Serial No	Date	Title	Page No	Teacher's Signature/Remarks

$\tau_{yx} \rightarrow$ force in the x-direction on a unit area ⊥er to y-direction

τ_{yx}: x → force direction

~~τ_{xy}~~ → ~~x-direction~~ force

Page No. | Jogmaya
Date

Monojit

→ **Continuity:-** $\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$

→ **Momentum balance:-**

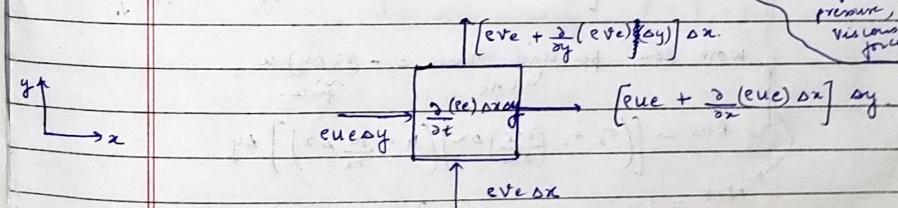
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) + \rho g_x$$

$$\rho \frac{Dv}{Dt} = -\nabla p - (\nabla \cdot \tau) + \rho g$$

– **Energy balance:-**

$$\begin{bmatrix}\text{rate of energy} \\ \text{accumulation} \\ \text{in CV}\end{bmatrix}_1 = \begin{bmatrix}\text{net transfer of} \\ \text{energy by} \\ \text{fluid flow}\end{bmatrix}_2 + \begin{bmatrix}\text{net transfer} \\ \text{by conduction}\end{bmatrix}_3 + \begin{bmatrix}\text{rate of} \\ \text{internal} \\ \text{heat generation} \\ \text{(Eg: electrical} \\ \text{---)}\end{bmatrix} - \begin{pmatrix}\text{net work transfer from} \\ \text{the control volume} \\ \text{to the environment}\end{pmatrix}_5$$

→ due to body force, pressure, viscous forces



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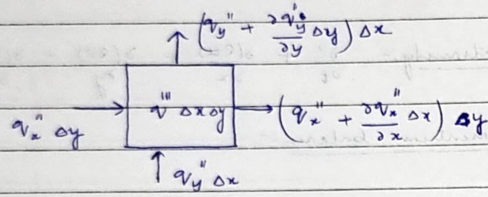
$F \cdot ds$ $\quad \frac{N}{t}$ $\quad F \cdot v$



e = specific ~~internal~~ energy - energy/mass.

① $\Delta x \Delta y \frac{\partial}{\partial t}(\rho e)$

$q_x'' \to$ heat flux in the x direction

② $-(\Delta x \Delta y)\left[\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e)\right]$

$q_y'' \to$ heat flux in y direction

③ $-(\Delta x \Delta y)\left[\frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y}\right]$

④ $(\Delta x \Delta y)\, q'''$

⑤ $(\Delta x \Delta y)\left[\sigma_x \frac{\partial u}{\partial x} - \tau_{xy}\frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} - \tau_{yx}\frac{\partial v}{\partial x}\right] +$

$(\Delta x \Delta y)\left[u\frac{\partial \sigma_x}{\partial x} - u\frac{\partial \tau_{xy}}{\partial y} + v\frac{\partial \sigma_y}{\partial y} - v\frac{\partial \tau_{yx}}{\partial x}\right].$

Work done per unit time = $-\underbrace{(\sigma_x \Delta y)}_{\text{force}} u$
↳ LHS

RHS $\to \left[\left(\sigma_x + \left(\frac{\partial \sigma_x}{\partial x}\right)\Delta x\right)\left(u + \frac{\partial u}{\partial x}\Delta x\right)\right]\Delta y$

Net $= \left[\sigma_x\left(\frac{\partial u}{\partial x}\right) + u\left(\frac{\partial \sigma_x}{\partial x}\right)\right]$

$\frac{\partial \psi}{\partial t}$

$$\rho \frac{De}{Dt} + e\left(\frac{D\rho}{Dt} + \rho \nabla \cdot V\right) = -\nabla \cdot q'' + q''' - P\nabla \cdot V + \underbrace{\mu\phi}_{\text{viscous dissipation}}$$

①

$-\rho g \nabla \cdot V$

$\nabla \psi$

$e = \hat{u} + \frac{1}{2} V_s^2$

$\hat{u}$ = specific internal energy

$\frac{1}{2}\hat{V}_s^2$ = specific KE

$g = -\nabla \psi \qquad \psi$ = Potential energy

Equation for total energy:

$$\rho \frac{D}{Dt}(e + \psi) = q''' - \nabla \cdot (P\vec{V}) - \nabla \cdot q'' + \mu\phi$$

viscous dissipation

$\to$ KE is neglected $\to$ $e = \hat{u}$ only.

$$\left[\rho \frac{De}{Dt} = q''' - \nabla \cdot (P\vec{V}) - \nabla \cdot q'' + \mu\phi\right] \to v$$

$H = U + PV$

$h = e + \frac{P}{\rho} \to$ specific enthalpy. $\quad e = h - \frac{P}{\rho}$

$\to$ ②

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} \to ③$$

Page No. | Date

→ $q'' = -k \nabla T$

$$\rho \frac{Dh}{Dt} = \nabla\cdot(k\nabla T) + q''' + \frac{DP}{Dt} + \mu\phi - \frac{P}{\rho}\cancel{\left(\frac{D\rho}{Dt} + \rho(\nabla\cdot V)\right)}$$

0 mass conservation.

$$\boxed{\rho \frac{Dh}{Dt} = \nabla\cdot(k\nabla T) + q''' + \frac{DP}{Dt} + \mu\phi}$$

$$dh = T\,ds + \frac{1}{\rho}dP$$

$$ds = \left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP$$

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial(1/\rho)}{\partial T}\right)_P = -\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P = -\beta/\rho$$

↳ maxwell's relation.

(β = coeff of thermal Expansion)

$\nabla\cdot V$

$dh = c_p dT$ → valid only for ideal gas.

$$dh = c_p\,dT + \frac{1}{\rho}(1-\beta T)\,dP$$

$$\rho\frac{Dh}{Dt} = \rho c_p \frac{DT}{Dt} + (1-\beta T)\frac{DP}{Dt}$$

$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$$** \left[\rho c_p \frac{DT}{Dt} = \nabla\cdot(k\nabla T) + q''' + \beta T\frac{DP}{Dt} + \mu\phi\right]$$

↳ No assumptions

Teacher's Signature

Page No. | Date

→ Incompressible, newtonian fluid

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot\nabla(kT) + q''' + \mu\phi$$

→ ~~For incompressible, Newtonian fluid~~.

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$ → incompressible 2D flow.

↓ viscous dissipation

→ Ideal gas :- $\beta = 1/T$

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot(k\nabla T) + q''' + \frac{DP}{Dt} + \mu\phi$$

→ Incompressible : $\beta = 0$

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot(k\nabla T) + q''' + \mu\phi$$

Teacher's Signature

Page No.
Date

Boundary layer

Incompressible
time independent

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_\infty}{L} \sim \frac{v}{\delta} \qquad v \sim \frac{U_\infty \delta}{L}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\text{Pressure } \frac{P}{\rho L}} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\approx 0} + \frac{\partial^2 u}{\partial y^2}\right)$$

Inertia $\left(\frac{U_\infty U_\infty}{L}\right)$, $v\frac{U_\infty}{\delta}$

Friction $\frac{\nu U_\infty}{L^2}$, $\nu\frac{U_\infty}{\delta^2}$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\underbrace{\frac{\partial^2 T}{\partial x^2}}_{\approx 0} + \frac{\partial^2 T}{\partial y^2}\right)$$

$u, v, P, T \to$ unknown.

→ 2 important things are

① Wall shear stress / net force exerted by the stream on the plate $F = \int_0^L \tau W dx$

② heat transfer rate between solid & fluid

$$q = \int_0^L q'' W dx$$

$$\tau = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad q'' = h(T_0 - T_\infty)$$

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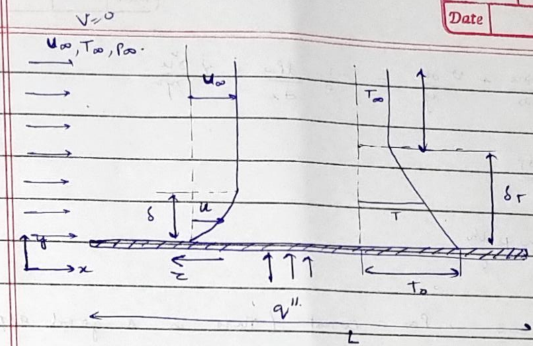
W = width of the plate

$$\frac{\partial T}{\partial y} \Rightarrow \frac{\Delta T}{\delta_T}$$

Page No.
Date



⇒ Near the wall

$$q'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = h(T_0 - T_\infty)$$

$$\boxed{h = \frac{-k(\partial T/\partial y)_{y=0}}{T_0 - T_\infty}}$$

δ = HBL thickness

δ_T = TBL thickness

$x \sim L$, $y \sim \delta$, $u \approx U_\infty$ → order of magnitude

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \underbrace{\frac{\partial P}{\partial y}}_{\approx 0}\frac{dy}{dx}$$

→ Pressure variation in the y-direction is negligible

$$\frac{\partial P}{\partial x} \sim \frac{\mu U_\infty}{\delta^2} \qquad \frac{dP}{dx} \approx \frac{\partial P}{\partial x}$$

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$$\frac{\partial P}{\partial x} = \frac{dP_\infty}{dx}$$

Pressure inside the BL is practically same as the pressure immediately outside it

Page No. — Jogmaya
Date

$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$

$\delta \neq \delta_T$

$\Rightarrow \tau \sim \mu \frac{U_\infty}{\delta}$

$\Rightarrow \frac{dP_\infty}{dx} = 0 \rightarrow P_\infty = \text{const}$ (This is a good approximation because pressure drop in the direction of flow is not significant over the longitudinal length L.

Inertia $\sim$ friction

$\frac{U_\infty^2}{L}, \frac{\vartheta U_\infty}{\delta} \sim \nu\frac{U_\infty}{\delta^2}$

$$\boxed{\delta \sim \left(\frac{\nu L}{U_\infty}\right)^{1/2}}$$

$\Rightarrow$ diffusion time scale $\rightarrow \frac{U_\infty}{t_D} \sim \nu\frac{U_\infty}{\delta^2}$

$t_D \sim \frac{\delta^2}{\nu}$

$\Rightarrow$ time of longitudinal convection $- t_L \sim \frac{L}{U_\infty}$

$t_D \sim t_L$ — at steady state -

$\delta \sim \sqrt{\frac{L\nu}{U_\infty}}$ ✓.

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$\mu\frac{dy}{dy} \quad \mu\frac{U_\infty}{\delta}$

Page No. — Jogmaya
Date

$\rightarrow \tau \sim \mu\frac{U_\infty}{L}Re_L^{1/2} \sim \rho U_\infty^2 Re_L^{-1/2}$

$\rightarrow C_f = \frac{\tau}{\frac{1}{2}\rho U_\infty^2}$ $C_f \sim Re_L^{-1/2}$

$\rightarrow h \sim \frac{k(\Delta T/\delta_T)}{\Delta T} \sim \frac{k}{\delta_T} \leftarrow q'' \sim h\,\Delta T$

convection $\sim$ conduction

$\frac{u\,\Delta T}{L}, \vartheta\frac{\Delta T}{\delta_T} \sim \alpha\frac{\Delta T}{\delta_T^2}$ } There is always a balance conduction from the wall into the stream & convection the to the wall

Case ①

Thick thermal boundary layer:- $\delta_T >> \delta$

$u \sim U_\infty$ + outside the BL

$\frac{\delta}{\delta_T} << 1$

$v \sim \frac{U_\infty \delta_T}{L}$

at the edge of TBL do order of magnitude analysis

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\frac{U_\infty}{L} \sim \frac{\vartheta}{\delta_T}$.

$u\frac{\partial T}{\partial x} + \vartheta\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$

$\sim U_\infty\frac{\Delta T}{L}$

$U_\infty\frac{\Delta T}{L} \sim \alpha^2\frac{\Delta T}{\delta_T^2}$

Peclet

$Pe_L = \frac{U_\infty L}{\alpha}$

$\delta_T \sim \sqrt{\frac{\alpha L}{U_\infty}}$

$\frac{\delta_T}{L} \sim Pe_L^{-1/2} \sim Pr^{-1/2} Re_L^{-1/2}$

$Pr = \nu/\alpha$ $Pe = Pr\,Re$ $\nu = \mu/\rho$ $\alpha = \frac{k}{\rho c_p}$

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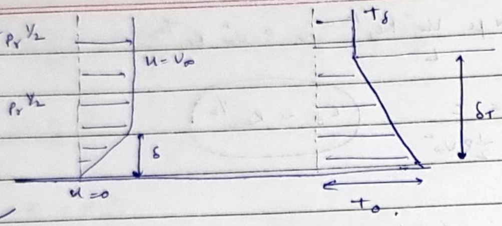
$h \sim \frac{k}{\delta_T} \sim \frac{k}{L} Re^{1/2} Pr^{1/2}$

$Nu = \frac{hL}{k} \sim Re^{1/2} Pr^{1/2}$

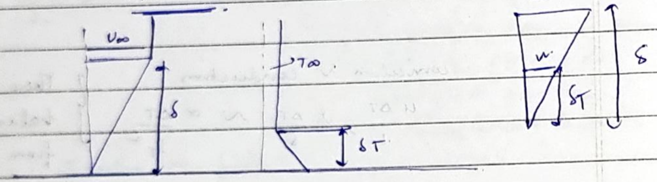
$\frac{\delta_T}{\delta} \sim Pr^{-1/2} >> 1$

$Pr << 1$ ✓



② Thin BL

$\delta_T << \delta$



Similar Δs → $\frac{U_\infty}{\delta} \sim \frac{u}{\delta_T}$

$u \sim \frac{U_\infty \delta_T}{\delta}$

$\frac{u}{L} \sim \frac{v}{\delta_T}$ → continuity at the edge of TBL

$v \sim \frac{u \delta_T}{L}$

$u \sim \frac{U_\infty \delta_T}{\delta}$

$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$

$\frac{u}{L} \Delta T \sim v \frac{\Delta T}{\delta_T} \sim \alpha \frac{\Delta T}{\delta_T^2}$

$\frac{u}{L} \sim \frac{\alpha}{\delta_T^2}$

$\delta_T^3 \propto \frac{L \alpha \delta}{U_\infty}$

$\left(\frac{\delta_T}{L}\right)^3 \sim \frac{\alpha L^2}{U_\infty} Re^{-1/2}$

$\boxed{\frac{\delta_T}{L} \sim Pr^{-1/3} Re^{-1/2}}$

$\frac{\delta_T}{\delta} \sim Pr^{-1/3} << 1$

$Pr >> 1$

$h \sim \frac{k}{L} Pr^{1/3} Re_L^{1/2}$

$Nu \sim Pr^{1/3} Re_L^{1/2}$

⇒ Reynold's number = $\frac{\text{Inertia}}{\text{friction}}$ → not true in case of BL → There is a balance between inertia & friction.

Physical interpretation
↳ geometric parameter

$Re_L^{1/2} = \frac{\text{Wall length}}{\text{Boundary layer thickness}}$

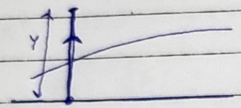
→ at the leading edge $\left(\frac{\delta}{L} \sim Re_L^{-1/2} \right.$, $\left. \frac{d\delta}{dx} \to \text{not defined} \right)$

$\frac{d}{dx}\int_0^Y (u V_\infty dy)$ $\quad V_\infty \int \frac{du}{dx} dy$ $\quad U_\infty$

Page No. — Jogmaya

Date

Integral Solution



$\boxed{Y > \max(\delta, \delta_T)}$

$$\left[\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}\right].$$

integrate

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha\frac{\partial^2 T}{\partial y^2}$$

$$\frac{d}{dx}\int_0^Y u^2 dy + u_Y v_Y - u_0 v_0 = -\frac{1}{\rho}Y\frac{dP_\infty}{dx} + \cancel{\nu\left(\frac{\partial u}{\partial y}\right)_Y}^{0} - \nu\left(\frac{\partial u}{\partial y}\right)_0.$$

$$\frac{d}{dx}\int_0^Y uT\,dy + v_Y T_Y - v_0 T_0 = \alpha\left(\frac{\partial T}{\partial y}\right)_Y - \alpha\left(\frac{\partial T}{\partial y}\right)_0.$$

$\left(\frac{\partial(\,)}{\partial y}\right)_Y = 0 \rightarrow$ free stream.

~~u_Y~~ $u_Y = U_\infty$ $\quad v_0 = 0 \rightarrow$ wall is impermeable

$T_Y = T_\infty$

$\rightarrow \frac{d}{dx}\int_0^Y u\,dy + v_Y - v_0 = 0 \rightarrow$ from continuity.

$$v_Y = -\frac{d}{dx}\int_0^Y u\,dy$$

$\frac{d}{dx}\int_0^Y u^2 dy + U_\infty\left(\frac{d}{dx}\int_0^Y u\,dy\right) - \frac{dU_\infty}{dx}\int + \frac{dU_\infty}{dx}\int$

$\frac{dU_\infty}{dx}\int$

$\frac{d}{dx}\int_0^Y u^2 dy - U_\infty \int u\,dy \quad U_\infty\frac{d}{dx}\int + \left(\frac{dU_\infty}{dx}\right)\int$

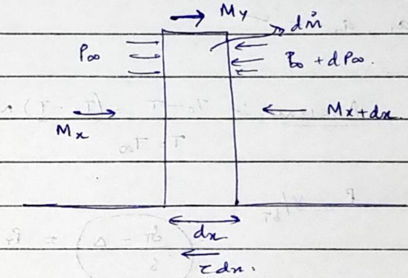
Page No. — Jogmaya

Date

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho}Y\frac{dP_\infty}{dx} + \left(\frac{dU_\infty}{dx}\right)\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0.$$

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0$$

$\Rightarrow$



~~Forces acting on the control volume~~

~~$M_x + \int_0^Y \rho u^2 dy$~~

$\Rightarrow$ Assume :- $\quad u = \begin{cases} U_\infty M(\eta), & 0 \le \eta \le 1 \\ U_\infty & 1 \le \eta \end{cases}$ $\quad \eta = y/\delta$

$P_\infty, U_\infty =$ const. $\quad M =$ shape function.

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \nu\left(\frac{\partial u}{\partial y}\right)_0.$$ $\quad d\eta = \frac{dy}{\delta}$

$$\delta\frac{d}{dx}\int_0^Y U_\infty M(U_\infty - U_\infty M)\,dy = \nu U_\infty\left(\frac{dM}{d\eta}\right)_{\eta=0}.$$

$$\delta\frac{d\delta}{dx}\left(\int_0^1 \left[M(1-M)\,d\eta\right]\right) = \frac{\nu}{U_\infty}\left(\frac{dM}{d\eta}\right)_{\eta=0}$$

1st order ODE for $\delta(x)$

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$\frac{\rho U_\infty x}{\mu}$

Page No.
Date

$$\frac{\delta}{x} = a_1 Re_x^{-1/2}$$

$$C_{f,x} = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} = a_2 Re_x^{-1/2}$$

find a_1, a_2

→ $\frac{dT_\infty}{dx}$ ⇒, Assume :- $T_0 - T = (T_0 - T_\infty) M(P)$ $0 \le P \le 1$

$T = T_\infty$ $1 \le P$

$P = y/\delta_T$

$$\frac{\delta_T}{\delta} = \Delta = P_r^{-1/3}$$

Similarity solutions

u, T profile look similar

$$\frac{u}{U_\infty} = g(\eta) \quad \eta \to \text{similarity parameter}$$

$$\eta = y/x \, Re_x^{1/2}$$

$$f' = \frac{df}{d\eta} \qquad \frac{u}{U_\infty} = f'(\eta)$$

Teacher's Signature

Page No.
Date

Teacher's Signature