Lecture-7
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§ Inverse Laplace Transform.

Suppose $\bar{f}(s)$ is known. To find $f(t)$.

Let $\bar{f}(s) = \frac{s}{s^2+9}$, what is $f(t)$?

Ans = $f(t) = \cos 3t$.

Let $L\{f(t)\} = \bar{f}(s)$. $\longrightarrow (1)$

Then inverse Laplace transform L^{-1} is defined such that $L^{-1}\{\bar{f}(s)\} = f(t) \longrightarrow (2)$.

Note that, operating L^{-1} on both sides of (1),

$$\underline{L^{-1}L\{f(t)\}} = L^{-1}\{\bar{f}(s)\} = \underline{f(t)}, \text{ by } (2).$$

$\therefore$ $L^{-1}L = I$ = identity operator.

Operating L on both sides of (2), get

$$LL^{-1}\{\bar{f}(s)\} = L\{f(t)\} = \bar{f}(s), \text{ by } (1).$$

$\therefore$ $LL^{-1} = I$ = identity operator.

$\therefore$ $L^{-1}L = LL^{-1} = I$.

__Note!__ ILT may not exist for all cases.

How can we find inverse L.T.

1) By observation. $\bar{f}(s) = \frac{s}{s^2+9}$

2) Use of some important properties. ~~f(t)~~

3) " " partial fractions $L^{-1}\{\bar{f}(s)\} = \cos 3t$,

4) " " convolution.

General properties of inverse LT.

1. Linearity property.
2. First shifting theorem.
3. second " "
4. Change of scale.
5. I.L.T. of derivatives $\rightarrow$ $L[t f(t)] = -\frac{d}{ds}\bar{f}(s)$ $\therefore L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right] = -t f(t)$
6. " " integrals $\rightarrow$ $L^{-1}\left[\int_s^\infty \bar{f}(u)\,du\right] = \frac{f(t)}{t}$.
7. Multiplication by powers of s $\rightarrow$ $L^{-1}\left[s\bar{f}(s) - f(0)\right] = f'(t)$
8. Division by s $\rightarrow$ $L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)\,du$.

1. Linearity property.

If $L[f_i(t)] = \bar{f}_i(s)$, then

$$L^{-1}\left[\sum_{i=1}^{n} c_i \bar{f}_i(s)\right] = \sum_{i=1}^{n} c_i L^{-1}\left[\bar{f}_i(s)\right].$$

Pf. We know, $L\left[\sum_{i=1}^{n} c_i f_i(t)\right] = \sum_{i=1}^{n} c_i L[f_i(t)]$.

$$\text{or, } L\left[\sum_{i=1}^{n} c_i f_i(t)\right] = \sum_{i=1}^{n} c_i \bar{f}_i(s) \rightarrow (1).$$

Operate L^{-1} on both sides, to get ..

$$L^{-1}L\left[\sum_{i=1}^{n} c_i f_i(t)\right] = L^{-1}\left(\sum_{i=1}^{n} c_i \bar{f}_i(s)\right).$$

$$\text{or, } I\left[\sum_{i=1}^{n} c_i f_i(t)\right] = R.H.S$$

$$\text{or, } \sum_{i=1}^{n} c_i L^{-1}\left\{\bar{f}_i(s)\right\} = L^{-1}\left\{\sum_{i=1}^{n} c_i \bar{f}_i(s)\right\}$$

Ex-1 Find $L^{-1}\left[\dfrac{2s+1}{s^2-4}\right]$

$$= 2L^{-1}\left(\frac{s}{s^2-4}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^2-4}\right).$$

$$= 2\cosh 2t + \frac{1}{2}\sinh 2t.$$

Ex-2 Find $L^{-1}\left\{\dfrac{3(s^2-2)^2}{2s^5}\right\}$.

$$= L^{-1}\left\{\frac{3(s^4-4s^2+4)}{2s^5}\right\}.$$

$$= \frac{3}{2}L^{-1}\left(\frac{1}{s}\right) - \frac{12}{2}L^{-1}\left(\frac{1}{s^3}\right) + \frac{12}{2}L^{-1}\left\{\frac{1}{s^5}\right\}$$

$$= \frac{3}{2} - 6\cdot\frac{t^2}{2} + \frac{\cancel{12}^{6}}{\cancel{2}}\times\frac{t^4}{\cancel{24}_{4}}$$

$$= \frac{3}{2} - 3t^2 + \frac{t^4}{4}.$$

$L[t^2] = \dfrac{2!}{s^3}$.

$\therefore L\left[\dfrac{t^2}{2}\right] = \dfrac{1}{s^3}$

$L[t^4] = \dfrac{4!}{s^5}$

$\therefore L\left[\dfrac{t^4}{24}\right] = \dfrac{1}{s^5}$

Exercise. If

$$L^{-1}\left[\frac{1}{s}e^{-\frac{1}{\sqrt{s}}}\right] = \sum_{n=0}^{\infty} f_n(t)$$

find $f_n(t)$. Ans. $\dfrac{(-1)^n}{n!}\cdot\dfrac{t^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$.

2. Change of scale property

$$L^{-1}\left\{\bar{f}(ks)\right\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

$$L\left\{f(at)\right\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) \longrightarrow (1).$$

take $a = \frac{1}{k}$. Then (1) becomes,

$$L\left\{f\left(\frac{t}{k}\right)\right\} = k\,\bar{f}(ks).$$

or, $\bar{f}(ks) = \frac{1}{k} L\left\{f\left(\frac{t}{k}\right)\right\}$.

$$\therefore \quad \bar{f}(ks) = L\left\{\frac{1}{k} f\left(\frac{t}{k}\right)\right\}.$$

Operating L^{-1} on both sides.

$$L^{-1}\left[\bar{f}(ks)\right] = L^{-1}L\left\{\frac{1}{k} f\left(\frac{t}{k}\right)\right\} = I\left\{\frac{1}{k} f\left(\frac{t}{k}\right)\right\}.$$

$$\therefore \; L^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k} f\left(\frac{t}{k}\right).$$

Ex-1. Find $L^{-1}\left\{\dfrac{64}{81s^4-256}\right\}$

$a^4=256$
$a=4$

Sol. Note,

$$\frac{1}{s^4-a^4}=\frac{1}{(s^2-a^2)(s^2+a^2)}.$$

$$=\frac{1}{2a^2}\left[\frac{1}{s^2-a^2}-\frac{1}{s^2+a^2}\right].$$

$$\boxed{81s^4=(3s)^4}$$

$$\frac{a^3}{s^4-a^4}=\frac{a^3}{2a^2}\left[\frac{1}{s^2-a^2}-\frac{1}{s^2+a^2}\right].$$

$$\therefore L^{-1}\left[\frac{a^3}{s^4-a^4}\right]=L^{-1}\left[\frac{a}{2}\left[\frac{1}{s^2-a^2}-\frac{1}{s^2+a^2}\right]\right].$$

$$=\frac{\cancel{a}}{2}L^{-1}\left[\frac{a}{s^2-a^2}\right]-\frac{\cancel{a}}{2}L^{-1}\left[\frac{a}{s^2+a^2}\right].$$

$$L^{-1}\left[\frac{a^3}{s^4-a^4}\right]=\frac{1}{2}\sinh at-\frac{1}{2}\sin at \cdots$$

We need, $L^{-1}\left[\dfrac{a^3}{(ls)^4-a^4}\right]$.

Here we've found $L^{-1}[\bar{f}(s)]$.

But we need to find, $L^{-1}[\bar{f}(ls)]$; $l=3$.

$$L^{-1}[\bar{f}(ls)]=\cancel{\frac{1}{l}f\left(\frac{t}{l}\right)}=\frac{1}{l}f\left(\frac{t}{l}\right).$$

$$L^{-1}\left[\frac{64}{s^4-256}\right] = \frac{1}{2}\sinh 4t - \frac{1}{2}\sin 4t.$$

$$\therefore L^{-1}\left[\frac{64}{81s^4-256}\right] = \frac{1}{3}\left[\frac{1}{2}\sinh\frac{4t}{3} - \frac{1}{2}\sin\frac{4t}{3}\right]$$

$$= \frac{1}{6}\left(\sinh\frac{4t}{3} - \sin\frac{4t}{3}\right).$$

- First shifting theorem:

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then

$$L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t) = e^{at}L^{-1}\{\bar{f}(s)\}.$$

Pf. We know, $L\{e^{at}f(t)\} = \bar{f}(s-a)$.

$\therefore L^{-1}L\{e^{at}f(t)\} = L^{-1}\bar{f}(s-a)$.

$\therefore e^{at}f(t) = L^{-1}\{\bar{f}(s-a)\}$.

Or, $e^{at}L^{-1}\{\bar{f}(s)\} = L^{-1}\{\bar{f}(s-a)\}$.

Ex-1. Find $L^{-1}\left[\dfrac{s}{(s+a)^2+b^2}\right]$.

$L^{-1}\bar{f}(s-a) = e^{at}L^{-1}\bar{f}(s)$.

$$= L^{-1}\left[\frac{s+a-a}{(s+a)^2+b^2}\right].$$

$$= L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] - a\,L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= e^{-at}L^{-1}\left[\frac{s}{s^2+b^2}\right] - \frac{a}{b}e^{-at}L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= e^{-at}\left[\cos bt - \frac{a}{b}\sin bt\right].$$

Ex-2. $L^{-1}\left\{\dfrac{3s+1}{(s+1)^4}\right\}$.

$$= L^{-1}\left\{\frac{3(s+1)-2}{(s+1)^4}\right\} = 3L^{-1}\left\{\frac{1}{(s+1)^3}\right\} - 2L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$$

$$= 3e^{-t}L^{-1}\left\{\frac{1}{s^3}\right\} - 2e^{-t}L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= 3e^{-t}\frac{t^2}{2} - 2e^{-t}\frac{t^3}{6}.$$

$L[t^2] = \dfrac{2}{s^3}$

$L[t^3] = \dfrac{3!}{s^4}$.

Ex-3. Find

$$L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}.$$

$$= e^{-3t}\left[\cos 4t - \frac{1}{2}\sin 4t\right].$$

2nd ~~sh~~ shifting th.

$$L\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s).$$

$$H(t-a) = \begin{cases} 1, & t>a. \\ 0, & t<a. \end{cases}$$

$$\therefore L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)H(t-a).$$

1) Find $L^{-1}\left[\dfrac{e^{-s}}{\sqrt{s+1}}\right] = L^{-1}\left[e^{-s}\bar{f}(s)\right]$

$$f(t) = L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{\sqrt{s+1}}\right] = e^{-t}L^{-1}\left[\frac{1}{\sqrt{s}}\right]$$

$$= e^{-t}\cdot\frac{1}{\sqrt{\pi t}}.$$

$$L\left[t^{-1/2}\right] = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma(\frac{1}{2})}{s^{1/2}}$$

$$\therefore L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$L^{-1}\left[\frac{e^{-s}}{\sqrt{s+1}}\right] = L^{-1}\left[e^{-s}\bar{f}(s)\right] = f(t-1)H(t-1)$$

$$= \begin{cases} \dfrac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}, & t>1. \\ 0, & t<1. \end{cases}$$

2. $L^{-1}\left[\dfrac{5-3e^{-3s}-2e^{-7s}}{s}\right]$

$= 5L^{-1}\left(\dfrac{1}{s}\right) - 3L^{-1}\left(\dfrac{e^{-3s}}{s}\right) - 2L^{-1}\left(\dfrac{e^{-7s}}{s}\right)$.

$= 5.1 - 3L^{-1}\left(e^{-3s}\bar{f}(s)\right) - 2L^{-1}\left(e^{-7s}\bar{f}(s)\right)$.

Here $\bar{f}(s) = \dfrac{1}{s}$ $\therefore f(t) = 1$.

$L^{-1}\left(\dfrac{e^{-3s}}{s}\right) = H(t-3)\, f(t-3) = H(t-3)$.

$L^{-1}\left(\dfrac{e^{-7s}}{s}\right) = H(t-7)\, f(t-7) = H(t-7)$.

$= 5 - 3H(t-3) - 2H(t-7)$.

$= \begin{cases} 5-3.0-2.0, & t<3. \\ 5-3.1-2.0, & 3<t<7 \\ 5-3.1-2.1, & t>7. \end{cases}$

$= \begin{cases} 5, & t<3 \\ 2, & 3<t<7 \\ 0, & t>7. \end{cases}$

Ex 3. $L^{-1}\left\{\dfrac{2+5s}{s^2 e^{4s}}\right\}$

$= 2L^{-1}\left\{\dfrac{e^{-4s}}{s^2}\right\} + 5L^{-1}\left\{\dfrac{e^{-4s}}{s}\right\} \to e^{-4s}\bar{g}(s)$ $\therefore \bar{g}(s) = \frac{1}{s},\ g(t)=1$ $\to f(t) = t.$

$= 2L^{-1}\{e^{-as}\bar{f}(s)\} \longrightarrow a=4,\ \bar{f}(s) = \dfrac{1}{s^2};$ $f(t)=t$, $L[f(t)] = \frac{1}{s^2}$

$= 2\,H(t-4)\,f(t-4) + 5\,H(t-4).g(t-4).$

$= 2\,\underline{H(t-4)}\,(t-4) + 5\,\underline{H(t-4)}.1.$

$= H(t-4)\{2t-3\} = \begin{cases} 2t-3, & t>4. \\ 0, & t<4 \end{cases}$

- <u>Inverse L.T. of derivatives.</u>

$L^{-1}\{\bar{f}^{(n)}(s)\} = (-1)^n t^n f(t).$

Ex 1. $L^{-1}\left\{\dfrac{s+1}{(s^2+\underline{2s}+2)^2}\right\}.$

$= L^{-1}\left[\dfrac{s+1}{\{(s+1)^2+1\}^2}\right]$

$= e^{-t}L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$

Let

$$\bar{f}(s) = \frac{1}{s^2+1}$$

$$L[\bar{f}'(s)] = -t f(t)$$

$$\frac{d\bar{f}(s)}{ds} = -\frac{2s}{(s^2+1)^2}$$

$$\therefore \quad L^{-1}\left(\frac{s}{(s^2+1)^2}\right) = -\frac{1}{2} L^{-1}\left[\bar{f}'(s)\right]$$

$$= -\frac{1}{2} \times -t f(t) = \frac{t}{2} L^{-1}\left[\bar{f}(s)\right].$$

$$= \frac{t}{2} \cdot L^{-1}\left[\frac{1}{s^2+1}\right] = \frac{t \sin t}{2}.$$

$$\therefore L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} = e^{-t} \cdot \frac{t \sin t}{2},$$

2. $$L^{-1}\left[\frac{1}{2} \ln \frac{s^2+b^2}{s^2+a^2}\right]$$

$$\bar{f}(s) = \frac{1}{2} \ln \frac{s^2+b^2}{s^2+a^2} = \frac{1}{2}\left[\ln s^2+b^2 - \ln s^2+a^2\right].$$

$$\frac{d}{ds}\bar{f}(s) = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2+b^2} - \frac{\cancel{2}s}{s^2+a^2}\right].$$

$$\therefore L^{-1}\left[\bar{f}'(s)\right] = L^{-1}\left(\frac{s}{s^2+b^2}\right) - L^{-1}\left(\frac{s}{s^2+a^2}\right)$$

$$-t f(t) = \cos bt - \cos at.$$

or, $$\therefore f(t) = (\cos at - \cos bt)/t.$$

3. Find

$$L^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\}.$$

$$= L^{-1}\left[\bar{f}(s)\right].$$

$$\bar{f}'(s) = \frac{d}{ds}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s+a}{b}\right)\right]$$

$$= -\frac{1}{1+\left(\frac{s+a}{b}\right)^2} \times \frac{1}{b}.$$

$$L^{-1}\left[\bar{f}'(s)\right] = L^{-1}\left[-\frac{b}{b^2+(s+a)^2}\right].$$

$$, -t\,f(t) = -e^{-at} L^{-1}\left[\frac{b}{s^2+b^2}\right].$$

$$-t\,f(t) = -e^{-at}\sin bt.$$

$$\therefore f(t) = \frac{e^{-at}\sin bt}{t}.$$