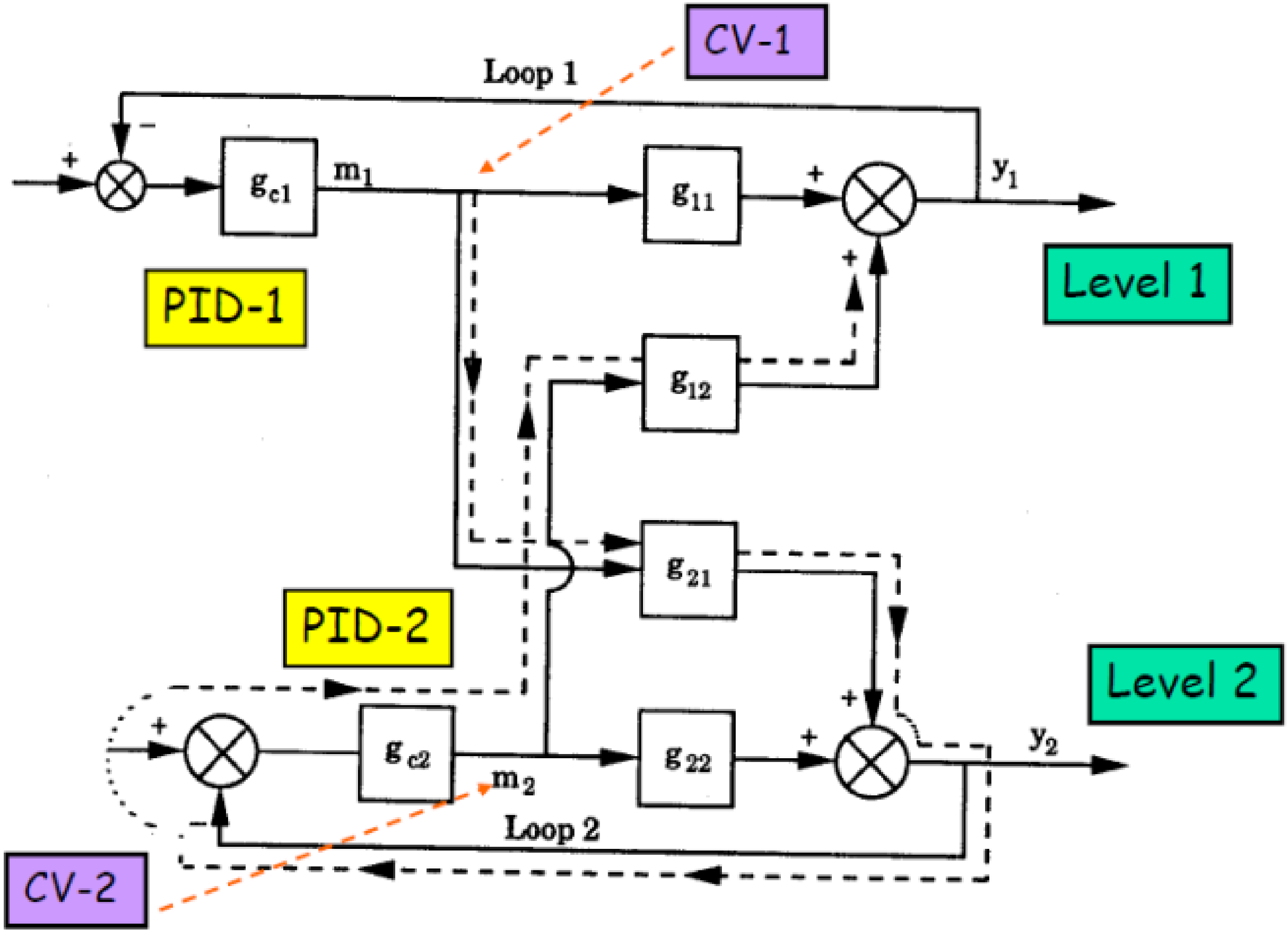
Quadruple Tank process



Block Diagram : Quad Tank



Loop Interactions : Quad Tank



Interaction Analysis

Assume a loop pairing say y_1-u_1 and perform the following experiments

- With all loops open, make a step change in u_1 to $u_1 + \Delta u$ and measure the change in output Δy_1.
- We will term this as a direct effect.
- With all loops except the u_1-y_1 loop closed, repeat the same change in u_1.
- There will be change in y_1 because of the direct effect but also there will be a retaliatory effect because u_2 changes to keep y_2 constant.
- We will term this change as $\Delta y_1 + \Delta y_{1r}$

Interaction Analysis

- Ratio of these two terms can be defined as λ_{11} (for the y1-u1 pairing) as

$$\lambda_{11} = \Delta y_1/(\ \Delta y_1+\Delta y_{1r})$$

This is called **relative gain**

- Compute relative gain for each assumed input/output pairing
- Depending on the values of this index for various assumed loop pairings (step 1), decision can be taken on the final loop pairing.
- This decision making is based on steady state analysis only.

Relative Gain Array (RGA)

RGA: A measure of loop interaction used for deciding loop pairing of SISO controllers

$$\lambda_{ij} = \frac{\left[\Delta y_i / \Delta u_j\right]_{\text{open\_loop}}}{\left[\Delta y_i / \Delta u_j\right]_{\text{all but } y_i - u_j \text{ loop closed}}}$$

y : Measured outputs

u : Manipulated Inputs

$\left[\Delta y_i / \Delta u_j\right]$: Steady state gain / sensitivity between i'th op and j'th i/p

Calculation of RGA

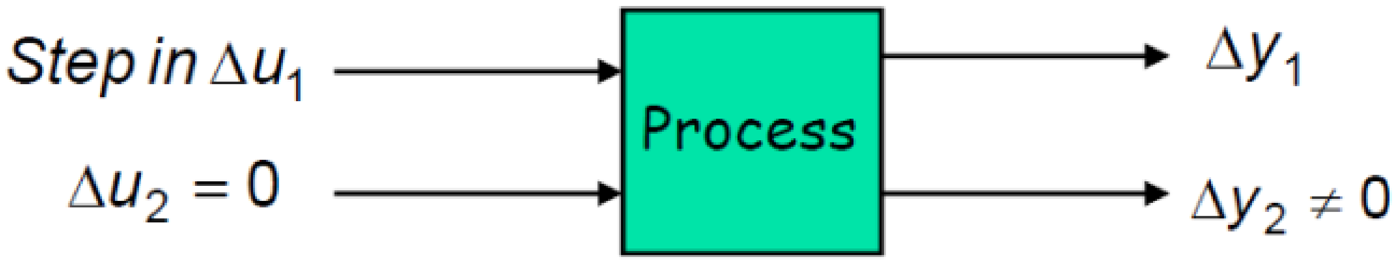
Let us assume pairing y_1-u_1

Steady state model

$$\Delta y_1 = k_{11}\Delta u_1 + k_{12}\Delta u_2$$
$$\Delta y_2 = k_{21}\Delta u_1 + k_{22}\Delta u_2$$

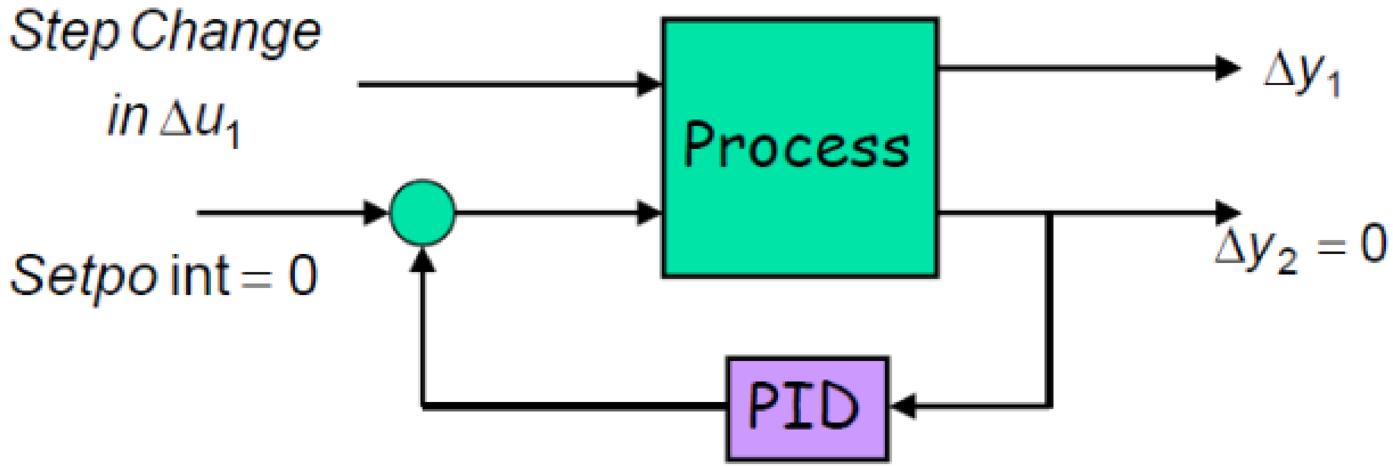
k_{ij} : Stesdy state gain / sensitivity between output i and input j

$$\left[\frac{\Delta y_1}{\Delta u_1}\right]_{Open-Loop} = k_{11}$$



Calculation of RGA

Now consider the situation where y_2-u_2 loop is closed and perfectly controlled



$$\Delta y_2 = k_{21}\Delta u_1 + k_{22}\Delta u_2 = 0$$

$$\Delta u_2 = -(k_{12} / k_{22})\Delta u_1$$

$$\Delta y_1 = k_{11}\Delta u_1 + k_{12}\Delta u_2 = \left[k_{11} - \frac{k_{12}k_{21}}{k_{22}}\right]\Delta u_1$$

Calculation of RGA

$$\left[\frac{\Delta y_1}{\Delta u_1}\right]_{y_2-u_2\,Loop\,Closed} = \left[k_{11} - \frac{k_{12}k_{21}}{k_{22}}\right]$$

$$\Rightarrow \lambda_{11} = \frac{1}{\left[1 - \dfrac{k_{12}k_{21}}{k_{11}k_{22}}\right]}$$

Other relative gains can be easily computed as follows

$$\lambda_{12} = \lambda_{21} = 1 - \lambda_{11}$$

$$\lambda_{22} = \lambda_{11}$$

$$RGA\ Matrix\,(\Lambda)\,for\ 2 \times 2\,system$$

$$\Lambda = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}$$

$$\boxed{\text{RGA is independent of variable scaling}}$$

Wood-Berry Column

Pilot scale distillation column

$$\begin{bmatrix} x_D(s) \\ x_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix}$$

Distillate Composition

Reflux Rate

Steady Stete Gain Matrix

$$\begin{bmatrix} \Delta x_D \\ \Delta x_B \end{bmatrix} = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix} \begin{bmatrix} \Delta R \\ \Delta S \end{bmatrix}$$

Bottom Composition

Steam Flow-rate

Wood-Berry Column

$$\lambda = 1/0.498$$

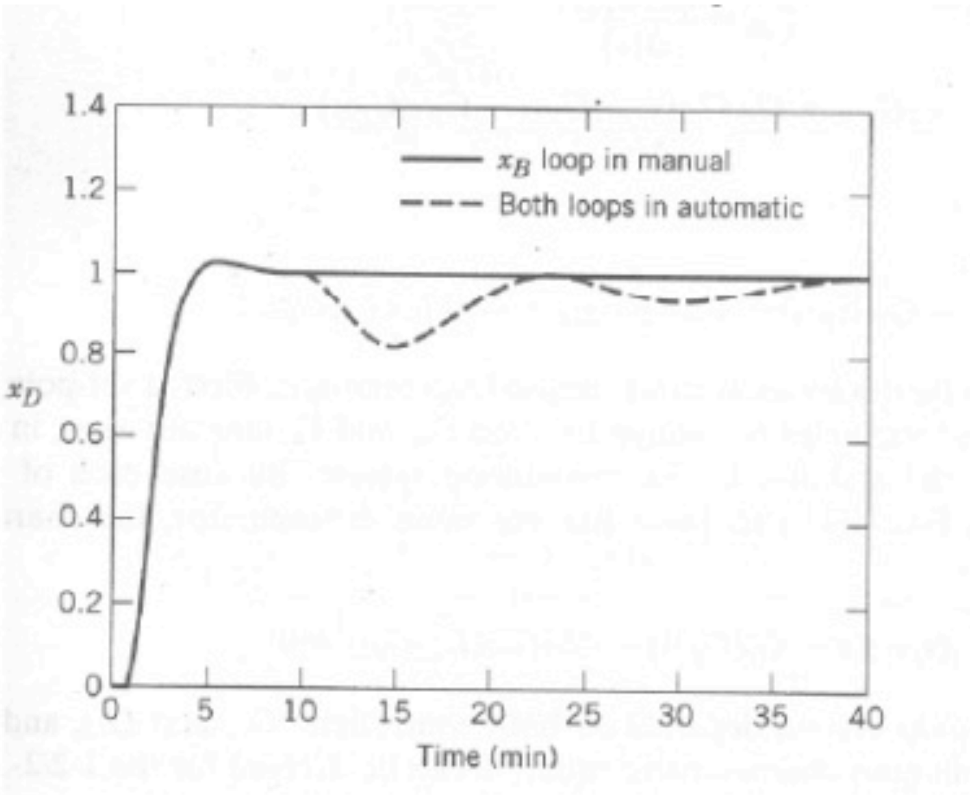
$$RGA(\Lambda) = \begin{bmatrix} 2.0019 & -1.0019 \\ -1.0019 & 2.0019 \end{bmatrix}$$

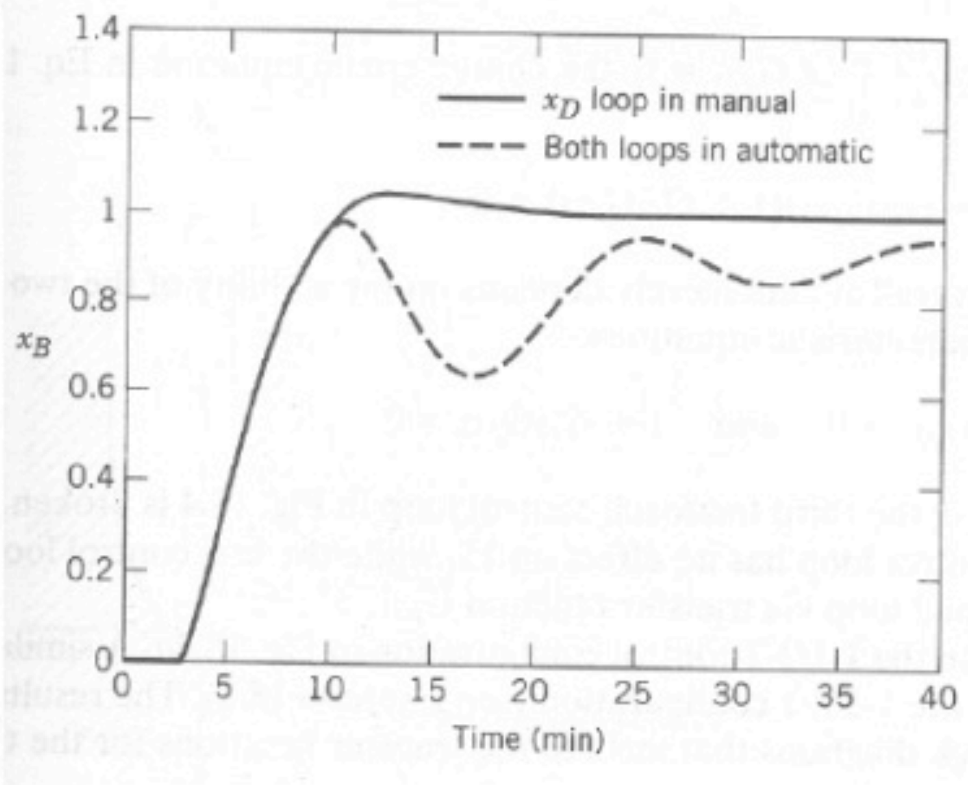
Any loop pairing will result in loop interactions

Pairing	k_c	τ_I(min)
$x_D - R$	0.604	16.37
$x_B - B$	-0.127	14.46

Wood-Berry Column

Multi-Loop PID Control
Change in closed loop behavior due to loop interactions





RGA calculation for MIMO Process

$\frac{\partial \mathbf{y}_i}{\partial \mathbf{u}_j} = \mathbf{K}_{ij}$ (steady state gain between input j and output i)

Steady state model : $\mathbf{y} = \mathbf{K}\mathbf{u} \Rightarrow \mathbf{u} = \mathbf{K}^{-1}\mathbf{y} = \widetilde{\mathbf{K}}\mathbf{y}$

When all but $\mathbf{y}_i - \mathbf{u}_j$ loop are closed and perfectly controlled

$$\mathbf{y} = [0 \quad .... \quad y_i \quad .... \quad 0]^T$$

$$\left(\frac{\partial \mathbf{u}}{\partial \mathbf{u}_j}\right)_c = \begin{bmatrix} \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}_j} \\ ..... \\ 1 \\ .... \\ \frac{\partial \mathbf{u}_n}{\partial \mathbf{u}_j} \end{bmatrix} = \widetilde{\mathbf{K}} \begin{bmatrix} 0 \\ ..... \\ \left(\frac{\partial \mathbf{y}_i}{\partial \mathbf{u}_j}\right)_c \\ .... \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{K}}_{1i} \\ ..... \\ ..... \\ .... \\ \widetilde{\mathbf{K}}_{ni} \end{bmatrix} \left(\frac{\partial \mathbf{y}_i}{\partial \mathbf{u}_j}\right)_c$$

i'th column of gain Matrix inverse

$$\Rightarrow \widetilde{\mathbf{K}}_{ji} \left(\frac{\partial \mathbf{y}_i}{\partial \mathbf{u}_j}\right)_c = 1 \Rightarrow \left(\frac{\partial \mathbf{y}_i}{\partial \mathbf{u}_j}\right)_c = \frac{1}{\widetilde{\mathbf{K}}_{ji}}$$

Relative gain $(\lambda_{ij}) = \mathbf{K}_{ij}\widetilde{\mathbf{K}}_{ji} = \mathbf{K}_{ij}\left[\mathbf{K}^{-1}\right]_{ij}^{T}$

MIMO RGA Calculation

For general $(n \times n)$ system

Given Steady Stage Gain Matrix $\mathbf{K}$

$$\text{RGA}\ (\Lambda) = \mathbf{K} \otimes \left(\mathbf{K}^{-1}\right)^{T}$$

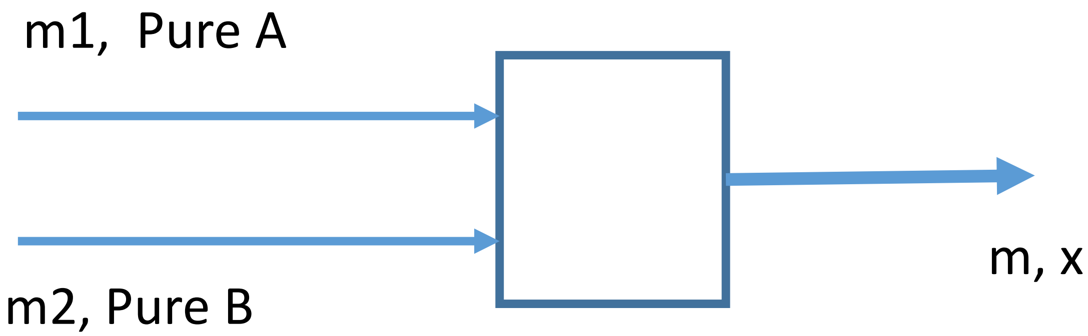
where $\otimes$ denotes Schur Product

(element by element multiplication of two matrices)

Properties

1. Summation of all elements in any row = 1
2. Summation of all elements in any column = 1
3. RGA is independent of scaling used for input of output variables

RGA for Blending Tank



Process: Blending of two process streams to get desired composition of A in the product.

- Control objective : m,x by manipulating m1 and m2
- Desired value for m and x is m* and x*. Find RGA

RGA for Blending Tank

- Total Mass Balance : $m = m_1 + m_2$
- Comp A balance: $x = \frac{m_1}{m_1+m_2}$
- $\left.\frac{\partial m}{\partial m_1}\right|_{both\ loops\ open} = 1 \quad \left.\frac{\partial m}{\partial m_1}\right|_{other\ loop\ closed} = ?$

- $x^* = \frac{m_1}{m_1+m_2}$ will give $m_2 = \frac{m_1}{x^*} - m_1$
- $m = m_1 + \frac{m_1}{x^*} - m_1 = \frac{m_1}{x^*}$
- so, $\left.\frac{\partial m}{\partial m_1}\right|_{other\ closed} = \frac{1}{x^*}$
- $\lambda_{11} = x^*$

$$\Lambda = \begin{bmatrix} x^* & 1 - x^* \\ 1 - x^* & x^* \end{bmatrix}$$

Analysis of RGA

- $\lambda = \frac{\Delta y}{\Delta y + \Delta y_r} = \frac{K_d}{K_d + K_r}$

- $\lambda = 1\, means\; K_r = 0$

- $\lambda < 1\, means\; K_r\; finite$ and in same direction to K_d

- $\lambda > 1\, when\; |K_d| > |K_r|$ but direction of Kr is opposite to Kd

Analysis of RGA

- If $\lambda_{11} = 1 \Rightarrow$ retaliatory action is not present. So assumed loop pairing is correct because there is no interaction from the other loop.

- If $0 < \lambda_{11} < 1 \Rightarrow$ retaliatory action is comparable to the direct action but is in the same direction. The assumed loop pairing may be chosen only if the index is closer to 1 (say 0.8).

- If $\lambda_{11} = 0 \Rightarrow$ Retaliatory action is much greater than the direct action. The assumed loop pairing is incorrect. The loop pairing u_1-y_2 is preferable

Analysis of RGA

- If $\lambda_{11} > 1 \Rightarrow$ Retaliatory action is in opposite direction to the direct action but is smaller in magnitude than the direct. The assumed loop pairing may be chosen only if the index is close to 1.

- If λ11 < 0, Retaliatory effect is larger and opposite in direction to the main effect. **Do not choose this loop pairing.**

Suggested Loop Pairing

Refinery Distillation Column

$$RGA = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0.931 & 0.15 & 0.08 & -0.164 \\ -0.011 & -0.429 & 0.286 & 1.154 \\ -0.135 & 3.314 & -0.27 & -1.91 \\ 0.215 & -2.03 & 0.9 & 1.919 \end{bmatrix} \end{array}$$

Outputs

Y_1: Top Composition Y_2: Bottom Composition

Y_3: Side stream 1 Composition Y_4: Side Stream 2 Composition

Inputs

u_1: Top Draw u_2: Bottom Draw

u_3: Side Draw 1 u_4: Side Draw 2

RGA analysis : Non-square System

Consider process with 2 measurements and 3 inputs

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.07 & 0.04 \\ 0.004 & -0.003 & -0.001 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Best Combination

$$RGA_1 = \begin{array}{c c} & \begin{array}{cc} \Delta u_1 & \Delta u_2 \end{array} \\ \begin{array}{c} \Delta y_1 \\ \Delta y_2 \end{array} & \begin{bmatrix} 0.84 & 0.16 \\ 0.16 & 0.84 \end{bmatrix} \end{array}$$

$$RGA_2 = \begin{array}{c c} & \begin{array}{cc} \Delta u_1 & \Delta u_3 \end{array} \\ \begin{array}{c} \Delta y_1 \\ \Delta y_2 \end{array} & \begin{bmatrix} 0.76 & 0.24 \\ 0.24 & 0.76 \end{bmatrix} \end{array}$$

$$RGA_3 = \begin{array}{c c} & \begin{array}{cc} \Delta u_2 & \Delta u_3 \end{array} \\ \begin{array}{c} \Delta y_1 \\ \Delta y_2 \end{array} & \begin{bmatrix} -1.4 & 2.4 \\ 2.4 & -1.4 \end{bmatrix} \end{array}$$

Singular Value Analysis

Powerful analytical tool for

- Selection of controlled, measured and manipulated variables
- Determination of best multi-loop configuration
- Evaluation of robustness (insensitivity to changes in plant behavior) of a control scheme

$$\Delta \mathbf{Y} = \mathbf{K} \Delta \mathbf{U} \quad (\mathbf{K}: \text{ Steady State Gain Matrix})$$

$$\text{Singular Values } (\sigma_1, \sigma_2, \ldots \sigma_r) \equiv$$

$$\text{+ve Square Root of Eigen-values of } \mathbf{K}^T \mathbf{K}$$

$$\text{Roots of polynomial}$$

$$\det(s\mathbf{I} - \mathbf{K}^T \mathbf{K}) = 0$$

$$\text{Condition Number (CN)} = \|\mathbf{K}\|_2 \|\mathbf{K}^{-1}\|_2 = \frac{\max(\sigma_i)}{\min(\sigma_i)}$$

Singular Value Analysis

- Non-square systems (number of inputs not equal to number of outputs)
  - SVA can be used to find a square subset with least difficulties in control

- Larger condition number implies difficulties in controlling a system
  - Among multiple possibilities, choose subset with minimum condition number

- Limitation: Singular values are dependent on scaling of input and output variables

Singular Value Decomposition

$$\boldsymbol{K} = \boldsymbol{W}\,\boldsymbol{\Sigma}\,\boldsymbol{V}^{\boldsymbol{T}}$$

Σ is diagonal matrix of singular values $(\sigma_1, \sigma_2, \ldots, \sigma_r)$

The singular values are the positive square roots of the eigenvalues of $K^T K$ (*r* = rank of $K^T K$)

W,V are input and output singular vectors
Columns of *W* and *V* are orthonormal.

$$WW^T = I \quad and \quad VV^T = I$$

Calculate Σ, *W, V* using MATLAB (svd = singular value decomposition)

Condition number (CN) is the ratio of the largest to the smallest singular value and indicates if *K* is ill-conditioned.

Singular Value Analysis

SVD of K : $K = W\,\Sigma\,V^T$

Pairing rule:

Output associated with the largest magnitude element (without regard to sign) of the W_1 vector with the manipulated variable associated with the largest magnitude element (without regard to sign) of the V_1 vector. Same logic will apply for $W_2 - V_2$ … $W_n - V_n$ vectors.

Alternate Method

- Arrange the singular values in order of largest to smallest and look for any $\sigma_i/\sigma_{i-1} > 10$; then one or more inputs (or outputs) can be deleted.
- Delete one row and one column of *K* at a time and evaluate the properties of the reduced gain matrix.

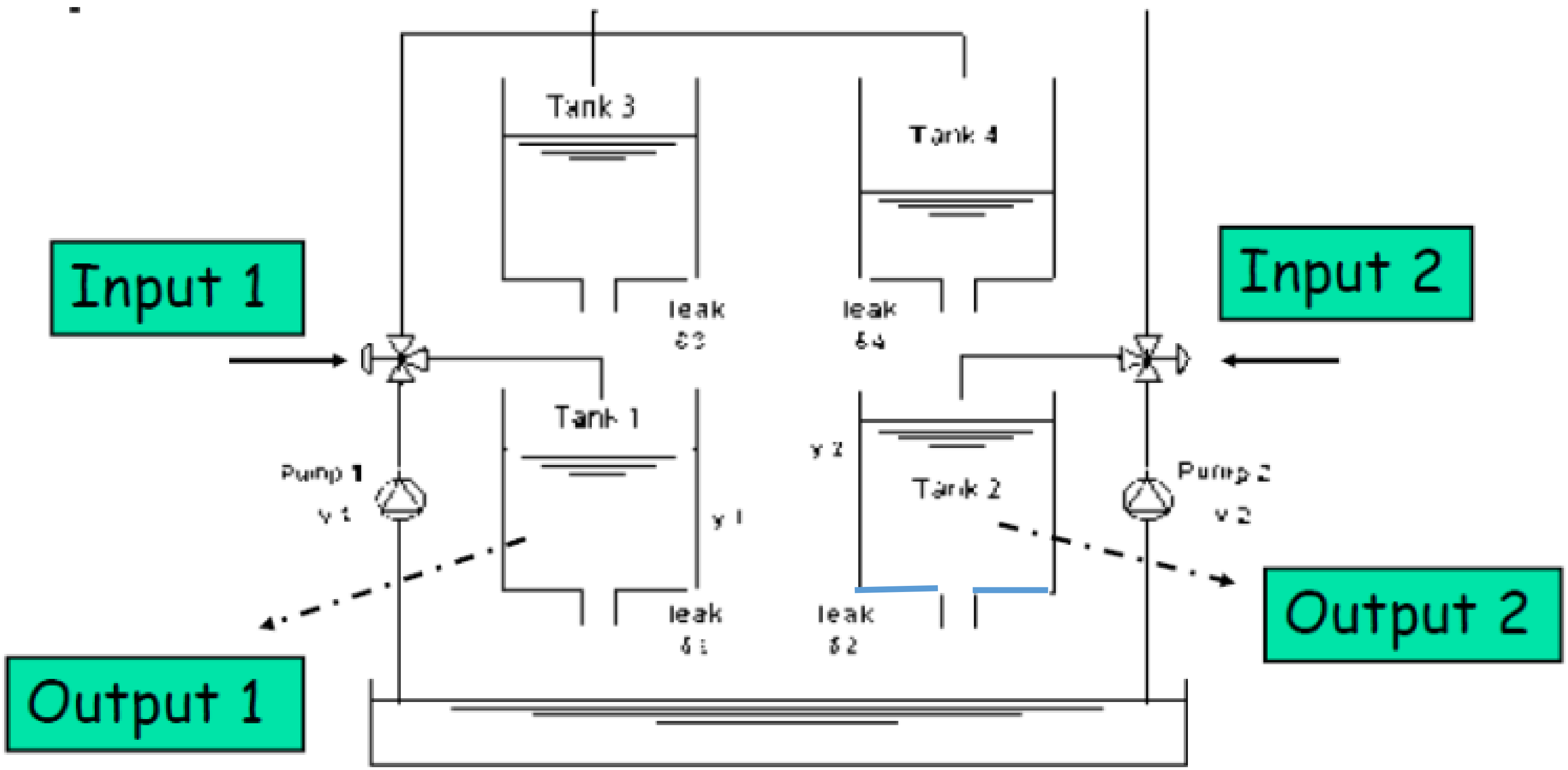
Example

$$K = \begin{bmatrix} 0.48 & 0.90 & -0.006 \\ 0.52 & 0.95 & 0.008 \\ 0.90 & -0.95 & 0.020 \end{bmatrix} \qquad \text{RGA}= \begin{bmatrix} -2.4376 & 3.0241 & 0.4135 \\ 1.2211 & -0.7617 & 0.5407 \\ 2.2165 & -1.2623 & 0.0458 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} 0.5714 & 0.3766 & 0.7292 \\ 0.6035 & 0.4093 & -0.6843 \\ -0.5561 & 0.8311 & 0.0066 \end{bmatrix} \qquad \sum = \begin{bmatrix} 1.618 & 0 & 0 \\ 0 & 1.143 & 0 \\ 0 & 0 & 0.0097 \end{bmatrix}$$

$$\boldsymbol{V} = \begin{bmatrix} 0.0541 & 0.9984 & 0.0151 \\ 0.9985 & -0.0540 & -0.0068 \\ -0.0060 & 0.0154 & -0.9999 \end{bmatrix}$$

Quadruple Tank Process



Schematic of Quadruple Tank Process

Quadruple Tank problem

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1} v_1$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2} v_2$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3} v_2$$

$$\frac{dh_2}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4} v_1$$

Control Objective

Both the levels of tank1 and tank2 should be controlled by manipulating voltages to the pumps.

Data for simulation:

A_1, A_3 = 28 cm^2

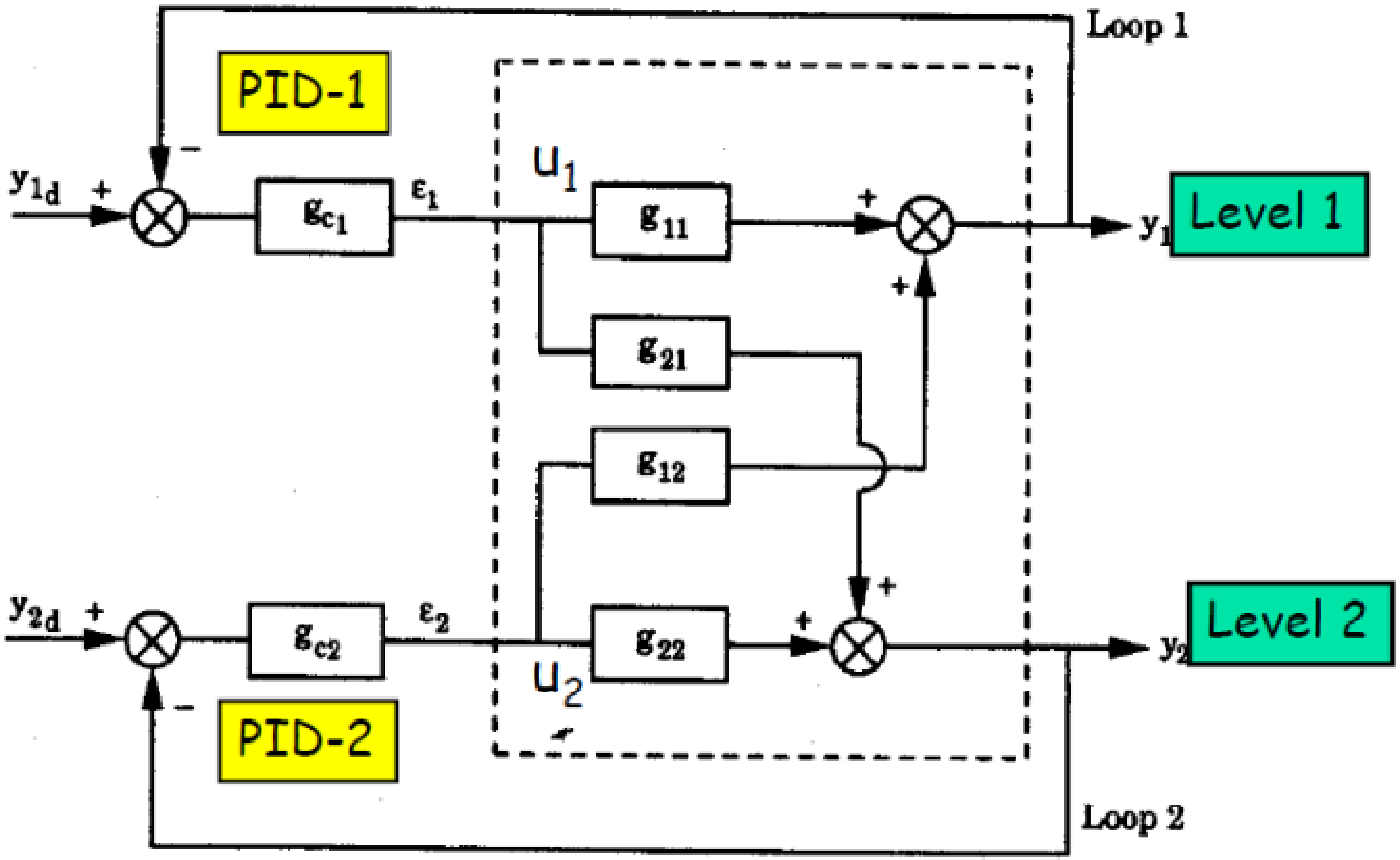
A_2, A_4 = 32 cm^2

a_1, a_3 = 0.071 cm^2

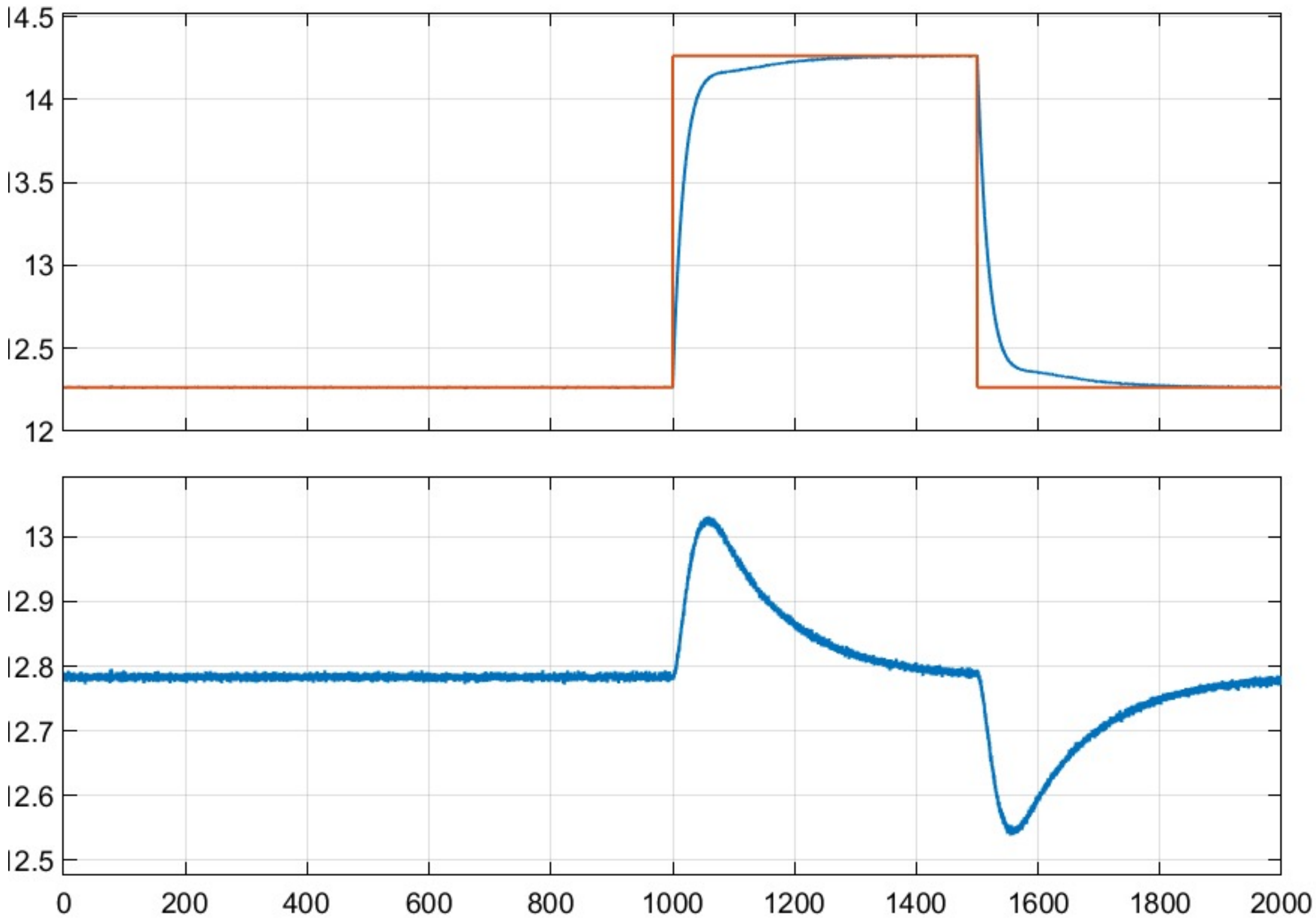
a_2, a_4 = 0.057 cm^2

Case	k1	k2	v1	v2	γ_1	γ_2
1	3.33	3.35	3.0	3.0	0.7	0.6
2	3.14	3.29	3.15	3.15	0.43	0.34
3	3.2	3.32	3.1	3.1	0.5	0.5

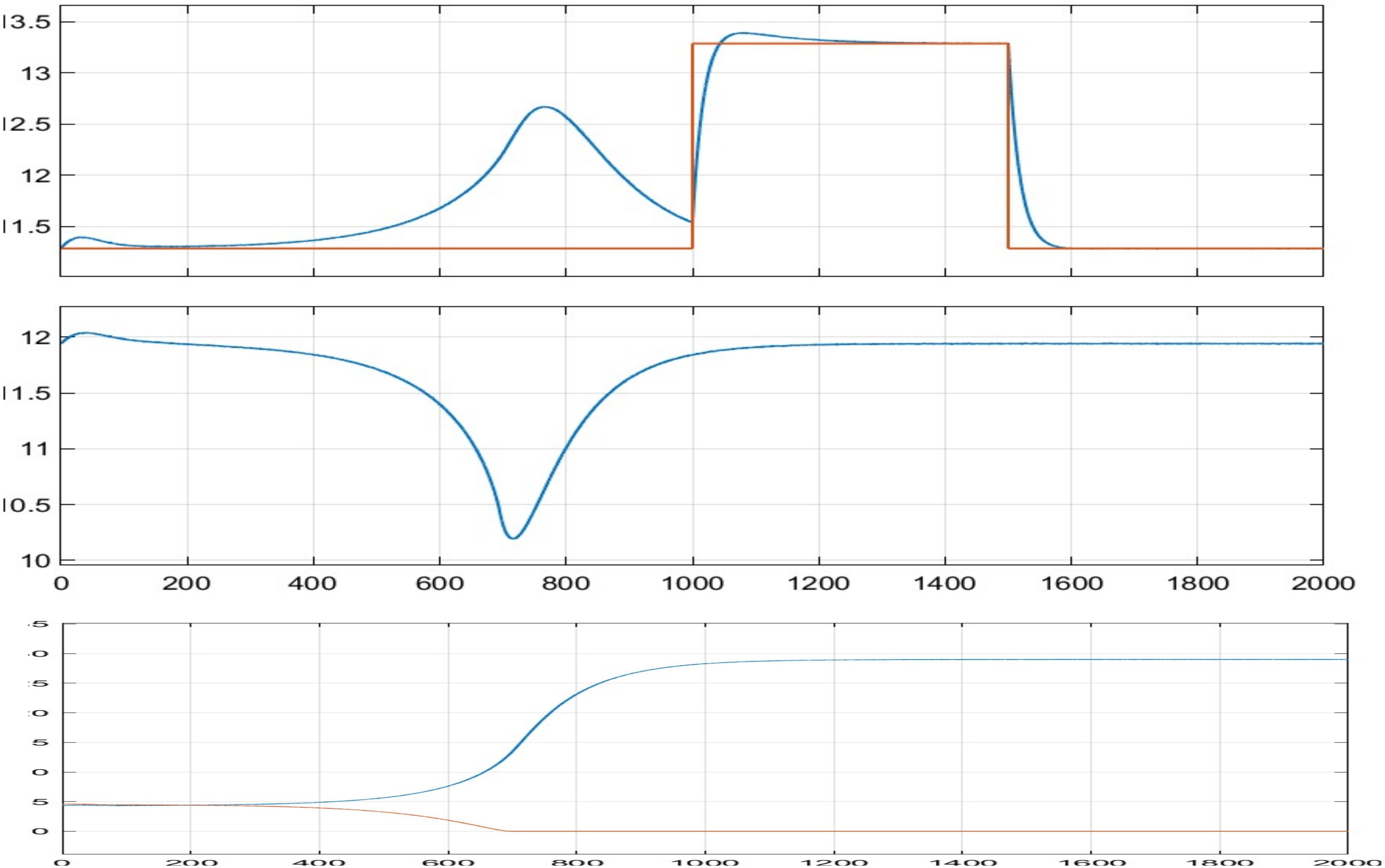
Block Diagram : Quad Tank



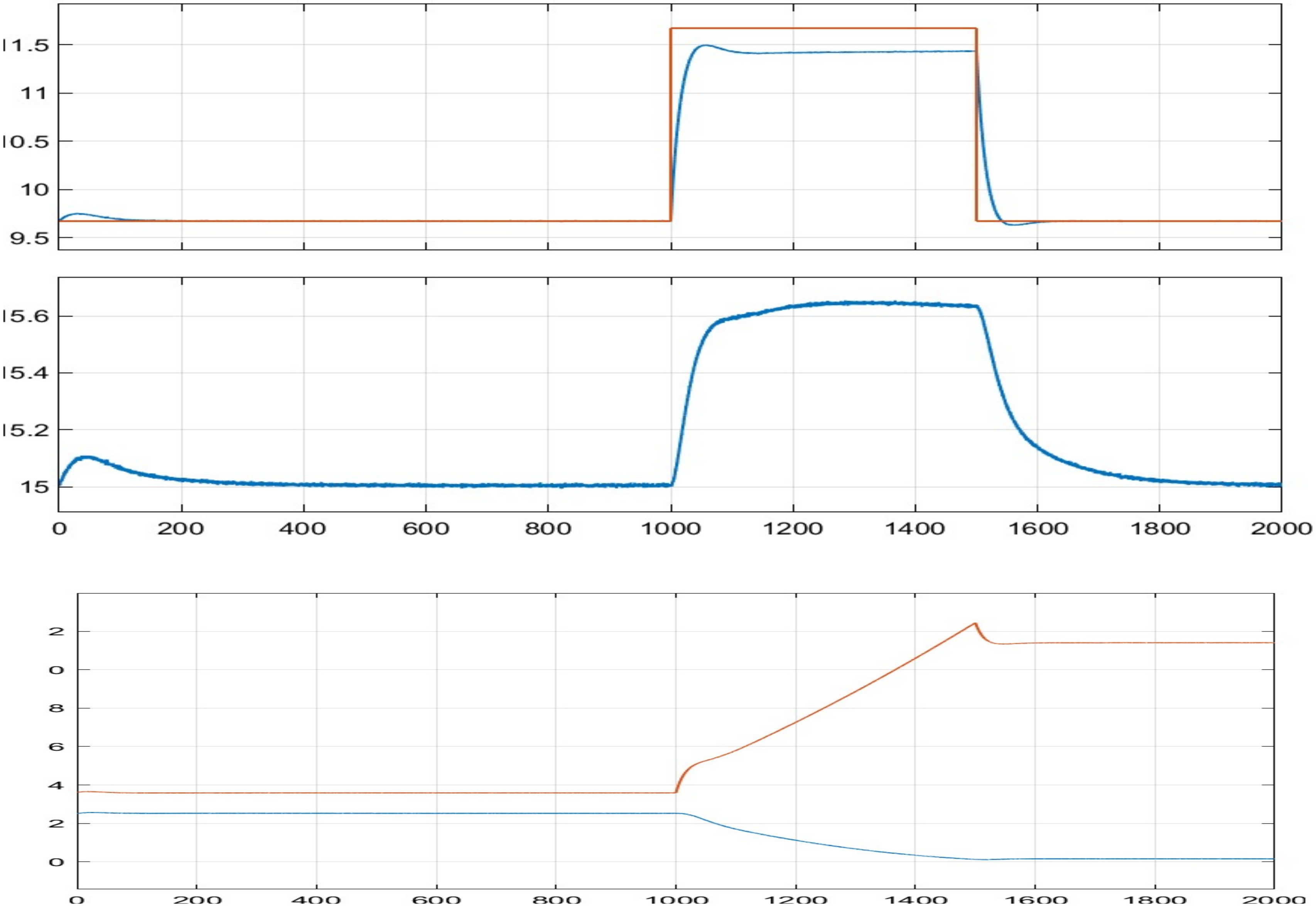
Quad Tank : Case 1



Quad Tank: Case 2



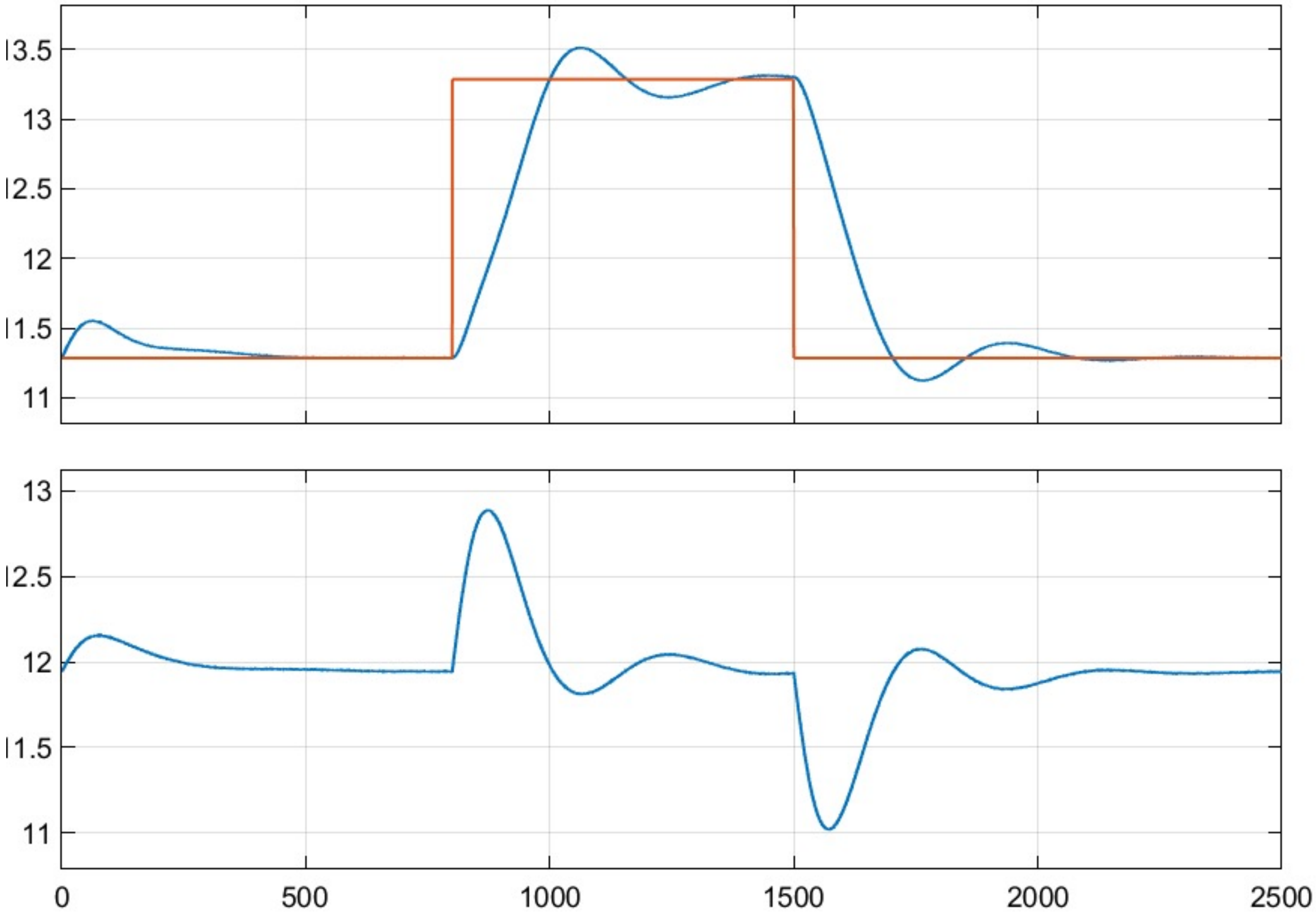
Quad Tank: Case 3



Quad Tank

- **Case 1**
- $G(s) = \begin{bmatrix} \dfrac{4.153}{62.36s+1} & \dfrac{3.036}{(62.36s+1)(22.76s+1)} \\ \dfrac{1.78}{(90.63s+1)(30.09s+1)} & \dfrac{4.554}{90.63s+1} \end{bmatrix}$
- $RGA = \begin{bmatrix} 1.4 & -0.4 \\ -0.4 & 1.4 \end{bmatrix}$
- **Case 2**
- $G(s) = \begin{bmatrix} \dfrac{2.306}{59.82s+1} & \dfrac{4.755}{(59.82s+1)(36.88s+1)} \\ \dfrac{3.059}{(87.6s+1)(53.91s+1)} & \dfrac{2.45}{87.6s+1} \end{bmatrix}$
- $RGA = \begin{bmatrix} -0.63 & 1.63 \\ 1.63 & -0.63 \end{bmatrix}$

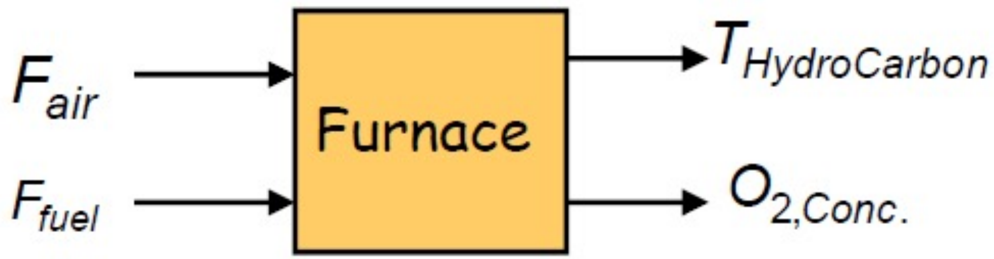
Quad Tank: $h_1 \rightarrow V_2 \;\; h_2 \rightarrow V_1$



Quad Tank

- **Case 3**
- $G(s) = \begin{bmatrix} \frac{2.53}{55.37s+1} & \frac{4.075}{(55.37s+1)(28.2s+1)} \\ \frac{2.53}{(98.19s+1)(48.19s+1)} & \frac{4.075}{98.19s+1} \end{bmatrix}$
- $RGA = undefined$

Furnace Control



$$\begin{bmatrix} T_{HC}(s) \\ O_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{-13.235e^{-4s}}{4.22s+1} & \dfrac{226.95e^{-4s}}{4.75s+1} \\ \dfrac{0.1445e^{-4s}}{4.434s+1} & \dfrac{-1.96e^{-4s}}{3.98s+1} \end{bmatrix} \begin{bmatrix} F_{air}(s) \\ F_{fuel}(s) \end{bmatrix}$$

$$\lambda = -3.778$$

$$RGA(\Lambda) = \begin{bmatrix} -3.778 & 4.778 \\ 4.778 & -3.778 \end{bmatrix}$$

Suggested Pairing

$\mathrm{T_{HC}} - F_{fuel}$ *and* $O_2 - F_{air}$

Singular Values

$\sigma_1 = 227.34 \quad \sigma_2 = 0.0302$

Condition Number $= 7530.5$

Large CN : Difficult to Control

4-Component Distillation Column

Table 18.2 Condition Numbers for the Gain Matrices Relating Column Controlled Variables to Various Sets of Manipulated Variables (Roat et al., 1986)

Controlled Variables

x_D = Mole fraction of propane in distillate D
x_{64} = Mole fraction of isobutane in tray 64 sidedraw
x_{15} = Mole fraction of n-butane in tray 15 sidedraw
x_B = Mole fraction of isopentane in bottoms B

Possible Manipulated Variables

L = Reflux flow rate	B = Bottoms flow rate
D = Distillate flow rate	S_{64} = Sidedraw flow rate at tray 64
V = Steam flow rate	S_{15} = Sidedraw flow rate at tray 15

Strategy Number[a]	Manipulated Variables	Condition Number
1	$L/D, S_{64}, S_{15}, V$	9,030
2	$V/L, S_{64}, S_{15}, V$	60,100
3	$D/V, S_{64}, S_{15}, V$	116,000
4	D, S_{64}, S_{15}, V	51.5
5	L, S_{64}, S_{15}, B	57.4
6	L, S_{64}, S_{15}, V	53.8

[a] In each control strategy, the first controlled variable is paired with the first manipulated variable, and so on. Thus, for Strategy 1, x_D is paired with L/D, and x_B is paired with V.

Niederlinsky Index

- Consider MIMO system whose inputs and outputs are paired as $y_1 - u_1, y_2 - u_2, \dots \dots \dots, y_n - u_n$ i.e, Transfer Function matrix G(s) is arranged such that transfer functions relating paired inputs and outputs are arranged along the diagonal.
- Further, let each element of G(s) be (a) rational and (b) open loop stable.
- Also, let n SISO feedback controllers with integral action be designed such that each SISO loop is stable when all the rest (n-1) loops are open.
- Under these assumptions, the multi-loop control system will be unstable for all possible values of controller parameters if the Niederlinsky Index (NI) defined as $N_i = \frac{\det[G(0)]}{\prod_{i=1}^{n} G_{ii}(0)} < 0$

Furnace Control

$$\begin{bmatrix} T_{HC}(s) \\ O_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{-13.235e^{-4s}}{4.22s+1} & \dfrac{226.95e^{-4s}}{4.75s+1} \\ \dfrac{0.1445e^{-4s}}{4.434s+1} & \dfrac{-1.96e^{-4s}}{3.98s+1} \end{bmatrix} \begin{bmatrix} F_{air}(s) \\ F_{fuel}(s) \end{bmatrix}$$

$$RGA(\Lambda) = \begin{bmatrix} -3.778 & 4.778 \\ 4.778 & -3.778 \end{bmatrix}$$

$$\text{Suggested Pairing}: \mathrm{T}_{HC} - F_{fuel} \quad and \quad O_2 - F_{air}$$

Rearrange tarnsfer function matrix such that paired variables are on main diagonal

$$\begin{bmatrix} T_{HC}(s) \\ O_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{226.95e^{-4s}}{4.75s+1} & \dfrac{-13.235e^{-4s}}{4.22s+1} \\ \dfrac{-1.96e^{-4s}}{3.98s+1} & \dfrac{0.1445e^{-4s}}{4.434s+1} \end{bmatrix} \begin{bmatrix} F_{fuel}(s) \\ F_{air}(s) \end{bmatrix}$$

$$Ni = \det\begin{bmatrix} 226.95 & -13.235 \\ -1.96 & 0.1445 \end{bmatrix} \times \frac{1}{226.96 \times 0.1445} = 0.209 > 0$$

$$\Rightarrow \text{Process is integral controllable}$$

Multi-Loop PID Control

- After selection of loop pairings with minimum interactions, one can design controllers for individual loops.

- Presence of interaction and retaliatory effects from other loops may require that the controller be detuned for acceptable performance.
  - BLT Detuning method
  - Sequential Loop Tuning
  - Independent Loop Tuning

- $F = \lambda - \sqrt{\lambda^2 - \lambda} \quad \lambda > 0$

$F = \lambda + \sqrt{|\lambda^2 - \lambda|} \quad \lambda < 0$

McAvoy method

Biggest Log Modulus Tuning (BLT)

SISO Loop Design Review:

- Characteristic Eqn. $1 + g_c(s)\, g_p(s) = 1 + G_{OL}(s) = 0$
- Nyquist Plot: depicts real part of $G_{OL}(s)$ on X-axis and imaginary part of $G_{OL}(s)$ on Y-axis as $\omega \rightarrow \infty$

Nyquist Stability Criteria:

- A feedback control system will be unstable if the Nyquist plot of $G(j\omega)$ encircles point (-1,0) as $\omega \rightarrow \infty$
- The number of encirclements correspond to the number of roots of the characteristic equation that lie in R.H.P. of s –plane assuming that the process is open loop stable.

BLT Method

A measure of distance of $G_{OL}(j\omega)$ contour from (-1,0) is given as

$$L_c(s) = 20\log\left|\frac{G_{OL}(s)}{1+G_{OL}(s)}\right|$$

Suggested design specification for Log Modulus: $L_c^{max} \leq 2\ dB$

Log Modulus Design:

Iteratively choose PI controller parameters K_c, τ_I such that the design specification is met.

Multivariable Process :

$$Y(s) = \left\{\left[I + G_c(s)G_p(s)\right]^{-1} G_p(s)G_c(s)\right\} R(s)$$

Characteristic Eqn. : $\det\left[I + G_c(s)G_p(s)\right] = 0$

BLT Method

Define $f(s) = -1 + \det\left[I + G_c(s)G_p(s)\right]$

Encirclement of (-1,0) by f(jω) would indicate instability.

Define a multivariable closed loop log modulus as

$$L_c^m = 20\log\left|\frac{f(s)}{1+f(s)}\right|$$

Suggested design specification for Log Modulus: $L_c^m \leq 2n\ dB$ for n-dimension system

Tuning Procedure

- Calculate Ziegler -Nichol's tuning for n individual PI controllers
- Assume a factor F such that 2 ≤ F ≤ 5
- De - tune PI controllers as follows:
- $K_{c,j} = K_j^{ZN}/F \quad \tau_{I,j} = \tau_I^{ZN}F$ for j=1,2,.. N
- Iteratively choose F such that criteria L_c^m = 2n is satisfied.

BLT: Example

TABLE 4.1.
Process Open-Loop Transfer Functions of 2 × 2 Systems

	TS (Tyreus stabilizer)	WB (Wood and Berry)	VL (Vinante and Luyben)	WW (Wardle and Wood)
G_{11}	$\dfrac{-0.1153(10S+1)e^{-0.1S}}{(4S+1)^3}$	$\dfrac{12.8e^{-S}}{16.7S+1}$	$\dfrac{-2.2e^{-S}}{7S+1}$	$\dfrac{0.126e^{-6S}}{60S+1}$
G_{12}	$\dfrac{0.2429e^{-2S}}{(33S+1)^2}$	$\dfrac{-18.9e^{-3S}}{21S+1}$	$\dfrac{1.3e^{-0.3S}}{7S+1}$	$\dfrac{-0.101e^{-12S}}{(48S+1)(45S+1)}$
G_{21}	$\dfrac{-0.0887e^{-12.6S}}{(43S+1)(22S+1)}$	$\dfrac{6.6e^{-7S}}{10.9S+1}$	$\dfrac{-2.8e^{-1.8S}}{9.5S+1}$	$\dfrac{0.094e^{-8S}}{38S+1}$
G_{22}	$\dfrac{0.2429e^{-0.17S}}{(44S+1)(20S+1)}$	$\dfrac{-19.4e^{-3S}}{14.4S+1}$	$\dfrac{4.3e^{-0.35S}}{9.2S+1}$	$\dfrac{-0.12e^{-8S}}{35S+1}$

BLT: Example

TABLE 4.2.
2 × 2 Systems

	TS (Tyreus stabilizer)	WB (Wood and Berry)	VL (Vinante and Luyben)	WW (Wardle and Wood
RGA	4.35	2.01	1.63	2.69
NI	+0.229	+0.498	+0.615	+0.372
empirical				
K_c	−30, 30	0.2, −0.04	−2.38, 4.39	18, −24
τ_I	∞	4.44, 2.67	3.16, 1.15	19, 24
L_c	1.74	10.1	13.3	8.4
Z-N				
K_c	−166.2, 706	0.96, −0.19	−2.40, 4.45	59, −28.5
τ_I	2.06, 8.01	3.25, 9.20	3.16, 1.15	19.3, 24.6
L_c	unstable	unstable	13.3	18.5
BLT				
F	10	2.55	2.25	2.15
K_c	−16.6, 70.6	0.375, −0.075	−1.07, 1.97	27.4, −13.3
τ_I	20.6, 80.1	8.29, 23.6	7.1, 2.58	41.4, 52.9

BLT: Example

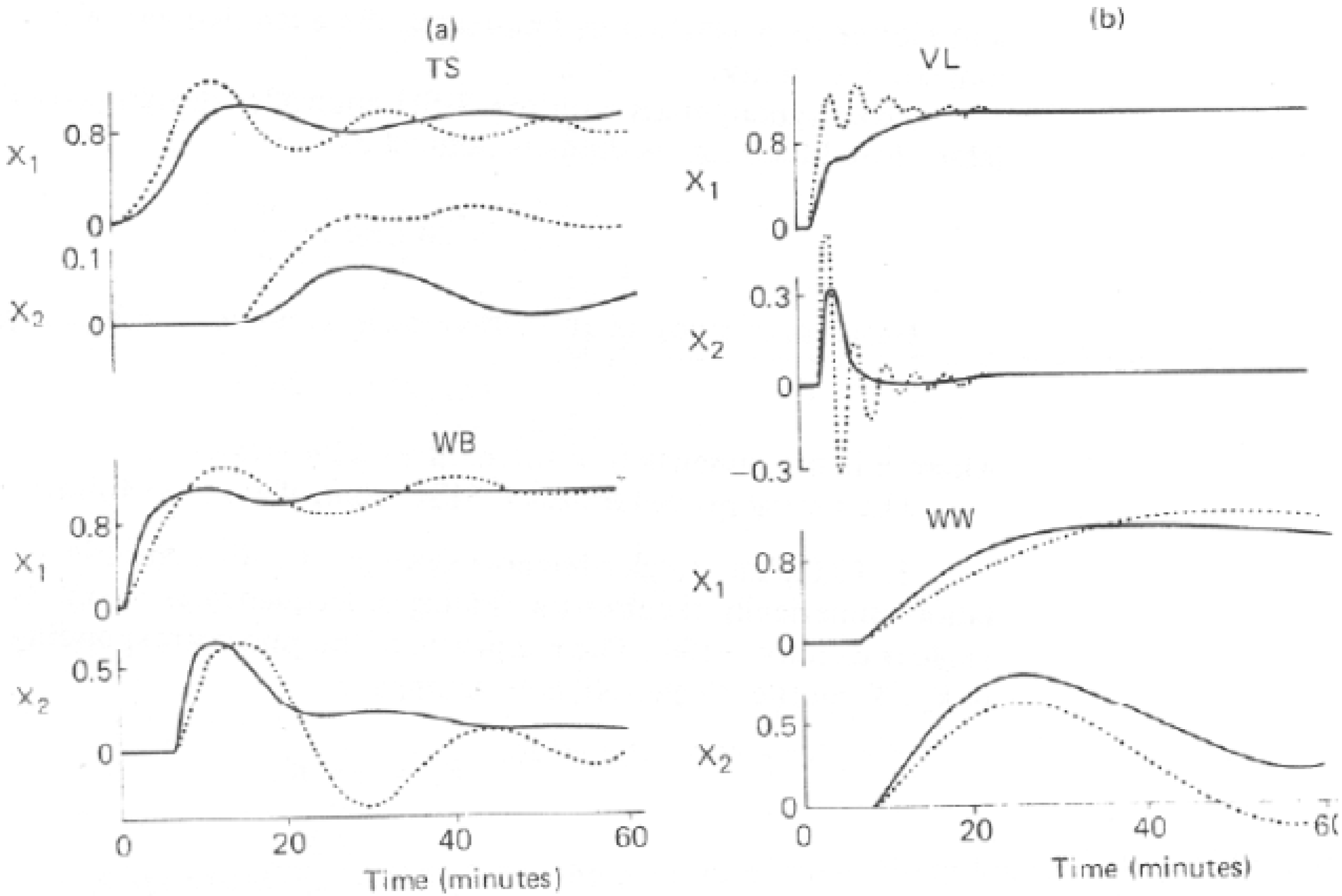


FIGURE 4.3. (a) X_1 Set Point Responses of TS and WB; (b) X_1 Set Point Response of VI and WW. Solid lines are BLT settings. Dashed lines are empirical settings.

Dynamic RGA

Process Model:

$$G(s) = \begin{bmatrix} -\frac{2e^{-s}}{10s+1} & \frac{1.5e^{-s}}{s+1} \\ \frac{1.5e^{-s}}{s+1} & \frac{2e^{-s}}{10s+1} \end{bmatrix} \quad K = \begin{bmatrix} -2 & 1.5 \\ 1.5 & 2 \end{bmatrix}$$

$$RGA = \begin{bmatrix} 0.64 & 0.36 \\ 0.36 & 0.64 \end{bmatrix}$$

Dynamic RGA $\lambda_{11} = \frac{1}{1-\frac{|G_{12}(jw)||G_{21}(jw)|}{|G_{11}(jw)||G_{22}(jw)|}}$

Principles of Decoupling

Main loop $y_1 — u_1$, $y_2 — u_2, \ldots, y_n — u_n$, couplings

- ➢desirable for control

Cross-couplings, $y_i — u_j (i \neq j)$

- ➢undesirable; loop interactions

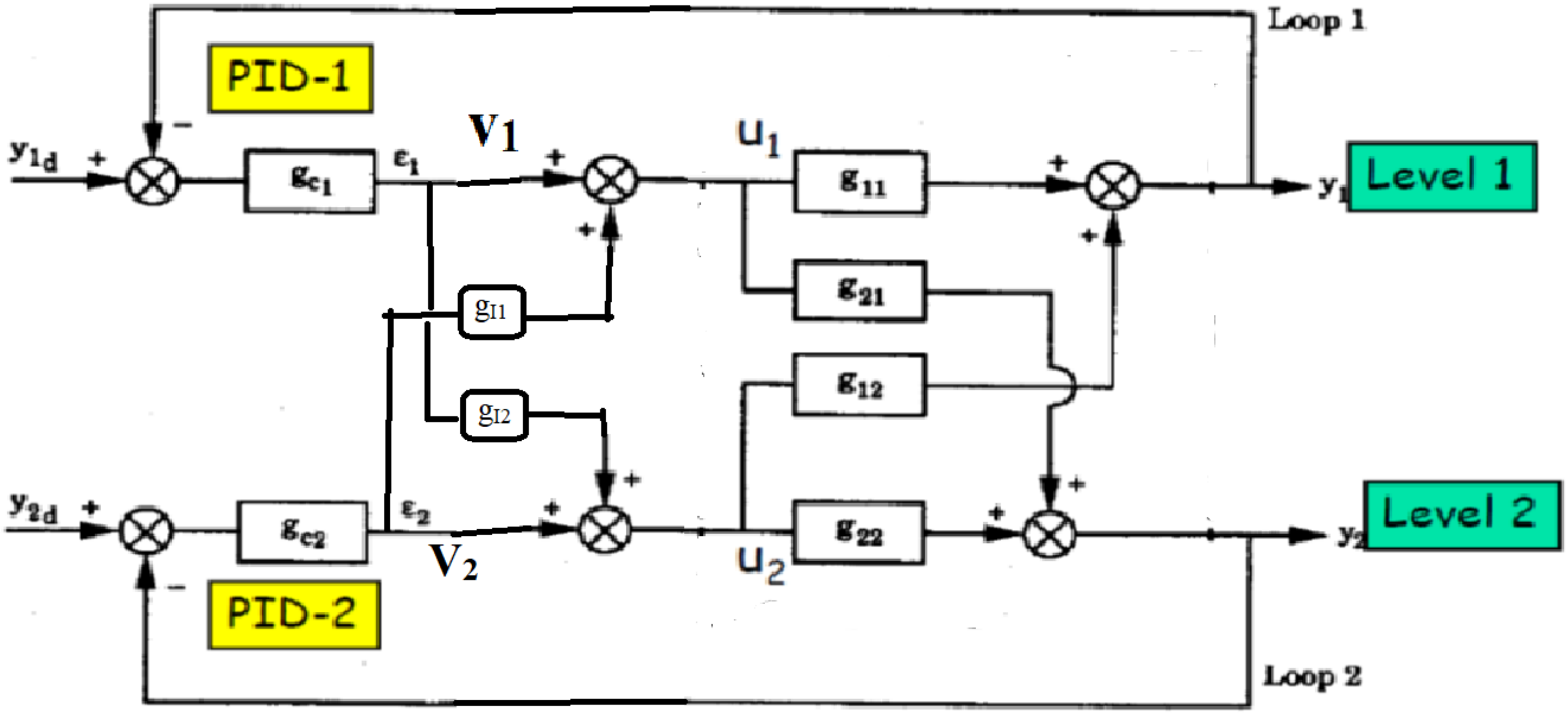
Eliminates the *effect* of the undesired cross-couplings

- ➢improve control performance.

Objective is to *compensate* for interactions by cross-couplings

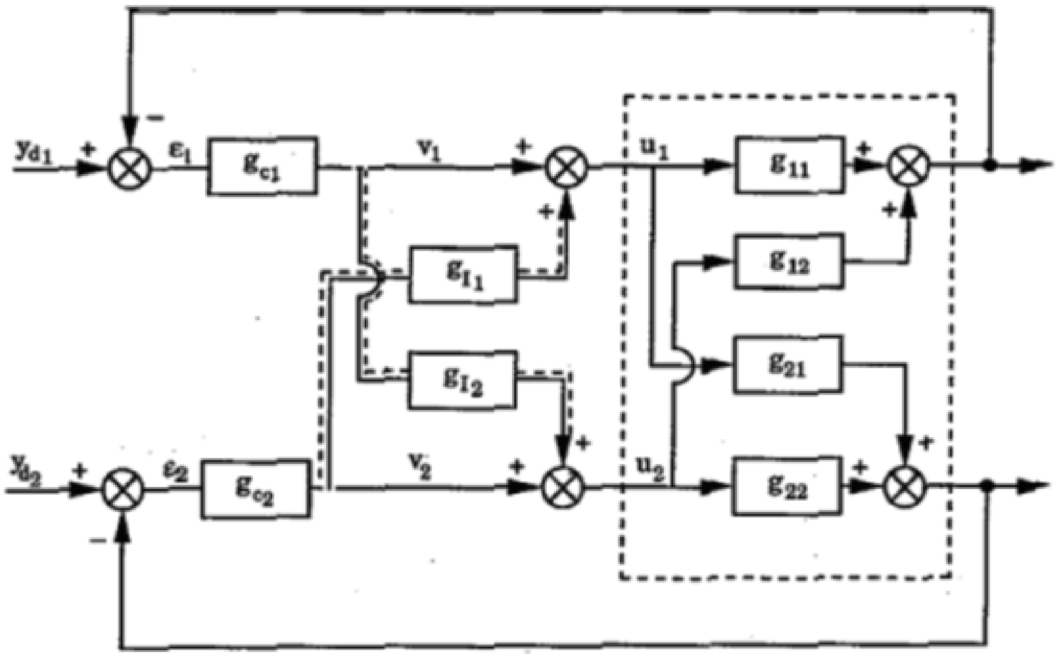
- ➢*not* to "eliminate" the cross-couplings; impossibility, require altering the physical nature of the system.

Block Diagram : Quad Tank



Simplified Decoupling

- Two compensator Blocks g_{I1} and g_{I2}
- Controller output v_1 and v_2
- Actual control on the process u_1 and u_2
- Without compensator
- $u_1 = v_1$ and $u_2 = v_2$
- Process model :
- $y_1 = g_{11}u_1 + g_{12}u_2$
- $y_2 = g_{21}u_1 + g_{22}u_2$



- With Compensator
- Loop 2 informed of changes
- of v_1 by g_{I2} and u_2 is adjusted
- Same for Loop 1

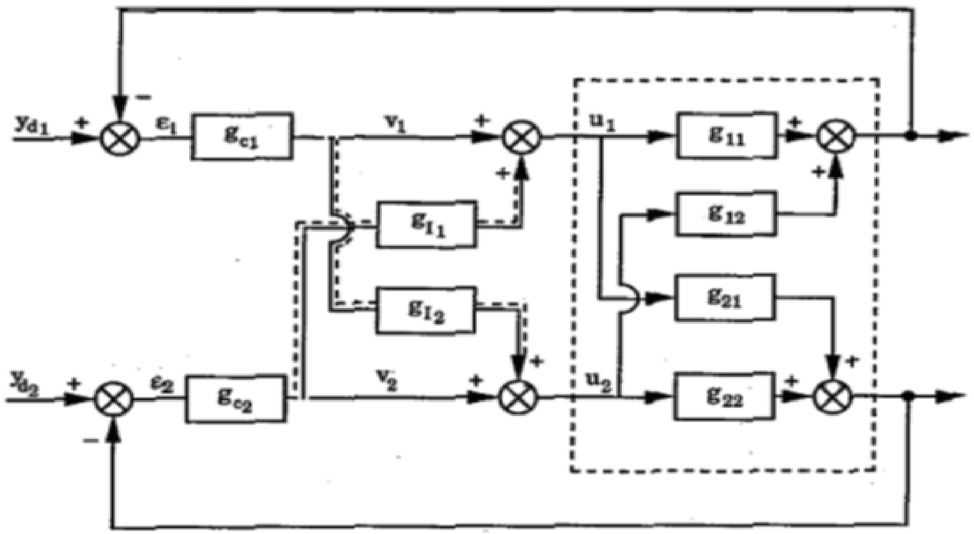
Simplified Decoupler Design

- Process Model

$$y_1 = g_{11}u_1 + g_{12}u_2$$
$$y_2 = g_{21}u_1 + g_{22}u_2$$

- u-v relation:

$$u_1 = v_1 + g_{I1}v_2$$
$$u_2 = v_2 + g_{I2}v_1$$



So,

$$y_1 = g_{11}(v_1 + g_{I1}v_2) + g_{12}(v_2 + g_{I2}v_1)$$
$$y_2 = g_{21}(v_1 + g_{I1}v_2) + g_{22}(v_2 + g_{I2}v_1)$$

Or,

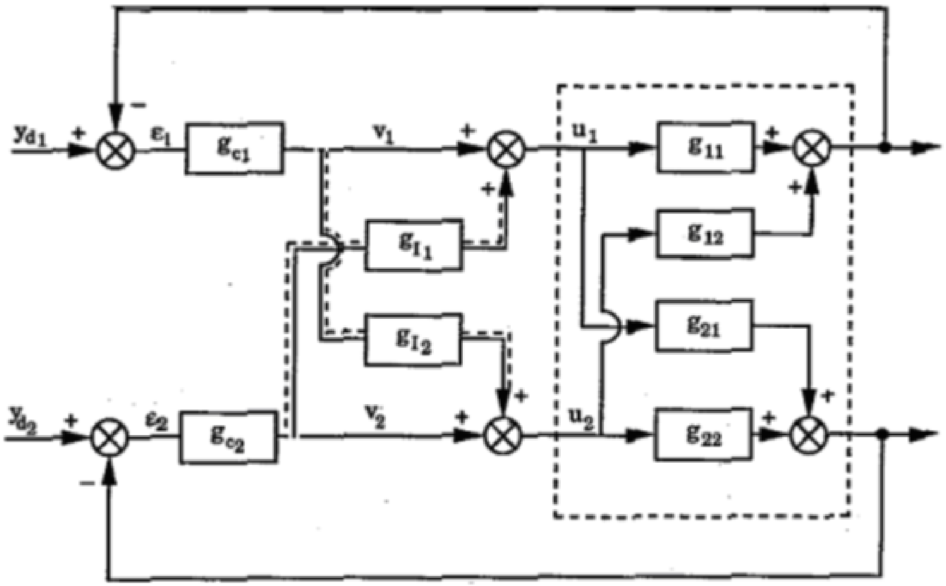
$$y_1 = (g_{11}+g_{12}g_{I2})v_1 + (g_{12} + g_{11}g_{I1})v_2$$
$$y_2 = (g_{21} + g_{22}g_{I2})v_1 + (g_{22} + g_{21}g_{I1})v_2$$

Simplified Decoupler Design

- $y_1 = (g_{11}+g_{12}g_{I2})v_1 + (g_{12} + g_{11}g_{I1})v_2$
- $y_2 = (g_{21} + g_{22}g_{I2})v_1 + (g_{22} + g_{21}g_{I1})v_2$

- Ideally, y_1 should be affected by v_1 only and y_2 should be affected by v_2 only. This means,
- $g_{12} + g_{11}g_{I1} = 0 \quad or, \quad \mathbf{g_{I1}} = -\frac{\mathbf{g_{12}}}{\mathbf{g_{11}}}$
- $g_{21} + g_{22}g_{I2} = 0 \quad or, \quad \mathbf{g_{I2}} = -\frac{\mathbf{g_{21}}}{\mathbf{g_{22}}}$
- So,
- $y_1 = \left(g_{11} - \frac{g_{12}g_{21}}{g22}\right)v_1$ and $y_2 = \left(g_{22} - \frac{g_{21}g_{12}}{g11}\right)v_1$

Simplified Decoupler Design



Decoupler Equation

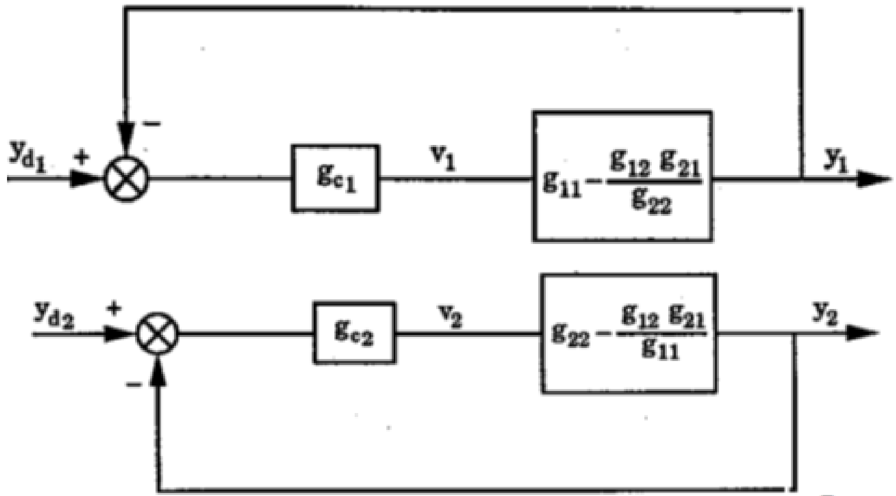
$$g_{I1} = -\frac{g_{12}}{g_{11}}$$

$$g_{I2} = -\frac{g_{21}}{g_{22}}$$

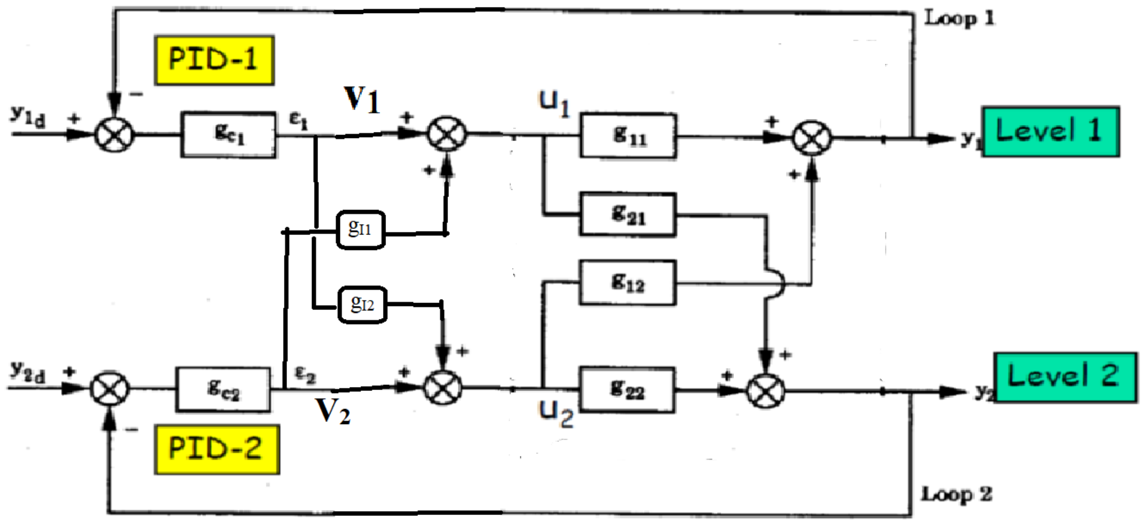
Steady State Decoupler

$$K_{I1} = -\frac{K_{12}}{K_{11}}$$

$$K_{I2} = -\frac{K_{21}}{K_{22}}$$



Quad tank : Decoupler Design

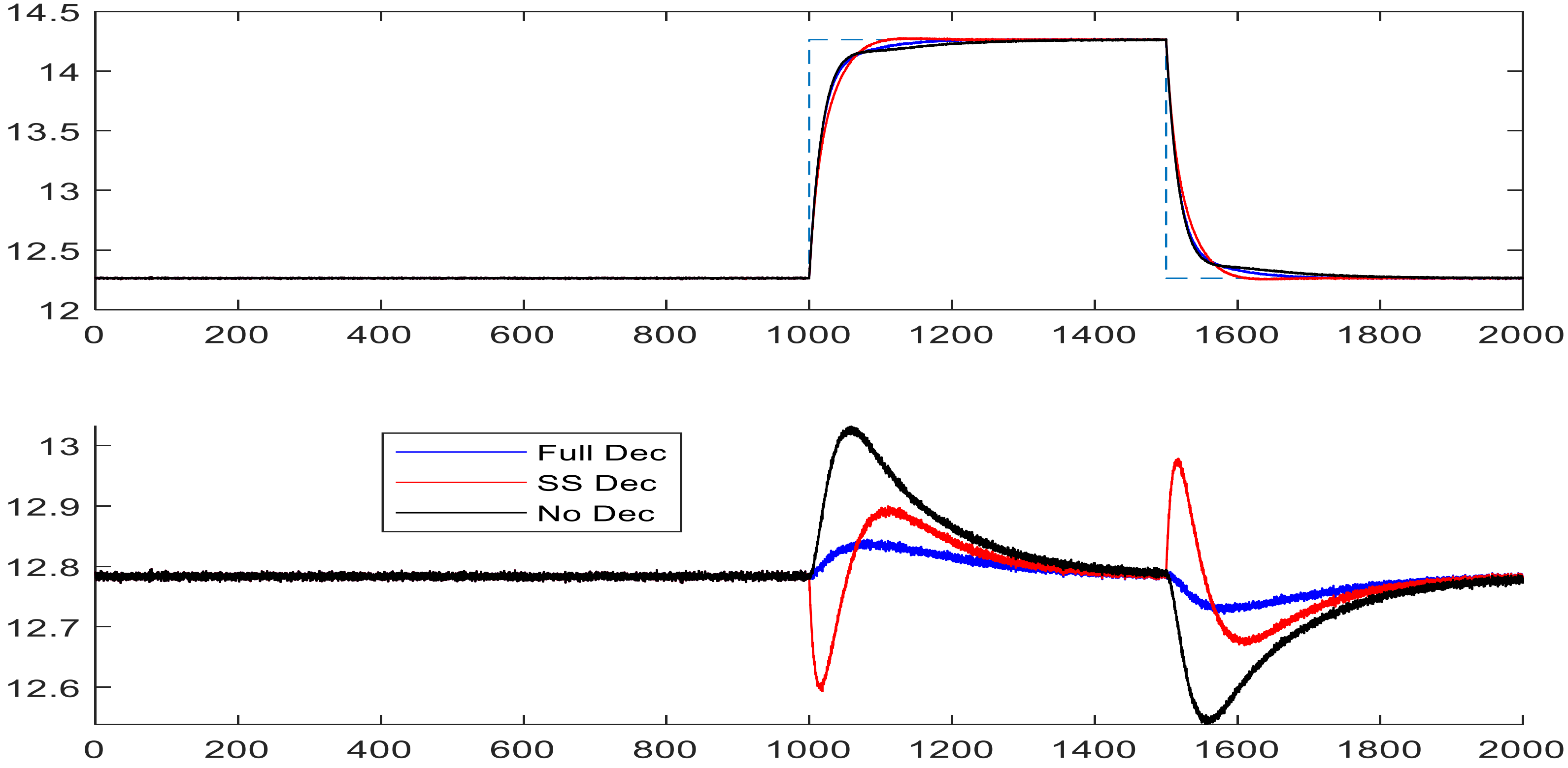


$$G(s) = \begin{bmatrix} \dfrac{4.153}{62.36s+1} & \dfrac{3.036}{(62.36s+1)(22.76s+1)} \\ \dfrac{1.78}{(90.63s+1)(30.09s+1)} & \dfrac{4.554}{90.63s+1} \end{bmatrix}$$

Steady State Decoupler : $K_{I1} = -\frac{3.036}{4.153} = -0.731 \quad K_{I2} = -\frac{1.78}{4.554} = -0.391$

Quad tank : Decoupler Design

- Full Decoupler : $g_{I1} = -\dfrac{0.731}{22.76s+1} \quad g_{I2} = -\dfrac{0.391}{30.09s+1}$



Example: Wood Berry Distillation Column

$$G(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

simplified decoupler

$$g_{I1} = 1.48\frac{(16.7s+1)e^{-2s}}{21.0s+1} \qquad g_{I2} = 0.34\frac{(14.4s+1)e^{-4s}}{10.9s+1}$$

actual implementation

$$u_1 = v_1 + 1.48\frac{(16.7s+1)e^{-2s}}{21.0s+1}v_2$$

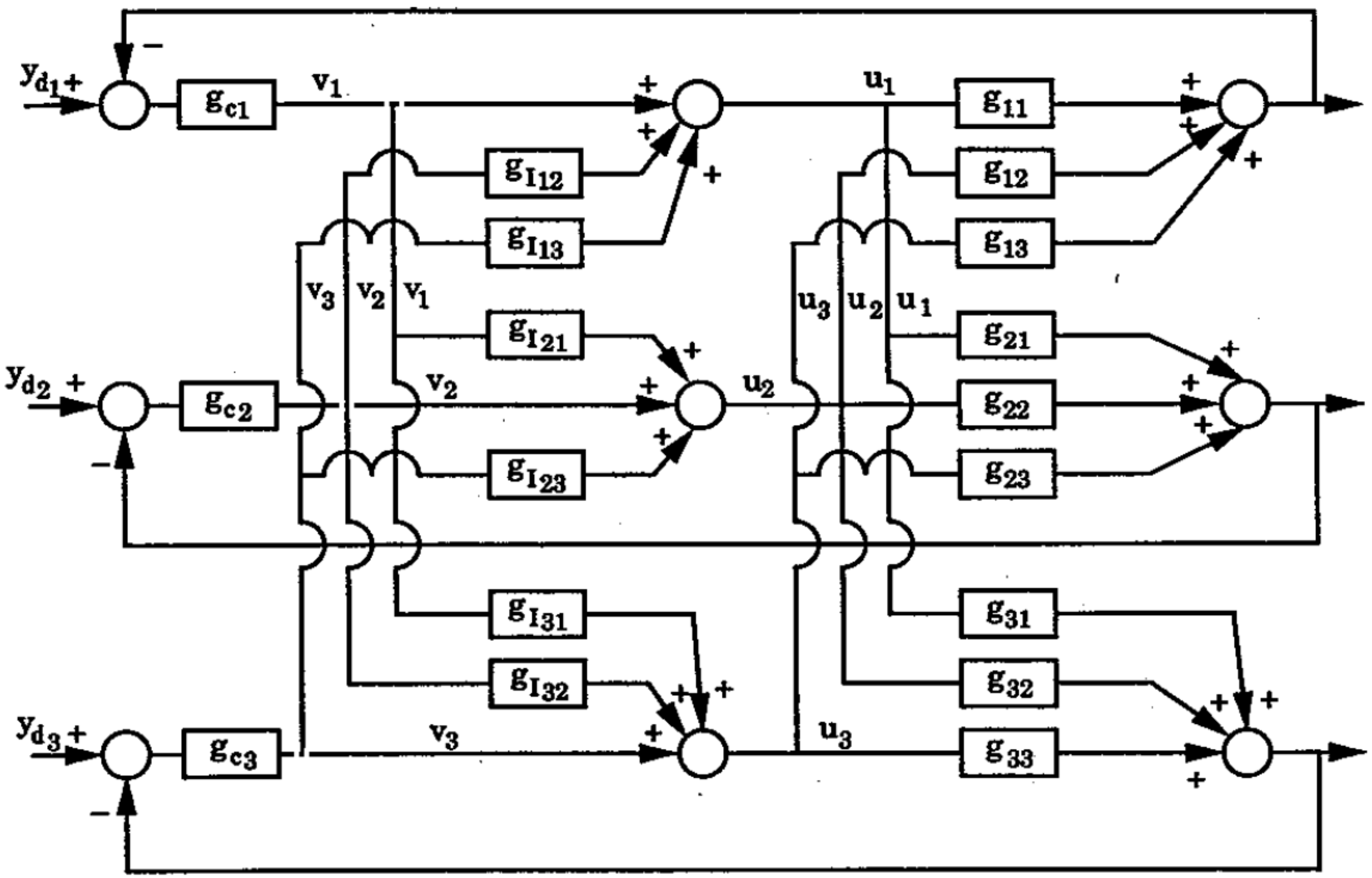
$$u_2 = v_2 + 0.34\frac{(14.4s+1)e^{-4s}}{10.9s+1}v_1$$

Difficulties in simplied Decoupler design

larger than 2 x 2, decoupling become tedious.

3x3 , six compensator.

*N*x*N*: (N^2-N) compensators.



Generalized Decoupler Design

MIMO process

$$y = G u \qquad u = G_I v \quad \Rightarrow \quad y = G G_I v$$

To eliminate interactions, *y* to *v* : a diagonal matrix; $G_R(s)$.

$$G G_I = G_R(s) \quad \Rightarrow \quad y = G_R(s) v$$

Choose G_I such that

$$G_I = G^{-1} G_R(s)$$

Selected to provide desired decoupled behavior with the simplest form

- A commonly employed choice

$$G_R(s) = Diag[G(s)]$$

Example: Wood Berry Distillation Column

Generalized decoupling:

$$G_R(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & 0 \\ 0 & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \qquad G^{-1}(s) = \frac{1}{\Delta}\begin{bmatrix} \dfrac{-19.4e^{-3s}}{14.4s+1} & \dfrac{18.9e^{-3s}}{21.0s+1} \\ \dfrac{-6.67e^{-7s}}{10.9s+1} & \dfrac{12.8e^{-s}}{16.7s+1} \end{bmatrix} \qquad G_I = \begin{bmatrix} g_{I11} & g_{I12} \\ g_{I21} & g_{I22} \end{bmatrix}$$

$$\Delta = \frac{-248.32(21.0s+1)(10.9s+1)e^{-4s} + 124.74(16.7s+1)(14.4s+1)e^{-10s}}{(21.0s+1)(10.9s+1)(16.7s+1)(14.4s+1)}$$

$$g_{I11} = \frac{-248.32(21.0s+1)(10.9s+1)}{124.74(16.7s+1)(14.4s+1)e^{-6s} - 248.32(21.0s+1)(10.9s+1)}$$

$$g_{I12} = \frac{-366.66(16.7s+1)(10.9s+1)e^{-2s}}{124.74(16.7s+1)(14.4s+1)e^{-6s} - 248.32(21.0s+1)(10.9s+1)}$$

$$g_{I21} = \frac{84.48(21.0s+1)(14.4s+1)}{124.74(16.7s+1)(14.4s+1)e^{-6s} - 248.32(21.0s+1)(10.9s+1)} \qquad g_{I22} = g_{I11}$$

The actual implementation:

$$u_1 = g_{I11}v_1 + g_{I12}v_2$$

$$u_2 = g_{I21}v_1 + g_{I22}v_2$$

Steady State Decoupler Design

Steady-state decoupling: uses steady-state gain of transfer function

2 x 2 system

Simplified steady-state decoupling $$g_{I1} = -\frac{K_{12}}{K_{11}}, \quad g_{I2} = -\frac{K_{21}}{K_{22}}$$

Generalized steady-state decoupling $$G_I = K^{-1} K_R$$

Very easy to design and implement, first technique to try;

- ideal decoupler only if dynamic interactions persistent
- big performance improvements with very little work or cost
- most often applied in practice.

Example: Wood Berry Distillation Column

$$K = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix} \quad \Leftarrow \quad G(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

Simplified steady-state decoupling

$$g_{I1} = -\frac{-18.9}{12.8} = 1.48, \quad g_{I2} = -\frac{6.6}{-19.4} = 0.34 \quad \Rightarrow \quad \begin{aligned} u_1 &= v_1 + 1.48v_2 \\ u_2 &= v_2 + 0.34v_1 \end{aligned}$$

Generalized steady-state decoupling

$$K_R = \begin{bmatrix} 12.8 & 0 \\ 0 & -19.4 \end{bmatrix} \Rightarrow G_I = \begin{bmatrix} 2.01 & 2.97 \\ 0.68 & 2.01 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} u_1 &= 2.01v_1 + 2.97v_2 \\ u_2 &= 0.68v_1 + 2.01v_2 \end{aligned}$$

Simplified Vs Generalized Decoupler

Simplified decoupling: “equivalent” open-loop decoupled system

$$y_1 = \left(g_{11} - \frac{g_{12}g_{21}}{g_{22}} \right) v_1 = \left(\frac{12.8e^{-s}}{(16.7s+1)} - \frac{18.9 \times 6.6(14.4s+1)e^{-7s}}{19.4(21.0s+1)(10.9s+1)} \right) v_1$$

$$y_2 = \left(g_{22} - \frac{g_{12}g_{21}}{g_{11}} \right) v_2 = \left(\frac{-19.4e^{-3s}}{(14.4s+1)} - \frac{18.9 \times 6.6(16.7s+1)e^{-9s}}{12.8(21.0s+1)(10.9s+1)} \right) v_2$$

much more complicated than G_R specified in the *Generalized* decoupling

- Difficult to tune controller

Generalized decoupling:

tuning and performance better than for *Simplified* decoupling

- complicated decoupler

Challenges in Decoupler Design

Perfect decouple if model perfect - impossible in practice.

The simplified decoupling similar to feedforward controllers

- realization problems, time delay elements

Perfect dynamic decouplers based on model inverses.

- can only be implemented if inverses *causal* and *stable.*

2 x 2 compensators, g_{I1} and g_{I2} must be causal (no $e^{+\alpha s}$ terms) and stable

- time delays in g_{11} smaller than time delays in g_{12}
- time delays in g_{22} smaller than time delays in g_{21}
- g_{11} and g_{22} no RHP zeros
- g_{12} and g_{21} must no RHP poles

Challenges in Decoupler Design

Adding delays to the inputs u_1, u_2, ..., u_n, by define: $G_m = GD$

$$D(s) = \begin{bmatrix} e^{-d_{11}s} & & & 0 \\ & e^{-d_{22}s} & & \\ & & \ddots & \\ 0 & & & e^{-d_{nn}s} \end{bmatrix}$$

Simplified decoupling: requiring the smallest delay in each row on the diagonal, designed by using G_m.

Generalized decoupling: use modified process G_m so that $G_I=(GD)^{-1}G_R$ are causal which requiring that $G_R^{-1}(GD)$ have the smallest delay in each row on the diagonal.

Example Problem

$$G(s) = \begin{bmatrix} \dfrac{12.8e^{-4s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.67e^{-10s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}$$

Smallest delay in each row is not on diagonal, simplified decoupling compensator becomes:

$$g_{I1} = 1.48\frac{(16.7s+1)e^{s}}{21.0s+1}$$

Design $D(s)$ to add a time delay of 1 minute to the input u_2, i.e.:

$$D(s) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-s} \end{bmatrix}$$

$$G_m = GD = \begin{bmatrix} \dfrac{12.8e^{-4s}}{16.7s+1} & \dfrac{-18.9e^{-4s}}{21.0s+1} \\ \dfrac{6.67e^{-10s}}{10.9s+1} & \dfrac{-19.4e^{-4s}}{14.4s+1} \end{bmatrix}$$

$$g_{I1} = 1.48\frac{(16.7s+1)}{21.0s+1}$$

$$g_{I2} = 0.34\frac{(14.4s+1)e^{-6s}}{10.9s+1}$$

Example Problem

Alternate Solution

$$G(s)=\begin{bmatrix} \dfrac{12.8e^{-4s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.67e^{-10s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \qquad g_{I1}=1.48\frac{(16.7s+1)e^{s}}{21.0s+1}$$

As time prediction term much small than time constant, drop prediction

$$g_{I1}=1.48\frac{(16.7s+1)}{21.0s+1}$$

Effective time constant of g_{12} and g_{11} are similar $\quad 16.7+4 \Leftrightarrow 21+3$

Steady-state decoupling

$$g_{I1}=1.48$$

Partial Decoupling

Consider partial decoupling if

- some of the loop interactions are weak
- some of the loops need not have high performance

Partial decoupling focused on a subset of control loops

- interactions are important, and/or
- high performance control is required.

Consider partial decoupling for 3x3 or higher systems

- main advantage: reduction of dimensionality.

Partial Decoupling Example

Grinding circuit analysis

- Least sensitive variables y_2
- Most interaction: Loops 1 and 3,
- Decouplers: loops 1 and 3,
- Loop 2 without decoupling.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{119}{217s+1} & \frac{153}{337s+1} & \frac{-21}{10s+1} \\ \frac{0.00037}{500s+1} & \frac{0.000767}{33s+1} & \frac{-0.00005}{10s+1} \\ \frac{930}{500s+1} & \frac{-667e^{-320s}}{166s+1} & \frac{-1033}{47s+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

the transfer function matrix for the subsystem

$$\begin{bmatrix} y_1 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{119}{217s+1} & \frac{-21}{10s+1} \\ \frac{930}{500s+1} & \frac{-1033}{47s+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix}$$

using the simplified decoupling approach

$$g_{I1} = \frac{\frac{21}{10s+1}}{\frac{119}{217s+1}} = \frac{0.176(217s+1)}{10s+1}; \qquad g_{I3} = \frac{\frac{930}{500s+1}}{\frac{1033}{47s+1}} = \frac{0.0(47s+1)}{500s+1}$$