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HEAT CONDUCTION

Fifth Edition

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To our families

This book is dedicated to the Memory of Yaman Yener

The only true enlightenment in life is science, technology.

Mustafa Kemal Atatürk

Heat, like gravity, penetrates every substance of the universe: its rays occupy all parts of space.

The theory of heat will hereafter form one of the most important branches of general physics.

J. B. Joseph Fourier, 1824

Contents

Preface to the Fifth Edition

It has been thirty-eight years since the publication of the first edition and ten years since the publication of the fourth edition of Heat Conduction. As with previous editions, the changes and additions that led our book to evolve to the fifth edition are a result of suggestions of our colleagues and students, and of the evolution of our teaching a graduate level course in heat conduction over the years.

In the current edition, we retained the overall objective of being a textbook for first year graduate engineering students and a reference book for engineers practicing in areas involving heat conduction problems. We added new problems at the end of each chapter to clarify the physically and theoretically important points and to supplement the text. We strongly believe that this is essential for a clear understanding of applications of the theoretical results. We also paid special attention to the derivation of basic equations and to their solutions in sufficient detail to help the reader better understand the subject matter. On the other hand, we knew we could not cover all topics in heat conduction and their solution techniques. Accordingly, the topics covered and the order and depth of the coverage represent our personal judgments. It has been our experience that the study of heat conduction as a one-semester graduate-level course in the manner followed in this book provides the best way for a first-year graduate student to develop an essential understanding of heat conduction.

We have expanded the fifth edition to include one new chapter. This is Chapter 13 Heat Conduction in Heterogeneous Media. In Chapter 13, we introduce a systematic derivation of the analytical solution of heat conduction problems involving space variable coefficients, introducing a generalization of the integral transform method, known as the Generalized Integral Transform Technique and the solution approach is described in different applications.

Moreover, we also expanded Chapter 2 and Chapter 10. In Chapter 2, we have added the section Hyperbolic Heat Conduction to include a discussion on the non-Fourier heat conduction model that considers the finite speed of heat propagation. In Chapter 10, we have added the section Coupled Integral Equations Approach to incorporate a very straightforward problem reformulation tool, which allows for the proposition of lumped-differential formulations as simple as the classical lumped system analysis, but with markedly improved accuracy.

Chapters 1, 3–9, and Chapters 11–12 of the fourth edition are mainly unchanged in the current edition except for the addition of new problems to each one. As in the previous editions, Chapter 1 serves as an introduction to the basic concepts and fundamentals of heat transfer. Chapter 2 is devoted to the derivation of the general forms of the heat conduction equation and to a discussion of initial and boundary conditions. Solutions of steady-state heat conduction problems in one-dimensional systems with and without internal energy sources, together with an extensive treatment of the steady-state performance of extended surfaces, are given in Chapter 3. The concepts of orthogonal functions, Fourier expansions, etc., which are fundamental to the application of the mathematical techniques used in this text, are introduced in Chapter 4. In Chapters 5 and 6, solutions of multi-dimensional steady-state and one- and multidimensional unsteady-state heat conduction problems are given, respectively, by the method of separation of variables. Chapter 7 is committed to the application of the Integral Transforms Method for finite, semi-infinite, and infinite regions.

In Chapter 8, we present the method of solution by Laplace transforms. Chapter 9 is dedicated to a class of problems which involve "local" sources of internal energy releasing heat in an infinitesimally small region of a system. Chapter 10 presents further analytical methods of solution. Chapter 11 is related to the analysis of heat conduction problems involving phase change and moving boundaries. Chapter 12 is intended to introduce the reader to the basics of the finite-difference formulation of heat conduction problems.

With few exceptions, no more engineering background than the usual undergraduate courses in thermodynamics, heat transfer, and advanced calculus is required of the reader.

In closing, we wish to express our sincere thanks to those people who helped us prepare the manuscript in its final form, including, among others, Prof. Renato M. Cotta and Prof. Anchasa Pramuanjaroenkij.

We also wish to express our thanks to Taylor & Francis group, whose competent work made this publication possible, especially Jonathan Plant, Jennifer Stair, Amy Rodriguez, and Amor Nanas, who are all very helpful to us in light of their expertise.

Finally, this book could have never been written if it were not for the support of our families. Their encouragement and patience have been an invaluable contribution.

Sadık Kakaç
Carolina P. Naveira-Cotta

Nomenclature

A	area, m^2
Bi	Biot number
c	specific heat, J/(kg·K)
c_p	specific heat at constant pressure, J/(kg·K)
c_v	specific heat at constant volume, J/(kg·K)
e	energy per unit mass, J/kg
E	energy, J
F_{ij}	radiation shape factor
Fo	Fourier number
$f(\mathbf{x})$	prescribed initial temperature distribution
g	gravitational acceleration, m/s^2
$g(\mathbf{x},t)$	boundary conditions source term
h	heat transfer coefficient, W/(m^2·K)
h	enthalpy per unit mass, J/kg
h_{sl}	latent heat of fusion, J/kg
l_v	modified Bessel function of the first kind of order ν
J_v	Bessel function of the first kind of order ν
k	thermal conductivity, W/(m·K)
k_{ij}	thermal conductivity coefficients, W/(m·K)
k_{in}	mean thermal conductivity, W/(m·K)
K_v	modified Bessel function of the second kind of order ν
m	mass, kg
$\hat{\mathbf{n}}$	unit vector
N_n	normalization integral
p	pressure, N/m^2
P_n	Legendre polynomial of order n
$P(\mathbf{x},t)$	heat conduction equation source term
q	heat transfer rate, W
q'	rate of heat transfer per unit length, W/m
q''	heat flux, W/m^2
$\mathbf{q}''$	heat flux vector, W/m^2
$\dot{q}$	rate of internal energy generation per unit volume, W/m^3
Q	heat, amount of heat J
Q_n	Legendre function of the second kind
$\mathbf{r}$	position vector
R_t	thermal resistance, K/W
s	entropy per unit mass. J/(kg·K)
S	entropy, J/K
Ste	Stefan number
t	time, s
T	temperature, K
u	internal energy per unit mass, J/kg
U	overall heat transfer coefficient, W/(m^2·K)
U	internal energy, J

ν	specific volume, m^3/kg
$\boldsymbol{\upsilon}$	volume, m^3
$\boldsymbol{V}$	velocity, m/s
$\mathbf{V}$	velocity vector, m/s
$\boldsymbol{W}$	work, J
$\dot{\boldsymbol{W}}$	power, W
$\boldsymbol{w}(\mathbf{x}), \boldsymbol{k}(\mathbf{x}), \boldsymbol{d}(\mathbf{x})$	space variable coefficients in heat conduction equation
$\boldsymbol{Y}_v$	Bessel function of the second kind of order v

Greek Letters

α	absorptivity; thermal diffusivity, m^2/s
α, β, γ	direction cosines of unit vector $\hat{\mathbf{n}}$
$\alpha(\mathbf{x}), \beta(\mathbf{x})$	boundary conditions coefficients
δ	penetration depth, m; velocity boundary layer thickness, m
δ_T	thermal boundary layer thickness, m
ε	emissivity
η	similarity variable
η_f	fin efficiency
λ	eigenvalue; molecular mean-free-path, m; dimensionless interface location, m
λ, μ	eigenvalues
μ	coefficient of dry friction
ϕ	fin effectiveness
$\phi(\mathbf{x}), \Omega(\mathbf{x})$	eigenfunctions
γ	temperature coefficient of thermal conductivity, 1/K
$\bar{\boldsymbol{\Phi}}$	eigenvectors
ρ	mass density, kg/m^3; reflectivity
σ	Stefan–Boltzmann constant, 5.6697×10^{-8} $W/(m^2{\cdot}K^4)$
τ	transmissivity
χ	reciprocal of neutron diffusion length, 1/m; volume percentage

Coordinates

$\mathbf{x}$	position vector
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	rectangular coordinates
$\boldsymbol{r}, \phi, \boldsymbol{z}$	cylindrical coordinates
$\boldsymbol{r}, \theta, \phi$	spherical coordinates

Unit vectors

$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$	unit vectors in x-, y-, and z-directions
$\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\mathbf{z}$	unit vectors in r-, ϕ-, and z-directions
$\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$	unit vectors in r-, θ-, and ϕ-directions

Subscripts

b	fin base value; bulk value
c	contact surface: center
cr	critical insulation value
cs	control surface
cv	control volume
e	electrical; eutectic point
f	fin value; fluid value; fusion temperature
h	homogeneous problem
l	liquid value
ln	line
m	mean value; mush
n	in the direction of unit vector $\hat{\mathbf{n}}$
pt	point
r	radiation
R	reference value
s	solid value; surface value
t	thermal
x	local condition
w	wall (surface) condition (value)
∞	ambient condition (value)

Notation

Δ	Change
∇	Amount
∇	Nabla (del) operator

The list presented here is intended to include a consolidated collection of symbols for the entire text. Some symbols that are used in only a specific development are not included here and are defined in the text where they are used.

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1

Foundations of Heat Transfer

1.1 Introductory Remarks

Heat transfer is that branch of engineering science which studies the transfer of energy solely as a result of temperature differences. Heat transfer problems confront engineers and researchers in nearly every branch of engineering and science. Although it is generally regarded as most closely related to mechanical engineering, much work in this field has also been done in chemical, nuclear, metallurgical, and electrical engineering, where heat transfer problems are equally important. It is probably this fundamental and widespread influence that has helped the field of heat transfer develop as an engineering science.

Thermodynamics, another branch of the engineering science, studies heat and work interactions of a system with its surroundings, and defines *heat* as the form of energy that crosses the boundary of a system by virtue of a temperature difference existing between the system and its surroundings. That is, heat is the energy in transition across the system boundary and temperature difference is the driving potential for its propagation. Since heat is energy in transit, one should really talk about the transfer or flow of heat. *Heat flow* is vectorial in the sense that it is in the direction of a negative temperature gradient, that is, from higher toward lower temperatures.

The *laws of thermodynamics* may be used to predict the gross amount of heat transferred to or from a system during a process in which the system goes from one thermodynamic state (i.e., mechanical and chemical, as well as thermal equilibrium) to another. In most instances, however, the overriding consideration may be the time period over which the transfer of heat occurs or, simply, the time rate at which it takes place. The laws of thermodynamics alone are not sufficient to provide such information; neither can they explain the mechanisms of heat transfer, which are not strictly restricted to equilibrium states. The science of heat transfer, on the other hand, studies the mechanisms of heat transfer and extends thermodynamic analysis, through the development of necessary empirical and analytical relations, in order to calculate heat transfer rates.

The science of heat transfer is based on foundations comprising both theory and experiment. As in other engineering disciplines, the theoretical part is constructed from one or more *physical* (or *natural*) *laws*. The physical laws are statements, in terms of various concepts, that have been found to be true through many years of experimental observations. A physical law is referred to as a *general law* if its application is independent of the nature of the medium under consideration. Otherwise, it is called a *particular law*. There are, in fact, four general laws, on which almost all the analyses concerning heat transfer, either directly or indirectly, depend:

1. the law of conservation of mass,
2. Newton's second law of motion,
3. the first law of thermodynamics, and
4. the second law of thermodynamics.

For most heat conduction problems, the use of the first and second laws of thermodynamics is sufficient. In addition to these general laws, certain particular laws have to be brought into the analysis. There are three such particular laws that are usually employed in the analysis of conduction heat transfer:

1. Fourier's law of heat conduction,
2. Newton's law of cooling, and
3. Stefan–Boltzmann's law of radiation.

1.2 Modes of Heat Transfer

The mechanism by which heat is transferred in a heat exchange or an energy conversion system is, in fact, quite complex. There appear, however, to be three rather basic and distinct *modes* of heat transfer:

1. conduction,
2. radiation, and
3. convection.

Conduction is the process of heat transfer by molecular motion, supplemented in some cases by the flow of free electrons and lattice vibrations, through a body (solid, liquid, or gas) from regions of high temperature to regions of low temperature. Heat transfer by conduction also takes place across the interface between two material bodies in contact when they are at different temperatures.

The mechanism of heat conduction in liquids and gases has been postulated as the transfer of kinetic energy of the molecular movement. Thermal energy transferred to a fluid increases its internal energy by increasing the kinetic energy of its vibrating molecules, which is measured by the increase in its temperature. A high temperature measurement would therefore indicate a high kinetic energy of the molecules, and heat conduction is thus the transfer of kinetic energy of the more active molecules in the high temperature regions by successive collisions to the molecules in the low molecular kinetic energy regions. On the other hand, heat conduction in solids with crystalline structures, such as quartz, depends on the energy transfer by molecular and lattice vibrations, and free-electron drift. In general, energy transfer by molecular and lattice vibrations is not so large as the energy transfer by free electrons, and it is for this reason that good electrical conductors are almost always good heat conductors, while electrical insulators are usually good heat insulators. In the case of amorphous solids, such as glass, heat conduction depends solely only on the molecular transport of energy.

Thermal radiation, or simply *radiation*, is the transfer of heat in the form of *electromagnetic waves*. All substances, solid bodies as well as liquids and gases, emit radiation as a result of their temperature, and they are also capable of absorbing such energy. Furthermore, radiation can pass through certain types of substances (i.e., *transparent* and *semitransparent* materials) as well as through vacuum, whereas for heat conduction to take place a material medium is absolutely necessary.

Conduction is the only mechanism by which heat can flow in *opaque* solids. Through certain transparent or semitransparent solids, such as glass and quartz, energy flow can be by radiation as well as by conduction. With gases and liquids, if there is no fluid motion, the heat transfer mechanism will be conduction (and also, if not negligible, radiation). However, if there is macroscopic fluid motion, energy can also be transported in the form of internal energy by the movement of the fluid itself. The process of energy transport by the combined effect of heat conduction (and radiation) and the movement of fluid is referred to as *convection* or *convective heat transfer*. Although we have classified convection as a mode of heat transfer, it is actually conduction (and radiation) in moving fluids. An analysis of convective heat transfer is, therefore, more involved than that of heat transfer by conduction alone, because the fluid motion must be studied simultaneously with the energy transfer process.

In reality, temperature distribution in a medium is controlled by the combined effect of these three modes of heat transfer. Therefore, it is not actually possible to entirely isolate one mode from interactions with other modes. For simplicity in the analysis, however, these three modes are almost always studied separately. In this book we focus on the study of conduction heat transfer only.

1.3 Continuum Concept

Matter, while seemingly continuous, is composed of molecules, atoms and electrons in constant motion and undergoing collisions. Since heat conduction is thought to come about through the exchange of kinetic energy among such particles, the most fundamental approach in analyzing the transfer of heat in a substance by conduction is, therefore, to apply the laws of motion to each individual particle or a statistical group of particles, subsequent to some initial states of affairs. Such an approach gives insight into the details of heat conduction; however, as one would predict, it would be too cumbersome for most situations arising in engineering.

In most engineering problems, primary interest lies not in the molecular behavior of a substance, but rather in how the substance behaves as a continuous medium. In our study of heat conduction, we will therefore ignore the molecular structure of the substance and consider it to be a continuous medium, or *continuum*, which is fortunately a valid approach to many practical problems where only macroscopic information is of interest. Such a model may be used provided that the size and the mean free path of molecules are small compared with other dimensions existing in the medium, so that a statistical average is meaningful. This approach, which is also known as the *phenomenological* approach in the study of heat conduction, is simpler than microscopic approaches and usually gives the answers required in engineering. On the other hand, to make up for the information lost by the neglect of the molecular structure, certain parameters, such as *thermodynamic state* (i.e., *thermophysical*) and *transport properties*, have to be introduced empirically. Parallel to the study of heat conduction by the continuum approach, molecular considerations can also be used to obtain information on thermodynamic and transport properties. In this book, we restrict our discussions to the phenomenological heat conduction theory.

1.4 Some Definitions and Concepts of Thermodynamics

In this section, we review some definitions and concepts of thermodynamics needed for the study of heat conduction. The reader, however, is advised to refer to textbooks on thermodynamics, such as References [4,18], for an in-depth discussion of these definitions and concepts, as well as the laws of thermodynamics.

In thermodynamics, a *system* is defined as an arbitrary collection of matter of fixed identity bounded by a closed surface, which can be a real or an imaginary one. All other systems that interact with the system under consideration are known as its *surroundings*. In the absence of any mass–energy conversion, not only does the mass of a system remain constant, but the system must be made up of exactly the same submolecular particles. The four general laws listed in Section 1.1 are always stated in terms of a system. In fact, one cannot meaningfully apply a general law until a definite system is identified.

A *control volume* is any defined region in space, across the boundaries of which matter, energy and momentum may flow, within which matter, energy and momentum storage may take place, and on which external forces may act. Its position and/or size may change with time. However, most often we deal with control volumes that are fixed in space and of fixed size and shape.

The dimensions of a system or a control volume may be finite or they may even be infinitesimal. The complete definition of a system or a control volume must include at least the implicit definition of a coordinate system, since the system or the control volume can be stationary or may even be moving with respect to the coordinate system.

The characteristic of a system we are most interested in is its *thermodynamic state*, which is described by a list of the values of all its *properties*. A property of a system is either a directly or an indirectly observable characteristic of that system which can, in principle, be quantitatively evaluated. Volume, mass, pressure, temperature, etc. are all properties. If all the properties of a system remain unchanged, then the system is said to be in an *equilibrium state*. A *process* is a change of state and is described in part by the series of states passed through by the system. A *cycle* is a process wherein the initial and final states of a system are the same.

If no energy transfer as heat takes place between any two systems when they are placed in contact with each other, they are said to be in *thermal equilibrium*. Any two systems are said to have the same *temperature* if they are in thermal equilibrium with each other. Two systems not in thermal equilibrium would have different temperatures, and energy transfer as heat would take place from one system to the other. Temperature is, therefore, the property of a system that measures its "thermal level."

The laws of thermodynamics deal with interactions between a system and its surroundings as they pass through equilibrium states. These interactions may be divided into two as (1) *work* or (2) *heat* interactions. Heat has already been defined as the form of energy that is transferred across a system boundary owing to a temperature difference existing between the system and its surroundings. Work, on the other hand, is a form of energy that is characterized as follows. When an energy form of one system (such as kinetic, potential, or internal energy) is transformed into an energy form of another system or surroundings without the transfer of mass from the system and not by means of a temperature difference, the energy is said to have been transferred through the performance of work.

The amount of change in the temperature of a substance with the amount of energy stored within that substance is expressed in terms of *specific heat*. Because of the different ways in which energy can be stored in a substance, the definition of specific heat depends on the nature of energy addition. The *specific heat at constant volume* is the change in *internal energy* (see Section 1.6) of a unit mass per degree change of temperature between two equilibrium states of the same volume. The *specific heat at constant pressure* is the change in *enthalpy* (see Section 1.6) of a unit mass between two equilibrium states at the same pressure per degree change of temperature.

1.5 Law of Conservation of Mass

The law of conservation of mass, when referred to a system, simply states that, *in the absence of any mass–energy conversion, the mass of the system remains constant*. Thus, for a system,

$$\frac{dm}{dt} = 0 \qquad \text{or} \qquad m = \text{constant} \tag{1.1}$$

where m is the mass of the system.

We now proceed to develop the form of this law as it applies to a control volume. Consider an arbitrary control volume fixed in space and of fixed shape and size, as illustrated in Fig. 1.1. Matter flows across the boundaries of this control volume.

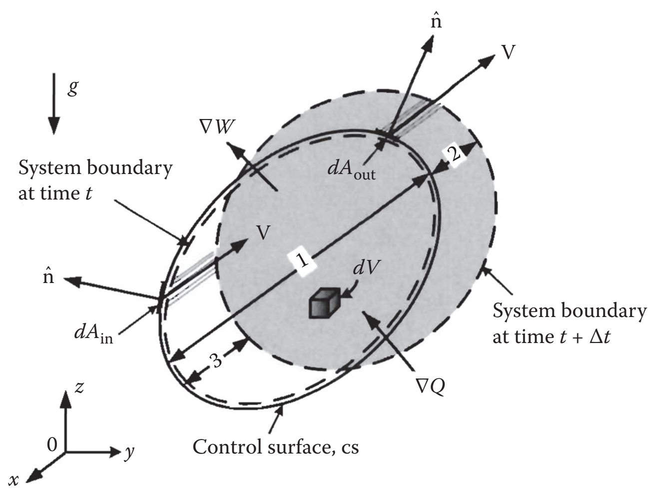

FIGURE 1.1
Flow of matter through a fixed control volume.

Define a system whose boundary at some time t happens to correspond exactly to that of the control volume. By definition, the control volume remains fixed in space, but the system moves and at some later time $t + \Delta t$ occupies different volume in space. The two positions of the system are shown in Fig. 1.1 by dashed lines. Since the mass of the system is conserved, we can write

$$m_1(t) = m_1(t+\Delta t) + m_2(t+\Delta t) - m_3(t+\Delta t) \tag{1.2}$$

where m_1, m_2, and m_3 represent the instantaneous values of mass contained in the three regions of space shown in Fig. 1.1. Dividing Eq. (1.2) by Δt and rearranging we get

$$\frac{m_1(t+\Delta t) - m_1(t)}{\Delta t} = \frac{m_3(t+\Delta t)}{\Delta t} - \frac{m_2(t+\Delta t)}{\Delta t} \tag{1.3}$$

As $\Delta t \to 0$, the left-hand side of Eq. (1.3) reduces to

$$\lim_{\Delta t\to 0} \frac{m_1(t+\Delta t) - m_1(t)}{\Delta t} = \left(\frac{\partial m}{\partial t}\right)_{\text{cv}} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho \, d\upsilon \tag{1.4}$$

where $d\upsilon$ is an element of the control volume and ρ is the local density of that element. In addition, "cv" designates the control volume bounded by the control surface, "cs." Equation (1.4) represents the time rate of change of the mass within the control volume. Furthermore,

$$\lim_{\Delta t\to 0} \frac{m_3(t+\Delta t)}{\Delta t} = \dot{m}_{\text{in}} = -\int_{A_{\text{in}}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, d A_{\text{in}} \tag{1.5}$$

and

$$\lim_{\Delta t\to 0} \frac{m_2(t+\Delta t)}{\Delta t} = \dot{m}_{\text{out}} = \int_{A_{\text{out}}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, d A_{\text{out}} \tag{1.6}$$

where $\dot{m}_{\text{in}}$ and $\dot{m}_{\text{out}}$ are the mass flow rates into and out of the control volume, V denotes the velocity vector, and $\hat{\mathbf{n}}$ is the outward-pointing unit vector normal to the control surface. Thus, as $\Delta t \to 0$, Eq. (1.3) becomes

$$\frac{\partial}{\partial t}\int_{\text{CV}} \rho \, d\upsilon = -\int_{A_{\text{in}}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA_{\text{in}} - \int_{A_{\text{out}}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA_{\text{out}} \tag{1.7a}$$

which can be rewritten as

$$\frac{\partial}{\partial t}\int_{\text{CV}} \rho \, d\upsilon = -\int_{\text{CS}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA \tag{1.7b}$$

where dA is an element of the control surface. Equation (1.7*b*) states that *the time rate of increase of mass within a control volume is equal to the net rate of mass flow into the control* *volume*. Since the control volume is fixed in location, size and shape, Eq. (1.7*b*) can also be written as

$$\int_{CV} \frac{\partial \rho}{\partial t} d\upsilon = -\int_{CS} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA \tag{1.7c}$$

The surface integral on the right-hand side of Eq. (1.7*c*) can be transformed into a volume integral by the *divergence theorem* [6]; that is,

$$\int_{CS} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA = \int_{CV} \nabla \cdot (\rho \mathbf{V}) d\upsilon \tag{1.8}$$

Hence, Eq. (1.7*c*) can also be written as

$$\int_{CV} \frac{\partial \rho}{\partial t} d\upsilon = -\int_{CV} \nabla \cdot (\rho \mathbf{V}) d\upsilon \tag{1.9a}$$

or

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] d\upsilon = 0 \tag{1.9b}$$

Since this result would be valid for all arbitrary control volumes, the integrand must be zero everywhere, thus yielding

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{1.10}$$

This equation is referred to as the *continuity equation*.

1.6 First Law of Thermodynamics

When a system undergoes a cyclic process, the first law of thermodynamics can be expressed as

$$\oint \delta Q = \oint \delta W \tag{1.11}$$

where the cyclic integral $\oint \delta Q$ represents the net heat transferred *to* the system, and the cyclic integral $\oint \delta W$ is the net work done *by* the system during the cyclic process.

Both heat and work are *path* functions; that is, the net amount of heat transferred to, and the net amount of work done by, a system when the system undergoes a change of state depend on the path that the system follows during the change of state. This is why the differentials of heat and work in Eq. (1.11) are *inexact differentials*, denoted by the symbols δQ and δW, respectively.

For a process that involves an infinitesimal change of state during a time interval dt, the first law of thermodynamics is given by

$$dE = \delta Q - \delta W \tag{1.12}$$

where δQ and δW are the differential amounts of heat added to the system and the work done by the system, respectively, and dE is the corresponding increase in the *energy* of the system during the time interval dt. The energy E is a property of the system and, like all other properties, is a *point* function. That is, dE depends on the initial and final states only, and not on the path followed between the two states. For a more complete discussion of point and path functions, see References [4,18]. The property E represents the total energy contained within the system and, in the absence of any mass–energy conversion and chemical reactions, is customarily separated into three parts as *bulk kinetic energy, bulk potential energy*, and *internal energy*; that is,

$$E = KE + PE + U \tag{1.13}$$

The internal energy, U, represents the energy associated with molecular and atomic behavior of the system.

Equation (1.12) can also be written as a rate equation:

$$\frac{dE}{dt} = \frac{\delta Q}{dt} - \frac{\delta W}{dt} \tag{1.14a}$$

or

$$\frac{dE}{dt} = q - \dot{W} \tag{1.14b}$$

where $q = \delta Q/dt$ represents the rate of heat transfer to the system and $\dot{W} = \delta W/dt$ is the rate of work done (power) by the system.

Following the approach used in the previous section, we now proceed to develop the control-volume form of the first law of thermodynamics. Referring to Fig. 1.1, the first law for the system under consideration can be written as

$$\Delta E = \nabla Q - \nabla W \tag{1.15}$$

where ∇Q is the amount of heat transferred to the system, ∇W is the work done by the system, and ΔE is the corresponding increase in the energy of the system during the time interval Δt. Dividing Eq. (1.15) by Δt we get

$$\frac{\Delta E}{\Delta t} = \frac{\nabla Q}{\Delta t} - \frac{\nabla W}{\Delta t} \tag{1.16}$$

The left-hand side of this equation can be written as

$$\frac{\Delta E}{\Delta t} = \frac{E_1(t+\Delta t) + E_2(t+\Delta t) - E_3(t+\Delta t) - E_1(t)}{\Delta t} \tag{1.17a}$$

or

$$\frac{\Delta E}{\Delta t} = \frac{E_1(t+\Delta t) - E_1(t)}{\Delta t} + \frac{E_2(t+\Delta t)}{\Delta t} - \frac{E_3(t+\Delta t)}{\Delta t} \tag{1.17b}$$

where E_1, E_2, and E_3 are the instantaneous values of energy contained in the three regions of space shown in Fig. 1.1.

As $\Delta t \to 0$, the first term on the right-hand side of Eq. (1.17*b*) becomes

$$\lim_{\Delta t \to 0} \frac{E_1(t+\Delta t) - E_1(t)}{\Delta t} = -\left(\frac{\partial E}{\partial t}\right)_{\mathrm{CV}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} e\rho\, d\upsilon \tag{1.18}$$

where e is the *specific total energy* (i.e., energy per unit mass). Equation (1.18) represents the time rate of change of energy within the control volume. Furthermore.

$$\lim_{\Delta t \to 0} \frac{E_2(t+\Delta t)}{\Delta t} = \int_{A_{\mathrm{out}}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA_{\mathrm{out}} \tag{1.19}$$

and

$$\lim_{\Delta t \to 0} \frac{E_3(t+\Delta t)}{\Delta t} = -\int_{A_{\mathrm{in}}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA_{\mathrm{in}} \tag{1.20}$$

which represent, respectively, the rates of energy leaving and entering the control volume at time t. Thus, as $\Delta t \to 0$, Eq. (1.17*b*) becomes

$$\begin{aligned} \frac{dE}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} \\ &= \frac{\partial}{\partial t}\int_{\mathrm{CV}} e\rho\, d\upsilon + \int_{A_{\mathrm{out}}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA_{\mathrm{out}} + \int_{A_{\mathrm{in}}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA_{\mathrm{in}} \end{aligned} \tag{1.21}$$

which can also be written as

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} e\rho\, d\upsilon + \int_{\mathrm{CS}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA \tag{1.22}$$

The first term on the right-hand side of Eq. (1.16) is the rate of heat transfer to the system, which is also the rate of heat transfer across the control surface as $\Delta t \to 0$; that is,

$$\lim_{\Delta t \to 0} \frac{\nabla Q}{\Delta t} = \left(\frac{\delta Q}{dt} \right)_{\text{CS}} = q_{\text{CS}} \tag{1.23}$$

As $\Delta t \to 0$, the second term on the right-hand side of Eq. (1.16) becomes

$$\lim_{\Delta t \to 0} \frac{\nabla W}{\Delta t} = \frac{\delta W}{dt} = \dot{W} \tag{1.24}$$

which is the rate of work done (power) by the matter in the control volume (i.e., the system) on its surroundings at time t. Hence, as $\Delta t \to 0$, Eq. (1.16) becomes

$$\frac{\partial}{\partial t} \int_{\text{CV}} e\rho \, d\upsilon + \int_{\text{CS}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA = q_{\text{CS}} - \dot{W} \tag{1.25}$$

The surface integral on the left-hand side of Eq. (1.25) can be transformed into a volume integral by the *divergence theorem* [6] to yield

$$\begin{aligned} \int_{\text{CS}} e\rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA &= \int_{\text{CV}} \nabla \cdot (e\rho \mathbf{V}) \, d\upsilon \\ &= \int_{\text{CV}} [e \nabla \cdot (\rho \mathbf{V}) + \rho \mathbf{V} \cdot \nabla e] \, d\upsilon \end{aligned} \tag{1.26}$$

Substituting this result into Eq. (1.25) and then rearranging the terms on the left-hand side we obtain*

$$\int_{\text{CV}} \rho \left(\frac{\partial e}{\partial t} + \mathbf{V} \cdot \nabla e \right) d\upsilon = q_{\text{CS}} - \dot{W} \tag{1.27}$$

where we have made use of the continuity relation (1.10).

Equation (1.25), or Eq. (1.27), is the control-volume form of the first law of thermodynamics. However, a final form can be obtained after further consideration of the power term $\dot{W}$.

Work can be done by the system in a variety of ways. In this analysis we consider the work done against normal stresses (*pressure*) and tangential stresses (*shear*), the work done by the system that could cause a shaft to rotate (*shaft work*), and the work done on the system due to power drawn from an external electric circuit. We neglect capillary and magnetic effects.

---

* Equation (1.27) can also be written as

$$\int_{\text{CV}} \rho \frac{De}{Dt} d\upsilon = q_{\text{CS}} - \dot{W}$$

where the derivative

$$\frac{De}{Dt} = \frac{\partial e}{\partial t} + \mathbf{V} \cdot \nabla e$$

is commonly referred to as the *substantial derivative* or the *material derivative.*

Consider now the work done by the system against pressure, which is also called *flow work*. The work done against the pressure p acting at a surface element $d\,A_{\text{out}}$ during the time interval Δt is $p\,dA_{\text{out}}\,\Delta n$, where $\Delta n = (V \cdot \hat{\mathbf{n}})\ \Delta t$ is the distance moved normal to dA_{out}. Hence, the *rate of work done by* the system against pressure at dA_{out} is $p dA_{\text{out}}\ \mathbf{V}\cdot\hat{\mathbf{n}}$. The *rate of work done against* pressure over A_{out} is, then, given by $\int A_{\text{out}}\, p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA_{\text{out}}$. Similarly, rate of work done on the system by pressure acting on A_{in} would be given by $-\int A_{\text{in}}\, p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA_{\text{in}}$. Thus, the net rate of work done by the system against pressure will be

$$\dot{W}_{\text{normal}} = \int_{A_{\text{out}}} \rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA_{\text{out}} + \int_{A_{\text{in}}} \rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA_{\text{in}} \tag{1.28a}$$

which can also be rewritten as

$$\dot{W}_{\text{normal}} = \int_{\text{CS}} \rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA \tag{1.28b}$$

Let $\dot{W}_{\text{shaft}}$ represent the rate at which the system does shaft work, and $\dot{W}_{\text{shear}}$ be the rate at which the system does work against shear stresses. The rate of work done on the system due to power drawn from an external electric circuit, on the other hand, can be written as $\int_{\text{cv}} \dot{q}_e\, d\upsilon$, where $\dot{q}_e$ is the rate of internal energy generation per unit volume due to the power drawn to the system from the external circuit.

Hence, the control-volume form of the first law thermodynamics, Eq. (1.25) or Eq. (1.27), can also be written as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\text{CV}} e\rho\, d\upsilon + \int_{\text{CS}} e\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA \\ = q_{\text{CS}} - \int_{\text{CS}} p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA - \dot{W}_{\text{shear}} - \dot{W}_{\text{shaft}} + \int_{\text{CV}} \dot{q}_e\, d\upsilon\end{aligned} \tag{1.29}$$

or

$$\begin{aligned}\int_{\text{CV}} \rho\left[\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla e\right] d\upsilon \\ = q_{\text{CS}} - \int_{\text{CS}} p\mathbf{V}\cdot\hat{\mathbf{n}}\, dA - \dot{W}_{\text{shear}} - \dot{W}_{\text{shaft}} + \int_{\text{CV}} \dot{q}_e\, d\upsilon\end{aligned} \tag{1.30}$$

The specific total energy is given by $e = u + \mathrm{V}^2/2 + gz$, where u is the internal energy per unit mass, $V^2/2$ is the bulk kinetic energy per unit mass, and gz is the bulk potential energy per unit mass. Hence, Eq. (1.29) can also be written as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\text{CV}} e\rho\, d\upsilon + \int_{\text{CS}} \left(h + \frac{1}{2}V^2 + gz\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA \\ = q_{\text{CS}} - \dot{W}_{\text{shear}} - \dot{W}_{\text{shaft}} + \int_{\text{CV}} \dot{q}_e\, d\upsilon\end{aligned} \tag{1.31}$$

where h is the *enthalpy* per unit mass defined as

$$h = u + \frac{p}{\rho} \tag{1.32}$$

Equation (1.31) is an alternative expression for the control-volume form of the first law of thermodynamics.

1.7 Second Law of Thermodynamics

The first law of thermodynamics, which embodies the idea of conservation of energy, gives means for quantitative calculation of changes in the state of a system due to interactions between the system and its surroundings, but it tells us nothing about the direction a process might take. In other words, physical observations such as the following cannot be explained by the first law:

- A cup of hot coffee placed in a cool room will always tend to cool to the temperature of the room, and once it is at room temperature it will never return spontaneously to its original hot state.
- Air will rush into a vacuum chamber spontaneously.
- The conversion of heat into work cannot be carried out on a continuous basis with a conversion efficiency of 100%.
- Water and salt will mix to form a solution, but separation of such a solution cannot be made without some external means.
- A vibrating spring will eventually come to rest all by itself, etc.

These observations concerning unidirectionality of naturally occurring processes have led to the formulation of the second law of thermodynamics. Over the years, many statements of the second law have been made. Here we give the following *Clausius* statement: *It is impossible for a self-acting system unaided by an external agency to move heat from one system to another at a higher temperature.*

The second law leads to the thermodynamic property of *entropy*. For any *reversible* process that a system undergoes during a time interval dt, the change in the entropy S of the system is given by

$$dS = \left(\frac{\delta Q}{T}\right)_{\text{rev}} \tag{1.33a}$$

For an *irreversible* process, the change, however, is

$$dS > \left(\frac{\delta Q}{T}\right)_{\text{irr}} \tag{1.33b}$$

where δQ is the small amount of heat transferred to the system during the time interval dt, and T is the temperature of the system at the time of the heat transfer. Equations (1.33*a*) and (1.33*b*) together may be taken as the mathematical statement of the second law, and they can also be written in rate form as

$$\frac{dS}{dt} \geq \frac{1}{T}\frac{\delta Q}{dt} \tag{1.34}$$

The control-volume form of the second law can be developed also by following a procedure similar to the one used in the previous section. Rather than going through the entire development, here we give the result [16]:

$$\frac{\partial}{\partial t}\int_{CV} s\rho\, dv + \int_{CS} s\rho \mathbf{V}\cdot\hat{\mathbf{n}}\, dA \geq \int_{CS} \frac{1}{T}\frac{\delta Q}{dt} \tag{1.35}$$

where s is the entropy per unit mass, and the equality applies to reversible processes and the inequality to irreversible processes.

1.8 Temperature Distribution

Since heat transfer takes place whenever there is a temperature gradient in a medium, a knowledge of the values of temperature at all points of the medium is essential in heat transfer studies. The instantaneous values of temperature at all points of the medium of interest is called the *temperature distribution* or *temperature field.*

An *unsteady* (or *transient*) temperature distribution is one in which temperature not only varies from point to point in the medium, but also with time. When the temperature at various points in a medium changes, the internal energy also changes accordingly at the same points. The following represents an unsteady temperature distribution:

$$T = T(\mathbf{r}, t) \tag{1.36a}$$

where $\mathbf{r} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y + \hat{\mathbf{k}}z$, and $\hat{\mathbf{i}}, \hat{\mathbf{j}},$ and $\hat{\mathbf{k}}$ are the unit vectors in the x, y, and z directions, respectively, in the rectangular coordinate system.

A *steady* temperature distribution is one in which the temperature at a given point never varies with time; that is, it is a function of space coordinates only. The following represents a steady temperature distribution:

$$T = T(\mathbf{r}), \qquad \frac{\partial T}{\partial t} = 0 \tag{1.36b}$$

Since the temperature distributions governed by Eqs. (1.36*a*) and (1.36*b*) are functions of three space coordinates $(\mathbf{r} = \hat{\mathbf{i}}x, \hat{\mathbf{j}}y + \hat{\mathbf{k}}z)$, they are called *three-dimensional*. When a temperature distribution is a function of two space coordinates, it is called *two-dimensional*. For example, in the rectangular coordinate system,

$$T = T(x, y, t), \qquad \frac{\partial T}{\partial z} = 0 \tag{1.36c}$$

represents an unsteady two-dimensional temperature distribution.

When a temperature distribution is a function of one space coordinate only, it is called *one-dimensional*. For example, in the rectangular coordinate system,

$$T = T(x, t), \qquad \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0 \tag{1.36d}$$

represents an unsteady one-dimensional temperature distribution.

If the points of a medium with equal temperatures are connected, then the resulting surfaces are called *isothermal surfaces*. The intersection of isothermal surfaces with a plane yields a family of *isotherms* on the plane surface. It is important to note that two isothermal surfaces never cut each other, since no part of the medium can have two different temperatures at the same time.

1.9 Fourier's Law of Heat Conduction

Fourier's law is the basic law governing heat conduction based on the continuum concept. It originated from experimental observations by J. B. Biot,* but was named after the well-known French scientist J. B. J. Fourier,† who used it in his remarkable work, *Théorie Analytique de la Chaleur*, published in Paris in 1822 [3]. This basic law may be illustrated best

* JEAN BAPTISTE V. BIOT, French physicist, was born in Paris on April 21, 1774. Through the influence of Laplace, he became Professor of Mathematical Physics at the Collège de France in 1800. J. B. Biot worked on heat conduction problems in early 1800s, earlier than Fourier. In 1804, he attempted, unsuccessfully, to deal with the problem of incorporating convection effects in the analysis of heat conduction. Fourier read Biot's work and by 1807, determined how to solve the problem. In 1820, with Felix Savart, he discovered the law known as *"Biot–Savart's Law."* He was especially interested in questions relating to the polarization of light and, for his achievements in this field, he was awarded the Rumford Medal of the Royal Society in 1840. He died in Paris on February 3, 1862.

† JEAN BAPTISTE JOSEPH FOURIER. French mathematician and physicist, was born in Auxerre, about 100 miles south of Paris, on March 21, 1768. Orphaned by the age of 9, he was educated at the monastery of Saint-Benoit-Sur-Loire. He taught at Ecole Normale in Paris from its founding in 1795, where his success soon led to the offer of the Chair of Analysis at the Ecole Polytechnique in Paris (1795–98). He was an active supporter of the French Revolution and, after the revolution, he joined the campaign of Napoleon in Egypt (1798–1802). In 1807 he was elected to the Académie des Sciences. He was named a baron by Napoleon in 1808. He spent the final years of his life in Paris where he was Secretary of the Académie des Sciences and succeeded Laplace as President of the Council of the Ecole Polytechnique. Fourier died at the age of 62 on May 16, 1830.

Fourier's fame rests on his mathematical theory of beat conduction. In his treatise *"Théorie Analytique de la Chaleur,"* one of the most important books published in the 19th century, he developed the theory of the series known by his name and applied it to the solution of heat conduction problems. Fourier series are now fundamental tools in science and engineering, and they are extensively used in this book for the solution of various heat conduction problems.

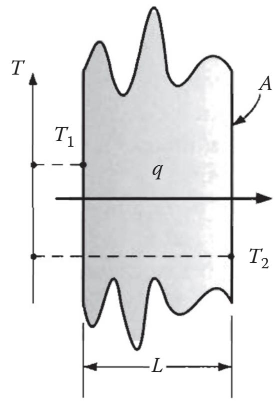

FIGURE 1.2
A flat plate of thickness L.

by considering a simple thought experiment: Consider a solid flat plate of thickness L such that the other two dimensions are very large compared to the thickness L, as shown in Fig. 1.2. Let A be the surface area of the plate, and T_1 and T_2 ($< T_1$) be the temperatures of the two surfaces. Since a temperature difference of $(T_1 - T_2)$ exists between the surfaces, heat will flow through the plate. From the second law of thermodynamics, we know that the direction of this flow is from the higher temperature surface to the lower one. According to the first law of thermodynamics, under steady-state conditions, this flow of heat will be at a constant rate (see Problem 1.3). Experiments with different solids would show that the rate of heat flow q is directly proportional to the temperature difference $(T_1 - T_2)$, the surface area A, and inversely proportional to the thickness L; that is,

$$q \sim A\frac{T_1 - T_2}{L} \tag{1.37}$$

This relation can be rewritten as an equation in the form

$$q = kA\frac{T_1 - T_2}{L} \tag{1.38}$$

where the positive proportionality constant k is called the *thermal conductivity* of the material of the plate. Note that Eq. (1.38) is, in fact, a relation which defines the thermal conductivity; that is,

$$k = \frac{q / A}{(T_1 - T_2) / L} \tag{1.39}$$

Consider now the same plate, and let the temperature of the isothermal surface at x be $T(x)$ and at $x + \Delta x$ be $T(x + \Delta x)$ as shown in Fig. 1.3. The rate of heat transfer through the plate can also be written as

$$q = kA\frac{T(x) - T(x + \Delta x)}{\Delta x} \tag{1.40}$$

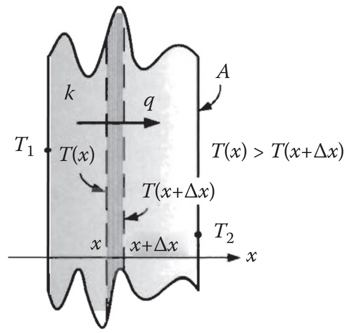

FIGURE 1.3
One-dimensional heat conduction through a flat plate.

If we rewrite Eq. (1.40) as $\Delta x \to 0$, we get

$$q = -kA \lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} \tag{1.41}$$

The limit in this relation is, by definition, the derivative of temperature with respect to the coordinate axis x. Hence, Eq. (1.41) reduces to

$$q = -kA\frac{dT}{dx} \tag{1.42}$$

which is Fourier's law of heat conduction for a one-dimensional steady system.

Thermal conductivity is a thermophysical property and has the units W/(m·K) in the SI system. A medium is said to be *homogeneous* if its thermal conductivity does not vary from point to point within the medium, and *heterogeneous* if there is such a variation. Further, a medium is said to be *isotropic* if its thermal conductivity at any point in the medium is the same in all directions, and *anisotropic* if it exhibits directional variation. Materials having porous structure, such as cork and glass wool, are examples of heterogeneous media, and those having fibrous structure, such as wood or asbestos, are examples of anisotropic media. If the material of the plate in Fig. 1.2 is heterogeneous, then k in Eq. (1.38) would represent the average thermal conductivity over the thickness L or over the temperature difference $(T_1 - T_2)$, and k in Eq. (1.42) would be the local thermal conductivity at x.

The quantity of heat transferred per unit time across a unit area is called the *heat flux*. The unit of heat flux is watts per square meter (W/m^2) in the SI system. Equation (1.42) can now be written in terms of heat flux as

$$q'' = \frac{q}{A} = -k\frac{dT}{dx} \tag{1.43}$$

Equation (1.43) can be interpreted as stating that if there is a negative temperature gradient dT/dx at a location x, then there will be positive heat flow (i.e., in the positive x direction) across the isothermal surface at the same location, and the magnitude of the heat flux q''

across the isothermal surface is given by Eq. (1.43). Since heat flow is considered positive when it is in the positive x direction, the minus sign in Eq. (1.43) is necessary to meet the requirement that heat must flow from a higher to a lower temperature. That is, if the temperature gradient is negative, then the heat flow is positive, and if, on the other hand, the gradient is positive, then the heat flow becomes negative (i.e., in the negative x direction) as illustrated in Fig. 1.4.

The foregoing discussions can now be extended to a medium with a two- or three-dimensional steady temperature distribution. Figure 1.5 shows a set of isothermal surfaces in a body, each differing in temperature by a small amount ΔT. The heat flux due to conduction across the isothermal surface at point P can readily be expressed as

$$q_n'' = -k\frac{\partial T}{\partial n} \tag{1.44}$$

where $\partial/\partial n$ represents the differentiation along the normal to the isothermal surface, which is characterized by the unit vector $\hat{\mathbf{n}}$ pointing in the direction of decreasing temperature.

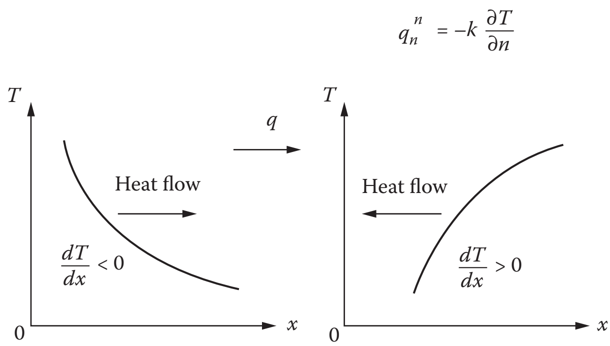

FIGURE 1.4
Interpretation of Fourier's law.

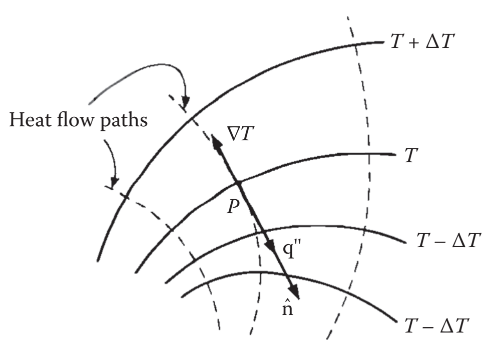

FIGURE 1.5
Isothermal surfaces and heat flow paths in a solid body.

Equation (1.44) can also be written, for example, in the rectangular coordinate system (x, y, z) as

$$q_n'' = -k\left(\frac{\partial T}{\partial x}\frac{dx}{dn} + \frac{\partial T}{\partial y}\frac{dy}{dn} + \frac{\partial T}{\partial z}\frac{dz}{dn}\right) \tag{1.45a}$$

or

$$q_n'' = -k\left(\frac{\partial T}{\partial x}\cos\alpha + \frac{\partial T}{\partial y}\cos\beta + \frac{\partial T}{\partial z}\cos\gamma\right) \tag{1.45b}$$

where (α, β, γ) are the *direction cosines* of the unit vector $\hat{\mathbf{n}}$; that is,

$$\hat{\mathbf{n}} = \hat{\mathbf{i}}\cos\alpha + \hat{\mathbf{j}}\cos\beta + \hat{\mathbf{k}}\cos\gamma \tag{1.46}$$

Therefore, using vector calculus. Eq. (1.45*b*) can be rewritten as

$$q_n'' = -k\left(\hat{\mathbf{i}}\frac{\partial T}{\partial x} + \hat{\mathbf{j}}\frac{\partial T}{\partial y} + \hat{\mathbf{k}}\frac{\partial T}{\partial z}\right)\cdot\hat{\mathbf{n}} \tag{1.47a}$$

or

$$q_n'' = -k\nabla T\cdot\hat{\mathbf{n}} \tag{1.47b}$$

where

$$\nabla T = \hat{\mathbf{i}}\frac{\partial T}{\partial x} + \hat{\mathbf{j}}\frac{\partial T}{\partial y} + \hat{\mathbf{k}}\frac{\partial T}{\partial z} \tag{1.48}$$

is the *gradient* of the temperature distribution.

The gradient of the temperature distribution ∇T at point P is a vector normal to the isothermal surface passing through P, which points in the direction of increasing temperature as shown in Fig. 1.5 (see Problem 1.6). The following are expressions for ∇T in various coordinate systems:

***Rectangular coordinates* (x, y, z):**

$$\nabla T = \hat{\mathbf{i}}\frac{\partial T}{\partial x} + \hat{\mathbf{j}}\frac{\partial T}{\partial y} + \hat{\mathbf{k}}\frac{\partial T}{\partial z}$$

***Cylindrical coordinates* (r, ϕ, z):**

$$\nabla T = \hat{\mathbf{e}}_r\frac{\partial T}{\partial r} + \hat{\mathbf{e}}_\phi\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{\mathbf{e}}_z\frac{\partial T}{\partial z}$$

$$\hat{\mathbf{e}}_r = \hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi$$
$$\hat{\mathbf{e}}_\phi = -\hat{\mathbf{i}}\sin\phi + \hat{\mathbf{j}}\cos\phi$$
$$\hat{\mathbf{e}}_z = \hat{\mathbf{k}}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$, and $\hat{\mathbf{e}}_z$ are the unit vectors in r, ϕ, and z directions, respectively (see Fig. 2.2).

***Spherical coordinates* (*r*, θ, ϕ):**

$$\nabla T = \hat{\mathbf{e}}_r \frac{\partial T}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial T}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$
$$\hat{\mathbf{e}}_r = \hat{\mathbf{i}}\sin\theta\cos\phi + \hat{\mathbf{j}}\cos\theta\sin\phi + \hat{\mathbf{k}}\cos\theta$$
$$\hat{\mathbf{e}}_\theta = \hat{\mathbf{i}}\cos\theta\cos\phi + \hat{\mathbf{j}}\cos\theta\sin\phi + \hat{\mathbf{k}}\sin\theta$$
$$\hat{\mathbf{e}}_\phi = -\hat{\mathbf{i}}\sin\phi + \hat{\mathbf{j}}\cos\phi$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\phi$ are the unit vectors in r, θ, and ϕ directions, respectively (see Fig. 2.3).

We now define a *heat flux vector* normal to the isothermal surface at P and pointing in the direction of decreasing temperature, as shown in Fig. 1.5, by

$$\mathbf{q}'' = \hat{\mathbf{n}} q_n'' \tag{1.49}$$

It can be shown from Eq. (1.47*b*) that

$$\mathbf{q}'' = -k\nabla T \tag{1.50}$$

which is the *vector form* of Fourier's law.

Fourier's law, Eq. (1.50), indicates that heat is transferred by conduction in a medium in the direction normal to isothermal surfaces from the higher temperature to the lower one. This law is, in fact, well established for heat conduction in isotropic solids, and practical applications of it for various problems require the laboratory measurement of the thermal conductivity of representative specimens. Equation (1.50) is also used in unsteady-state problems as a valid particular law as it has never been refuted.

The magnitude of the heat flux across any arbitrary surface passing through P and having the unit direction vector $\hat{\mathbf{s}}$ as its normal will be equal to the magnitude of the component of $\mathbf{q}''$ in the s direction; that is,

$$q_s'' = \mathbf{q}'' \cdot \hat{\mathbf{s}} = k\nabla T \cdot \hat{\mathbf{s}} \tag{1.51}$$

Since it is also true (see Problem 1.7) that

$$\nabla T \cdot \hat{\mathbf{s}} = \frac{\partial T}{\partial s} \tag{1.52}$$

Eq. (1.51) can also be written as

$$q_s'' = -k\frac{\partial T}{\partial s} \tag{1.53}$$

where $\partial/\partial s$ represents the differentiation in the direction of the normal $\hat{\mathbf{s}}$.

In the rectangular coordinate system, for example, the three components of the heat flux vector $\mathbf{q}''$ are given by

$$q_x'' = -k\frac{\partial T}{\partial x}, \qquad q_y'' = -k\frac{\partial T}{\partial y}, \qquad \text{and} \qquad q_z'' = -k\frac{\partial T}{\partial z} \tag{1.54a,b,c}$$

which are the magnitudes of the heat fluxes at P across the surfaces perpendicular to the directions x, y, and z, respectively.

In anisotropic solids, the heat flux vector may not necessarily be parallel to the temperature gradient ∇T. That is, the heat flux due to conduction in a given direction can also be proportional to the temperature gradients in other directions, and therefore Eq. (1.44), or Eq. (1.50), may not be valid. Fourier's law can be generalized for anisotropic media by assuming each component of the heat flux vector at a point to be linearly dependent on all components of the temperature gradient at that point. Thus, referred to a set of rectangular axes ox_1, ox_2, and ox_3, the components of the heat flux vector can be written as

$$q_1'' = -k_{11}\frac{\partial T}{\partial x_1} - k_{12}\frac{\partial T}{\partial x_2} - k_{13}\frac{\partial T}{\partial x_3} \tag{1.55a}$$

$$q_2'' = -k_{21}\frac{\partial T}{\partial x_1} - k_{22}\frac{\partial T}{\partial x_2} - k_{23}\frac{\partial T}{\partial x_3} \tag{1.55b}$$

$$q_3'' = -k_{31}\frac{\partial T}{\partial x_1} - k_{32}\frac{\partial T}{\partial x_2} - k_{33}\frac{\partial T}{\partial x_3} \tag{1.55c}$$

where k_{ij} are the *thermal conductivity coefficients*. They are the components of the *thermal conductivity tensor*,

$$[k_{ij}] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{1.56}$$

Equations (1.55) can be written in a more compact form by using the Cartesian tensor notation as

$$q_i'' = -k_{ij}\frac{\partial T}{\partial x_j}, \qquad i, j = 1, 2, 3 \tag{1.57}$$

In this book we limit our discussions to heat conduction in isotropic media only. Those who are interested in heat conduction in anisotropic media may refer, for example, to Reference [13].

Fourier's law assumes that heat is propagated with an infinite speed. Therefore, it cannot be applied to certain physical situations, such as in heat transfer at the nanoscale or in extremely fast phenomena, for instance, in pulsating laser heating. For such cases, the non-Fourier heat conduction model considers that a relaxation time exists, which describes the time lag in the response of the heat flux to a temperature gradient. This thermal relaxation time is extremely short compared to the time scales of events in most engineering applications, thus making the classical Fourier model applicable in the majority of the heat transfer problems. The reader, however, is referred to recent textbooks on microscale and nanoscale heat transfer, such as Ref. [19], for an in-depth discussion of this subject.

1.10 Thermal Conductivity

Thermal conductivity is a thermophysical property, and it can be interpreted from Eq. (1.39) as being equal to the heat transfer rate across a unit area through a unit thickness per unit temperature difference. The magnitude of thermal conductivity varies over wide ranges for different materials; for example, from 0.0152 W/(m·K) for carbon dioxide at 300 K to 429 W/(m·K) for pure silver at 300 K. Figure 1.6 delineates the range of thermal conductivity of various substances at normal temperatures and pressures. The thermal conductivity of a material depends on its chemical composition, physical structure, and state. It also varies with the temperature and pressure to which the material is subjected. In most cases, however, thermal conductivity is much less dependent on pressure than on temperature, so that the dependence on pressure may be neglected and thermal conductivity can be tabulated as a function of temperature. In some cases, thermal conductivity may also vary with the direction of heat flow; these are referred to as anisotropic materials.

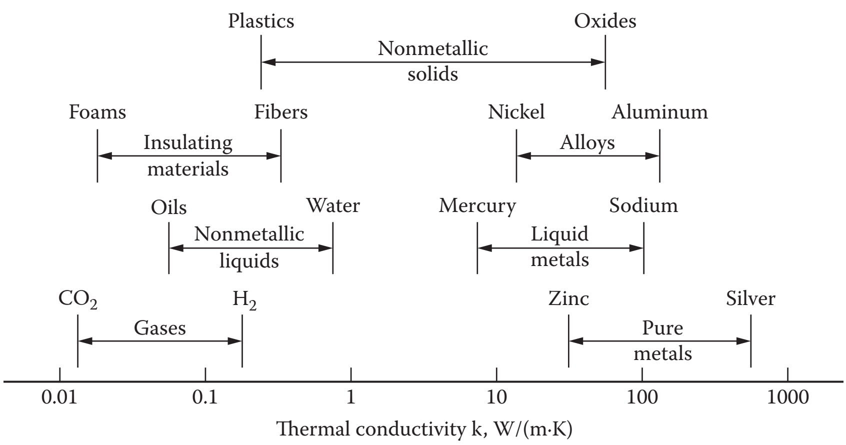

FIGURE 1.6
Range of thermal conductivity of various substances at normal pressure.

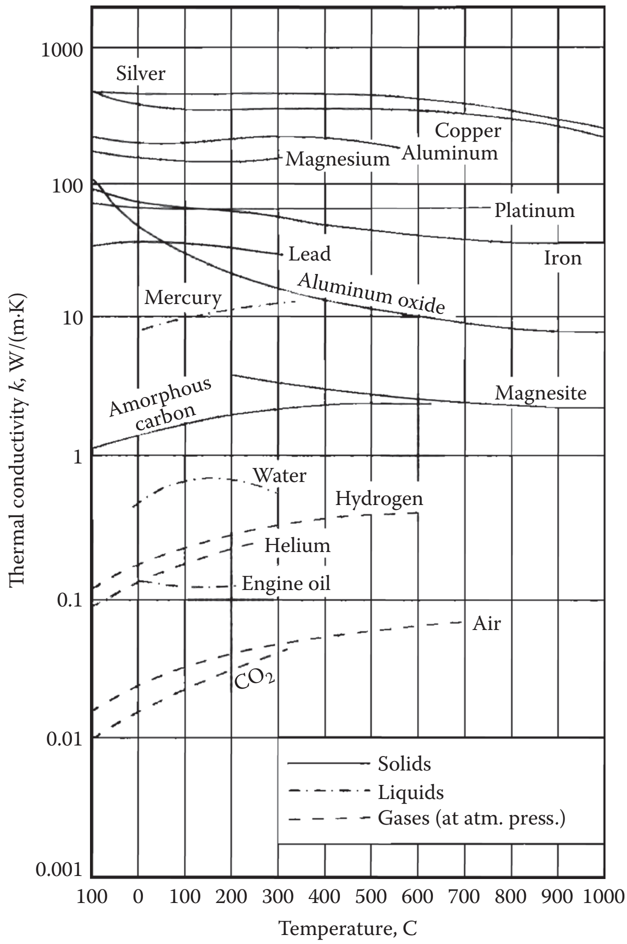

FIGURE 1.7
Variation of thermal conductivity of various substances with temperature.

Figure 1.7 gives the thermal conductivity as a function of temperature for various substances. Obviously, the variation of thermal conductivity with temperature may be neglected when the temperature range under consideration is not too large or the dependence of thermal conductivity on temperature is not too severe. For numerous materials, especially within a small temperature range, the variation of thermal conductivity with temperature can be represented by the linear function

$$k(T) = k_R[1 + \gamma(T - T_R)] \tag{1.58}$$

where $k_R = k(T_R)$, T_R is a reference temperature, and γ is a constant called the *temperature coefficient of thermal conductivity.*

Heat conduction in gases and vapors depends mainly on the molecular transfer of kinetic energy of the molecular movement. That is, heat conduction is transmission of kinetic energy by the more active molecules in high temperature regions to the molecules in low molecular kinetic energy regions by successive collisions. According to the kinetic theory of gases, the temperature of an element of gas is proportional to the mean kinetic energy of its constituent molecules. Clearly, the faster the molecules move, the faster they will transfer energy. This implies, therefore, that thermal conductivity of a gas should be dependent on its temperature. For gases at moderately low temperatures, kinetic theory of gases may be used to accurately predict the experimentally observed values. A very simple model of kinetic theory (traffic model) leads to the following approximate relation for gases [14]:

$$k = \frac{\rho c_v \bar{V} \lambda}{3} \tag{1.59}$$

where ρ is the gas density, c_v is the specific heat at constant volume, $\bar{V}$ is the mean molecular velocity, and λ is the molecular mean free path between collisions. A more accurate analysis gives a numerical constant in Eq. (1.59) slightly different from 1/3.

In liquids, molecules are more closely spaced than in gases, and therefore molecular force fields exert a strong influence on the energy exchange during molecular collisions. Because of this, liquids have much higher values of thermal conductivity than gases. The thermal conductivity of nonmetallic liquids generally decreases with increasing temperature; water and glycerine are exceptions. Liquid metals, which are useful as heat transfer media in nuclear reactors where high heat removal rates are essential, have relatively high thermal conductivities.

Solid materials may have solely *crystalline structures* (such as quartz), may be in *amorphous* solid state (such as glass), may be a mixture of the two, or may be somewhat porous in structure with air or other gases in the pores. Heat conduction in solids with crystalline structures depends on the energy transfer by molecular and lattice vibrations and free electrons. In general, energy transfer by molecular and lattice vibrations is not as large as the energy transported by free electrons. It is for this reason that good electrical conductors are almost always good heat conductors, while electrical insulators are usually good heat insulators. Materials having high thermal conductivities are called *conductors*, while materials of low thermal conductivity are referred to as *insulators*.

In the case of amorphous solids, heat conduction depends on the molecular energy transport. Thus, thermal conductivities of such solids are of the same order of magnitude as those observed for liquids. That is, amorphous solids have smaller thermal conductivities than solids with crystalline structure. The thermal conductivities of amorphous solids increase with temperature.

For pure crystalline metals, the ratio of the thermal conductivity k to electrical conductivity k_e is found to be nearly proportional to the absolute temperature. A modified Lorenz equation expressing this relation is $k/k_e = 783 \times 10^{-9}\ T$, where T is in°R [14]. This equation does not hold for amorphous materials or alloys of metals. Thermal conductivities of alloys may be less than that of any constituent; for example, constantan is an alloy of 55% copper (Cu) and 45% nickel (Ni) and has $k = 23$ W/(m·K), while for pure copper $k = 401$ W/(m·K) and for nickel $k = 90.7$ W/(m·K). Thermal conductivities of selected typical solids are given in Appendix A.

Solids containing pores filled with gases exhibit rather low values of thermal conductivity compared to more dense nonporous materials. In general, thermal conductivities of solids increase with density. They also increase with moisture content. Moisture in solids presents a special problem in the presence of a temperature gradient. It usually migrates toward colder regions, thus changing the thermal conductivity and, perhaps, damaging the material or the surrounding structure. For solids with porous structure or loosely packed fibrous materials (such as cork and glass wool), one can only talk about an *apparent* thermal conductivity. The apparent thermal conductivities of such materials usually go through a minimum value as the apparent density decreases. At very low densities, the gas spaces may be so large that an internal convection flow may result which, in turn, increases heat transfer and, therefore, thermal conductivity. In cellular or porous type materials, internal radiation may also be important. If internal radiation is very significant, then the curve of k versus T will be concave upward.

1.11 Newton's Cooling Law

Convection has already been defined in Section 1.2 as the process of heat transport in a fluid by the combined action of heat conduction (and radiation) and macroscopic fluid motion. As a mechanism of heat transfer it is important not only between the layers of a fluid but also between a fluid and a solid surface when they are in contact.

When a fluid flows over a solid surface, as illustrated in Fig. 1.8*a*, the fluid particles adjacent to the surface stick to it and, therefore, have zero velocity relative to the surface. Other fluid particles attempting to slide over the stationary ones at the surface are retarded as a result of viscous forces (i.e., friction) between the fluid particles. The velocity of the fluid particles thus asymptotically approaches that of the undisturbed free stream over a short distance δ (*velocity boundary-layer thickness*) from the surface.

As further illustrated in Fig. 1.8*b*, if $T_w > T_\infty$, then heat will flow from the solid to the fluid particles at the surface. The energy thus transmitted increases the internal energy of the fluid particles (*sensible* heat storage) and is carried away by the motion of the fluid. The temperature distribution in the fluid adjacent to the surface will then appear as shown in Fig. 1.8*b*, asymptotically approaching the free-stream value T_∞ in a short distance δ_T (*thermal boundary-layer thickness*) from the surface.

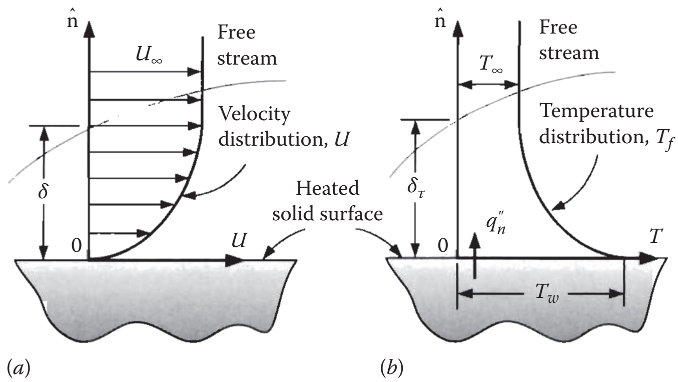

FIGURE 1.8
Velocity (*a*) and thermal (*b*) boundary layers along a solid surface.

Since the fluid particles at the surface are stationary, the heat flux from the surface to the fluid will be given by

$$q_n'' = -k_f \left(\frac{\partial T_f}{\partial n} \right)_w \tag{1.60}$$

where k_f is the thermal conductivity of the fluid, T_f is the temperature distribution in the fluid, the subscript w means that the derivative is evaluated at the surface, and n denotes the normal direction from the surface.

In 1701, Newton* expressed the heat flux from a solid surface to a fluid by the equation

$$q_n'' = h(T_w - T_\infty) \tag{1.61}$$

where h is called *heat transfer coefficient*, *film conductance*, or *film coefficient*. In the literature, Eq. (1.61) is known as *Newton's law of cooling*. In fact, it is a relation which defines the heat transfer coefficient; that is.

$$h = \frac{q_n''}{T_w - T_\infty} = \frac{-k_f (\partial T_f / \partial n)_w}{T_w - T_\infty} \tag{1.62}$$

The heat transfer coefficient has the units W/(m^2·K) in the SI system. Note that h is also given by

$$h = \frac{-k_s (\partial T_s / \partial n)_w}{T_w - T_\infty} \tag{1.63}$$

where k_s is the thermal conductivity of the solid and T_s is the temperature distribution in the solid.

If the fluid motion involved in the process is induced by some external means such as a pump, blower, or fan, then the process is referred to as *forced convection*. If the fluid motion is caused by any body force within the system, such as those resulting from the density gradients near the surface, then the process is called *natural* (or *free*) *convection*. Certain convective heat transfer processes, in addition to *sensible* heat storage, may also involve *latent* heat storage (or release) due to phase change. Boiling and condensation are two such cases.

The heat transfer coefficient is actually a complicated function of the flow conditions, thermophysical properties (i.e., viscosity, thermal conductivity, specific heat and density)

* SIR ISAAC NEWTON. English mathematician and natural philosopher (physicist), was born on January 4, 1643. He is considered to be the greatest scientist who ever lived. His accomplishments in mathematics, optics and physics laid the foundations for modern science and revolutionized the world. He studied at Cambridge and was a professor there from 1669 to 1701. His most important discoveries were made during the two-year period from 1664 to 1666. During this period he discovered the law of universal gravitation, discovered that white light is composed of all colors of the spectrum, and began to study calculus. Newton summarized his discoveries in terrestrial and celestial mechanics in his "Philosophiae Naturalis Principia Mathematica (1687)," one of the greatest milestones in the history of science. He outlined his discoveries in optics in his "Opticks (1704)," in which he elaborated his theory that light is composed of particles. In his later years, Newton considered mathematics and physics a recreation and turned much of his energy toward theology, history and alchemy. He died in London on March 31, 1727.

TABLE 1.1

Approximate Values of *h*, W/(m²·K)

Fluid	Free Convection	Forced Convection
Gases	5–30	30–300
Water	30–300	300–10,000
Viscous oils	5–100	30–3,000
Liquid metals	50–500	500–20,000
Boiling water	2,000–20,000	3,000–100,000
Condensing water vapor	3,000–30,000	3,000–200,000

of the fluid, and geometry and dimensions of the surface. Its numerical value, in general, is not uniform over the surface. Table 1.1 gives the *order of magnitude* of the range of values of the heat transfer coefficient under various conditions.

1.12 Stefan–Boltzmann Law of Radiation

As mentioned in Section 1.2, in contrast to the mechanisms of conduction and convection where energy transfer through a material medium takes place, experimental observations show that energy may also be transported in the absence of a physical medium. This transfer takes place in the form of *electromagnetic waves* and is known as *thermal radiation*.

All substances, solid bodies as well as liquids and gases, at normal and especially at elevated temperatures emit energy as a result of their temperature in the form of electromagnetic waves, and are also capable of absorbing such energy. When radiation is incident on a body, part of it is reflected by the surface. The remainder penetrates into the body, which may then be absorbed as it travels through the body as illustrated in Fig. 1.9. If the material of the body is a strong absorber of thermal radiation, then the energy that penetrates into the body will all be absorbed and converted into internal energy within a very thin layer adjacent to the surface. Such a body is called *opaque*. If the material thickness required to substantially absorb radiation is large compared to the thickness of the body,

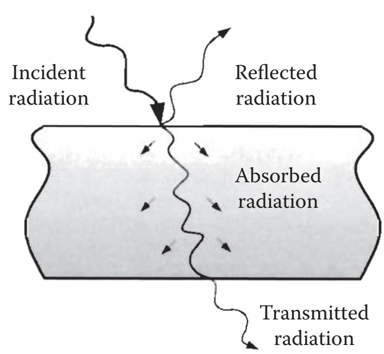

FIGURE 1.9
Absorption, reflection, and transmission of incident radiation.

then most of the radiation will pass through the body without being absorbed, and such a body is called *transparent*.

When radiation impinges on a surface, the fraction that is reflected back is defined as the *reflectivity* ρ, the fraction absorbed is the *absorptivity* α, and the fraction transmitted is the *transmissivity* τ. Thus,

$$\rho + \alpha + \tau = 1 \tag{1.64}$$

For opaque substances, $\tau = 0$, and Eq. (1.64) reduces to

$$\rho + \alpha = 1 \tag{1.65}$$

An ideal body which absorbs all the impinging radiation energy without reflection and transmission is called a *blackbody*. Therefore, for a blackbody, Eq. (1.64) reduces to $\alpha = 1$. Only a few materials, such as those painted with carbon black and platinum black, approach the blackbody in their ability to absorb radiation energy. A blackbody also emits the maximum possible thermal radiation [12,17]. The total emission of radiation per unit surface area and per unit time from a blackbody is related to the fourth power of the absolute temperature T of the surface by the *Stefan–Boltzmann law of radiation*, which is

$$q''_{r,b} = \sigma T^4 \tag{1.66}$$

where σ is the *Stefan–Boltzmann constant* with the value 5.6697×10^{-8} W/(m²·K⁴) in the SI system. The basic equation (1.66) for the total thermal radiation from a blackbody was proposed by Stefan* in 1879 based on experimental evidence, and developed theoretically by Boltzmann† in 1884.

Real bodies (surfaces) do not meet the specifications of a blackbody, but emit radiation at a lower rate than a blackbody of the same size and shape and at the same temperature. If q''_r is the radiative flux (i.e., radiation emitted per unit surface area and per unit time) from

* JOSEF STEFAN, Austrian physicist, was born on March 24, 1835. Stefan was educated at the University of Vienna. After receiving his PhD degree in 1858, he was appointed Privatdozent of mathematical physics and in 1863 he became Professor Ordinarius of physics there. In 1866 he became Director of the Physical Institute at Vienna. He was a distinguished member of the Academy of Sciences Vienna, of which he was appointed secretary in 1875. Stefan's contributions ranged over several important fields of science and engineering, including the kinetic theory of gases, hydrodynamics and, in particular, radiation. Before Stefan's work, G. R. Kirchhoff had already described the *"blackbody."* Stefan showed empirically, in 1879, that the radiation from such a body was proportional to the fourth power of its absolute temperature. In 1891, Stefan published his work on the formation of Polar ice, giving a special solution of this nonlinear conduction problem with phase change (see Chapter 12). He died in Vienna on January 7, 1893.

† LUDWIG BOLTZMANN, born on February 20, 1844 in Vienna, Austria, was awarded a doctorate degree from the University of Vienna in 1866 for a thesis on the kinetic theory of gases supervised by Josef Stefan. After receiving his doctorate degree, he became an assistant to his advisor Josef Stefan. He was one of the first to recognize the importance of Maxwell's electromagnetic theory, and obtained, in 1871, the Maxwell–Boltzmann distribution, namely the average energy of motion of a molecule is the same for each direction. In 1884, Boltzmann derived Stefan's empirical T^4 law for blackbody radiation from the principles of thermodynamics. Boltzmann's fame is mostly due to his work on statistical mechanics using probability to describe how the properties of atoms determine the properties of matter. In particular, his work relates to the second law of thermodynamics which he derived from the principles of mechanics. Depressed and in bad health, Boltzmann committed suicide in Duino near Trieste, Austria (now Italy) on October 5, 1906.

a real surface maintained at the absolute temperature T then the *emissivity* of the surface is defined as

$$\varepsilon = \frac{q_r''}{\sigma T^4} \tag{1.67}$$

Thus, for a blackbody, $\varepsilon = 1$. For a real body exchanging radiation only with other bodies at the same temperature (i.e., for thermal equilibrium) it can be shown that $\alpha = \varepsilon$, which is a statement of *Kirchhoff's law* in thermal radiation [17]. The magnitude of emissivity varies from material to material, and for a given material it depends on the state, temperature, and surface conditions of the material. Emissivities of various typical materials are given in Appendix A.

If two isothermal surfaces A_1 and A_2, having emissivities ε_1 and ε_2 and absolute temperatures T_1 and T_2, respectively, exchange heat by radiation only, then the net rate of heat exchange between these two surfaces is given by

$$q_r = \sigma A_1 \Im_{12}\left(T_1^4 - T_2^4\right) \tag{1.68}$$

where Kirchhoff's law is assumed to be valid. If A_1 and A_2 are two large parallel surfaces with negligible heat losses from the edges as shown in Fig. 1.10*a*, then the factor $\Im_{12}$ in Eq. (1.68) is given by

$$\frac{1}{\Im_{12}} = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1 \tag{1.69}$$

If A_1 is completely enclosed by the surface A_2 as shown in Fig. 1.10*b*, then

$$\frac{1}{\Im_{12}} = \frac{1}{F_{12}} + \frac{1}{\varepsilon_1} - 1 + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right) \tag{1.70}$$

where F_{12} is a purely geometric factor called *radiation shape factor* or *view factor* between the surfaces A_1 and A_2, and is equal to the fraction of the radiation leaving surface A_1

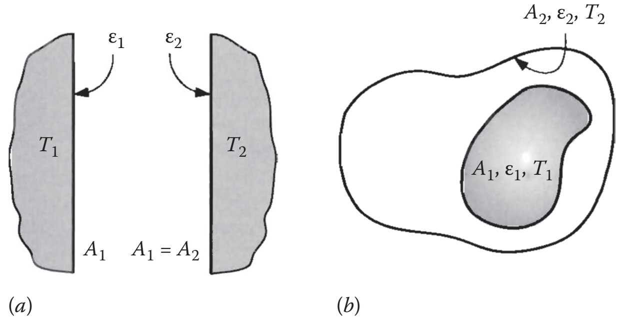

FIGURE 1.10
Two isothermal surfaces A_1 and A_2 exchanging heat by radiation. (*a*) A_1 and A_2 are two large parallel surfaces. (*b*) A_1 is completely enclosed by surface A_2

that directly reaches surface A_2. Radiation shape factors, in the form of equations and/or charts, can be found in the literature [12,17]. For the surfaces A_1 and A_2, it is obvious that

$$\sum_{j=1}^{2} F_{ij} = 1, \quad i = 1.2 \tag{1.71}$$

Obviously, if A_i is a completely convex or a plane surface, then $F_{ii} = 0$, and Eq. (1.70) reduces to

$$\frac{1}{\Im_{12}} = \frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right) \tag{1.72}$$

In certain applications it may be convenient to define a *radiation heat transfer coefficient, h_r*, by

$$q_r = h_r A_1 (T_1 - T_2) \tag{1.73}$$

When this is applied to Eq. (1.68), h_r is given by

$$h_r = \sigma \Im_{12} (T_1 + T_2)\left(T_1^2 + T_2^2\right) \tag{1.74}$$

For configurations involving more than two surfaces, the evaluation of heat transfer by radiation becomes involved, and interested readers may refer to books on thermal radiation such as References [12,17] for more information.

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Problems

1.1 In most commonly encountered engineering problems involving steady flows of fluids through various devices, the inlet and outlet flows are usually regarded as one dimensional. Develop the first law of thermodynamics (i.e., the steady-state steady-flow energy equation) for the device illustrated in Fig. 1.11.

1.2 Apply the first law of thermodynamics, Eq. (1.30), to a finite volume element in a stationary solid. Let $T(\mathbf{r}, t)$ be the temperature distribution in the solid. Simplify the relation you obtain to reduce it to an expression which relates the temperature distribution to heat fluxes in the solid.

1.3 Prove that, under steady-state conditions and in the absence of internal energy sources or sinks, the rate of heat transfer by conduction through a solid flat plate whose surfaces are maintained at constant temperatures T_1 and T_2 is constant.

1.4 Obtain an expression for the steady-state temperature distribution in the flat plate of Problem 1.3 if the thickness of the plate is L and its thermal conductivity k is constant.

1.5 The plane wall shown in Fig. 1.12 has one surface maintained at T_1 and the other at T_2. The temperature at the center plane is measured to be T_3, and the rate of heat flow through the wall is q. Assuming that the thermal conductivity of the wall varies linearly with temperature, find an expression for the thermal conductivity as a function of temperature and the rate of heat flow through the wall.

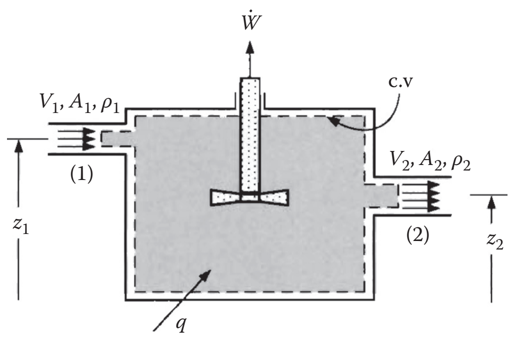

FIGURE 1.11
Figure for Problem 1.1.

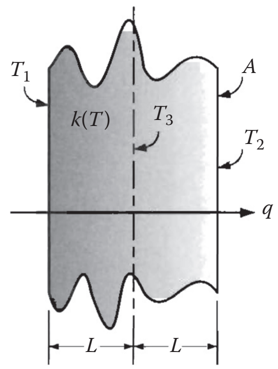

FIGURE 1.12
Figure for Problem 1.5.

1.6 Show that the gradient of the temperature distribution, ∇T at point P shown in Fig. 1.5, is a vector normal to the isothermal surface passing through P and pointing in the direction of increasing temperature.

1.7 Obtain the relation given by Eq. (1.52).

1.8 The steady-state temperature distribution in a flat plate is given by

$$T(°\mathrm{C}) = 150 - 400x^2$$

where x is the distance in meters (m) along the width of the plate and is measured from the surface at 150°C. Determine the heat fluxes at the two surfaces of the plate. The thermal conductivity of the wall material is 40 W/(m·K), and the thickness of the wall is 0.25 m.

1.9 The instantaneous temperature distribution in a plane wall at a specific time t_1 during a transient heat transfer process is shown in Fig. 1.13. Is the wall being heated or cooled at t_1? Explain.

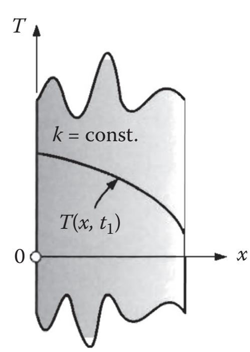

FIGURE 1.13
Figure for Problem 1.9.

1.10 The temperature profile at a location in water flowing over a flat surface is experimentally measured to be

$$T(°C) = 20 + 80e^{-800y}$$

where y is the distance in meters (m) measured normal to the surface with $y = 0$ corresponding to the surface. What is the value of heat transfer coefficient at this location? Assume that the thermal conductivity of water is $k = 0.62$ W/(m·K).

1.11 Estimate the equilibrium temperature of a long rotating cylinder of diameter D, and oriented in space with its axis normal to the sun's rays. The cylinder is at a location in space where the irradiation from the sun (i.e., energy incident on a surface perpendicular to the sun's rays per unit time per unit area) is 1500W/m^2. Assume that the absorptivity of the surface of the cylinder to solar radiation, α_s, is equal to its emissivity ε. and the outer space is a blackbody at 0 K.

1.12 A freighter vessel is adapted for the transportation of liquefied gas using spherical tanks. The tanks have a radius of 5 m and receive liquefied natural gas at a temperature of −160°C. Assuming a heat transfer coefficient $h = 25$ W/m^{2}C on the external surface of the tank (without insulation) and the outside air at 20°C, what is the heat transfer rate through the tank external surface, assuming the internal temperature in the tank wall remains at −160°C and neglecting the thermal resistance of the tank wall?

2

General Heat Conduction Equation

2.1 Introduction

The mechanism of heat conduction, as discussed in Chapter 1, is visualized as the exchange of kinetic energy at the molecular level between the microparticles in the high and low temperature regions. In phenomenological heat conduction studies, however, the molecular structure of the medium is disregarded and the medium is considered to be a continuum. Analytical investigations into heat conduction based on the continuum concept usually start with the derivation of the heat conduction equation. Expressed by a differential equation, the heat conduction equation is a mathematical expression which relates temperature to time and space coordinates.

2.2 General Heat Conduction Equation

To derive the general heat conduction equation we consider a stationary and opaque solid as shown in Fig. 2.1*a*. Let $T(\mathbf{r}, t)$ represent the temperature distribution in this solid, and k and ρ be the thermal conductivity and density of the solid, respectively, both of which may be functions of space coordinates and/or temperature. Consider a point P at any location $\mathbf{r}$ in the solid. Suppose that the point P is enclosed by any surface S lying entirely within the solid. Let υ be the volume of the space enclosed by S, as illustrated in Fig. 2.1*b*.

Assume that internal energy is generated in the solid due to a power drawn from an external electric circuit at a rate of $\dot{q}_e = \dot{q}_e(\mathbf{r}, t)$ per unit volume. Since

$$\mathbf{V} = 0, \quad \dot{W}_{\text{shaft}} = 0 \quad \text{and} \quad \dot{W}_{\text{shear}} = 0$$

the first law of thermodynamics given by Eq. (1.30) reduces to

$$\int_{\upsilon} \rho \frac{\partial e}{\partial t} d\upsilon = q_s + \int_{\upsilon} \dot{q}_e \, d\upsilon \tag{2.1a}$$

where e is the *total energy* per unit mass (or the specific energy) of the solid and q_s represents the net rate of heat conducted into the volume υ across its bounding surface S. Since

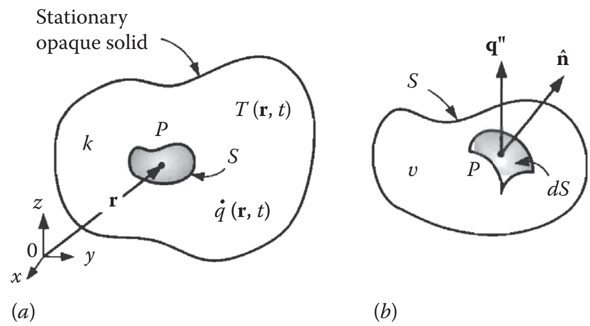

FIGURE 2.1
(a) A stationary and opaque solid, and (b) the control volume υ for the derivation of the general heat conduction equation.

the solid is stationary and there are no nuclear and chemical reactions, $de = du$, where u is the *internal energy* per unit mass of the solid. Therefore, Eq. (2.1a) can also be written as

$$\int_{\upsilon} \rho \frac{\partial u}{\partial t} d\upsilon = q_s + \int_{\upsilon} \dot{q}_e \, d\upsilon \tag{2.1b}$$

In general, for a substance that is *homogeneous* and *invariable* in composition

$$du = \left(\frac{\partial u}{\partial \upsilon} \right)_T d\upsilon + c_\upsilon dT \tag{2.2a}$$

and

$$dh = \left(\frac{\partial h}{\partial p} \right)_T dp + c_p dT \tag{2.2b}$$

where h is the *enthalpy* per unit mass and p is the pressure. In addition, υ (= $1/\rho$) denotes *specific volume*, and c_υ and c_p are the *specific heats* at *constant volume* and *pressure*, respectively. For solids (and incompressible fluids) the specific volume υ = constant. On the other hand, if the pressure p = constant, then from Eq. 1.32 we have

$$dh = du \tag{2.3}$$

Therefore, for solids (and incompressible fluids) from Eqs. (2.2) we get

$$c_\upsilon = c_p = c \tag{2.4}$$

If p is not constant, then Eq. (2.4) still holds, but only approximately, since the difference ($c_p - c_\upsilon$) for solids (and incompressible fluids) is negligibly small.

Thus, introducing $du = c\,dT$ in Eq. (2.1*b*) we get

$$\int_{\upsilon} \rho c \frac{\partial T}{\partial t} d\upsilon = q_s + \int_{\upsilon} \dot{q}_e \, d\upsilon \tag{2.5}$$

In a *fissionable* material, internal energy is generated as a result of nuclear reactions which consist of continuous changes in the composition of the material as fissionable material is turned into internal energy. Since these composition changes are generally small, the effects on the thermophysical properties of such a material can be assumed to be insignificant. Although this internal energy generation in a fissionable material cannot be identified as a power input from an external power source, the time rate of change of internal energy per unit mass, in the absence of chemical reactions, can be expressed as

$$\rho \frac{\partial u}{\partial t} = \rho c \frac{\partial T}{\partial t} - \dot{q}_n \tag{2.6a}$$

where $\dot{q}_n = \dot{q}_n(\mathbf{r}, t)$ represents the rate of internal energy generation per unit volume due to nuclear reactions.

If the solid under consideration is a fissionable material and the internal energy generation is solely due to nuclear reactions, then substituting Eq. (2.6*a*) into Eq. (2.1*b*) we get

$$\int_{\upsilon} \rho c \frac{\partial T}{\partial t} d\upsilon = q_s + \int_{\upsilon} \dot{q}_n \, d\upsilon \tag{2.6b}$$

A similar argument can also be given for the case of internal energy sources or sinks resulting from *exothermic* and *endothermic* chemical reactions. Hence, Eq. (2.5), or Eq. (2.6*b*), can, in general, be written as

$$\int_{\upsilon} \rho c \frac{\partial T}{\partial t} d\upsilon = q_s + \int_{\upsilon} \dot{q} \, d\upsilon \tag{2.7}$$

where $\dot{q}(\mathbf{r}, t)$ represents the rate of internal energy generation in the solid per unit volume, and this generation may be due to electrical, nuclear, chemical, as well as other sources, such as infrared sources.

The term q_s in Eq. (2.7), which represents the net rate of heat conducted into the volume υ across its bounding surface S, can be written as

$$q_s = -\int_S \mathbf{q}'' \cdot \hat{\mathbf{n}} \, dS \tag{2.8}$$

where fi is the outward-drawn unit vector normal to the surface element dS, as indicated in Fig. 2.1*b*, and $\mathbf{q}''$ is the heat flux vector due to conduction. Hence, substituting Eq. (2.8) into Eq. (2.7) yields

$$\int_{\upsilon} \rho c \frac{\partial T}{\partial t} d\upsilon = -\int_S \mathbf{q}'' \cdot \hat{\mathbf{n}} \, dS + \int_{\upsilon} \dot{q} \, d\upsilon \tag{2.9}$$

The surface integral in the above equation can be converted into a volume integral by using the *divergence theorem* as [2]. Thus,

$$\int_S \mathbf{q}'' \cdot \hat{\mathbf{n}}\, dS = \int_\upsilon \nabla \cdot \mathbf{q}''\, d\upsilon \tag{2.10}$$

Substituting Eq. (2.10) into Eq. (2.9), we get

$$\int_\upsilon \rho c \frac{\partial T}{\partial t} d\upsilon = -\int_U \nabla \cdot \mathbf{q}''\, d\upsilon + \int_\upsilon \dot{q}\, d\upsilon \tag{2.11a}$$

or

$$\int_\upsilon \left[\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}'' - \dot{q} \right] d\upsilon = 0 \tag{2.11b}$$

Since the integral in the above relation vanishes for every volume element υ, its integrand must vanish everywhere, thus yielding

$$-\nabla \cdot \mathbf{q}'' + \dot{q} = \rho c \frac{\partial T}{\partial t} \tag{2.12}$$

Assume that the solid under consideration is *isotropic*. Fourier's law then gives

$$\mathbf{q}'' = -k\nabla T \tag{2.13}$$

Substituting the relation (2.13) into Eq. (2.12), we obtain

$$\nabla \cdot (k\nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t} \tag{2.14}$$

This relation is referred to as the *general heat conduction equation* for isotropic solids, which may be rearranged to give

$$k\nabla^2 T + \nabla k \cdot \nabla T + \dot{q} = \rho c \frac{\partial T}{\partial t} \tag{2.15}$$

where $\nabla^2 = \nabla \cdot \nabla$ is the *Laplacian* operator,

If the thermophysical properties k, ρ, and c are functions of space coordinates only, then Eq. (2.15) is a *linear* partial differential equation. On the other hand, if, for example, any thermophysical property, k, ρ, or c depends on temperature, Eq. (2.15) becomes a *nonlinear* partial differential equation.

For a *homogeneous isotropic* solid, k is constant and the general heat conduction equation (2.15) reduces to

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.16}$$

where $\alpha = k/\rho c$ is the so-called *thermal diffusivity* of the solid. Equation (2.16) is also known as the *Fourier–Biot equation*.

Thermal diffusivity is a thermophysical property. Its units are square meters per second (m^2/s) in the SI system. A high value of thermal diffusivity can result either from a high value of thermal conductivity k, which indicates a higher rate of heat transfer, or from a low value of thermal capacity ρc, which means that less thermal energy moving through the medium will be absorbed and used to raise the temperature. Therefore, the larger the value of α, the faster will the heat diffuse through a medium. In the absence of internal energy sources and power drawn to the system from an external electric circuit (both will also be named as internal heat sources), the heat conduction equation (2.16) takes the form

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.17}$$

which is the so-called *heat diffusion equation*.

For steady-state conditions and in the presence of internal heat sources, we get

$$\nabla^2 T + \frac{\dot{q}}{k} = 0 \tag{2.18}$$

which is known as the *Poisson equation*.

Under steady-state conditions and in the absence of internal heat sources, the heat conduction equation (2.16) reduces to

$$\nabla^2 T = 0 \tag{2.19}$$

which is the *Laplace equation*.

From the Laplace equation (2.19) we conclude that, in a stationary solid with constant thermal conductivity and without internal heat sources, the temperature distribution under steady-state conditions does not depend on the thermophysical properties of the solid, but is determined only by the solid's shape and the temperature distribution along its boundaries.

A summary of the special cases of Eq. (2.16) along with the conditions that apply to each case is presented in Table 2.1.

In the above relations, $\nabla^2 T$ represents the *Laplacian* of the temperature distribution. In *rectangular* coordinates it is given by

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.20}$$

where $T = T\,(x, y, z, t)$.

TABLE 2.1

Special Cases of the General Heat Conduction Equation with Constant Thermal Conductivity

Equation Name	Conditions	Equation
Fourier–Biot	Constant thermophysical properties	$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
Diffusion	Constant thermophysical properties, no internal heat sources	$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
Poisson	Steady state, k = constant	$\nabla^2 T + \frac{\dot{q}}{k} = 0$
Laplace	Steady state, no internal heat sources, k = constant	$\nabla^2 T = 0$

In *cylindrical* coordinates for $\nabla^2 T$ we have

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.21}$$

where $T = T(r, \phi, z, t)$. The relationships between the rectangular and cylindrical coordinates of a point P, as shown in Fig. 2.2, are given by

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z \tag{2.22a,b,c}$$

In *spherical* coordinates, $\nabla^2 T$ is given by

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r} + \left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{2.23}$$

where $T = T(r, \theta, \phi, t)$. The spherical coordinates (r, θ, ϕ) of a point P are indicated in Fig. 2.3, and the relationships between the rectangular and spherical coordinates of the point P are given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta \tag{2.24a,b,c}$$

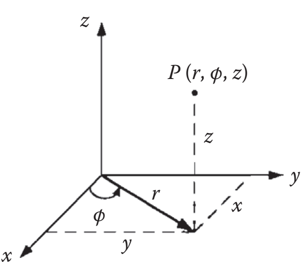

FIGURE 2.2
Cylindrical coordinates r, ϕ, and z of a point P.

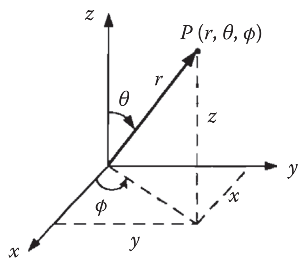

FIGURE 2.3
Spherical coordinates r, θ. and ϕ of a point P.

A summary of the Laplacian of temperature in various coordinate systems is given in Table 2.2.

If the thermal conductivity is not constant, that is, if the substance is not homogeneous, then the general heat conduction equation is given by Eq. (2.14). Table 2.3 presents the specific forms of this equation in various coordinate systems.

TABLE 2.2

Laplacian of Temperature in Various Coordinates Systems

Coordinate System	$\nabla^2 T$
Rectangular	$\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}+\dfrac{\partial^2 T}{\partial z^2}$
Cylindrical	$\dfrac{\partial^2 T}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial T}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial \phi^2}+\dfrac{\partial^2 T}{\partial z^2}$
Spherical	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}+\left(r^2\dfrac{\partial T}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial T}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 T}{\partial\phi^2}$ or $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}+\left(r^2\dfrac{\partial T}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial}{\partial\mu}\left((1-\mu^2)\dfrac{\partial T}{\partial\mu}\right)+\dfrac{1}{r^2(1-\mu^2)}\dfrac{\partial^2 T}{\partial\phi^2}$ where $\mu = \cos\theta$

TABLE 2.3

General Heat Conduction Equation with Variable Thermal Conductivity in Various Coordinate Systems

Coordinate System	$\nabla\cdot(k\nabla T)+\dot{q}=\rho c\dfrac{\partial T}{\partial t}$
Rectangular	$\dfrac{\partial}{\partial x}\left(k\dfrac{\partial T}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right)+\dot{q}=\rho c\dfrac{\partial T}{\partial t}$
Cylindrical	$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(kr\dfrac{\partial T}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial}{\partial\phi}\left(k\dfrac{\partial T}{\partial\phi}\right)+\dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right)+\dot{q}=\rho c\dfrac{\partial T}{\partial t}$
Spherical	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(kr^2\dfrac{\partial T}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(k\sin\theta\dfrac{\partial T}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial}{\partial\phi}\left(k\dfrac{\partial T}{\partial\phi}\right)+\dot{q}=\rho c\dfrac{\partial T}{\partial t}$

2.3 Initial and Boundary Conditions

In general, regardless of the mathematical method employed, a solution to the general heat conduction equation will contain "seven constants of integration" because of the first-order derivative with respect to the time variable and second-order derivatives with respect to each space variable. To complete the formulation of a problem, we need to have certain conditions to determine the constants of integration. These will be the initial and boundary conditions of the problem. That is, there is an infinite number of solutions to the general heat conduction equation (2.14), or (2.16), but there is usually only one solution for the prescribed initial and boundary conditions (see, for example, Problems 2.10 and 2.11). The number of conditions in the direction of each independent variable is equal to the order of the highest derivative of the governing differential equation in the same direction. That is, we need to specify one initial condition (for time-dependent problems) and two boundary conditions in each coordinate direction.

2.3.1 Initial Condition

The initial condition for a time-dependent problem is the given or known temperature distribution in the medium under consideration at some instant of time, usually at the beginning of the heating or cooling process; that is, at $t = 0$. Mathematically speaking, if the initial condition is given by $T_0(\mathbf{r})$, then the solution $T(\mathbf{r}, t)$ of the problem must be such that, at all points of the medium,

$$T(\mathbf{r},t)\big|_{t\to 0} = T_0(\mathbf{r}) \tag{2.25}$$

where $\mathbf{r}$ is the position vector.

2.3.2 Boundary Conditions

Boundary conditions specify the temperature or the heat flow at the boundary of the region under consideration. For convenience in the analysis we separate the boundary conditions into the following categories: prescribed boundary temperature, prescribed heat flux, heat transfer by convection, heat transfer by radiation, and interface conditions.

Prescribed boundary temperature. The distribution or the value of temperature may be prescribed at a boundary surface, as shown in Fig. 2.4. This prescribed surface temperature may, in general, be a function of time and space variables; that is,

$$T(\mathbf{r},t)\big|_{r=r_s} = T_s(\mathbf{r}_s,t) \tag{2.26}$$

A boundary condition of this form is also called the *boundary condition of the first kind*. The prescribed temperature distribution at the boundary surface S, in special cases, can be a function only of position or time or it can be a constant. If the temperature on the boundary surface vanishes, that is, if

$$T(\mathbf{r},t)\big|_{r=r_s} = 0 \tag{2.27}$$

then the boundary condition is called a *homogeneous boundary condition of the first kind*.

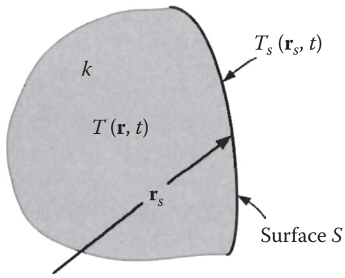

FIGURE 2.4
Prescribed surface temperature $T_s(\mathbf{r}_s, t)$ at a boundary surface S.

Prescribed heat flux. The distribution or the value of the heat flux over a boundary surface may be specified to be constant or a function of space variables and/or time. Consider a boundary surface at $\mathbf{r} = \mathbf{r}_s$, and let $\hat{\mathbf{n}}$ be the outward-drawn unit vector normal to this surface, as shown in Fig. 2.5*a*. If the heat flux q_s'' leaving the surface is specified, then the boundary condition can be written as

$$-k\left(\frac{\partial T}{\partial n}\right)_s = q_s'' \tag{2.28a}$$

which is a statement of energy balance on a system of infinitesimal thickness at any location $\mathbf{r}_s$ on the surface.

When the heat flux q_s'' crossing the surface into the region is prescribed, as shown in Fig. 2.5*b*, the boundary condition can be stated as

$$k\left(\frac{\partial T}{\partial n}\right)_s = q_s'' \tag{2.28b}$$

In writing the conditions (2.28*a*) and (2.28*b*) it has been assumed that the thermal conductivity k is constant in the region. If the thermal conductivity happens to be a function of

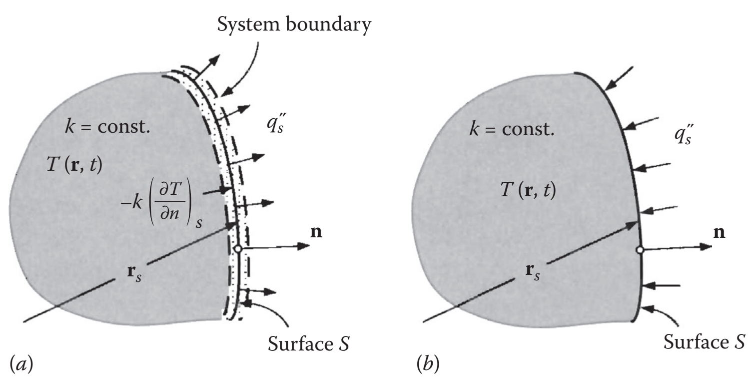

FIGURE 2.5
Prescribed heat flux q_s'' at a boundary surface 5.

space coordinates and/or temperature, then k in these relations would represent the local value of the thermal conductivity k_s at $\mathbf{r}_s$.

When the heat flux is specified either from the surface or to the surface, it mathematically means that we are given the normal derivative of temperature at the boundary. A boundary condition of this form is also referred to as a *boundary condition of the second kind*.

If the derivative of temperature normal to the boundary surface is zero, that is, if

$$\left(\frac{\partial T}{\partial n}\right)_s = 0 \tag{2.29}$$

then the boundary condition is referred to as a *homogeneous boundary condition of the second kind*. Such a boundary condition indicates either a *thermally insulated boundary* (i.e., no heat transfer) or a *thermal symmetry* condition at the boundary.

Heat transfer by convection. When a boundary surface exchanges heat by convection with an ambient fluid at a prescribed temperature T_∞, as shown in Fig. 2.6*a*, the boundary condition at that boundary can be expressed, by using Newton's law of cooling, as

$$-k_s\left(\frac{\partial T}{\partial n}\right)_s = h[T(\mathbf{r}_s, t) - T_\infty] \tag{2.30a}$$

or

$$\left[k\frac{\partial T}{\partial n} + hT(\mathbf{r}, t)\right]_s = hT_\infty \tag{2.30b}$$

where h is the heat transfer coefficient between the surface and the surrounding fluid,

If $\hat{\mathbf{n}}$ represents the inward-drawn normal of the boundary surface as shown in Fig. 2.6*b*, the boundary condition can then be written as

$$k_s\left(\frac{\partial T}{\partial n}\right)_s = h[T(\mathbf{r}_s, t) - T_\infty] \tag{2.31a}$$

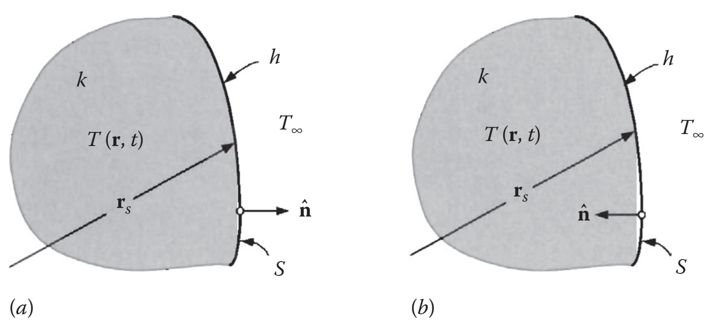

FIGURE 2.6
Boundary surface exchanging heat by convection with an ambient fluid at a prescribed temperature T_∞ and with a heat transfer coefficient h.

or

$$\left[-k\frac{\partial T}{\partial n} + hT(\mathbf{r},t) \right]_s = hT_\infty \tag{2.31b}$$

A boundary condition of the form of either Eq. (2.30*b*) or (2.31*b*) is also called a *boundary condition of the third kind*. Here we note that when $h \to \infty$, a boundary condition of the third kind reduces to a boundary condition of the first kind.

The ambient fluid temperature T_∞ may be a constant or a function of space variables and/or time. If the ambient fluid temperature $T_\infty = 0$, that is, if

$$\left[\pm k\frac{\partial T}{\partial n} + hT(\mathbf{r},T) \right]_s = 0 \tag{2.32}$$

where the plus and minus signs correspond to the differentiations along outward and inward normals, respectively, then the boundary condition is called a *homogeneous boundary condition of the third kind*.

As an example, let us consider the cooling of an electrically heated tube on both sides as shown in Fig. 2.7. The boundary conditions can be written as

$$\left[-k\frac{\partial T}{\partial r} + h_1T(r,t) \right]_{r=r_1} = h_1T_{f1} \tag{2.33a}$$

and

$$\left[k\frac{\partial T}{\partial r} + h_2T(r,t) \right]_{r=r_2} = h_2T_{f2} \tag{2.33b}$$

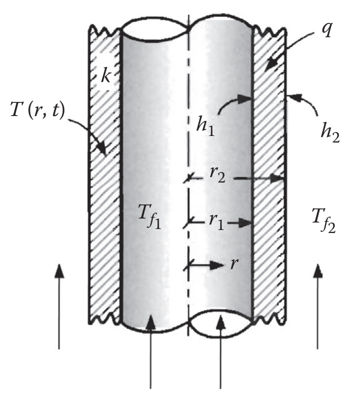

FIGURE 2.7
Electrically heated tube.

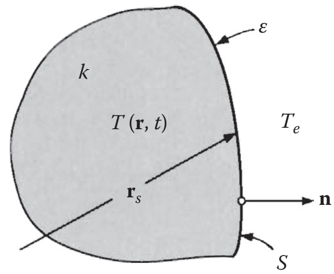

FIGURE 2.8
Boundary surface exchanging heat only by radiation with an environment maintained at an *"effective"* blackbody temperature T_e.

Heat transfer by radiation. When the boundary surface exchanges heat only by radiation with an environment at an *"effective"* blackbody temperature T_e as shown in Fig. 2.8, the boundary condition at this boundary can be written, by using Eq. (1.68), as

$$\left[k\frac{\partial T}{\partial n} + \sigma\varepsilon T^4(\mathbf{r},t) \right]_s = \sigma\varepsilon T_e^4 \tag{2.34}$$

where ε is the emissivity of the surface and σ is the Stefan–Boltzmann constant. Since it involves the fourth power of the surface temperature (*dependent variable*), Eq. (2.34) is a *nonlinear* boundary condition.

Interface conditions. Consider the interface between two solids, such as shown in Fig. 2.9. If there is no relative motion between the solids, the rate of heat flow must be continuous since energy cannot be destroyed or generated there; that is,

$$-k_1\left(\frac{\partial T_1}{\partial n}\right)_s = -k_2\left(\frac{\partial T_2}{\partial n}\right)_s \tag{2.35}$$

If the two solids are in perfect thermal contact, the temperature of the two surfaces at the interface will be equal to each other; that is,

$$[T_1(\mathbf{r},t)]_s = [T_2(\mathbf{r},t)]_s \tag{2.36}$$

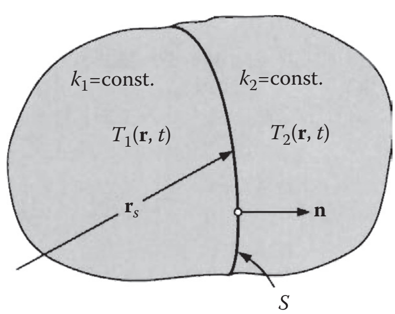

FIGURE 2.9
Two solids in contact.

If the contact is not thermally perfect, some form of thermal contact resistance or heat transfer coefficient must be introduced at the interface. We defer the discussion of thermal contact resistance to Section 3.2.9.

Other types of boundary conditions, such as change of phase (moving interface of two media), the interface of two solids in relative motion, or a free convection boundary condition with the heat transfer being proportional to the 5/4th power of temperature difference, can be written following the same procedure that we implemented in formulating the above boundary conditions.

2.4 Temperature-Dependent Thermal Conductivity and Kirchhoff Transformation

Consider the general heat conduction equation for solids with temperature-dependent thermal conductivity in the form

$$\nabla \cdot [k(T)\nabla T] + \dot{q}(\mathbf{r},t) = \rho(T)c(T)\frac{\partial T}{\partial t} \tag{2.37}$$

Because of the dependence of thermal thermophysical properties k, ρ, and c, on temperature T, Eq. (2.37) is a *nonlinear* differential equation. Provided that the thermal diffusivity is independent of temperature, Eq. (2.37) can be reformulated by introducing a new temperature function $\theta(\mathbf{r}, t)$ by means of the *Kirchhoff transformation* as

$$\theta(\mathbf{r},t) = \frac{1}{k_R}\int_{T_R}^{T(\mathbf{r},t)} k(T')\,dT' \tag{2.38}$$

where T_R is a reference temperature and $k_R = k(T_R)$. From Eq. (2.38), it follows that

$$\nabla\theta = \frac{k(T)}{k_R}\nabla T \tag{2.39a}$$

and

$$\frac{\partial\theta}{\partial t} = \frac{k(T)}{k_R}\frac{\partial T}{\partial t} \tag{2.39b}$$

Therefore, Eq. (2.37) can be rearranged as

$$\nabla^2\theta + \frac{\dot{q}(\mathbf{r},t)}{k_R} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.40}$$

where $\alpha(T) = k(T)/\rho(T)c(T)$ is the thermal diffusivity. Since α is temperature dependent, then Eq. (2.40) is still nonlinear. For many solids, however, the dependence of α on temperature can usually be neglected compared to that of k. If the variation of α with temperature is not significant and. hence, it can be approximated to be constant, then Eq. (2.40) becomes linear. For steady-state problems, since the right-hand side vanishes identically, Eq. (2.40) is a linear differential equation regardless of whether α is temperature dependent or not.

The transformations of the boundary conditions of the first and second kinds, which prescribe T or $\partial T/\partial n$ at a boundary, by means of the Kirchhoff transformation pose no difficulty and yield again boundary conditions of the first and second kinds, respectively. The transformation of a boundary condition of the third kind is, in general, not possible; only under certain restrictions on the heat transfer coefficient h may the transformation become possible [4] (see Problem 2.13).

2.5 Hyperbolic Heat Conduction

The hypothesis of an instantaneous response of the heat flux to a temperature gradient, intrinsic to the classical Fourier's law, may fail for some special applications, such as in heat transfer at the nanoscale and heat conduction problems related to very short time scales, such as in fast pulsating laser heating and rapidly contacting surfaces in electronic devices [6]. Therefore, a non-Fourier heat conduction model that considers the finite speed of heat propagation should be addressed in such situations. In this sense, the so-called Cattaneo equation, first discussed by Carlo Cattaneo in 1948 [7] and later on derived by both Cattaneo [8] and Vernotte [9], originated from the gas kinetic theory, is given by

$$\mathbf{q}'' + \tau \frac{\partial \mathbf{q}''}{\partial t} = -k\nabla T \tag{2.41}$$

where τ is a relaxation time, i.e., the average time between heat carriers collisions. If an infinite speed of heat propagation, i.e., instantaneous collisions, is assumed ($\tau \to 0$), Eq. (2.41) reduces to Fourier's law. Substituting the relation (2.41) into Eq. (2.12), we obtain the *hyperbolic heat conduction equation* as

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau}{\alpha}\frac{\partial^2 T}{\partial t^2} \tag{2.42}$$

in contrast to the parabolic heat conduction equation, or Fourier–Biot equation presented in Eq. (2.16), both for a homogeneous isotropic solid with constant thermophysical properties. Since the solution of the hyperbolic heat conduction equation results in a propagating wave, the amplitude of which decays exponentially as it travels, this relaxation time can be defined through the speed of this *temperature wave*, v_{tw}.

$$v_{tw} = \sqrt{\frac{\alpha}{\tau}} \tag{2.43}$$

Substituting Eq. (2.43) into Eq. (2.42) results in Eq. (2.44), which is also reduced to the parabolic heat conduction equation for an infinite speed of the temperature wave ($v_{tw} \to \infty$).

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{1}{v_{tw}^2}\frac{\partial^2 T}{\partial t^2} \tag{2.44}$$

The boundary conditions for this class of problems are similar to those used for the Fourier–Biot heat conduction equation, but an additional initial condition must be prescribed due to the second-order time derivative of the temperature field.

Although the Cattaneo equation is mathematically more general than the Fourier–Biot equation, it should not be taken as more realistic, since the Cattaneo equation was not justified on a fundamental basis. Some authors have investigated the hyperbolic heat conduction equation based on the second law of thermodynamics, but this approach will not be detailed here, and interested readers should refer to Ref. [10].

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Problems

2.1 Consider an opaque solid which is homogeneous and isotropic. Let the temperature distribution in this solid be expressed in terms of rectangular coordinates as $T = T(x, y, z, t)$. By applying the first law of thermodynamics (i.e., conservation of thermal energy) to a differential volume element $\Delta x \cdot \Delta y \cdot \Delta z$ at any point (x, y, z) in this solid as shown in Fig. 2.10, obtain the diffusion equation in rectangular coordinates.

2.2 For the analysis of heat conduction in cylindrical and spherical coordinates, the differential volume elements, respectively, $\Delta r \cdot r\ \Delta\phi \cdot \Delta z$ and $\Delta r \cdot r \sin\theta\ \Delta\phi \cdot \Delta\theta$ shown in Figs. 2.11*a,b* can be used. Repeat Problem 2.1 to derive the diffusion equation (*a*) in cylindrical coordinates, and (*b*) in spherical coordinates.

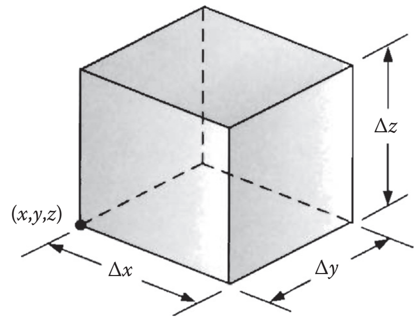

FIGURE 2.10
Differential volume element $\Delta x \cdot \Delta y \cdot \Delta z$ for the analysis of heat conduction in rectangular coordinates.

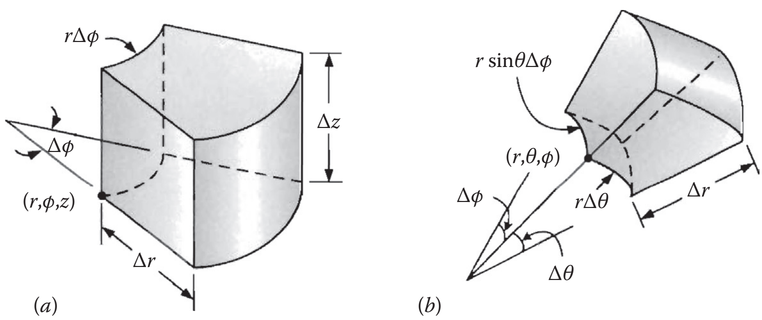

FIGURE 2.11
Differential volume elements $\Delta r \cdot r\Delta\phi \cdot \Delta z$ and $\Delta r \cdot r \sin\theta\, \Delta\phi \cdot r\Delta\theta$ for the analysis of heat conduction in (a) cylindrical (r, ϕ, z) and (b) spherical (r, θ, ϕ) coordinates.

2.3 The general heat conduction equation in rectangular coordinates for constant thermal conductivity is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Making coordinate transformations, obtain the general heat conduction equation in (a) cylindrical coordinates, and (b) spherical coordinates.

2.4 A plane wall of thermal conductivity k receives solar radiation on one of its surfaces (all of the radiation is absorbed at the surface). At the same time it transfers heat from the same surface to the surrounding air at temperature T_∞ with a heat transfer coefficient h. Let q_s'' be the radiative heat flux received from the sun. Give an expression for the boundary condition on this surface.

2.5 Consider two solids which are in perfect contact, one moving relative to the other. The local pressure on the common boundary is p, the coefficient of dry friction is

μ, and the relative velocity is V. How can you express the boundary conditions at the interface of these two solids?

2.6 Consider a solid spherical ball of radius r_0 and of constant thermophysical properties. The ball is first heated to a uniform temperature T_l in an oven and then suddenly immersed, at time $t = 0$, in a large oil bath maintained at temperature T_∞. Assuming a constant heat transfer coefficient h, formulate the problem (i.e., give the applicable differential equation, and the initial and boundary conditions) which can be solved to determine the unsteady-state temperature distribution in the ball for times $t > 0$ as a function of space variables and time.

2.7 The lower circular surface of a solid cylindrical bar, of constant thermal conductivity k, radius r_0 and height H, is in contact with boiling water at temperature T_f. The upper surface is insulated. The bar, in the meantime, loses heat by convection from its peripheral surface to an environment maintained at temperature T_∞. Let the heat transfer coefficient h be constant on this surface. Give the differential equation and the boundary conditions that can be solved to determine the steady-state temperature distribution in the bar.

2.8 Consider a long bar of rectangular cross section and of dimensions a and b in the x and y directions, respectively. The thermal conductivity of the material of the bar is direction dependent such that the thermal conductivity in the x direction, k_x, is two times greater than the thermal conductivity in the y direction, k_y. The surfaces at $x = 0$, $x = a$ and $y = 0$ are all maintained at a constant temperature T_1, while the surface at $y = b$ is kept at constant T_2. There are no heat sources or sinks in the bar. Give the formulation of the problem for the steady-state temperature distribution $T(x, y)$.

2.9 A slab, which extends from $x = -L$ to $x = L$, is initially at a uniform temperature T_i. For times $t \geq 0$, internal energy is generated in the slab at an exponential decay rate per unit volume according to

$$\dot{q} = \dot{q}_0 e^{-\beta t}$$

where $\dot{q}_0$ and β are two given positive constants, while the surfaces at $x = \mp L$, are kept at the initial temperature T_j. Assuming constant thermophysical properties, formulate the problem for the unsteady-state temperature distribution $T(x, t)$ in the slab for times $t > 0$.

2.10 Show that the following one-dimensional heat conduction problem has a unique solution:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i(x)$$

$$T(0,t) = T_1 \quad \text{and} \quad T(L,t) = T_2$$

where T_1 and T_2 are two constant temperatures.

2.11 Consider the following problem formulated for the two-dimensional steady-state temperature distribution $T(x, y)$ in a long bar of rectangular cross section of dimensions a and b in the x and y directions, respectively:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial T}{\partial x}\right|_{x=a} = -\frac{q_1''}{k}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial T}{\partial y}\right|_{y=a} = -\frac{q_2''}{k}$$

where q_1'' and q_2'' are given constant heat fluxes at the surfaces at $x = a$ and $y = b$, respectively.

(*a*) In order for the temperature distribution to be steady, what must be the relationship between q_1'' and q_2''?

(*b*) Show that this problem does not have a unique solution for $T(x, y)$.

2.12 Transform the following one-dimensional nonlinear steady-state heat conduction problem

$$\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] + \dot{q} = 0$$

$$\frac{dT(0)}{dx} = 0 \quad \text{and} \quad T(L) = T_w$$

into a linear problem in terms of a new temperature function defined as

$$\theta(x) = \frac{1}{k_w}\int_{T_w}^{T(x)} k(T')dT'$$

where $k_w = k(T_w)$.

2.13 Determine the restriction on the heat transfer coefficient h, so that the boundary condition of the third kind given by Eq. (2.30*a*) can be transformed by the Kirchhoff transformation (2.38).

2.14 A PVC plate with 2 mm thickness, 4 cm width, and 8 cm height, vertically arranged, is heated by an electric resistance fed by direct current on its face $y = 0$, as shown in Fig. 2.12, and exchanges heat by convection with the external environment at $y = Ly$, with variable heat transfer coefficient along the height, $h(x)$. The other thinner faces of the plate may be considered thermally insulated. Assuming constant thermophysical properties (k, ρ, c), formulate the transient heat conduction problem that governs the three-dimensional temperature field, $T(x, y, z, t)$, on the PVC plate, from an initial temperature distribution equal to the surrounding air temperature, T_∞.

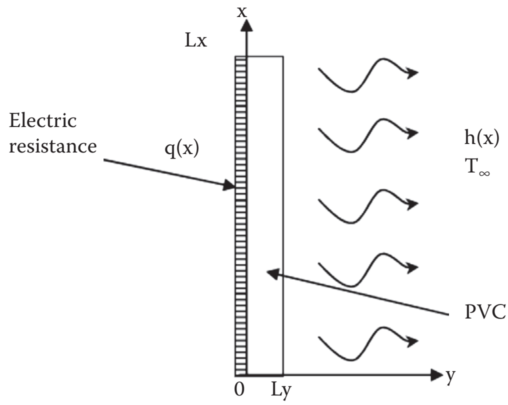

FIGURE 2.12
Figure for Problem 2.14.

2.15 Consider a solid and cylindrical resistive element of diameter D and height H that generates power at uniform and constant rate g_0 (W), having thermal conductivity k (W/m °C). The element initially at the temperature T_∞ is immersed in a fluid at the same uniform and constant temperature T_∞. The heat generation within the cylinder is then started, while the wall of the cylinder exchanges heat by convection with the fluid through a constant and uniform heat transfer coefficient h (W/m^2 °C). Formulate the two-dimensional transient problem for the temperature distribution in the resistive element, $T(r, z, t)$.

2.16 An iron-cast sphere 5 cm in diameter (ρ = 7800 kg/m^3, c = 460 J/kg K, k = 60 W/m K) is initially at a uniform temperature of 700°C. The sphere is then cooled by air at 100°C, with combined convective heat transfer coefficient and radiation of 80 W/m^2 K. Formulate the heat transfer problem for the temperature distribution in the sphere using spherical coordinates.

2.17 Pitot tubes are devices commonly used in the aeronautical industry to measure in-flight speeds. To avoid freezing at high altitudes, it is common to use electrically heated Pitot tubes. To do so, it is necessary to thermally design the instrument, conservatively establishing the power to be dissipated in the tube to avoid ice growth on its surface and the dangerous blockage in the stagnation region. Therefore, the Pitot tube is shown in Fig. 2.13, with external radius r_e and internal radius r_i, made of a material with thermal conductivity k, density ρ, and specific heat c. The horizontal length is L_1 and the vertical length is L_2. Only the horizontal section of the tube will be heated by Joule effect, with uniform volumetric heat generation rate g_0 (W/m^3). The tube exchanges heat with the outside air at the temperature T_∞ flowing along the sensor, with an x-variable heat transfer coefficient $h_1(x)$ at the horizontal tube and a constant heat transfer coefficient h_2 at the vertical tube. The leftmost tip of the tube (x = 0) exchanges heat by convection with a constant coefficient h_e, while the top end at $x = L = L_1 + L_2$ is thermally insulated from the structure of the airplane. Considering that the wall of the Pitot tube is thermally thin in the radial direction, formulate the one-dimensional transient heat conduction along the length of the tube, assuming that the two segments L_1 and L_2 are aligned and with total length $L = L_1 + L_2$. Consider the initial condition that the whole sensor is at the temperature of the external air T_∞, and neglecting

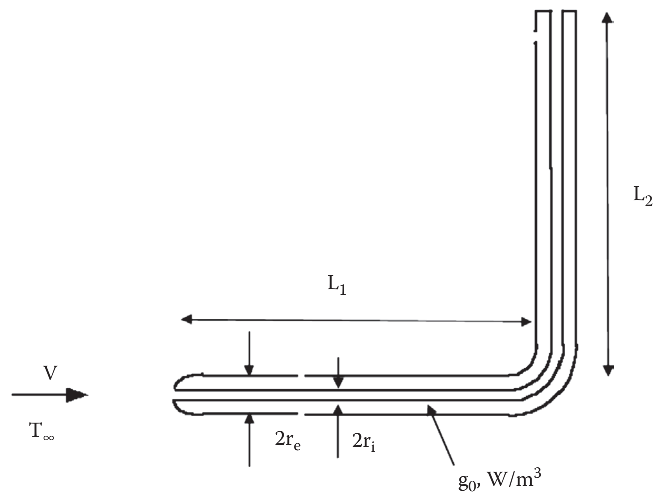

FIGURE 2.13
Figure for Problem 2.17.

the thermal exchange for the stagnant air inside the sensor. Write the corresponding heat conduction equation, with its respective boundary and initial conditions, assuming constant thermophysical properties.

2.18 A metallic pipe of constant thermal conductivity k, with length L, external radius r_e, and internal radius r_i, is heated on the outer surface ($r = r_e$) by an electric resistance that provides a heat flux of q_w to the pipe, discounting the losses to the external environment. Inside the tube, a water flow exchanges heat with the inner surface ($r = r_i$), with a variable heat transfer coefficient along the longitudinal coordinate, $h(z)$, and with internal fluid temperature also variable along z, $T_f(z)$. At the two ends of the pipe ($z = 0$ and $z = L$), there is no heat exchange in the axial direction. Formulate the steady-state two-dimensional heat conduction problem for the pipe wall temperature.

3

One-Dimensional Steady-State Heat Conduction

3.1 Introduction

In the preceding chapter we established the general formulation of conduction heat transfer. In this chapter, we study a number of simple representative systems in which the temperature distribution and, therefore, heat flow are functions of one space variable only. For such problems, the heat conduction equation can readily be obtained directly from one of the forms of the general heat conduction equation developed in Chapter 2 by neglecting unnecessary terms to suit the given problem. This, however, may not always be convenient. The heat conduction equation, on the other hand, can also be derived for each specific problem individually from the basic principles. By doing so we bring the physics of the problem into each phase of the derivation of the heat conduction equation.

3.2 One-Dimensional Steady-State Heat Conduction Without Heat Sources

In this section we discuss various one-dimensional steady-state heat conduction problems without heat sources in rectangular, cylindrical, and spherical coordinates. We also introduce a number of physical and mathematical facts in terms of representative examples.

3.2.1 Plane Wall

Consider a plane wall, or a slab, of a homogeneous isotropic material, of thickness L in the x direction as shown in Fig. 3.1. Let the surfaces at $x = 0$ and $x = L$ be maintained at constant temperatures T_1 and T_2, respectively. If the depth and height of the wall are very large compared to its thickness L, then heat transfer (away from the edges) will be in the x direction and the temperature distribution within the wall will depend on x only.

For steady-state heat conduction through such a wall with constant thermal conductivity, Eq. (2.19) reduces to

$$\frac{d^2T}{dx^2} = 0 \tag{3.1}$$

Equation (3.1) can also be obtained from the basic principles. In order to do this, we first define a system, or a control volume, as shown in Fig. 3.1. Since the heat transfer process

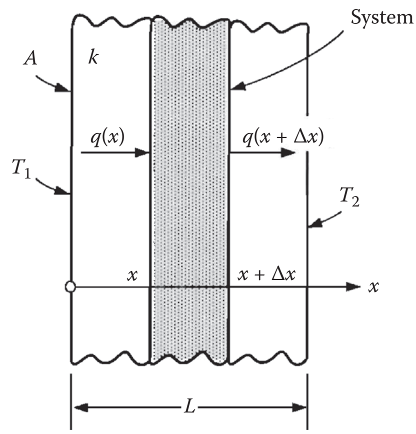

FIGURE 3.1
Plane wall of thickness L in x direction.

is steady, the time rate of change of internal energy of the system will be zero. Also, there is no work done by or on the system. Therefore, the application of the first law of thermodynamics, Eq. (1.14*b*), to the system shown, in the absence of internal heat sources, yields

$$q(x) - q(x + \Delta x) = 0 \tag{3.2}$$

where $q(x)$ is the rate of heat transfer across the isothermal surface at x. On the other hand, as $\Delta x \to 0$, the rate of heat transfer at $x + \Delta x$ can be written as

$$q(x + \Delta x) = q(x) + \frac{dq}{dx}\Delta x \tag{3.3}$$

Substitution of Eq. (3.3) into Eq. (3.2) yields

$$\frac{dq}{dx} = 0 \tag{3.4}$$

Hence, the first law gives only $q(x)$ = constant. Substituting the value of $q(x)$ from Fourier's law, Eq. (1.42), into Eq. (3.4), we get

$$\frac{d}{dx}\left(-kA\frac{dT}{dx}\right) = 0 \tag{3.5}$$

Since k and A are constants, Eq. (3.5) reduces to Eq. (3.1).

In addition to the differential equation, boundary conditions have to be specified to complete the formulation of the problem. For the problem under consideration, the boundary conditions can be written as

$$T(0) = T_1 \quad \text{and} \quad T(L) = T_2 \tag{3.6a,b}$$

As seen from this example, to formulate a problem from the basic principles the following five steps need to be followed:

1. Select a coordinate system appropriate to the geometry of the problem.
2. Define a system or a control volume suited to the one-, two-, or three-dimensional nature of the problem.
3. State the first law of thermodynamics for the system, or the control volume, defined in step 2.
4. Introduce Fourier's law of heat conduction into the resulting equation in step 3, and obtain the governing differential equation for the temperature distribution.
5. Specify the origin of the coordinate system and state the necessary boundary conditions (and an initial condition for time-dependent problems) on the temperature.

Now, by integrating Eq. (3.1) twice we get

$$T(x) = C_1 x + C_2 \tag{3.7a}$$

Application of the boundary conditions (3.6*a,b*) yields

$$C_1 = \frac{T_2 - T_1}{L} \quad \text{and} \quad C_2 = T_1 \tag{3.7b,c}$$

Substituting C_1 and C_2 into Eq. (3.7*a*), the temperature distribution in the plane wall is found to be

$$T(x) = T_1 - (T_1 - T_2)\frac{x}{L} \tag{3.8}$$

It follows that, with constant thermal conductivity, the temperature distribution in the wall is a linear function of x.

The rate of heat transfer through the wall is obtained by applying Fourier's law as

$$q = -kA\frac{dT}{dx} = kA\frac{T_1 - T_2}{L} \tag{3.9}$$

which is, as expected, consistent with Eq. (1.38).

If $k = k(x)$, then the one-dimensional heat conduction equation from Eq. (3.5) is given by

$$\frac{d}{dx}\left[k(x)\frac{dT}{dx}\right] = 0 \tag{3.10}$$

Integrating this equation twice and using the boundary conditions (3.6*a,b*) we get

$$T(x) = T_1 - \frac{T_1 - T_2}{\int_0^L dx/k(x)} \int_0^x \frac{dx}{k(x)} \tag{3.11}$$

The heat transfer rate through the wall is then given by

$$q = \frac{A(T_1 - T_2)}{\int_0^L dx/k(x)} \tag{3.12}$$

Equations (3.11) and (3.12) reduce, as expected, to Eqs. (3.8) and (3.9), respectively, when k = constant.

If $k = k(T)$, then Eq. (3.5) becomes

$$\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] = 0 \tag{3.13}$$

which is a nonlinear differential equation. Integrating this equation once yields

$$k(T)\frac{dT}{dx} = C_1 = -\frac{q}{A} \tag{3.14}$$

One more integration from $x = 0$ ($T = T_1$) to $x = L$ ($T = T_2$) gives

$$\int_{T_1}^{T_2} k(T)\,dT = -\frac{q}{A}L \tag{3.15}$$

from which we obtain

$$q = \frac{A}{L}\int_{T_2}^{T_1} k(T)\,dT \tag{3.16a}$$

This result can also be written in terms of a *mean thermal conductivity*, k_m, *as*

$$q = k_m A \frac{T_1 - T_2}{L} \tag{3.16b}$$

where k_m is defined as

$$k_m = \frac{1}{T_1 - T_2}\int_{T_2}^{T_1} k(T)\,dT \tag{3.17}$$

The temperature distribution in the wall can be obtained by integrating Eq. (3.14) as follows:

$$\int_{T_1}^{T(x)} k(T)\,dT = -\frac{q}{A}\int_0^x dx = -\frac{q}{A}x \tag{3.18}$$

On the other hand, in order for this expression to be written explicitly for $T(x)$, the relation $k = k(T)$ needs to be specified.

As discussed in Section 1.10, for many materials the thermal conductivity k is not constant, but varies in a nearly linear manner with temperature as

$$k(T) = k_R[1 + \gamma(T - T_R)] \tag{3.19}$$

where $k_R = k(T_R)$ and T_R is a reference temperature. The linear thermal conductivity relation (3.19) may also be written as

$$k(T) = k_0(1 + \beta T) \tag{3.20}$$

where

$$k_0 = k_R(1 - \gamma T_R) \quad \text{and} \quad \beta = \frac{k_R}{k_0}\gamma \tag{3.21a,b}$$

The mean thermal conductivity then becomes

$$\begin{aligned} k_m &= \frac{1}{T_1 - T_2}\int_{T_2}^{T_1} k_0[1 + \beta T]\,dT \\ &= k_0\left[1 + \frac{1}{2}\beta(T_1 - T_2)\right] \end{aligned} \tag{3.22}$$

Moreover, substituting Eq. (3.20) into Eq. (3.18) yields the following relation for the temperature distribution $T(x)$:

$$\beta[T(x) - T_1]^2 + 2(1 + \beta T_1)[T(x) - T_1] - 2(T_2 - T_1)\frac{k_m}{k_0}\frac{x}{L} = 0 \tag{3.23}$$

If $\beta = 0$, Eq. (3.23) results in the linear temperature distribution given by Eq. (3.8) as expected. When $\beta \neq 0$, however, Eq. (3.23) is a *quadratic* equation and the temperature distribution is no longer linear. Solving Eq. (3.23) for $T(x)$ and neglecting the physically meaningless root, we find

$$\frac{T(x) - T_2}{T_1 - T_2} = 1 - \frac{1 + \beta T_1}{\beta(T_1 - T_2)}\left[1 - \sqrt{1 + \beta\frac{T_1 - T_2}{1 + \beta T_1}\left(2 - \beta\frac{T_1 - T_2}{1 + \beta T_1}\right)\frac{x}{L}}\right] \tag{3.24}$$

To illustrate the effect of the linear thermal conductivity on the temperature distribution, Eq. (3.24) is plotted against x/L for various values of $\beta(T_1 - T_2)/(1 + \beta T_1)$ in Fig. 3.2.

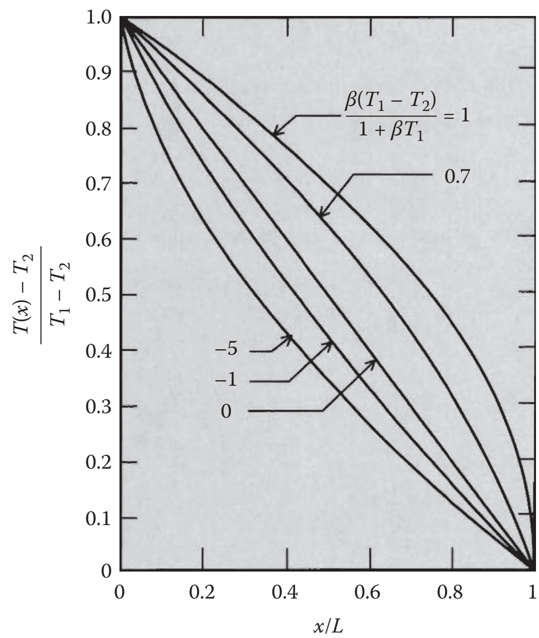

FIGURE 3.2
Temperature distribution in a plane wall with $k(T) = k_0(1 + \beta T)$.

Example 3.1

A plane wall, 50 cm in thickness, is constructed from a material whose thermal conductivity varies linearly with temperature according to the relation $k = 1 + 0.0015\ T$, where T is in °C and k in W/(m·K). Calculate the rate of heat transfer through this wall per unit surface area if one side of the wall is maintained at 1000°C and the other side at 0°C.

SOLUTION

Equation (3.22) may be used first to calculate the mean thermal conductivity, where k_0 = 1.0W/(m·K) and β = 0.0015 1/K. Hence,

$$k_m = 1.0 \times \left(1 + \frac{1}{2} \times 0.0015 \times 1000\right) = 1.75\text{W}/(\text{m·K})$$

Then, the rate of heat transfer through this wall per unit area is given, from Eq. (3.16*b*), by

$$\frac{q}{A} = \frac{k_m(T_1 - T_2)}{L} = \frac{1.75(1000 - 0)}{0.50} = 3500\text{W}/\text{m}^2$$

Note that if $\beta = 0$, then k = 1 W/(m·K) and the rate of heat transfer per unit area would be 2000 W/m^2.

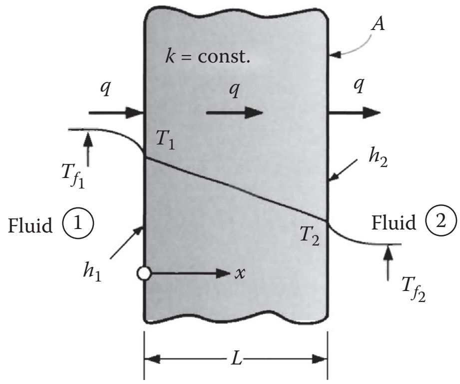

FIGURE 3.3
Heat transfer through a homogeneous plane wall from one fluid to another.

3.2.2 Conduction through a Plane Wall from One Fluid to Another

Consider a homogeneous plane wall separating two fluids as shown in Fig. 3.3. Heat is convected from the fluid of higher temperature to the wall, conducted through the wall, and then convected again from the wall to the fluid of lower temperature. Under steady-state conditions and in the absence of internal energy sources, the rate of heat transfer will be the same on both surfaces and through the wall. In terms of the thickness L, thermal conductivity k, surrounding fluid temperatures T_{f1} and T_{f2} and the constant heat transfer coefficients h_1 and h_2, we can write the following system of equations:

$$q = Ah_1(T_{f1} - T_1) \quad \text{for convection at surface } x = 0$$

$$q = A\frac{k}{L}(T_1 - T_2) \quad \text{for conduction through the wall}$$

$$q = Ah_2(T_2 - T_{f2}) \quad \text{for convection at surface } x = L$$

where T_1 and T_2 are the unknown surface temperatures. After solving these equations for the temperature differences, elimination of T_1 and T_2 yields

$$q = \frac{A(T_{f_1} - T_{f_2})}{1/h_1 + L/k + 1/h_2} \tag{3.25}$$

Equation (3.25) gives the heat transfer rate through the wall in terms of the surrounding fluid temperatures.

3.2.3 Hollow Cylinder

Consider a hollow circular cylinder of inside and outside radii r_1 and r_2, respectively, and length L, whose cross section is shown in Fig. 3.4. The inside and outside surfaces are maintained at the uniform temperatures T_1 and T_2, respectively. Let the material of

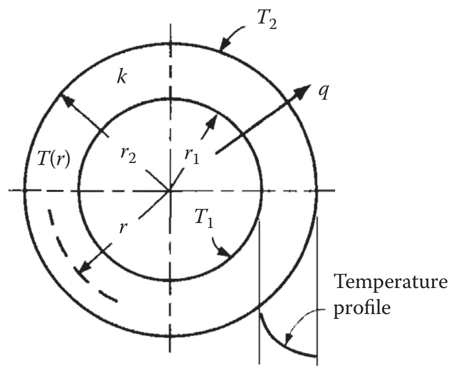

FIGURE 3.4
Heat flow through a hollow cylinder.

the cylinder be homogeneous. If the cylinder is sufficiently long so that the end effects may be neglected, or if the ends are perfectly insulated, then the temperature distribution will be one-dimensional; that is, the temperature distribution will be a function of the radial coordinate r only. For such a case, the heat conduction Equation (2.19) reduces to (see Table 2.2)

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \tag{3.26a}$$

or

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{3.26b}$$

Integrating Eq. (3.26*b*) twice yields

$$T(r) = C_1 \ln r + C_2 \tag{3.27}$$

The boundary conditions are

$$T(r_1) = T_1 \quad \text{and} \quad T(r_2) = T_2 \tag{3.28a,b}$$

Application of these boundary conditions gives

$$T_1 = C_1 \ln r_1 + C_2 \quad \text{and} \quad T_2 = C_1 \ln r_2 + C_2 \tag{3.29a,b}$$

Hence, we obtain

$$C_1 = -\frac{T_1 - T_2}{\ln(r_2/r_1)} \quad \text{and} \quad C_2 = T_1 + \frac{T_1 - T_2}{\ln(r_2/r_1)}\ln r_1 \tag{3.29c,d}$$

Finally, substituting the constants C_1 and C_2 into the solution (3.27), we get

$$T(r) = T_1 - \frac{T_1 - T_2}{\ln(r_2/r_1)} \ln \frac{r}{r_1} \tag{3.30}$$

Thus, the steady-state temperature distribution in a hollow circular cylinder is a logarithmic function of the radial coordinate r.

The rate of heat transfer through the cylinder wall is obtained from Fourier's law of heat conduction:

$$q = -k2\pi rL \frac{dT}{dr} = k2\pi rL \frac{T_1 - T_2}{r \ln(r_2/r_1)} = \frac{2\pi Lk(T_1 - T_2)}{\ln(r_2/r_1)} \tag{3.31}$$

If k is a function of the radial coordinate, that is, if $k = k(r)$, then from Table 2.3 we get

$$\frac{d}{dr}\left[rk(r) \frac{dT}{dr} \right] = 0 \tag{3.32}$$

Integrating this equation over r yields

$$rk(r) \frac{dT}{dr} = C_1 = -\frac{q}{2\pi L} \tag{3.33}$$

One more integration from r_1 to r_2 gives

$$q = \frac{2\pi L(T_1 - T_2)}{\displaystyle\int_{r_1}^{r_2} \frac{dr}{rk(r)}} \tag{3.34}$$

For k = constant, Eq. (3.34) reduces, as expected, to Eq. (3.31).

3.2.4 Spherical Shells

Consider a hollow sphere of inside and outside radii r_1 and r_2, and with constant inside and outside surface temperatures T_1 and T_2, respectively. If the material of the sphere is homogeneous, then the temperature distribution will be a function of r only; that is, $T = T(r)$, and the heat conduction equation (2.19) takes the form

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0 \tag{3.35a}$$

or

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr} \right) = 0 \tag{3.35b}$$

The boundary conditions are given by

$$T(r_1) = T_1 \quad \text{and} \quad T(r_2) = T_2 \tag{3.36a,b}$$

Integrating Eq. (3.35*b*) twice yields

$$T(r) = -\frac{C_1}{r} + C_2 \tag{3.37}$$

Imposing the boundary conditions (3.36*a,b*), we get

$$T_1 = -\frac{C_1}{r_1} + C_2 \quad \text{and} \quad T_2 = -\frac{C_1}{r_2} + C_2 \tag{3.38a,b}$$

from which we obtain

$$C_1 = \frac{T_1 - T_2}{1/r_2 + 1/r_1} \quad \text{and} \quad C_2 = T_1 + \frac{T_2 + T_1}{r_1(1/r_2 + 1/r_1)} \tag{3.39a,b}$$

Substitution of these constants into Eq. (3.37) yields the temperature distribution in the sphere:

$$T(r) = T_1 + \frac{T_1 - T_2}{1/r_2 - 1/r_1}\left(\frac{1}{r_1} - \frac{1}{r}\right) \tag{3.40}$$

The rate of heat transfer through the spherical wall is obtained from Fourier's law:

$$\begin{aligned} q &= -k4\pi r^2 \frac{dT}{dr} = -k4\pi r^2 \frac{T_1 - T_2}{r^2(1/r_2 - 1/r_1)} \\ &= \frac{4\pi k(T_1 - T_2)}{1/r_2 - 1/r_1} = \frac{kA_m(T_1 - T_2)}{r_2 - r_1} \end{aligned} \tag{3.41}$$

where

$$A_m = 4\pi r_1 r_2 = \left(4\pi r_1^2 \times 4\pi r_2^2\right)^{1/2} = \sqrt{A_1 A_2}$$

and $A_1 = 4\pi r_1^2$ and $A_2 = 4\pi r_2^2$ are the inside and outside surface areas, respectively.

3.2.5 Thermal Resistance Concept

Temperature is the driving force or potential for heat flow. The flow of heat over a "*heat flow path*" should then be governed by the thermal potential difference, ΔT, across the path and the resistance of it. This suggests that heat flow is analogous to electric current flow across an electrical resistance. If we consider steady flows, then the heat flow can be written as

$$q = \frac{\Delta T}{R_t}$$

where R_t is the so-called *thermal resistance* of the heat flow path. Comparison of this definition with Eqs. (1.61), (3.9), (3.31) and (3.41) results in various expressions for the thermal resistance as follows:

$$\text{Surface to fluid:} \quad R_t = \frac{1}{hA} \tag{3.42a}$$

$$\text{Plane wall:} \quad R_t = \frac{L}{kA} \tag{3.42b}$$

$$\text{Hollow cylinder:} \quad R_t = \frac{\ln(r_2/r_1)}{2\pi Lk} \tag{3.42c}$$

$$\text{Spherical shell:} \quad R_t = \frac{r_2 - r_1}{4\pi k r_1 r_2} \tag{3.42d}$$

The thermal resistance concept, although very useful, is strictly valid only for steady-state conditions and negligible lateral heat conduction. The thermal resistance (3.42*a*) is called *convective* or *surface* resistance. The resistances (3.42*b*,*c*,*d*), on the other hand, are called *conductive* or *internal* resistances of a plane wall of thickness L, of a hollow cylinder of inside and outside radii r_1 and r_2, and of a spherical shell of inside and outside radii r_1 and r_2, respectively.

3.2.6 Composite Plane Walls

By using the concept of thermal resistance, we can write the following system of equations for heat flow through the composite plane wall of Fig. 3.5:

$$T_{f_1} - T_1 = qR_{t_1}, \quad T_1 - T_2 = qR_{t_1} \tag{3.43a,b}$$

$$T_2 - T_3 = qR_{t_2}, \quad T_3 - T_{f_2} = qR_{t_0} \tag{3.43c,d}$$

Adding these four equations side by side we get

$$T_{f_1} - T_{f_2} = q(R_{t_1} + R_{t_1} + R_{t_2} + R_{t_0}) \tag{3.44}$$

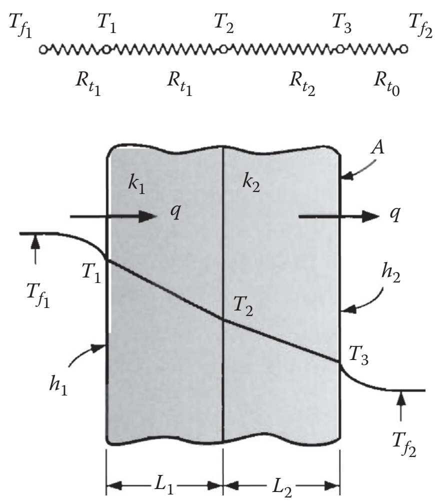

FIGURE 3.5
Composite plane wall and electrical analogy.

or

$$q = \frac{T_{f_1} - T_{f_2}}{R_{t_1} + R_{t_1} + R_{t_2} + R_{t_0}} \tag{3.45}$$

This result shows that the overall thermal resistance is equal to the sum of individual resistances, which would also be the case for electrical resistances. For the composite plane wall of Fig. 3.5, the resistances in Eq. (3.45) are given by Eqs. (3.42*a,b*), and therefore Eq. (3.45) can be rewritten as

$$q = \frac{A(T_{f_1} - T_{f_2})}{1/h_1 + L_1/k_1 + L_2/k_2 + 1/h_2} \tag{3.46}$$

Since the overall thermal resistance of a composite wall is the sum of the thermal resistances of individual layers, it is obvious that the rate of heat transfer through a composite wall of n layers would be given by

$$q = \frac{A(T_{f_1} - T_{f_2})}{\dfrac{1}{h_1} + \displaystyle\sum_{i=1}^{n} \frac{L_i}{k_i} + \dfrac{1}{h_2}} \tag{3.47}$$

The thermal resistance concept may also be used to solve more complex steady-state problems involving both series and parallel thermal resistances using the same rules for combining electrical resistances as illustrated in Fig. 3.6. However, some care must be exercised in representing thermal systems with parallel resistances since multidimensional effects are likely to be present.

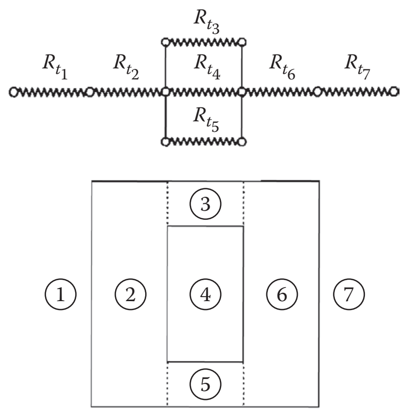

FIGURE 3.6
Composite wall with both series and parallel thermal resistances.

3.2.7 Cylindrical Composite Walls

The reasoning that has led to Eq. (3.46) is also applicable for the cylindrical composite wall of Fig. 3.7. The resistances are given by Eqs. (3.42*a,c*), and, therefore, the equation for the rate of heat transfer q becomes

$$q = \frac{2\pi L(T_{f_1} - T_{f_2})}{\dfrac{1}{h_1 r_1} + \dfrac{\ln(r_2/r_1)}{k_1} + \dfrac{\ln(r_3/r_2)}{k_2} + \dfrac{1}{h_2 r_3}} \tag{3.48}$$

where L is the length of the composite cylinder.

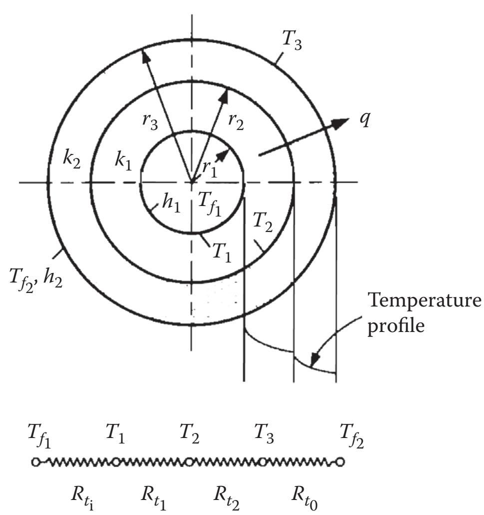

FIGURE 3.7
Heat conduction through a cylindrical composite wall.

The rate of heat flow through a cylindrical composite wall of n layers can then be written as

$$q = \frac{2\pi L(T_{f_1} - T_{f_2})}{\dfrac{1}{h_1 r_1} + \displaystyle\sum_{i=1}^{n} \frac{\ln(r_{i+1}/r_i)}{k_i} + \dfrac{1}{h_2 r_{n+1}}} \tag{3.49}$$

3.2.8 Overall Heat Transfer Coefficient

It sometimes becomes convenient if the equations for the rate of heat transfer through composite walls are simplified by rewriting them in terms of the so-called *overall heat transfer coefficient, U,* which is defined by the equation

$$q = AU(T_{f_1} - T_{f_2}) \tag{3.50}$$

Comparing this equation with Eqs. (3.46) and (3.48), we get the following expressions for U for composite plane and cylindrical walls:

$$\frac{1}{U} = \frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h_2} \tag{3.51}$$

$$\frac{1}{U} = \frac{A}{2\pi L}\left[\frac{1}{r_1 h_1} + \frac{1}{k_1}\ln\frac{r_2}{r_1} + \frac{1}{k_2}\ln\frac{r_3}{r_2} + \frac{1}{r_3 h_2}\right] \tag{3.52}$$

Equation (3.50) defines U in terms of a heat transfer area A. In the cylindrical wall case, A is not constant, but varies from $2\pi r_1 L$ to $2\pi r_3 L$. Therefore, the definition of U in this case depends on the area selected. There is no accepted practice in this matter. If, for example, $A = A_1 = 2\pi r_1 L$, then Eq. (3.52) becomes

$$\frac{1}{U_1} = \frac{1}{h_1} + \frac{r_1}{k_1}\ln\frac{r_2}{r_1} + \frac{r_1}{k_2}\ln\frac{r_3}{k_2} + \frac{r_1}{r_3}\frac{1}{h_2} \tag{3.53}$$

This definition of U was arbitrarily based on A_1; it can also be based on either A_2 or A_3, or any area in between. However, in any case,

$$A_1 U_1 = A_2 U_2 = A_3 U_3 \tag{3.54}$$

3.2.9 Thermal Contact and Fouling Resistances

In addition to the thermal resistances discussed above, two other resistances are often encountered in thermal systems, namely, *thermal contact resistance* and *fouling resistance.* All machined surfaces that are supposed to be smooth are, in fact, wavy with a regular pitch owing to the periodic nature of the machining processes. When two such surfaces

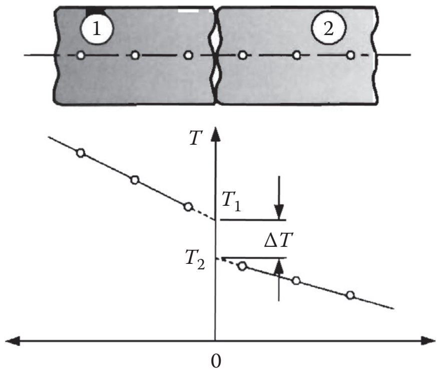

FIGURE 3.8
Temperature drop at an interface due to thermal contact resistance.

are brought into contact, they actually touch only at a limited number of spots, the total of which is usually only a small fraction of the *apparent* contact area, as illustrated in Fig. 3.8. The remainder of the space between the surfaces may be filled with air or another fluid, or may even be a vacuum. When heat flows from one solid to the other, heat flow paths converge toward the actual contact spots, since the thermal conductivities of solids are generally greater than those of fluids. This creates an additional resistance to the heat flow at the interface.

Thermal contact resistance has been the subject of many investigations, both theoretical and experimental, since such resistances must be accurately estimated for a reliable heat transfer analysis of a given system. In theoretical studies, usually geometrically simple *contact elements* are considered. Some exact solutions for thermal resistances of planar and circular contacts are presented in Reference [18].

Thermal contact resistance depends on several interdependent parameters. The actual contact area is the most important one. In addition to the holes and vacant spaces, traces of poorly conducting materials, such as oxide films, may also be present between the surfaces. It is usually difficult to estimate accurately the thickness of such films. The actual contact area, on the other hand, strongly depends on the applied load. At the interface of two solids a *contact heat transfer coefficient* is defined as

$$h_c = \frac{q/A}{\Delta T} \tag{3.55}$$

where ΔT is the temperature drop and q/A is the heat flux across the interface. A thermal contact resistance can, therefore, be defined as

$$R_c = \frac{1}{h_c A} \tag{3.56}$$

where A is the apparent contact area.

One of the first analytical studies of thermal contact resistance was presented by Çetinkale (Veziroğlu) and Fishenden [2] in 1951. An experimental investigation was also conducted by Veziroğlu and Fishenden at atmospheric conditions using steel, brass, and aluminum specimens with varying degrees of surface roughnesses. They used air, spindle oil, and glycerol as interstitial fluids. Literature on thermal contact resistance has grown considerably during the past three decades. A good review of these works is given in References [3,14,15,19].

It is important to recognize that there may also be an additional thermal resistance at fluid-solid interfaces, which is called *scale* resistance or *fouling resistance.* The fouling of heat transfer surfaces may be defined as the deposition of unwanted material on such surfaces causing a degradation in performance. The fouling of solid–fluid interfaces can be described in six categories:

1. *Precipitation fouling*, the crystallization from solutions of dissolved salts on heat transfer surfaces. This is sometimes called *scaling.* Normal solubility salts precipitate on subcooled surfaces, while the more troublesome inverse solubility salts precipitate on superheated surfaces.
2. *Particulate fouling,* the accumulation of finely divided solids suspended in process fluids on heat transfer surfaces. In some instances settling by gravity also prevails, and the fouling is then referred to as *sedimentation fouling.*
3. *Chemical reaction fouling,* the formation of deposits at heat transfer surfaces by chemical reactions in which the surface material itself is not a reactant (e.g., in petroleum refining, polymer production, food processing).
4. *Corrosion fouling,* the accumulation of indigenous corrosion products on heat transfer surfaces.
5. *Biological fouling,* the attachment of macroorganisms (*macrobiofouling*) and/or microorganisms (*microbiofouling or microbial fouling*) to a heat transfer surface, along with the adherent slimes often generated by the latter.
6. *Solidification fouling,* the *freezing* of a pure liquid or a higher melting point constituent of a multicomponent solution on a subcooled heat transfer surface.

The functional effect of fouling on a heat transfer surface may be expressed by a *fouling resistance* defined as

$$R_f = \frac{x_f}{k_f A} \tag{3.57}$$

where x_f is the thickness and k_f is the thermal conductivity of the deposit. However, it is seldom practical to measure the thickness of the fouling deposits. In addition, the thermal conductivity k_f of the deposit may not be known, and it may also vary with the thickness x_f.

Over the past three decades, increasing efforts have also been directed toward a better understanding of fouling, and several models have been proposed to predict the fouling and fouling resistance for use in the design and operation of heat exchangers [10,11]. For the purposes of deduction of the overall heat transfer coefficient, a fouling factor, F_f (also referenced as a scale coefficient), can be alternatively defined as the inverse of a *fouling heat transfer coefficient* h_f, as presented below, and thus this fouling factor has dimensions of m^2K/W:

$$F_f = \frac{1}{h_f}.$$

When contact and fouling resistances are significant, the total thermal resistance must also include these, and therefore terms like $1/Ah_c$ and $1/Ah_f$ should appear, for example, in Eqs. (3.47) and (3.49).

Example 3.2

Calculate the overall heat transfer coefficient for the following cases:

(*a*) Heat is transferred across 1m² of a 3-cm thick plate made of steel of k = 36.4 W/(m·K). A liquid flows on one side with a heat transfer coefficient h_1 = 4000 W/(m²·K) and the other side is exposed to air with h_2 = 12W/(m²·K). Assume a fouling heat transfer coefficient of h_f = 3000 W/(m²·K) on the liquid side.
(*b*) Same as part (*a*), with the air replaced by condensing steam, h_2 = 6000 W/(m²·K).
(*c*) Same as part (*a*) with the liquid replaced by a flowing gas, h_1 = 6 W/(m²·K).

SOLUTION

The overall heat transfer coefficient, U, can be calculated as follows:

(*a*)
$$\frac{1}{U} = \frac{1}{h_1} + \frac{L}{k} + \frac{1}{h_2} + \frac{1}{h_f} = \frac{1}{4000} + \frac{0.03}{36.4} + \frac{1}{12} + \frac{1}{3000}$$
$$= 0.00025 + 0.000824 + 0.08333 + 0.000333$$
$$U = 11.8\mathrm{W/(m^2 \cdot K)}$$

(*b*)
$$\frac{1}{U} = \frac{1}{4000} + \frac{0.03}{36.4} + \frac{1}{6000} + \frac{1}{3000}$$
$$= 0.00025 + 0.000824 + 0.00016667 + 0.000333$$
$$U = 635.25\mathrm{W/(m^2 \cdot K)}$$

(*c*)
$$\frac{1}{U} = \frac{1}{6} + \frac{0.03}{36.4} + \frac{1}{12} + \frac{1}{3000}$$
$$= 0.16667 + 0.000824 + 0.08333 + 0.000333$$
$$U = 4\mathrm{W/(m^2 \cdot K)}$$

Note that in (*a*) the only significant thermal resistance is on the air side surface; in (*b*) none of the thermal resistances is negligible compared to the others; but in (*c*) the wall and fouling resistances are negligible compared to the others.

3.2.10 Biot Number

Consider a plane wall, of thickness $2L$ and thermal conductivity k, that separates two fluids having temperatures T_{f1} and T_{f2} as shown in Fig. 3.9. Let the heat transfer coefficients on both surfaces be the same h. The rate of heat transfer from fluid ① to fluid ② through the wall is

$$q = \frac{T_{f_1} - T_{f_2}}{\sum R_t} \tag{3.58a}$$

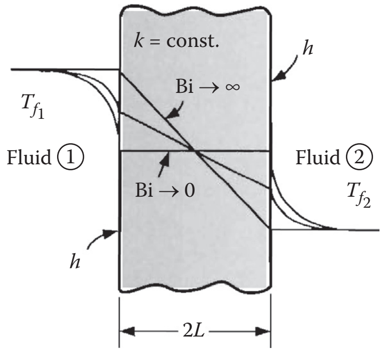

FIGURE 3.9
Temperature distribution in a plane wall for two limiting cases of Bi number.

where the total thermal resistance is given by

$$\sum R_t = \frac{2}{hA} + \frac{2L}{kA} \tag{3.58b}$$

As seen, there are two kinds of resistances to the heat flow from fluid ① to fluid ②; namely, the surface resistance $2/hA$ and the internal resistance $2L/kA$. The ratio of these two resistances is named the Biot number:

$$\text{Biot number} = \text{Bi} = \frac{\text{internal resistance}}{\text{surfacer resistance}} = \frac{hL}{k}$$

Two limiting cases of Bi are important:

1. The Biot number may be very large; that is, Bi → ∞. In this case, the total internal resistance is very large compared to the total surface resistance. That is,

$$\sum R_t \cong \frac{2L}{kA}$$

   Therefore, there will be almost no temperature drop on the surfaces. The temperatures of surfaces and fluids will be the same on both sides, and the temperature distribution in the wall will appear as shown in Fig. 3.9.
2. The Biot number may be very small; that is, Bi → 0. In this case, the total surface resistance is very large compared to the total internal resistance. That is,

$$\sum R_t \cong \frac{2}{hA}$$

   Therefore, the temperature drop in the wall will be negligible, and the temperature distribution will appear as indicated in Fig. 3.9.

3.2.11 Critical Thickness of Cylindrical Insulation

Adding insulation to pipes of small outside diameter does not always reduce heat loss. As the thickness of insulation around a pipe is increased, thermal resistance of the insulation layer increases logarithmically, but at the same time outer surface resistance decreases because of the increase in surface area. Since the total thermal resistance is proportional to the summation of these two resistances, heat transfer may increase instead of decreasing.

Let us consider the influence of thickness of cylindrical insulation on heat transfer. For simplicity, let us consider a single layer of insulation as shown in Fig. 3.10. The rate of heat flow per unit length of the cylinder can be written as

$$\frac{q}{L} = \frac{T_f - T_\infty}{R_t} \tag{3.59a}$$

where the total thermal resistance per unit length is given by

$$R_t = \frac{1}{2\pi}\left(\frac{1}{h_1 r_1} + \frac{1}{k_1}\ln\frac{r_2}{r_1} + \frac{1}{k}\ln\frac{r}{r_2} + \frac{1}{hr}\right) \tag{3.59b}$$

where r is the outer radius of the insulation layer. Differentiating this expression with respect to r, while holding h constant, we get

$$\frac{dR_t}{dr} = \frac{1}{2\pi r}\left(\frac{1}{k} - \frac{1}{hr}\right) \tag{3.60}$$

We see that for $k = hr$ this derivative vanishes. The radius at which $dR_t/dr = 0$ is called *critical radius*; that is,

$$r_{cr} = \frac{k}{h} \tag{3.61}$$

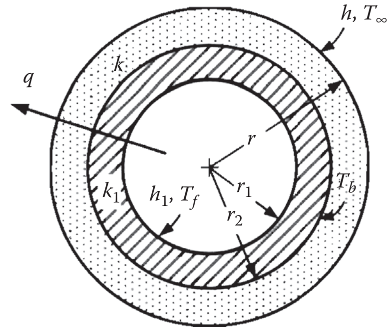

FIGURE 3.10
Single layer of insulation around a pipe.

One more differentiation of Eq. (3.60) yields

$$\frac{d^2R_1}{dr^2} = \frac{1}{2\pi r^2}\left(\frac{2}{hr} - \frac{1}{k}\right) \tag{3.62a}$$

At $r = r_{cr} = k/h$, this second derivative is

$$\frac{d^2R_t}{dr^2} = \frac{1}{2\pi}\frac{h^2}{k^3} > 0 \tag{3.62b}$$

Thus, at $r = r_{cr}$ the thermal resistance is a minimum, and therefore q/L is a maximum.

If $r_2 > k/h$, then adding insulation decreases the heat loss as illustrated in Fig. 3.11. On the other hand, if $r_2 < k/h$, then the heat loss increases with the increase in the thickness of insulation until $r_2 = k/h$. In other words, the addition of insulation has a deinsulating effect when $r_2 < k/h$.

Example 3.3

Does heat loss from a 2-in. outer diameter pipe decrease if asbestos of $k = 0.151$ W/(m·K) is added to insulate it? Assume that the heat transfer coefficient on the outer surface is $h = 5$ W/(m²·K).

SOLUTION

Since

$$r_{cr} = \frac{k}{h} = \frac{0.151}{5} = 0.032 \text{ m} = 3.2 \text{ cm}$$

and the outer radius of the pipe is 1 in (=2.54 cm) < 3.2 cm. heat loss from the pipe will increase until the insulation thickness is made greater than 0.66 cm.

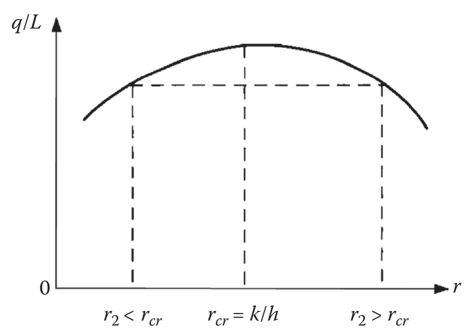

FIGURE 3.11
Relationship between heat loss and thickness of insulation.

3.3 One-Dimensional Steady-State Heat Conduction with Heat Sources

Heat conduction problems with internal energy sources are frequently encountered in several applications. For example, an electric current flowing through a body has the effect of an external energy addition (power input) to the internal portions of the body because of the dissipation due to electrical resistance. This is measured by the quantity I^2R_e [(current)^2 × electrical resistance]. Since the dissipated energy is to be transferred out of the body by some heat transfer mechanism, the effect is said to be a heat source distributed throughout the body. Processes that produce similar effects are chemical reactions distributed throughout a body, nuclear reactions in a fissionable material exposed to a neutron flux, change of phase, and biological problems of fermentation. Internal energy generation in the latter cases, however, cannot be identified as power input from an external source.

In the following sections we consider typical one-dimensional steady-state heat conduction problems that are idealizations of more involved problems frequently encountered in practice.

3.3.1 Plane Wall

Consider a plane wall of thickness of $2L$, as shown in Fig. 3.12. Internal energy is generated at a uniform rate of $\dot{q}$ per unit volume throughout this wall. It is exposed to a fluid at temperature T_f with a constant heat transfer coefficient h on both surfaces. It is assumed that the dimensions of the wall in the other two directions are sufficiently large so that heat flow may be considered as one dimensional. We also assume that the thermal conductivity k of the material of the wall is constant. The heat conduction equation that governs the temperature distribution in the wall under steady-state conditions can be obtained from the general heat conduction equation (2.18) as

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \tag{3.63}$$

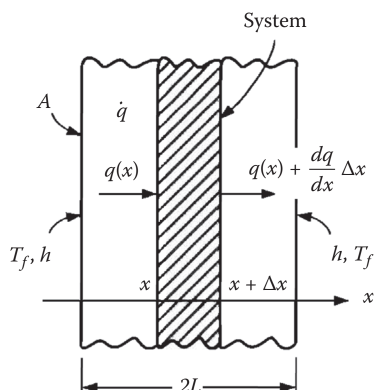

FIGURE 3.12
Plane wall with internal energy sources.

As mentioned in Section 3.2.1, the heat conduction equation in any problem may also be obtained starting from the basic principles. By doing so, one brings the physics of the problem into each phase of the derivation of the heat conduction equation. Proceeding according to the five steps outlined in Section 3.2.1, we now derive Eq. (3.63):

a. *Coordinate system*: Let the horizontal rightward direction be denoted by the positive x axis.
b. *System*: Consider the one-dimensional differential system of thickness Δx shown in Fig. 3.12.
c. *First law*: If internal energy generation is due to an electric current passing through this wall, then application of the first law of thermodynamics, Eq. (1.14*b*), to the system shown gives

$$q(x) - \left[q(x) + \frac{dq}{dx}\Delta x \right] + \dot{q}A\Delta x = 0 \tag{3.64a}$$

because $dE/dt = 0$ and $\dot{W} = -\dot{q}\,A\Delta x$.

If the generation of internal energy is due to chemical or nuclear reactions distributed throughout the body, then application of Eq. (1.14*b*) yields

$$q(x) - \left[q(x) + \frac{dq}{dx}\Delta x \right] = -\dot{q}A\Delta x \tag{3.64b}$$

because $dE/dt = -\dot{q}A\Delta x$ and $\dot{W} = 0$. In Eqs. (3.64*a*) and (3.64*b*), $q(x)$ represents the rate of heat transfer across the isothermal surface at location x.

Whatever the source of internal energy generation may be, the first law of thermodynamics gives

$$-\frac{dq}{dx} + \dot{q}A = 0 \tag{3.65}$$

d. *Fourier's law and differential equation*: Introduce Fourier's law of heat conduction:

$$q(x) = -kA\frac{dT}{dx} \tag{3.66}$$

Substitution of Eq. (3.66) into Eq. (3.65) yields

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \tag{3.67}$$

which is the same as Eq. (3.63).

e. *Boundary conditions*: Since the problem is a steady-state problem, no initial condition is required. The order of the x derivative in Eq. (3.67) requires that two boundary conditions be specified in the x direction. Before we specify the boundary conditions, the origin of the coordinate system must be identified. Noting the thermal as well as the geometric symmetries of the problem, the origin of the coordinate system is selected as shown in Fig. 3.13. Thus, the boundary conditions can be written in the form

$$\left(\frac{dT}{dx}\right)_{x=0} = 0 \quad \text{and} \quad \left(k\frac{dT}{dx} + hT\right)_{x=L} = hT_f \tag{3.68a,b}$$

Equation (3.67) and the boundary conditions (3.68*a*,*b*) complete the formulation of the problem. Integrating Eq. (3.67) twice yields

$$T(x) = -\frac{\dot{q}x^2}{2k} + C_1 x + C_2 \tag{3.69}$$

Applying the boundary condition (3.68*a*) we get $C_1 = 0$, and the boundary condition (3.68*b*) gives

$$C_2 = \frac{\dot{q}L^2}{2k} + \frac{\dot{q}L}{h} + T_f$$

Substituting these constants into Eq. (3.69), we obtain the following expression for the temperature distribution in the wall:

$$T(x) - T_f = \frac{\dot{q}L^2}{2k}\left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{\dot{q}L}{h} \tag{3.70}$$

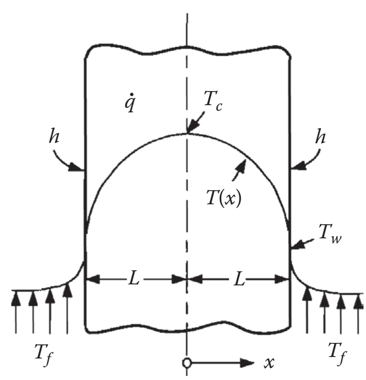

FIGURE 3.13
Temperature distribution in a plane wall with uniform internal energy generation.

The temperature at the midplane is then given by

$$T_c + T_f = \frac{\dot{q}L^2}{2k} + \frac{\dot{q}L}{h} \tag{3.71}$$

and the surface temperature is

$$T_w - T_f = \frac{\dot{q}L}{h} \tag{3.72}$$

Hence, the temperature drop from the midplane to the surface is

$$T_c - T_w = \frac{\dot{q}L^2}{2k} \tag{3.73}$$

The temperature distribution, Eq. (3.70), can be rewritten in dimensionless form as

$$\frac{T - T_f}{\dot{q}L^2/2k} = 1 - \left(\frac{x}{L}\right)^2 + \frac{1}{\text{Bi}} \tag{3.74}$$

where Bi = hL/k. If Bi $\gg$ 1, then Eq. (3.74) reduces to

$$\frac{T - T_f}{\dot{q}L^2/2k} = 1 - \left(\frac{x}{L}\right)^2 \tag{3.75}$$

Note that in this case $T_w \to T_f$, and Eq. (3.75) is the solution of the heat conduction equation (3.67) for the following boundary conditions:

$$\left(\frac{dT}{dx}\right)_{x=0} = 0 \quad \text{and} \quad T(L) = t_f \tag{3.76a,b}$$

The total internal energy generated in the wall should be equal to the energy lost to the surrounding fluid on both sides. Therefore, the energy lost to the surrounding fluid per unit time is given by

$$q_L = 2LA\dot{q} \tag{3.77}$$

The energy lost to the surrounding fluid per unit time can also be calculated as follows:

$$q_L = 2\left(-kA\frac{dT}{dx}\right)_{x=L} = -2kA\left(-\frac{L\dot{q}}{k}\right) = 2LA\dot{q} \tag{3.78}$$

which is, as expected, the same result as Eq. (3.77).

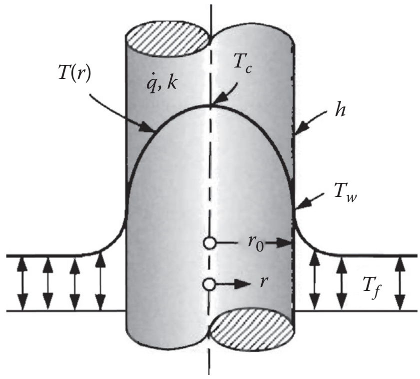

FIGURE 3.14
Solid cylinder with uniform internal energy generation.

3.3.2 Solid Cylinder

Consider a long solid cylinder of radius r_0 with uniformly distributed heat sources and of constant thermal conductivity, as shown in Fig. 3.14. This cylinder is exposed to a fluid at temperature T_f with a constant heat transfer coefficient h on the surface. The governing heat conduction equation from Eq. (2.18) is given by

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{\dot{q}}{k}=0 \tag{3.79}$$

where $\dot{q}$ is the uniform rate of internal energy generation per unit volume. The boundary conditions can be written as

$$\left(\frac{dT}{dr}\right)_{r=0}=0 \quad \text{and} \quad -k\left(\frac{dT}{dr}\right)_{r=r_0}=h\left[T(r)-T_1\right]_{r=r_0} \tag{3.80a,b}$$

Integrating Eq. (3.79) twice yields

$$T(r)=\frac{\dot{q}r^2}{4k}+C_1\ln r+C_2 \tag{3.81}$$

The boundary condition (3.80*a*), or the fact that the temperature should be finite at $r = 0$, requires that $C_1 = 0$. The second constant C_2 follows from the boundary condition (3.80*b*):

$$C_2=T_f+\frac{\dot{q}r_0^2}{4k}\left(1+\frac{2k}{hr_0}\right)$$

Thus, for the temperature distribution, we obtain

$$T(r)-T_f=\frac{\dot{q}r_0^2}{4k}\left[1-\left(\frac{r}{r_0}\right)^2+\frac{2k}{hr_0}\right] \tag{3.82}$$

The temperature drop from the centerline to the surface of the cylinder is then given by

$$(\Delta T)_{\max} = T_c - T_w = \frac{\dot{q} r_0^2}{4k} \tag{3.83}$$

In the limiting case when $h \to \infty$, $T_w \to T_f$, and the temperature distribution (3.82) reduces to

$$T(r) - T_w = \frac{\dot{q} r_0^2}{4k}\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{3.84}$$

The rate of heat release per unit length of the cylinder is

$$q'_L = \dot{q}\pi r_0^2 \tag{3.85}$$

Hence, Eq. (3.83) can be written as

$$(\Delta T)_{\max} = T_c - T_w = \frac{q'_L}{4\pi k} \tag{3.86}$$

Thus, for a given heat release rate per unit length (for example, from a fuel rod in a nuclear reactor) the maximum radial temperature difference in the rod is independent of the rod diameter. Conversely, for a fixed surface temperature T_w, the maximum heat release rate per unit length is determined by the maximum permissible temperature (i.e., the centerline temperature) and is independent of the rod diameter.

Example 3.4

A Cr-Ni steel wire, 2.5 mm in diameter and 30 cm in length, has a voltage of 10 V applied on it, while its surface is maintained at 90°C. Assuming that the resistivity of the wire is 70 μohm-cm and the thermal conductivity is 17.3 W/(m·K), calculate the centerline temperature.

SOLUTION

The temperature difference between the centerline and the surface is given by Eq. (3.83), where $\dot{q}$ is calculated from

$$\dot{q} = \frac{I^2 R_e}{\pi r_0^2 L} = \left(\frac{V}{R_e}\right)^2 \frac{R_e}{\pi r_0^2 L} = \frac{V^2}{\pi R_e r_0^2 L}$$

The electrical resistance, R_e, of the wire, on the other hand, is given by

$$R_e = \rho \times \frac{L}{A} = (70 \times 10^{-6}) \times \frac{30}{\pi (0.25/2)^2} = 4.27 \times 10^{-2} \text{ ohm}$$

Thus,

$$\dot{q} = \frac{(10)^2}{\pi \times 4.27 \times 10^{-2} \times (1.25 \times 10^{-3}) \times 0.3} = 1.59 \times 10^9 \, \mathrm{W/m^3}$$

The centerline temperature of the wire is then found to be

$$\begin{aligned} T_c &= T_w + \frac{\dot{q} r_0^2}{4k} = 90 + \frac{1.59 \times 10^9 \times (1.25 \times 10^{-3})^2}{4 \times 17.3} \\ &= 90 + 35.9 = 126°\mathrm{C} \end{aligned}$$

In general, the following relationship exists between the surface heat flux and the heat generation rate:

$$q_s'' = \frac{\dot{q} V}{A} \tag{3.87}$$

where V is the volume, A is total surface area and q_s'' is the surface heat flux, respectively. It follows from Eq. (3.87) that the surface heat flux depends only on the strength of the internal energy source and the ratio of volume to surface area.

Furthermore, it can also be shown that

$$(\Delta T)_{\max} = T_c - T_w = \frac{\dot{q} R^2}{mk} \tag{3.88}$$

where

Geometry	**m**	**R**	**V/A**
Slab	2	L	L
Cylinder	4	r_0	$r_0/2$
Sphere	6	r_0	$r_0/3$

3.3.3 Effect of Cladding

Nuclear reactors use mostly cylindrical fuel elements. The simplest arrangement, found in graphite-moderated and gas-cooled reactors fueled with natural uranium, consists of solid uranium rods of approximately 1-in, diameter stacked inside sealed cladding tubes of magnesium alloy (magnox). Consider now a fuel element as shown in Fig. 3.15. The thickness of the cladding material is $r_2 - r_1$. Internal energy is generated in the fuel rod due to fission reactions only. This is a multidomain problem. If we denote the properties of fuel and cladding by subscripts 1 and 2, respectively, we have

$$\frac{d^2 T_1}{dr^2} + \frac{1}{r}\frac{dT_1}{dr} + \frac{\dot{q}}{k_1} = 0, \quad 0 < r < r_1 \tag{3.89}$$

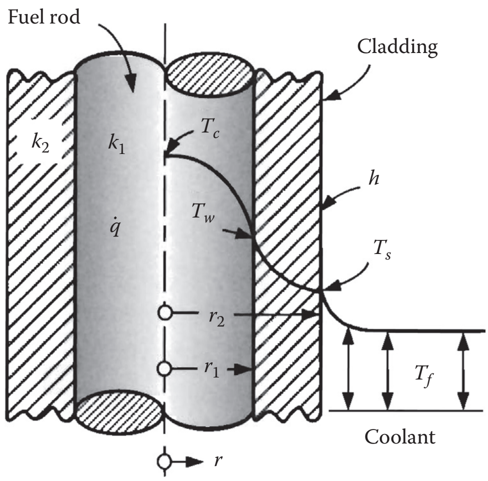

FIGURE 3.15
Fuel element with cladding.

$$\frac{d^2T_2}{dr^2} + \frac{1}{r}\frac{dT_2}{dr} = 0, \quad r_1 < r < r_2 \tag{3.90}$$

where $\dot{q}$ is the uniform rate of internal energy generation per unit volume in the fuel rod. The boundary conditions can be written as

$$\frac{dT_1(0)}{dr} = 0 \tag{3.91a}$$

$$T_1(r_1) = T_2(r_1) = T_w \tag{3.91b}$$

$$k_1\frac{dT_1(r_1)}{dr} = k_2\frac{dT_2(r_1)}{dr} \tag{3.91c}$$

$$T_2(r_2) = T_s \tag{3.91d}$$

where we have assumed perfect thermal contact between the fuel rod and the cladding, and the interface temperature is denoted by T_w. Generally, we do not expect to achieve perfect thermal contact. In some fuel element designs, in fact, there is usually a space between the fuel rod and the cladding which may, for example, be filled with helium. In addition, the surface temperature of the cladding, T_s, is assumed to be given. By solving Eqs. (3.89) and (3.90) with the boundary conditions (3.91) we obtain

$$T_w - T_s = \frac{q_L'}{2\pi k_2}\ln\frac{r_2}{r_1} \tag{3.92a}$$

and

$$T_c - T_s = \frac{q_L'}{4\pi k_1} + \frac{q_L'}{2\pi k_2} + \ln\frac{r_2}{r_1} \tag{3.92b}$$

where $q_L' = \dot{q}\pi r_1^2$ is the rate of heat dissipation per unit length of the fuel element.

In general, however, some allowance must be given for thermal contact resistance between the fuel rod and the cladding. The overall temperature drop from the center-line to the surface may then be expressed in the form

$$T_c - T_s = \frac{q_L'}{4\pi k_1} + \Delta T_c + \frac{q_L'}{2\pi k_2} + \ln\frac{r_2}{r_1} \tag{3.93}$$

where ΔT_c is the temperature drop at the interface.

If the coolant temperature T_f and the surface heat transfer coefficient h are known, then the rate of heat dissipation per unit length can be written as

$$q_L' = 2\pi r_2 h (T_s - T_f) \tag{3.94a}$$

which yields

$$T_s - T_f = \frac{q_L'}{2\pi r_2 h} = \frac{\dot{q} r_1^2}{2 h r_2} \tag{3.94b}$$

Combining Eqs. (3.93) and (3.94*b*), we find the temperature difference between the center of the rod and the coolant as

$$T_c - T_f = \frac{q_L'}{4\pi k_1} + \Delta T_c + \frac{q_L'}{2\pi k_2} + \ln\frac{r_2}{r_1} + \frac{q_L'}{2\pi r_2 h} \tag{3.95}$$

Similarly, for a large flat plate of a fissionable material with cladding on both sides, the following relation is obtained:

$$T_c - T_f = \frac{\dot{q}L^2}{2k_1} + \frac{\dot{q}L}{k_2} a + \frac{\dot{q}L}{h} + \Delta T_c \tag{3.96}$$

where $2L$ and a are the thicknesses of the plate and cladding, respectively.

3.4 Temperature-Dependent Thermal Conductivity

When the thermal conductivity is temperature dependent, the heat conduction equation becomes nonlinear and, under steady-state conditions is given, from Eq. (2.14), by

$$\nabla \cdot (k \nabla T) + \dot{q} = 0 \tag{3.97}$$

Although this is a nonlinear partial differential equation, it can be reduced to a linear differential equation by means of the Kirchhoff transformation we introduced in Section 2.4 by defining a new temperature function θ as

$$\theta = \frac{1}{k_0}\int_{T_0}^{T} k(T')\,dT' \tag{3.98}$$

where T_0 denotes a convenient reference temperature and $k_0 = k(T_0)$. It follows from Eq. (3.98) that

$$\nabla\theta = \frac{k}{k_0}\nabla T \tag{3.99}$$

Therefore, Eq. (3.97) can be rewritten as

$$\nabla^2\theta + \frac{\dot{q}}{k_0} = 0 \tag{3.100}$$

which is similar to the heat conduction equation for constant k. Hence, a steady-state problem with temperature-dependent thermal conductivity poses no problem because the transformed equation can be solved with the usual techniques, provided that the boundary conditions can also be transformed. As we discussed in Section 2.4, if a boundary condition is of either the first or second kind, then it can be transformed. A boundary condition of the third kind, in general, cannot be transformed. Problems involving such boundary conditions are usually solved using numerical techniques.

We have already discussed in Section 3.2.1 a steady-state, one-dimensional heat conduction problem with temperature-dependent thermal conductivity, but without heat sources. Here we consider another one-dimensional problem with temperature-dependent thermal conductivity that contains uniform heat sources and implement the Kirchhoff transformation.

We now consider a long solid rod of radius r_0. Assume that the internal energy is generated at a uniform rate $\dot{q}$ per unit volume and the surface is maintained at a uniform temperature T_w. The formulation of the problem is then given by

$$\frac{1}{r}\frac{d}{dr}\left[rk(T)\frac{dT}{dr}\right] + \dot{q} = 0 \tag{3.101}$$

with

$$\left(\frac{dT}{dr}\right)_{r=0} = 0 \quad \text{and} \quad T(r_0) = T_w \tag{3.102a,b}$$

Defining a new temperature function $\theta(r)$ as

$$\theta(r) = \frac{1}{k_w}\int_{T_w}^{T(r)} k(T')\,dT' \tag{3.103}$$

where $k_w = k(T_w)$, we can transform Eq. (3.101) and the conditions (3.102) to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{\dot{q}}{k_w}=0 \tag{3.104}$$

$$\left(\frac{d\theta}{dr}\right)_{r=0}=0 \quad \text{and} \quad \theta(r_0)=0 \tag{3.105a,b}$$

The solution of this problem for $\theta(r)$ is given by

$$\theta(r)=\frac{\dot{q}r_0^2}{4k_w}\left[1-\left(\frac{r}{r_0}\right)^2\right] \tag{3.106}$$

Introducing Eq. (3.106) into Eq. (3.103), we obtain

$$\int_{T_w}^{T(r)} k(T')\,dT'=\frac{\dot{q}r_0^2}{4}\left[1-\left(\frac{r}{r_0}\right)^2\right] \tag{3.107}$$

This relation cannot be written explicitly for $T(r)$ unless the relation $k = k(T)$ is given. At $r = 0$, this equation yields

$$\int_{T_w}^{T_c} k(T)\,dT=\frac{\dot{q}r_0^2}{4} \tag{3.108}$$

where T_c is the centerline temperature. For constant k, Eq. (3.107) reduces to Eq. (3.84), as expected.

Example 3.5

Find the rate of heat generation per unit volume in a rod that will produce a centerline temperature of 2000°C for the following conditions:

$$r_0 = 1\,\text{cm},\ T_w = 350°\text{C and } k=\frac{3167}{T+273}$$

where T is in °C and k in W/(m·K). Also, calculate the surface heat flux.

SOLUTION

From Eq. (3.108) we have

$$\dot{q}=\frac{4}{r_0^2}\int_{350}^{2000}\frac{3167}{T+273}dT=\frac{4\times 3167}{(0.01)^2}\ln\frac{2273}{623}=1.64\times 10^8\,\text{W/m}^3$$

The surface heat flux, from Eq. (3.87), is then given by

$$q_s'' = \frac{\dot{q}\pi r_0^2 L}{2\pi r_0 L} = \frac{\dot{q} r_0}{2} = \frac{1.64 \times 10^8 \times 0.01}{2} = 8.2 \times 10^6 \, \mathrm{W/m^3}$$

3.5 Space-Dependent Internal Energy Generation

Internal energy generation, or simply heat generation, may in general be a function of space coordinates and/or time. In a cylindrical nuclear fuel element, for example, the rate of heat generation per unit volume will be in the form

$$\dot{q}(r) = \dot{q}_0 I_0(\chi r) \tag{3.109}$$

where $\dot{q}_0$ is the heat generation rate per unit volume at the centerline of the fuel rod, $I_0(\chi r)$ is the modified Bessel function of the first kind of order zero (see Appendix B), and χ is the reciprocal of the "neutron diffusion length." If we assume that the thermal conductivity of the fuel is constant, then by substituting Eq. (3.109) into Eq. (3.79), we obtain

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = \frac{\dot{q}_0 I_0(\chi r)}{k} \tag{3.110}$$

The solution of Eq. (3.110) can be written as

$$T(r) = -\frac{\dot{q}_0 I_0(\chi r)}{\chi^2 k} + A \ln r + B \tag{3.111}$$

Therefore, the temperature distribution $T(r)$ in the fuel is given, in terms of the surface temperature T_w at $r = r_0$, by

$$T(r) - T_w = \frac{\dot{q}_0}{\chi^2 k}\left[I_0(\chi r_0) - I_0(\chi r)\right] \tag{3.112}$$

The relationship between the surface heat flux, q_s'', and the rate of heat generation per unit volume, $\dot{q}$, on the other hand is

$$2\pi r_0 q_s'' = \int_0^{r_0} 2\pi r \dot{q}(r)\, dr \tag{3.113}$$

Substituting Eq. (3.109) into Eq. (3.113), and combining the result with Eq. (3.112), we get the following expression for the temperature distribution in terms of the surface heat flux:

$$T(r) - T_w = -\frac{q_s''}{k}\frac{I_0(\chi r_0) - I_0(\chi r)}{\chi I_1(\chi r_0)} \tag{3.114}$$

If $k = k(T)$, it can be shown that

$$\int_{T_w}^{T(r)} k(T')dT' = q_s'' \frac{I_0(\chi r_0) - I_0(\chi r)}{\chi I_1(\chi r_0)} \tag{3.115}$$

which, of course, cannot be written explicitly for $T(r)$ unless the relation $k = k(T)$ is specified. Equation (3.115) reduces, as it should, to Eq. (3.114) when k = constant.

3.6 Extended Surfaces: Fins and Spines

In conventional heat exchangers, heat is transferred from one fluid to another through a metal wall, and the rate of heat transfer is directly proportional to the surface area of the wall and the temperature difference between the fluids. In most cases, however, the temperature difference is limited. Therefore, increasing the rate of heat transfer depends on increasing the *effective* heat transfer area. The effective heat transfer area on a solid surface can be increased by attaching thin metal strips, called fins, or spines to the surface. Although attachment of metal fins or spines increases the effective heat transfer area, the average surface temperature of the fins or spines will not be the same as the original surface temperature, but will be closer to the surrounding fluid temperature. This causes the rate of heat transfer to be somewhat less than proportional to the extent of total heat transfer area. Fins and spines can be of several types. Some commonly used ones are shown in Fig. 3.16.

In this section we restrict our discussions to the steady-state performances of one-dimensional extended surfaces under the following assumptions:

a. Heat flow in the extended surface is steady.

b. Thermal conductivity of the material of the extended surface is constant.

c. There are no heat sources within the extended surface.

d. Temperature of the surrounding fluid is uniform and constant.

e. The heat transfer coefficient between the extended surface and the surrounding fluid is constant, and it includes the combined effect of convection and radiation.

f. Temperature of the base of the extended surface is constant.

g. Thickness of the extended surface is so small compared to its length such that the temperature gradients normal to the surface may be neglected; that is, temperature distribution in the extended surface is one dimensional.

Of these assumptions, *e*, *f*, and *g* may be questionable. Although the value of heat transfer coefficient on the surface of a fin or spine varies from point to point, the use of an average value in analytical studies gives heat transfer results that are in good agreement with experimental measurements. For most fins or spines of practical interest, on the other hand, the error introduced by assumption *g* is less than 1%.

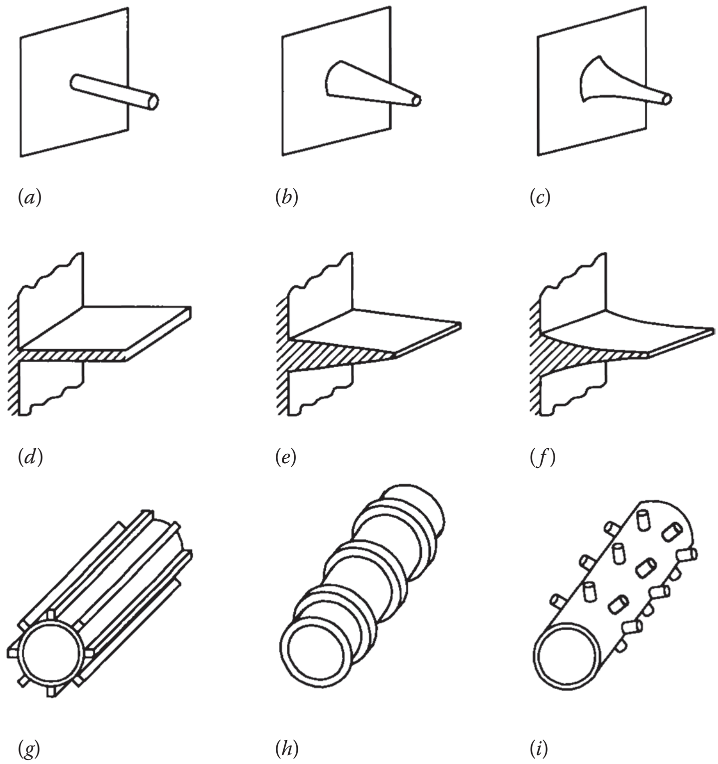

FIGURE 3.16
Several types of extended surfaces: (*a*) cylindrical spine; (*b*) truncated conical spine; (*c*) parabolic spine; (*d*) longitudinal fin of rectangular profile; (*e*) longitudinal fin of trapezoidal profile; (*f*) longitudinal fin of parabolic profile; (*g*) cylindrical tube equipped with straight fins of rectangular profile; (*h*) cylindrical tube equipped with annular fins of rectangular profile; (*i*) cylindrical tube equipped with cylindrical spines.

Consider now the diffusion of heat in the extended surface shown in Fig. 3.17. An energy balance (i.e., the first law of thermodynamics), Eq. (1.14*b*), when applied to the system shown in Fig. 3.17 gives

$$q(x) = q(x + \Delta x) + q_{\text{conv.}} \tag{3.116}$$

Since, as $\Delta x \to 0$,

$$q(x + \Delta x) = q(x) + \frac{dq}{dx}\Delta x \tag{3.117}$$

and

$$q_{\text{conv}} = hP(x)\Delta x(T - T_\infty) \tag{3.118}$$

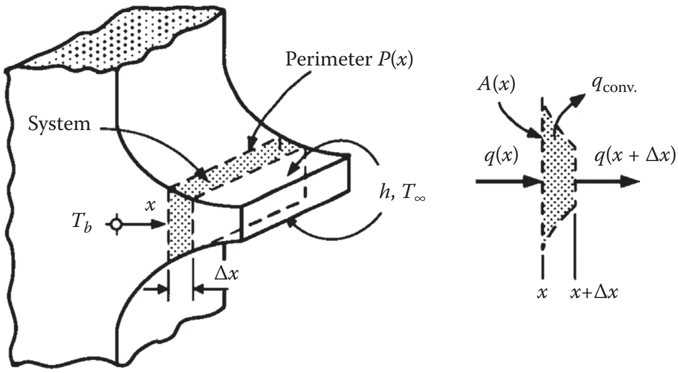

FIGURE 3.17
Energy balance on a system in a one-dimensional extended surface.

Eq. (3.116) reduces to

$$\frac{dq}{dx} + hP(x)(T - T_\infty) = 0 \tag{3.119}$$

where $P(x)$ is the perimeter of the extended surface at x, T_∞ is the surrounding fluid temperature, and h is the heat transfer coefficient. On the other hand, from Fourier's law of heat conduction, we have

$$q(x) = -kA(x)\frac{dT}{dx} \tag{3.120}$$

where $A(x)$ is the cross-sectional area of the extended surface normal to the x direction. Substitution of Eq. (3.120) into Eq. (3.119) yields

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T - T_\infty) = 0 \tag{3.121}$$

Defining a new temperature function by $\theta(x) = T(x) - T_\infty$, and since T_∞ is constant, Eq. (3.121) can be rewritten as

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right] - \frac{hP(x)}{k}\theta = 0 \tag{3.122}$$

This is the heat conduction equation that governs the variation of the temperature distribution in the extended surface. Since this is a second-order ordinary differential equation, two boundary conditions are needed in the x direction, one related to the base and the other to the tip of the extended surface.

3.6.1 Extended Surfaces with Constant Cross Sections

For an extended surface with constant cross section, Eq. (3.122) reduces to

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{3.123}$$

where $m^2 = hP/kA$. The general solution of this differential equation can be written in the form

$$\theta(x) = C_1 e^{-mx} + C_2 e^{mx} \tag{3.124a}$$

or

$$\theta(x) = C_3 \sinh mx + C_4 \cosh mx \tag{3.124b}$$

where C_1 and C_2, or C_3, and C_4, are constants of integration to be determined from the boundary conditions. Since the base temperature, T_b, was assumed to be constant, the boundary condition at $x = 0$ is

$$T(x)\Big|_{x=0} = T_b \quad \text{or} \quad \theta(x)\Big|_{x=0} = T_b - T_\infty = \theta_b \tag{3.125}$$

The second boundary condition at the tip of the extended surface depends on the nature of the problem. Several cases may be considered:

Case 1. The extended surface under consideration can be very long as shown in Fig. 3.18. In this case, the temperature at the tip is essentially equal to the temperature of the surrounding fluid. The second boundary condition can therefore be written as

$$T(x)\Big|_{x\to\infty} = T_\infty \quad \text{or} \quad \theta(x)\Big|_{x\to\infty} = 0 \tag{3.126}$$

The boundary conditions (3.125) and (3.126) give $C_1 = \theta_b$ and $C_2 = 0$ in Eq. (3.124a). Hence, the temperature distribution is found to be

$$\theta(x) = \theta_b e^{-mx} \tag{3.127a}$$

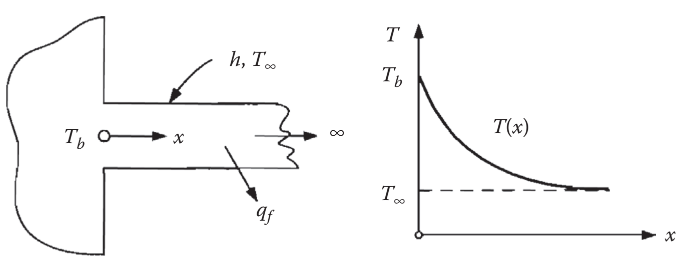

FIGURE 3.18
Long extended surface.

or

$$\frac{T - T_\infty}{T_b - T_\infty} = e^{-mx} \tag{3.127b}$$

The rate of heat loss from the extended surface can now be determined by integrating the local convective heat transfer over the whole length:

$$\begin{aligned} q_f &= \int_0^\infty hP\,dx(T - T_\infty) = hP\int_0^\infty \theta(x)\,dx = hP\theta_b \int_0^\infty e^{-mx}dx \\ &= \frac{hP\theta_b}{m} = \sqrt{hPkA}\,\theta_b \end{aligned} \tag{3.128}$$

Under steady-state conditions, the heat transferred from the extended surface by convection to the surrounding fluid must be equal to the heat conducted to the extended surface at the base. Hence, we may also evaluate the heat transfer rate from the extended surface by applying Fourier's law at the base:

$$\begin{aligned} q_f &= -kA\left(\frac{dT}{dx}\right)_{x=0} = -kA\left(\frac{d\theta}{dx}\right)_{x=0} \\ &= -kA\theta_b \frac{d}{dx}(e^{-mx})_{x=0} = kA\theta_b m = \sqrt{hPkA}\,\theta_b \end{aligned} \tag{3.129}$$

Since it involves differentiation, this method is usually easier to apply than the first method which requires integration.

Example 3.6

A very long rod of 2.5 cm diameter is heated at one end. Under steady-state conditions, the temperatures at two different locations along the rod, which are 7.5 cm apart, are measured to be 125°C and 90°C, while the surrounding air temperature is 25°C. Assuming that the heat transfer coefficient is 20 W/(m^2·K), estimate the value of thermal conductivity of the rod.

SOLUTION

Since the rod is very long, the temperature distribution will be given by Eq. (3.127*b*). Therefore, we can write

$$T(x_1) - T_\infty = (T_b - T_\infty)e^{-mx_1} \quad \text{and} \quad T(x_2) - T_\infty = (T_b - T_\infty)e^{-mx_2}$$

from which we obtain

$$\frac{T(x_1) - T_\infty}{T(x_2) - T_\infty} = e^{m(x_2 - x_1)}$$

or substituting the numerical values we get

$$\frac{125-25}{90-25} = e^{7.5m}$$

which gives

$$m = 5.74\,\mathrm{m}^{-1}$$

Since

$$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{4h\pi D}{k\pi D^2}} = \sqrt{\frac{4h}{kD}}$$

where D is the diameter of the rod, we obtain

$$k = \frac{4h}{m^2 D} = \frac{4\times 20}{(5.76)^2 \times 2.5\times 10^{-2}} = 97\,\mathrm{W/(m{\cdot}k)}$$

Case 2. The extended surface may be of finite length L as shown in Fig. 3.19, and heat is lost to the surrounding fluid by convection from its end. In this case, the boundary condition at $x = L$ is given by

$$-k\left(\frac{dT}{dx}\right)_{x=L} = h_e[T(L) - T_\infty] \tag{3.130}$$

where h_e is the heat transfer coefficient at the end of the extended surface, which may or may not be equal to h. From the boundary condition (3.125) we have $C_4 = \theta_b$ in the general solution (3.124b), and the boundary condition (3.130) yields

$$C_3 = -\theta_b \frac{mk \sinh mL + h_e \cosh mL}{mk \cosh mL + h_e \sinh mL} \tag{3.131}$$

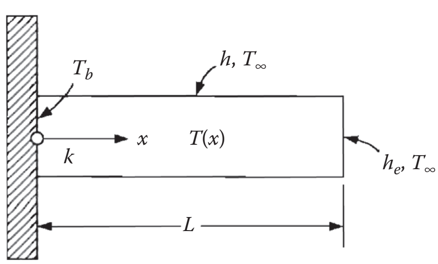

FIGURE 3.19
Extended surface of finite length L.

Substituting C_3 and $C_4 = \theta_b$ into Eq. (3.124*b*), we obtain

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh[m(L-x)] + N\sinh[m(L-x)]}{\cosh mL + N\sinh mL} \tag{3.132}$$

where $N = h_e/mk$.

Following the same procedure as in the previous case, one can easily show that the rate of heat transfer from the extended surface to the surrounding fluid is given by

$$q_f = \sqrt{hPkA}\,\theta_b \frac{\sinh mL + N\cosh mL}{\cosh mL + N\sinh mL} \tag{3.133}$$

The temperature at the tip of the extended surface (i.e., at $x = L$) is found from Eq. (3.132) to be

$$\frac{\theta(L)}{\theta_b} = \frac{T(L) - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL + N\sinh mL} \tag{3.134}$$

If the heat loss from the tip of the extended surface under consideration is negligible, or if the end is insulated, then the temperature distribution can be obtained from Eq. (3.132) by letting $N = 0$ (why?) as

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh[m(L-x)]}{\cosh mL} \tag{3.135}$$

and the rate of heat loss from the extended surface becomes

$$q_f = \sqrt{hPkA}\,\theta_b \tanh mL \tag{3.136}$$

The tip temperature at $x = L$ is then given by

$$\frac{\theta(L)}{\theta_b} = \frac{T(L) - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL} \tag{3.137}$$

Equations (3.134) and (3.137) may be applied to the problem of temperature measurement. That is, they can be used, for example, to determine the tip temperature of a thermocouple immersed in a gas stream at a temperature different from that of the duct wall which supports the thermocouple (see Problem 3.16).

Example 3.7

An aluminum rod, $k = 206$ W/(m·K), 2.5 cm in diameter and 15 cm in length, protrudes from a wall which is at 260°C. The rod is surrounded by a fluid at 16°C, and the heat transfer coefficient on the surface of the rod is 15 W/(m^2·K). Calculate the rate of heat loss from the rod.

SOLUTION

The rate of heat loss from the rod can be calculated from Eq. (3.133). We have

$$P = \pi D = \pi \times 2.5 \times 10^{-2} = 7.85 \times 10^{-2}\,\mathrm{m}$$
$$A = \frac{\pi D^2}{4} = \frac{\pi \times (2.5 \times 10^{-2})^2}{4} = 4.91 \times 10^{-4}\,\mathrm{m}^2$$

Hence.

$$m = \sqrt{\frac{hP}{kA}} = \left(\frac{15 \times 7.85 \times 10^{-2}}{206 \times 4.91 \times 10^{-4}}\right)^{1/2} = 3.412\mathrm{m}^{-1}$$

$$mL = 3.412 \times 0.15 = 0.512$$

$$N = \frac{h}{km} = \frac{15}{206 \times 3.412} = 0.0213$$

$$\sinh mL = 0.5346$$

$$\cosh mL = 1.1340$$

$$\sqrt{hPkA} = (15 \times 7.85 \times 10^{-2} \times 206 \times 4.91 \times 10^{-4})^{1/2} = 0.345\mathrm{W/K}$$

Substituting these values into Eq. (3.133). we obtain

$$q_f = 0.345 \times (260 - 16) \times \frac{0.5346 + 0.0213 \times 1.1340}{1.1340 + 0.0212 \times 0.5346} = 41.07\mathrm{W}$$

3.6.2 Rectangular Fin of Least Material

Consider the straight fin of rectangular profile shown in Fig. 3.20. When heat transfer from the tip is negligible, the rate of heat loss from this fin is given by

$$q_f = \sqrt{hPk\delta l}\theta_b \tanh mL \tag{3.138}$$

Since the width of the fin is much larger than its thickness (i.e., $\delta \ll 1$), $P = 2l$. Therefore,

$$q_f = \sqrt{2hk\delta l^2}\theta_b \tanh\left(\sqrt{\frac{2h}{k}}\frac{L\delta}{\delta^{3/2}}\right) \tag{3.139}$$

Schmidt [17] has suggested that the criterion for the most favorable fin dimensions might be maximum value of q_f/θ_b, for a given amount of fin material (which is proportional to the

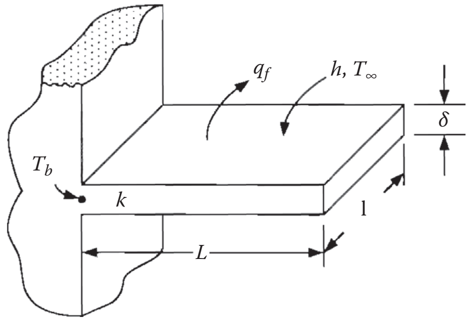

FIGURE 3.20
Straight fin of rectangular profile.

product $L\delta$). Taking h, k, and the product $L\delta$ to be constants, if we differentiate Eq. (3.139) with respect to δ and set the result equal to zero, we obtain

$$M = \frac{1}{6}\sinh 2M \tag{3.140}$$

where

$$M = mL = \sqrt{\frac{2h}{k}}\frac{L\delta}{\delta^{3/2}} \tag{3.141}$$

Equation (3.140) is a transcendental equation which can be solved numerically or graphically by plotting both sides against M and determining the point of intersection of the two curves. Thus, the value $M = mL = 1.4192$ that satisfies Eq. (3.140) is obtained. The most favorable thickness-to-length ratio is then given by

$$\frac{\delta}{L} = \frac{2}{(1.4192)^2}\frac{hL}{k} \tag{3.142}$$

Combining this result with Eq. (3.139) we obtain

$$L = \frac{0.7978}{h}\frac{q_f}{\theta_b} \tag{3.143}$$

and

$$\delta = \frac{0.6321}{hk}\left(\frac{q_f}{\theta_b}\right)^2 \tag{3.144}$$

The temperature at the tip of this favorable fin from Eq. (3.137) will be

$$\frac{\theta(L)}{\theta_b} = \frac{1}{\cosh mL} = 0.45706 \tag{3.145}$$

Equation (3.145) provides a test as to whether or not a given L is the optimum length by measurement of $\theta(L)$ and θ_b.

3.6.3 Performance Factors

To compare and evaluate extended surfaces in augmenting heat transfer from the base area two factors are used: *fin effectiveness* and *fin efficiency.* Fin effectiveness ϕ is defined as the ratio of the rate of heat transfer from an extended surface to the rate of heat transfer that would take place from the same base area A_b, without the extended surface, with the base temperature T_b remaining constant; that is,

$$\phi = \frac{q_{\text{fin}}}{q_{\text{base}}} = \frac{\int_{A_f} h\theta(x)\,dA}{hA_b\theta_b} = \frac{\int_{A_f} \theta(x)\,dA}{A_b\theta_b} \tag{3.146}$$

where A_f is the total surface area over which the extended surface transfers heat to its surrounding fluid.

The fin efficiency η is defined as the ratio of the rate of heat transfer from an extended surface to the rate of heat transfer if the extended surface were uniformly at the base temperature, T_b, throughout its length; that is,

$$\eta_f = \frac{q_{\text{fin}}}{q_{\text{fin}}(\text{when } \theta = \theta_b \text{ along the fin})} = \frac{\int_{A_f} h\theta(x)\,dA}{hA_f\theta_b} = \frac{\int_{A_f} \theta(x)\,dA}{A_f\theta_b} \tag{3.147}$$

A comparison of the expressions for ϕ and η_f indicates that

$$\phi = \frac{A_f}{A_b}\eta_f \tag{3.148}$$

For example, the effectiveness and efficiency of the extended surface shown in Fig. 3.19 are obtained by combining Eq. (3.133) with Eqs. (3.146), and (3.147) as

$$\phi = \sqrt{\frac{kP}{hA}}\,\frac{\sinh mL + N\cosh mL}{\cosh mL + N\sinh mL} \tag{3.149}$$

and

$$\eta_f = \frac{\sqrt{hPkA}}{hPL + h_e A}\,\frac{\sinh mL + N\cosh mL}{\cosh mL + N\sinh mL} \tag{3.150}$$

If heat transfer from the tip is negligible or if the tip is insulated (i.e., $N \cong 0$), then Eqs. (3.149) and (3.150) reduce to

$$\phi = \sqrt{\frac{kP}{hA}}\tanh mL \tag{3.151}$$

and

$$\eta_f = \frac{\tanh mL}{mL} \tag{3.152}$$

Equations (3.151) and (3.152) can also be obtained from Eq. (3.136).

In engineering applications, the fin efficiency η_f is more widely used. In literature, the efficiencies for various fin configurations have been presented in chart forms [5,6].

For a fin of given material and shape, the efficiency decreases as h increases. For example, a fin that is highly efficient when used with a gas coolant will usually be found inefficient when used with water where the value of h is usually much larger.

3.6.4 Heat Transfer from a Finned Wall

The rate of heat transfer from the fins on a wall, from Eq. (3.147), would be

$$q_f = \eta_f h a_f (T_b - T_\infty) \tag{3.153}$$

where a_f is the *total* heat transfer surface area of the fins. On the other hand, the rate of heat removed from the wall between the fins is given by

$$q_w = h a_w (T_b - T_\infty) \tag{3.154}$$

where *aw* is the *total* wall surface area between the fins. Therefore, the total rate of heat transfer is

$$q_T = q_f + q_w = h(a_w + \eta_f a_f)(T_b - T_\infty) = h\, a_{\text{eff}}(T_b - T_\infty) \tag{3.155}$$

where $a_{\text{eff}} = a_w + \eta_f a_f$ is the *effective heat transfer area* of the wall. Consequently, if h = constant, the rate of heat transfer is increased by a factor of $(a_w + \eta_f a_f)/(a_w + a_b)$, where a_b is the total base area of the fins.

Example 3.8

Fuel elements in a nuclear reactor consist of 0.25-cm thick fuel plates of fissionable material (alloy of uranium and zirconium) with a 0.050-cm thick protective cladding of zirconium on each side as illustrated in Fig. 3.21*a*. The coolant flows over the outside surface of the cladding at 260°C, and the heat transfer coefficient is 2500 W/(m²·K). The thermal conductivities of the fuel and the cladding are 25 W/(m·K) and 21 W/(m·K), respectively.

(*a*) Determine the maximum temperature within the fuel elements and the temperature of the outer surface of the cladding when the fuel plates are operating with a uniform internal heat generation rate of $\dot{q} = 8 \times 10^8$ W/m³.

(*b*) To increase the heat transfer to the coolant, one engineer proposed that fins be added to the fuel elements as illustrated in Fig. 3.21*b*. Suppose that 0.25-cm thick and 0.625-cm long zirconium fins are added to the cladding surface, spaced so as to provide two fins per cm. By what factor is the heat transfer rate increased? Assume that the ends of the fins are insulated.

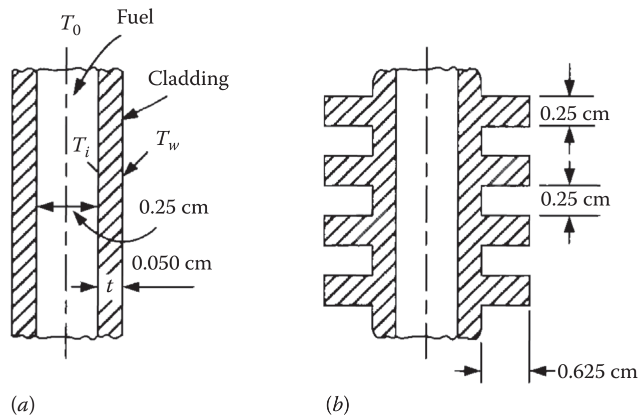

FIGURE 3.21
Figure for Example 3.8.

SOLUTION

(*a*) The rate of heat transfer through the cladding is given by

$$q = UA(T_i - T_\infty)$$

with

$$\frac{1}{U} = \frac{1}{h} + \frac{t}{k} = \frac{1}{2500} + \frac{0.050 \times 10^{-2}}{21} = 0.424 \times 10^{-3}\,\mathrm{m^2 \cdot K/W}$$

where T_∞ = coolant temperature, and t = thickness of the cladding. The surface heat flux, on the other hand, is

$$q_s'' = \dot{q}L = 8 \times 10^8 \times \frac{0.25 \times 10^{-2}}{2} = 10^6\,\mathrm{W/m^2}$$

Therefore,

$$T_1 - T_\infty = \frac{q}{UA} = \frac{q_s''}{U} = 10^6 \times 0.424 \times 10^{-3} = 424°\mathrm{C}$$

The temperature of the midplane is then given by

$$T_0 - T_i = \frac{\dot{q}L^2}{2k} = \frac{8 \times 10^8 \times (0.125 \times 10^{-2})^2}{2 \times 24} = 26°\mathrm{C}$$

Thus, the maximum temperature within the fuel elements is

$$T_0 = 26 + 424 + 260 = 710°\mathrm{C}$$

The outer surface temperature of the cladding is obtained from

$$T_w = \frac{q_s''}{h} + T_\infty = \frac{10^6}{2500} + 260 = 400 + 260 = 660°\mathrm{C}$$

(*b*) Under steady-state conditions, the rate of heat transfer from the fuel elements must be equal to the rate of internal energy generation in the fuel plates. Therefore, the rate of heat transfer will not be increased by adding fins. It will be the same as long as $\dot{q}$ and the thickness of the fuel plates remain unchanged.
Is the maximum fuel element temperature reduced by adding fins?

3.6.5 Limit of Usefulness of Fins

When the heat transfer coefficient h is high compared to k/δ for a straight rectangular fin as shown in Fig. 3.22, the addition of such a fin to a solid surface may, in fact, decrease the heat transfer rate. From the definition of fin effectiveness, Eq. (3.146), this means that ϕ would be less than unity. To illustrate, consider the fin shown in Fig. 3.22. For this fin

$$N = \frac{h}{mk} = \sqrt{\frac{hA}{kP}} = \sqrt{\frac{h}{k}\delta}$$

Substituting this into Eq. (3.149), we get

$$\phi = \frac{1}{N}\frac{\sinh(NL/\delta) + N\cosh(NL/\delta)}{\cosh(NL/\delta) + N\sinh(NL/\delta)}$$

The values of ϕ for various values of L/δ and N are given in Table 3.1.

Table 3.1 shows that when $N \geq 1$, the effectiveness $\phi < 1$; that is, the addition of fins to a solid surface will decrease the heat transfer from the surface. If, on the other hand, $N < 1$ (and $L > \delta$), that is, if

$$\frac{h\delta}{k} < 1$$

the provision of fins will be worthwhile, and fins will improve the heat transfer from the surface.

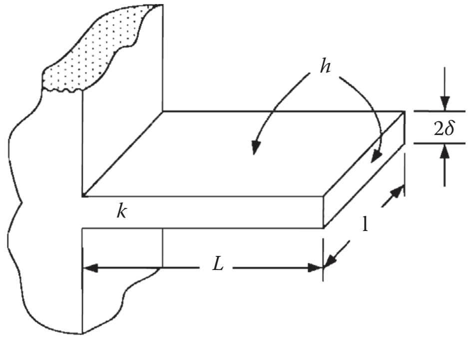

FIGURE 3.22
Rectangular fin of thickness 2δ.

TABLE 3.1

Effectiveness ϕ for Various Values of L/δ and N for The Fin of Fig. 3.22

L/δ	$N = 2$	$N = 1$	$N = 0.5$	$N = 0.25$
1	0.4153	0.6321	0.8427	0.9518
2	0.4879	0.8646	1.3580	1.7585
5	0.4999	0.9932	1.9909	3.8370
10	0.5000	0.9999	1.9992	3.9597
15	0.5000	0.9999	1.9992	3.9597
20	0.5000	0.9999	1.9994	3.9890

3.6.6 Extended Surfaces with Variable Cross Sections

In determining an optimum fin for a given application, one may ask whether or not weight advantage can be gained by using fins of shapes other than the constant cross section considered thus far. The differential equation that will be satisfied by the temperature distribution in extended surfaces with variable cross sections has already been given by Eq. (3.122). Since A and P are no longer constants, this equation is now a differential equation with variable coefficients. The two boundary conditions in the x direction, namely one at the base and the other at the tip of the fin, will complete the formulation.

Triangular fins. In the discussion that follows, consider a straight fin of triangular profile as shown in Fig. 3.23. The mathematical treatment in this case is similar to the case of fins of rectangular profile except that the cross-sectional area normal to the heat flow is a function of the distance along the fin, decreasing as the fin length increases. From Fig. 3.23, we see that

$$A(x) = \frac{bx}{L}l \quad \text{and} \quad P(x) = 2\left(\frac{bx}{L} + l\right) \tag{3.156a,b}$$

where l is the width of the fin. If we assume that $b \ll l$, then $P(x) \cong 21$. Inserting these values into Eq. (3.122) we get

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0 \tag{3.157a}$$

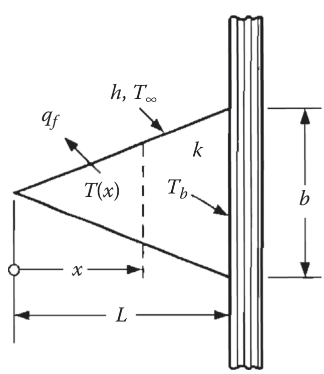

FIGURE 3.23
A straight fin of triangular profile.

or

$$x^2 \frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2 x\theta = 0 \tag{3.157b}$$

where $m^2 = 2hL/kb$. Defining a new independent variable η as

$$\eta = \sqrt{x} \tag{3.158}$$

Eq. (3.157*b*) can be rewritten as

$$\eta^2 \frac{d^2\theta}{d\eta^2} + \eta\frac{d\theta}{d\eta} - 4m^2\eta^2\theta = 0 \tag{3.159}$$

The solution of Eq. (3.159) can be written as (see Appendix B)

$$\theta(\eta) = C_1 I_0(2m\eta) + C_2 K_0(2m\eta) \tag{3.160a}$$

or

$$\theta(x) = C_1 I_0(2m\sqrt{x}) + C_2 K_0(2m\sqrt{x}) \tag{3.160b}$$

which could also be obtained directly from Eq. (B.22) in Appendix B. The boundary conditions are given by

$$T(0) = \mathit{finite} \Rightarrow \theta(0) = \mathit{finite} \tag{3.161a}$$

$$T(L) = T_b \Rightarrow \theta(L) = T_b - T_\infty = \theta_b \tag{3.161b}$$

Since $K_0(0) \to \infty$ (see Fig. B.4 in Appendix B), the application of the boundary condition (3.161*a*) yields $C_2 = 0$. The second boundary condition (3.161*b*), on the other hand, gives

$$C_1 = \frac{\theta_b}{I_0\left(2m\sqrt{L}\right)} \tag{3.162}$$

Thus, the temperature distribution in the fin becomes

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{I_0\left(2m\sqrt{x}\right)}{I_0\left(2m\sqrt{L}\right)} \tag{3.163}$$

The rate of heat transfer from the fin can be obtained by considering the conduction across its base; that is.

$$q_f = kA\left(\frac{dT}{dx}\right)_{x=L} = kA\left(\frac{d\theta}{dx}\right)_{x=L} \tag{3.164}$$

Since, from Appendix B,

$$\frac{d}{dx}\left[I_0(\alpha x)\right] = \alpha I_1(\alpha x) \tag{3.165}$$

Eq. (3.164) yields

$$q_f = l\sqrt{2hkb}\,\theta_b \frac{I_1(2m\sqrt{L})}{I_0(2m\sqrt{L})} \tag{3.166}$$

Circular fins. Let us now consider the circular fins arranged radially around a tube as shown in Fig. 3.24. Equation (3.122) is also applicable for this case. For one of the fins in Fig. 3.24, we have

$$A(r) = 2\pi r\delta \quad \text{and} \quad P(r) = 4\pi r \tag{3.167a,b}$$

Substituting these into Eq. (3.122) yields

$$\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) - \frac{2hr}{k\delta}\theta = 0 \tag{3.168a}$$

or

$$r^2\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - m^2r^2\theta = 0 \quad \text{with} \quad m^2 = \frac{2h}{k\delta} \tag{3.168b}$$

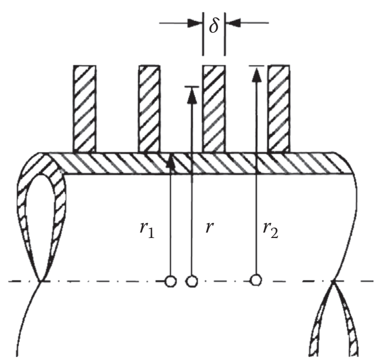

FIGURE 3.24
Circular fins of rectangular profile.

If heat loss from the tip of the fins is assumed to be negligible then the boundary conditions can be stated as follows:

$$\theta(r_1) = \theta_b = T_b - T_\infty \tag{3.169a}$$

$$\frac{d\theta(r_2)}{dr} = 0 \tag{3.169b}$$

where the base temperature T_b is assumed to be known.

The solution of Eq. (3.168) can be written (see Appendix B) as

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr) \tag{3.170}$$

Once the constants C_1 and C_2 are determined by imposing the boundary conditions (3.169), the solution is given by

$$\frac{\theta(r)}{\theta_b} = \frac{I_0(mr)K_1(mr_2) + K_0(mr)I_1(mr_2)}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)} \tag{3.171}$$

The rate of heat loss from one fin is determined by evaluating the heat conduction at the base; that is,

$$\begin{aligned} q_f &= -k2\pi r_1 \delta \left(\frac{d\theta}{dr} \right)_{r_1} \\ &= 2\pi r_1 \sqrt{2hk\delta}\theta_b = \frac{I_1(mr_2)K_1(mr_1) - I_1(mr_1)K_1(mr_2)}{I_0(mr_1)K_1(mr_2) + I_1(mr_2)K_0(mr_1)} \end{aligned} \tag{3.172}$$

Expressions (3.166) and (3.172), and the similar ones that are available in the literature for various fins of variable cross sections may not be convenient for practical use. Therefore, in practice, fin efficiencies given in chart forms are mostly used. Figures 3.25 and 3.26 are two such charts giving the efficiencies of several types of fins.

Harper and Brown [6] demonstrated that if a corrected length L_c is defined by increasing the length of a fin by one-half the thickness, then, for the heat loss, the equation that applies to the insulated tip case can be used. The error introduced by this approximation in most cases is usually very small, especially when the fin is surrounded by a gas so that h is not too high. They also showed that the error introduced by such an approximation is less than 8% when

$$\left(\frac{ht}{2k} \right)^{1/2} = \frac{1}{2} \tag{3.173}$$

where t is the fin thickness.

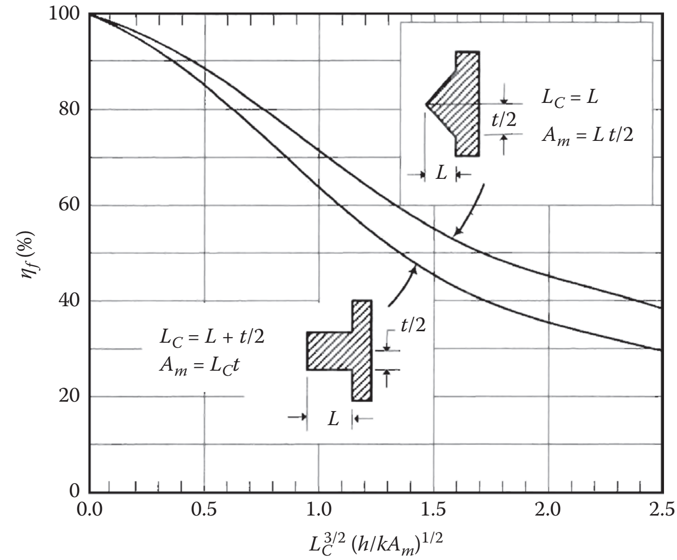

FIGURE 3.25
Efficiencies of straight fins of rectangular and triangular profiles [6].

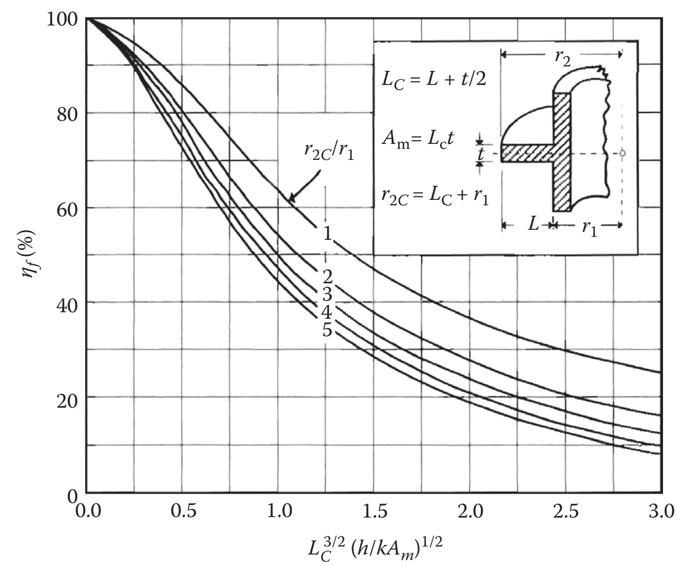

FIGURE 3.26
Efficiencies of circular fins of rectangular profile [6].

Example 3.9

Re-solve the problem of Example 3.7, (*a*) by the use of the corrected length L_c concept, and (*b*) by the use of efficiency curves.

SOLUTION

(*a*) We can use Eq. (3.136) by replacing L by a corrected length L_c; that is,

$$q_f \cong \sqrt{hPkA}\;\theta_b \tanh mL_c$$

where

$$\begin{aligned} L_c &\cong L + d/2 = 0.15 + 0.025/2 = 0.1625\,\text{m} \\ m &= 3.412\ \text{m}^{-1} \\ mL_c &= 0.554 \end{aligned}$$

Thus,

$$\begin{aligned} q_f &\cong (15 \times 7.85 \times 10^{-2} \times 206 \times 4.91 \times 10^{-4})^{1/2} \\ &\quad \times (260 - 16) \times \tanh(0.554) \cong 42.42\,\text{W} \end{aligned}$$

(*b*) The heat loss can also be found from

$$q_f = hA_f \eta_f (T_b - T_\infty)$$

Here we will use Fig. 3.25 to obtain η_f approximately, and the parameters needed are

$$A_m = L_c d = 0.1625 \times 0.025 = 0.00406\,\text{m}^2$$

$$A_f = \pi d L_C = \pi \times 0.025 \times 0.1625 = 0.01276\,\text{m}^2$$

$$\left(\frac{h}{kA_m}\right)^{1/2} = \left(\frac{15}{206 \times 4.06 \times 10^{-3}}\right)^{1/2} = 4.234$$

$$L_c^{3/2} = 0.0655$$

$$L_c^{3/2}\left(\frac{h}{kA_m}\right)^{1/2} = 0.0655 \times 4.234 = 0.272$$

From Fig. 3.25 we then have $\eta_f = 0.925$. Thus,

$$q_f \cong 15 \times 0.01276 \times 0.925 \times 244 \cong 43.21\,\text{W}$$

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Problems

3.1 The walls of a furnace are constructed from 12-cm thick fire brick on the inside and from 25-cm thick red brick on the outside. During steady-state operation of the furnace the surface temperature on the flame side of the fire brick is 680°C and the outside surface temperature is 120°C. To reduce the heat loss, a 5-cm layer of magnesia insulation, $k = 0.085$ W/(m·K), is added onto the outside of the red brick. When steady-state conditions are reached again, the following temperatures are measured: on the flame side of the fire brick, 700°C; at the interface between fire brick and red brick, 650°C; at the junction between the red brick and the magnesia, 490°C; on the outer surface of the magnesia insulation, 75°C. Calculate the rate of heat loss from the furnace, expressed as a percentage of the earlier rate of heat loss without the insulation.

3.2 A long hollow cylinder is constructed from a material whose thermal conductivity is a function of temperature, given by $k = 0.137 + 0.0027\,T$, where T is in °C and

k in W/(m·K). The inner and outer radii of the cylinder are 12.5 cm and 25 cm, respectively. Under steady-state conditions, the temperature of the inside surface is 800°C, and the temperature of the outside surface is 80°C.

(*a*) Calculate the rate of heat loss from the cylinder per meter length.

(*b*) If the heat transfer coefficient on the outside surface is 12W/(m^2·K), estimate the surrounding air temperature.

3.3 The main steam line of a proposed power plant will carry steam at 113 bar and 400°C. To insulate the pipe, 85% magnesia, k = 0.078 W/(m·K), will be used. Since magnesia is not an effective insulator at temperatures above 300°C, it is recommended that a layer of an expensive high-temperature insulation, k = 0.2W/(m·K), be placed between the pipe and the magnesia layer. Enough insulation must be used so that the outside surface temperature of the magnesia layer is 48°C. The pipe is a 12 in. standard diameter carbon steel pipe (OD = 12.75 in., wall thickness = 1.312 in.) of k = 40 W/(m·K). The heat transfer coefficients on the steam and air sides may be taken as 4500 W/(m^2·K) and 12W/(m^2·K), respectively. Recommend a thickness for the high-temperature insulation and magnesia layers for an ambient air temperature of 30°C.

3.4 Determine an expression for the critical radius of insulation for a sphere.

3.5 What would be the radius of an asbestos insulation, k = 0.151 W/(m·K). so that the heat loss from a pipe of 0.5-in. outer radius become the same as the heat loss without insulation? Assume that the heat transfer coefficient on the outer surface is the same in both cases and given by h = 6W/(m^2·K).

3.6 Derive an expression for the steady-state temperature distribution $T(x)$ in a slab of radioactive material in which internal energy is generated at a uniform rate of $\dot{q}$ per unit volume when it is active. The surface of the slab at $x = 0$ is perfectly insulated, and the other surface at $x = L$ is exposed to a fluid medium maintained at constant temperature T_∞, with a heat transfer coefficient h. Also, obtain an expression for the rate of heat transfer to the fluid per unit surface area of the slab. Assume constant thermal conductivity for the radioactive material.

3.7 In a plane wall, 3 in. thick, internal energy is generated at the rate of 5×10^4 W/m^3. One side of the wall is insulated and the other side is exposed to an environment at 30°C. The heat transfer coefficient between the wall and the environment is 600 W/(m^2·K). The thermal conductivity of the wall is 17 W/(m·K). What is the maximum temperature in the wall?

3.8 Derive an expression for the steady-state temperature distribution $T(x)$ in a plane wall, $0 \le x \le L$, having uniformly distributed heat sources of strength $\dot{q}$ (W/m^3). The surface of the wall at $x = 0$ is maintained at a constant temperature T_1, while the other at $x = L$ is at T_2. The thermal conductivity of the material of the wall can be assumed to be constant.

3.9 A rocket motor, which is in the form of hollow tube of inner and outer radii r_i and r_o, respectively, is constructed from a radioactive material having an equivalent heat source of strength $\dot{q}$ (W/m^3). The outer surface of the motor is insulated, while the inner surface transfers heat to a flowing gas at temperature T_{f1}. through a film coefficient h. Derive an expression for the temperature distribution in the radioactive material. What is the rate of heat transfer to the gas per unit length of the motor?

3.10 The fuel elements in a nuclear reactor are in the form of hollow tubes of inner and outer radii r_i and r_o, respectively. The neutron flux results in uniformly distributed heat sources of strength $\dot{q}$ (W/m^3) in the fuel elements. The cooling fluid temperatures on the inside and outside of the fuel elements are T_{f1} and T_{f2}, and the respective surface heat transfer coefficients are h_1 and h_2. Obtain an expression for the surface temperature of the fuel elements. What is the maximum temperature within the fuel elements?

3.11 Obtain the steady-state temperature distribution $T(x)$ in a plane wall of thickness $2L$ and constant thermal conductivity k in which internal energy is generated at a rate of

$$\dot{q}(x) = \dot{q}_0\left[1 - \left(\frac{x}{L}\right)^2\right]$$

where $\dot{q}_0$ is the known constant rate of internal energy generation per unit volume at the midplane at $x = 0$. The surfaces of the wall at $x = \pm L$ are maintained at a constant temperature T_w.

3.12 Derive an expression for the steady-state temperature distribution $T(r)$ in a solid cylinder of radius r_0 and constant thermal conductivity k in which internal energy is generated at a rate of

$$\dot{q} = \dot{q}_0\left[1 - \frac{r}{r_0}\right]$$

where $\dot{q}_0$ is the rate of internal energy generation per unit volume at the centerline and is a given constant. The surface of the cylinder at $r = r_0$ is maintained at a constant temperature T_w.

3.13 Derive an expression for the steady-state temperature distribution $T(r)$ in a long solid cylinder of thermal conductivity k and radius r_0, in which internal energy is generated at a constant rate of $\dot{q}$ per unit volume. The outer surface at $r = r_0$, is exposed to a fluid at temperature T_∞, with a constant heat transfer coefficient h.

3.14 The core of a nuclear reactor is made of cylindrical fuel elements, each composed of a uranium rod of radius r_1 and a stainless steel cladding of thickness $(r_2 - r_1)$. Internal energy is generated in the uranium rods according to

$$\dot{q} = \dot{q}_0\left[1 - \left(\frac{r}{r_1}\right)^2\right], \quad 0 \le r \le r_1$$

where $\dot{q}_0$ is the rate of internal energy generation per unit volume at the centerline and is a given constant. The temperature of the coolant is held constant at T_f, and the heat transfer coefficient h can be assumed to be constant. Find the temperature distribution in the fuel elements under steady-state conditions.

3.15 Consider a thin rod of cross-sectional area A, perimeter P, length L and thermal conductivity k. The rod is supported at its ends by two plates maintained at constant temperatures T_1 and T_2 and exposed to a fluid at temperature T_∞ with a constant heat transfer coefficient h at its peripheral surface. Assume perfect thermal contact between the rod and the plates.

(*a*) Obtain an expression for the steady-state temperature distribution in the rod as a function of distance x along the rod.

(*b*) What is the rate of heat loss from the rod to the surrounding fluid?

3.16 A thermometer well mounted through the wall of a steam pipe is a steel tube with 0.1 in. wall thickness, 0.5 in. outer diameter, 2 in. length, and of $k = 26$ W/(m·K). The steam flow produces an average heat transfer coefficient of 100 W/(m^2·K) on the outside surface of the well. If the thermometer reads 149°C and the temperature of the pipe is 65°C, estimate the average steam temperature.

3.17 Starting from the general heat conduction equation (2.16), obtain the heat conduction equation (3.123) which governs the steady-state temperature distribution in one-dimensional extended surfaces with constant cross section.

3.18 A straight fin of triangular profile, 10 cm long and 2 cm wide at its base, is constructed of mild steel, $k = 50$ W/(m·K), and is attached to a wall maintained at 40°C. Air at 200°C flows past the fin and the average heat transfer coefficient is 250 W/(m^2·K). Calculate the rate of heat removed by the fin per unit depth.

3.19 Hot water at an average temperature of 95°C flows inside a steel tube of 1-in. diameter and $k = 41$ W/(m·K). As illustrated in Fig. 3.27, there are eight longitudinal steel fins of rectangular profile on the outside of the tube, and each fin is 0.75 in. long. The outside surface of the tube and the fins are exposed to air at 15°C. Heat transfer coefficients on the inside and outside are 500 W/(m^2·K) and 12 W/(m^2·K), respectively. Calculate the heat transfer rate from the hot water to the air per length of the tube.

3.20 A thin wire of cross-sectional area A and perimeter P is extruded at a fixed velocity V through an extrusion nozzle as shown in Fig. 3.28. The temperature of the wire at the extrusion nozzle is T_0, high enough to make the metal extrudable. The wire, after extrusion, passes through air at temperature T_∞ for some distance L and

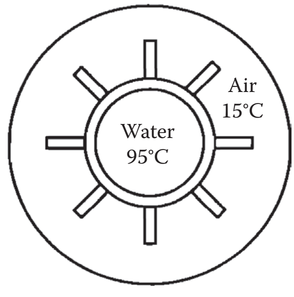

FIGURE 3.27
Figure for Problem 3.19.

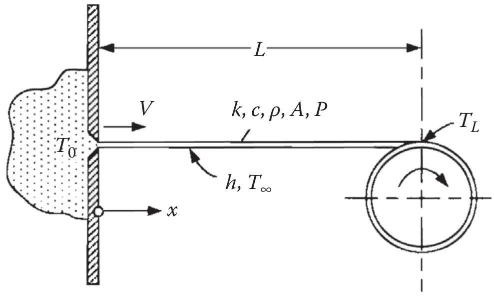

FIGURE 3.28
Figure for Problem 3.20.

then is rolled onto a large spool, where the temperature reduces to T_L. Let the heat transfer coefficient be constant h.

(*a*) Starting from the basic principles, derive a differential equation that governs the variation of the temperature of the wire as a function of the distance x from the extrusion nozzle. Also specify the boundary conditions.

(*b*) Solve the problem formulated in (*a*) and obtain an expression for the variation of the wire temperature as a function of x.

3.21 Using the solution obtained in Problem 3.20, develop an expression for the wire velocity V in terms of L and the other parameters of the problem.

3.22 A thin-walled disk rotates with angular velocity ω on a stationary disk as illustrated in Fig. 3.29. Both sides have the same dimensions and are made of the same material with constant thermal conductivity k. The interface pressure is p. the coefficient of dry friction at the interface is μ, and the ambient temperature is T_∞. The heat transfer coefficient is the same h on both sides, and heat transfer from the edges at $r = R$ is negligible. Assuming constant wear at the interface so that $\rho r = C$, where C is a given constant, determine the steady-state temperature distribution in the system.

3.23 Solve Problem 3.22 for ρ = *constant*.

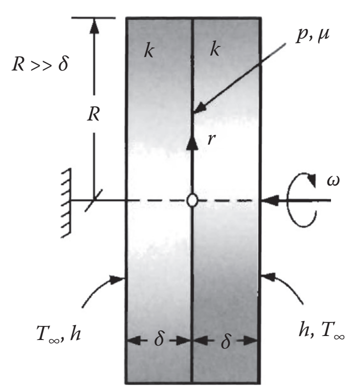

FIGURE 3.29
Figure for Problem 3.22.

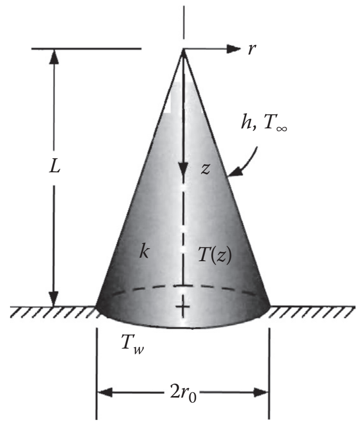

FIGURE 3.30
Figure for Problem 3.24.

3.24 A spine attached to a wall maintained at a uniform temperature T_w has the shape of a circular cone with base radius r_0 and height L as illustrated in Fig. 3.30, and is exposed to a fluid at a uniform temperature T_∞. Assuming constant thermal conductivity k and heat transfer coefficient h, and that the variation of the temperature in the r direction is negligible, obtain

(*a*) an expression for the steady-state temperature distribution $T(z)$ in the spine, and

(*b*) an expression for the rate of heat loss from the spine to the surrounding fluid.

3.25 Consider the straight fin of parabolic profile shown in Fig. 3.31. The surrounding fluid temperature and heat transfer coefficient are T_∞ and h, respectively, while the base temperature is T_0. Assuming that the side surfaces of the fin are insulated, the thermal conductivity k of the material of the fin is constant and $b \ll L$, obtain an expression for the steady-state temperature distribution $T(x)$ in the fin. Also, determine the rate of heat loss from the fin per unit depth.

3.26 A solid rod of radius r_1 rotates, as illustrated in Fig. 3.32, steadily with angular velocity ω in a stationary hollow cylinder of outer radius r_2 and of constant thermal conductivity k. The pressure and the coefficient of friction at the interface of

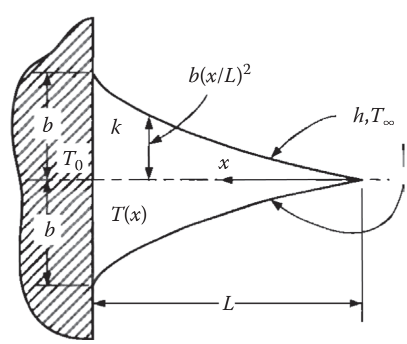

FIGURE 3.31
Figure for Problem 3.25.

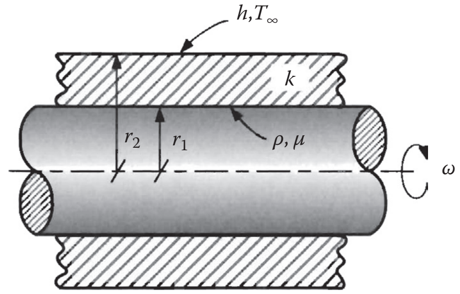

FIGURE 3.32
Figure for Problem 3.26.

the cylinders are ρ and μ, respectively. Assume that the surrounding air temperature T_∞ and the heat transfer coefficient h are constants. For what specific value of r_2 will the rod have the lowest temperature?

3.27 A thin cylindrical shaft is rotating with an angular velocity of ω inside a fixed sleeve of negligible thickness as illustrated in Fig. 3.33. The pressure and the coefficient of dry friction between the sleeve and the shaft are ρ and μ, respectively. Find the steady-state temperature distribution in the shaft. The two ends of the shaft can be approximated as adiabatic.

3.28 Consider a thin circular rod of length $2L$. Internal energy is generated over the middle half section of the rod as illustrated in Fig. 3.34. Find the steady-state temperature in the rod. The two ends of the rod can be approximated as adiabatic.

3.29 A laser source is used to provide a constant heat flux, q_0'', across the top surface of a thin adhesive-backed sheet of plastic film to affix it to a metal strip as illustrated

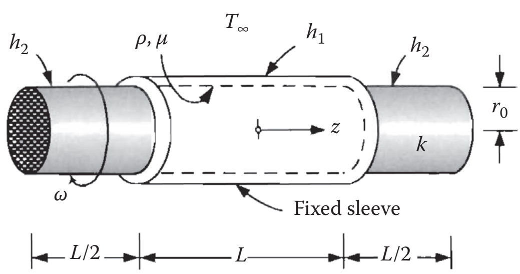

FIGURE 3.33
Figure for Problem 3.27.

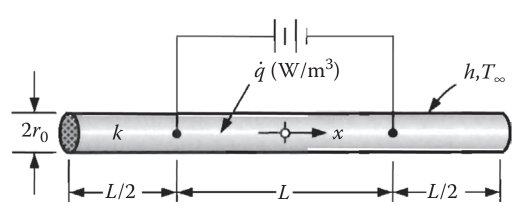

FIGURE 3.34
Figure for Problem 3.28.

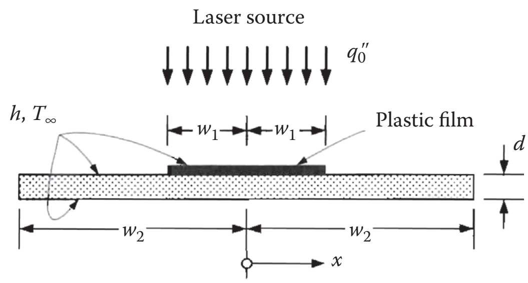

FIGURE 3.35
Figure for Problem 3.29.

in Fig. 3.35. The metal strip and film are very long in the direction normal to the page. Assuming $d \ll w_2$, obtain the steady-state temperature distribution in the metal strip. The edges of the metal strip can be approximated as adiabatic.

3.30 Consider a long thin-walled cylinder of radius r_0. The cylinder rotates steadily with an angular velocity ω as illustrated in Fig. 3.36. One half of the cylinder passes through a furnace where it is heated by exposing it to a hot gas at temperature T_g, while the other half is cooled by convection in open air at temperature T_∞, with the constant heat transfer coefficients h_1 and h_2, respectively. Assuming that both temperatures T_g and T_∞ are constant, obtain the steady-state temperature $T(x)$ in the cylinder.

3.31 Consider a circular disk of radius R and thickness δ ($\ll R$). On one side, a laser source is used to provide a constant heat flux, q_0'', over a circular section of radius r_0 in the middle as illustrated in Fig. 3.37. Assuming that the surrounding air temperature T_∞ and the heat transfer coefficient h are constants, and that heat transfer from the edge at $r = R$ is negligible, determine the steady-state temperature distribution in the disk.

3.32 Obtain an expression for the steady-state temperature distribution $T(z)$ in the thin semi-infinite cylindrical solid rod shown in Fig. 3.38. Internal energy is generated in the rod according to $\dot{q} = \dot{q}_0 e^{-\beta z}$, where $\dot{q}_0$ and β are two given positive constants, while the peripheral surface at r_0 is exposed to a fluid maintained at a constant temperature T_∞ with a constant heat transfer coefficient h.

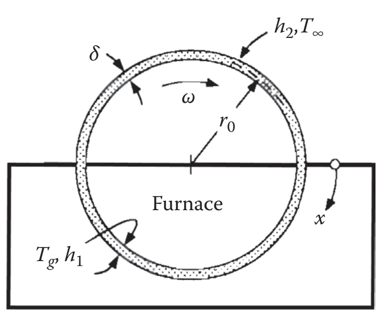

FIGURE 3.36
Figure for Problem 3.30.

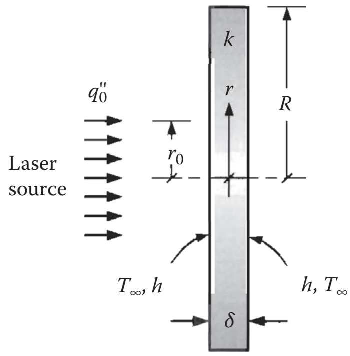

FIGURE 3.37
Figure for Problem 3.31.

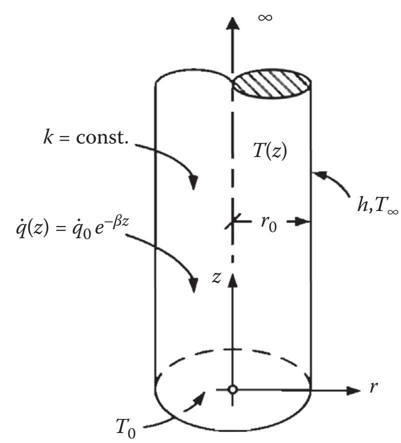

FIGURE 3.38
Figure for Problem 3.32.

3.33 A steam pipe with external radius r_0 = 4 cm should be covered with two layers of insulation (see Fig. 3.39). The most expensive insulation for the internal layer is available for installation with thickness e_1 = 1 cm and has thermal conductivity equal to k_1 = 0.05 W/(m °C). The second layer of insulation will have thickness e_2 to be determined, having a thermal conductivity of k_2 = 0.15 W/(m °C). The internal surface of the steam pipe is maintained at a temperature T_0 = 330°C, while the thermal resistance of its metallic wall is negligible. Moreover, it is desired to reduce the heat losses by 50% after application of the second insulation in relation to the losses if only the first insulating layer were used, exchanging heat with the environment with heat transfer coefficient h = 10 W/m^{2}C and external air temperature T_∞ = 25°C. Under the restriction that the temperature of the outer surface of the second insulation should be maximum at 45°C, determine the heat transfer rate per meter of length of tubing, the thickness of the second insulation, and the temperature at the interface between the two insulations. Assume perfect thermal contact between the two insulation layers.

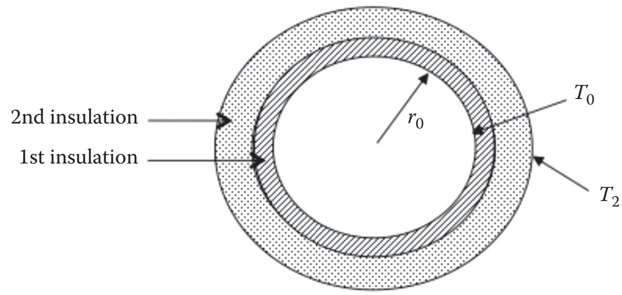

FIGURE 3.39
Figure for Problem 3.33.

3.34 Consider a thin aluminum can of a cold drink, which is initially at 4°C in thermal equilibrium with the fluid that is inside it. The can has a height of 12.5 cm and a diameter of 6 cm. The combined coefficient of heat transfer by convection and radiation, h, between the can and the surrounding air, which is at 25°C, is 10 W/m²C. To maintain the beverage cold for a longer period, a cylindrical Styrofoam insulation is used, which coats the can, with thermal conductivity k = 0.035 W/m °C, with a thickness of 1 cm. The contact between the insulation and the can has an estimated contact heat transfer coefficient h_c = 125 W/m²°C. Assuming that the heating of the can is sufficiently slow to neglect the transient effects, estimate the rate of heat transfer to the can (a) without and (b) with the insulation material. What is the percentage reduction in the rate of heat transfer after insulation? OBS: Neglect heat exchange by the base and top of the can.

3.35 Cylindrical pin fins with a diameter of 2 cm and 8 cm in length, made of aluminum (k = 205 W/m °C), are installed in a wall that is at 150°C. Air at 26°C flows between them, with a mean heat transfer coefficient h = 120 W/m²C. Determine how much represents the heat exchange at the end of the fin, opposite to the base, relative to the exchange along the length of the fin. Consider that the heat transfer coefficient at the tip is the same as along the fin.

3.36 An electric wire of 3 mm in diameter and 5 m in length is firmly covered with a plastic layer of 2 mm in thickness, with thermal conductivity k = 0.15 W/m °C. Electrical measurements indicate that a current of 10 A passes through the wire, and there is a voltage drop of 8 V along the wire. If the electrically insulated wire is exposed to ambient air at T_∞ = 30°C with an estimated natural convection heat transfer coefficient of h = 25 W/m²°C, determine the temperature at the interface between the wire and the plastic cover and at the external surface of the plastic cover, under steady state. Also, determine if by doubling the thickness of the plastic cover, this interface temperature will increase or decrease, and explain the behavior.

3.37 Consider that human skin is formed by three tissue layers as shown in Fig. 3.40, with the outer wall of the epidermis at $x = 0$ and the wall of the subcutaneous tissue in contact with the inner tissue at $x = L = L_1 + L_2 + L_3$. The internal tissue maintains its uniform temperature at T_L = 37°C due to blood perfusion. The three outermost layers are subject to variations in temperature and heat fluxes imposed by the external environment. Knowing that temperatures greater than 42°C lead to cell death, we want to know whether a steady-state hyperthermia treatment with a

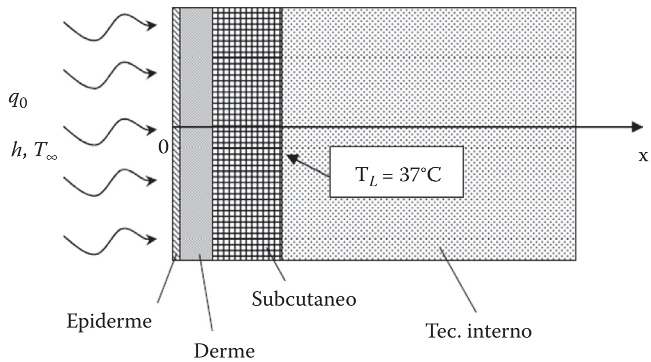

FIGURE 3.40
Figure for Problem 3.37.

heat flux of $q_0 = 420$ W/m² can lead to cellular damage in any of these three layers. It is considered that the epidermis exchanges heat with the external medium at $T_\infty = 20°C$ and a heat transfer coefficient of $h = 15$ W/m²C. Consider perfect thermal contact between the layers, whose properties are as follows: epidermis ($k_1 = 0.24$ W/mC, $L_1 = 8 \times 10^{-5}$ m), dermis ($k_2 = 0.45$ W/mC, $L_2 = 0.002$ m), and subcutaneous tissue ($k_3 = 0.19$ W/mC, $L_3 = 0.01$ m).

3.38 A square plate of composite material with a polymeric matrix (high density polyethylene) impregnated with alumina particles, with side $w = 0.3$ m, was designed for a gradual increase in the thermal conductivity along the thickness of the plate, L (0.05 m). A space variable thermal conductivity was obtained along the thickness, x, in the form

$$k(x) = ae^{bx}, \quad \text{where} \quad a = 0.551 \quad \text{and} \quad b = 26.06.$$

The more conductive plate face ($x = L$) is in perfect thermal contact with a wall at the temperature $T_L = 80°C$, and its less conductive face ($x = 0$) exchanges heat by convection with a heat transfer coefficient $h = 12$ W/m²C and an ambient at T_∞ = 25°C. Considering steady state and one-dimensional heat conduction in the plate:

(*a*) Calculate the heat transfer rate and the temperature on the face $x = 0$ of the plate, when considering a constant effective thermal conductivity in the whole plate, given by the average value k_{av}, $k_{av} = \frac{1}{L}\int_0^L k(x)\,dx$.

(*b*) Calculate the heat transfer rate and the temperature at the face $x = 0$, using the variable conductivity $k(x)$ across the plate thickness. Compare the values found in the two situations.

3.39 A microelectromechanical system consists of two superimposed flat plates of silicon and glass, with thermal conductivities of $k_s = 147$ W/mC and $k_g = 0.8$ W/mC, respectively (see Fig. 3.41). The silicon plate has a thickness of $e_s = 500$ μm and the glass plate a thickness of $e_g = 2$ mm. Both can be considered infinite in the other two directions, but for calculating the volumetric rate of heat generation, g, we take the width as $w = 2$ cm and the length as $L = 7$ cm. The silicon plate will

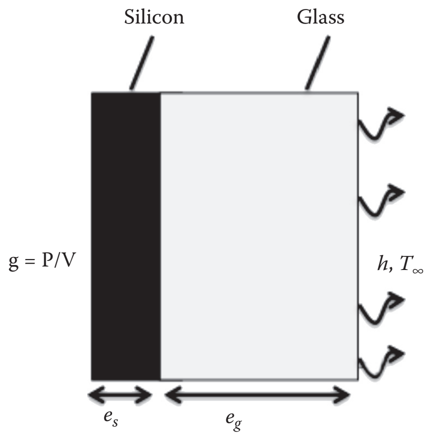

FIGURE 3.41
Figure for Problem 3.39.

be heated by Joule effect with direct current, for a power of $P = 1$ W generated uniformly throughout the plate dimensions, being thermally isolated from the external environment at $x = 0$. However, the glass plate has no internal generation, and it exchanges heat by convection with the ambient air at the temperature of $T_\infty = 25$°C and heat transfer coefficient $h = 30$ W/mC, at $x = e_s + e_g$. Determine the temperature distributions through the thicknesses of the two materials, assuming perfect thermal contact at the silicon–glass interface. What is the heat flow on the outer surface of the glass? Determine the temperature drop across the glass slide.

4

The Sturm-Liouville Theory and Fourier Expansions

4.1 Introduction

We have, so far, discussed one-dimensional steady-state heat conduction problems and observed that formulations of such problems involve ordinary differential equations. Formulations of two- or three-dimensional steady-state and one-dimensional or multi-dimensional unsteady-state heat conduction problems, on the other hand, involve partial differential equations. When the boundary surfaces correspond to the coordinate surfaces in a specific system of coordinates, such as rectangular, cylindrical, or spherical coordinates, we may then employ, among several different techniques, *separation of variables*, *Fourier transforms,* or *Laplace transforms* as the method of solution to obtain analytical solutions. In Chapters 5 and 6, we will obtain the solutions of some typical linear heat conduction problems using the method of separation of variables, which was first introduced by d'Alembert, Bernoulli, and Euler in the middle of the 18th century. This method is still of great value and also lies at the heart of the Fourier transforms method which we will discuss in Chapter 7. In this present chapter we discuss the basic mathematical concepts related to these two methods. We will introduce and solve some representative time-dependent problems by Laplace transforms in Chapter 8.

4.2 Characteristic-Value Problems

Let us first review some mathematical concepts. A differential equation or a boundary condition is said to be *linear* if, when rationalized and cleared of fractions, it contains no products of the dependent variable or its derivatives. The heat conduction equation with temperature-dependent thermal conductivity is a good example of a *nonlinear* differential equation:

$$\nabla \cdot \left[k(T) \nabla T \right] + \dot{q} = \rho c \frac{\partial T}{\partial t} \tag{4.1a}$$

An example of a nonlinear condition is a radiation boundary condition as illustrated in Fig. 4.1:

$$-k \left(\frac{dT}{dx} \right)_{x=L} = \varepsilon \sigma \left[T^4(L) - T_e^4 \right] \tag{4.1b}$$

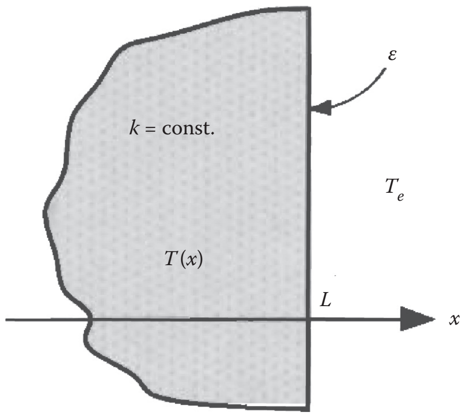

FIGURE 4.1
Solid exchanging heat by radiation at the surface at $x = L$ with an environment maintained at an effective blackbody temperature T_e.

where ε is the emissivity of the surface, σ is the Stefan–Boltzmann constant, and T_e is the *effective* blackbody temperature of the environment.

A linear differential equation or a linear boundary condition is said to be *homogeneous* if, when satisfied by a particular function v, it is also satisfied by Cy, where C is an arbitrary nonzero constant. Thus, for example,

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = f_3(x) \tag{4.2a}$$

is a *nonhomogeneous* linear and second-order ordinary differential equation. On the other hand, the equation

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0 \tag{4.2b}$$

is a homogeneous linear and second-order ordinary differential equation. At the boundary $x = a$,

$$y(a) = C_1, \quad \frac{dy(a)}{dx} = C_2, \quad \text{or} \quad \alpha y(a) + \beta\frac{dy(a)}{dx} = C_3 \tag{4.2c}$$

denote *nonhomogeneous* linear boundary conditions, whereas

$$y(a) = 0, \quad \frac{dy(a)}{dx} = 0, \quad \text{or} \quad \alpha y(a) + \beta\frac{dy(a)}{dx} = 0 \tag{4.2d}$$

represent *homogeneous* linear boundary conditions. Here C_1, C_2, C_3, α, and β are prescribed nonzero constants.

As in the case of many physical problems, in heat conduction problems certain specified conditions must be satisfied by the solution of the relevant form of the heat conduction equation. These conditions often enable the arbitrary constants of integration appearing in the solution of the heat conduction equation to be determined, leading to the solution of the problem. If the conditions are specified at two* different values of the independent variable (or variables), the problem is called a *boundary-value problem,* as distinct from *initial-value problems,* wherein all conditions are specified at one value of the independent variable. The heat conduction equation for a one-dimensional steady-state problem together with two boundary conditions, for example, forms a boundary-value problem, whereas the heat conduction equation for a one-dimensional unsteady-state problem together with an initial condition and two boundary conditions forms an *initial-and-boundary-value problem.*

Let us now consider the following linear and homogeneous boundary-value problem:

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0 \tag{4.3a}$$

$$\alpha_1 y(a) + \beta_1 \frac{dy(a)}{dx} = 0, \quad \alpha_1^2 + \beta_1^2 > 0 \tag{4.3b}$$

$$\alpha_2 y(b) + \beta_2 \frac{dy(b)}{dx} = 0, \quad \alpha_2^2 + \beta_2^2 > 0 \tag{4.3c}$$

Since the differential equation (4.3*a*) is linear and homogeneous, its general solution can be written as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{4.4}$$

where $y_1(x)$ and $y_2(x)$ are the two linearly independent solutions of the differential equation (4.3*a*), and C_1 and C_2 are two arbitrary nonzero constants. Application of the boundary conditions (4.3*b,c*) yields the following set of algebraic equations for the two unknown constants:

$$B_{11}C_1 + B_{12}C_2 = 0 \tag{4.5a}$$

$$B_{21}C_1 + B_{22}C_2 = 0 \tag{4.5b}$$

where we have introduced the following constants:

$$B_{1i} = \alpha_1 y_i(a) + \beta_1 \frac{dy_i(a)}{dx} = 0, \quad i = 1,2 \tag{4.6a}$$

* Boundary conditions may, of course, be given at more than two values of the independent variable. We shall, however, consider only "two-point" boundary-value problems.

$$B_{2i} = \alpha_2 y_i(b) + \beta_2 \frac{dy_i(b)}{dx} = 0, \quad i = 1,2 \tag{4.6b}$$

One possible solution of Eqs. (4.5*a,b*) for the unknown constants is $C_1 = C_2 = 0$, leading to the trivial solution $y(x) \equiv 0$. By Cramer's rule, in order to have a nontrivial solution, the determinant of coefficients of C_1 and C_2 must vanish; that is,

$$\begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = 0 \tag{4.7a}$$

or

$$B_{11}B_{22} - B_{12}B_{21} = 0 \tag{4.7b}$$

If this condition exists, Eqs. (4.5*a*) and (4.5*b*) become equivalent, and one of the constants can be expressed as a multiple of the other by the use of either equation, the second constant then being arbitrary. The use of Eq. (4.5*a*), for example, yields

$$y(x) = A[B_{12}y_1(x) - B_{11}y_2(x)] \tag{4.8}$$

where A is an arbitrary constant defined as

$$A = \frac{C_1}{B_{12}} = -\frac{C_2}{B_{11}} \tag{4.9}$$

Equation (4.8) can easily be shown to satisfy the boundary conditions (4.3*b,c*) (see Problem 4.1). Notice that Eq. (4.8) is a nontrivial solution only if B_{11} and B_{12} are not both zero. If $B_{11} = B_{12} = 0$, then Eq. (4.5*a*) is a trivial relation, and in that case we must use Eq. (4.5*b*) to relate C_1 and C_2. This leads to a nontrivial solution of the form

$$y(x) = B\left[B_{22}y_1(x) - B_{21}y_2(x)\right] \tag{4.10}$$

where B is an arbitrary constant defined as

$$B = \frac{C_1}{B_{22}} = -\frac{C_2}{B_{21}} \tag{4.11}$$

Equation (4.10) will be the nontrivial solution only if B_{21} and B_{22} are not both zero. If, on the other hand, $B_{ij} = 0$, for $i, j = 1, 2$, then Eq. (4.4) will be the solution for any nonzero C_1 and C_2.

Example 4.1

Find the nontrivial solution of the following boundary-value problem:

$$\frac{d^2y}{dx^2} + y = 0$$

$$y(0) = 0 \quad \text{and} \quad y(\pi) = 0$$

SOLUTION

The general solution of the differential equation is

$$y(x) = A \sin x + B \cos x$$

Substituting into the boundary conditions, we obtain

$$A \cdot 0 + B \cdot 1 = 0 \quad \text{or} \quad B = 0$$

$$A \cdot 0 - B \cdot 1 = 0 \quad \text{or} \quad B = 0$$

Clearly, the solution of these equations gives $\beta = 0$ and A = arbitrary. Hence,

$$y(x) = A \sin x$$

is the solution of the boundary-value problem for any nonzero arbitrary constant A.

One or both of the coefficients $f_1(x)$ and $f_2(x)$ in Eq. (4.3*a*) and hence the solutions $y_1(x)$ and $y_2(x)$ may depend on a constant parameter λ. In such problems, the determinant (4.7*a*) may vanish for certain values of λ, say $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots$ These values of λ are called the *characteristic values,* or *eigenvalues,* and the corresponding solutions are called the *characteristic functions,* or *eigenfunctions,* of the problem. Boundary-value problems of this kind are known as *characteristic-value,* or *eigenvalue,* problems. Let us now consider Example 4.2.

Example 4.2

Find the characteristic values and the characteristic functions of the following characteristic-value problem:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$$y(0) = 0 \quad \text{and} \quad y(L) = 0$$

SOLUTION

The general solution of the differential equation is

$$y(x) = A\sin\lambda x + B\cos\lambda x$$

The boundary condition at $x = 0$, that is, $v(0) = 0$, gives $B = 0$, and the boundary condition at $x = L$, that is, $y(L) = 0$, yields $A \sin \lambda L = 0$. Hence, a nontrivial solution of this problem exists only if λ has a value such that $\sin \lambda L = 0$, and this is possible if λ is equal to one of the values of

$$\lambda_n = \frac{n\pi}{L}, \quad n = 1,\ 2,\ 3\ldots.$$

which are the characteristic values of the problem. Thus, the foregoing boundary-value problem has no solution other than the trivial solution $y(x) \equiv 0$, unless λ assumes one of the characteristic values λ_n given above. Corresponding to each characteristic value λ_n, the solution of the problem can be written as

$$y_n(x) = A_n\phi_n(x)$$

where A_n is an arbitrary nonzero constant and

$$\phi_n(x) = \sin\frac{n\pi}{L}x$$

is the characteristic function corresponding to the characteristic value λ_n. Here it should be noted that no new solutions are obtained when n assumes negative integer values. Also, $n = 0$ leads to the trivial solution $y(x) \equiv 0$.

4.3 Orthogonal Functions

Two real-valued functions $\phi_m(x)$ and $\phi_n(x)$ are said to be *orthogonal* with respect to a *weight function* (or *density function*) $w(x)$ on an interval (a, b) if

$$\int_a^b \phi_m(x)\phi_n(x)w(x)\,dx = 0, \quad m \neq n \tag{4.12}$$

Furthermore, a set of real-valued functions $\{\phi_n(x); n = 1, 2,...\}$ is called *orthogonal* with respect to a weight function $w(x)$ on an interval (a, b) if *all* the pairs of distinct functions in this set satisfy the orthogonality condition (4.12). If there exists some function $f(x)$, different from zero, which is orthogonal to *all* members of the set; that is, if

$$\int_a^b f(x)\phi_n(x)w(x)\,dx = 0, \quad n = 1, 2, 3, \ldots$$

then the set is called *incomplete*, or otherwise it is said to be *complete.*

For example, the set $\{\sin(n\pi/L)x;\ n = 1, 2, 3,...\}$ is an orthogonal set with respect to the weight function unity on the interval $(0, L)$ because

$$\int_0^L \sin\frac{m\pi}{L}x\sin\frac{n\pi}{L}x\,dx = 0,\quad m \neq n$$

which can be verified by direct integration. It can also be shown that this set is, in fact, a complete orthogonal set, but such a proof is beyond the scope of this text. However, the matter of completeness of certain sets of orthogonal functions will be clarified in the following section.

4.4 Sturm–Liouville Problem

Consider the following characteristic-value problem, composed of the linear and homogeneous second-order differential equation of the general form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y = 0 \tag{4.13a}$$

and the two homogeneous linear boundary conditions

$$\alpha_1 y(a)+\beta_1\frac{dy(a)}{dx} = 0,\quad \alpha_1^2+\beta_1^2 > 0 \tag{4.13b}$$

$$\alpha_2 y(b)+\beta_2\frac{dy(b)}{dx} = 0,\quad \alpha_2^2+\beta_2^2 > 0 \tag{4.13c}$$

prescribed at the ends of the finite interval (a, b). The functions $p(x)$, $q(x)$ and $w(x)$ are *real-valued* and *continuous* (including dp/dx), and $p(x)$ and $w(x)$ are *positive* over the entire interval (a, b), including the end points, while $q(x) \leq 0$ in the same interval. Furthermore, α_1, α_2, β_1 and β_2 are given real constants, and λ is an unspecified parameter, independent of x. Characteristic-value problems of this type are known as (regular) *Sturm–Liouville problems* or *systems.* The first extensive development of the theory of such systems was published by J. C. F. Sturm (1803–1855), a Swiss mathematician, and J. Liouville (1809–1882), a French mathematician, in the first three volumes of *Journal de Mathématique,* 1836–1838.

As we shall see in Chapters 5, 6, 7, and 13, when the method of separation of variables or finite integral transforms is used to solve a heat conduction problem, the problem is reduced, at one point in the solution, to a Sturm–Liouville system. Depending on the coordinate system used, the functions $p(x)$, $q(x)$, and $w(x)$ in Eq. (4.13*a*) will be of certain forms. The homogeneous boundary conditions of the third kind, Eqs. (4.13*b*,*c*), result if the heat conduction problem has boundary conditions of the third kind both at $x = a$ and at $x = b$.

If the problem has a boundary condition of the second kind, for example, at $x = a$, then α_1 in Eq. (4.13*b*) will be zero, and if the boundary condition at $x = a$ is of the first kind, then β_1 will be zero.

Nontrivial solutions of the problem (4.13) exist, in general, for a particular set of values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the parameter λ. These are the characteristic values, (or the eigenvalues), and the corresponding solutions are the characteristic functions (or the eigenfunctions) of the problem. Let λ_m and λ_n be any two distinct characteristic values, and $\phi_m(x)$ and $\phi_n(x)$ be the corresponding characteristic functions, respectively. These functions satisfy the differential equation (4.13*a*):

$$\frac{d}{dx}\left[p(x)\frac{d\phi_m}{dx}\right]+\left[q(x)+\lambda_m w(x)\right]\phi_m=0 \tag{4.14a}$$

$$\frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right]+\left[q(x)+\lambda_n w(x)\right]\phi_n=0 \tag{4.14b}$$

Multiplying Eq. (4.14*a*) by $\phi_n(x)$ and Eq. (4.14*b*) by $\phi_m(x)$ and subtracting the resultant equations from each other, we obtain

$$\phi_n\frac{d}{dx}\left[p(x)\frac{d\phi_m}{dx}\right]-\phi_m\frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right]+(\lambda_m-\lambda_n)\phi_m\phi_n w(x)=0 \tag{4.15a}$$

or, after simplifying, we get

$$(\lambda_n-\lambda_m)\phi_m\phi_n w(x)=\frac{d}{dx}\left[p(x)\left(\phi_n\frac{d\phi_m}{dx}-\phi_m\frac{d\phi_n}{dx}\right)\right] \tag{4.15b}$$

Integrating this result over the interval (a, b) yields

$$(\lambda_n-\lambda_m)\int_a^b \phi_m(x)\phi_n(x)w(x)\,dx=\left[p(x)\left(\phi_n(x)\frac{d\phi_m}{dx}-\phi_m(x)\frac{d\phi_n}{dx}\right)\right]_a^b \tag{4.16}$$

Moreover, the characteristic functions $\phi_m(x)$ and $\phi_n(x)$ also satisfy the conditions given by Eqs. (4.13*b*) and (4.13*c*). For example, at $x = a$,

$$\alpha_1\phi_m(a)+\beta_1\frac{d\phi_m(a)}{dx}=0 \tag{4.17a}$$

$$\alpha_1\phi_n(a)+\beta_1\frac{d\phi_n(a)}{dx}=0 \tag{4.17b}$$

Multiplying Eq. (4.17*a*) by $\phi_n(a)$ and Eq. (4.17*b*) by $\phi_m(a)$ and subtracting the resultant equations from each other, we find

$$\beta_1\left[\phi_n(a)\frac{d\phi_m(a)}{dx}-\phi_m(a)\frac{d\phi_n(a)}{dx}\right]=0 \tag{4.17c}$$

If $\beta_1 \neq 0$, that is, if the homogeneous boundary condition (4.13*b*) is either of the second or the third kind, then the term in the bracket in Eq. (4.17*c*) will vanish. If, on the other hand, $\beta_1 = 0$, that is, if the homogeneous boundary condition (4.13*b*) is of the first kind, then $\phi_m(a) = \phi_n(a) = 0$, and hence the term in the bracket in Eq. (4.17*c*) will again vanish, provided that $dy(a)/dx$ is finite. In either case,

$$\phi_n(a)\frac{d\phi_m(a)}{dx}-\phi_m(a)\frac{d\phi_n(a)}{dx}=0 \tag{4.18}$$

Similarly, one can also show that

$$\phi_n(b)\frac{d\phi_m(b)}{dx}-\phi_m(b)\frac{d\phi_n(b)}{dx}=0 \tag{4.19}$$

In view of Eqs. (4.18) and (4.19), the right-hand side of Eq. (4.16) vanishes; that is,

$$(\lambda_n-\lambda_m)\int_a^b \phi_m(x)\phi_n(x)w(x)dx=0 \tag{4.20a}$$

or

$$\int_a^b \phi_m(x)\phi_n(x)w(x)dx=0, \quad m\neq n \tag{4.20b}$$

Thus, we see that the characteristic functions of the Sturm–Liouville problem (4.13) form an orthogonal set with respect to the weight function $w(x)$ on the interval (a, b). The following extensions of the Sturm–Liouville problem are also important in certain applications:

Singular end points. If it so happens that $p(a) = 0$, then the right-hand side of Eq. (4.16) vanishes at $x = a$, provided that $y(x)$ and dy/dx are both finite at $x = a$. With this condition, the characteristic functions corresponding to different characteristic values will be orthogonal with respect to the weight function $w(x)$ on (a, b), even if Eq. (4.18) does not hold. In this case, the boundary condition (4.13*b*) is replaced by the requirement that $y(x)$ and dy/dx be finite at $x = a$ when $p(a) = 0$.

Similarly, if $p(b) = 0$, we can then replace the second boundary condition (4.13*c*) by the requirement that $y(x)$ and dy/dx be finite at $x = b$. If both $p(a) = 0$ and $p(b) = 0$, then neither boundary condition as specified by Eqs. (4.13*b,c*) is needed to ensure the orthogonality of the characteristic functions, provided that both $y(x)$ and dy/dx are finite at $x = a$ and $x = b$.

Periodic boundary conditions. If $p(a) = p(b)$, then the right-hand side of Eq. (4.16) would vanish if the boundary conditions (4.13*b*,*c*) were replaced by

$$y(a) = y(b) \tag{4.21a}$$

$$\frac{dy(a)}{dx} = \frac{dy(b)}{dx} \tag{4.21b}$$

These are called *periodic boundary conditions* and they are satisfied, in particular, if the solution $y(x)$ is required to be periodic, of period $b - a$.

It can be shown that the characteristic values of the Sturm–Liouville problem are all *real* and *nonnegative,* and the corresponding characteristic functions are *real.* The parameter λ in Eq. (4.13*a*) can, therefore, be replaced by λ^2 with no loss in the generality of the problem. In addition, there is *only one* characteristic function $\phi_n(x)$ which corresponds to each characteristic value λ_n, except when the periodic boundary conditions (4.21*a*,*b*) are involved (see Problem 4.13). More importantly, the characteristic functions form a *complete* orthogonal set. Proofs of these statements are beyond the scope of this text.

4.5 Generalized Fourier Series

Consider a set of functions $\{\phi_n(x); n = 0, 1, 2,...\}$ orthogonal with respect to a weight function $w(x)$ on the finite interval (a, b). Expand an arbitrary function $f(x)$ in a series of these functions as

$$f(x) = A_0\phi_0(x) + A_1\phi_1(x) + \ldots + A_n\phi_n(x) + \ldots \tag{4.22a}$$

or

$$f(x) = \sum_{n=0}^{\infty} A_n\phi_n(x) \tag{4.22b}$$

Assuming that such an expansion exists, the unknown coefficients A_n can be evaluated by multiplying Eq. (4.22*a*), or (4.22*b*), by $\phi_n(x)w(x)$ and then integrating the resulting equation over the interval (a, b):

$$\int_a^b f(x)\phi_n(x)w(x)dx = \int_a^b \left[\sum_{k=0}^{\infty} A_k\phi_k(x)\right]\phi_n(x)w(x)dx$$

Because of the orthogonality property of the set, all of the terms on the right-hand side vanish, except the one for which $k = n$. Hence, we get

$$\int_a^b f(x)\phi_n(x)w(x)dx = A_n\int_a^b [\phi_n(x)]^2 w(x)dx$$

which yields

$$A_n = \frac{1}{N_n}\int_a^b f(x)\phi_n(x)w(x)\,dx \tag{4.23a}$$

where N_n are the *normalization integrals** of the functions $\phi_n(x)$ defined as

$$N_n = \int_a^b \left[\phi_n(x)\right]^2 w(x)\,dx \tag{4.23b}$$

The expansion (4.22*a*), or (4.22*b*), with coefficients (4.23*a*), is a *formal* series representation of the function $f(x)$ on the interval (a, b) and is referred to as the *generalized Fourier series* (or *expansion*) of $f(x)$ corresponding to the orthogonal set $\{\phi_n(x); n = 0, 1, 2, \ldots\}$. The coefficients A_n are called *Fourier constants* of the function $f(x)$.

The problem of determining whether the expansion (4.22*a*), or (4.22*b*), actually represents the given function in the interval (a, b) is beyond the scope of this text. On the other hand, if $f(x)$ is a *piecewise-differentiable*† function in the interval (a, b), and $\{\phi_n(x); n = 0, 1, 2, \ldots\}$ is the complete set of all the characteristic functions of a Sturm–Liouville problem, then the series representation of $f(x)$ converges inside (a, b) to $f(x)$ at all points where $f(x)$ is continuous and converges to the mean value $[f(x^+) + f(x^-)]/2$ at the points where finite jumps occur. In addition, this series representation may or may not converge to the given value of $f(x)$ at one or both end points $x = a$ and $x = b$.

4.6 Ordinary Fourier Series

The *ordinary Fourier series,* or simply the *Fourier series,* are developed from the characteristic functions of the following characteristic-value problem for different combinations of the boundary conditions:

$$\frac{d^2y}{dx^2} + \lambda^2 y(x) = 0 \tag{4.24a}$$

$$\alpha_1 y(0) + \beta_1 \frac{dy(0)}{dx} = 0, \quad \alpha_1^2 + \beta_1^2 > 0 \tag{4.24b}$$

$$\alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0, \quad \alpha_2^2 + \beta_2^2 > 0 \tag{4.24c}$$

* The constants N_n are also known as the *norm* (or sometimes as the square of the norm) of $\phi_n(x)$ with respect to the weight function $w(x)$.

† A function $f(x)$ is said to be *piecewise differentiable* in (a, b) if there exist at most a finite number of points $x_1, x_2, \ldots, x_N$ such that $f(x)$ is differentiable in each subinterval $a < x < x_1, x_1 < x < x_2, \ldots, x_n < x < b$, and has a right-hand derivative at the initial point and a left-hand derivative at the terminal point.

This characteristic-value problem is a special case of the Sturm–Liouville system (4.13) with $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, and λ replaced by λ^2. Replacement of λ by λ^2 is merely for convenience and has no effect on the generality of the problem as the system (4.24) is a Sturm–Liouville problem and, hence, would not have negative characteristic values. The characteristic functions $\phi_n(x)$, which are sinusoidal functions, will therefore form a complete orthogonal set in the interval (0, L) with respect to the weight function *unity.* An arbitrary function $f(x)$, which is piecewise differentiable in the interval (0, L), can then be expanded in this interval in a series of these functions as

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x), \quad 0 < x < L \tag{4.25}$$

where the coefficients A_n are given by

$$A_n = \frac{1}{N_n} \int_0^L f(x)\phi_n(x)\,dx \tag{4.26a}$$

with

$$N_n \int_0^L \left[\phi_n(x)\right]^2 dx \tag{4.26b}$$

Equation (4.25), where the functions $\phi_n(x)$ are the characteristic functions of the characteristic-value problem (4.24), is called an *ordinary Fourier series,* or simply a *Fourier series,* of $f(x)$ on the interval (0, L).

There are nine different combinations of the boundary conditions (4.24*b,c*). Accordingly, corresponding to each combination there will be a series expansion of the form given by Eq. (4.25). We now discuss the two most common cases in detail.

4.6.1 Fourier Sine Series

Consider the following characteristic-value problem:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \tag{4.27a}$$

$$y(0) = 0 \quad \text{and} \quad y(L) = 0 \tag{4.27b,c}$$

We saw in Section 4.2 (see Example 4.2) that this problem has the following characteristic functions and characteristic values:

$$\phi_n(x) = \sin\lambda_n \quad \text{and} \quad \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

The characteristic-value problem (4.27) is a special case of the system (4.24) with $\beta_1 = \beta_2 = 0$. Therefore, Eq. (4.25) can now be written as

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x, \quad 0 < x < L \tag{4.28}$$

where the coefficients A_n are given by

$$A_n = \frac{1}{N_n} \int_0^L f(x) \sin \frac{n\pi}{L} x \; dx \tag{4.29a}$$

with

$$N_n = \int_0^L \sin^2 \frac{n\pi}{L} x \, dx = \frac{L}{2} \tag{4.29b}$$

Equation (4.28), with coefficients (4.29*a*), is known as the *Fourier sine series* representation of $f(x)$ on the interval $(0, L)$.

Note that the right-hand side of Eq. (4.28) is *a periodic* function of period $2L$, and also is an *odd* function of x. Therefore, if Eq. (4.28) converges to $f(x)$ in $(0, L)$, it will also converge to $-f(-x)$ in $(-L, 0)$. In other words, if $f(x)$ is an *odd* function of x, then Eq. (4.28) will represent $f(x)$ not only in $(0, L)$, but also in $(-L, L)$.

Example 4.3

Expand $f(x) = x$ in a Fourier sine series of period $2L$ over the interval $(0, L)$.

SOLUTION

The expansion

$$x = \sum_{n=t}^{\infty} A_n \sin \frac{n\pi}{L} x$$

is a Fourier sine series expansion of $f(x) = x$ of period $2L$ on the interval $(0, L)$. The expansion coefficients A_n are given by

$$A_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi}{L} x \, dx = -\frac{2L}{n\pi} \cos n\pi = \frac{2L}{\pi} \frac{(-1)^{n+1}}{n}$$

Hence,

$$x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} x, \quad 0 \le x < L$$

Here, we note that this series converges to $f(x) = x$ at $x = 0$, but not at $x = L$ as it converges to zero at $x = L$. Moreover, since $f(x) = x$ is an odd function, this series is not only valid on the interval $(0, L)$, but also on the interval $(-L, L)$.

4.6.2 Fourier Cosine Series

A similar series expansion involving cosine terms, rather than sine terms, may be obtained by considering the following characteristic-value problem:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \tag{4.30a}$$

$$\frac{dy(0)}{dx} = 0 \quad \text{and} \quad \frac{dy(L)}{dx} = 0 \tag{4.30b,c}$$

This is another special case of the Sturm–Liouville problem (4.24) with $\alpha_1 = \alpha_2 = 0$. Therefore, the characteristic functions form a complete orthogonal set with respect to the weight function *unity* over the interval $(0, L)$. It can easily be shown that the characteristic functions and characteristic values are (see Problem 4.5)

$$\phi_n(x) = \cos\lambda_n x \quad \text{and} \quad \lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots$$

Here it should be noted that $\phi_n(x) = 1$ is a member of the set of characteristic functions corresponding to $\lambda_0 = 0$. Thus, Eq. (4.25) can now be written as

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L}x, \quad 0 < x < L \tag{4.31}$$

where the coefficients A_n are given by

$$A_n = \frac{1}{N_n}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx \tag{4.32a}$$

with

$$N_n = \int_0^L \cos^2\frac{n\pi}{L}x\,dx = \begin{cases} L, & n = 0 \\ \dfrac{L}{2}, & n = 1,2,3,\ldots \end{cases} \tag{4.32b}$$

Therefore,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \tag{4.33a}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots \tag{4.33b}$$

Equation (4.31) is known as the *Fourier cosine series* representation of $f(x)$ on the interval $(0, L)$.

We also note that the right-hand side of Eq. (4.31) is a *periodic* function of period $2L$ and also an *even* function of x. Therefore, if Eq. (4.31) converges to $f(x)$ in $(0, L)$, it will converge to $f(-x)$ in $(-L, 0)$. In other words, if $f(x)$ is an *even* function of x, then Eq. (4.31) will represent $f(x)$ not only on $(0, L)$, but also on $(-L, L)$.

Example 4.4

Expand $f(x) = x$ in a Fourier cosine series of period $2L$ over the interval $(0, L)$.

SOLUTION

The expansion

$$x = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L}x$$

is a Fourier cosine series representation of $f(x) = x$ of period $2L$ on the interval $(0, L)$. The expansion coefficients A_n are given by

$$A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

and

$$A_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi}{L}x\,dx = -\left[1-(-1)^n\right]\frac{2L}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

Hence, the expansion becomes

$$x = \frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos\frac{n\pi}{L}x, \quad 0 \le x \le L$$

Since x is an odd function, this expansion is valid only on the interval $(0, L)$. In addition, it converges to $f(x) = x$ both at $x = 0$ and $x = L$, which can readily be demonstrated by the use of the relation (see Problem 4.11)

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}$$

The Fourier sine and cosine series that we have discussed in this section can be used in solving certain heat conduction problems. These two series were developed by considering two special cases of the boundary conditions of the Sturm–Liouville problem (4.24). Other series similar to these can also be developed by considering different combinations of the boundary conditions. The characteristic values and the characteristic functions for the remaining seven cases can be obtained by following the same procedure. The characteristic values, characteristic functions, and the corresponding normalization integrals for all of these cases were evaluated and tabulated by Özışık [6–8]. They are summarized here in Table 4.1.

We will discuss the application of Fourier expansions in Chapters 5 and 6 where we solve a number of representative linear heat conduction problems by the method of separation of variables. In Chapter 7, we will further extend the Fourier expansions discussed in this chapter and develop Fourier transforms, which we implement as another method of solution for linear heat conduction problems. In Chapter 13, a more general perspective of the finite Fourier transforms method (also called the integral transform technique) is presented and then applied in the solution of heat conduction in heterogeneous media.

4.7 Complete Fourier Series

In the previous section we saw that an arbitrary function $f(x)$ can be expanded on the interval $(0, L)$ in a series of sine functions as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x \tag{4.34}$$

If $f(x)$ is an *odd* function, that is, if

$$f(-x) = -f(x)$$

then Eq. (4.34) will be a valid expansion not only on the interval $(0, L)$ but also on the interval $(-L, L)$.

We also saw that we can expand another arbitrary function $g(x)$ on the same interval $(0, L)$ in a series of cosine functions as

$$g(x) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi}{L} x \tag{4.35}$$

If $g(x)$ is an *even* function, that is, if

$$g(-x) = g(x)$$

then Eq. (4.35) will be a valid expansion not only on the interval $(0, L)$ but also on the interval $(-L, L)$.

Any function of x, say $F(x)$, can be written as

$$F(x) = \frac{1}{2}\left[F(x) - F(-x)\right] + \frac{1}{2}\left[F(x) + F(-x)\right] \tag{4.36}$$

TABLE 4.1

Fourier Series in the Finite Internal (0, L)

Fourier expansion: $f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x),\ 0 < x < L$

Expansion coefficients: $A_n = \frac{1}{N_n} \int_0^L f(x)\phi_n(x)\,dx$

Characteristic-value problem; $\left\{ \begin{array}{c} \frac{d^2y}{dx^2} + \lambda^2 y = 0 \\ \alpha_1 y(0) + \beta_1 \frac{dy(0)}{dx} = 0 \\ \alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0 \end{array} \right\}$

Boundary conditions: At $x = 0$	Boundary conditions: At $x = L$	Characteristic function $\phi_n(x)$[†]	$N_n = \int_0^L [\phi_n(x)]^2\,dx$	Characteristic values λ_n are positive roots of[†]
Third kind[‡] ($\alpha_1 \neq 0, \beta_1 = 0$)	Third kind[‡] ($\alpha_2 \neq 0$, $\beta_2 \neq 0$)	$\lambda_n \cos \lambda_n x - H_1 \sin \lambda_n x$	$\frac{1}{2}\left\{\left(\lambda_n^2 + H_1^2\right)\left[L + \frac{H_2}{\lambda_n^2 + H_2^2}\right] - H_1\right\}$	$\tan \lambda L = \frac{\lambda(H_2 - H_1)}{\lambda^2 + H_1 H_2}$
Third kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Second kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\cos \lambda_n (L - x)$	$\frac{1}{2\lambda_n}(\lambda_n L + \sin \lambda_n L \cos \lambda_n L)$	$\lambda \tan \lambda L = -H_1$
Third kind[§] ($\alpha_1 \neq 0, \beta_1 \neq 0$)	First kind[§] ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sin \lambda_n (L - x)$	$\frac{1}{2\lambda_n}(\lambda_n L + \sin \lambda_n L \cos \lambda_n L)$	$\lambda \tan \lambda L = H_1$
Second kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Third kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\cos \lambda_n x$	$\frac{1}{2\lambda_n}(\lambda_n L + \sin \lambda_n L \cos \lambda_n L)$	$\lambda \tan \lambda L = -H_2$
Second kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Second kind ($\alpha_2 = 0, \beta_2 \neq 0$)	$\cos \lambda_n x$	$\frac{L^{\|}}{2}$	$\sin \lambda L = 0$ $\left(\lambda_n = \frac{n\pi}{L}, n = 0, 1, 2, \ldots\right)$
Second kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	First kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\cos \lambda_n x$	$\frac{L}{2}$	$\cos \lambda L = 0$ $\left(\lambda_n = \frac{2n-1}{L}\frac{\pi}{2}\ n = 1, 2, 3, \ldots\right)$

(*Continued*)

TABLE 4.1 (CONTINUED)

Fourier Series in the Finite Internal (0, L)

First kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Third kind[#] ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sin \lambda_n x$	$\frac{1}{2\lambda_n}(\lambda_n L - \sin \lambda_n L \cos \lambda_n L)$	$\lambda \cot \lambda L = -H_2$
First kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Second kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sin \lambda_n x$	$\frac{L}{2}$	$\cos \lambda L = 0$ $\left(\lambda_n = \frac{2n-1}{L}\frac{\pi}{2} \quad n = 1, 2, 3, \ldots\right)$
First kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	First kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sin \lambda_n x$	$\frac{L}{2}$	$\sin \lambda L = 0$ $\left(\lambda_n = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots\right)$

† $H_1 = \alpha_1/\beta_1$ and $H_2 = \alpha_2/\beta_2$.
‡ $\lambda_0 = 0$ is a characteristic value if $L = 1/H_1 - 1/H_2 > 0$. The corresponding characteristic function is $\phi_0(x) = x - 1/H_1$.
§ $\lambda_0 = 0$ is a characteristic value if $L = 1/H_1 > 0$. The corresponding characteristic function is $\phi_0(x) = L - x$.
|| When $n = 0$, replace $L/2$ by L.
[#] $\lambda_0 = 0$ is a characteristic value if $L = -1/H_2 > 0$. The corresponding characteristic, function is $\phi_0(x) = x$.

where the first term on the right side is an odd function, and the second term is an even function, of x. Therefore, the function $F(x)$ can be expanded on the interval $(-L, L)$ in a series of sine and cosine functions as

$$F(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x + \sum_{n=0}^{\infty} b_n \cos\frac{n\pi}{L}x, \quad -L < x < L \tag{4.37a}$$

or

$$F(x) = b_0 + \sum_{n=1}^{\infty}\left[a_n \sin\frac{n\pi}{L}x + b_n \cos\frac{n\pi}{L}x\right], \quad -L < x < L \tag{4.37b}$$

where the coefficients a_n and b_n are given by

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \left\{\frac{1}{2}[F(x) - F(-x)]\right\}\sin\frac{n\pi}{L}x\,dx \\ &= \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{4.38a}$$

$$b_0 = \frac{1}{L}\int_0^L \left\{\frac{1}{2}[F(x) + F(-x)]\right\}dx = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx \tag{4.38b}$$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \left\{\frac{1}{2}\left[F(x) + F(-x)\right]\right\}\cos\frac{n\pi}{L}x\,dx \\ &= \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{4.38c}$$

Equation (4.37*a*), or (4.37*b*), is called the *complete Fourier* series of function $F(x)$ in the interval $(-L, L)$. If $F(x)$ is a periodic function with period $2L$, then the coefficients a_n and b_n can be determined equivalently from

$$a_n = \frac{1}{L}\int_c^{c+2L} F(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots \tag{4.39a}$$

$$b_0 = \frac{1}{2L}\int_c^{c+2L} F(x)\,dx \tag{4.39b}$$

$$b_n = \frac{1}{L}\int_c^{c+2L} F(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots \tag{4.39c}$$

where c is any real constant. Problem 4.13 describes an alternative way to obtain the complete Fourier series.

4.8 Fourier–Bessel Series

Series expansions in terms of Bessel functions arise most frequently in connection with the following characteristic-value problem:

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \tag{4.40a}$$

$$\alpha_1 R(a) + \beta_1\frac{dR(a)}{dr} = 0, \quad \alpha_1^2 + \beta_1^2 > 0 \tag{4.40b}$$

$$\alpha_2 R(b) + \beta_2\frac{dR(b)}{dr} = 0, \quad \alpha_2^2 + \beta_2^2 > 0 \tag{4.40c}$$

This is a special case of the Sturm–Liouville system (4.13) with (see Problem 4.4)

$$p(r) = r,\ q(r) = -\frac{v^2}{r}, \quad \text{and} \quad w(r) = r$$

Hence, the characteristic functions of this problem form a complete orthogonal set with respect to the weight function r on the interval (a, b).

As we shall see in the following chapters, solutions of certain types of heat conduction problems, especially of those in the cylindrical coordinates with homogeneous boundary conditions in the r-direction, can be obtained as an expansion in terms of the characteristic functions of the above characteristic-value problem. In this text, however, we shall restrict our discussions mostly to solid cylinders, and such problems, in general, involve characteristic-value problems of the following form:

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \tag{4.41a}$$

$$R(0) = \textit{finite} \tag{4.41b}$$

$$\alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0, \quad \alpha^2 + \beta^2 > 0 \tag{4.41c}$$

The general solution of Eq. (4.41*a*) can be written as (see Appendix B)

$$R(r) = AJ_v(\lambda r) + BY_v(\lambda r) \tag{4.42}$$

The boundary condition (4.41*b*) yields $B \equiv 0$. Hence, the characteristic functions are $J_v(\lambda r)$ and the characteristic values λ_n are the roots of the *characteristic-value equation*

$$\alpha J_v(\lambda r_0) + \beta \frac{dJ_v(\lambda r_0)}{dr} = 0 \quad \Rightarrow \quad \lambda_n, n = 1, 2, 3, \ldots \tag{4.43}$$

which is obtained from the application of the boundary condition (4.41*c*). In view of the fact that $p(0) = 0$ and $R(0)$ = finite (and also $dR(0)/dr$ = finite), the characteristic functions of the system (4.41) form an orthogonal set with respect to the weight function $w(r) = r$ over the interval $(0, r_0)$; that is,

$$\int_0^{r_0} J_v(\lambda_m r) J_v(\lambda_n r) r\, dr = 0, \quad \lambda_m \neq \lambda_n \tag{4.44}$$

and the set $\{J_v(\lambda_n r);\, n = 1, 2, \ldots\}$ is a complete orthogonal set. We can, therefore, expand an arbitrary function $f(r)$, which is piecewise differentiable on the interval $(0, r_0)$, in a series of these characteristic functions in the same interval as

$$f(r) = \sum_{n=1}^{\infty} A_n J_v(\lambda_n r), \quad 0 < r < r_0 \tag{4.45}$$

which is known as the *Fourier–Bessel* series of $f(r)$ on the interval $(0, r_0)$. Making use of the orthogonality relation of the Bessel functions (4.44), the coefficients A_n may readily be obtained as

$$A_n = \frac{1}{N_n} \int_0^{r_0} f(r) J_v(\lambda_n r) r\, dr \tag{4.46a}$$

with

$$N_n = \int_0^{r_0} J_v^2(\lambda_n r) r\, dr \tag{4.46b}$$

The normalization integral (4.46b) can be evaluated as follows. The function $J_v(\lambda_n r)$ satisfies the differential equation (4.41a) when $\lambda = \lambda_n$, that is

$$r^2 \frac{d^2 J_v(\lambda_n r)}{dr^2} + r\frac{dJ_v(\lambda_n r)}{dr} + \left(\lambda_n^2 r^2 - v^2\right) J_v(\lambda_n r) = 0 \tag{4.47a}$$

which can be rewritten as

$$r\frac{d}{dr}\left[r\frac{dJ_v(\lambda_n r)}{dr}\right] + \left(\lambda_n^2 r^2 - v^2\right) J_v(\lambda_n r) = 0 \tag{4.47b}$$

Multiplying Eq. (4.47b) by $2[dJ_v(\lambda_n r)/dr]$ and rearranging the resulting expression, we get

$$\frac{d}{dr}\left[r\frac{dJ_v(\lambda_n r)}{dr}\right]^2 = -\left(\lambda_n^2 r^2 - v^2\right)\frac{dJ_v^2(\lambda_n r)}{dr} \tag{4.48}$$

Integrating Eq. (4.48) with respect to r over $(0, r_0)$ and rearranging the right-hand side by integration by parts, we obtain

$$N_n = \frac{r_0^2}{2\lambda_n^2}\left\{\left[\lambda_n^2 - \left(\frac{v}{r_0}\right)^2\right] J_v^2(\lambda_n r_0) + \left[\frac{dJ_v(\lambda_n r_0)}{dr}\right]^2\right\} \tag{4.49}$$

This general result can be simplified for the special cases of the boundary condition at $r = r_0$ as follows:

Case 1: $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$. For this special case the characteristic values are the roots of

$$J_v(\lambda_n r_0) = 0 \tag{4.50}$$

and, therefore, Eq. (4.49) reduces to

$$N_n = \frac{r_0^2}{2\lambda_n^2}\left[\frac{dJ_v(\lambda_n r_0)}{dr}\right]^2 \tag{4.51}$$

Noting that (see Appendix B)

$$\frac{dJ_v(\lambda_n r)}{dr} = -\lambda_n J_{v+1}(\lambda_n r) + \frac{v}{r} J_v(\lambda_n r) \tag{4.52}$$

Eq. (4.51) can also be rewritten as

$$N_n = \frac{r_0^2}{2} J_{v+1}(\lambda_n r_0) \tag{4.53}$$

Case 2: $\alpha \neq 0$, $\beta \neq 0$. For this special case the characteristic values are the roots of

$$\frac{dJ_v(\lambda_n r_0)}{dr} = 0 \tag{4.54}$$

and, therefore, Eq. (4.49) becomes

$$N_n = \frac{r_0^2}{2}\left[1 - \left(\frac{v}{\lambda_n r_0}\right)^2\right] J_v^2(\lambda_n r_0) \tag{4.55}$$

We should, however, note that if $v = 0$ then $\lambda_0 = 0$ is a characteristic value and the corresponding characteristic function is $J_0(\lambda_0 r) = 1$. Thus, when $\nu = 0$, the function $J_0(\lambda_0 r) = 1$ must be included in the set of characteristic functions. The Fourier–Bessel series of f(r) then becomes

$$f(r) = A_0 + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \tag{4.56}$$

where

$$A_0 = \frac{2}{r_0^2}\int_0^{r_0} f(r) r\, dr \tag{4.57}$$

Case 3: $\alpha \neq 0$, $\beta \neq 0$. For this special case the characteristic values are the roots of

$$HJ_v(\lambda_n r_0) + \frac{dJ_v(\lambda_n r_0)}{dr} = 0 \tag{4.58}$$

where we have defined $H = \alpha/\beta$, and, therefore, Eq. (4.49) reduces to

$$N_n = \frac{r_0^2}{2}\left[1 - \frac{1}{\lambda_n^2}\left(H^2 - \frac{v^2}{r_0^2}\right)\right] J_v^2(\lambda_n r_0) \tag{4.59}$$

The Bessel function of the first kind, $J_v(\lambda_n r)$, is defined by (see Appendix B)

$$J_v(\lambda r) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda r/2)^{2k+v}}{k!\,\Gamma(k+v+1)} \tag{4.60}$$

Therefore,

$$J_v(-\lambda r) = (-1)^v J_v(\lambda r) \tag{4.61}$$

So, replacing λ_n by $-\lambda_n$ in $J_v(\lambda_n r)$ either does not change it (when v is zero or an even integer) or multiplies $J_v(\lambda_n r)$ by -1 (when v is an odd integer). Consequently, the roots of any one of the above three characteristic-value equations (4.50), (4.54), and (4.58) exist in pairs symmetrically located with respect to $r = 0$. However, we do not need to consider the negative values of λ_n, as both $\pm\lambda_n$ would lead to the same characteristic function $J_v(\lambda_n r)$.

We summarize the Fourier–Bessel series expansions obtained in this section in Table 4.2.

Example 4.5

Expand the function $f(r) = 1$ on the interval $(0. r_0)$ in a Fourier–Bessel series of the form

$$1 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

where λ_n are the positive roots of $J_0(\lambda_n r) = 0$.

SOLUTION

From Eqs. (4.46*a*) and (4.53), or from Table 4.2, we have

$$A_n = \frac{1}{N_n}\int_0^{r_0} J_0(\lambda_n r) r\,dr \quad \text{with} \quad N_n = \frac{r_0^2}{2} J_1^2(\lambda_n r_0)$$

TABLE 4.2

Fourier–Bessel Series in the Finite Interval $(0, r_0)$

Fourier–Bessel expansion: $f(r) = \sum_{n=1}^{\infty} A_n J_v(\lambda_n r), \quad 0 < r < r_0$

Expansion coefficients: $A_n = \frac{1}{N_n}\int_0^{r_0} f(r) J_v(\lambda_n r) r\,dr$

$$\left\{\begin{array}{c} r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \\ R(0) = \textit{finite} \\ \alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0 \end{array}\right\}$$

Boundary condition at $r = r_0$	$N_n = \int_0^{r_0} J_v^2(\lambda_n r) r\,dr$ †	Characteristic values λ_n are positive roots of†
Third kind‡ $(\alpha \neq 0, \beta \neq 0)$	$\frac{r_0^2}{2}\left[1 + \frac{1}{\lambda_n^2}\left(H^2 - \frac{v^2}{r_0^2}\right)\right] J_v^2(\lambda_n r_0)$	$HJ_v(\lambda r_0) + \frac{dJ_v(\lambda r_0)}{dr} = 0$
Second kind $(\alpha \neq 0, \beta \neq 0)$	$\frac{r_0^2}{2}\left[1 - \left(\frac{v}{\lambda_n r_0}\right)^2\right] J_v^2(\lambda_n r_0)$	$\frac{dJ_v(\lambda r_0)}{dr} = 0$§
First kind $(\alpha \neq 0, \beta \neq 0)$	$\frac{r_0^2}{2} J_{v+1}^2(\lambda_n r_0)$	$J_v(\lambda r_0) = 0$

† $H = \alpha/\beta$.

‡ When $v > 0$ and $r_0 = -\nu/H > 0$, $\lambda_0 = 0$ is a characteristic value for this case with the corresponding characteristic function $\phi_0(r) = r^v$.

§ When $v = 0$, $\lambda_0 = 0$ is also a characteristic value for this case with the corresponding characteristic function $\phi_0(r) = 1$.

Since (see Appendix B)

$$\frac{d}{dr}\left[rJ_1(\lambda_n r)\right] = \lambda_n r J_0(\lambda_n r)$$

then

$$\int_0^{r_0} J_0(\lambda_n r) r\, dr = \frac{1}{\lambda_n}\int_r^{r_0} \frac{d}{dr}\left[rJ_1(\lambda_n r)\right] dr = \frac{r_0}{\lambda_n} J_1(\lambda_n r_0)$$

and, therefore, we get

$$A_n = \frac{2}{(\lambda_n r_0) J_1(\lambda_n r_0)}, \quad n = 1, 2, 3, \ldots$$

Thus, the desired Fourier–Bessel series is given by

$$1 = \frac{2}{r_0}\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)} = \frac{2}{\lambda_1 r_0}\frac{J_0(\lambda_1 r)}{J_1(\lambda_1 r_0)} + \frac{2}{\lambda_2 r_0}\frac{J_0(\lambda_2 r)}{J_1(\lambda_2 r_0)} + \ldots$$

A table of the first 40 zeros, α_n, of $J_0(\alpha)$ and the corresponding values of $J_0(\alpha)$ is given in Appendix B. The first three zeros and the corresponding values of $J_1(\alpha_n)$ are

$$\begin{aligned} \alpha_1 &= 2.4048, \quad & J_1(\alpha_1) &= 0.5191 \\ \alpha_2 &= 5.5201, \quad & J_1(\alpha_2) &= -0.3403 \\ \alpha_3 &= 8.6537, \quad & J_1(\alpha_3) &= 0.2715 \end{aligned}$$

So, the first three leading terms of the expansion become

$$\begin{aligned} 1 = 1.602\, J_0\left(2.4048\frac{r}{r_0}\right) &- 1.065\, J_0\left(5.5201\frac{r}{r_0}\right) \\ &+ 0.8512 J_0\left(8.6537\frac{r}{r_0}\right) + \ldots \end{aligned}$$

Since $J_0(a_n)$ are even functions, the above series represents 1 not only on the interval $(0, r_0)$ but also on the symmetrical interval $(-r_0, r_0)$. At the end points $r = \pm r_0$, however, the series does not converge to 1, because all the terms in the series vanish at $r = \pm r_0$.

As we shall discuss in Chapter 5, two-dimensional steady-state problems in the cylindrical coordinates can be in one of the following forms:

$$T = f_1(r, \phi), \quad T = f_2(r, z), \quad \text{and} \quad T = f_3(\phi, z)$$

The problems of the form $T = f_1(r, \phi)$, usually lead to solutions in the form of ordinary Fourier expansions in the ϕ-direction, while the second class of problems of the form

$T = f_2(r, z)$, when the boundary conditions in the r direction are homogeneous or can be made homogeneous, lead to solutions in the form of Fourier–Bessel series in the r-direction. The problems of the form $T = f_3(\phi, z)$, on the other hand, are of no physical importance, except in thin-walled cylinders.

If the temperature distribution in a two-dimensional heat conduction problem in the spherical coordinates depends on the polar angle, θ, then the solution usually leads to an expansion in terms of *Legendre polynomials.* The necessary mathematical background for such expansions is deferred to Section 5.4.

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Problems

4.1 Show that Eqs. (4.8) and (4.10) satisfy the boundary conditions (4.3*b,c*).

4.2 Show that the characteristic-value problem

$$\frac{d^2y}{dx^2} - \lambda^2 y = 0$$
$$y(0) = 0 \quad \text{and} \quad y(L) = 0$$

cannot have nontrivial solutions for real values of λ.

4.3 Consider the following functions

$$A_0,\ A_1 + A_2x \quad \text{and} \quad A_3 + A_4x + A_5x^2$$

where A_o,..., A_5 are constants. Determine the constants so that these three functions form an orthogonal set on the interval (0. 1) with respect to the weight function unity.

4.4 Show that any equation having the form

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+\left[a_2(x)+\lambda a_3(x)\right]y=0$$

can be written in the form of Eq. (4.13*a*) with

$$p(x)=\exp\left[\int\frac{a_1}{a_0}dx\right],\quad q(x)=\frac{a_2}{a_0}p(x),\quad \text{and}\quad w(x)=\frac{a_3}{a_0}p(x)$$

4.5 Find the characteristic values and the characteristic functions of the following characteristic-value problem:

$$\frac{d^2y}{dx^2}+\lambda^2y=0$$

$$\frac{dy(0)}{dx}=0 \quad \text{and} \quad \frac{dy(L)}{dx}=0$$

4.6 (*a*) Find the characteristic values and the characteristic functions of the following characteristic-value problem:

$$\frac{d^2y}{dx^2}+\lambda^2y=0$$

$$\frac{dy(0)}{dx}=0 \quad \text{and} \quad \alpha y(L)+\beta\frac{dy(L)}{dx}=0$$

where α and β are two nonzero real constants.

(*b*) Expand an arbitrary piecewise-differentiable function $f(x)$ on the interval $(0, L)$ in a series of the characteristic functions obtained in part (*a*) and determine the expansion coefficients.

4.7 (*a*) Find the characteristic values and the characteristic functions of the following characteristic-value problem:

$$\frac{d^2y}{dx^2}+\lambda^2y=0$$

$$y(0)=0 \quad \text{and} \quad y(L)-L\frac{dy(L)}{dx}=0$$

where L is a nonzero real constant.

(*b*) Expand an arbitrary piecewise-differentiable function $f(x)$ on the interval $(0, L)$ in a series of the characteristic functions found in part (*a*) and determine the expansion coefficients.

4.8 Expand the following function into a Fourier sine series of period $2L$ on the interval $(0, L)$.

$$f(x) = \begin{cases} 1, & \text{when} \quad x < \dfrac{L}{2} \\ 0, & \text{when} \quad x > \dfrac{L}{2} \end{cases}$$

4.9 Expand the function given in Problem 4.8 in a Fourier cosine series of period $2L$ on the interval $(0, L)$.

4.10 Consider the Fourier expansion,

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x), \quad a < x < b$$

where $\{\phi n(x);\ n = 0, 1, 2, \ldots\}$ is the complete set of all the characteristic functions of the Sturm–Liouville problem (4.13). and $f(x)$ is any piecewise-differentiable function on the interval (a, b) Show formally that

$$\int_a^b \left[f(x)\right]^2 w(x)\,dx = \sum_{n=0}^{\infty} N_n A_n^2$$

where N_n are normalization integrals of the characteristic functions $\phi_n(x)$.

4.11 (*a*) Find the expansion coefficients of the sine series.

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x, \quad 0 < x < L$$

(*b*) From the result of part (*a*) and from the result of Problem 4.10, deduce the following relation:

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{\left[1 - (-1)^n\right]}{n^2}$$

4.12 Expand the function given in Problem 4.8 in a complete Fourier series of period $2L$ on the interval $(-L, L)$.

4.13 (*a*) Determine the characteristic values and the characteristic functions of the following Sturm–Liouville problem:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$$y(-L) = y(L) \quad \text{and} \quad \frac{y(-L)}{dx} = \frac{y(L)}{dx}$$

where L is a nonzero real constant.

(*b*) Show that the expansion of an arbitrary piecewise-differentiable function $f(x)$ on the interval $(-L, L)$ in terms of the characteristic functions found in (*a*) leads to the *complete Fourier series* representation of $f(x)$.

4.14 Expand the function $f(r) = r$ on the interval $(0, r_0)$ in a series of the characteristic functions of the following characteristic-value problem:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0$$
$$R(0) = \textit{finite} \quad \text{and} \quad R(r_0) = 0$$

4.15 Show that the characteristic-value problem

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - (\lambda^2 r^2 + v^2)R = 0$$
$$R(0) = \textit{finite} \quad \text{and} \quad R(r_0) = 0$$

cannot have nontrivial solutions for real values of λ.

4.16 (*a*) Find the characteristic values and the characteristic functions of the following Sturm–Liouville problem:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0$$
$$R(a) = 0 \quad \text{and} \quad R(b) = 0$$

(*b*) Expand an arbitrary piecewise-differentiable function $f(r)$ on (a, b) in a series of the characteristic functions found in part (*a*) and determine the expansion coefficients.

4.17 Write the analytical solution of the boundary value problem below by direct comparison with the generalized Bessel equation solution:

$$\frac{d}{dx}\left[K(x)\frac{d\theta(x)}{dx}\right] - M^2W(x)\big(\theta(x)+1\big) = 0 \qquad 0 < x < 1$$

$$\theta(0) = 0 \qquad \left.\frac{d\theta(x)}{dx}\right|_{x=1} = 0$$

for the following choice of coefficients:

$$K(x) = x^2 \qquad M(x) = x$$

Propose a solution by expanding the potential in terms of eigenfunctions from the following choice of auxiliary eigenvalue problem:

$$\frac{d^2\varphi}{dx^2} + \mu^2\varphi(x) = 0 \qquad 0 < x < 1$$

$$\varphi(0) = 0 \quad \text{and} \quad \left.\frac{d\varphi}{dx}\right|_{x=1} = 0$$

5

Steady-State Two- and Three-Dimensional Heat Conduction: Solutions with Separation of Variables

5.1 Introduction

In this chapter we obtain the solutions of various two- and three-dimensional steady-state linear heat conduction problems by the method of *separation of variables* and introduce the method in terms of examples. We first consider some representative examples in the rectangular coordinate system and investigate the conditions under which the method of separation of variables is applicable. Next, we consider similar problems in the cylindrical and spherical coordinate systems.

5.2 Two-Dimensional Steady-State Problems in the Rectangular Coordinate System

Consider, as an example, a solid bar of rectangular cross section as shown in Fig. 5.1. Let the bar be free of internal heat sources or sinks and have a constant thermal conductivity. If there are no temperature gradients in the z direction (i.e., either it is very long in the z direction or its surfaces perpendicular to the z direction at the two ends are perfectly insulated), then under steady-state conditions the temperature distribution $T(x, y)$ at any cross section in the bar must satisfy the *Laplace equation* in two dimensions:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{5.1}$$

Assume that the surfaces at $x = 0$, $x = a$ and $y = 0$ are maintained at zero temperature, while the temperature of the surface at $y = b$ is given as a function of the x coordinate; that is, $T(x, b) = f_1(x)$. The boundary conditions can then be written as

$$T(0,y)=0, \quad 0<y<b \tag{5.2a}$$

$$T(a,y)=0, \quad 0<y<b \tag{5.2b}$$

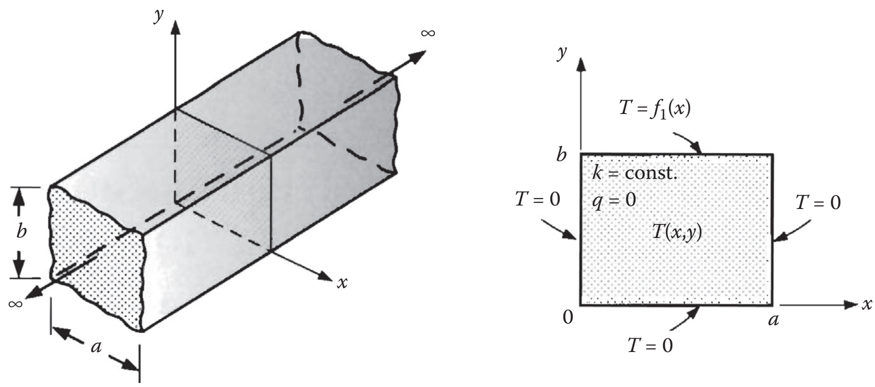

FIGURE 5.1
Solid bar of rectangular cross section.

$$T(x,0)=0, \qquad 0<x<b \tag{5.2c}$$

$$T(x,b)=f_1(x), \qquad 0<x<a \tag{5.2d}$$

The formulation of the problem with the differential equation (5.1) and the boundary conditions (5.2) is now complete. To find the temperature distribution in the bar, we must determine that solution of the Laplace equation (5.1) which will satisfy the prescribed boundary values (5.2). We now seek a solution by the method of *separation of variables,* which requires the assumption of the existence of a product solution of the form

$$T(x,y)=X(x)Y(y) \tag{5.3}$$

where X is a function of x alone and Y is a function of y alone. Substituting the assumption (5.3) into Eq. (5.1), there follows

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0 \tag{5.4a}$$

or, *"separating the variables"*, we get

$$-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{5.4b}$$

Since each term in Eq. (5.4*b*) is a function of only one of the variables and each variable can be changed independently, the right- and left-hand sides of Eq. (5.4*b*) will be equal to each other if and only if they are equal to some constant, say γ. Equation (5.4*b*) can now be written as two ordinary differential equations as follows:

$$\frac{d^2X}{dx^2} + \gamma X(x) = 0 \tag{5.5a}$$

$$\frac{d^2Y}{dy^2} - \gamma Y(y) = 0 \tag{5.5b}$$

where the *"separation constant"* γ is yet to be determined. Let us now consider the following possible cases:

Case 1. Let γ be *zero*. In this case, the solutions of Eqs. (5.5*a,b*) are given by

$$X(x) = Ax + B \quad \text{and} \quad Y(y) = Cy + D \tag{5.6a,b}$$

Hence,

$$T(x,y) = (Ax + B)(Cy + D) \tag{5.6c}$$

Since $Y(y) = Cy + D$ cannot vanish, as it would lead to $T(x, y) = 0$ which does not satisfy the boundary condition (5.2*d*), then imposing the boundary conditions (5.2*a,b*) we get $A = B = 0$. This, in turn, leads to $T(x, y) = 0$. Thus, for $\gamma = 0$ there is no solution.

Case 2. Let $\gamma < 0$. The solutions of Eqs. (5.5*a,b*) are then given by

$$X(x) = A\sinh\lambda x + B\cosh\lambda x \quad \text{and} \quad Y(y) = C\sin\lambda y + D\cos\lambda y \tag{5.7a,b}$$

where we have substituted $\gamma = -\lambda^2$. Hence,

$$T(x,y) = (A\sinh\lambda x + B\cosh\lambda x)(C\sin\lambda y + D\cos\lambda y) \tag{5.7c}$$

Applying the boundary conditions (5.2*a,b*) yields $A = B = 0$. Thus, as in the previous case, again there is no solution for $\gamma < 0$.

Case 3. Let $\gamma > 0$. In this case, we get

$$X(x) = A\sin\lambda x + B\cos\lambda x \quad \text{and} \quad Y(y) = C\sinh\lambda y + D\cosh\lambda y \tag{5.8a,b}$$

where we have substituted $\gamma = \lambda^2$. Therefore,

$$T(x,y) = (A\sin\lambda x + B\cos\lambda x)(C\sinh\lambda y + D\cosh\lambda y) \tag{5.8c}$$

Applying the boundary condition (5.2*a*) yields

$$T(0,y) = B(C\sinh\lambda y + D\cosh\lambda y) = 0$$

so that $B = 0$, as $(C \sinh \lambda y + D \cosh \lambda y) = 0$ would lead to $T(x, y) = 0$, which is not a solution. Similarly, the boundary condition (5.2*c*) gives $D = 0$. Thus, Eq. (5.8*c*) reduces to

$$T(x, y) = E \sin \lambda x \sinh \lambda y \tag{5.9}$$

where $E = AC$. Imposing the boundary condition (5.2*b*) we obtain

$$T(a, y) = E \sin \lambda a \sinh \lambda y = 0$$

In order not to have $T(x, y) = 0$, it follows that $\sin \lambda a = 0$. This, in turn, suggests that λ can have any of the following values:

$$\lambda_n = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots \tag{5.10}$$

Note that $n = 0$ has been excluded because it leads to $T(x, y) = 0$. Furthermore, *negative* integers, $n = -1, -2, \ldots$, can be omitted with no loss of generality. For instance, if we substitute $\lambda = 5\pi/a$ and $\lambda = -5\pi/a$ in Eq. (5.9), the results differ only in a minus sign out in front, which can be absorbed by the arbitrary constant E. Thus, a solution of the heat conduction equation (5.1) which satisfies the boundary conditions (5.2*a,b,c*) can be written as

$$T_n(x, y) = E_n \sin \lambda_n x \sinh \lambda_n y \tag{5.11}$$

But is Eq. (5.11) the solution of the problem for any value of n? That is, does it also satisfy the boundary condition (5.2*d*)? In general,

$$f_1(x) \neq E_n \sin \lambda_n x \sinh \lambda_n b$$

Therefore, Eq. (5.11) does not satisfy the boundary condition at $y = b$. On the other hand, since the differential equation (5.1) and the boundary conditions (5.2*a,b,c*) are linear, it follows that a linear combination in the form

$$T(x, y) = \sum_{n=1}^{\infty} E_n \sin \lambda_n x \sinh \lambda_n y \tag{5.12}$$

will also satisfy the differential equation (5.1) and the boundary conditions (5.2*a,b,c*). Now, imposing the boundary condition (5.2*d*) on Eq. (5.12), we get

$$f_1(x) = \sum_{n=1}^{\infty} E_n \sin \lambda_n x \sinh \lambda_n b \tag{5.13a}$$

or

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \lambda_n x \tag{5.13b}$$

where $a_n = E_n \sinh \lambda_n b$. Is it possible to find the constants a_n that satisfy Eq. (5.13*b*)? If so, how? Observe that

$$T(0,y) = X(0)Y(y) \quad \Rightarrow \quad X(0) = 0$$

and

$$T(a,y) = X(a)Y(y) \quad \Rightarrow \quad X(a) = 0$$

So, $X(x)$ satisfies

$$\frac{d^2X}{dx^2} + \lambda^2 X(x) = 0 \tag{5.14a}$$

$$X(0) = 0 \tag{5.14b}$$

$$X(a) = 0 \tag{5.14c}$$

Recalling Section 4.6.1, we see that this boundary-value problem is a Sturm–Liouville system with the following characteristic functions and characteristic values:

$$\phi_n(x) = \sin \lambda_n x \quad \text{and} \quad \lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots$$

Therefore, $f_1(x)$ can be expanded in a Fourier series in terms of the characteristic functions $\sin(n\pi/a)x$; in fact, Eq. (5.13*b*) is the *Fourier sine expansion* of $f_1(x)$ on the interval $(0, a)$, where the expansion coefficients a_n can be determined by using Eqs. (4.29*a,b*) or Table 4.1 as

$$a_n = \frac{2}{a}\int_0^a f_1(x) \sin \lambda_n x \, dx \tag{5.15}$$

Thus,

$$E_n = \frac{2}{a \sinh \lambda_n b}\int_0^a f_1(x) \sin \lambda_n x \, dx \tag{5.16}$$

The solution for the temperature distribution may then be written as

$$T(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin(n\pi/a)x \sinh(n\pi/a)y}{\sin(n\pi/a)b}\int_0^a f_1(x') \sin \frac{n\pi}{a} x' \, dx' \tag{5.17}$$

If, in particular, $f_1(x) = T_0$ = constant, then

$$E_n = \frac{2}{a \sinh \lambda_n b} \int_0^a T_0 \sin \frac{n\pi}{a} x\, dx = \frac{2T_0}{n\pi} \frac{1-(-1)^n}{\sinh \lambda_n b} \tag{5.18}$$

Substituting this result into Eq. (5.12), we get

$$\frac{T(x,y)}{T_0} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \frac{\sin(n\pi / a)x \sinh(n\pi / a)y}{\sin(n\pi / a)b} \tag{5.19}$$

Except for very small values of y/a, the series (5.19) converges rapidly, and only the first few terms would be sufficient to find the temperature at any point numerically. As a numerical example, let us consider the problem given in Example 5.1.

Example 5.1

The temperature is maintained at 0°C along the three surfaces of the rectangular bar shown in Fig. 5.1, while the fourth surface at $y = b$ is held at 100°C. If $a = 2b$, calculate the centerline temperature under steady-state conditions.

SOLUTION

Equation (5.19) with $x = a/2$ and $y = b/2 = a/4$ reduces to

$$\begin{aligned} T\left(\frac{a}{2}, \frac{a}{4}\right) &= \frac{2 \times 100}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n \sin(n\pi / 2) \sinh(n\pi / 4)}{\sin(n\pi / 2)} \\ &= 48.061 - 3.987 + 0.502 - \ldots \\ &= 44.576°\text{C} \end{aligned}$$

As it is seen, this series converges rather rapidly at $x = a/2$ and $y = a/4$.

It was not obvious, a priori, that Eq. (5.1) would possess a separable solution of the form (5.11) or that a solution built up from such solutions could be made to satisfy the prescribed boundary conditions (5.2). In view of the above example, we can now conclude that the method of separation of variables is applicable to steady-state two-dimensional problems if and when (*a*) the differential equation is linear and homogeneous, and (*b*) the four boundary conditions are linear, and three of them are homogeneous, so that one of the directions of the problem is reduced to a boundary-value problem consisting of a homogeneous differential equation with two homogeneous boundary conditions. The *sign of the separation constant* is chosen so that this boundary-value problem becomes a Sturm–Liouville type characteristic-value problem such as the one given by Eqs. (5.14).

Let us now consider a long rectangular bar as shown in cross section in Fig. 5.2. The formulation of the problem for the steady-state temperature distribution $T(x, y)$ over the cross section of the bar is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{5.20}$$

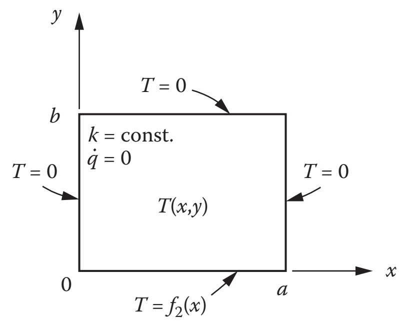

FIGURE 5.2
Cross section of a long solid bar.

$$T(0,y) = 0, \qquad 0 < y < b \tag{5.21a}$$

$$T(a,y) = 0, \qquad 0 < y < b \tag{5.21b}$$

$$T(x,0) = f_2(x), \qquad 0 < x < a \tag{5.21c}$$

$$T(x,b) = 0, \qquad 0 < x < a \tag{5.21d}$$

Assuming the existence of a product solution of the form

$$T(x,y) = X(x)Y(y) \tag{5.22}$$

the solution of the differential equation (5.20) can again be written as

$$T(x,y) = (A \sin \lambda x + B \cos \lambda x)(C \sinh \lambda y + D \cosh \lambda y) \tag{5.23}$$

where the sign of the separation constant has been so chosen that the homogeneous x direction results in a Sturm–Liouville type characteristic-value problem. Application of the boundary condition at $x = 0$ yields $B = 0$, and the boundary condition at $y = b$ gives

$$A \sin \lambda x (C \sinh \lambda b + D \cosh \lambda b) = 0$$

from which we obtain

$$D = -C \frac{\sinh \lambda b}{\cosh \lambda b}$$

Substituting D into Eq. (5.23), together with $B = 0$, gives

$$T(x,y) = AC \sin \lambda x \left(\sin \lambda y - \frac{\sinh \lambda b}{\cosh \lambda b} \cosh \lambda y \right) \tag{5.24a}$$

which can also be written as

$$T(x,y) = E \sin \lambda x \sinh \lambda (b - y) \tag{5.24b}$$

where $E = -AC/\cosh \lambda b$. Applying the boundary condition at $x = a$ yields

$$E \sin \lambda a \sinh \lambda (b - y) = 0$$

or

$$\sin \lambda a = 0$$

which is satisfied only if λ assumes one of the following:

$$\lambda_n = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots$$

Equation (5.24*b*) satisfies, for each value of λ_n, the differential equation (5.20) and the boundary conditions (5.21*a,b,d*). The solution of Eq. (5.20) that will satisfy all the boundary conditions can now be constructed as the linear combination of these individual solutions; that is,

$$T(x,y) = \sum_{n=1}^{\infty} E_n \sin \lambda_n x \sinh \lambda_n y (b - y) \tag{5.25}$$

Finally, applying the condition $T(x, 0) = f_2(x)$, we get

$$f_2(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{a} x \tag{5.26}$$

where $a_n = E_n \sin \lambda_n b$. Equation (5.26) is the Fourier sine expansion of $f_2(x)$ on $(0, a)$, and the expansion coefficients a_n are given by (see Table 4.1):

$$a_n = \frac{2}{a} \int_0^a f_2(x) \sin \frac{n\pi}{a} x \, dx$$

Hence, Eq. (5.25) can be rewritten as

$$T(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin(n\pi/a)x\sinh(n\pi/a)(b-y)}{\sin(n\pi/a)b}\int_0^a f_2(x')\sin\frac{n\pi}{a}x'\,dx' \tag{5.27}$$

Here, we should note that this result could also be obtained directly from the solution (5.17) by substituting $b - y$ for y and $f_2(x)$ for $f_1(x)$.

Similarly, the two-dimensional steady-state temperature distribution in the bar shown in cross section in Fig. 5.3 would be obtained to be (see Problem 5.2)

$$T(x,y) = \frac{2}{b}\sum_{n=1}^{\infty}\frac{\sinh(n\pi/b)x\sin(n\pi/b)y}{\sinh(n\pi/b)a}\int_0^b f_3(y')\sin\frac{n\pi}{a}y'\,dy' \tag{5.28}$$

It can also be shown that the temperature distribution $T(x, y)$ over the cross section shown in Fig. 5.4 is given by (see Problem 5.3)

$$T(x,y) = \frac{2}{b}\sum_{n=1}^{\infty}\frac{\sinh(n\pi/b)(a-x)\sinh(n\pi/a)y}{\sin(n\pi/b)a}\int_0^b f_4(y')\sin\frac{n\pi}{a}y'\,dy' \tag{5.29}$$

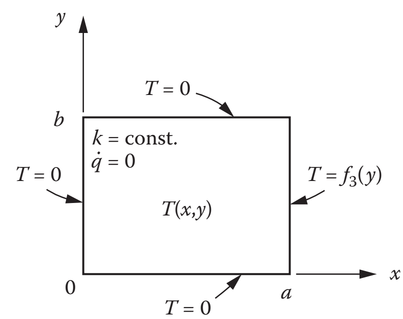

FIGURE 5.3
Cross section of a long solid bar.

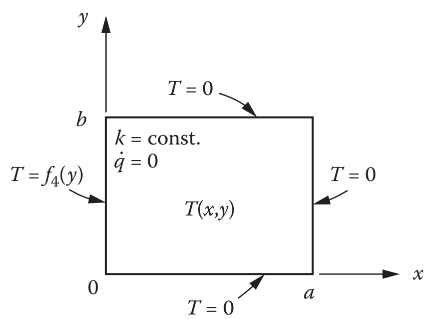

FIGURE 5.4
Cross section of a long solid bar.

5.2.1 Nonhomogeneity in Boundary Conditions

We have seen that the method of separation of variables is readily applicable to two-dimensional steady-state linear problems consisting of a homogeneous differential equation subject to three homogeneous and one nonhomogeneous boundary conditions. Most two-dimensional steady-state problems do not satisfy these requirements. Nonhomogeneities may result, for example, from more than one nonhomogeneous boundary condition and/or a nonhomogeneous differential equation. In such cases, it may be possible to divide the problem into a number of simpler problems such that each simpler problem can be solved by the method of separation of variables. The solution to the original problem is then obtained by superposing the solutions of the simpler problems.

As an example of the problems with more than one nonhomogeneous boundary conditions, let us consider the following two-dimensional steady-state problem:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{5.30}$$

$$T(0,y) = f_4(y), \quad 0 < y < b \tag{5.31a}$$

$$T(0,y) = f_3(y), \quad 0 < y < b \tag{5.31b}$$

$$T(x,0) = f_2(x), \quad 0 < x < a \tag{5.31c}$$

$$T(x,b) = f_1(x), \quad 0 < x < a \tag{5.31d}$$

This is a linear problem with a homogeneous differential equation and four nonhomogeneous boundary conditions. Since it is linear, by the *principle of superposition*, the solution of this problem can be obtained as a sum of the four solutions given by Eqs. (5.17) and (5.27) through (5.29); that is (see Fig. 5.5),

$$T(x,y) = T_1(x,y) + T_2(x,y) + T_3(x,y) + T_4(x,y)$$

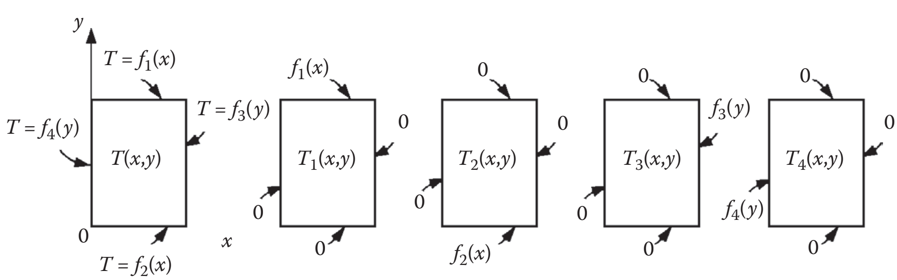

FIGURE 5.5
Principle of superposition.

As demonstrated by the example above, the solution of linear problems, such as heat conduction problems with constant properties, may often be reduced to the solution of a number of simpler problems by employing the principle of superposition.

In some cases, however, the problem can be handled by the method of separation of variables by simply shifting the temperature level. As an example, let us consider the long two-dimensional fin shown in Fig. 5.6, which exchanges heat by convection with a surrounding fluid of constant temperature T_∞. The heat transfer coefficient h is specified to be constant over the surfaces of the fin. Taking into account the geometric and thermal symmetries with respect to the x axis, the formulation of the problem for the temperature distribution $T(x, y)$ under steady-state conditions is given by

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{5.32}$$

$$T(0,y)=f(y), \qquad T(x,y)\Big|_{x\to\infty}=T_\infty \tag{5.33a,b}$$

$$\frac{\partial T(x,0)}{\partial y}=0, \qquad -k\frac{\partial T(x,b)}{\partial y}=h[T(x,b)-T_\infty] \tag{5.33c,d}$$

The problem as formulated above does not satisfy the requirement that three of the four boundary conditions be homogeneous. Therefore, the method of separation of variables would not be directly applicable. On the other hand, if we introduce

$$\theta(x,y)=T(x,y)-T_\infty \tag{5.34}$$

then the formulation of the problem can be rewritten as

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0 \tag{5.35}$$

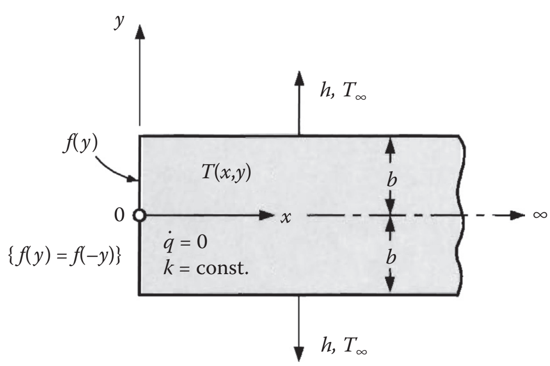

FIGURE 5.6
Infinitely long two-dimensional fin with convection boundary conditions.

$$\theta(0,y) = f(y) - T_\infty = F(y), \qquad \theta(x,y)\big|_{x\to\infty} = 0 \tag{5.36a,b}$$

$$\frac{\partial\theta(x,0)}{\partial y} = 0, \qquad -k\frac{\partial\theta(x,b)}{\partial y} = h\theta(x,b) \tag{5.36c,d}$$

which satisfies the conditions required by the method of separation of variables. Therefore, first assuming the existence of a product solution of the form

$$\theta(x,y) = X(x)Y(y) \tag{5.37}$$

and then introducing this assumption into Eq. (5.35) and dividing each term of the resulting expression by XY, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \tag{5.38}$$

or

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0 \tag{5.39a}$$

and

$$\frac{d^2Y}{dy^2} + \lambda^2 Y = 0 \tag{5.39b}$$

which lead to

$$\theta(x,y) = (Ae^{-\lambda x} + Be^{\lambda x})(C\cos\lambda y + D\sin\lambda y) \tag{5.40}$$

for the solution of the differential equation (5.35). Here, we have chosen the sign of the separation constant λ^2 such that the homogeneous y direction results in a characteristic-value problem; that is,

$$\frac{d^2Y}{dy^2} + \lambda^2 Y = 0 \tag{5.41a}$$

$$\frac{dT(0)}{dy} = 0 \tag{5.41b}$$

$$k\frac{dY(b)}{dy} + hY(b) = 0 \tag{5.41c}$$

This is a Sturm–Liouville system with the following characteristic functions (see Table 4.1 or Problem 4.6):

$$\phi_n(y) = \cos\lambda_n y \tag{5.42}$$

where the characteristic values are the positive roots of the following transcendental equation:

$$\lambda_n \tan\lambda_n b = \frac{h}{k}, \qquad n = 1, 2, 3, \ldots \tag{5.43}$$

which is obtained from the application of the boundary condition (5.41*c*).

Finally, after the application of the boundary conditions (5.36*b*,*c*,*d*), the solution for $\theta(x, y)$ can be constructed as

$$\theta(x,y) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n x} \cos\lambda_n y \tag{5.44}$$

The nonhomogeneous boundary condition (5.36*a*) requires that

$$F(y) = f(y) - T_\infty = \sum_{n=1}^{\infty} a_n \cos\lambda_n y \tag{5.45}$$

where the expansion coefficients a_n are given by (see Table 4.1)

$$a_n = \frac{2\lambda_n}{\lambda_n b + \sin\lambda_n b\cos\lambda_n b} \int_0^b [f(y) - T_\infty]\cos\lambda_n y\, dy \tag{5.46}$$

Hence, the temperature distribution becomes

$$\begin{aligned} \theta(x,y) &= T(x,y) - T_\infty \\ &= 2\sum_{n=1}^{\infty} \frac{\lambda_n e^{-\lambda_n x}\cos\lambda_n y}{\lambda_n b + \sin\lambda_n b\cos\lambda_n b} \int_0^b \left[f(y') - T_\infty\right]\cos\lambda_n y'\, dy' \end{aligned} \tag{5.47}$$

If, in particular, $f(y) = T_0$ = constant, then the temperature distribution reduces to

$$\frac{T(x,y) - T_\infty}{T_0 - T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n b}{\lambda_n b + \sin\lambda_n b\cos\lambda_n b} e^{-\lambda_n x}\cos\lambda_n y \tag{5.48}$$

The characteristic values, λ_n, in Eqs. (5.47) and (5.48) are to be obtained from the transcendental equation (5.43), which may be rearranged as

$$\tan \lambda_n b = \frac{\mathrm{Bi}}{\lambda_n b} \qquad \text{or} \qquad \cot \lambda_n b = \frac{\lambda_n b}{\mathrm{Bi}} \tag{5.49a,b}$$

where Bi = *hb*/*k*. Here, it is to be noted that the number of roots of Eq. (5.49*a*), or Eq. (5.49*b*), is infinite. Although they cannot be obtained by ordinary algebraic methods, they can be found numerically or determined graphically as illustrated in Fig. 5.7. A tabulation of the roots of Eq. (5.49*b*) is given in References [8,9].

5.2.2 Nonhomogeneity in Differential Equations

So far, we have considered nonhomogeneities in boundary conditions. Let us now discuss nonhomogeneities in differential equations.

As an example, consider the rectangular bar shown in cross section in Fig. 5.8. Internal energy is generated in this bar at a constant rate $\dot{q}$ per unit volume (W/m³). Assuming that there are no temperature gradients in the *z* direction and the thermal conductivity of the material of the bar is constant, the heat conduction equation for the steady-state temperature distribution $T(x, y)$ can be written as

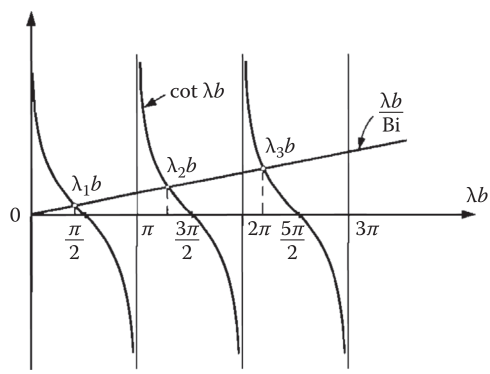

FIGURE 5.7
Graphical determination of the roots of the transcendental equation (5.49*b*).

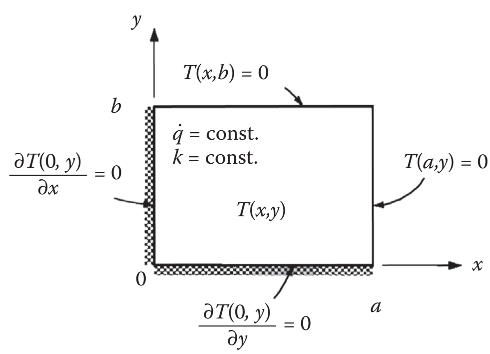

FIGURE 5.8
Bar of rectangular cross section with internal energy generation.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{k} = 0 \tag{5.50}$$

which is linear but not homogeneous and, therefore, not separable. The boundary conditions as given by

$$\frac{\partial T(0,y)}{\partial x} = 0, \qquad T(a,y) = 0 \tag{5.51a,b}$$

$$\frac{\partial T(x,0)}{\partial y} = 0, \qquad T(x,b) = 0 \tag{5.51c,d}$$

are all linear and homogeneous. If the solution is assumed to be in the form

$$T(x,y) = \psi(x,y) + \phi(x) \tag{5.52}$$

then the problem can be written as a superposition of the following two simpler problems:

$$\frac{d^2\phi}{dx^2} + \frac{\dot{q}}{k} = 0 \tag{5.53}$$

$$\frac{d\phi(0)}{dx} = 0, \qquad \phi(a) = 0 \tag{5.54a,b}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{5.55}$$

$$\frac{\partial \psi(0,y)}{\partial x} = 0, \qquad \psi(a,y) = 0 \tag{5.56a,b}$$

$$\frac{\partial \psi(x,0)}{\partial y} = 0, \qquad \psi(x,b) = -\phi(x) \tag{5.56c,d}$$

The solution of the one-dimensional $\phi(x)$ problem can easily be found to be

$$\phi(x) = \frac{\dot{q}a^2}{2k}\left[1 - \left(\frac{x}{a}\right)^2\right] \tag{5.57}$$

The problem for $\psi(x, y)$ can readily be solved by the method of separation of variables, and the solution is given by (see Problem 5.6)

$$\psi(x,y) = -\frac{2\dot{q}}{ak}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\frac{\cos\lambda_n x\cosh\lambda_n y}{\cosh\lambda_n b} \tag{5.58a}$$

where

$$\lambda_n = \frac{(2n+1)\pi}{2a}, \qquad n = 1,2,3,\ldots \tag{5.58b}$$

Combining the two solutions for $\phi(x)$ and $\psi(x, y)$, we get the solution for $T(x, y)$ as

$$\frac{T(x,y)}{\dot{q}a^2/k} = \frac{1}{2}\left[1-\left(\frac{x}{a}\right)^2\right] - \frac{2}{a^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\frac{\cos\lambda_n x\cosh\lambda_n y}{\cosh\lambda_n b} \tag{5.59}$$

Note that this problem can also be solved by assuming

$$T(x,y) = \psi(x,y) + \phi(y) \tag{5.60}$$

However, if $\dot{q} = \dot{q}(x)$, then only the first assumption (5.52) would work. Similarly, if $\dot{q} = \dot{q}(y)$, then only the second assumption (5.60) can be used. On the other hand, if $\dot{q} = \dot{q}(x,y)$ then, in general, the problem cannot be separated into simpler problems. In such a case, the solution may be obtained, for example, by following the procedure outlined in Problem 5.12, or by integral transforms that we shall discuss in Chapters 7 and 13. The superposition techniques we have introduced in this section are general in the sense that they can equally be applied to problems in the cylindrical and spherical geometries, as well as to unsteady problems.

5.3 Two-Dimensional Steady-State Problems in the Cylindrical Coordinate System

Under steady-state conditions and without internal energy sources or sinks, the heat conduction equation in the cylindrical coordinates with constant thermal conductivity is given, from Eqs. (2.19) and (2.21), by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{5.61}$$

where $T = T(r, \phi, z)$. As is obvious from this differential equation, there can be three different types of two-dimensional steady-state problems in cylindrical systems in the forms

$$T = T(r,\phi), \quad T = T(r,z) \quad \text{and} \quad T = T(\phi,z)$$

Problems of the form $T(\phi, z)$ have no physical significance, except in thin-walled cylinders. Accordingly, in the following sections we discuss problems of the other two forms, namely, $T(r, \phi)$ and $T(r, z)$, in terms of representative examples.

5.3.1 Two-Dimensional Steady-State Problems in (r, ϕ) Variables

As an example to the problems of the form $T(r, \phi)$, consider a long solid cylinder of semicircular cross section as shown in Fig. 5.9. Assume that the cylindrical surface at $r = r_0$ is held at an arbitrary temperature $f(\phi)$ and the planar surfaces at $\phi = 0$ and $\phi = \pi$ are both maintained at the same constant temperature T_0. In the absence of internal energy sources or sinks, the differential equation that governs the two-dimensional steady-state temperature distribution $T(r, \phi)$ in the cylinder, from Eq. (5.61), is governed by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0 \tag{5.62}$$

with the boundary conditions

$$T(0,\phi) = T_0, \quad T(r_0,\phi) = f(\phi) \tag{5.63a,b}$$

$$T(r,0) = T_0 \quad T(r,\pi) = T_0 \tag{5.63c,d}$$

Introducing a new temperature function as $\theta(r, \phi) = T(r, \phi) - T_0$ and assuming a product solution in the form.

$$\theta(r,\phi) = R(r)\psi(\phi) \tag{5.64}$$

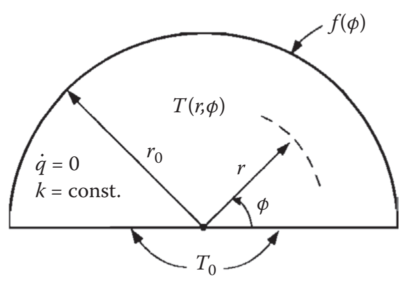

FIGURE 5.9
Long solid cylinder of semicircular cross section.

from the differential equation (5.62) we get

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0 \tag{5.65a}$$

and

$$\frac{d^2\psi}{d\phi^2} + \lambda^2\psi = 0 \tag{5.65b}$$

where the sign of the separation constant is consistent with the fact that the ϕ direction is the homogeneous direction for $\theta(r, \phi)$. Hence, noting that Eq. (5.65*a*) is a Cauchy–Euler equation (see Appendix B), we obtain*

$$\theta(r,\phi) = (A_1 r^{\lambda} + A_2 r^{-\lambda})(B_1 \sin\lambda\phi + B_2\cos\lambda\phi) \tag{5.66}$$

Since $T(0, \phi) = T_0$ and, therefore, $\theta(0, \phi) = 0$, then $r^{-\lambda}$ cannot exist in the solution as it is not bounded at $r = 0$. Setting $A_2 = 0$ so that $\theta(r, \phi)$ is bounded as $r \to 0$ and also employing the other boundary conditions, we get

$$\theta(r,\phi) = \sum_{n=1}^{\infty} a_n r^{\lambda_n} \sin\lambda_n\phi, \qquad 0 < \phi < \pi \tag{5.67a}$$

with

$$\lambda_n = \frac{n\pi}{\pi} = n, \qquad n = 1, 2, 3, \ldots \tag{5.67b}$$

Finally, imposing the nonhomogeneous boundary condition (5.63*b*) yields

$$F(\phi) = \sum_{n=1}^{\infty} a_n r_0^n \sin n\phi, \qquad 0 < \phi < \pi \tag{5.68}$$

where we have introduced

$$F(\phi) = f(\phi) - T_0 \tag{5.69}$$

* Equation (5.66) will be the solution if $\lambda \neq 0$. In this problem λ is, in fact, not equal to zero as will be shown; see Eq. (5.67b).

Equation (5.68) is the Fourier sine expansion of $F(\phi)$ on the interval $(0, \pi)$, with the expansion coefficients given by (see Table 4.1)

$$a_n = \frac{2}{\pi r_0^n} \int_0^{\pi} F(\phi') \sin n\phi' \, d\phi' \tag{5.70}$$

Thus, the solution for the temperature distribution $T(r, \phi)$ can be written as

$$T(r,\phi) = T_0 + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^n \sin n\phi \int_0^{\pi} [f(\phi') - T_0] \sin n\phi' d\phi' \tag{5.71}$$

If, in particular, $f(\phi) = T_1$ = constant, then the temperature distribution becomes

$$\frac{T(r,\phi) - T_0}{T_1 - T_0} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \left(\frac{r}{r_0} \right)^n \sin n\phi \tag{5.72}$$

As a second example, consider a long solid cylinder of circular cross section as shown in Fig. 5.10. Assume that the surface of the cylinder is held at an arbitrary temperature $f(\phi)$. Under steady-state conditions, this problem can be formulated as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0 \tag{5.73}$$

$$T(0,\phi) = \textit{finite} \tag{5.74a}$$

$$T(r_0,\phi) = f(\phi) \tag{5.74b}$$

$$T(r,\phi) = T(r,\phi + 2\pi) \tag{5.74c}$$

$$\frac{\partial T(r,\phi)}{\partial \phi} = \frac{T(r,\phi + 2\pi)}{\partial \phi} \tag{5.74d}$$

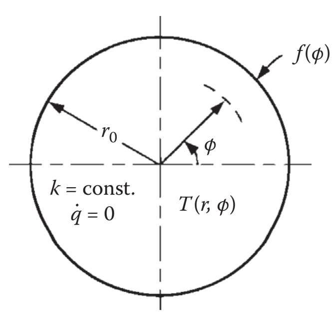

FIGURE 5.10
Long solid cylinder of circular cross section with an arbitrary surface temperature $f(\phi)$.

Seeking a product solution in the form

$$T(r,\phi) = R(r)\psi(\phi) \tag{5.75}$$

yields the following solution

$$T(r,\phi) = (A_1 r^{\lambda} + A_2 r^{-\lambda})(B_1 \sin\lambda\phi + B_2 \cos\lambda\phi) \tag{5.76}$$

for the differential equation (5.73), where we have chosen the sign of the separation constant such that the homogeneous ϕ direction results in a Sturm–Liouville type characteristic-value problem, that is

$$\frac{d^2\psi}{d\phi^2} + \lambda^2\psi = 0 \tag{5.77a}$$

$$\psi(0) = \psi(\phi + 2\pi) \tag{5.77b}$$

$$\frac{d\psi(0)}{d\phi} = \frac{d\psi(\phi + 2\pi)}{d\phi} \tag{5.77c}$$

This is a special kind of the Sturm–Liouville problem with periodic boundary conditions of period 2π.

If we impose the boundary conditions (5.74*c*,*d*) on the solution (5.76), we obtain

$$[\sin\lambda\phi - \sin\lambda(\phi + 2\pi)]B_1 + [\cos\lambda\phi - \cos\lambda(\phi + 2\pi)]B_2 = 0 \tag{5.78a}$$

$$[\cos\lambda\phi - \cos\lambda(\phi + 2\pi)]B_1 - [\sin\lambda\phi - \sin\lambda(\phi + 2\pi)]B_2 = 0 \tag{5.78b}$$

In order to have nontrivial solutions for B_1 and B_2, the determinant of the coefficients must vanish, which yields

$$\cos 2\lambda\pi = 1 \tag{5.79}$$

This is possible only if λ is equal to one of the values of

$$\lambda_n = n, \qquad n = 0, 1, 2, \ldots \tag{5.80}$$

which are, in fact, the characteristic values of the problem (5.77). This result means that λ in Eq. (5.76) can only be zero or an integer if that solution is to satisfy the conditions (5.74*c*,*d*). Here we have discarded negative integers in Eq. (5.80) because they would lead to the same solutions as the positive ones. Setting $A_2 = 0$ so that the solution will satisfy the condition (5.74*a*) and employing superposition, we can write

$$T(r,\phi) = \sum_{n=0}^{\infty} r^n (a_n \sin n\phi + b_n \cos n\phi) \tag{5.81a}$$

or

$$T(r,\phi) = b_0 + \sum_{n=1}^{\infty} r^n (a_n \sin n\phi + b_n \cos n\phi) \tag{5.81b}$$

Note that for $\lambda = 0$ the product solution (5.75) yields

$$T_0(r,\pi) = (A_{10} + A_{20} \ln r)(B_{10} + B_{20}\phi) \tag{5.82}$$

But boundedness, that is the condition (5.74*a*), implies that $A_{20} = 0$, and 2π periodicity, that is the condition (5.74*c*), implies that $B_{20} = 0$. Therefore, the omission of the ϕ and ln r terms Eq. (5.81*a*) caused no problem as b_0 corresponds to $A_{10}B_{10}$.

Equation (5.81*a*) satisfies both the differential equation (5.73) and the conditions (5.74*a,c,d*). It also satisfies the condition (5.74*b*), because imposing this condition yields

$$f(\phi) = b_0 + \sum_{n=1}^{\infty} r_0^n (a_n \sin n\phi + b_n \cos n\phi) \tag{5.83}$$

which is the *complete Fourier series* representation of $f(\phi)$ on the interval $(0, 2\pi)$. Since $f(\phi)$ is a periodic function with period 2π. the coefficients in Eq. (5.83) can be determined from Eqs. (4.39) as

$$b_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi \tag{5.84a}$$

$$a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\phi) \sin n\phi\, d\phi, \qquad n = 1,2,3,\ldots \tag{5.84b}$$

$$b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\phi) \cos n\phi\, d\phi, \qquad n = 1,2,3,\ldots \tag{5.84c}$$

where we set $c = 0$ in Eqs. (4.39). Thus, the final solution can then be written as

$$T(r,\phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi')\, d\phi' + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^n \left[\sin n\phi \int_0^{2\pi} f(\phi') \sin n\phi'\, d\phi' + \cos n\phi \int_0^{2\pi} f(\phi') \cos n\phi'\, d\phi' \right] \tag{5.85a}$$

or

$$T(r,\phi)=\frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n\cos n(\phi-\phi')\right]f(\phi')d\phi' \tag{5.85b}$$

This solution can also be written as

$$T(r,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\frac{r_0^2-r^2}{r_0^2-2rr_0\cos(\phi-\phi')+r^2}f(\phi')d\phi' \tag{5.86}$$

where we have used the relation [4]

$$\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n\cos n(\phi-\phi')=\frac{r_0^2-rr_0\cos(\phi-\phi')}{r_0^2-2rr_0\cos(\phi-\phi')+r^2}-1 \tag{5.87}$$

The solution (5.86) is also known as the *Poisson integral formula.*

At $r = 0$, Eq. (5.86) reduces to

$$T(0,\phi)=\frac{1}{2\pi}\int_0^{2\pi}f(\phi')d\phi' \tag{5.88}$$

Hence, the centerline temperature is the *average* of the surface temperature distribution. This striking result must apply *locally* as well; that is, the temperature at any point P is equal to the average temperature around the edge of each circle that is centered at P and that lies within $r < r_0$.

5.3.2 Steady-State Two-Dimensional Problems in (*r, z*) Variables

As an example to the problems of the form $T(r, z)$, consider a fin of circular cross section, with radius r_0 and length L, protruding from a hot wall as shown in Fig. 5.11. The base temperature is specified as $f(r)$. The fin is cooled by a fluid stream maintained at a constant temperature T_∞. Assuming constant thermal conductivity, an infinite heat transfer coefficient and no sources or sinks of internal energy, the problem can be formulated under steady-state conditions in terms of $\theta(r, z) = T(r, z) - T_\infty$ as

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\theta}{\partial z^2}=0 \tag{5.89}$$

$$\theta(0,z)=\textit{finite},\qquad \theta(r_0,z)=0 \tag{5.90a,b}$$

$$\theta(r,0)=f(r)-T_\infty=F(r),\qquad \theta(r,L)=0 \tag{5.90c,d}$$

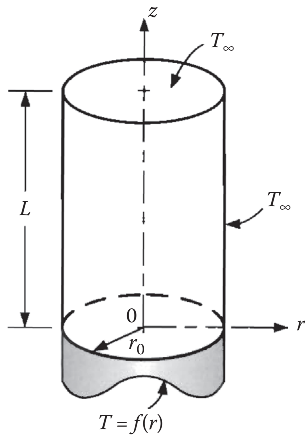

FIGURE 5.11
Fin of circular cross section.

Assume a product solution in the form

$$\theta(r, z) = R(r)Z(z) \tag{5.91}$$

Substituting Eq. (5.91) into Eq. (5.89) and observing that the r direction is the homogeneous direction, we get the following two ordinary differential equations:

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 rR = 0 \tag{5.92a}$$

and

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0 \tag{5.92b}$$

Thus, Bessel functions of order zero in r (see Appendix B) and hyperbolic functions in z are obtained, and the solution to Eq. (5.89) can be written as

$$\theta(r, z) = [A_1 J_0(\lambda r) + A_2 Y_0(\lambda r)][B_1 \sinh \lambda z + B_2 \cosh \lambda z] \tag{5.93}$$

Imposing the boundary conditions (5.90*a*,*b*,*d*), the solution for $\theta(r, z)$ can be developed as

$$\theta(r, z) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \sinh \lambda_n (L - z) \tag{5.94}$$

where λ_n are the positive roots of

$$J_0(\lambda_n r_0) = 0, \qquad n = 1, 2, 3, \ldots \tag{5.95}$$

Now, we have to determine the coefficients a_n in Eq. (5.94) such that the boundary condition (5.90*c*) is also satisfied; that is,

$$F(r) = \sum_{n=1}^{\infty} a_n \sinh \lambda_n L \, J_0(\lambda_n r) \tag{5.96}$$

which is a Fourier–Bessel expansion of $F(r)$ on the interval $(0, r_0)$. Making use of either Eq. (4.46*a*), together with Eq. (4.53), or Table 4.2, we obtain

$$a_n \sinh \lambda_n L = \frac{2}{r_0^2 J_1^2(\lambda_n r_0)} \int_0^{r_0} F(r) J_0(\lambda_n r) r \, dr \tag{5.97}$$

The solution for $\theta(r, z)$ can then be written as

$$\theta(r,z) = \frac{2}{r_0^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0)} \frac{\sinh \lambda_n (L-z)}{\sinh \lambda_n L} \int_0^{r_0} F(r') J_0(\lambda_n r') r' \, dr' \tag{5.98}$$

If, in particular, $f(r) = T_0 =$ constant, the integral in Eq. (5.98) can easily be evaluated (see Example 4.5) and the temperature distribution becomes

$$\frac{\theta(r,z)}{\theta_0} = \frac{2}{r_0} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{J_0(\lambda_n r)}{J_1(\lambda_n r_0)} \frac{\sinh \lambda_n (L-z)}{\sinh \lambda_n L} \tag{5.99}$$

where we have introduced $\theta_0 = T_0 - T_\infty$.

Example 5.2

For the solid cylinder shown in Fig. 5.11, calculate the value of $\theta(r, z)/\theta_0$ at $z/L = 0.75$, $r/r_0 = 0.50$. Take $L/r_0 = 2.0$.

SOLUTION

From Eq. (5.99) we have

$$\frac{\theta(r/r_0, z/L)}{\theta_0} = 2 \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \frac{J_0(\alpha_n r/r_0)}{J_1(\alpha_n)} \frac{\sinh[a_n (L/r_0)(1 - z/L)]}{\sinh(\alpha_n L/r_0)}$$

which, at $z/L = 0.75$ and $r/r_0 = 0.50$ with $L/r_0 = 2.0$, becomes

$$\frac{\theta(0.5, 0.75)}{\theta_0} = 2 \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \frac{J_0(0.5\alpha_n)}{J_1(\alpha_n)} \frac{\sinh(0.5\alpha_n)}{\sinh(2\alpha_n)}$$

where $\alpha_n = \lambda_n r_0$. The first four zeros of $J_0(\alpha)$ and the corresponding values of $J_1(\alpha)$ and $J_0(0.5\alpha)$ are (see Appendix B):

$$\begin{array}{lll}
\alpha_1 = 2.4048, & J_1(\alpha_1) = 0.5191, & J_0(0.5\alpha_1) = 0.6711 \\
\alpha_2 = 5.5201, & J_1(\alpha_2) = -0.3403, & J_0(0.5\alpha_2) = -0.1680 \\
\alpha_2 = 8.4048, & J_1(\alpha_3) = -0.2715, & J_0(0.5\alpha_3) = -0.3560 \\
\alpha_4 = 11.7915, & J_1(\alpha_4) = -0.2325, & J_0(0.5\alpha_4) = -0.1207
\end{array}$$

Substituting these numerical values we get

$$\frac{\theta(0.5, 0.75)}{\theta_0} = 0.02654 + 4.527 \times 10^{-5} - 2.629 \times 10^{-10}$$
$$-1.809 \times 10^{-9} \ldots \cong 0.02654$$

As it is seen, the series (5.99) converges rather rapidly for $z/L = 0.75$ and $r/r_0 = 0.5$ when $L/r_0 = 2.0$.

Let us now consider the solution of the heat conduction equation (5.89) with the following boundary conditions:

$$\theta(0, z) = \textit{finite}, \qquad \theta(r_0, z) = F(z) \tag{5.100a,b}$$

$$\theta(r, 0) = 0 \qquad \theta(r, L) = 0 \tag{5.100c,d}$$

Since the boundary conditions in the z direction are homogeneous, if we assume a product solution in the form

$$\theta(r, z) = R(r)Z(z) \tag{5.101}$$

then the differential equation (5.89) yields the following two differential equations:

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda^2 rR = 0 \tag{5.102a}$$

and

$$\frac{d^2Z}{dz^2} + \lambda^2 Z = 0 \tag{5.102b}$$

Thus, the solution to Eq. (5.89) can be written as (see Appendix B)

$$\theta(r,z) = [A_1 I_0(\lambda r) + B_1 K_0(\lambda r)][A_2 \sin\lambda z + B_2 \cos\lambda z] \tag{5.103}$$

where I_0 and K_0 are the zeroth-order modified Bessel functions of the first and second kinds, respectively. Since $K_0(0) \to \infty$ and $\theta \neq \infty$ at $r = 0$, $B_1 = 0$. On the other hand, the condition $\theta(r, 0) = 0$ requires that $B_2 = 0$. Hence, Eq. (5.103) reduces to

$$\theta(r,z) = aI_0(\lambda r)\sin\lambda z \tag{5.104}$$

where $a = A_1A_2$. Imposing the condition (5.100*d*) yields $\sin\lambda L = 0$, which is satisfied if λ is equal to any of the following:

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1,2,3,\ldots$$

Therefore, the solution of the problem can be constructed as

$$\theta(r,z) = \sum_{n=1}^{\infty} a_n I_0\left(\frac{n\pi r}{L}\right)\sin\frac{n\pi z}{L} \tag{5.105}$$

Application of the boundary condition (5.100*b*) yields

$$F(z) = \sum_{n=1}^{\infty} a_n I_0\left(\frac{n\pi r_0}{L}\right)\sin\frac{n\pi z}{L} \tag{5.106}$$

which is the Fourier sine series expansion of $F(z)$ on the interval $(0, L)$. Hence, we have (see Table 4.1)

$$a_n = \frac{2}{L}\frac{1}{I_0(n\pi r_0 / L)}\int_0^L F(z)\sin\frac{n\pi z}{L}dz \tag{5.107}$$

Substitution of this result into Eq. (5.105) gives

$$\theta(r,z) = \frac{2}{L}\sum_{n=1}^{\infty}\frac{I_0(n\pi r / L)}{I_0(n\pi r_0 / L)}\sin\frac{n\pi z}{L}\int_0^L F(z')\sin\frac{n\pi z'}{L}dz' \tag{5.108}$$

In the problems discussed in this section, we were able to apply the method of separation of variables directly because the heat conduction equations were linear and homogeneous. Moreover, the boundary conditions were also linear, and three of them were either homogeneous or could be made homogeneous by defining a new temperature function. If these problems involved internal energy generation and/or more than one nonhomogeneous boundary conditions, then the principle of superposition would be applied as discussed in Section 5.2 (for an example, see Problem 5.16).

5.4 Two-Dimensional Steady-State Problems in the Spherical Coordinate System

When the temperature distribution in a heat conduction problem posed in spherical coordinates is a function of r and θ variables only (Fig. 5.12), the solution for the temperature distribution may be expressed in terms of *Legendre polynomials*. Accordingly, we will first consider the solutions of Legendre's differential equation and then study briefly the expansion of an arbitrary function in a series of Legendre polynomials.

5.4.1 Legendre Polynomials

Consider the following linear and homogeneous second-order ordinary differential equation with variable coefficients

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha(\alpha+1)y = 0 \tag{5.109a}$$

or, equivalently,

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \alpha(\alpha+1)y = 0 \tag{5.109b}$$

where α is *real* and *non-negative*. This equation is known as *Legendre's differential equation*, and its two linearly independent solutions are called *Legendre functions* of the *first* and *second kind*, respectively. When $\alpha = n$, where n is *zero* or a positive integer, which are the cases commonly arising in practice, the Legendre function of the first kind becomes a polynomial in x of degree n, whereas the second kind can be expressed as an infinite series. The general solution of Legendre's equation for $n = 0, 1, 2, \ldots$ can be written as

$$y(x) = C_1P_n(x) + C_2Q_n(x) \tag{5.110}$$

where $P_n(x)$ are polynomials called *Legendre polynomials of degree n*, and $Q_n(x)$ are the *Legendre functions of the second kind*.

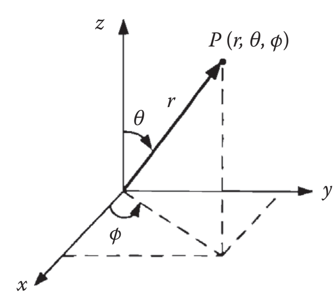

FIGURE 5.12
Spherical coordinates.

The Legendre polynomials are given by

$$P_0(x) = 1 \tag{5.111a}$$

$$P_n(x) = (-1)^{n/2} \frac{1.3.5\ldots(n-1)}{2.4.6\ldots n} U_n(x), \qquad n = 2,4,6,\ldots \tag{5.111b}$$

and

$$P_1(x) = x \tag{5.111c}$$

$$P_n(x) = (-1)^{(n-1)/2} \frac{1.3.5\ldots n}{2.4.6\ldots(n-1)} V_n(x), \qquad n = 3,5,7,\ldots \tag{5.111d}$$

Here, we introduced

$$U_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots \tag{5.112a}$$

$$V_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \ldots \tag{5.112b}$$

Note that if n is an *even* positive integer, the series (5.112*a*) terminates with the term involving x^n, and hence is a polynomial of degree n. Similarly, if n is an *odd* positive integer, the series (5.112*b*) terminates with the term involving x^n. Otherwise, these expressions are infinite series convergent only when $|x| < 1$.

The first six Legendre polynomials are readily found to be

$$\begin{aligned} &P_0(x) = 1 \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) \\ &P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad\qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned} \tag{5.113}$$

In all cases, $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

The Legendre polynomials can also be expressed by *Rodrigues' formula*:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n \tag{5.114}$$

A useful *recurrence* formula can be established at once by observing that each of these polynomials is a function of the polynomials preceding and succeeding it. Thus,

$$P_0(x) = \frac{1}{x}P_1(x), \qquad P_1(x) = \frac{1}{3x}[P_0(x) + 2P_2(x)], \ldots$$

or, in general,

$$P_n(x) = \frac{nP_{n-1}(x) + (n+1)P_{n+1}(x)}{(2n+1)x} \tag{5.115}$$

The following property of Legendre polynomials is also used frequently:

$$\frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx} = (2n+1)P_n(x), \qquad n = 1, 2, 3, \ldots \tag{5.116}$$

If $|x| < 1$, the Legendre functions of the second kind are given by

$$Q_0(x) = V_0(x) \tag{5.117a}$$

$$Q_n(x) = (-1)^{n/2}\frac{2.4.6\ldots n}{1.3.5\ldots(n-1)}V_n(x), \qquad n = 2, 4, 6, \ldots \tag{5.117b}$$

and

$$Q_1(x) = -U_1(x) \tag{5.117c}$$

$$Q_n(x) = (-1)^{(n+1)/2}\frac{2.4.6\ldots(n-1)}{1.3.5\ldots n}U_n(x), \qquad n = 3, 5, 7, \ldots \tag{5.117d}$$

Note that the Legendre functions of the second kind are infinite series, which are convergent when $|x| < 1$ but diverge as $x \to \pm 1$.

It can be shown that the Legendre functions of the second kind $Q_n(x)$ also satisfy the recurrence formulas (5.115) and (5.116). It can further be verified that, when $|x| < 1$,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x} = \tanh^{-1} x \tag{5.118}$$

The use of the recurrence formula (5.115) permits the determination of $Q_n(x)$ for any positive integer value of n:

$$Q_1(x) = P_1(x)Q_0(x) - 1, \qquad Q_2(x) = P_2(x)Q_0(x) - \frac{3}{2}x,$$

$$Q_1(x) = P_1(x)Q_0(x) - \frac{5}{2}x^2 + \frac{2}{3}, \qquad Q_4(x) = P_4(x)Q_0(x) - \frac{35}{8}x^3 + \frac{55}{24}x,$$

$$Q_5(x) = P_5(x)Q_0(x) - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}$$

or, in general,

$$Q_n(x) = P_n(x)Q_0(x) - \frac{(2n-1)}{1-n}P_{n-1}(x) - \frac{(2n-5)}{3(n-1)}P_{n-3}(x)\ldots \tag{5.119}$$

The substitution of $x = \cos\theta$ transforms Legendre's differential equation from the form (5.109*b*) into the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + n(n+1)y = 0 \tag{5.120a}$$

or, equivalently,

$$\frac{d^2y}{d\theta^2} + \cot\theta\frac{dy}{d\theta} + n(n+1)y = 0 \tag{5.120b}$$

Hence, when $\alpha = n$, the general solution of Eq. (5.120*a*) or (5.120*b*) is

$$y(\theta) = C_1P_n(\cos\theta) + C_2Q_n(\cos\theta) \tag{5.121}$$

Equations of such a form arise in connection with solutions of various heat conduction problems in spherical coordinates. Here, it should be noted that $Q_n(\cos\theta)$ is not finite when $\cos\theta = \pm 1$; that is, when $\theta = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, whereas $P_n(\cos\theta)$ is merely a polynomial of degree n in $\cos\theta$. In particular, we have

$$P_0(\cos\theta) = 1, \qquad P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^3\theta - 1) = \frac{1}{4}(3\cos 2\theta + 1)$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta)$$

When $|x| < 1$, the functions

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}, \qquad P_n^0(x) = P_n(x) \tag{5.122a}$$

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m Q_n(x)}{dx^m}, \qquad Q_n^0(x) = Q_n(x) \tag{5.122b}$$

are called *associated Legendre functions* of *degree n* and *order m* of the *first* and *second kinds*, respectively. They satisfy the differential equation

$$(1-x)^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{5.123}$$

For $m = 0$, Eq. (5.123) reduces to Eq. (5.109*a*).

5.4.2 Fourier–Legendre Series

Equation (5.109*b*), when compared with Eq. (4.13*a*), gives

$$p(x) = 1 - x^2, \qquad q(x) = 0, \qquad w(x) = 1, \qquad \lambda = \alpha(\alpha+1)$$

Since the function $p(x)$ vanishes at $x = \pm 1$, from the results of Section 4.4 we conclude that any two distinct solutions of Eq. (5.109*a*), or (5.109*b*), which are finite and have finite first derivatives at $x = \pm 1$, will be orthogonal with respect to the weight function $w(x) = 1$ on the interval (–1, 1). Therefore, no specific boundary conditions are needed for Eq. (5.109*a*) or Eq. (5.109*b*) to lead to a Sturm–Liouville type characteristic-value problem over the interval (–1, 1).

Equation (5.109*a*) possesses solutions that are finite at $x = \pm 1$ only if α is *zero* or *a positive integer*. This condition of finiteness determines the permissible values of α in the form

$$\alpha = n, \qquad n = 0, 1, 2, \ldots \tag{5.124}$$

The corresponding solutions that are finite when $x = \pm 1$ are proportional to the Legendre polynomials $P_n(x)$. Thus, we conclude that

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad m \neq n \tag{5.125}$$

and the set $\{P_n(x);\ n = 0, 1, 2, \ldots\}$ is a complete orthogonal set with respect to the weight function *unity* over the interval (–1, 1). Therefore any piecewise differentiable function $f(x)$ can be represented in the interval (–1, 1) by a series of the Legendre polynomials as

$$\begin{aligned} f(x) &= A_0 P_0(x) + A_1 P_1(x) + A_2 P_a(x) + \ldots \\ &= \sum_{n=0}^{\infty} A_n P_n(x), \quad -1 < x < 1 \end{aligned} \tag{5.126}$$

which is called the *Fourier–Legendre series expansion* of $f(x)$ on the interval (–1, 1). The expansion coefficients A_n are given by

$$A_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} [P_n(x)]^2\, dx} \tag{5.127}$$

To evaluate the integrals appearing in Eq. (5.127), it is convenient to express $P_n(x)$ by Rodrigues' formula (5.114). Hence,

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n\, dx \tag{5.128}$$

If the right-hand side of Eq. (5.128) is integrated by parts N times, where $N \leq n$, then Eq. (5.128) becomes

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^N}{2^n n!} \int_{-1}^{1} \left[\frac{d^N}{dx^N} f(x) \right] \left[\frac{d^{n-N}}{dx^{n-N}} (x^2 - 1)^n \right] dx, \quad N \leq n \tag{5.129}$$

where we assumed that the first N derivatives of $f(x)$ are continuous in (–1, 1); otherwise, integration by parts may not be valid. Also, we note that the first $n - 1$ derivatives of $(x^2 - 1)^n$ vanish at $x = \pm 1$. In particular, when $N = n$, Eq. (5.129) reduces to

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} (1 - x^2)^n \frac{d^n f(x)}{dx^n}\, dx \tag{5.130}$$

Replacing $f(x)$ by $P_n(x)$ in Eq. (5.130) and noticing from Eq. (5.114) that

$$\frac{d^n P_n(x)}{dx^n} = \frac{1}{2^n n!}\frac{d^2 n}{dx^{2n}}(x^2-1)^n = \frac{1}{2^n n!}\frac{d^{2n}}{dx^{2n}}(x^{2n} - nx^{2n-2} + \ldots) = \frac{(2n)!}{2^n n!} \tag{5.131}$$

we obtain, from Eq. (5.130),

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{(2n)!}{2^n (n!)2}\int_{-1}^{1}(1-x^2)^n\,dx \tag{5.132}$$

The integral on the right-hand side of Eq. (5.132) can be evaluated by successive reductions to yield [5]

$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\frac{(2n)(2n-2)\ldots\;\ldots4.2}{(2n+1)(2n-1)\ldots5.3} = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \tag{5.133}$$

Thus, Eq. (5.132) takes the form

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1} \tag{5.134}$$

The expansion coefficients A_n, from Eq. (5.127), can now be written as

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \tag{5.135}$$

By making use of Eq. (5.130) an alternative expression for A_n can also be written as

$$A_n = \frac{2n+1}{2^{n+1} n!}\int_{-1}^{1}(1-x^2)^n \frac{d^n f(x)}{dx^n}\,dx \tag{5.136}$$

Equation (5.136) would be useful only if $f(x)$ and its first n derivatives were continuous on (–1, 1).

Since $P_n(x)$ is an *even* function of x when n is *even*, and an *odd* function when n is *odd*, it follows that if $f(x)$ is an *even* function of x, the coefficients A_n will vanish when n is *odd*; whereas if $f(x)$ is an *odd* function of x, the coefficients A_n will vanish when n is *even*. Thus, for an *even* function $f(x)$, there follows

$$A_n = \begin{cases} 0, & n \text{ odd} \\ (2n+1)\displaystyle\int_0^1 f(x)P_n(x)\,dx, & n \text{ even} \end{cases} \tag{5.137}$$

whereas for an odd function $f(x)$ the coefficients A_n would be given by

$$A_n = \begin{cases} (2n+1)\int_0^1 f(x)P_n(x)\,dx, & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{5.138}$$

It is obvious that an expansion of a given function $f(x)$ into a series containing only *even* Legendre polynomials with the expansion coefficients given by Eq. (5.137), or a series containing only *odd* Legendre polynomials with the expansion coefficients given by Eq. (5.138), would both represent $f(x)$ on the interval (0, 1). However, the first series in terms of even polynomials would represent $f(-x)$ and the second series $-f(-x)$ on the interval (−1, 0).

Furthermore, if $f(x)$ is a polynomial of degree k, all derivatives of $f(x)$ of order n vanish identically when $n > k$. Hence, it follows from Eqs. (5.126) and (5.136) that any polynomial of degree k can be expressed as a linear combination of the first $(k + 1)$ Legendre polynomials.

Expansions valid in the more general interval $(-a, a)$ are readily obtained by replacing x by x/a in the preceding development, which leads to the expansion

$$f(x) = \sum_{n=0}^{\infty} A_n P_n\left(\frac{x}{a}\right), \qquad -a < x < a \tag{5.139}$$

where

$$A_n = \frac{2n+1}{2a}\int_{-a}^{a} f(x)P_n\left(\frac{x}{a}\right)dx \tag{5.140}$$

or

$$A_n = \frac{2n+1}{2^{n+1}n!a^{n+1}}\int_{-a}^{a}(a^2 - x^2)^n \frac{d^n f(x)}{dx^n}dx \tag{5.141}$$

which are the general forms of Eqs. (5.135) and (5.136).

5.4.3 Solid Sphere

As an example to the solution of Laplace's equation in spherical coordinates by the application of Fourier–Legendre series, we now consider the steady-state temperature distribution in a solid sphere of radius r_0. Assume that the surface of the sphere at $r = r_0$ is maintained at some arbitrary temperature distribution $f(\theta)$, the thermal conductivity of the material of the sphere is constant, and there are no internal energy sources or sinks in the sphere.

The heat conduction equation from Eqs. (2.19) and (2.23) and the boundary conditions are given by

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = 0 \tag{5.142}$$

$$T(0,\theta) = \textit{finite} \tag{5.143a}$$

$$T(r_0,\theta) = f(\theta) \tag{5.143b}$$

Assuming a product solution in the form

$$T(r,\theta) = R(r)\phi(\theta) \tag{5.144}$$

Eq. (5.142) can be separated into two as follows:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \lambda^2\phi = 0 \tag{5.145}$$

and

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\lambda^2}{r^2}R = 0 \tag{5.146}$$

where λ^2 is a *separation constant.*

The differential equation (5.145) can be transformed into Legendre's equation by redefining the independent variable as $x = \cos\theta$. This gives

$$\frac{d}{dx}\left[(1-x^2)\frac{d\phi}{dx}\right] + \alpha(\alpha+1)\phi = 0 \tag{5.147}$$

where we replaced $\lambda^2 = \alpha(\alpha + 1)$. If $\alpha = n$, where n is *zero* or a *positive integer,* we know that the solutions of Legendre's equation (5.147), which are finite at $x = \pm 1$ (i.e., at $\theta = 0$ and $\theta = \pi$), are the Legendre polynomials. Thus,

$$\phi_n(x) = A_n P_n(x), \qquad n = 0,1,2,\ldots \tag{5.148a}$$

or

$$\phi_n(\theta) = A_n P_n(\cos\theta), \qquad n = 0,1,2,\ldots \tag{5.148b}$$

The solution of the Cauchy–Euler equation (5.146), on the other hand, can be written, with $\lambda^2 = \alpha(\alpha + 1)$, as

$$R_n(r) = B_n r^n + C_n r^{-(n+1)} \tag{5.149}$$

The general solution of Laplace's equation (5.142) which satisfies the boundary conditions (5.145*a*,*b*) can then be constructed as

$$T(r,\theta) = \sum_{n=0}^{\infty} A_n\left[C_n r^n + B_n r^{-(n+1)}\right]P_n(\cos\theta) \tag{5.150}$$

Here, it should be noted that the finiteness of the solution defines the characteristic functions and the characteristic values, which is equivalent to stating two boundary conditions in the θ direction as $T(r, 0)$ = *finite*, and $T(r, \pi)$ = *finite*. Moreover, the boundary condition (5.143*a*) gives $B_n = 0$. Therefore, Eq. (5.150) reduces to

$$T(r,\theta) = \sum_{n=0}^{\infty} K_n r^n P_n(\cos\theta) \tag{5.151}$$

where $K_n = A_n B_n$. Imposing the second boundary condition (5.143*b*) on Eq. (5.151) we now obtain

$$f(\theta) = \sum_{n=0}^{\infty} K_n r_0^n P_n(\cos\theta), \qquad 0 < \theta < \pi \tag{5.152a}$$

which is the Fourier–Legendre series expansion of the surface temperature distribution $f(\theta)$. If we introduce $x = \cos\theta$, the expansion (5.152*a*) can also be written as

$$F(x) = \sum_{n=0}^{\infty} K_n r_0^n P_n(x), \qquad -1 < x < 1 \tag{5.152b}$$

where we introduced

$$F(x) = f(\cos^{-1} x) \tag{5.153}$$

The coefficients $K_n r_0^n$ can now be determined by employing Eq. (5.135) as

$$K_n r_0^n = \frac{2n+1}{2}\int_{-1}^{1} F(x)P_n(x)\,dx \tag{5.154a}$$

or, alternatively,

$$K_n r_0^n = \frac{2n+1}{2}\int_{0}^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta \tag{5.154b}$$

Hence, the temperature distribution $T(r, \theta)$ in the sphere can be written as

$$T(r,\theta) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{r_0}\right)^n P_n(\cos\theta)\int_0^{\pi} f(\theta')P_n(\cos\theta')\sin\theta'\,d\theta' \tag{5.155}$$

As an application, assume that the surface temperature is specified as

$$f(\theta) = \begin{cases} T_0, & 0 < \theta < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < \theta < \pi \end{cases} \tag{5.156a}$$

which can also be written as

$$F(x) = \begin{cases} T_0, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases} \tag{5.156b}$$

where $x = \cos\theta$, and $F(x) = f(\cos^{-1} x)$. The expansion coefficients from Eq. (5.154*a*) are then given by

$$K_n r_0^n = T_0 \frac{2n+1}{2}\int_0^1 P_n(x)\,dx \tag{5.157}$$

Integrating Eq. (5.157) we obtain

$$K_0 = \frac{1}{2}T_0\int_0^1 dx = \frac{1}{2}T_0$$

$$K_1 r_0 = \frac{3}{2}T_0\int_0^1 x\,dx = \frac{3}{4}T_0$$

$$K_2 r_0^2 = \frac{5}{2}\frac{1}{2}T_0\int_0^1 (3x^2-1)\,dx = 0$$

$$K_0 r_0^3 = \frac{7}{2}\frac{1}{2}T_0\int_0^1 (5x^2-3x)\,dx = \frac{7}{16}T_0$$

Hence, the temperature distribution from Eq. (5.151) becomes

$$\frac{T(r,\theta)}{T_0}=\frac{1}{2}+\frac{3}{4}\left(\frac{r}{r_0}\right)P_1(\cos\theta)-\frac{7}{16}\left(\frac{r}{r_0}\right)^3 P_3(\cos\theta)+\frac{11}{32}\left(\frac{r}{r_0}\right)^5 P_5(\cos\theta)+\ldots \tag{5.158}$$

The results obtained in this section can readily be extended to various problems in the spherical coordinates. Readers are referred to the problems given at the end of this chapter for further examples.

5.5 Three-Dimensional Steady-State Systems

In order to illustrate the solution of Laplace's equation in three-dimensional steady-state problems, let us consider the solid rectangular parallelepiped shown in Fig. 5.13, with the boundary conditions approximated as

$$T(0,y,z)=0, \quad T(x,0,z)=0, \quad T(x,y,0)=f(x,y) \qquad (5.159a,b,c)$$

$$T(a,y,z)=0, \quad T(x,b,z)=0, \quad T(x,y,c)=0 \qquad (5.159d,e,f)$$

Assume that the material of the parallelepiped is homogeneous and that there are no internal energy sources or sinks. The three-dimensional steady-state temperature distribution in the parallelepiped will then satisfy Laplace's equation in three dimensions,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \tag{5.160}$$

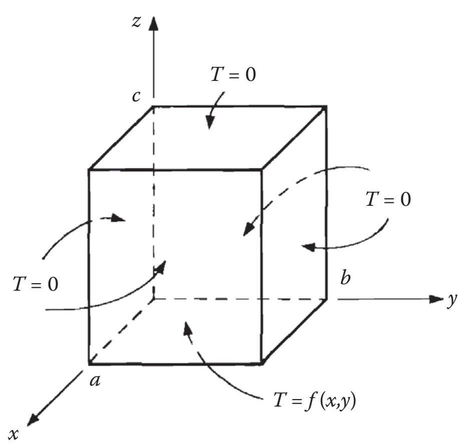

FIGURE 5.13
Rectangular solid parallelepiped.

If we assume a product solution in the form

$$T(x,y,z) = X(x)Y(y)Z(z) \tag{5.161}$$

then Eq. (5.160) can be separated as

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} \tag{5.162}$$

The left-hand side of Eq. (5.162) is independent of both y and z, and the right-hand side is independent of x. Therefore, the right- and left-hand sides of Eq. (5.162) can be equal to each other if and only if they are equal the some constant. This constant, in view of the homogeneous boundary conditions in the x direction, is a positive real constant; that is, λ^2. Hence, we have

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \tag{5.163}$$

Similarly,

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{Z}\frac{d^2Z}{dz^2} - \lambda^2 = \beta^2 \tag{5.164}$$

which yields

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0 \tag{5.165}$$

and, therefore, we also get

$$\frac{d^2Z}{dz^2} - (\lambda^2 + \beta^2)Z = 0 \tag{5.166}$$

The solutions of the differential equations (5.163), (5.165), and (5.166) lead to the following solution for Eq. (5.161):

$$\begin{aligned} T(x,y,z) = {} & (A_1 \cos\lambda x + A_2 \sin\lambda x)(B_1 \cos\beta y + B_2 \sin\beta y) \\ & \times\left[C_1 \cosh\sqrt{\lambda^2+\beta^2}\,(c-z) + C_2 \sinh\sqrt{\lambda^2+\beta^2}\,(c-z)\right] \end{aligned} \tag{5.167}$$

The boundary conditions at $x = 0$, $y = 0$, and $z = c$ give $A_1 = B_1 = C_1 = 0$. Thus, substituting these into Eq. (5.167) we obtain

$$T(x,y,z) = A \sin\lambda x \sin\beta y \sinh\sqrt{\lambda^2+\beta^2}\,(c-z) \tag{5.168}$$

where $A = A_2B_2C_2$. The boundary conditions at $x = a$ and $y = b$ yield

$$\sin \lambda a = 0 \rightarrow \lambda_m = \frac{m\pi}{a}, \qquad m = 1,2,3,\ldots \tag{5.169}$$

and

$$\sin \beta b = 0 \rightarrow \beta_n = \frac{n\pi}{b}, \qquad n = 1,2,3,\ldots \tag{5.170}$$

Hence, the solution of Eq. (5.160) which will satisfy all the boundary conditions (5.159) can be written as

$$T(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \times \sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}(c-z)\right] \tag{5.171}$$

Applying the boundary condition (5.159*c*) gives

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \times \sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, c\right] \tag{5.172a}$$

or

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{5.172b}$$

where we have introduced

$$B_{mn} = A_{mn} \sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, c\right] \tag{5.173}$$

Here, the constants B_{mn} are the coefficients of the double Fourier sine series expansion of $f(x, y)$ on the rectangular domain $0 < x < a$, $0 < y < b$.

If both sides of Eq. (5.172*b*) are multiplied by $\sin(p\pi x/a)\sin(q\pi y/b)$, where p and q are arbitrary positive integers, and the result is integrated over the rectangular domain $0 < x < a$, $0 < y < b$, there follows

$$\begin{aligned}&\int_0^a\int_0^b f(x,y)\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\,dx\,dy\\&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\int_0^a\int_0^b\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy\end{aligned}\tag{5.174}$$

The double integrals in this result can be written as

$$\left(\int_0^a\sin\frac{p\pi x}{a}\sin\frac{m\pi x}{a}\,dx\right)\left(\int_0^b\sin\frac{q\pi y}{b}\sin\frac{n\pi y}{b}\,dy\right)$$

This product vanishes unless $p = m$ and $q = n$, in which case it has the value $(a/2)(b/2) = ab/4$. Thus, the double series in Eq. (5.174) reduces to a single term, for which $m = p$ and $n = q$, leading to the result

$$\begin{aligned}A_{mn}&=\frac{4}{ab}\left\{\sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\,c\right]\right\}^{-1}\\&\quad\times\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy\end{aligned}\tag{5.175}$$

Finally, substitution of A_{mn} from Eq. (5.175) into Eq. (5.171) yields the temperature distribution $T(x, y, z)$ inside the parallelepiped.

5.6 Heat Transfer Rates

In the preceding sections, temperature distributions in several representative problems have been obtained. Once the temperature distribution is known, the heat transfer rate across any area A can be calculated by using Fourier's law of heat conduction:

$$q_n = -\int_A k\frac{\partial T}{\partial n}\,dA\tag{5.176}$$

where n represents the direction normal to dA. Across a boundary surface A exposed to a fluid at temperature T_∞, with a finite heat transfer coefficient h, the heat transfer rate may also be calculated by the use of Newton's law of cooling:

$$q = \int_A h(T_w - T_\infty)\,dA\tag{5.177}$$

where T_w is the temperature of dA.

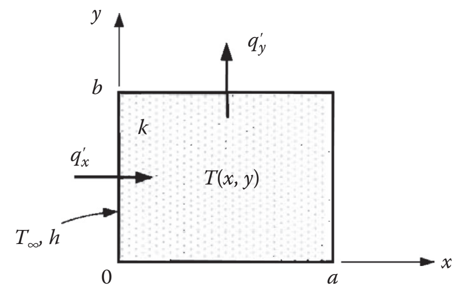

FIGURE 5.14
Solid bar of rectangular cross section.

For example, the rate of heat transfer per unit depth across the surface at $y = b$ of the rectangular bar shown in cross section in Fig. 5.14 can be calculated as

$$q'_y = -\int_0^a k \frac{\partial T(x,b)}{\partial y} dx \tag{5.178}$$

and the rate of heat transfer per unit depth across the surface at $x = 0$ into the bar is given by

$$q'_x = -\int_0^b k \frac{\partial T(0,y)}{\partial x} dy \tag{5.179}$$

Similarly, in Fig. 5.13 the rate of heat transfer into the parallelepiped across the surface at $z = 0$ is given by

$$q_z = -\int_0^a \int_0^b k \frac{\partial T(x,y,0)}{\partial z} dx\,dy \tag{5.180}$$

Furthermore, the rate of heat transfer from the surface at $z = L$ of the cylinder in Fig. 5.15 is

$$q_z = -\int_0^{r_0} k \frac{\partial T(r,L)}{\partial z} 2\pi r\,dr \tag{5.181}$$

Alternatively, if the heat transfer coefficient h and the surrounding fluid temperature T_∞ are given, Eqs. (5.179) and (5.181) can equally be replaced, respectively, by

$$q'_x = \int_o^b h[T_\infty - T(0,y)]dy \tag{5.182}$$

and

$$q_z = \int_0^{r_0} h[T(r,L) - T_\infty]2\pi r\,dr \tag{5.183}$$

Equations (5.182) and (5.183), however, can be used only if h is finite.

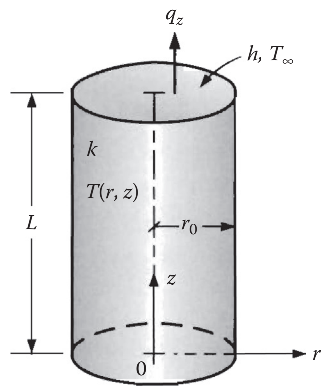

FIGURE 5.15
Solid cylinder of finite length L and radius r_0.

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Problems

5.1 Three surfaces of a long bar of square cross section are maintained at 0°C, while the fourth surface is at 100°C. Calculate the centerline temperature under steady-state conditions.

5.2 Temperature is maintained at 0°C along three surfaces of the rectangular bar shown in Fig. 5.1, while the fourth surface at $y = 0$ is held at a constant and uniform temperature T_0. Calculate $T(x, y)/T_0$ at $x/a = 0.25$ and $y/b = 0.75$, Take $b/a = 2$.

5.3 Applying the method of separation of variables obtain the steady-state temperature distribution $T(x, y)$ in the long bar of rectangular cross section shown in Fig. 5.16.

5.4 Using the temperature distribution found in Problem 5.3, obtain the steady-state temperature distribution $T(x, y)$ in the long bar of rectangular cross section shown in Fig. 5.17.

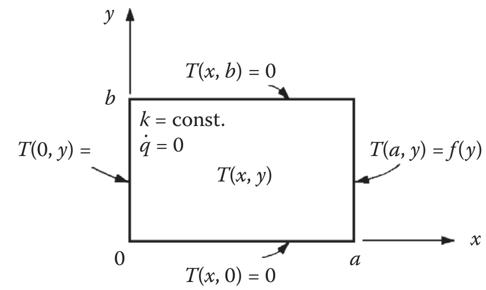

FIGURE 5.16
Long bar of rectangular cross section of Problem 5.3.

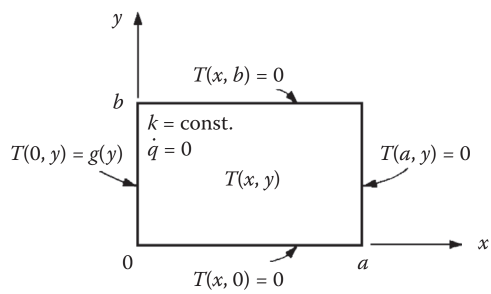

FIGURE 5.17
Long bar of rectangular cross section of Problem 5.4.

5.5 Assuming constant thermal conductivity and no internal energy sources, develop an expression for the steady-state temperature distribution $T(x, y)$ in a long bar of rectangular cross section for the following boundary conditions:

$$T(x,b) = T_1, \qquad T(x,0) = T_2$$

$$T(a,y) = T_3, \qquad T(0,y) = T_4$$

where T_1, T_2, T_3, and T_4 are given constant temperatures.

5.6 Obtain an expression for the steady-temperature distribution $T(x, y)$ in the long bar of rectangular cross section shown in Fig. 5.18, where T_1 and T_2 are two given constant temperatures.

5.7 The combustion chamber of a jet engine is cooled by water flowing in an annular jacket around it. Fins running spirally in the annular jacket are cast integral with the combustion chamber wall and guide the cooling water. As an approximation, the fins may be considered straight and of rectangular cross section as shown in Fig. 5.19. In addition, the inside surface of the combustion chamber wall may be assumed to be at the temperature of the combustion gases T_g. The cooling

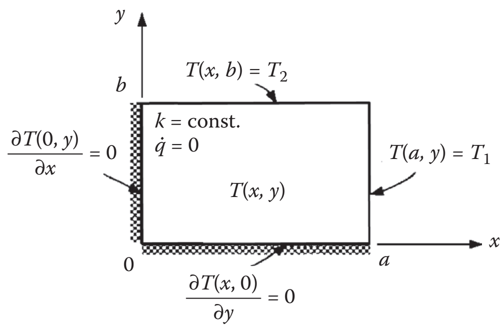

FIGURE 5.18
Long bar of rectangular cross section of Problem 5.6.

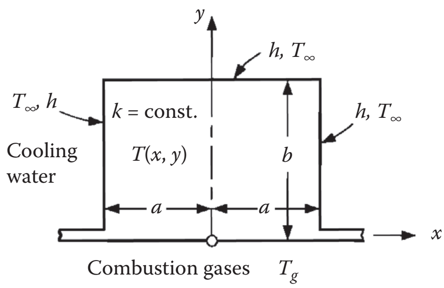

FIGURE 5.19
Figure for Problem 5.7.

water temperature T_∞ and the heat transfer coefficient h may also be considered constants.

(*a*) Develop an expression for the steady-state temperature distribution $T(x, y)$ in the fins.

(*b*) Obtain an expression for the rate of heat dissipated by each fin per unit depth.

5.8 Solve the problem formulated in Problem 2.8 for the temperature distribution $T(x, y)$.

5.9 Consider a straight fin of rectangular profile and of constant thermal conductivity. The fin has a thickness a in the x direction and is very long in the y direction. Obtain an expression for the steady-state temperature distribution $T(x, y)$ in this fin under the following boundary conditions:

$$\frac{\partial T(0,y)}{\partial x}, \quad T(a,y) = 0, \quad \text{and} \quad T(x,0) = f(x)$$

What is the rate of heat dissipated by the fin from the surface at $x = 0$ per unit depth?

5.10 Obtain an expression for the steady-state temperature distribution $T(x, y)$ in the long rod of triangular cross section shown in Fig. 5.20. Assume that the thermal conductivity k of the material of the rod is constant.

5.11 Obtain an expression for the steady-state temperature distribution $T(x, y)$ in the long rod of triangular cross section shown in Fig. 5.21. Assume that the thermal conductivity k of the material of the rod is constant.

5.12 Consider the long bar of rectangular cross section shown in Fig. 5.22, in which internal energy is generated at a rate $\dot{q}(x)$ per unit volume. Assuming that the thermal conductivity of the material of the bar is constant, obtain an expression for the steady-state temperature distribution $T(x, y)$ under the boundary conditions given in the figure in the following ways:

(*a*) By separating the problem into two as

$$T(x,y) = \psi(x,y) + \phi(x)$$

(*b*) By seeking a solution as

$$T(x,y) = \sum_{n=0}^{\infty} A_n(y) \cos\frac{(2n+1)\pi}{2a} x \qquad \text{with} \qquad \frac{dA_n(0)}{dy} = A_n(b) = 0$$

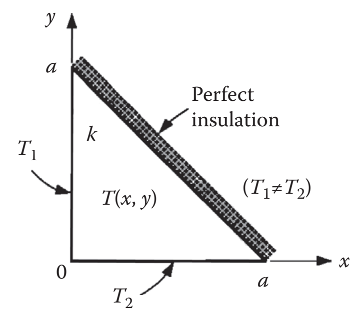

FIGURE 5.20
Long rod of triangular cross section of Problem 5.10.

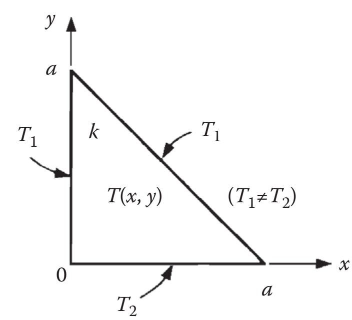

FIGURE 5.21
Long rod of triangular cross section of Problem 5.11.

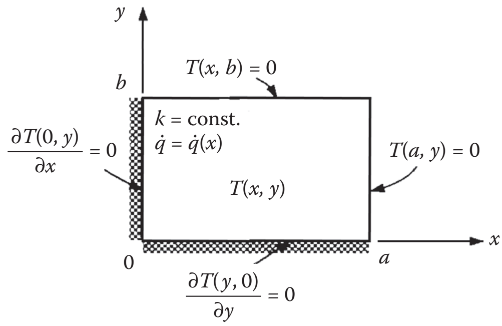

FIGURE 5.22
Long bar of rectangular cross section of Problem 5.12.

Hint: Also expand

$$\frac{\dot{q}(x)}{k} = \sum_{n=0}^{\infty} B_n \cos\frac{(2n+1)\pi}{2a} x$$

in $0 < x < a$ and substitute these two expansions for $T(x, y)$ and $\dot{q}(x)/k$ into the heat conduction equation to relate the unknown coefficients $A_n(y)$ to the known constants B_n through a set of ordinary differential equations.

(c) By seeking a solution as

$$T(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos\frac{(2m+1)\pi}{2a} x \cos\frac{(2n+1)\pi}{2b} y$$

Hint: Also expand

$$\frac{\dot{q}(x)}{k} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \cos\frac{(2m+1)\pi}{2a} x \cos\frac{(2n+1)\pi}{2b} y$$

in $0 < x < a$ and $0 < y < b$ and relate the unknown constants A_{mn} to the known constants B_{mn} by the use of the differential equation of the problem.

5.13 Re-solve Problem 5.12 if the rate of internal energy generation is given as $\dot{q}(x,y)$. *Hint*: See the hint in part (c) of Problem 5.12.

5.14 Determine the steady-state temperature distribution $T(r, z)$ in the solid cylinder shown in Fig. 5.23 for the following boundary conditions:

$$T(r_0,z) = T_0, \quad T(r,0) = 0 \quad \text{and} \quad T(r,L) = 0$$

where T_0 is a given constant temperature. Assume that the thermal conductivity k of the material of the cylinder is constant and that there are no internal energy sources or sinks in the cylinder.

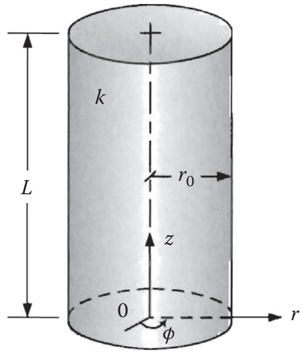

FIGURE 5.23
Solid cylinder of Problem 5.14.

5.15 Determine the steady-state temperature distribution $T(r, z)$ in the solid cylinder of Problem 5.14 (see Fig. 5.23) for the following boundary conditions:

$$T(r_0, z) = 0, \quad T(r, 0) = 0 \quad \text{and} \quad T(r, L) = f(r)$$

5.16 (*a*) Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the solid cylinder of Problem 5.14 (see Fig. 5.23) for the following boundary conditions:

$$T(r_0, z) = f_1(z), \quad T(r, 0) = f_2(r) \quad \text{and} \quad T(r, L) = f_3(r)$$

(*b*) If $f_1(z) = T_0$ and $f_2(r) = f_3(r) = T_1$, where T_0 and T_1 are two given constant temperatures, determine the steady-state temperature distribution in the cylinder in terms of $\theta = T - T_0$ and $\phi = T - T_1$.

5.17 Consider the solid cylinder of Problem 5.14 shown in Fig. 5.23. Obtain an expression for the steady-state temperature distribution $T(r, z)$ in this cylinder, if the peripheral surface at $r = r_0$ and the circular surface at $z = L$ are both exposed to a fluid maintained at a uniform temperature T_∞, with a constant heat transfer coefficient h, while the temperature of the circular surface at $z = 0$ is given as a function of the r coordinate: that is, $T(r, 0) = f(r)$.

5.18 Consider the solid cylinder of Problem 5.14 (see Fig. 5.23). Obtain an expression for the steady-state temperature distribution $T(r, z)$ in this cylinder, if the two circular surfaces at $z = 0$ and $z = L$ are maintained at a constant temperature T_0, while the temperature of the peripheral surface at $r = r_0$ is given as

$$T(r_0, z) = T_0 + \sin\frac{\pi z}{L}$$

5.19 Determine the steady-state temperature distribution $T(r, z)$ in a solid rod of radius r_0, height H, and of constant thermal conductivity k under the following boundary conditions:

$$\frac{\partial T(r_0, z)}{\partial r} = 0, \qquad T(r, 0) = T_1, \qquad \text{and} \qquad T(r, H) = T_2$$

where T_1 and T_2 are two given constant temperatures.

5.20 Consider a circular rod of radius r_0, length L, and constant thermal conductivity k as shown in Fig. 5.24. The rod is supported at its ends by two plates maintained at a constant temperature T_0 and is exposed to a fluid maintained at a constant temperature T_∞, with a constant heat transfer coefficient h at its peripheral surface. Assume perfect thermal contact between the rod and the plates.

(*a*) Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the rod.

(*b*) What is the rate of heat loss from the rod to the surrounding fluid?

5.21 Consider a solid cylinder of length L, radius r_0, and constant thermal conductivity k. Internal energy is generated in this cylinder at a constant rate $\dot{q}$ per unit volume. Obtain an expression for the steady-state temperature distribution $T(r, z)$ under the following boundary conditions:

$$T(r_0, z) = T(r, 0) = T(r, L) = T_0 = \textit{constant}$$

5.22 Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the solid cylinder of Problem 5.21 if the rate of internal energy generation per unit volume is given as $\dot{q} = \dot{q}(r, z)$.

5.23 Solve the two-dimensional steady-state problem formulated in Problem 2.11 for $T(x, y)$ (up to a constant).

5.24 The cross section of a long rectangular bar, in which internal energy is generated at a constant rate $\dot{q}$ per unit volume, is shown in Fig. 5.25. The surfaces at $x = 0$

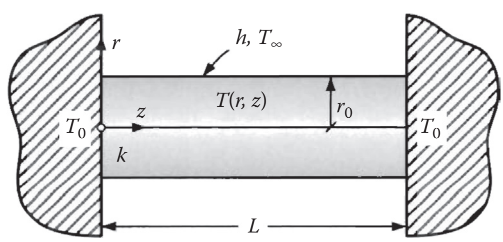

FIGURE 5.24
Circular rod of Problem 5.20.

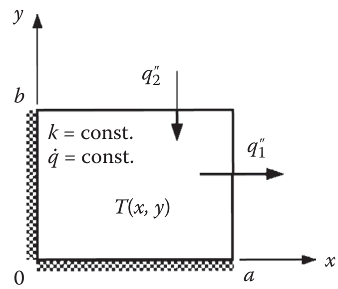

FIGURE 5.25
Figure for Problem 5.24.

and $y = 0$ are perfectly insulated. Assume that the thermal conductivity k of the material of the bar is constant.

(*a*) Formulate the problem for the two-dimensional steady-state temperature distribution $T(x, y)$ under the boundary conditions given in the figure, where q_1'' and q_2'' are given constant heat fluxes out of and into the bar at $x = a$ and $y = b$, respectively.

(*b*) In order for the temperature distribution to be steady, what must be the relationship between q_1'', q_2'' and $\dot{q}$?

(*c*) Show that this problem does not have a unique solution for $T(x, y)$.

(*d*) Obtain an expression for $T(x, y)$.

5.25 A solid cylinder of radius r_0, height H, and constant thermal conductivity k is subjected to the boundary conditions shown in Fig. 5.26.

(*a*) Show that there is no unique solution for the steady-state temperature distribution $T(r, z)$ in this problem.

(b) Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the cylinder (up to a constant).

5.26 (*a*) Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the semi-infinite solid cylinder shown in Fig. 5.27. Assume that the thermal conductivity k of the material of the cylinder is constant.

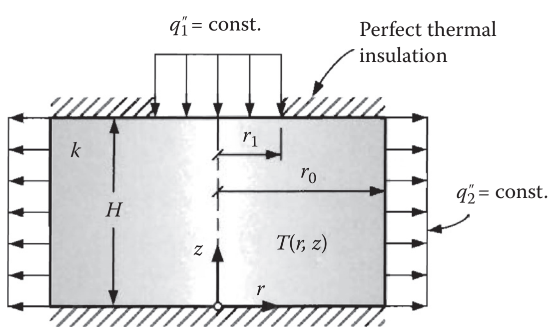

FIGURE 5.26
Solid cylinder of Problem 5.25.

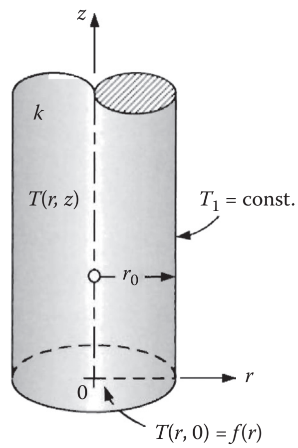

FIGURE 5.27
Semi-infinite solid cylinder of Problem 5.26.

(*b*) Simplify the solution obtained in Part (*a*) for the particular case of $f(r) = T_0 =$ constant, where $T_0 \neq T_1$.

(*c*) What is the rate of heat transfer into the cylinder at $z = 0$ when $f(r) = T_0$?

5.27 Consider the long circular solid rod of radius r_0 and constant thermal conductivity k shown in Fig. 5.28. The rod is attached to a wall maintained at a constant temperature T_0 and is exposed to a fluid maintained at a constant temperature T_∞ with a constant heat transfer coefficient h at its peripheral surface. Assume perfect thermal contact between the rod and the plate.

(*a*) Obtain an expression for the steady-state temperature $T(r, z)$ in the rod.

(*b*) What is the rate of heat loss, q_{loss}, from the rod to the surrounding fluid?

5.28 Consider a semi-infinite solid cylinder, $0 \leq r \leq r_0$ and $0 \leq z < \infty$, of constant thermal conductivity k. The peripheral surface at $r = r_0$ is maintained at a constant temperature T_1, while the circular surface at $z = 0$ is subjected to a uniform heat flux q''_w. Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the cylinder.

5.29 Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the semi-infinite cylinder of Problem 5.28 if the peripheral surface at $r = r_0$ is

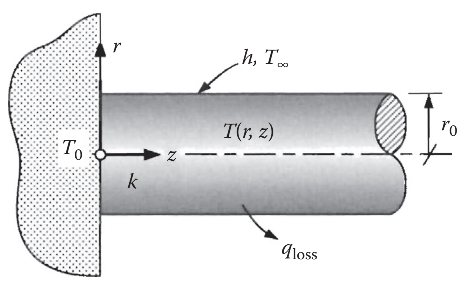

FIGURE 5.28
Long circular rod of Problem 5.27.

exposed to a fluid maintained at temperature T_∞, with a constant heat transfer coefficient h.

5.30 Obtain an expression for the steady-state temperature distribution $T(x, y)$ in the semi-infinite rectangular strip shown in Fig. 5.29.

5.31 Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the semi-infinite cylindrical rod shown in Fig. 5.30.

5.32 Consider a hollow solid cylinder, $r_1 \le r \le r_2$ and $0 \le z \le L$, of constant thermal conductivity k. Internal energy is generated in this cylinder at a constant rate $\dot{q}$ per unit volume. Obtain an expression for the steady-state temperature distribution $T(r, z)$ if the surface at $r = r_1$ is insulated and the other surfaces are maintained at a constant temperature T_0.

5.33 Consider a circular rod of length $2L$. Internal energy is generated uniformly over the middle half section of the rod as illustrated in Fig. 5.31. Determine the steady-state temperature in the rod. The two ends of the rod can be approximated as adiabatic.

5.34 A cylindrical shaft is rotating with an angular velocity of ω inside a fixed sleeve of negligible thickness as illustrated in Fig. 5.32. The pressure and the coefficient

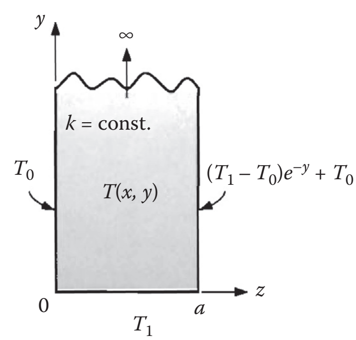

FIGURE 5.29
Semi-infinite rectangular strip of Problem 5.30.

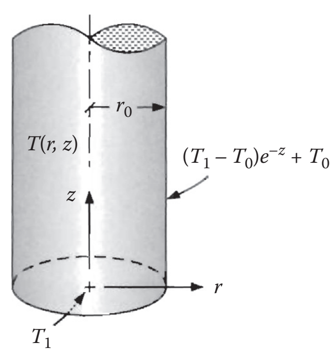

FIGURE 5.30
Semi-infinite cylindrical rod of Problem 5.31.

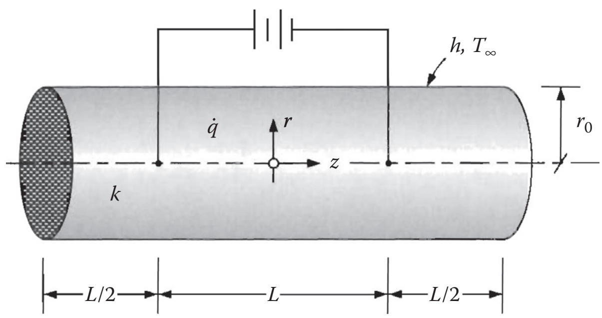

FIGURE 5.31
Circular rod of Problem 5.33.

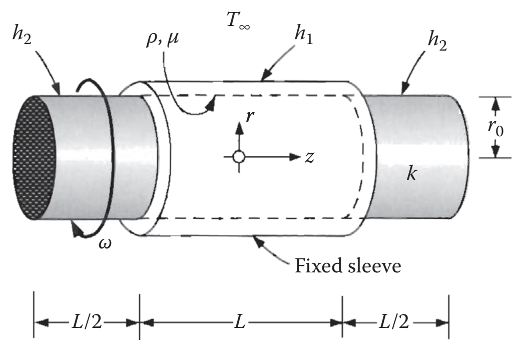

FIGURE 5.32
Rotating cylindrical shaft of Problem 5.34.

of dry friction between the sleeve and the shaft are p and μ, respectively. Find the steady-state temperature distribution in the shaft. The two ends of the shaft can be approximated as adiabatic.

5.35 (*a*) Determine the steady-state temperature distribution $T(r, \phi)$ inside a long solid rod whose cross section is a sector of a circle, $0 \le r \le r_0$ and $0 \le \phi \le \alpha$. The temperature is maintained at zero along the straight edges at $\phi = 0$ and $\phi = \alpha$ and at a prescribed distribution $T(r_0, \phi) = f(\phi)$ along the curved edge.

(*b*) What will be the temperature distribution if $f(\phi) = T_1 = constant$?

5.36 Determine the steady-state temperature distribution $T(r, \phi)$ inside the long rod of Problem 5.35 if the temperature is maintained at a prescribed constant value T_0 along the straight edge at $\phi = 0$ and at *zero* along both the straight edge at $\phi = \alpha$ and the curved edge at $r = r_0$.

5.37 Consider a long solid rod whose cross section is a sector of a circle of radius r_0 as shown in Fig. 5.33. A uniform radiation heat flux q_s'' falls onto the peripheral surface, while this surface loses heat by convection to the ambient maintained at constant temperature T_∞. The two plane surfaces at $\phi = \pm\alpha$ are kept isothermal at the ambient temperature T_∞. The peripheral surface at $r = r_0$ absorbs all the incident energy, and the heat loss by radiation from this surface is negligible relative to the loss by convection. Assuming that the thermal conductivity k and the heat

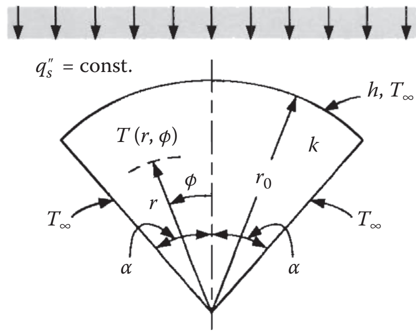

FIGURE 5.33
Cross section of rod of Problem 5.37.

transfer coefficient h are both constants, obtain an expression for the steady-state temperature distribution $T(r, \phi)$ in the rod.

5.38 Determine the two-dimensional steady-state temperature distribution $T(r, \phi)$ inside a large plate with a circular hole of radius r_0 as illustrated in Fig. 5.34, where the circular surface at r_0 is held at a prescribed temperature $f(\phi)$.

5.39 Use the Poisson integral formula (5.86) to solve the problem defined by Eqs. (5.62) and (5.63*a*–*d*).

5.40 Use the Poisson integral formula (5.86) to solve the following two-dimensional steady-state heat conduction problem:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0, \qquad 0 < r < r_0, \qquad 0 < \phi < \pi$$

$$T(r_0, \phi) = f(\phi)$$

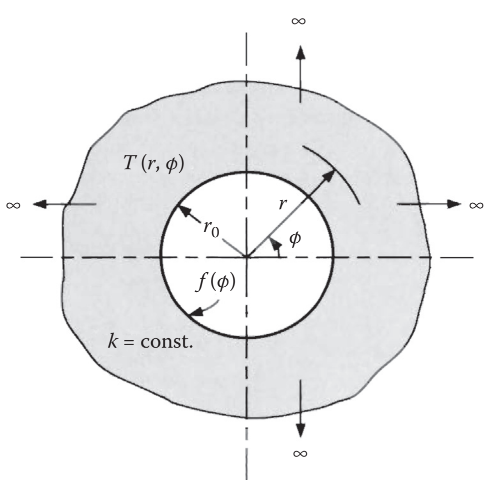

FIGURE 5.34
Large plate with circular hole of Problem 5.38.

$$\frac{\partial T(r,0)}{\partial \phi} = \frac{\partial T(r,\pi)}{\partial \phi} = 0$$

5.41 Find the first three coefficients in the expansion of the function

$$f(\theta) = \begin{cases} \cos\theta, & 0 < \theta < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < \theta < \pi \end{cases}$$

in a series of Legendre polynomials in cos θ over the interval (0, π).

5.42 The temperature of the surface of a solid sphere of constant thermal conductivity k and radius r_0 is maintained at

$$f(\theta) = \begin{cases} T_0\cos\theta, & 0 < \theta < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < \theta < \pi \end{cases}$$

where θ is the polar angle. Find an expression for the steady-state temperature distribution $T(r, \theta)$ inside the sphere.

5.43 Consider a hemispherical solid of constant thermal conductivity k and radius r_0 as illustrated in Fig. 5.35. The solid is heated by applying a uniformly distributed constant heat flux q_s'' to its spherical surface, while the flat surface is maintained at the constant temperature T_0. Find an expression for the steady-state temperature distribution inside the solid.

5.44 Consider a hollow solid sphere of constant thermal conductivity k, and inner and outer radii r_1 and r_2, respectively. Let the inner surface be maintained at a constant temperature T_0 and the outer surface at $f(\theta)$, where θ is the polar angle. Find an

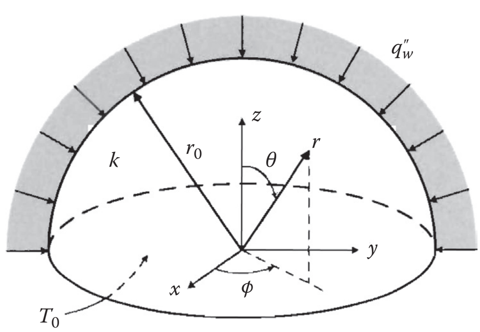

FIGURE 5.35
Hemispherical solid of Problem 5.43.

expression for the steady-state temperature distribution $T(r, \theta)$ inside the hollow sphere.

5.45 (*a*) Obtain an expression for the three-dimensional steady-state temperature distribution $T(r, \phi, z)$ in the solid cylinder of Problem 5.14 (see Fig. 5.23) for the following boundary conditions:

$$T(r_0,\phi,z) = f(\phi), \quad T(r,\phi,0) = T_1 \quad \text{and} \quad T(r,\phi,L) = T_1$$

where T_1 is a given constant temperature.

(*b*) Simplify the solution obtained in Part (a) if $f(\phi) = T_0$ = constant.

5.46 Consider the square plate shown in Fig. 5.36. Assume that the variation in temperature in the z direction over the thickness δ is negligible.

(*a*) Starting from the basic principles obtain the differential equation that governs the variation of the steady-state temperature distribution in the plate as a function of the x and y variables.

(*b*) Determine the temperature distribution $T(x, y)$ in the plate under steady-state conditions.

5.47 Re-solve Problem 5.46 if internal energy is generated in the plate at a constant rate $\dot{q}$ per unit volume (W/m³).

5.48 Obtain an expression for the steady-state temperature distribution $T(r, z)$ in the semi-infinite cylindrical solid rod shown in Fig. 5.37. Internal energy is generated in the rod according to $\dot{q} = \dot{q}_0 \, e^{-\beta z}$, where $\dot{q}_0$ and β are two given positive constants, while the surfaces are maintained at a constant temperature T_0.

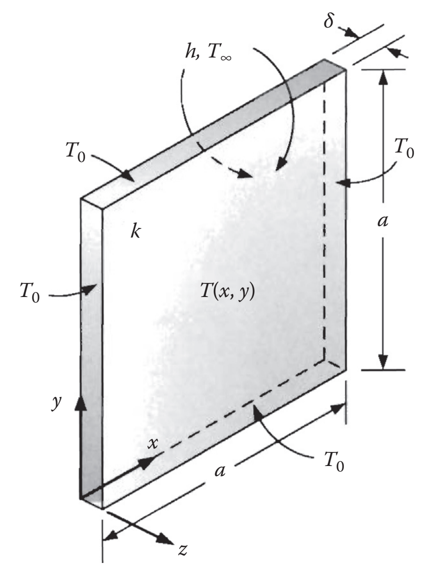

FIGURE 5.36
Square plate of Problem 5.46.

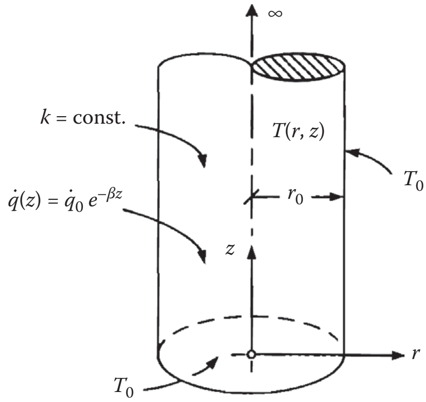

FIGURE 5.37
Figure for Problem 5.48.

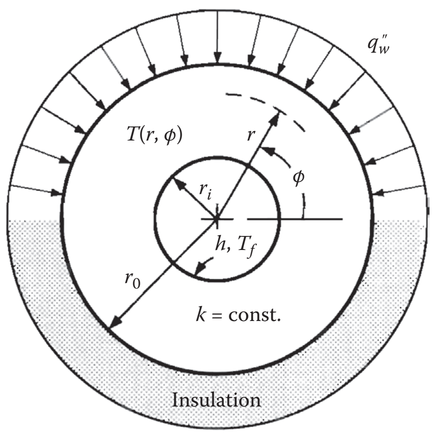

FIGURE 5.38
Figure for Problem 5.49.

5.49 Consider a long circular pipe of inner radius r_l, outer radius r_o, and constant thermal conductivity k. One half of the outer surface is subjected to a uniform heat flux q''_w, while the other half is perfectly insulated as shown in cross section in Fig. 5.38. The pipe transfers heat by convection at its inside surface to a fluid maintained at a uniform temperature T_f with a constant heat transfer coefficient h. Obtain an expression for the steady-state temperature distribution $T(r, \phi)$ in the pipe.

5.50 The mathematical formulation for a three-dimensional steady-state heat conduction problem in a cubic region is given by

$$\alpha\left[\frac{\partial^2 T(x,y,z)}{\partial x^2}+\frac{\partial^2 T(x,y,z)}{\partial y^2}+\frac{\partial^2 T(x,y,z)}{\partial z^2}\right]=0, 0<x<1, 0<y<1, 0<z<1$$

with boundary conditions given by

$$T(0,y,z)=1; \quad T(1,y,z)=0$$

$$\left.\frac{\partial T(x,y,z)}{\partial y}\right|_{y=0}=0; \quad T(x,1,z)=0$$

$$\left.\frac{\partial T(x,y,z)}{\partial z}\right|_{z=0}=0; \quad T(x,y,1)=0$$

Find the analytical solution of this problem by separation of variables.

6

Unsteady-State Heat Conduction: Solutions with Separation of Variables

6.1 Introduction

In the previous chapters, we considered various steady-state heat conduction problems. In such problems, the temperature distribution depends only on the space coordinates. A great majority of engineering problems, however, involve time-dependent thermal conditions. For example, the boundary conditions may vary periodically, suddenly (step-wise), or irregularly with time. The initial temperature distribution may not be in thermal equilibrium with the boundary temperatures. Internal energy generation may be started or stopped suddenly, or the internal energy generation may be time dependent. The temperature at each point in such systems changes with time accordingly. Some examples of engineering interest include the temperature distribution in the turbine wheel or a turbine blade during the start-up and shut-down of a gas turbine when it is subjected to sudden changes in gas temperature; the temperature distribution in a piece of steel during a quench-hardening operation when it is removed from a furnace and plunged suddenly into a cold bath of air, water, or oil: the temperature distribution in the cylinder walls of a reciprocating internal combustion engine that is subjected to periodically varying gas temperature; and the temperature distribution in the combustion chamber wall of a jet engine or a rocket motor during start-up. From the knowledge of the temperature distribution at any instant, thermal stress distributions can be evaluated, and from the knowledge of how the temperature at a point varies with time, metallurgical conditions may be predicted.

Such unsteady-state, or transient, problems can be grouped into two; *nonperiodic* and *periodic*. Nonperiodic problems include those involving heating or cooling of bodies in a medium of a different initial thermal state, such as the heating of steel ingots, cooling of iron bars in steelworks, hardening of steel by quenching in an oil bath, or starting up or shutting down a nuclear reactor or a furnace. Periodic problems are those such as the heating process of regenerators whose packings are periodically heated by flue gases and cooled by air, daily periodic variation of the heating and cooling of the earth's surface, and temperature fluctuations in the walls of internal combustion engines.

In nonperiodic transient problems, the temperature at any point within the body under consideration changes as some nonlinear function of time. With periodic transient cases, the temperature undergoes a periodic change that is either regular or irregular, but is always cyclic. A regular periodic variation is characterized by a harmonic function and an irregular periodic variation by any function that is cyclic but not necessarily harmonic.

In any heating or cooling problem, the heat transfer process between a body and the fluid surrounding it is influenced by both the internal resistance of the body and the surface resistance. In Chapter 3, we defined the ratio of internal resistance to surface resistance as the Biot number and discussed how to interpret the spatial variation of temperature in a solid in terms of the Biot number under steady-state conditions (see Fig. 3.9). We now extend this interpretation to spatial temperature distribution in unsteady-state problems. In a thin or small body with a large thermal conductivity, the internal resistance may be negligible compared to the surface resistance. If this is the case, that is, if $\mathrm{Bi} \ll 1$, then the spatial variation of temperature can be neglected as illustrated in Fig. 6.1. Such a case is referred to as a *lumped-heat-capacity system*.

When the surface resistance is negligible compared to the internal resistance, that is, when $\mathrm{Bi} \gg 1$, the surface (boundary) temperature will practically be equal to the surrounding fluid temperature at all times as illustrated in Fig. 6.2. This is usually the case with boiling, condensation and highly turbulent flows, where the heat transfer coefficient h is very large.

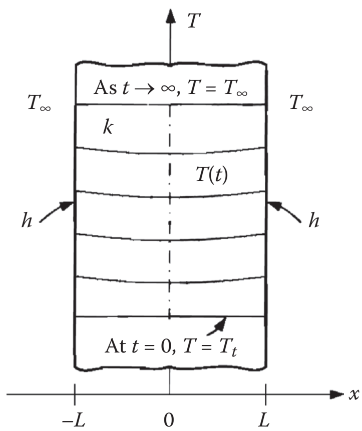

FIGURE 6.1
Spatial variation of temperature in a flat plate with time when $\mathrm{Bi} = hL/k \ll 1$.

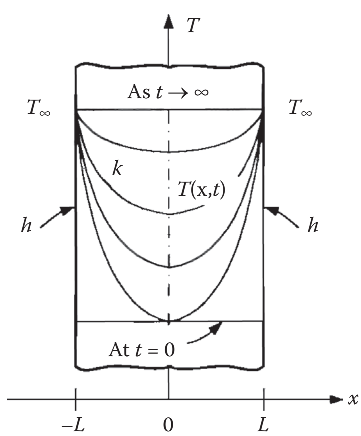

FIGURE 6.2
Spatial variation of temperature in a flat plate with time when $\mathrm{Bi} = hL/k \gg 1$.

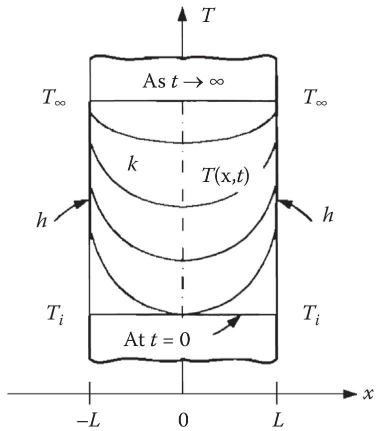

FIGURE 6.3
Spatial variation of temperature in a flat plate with time when $\mathrm{Bi} = hL/k \approx 1$.

When the internal and surface resistances are comparable, that is, when $\mathrm{Bi} \cong 1$, the spatial variation of the temperature with time would be as illustrated in Fig. 6.3. The last two cases correspond to what is known as *distributed system*.

In this chapter we first study lumped-heat-capacity systems where the elimination of the spatial variation of temperature considerably simplifies the problems. Next, we consider some representative nonperiodic heating and/or cooling problems and obtain solutions by the method of separation of variables. Finally, we also consider a periodic problem and introduce the *concept of complex temperature*.

6.2 Lumped-Heat-Capacity Systems

The simplest example of transient cooling or heating is the case in which the internal resistance of a solid body, into or out of which heat transfer takes place, is negligibly small compared to the resistance to heat flow at its surface. This would be the case, for example, if the thermal conductivity k is infinitely large. Actually, the underlying assumption of this model is that the heat transferred to a body is instantaneously and uniformly distributed throughout the body, resulting in a uniform temperature distribution. Such a system, as already mentioned, is called a lumped-heat-capacity system. It is obviously an idealized system, since the thermal conductivity of even the best conductor is not infinitely large. In fact, a temperature gradient must exist in a material if heat is to be conducted into or out of it. However, investigations of instantaneous temperature distribution in various systems during transient heating or cooling processes show that only small temperature differences exist if the Biot number is very small, that is, if $\mathrm{Bi} = h(V/A)/k < 0.1$, where V is the volume and A is the total heat transfer area.

Consider, for example, a long and thin metal bar of constant thermophysical properties (k, ρ, c) as shown in Fig. 6.4. The bar is initially at a uniform temperature T_i and immersed, for times $t \geq 0$, in a large body of cool water maintained at a constant temperature T_∞. Let the heat transfer coefficient on both surfaces be the same constant h. The lumped-heat-capacity

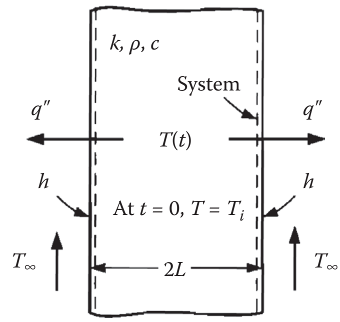

FIGURE 6.4
Long metal bar modeled as a lumped-heat-capacity system.

method of analysis may be used if we can justify the assumption of uniform bar temperature at any time during the cooling process. Accordingly, let us assume that

$$\mathrm{Bi} = \frac{h(V/A)}{k} = \frac{hL}{k} < 0.1$$

where V is the volume and A is the total heat transfer area of the bar. Clearly, in addition to its thickness, the temperature distribution in the bar will depend on its thermal conductivity, as well as on the heat transfer conditions from its surfaces to the surrounding fluid, that is, on the heat transfer coefficient h.

In order to proceed with a lumped analysis, the whole bar is taken to be the system as shown in Fig. 6.4. The first law of thermodynamics, when applied to this system at any instant t, yields

$$\frac{dU}{dt} = -Aq'' \tag{6.1}$$

where U is the internal energy of the bar and q'' is the rate of heat loss from the bar per unit surface area. In terms of the definition of the specific heat c, the left-hand side of Eq. (6.1) is given by

$$\frac{dU}{dt} = \rho Vc\frac{dT}{dt} \tag{6.2a}$$

where ρ is the density of the material of the bar. The heat flux q'' on the right-hand side of Eq. (6.1) can be written, from Newton's law, as

$$q'' = h(T - T_\infty) \tag{6.2b}$$

Combining Eqs. (6.1), (6.2*a*), and (6.2*b*) yields

$$\rho Vc\frac{dT}{dt} = -Ah(T - T_\infty) \tag{6.3a}$$

which is a first-order ordinary differential equation, for which we have the following initial condition:

$$T(0) = T_i \tag{6.3b}$$

Equation (6.3*a*) can be rewritten as

$$\frac{dT}{T - T_\infty} = -\frac{hA}{\rho Vc}dt \tag{6.4}$$

Integrating Eq. (6.4) we obtain

$$\ln(T - T_\infty) = -\frac{hA}{\rho Vc}t + C \tag{6.5a}$$

Applying the initial condition (6.3*b*) yields

$$C = \ln(T_i - T_\infty) \tag{6.5b}$$

Thus, combining Eqs. (6.5*a*) and (6.5*b*) we get

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-[hA/(\rho Vc)]t} \tag{6.6}$$

In terms of the *"characteristic length"* L_c (= V/A), Eq. (6.6) can be rewritten as

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-(hL_c/k)(\alpha t/L_c^2)} \tag{6.7a}$$

or

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\mathrm{BiFo}} \tag{6.7b}$$

where, in addition to the Biot number. Bi = hL_c/k, we have introduced the following dimensionless number:

$$\mathrm{Fo}(\textit{Fourier number}) = \frac{\pi t}{L_c^2}$$

The physical significance of the Fourier number is discussed in Section 6.3.1.

At any instant t, the instantaneous rate of heat loss from the bar would be

$$q = hA[T(t) - T_\infty] = hA(T_i - T_\infty)e^{-[hA/(\rho Vc)]t} = hA(T_i - T_\infty)e^{-\mathrm{BiFo}} \tag{6.8}$$

The total heat loss from the bar from $t = 0$ to any instant t is then given by

$$\begin{aligned} Q &= \int_0^1 q\,dt = hA(T_i - T_\infty)\int_0^1 e^{-[hA/(\rho Vc)]t}\,dt \\ &= \rho Vc(T_i - T_\infty)\left\{1 - e^{-[hA/(\rho Vc)]t}\right\} = \rho Vc(T_i - T_\infty)[1 - e^{-\mathrm{BiFo}}] \end{aligned} \tag{6.9a}$$

This result can be rewritten as

$$\frac{Q}{Q_i} = 1 - e^{-\mathrm{BiFo}} \tag{6.9b}$$

where $Q_i = \rho Vc\,(T_i - T_\infty)$, which is the initial internal energy content of the bar with reference to the surrounding fluid temperature T_∞.

Equations (6.7*b*), (6.8), and (6.9*b*) are applicable for any geometry as long as the assumption Bi < 0.1 holds. For simple geometries, the characteristic length L_c is given by

Sphere: $L_c = r/3$ (r = radius of sphere)
Cylinder: $L_c = r/2$ (height $\gg r$; r = radius of cylinder)
Cube: $L_c = a/6$ (a = side of cube)

Example 6.1

During a heat treatment process, a cylindrical steel rod, 3 cm in diameter and 15 cm in length, is first heated uniformly to 650°C in a furnace and then immersed in an oil bath maintained at 40°C. The heat transfer coefficient can be taken as 110 W/(m²·K). The density, specific heat, and thermal conductivity of the steel are $\rho = 7.83$ g/cm³, $c = 465$ J/(kg·K), and $k = 45$ W/(m·K), respectively. Assuming a uniform temperature variation with time (i.e., a lumped-heat-capacity system), calculate the time required for the rod to reach 345°C.

SOLUTION

From the given data we have

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{hr/2}{k} = \frac{110 \times 1.5 \times 10^{-2}}{45 \times 2} = 0.0183$$

Hence, the lumped-heat-capacity system assumption is justifiable. Furthermore,

$$\alpha = \frac{k}{\rho c} = \frac{45 \times 3600}{7830 \times 465} = 0.0445\,\mathrm{m^2/h}$$

and

$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{345 - 40}{650 - 40} = 0.5$$

Substituting these values into Eq. (6.7*b*), we obtain

$$\mathrm{Fo} = \frac{1}{\mathrm{Bi}} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{1}{0.0183} \ln 2 = 37.877$$

Thus, the corresponding time is

$$t = \frac{\mathrm{Fo}\, L_c^2}{\alpha} = \frac{\mathrm{Fo}(r/2)^2}{\alpha} = \frac{37.877 \times (1.5 \times 10^{-2})^2}{0.0445 \times 4} = 0.048\,\mathrm{h} = 2.87\,\mathrm{min}$$

6.3 One-Dimensional Distributed Systems

In engineering applications, a number of situations can be idealized either as an infinite flat plate with no temperature gradients, say, along the y and z coordinates (i.e., $\partial T/\partial y = \partial T/\partial z = 0$) as illustrated in Fig. 6.5*a*, or as a perfectly insulated rod as shown in Fig. 6.5*b*. In such cases, the unsteady-state temperature distribution will be one-dimensional and, when the thermophysical properties (k, ρ, c) are constant, it will satisfy the following diffusion equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{6.10}$$

Assume that the following initial and boundary conditions exist:

$$T(x,0) = f(x) \tag{6.11a}$$

$$T(0,t) = 0, \qquad T(L,t) = 0 \tag{6.11b,c}$$

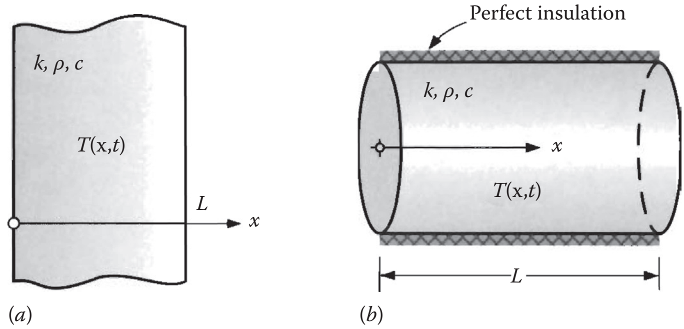

FIGURE 6.5
(*a*) A flat plate, and (*b*) an insulated rod with one-dimensional unsteady temperature distributions.

Since the boundary conditions are homogeneous, the assumption of the existence of a product solution of the form $T(x, t) = X(x)\,\Gamma(t)$ yields

$$T(x,t) = e^{-\alpha\lambda^2 t}(A\sin\lambda x + B\cos\lambda x) \tag{6.12}$$

for the solution of the differential equation (6.10). Here, we note that the sign of the separation constant λ^2 is also consistent with the fact that the temperature distribution will asymptotically approach *zero* as time increases indefinitely. After imposing the boundary conditions (6.11*b*,*c*), the solution for the temperature distribution can be constructed as

$$T(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t}\sin\lambda_n x \tag{6.13}$$

where $\lambda_n = n\pi/L$, $n = 1, 2, 3,...$ Introducing the initial condition (6.11*a*) into Eq. (6.13) we get

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \tag{6.14}$$

which is the *Fourier sine expansion* of $f(x)$ on the interval $(0, L)$. By making use of Table 4.1, the coefficients A_n are obtained as

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \tag{6.15}$$

Thus, the solution for $T(x, t)$ is given by

$$T(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha(n\pi/L)^2 t}\sin\frac{n\pi}{L}x\int_0^L f(x')\sin\frac{n\pi}{L}x'\,dx' \tag{6.16}$$

Let us now consider the same problem with the following initial and boundary conditions:

$$T(x,0) = f(x) \tag{6.17a}$$

$$T(0,t) = T_1, \qquad T(L,t) = T_2 \tag{6.17b,c}$$

Since the problem is linear, the superposition technique can be used to obtain the solution for the temperature distribution $T(x, t)$ in the form:

$$T(x,t) = T_s(x) + T_t(x,t) \tag{6.18}$$

Here $T_s(x)$ is the *steady-slate solution* of the problem and satisfies

$$\frac{d^2T_s}{dx^2} = 0 \tag{6.19a}$$

$$T_s(0) = T_1, \qquad T_s(L) = T_2 \tag{6.19b,c}$$

and $T_t(x, t)$ is the *transient solution* and satisfies

$$\frac{\partial^2 T_t}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_t}{\partial t} \tag{6.20a}$$

$$T_t(x,0) = f(x) - T_s(x) \tag{6.20b}$$

$$T_t(0,t) = 0, \qquad T_t(L,t) = 0 \tag{6.20c,d}$$

The solution of the steady-state problem (6.19) is given by

$$T_s(x) = T_1 - (T_1 - T_2)\frac{x}{L} \tag{6.21}$$

and the solution of the transient problem (6.20) is

$$T_t(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha(n\pi/L)^2 t} \sin\frac{n\pi}{L}x \tag{6.22}$$

Hence, the solution for $T(x, t)$ becomes

$$T(x,t) = T_1 - (T_1 - T_2)\frac{x}{L} + \sum_{n=1}^{\infty} A_n e^{-\alpha(n\pi/L)^2 t} \sin\frac{n\pi}{L}x \tag{6.23}$$

Applying the initial condition (6.17*a*) we get

$$f(x) = T_1 - (T_1 - T_2)\frac{x}{L} + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \tag{6.24a}$$

which is the *Fourier sine expansion* of

$$F(x) = f(x) - T_1 + (T_1 - T_2)\frac{x}{L} \tag{6.24b}$$

on the interval (0, L). Thus, the expansion coefficients A_n are given by

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi}{L}x\,dx \\ &= \frac{2}{n\pi}\left[(-1)^n T_2 - T_1\right] + \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \end{aligned} \tag{6.24c}$$

Now, substituting the expansion coefficients from Eq. (6.24c) into Eq. (6.23), we obtain

$$\begin{aligned} T(x,t) = T_1 - (T_1 - T_2)\frac{x}{L} - \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x \\ \times\left\{\frac{T_1 - (-1)^n T_2}{\lambda_n} - \int_0^L f(x')\sin\lambda_n x'\,dx'\right\} \end{aligned} \tag{6.25}$$

where we have substituted $\lambda_n = n\pi/L$.

It is now obvious that time-dependent linear problems can also be solved using the method of separation of variables, provided that the differential equation and the boundary conditions are homogeneous. Difficulties resulting from various nonhomogeneities, however, can be circumvented by the use of the principle of superposition. In cases where the nonhomogeneity in a boundary condition is due to a nonzero heat flux, then the "nature" of the problem may require a superposition of the form

$$T(x,t) = T_1(x,t) + T_2(x) + T_3(t) \tag{6.26}$$

instead of the form (6.18) (for example, see Problem 6.20).

6.3.1 Cooling (or Heating) of a Large Flat Plate

The unsteady-state temperature distribution in a relatively thin slab of material, which is cooled (or heated) on one or both sides, can be approximated, away from the edges, as one dimensional. Such situations are frequently encountered in engineering applications. As an example, consider a flat plate, initially at $t = 0$ at a uniform temperature T_i, which is subjected to the same cooling (or heating) conditions on both sides for times $t \geq 0$ as illustrated in Fig. 6.6. This plate is considered to be sufficiently large in the y and z directions compared to its thickness $2L$ in the x direction. The heat transfer coefficient h on both surfaces is assumed to be constant. It is further assumed that the surrounding fluid temperature remains constant at T_∞ during the whole cooling (or heating) process. Under these conditions heat flow through the plate will be one-dimensional in the x direction. If the thermophysical properties (k, ρ, c) are also assumed to be constant, the temperature distribution, redefined as $\theta(x, t) = T(x, t) - T_\infty$, will then satisfy

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{6.27}$$

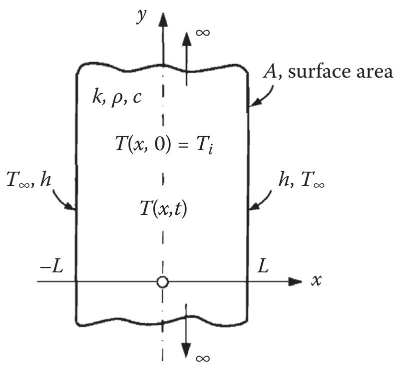

FIGURE 6.6
Transient cooling (or heating) of a flat plate of thickness 2L.

with the initial condition

$$\theta(x,0) = T_i - T_\infty = \theta_i \tag{6.28a}$$

and boundary conditions

$$\frac{\partial\theta(0,t)}{\partial x} = 0, \qquad k\frac{\partial\theta(L,t)}{\partial x} + h\theta(L,t) = 0 \tag{6.28b,c}$$

Here, it should be noticed that the problem under consideration is mathematically the same as the problem for a slab of the same material, but of thickness L, which is perfectly insulated on one side and cooled (or heated) on the other, as illustrated in Fig. 6.7.

Since the above problem is linear with two homogeneous boundary conditions, it can be shown, by assuming a product solution of the form $\theta(x, t) = X(x)\,\Gamma(t)$, that

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t} \cos\lambda_n x \tag{6.29}$$

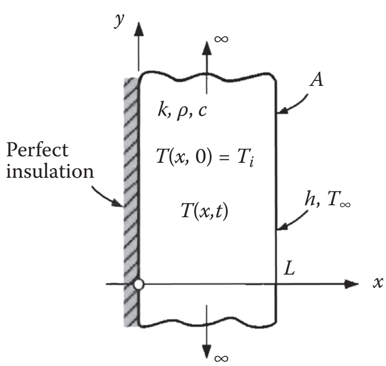

FIGURE 6.7
Transient cooling (or heating) of a flat plate insulated on one side.

where the characteristic values λ_n are the positive roots of

$$\cot \lambda L = \frac{\lambda L}{\text{Bi}} \tag{6.30}$$

with Bi = hL/k. Equation (6.30) is a transcendental equation, the roots of which can be found numerically or graphically, as discussed in Section 5.2.1 (see Fig. 5.7). The constants A_n in Eq. (6.29) are determined from the application of the initial condition (6.28*a*): that is,

$$\theta_i = \sum_{n=1}^{\infty} A_n \cos \lambda_n x \tag{6.31}$$

which is a *Fourier cosine expansion* of θ_i, with the values of λ_n obtained from the characteristic-value equation (6.30). Hence, making use of Table 4.1 we get

$$A_n = \frac{2\lambda_n}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L} \int_0^L \theta_i \cos \lambda_n x \, dx \tag{6.32a}$$

which reduces to

$$A_n = \frac{2\theta_i \sin \lambda_n L}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L} \tag{6.32b}$$

The solution for the temperature distribution is then given by

$$\frac{\theta(x,t)}{\theta_i} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin \lambda_n L \cos \lambda_n x}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L} e^{-\alpha \lambda_n^2 t} \tag{6.33}$$

For the case in which $h \to \infty$ (i.e., when Bi $\to \infty$) the boundary condition (6.28*c*) reduces to $\theta(L, t) = 0$ (i.e.. $T = T_\infty$ at $x = L$). This condition leads to the characteristic-value equation cos $\lambda L = 0$, resulting in the following characteristic values:

$$\lambda_n = \frac{2n-1}{L}\frac{\pi}{2}, \qquad n = 1, 2, 3, \ldots \tag{6.34}$$

Then, the solution would be given by

$$\frac{\theta(x,t)}{\theta_i} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} e^{-\alpha[(2n-1)/2L]^2 \pi^2 t} \cos \frac{(2n-1)\pi}{2L} x \tag{6.35}$$

It often becomes necessary to know both the instantaneous rate of heat transfer to or from the plate and the total amount of heat that has been transferred over a period of time after the transient period has started. The instantaneous rate of heat loss from both surfaces can be obtained from Fourier's law as

$$q = -2kA\frac{\partial T(L,t)}{\partial x} \tag{6.36}$$

By the use of Eq. (6.33), we get

$$q = 4kA\theta_i \sum_{n=1}^{\infty} \frac{\lambda_n \sin^2 \lambda_n L}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L} e^{-\alpha\lambda_n^2 t} \tag{6.37}$$

The total amount of heat loss from the plate over a period of time t, after the transient period has started, is then given by

$$Q = \int_0^t q\,dt = 4kA\theta_i \int_0^t \sum_{n=1}^{\infty} \frac{\lambda_n \sin^2 \lambda_n L}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L} e^{-\alpha\lambda_n^2 t'} dt' \tag{6.38a}$$

When the integral is evaluated, we obtain

$$Q = 4\rho c A\theta_i \sum_{n=1}^{\infty} \frac{\sin^2 \lambda_n L}{\lambda_n(\lambda_n L + \sin \lambda_n L \cos \lambda_n L)} \left(1 - e^{-\alpha\lambda_n^2 t}\right) \tag{6.38b}$$

This result can also be written as

$$\frac{Q}{Q_i} = 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n L} \frac{\sin^2 \lambda_n L}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L} \left(1 - e^{-\alpha\lambda_n^2 t}\right) \tag{6.38c}$$

where $Q_i = 2AL\rho c\,(T_i - T_\infty)$, which is the initial internal energy content of the plate with reference to the surrounding fluid temperature T_∞.

In the case of a plate of thickness L that is cooled (or heated) on one side and insulated on the other, Eqs. (6.37) and (6.38*b*) must be divided by 2, and also, in Eq. (6.38*c*), $Q_i = AL\rho c\,(T_i - T_\infty)$.

Except for very small values of time, the infinite series in the results above converges rather rapidly, and satisfactory accuracy can be achieved by considering only a few terms of the series.

In the form in which they are presented, the results above apply only to the particular plate considered with the given initial temperature T_i, the heat transfer coefficient h, and the surrounding fluid temperature T_∞. Obviously, more general results would be desirable. This is possible because Eqs. (6.33) and (6.38*c*) can be rewritten as functions of only three dimensionless variables. If solutions over wide ranges of these dimensionless variables are presented in the form of charts or tables, the answers to a specific problem can then be determined readily from the appropriate chart or table: Now, if we let $\gamma_n = \lambda_n L$, then the characteristic-value equation (6.30) becomes

$$\cot \gamma_n = \frac{\gamma_n}{\text{Bi}} \tag{6.39}$$

Therefore,

$$\gamma_n = \gamma_n(\text{Bi}) = \gamma_n\left(\frac{hL}{k}\right) \tag{6.40}$$

and Eq. (6.33) can be rewritten as

$$\frac{\theta(x,t)}{\theta_i} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin \gamma_n \cos(\gamma_n x / L)}{\gamma_n + \sin \gamma_n \cos \gamma_n} e^{-\gamma_n^2 Fo} \tag{6.41}$$

where $\text{Fo} = \alpha t/L^2$ is the Fourier number, the magnitude of which is a measure of the degree of cooling or heating effects generated through the plate. If, for example, α/L^2 is small, a large value of t is required before significant temperature changes occur through the body. In view of Eq. (6.40), we now conclude that the dimensionless temperature ratio $\theta(x, t)/\theta_i$ is a function of the Biot number Bi, the Fourier number Fo, and the dimensionless position x/L; that is,

$$\frac{T(x,t) - T_\infty}{T_i - T_\infty} = \psi\left(\text{Bi},\ \text{Fo}, \frac{x}{L}\right) = \psi\left(\frac{hL}{k}, \frac{\alpha t}{L^2}, \frac{x}{L}\right) \tag{6.42}$$

Similarly, Eq. (6.38*c*) can be rewritten as

$$\frac{Q}{Q_i} = 2\sum_{n=1}^{\infty} \frac{1}{\gamma_n} \frac{\sin^2 \gamma_n}{\gamma_n + \sin \gamma_n \cos \gamma_n}\left(1 - e^{-\gamma_n^2 \text{Fo}}\right) \tag{6.43}$$

Therefore,

$$\frac{Q}{Q_i} = \phi(\text{Bi}, \text{Fo}) = \phi\left(\frac{hL}{k}, \frac{\alpha t}{L^2}\right) \tag{6.44}$$

These solutions have already been presented in the form of charts in the literature. Here we describe the ones presented by Heisler [7] and Gröber [6] since they are the most complete and best known.

Figure 6.8 is a chart for determining the variation of the midplane temperature T_c of a large plate of thickness $2L$ or the temperature of the insulated surface of a large plate of thickness L. In Fig 6.8, the dimensionless temperature $(T_c - T_\infty)/(T_i - T_\infty)$ is plotted versus the Fourier number $\text{Fo} = \alpha t/L^2$ for values of $1/\text{Bi}$ up to 100. The temperatures at other locations

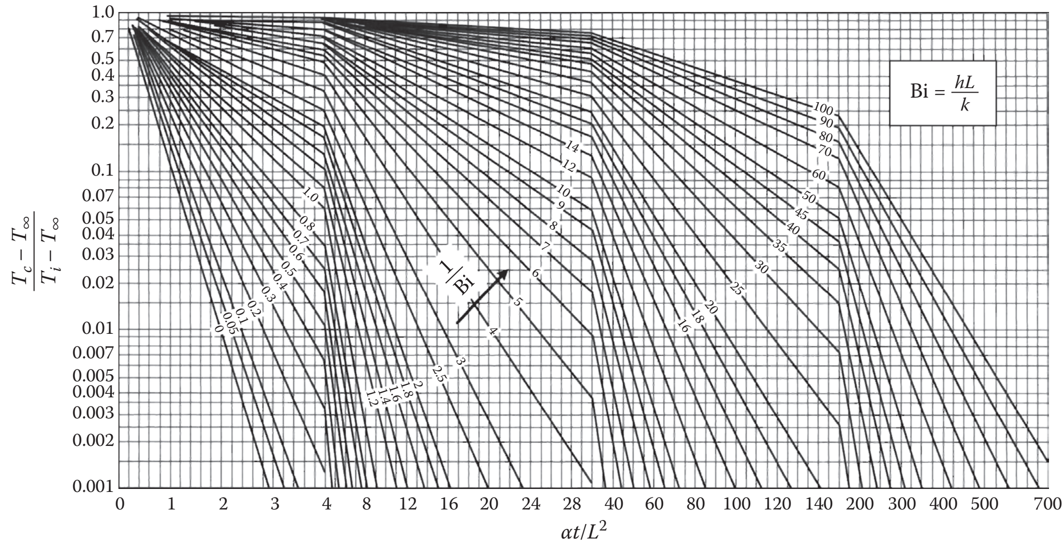

FIGURE 6.8
Temperature–time history at midplane of infinite plate of thickness $2L$ [7].

are obtained by multiplying the temperature ratio from Fig. 6.8 by the *position-correction factors* taken from Fig. 6.9.

Here we note that the curve $1/\text{Bi} = k/hL = 0$ in Fig. 6.8 corresponds to the case with negligible surface resistance (i.e., $h \to \infty$). This would be equivalent to the case in which the temperature of both surfaces is suddenly changed and held at the surrounding fluid temperature T_∞.

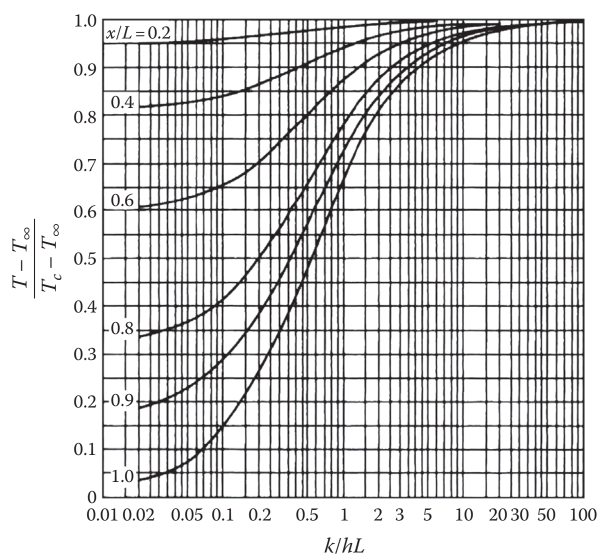

FIGURE 6.9
Position-correction chart for infinite plate of thickness $2L$ [7].

In problems where k/hL is greater than 100 (and Fo > 0.2), Heisler showed that the midplane temperature could be satisfactorily represented by [7]

$$\frac{T_c - T_\infty}{T_i - T_\infty} = e^{-(hL/k)(\alpha t/L^2)} \tag{6.45}$$

That is, for these values of Bi and Fo numbers, the internal thermal resistance may be considered negligible compared to the surface resistance.

The dimensionless heat loss Q/Q_i is given in Fig. 6.10 as a function of the dimensionless quantity $\text{Bi}^2\text{Fo} = h^2\alpha t/k^2$ for various values of Bi.

Example 6.2

A 60-cm-thick plane wall at a uniform temperature of 21°C is suddenly exposed on both sides to a hot gas stream at 577°C. The heat transfer coefficient is 10 W/(m²·K). The density, specific heat and thermal conductivity of the wall are ρ = 2600 kg/m³, c = 1256 J/(kg·K), and k = 15.5 W/(m·K). After 27 h of heating, calculate the temperatures (*a*) at the midplane, and (*b*) at a depth of 12 cm from one of the surfaces.

SOLUTION

(a) From the given data we have

$$\alpha = \frac{k}{\rho c} = \frac{15.5}{2600 \times 1256} = 4.746 \times 10^{-6}\, m^2/s$$

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{4.746 \times 10^{-6} \times 27 \times 3600}{(0.3)^2} = 5.13$$

$$\frac{1}{\text{Bi}} = \frac{k}{hL} = \frac{15.5}{10 \times 0.3} = 5.17$$

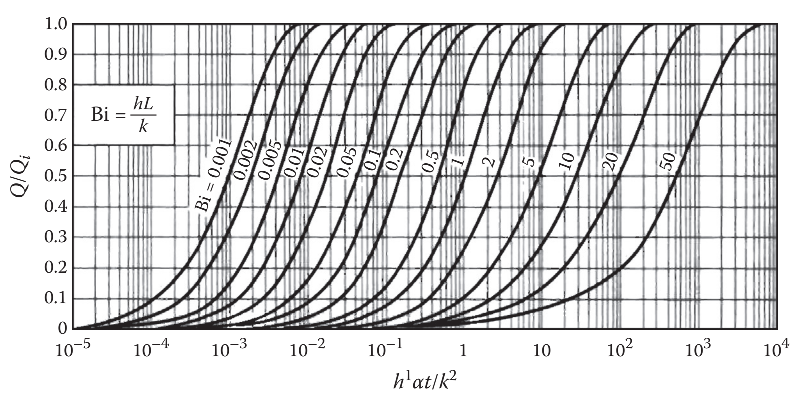

FIGURE 6.10
Dimensionless heat loss Q/Q_i from infinite plate of thickness $2L$ [6].

From Fig. 6.8 for 1/Bi = 5.17 and Fo = 5.13, we get

$$\frac{T_c - T_\infty}{T_i - T_\infty} \cong 0.4 \quad \Rightarrow \quad T_c \cong 355°\text{C}$$

(*b*) From Fig. 6.9, the dimensionless temperature at $x/L = 18/30 = 0.6$ is

$$\frac{T - T_\infty}{T_c - T_\infty} \cong 0.97 \Rightarrow \quad T \cong 362°\text{C}$$

6.3.2 Cooling (or Heating) of a Long Solid Cylinder

Equally important as the plate configuration we have just discussed is a circular solid cylinder. To ensure one-dimensional heat conduction, so that the transient problem will involve only two independent variables, we assume that the cylinder is infinitely long and that axial symmetry exists. Thus, the heat flow can be considered to be in the radial direction only.

Let the solid cylinder under consideration be heated initially to some known axially symmetric temperature distribution $f(r)$ and then, suddenly at $t = 0$, be placed in contact with a fluid at temperature T_∞ as shown in Fig. 6.11. The heat transfer coefficient h at the surface of the cylinder is constant. In addition, we assume that the temperature of the surrounding fluid remains constant at T_∞ during the whole cooling (or heating) process for $t \geq 0$. If the thermophysical properties (k, p, c) are also assumed to be constant, then the formulation of the problem in terms of $\theta(r, t) = T(r, t) - T_\infty$ is given by the heat conduction equation

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{6.46}$$

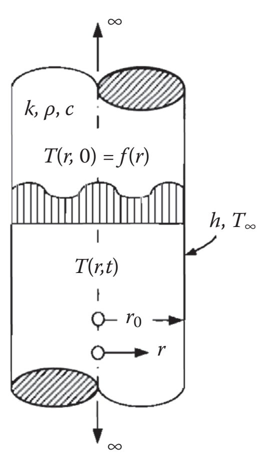

FIGURE 6.11
Transient cooling (or heating) of an infinitely long solid cylinder.

together with the initial condition

$$\theta(r,0) = f(r) - T_\infty = F(r) \tag{6.47a}$$

and the boundary conditions

$$\theta(0,t) = \mathit{finite}, \quad k\frac{\partial\theta(r_0,t)}{\partial r} + h\theta(r_0,t) = 0 \tag{6.47b,c}$$

Since the boundary conditions are homogeneous, the assumption of a product solution of the form $\theta(r, t) = R(r)\,\Gamma(t)$ yields the following solution:

$$\theta(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t} J_0(\lambda_n r) \tag{6.48}$$

where the characteristic values λ_n are the positive roots of

$$\lambda k J_1(\lambda r_0) - h J_0(\lambda r_0) = 0 \tag{6.49}$$

which is a transcendental equation obtained from the application of the boundary condition (6.47*c*). The characteristic-value equation (6.49) can be rearranged in the following form:

$$\frac{J_1(\lambda r_0)}{J_0(\lambda r_0)} = \frac{\mathrm{Bi}}{\lambda r_0} \tag{6.50}$$

where $\mathrm{Bi} = hr_0/k$. Equation (6.50) can be solved numerically to obtain the characteristic values λ_n. The first six characteristic values are tabulated in Reference [10] for several values of the Biot number.

Applying the initial condition (6.47*a*) we now obtain

$$F(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \tag{6.51}$$

which is a *Fourier–Bessel expansion* of $F(r)$ with the values of λ_n obtained from Eq. (6.50). Hence, from Table 4.2, we have

$$A_n = \frac{1}{N_n}\int_0^{r_0} F(r) J_0(\lambda_n r) r\, dr \tag{6.52}$$

where

$$N_n = \frac{r_0^2}{2}\left[1 + \frac{1}{\lambda_n^2}\left(\frac{h}{k}\right)^2\right] J_0^2(\lambda_n r_0) \tag{6.53a}$$

which, by the use of Eq. (6.49), can be rearranged as

$$N_n = \frac{r_0^2}{2}\left[J_0^2(\lambda_n r_0) + J_1^2(\lambda_n r_0)\right] \tag{6.53b}$$

Thus, the solution for the temperature distribution becomes

$$\begin{aligned}\theta(r,t) &= T(r,t) - T_\infty \\ &= \frac{2}{r_0^2}\sum_{n=1}^{\infty}\frac{e^{-\alpha\lambda_n^2 t}J_0(\lambda_n r)}{\left[J_0^2(\lambda_n r_0)+J_1^2(\lambda_n r_0)\right]}\int_0^{r_0} F(r')J_0(\lambda_n r')r'\,dr'\end{aligned} \tag{6.54}$$

Like the flat plate, a long circular solid bar initially heated to a uniform temperature is often of practical interest. So, with $f(r) = T_i$ = constant, the expression (6.54) reduces to

$$\frac{\theta(r,t)}{\theta_i} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = \frac{2}{r^0}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{J_1(\lambda_n r_0)J_0(\lambda_n r)}{\left[J_0^2(\lambda_n r_0)+J_1^2(\lambda_n r_0)\right]}e^{-\alpha\lambda_n^2 t} \tag{6.55}$$

which can also be rewritten as

$$\frac{\theta(r,t)}{\theta_i} = \frac{T(r,t)-T_\infty}{T_i - T_\infty} = 2\sum_{n=1}^{\infty}\frac{1}{\gamma_n}\frac{J_1(\gamma_n)J_0(\gamma_n(r/r_0))}{\left[J_0^2(\gamma_n)+J_1^2(\gamma_n)\right]}e^{-\gamma_n^2 Fo} \tag{6.56}$$

where $\gamma_n = \lambda_n r_0$ and $\mathrm{Fo} = \alpha t/r_0^2$.

It is seen from Eqs. (6.50) and (6.56) that the dimensionless temperature distribution $\theta(r, t)/\theta_i$ is a function of the Fourier number Fo, the Biot number Bi, and the dimensionless position r/r_0; that is,

$$\frac{T(r,t)-T_\infty}{T_i - T_\infty} = \psi\left(\mathrm{Bi}, \mathrm{Fo}, \frac{r}{r_0}\right) = \psi\left(\frac{hr_0}{k}, \frac{\alpha t}{r_0^2}, \frac{r}{r_0}\right) \tag{6.57}$$

Charts sitilar to those for the large plate have also been presented for long solid cylinders in the literature. Figures 6.12 and 6.13, together, give the dimensionless temperatures from Eq. (6.56) in graphical forms. Thus, the transient temperature variations in a long solid cylinder which is initially at a uniform temperature T_i and suddenly exposed (at $t = 0$) to convective cooling or heating can be determined by the same procedure as in the large plate case.

The total heat loss from a length L of the cylinder over a time interval $(0, t)$ can be evaluated as

$$Q = -2\pi r_0 Lk\int_0^t \frac{\partial T(r_0, t')}{\partial r}\,dt' \tag{6.58}$$

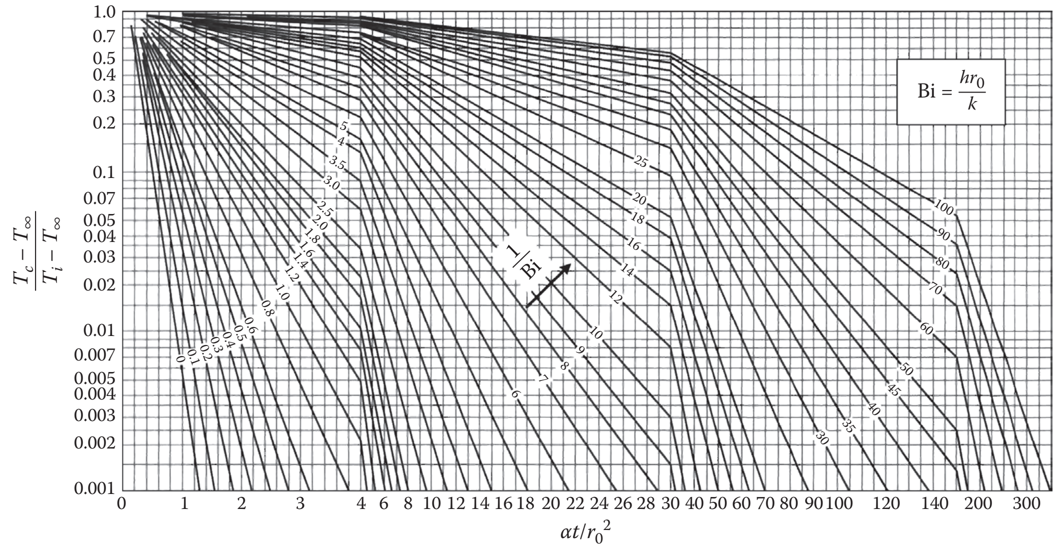

FIGURE 6.12
Temperature–time history at centerline of infinitely long cylinder of radius r_0 [7].

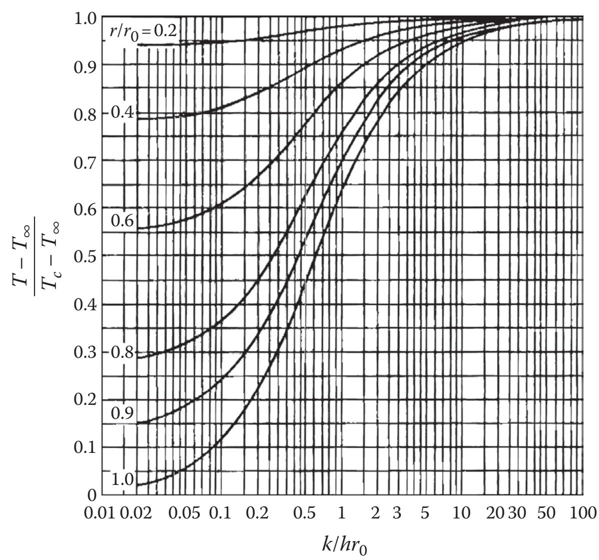

FIGURE 6.13
Position-correction chart for infinitely long cylinder of radius r_0 [7].

Substituting Eq. (6.55) into Eq. (6.58), we obtain

$$\frac{Q}{Q_i} = 4\sum_{n=1}^{\infty} \frac{J_1^2(\gamma_n)}{J_0^2(\gamma_n) + J_1^2(\gamma_n)} \left(1 - e^{-\gamma_n^2 \mathrm{Fo}}\right) \tag{6.59a}$$

where $Q_i = \pi r_0^2 L\rho c(T_i - T_\infty)$ is the initial energy content of the cylinder relative to the surrounding fluid temperature T_∞. Making use of the relation (6.50), Eq. (6.59*a*) can also be written as

$$\frac{Q}{Q_i} = 4\sum_{n=1}^{\infty} \frac{\mathrm{Bi}^2}{\gamma_n^2\left(\gamma_n^2 + \mathrm{Bi}^2\right)} \left(1 - e^{-\gamma_n^2 \mathrm{Fo}}\right) \tag{6.59b}$$

It is obvious from Eqs. (6.50) and (6.59*b*) that

$$\frac{Q}{Q_i} = \phi(\mathrm{Bi}, \mathrm{Fo}) = \phi\left(\frac{hr_0}{k}, \frac{\alpha t}{r_0^2}\right) \tag{6.60}$$

In Fig. 6.14, the dimensionless heat loss Q/Q_i is plotted as a function of $\mathrm{Bi}^2\mathrm{Fo} = h^2\alpha t/k^2$ for several values of Bi.

Example 6.3

A solid cylinder made of steel, 20 cm in diameter and 5 m in length, is initially at 500°C. It is suddenly immersed in an oil bath maintained at 20°C. The properties of steel are: ρ = 7700 kg/m^3, c = 500 J/(kg·K), and k = 41 W/(m·K). Assuming a heat transfer coefficient of 1200 W/(m^2·K), calculate, after the cylinder has been exposed to the cooling process for 12 min, (*a*) the temperature at the centerline, and (*b*) the heat lost per unit length of the cylinder.

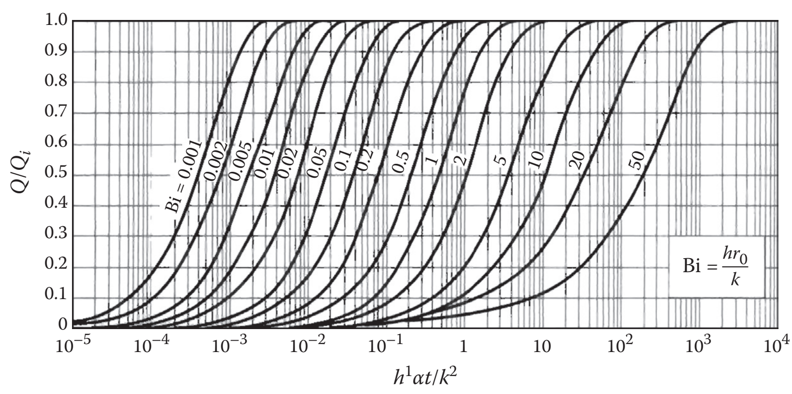

FIGURE 6.14
Dimensionless heat loss Q/Q_i from infinitely long cylinder of radius r_0 [6].

SOLUTION

From the given data, we have

$$\alpha = \frac{k}{\rho c} = \frac{41}{7700 \times 500} = 1.065 \times 10^{-5}\,\mathrm{m^2/s}$$

$$\mathrm{Fo} = \frac{1.065 \times 10^{-5} \times 12 \times 60}{(0.1)^2} = 0.767$$

$$\mathrm{Bi} = \frac{1200 \times 0.1}{41} = 2.93$$

a) For 1/Bi = 0.341 and Fo = 0.767, from Fig. 6.12 we get

$$\frac{T_c - T_\infty}{T_i - T_\infty} \cong 0.12 \Rightarrow \quad T_c \cong 78°\mathrm{C}$$

b) We calculate the heat lost from the cylinder by the use of Fig. 6.14. Since

$$\frac{h^2 \alpha t}{k^2} = \frac{(1200)^2 \times 1.065 \times 10^{-5} \times 12 \times 60}{(41)^2} = 6.57$$

from Fig. 6.14, for Bi = 2.93, we get $Q/Q_i \cong 0.85$. Thus, for unit length

$$\begin{aligned} Q_i &= \rho c \pi r_0^2 (T_i - T_\infty) = 7700 \times 500 \times \pi \times (0.1)^2 \times (500 - 20) \\ &= 5.805 \times 10^7\,\mathrm{J} \end{aligned}$$

Thus, the heat lost per unit length is

$$Q = 0.85 Q_i = 0.85 \times 5.805 \times 10^7 = 4.934 \times 10^7\,\mathrm{J}$$

6.3.3 Cooling (or Heating) of a Solid Sphere

For heat conduction problems posed in spherical coordinate systems which involve spherical symmetry (so that the derivatives of the temperature with respect to the ϕ and θ variables vanish), the general heat conduction equation, in the absence of internal energy generation and for constant thermophysical properties, reduces to the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial T}{\partial t} \tag{6.61}$$

This diffusion equation can be transformed to the Cartesian form by redefining the dependent variable as

$$U(r,t) = rT(r,t) \tag{6.62}$$

By substituting $T(r, t)$ from Eq. (6.62) into Eq. (6.61), it can be shown that U satisfies

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \tag{6.63}$$

Let us now consider the cooling (or heating) of a solid sphere of radius r_0, for times $t \geq 0$, in a medium maintained at a constant temperature T_∞ with a constant heat transfer coefficient h. Assume that the sphere under consideration has an initial temperature distribution given by $f(r)$ at $t = 0$. The formulation of the problem for the unsteady-state temperature distribution in terms of $\theta(r, t) = T(r, t) - T_\infty$ is then given by

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{6.64a}$$

$$\theta(r,0) = f(r) - T_\infty \tag{6.64b}$$

$$\theta(0,t) = \mathit{finite}, \qquad k\frac{\partial \theta(r_0,t)}{\partial r} + h\theta(r_0,t) = 0 \tag{6.64c,d}$$

In terms of $U(r, t) = r\theta(r, t)$, this problem becomes

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \tag{6.65a}$$

$$U(r,0) = r[f(r) - T_\infty] \tag{6.65b}$$

$$U(0,t) = 0, \qquad \frac{\partial U(r_0,t)}{\partial r} = \left[\frac{1}{r_0} - \frac{h}{k}\right]U(r_0,t) \tag{6.65c,d}$$

Since the boundary conditions (6.65*c*,*d*) are homogeneous, the assumption of the existence of a product solution of the form $U(r, t) = R(r)\,\Gamma(t)$ yields

$$U(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n^2 t}\sin\lambda_n r \tag{6.66}$$

where the characteristic values λ_n are the positive roots of

$$\tan \lambda r_0 = \frac{\lambda r_0}{\text{Bi} - 1}, \quad \text{Bi} = \frac{hr_0}{k} \tag{6.67}$$

which is a transcendental equation obtained from the application of the boundary condition (6.65*d*). The characteristic-value equation (6.67) can be solved numerically to obtain the characteristic values λ_n. The use of the initial condition (6.65*b*) yields

$$r[f(r) - T_\infty] = \sum_{n=1}^{\infty} A_n \sin \lambda_n r \tag{6.68}$$

where the expansion coefficients A_n, from Table 4.1, are given by

$$A_n = \frac{2\lambda_n \int_0^r r[f(r) - T_\infty] \sin \lambda_n r\, dr}{\lambda_n r_0 - \sin \lambda_n r_0 \cos \lambda_n r_0} \tag{6.69}$$

Hence, the solution for $\theta(r, t)$ becomes

$$\theta(r,t) = \frac{1}{r} \sum_{n=1}^{\infty} A_n e^{-\alpha \lambda_n^2 t} \sin \lambda_n r \tag{6.70}$$

with A_n given by Eq. (6.69).

When $f(r) = T_i$ = constant, the expansion coefficients A_n are given by

$$A_n = 2(T_i - T_\infty) \frac{\sin \lambda_n r_0 - \lambda_n r_0 \cos \lambda_n r_0}{\lambda_n(\lambda_n r_0 - \sin \lambda_n r_0 \cos \lambda_n r_0)} \tag{6.71}$$

Then, the transient temperature distribution in the sphere becomes

$$\frac{\theta(r,t)}{\theta_i} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = \frac{2}{r} \sum_{n=1}^{\infty} \frac{\sin \lambda_n r_0 - \lambda_n r_0 \cos \lambda_n r_0}{\lambda_n r_0 - \sin \lambda_n r_0 \cos \lambda_n r_0} \frac{\sin \lambda_n r}{\lambda_n} e^{-\alpha \lambda_n^2 t} \tag{6.72a}$$

which can be rewritten as

$$\frac{\theta(r,t)}{\theta_i} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = 2\frac{r_0}{r} \sum_{n=1}^{\infty} \frac{\sin \gamma_n - \gamma_n \cos \gamma_n}{\gamma_n[\gamma_n - \sin \gamma_n \cos \gamma_n]} \sin\left(\gamma_n \frac{r}{r_0}\right) e^{-\gamma_n^2 \text{Fo}} \tag{6.72b}$$

where $\gamma_n = \lambda_n r_0$ and $\text{Fo} = \alpha t / r_0^2$. As in the previous cases, Eq. (6.72*b*) can be represented as

$$\frac{T(r,t) - T_\infty}{T_i - T_\infty} = \psi\left(\text{Bi}, \text{Fo}, \frac{r}{r_0}\right) = \psi\left(\frac{hr_0}{k}, \frac{\alpha t}{r_0^2}, \frac{r}{r_0}\right) \tag{6.73}$$

Charts similar to those for the large plate and the long solid cylinder have also been presented in the literature for the dimensionless temperature distribution in a solid sphere from Eq. (6.72*b*). These charts are given here in Figs. 6.15 and 6.16.

Next, the total heat loss Q from the sphere over the time interval (0, t) can be shown to be

$$\frac{Q}{Q_i} = 6\sum_{n=1}^{\infty} \frac{1}{\gamma_n^3} \frac{\sin\gamma_n - \gamma_n \cos\gamma_n}{\gamma_n - \sin\gamma_n + \cos\gamma_n}\left(1 - e^{-\alpha\gamma_n^2 \mathrm{Fo}}\right) \tag{6.74}$$

where $Q_i = \frac{4}{3}\pi r_0^3 \rho c(T_i - T_\infty)$. An examination of Eq. (6.74). with Eq. (6.67). reveals that

$$\frac{Q}{Q_i} = \phi(\mathrm{Bi}, \mathrm{Fo}) = \phi\left(\frac{hr_0}{k}, \frac{\alpha t}{r_0^2}\right) \tag{6.75}$$

Figure 6.17 gives the dimensionless heat loss Q/Q_i as a function of $\mathrm{Bi}^2\mathrm{Fo} = h^2\alpha t/k^2$ for several values of Bi.

Example 6.4

A solid sphere made of steel, 3 in. in diameter, is suddenly immersed, for hardening purposes, in a water bath maintained at a uniform temperature of 35°C after it has been initially heated to 900°C in a furnace. How long will it take for the surface of the sphere to cool down to 200°C? The heat transfer coefficient h is 60 W/(m²·K) and the properties of the steel are ρ = 8100 kg/m³, c = 465 J/(kg·K), and k = 44 W/(m·K).

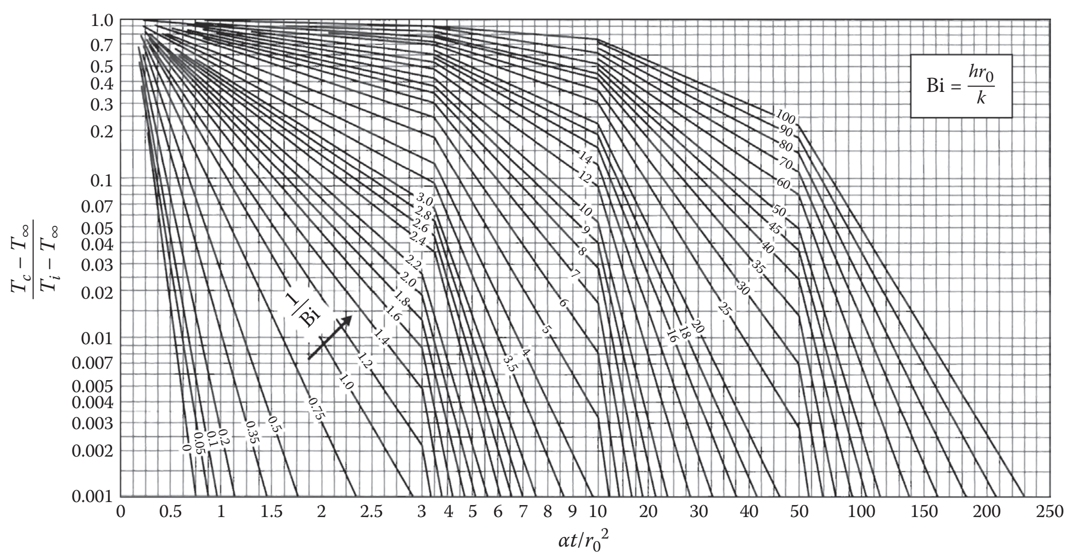

FIGURE 6.15
Temperature–time history at the center of a solid sphere of radius r_0 [7].

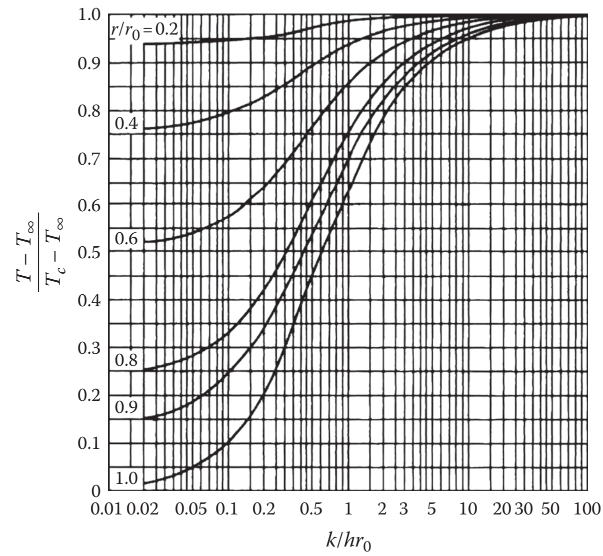

FIGURE 6.16
Temperature as a function of center temperature for a sphere of radius r_0 [7].

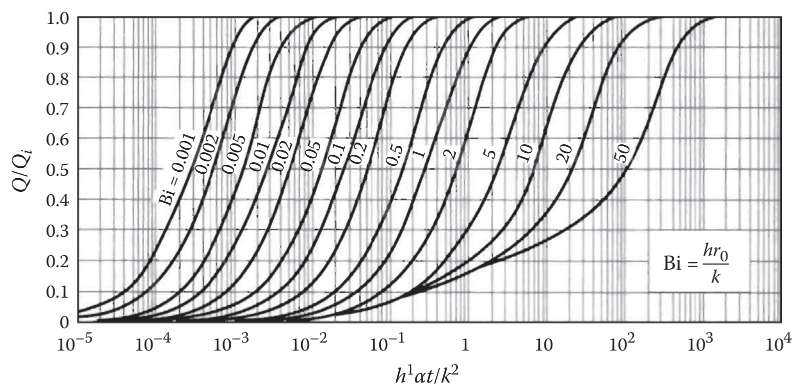

FIGURE 6.17
Dimensionless heat loss Q/Q_i from a solid sphere of radius r_0 [6].

SOLUTION

From the given data, we have

$$\alpha = \frac{k}{\rho c} = \frac{44}{8100 \times 465} = 1.168 \times 10^{-5}\,\text{m}^2/\text{s}$$

$$\frac{1}{\text{Bi}} = \frac{k}{hr_0} = \frac{44 \times 100}{60 \times 1.5 \times 2.54} = 19.25$$

Thus, at $r/r_0 = 1$, with $1/\text{Bi} = 19.25$, from Fig. 6.16 we obtain

$$\frac{T_s - T_\infty}{T_c - T_\infty} \cong 0.965 \quad \Rightarrow \quad T_c - T_\infty \cong 171°\text{C}$$

and the dimensionless temperature ratio at $r/r_0 = 0$ is then given by

$$\frac{T_c - T_\infty}{T_i - T_\infty} = \frac{\theta_c}{\theta_i} = \frac{171}{865} \cong 0.198$$

With $\theta_c/\theta_i = 0.198$ and $1/\text{Bi} = 19.25$, from Fig. 6.15 we now get

$$\alpha t / r_0^2 \cong 11$$

Thus, the time required for the surface of the sphere to cool to 200°C is

$$t = \frac{11 \times (1.5 \times 2.54 \times 10^{-2})^2}{1.168 \times 10^{-5} \times 3600} = 0.38\,\text{h} = 22.8\text{ min}$$

6.4 Multidimensional Systems

A two- or three-dimensional unsteady-state problems can be solved by the method of separation of variables in the same way as the one-dimensional problems if such problems are linear and consist of a homogeneous heat conduction equation together with homogeneous boundary conditions. Nonhomogeneities both in the heat conduction equation and the boundary conditions can be handled by separating the problem into simpler ones by the use of the principle of superposition. The results, in such cases, will be in the form of double or triple series. Under some special conditions, however, solutions of two- and three-dimensional unsteady-state heat conduction problems may be obtained by a simple product superposition of the solutions of certain one-dimensional problems.

6.4.1 Cooling (or Heating) of a Long Rectangular Bar

Consider the cooling (or heating) of the long rectangular bar shown in Fig. 6.18. The bar has a thickness $2a$ in the x direction and $2b$ in the y direction. It is initially at a uniform temperature T_i and, suddenly at $t = 0$, is placed in a fluid of constant temperature T_∞, which remains at this value during the whole cooling (or heating) period. Let the heat transfer coefficient be the same constant h on all surfaces.

Assuming constant thermophysical properties (k, ρ, c), the problem can be formulated in terms of the temperature function $\theta(x, y, t) = T(x, y, t) - T_\infty$, as follows. The differential equation to be solved is

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{6.76}$$

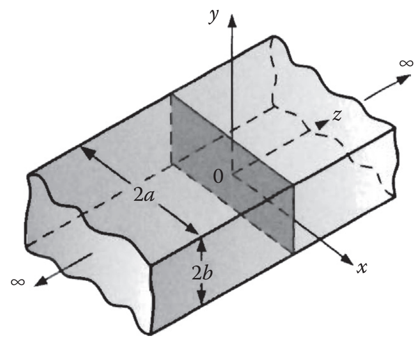

FIGURE 6.18
Long rectangular bar.

with the initial condition

$$\theta(x,y,0) = T_i - T_\infty = \theta_i \tag{6.77a}$$

and the boundary conditions

$$\frac{\partial\theta(0,y,t)}{\partial x} = 0, \qquad \frac{\partial\theta(a,y,t)}{\partial x} = -\frac{h}{k}\theta(a,y,t) \tag{6.77b,c}$$

$$\frac{\partial\theta(x,0,t)}{\partial y} = 0, \qquad \frac{\partial\theta(x,b,t)}{\partial y} = -\frac{h}{k}\theta(x,b,t) \tag{6.77d,e}$$

With these initial and boundary conditions it can be shown that the solution for the temperature distribution is given by

$$\begin{aligned}\frac{\theta(x,y,t)}{\theta_i} = \frac{T(x,y,t) - T_\infty}{T_i - T_\infty} &= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-\alpha\left(\lambda_n^2 + \beta_m^2\right)t} \\ &\times \frac{(\sin\lambda_n a\cos\lambda_n x)(\sin\beta_m b\cos\beta_m y)}{(\lambda_n a + \sin\lambda_n a\cos\lambda_n a)(\beta_m b + \sin\beta_m b\cos\beta_m b)}\end{aligned} \tag{6.78}$$

where λ_n and β_n are, respectively, the positive roots of

$$\cot\lambda_a = \frac{\lambda k}{h} \quad \text{and} \quad \cot\beta b = \frac{\beta k}{h} \tag{6.79a,b}$$

Here we note that it is possible to rewrite the solution (6.78) as

$$\frac{\theta(x,y,t)}{\theta_i}=2\sum_{n=1}^{\infty}\frac{\sin\lambda_n a\cos\lambda_n x}{\lambda_n a+\sin\lambda_n a\cos\lambda_n a}e^{-\alpha\lambda_n^2 t}\times 2\sum_{m=1}^{\infty}\frac{\sin\beta_m b\cos\beta_m y}{\beta_m b+\sin\beta_m b\cos\beta_m b}e^{-\alpha\beta_m^2 t} \tag{6.80}$$

When Eq. (6.80) is compared with Eq. (6.33), it is seen that

$$\left[\frac{\theta(x,y,t)}{\theta_i}\right]_{2a\times 2b\,\text{bar}}=\left[\frac{\theta(x,t)}{\theta_i}\right]_{2a\,\text{plate}}\times\left[\frac{\theta(y,t)}{\theta_i}\right]_{2b\,\text{plate}} \tag{6.81a}$$

or

$$\left[\frac{T(x,y,t)-T_\infty}{T_i-T_\infty}\right]_{2a\times 2b\,\text{bar}}=\left[\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right]_{2a\,\text{plate}}\times\left[\frac{T(y,t)-T_\infty}{T_i-T_\infty}\right]_{2b\,\text{plate}} \tag{6.82}$$

Thus, the dimensionless temperature distribution in the large rectangular bar shown in Fig. 6.18 is the product of the dimensionless temperature distributions in two large plates whose intersection forms the bar in question, as illustrated in Fig. 6.19.

The Heisler charts, Figs. 6.8 and 6.9, can therefore be used to find the temperature at any location in the two-dimensional bar. It should be remembered that the length L used in the formulas and charts for the flat plate is the half thickness of the plate, so that L will have to be replaced by a and b before the results of the previous section are applied to the bar problem discussed in this section. The Heisler charts also will be applicable even if the heat transfer coefficient h is different on the x and y surfaces but the same on each pair of parallel surfaces, that is if h in Eq. (6.77c), say, is h_1 and h in Eq. (6.77e) is h_2.

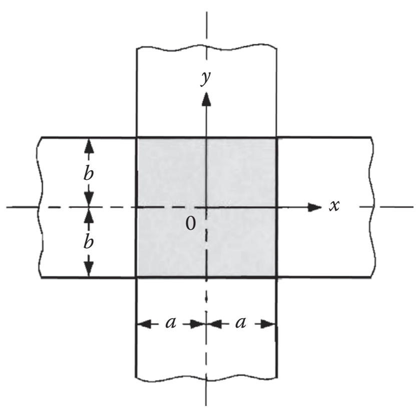

FIGURE 6.19
Rectangular bar formed by intersection of two infinite plates.

Although we have demonstrated the validity of the product solution after we obtained the solution (6.78) for the bar, this can also be shown directly from the formulation of the problem. Thus, for example, if we reformulate the problem in terms of $\phi = \theta/\theta_i$, we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\phi}{\partial t} \tag{6.83}$$

$$\phi(x, y, 0) = 1 \tag{6.84a}$$

$$\frac{\partial\phi(0, y, t)}{\partial x} = 0, \qquad \frac{\partial\phi(a, y, t)}{\partial x} = -\frac{h}{k}\phi(a, y, t) \tag{6.84b,c}$$

$$\frac{\partial\theta(x, 0, t)}{\partial y} = 0, \qquad \frac{\partial\phi(x, b, t)}{\partial y} = -\frac{h}{k}\phi(x, b, t) \tag{6.84d,e}$$

Now, letting

$$\phi(x, y, t) = \phi_1(x, t)\phi_2(y, t) \tag{6.85}$$

it can readily be shown that $\phi_1(y, t)$ is the solution of

$$\frac{\partial^2\phi_1}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\phi_1}{\partial t} \tag{6.86a}$$

$$\phi_1(x, 0) = 1 \tag{6.86b}$$

$$\frac{\partial\phi_1(0, t)}{\partial x} = 0, \qquad \frac{\partial\phi_1(a, t)}{\partial x} = -\frac{h}{k}\phi_1(a, t) \tag{6.86c,d}$$

and $\phi_2(y, t)$ is the solution of

$$\frac{\partial^2\phi_2}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\phi_2}{\partial t} \tag{6.87a}$$

$$\phi_2(y, 0) = 1 \tag{6.87b}$$

$$\frac{\partial\phi_2(0, t)}{\partial y} = 0, \qquad \frac{\partial\phi_2(b, t)}{\partial y} = -\frac{h}{k}\phi_2(b, t) \tag{6.87c,d}$$

where both are one-dimensional plate problems.

This method of obtaining the solution of a multidimensional unsteady-state problem in terms of one-dimensional solutions is called the *Newman method*. It should, however, be used with caution as it is applicable only to certain special cases. In general, to apply the method, the following must hold:

a. The problem has to be linear, consisting of a homogeneous heat conduction equation together with homogeneous boundary conditions.

b. The initial temperature distribution has to be uniform or be in a form that can be factored into a product of terms, each involving only one of the space variables.

c. If one (or more) of the one-dimensional problems is a large plate case and, therefore, the Heisler chart (Fig. 6.8) is to be used, then the problem has to have thermal symmetry in the respective direction; that is, the heat transfer coefficient should be the same on each pair of parallel surfaces.

Other examples of the Newman method are given in the following sections.

6.4.2 Cooling (or Heating) of a Parallelepiped and a Finite Cylinder

The product superposition, just illustrated for a two-dimensional transient heat conduction in a rectangular bar, can be extended to other configurations as well. Consider, for example, the rectangular parallelepiped of dimensions $2a \times 2b \times 2c$ shown in Fig. 6.20*a*, which is initially at a uniform temperature T_i and suddenly at $t = 0$ is cooled (or heated) on all sides by a fluid maintained at a constant temperature T_∞ with a constant heat transfer coefficient h. As in the preceding case, it can easily be shown that the dimensionless temperature at any point (x, y, z) in the rectangular parallelepiped is given by

$$\left[\frac{T(x,y,z,t)-T_\infty}{T_i-T_\infty}\right]_{\substack{\text{rectangular}\\ \text{parallelepiped}}} = \left[\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right]_{2a\,\text{plate}} \times \left[\frac{T(y,t)-T_\infty}{T_i-T_\infty}\right]_{2b\,\text{plate}} \times \left[\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right]_{2c\,\text{plate}} \tag{6.88}$$

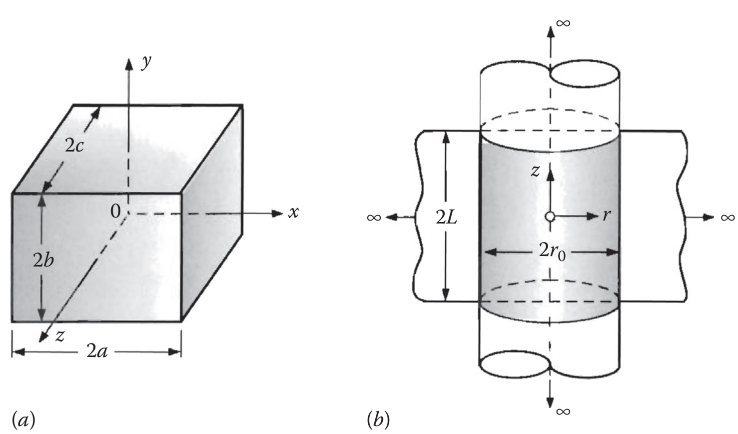

FIGURE 6.20
(*a*) Rectangular parallelepiped, and (*b*) solid cylinder of height 2*L*.

Similarly, as illustrated in Fig. 6.20*b*, the solution for a short cylinder of radius r_0 and height $2L$ can be shown to be

$$\left[\frac{T(r,z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{finite cylinder}} = \left[\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right]_{2L\text{ plate}} \times \left[\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right]_{\text{infinite cylinder}} \tag{6.89}$$

Example 6.5

A solid cylinder made of brass, 5 cm in diameter and 5 cm long, is initially at a uniform temperature of 316°C. It is immersed in a tank of water maintained at 21°C. Calculate the temperature, after it cools for 15 s, at the center and at a radial position of 2 cm and a distance of 2 cm from one end of the cylinder. The heat transfer coefficient h can be taken as 2000 W/(m²·K). For brass, ρ = 8520 kg/m³, c = 385 J/(kg·K), and k = 111 W/(m·K).

SOLUTION

The two-dimensional temperature distribution in the cylinder is given by Eq. (6.88): that is, the Heisler charts for an infinite plate and an infinite cylinder are to be used together. From the given data we obtain

$$\alpha = \frac{k}{\rho c} = \frac{111}{8520\times 385} = 3.38\times 10^{-5}\,\text{m}^2/\text{s}$$

For both a flat plate of thickness 5 cm and a long cylinder of radius 2.5 cm we have

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{3.38\times 10^{-5}\times 15}{(0.025)^2} = 0.811$$

$$\frac{1}{\text{Bi}} = \frac{h}{kL} = \frac{111}{2000\times 0.025} = 2.22$$

With 1/Bi = 2.22 and Fo = 0.811, from Fig. 6.8 we get

$$\left(\frac{T_c-T_\infty}{T_i-T_\infty}\right)_{2L\text{ plate}} \cong 0.8$$

and Fig. 6.12 gives

$$\left(\frac{T_c-T_\infty}{T_i-T_\infty}\right)_{2r_0\text{ cylinder}} \cong 0.6$$

Hence, the dimensionless temperature at the center of the finite cylinder is

$$\left(\frac{T_c-T_\infty}{T_i-T_\infty}\right)_{\text{finite cylinder}} \cong 0.8\times 0.6 \cong 0.48$$

which yields $T_c \cong 163$°C.

The temperature at $r/r_0 = 0.8$ and $z/L = 0.2$: From Fig. 6.9, with $z/L = 0.2$ and $1/\text{Bi} = 2.22$, we have

$$\left(\frac{T_z - T_\infty}{T_c - T_\infty}\right)_{2L\,\text{plate}} \cong 0.985$$

and from Fig. 6.13, for $r/r_0 = 0.8$ and $1/\text{Bi} = 2.22$ we get

$$\left(\frac{T_r - T_\infty}{T_c - T_\infty}\right)_{\text{infinite cylinder}} \cong 0.87$$

The dimensionless temperature at $r/r_0 = 0.8$ and $z/L = 0.2$ is then given by

$$\left(\frac{T - T_\infty}{T_c - T_\infty}\right)_{\text{finite cylinder}} \cong 0.985 \times 0.87 \cong 0.857$$

which yields $T \cong 143°\text{C}$.

6.4.3 Semi-Infinite Body

In practice, a number of cases can be idealized as semi-infinite solids in which, at a given instant, there will be regions which are still at the initial temperature after a temperature change occurs on one of its surfaces. A thick plate, for example, can be treated as a semi-infinite solid if the transient temperature response of the plate is to be investigated for short periods of time after a change of temperature of one of its surfaces (such as during a welding process). Another typical example is the earth's surface. After a change of the surface temperature (following the weather conditions), there will always be some regions below the earth's surface where the temperature is not affected by the change at the surface.

Consider a semi-infinite solid as shown in Fig. 6.21, which is initially at a uniform temperature T_i. The surface at $x = 0$ is suddenly exposed at $t = 0$ to a fluid maintained at a constant temperature T_∞ with a constant heat transfer coefficient h. Assuming constant

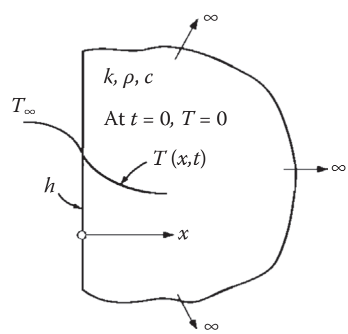

FIGURE 6.21
A semi-infinite solid.

thermophysical properties (k, ρ, c), the problem can be formulated in terms of $\theta(x, t) = T(x, t) - T_\infty$, as follows:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{6.90}$$

$$\theta(x,0) = T_i - T_\infty = \theta_i \tag{6.91a}$$

$$\frac{\partial\theta(0,t)}{\partial x} = \frac{h}{k}\theta(0,t), \qquad \theta(\infty,t) = \theta_i \tag{6.91b,c}$$

This problem cannot be solved readily by the application of the method of separation of variables. It can, however, be solved easily by the use of Fourier or Laplace transforms that we shall discuss in Chapters 7 and 8, and the solution is given by (see Problem 8.6).

$$\begin{aligned}\frac{\theta(x,t)}{\theta_i} &= \frac{T(x,t) - T_\infty}{T_i - T_\infty} \\ &= \operatorname{erf}\frac{x}{2\sqrt{\alpha t}} + \exp\left[\frac{hx}{k} + \left(\frac{h}{k}\right)^2 \alpha t\right]\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right)\end{aligned} \tag{6.92}$$

where the *error/unction* erf and the *complementary error function* erfc are defined as

$$\operatorname{erf}(x) = 1 - \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}\, d\beta \tag{6.93}$$

The values of erf(x) are available in the literature. A short table of the values of erf(x) is given in Appendix C.

If $h \to \infty$, then the surface temperature $T_w \to T_\infty$ and Eq. (6.92) reduces to

$$\frac{T(x,t) - T_w}{T_i - T_w} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{6.94}$$

In Section 8.8, this solution is also obtained by Laplace transforms. Figure 6.22 gives the solution (6.93) in graphical form. Similarly, the solution (6.92) is given in graphical form in Figs. 6.23*a* and *b*.

Example 6.6

On a cold winter day, snow was removed from parts of a city. Just after the removal of snow, the air temperature dropped to – 15°C and stayed at this value for 2 days. The ground near the earth's surface was initially at 4°C. The ground is made of coarse gravel with the properties α = 0.0005 m²/h and k = 0.52 W/(m·K). Calculate the earth's

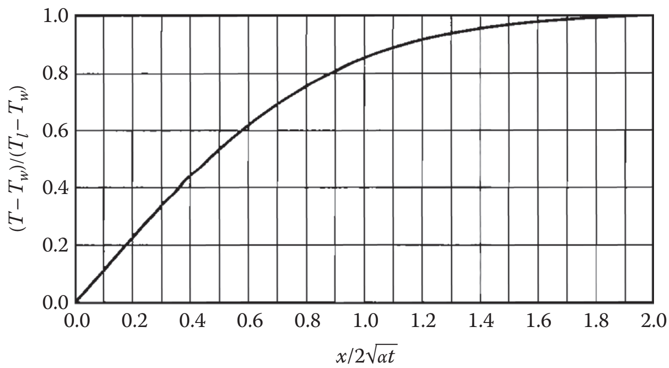

FIGURE 6.22
Temperature-time history in a semi-infinite solid ($h \to \infty$) [1].

temperature at a depth of 25 cm at the end of the cold period. Assume a heat transfer coefficient of 6W/(m^2·K) from the surface.

SOLUTION

We may use either Eq. (6.92) or Fig. 6.23 for the solution of this problem. Here we shall use Eq. (6.91). From the given data of the problem we have

$$\frac{hx}{k} = \frac{6 \times 0.25}{0.52} = 2.885$$

$$\frac{x}{2\sqrt{\alpha t}} = \frac{0.25}{2 \times \sqrt{0.0005 \times 48}} = 0.807$$

$$\frac{hx}{k} + \left(\frac{h}{k}\right)^2 \alpha t = \left(\frac{6}{0.52}\right)^2 \times 0.0005 \times 48 + 2.885 = 6.08$$

$$\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t} = 0.807 + \frac{6}{0.52}(0.0005 \times 48)^{1/2} = 2.595$$

Substituting these values into Eq. (6.92), we get

$$\frac{T + 15}{4 + 15} = \mathrm{erf}\,(0.807) + e^{6.08}[1 - \mathrm{erf}(2.595)]$$

The values of the error function can be taken from Appendix C (or read from Fig. 6.22). Thus,

$$\frac{T + 15}{4 + 15} = 0.851 \quad \Rightarrow \quad T = 1.17°\mathrm{C}$$

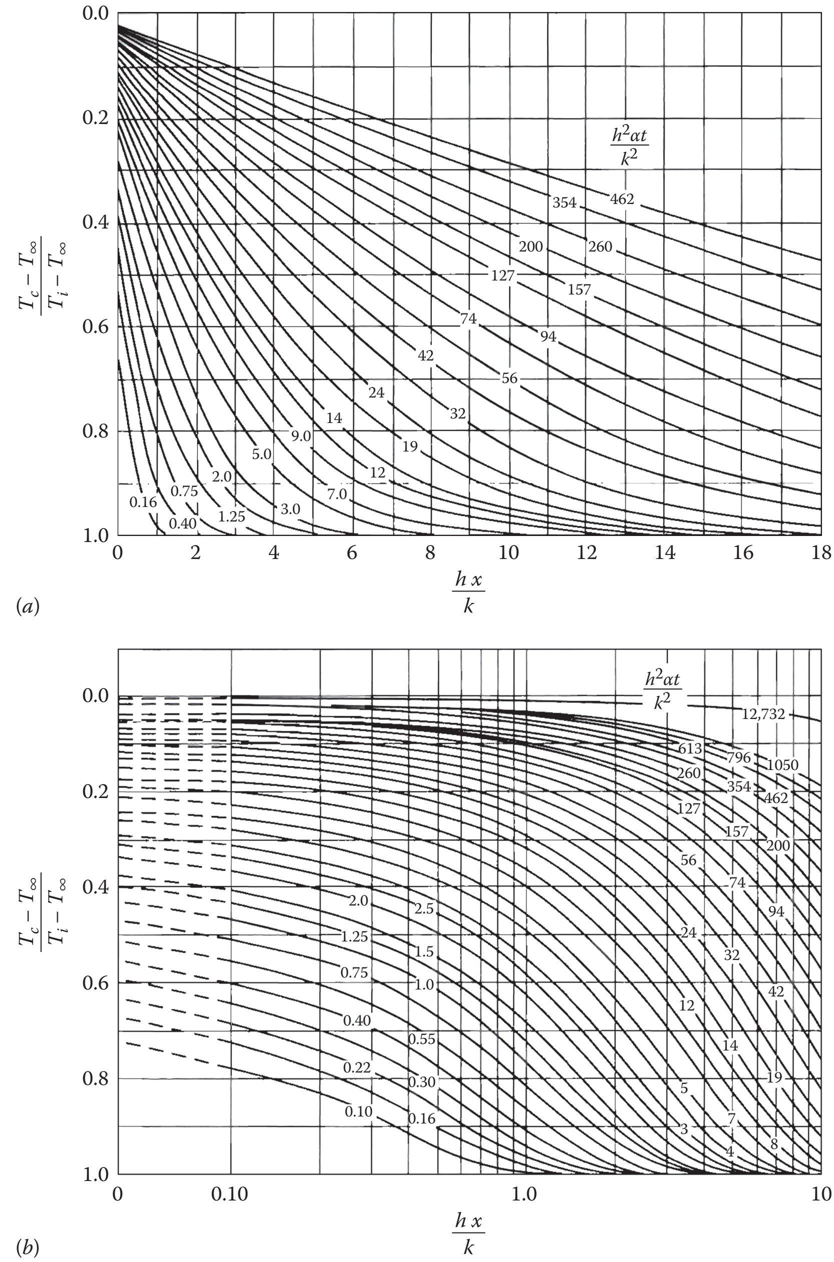

FIGURE 6.23
Temperature–time history in a semi-infinite solid (h = finite) [3]. (*a*) Linear plot. (*b*) Semi-log plot.

6.4.4 Cooling (or Heating) of Semi-Infinite Bars, Cylinders, and Plates

Consider the cooling (or heating) of a semi-infinite rectangular bar, a semi-infinite cylinder, and a semi-infinite plate, as shown in Fig. 6.24, in a surrounding fluid maintained at a constant temperature T_∞. Let the heat transfer coefficient at the base and the other surfaces be the same or satisfy the Newman condition. At the beginning of the cooling

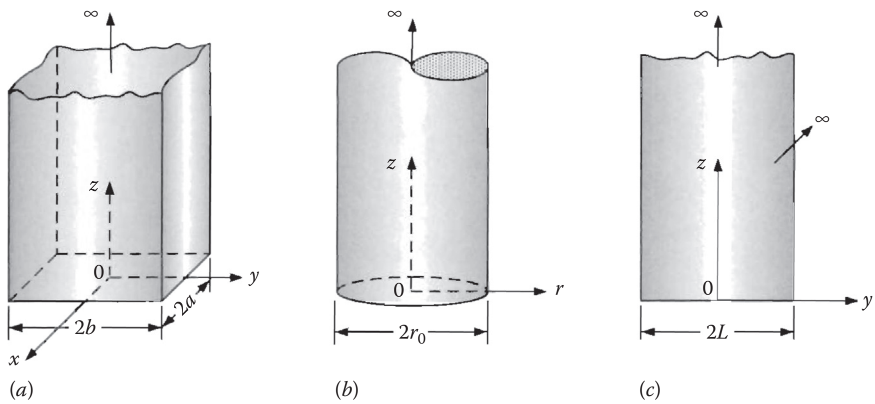

FIGURE 6.24
Semi-infinite rectangular bar (*a*), cylinder (*b*), and plate (*c*).

(or heating) process at $t = 0$, all points in these systems are at the same temperature T_i. The temperature distributions for these transient heat conduction problems can be obtained in dimensionless form as products of the dimensionless temperature distributions for the previous appropriate one-dimensional solids, whose intersection forms the semi-infinite solid in question.

For example, the dimensionless temperature distribution in the semi-infinite rectangular bar of Fig. 6.24*a* can be shown to be

$$\left[\frac{T(x,y,z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{semi-infinite bar}} = \left[\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right]_{2a\text{ plate}} \times \left[\frac{T(y,t)-T_\infty}{T_i-T_\infty}\right]_{2b\text{ plate}} \times \left[\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{semi-infinite solid}} \tag{6.95a}$$

The dimensionless temperature in the semi-infinite cylinder of Fig. 6.24*b* is

$$\left[\frac{T(r,z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{semi-infinite cylinder}} = \left[\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{semi-infinite solid}} \times \left[\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right]_{r_0\text{ cylinder}} \tag{6.95b}$$

and the dimensionless temperature in the semi-infinite plate of Fig. 6.24*c* is

$$\left[\frac{T(y,z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{semi-infinite plate}} = \left[\frac{T(z,t)-T_\infty}{T_i-T_\infty}\right]_{\text{semi-infinite plate}} \times \left[\frac{T(y,t)-T_\infty}{T_i-T_\infty}\right]_{2L\text{ plate}} \tag{6.95c}$$

Example 6.7

A semi-infinite steel plate of thickness 5 cm is initially at 500°C. It is suddenly immersed in an oil bath maintained at 20°C. The properties of the steel are ρ = 7833 kg/m^3, c = 465 J/(kg·K), and k = 45 W/(m·K). Assuming a heat transfer coefficient of 1200 W/(m^2·K), calculate the temperature on the axis of the plate, 20 cm from the end, 6 min after the cooling process has started.

SOLUTION

For the solution of this problem we will take the product of an infinite plate and a semi-infinite solid in accordance with Eq. (6.94*c*). From the given data we have

$$\alpha = \frac{k}{\rho c} = \frac{45}{7833 \times 465} = 1.236 \times 10^{-5}\,\text{m}^2/\text{s}$$

For the semi-infinite solid, Fig. 6.23 will be used. The parameters to be used with x = 20 cm are the following:

$$\frac{hz}{k} = \frac{1200 \times 0.20}{45} = 5.33$$

$$\left(\frac{h}{k}\right)^2 \alpha t = \left(\frac{1200}{45}\right)^2 \times 1.236 \times 10^{-5} \times 6 \times 60 = 3.16$$

From Fig. 6.23 we obtain

$$\left[\frac{T(z,t) - T_\infty}{T_i - T_\infty}\right]_{\text{semi-infinite solid}} \cong 0.98$$

For the infinite plate with L = 2.5 cm we have

$$\frac{k}{hL} = \frac{45}{1200 \times 0.025} = 1.5$$

$$\frac{\alpha t}{L^2} = \frac{1.236 \times 10^{-5} \times 6 \times 60}{(0.025)^2} = 7.12$$

From Fig. 6.8 we get

$$\left[\frac{T(y,t) - T_\infty}{T_i - T_\infty}\right]_{2L\ \text{plate}} \cong 0.025$$

Combining the above results according to Eq. (6.94*c*) yields

$$\frac{T - T_\infty}{T_i - T_\infty} \cong 0.98 \times 0.025 = 0.0246$$

which gives $T \cong 32$°C.

6.5 Periodic Surface Temperature Change

In this section we discuss the determination of the temperature distribution in the semi-infinite solid shown in Fig. 6.25, which has a periodically varying surface temperature. Since the thickness of the semi-infinite solid in the x direction is so large, the temperature-time variation at any depth x will depend only on the conditions imposed on the surface at $x = 0$. We will assume that the periodic temperature variations have occurred for a sufficiently long period of time so that, at locations below the surface, succeeding cycles are identical. This is equivalent to neglecting the initial transient heating-up period that would result if a uniformly cold solid were suddenly subjected to a higher but periodically varying surface temperature. The type of conduction just described here is referred to as *quasi-steady conduction*. By this it is meant that the unsteady temperature changes repeat themselves steadily.

In practice, the daily temperature variation of the earth exposed to solar radiation or the variation of temperature in the cylinder walls of internal combustion engines are of this type of problem.

The boundary condition at the surface of the solid at $x = 0$ will be taken to be a sinusoidal surface temperature variation as shown in Fig. 6.25; that is,

$$T(0,t) = T_m + (\Delta T)_0 \sin(2\pi n t) \tag{6.96}$$

where

$$T_m = \frac{T_{max} + T_{min}}{2} \quad \text{and} \quad (\Delta T)_0 = \frac{T_{max} - T_{min}}{2} \tag{6.97a,b}$$

are the mean and amplitude of the surface temperature oscillations, respectively, and n is the frequency of the oscillations. Thus, $1/n$ is the period of the surface temperature oscillations.

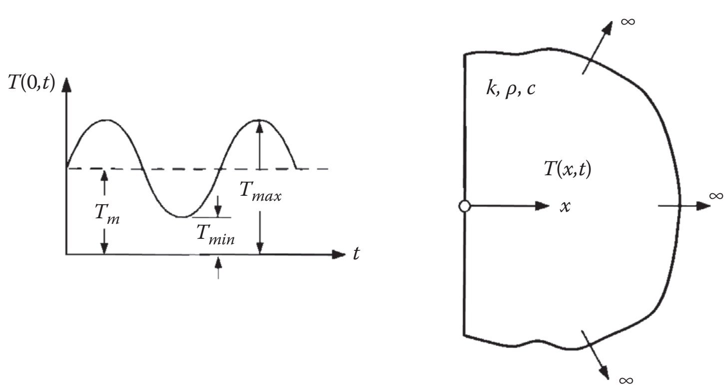

FIGURE 6.25
Semi-infinite solid with a periodic surface temperature variation.

Defining a temperature variable as $\theta(x, t) = T(x, t) - T_m$, the periodic boundary condition (6.95) can be rewritten as

$$\theta(0,t) = (\Delta T)_0 \sin(2\pi nt) \tag{6.98}$$

If the thermophysical properties (k, ρ, c) are assumed to be constant, then $\theta(x, t)$ will satisfy the following one-dimensional diffusion equation:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{6.99}$$

Now, consider an auxiliary problem with the differential equation

$$\frac{\partial^2 \tilde{\theta}}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \tilde{\theta}}{\partial t} \tag{6.100}$$

and the periodic boundary condition

$$\tilde{\theta}(0,t) = (\Delta T)_0 \cos(2\pi nt) \tag{6.101}$$

Let us define a (*complex*) temperature function as

$$\theta_c(x,t) = \tilde{\theta}(x,t) + i\theta(x,t) \tag{6.102}$$

where $i = \sqrt{-1}$. The complex function $\theta_c(x, t)$ can be shown to satisfy the following differential equation and boundary condition:

$$\frac{\partial^2 \theta_c}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta_c}{\partial t} \tag{6.103}$$

$$\theta_c(0,t) = (\Delta T)_0 e^{2i\pi nt} \tag{6.104}$$

Assume a product solution in the form

$$\theta_c(x,t) = X(x)e^{2i\pi nt} \tag{6.105}$$

Substitution of Eq. (6.104) into Eqs. (6.102) and (6.103) yields

$$\frac{d^2X}{dx^2} - 2i\frac{\pi n}{\alpha}X(x) = 0 \tag{6.106}$$

$$X(0) = (\Delta T)_0 \tag{6.107a}$$

Since, as $x \to \infty$, both $\theta \neq \infty$ and $\tilde{\theta} \neq \infty$, we also have

$$X(\infty) \neq \infty \tag{6.107b}$$

Noting that

$$\sqrt{2i\frac{\pi n}{\alpha}} = \pm(1+i)\sqrt{\frac{\pi n}{\alpha}} \tag{6.108}$$

the solution of Eq. (6.105) can be written as

$$X(x) = A_1 e^{-(1+i)\sqrt{\pi n/\alpha}\,x} + A_2 e^{(1+i)\sqrt{\pi n/\alpha}x} \tag{6.109}$$

Use of the condition (6.106*b*) gives $A_2 = 0$. Thus,

$$X(x) = A_1 e^{-(1+i)\sqrt{\pi n/\alpha}x} \tag{6.110}$$

Hence, the product solution (6.104) gives

$$\theta_c(x,t) = A_1 e^{-x\sqrt{\pi n/\alpha}} \exp\left[i\left((2\pi nt - \sqrt{\frac{\pi n}{\alpha}}x\right)\right] \tag{6.111}$$

Since $e^{i\beta} = \cos\beta + i\sin\beta$, Eq. (6.110) can also be written as

$$\theta_c(x,t) = A_1 e^{-x\sqrt{\pi n/\alpha}}\left[\cos\left(2\pi nt - \sqrt{\frac{\pi n}{\alpha}}x\right) + i\sin\left(2\pi nt - \sqrt{\frac{\pi n}{\alpha}}x\right)\right] \tag{6.112}$$

Hence, from the definition (6.101), we see that

$$\theta(x,t) = A_1 e^{-x\sqrt{\pi n/\alpha}} \sin\left(2\pi nt - \sqrt{\frac{\pi n}{\alpha}}x\right) \tag{6.113}$$

Application of the boundary condition (6.97) yields $A_1 = (\Delta T)_0$. Thus,

$$\theta_c(x,t) = (\Delta T)_0 e^{-x\sqrt{\pi n/\alpha}} \sin\left(2\pi nt - \sqrt{\frac{\pi n}{\alpha}}x\right) \tag{6.114}$$

Comparing this result with the imposed variation (6.97) at the surface, we observe that the temperature variation at a depth x from the surface is a sinusoidal variation of the same period, but of an amplitude that decreases exponentially with distance as illustrated in Fig. 6.26. That is, the amplitude at a depth x is

$$(\Delta T)_x = (\Delta T)_0 e^{-x\sqrt{\pi n/\alpha}} \tag{6.115}$$

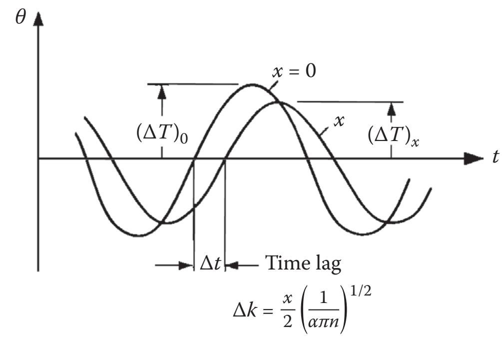

FIGURE 6.26
Temperature variation at the surface and at a depth x in a semi-infinite solid subjected to a periodically varying surface temperature.

Also, the period at a depth x is the same as that at the surface, but lags by a phase difference equal to

$$\Delta t = \frac{x}{2}\left(\frac{1}{\alpha\pi n}\right)^{1/2} \tag{6.116}$$

Figure 6.27 illustrates the temperature distribution within the semi-infinite solid at two different times. From the solution (6.113), the wavelength is given by

$$x_0 = 2\sqrt{\frac{\pi\alpha}{n}} \tag{6.117}$$

From the wave theory we know that the velocity of wave propagation is equal to the wavelength divided by the period. Hence, the propagation velocity of the temperature waves through the semi-infinite solid is given by

$$V(\text{velocity of wave}) = \frac{2(\pi\alpha / n)^{1/2}}{1/n} = 2(\pi\alpha n)^{1/2} \tag{6.118}$$

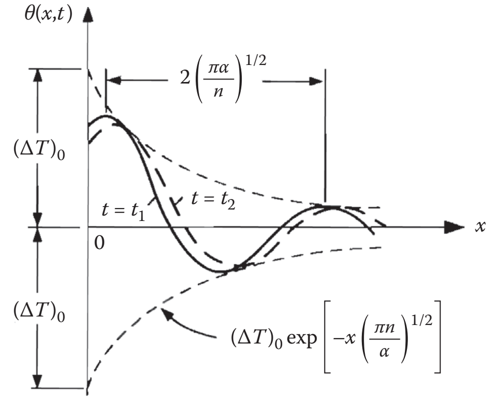

FIGURE 6.27
Penetration of surface temperature oscillations through a semi-infinite solid.

Thus, the wave at t_2 is displaced from the wave at t_1 by a distance

$$(t_2 - t_1)2(\pi\alpha n)^{1/2} \tag{6.119}$$

In Eq. (6.113), the exponential factor has no effect on the phase difference, wavelength, or propagation velocity, but results in a rapid decrease in the amplitude of the waves with increasing depth x.

The instantaneous heat flow rate per unit area across an isothermal surface at a depth x is

$$q'' = -k\frac{\partial T}{\partial x} = -k\frac{\partial \theta}{\partial x} \tag{6.120}$$

Substituting the temperature distribution (6.113) into Eq. (6.119) we obtain

$$q'' = k(\Delta T)_0 e^{-x\sqrt{\pi n/\alpha}}\sqrt{\frac{n\pi}{\alpha}}\left[\cos\left(2\pi nt - x\sqrt{\frac{n\pi}{\alpha}}\right) + \sin\left(2\pi nt - x\sqrt{\frac{n\pi}{\alpha}}\right)\right] \tag{6.121a}$$

which can be rewritten as

$$q'' = k(\Delta T)_0 e^{-x\sqrt{\pi n/\alpha}}\sqrt{\frac{2n\pi}{\alpha}}\sin\left(2\pi nt + \frac{\pi}{4} - x\sqrt{\frac{n\pi}{\alpha}}\right) \tag{6.121b}$$

Thus, the heat flux at a depth x is also a periodic function of time with the same period as the temperature variation. It is seen from Eq. (6.120*b*) that the heat flow per unit area at the surface (i.e., at $x = 0$) is given by

$$q_0'' = k(\Delta T)_0\sqrt{\frac{2n\pi}{\alpha}}\left[\sin\left(2\pi nt + \frac{\pi}{4}\right)\right] \tag{6.122}$$

The phase lag between the temperature and heat flux variations at the surface is, therefore, $1/8n$.

Example 6.8

On cold winter days, the earth's daily surface temperature variation in the city of Ankara is sinusoidal between –8 and 10°C. Calculate the depth below which a water pipe must be placed to prevent it from freezing. Take $\alpha = 0.0025$ m^2/h.

SOLUTION

From the given data we have

$$(\Delta T)_0 = \frac{T_{max} - T_{min}}{2} = \frac{10 - (-8)}{2} = \frac{18}{2} = 9°\text{C}$$

$$T_m = \frac{T_{max} - T_{min}}{2} = \frac{10 - 8}{2} = 1°\text{C}$$

$$n = \frac{1}{24}\,\text{h}^{-1} \quad \Leftrightarrow \quad \sqrt{\frac{\pi n}{\alpha}} = 7.237\,\text{m}^{-1}$$

Since T_m = 1°C, for a water pipe at a depth x not to freeze, the amplitude must be $(\Delta T)_0 \leq$ 1°C. Thus. Eq. (6.114) gives

$$x \geq \frac{\ln[(\Delta T)_0 / (\Delta T)_x]}{\sqrt{\pi n / \alpha}} = \frac{\ln(9/1)}{7.237} = 0.304\,\text{m}$$

$$x \geq 30.4\,\text{cm}$$

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Problems

6.1 Consider a solid body of volume V and surface area A. which is initially at a uniform temperature T_∞. For times $t \geq 0$, while surrounded by a coolant maintained at the temperature T_∞, internal energy is generated in the solid at an exponential decay rate per unit volume according to

$$\dot{q} = \dot{q}_0 e^{-\beta t}$$

where $\dot{q}_0$ and β are two given positive constants. Let the heat transfer coefficient between the solid body and the coolant be constant h. Assuming constant thermophysical properties and neglecting the spatial variation of the temperature, obtain an expression for the temperature of the solid body as a function of time for $t > 0$. What will be the maximum solid temperature and when is it reached?

6.2 Consider a thin-walled hollow metal sphere initially at a uniform temperature T_i. For times $t \geq 0$, the sphere is exposed to a low-speed air stream, whose temperature T_f varies linearly with time t according to

$$T_f(t) = T_i + At$$

where A is a given constant. Let the heat transfer coefficient between the air stream and the sphere be constant h. The inside of the sphere is evacuated so that the internal heat capacity and conductivity can be neglected. Assume constant thermophysical properties.

(*a*) Obtain an expression for the temperature of the sphere as a function of time for $t > 0$.

(*b*) What is the asymptotic solution for the temperature of the sphere for large times?

6.3 A slab extending from $x = 0$ to $x = L$ and infinite in extent in the ν and z directions is initially at a uniform temperature T_i. The surface temperatures at $x = 0$ and $x = L$ are suddenly changed to T_1 and T_2, respectively, at $t = 0$ and are held constant at these values for times $t > 0$. Assuming constant thermophysical properties for the material of the slab, find an expression for the unsteady-state temperature distribution $T(x, t)$ in the slab for $t > 0$.

6.4 The combustion chamber wall of a jet engine may be assumed to be a large flat plate of thickness L. Initially the wall is at the temperature T_∞ of the surrounding air. As a first approximation, assume that, when combustion begins, the combustion gases inside the chamber flow at a high velocity resulting in a very high surface heat transfer coefficient, so that the temperature of the inner face of the wall immediately takes on the flame temperature T_f. Let the heat transfer coefficient h between the wall and the surrounding air be constant. Obtain an expression for the transient temperature distribution in the chamber wall. The thermophysical properties of the wall material may be taken as constant.

6.5 A slab, of thickness L and surface area A, is initially at a uniform temperature T_i. For times $t \geq 0$, the temperature of the surface at $x = L$ is maintained constant at T_w, while the other surface at $x = 0$ is kept perfectly insulated. Obtain

(*a*) an expression for the unsteady temperature distribution $T(x, t)$ in the slab for $t > 0$,

(*b*) an expression for the average temperature of the slab as a function of time.

(*c*) an expression for the instantaneous rate of heat loss from the slab, and

(*d*) an expression for the total heat loss over a period of time t from the start of the transient period.

6.6 A flat plate of thickness L in the x direction and of infinite extent in the y and z directions is initially at a uniform temperature T_i. At time $t = 0$, the temperature of the surface at $x = 0$ is changed to T_w, and subsequently kept at this value for $t > 0$, while the other surface at $x = L$ is maintained at T_i. Assuming constant thermophysical properties, obtain

(*a*) an expression for the unsteady temperature distribution $T(x. t)$ in the plate for $t > 0$,

(*b*) an expression for the average temperature of the plate as a function of time, and

(c) an expression for the heat flux across each surface as a function of time.

6.7 A long solid rod of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. At time $t = 0$, the temperature of the peripheral surface at $r = r_0$ is changed to T_w, and is subsequently maintained constant at this value for $t > 0$. Obtain

(*a*) an expression for the unsteady temperature distribution $T(r, t)$ in the rod,

(*b*) an expression for the average temperature of the rod as a function of time,

(*c*) an expression for the instantaneous rate of heat loss per unit length of the rod, and

(*d*) an expression for the total heat loss per unit length over a period of time t from the start of the transient period.

6.8 A large steel plate, 1.5 cm thick and initially at a uniform temperature of 24°C, is placed in a furnace maintained at 930°C. The mean heat transfer coefficient for combined convection and radiation may be taken as 90 W/(m^2·K). Estimate the time required for the midplane of the plate to reach 540°C and find the corresponding surface temperature.

6.9 The cylindrical combustion chamber of a rocket engine is made of 1-cm-thick stainless steel. The heat transfer coefficient on the gas side is 4000 W/(m^2·K), and the temperature of the combustion gases is 2320°C. The temperature of the engine prior to firing is 28°C. Estimate the minimum possible combustion period if the temperature anywhere in the combustion wall is not to exceed 1100°C. The wall thickness may be assumed small in comparison with the diameter of the combustion chamber.

6.10 If the combustion chamber of the rocket engine discussed in Problem 6.9 is surrounded by a jacket through which liquid oxygen is circulated at –185°C, calculate the gain in the permissible firing time. Assume that the combustion chamber wall is precooled to –185°C and that the heat transfer coefficient on the oxygen side is 9800 W/(m^2·K).

6.11 A large slab of hard rubber, 1 in. thick and initially at 20°C, is placed between two heated steel plates maintained at 140°C. The heating is to be discontinued when the temperature of the midplane of the slab reaches 130°C. The rubber has a thermal diffusivity of 8.8×10^{-8}m^2/s. The thermal contact resistance between the steel plates and the rubber is negligible.

(*a*) Estimate the length of the heating period.

(*b*) Estimate the temperature of the rubber at a distance 1/4 in. from one of the steel plates at the end of the heating period.

(*c*) Estimate the time required for the temperature at a distance 1/4 in. from one of the steel plates to reach 130°C.

6.12 A long cylindrical steel bar, 20 cm in diameter, is first heated to 980°C and then quenched in an oil bath maintained at 40°C. The heat transfer coefficient can be assumed to be 500 W/(m²·K). How long will it take for the temperature of the centerline of the bar to reach 260°C?

6.13 A solid steel sphere, 8 cm in diameter, is heated in a furnace to a uniform temperature of 800°C. It is to be quenched until its center reaches 510°C by immersion in a well-stirred lead bath, maintained at the melting point of 327°C. At this point the sphere will be removed from the lead bath and quenched in oil. Calculate the required period of immersion in the lead bath. The properties of steel are: k = 36.4 W/(m·K), ρ = 7753 kg/m³, and c = 486 J/(kg·K).

6.14 A copper sphere [k_c = 390 W/(m·K), ρ_c = 8990 kg/m³, and c_c = 381 J/(kg·K)], which is 6cm in diameter and initially at 100°C, is placed in an airstream at 20°C. The surface and center temperatures of the sphere are both recorded simultaneously and the following data are obtained:

Time (min)	**Center Temperature (°C)**	**Surface Temperature** (°C)
0	100	100
5	50	50
10	31.2	31.2

Another sphere 6 cm in diameter, made from rubber [k_r = 0.43 W/(m·K), ρ_r = 1070 kg/m³, and c_r = 1630 J/(kg·K)] and initially at 100°C uniform temperature, is also placed in the same airstream as the copper sphere. In this case we are not able to record the temperatures. Estimate the center and surface temperatures 30 min after the rubber sphere has been placed in the airstream.

6.15 Circular stainless steel bars, each 12 cm in diameter, are to be quenched in a large oil bath maintained at 38°C. The initial temperature of the bars is 870°C. The maximum temperature within the bars at the end of the quenching process will have to be 200°C. The heat transfer coefficient can be taken as 400 W/(m²·K). How long must the bars be kept in the oil bath, if

(*a*) the bars are infinitely long, or

(*b*) the length of the bars is twice their diameter? The properties of the stainless steel are: k = 41 W/(m·K), ρ = 7865 kg/m³, and c = 460 J/(kg·K).

6.16 In an aluminum mill, aluminum bars 2.5 cm × 5 cm in cross section are extruded at 500°C as illustrated in Fig. 6.28. In order to prepare a large quantity of extruded aluminum bars for shipping soon after extrusion, rapid cooling is necessary. A cooling system has been proposed wherein bars pass through a water spray bath after they have been extruded. At the end of the water bath the temperature of the bars must be low enough to permit handling. The water spray is available at a temperature of 25°C. The heat transfer coefficient between the cooling water and the surface of the bars can be taken as 5000 W/(m²·K). The properties of aluminum are: k = 230 W/(m·K), ρ = 2707 kg/m³, and c = 896 J/(kg·K).

(*a*) If the bars are extruded at a velocity of 0.5 m/s, determine the length L of the cooling tank required to reduce the centerline temperature of the bars to 150°C.

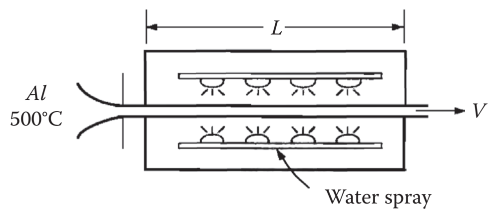

FIGURE 6.28
Figure for Problem 6.16.

(*b*) As the bars emerge from the cooling tank the surfaces will be cooler than the center. What is the maximum possible surface temperature that can be reached on the extruded bars after they leave the cooling tank?

6.17 A long rectangular steel rod, 10 cm × 20 cm in cross section, is initially at a uniform temperature of 16°C. At time $t = 0$, the temperature of its surfaces is raised to 100°C by immersing it in boiling water and is subsequently maintained at this temperature. Calculate the temperature at the axis of the rod after 40 s have elapsed since the beginning of the heating process.

6.18 The surface temperature of a very thick wall, which is initially at the uniform temperature of 0°C, is changed suddenly from 0°C to 1500°C and then maintained at the new value. The wall is made of concrete with the properties: $k = 0.814$ W/(m·K), $c = 879$ J/(kg·K) and $\rho = 1906$ kg/m^3.

(*a*) Calculate the temperature at a depth of 15 cm from the surface after 6 h have elapsed.

(*b*) How long will it take for the temperature at a depth of 30cm from the surface to reach the value calculated in (a)?

6.19 A solid rod of radius r_0 and height H is perfectly insulated against radial heat flow at the peripheral surface $r = r_0$. The ends of the rod at $z = 0$ and $z = H$ are maintained at the uniform temperatures T_1 and T_2, respectively, and the temperature distribution in the rod is steady. At time $t = 0$, the end at $z = 0$ is perfectly insulated against heat flow and kept insulated for $t > 0$, while the other end at $z = H$ is still maintained at T_2. Determine the subsequent change in the temperature distribution in the rod for $t > 0$.

6.20 A slab, extending from $x = 0$ to $x = L$ and of infinite extent in the y and z directions, is initially at a uniform temperature T_i. For times $t \geq 0$, a constant heat flux q''_w, is applied to the surface at $x = L$, while the surface at $x = 0$ is kept perfectly insulated. Assume that the thermophysical properties of the slab are constant. Obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the slab for $t > 0$.

6.21 A long solid cylinder of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. For times $t \geq 0$, a constant heat flux q''_w is applied to the peripheral surface at $r = r_0$. Obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the cylinder for $t > 0$.

6.22 A solid sphere, $0 \leq r \leq r_0$, of constant thermophysical properties is initially at a uniform temperature T_i. For times $t \geq 0$, the sphere is heated by applying a constant heat flux q''_w to its surface at $r = r_0$. Obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the sphere for $t > 0$.

6.23 A hollow sphere. $r_1 \leq r \leq r_2$, of constant thermophysical properties is initially at a uniform temperature T_i, For times $t > 0$, the inner boundary surface at $r = r_1$ is kept at *zero* temperature, and the outer boundary surface at $r = r_2$ exchanges heat by convection with a medium maintained at *zero* temperature with a constant heat transfer coefficient h. Determine the unsteady-state temperature distribution $T(r, t)$ in the hollow sphere for $t > 0$.

6.24 A slab, which extends from $x = 0$ to $x = L$, has an initial temperature distribution $T_i(x)$ at $t = 0$. For times $t \geq 0$, internal energy is generated in the slab at a constant rate $\dot{q}$ per unit volume, while the surfaces at $x = 0$ and $x = L$ are maintained at a constant temperature T_w. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the slab for $t > 0$.

6.25 Re-solve Problem 6.24, if the initial temperature distribution $T_i(x) = T_i =$ constant.

6.26 Re-solve Problem 6.24, if $\dot{q} = \dot{q}(x,t)$ and $T_w = T_w(t)$. *Hint:* use the technique described in Problem 5.12*b*.

6.27 A long solid cylinder of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. For times $t > 0$, internal energy is generated in the rod at a constant rate q per unit volume, while the peripheral surface at $r = r_0$ is maintained at T_i. Obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the cylinder for $t > 0$.

6.28 A solid sphere of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. For times $t \geq 0$, internal energy is generated in the sphere at a constant rate $\dot{q}$ per unit volume, while the surface at $r = r_0$ is maintained at T_i. Obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the sphere. What is the steady-state temperature distribution in the sphere?

6.29 Solve the problem formulated in Problem 2.9.

6.30 A slab that extends from $x = -L$ to $x = L$ is initially at a uniform temperature T_i. For times $t \geq 0$, internal energy is generated in the slab at a rate

$$\dot{q} = \dot{q}_0[1 + \beta(T - T_i)]$$

where $\dot{q}_0$ and β are given constants, while the surfaces at $x = \pm L$ are maintained at the initial temperature T_i. Assuming constant thermophysical properties, obtain, for $t > 0$, expressions for the unsteady-state temperature distribution $T(x, t)$ in the slab for the cases of $\dot{q}_0\beta < 0$ and $\dot{q}_0\beta > 0$. What are the steady-state temperature distributions for these two cases?

6.31 Re-solve Problem 6.29 for a long solid cylinder of radius r_0.

6.32 Re-solve Problem 6.29 for a solid sphere of radius r_0.

6.33 Two flat plates, made of the same material and with thicknesses a and b, are initially at the uniform temperatures T_1 and T_2, respectively. The plates are suddenly brought into contact at $t = 0$ as illustrated in Fig. 6.29. The external surfaces at $x = 0$ and $x = a + b$ are perfectly insulated. Assuming perfect thermal contact at the interface and constant thermophysical properties, obtain an expression for the temperature distribution in the system for times $t > 0$, What is the steady-state temperature of the plates?

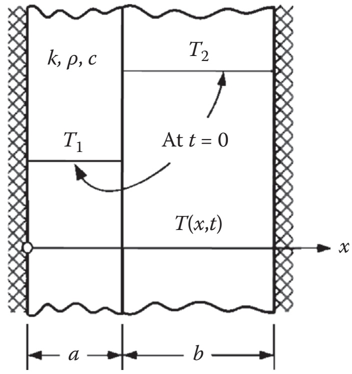

FIGURE 6.29
Figure for Problem 6.33.

6.34 A slab, which extends from $x = 0$ to $x = L$ and is of infinite extent in the y and z directions, is initially at a uniform temperature T_i. For times $t \geq 0$, the surface temperature at $x = L$ varies according to

$$T(L,t) = T_i + At$$

where A is a given constant, while the surface at $x = 0$ is kept perfectly insulated. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the slab for $t > 0$.

6.35 Re-solve Problem 6.34 for a long solid cylinder of radius r_0.

6.36 Re-solve Problem 6.34 for a solid sphere of radius r_0.

6.37 An insulated wire of length $2L$ is bent into a circle as shown in Fig. 6.30. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(s, t)$ in this wire for times $t > 0$, if it has a given

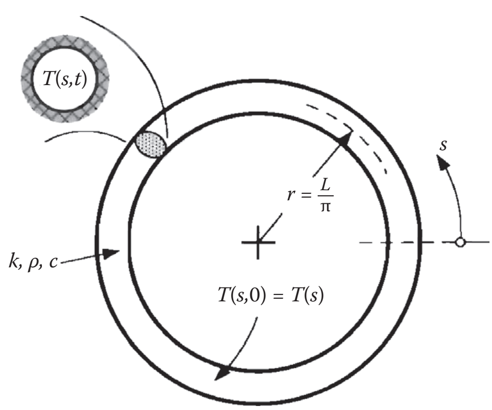

FIGURE 6.30
Figure for Problem 6.37.

temperature distribution $T_i(s)$ along it initially at $t = 0$. What would be the final steady-state temperature of the wire*?

6.38 Consider a two-dimensional fin of length L in the x direction and thickness $2b$ in the y direction. The fin is initially at a uniform temperature T_i. At $t = 0$, while the temperature of the base at $x = 0$ is kept at T_i, the temperatures of the surfaces at $x = L$ and $\nu = \pm b$ are suddenly changed to T_∞ and subsequently maintained constant at this value for times $t > 0$. Obtain an expression for the unsteady-state temperature distribution $T(x, y, t)$ in the fin for $t > 0$. Assume constant thermophysical properties.

6.39 A long half-cylinder, $0 \le r \le r_0$ and $0 \le \phi \le \pi$, has an initial temperature distribution given by $F(r, \phi)$. For times $t \ge 0$, the boundary surfaces at $r = r_0$, $\phi = 0$, and $\phi = \pi$ are all maintained at a constant temperature T_w. Determine the unsteady-state temperature distribution $T(r, \phi, t)$ in the cylinder for times $t > 0$. Assume constant thermophysical properties.

6.40 A solid rod, of constant thermophysical properties, radius r_0 and height H, is initially at a uniform temperature T_i. For times $t \ge 0$, a constant heat flux, q''_w, is applied to the peripheral surface at $r = r_0$. while the circular surfaces at $z = 0$ and $z = H$ are maintained at T_i. Obtain an expression for the unsteady temperature distribution $T(r, z, t)$ in the cylinder for $t > 0$.

6.41 Re-solve Problem 6.40 if a constant heat flux q''_0 is applied to the circular surface at $z = 0$, while the peripheral surface at $r = r_0$ and the circular surfaces at $z = H$ are maintained at the initial temperature T_i.

6.42 Consider a long bar of rectangular cross section, $0 \le x \le a$ and $0 \le y \le b$, initially at a uniform temperature T_i. For times $t \ge 0$, internal energy is generated in the bar at a constant rate $\dot{q}$ per unit volume, while its surfaces are maintained at the initial temperature T_i. Assuming constant thermophysical properties obtain an expression for the unsteady-state temperature distribution $T(x, y, t)$ in the bar for $t > 0$.

6.43 A solid rod of constant thermophysical properties, radius r_0 and height H, is initially at a uniform temperature T_i. For times $t \ge 0$, internal energy is generated in the rod at a constant rate $\dot{q}$ per unit volume, while the peripheral surface at $r = r_0$ and the end surfaces at $z = 0$ and $z = H$ are maintained at T_i. Obtain an expression for the unsteady-state temperature distribution $T(r, z, t)$ in the cylinder for $t > 0$.

6.44 Obtain the quasi-steady temperature distribution $T(x, t)$ in a semi-infinite solid of constant properties when the surface temperature varies according to

$$T(0,t) = T_m + (\Delta T)_0 \cos(2\pi n t)$$

where n is the frequency and T_m and $(\Delta T)_0$ are the mean and the amplitude of the surface temperature variation, respectively.

6.45 Using the temperature distribution obtained in Problem 6.44, obtain an expression for the time lag between the temperature waves at the surface and at a depth x.

6.46 On cold days, at a certain locality, the daily variation of the earth's surface temperature becomes sinusoidal between 5 and 15°C. Calculate the amplitude of the

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* This problem was of considerable interest to Fourier himself, and the wire has come to be known as *Fourier's ring* [2].

temperature variation at a depth of 10 cm from the earth's surface. Take $\alpha = 0.0036$ m^2/h for the earth's material.

6.47 Consider a steel sphere of 6 cm in diameter ($k = 61$ W/mC, $\rho = 7865$ kg/m^3, $c = 460$ J/kg C) that comes out of a furnace at 800°C and should be tempered by an oil bath at 50°C. If the balls are to be removed when they reach the temperature of 100°C and the heat transfer coefficient between the ball and the oil is $h = 500$ W/m^{2}C, determine when it should be removed (use the lumped system analysis, but first check its validity).

6.48 A mercury thermometer, initially at 15°C, is fully immersed in a reservoir with water at 75°C. The thermometer is 20 cm in length and 3 mm in radius, assuming it to be essentially a solid glass cylinder. One needs to know after how many seconds the thermometer will indicate a water temperature in the tank with a 5% error. Consider the convective heat transfer coefficient equal to $h = 180$ W/m^{2}C.

6.49 The dimensionless mathematical formulation for a three-dimensional transient heat conduction problem in a cubic region is given by

$$\frac{\partial T(x,y,z,t)}{\partial t} = \left[\frac{\partial^2 T(x,y,z,t)}{\partial x^2} + \frac{\partial^2 T(x,y,z,t)}{\partial y^2} + \frac{\partial^2 T(x,y,z,t)}{\partial z^2}\right], 0 < x < 1, 0 < y < 1, 0 < z < 1, t > 0$$

with initial and boundary conditions given by

$$T(x,y,z,0) = 1, 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$$

$$T(0,y,z,t) = 0; \quad T(1,y,z,t) = 0, \quad t > 0$$

$$\left.\frac{\partial T(x,y,z,t)}{\partial y}\right|_{y=0} = 0; \quad T(x,1,z,t) = 0, \quad t > 0$$

$$\left.\frac{\partial T(x,\mathrm{y},z,t)}{\partial z}\right|_{z=0} = 0; \quad T(x,y,1,t) = 0, \quad t > 0$$

Find the analytical solution of this problem by separation of variables.

7

Solutions with Integral Transforms

7.1 Introduction

In the preceding two chapters, we discussed solutions of various linear heat conduction problems by the application of the classical method of *separation of variables.* When separation is possible, this method turns out to be an effective and simple method to implement. However, in most cases, separation may not be easily achievable or it may not even be possible. In this chapter, we study the method of solution of linear heat conduction problems by the application of various *integral transforms,* such as *Fourier* and *Hankel transforms.* These transforms remove the partial derivatives with respect to space variables and are equally attractive for both steady- and unsteady-state problems. The method of *Laplace transforms,* which is also an integral transform method and is typically used to remove the partial derivative with respect to the time variable, is studied in the next chapter.

As we shall see in the following sections, *finite integral transforms* and the corresponding *inversion relations* are obtained by rearranging in two parts the Fourier expansions introduced in Chapter 4. Using this approach, Churchill [1] and Eringen [5] developed the theory of finite integral transforms. Thus, by expanding an arbitrary function in an infinite series of the characteristic functions of various Sturm–Liouville systems, they introduced several finite integral transforms. These transforms, following Eringen [5], are also called *finite Sturm–Liouville transforms.* The *finite Fourier transforms*, *finite Hankel transforms*, *finite Legendre transforms*, etc., are all examples of finite Sturm–Liouville transforms. In this chapter, we further consider the extensions of the theory and introduce integral transforms in the *semi-infinite* and *infinite regions.* More fundamental and extensive treatment of the theory of integral transforms and their applications to the solution of heat conduction problems can be found in Chapter 13 and in the books Özışık [7,8], Sneddon [10], Cotta [12] and Cotta and Mikhailov [13].

7.2 Finite Fourier Transforms

In Chapter 4 we saw that any function $f(x)$, which is piecewise differentiable on the interval $(0, L)$, can be expanded in a *Fourier sine series* as (see Section 4.6.1)

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x, \quad 0 < x < L \tag{7.1}$$

where the expansion coefficients A_n are given by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \tag{7.2}$$

Substituting A_n from Eq. (7.2) into Eq. (7.1), we obtain

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x')\sin\frac{n\pi}{L}x'\,dx'\right]\sin\frac{n\pi}{L}x \tag{7.3}$$

which can be rearranged in two parts as

$$\bar{f}_n = \int_0^L f(x)K_n(x)\,dx \tag{7.4a}$$

and

$$f(x) = \sum_{n=1}^{\infty}\bar{f}_n K_n(x) \tag{7.4b}$$

where $K_n(x)$ are given by

$$K_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x \tag{7.4c}$$

Equations (7.4*a*) and (7.4*b*) can also be written as

$$\bar{f}_n = \sqrt{\frac{2}{L}}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \tag{7.5a}$$

and

$$f(x) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\bar{f}_n\sin\frac{n\pi}{L}x \tag{7.5b}$$

Equation (7.4*a*), or (7.5*a*), is called the *finite Fourier sine transform* of the function $f(x)$ over the interval (0, L), and Eq. (7.4*b*), or (7.5*b*), is the corresponding *inversion* formula. The functions $K_n(x)$, defined by Eq. (7.4*c*), are the *kernels* of the transform.

The same function $f(x)$ can also be expanded in a *Fourier cosine series* on the interval (0, L) as (see Section 4.6.2)

$$f(x) = \sum_{n=0}^{\infty}A_n\cos\frac{n\pi}{L}x = A_0 + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{L}x,\ \ 0 < x < L \tag{7.6}$$

where the coefficients A_n are given by

$$A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx, & n=0 \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi}{L}x\,dx, & n=1,2,3,\ldots \end{cases} \tag{7.7}$$

Substituting A_n from Eq. (7.7) into Eq. (7.6), we get

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{N_n}\left[\int_0^L f(x')\cos\frac{n\pi}{L}x'\,dx'\right]\cos\frac{n\pi}{L}\pi \tag{7.8}$$

where

$$N_n = \begin{cases} L, & n=0 \\ \dfrac{L}{2}, & n=1,2,3,\ldots \end{cases} \tag{7.9}$$

The expansion (7.8) can now be rewritten in two parts as

$$\bar{f}_n = \int_0^L f(x)K_n(x)\,dx \tag{7.10a}$$

and

$$f(x) = \sum_{n=0}^{\infty} \bar{f}_n K_n(x) \tag{7.10b}$$

where $K_n(x)$ are given by

$$K_n(x) = \frac{1}{\sqrt{N_n}}\cos\frac{n\pi}{L}x \tag{7.10c}$$

Equation (7.10*a*) is called the *finite Fourier cosine transform* of the function $f(x)$ over the interval $(0, L)$, and Eq. (7.10*b*) is the corresponding *inversion* formula. The functions $K_n(x)$, defined by Eq. (7.10*c*), are the *kernels* of the transform.

The *finite Fourier sine* and *cosine* transforms that we introduced here were, in fact, developed from the expansions of an arbitrary function $f(x)$ in series of the characteristic functions of the two special cases of the following characteristic-value problem:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \tag{7.11a}$$

$$\alpha_1 y(0) + \beta_1\frac{dy(0)}{dx} = 0,\ \alpha_1^2 + \beta_1^2 > 0 \tag{7.11b}$$

$$\alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0, \ \alpha_2^2 + \beta_2^2 > 0 \tag{7.11c}$$

Since the problem (7.11) is a special case of the Sturm–Liouville system (4.13), the characteristic functions $\phi_n(x)$ form a complete orthogonal set with respect to the weight function unity on the interval (0, L). As we discussed in Section 4.6, an arbitrary function $f(x)$, which is piecewise differentiable in the interval (0, L), can be expanded in a series of these characteristic functions as

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x), \ 0 < x < L \tag{7.12}$$

where the coefficients A_n are given by

$$A_n = \frac{1}{N_n} \int_0^L f(x)\phi_n(x)\,dx \tag{7.13a}$$

with the *normalization* integrals

$$N_n = \int_0^L \left[\phi_n(x)\right]^2 dx \tag{7.13b}$$

The Fourier expansion (7.12) can now be rearranged as a *finite integral transform* of the function $f(x)$ in the form

$$\bar{f}_n = \int_0^L f(x) K_n(x)\,dx \tag{7.14a}$$

with the *inversion*

$$f(x) = \sum_{n=0}^{\infty} \bar{f}_n K_n(x) \tag{7.14b}$$

where the *kernels* $K_n(x)$ are defined as

$$K_n(x) = \frac{\phi_n(x)}{\sqrt{N_n}} \tag{7.14c}$$

Here it should be noted that since the kernels $K_n(x)$ are, by definition, the normalized characteristic functions, they satisfy the differential equation (7.11*a*) and the boundary conditions (7.11*b*,*c*) when $\lambda = \lambda_n$.

The *kernels of the finite Fourier sine* and *cosine* transforms are, in fact, normalized characteristic functions corresponding to $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = 0$, respectively. To illustrate further, the kernels of the finite Fourier transform corresponding to the case where the boundary conditions (7.11*b*,*c*) are both of the third kind (i.e., $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_1 \neq 0$, and $\beta_2 \neq 0$) can be obtained as follows. The characteristic functions of the system (7.11) are then given by (see Table 4.1)

TABLE 7.1

Characteristic Values and Kernels for Use in Finite Fourier Transforms

Integral Transform:

$$\bar{f}_n = \int_0^L f(x) K_n(x)\, dx$$

Inversion Formula:

$$f(x) = \sum_{n=0}^{\infty} \bar{f}_n K_n(x)$$

Characteristic–value problem:

$$\left\{ \begin{array}{c} \dfrac{d^2y}{dx^2} + \lambda^2 y = 0 \\ \alpha_1 y(0) + \beta_1 \dfrac{dy(0)}{dx} = 0 \\ \alpha_2 y(L) + \beta_2 \dfrac{dy(L)}{dx} = 0 \end{array} \right\}$$

Boundary conditions: At $x = 0$	Boundary conditions: At $x = L$	Kernel $K_n(x)$†	Characteristic values λ_n's are positive roots of†
Third kind‡ ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Third kind‡ ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{2}\dfrac{\lambda_n \cos \lambda_n x - H_1 \sin \lambda_n x}{\left\{(\lambda_n^2 + H_1^2)\left[L + H_2/(\lambda_n^2 + H_2^2)\right] - H_1\right\}^{1/2}}$	$\tan \lambda L = \dfrac{\lambda(H_2 - H_1)}{\lambda^2 + H_1 H_2}$
Third kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Second kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{2\lambda_n}\dfrac{\cos \lambda_n (L - x)}{(\lambda_n L + \sin \lambda_n L \cos \lambda_n L)^{1/2}}$	$\lambda \tan \lambda L = -H_1$
Third kind§ ($\alpha_1 \neq 0, \beta_1 \neq 0$)	First kind§ ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{2\lambda_n}\dfrac{\sin \lambda_n (L - x)}{(\lambda_n L - \sin \lambda_n L \cos \lambda_n L)^{1/2}}$	$\lambda \cot \lambda L = H_1$
Second kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Third kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{2\lambda_n}\dfrac{\cos \lambda_n x}{(\lambda_n L - \sin \lambda_n L \cos \lambda_n L)^{1/2}}$	$\lambda \tan \lambda L = H_2$
Second kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	Second kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{\dfrac{2}{L}} \cos \lambda_n x$	$\sin \lambda L = 0$ $\left(\lambda_n = \dfrac{n\pi}{L},\ n = 1, 2, 3, \ldots\right)$
Second kind ($\alpha_1 \neq 0, \beta_1 \neq 0$)	First kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{\dfrac{2}{L}} \cos \lambda_n x$	$\cos \lambda L = 0$ $\left(\lambda_n = \dfrac{2n-1}{L}\dfrac{\pi}{2},\ n = 1, 2, 3, \ldots\right)$
First kind# ($\alpha_1 \neq 0, \beta_1 = 0$)	Third kind# ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{2\lambda_n}\dfrac{\sin \lambda_n x}{(\lambda_n L + \sin \lambda_n L \cos \lambda_n L)^{1/2}}$	$\lambda \cot \lambda L = -H_2$
First kind ($\alpha_1 \neq 0, \beta_1 = 0$)	Second kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{\dfrac{2}{L}} \sin \lambda_n x$	$\cos \lambda L = 0$ $\left(\lambda_n = \dfrac{2n-1}{L}\dfrac{\pi}{2},\ n = 1, 2, 3, \ldots\right)$
First kind ($\alpha_1 \neq 0, \beta_1 = 0$)	First kind ($\alpha_2 \neq 0, \beta_2 \neq 0$)	$\sqrt{\dfrac{2}{L}} \sin \lambda_n x$	$\sin \lambda L = 0$ $\left(\lambda_n = \dfrac{n\pi}{L},\ n = 1, 2, 3, \ldots\right)$

† $H_1 = \alpha_1/\beta_1$ and $H_2 = \alpha_2/\beta_2$.

‡ See footnote (‡) in Table 4.1, and modify the transform accordingly.

§ See footnote (§) in Table 4.1, and modify the transform accordingly.

|| When n = 0, replace $\sqrt{2/L}$ by $\sqrt{1/L}$.

See footnote (#) in Table 4.1, and modify the transform accordingly.

$$\phi_n(x) = \lambda_n \cos \lambda_n x - H_1 \sin \lambda_n x \tag{7.15a}$$

where the characteristic values λ_n are the positive roots of the transcendental equation

$$\tan \lambda L = \frac{\lambda(H_2 - H_1)}{\lambda^2 + H_1 H_2} \tag{7.15b}$$

with $H_1 = \alpha_1/\beta_1$ and $H_2 = \alpha_2/\beta_2$. Furthermore, the normalization integrals N_n are given by

$$N_n = \frac{1}{2}\left[\left(\lambda_n^2 + H_1^2\right)\left(L + \frac{H_2}{\lambda_n^2 + H_2^2}\right) - H_1\right] \tag{7.15c}$$

The kernels can then be written, from Eq. (7.14*c*), as

$$K_n(x) = \sqrt{2}\,\frac{\lambda_n \cos \lambda_n x - H_1 \sin \lambda_n x}{\left\{\left(\lambda_n^2 + H_1^2\right)\left[L + H_2/\left(\lambda_n^2 + H_2^2\right)\right] - H_1\right\}^{1/2}} \tag{7.15d}$$

The kernels for the remaining cases can be obtained following the same procedure. In fact, they can also be obtained readily, as limiting cases, from Eq. (7.15*d*). The kernels for all of the nine possible cases, which were also tabulated by Özışık [7], are summarized here in Table 7.1.

The transforms introduced here are called finite Fourier transforms simply because they are derived from Fourier expansions of an arbitrary function $f(x)$ within the finite interval $(0, L)$.

7.3 An Introductory Example

As an introductory example, let us consider the plane wall shown in Fig. 7.1, which has a thickness L in the x direction. Let the initial temperature distribution be $T_i(x)$ at $t = 0$. Assume that internal energy is generated in this wall at a rate of $\dot{q}\,(x, t)$ per unit volume

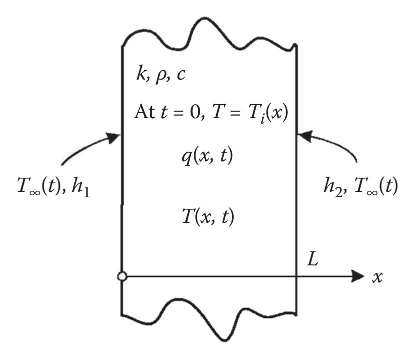

FIGURE 7.1
Plane wall containing time- and space-dependent distributed internal energy source and exchanging heat with a surrounding medium of time-dependent temperature $T_\infty(t)$.

for times $t \geq 0$, and heat is dissipated by convection from the surfaces at $x = 0$ and $x = L$ into a surrounding medium, whose temperature T_∞ varies with time. We also assume that the thermal conductivity k and the thermal diffusivity α are constants, and that the heat transfer coefficients h_1 and h_2 are very large.

The unsteady-state temperature distribution $T(x, t)$ in this wall will satisfy the following *initial-and-boundary-value problem* for $t > 0$:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}(x,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.16a}$$

$$T(x,0) = T_i(x) \tag{7.16b}$$

$$T(0,t) = T(L,t) = T_\infty(t) \tag{7.16c,d}$$

This problem cannot be solved readily by the method of separation of variables (see Problem 6.26). Here we solve it by the method of integral transforms and, thus, illustrate the method of solution. The basic idea behind the application of this method is to remove the partial derivative with respect to the space variable x from the formulation of the problem. We note that x varies from *zero* to L. Accordingly, an integral transform $\bar{T}_n(t)$ of the temperature distribution $T(x, t)$ with respect to the space variable x is defined on the finite interval $(0, L)$ as

$$\bar{T}_n(t) = \int_0^L T(x,t) K_n(x)\, dx \tag{7.17a}$$

with the inversion

$$T(x,t) = \sum_{n=1}^{\infty} \bar{T}_n(t) K_n(x) \tag{7.17b}$$

At this stage, to determine $K_n(x)$, we consider the following *auxiliary* problem:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \psi}{\partial t} \tag{7.18a}$$

$$\psi(x,0) = T_i(x) \tag{7.18b}$$

$$\psi(0,t) = \psi(L,t) = 0 \tag{7.18c,d}$$

which is obtained by removing the nonhomogeneous terms from the differential equation and boundary conditions of the original problem (7.16). This auxiliary problem can be separated into two by letting $\psi(x, t) = X(x)\Gamma(t)$. Then, $X(x)$ will satisfy the following characteristic-value problem:

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \tag{7.19a}$$

$$X(0) = X(L) = 0 \tag{7.19b,c}$$

We now let $K_n(x)$ be the normalized characteristic functions of this characteristic-value problem, which are (see Section 4.6.1):

$$K_n(x) = \frac{\phi_n(x)}{\sqrt{N_n}} = \sqrt{\frac{2}{L}} \sin \lambda_n x \tag{7.20a}$$

with

$$\lambda_n = \frac{n\pi}{L}, \; n = 1, 2, 3, \ldots \tag{7.20b}$$

Thus, Eq. (7.17*a*) becomes the *finite Fourier sine transform* of the temperature distribution $T(x, t)$.

Next, we obtain the finite Fourier sine transform of Eq. (7.16*a*) by first multiplying it by $K_n(x)$ and then integrating the resultant expression over x from *zero* to L:

$$\int_0^L K_n(x) \frac{\partial^2 T}{\partial x^2} dx + \frac{1}{k} \int_0^L K_n(x) \dot{q}(x,t) dx = \frac{1}{\alpha} \int_0^L K_n(x) \frac{\partial T}{\partial t} dx \tag{7.21}$$

The integral on the right-hand side of Eq. (7.21) can be written as

$$\int_0^L K_n(x) \frac{\partial T}{\partial t} dx = \frac{d}{dt} \int_0^L K_n(x) T(x,t) dx = \frac{d\bar{T}_n}{dt} \tag{7.22}$$

The first integral on the left-hand side of Eq. (7.21) can be evaluated by integrating it by parts as follows:

$$\begin{aligned} \int_0^L K_n(x) \frac{\partial^2 T}{\partial x^2} dx &= \underbrace{K_n(L)}_{=0} \left. \frac{\partial T}{\partial x} \right|_{x=L} - \underbrace{K_n(0)}_{=0} \left. \frac{\partial T}{\partial x} \right|_{x=0} \\ &- \int_0^L \frac{dK_n}{dx} \frac{\partial T}{\partial x} dx = - \int_0^L \frac{dK_n}{dx} \frac{\partial T}{\partial x} dx \end{aligned} \tag{7.23a}$$

One more integration by parts yields

$$\begin{aligned} \int_0^L K_n(x) \frac{\partial^2 T}{\partial x^2} dx &= -\underbrace{T(L,t)}_{T_\infty(t)} \left. \frac{dK_n}{dx} \right|_{x=L} \\ &+ \underbrace{T(0,t)}_{T_\infty(t)} \left. \frac{dK_n}{dx} \right|_{x=0} + \int_0^L \frac{d^2 K_n}{dx^2} T(x,t) dx \end{aligned} \tag{7.23b}$$

Note that

$$\frac{dK_n}{dx} = \sqrt{\frac{2}{L}} \lambda_n \cos \lambda_n x \text{ and } \frac{d^2 K_n}{dx^2} = -\lambda_n^2 K_n(x)$$

Hence, Eq. (7.23*b*) can be rewritten as

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx = \sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_\infty(t) - \lambda_n^2\bar{T}_n(t) \tag{7.23c}$$

When Eqs. (7.22) and (7.23*c*) are substituted into Eq. (7.21), we get

$$\frac{1}{\alpha}\frac{d\bar{T}_n}{dt} + \lambda_n^2\bar{T}_n(t) = \sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_\infty(t) + \frac{1}{k}\bar{\dot{q}}_n(t) \tag{7.24}$$

where $\bar{\dot{q}}_n(t)$ is the *finite Fourier sine transform* of $\dot{q}$ (*x*, *t*); that is,

$$\bar{\dot{q}}_n(t) = \int_0^L \dot{q}(x,t)K_n(x)dx \tag{7.25}$$

Thus, we have reduced the original partial differential equation (7.16*a*) to an ordinary differential equation (7.24) by removing the space variable *x*. Equation (7.24) can be solved as follows: multiply both sides of this equation by $\exp\left(\alpha\lambda_n^2 t\right)$ and then integrate the resultant expression from *zero* to *t*,

$$\int_0^t e^{\alpha\lambda_n^2 t'}\left[\frac{1}{\alpha}\frac{d\bar{T}_n}{dt'} + \lambda_n^2\bar{T}_n(t')\right]dt' = \int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt' \tag{7.26}$$

where

$$F_n(t) = \sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_\infty(t) + \frac{1}{k}\bar{\dot{q}}_n(t) \tag{7.27}$$

Equation (7.26) can be rewritten as

$$\frac{1}{\alpha}\int_0^t \frac{d}{dt'}\left[e^{\alpha\lambda_n^2 t'}\bar{T}_n(t')\right]dt' = \int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt' \tag{7.28}$$

which yields

$$\bar{T}_n(t) = e^{-\alpha\lambda_n^2 t}\left[\bar{T}_n(0) + \alpha\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'\right] \tag{7.29}$$

where $\bar{T}_n(0)$ is the *finite Fourier sine transform* of the initial condition; that is,

$$\bar{T}_n(0) = \int_0^L T_i(x)K_n(x)dx \tag{7.30}$$

We now invert $\bar{T}_n(t)$ given by Eq. (7.29) by using the inversion formula (7.17*b*) to obtain the temperature distribution *T*(*x*, *t*) as follows:

$$T(x,t) = \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\left[\bar{T}_n(0) + \alpha\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'\right]K_n(x) \tag{7.31a}$$

which can be rewritten as

$$T(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^L T_i(x')\sin\lambda_n x'\,dx'\right.$$
$$+\alpha\lambda_n\left[1-(-1)^n\right]\int_0^t e^{\alpha\lambda_n^2 t'}T_\infty(t')dt' \tag{7.31b}$$
$$\left.+\frac{1}{\rho c}\int_0^t e^{\alpha\lambda_n^2 t'}\int_0^L \dot{q}(x',t')\sin\lambda_n x'\,dx'\,dt'\right\}$$

Here, we note that, although Eq. (7.31*b*) is an exact solution of the problem, it is valid only inside the interval $0 < x < L$. but not at the boundaries (i.e., it is not valid at $x = 0$ and $x = L$). This is because Eq. (7.31*b*) is an infinite series solution in terms of the functions $\sin(n\pi/L)$ x, $n = 1, 2, 3,\ldots$, which vanish both at $x = 0$ and $x = L$; hence, difficulty arises in the computation of temperature at the boundaries. This difficulty, however, can be alleviated by developing an *alternative form* of the solution as described below.

We obtain the *alternative form* as follows: first we evaluate the integral in the second term of the solution (7.31*b*) by parts to obtain

$$\int_0^t e^{\alpha\lambda_n^2 t'}T_\infty(t')dt'=\frac{1}{\alpha\lambda_n^2}\left[e^{\alpha\lambda_n^2 t}T_\infty(t)-T_\infty(0)-\int_0^t e^{\alpha\lambda_n^2 t'}\frac{dT_\infty}{dt'}dt'\right] \tag{7.32}$$

Next, by substituting Eq. (7.32) into Eq. (7.31*b*), we get the following new expression for the temperature distribution:

$$T(x,t)-T_\infty(t)=\frac{2}{L}\sum_{n=1}^{\infty}e^{-\alpha(n\pi/L)^2 t}\sin\frac{n\pi}{L}x\left\{\int_0^L T_i(x')\sin\frac{n\pi}{L}x'\,dx'\right.$$
$$-\frac{L}{n\pi}\left[1-(-1)^n\right]\left[T_\infty(0)+\int_0^t e^{\alpha(n\pi/L)^2 t'}\frac{dT_\infty}{dt'}dt'\right] \tag{7.33}$$
$$\left.+\frac{1}{\rho c}\int_0^t e^{\alpha(n\pi/L)^2 t'}\int_0^L \dot{q}(x',t')\sin\frac{n\pi}{L}x'\,dx'\,dt'\right\}$$

where we have made use of the expansion (see Problem 4.11)

$$1=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n}\sin\frac{n\pi}{L}x,\ \ 0<x<L \tag{7.34}$$

The alternative form of the solution given by Eq. (7.33) converges to $T_\infty(t)$ both at $x = 0$ and $x = L$. Here, we note that it is possible to obtain this alternative form directly if one first formulates the problem in terms of $\theta(x, t) = T(x, t) - T_\infty(t)$ and then solves it following the same procedure outlined above (see Problem 7.3).

If, in particular, both $T_\infty(t)$ = constant and $\dot{q}$ (x, t) = constant, then the solution (7.33) reduces to

$$T(x,t)-T_\infty=\frac{2}{L}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^L T_i(x')\sin\lambda_n x'\,dx'-\frac{\left[1-(-1)^n\right]}{\lambda_n}\left[T_\infty+\frac{\dot{q}}{k}\frac{e^{\alpha\lambda_n^2 t}-1}{\lambda_n^2}\right]\right\} \quad (7.35a)$$

where $\lambda_n = n\pi/L$. Equation (7.35*a*) can also be written as

$$\begin{aligned}T(x,t)-T_\infty&=\frac{\dot{q}L^2}{2k}\left[\frac{x}{L}-\left(\frac{x}{L}\right)^2\right]+\frac{2}{L}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\\&\times\left\{\int_0^L T_i(x')\sin\lambda_n x'\,dx'-\frac{[1-(-1)^n]}{\lambda_n}\left[T_\infty-\frac{\dot{q}}{k}\frac{1}{\lambda_n^2}\right]\right\}\end{aligned} \quad (7.35b)$$

where we have made use of the following expansion (see Problem 7.4*a*):

$$\frac{x}{L}-\left(\frac{x}{L}\right)^2=\frac{4}{L^3}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{\lambda_n^3}\sin\lambda_n x,\ \ 0<x<L \quad (7.36)$$

Equation (7.35*b*) is the form of the solution that one would obtain if the method of separation of variables were used to solve the same problem (see Problem 6.24).

In this section we outlined the steps typically followed in solving linear heat conduction problems by integral transforms. The procedure is straightforward. The determination of the kernel of the transform from an auxiliary characteristic-value problem poses no difficulty. As we shall further see in the following sections, the auxiliary characteristic-value problem can readily be obtained merely by considering a simplified version of the formulation of the original problem.

7.4 Fourier Transforms in the Semi-Infinite and Infinite Regions

In Section 7.2, we introduced finite Fourier transforms in the interval (0, *L*). These can be used for the solution of various heat conduction problems in finite regions. For problems posed in the semi-infinite region, however, we need transforms in the semi-infinite interval (0, ∞). Such transforms can *formally* be obtained from finite Fourier transforms by taking limits as $L \to \infty$.

Consider, for example, the *finite Fourier sine transform*

$$\bar{f}_n=\sqrt{\frac{2}{L}}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \quad (7.37a)$$

with the *inversion*

$$f(x)=\sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\bar{f}_n \sin\frac{n\pi}{L}x\,dx \tag{7.37b}$$

Substitution of Eq. (7.37*a*) into Eq. (7.37*b*) results in

$$f(x)=\frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L f(x')\sin\frac{n\pi}{L}x'\,dx'\right\}\sin\frac{n\pi}{L}x \tag{7.37c}$$

which is the *Fourier sine expansion* of $f(x)$ in the interval $(0, L)$. Calling $\Delta\omega = \pi/L$, Eq. (7.37*c*) can be rewritten as

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^L f(x')\sin(n\Delta\omega)x'\,dx'\right\}\sin(n\Delta\omega)x\Delta\omega \tag{7.38a}$$

or

$$f(x)=\frac{2}{\pi}\int_0^L f(x')\left\{\sum_{n=1}^{\infty}\sin(n\Delta\omega)x'\sin(n\Delta\omega)x\Delta\omega\right\}dx' \tag{7.38b}$$

Now, consider the *infinite integral*,

$$\int_0^{\infty}\sin\omega x'\sin\omega x\,d\omega \equiv \lim_{\omega\to 0}\sum_{n=1}^{\infty}\sin(n\Delta\omega)x'\sin(n\Delta\omega)x\Delta\omega \tag{7.39}$$

Thus, as $L \to \infty$ (or $\Delta\omega \to 0$), Eq. (7.38*b*) can be written as

$$f(x)=\frac{2}{\pi}\int_0^{\infty} f(x')\int_0^{\infty}\sin\omega x'\sin\omega x\,d\omega\,dx', \qquad x>0 \tag{7.40}$$

Switching the order of integration in Eq. (7.40), we finally get

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\omega x\int_0^{\infty} f(x')\sin\omega x'\,dx'\,d\omega,\ x>0 \tag{7.41}$$

which is called the *Fourier sine integral representation** of $f(x)$, and represents $f(x)$ at points of continuity in the interval $(0, \infty)$ and converges to the mean value $1/2\,[f(x^+)+f(x^-)]$ at points where $f(x)$ has finite jumps. It is also clear that Eq. (7.41) represents $-f(-x)$ when $x<0$. Thus, if $f(x)$ is an *odd* function, Eq. (7.41) is valid for all values of x.

* It can be shown that the Fourier sine integral representation (7.41) is valid when $x > 0$ if $f(x)$ is piecewise differentiable in every finite positive interval and $f(x)$ is absolutely integrable from *zero* to ∞.

If we now let

$$\bar{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \tag{7.42a}$$

then Eq. (7.41) can be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(\omega) \sin \omega x \, d\omega \tag{7.42b}$$

Equation (7.42*a*) is known as the *Fourier sine transform* of f(x) in the interval (0, ∞), and Eq. (7.42*b*) is the corresponding *inversion* formula. The kernel of this transform is

$$K(\omega, x) = \sqrt{\frac{2}{\pi}} \sin \omega x \tag{7.43}$$

which satisfies the following characteristic-value problem with $\beta = 0$:

$$\frac{d^2 y}{dx^2} + \omega^2 y = 0 \tag{7.44a}$$

$$\alpha y(0) + \beta \frac{dy(0)}{dx} = 0, \ \alpha^2 + \beta^2 \neq 0 \tag{7.44b}$$

$$|y(x)| \leq M \text{ for } x > 0 \tag{7.44c}$$

where M is some finite constant. In this case the characteristic values, ω, are continuous from *zero* to ∞ rather than discrete.

In a similar way, it can be shown that (see Problem 7.5)

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty f(x') \cos \omega x' \, dx' \, d\omega, \ x > 0 \tag{7.45}$$

which is called the *Fourier cosine integral representation* of $f(x)$. It is obvious that Eq. (7.45) represents $f(-x)$ when $x < 0$. Therefore, if $f(x)$ is an *even* function, then Eq. (7.45) is valid for all values of x.

The Fourier cosine integral representation of $f(x)$, Eq. (7.45), can also be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(\omega) \cos \omega x \, d\omega \tag{7.46a}$$

with

$$\bar{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \tag{7.46b}$$

Equation (7.46*b*) is known as the *Fourier cosine transform* of $f(x)$ in the interval $(0, \infty)$, and Eq. (7.46*a*) is the corresponding *inversion* formula. The kernel of this transform is

$$K(\omega, x) = \sqrt{\frac{2}{\pi}}\cos\omega x \tag{7.47}$$

which satisfies the characteristic-value problem (7.44) with $\alpha = 0$.

A third integral transform in the semi-infinite interval $(0, \infty)$ can be developed in a similar way by considering the case where the kernel satisfies the characteristic-value problem (7.44) for both $\alpha \neq 0$ and $\beta \neq 0$. Table 7.2 gives a summary of these transforms for the three different kinds of boundary conditions at $x = 0$ that the kernels satisfy.

Moreover, if $f(x)$ is defined for all values of x, then it can be rewritten as

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{f_e(x)} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{f_o(x)}, \quad -\infty < x < \infty \tag{7.48}$$

The first bracket is an *even* function of x and can be represented as

$$\begin{aligned} f_e(x) &= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f_e(x')\cos\omega x'\,dx'\,d\omega \\ &= \frac{1}{\pi}\int_0^\infty \cos\omega x \int_{-\infty}^\infty f(x')\cos\omega x'\,dx'\,d\omega, \quad -\infty < x < \infty \end{aligned} \tag{7.49a}$$

TABLE 7.2

Kernels For Use in Fourier Transforms in the Semi-Finite Interval

Transform: $\bar{f}(\omega) = \int_0^\infty f(x)K(\omega, x)dx$ Inversion: $f(x) = \int_0^\infty \bar{f}(x)K(\omega, x)d\omega$	$\left\{\begin{array}{c} \dfrac{d^2y}{dx^2} + \omega^2 y = 0 \\ \alpha y(0) + \beta\dfrac{dy(0)}{dx} = 0 \\ \lvert y(x)\rvert \leq M \text{ for } x > 0 \end{array}\right\}$
Boundary condition at $x = 0$	Kernel, $K(\omega, x)$†
Third kind ($\alpha \neq 0, \beta, \neq 0$)	$\sqrt{\dfrac{2}{\pi}}\dfrac{\omega\cos\omega x - H\sin\omega x}{\sqrt{\omega^2 + H^2}}$
Second kind ($\alpha_1 = 0, \beta \neq 0$)	$\sqrt{\dfrac{2}{\pi}}\cos\omega x$
First kind ($\alpha_1 \neq 0, \beta, = 0$)	$\sqrt{\dfrac{2}{\pi}}\sin\omega x$

† $H = \alpha/\beta$.

Similarly, the second bracket, which is an *odd* function of x, can be represented as

$$f_o(x) = \frac{1}{\pi}\int_0^\infty \sin\omega x \int_{-\infty}^\infty f(x')\sin\omega x'\,dx'\,d\omega, \quad -\infty < x < \infty \tag{7.49b}$$

Thus, by introducing Eqs. (7.49a) and (7.49*b*) into Eq. (7.48), we obtain

$$\begin{aligned} f(x) = {} & \frac{1}{\pi}\int_0^\infty \cos\omega x \int_{-\infty}^\infty f(x')\cos\omega x'\,dx'\,d\omega \\ & + \frac{1}{\pi}\int_0^\infty \sin\omega x \int_{-\infty}^\infty f(x')\sin\omega x'\,dx'\,d\omega, \quad -\infty < x < \infty \end{aligned} \tag{7.50a}$$

which can be rewritten as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(x')\cos\omega(x'-x)\,dx'\,d\omega, \quad -\infty < x < \infty \tag{7.50b}$$

or, equivalently,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(x')\cos\omega(x'-x)\,dx'\,d\omega, \quad -\infty < x < \infty \tag{7.50c}$$

This expression is called the complete *Fourier integral representation* of $f(x)$ for all values of x. Equation (7.50*b*) can also be rewritten in the alternative *complex form*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega x}\int_{-\infty}^\infty e^{-i\omega x} f(x')\,dx'\,d\omega, \quad -\infty < x < \infty \tag{7.50d}$$

where we have used the relation

$$\cos\omega(x'-x) = \frac{1}{2}\left[e^{i\omega(x'-x)} + e^{-i\omega(x'-x)}\right], \; i = \sqrt{-1}$$

Equation (7.50*d*) can be rearranged in two parts as

$$\bar{f}(\omega) = \int_{-\infty}^\infty e^{-i\omega x'} f(x')\,dx', \quad -\infty < \omega < \infty \tag{7.51a}$$

and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega x}\bar{f}(\omega)\,d\omega, \quad -\infty < x < \infty \tag{7.51b}$$

The function $\bar{f}(\omega)$ defined by Eq. (7.51*a*) is known as the *Fourier transform* of $f(x)$, and Eq. (7.51*b*) is the corresponding *inversion* formula.

7.5 Unsteady-State Heat Conduction in Rectangular Coordinates

In this section, we examine the solution of unsteady-state linear heat conduction problems posed in the rectangular coordinate system by the application of Fourier transforms. In Section 7.3, we have already introduced the method of solution by solving a one-dimensional unsteady-state problem. For two- and three-dimensional unsteady-state problems the method of solution is the same: partial derivatives with respect to space variables are removed from the problem by repeated application of Fourier transforms. Thus, an ordinary differential equation with an initial condition is obtained, the solution of which yields the multiple transform of the temperature distribution. The temperature distribution is then obtained by multiple inversions of the transform. We now consider two representative examples of unsteady-state heat conduction posed in the rectangular coordinate system.

7.5.1 A Semi-Infinite Rectangular Strip

Let the temperature distribution in the semi-infinite rectangular strip shown in Fig. 7.2 initially, at $t = 0$, be $T_i(x, y)$. The surface at $x = 0$ is perfectly insulated. For times $t \geq 0$, the surfaces at $x = L$ and $y = 0$ are kept at temperatures $T_1(y, t)$ and $T_2(x, t)$, respectively, which are functions of both time and position along the respective surfaces. Assuming that both $T_i(x, y)$ and $T_1(y, t)$ vanish as $y \to \infty$, we wish to find the unsteady-state temperature distribution $T(x, y, t)$ in the strip for $t > 0$. The formulation of the problem for $t > 0$ is then given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.52a}$$

$$T(x,y,0) = T_i(x,y) \tag{7.52b}$$

$$\frac{\partial T(0,y,t)}{\partial x} = 0, \; T(L,y,t) = T_1(y,t) \tag{7.52c,d}$$

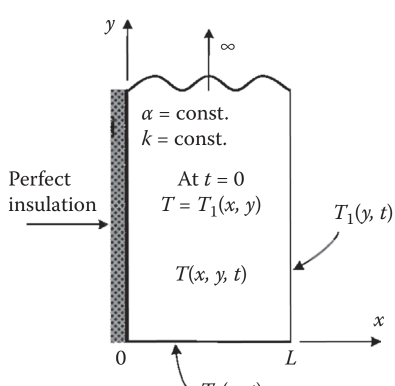

FIGURE 7.2
Semi-infinite rectangular strip.

$$T(x,0,t) = T_2(x,t) \tag{7.52e}$$

We note that the range of the space variable x is finite; that is, it changes from *zero* to L. In this finite interval $(0, L)$, the integral transform $\bar{T}_n(y, t)$ of the temperature distribution $T(x, y, t)$ with respect to the space variable x can be defined as

$$\bar{T}_n(y,t) = \int_0^L T(x,y,t)K_n(x)\,dx \tag{7.53a}$$

with the inversion

$$T(x,y,t) = \sum_{n=1}^{\infty} \bar{T}_n(y,t)K_n(x) \tag{7.53b}$$

The kernels $K_n(x)$ needed in this transform can be determined by following the procedure outlined in Section 7.3. On the other hand, the "rectangular" nature of the partial derivative of $T(x, y, t)$ with respect to x in Eq. (7.52*a*) and the types of the boundary conditions (7.52*c*,*d*) in the x direction would readily indicate that the kernels are the normalized characteristic functions of the following characteristic-value problem:

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \tag{7.54a}$$

$$\frac{dy(0)}{dx} = 0 \text{ and } y(L) = 0 \tag{7.54b,c}$$

The kernels, which can be taken from Table 7.1, are then given by

$$K_n(x) = \sqrt{\frac{2}{L}}\cos\lambda_n x \tag{7.55}$$

with

$$\lambda_n = \frac{(2n-1)\pi}{2L},\ n = 1, 2, 3, \ldots$$

We now obtain the transform of Eq. (7.52*a*) with respect to variable x as

$$\int_0^L K_n(x)\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)dx = \frac{1}{\alpha}\int_0^L K_n(x)\frac{\partial T}{\partial t}\,dx \tag{7.56a}$$

which can be rewritten in the form

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}\,dx + \frac{\partial^2 \bar{T}_n}{\partial y^2} = \frac{1}{\alpha}\frac{\partial \bar{T}_n}{\partial t} \tag{7.56b}$$

The integral on the left-hand side can be evaluated by integrating it by parts twice, resulting in

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx = (-1)^n\sqrt{\frac{2}{L}}\lambda_n T_1(y,t) - \lambda_n^2\overline{T}_n(y,t) \tag{7.57}$$

Substitution of Eq. (7.57) into Eq. (7.56*b*) gives

$$\frac{\partial^2 \overline{T}_n}{\partial y^2} - \lambda_n^2\overline{T}_n(y,t) + (-1)^n\sqrt{\frac{2}{L}}\lambda_n T_1(y,t) = \frac{1}{\alpha}\frac{\partial \overline{T}}{\partial t} \tag{7.58}$$

Equation (7.58) is still a partial differential equation. We now remove the space variable y from Eq. (7.58). Note that the range of the variable y is $(0, \infty)$. In this semi-infinite region, the integral transform $\overline{\overline{T}}_n(\omega, t)$ of the function $\overline{T}_n(y, t)$ with respect to the variable y can be defined as

$$\overline{\overline{T}}_n(\omega,t) = \int_0^\infty \overline{T}_n(y,t)K(\omega,y)dy \tag{7.59a}$$

with the inversion

$$\overline{T}_n(y,t) = \int_0^\infty \overline{\overline{T}}_n(\omega,t)K(\omega,y)d\omega \tag{7.59b}$$

where the kernel $K(\omega, y)$ satisfies the following characteristic-value problem:

$$\frac{d^2\phi}{dy^2} + \omega^2\phi = 0 \tag{7.60a}$$

$$\phi(0) = 0 \text{ and } |\phi(y)| \leq M \text{ for } y > 0 \tag{7.60b,c}$$

where M is some finite constant. Note that, in writing the auxiliary characteristic-value problem (7.60), the homogeneous boundary condition at $y = 0$ is taken to be of the same kind as the boundary condition (7.52*e*) at $y = 0$ of the original problem; that is, they are both of the first kind. In fact, the kernel $K(\omega, y)$, from Table 7.2, is

$$K(\omega,y) = \sqrt{\frac{2}{\pi}}\sin\omega y \tag{7.61}$$

We also note that the kernel $K(\omega, y)$ satisfies the problem (7.60) for all values of ω from *zero* to *infinity*.

We now obtain the transform of Eq. (7.58) with respect to the space variable y:

$$\begin{aligned}\int_0^\infty K(\omega,y)\frac{\partial^2\overline{T}_n}{\partial y^2}dy - \lambda_n^2\int_0^\infty K(\omega,y)\overline{T}_n(y,t)dy \\ +(-1)^n\sqrt{\frac{2}{L}}\lambda_n\int_0^\infty K(\omega,y)T_1(y,t)dy = \frac{1}{\alpha}\int_0^\infty K(\omega,y)\frac{\partial\overline{T}_n}{\partial t}dy\end{aligned} \tag{7.62a}$$

which can also be written as

$$\int_0^\infty K(\omega,y)\frac{\partial^2 \bar{T}_n}{\partial y^2}dy - \lambda^2 \bar{\bar{T}}_n(\omega,t) + (-1)^n\sqrt{\frac{2}{L}}\lambda_n \bar{T}_1(\omega,t) = \frac{1}{\alpha}\frac{d\bar{\bar{T}}_n}{dt} \tag{7.62b}$$

where we have defined

$$\bar{T}_1(\omega,t) = \int_0^\infty T_1(y,t)K(\omega,y)\,dy \tag{7.62c}$$

The integral on the left-hand side of Eq. (7.62*b*) can be evaluated by integrating it by parts twice:

$$\int_0^\infty K(\omega,y)\frac{\partial^2 \bar{T}_n}{\partial y^2}dy = -\omega^2 \bar{\bar{T}}_n(\omega,t) - \sqrt{\frac{2}{L}}\omega \bar{T}_{2n}(t) \tag{7.63a}$$

where

$$\bar{T}_{2n}(t) = \bar{T}_n(0,t) = \int_0^L T_2(x,t)K_n(x)\,dx \tag{7.63b}$$

In obtaining the result (7.63*a*), because $T_i(x, y)$ and $T_1(y, t)$ both vanish as $y \to \infty$, we noted that both the temperature and its first derivative with respect to y also vanish as $y \to \infty$. Substituting Eq. (7.63*a*) into Eq. (7.62*b*) now gives

$$\frac{d\bar{\bar{T}}_n}{dt} + \alpha\left(\omega^2 + \lambda_n^2\right)\bar{\bar{T}}_n(\omega,t) = F_n(\omega,t) \tag{7.64a}$$

where

$$F_n(\omega,t) = \alpha\left[(-1)^n\sqrt{\frac{2}{L}}\lambda_n \bar{T}_1(\omega,t) + \sqrt{\frac{2}{\pi}}\omega \bar{T}_{2n}(t)\right] \tag{7.64b}$$

The first-order differential equation (7.64*a*) for the double transform $\bar{\bar{T}}_n(\omega, t)$ can be solved easily, and the solution is given by

$$\bar{\bar{T}}_n(\omega,t) = e^{-\alpha\left(\omega^2+\lambda_n^2\right)t}\left[\bar{\bar{T}}_{in}(\omega) + \int_0^t e^{\alpha(\omega^2-\lambda_n^2)t'}F_n(\omega,t')\,dt'\right] \tag{7.65a}$$

where

$$\bar{\bar{T}}_{in}(\omega) = \int_0^L K_n(x)\left\{\int_0^\infty K(\omega,y)T_i(x,y)\,dy\right\}dx \tag{7.65b}$$

Inverting Eq. (7.65*a*) twice, first with the inversion formula (7.59*b*) and then with (7.53*b*), we obtain the solution for $T(x. y, t)$:

$$T(x,y,t) = \sum_{n=1}^{\infty} K_n(x)\left\{\int_0^{\infty} K(\omega,y)e^{-\alpha\left(\omega^2+\lambda_n^2\right)t} \times\left[\bar{\bar{T}}_{in}(\omega) + \int_0^t e^{\alpha(\omega^2+\lambda_n^2)t'} F_n(\omega,t')dt'\right]d\omega\right\} \tag{7.66}$$

where all the terms have been defined previously.

This example demonstrates the procedure in solving a two-dimensional unsteady-state problem with the application of Fourier transforms. The method of solution of three-dimensional problems is exactly the same: by repeated application of Fourier transforms three times, the problem is reduced to an ordinary differential equation for the transform of the temperature distribution with an initial condition. Solution of this ordinary differential equation yields the transform of the temperature distribution, which can readily be inverted through the use of relevant inversion relations.

7.5.2 Infinite Medium

We now consider an infinite medium, $-\infty < x < \infty$, of constant thermal conductivity k and thermal diffusivity α. Initially, the region $-L < x < L$ is at a nonuniform temperature $T_i(x)$ at $t = 0$, and everywhere outside this region is at zero temperature. We wish to find the unsteady-state temperature distribution $T(x, t)$ in the medium for times $t > 0$. The formulation of the problem is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.67a}$$

$$T(x,0) = \begin{cases} T_i(x), & -L < x < L \\ 0, & x < -L \text{ and } x > L \end{cases} \tag{7.67b}$$

The range of the x variable is $(-\infty, \infty)$. Accordingly, we take the Fourier transform of Eq. (7.67*a*) by applying Eq. (7.51*a*),

$$\int_{-\infty}^{\infty} e^{-i\omega x}\frac{\partial^2 T}{\partial x^2}dx = \frac{1}{\alpha}\frac{d\bar{T}}{dt} \tag{7.68a}$$

where

$$\bar{T}(w,t) = \int_{-\infty}^{\infty} e^{-i\omega x}T(x,t)dx \tag{7.68b}$$

is the Fourier transform of the temperature distribution $T(x, t)$. The integral on the left-hand side of Eq. (7.68*a*) can be evaluated by parts twice to yield

$$\int_{-\infty}^{\infty} e^{-i\omega x}\frac{\partial^2 T}{\partial x^2}dx = -\omega^2\bar{T}(\omega,t) \tag{7.69}$$

In obtaining the result (7.69), we let both $T(x, t)$ and $\partial T/\partial x$ vanish as $x \to \pm\infty$. Substituting Eq. (7.69) into Eq. (7.68*a*), we get

$$\frac{d\bar{T}}{dt} + \alpha\omega^2 \bar{T}(\omega, t) = 0 \tag{7.70}$$

The Fourier transform of the initial condition (7.67*b*), on the other hand, yields

$$\begin{aligned}\bar{T}(\omega, 0) &= \int_{-\infty}^{\infty} e^{-i\omega x} T(x, 0)\, dx \\ &= \int_{-L}^{L} e^{-i\omega x} T_i(x)\, dx\end{aligned} \tag{7.71}$$

The solution of Eq. (7.70), subject to the initial condition (7.71), is

$$\bar{T}(\omega, t) = e^{-\alpha\omega^2 t} \int_{-L}^{L} e^{-i\omega x} T(x, t)\, dx \tag{7.72}$$

Inverting $\bar{T}(\omega, t)$ by the use of the inversion formula (7.51*b*) we obtain

$$T(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x - \alpha\omega^2 t} \int_{-L}^{L} e^{-i\omega x'} T_i(x')\, dx'\, d\omega \tag{7.73}$$

or, after changing the order of integration, Eq. (7.73) becomes

$$T(x, t) = \frac{1}{2\pi} \int_{-L}^{L} T_i(x') \int_{-\infty}^{\infty} e^{-\alpha\omega^2 t - i\omega(x' - x)}\, d\omega\, dx' \tag{7.74}$$

In this relation, the integral with respect to ω can be evaluated by making use of the relation [4]

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha\omega^2 t - i\omega(x' - x)}\, d\omega = \frac{1}{\sqrt{4\pi\alpha t}} e^{-\frac{(x' - x)^2}{4\alpha t}} \tag{7.75}$$

Thus, Eq. (7.74) reduces to

$$T(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-L}^{L} T_i(x') e^{-\frac{(x' - x)^2}{4\alpha t}}\, dx' \tag{7.76}$$

If, in particular, $T_i(x) = T_i =$ constant, then the solution is given by

$$T(x, t) = \frac{T_i}{\sqrt{4\pi\alpha t}} \int_{-L}^{L} e^{-\frac{(x' - x)^2}{4\alpha t}}\, dx' \tag{7.77}$$

By introducing a new variable,

$$\eta = \frac{x' - x}{\sqrt{4\alpha t}} \quad \Rightarrow \quad dx' = \sqrt{4\alpha t}\, d\eta$$

Eq. (7.77) can be rewritten as

$$T(x,t) = \frac{T_i}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{(L-x)/\sqrt{4\alpha t}} e^{-\eta^2}\, d\eta + \frac{2}{\sqrt{\pi}}\int_0^{(L+x)/\sqrt{4\alpha t}} e^{-\eta^2}\, d\eta\right] \tag{7.78}$$

or, in the form

$$\frac{T(x,t)}{T_i} = \frac{1}{2}\left[\operatorname{erf}\left(\frac{L-x}{\sqrt{4\alpha t}}\right) + \operatorname{erf}\left(\frac{L+x}{\sqrt{4\alpha t}}\right)\right] \tag{7.79}$$

where erf(x) is the *error function*, which is defined by Eq. (6.92).

7.6 Steady-State Two- and Three-Dimensional Problems in Rectangular Coordinates

Steady-state two- and three-dimensional linear heat conduction problems in rectangular coordinates can also be solved by applying Fourier transforms. The procedure is the same as in solving unsteady-state problems; the partial derivatives with respect to space variables are removed by repeated application of Fourier transforms, and thus the partial differential equation is reduced to an algebraic equation which is then inverted to find the temperature distribution. However, we may prefer not to remove all the partial derivatives, but to leave one space variable and thus reduce the original partial differential equation to an ordinary differential equation. Solution of this differential equation gives the transform of the temperature distribution, which is then inverted to find the temperature distribution. We now discuss the solution of one example problem in both ways to explain the method of solution.

Let us re-solve the problem discussed in Section 5.2 by applying Fourier transforms. The formulation of the problem is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{7.80a}$$

$$T(0,y) = T(a,y) = 0 \tag{7.80b,c}$$

$$T(x,0) = 0, \; T(x,b) = f(x) \tag{7.80d,e}$$

In the finite interval (0, *a*), the Fourier transform of the temperature distribution $T(x, y)$ with respect to variable x can be defined as

$$\bar{T}_n(y) = \int_0^a T(x,y)K_n(x)dx \tag{7.81a}$$

with the inversion formula

$$T(x,y) = \sum_{n=1}^{\infty} \bar{T}_n(y)K_n(x) \tag{7.81b}$$

In view of the type of the boundary conditions of the problem at $x = 0$ and $x = a$, we take the kernels from Table 7.1 as

$$K_n(x) = \sqrt{\frac{2}{a}}\sin\lambda_n x \tag{7.82}$$

with

$$\lambda_n = \frac{n\pi}{a},\ n = 1, 2, 3, \ldots$$

We now obtain the transform of the heat conduction equation (7.80*a*) as follows:

$$\int_0^a K_n(x)\frac{\partial^2 T}{\partial x^2}dx + \int_0^a K_n(x)\frac{\partial^2 T}{\partial y^2}dx = 0 \tag{7.83a}$$

which can be written as

$$\int_0^a K_n(x)\frac{\partial^2 T}{\partial x^2}dx + \frac{d^2\bar{T}_n}{dy^2} = 0 \tag{7.83b}$$

Integrating by parts twice, it can easily be shown that the integral on the left-hand side of Eq. (7.83*b*) reduces to

$$\int_0^a K_n(x)\frac{\partial^2 T}{\partial x^2}dx = -\lambda_n^2\bar{T}_n(y) \tag{7.84}$$

Substitution of Eq. (7.84) into Eq. (7.836) yields

$$\frac{d^2\bar{T}_n}{dy^2} - \lambda_n^2\bar{T}_n(y) = 0 \tag{7.85}$$

Equation (7.85) is a second-order ordinary differential equation, the solution of which can be written as

$$\bar{T}_n(y) = A_n \sinh \lambda_n y + B_n \cosh \lambda_n y \tag{7.86}$$

The transforms of the boundary conditions at $y = 0$ and at $y = b$ yield

$$\bar{T}_n(0) = \int_0^a T(x,0)K_n(x)dx = 0 \tag{7.87a}$$

and

$$\bar{T}_n(b) = \int_0^a T(x,b)K_n(x)dx = \int_0^a f(x)K_n(x)dx = \bar{f}_n \tag{7.87b}$$

Application of the condition (7.87*a*) gives $B_n = 0$, and the condition (7.87*b*) gives

$$A_n = \frac{\bar{f}_n}{\sinh \lambda_n b}$$

Thus, the transform of the temperature distribution is given by

$$\bar{T}_n(y) = \bar{f}_n \frac{\sinh \lambda_n y}{\sinh \lambda_n b} \tag{7.88}$$

Equation (7.88) can now inverted to yield

$$T(x,y) = \sum_{n=1}^{\infty} \bar{f}_n \frac{\sinh \lambda_n y}{\sinh \lambda_n b} K_n(x) \tag{7.89}$$

which can also be written as

$$T(x,y) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(n\pi/a)x \sinh(n\pi/a)y}{\sinh(n\pi/a)b} \int_0^a f(x') \sin \frac{n\pi}{a} x' dx' \tag{7.90}$$

This is the same result as Eq. (5.17).

As an *alternative* approach, we define the Fourier transform of $\bar{T}_n(y)$ with respect to variable y in the finite interval (0, *b*) as

$$\bar{\bar{T}}_{nm} = \int_0^b \bar{T}_n(y)K_m(y)dy \tag{7.91a}$$

with the inversion

$$\bar{T}_n(y) = \sum_{m=1}^{\infty} \bar{\bar{T}}_{nm} K_m(y) \tag{7.91b}$$

where the kernels $K_m(y)$ are given by

$$K_m(y) = \sqrt{\frac{2}{b}} \sin \beta_m y \tag{7.91c}$$

with

$$\beta_m = \frac{m\pi}{b}, \; m = 1, 2, 3, \ldots$$

In taking $K_m(y)$ and the characteristic values β_m from Table 7.1, we made use of the type of the boundary conditions at $y = 0$ and $y = b$ of the original problem.

We now obtain the transform of Eq. (7.85) with respect to variable y as follows:

$$\int_0^b K_m(y) \frac{d^2 \bar{T}_n}{dy^2} dy - \lambda_n^2 \int_0^b K_m(y) \bar{T}_n(y) dy = 0 \tag{7.92a}$$

which can be rewritten as

$$\int_0^b K_m(y) \frac{d^2 \bar{T}_n}{dy^2} dy - \lambda_n^2 \bar{\bar{T}}_{nm} = 0 \tag{7.92b}$$

The integral in Eq. (7.92*b*) can be evaluated by parts to yield

$$\int_0^b K_m(y) \frac{d^2 \bar{T}_n}{dy^2} dy = -\beta_m^2 \bar{\bar{T}}_{nm} + (-1)^{m+1} \sqrt{\frac{2}{b}} \beta_m \bar{f}_n \tag{7.93}$$

Substitution of Eq. (7.93) into Eq. (7.92*b*) yields the following algebraic equation:

$$(\lambda_n^2 + \beta_m^2) \bar{\bar{T}}_{nm} = (-1)^{m+1} \sqrt{\frac{2}{b}} \beta_m \bar{f}_n \tag{7.94a}$$

or

$$\bar{\bar{T}}_{nm} = (-1)^{m+1} \sqrt{\frac{2}{b}} \frac{\beta_m \bar{f}_n}{\lambda_n^2 + \beta_m^2} \tag{7.94b}$$

Inversion of Eq. (7.94*b*), through Eq. (7.81*b*), yields

$$\bar{T}_n(y) = \sqrt{\frac{2}{b}} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\beta_m \bar{f}_n}{\lambda_n^2 + \beta_m^2} K_m(y) \tag{7.95}$$

and the inversion of Eq. (7.95), through Eq. (7.53*b*), gives the temperature distribution

$$T(x,y) = \sqrt{\frac{2}{b}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\beta_m \bar{f}_n}{\lambda_n^2 + \beta_m^2} K_m(y) K_n(x) \tag{7.96}$$

which can also be written as

$$\begin{aligned} T(x,y) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(m\pi/b)\sin(n\pi/a)x \sin(m\pi/b)y}{(n\pi/a)^2 + (m\pi/b)^2} \\ \times \int_0^a f(x') \sin \frac{n\pi}{a} x' \, dx' \end{aligned} \tag{7.97}$$

We have thus obtained another expression for the same temperature distribution. Now the question arises: are these two expressions the same? The answer is yes, because it can easily be shown that (see Problem 7.4*b*)

$$\frac{\sinh(n\pi/a)y}{\sinh(n\pi/a)b} = \frac{2}{b} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(m\pi/b)\sin(m\pi/b)y}{(n\pi/a)^2 + (m\pi/b)^2} \tag{7.98}$$

One disadvantage of the second approach might be the presence of an additional infinite series in the result.

The method of solution of three-dimensional steady-state problems is exactly the same as for two-dimensional problems discussed in this section.

7.7 Hankel Transforms

Integral transforms whose kernels are Bessel functions are called *Hankel* transforms, and they are obtained from the expansion of an arbitrary function in an infinite series of Bessel functions. They are also referred to as *Bessel transforms.* There are a great variety of these transforms, first because of the variety of Bessel functions that are the solutions of Bessel's differential equation

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0$$

containing two parameters λ and v. Furthermore, the Hankel transforms may be developed over either a finite interval or a semi-infinite region with various boundary conditions, or even over the infinite region. The mathematical theory of Hankel transforms can be found in References [2] and [10], and the applications of these transforms to various heat conduction problems are given in Reference [7].

In this text, we restrict our discussions to *finite Hankel transforms.* The use of finite Hankel transforms was first suggested by Sneddon [9] in 1946, and later by Eringen [5] in 1954. These transforms were extensively used by Özışık [7] in the solution of various heat conduction problems posed in cylindrical coordinates.

The *finite Hankel transform* of an arbitrary function $f(r)$ on the region (a, b) is defined as

$$\bar{f}_n = \int_a^b f(r) K_v(\lambda_n, r) r\, dr \tag{7.99a}$$

with the inversion

$$f(r) = \sum_{n=1}^{\infty} \bar{f}_n K_v(\lambda_n, r) \tag{7.99b}$$

where the kernels $K_v(\lambda_n, r)$ are the normalized characteristic functions of the following characteristic-value problem:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \tag{7.100a}$$

$$\alpha_1 R(a) + \beta_1 \frac{dR(a)}{dr} = 0, \;\; \alpha_1^2 + \beta_1^2 \neq 0 \tag{7.100b}$$

$$\alpha_2 R(b) + \beta_2 \frac{dR(b)}{dr} = 0, \;\; \alpha_2^2 + \beta_2^2 \neq 0 \tag{7.100c}$$

The kernels $K_v(\lambda_n, r)$ and the characteristic values λ_n have been evaluated and listed by Özışık [7] for the nine different combinations of the boundary conditions at $r = a$ and $r = b$.

When the region of the transform is $(0, r_0)$, the kernels $K_\nu(\lambda_n, r)$ are the normalized characteristic functions of the following characteristic-value problem:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \tag{7.101a}$$

$$R(0) = \mathit{finite} \tag{7.101b}$$

$$\alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0, \;\; \alpha^2 + \beta^2 \neq 0 \tag{7.101c}$$

As we discussed in Section 4.8, the characteristic functions of this problem are $J_v(\lambda_r)$. In Section 4.8, we also evaluated the characteristic values λ_n and the normalization integrals

N_n for each special case of the boundary condition at $r = r_0$. For the case where the boundary condition (7.101*c*) is of the third kind (i.e., $\alpha \neq$ and $\beta \neq 0$), the normalization integrals N_n are given by (see Table 4.2)

$$N_n = \frac{r_0^2}{2}\left[1 + \frac{1}{\lambda_n^2}\left(H^2 - \frac{v^2}{r_0^2}\right)\right] J_v^2(\lambda_n r_0) \tag{7.102a}$$

where the characteristic values λ_n are the positive roots of the transcendental equation

$$HJ_v(\lambda r_0) + \frac{dJ_v(\lambda r_0)}{dr} = 0 \tag{7.102b}$$

with $H = \alpha/\beta$. The kernels $K_v(\lambda_n, r)$ are then obtained as

$$K_v(\lambda_n, r) = \frac{J_v(\lambda_n r)}{\sqrt{N_n}} = \frac{\sqrt{2}}{r_0} \frac{1}{\left[1 + \left(H^2 - v^2/r_0^2\right)/\lambda_n^2\right]^{1/2}} \frac{J_v(\lambda_n r)}{J_v(\lambda_n r_0)} \tag{7.102c}$$

The kernels for the remaining two cases (i.e., $\alpha = 0, \beta \neq 0$ and $\alpha \neq 0, \beta = 0$) can be obtained following the same procedure. They can also be obtained readily as limiting cases from Eq. (7.102*c*). The kernels for these cases are summarized in Table 7.3.

The theory of Hankel transforms for the half-space can be found in Reference [10]. Applications to heat conduction problems are given in Reference [7].

TABLE 7.3

Kernels and Characteristic Values for Use in Finite Hankel Transforms in the Finite Region $(0, r_0)$

Transform: $\bar{f}_n = \int_a^{r_0} f(r) K_v(\lambda_n, r) r\, dr$

Inversion: $f(r) = \sum_{n=1}^{\infty} \bar{f}_n K_v(\lambda_n, r)$

$$\left\{\begin{aligned} & r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \\ & R(0) = \textit{finite} \\ & \alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0 \end{aligned}\right\}$$

Boundary condition at $r = r_0$	Kernel, $K_v(\lambda_n, r)$†	Characteristic values λ_n's are positive roots of†
Third kind‡ ($\alpha \neq 0, \beta \neq 0$)	$\frac{\sqrt{2}}{r_0} \frac{1}{\left[1 + 1/\lambda_n^2\right]\left(H^2 - v^2/r_0^2\right)^{1/2}} \frac{J_v(\lambda_n r)}{J_v(\lambda_n r_0)}$	$HJ_v(\lambda r_0) + \frac{dJ_v(\lambda r_0)}{dr} = 0$
Second kind ($\alpha \neq 0, \beta \neq 0$)	$\frac{\sqrt{2}}{r_0} \frac{1}{\left(1 - v^2/\lambda_n r_0^2\right)^{1/2}} \frac{J_v(\lambda_n r)}{J_v(\lambda_n r_0)}$	$\frac{dJ_v(\lambda r_0)}{dr} = 0$§
First kind ($\alpha \neq 0, \beta \neq 0$)	$\frac{\sqrt{2}}{r_0} \frac{J_v(\lambda_n r)}{J_{v+1}(\lambda_n r_0)}$	$J_v(\lambda r_0) = 0$

† $H = \alpha/\beta$.
‡ See footnote ‡ in Table 4.2, and modify the transform accordingly.
§ When $v = 0$, $\lambda_0 = 0$ is also a characteristic value for this case.

7.8 Problems in Cylindrical Coordinates

Linear heat conduction problems with constant thermal conductivity and posed in cylindrical coordinates, in general, will involve the following terms

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

in the heat conduction equation. The partial derivative with respect to z can be removed from the problem by applying Fourier transforms in the z direction. The partial derivative with respect to ϕ can also be removed by applying Fourier transforms in the ϕ direction, provided that the range of ϕ is $(0, \phi_0)$, where $\phi_0 < 2\pi$. If the range of ϕ is $(0, 2\pi)$, then development of a new transform is necessary to remove the partial derivative with respect to the variable ϕ [7]. The partial derivatives with respect to r are removed by the application of Hankel transforms. If the problem involves both variables r and ϕ, the partial derivative with respect to the variable ϕ need to be removed first because of the coefficient $1/r^2$ of this derivative.

As an example, consider a half cylinder of semi-infinite length, $0 \le r \le r_0$, $0 \le \phi \le \pi$ and $0 \le z < \infty$ as illustrated in cross-section in Fig. 7.3. Internal energy is generated in this cylinder at a rate of $q(r, \phi)$ per unit volume. The surfaces at $\phi = 0$, $\phi = \pi$, and $z = 0$ are at zero temperature, while the surface at $r = r_0$ is kept at temperature $T_w(\phi)$. We wish to find the steady-state temperature distribution $T(r, \phi, z)$ in the cylinder. Assuming constant thermophysical properties, the problem can be formulated as:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}(r,\phi)}{k} = 0 \tag{7.103a}$$

$$T(0,\phi,z) = 0 \text{ and } T(r_0,\phi,z) = T_w(\phi) \tag{7.103b,c}$$

$$T(r,0,z) = T(r,\pi,z) = 0 \tag{7.103d,e}$$

$$T(r,\phi,0) = 0 \text{ and } T(r,\phi,\infty) = \textit{finite} \tag{7.103f,g}$$

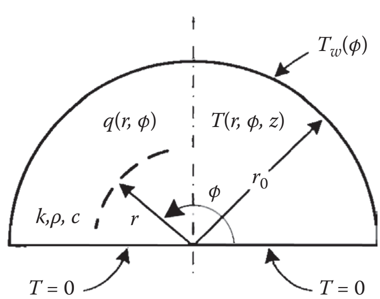

FIGURE 7.3
Half-cylinder of semi-infinite length.

The partial derivative with respect to the variable ϕ can be removed by Fourier transforms.

The range of ϕ is $(0, \pi)$, and in this finite interval the finite Fourier transform $\bar{T}_n(r, z)$ of $T(r, \phi, z)$ with respect to the variable ϕ can be defined as

$$\bar{T}_n(r,z) = \int_0^{\pi} T(r,\phi,z) K_n(\phi)\, d\phi \tag{7.104a}$$

with the inversion

$$T(r,\phi,z) = \sum_{n=1}^{\infty} \bar{T}_n(r,z) K_n(\phi) \tag{7.104b}$$

where the kernels $K_n(\phi)$ are the normalized characteristic functions of the following characteristic-value problem:

$$\frac{d^2\psi}{d\phi^2} + n^2\psi = 0 \tag{7.105a}$$

$$\psi(0) = \psi(\pi) = 0 \tag{7.105b,c}$$

The kernels $K_n(\phi)$ can, however, be taken from Table 7.1:

$$K_n(\phi) = \sqrt{\frac{2}{\pi}} \sin n\phi, \quad n = 1, 2, 3, \ldots$$

The transform of the heat conduction equation (7.103*a*) with respect to ϕ, through the use of Eq. (7.104*a*), yields

$$\frac{\partial^2 \bar{T}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_n}{\partial r} - \frac{n^2}{r^2}\bar{T}_n(r,z) + \frac{\partial^2 \bar{T}_n}{\partial z^2} + \frac{\bar{\dot{q}}_n(r)}{k} = 0 \tag{7.106}$$

where we have defined

$$\bar{\dot{q}}_n(r) = \int_0^{\pi} \dot{q}(r,\phi) K_n(\phi)\, d\phi \tag{7.107}$$

Equation (7.106) involves the differential operator

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}$$

In order to remove this differential operator, we define the finite Hankel transform $\bar{\bar{T}}_n(\lambda_m, z)$ of the function $\bar{T}_n(r, z)$ in the finite interval $(0, r_0)$ as

$$\bar{\bar{T}}_n(\lambda_m, z) = \int_0^{r_0} \bar{T}_n(r, z) K_n(\lambda_m, r) r\, dr \tag{7.108a}$$

with the inversion

$$\bar{T}_n(r, z) = \sum_{m=1}^{\infty} \bar{\bar{T}}_n(\lambda_m, z) K_n(\lambda_m, r) \tag{7.108b}$$

where the kernels $K_n(\lambda_m, r)$ are the normalized characteristic functions of the following characteristic value problem:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - n^2)R = 0 \tag{7.109a}$$

$$R(0) = \textit{finite} \text{ and } R(r_0) = 0 \tag{7.109b,c}$$

The kernels $K_n(\lambda_m, r)$, from Table 7.3, are given by

$$K_n(\lambda_m, r) = \frac{\sqrt{2}}{r_0} \frac{J_n(\lambda_m r)}{J_{n+1}(\lambda_m r_0)} \tag{7.110a}$$

where the characteristic values λ_m are positive roots of

$$J_n(\lambda r_0) = 0 \tag{7.110b}$$

Now, the transform of Eq. (7.106) with respect to r, through the use of transform (7.108a), yields

$$\frac{d^2\bar{\bar{T}}_n}{dz^2} - \lambda_m^2 \bar{\bar{T}}_n(\lambda_m, z) = r_0 \frac{dK_n(\lambda_m, r_0)}{dr} \bar{T}_{wn} - \frac{1}{k} \bar{\bar{\dot{q}}}_n(\lambda_m) \tag{7.111}$$

where we have defined

$$\bar{T}_{wn} = \int_0^{\pi} K_n(\phi) T_w(\phi) d\phi \tag{7.112a}$$

and

$$\bar{\bar{\dot{q}}}_n(\lambda_m) = \int_0^{r_0} \bar{\dot{q}}_n(r) K_n(\lambda_m, r) r\, dr \tag{7.112b}$$

Equation (7.111) can further be transformed with respect to z in the semi-infinite interval (0, ∞) to reduce it to an algebraic equation. However, here we prefer to solve this ordinary differential equation. The solution can be written as

$$\bar{\bar{T}}_n(\lambda_m, z) = A_n^m e^{-\lambda_m z} + B_n^m e^{\lambda_m z} + \bar{\bar{T}}_{pn}(\lambda_m) \tag{7.113}$$

where the particular solution $\bar{\bar{T}}_{pn}(\lambda_m)$ is given by

$$\bar{\bar{T}}_{pn}(\lambda_m) = -\frac{1}{\lambda_m^2}\left[r_0 \frac{dK_n(\lambda_m, r_0)}{dr}\bar{T}_{wn} - \frac{1}{k}\bar{\bar{\dot{q}}}_n(\lambda_m)\right] \tag{7.114}$$

Since the temperature distribution $T(r, \phi, z)$ is to be finite as $z \to \infty$, we have

$$\lim_{z\to\infty} \bar{\bar{T}}_n(\lambda_m, z) = \mathit{finite}$$

which yields $B_n^m = 0$. On the other hand, since $T(r, \phi, 0) = 0$,

$$\bar{\bar{T}}_n(\lambda_m, z) = 0$$

which gives

$$A_n^m = -\bar{\bar{T}}_p(\lambda_m, n)$$

The solution for $\bar{\bar{T}}_n(\lambda_m, z)$ now becomes

$$\bar{\bar{T}}_n(\lambda_m, z) = (1 - e^{-\lambda_m z})\bar{\bar{T}}_{pn}(\lambda_m) \tag{7.115}$$

When this double transform is inverted successively through the use of inversion relations (7.108*b*) and (7.104*b*), we obtain the temperature distribution as

$$T(r, \phi, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(1 - e^{-\lambda_m z})\bar{\bar{T}}_{pn}(\lambda_m)K_n(\lambda_m, r)K_n(\phi) \tag{7.116a}$$

which can also be written as

$$\begin{aligned} T(r, \phi, z) = \frac{4}{\pi r_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1 - e^{-\lambda_m z}}{\lambda_m^2}\sin n\phi \frac{J_n(\lambda_m r)}{J_{n+1}(\lambda_m r_0)}\Bigg[\lambda_m \int_0^{\pi} \sin n\phi' T_w(\phi')\, d\phi' \\ + \frac{1}{k r_0 J_{n+1}(\lambda_m r_0)}\int_0^{r_0}\int_0^{\pi} J_n(\lambda_m r')\sin n\phi' \dot{q}(r', \phi') r'\, dr' d\phi'\Bigg] \end{aligned} \tag{7.116b}$$

The method of solution of unsteady-state heat conduction problems in cylindrical coordinates follows the same procedure as in rectangular coordinates.

7.9 Problems in Spherical Coordinates

As we discussed in Chapter 6, one-dimensional linear heat conduction problems posed in spherical coordinates may be transformed into rectangular coordinates by introducing a new temperature function. Fourier transforms can then be used to solve such problems. The following example illustrates the method of solution.

Consider the hollow sphere shown in Fig. 7.4, which is initially at temperature $T_i(r)$. The surfaces at $r = a$ and $r = b$ are maintained at temperatures $T_1(t)$ and $T_2(t)$, respectively, for times $t \geq 0$. We wish to find the unsteady-state temperature distribution $T(r, t)$ in this spherical shell for $t > 0$. Assuming constant thermophysical properties, the formulation of the problem is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.117a}$$

$$T(r,0) = T_i(r) \tag{7.117b}$$

$$T(a,t) = T_1(t) \text{ and } T(b,t) = T_2(t) \tag{7.117c,d}$$

We now define a new temperature function $\theta(r, t)$ as

$$\theta(r,t) = rT(r,t) \tag{7.118}$$

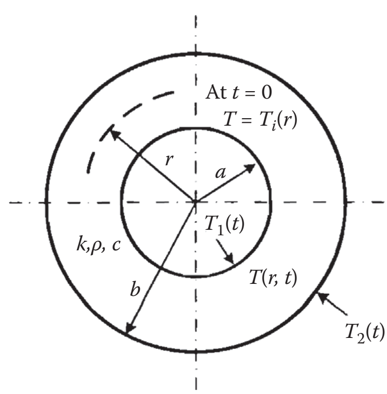

FIGURE 7.4
Hollow sphere.

The formulation of the problem can then be rewritten in terms of $\theta(r, t)$ as

$$\frac{\partial^2\theta}{\partial r^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{7.119a}$$

$$\theta(r,0) = rT_i(r) \tag{7.119b}$$

$$\theta(a,t) = aT_1(t) \text{ and } \theta(b,t) = bT_2(t) \tag{7.119c,d}$$

Now, a change of variable $r = x + a$ yields

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{7.120a}$$

$$\theta(x,0) = (x+a)T_i(x+a) \tag{7.120b}$$

$$\theta(0,t) = aT_1(t) \text{ and } \theta(b-a,t) = bT_2(t) \tag{7.120c,d}$$

This problem can now be solved readily by using Fourier transforms. We note that the range of variable x is $(0, b-a)$, and in this finite interval the Fourier transform $\bar{\theta}\ n\ (t)$ of the function $\theta(x, t)$ can be defined as

$$\bar{\theta}_n(t) = \int_0^{b-a} \theta(x,t)K_n(x)dx \tag{7.121a}$$

with the inversion

$$\theta(x,t) = \sum_{n=1}^{\infty} \bar{\theta}_n(t)K_n(x) \tag{7.121b}$$

where the kernels $K_n(x)$ are given by

$$K_n(x) = \sqrt{\frac{2}{b-a}}\sin\lambda_n x \tag{7.121c}$$

with the characteristic values

$$\lambda_n = \frac{n\pi}{b-a},\ n = 1, 2, 3, \ldots \tag{7.121d}$$

The transform of Eq. (7.120*a*), through the use of Eq. (7.121*a*), yields

$$\frac{d\bar{\theta}_n}{dt} + \alpha\lambda_n^2\bar{\theta}_n(t) = \alpha\sqrt{\frac{2}{b-a}}\left[(-1)^n aT_1(t) - bT_2(t)\right] \tag{7.122}$$

the solution of which can be written as

$$\bar{\theta}_n(t) = e^{-\alpha\lambda_n^2 t}\left\{\bar{\theta}_n(0) + \alpha\sqrt{\frac{2}{b-a}}\int_0^t \left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\} \tag{7.123}$$

where

$$\bar{\theta}_n(0) = \int_0^{b-a}(x+a)T_i(x+a)K_n(x)\,dx \tag{7.124}$$

Equation (7.123) can now be inverted using Eq. (7.121*b*) to obtain $\theta(x, t)$:

$$\begin{aligned}\theta(x,t) = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^{b-a}(x'+a)T_i(x'+a)\sin\lambda_n x'\,dx'\right.\\ \left.+\alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}\end{aligned} \tag{7.125a}$$

which can also be written as

$$\begin{aligned}\theta(r,t) = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n(r-a)\left\{\int_0^{b} r'T_i(r')\sin\lambda_n(r'-a)\,dr'\right.\\ \left.+\alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}\end{aligned} \tag{7.125b}$$

The temperature distribution $T(r, t)$ then becomes

$$\begin{aligned}T(r,t) = \frac{2}{r(b-a)}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n(r-a)\left\{\int_0^{b} T_i(r')\sin\lambda_n(r'-a)\,r'\,dr'\right.\\ \left.+\alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}\end{aligned} \tag{7.126}$$

In order to obtain the solutions of two- and three-dimensional heat conduction problems in spherical coordinates by using integral transforms, new transforms may need to be developed to remove the partial derivatives with respect to the r-variable. These transforms can be obtained from the expansion of an arbitrary function $f(r)$ in an infinite series of *Legendre polynomials.* Such transforms were first discussed by Tranter [11] and Churchill [3], and were later applied to the solutions of linear heat conduction problems in spherical coordinates by Özışık [7].

7.10 Observations on the Method

As we have demonstrated by means of several representative cases in this chapter, in solving a heat conduction problem by the application of an integral transform, the problem is first reduced to a much simpler problem consisting of an ordinary differential equation together with an initial condition for unsteady-state problems or with two boundary conditions for steady-state problems. We have also seen that steady-state problems, if preferred, can be reduced to an algebraic equation for the transform of the temperature distribution. Once the solution of the simpler problem, or the algebraic equation, is found, the inversion relations, which are given at the onset of the problem, are used to obtain the solution of the original problem. Figure 7.5 describes diagrammatically the implementation of the integral transform methods. In the next chapter we will see that a similar procedure is followed in the application of Laplace transforms for the solution of unsteady-state heat conduction problems. The path in Fig. 7.5 from the formulation to the solution is shown by a broken line because it can seldom be realized and the usual path is along the full line.

The final integral transform solutions for multidimensional problems are expressed as nested summations, as given by Eq. 7.96 or 7.116. However, from a computational point of view, only a truncated version of such summations can actually be evaluated. The truncation of the individual sums to finite orders in each direction leads the total number of terms in these series to become the product of the individual truncation orders. Among all such terms, there can be several ones that have practically no contribution to the final converged result, while others have been neglected that would still alter the final result. In this sense, an appropriate way of performing this multiple summations must involve the reorganization of the multiple series into a single series representation, through an adequate ordering scheme, that progressively accounts for the most important terms. The most common choice of reordering strategy is based on rearranging in increasing order the

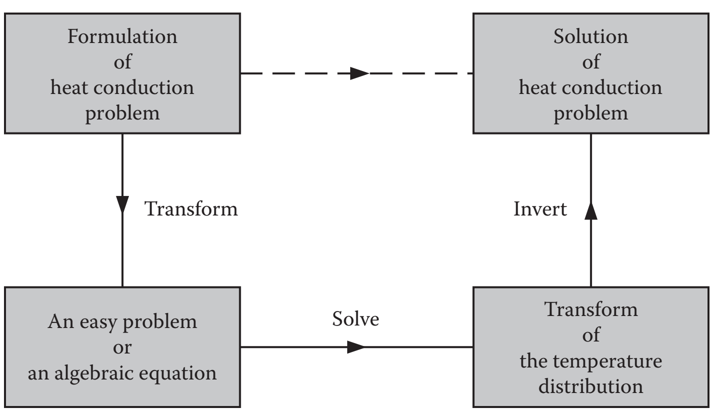

FIGURE 7.5
From the formulation to the solution.

sum of the squared eigenvalues in each spatial coordinate. However, individual applications may require more elaborate reordering that accounts for the influence of transformed initial conditions or source terms. The aim of the reordering procedure consists of keeping the evaluation of additional eigenvalues and related quantities to a minimum, avoiding increase on the computational cost. For further information on reordering schemes for multidimensional eigenfunction expansions, the interested readers may consult Ref. [13].

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Problems

7.1 From the expansion obtained in part (*b*) of Problem 4.6, develop a finite Fourier transform of the function $f(x)$ on $(0, L)$.

7.2 From the expansion obtained in part (*b*) of Problem 4.16, develop a finite Hankel transform of the function $f(r)$ on (a, b).

7.3 Obtain the solution given by Eq. (7.33) directly after reformulating the problem given by Eqs. (7.16) in terms of $\theta(x, t) = T(x, t) - T_\infty(t)$.

7.4 Show that the following are two valid expansions:

$$(a)\ \frac{x}{L} - \left(\frac{x}{L}\right)^2 = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^3}\sin\frac{n\pi}{L}x,\ 0 < x < L$$

$$(b)\ \frac{\sinh(n\pi/a)y}{\sinh(n\pi/a)b} = \frac{2}{b}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(m\pi/b)\sin(m\pi/b)y}{(n\pi/a)^2+(m\pi/b)^2},\ 0 < y < b$$

7.5 Let $f(x)$ be a piecewise-differentiable function in every finite interval of the semi-infinite region $x > 0$ and be absolutely integrable from *zero* to ∞. Show that (*a*) for $x > 0$, the function $f(x)$ can be represented by the *Fourier cosine integral* formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(x')\cos\omega x'\,dx'\,d\omega, \ x > 0$$

(*b*) when $f(x)$ is *even*, the above representation is valid for all values of x,

(*c*) the Fourier cosine integral representation of $f(x)$ can be rearranged as the *Fourier cosine transform* on the interval $(0, \infty)$ as

$$\bar{f}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx$$

with the *inversion*

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}(\omega)\cos\omega x\,d\omega$$

and

(*d*) the kernel of the Fourier cosine transform satisfies, for all values of ω from *zero* to ∞, the following characteristic-value problem:

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

$$\frac{dy(0)}{dx} = 0 \text{ and } |y(x)| \le M \text{ for } x > 0$$

where M is some finite constant.

7.6 Re-solve Problem 5.13 by integral transforms.

7.7 Re-solve Problem 5.30 by integral transforms.

7.8 Re-solve Problem 5.31 by integral transforms.

7.9 Re-solve Problem 5.33 by integral transforms.

7.10 Re-solve Problem 5.34 by integral transforms.

7.11 Re-solve Problem 5.46 by integral transforms.

7.12 Re-solve Problem 5.47 by integral transforms.

7.13 Consider a plane wall of thickness L, which is initially at a nonuniform temperature $T_i(x)$. Assume that, for times $t > 0$, internal energy is generated in this wall at a rate of $\dot{q}\,(x, t)$ per unit volume, and heat is dissipated by convection from the boundaries at $x = 0$ and $x = L$ into a surrounding medium whose temperature $T_\infty(t)$ varies with time. The thermophysical properties (k, ρ, c) and the heat transfer coefficients h_1 and h_2 at the surfaces at $x = 0$ and $x = L$, respectively, are constant. Obtain an expression for the unsteady-state temperature distribution in the wall for $t > 0$.

7.14 A slab, which extends from $x = 0$ to $x = L$, has an initial distribution $T_i(x)$. For times $t \geq 0$, internal energy is generated within this slab at a rate of $\dot{q}\,(x, t)$ per unit volume, while the surface at $x = 0$ is kept perfectly insulated and the surface at $x = L$ is exposed to a fluid maintained at constant temperature T_∞ with a constant heat transfer coefficient h. Assuming constant thermophysical properties (k, ρ, c), obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the slab for $t > 0$.

7.15 Re-solve Problem 6.29 by integral transforms.

7.16 Re-solve Problem 6.30 by integral transforms.

7.17 Consider an infinite medium, $-\infty < x < \infty$, of constant thermophysical properties and initially at a uniform temperature T_i at $t = 0$. For times $t > 0$, internal energy is generated at a constant rate $\dot{q}$ per unit volume within the region $-L < x < L$. Obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the medium for $t > 0$.

7.18 Consider a long bar of rectangular cross-section, $0 < x < a$ and $0 < y < b$, initially at a nonuniform temperature $T_i(x, y)$. For times $t \geq 0$, internal energy is generated in the bar at a rate $\dot{q}\,(x, y, t)$ per unit volume, while heat is dissipated by convection from its surfaces into a surrounding medium whose temperature $T_\infty(t)$ varies with time. Let the heat transfer coefficient be the same constant h on all the surfaces. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(x, y, t)$ in the bar for $t > 0$.

7.19 Consider a long solid cylinder of radius r_0, which is initially at a uniform temperature T_i at $t = 0$. Assume that, for times $t \geq 0$, internal energy is generated within the cylinder at a rate of $\dot{q}\,(r, t)$ per unit volume, while the surface at $r = r_0$ is kept at the initial temperature T_i. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the cylinder for $t > 0$.

7.20 Consider a long solid cylinder of radius r_0, which is initially at a nonuniform temperature $T_i(r)$. Assume that, for times $t \geq 0$, internal energy is generated in the cylinder at a rate of $\dot{q}\,(r, t)$ per unit volume, and heat is dissipated by convection from the boundary surface at $r = r_0$ with a constant heat transfer coefficient h into a medium whose temperature $T_\infty(t)$ varies with time. Assuming constant thermophysical properties (k, ρ, c), develop an expression for the unsteady-state temperature distribution $T(r, t)$ in the cylinder for $t > 0$.

7.21 Consider a long solid hollow cylinder of radius $r_1 < r < r_2$, with an initial temperature distribution $T_¿(r)$. For times $t \geq 0$, internal energy is generated in the cylinder at a rate of $\dot{q}\,(r, t)$ per unit volume, while the surfaces at $r = r_1$ and $r = r_2$ are maintained, respectively, at temperatures $T_1(t)$ and $T_2(t)$. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the cylinder for $t > 0$. *Hint:* Make use of the transform developed in Problem 7.2.

7.22 A solid rod of constant thermophysical properties, radius r_0 and height H, is initially at a uniform temperature T_i. For times $t > 0$, internal energy is generated in the rod at a rate $\dot{q}\,(r, z, t)$ per unit volume, while the peripheral surface at $r = r_0$ and the end surfaces at $z = 0$ and $z = H$ are kept at the initial temperature T_i. Obtain an expression for the unsteady-state temperature distribution $T(r, z, t)$ in the cylinder for $t > 0$.

7.23 A long solid cylinder, $0 \le r \le b$ and $0 \le \phi \le \pi/2$ is at a nonuniform temperature $T_i(r, \phi)$ at $t = 0$. For times $t \ge 0$, the boundary surfaces are maintained at temperatures $T_1(r, t)$ at $\phi = 0$, $T_2(r, t)$ at $\phi = \pi/2$, and $T_3(\phi, t)$ at $r = b$. Find an expression for the unsteady-state temperature distribution $T(r, \phi, t)$ in the cylinder for times $t > 0$.

7.24 Find an expression for the steady-state temperature distribution $T(r, \phi)$ in the cylinder of Problem 7.23, if the surface temperatures are independent of time.

7.25 A solid sphere, of radius $r0$ and constant thermophysical properties, is at a uniform temperature T_i at $t = 0$. For times $t \ge 0$, internal energy is generated in the sphere at a rate $\dot{q}\,(r, t)$ per unit volume, while the surface at $r = r0$ is kept at the initial temperature T_i. Obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the sphere for $t > 0$.

7.26 A solid sphere, of radius r0 and constant thermophysical properties, is at a nonuniform temperature $T_i\,(r)$. For times $t \ge 0$, internal energy is generated in the sphere at a rate $\dot{q}\,(r, t)$ per unit volume, while it exchanges heat by convection, with a constant heat transfer coefficient h, at the surface at $r = r0$ with a medium whose temperature $T\infty(t)$ varies with time. Obtain an expression for the unsteady-state temperature distribution $T\,(r, t)$ in the sphere for $t > 0$.

7.27 Consider an infinite solid medium, $r_0 < r < \infty$, internally bounded by a spherical surface at $r = r_0$ as illustrated in Fig. 7.6. Initially, the medium is at a uniform temperature T_i. For times $t \ge 0$, the surface temperature at $r = r_0$ is kept at $f(t)$. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the medium for $t > 0$.

7.28 Consider an infinite solid medium, $0 < r < \infty$. Initially, at $t = 0$, the spherical region $0 < r < r_0$ in the medium is at a uniform temperature T_i, and everywhere outside this region is at zero temperature. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the medium for $t > 0$.

7.29 Consider an infinite solid medium, $0 < r < \infty$, which is initially, at $t = 0$, at a uniform temperature T_i. For times $t \ge 0$, internal energy is generated at a constant rate $\dot{q}$ per unit volume within the spherical region $0 < r < r_0$. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the medium for $t > 0$.

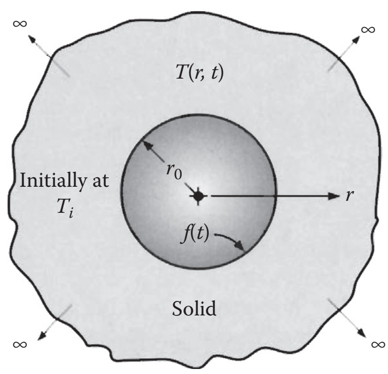

FIGURE 7.6
For Problem 7.27.

7.30 Re-solve Problem 5.48 by integral transforms.

7.31 Find the analytic solution for the heat conduction problem below by integral transformation, where m, k, g_0, and T_0 are constant parameters. Find the expressions for the corresponding eigenfunctions and eigenvalues. It is recommended to homogenize the boundary condition at $x = 0$ from the definition of a new temperature, $T(x, t) = \theta(x, t) + T_w$, before determining the analytical solution.

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} - m^2\left[T(x,t) - T_0\right] + g_0/k, \; 0 \le x \le 1, \; t > 0$$

$$T(x,0) = T_0, \qquad 0 < x < 1$$

$$T(0,t) = T_w, \qquad t > 0$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=1} = 0, \qquad t > 0$$

7.32 The energy equation for cooling a porous wall by transpiration can be written in the dimensionless form as

$$\frac{\partial T(x,t)}{\partial t} + u\frac{\partial T(x,t)}{\partial x} = \frac{\partial^2 T(x,t)}{\partial x^2}, \; 0 < x < 1, \; t > 0$$

or in the equivalent and most convenient form below, in the case of a constant value for the velocity u,

$$e^{-ux}\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[e^{-ux}\frac{\partial T(x,t)}{\partial x}\right], \; 0 < x < 1, \; t > 0$$

with initial and boundary conditions given by

$$T(x,0) = 0, \; 0 \le x \le 1$$

$$uT(0,t) - \frac{\partial T(0,t)}{\partial x} = 0; \; \delta T(1,t) + \gamma\frac{\partial T(1,t)}{\partial x} = \varphi, \; t > 0$$

Find the exact solution to the above problem using the integral transform technique.

8

Solutions with Laplace Transforms

8.1 Introduction

In Chapter 7, we saw how various heat conduction problems can be solved by removing the space variable(s) by means of integral transforms. In solving unsteady-state problems, on the other hand, the time variable may be removed from the problem by *Laplace transforms.* The Laplace transform method is also an integral transform method. However, in this chapter, and through the rest of the book, when we talk about integral transform methods, we will be referring to the transforms we introduced in Chapter 7. There are both advantages and disadvantages of the Laplace transform method as compared to the other integral transform methods. The Laplace transform method is particularly applicable to time-dependent problems and, in fact, is attractive for one-dimensional problems. If the method of Laplace transforms is used to solve two- or three-dimensional time-dependent problems, the resulting simpler problem would again involve partial derivatives with respect to space variables. Furthermore, with Laplace transforms, inversion may not be as easy as with integral transforms. On the other hand, Laplace transforms may yield closed form solutions, whereas integral transforms normally give solutions in the form of series expansions.

In this chapter, we first examine the basic properties of Laplace transforms. Next, we discuss the method of solution of various unsteady-state heat conduction problems by Laplace transforms in terms of various representative examples.

8.2 Definition of the Laplace Transform

Let $f(t)$ be a function of t defined for $t > 0$. Multiply $f(t)$ by e^{-pt} and then integrate it with respect to t from *zero* to *infinity*,

$$\bar{f}(p) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt \tag{8.1}$$

If this integral, denoted by $\bar{f}(p)$ or $\mathcal{L}\{f(t)\}$, converges for some value(s) of the *parameter p*, it is called the *Laplace transform* of $f(t)$; otherwise, the Laplace transform of $f(t)$ does not exist. Some simple examples can be given to illustrate the calculation of Laplace transforms. For instance, let $f(t) = 1$ when $t > 0$, and then

$$E\{1\} = \int_0^\infty e^{-pt} dt = \frac{1}{p}, \quad (p > 0) \tag{8.2}$$

Another example is the transform of $f(t) = t$ when $t > 0$:

$$\mathcal{L}\{t\} = \int_0^\infty e^{-pt} t dt = \frac{1}{p^2}, \quad (p > 0) \tag{8.3}$$

The Laplace transform of $f(t) = e^{\alpha t}$ when $t > 0$, where α is a constant, is

$$\mathcal{L}\{e^{\alpha t}\} = \int_0^\infty e^{-pt} e^{\alpha t} dt = \frac{1}{p - \alpha}, \quad (p > \alpha) \tag{8.4}$$

Let us now consider another example where

$$f(t) = \begin{cases} 1 & \text{for} \quad 0 < t < 2 \\ t & \text{for} \quad t > 2 \end{cases}$$

We find that

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-pt} f(t) dt = \int_0^2 e^{-pt} dt + \int_2^\infty e^{-pt} t dt \\ &= \frac{1}{p} + e^{-2p}\left(\frac{1}{p} + \frac{1}{p^2}\right), \; (p > 0) \end{aligned} \tag{8.5}$$

With the help of elementary methods of integration, Laplace transforms of many other functions can easily be found. For example,

$$\mathcal{L}\{\sin \omega t\} = \int_0^\infty e^{-pt} \sin \omega t \, dt = \frac{\omega}{P^2 + \omega^2}, \quad (p > 0) \tag{8.6}$$

and

$$\mathcal{L}\{\cos \omega t\} = \int_0^\infty e^{-pt} \cos \omega t \, dt = \frac{\omega}{p^2 + \omega^2}, \quad (p > 0) \tag{8.7}$$

It follows from the elementary properties of integrals that the Laplace transformation is *linear*. That is, if C_1 and C_2 are two constants while $f_1(t)$ and $f_2(t)$ are functions of t with Laplace transforms $\bar{f}_1(p)$ and $\bar{f}_2(p)$, respectively, then

$$\begin{aligned} \mathcal{L}\{C_1 f_1(t) + C_2 f_2(t)\} &= C_1 \mathcal{L}\{f_1(t) + C_2 \mathcal{L}(f_2(t)\} \\ &= C_1 \bar{f}_1(p) + C_2 \bar{f}_2(p) \end{aligned} \tag{8.8}$$

Thus, the Laplace transform of the sum of two functions is the sum of the Laplace transforms of each individual function.

The Laplace transform of $f(t)$ exists – that is, the integral of Eq. (8.1) converges – if the following conditions are satisfied:

a. The function $f(t)$ is *continuous* or *piecewise continuous* in every interval $t_1 \le t \le t_2$, where $t_1 > 0$.

b. $t^n|f(t)|$ is bounded near $t = 0$ for some number n in $0 < n < 1$.

c. The function $f(t)$ is of *exponential order* γ as $t \to \infty$. That is, two real constants $M > 0$ and $\gamma > 0$ should exist such that

$$\left|e^{-\gamma t} f(t)\right| < M \quad \text{or} \quad \left|f(t)\right| < M\, e^{\gamma t}$$

for all t greater than some finite t_0. Intuitively, the absolute value of functions of exponential order cannot grow more rapidly than $M\,e^{\gamma t}$ as t increases. In applications, however, this is not a restriction since M and γ can be as large as desired. The bounded functions, such as $\sin \omega t$ or $\cos \omega t$, are of exponential order.

8.3 Introductory Example

In this section, we solve a simple heat conduction problem by using the definition of the Laplace transform, and thus introduce the method of solution. Consider a solid body of constant thermophysical properties and initially maintained at a uniform temperature T_i, as shown in Fig. 8.1. At time $t = 0$, this solid body is exposed to a fluid whose temperature varies periodically with time as

$$T_\infty(t) = T_0 + (\Delta T)_0 \sin \omega t \tag{8.9}$$

We assume that the thermal conductivity k of the solid, the surface heat transfer coefficient h, and the dimensions of the body are such that $\mathrm{Bi} = h(V/A)/k \ll 1$. Accordingly, the heat transferred to the solid body is distributed instantaneously and uniformly throughout it,

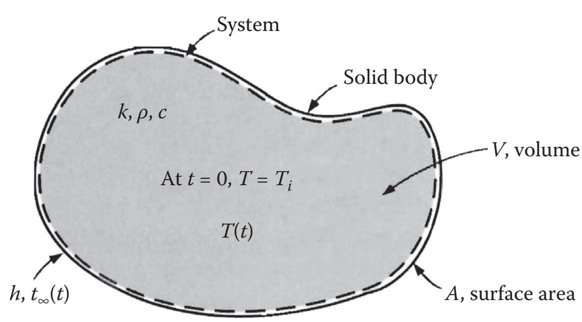

FIGURE 8.1
Lumped-heat capacity system.

which results in a uniform temperature in the body at any time. The first law of thermodynamics as applied to the system shown in Fig. 8.1 yields

$$\frac{d\theta}{dt} + m\theta(t) = m \sin \omega t \tag{8.10a}$$

with the initial condition

$$\theta(0) = \theta_i = \frac{T_i - T_0}{(\Delta T)_0} \tag{8.10b}$$

where ρ and c are, respectively, the density and specific heat of the solid, and

$$\theta(t) = \frac{T(t) - T_0}{(\Delta T)_0} \quad \text{and} \quad m = \frac{hA}{\rho c V}$$

To find the temperature in the solid at any instant t by using the definition of the Laplace transform, we first multiply both sides of Eq. (8.10*a*) by e^{-pt} and then integrate the resultant expression with respect to t from *zero* to *infinity*:

$$\int_0^\infty e^{-pt} \frac{d\theta}{dt} dt + m \int_0^\infty e^{-pt} \theta(t)\, dt = m \int_0^\infty e^{-pt} \sin \omega t\, dt \tag{8.11}$$

The first integral on the left-hand side can be integrated by parts:

$$\begin{aligned} \int_0^\infty e^{-pt} \frac{d\theta}{dt} dt &= \theta(t) e^{-pt} \Big|_0^\infty - \int_0^\infty \theta(t)(-p e^{-pt})\, dt \\ &= -\theta_i + p \int_0^\infty \theta(t) e^{-pt}\, dt \end{aligned} \tag{8.12a}$$

The integral on the right-hand side of Eq. (8.11) is the Laplace transform of sin ωt, and, from Eq. (8.6), we have

$$\int_0^\infty e^{-pt} \sin \omega t\, dt = \frac{\omega}{p^2 + \omega^2}, \quad (p > 0) \tag{8.12b}$$

Now, inserting these results into Eq. (8.11), we obtain

$$\int_0^\infty \theta(t) e^{-pt} dt = \frac{m\omega}{\left(p^2 + \omega^2\right)\left(p + m\right)} + \frac{\theta_i}{p + m} \tag{8.13a}$$

The problem is now to find a function $\theta(t)$ whose Laplace transform is the right-hand side of Eq. (8.13a). To facilitate the procedure of finding this function, we rearrange the above equation as (see Section 8.6)

$$\int_0^\infty \theta(t)e^{-pt}dt = \frac{m}{m^2+\omega^2}\left(\frac{m\omega}{p^2+\omega^2} - \frac{\omega p}{p^2+\omega^2}\right) + \left(\frac{m\omega}{p^2+\omega^2} + \theta_i\right)\frac{1}{p+m} \tag{8.13b}$$

A close examination of the right-hand side of Eq. (8.13*b*), with the help of Eqs. (8.4), (8.6), and (8.7), reveals that the function $\theta(t)$ is

$$\theta(t) = \frac{m}{m^2+\omega^2}(m\sin\omega t - \omega\cos\omega t) + \left(\frac{m\omega}{m^2+\omega^2} + \theta_i\right)e^{-mt} \tag{8.14a}$$

which can also be written as

$$\theta(t) = \frac{m\omega}{\sqrt{m^2+\omega^2}}\sin(\omega t - \phi) + \left(\frac{m\omega}{m^2+\omega^2} + \theta_i\right)e^{-mt} \tag{8.14b}$$

where

$$\phi = \tan^{-1}\frac{\omega}{m} \tag{8.14c}$$

We have solved a simple *lumped-heat-capacity-system* problem by using the definition of the Laplace transform. Fortunately, most of the steps in this procedure can be eliminated if certain properties of Laplace transforms are implemented during the solution process. Accordingly, in the following section we review some of the important properties of Laplace transforms.

8.4 Some Important Properties of Laplace Transforms

Some of the important properties of Laplace transforms that we will use in solving heat conduction problems are as follows:

a. First shifting property. If $\bar{f}(p) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \bar{f}(p-a) \tag{8.15}$$

b. Second shifting property, if $\bar{f}(p) = \mathcal{L}\{f(t)\}$ and

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

then

$$\pounds\left\{g(t)\right\} = e^{-ap}\bar{f}(p) \tag{8.16}$$

c. **Change of scale property.** If $\bar{f}(p) = \pounds\{f(t)\}$, then

$$\pounds\left\{f(at)\right\} = \frac{1}{a}\bar{f}\left(\frac{p}{a}\right) \tag{8.17}$$

d. **Laplace transform of derivatives.** Let $f(t)$ be continuous with a piecewise continuous derivative df/dt over every interval $0 \le t \le t_1$. Also, let $f(t)$ be of exponential order for $t > t_1$. Then

$$\pounds\left\{\frac{df}{dt}\right\} = p\,\bar{f}(p) - f(0) \tag{8.18a}$$

If $f(t)$ fails to be continuous at $t = 0$ but $\lim\limits_{t \to 0^+} f(t) = f(0^+)$ exists, then

$$\pounds\left\{\frac{df}{dt}\right\} = p\,\bar{f}(p) - f(0^+) \tag{8.18b}$$

Here it should be noted that $f(0^+)$ is not equal to $f(0)$, which may or may not exist. If $f(t)$ fails to be continuous at $t = a$, then

$$\pounds\left\{\frac{df}{dt}\right\} = p\,\bar{f}(p) - f(0) - e^{-ap}\left[f(a^+) - f(a^-)\right] \tag{8.18c}$$

where $f(a^+) - f(a^-)$ is the jump at the discontinuity at $t = a$. For more than one discontinuity, appropriate modifications can be made.

If $f(t)$, $df/dt, \ldots, d^{n-1}f/dt^{n-1}$ are continuous in the interval $0 \le t < t_1$, and are of exponential order for $t > t_1$, while d^nf/dt^n is piecewise continuous for $0 \le t \le t_1$, then

$$\pounds\left\{\frac{d^n f}{dt^n}\right\} = p^n\,\bar{f}(p) - p^{n-1}f(0) - p^{n-2}\frac{df(0)}{dt} - \ldots - p\frac{d^{n-2}f(0)}{dt^{n-2}} - \frac{d^{n-1}f(0)}{dt^{n-1}} \tag{8.18d}$$

If $f(t)$, $df/dt, \ldots, d^{n-1}f/dt^{n-1}$ have discontinuities, appropriate modification in Eq. (8.18d) can be made as in Eq. (8.18b) or Eq. (8.18c).

e. Laplace transforms of integrals. If $\bar{f}(p) = \pounds\{f(t)\}$, then

$$\pounds\left\{\int_0^t f(t')\,dt'\right\} = \frac{1}{p}\bar{f}(p) \tag{8.19}$$

f. Multiplication by t^n. If $\bar{f}(p) = \pounds\{f(t)\}$, then

$$\pounds\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{dp^n}\bar{f}(p) \tag{8.20}$$

g. Division by t. If $\bar{f}(p) = \pounds\{f(t)\}$, then

$$\pounds\left\{\frac{f(t)}{t}\right\} = \int_p^\infty \bar{f}(p')\,dp' \tag{8.21}$$

provided that

$$\lim_{t\to 0}\frac{f(t)}{t}$$

exists.

8.5 The Inverse Laplace Transform

As seen in Section 8.3, in solving heat conduction problems by Laplace transforms, we encounter the problem of determining the function that has a given transform. The notation $\pounds^{-1}\{\bar{f}(p)\}$ is conveniently used for a function whose Laplace transform is $f(p)$. That is, if

$$\pounds\{f(t)\} = \bar{f}(p) \quad \text{then} \quad f(t) = \pounds^{-1}\{\bar{f}(p)\}$$

Therefore, in order to determine the inverse transform of $\bar{f}(p)$, it is necessary to find a function $f(t)$ that satisfies the equation

$$\int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p) \tag{8.22}$$

Since the unknown function $f(t)$ appears under an integral sign, Eq. (8.22) is an integral equation. If we restrict ourselves to functions $f(t)$ that are piecewise continuous in every finite interval $0 \le t \le t_1$ and of exponential order for $t > t_1$, then if there exists a solution to Eq. (8.22) that solution is unique [4]. Thus, for example, Eq. (8.4) shows that the solution of

$$\int_0^\infty e^{-pt} f(t)dt = \frac{1}{p-a} \tag{8.23a}$$

is $f(t) = e^{at}$; that is,

$$£^{-1}\left\{\frac{1}{p-a}\right\} = e^{at} \tag{8.23b}$$

Although there are methods for direct determination of the inverse transform of a given function of p, the most obvious way is to read the result from a table of Laplace transforms. Extensive tables of functions and corresponding transforms are available in the literature, and their use is sufficient for our purpose [1,2,4,5,8]. A short table of Laplace transforms is also given in Appendix D. If the transform we are looking for cannot be found in the Laplace transform tables, the methods discussed in the following two sections are very useful in obtaining both direct and inverse transforms.

8.5.1 Method of Partial Fractions

The method of partial fractions is a useful tool to employ in conjunction with the table of transforms to determine both direct and inverse transforms. This method is used to split complicated fractions whose inverse transforms cannot be obtained directly from the Laplace transform tables into simpler ones that can be found in the tables. To illustrate how the method of partial fractions can be used to obtain inverse transforms of quotients of polynomials in p, consider

$$£^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$$

To find the inverse transform, the fraction can be written as the sum of several fractions whose least common denominator is the denominator in this fraction. Thus, we first assume an expansion of the form

$$\frac{1}{(p+1)(p^2+1)} = \frac{A}{p+1} + \frac{Bp+C}{p^2+1}$$

After clearing fractions, we require this equation to be an identity and obtain $A = -B = C = 1/2$. Hence,

$$\frac{1}{(p+1)(p^2+1)} = \frac{1}{2}\frac{1}{p+1} - \frac{1}{2}\frac{p-1}{p^2+1}$$

and the use of Appendix D gives the inverse transform as

$$\pounds^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\} = \frac{1}{2}(e^{-t} + \sin t - \cos t)$$

If, in general, the transform is in the form of a rational function $P(p)/Q(p)$, where $P(p)$ and $Q(p)$ are polynomials as in the above example, then it can be rewritten as the sum of rational functions, called *partial fractions*, having the form

$$\frac{A}{(ap+b)^r} \quad \text{and} \quad \frac{Ap+B}{(ap^2+bp+c)^r}$$

where $r = 1, 2, 3, \ldots$. By finding the inverse Laplace transform of each partial fraction, we can then determine $\pounds^{-1}\{P(p)/Q(p)\}$.

As a second example, let us consider

$$\frac{2p-5}{(3p-4)(2p+1)^3} = \frac{A}{3p-4} + \frac{B}{2p+1} + \frac{C}{(2p+1)^2} + \frac{D}{(2p+1)^3}$$

The constants A, B, C, and D can be found by clearing fractions and equating the coefficients of like powers of p on both sides of the resulting equation.

If the degree of the polynomial $P(p)$ is less than that of the polynomial $Q(p)$, and if $Q(p)$ has n *distinct real* zeros p_k, $k = 1, 2, \ldots, n$, then

$$\pounds^{-1}\left\{\frac{P(p)}{Q(p)}\right\} = \sum_{k=1}^{n} \frac{P(p_k)}{dQ(p_k)/dp}\, e^{p_k t} \tag{8.24}$$

This relation is often referred to as the *Heaviside's expansion theorem*.

8.5.2 Convolution Theorem

There are several other properties and theorems regarding Laplace transforms. The most useful one of these is the *convolution theorem*. It frequently happens that although a given $\bar{f}(p)$ is not the transform of a known function, it can be expressed as the product of two functions, each of which is the transform of a known function; that is,

$$\bar{f}(p) = \bar{g}(p)\bar{h}(p)$$

where $\bar{g}(p)$ and $\bar{h}(p)$ are the transforms of the known functions $g(t)$ and $h(t)$, respectively. In this case

$$\pounds^{-1}\left\{\bar{f}(p)\right\} = \pounds^{-1}\{\bar{g}(p)\bar{h}(p)\} = \int_{t'=0}^{t} g(t-t')h(t')\,dt' \tag{8.25}$$

This integral, sometimes denoted by $g * h$, is called the convolution of $g(t)$ and $h(t)$. It has the following property (see Problem 8.4):

$$g * h = h * g$$

To illustrate the use of the convolution theorem, let us reconsider the example we discussed in Section 8.5.1, where the inverse of the following expression was desired:

$$\bar{f}(p) = \frac{1}{(p+1)(p^2+1)}$$

This expression can be treated as a product of

$$\bar{g}(p) = \frac{1}{p+1} \quad \text{and} \quad \bar{h}(p) = \frac{1}{p^2+1}$$

The inverses of these are given in Appendix D by $g(t) = e^{-t}$ and $h(t) = \sin t$. Thus, the convolution theorem gives

$$f(t) = \int_0^t e^{-(t-t')}\sin t'dt' = e^{-1}\int_0^t e^{t'}\sin t'dt'$$

The integral may be evaluated using integral tables to give

$$\bar{f}(t) = \frac{1}{2}(e^{-t} + \sin t - \cos t)$$

which is the same result we obtained previously by the method of partial fractions.

8.6 Laplace Transforms and Heat Conduction Problems

The Laplace transform method is particularly convenient and efficient for solving one-dimensional unsteady-state heat conduction problems. The procedure is very much the same as that followed in the solution of the problem discussed in Section 8.3. Briefly, the procedure for solving one-dimensional time-dependent problems can be outlined as shown in Fig. 8.2. This procedure will be illustrated in the following sections in terms of several representative examples.

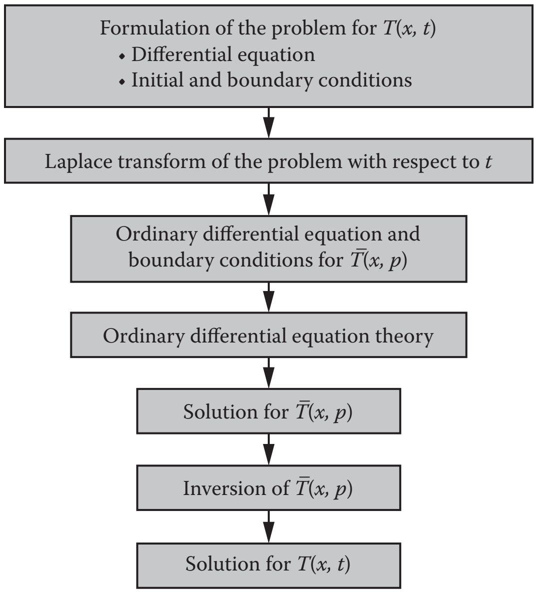

FIGURE 8.2
Procedure for solving one-dimensional unsteady-state problems by Laplace transforms.

8.7 Plane Wall

Consider a plane wall of constant thermophysical properties and thickness $2L$, which is initially at a uniform temperature T_i. Assume that the temperatures of the surfaces are changed to T_w at $t = 0$ and are kept constant at this value during the whole heat transfer process (heating and cooling). We wish to find the unsteady-state temperature distribution $T(x, t)$ in the wall for times $t > 0$. If we place the origin of the x coordinate at the center of the wall as shown in Fig. 8.3, then the problem can be formulated as follows:

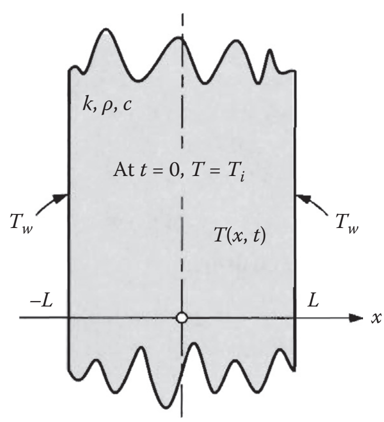

FIGURE 8.3
Plane wall of thickness $2L$.

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.26a}$$

$$T(x,0) = T_i \tag{8.26b}$$

$$\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{and} \quad T(L,t) = T_w \tag{8.26c,d}$$

Let us first find the Laplace transform of Eq. (8.26*a*); that is,

$$\pounds\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{1}{\alpha}\pounds\left\{\frac{\partial T}{\partial t}\right\} \tag{8.27}$$

Use of the properties of Laplace transforms yields

$$\pounds\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{\partial^2 \bar{T}}{\partial x^2} \tag{8.28a}$$

and

$$\pounds\left\{\frac{\partial T}{\partial x}\right\} = p\bar{T}(x,p) - T_i \tag{8.28b}$$

where

$$\bar{T} = \bar{T}(x,p) = \pounds\{T(x,t)\} = \int_0^\infty e^{-pt}T(x,t)dt \tag{8.28c}$$

Thus, Eq. (8.27) can be written as

$$\frac{d^2\bar{T}}{dx^2} - \frac{p}{\alpha}\bar{T} = -\frac{T_i}{\alpha} \tag{8.29}$$

The Laplace transform of the boundary conditions (8.26*c*,*d*) are

$$\frac{\bar{T}(0,p)}{dx} = 0 \quad \text{and} \quad \bar{T}(L,p) = \frac{T_w}{p} \tag{8.30a,b}$$

Equation (8.29) is an ordinary differential equation for $\bar{T}(x,p)$, which is considered to be a function of x with the constant parameter p carried along. The solution of this equation can be written as

$$\bar{T}(x,p) = C_1\cosh mx + C_2\sinh mx + \frac{T_i}{p} \tag{8.31}$$

where $m^2 = p/\alpha$, and C_1 and C_2 are the constants of integration to be determined from the boundary conditions. Application of the boundary condition at $x = 0$ yields $C_2 = 0$, and the boundary condition at $x = L$ gives

$$C_1 = \frac{T_i - T_w}{p\cosh mL} \tag{8.32}$$

Thus, the solution for $\bar{T}(x,p)$ may now be written as

$$\bar{T}(x,p) = \frac{T_i}{p} - (T_i - T_w)\frac{\cosh mx}{p\cosh mL} \tag{8.33}$$

The final step now is to sinvert $\bar{T}(x,p)$ to obtain $T(x, t)$; that is,

$$T(x,p) = T_i \mathcal{L}^{-1}\left\{\frac{1}{p}\right\} - (T_i - T_w)\mathcal{L}^{-1}\left\{\frac{\cos mx}{p\cosh mL}\right\} \tag{8.34}$$

A check of Appendix D shows that both transforms in Eq. (8.34) are available in the table of Laplace transforms. Thus, from transform No. 1, we have,

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\right\} = 1 \tag{8.35}$$

Moreover, since $m = \sqrt{p/\alpha}$, from transform No. 40 we get

$$\mathcal{L}^{-1}\left\{\frac{\cosh mx}{p\cosh mL}\right\} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp\left[-\frac{(2n-1)^2\pi^2\alpha t}{4L^2}\right]\cos\frac{(2n-1)\pi x}{2L} \tag{8.36}$$

Thus, the solution for $T(x, t)$ becomes

$$\begin{aligned} T(x,t) = T_i - (T_i - T_w)\Bigg\{1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp\left[-\frac{(2n-1)^2\pi^2\alpha t}{4L^2}\right] \\ \times \cos\frac{(2n-1)\pi x}{2L}\Bigg\} \end{aligned} \tag{8.37a}$$

which can be rewritten as

$$\frac{T(x,t) - T_w}{T_i - T_w} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}e^{-[(2n-1)/2L]^2\pi^2\alpha t}\cos\frac{(2n-1)\pi x}{2L} \tag{8.37b}$$

This is the same solution as the one given by Eq. (6.35), which we obtained previously by the method of separation of variables.

An alternative expression for $T(x, t)$ can be obtained by following a different approach in inverting the second term of Eq. (8.34). Note that

$$\begin{aligned}\frac{\cosh mx}{\cosh mL} &= \frac{e^{mx}+e^{-mx}}{e^{mL}+e^{e-mL}} = e^{-mL}(e^{mx}+e^{-mx})\frac{1}{1+e^{-2mL}}\\ &= \left[e^{-m(L-x)}+e^{-m(L+x)}\right]\sum_{k=0}^{\infty}(-1)^k e^{-2mL}\\ &= \sum_{k=0}^{\infty}(-1)^k\{\exp[-m(nL-x)]+\exp[-m(nL+x)]\}\end{aligned} \tag{8.38}$$

where $n = 2k + 1$. From Appendix D, transform No. 27, we have

$$\frac{1}{p}e^{-a\sqrt{p}} = \pounds\left\{\text{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right)\right\} \tag{8.39}$$

It then follows that

$$\pounds^{-1}\left\{\frac{\cosh mx}{p\cosh mL}\right\} = \sum_{k=0}^{\infty}(-1)^k\left\{\text{erfc}\left[\frac{(2k+1)L-x}{2\sqrt{\alpha t}}\right]+\text{erfc}\left[\frac{(2k+1)L+x}{2\sqrt{\alpha t}}\right]\right\} \tag{8.40}$$

Hence, the solution for $T(x, t)$ can be written as

$$\frac{T(x,t)-T_w}{T_i-T_w} = 1-\sum_{k=0}^{\infty}(-1)^k\left\{\text{erfc}\left[\frac{(2k+1)L-x}{2\sqrt{\alpha t}}\right]+\text{erfc}\left[\frac{(2k+1)L+x}{2\sqrt{\alpha t}}\right]\right\} \tag{8.41}$$

We have thus obtained another expression for $T(x, t)$ in the form of a series expansion in terms of the complementary error function. Since the value of the complementary error function decreases rapidly as the argument increases, the solution (8.41) will converge rapidly for small values of t, whereas the convergence of the solution (8.37*b*) is fast for large values of t because of the negative argument of the exponential term.

8.8 Semi-Infinite Solid

We now consider the semi-infinite solid shown in Fig. 8.4, which is initially at a uniform temperature T_i. The surface temperature is changed to T_w at $t = 0$ and is maintained constant at this value for times $t > 0$. Assuming constant thermophysical properties, let us find the unsteady-state temperature distribution $T(x, t)$ in the solid for $t > 0$.

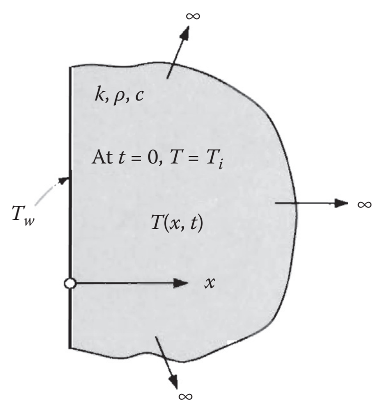

FIGURE 8.4
Semi-infinite solid with constant surface temperature T_w.

The formulation of the problem in terms of $\theta(x, t) = T(x, t) - T_i$ is given by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{8.42a}$$

$$\theta(x,0) = 0 \tag{8.42b}$$

$$\theta(0,t) = T_w - T_i = \theta_w \quad \text{and} \quad \lim_{x\to\infty}\theta(x,t) = 0 \tag{8.42c,d}$$

The transformed differential equation and boundary conditions are

$$\frac{d^2\bar{\theta}}{dx^2} - \frac{p}{\alpha}\bar{\theta} = 0 \tag{8.43a}$$

$$\bar{\theta}(0,p) = \frac{\theta_w}{p} \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0 \tag{8.43b,c}$$

where $\bar{\theta}(x,p)$ is the Laplace transform of $\theta(x, t)$. The general solution of Eq. (8.43*a*) can be written as

$$\bar{\theta}(x,p) = C_1 e^{-mx} + C_2 e^{mx} \tag{8.44}$$

where C_1 and C_2 are the constants of integration, and $m^2 = p/\alpha$. Application of the boundary condition (8.43*c*) yields $C_2 = 0$, and the boundary condition (8.43*b*) gives $C_1 = \theta_w/p$. Then, it follows that

$$\frac{\bar{\theta}(x,p)}{\theta_w} = \frac{e^{-mx}}{p} = \frac{-e^{x\sqrt{p/\alpha}}}{p} \tag{8.45}$$

The inverse transform of Eq. (8.45) is given in terms of the *complementary error function* in Appendix D. Thus, we get

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{8.46}$$

which is identical to the solution given by Eq. (6.93), because erfc(x) = 1 – erf(x).

As a second example, let us now obtain the unsteady-state temperature distribution in the same semi-infinite solid when the surface temperature variation is specified as a prescribed function $f(t)$ of time for $t \geq 0$ as shown in Fig. 8.5. The formulation of the problem in terms of $\theta(x, t) = T(x, t) - T_i$ is given by

$$\frac{\partial^2 \theta}{\theta x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{8.47a}$$

$$\theta(x,0) = 0 \tag{8.47b}$$

$$\theta(0,t) = F(t) \quad \text{and} \quad \lim_{x \to \infty}(x,t) = 0 \tag{8.47c,d}$$

where $F(t) = f(t) - T_i$. The transformed differential equation and boundary conditions in terms of the transform $\bar{\theta}(x,p)$ of $\theta(x, t)$ are given by

$$\frac{\partial^2 \bar{\theta}}{\partial x^2} - \frac{p}{\alpha}\bar{\theta} = 0 \tag{8.48a}$$

$$\bar{\theta}(0,p) = \bar{F}(p) \quad \text{and} \quad \lim_{x \to \infty}\bar{\theta}(x,p) = 0 \tag{8.48b,c}$$

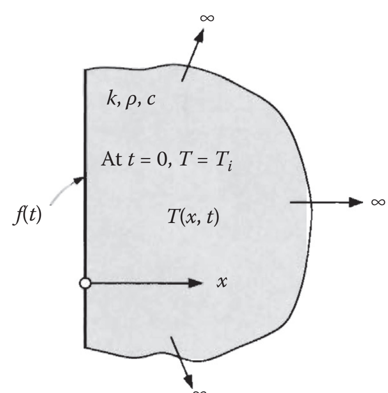

FIGURE 8.5
Semi-infinite solid with time-dependent surface temperature $T(0, t) = f(t)$.

where $\bar{F}(p)$ is the Laplace transform of $F(t)$; that is,

$$\bar{F}(p) = \pounds\{F(t)\} = \bar{f}(p) - \frac{T_i}{p} \tag{8.49a}$$

with

$$\bar{f}(p) = \pounds\{f(t)\} = \int_0^\infty e^{-pt} f(t)dt \tag{8.49b}$$

The solution of Eq. (8.48*a*) can be written as

$$\theta(x,p) = C_1 e^{-mx} + C_2 e^{mx} \tag{8.50a}$$

where C_1 and C_2 are the constants of integration, and $m^2 = p/\alpha$. Application of Eq. (8.48*c*) gives $C_2 = 0$, and Eq. (8.48*b*) yields $C_1 = \bar{F}(p)$. Thus.

$$\bar{\theta}(x,p) = \bar{F}(p)e^{-mx} = \bar{F}(p)e^{-x\sqrt{p/\alpha}} = \left[\bar{f}(p) - \frac{T_i}{p}\right]e^{-x\sqrt{p/\alpha}} \tag{8.50b}$$

Thus,

$$\theta(x,t) = T(x,t) - T_i = \pounds^{-1}\left\{\bar{f}(p)e^{-x\sqrt{p/\alpha}}\right\} - T_i \pounds^{-1}\left\{\frac{1}{p}e^{-x\sqrt{p/\alpha}}\right\} \tag{8.51a}$$

The first term on the right-hand side of Eq. (8.51*a*) is the inverse of a product of two functions of *p*, each of which can be inverted easily. The convolution theorem can, therefore, be used to find this inverse. From Appendix D, transform No. 25, we have

$$\pounds^{-1}\left\{e^{-x\sqrt{p/\alpha}}\right\} = \frac{x}{2\sqrt{4\alpha t^3}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$

Thus, by the convolution theorem, we get

$$\pounds^{-1}\left\{\bar{f}(p)e^{-x\sqrt{p/\alpha}}\right\} = \frac{x}{2\sqrt{\pi\alpha}}\int_0^t \frac{f(t-t')}{(t')^{3/2}}\exp\left(-\frac{x^2}{4\alpha t'}\right)dt' \tag{8.51b}$$

Furthermore, from Appendix D, transform No. 27, we also have

$$\pounds^{-1}\left\{\frac{1}{p}e^{-x\sqrt{p/\alpha}}\right\} = \text{erfc}\left(-\frac{x}{2\sqrt{\alpha t}}\right) \tag{8.51c}$$

Hence, substituting Eqs. (8.51*b*,*c*) into Eq. (8.51*a*), we obtain

$$T(x,t) = T_i - T_i \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{x}{2\sqrt{\pi\alpha}} \int_0^t \frac{f(t-t')}{(t')^{3/2}} \exp\left(-\frac{x^2}{4\alpha t}\right) dt' \tag{8.52}$$

If, in particular, the surface temperature is constant, that is, if $f(t) = T_w$, then the temperature distribution in the solid would be given by

$$T(x,t) = T_i - T_i \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + T_w \frac{x}{2\sqrt{\pi\alpha}} \int_0^t \frac{1}{(t')^{3/2}} \exp\left(-\frac{x^2}{4\alpha t'}\right) dt' \tag{8.53a}$$

The integral in the above result can be written in terms of the complementary error function. Then, Eq. (8.53*a*) becomes

$$\frac{T(x,t) - T_i}{T_w - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{8.53b}$$

This is the same result as the one given by Eq. (8.46).

8.9 Solid Cylinder

Let us consider a solid cylinder of constant thermophysical properties and radius r_0 as shown in Fig. 8.6, which is initially at a uniform temperature T_i. Assume that the temperature of the surface is suddenly changed to T_w at $t = 0$, and is subsequently maintained constant at this value for times $t > 0$. We wish to find the unsteady-state temperature distribution $T(r, t)$ in the cylinder during the heating (or cooling) process for $t > 0$.

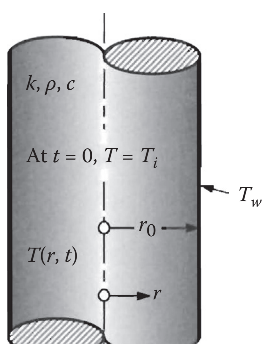

FIGURE 8.6
Solid cylinder.

The formulation of the problem in terms of $\theta(r, t) = T(r, t) - T_i$ is given by

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{8.54a}$$

$$\theta(r,0) = 0 \tag{8.54b}$$

$$\theta(0,t) = \textit{finite} \quad \text{and} \quad \theta(r_0,t) = \theta_w \tag{8.54c,d}$$

where $\theta_w = T_w - T_i$.

If $\bar{\theta}(x,p)$ denotes the Laplace transform of $\theta(x, t)$, then it satisfies the following boundary-value problem:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\theta}}{\partial r} = \frac{p}{\alpha}\bar{\theta} \tag{8.55a}$$

$$\bar{\theta}(0,p) = \textit{finite} \quad \text{and} \quad \bar{\theta}(r_0,p) = \frac{\theta_w}{p} \tag{8.55b,c}$$

The solution of this problem is easily found to be

$$\frac{\bar{\theta}(r,p)}{\theta_w} = \frac{I_0(r\sqrt{p/\alpha})}{pI_0(r_0\sqrt{p/\alpha})} \tag{8.56}$$

This expression is not found in Appendix D (nor in most of the tables in the references). The inverse transform, however, can be obtained by the use of the Heaviside expansion theorem (8.24) as follows. Note that Eq. (8.56) can be written as

$$\frac{\bar{\theta}(r,p)}{\theta_w} = \frac{P(p)}{Q(p)} \tag{8.57a}$$

where the polynomials $P(p)$ and $Q(p)$ are given by

$$P(p) = I_0 r(r\sqrt{p/\alpha}) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{r}{2}\right)^{2k}\left(\frac{p}{\alpha}\right)^k \tag{8.57b}$$

$$Q(p) = pI_0(r_0\sqrt{p/\alpha}) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{r_0}{2}\right)^{2k}\left(\frac{p^{k+1}}{\alpha^k}\right) \tag{8.57c}$$

The Heaviside expansion theorem (8.24), therefore, gives

$$\pounds^{-1}\left\{\frac{I_0(r_0\sqrt{p/\alpha})}{pI_0(r_0\sqrt{p/\alpha})}\right\}=\sum_{k=0}^{\infty}\frac{P(p_k)}{dQ(p_k)/dp}\,e^{p_k t} \tag{8.58}$$

where p_k are the zeros of the polynomial $Q(p)$; that is, they are the zeros of

$$pI_0(r_0\sqrt{p/\alpha})=0 \tag{8.59a}$$

which yields

$$p_0=0 \tag{8.59b}$$

and

$$I_0\left(r_0\sqrt{p/\alpha}\right)=0,\quad k=1,2,3,\ldots \tag{8.59c}$$

Equation (8.59c) is also equivalent to stating that

$$J_0\left(ir_0\sqrt{p/\alpha}\right)=0,\quad k=1,2,3,\ldots \tag{8.59d}$$

If λ_k represent the zeros of the Bessel function $J_o(z)$, then

$$ir_0\sqrt{p/\alpha}=\lambda_k\quad or\quad p_k=-\frac{\alpha\lambda_k^2}{r_0^2},\quad k=1,2,3,\ldots \tag{8.59e}$$

The zeros of the Bessel function $J_o(z)$ are given in Appendix B, and the first four zeros are: $\lambda_1 = 2.4048$, $\lambda_2 = 5.5201$, $\lambda_3 = 8.6537$ and $\lambda_4 = 11.7915$. The functions $P(p_k)$ are now found to be

$$P(0)=1,\quad P(p_k)=I_0\left(-\frac{ir\lambda_k}{r_0}\right)=J_0\left(\lambda_k\frac{r}{r_0}\right)$$

Since

$$\frac{dQ}{dp}=I_0\left(r_0\sqrt{p/\alpha}\right)+\frac{r_0}{2}\sqrt{\frac{p}{\alpha}}I_1\left(r_0\sqrt{p/\alpha}\right)$$

we obtain

$$\frac{dQ(0)}{dp}=1$$

and

$$\frac{dQ(p_k)}{dp} = -\frac{\lambda_k}{2} I_1(-i\lambda_k) = -\frac{\lambda_k}{2} J_1(\lambda_k), \quad k = 1,2,3,\ldots$$

Thus, the solution can be written as

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = 1 - 2\sum_{k=1}^{\infty} \frac{J_0[\lambda_k (r/r_0)]}{\lambda_k J_1(\lambda_k)} \exp\left(-\frac{\alpha\lambda_k^2}{r_0^2} t\right) \tag{8.60}$$

which gives the temperature distribution in the cylinder for times $t > 0$.

8.10 Solid Sphere

Consider now a solid sphere of radius r_0, which is initially at a uniform temperature T_i. The temperature of the surface at $r = r_0$ is changed to T_w at $t = 0$ and is subsequently maintained constant at this value for times $t > 0$. We wish to find the unsteady-state temperature distribution $T(r, t)$ in the sphere.

Assuming constant thermophysical properties, the formulation of the problem in terms of $\bar{\theta}(r,t) = T(r,t) - T_i$ *is* given by

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{8.61a}$$

$$\theta(r,0) = 0 \tag{8.61b}$$

$$\theta(0,t) = \textit{finite} \quad \text{and} \quad \theta(r_0,t) = \theta_w \tag{8.61c,d}$$

where $\theta_w = T_w - T_i$.

The Laplace transform of the differential equation (8.61*a*) and the boundary conditions (8.61*c*,*d*) are

$$\frac{\partial^2\bar{\theta}}{\partial r^2} + \frac{2}{r}\frac{\partial\bar{\theta}}{\theta r} = \frac{p}{\alpha}\bar{\theta} = 0 \tag{8.62a}$$

$$\theta(0,p) = \textit{finite} \quad \text{and} \quad \theta(r_0,p) = \frac{\theta_w}{p} \tag{8.62b,c}$$

where $\bar{\theta}(r,p)$ is the Laplace transform of $\bar{\theta}(r,t)$. The differential equation (8.62a) can be written as

$$\frac{d^2}{dr^2}(r\bar{\theta}) - \frac{p}{\alpha}(r\bar{\theta}) = 0 \tag{8.63}$$

the solution of which gives

$$\bar{\theta}(r,p) = \frac{1}{r}(C_1 \cosh mr + C_2 \sinh mr) \tag{8.64}$$

where $m^2 = p/\alpha$. Since $\bar{\theta}(0,p) = \mathit{finite}$, it follows that $C_1 = 0$. Furthermore, the boundary condition (8.62*c*) yields

$$C_2 = \theta_w \frac{r_0}{p \sinh mr_0} \tag{8.65}$$

Then, the solution for $\bar{\theta}(r,p)$ becomes

$$\frac{\bar{\theta}}{\theta_w} = \frac{r_0}{r}\frac{\sinh mr}{p \sinh mr_0} = \frac{r_0}{r}\frac{\sinh\sqrt{p/\alpha}}{p \sinh\sqrt{p/\alpha}\, r_0} \tag{8.66}$$

The use of transform No. 39 from Appendix D gives

$$\frac{\theta(r,t)}{\theta_w} = \frac{T(r,t) - T_i}{T_w - T_i} = 1 + \frac{2r_0}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n^2\pi^2\alpha t/r_0^2} \sin\frac{n\pi r}{r_0} \tag{8.67}$$

Equation (8.67) is a representation of the unsteady temperature distribution in the sphere in the form of an infinite series. Because of the negative argument of the exponential term, the convergence of this series is fast for large values of time. However, an alternative expression that converges quickly for small values of time can be developed by expanding the transform (8.66) as an asymptotic series in negative exponentials and then inverting the resulting expansion term by term. The development of this alternative form is left as an exercise (see Problem 8.19).

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Problems

8.1 Prove that £{sin ωt} = $\omega/(p^2 + \omega^2)$ and £ {cos ωt} = $p/(p^2 + \omega^2)$.

8.2 Prove that if $\pounds\{f(t)\} = \bar{f}(p)$ then $\pounds\{df/dt\} = p\bar{f}(p) - f(0)$.

8.3 Prove that if $\pounds\{f(t)\} = \bar{f}(p)$ then $\pounds\left\{\int_0^t f(t')dt'\right\} = (1/p)\bar{f}(\rho)$.

8.4 Prove that $g * h = h * g$.

8.5 Find the inverse of

$$\frac{\cosh\sqrt{(p/\alpha)}x}{p\cosh\sqrt{(p/\alpha)}L}$$

by the Heaviside expansion theorem and obtain the expression (8.36).

8.6 The semi-infinite solid shown in Fig. 8.7 is initially at a uniform temperature T_i and has constant thermophysical properties. For times $t \geq 0$, the surface at $x = 0$ is exposed to a fluid maintained at a constant temperature $T_\infty(\neq T_i)$. Assuming that the heat transfer coefficient h is constant, obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the solid for times $t > 0$.

8.7 A semi-infinite solid, $x \geq 0$, is initially at a uniform temperature T_i. For times $t \geq 0$, a constant heat flux q''_w is applied to the surface at $x = 0$. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the solid for times $t > 0$.

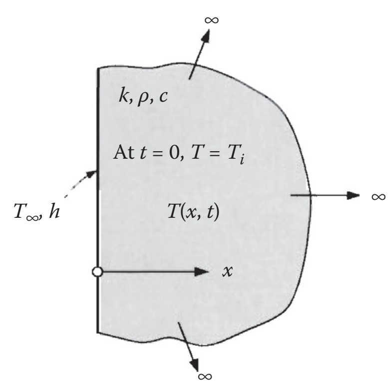

FIGURE 8.7
Figure for Problem 8.6.

8.8 A plane wall of thickness L is initially at a uniform temperature T_i. For times $t \geq 0$, the temperature of the surface at $x = 0$ is varied according to $T(0, t) = F(t)$, where $F(t)$ is a prescribed function of time, and the surface at $x = L$ is kept at T_i. Assuming constant thermo-physical properties, obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the wall for $t > 0$.

8.9 Two semi-infinite solids are initially at uniform temperatures T_1 and T_2. Assume constant thermophysical properties for the two solids.

(*a*) Find the temperature distributions in the solids for times $t > 0$ after they have been put into contact at $t = 0$ as illustrated in Fig. 8.8. Assume perfect thermal contact at the interface.

(*b*) Obtain an expression for the variation of the interface temperature with time.

8.10 Re-solve Problem 6.1 using Laplace transforms.

8.11 Re-solve Problem 6.2 using Laplace transforms.

8.12 Re-solve Problem 6.20 using Laplace transforms and obtain an expression for the unsteady-state temperature distribution that will converge rapidly for small values of time.

8.13 Re-solve Problem 6.21 using Laplace transforms.

8.14 Re-solve Problem 6.22 using Laplace transforms and obtain an expression for the unsteady-state temperature distribution that will converge rapidly for small values of time.

8.15 Re-solve Problem 6.24 using Laplace transforms and obtain an expression for the unsteady-state temperature distribution that will converge rapidly for small values of time.

8.16 Re-solve Problem 6.27 using Laplace transforms.

8.17 Re-solve Problem 6.28 using Laplace transforms and obtain an expression for the unsteady-state temperature distribution that will converge rapidly for small values of time.

8.18 Re-solve Problem 6.29 using Laplace transforms.

8.19 A solid sphere of radius r_0 is initially at a uniform temperature T_i. The temperature of the surface at $r = r_0$ is maintained at a constant temperature T_w ($\neq T_i$) for times $t \geq 0$. Assume that the thermophysical properties are constant. Obtain an expression for the unsteady temperature distribution $T(r, t)$ in the sphere that will converge rapidly for small values of time.

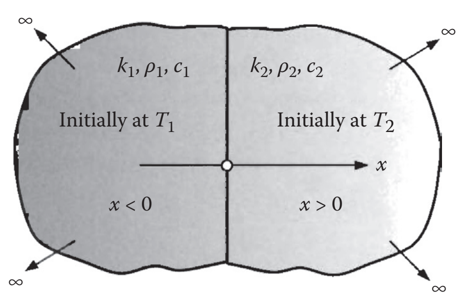

FIGURE 8.8
Figure for Problem 8.9.

8.20 A semi-infinite solid, $x \geq 0$, is initially at a uniform temperature T_i. Obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the solid, if the temperature of the surface at $x = 0$ is given as $T(0, t) = T_0 + (\Delta T)_0 \sin \omega t$ for times $t \geq 0$. Assume constant thermophysical properties for the solid.

8.21 Re-solve Problem 6.30 using Laplace transforms.

8.22 Re-solve Problem 6.31 using Laplace transforms.

8.23 Re-solve Problem 6.32 using Laplace transforms.

8.24 Re-solve Problem 6.33 using Laplace transforms.

8.25 Re-solve Problem 6.34 using Laplace transforms and obtain an expression for the unsteady-state temperature distribution that will converge rapidly for small values of time.

8.26 Re-solve Problem 6.35 using Laplace transforms.

8.27 Re-solve Problem 6.36 using Laplace transforms and obtain an expression for the unsteady-state temperature distribution that will converge rapidly for small values of time.

9

Heat Conduction with Local Heat Sources

9.1 Introduction

Our discussions in the previous chapters included several heat conduction problems involving *"distributed"* internal energy sources (or sinks). In this chapter, we focus our attention on a class of problems which involve *"local"* sources of internal energy such that heat is released continuously or spontaneously in a region of a system of interest that is infinitesimally small compared to the dimensions of the system. Such local sources can be in the form of a plane, a cylindrical or a spherical shell, a line or even a point. They can be stationary or moving in a system.

In solving such problems, the use of the *Dirac delta function* significantly helps to reduce the steps in obtaining the solutions. Accordingly, we first introduce the Dirac delta function in Section 9.2 and then use it in the solution of a number of representative examples involving local sources releasing heat continuously or spontaneously in finite regions. We next consider two representative cases involving line and point sources of heat in infinite regions, where we demonstrate the use of Laplace transforms as another method of solution. Finally, we consider an example involving a moving heat source, as a model of such engineering applications as welding, grinding, metal cutting, flame or laser hardening.

In the following sections we consider only those cases that involve heat sources, with the understanding that the study of the cases with heat sinks follows the same solution procedures.

9.2 The Delta Function

The so-called *Dirac delta function* $\delta(x)$, first introduced by the physicist P. M. A. Dirac [1], is defined by

$$\delta(x-a) = 0 \qquad \text{when} \qquad x \neq a \tag{9.1a}$$

and

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1 \tag{9.1b}$$

Although these relations do not give a clear picture of the Dirac delta function, it can be regarded, in a rough sense, as a concentrated "spike" of unit area. For example, let us define a function $\Delta_\varepsilon(x)$ by the relation

$$\Delta_\varepsilon(x) = \begin{cases} 0, & x < a \\ \dfrac{1}{\varepsilon} & a \le x \le a + \varepsilon \\ 0, & x > a + \varepsilon \end{cases} \tag{9.2}$$

As $\varepsilon \to 0$, the function $\Delta_\varepsilon(x)$ becomes taller and narrower as illustrated Fig. 9.1. On the other hand, since

$$\int_{-\infty}^{\infty} \Delta_\varepsilon(x)dx = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} dx = 1$$

it follows that $\delta(x - a)$ may be regarded as being the limiting form of $\Delta_\varepsilon(x)$ as $\varepsilon \to 0$; that is,

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon(x) = \delta(x - a) \tag{9.3}$$

Furthermore, consider a function $f(x)$ that is *continuous* on the internal $a \le x \le \alpha + \varepsilon$. Then,

$$\int_{-\infty}^{\infty} f(x)\Delta_\varepsilon(x)dx = \int_a^{a+\varepsilon} f(x)\Delta_\varepsilon(x)dx = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(x)dx \tag{9.4}$$

Since $f(x)$ is continuous on $a \le x \le a + \varepsilon$, the "*mean value theorem*" of integral calculus [2] tells us that

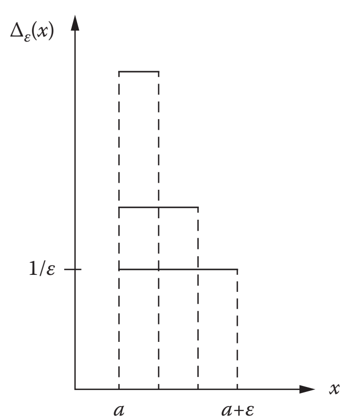

FIGURE 9.1
The $\Delta_\varepsilon(x)$ function defined by Eq. (9.2).

$$\int_a^{a+\varepsilon} f(x)\,dx = \varepsilon f(\xi) \tag{9.5}$$

where ξ is a point in $a \le x \le a + \varepsilon$. If we now substitute Eq. (9.5) into Eq. (9.4) and let $\varepsilon \to 0$, noting that $\lim_{\varepsilon \to 0} f(\xi) = f(a)$, we obtain

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a) \tag{9.6a}$$

or

$$\int_b^c f(x)\delta(x-a)\,dx = f(a) \tag{9.6b}$$

where $b < a < c$. Thus, the action of $\delta(x - a)$ on $f(x)$ is to "pick out" its value at $x = a$. As a consequence of Eq. (9.6*a*), or Eq. (9.6*b*), we also note that

$$f(x)\delta(x-a) = f(a)\delta(x-a) \tag{9.7}$$

The Dirac delta function $\delta(x - a)$, as introduced here, may be regarded to be zero everywhere except at $x = a$ and infinite at $x = a$, in such a way as to have unit area. This definition, however, does not qualify it as a function within the framework of ordinary function theory. It belongs to a group of generalized functions, called *distributions*. Dirac referred to it as an "improper function." It may effectively be used in analysis only when it is obvious that no inconsistency will follow from its use. In this chapter we will use it to a limited extent in the solution of a number of representative problems involving local heat sources.

9.2.1 Plane Heat Source

Consider the slab shown in Fig. 9.2. Assume that internal energy is generated at $x = a$ continuously over a planar section of infinitesimally small thickness. Let q''_{pl} (W/m^2) represent the *total of the heat fluxes* from the two surfaces of this planar section. The rate of internal energy generation per unit volume $\dot{q}(x)$ within the slab (i.e., *volumetric heat source strength*, W/m^3) can then be defined as

$$\dot{q}(x) = q''_{pl}\delta(x-a) \tag{9.8}$$

Accordingly, Eq. (9.8) leads to the correct calculation of the total rate of internal energy generation within the slab per unit surface area; that is,

$$\int_0^L \dot{q}(x)\,dx = \int_0^L q''_{pl}\delta(x-a)\,dx = q''_{pl} \tag{9.9}$$

A heat source of this type is referred to as a *plane heat source*, and q''_{pl} is called its *strength per unit area*. It should be noted that the plane heat source strength does not have to be a constant, but it can be a function of time t; that is, $q''_{pl} = q''_{pl}(t)$.

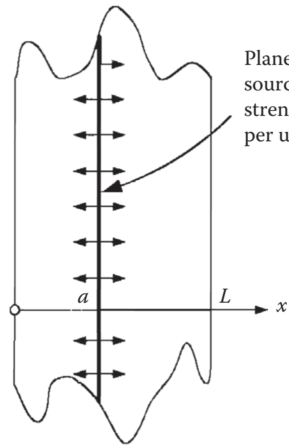

FIGURE 9.2
Slab with a plane heat source at $x = a$.

If the plane heat source releases heat instantaneously at $t = t_0$ with a strength $q''_{pl,i}$ (J/m²), then a similar argument would lead to

$$q''_{pl}(t) = q''_{pl,i}\delta(t - t_0) \tag{9.10}$$

or

$$\dot{q}(x, t) = q''_{pl,i}\delta(x - a)\delta(t - t_0) \tag{9.11}$$

9.2.2 Cylindrical and Spherical Shell Heat Sources

Let us now consider a cylinder with a *cylindrical shell heat source* concentrically situated at $r = r_1$ as shown in Fig. 9.3a. If $q'_{cy}(t)$ (W/m) represents the *strength* of the source *per unit length*, then a similar argument as in the plane heat source case would lead to a definition of the volumetric heat source in the cylinder as

$$\dot{q}(r, t) = \frac{q'_{cy}(t)}{2\pi r_1}\delta(r - r_1) \tag{9.12a}$$

If the cylindrical shell heat source releases heat instantaneously at $t = t_0$ with a strength $q'_{cy,i}$ (J/m), then $\dot{q}(r,t)$ would be given by

$$\dot{q}(r, t) = \frac{q'_{cy,i}}{2\pi r_1}\delta(r - r_1)\delta(t - t_0) \tag{9.12b}$$

Similarly, in a sphere with a concentrically situated *spherical shell heat source* of radius r_1 and strength $q_{sp}(t)$ (W), as shown in Fig. 9.3*b*, the volumetric heat source strength in the sphere is defined by

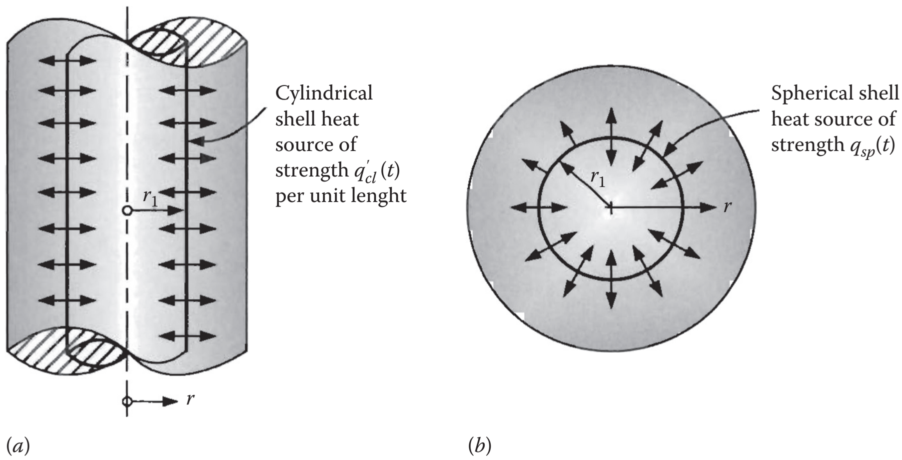

FIGURE 9.3
Cylindrical (*a*) and spherical (*b*) shell heat sources.

$$\dot{q}(r,t) = \frac{q_{sp}(t)}{4\pi r_1^2}\delta(r - r_1) \tag{9.13a}$$

If the spherical shell heat source releases heat instantaneously at $t = t_0$, then

$$\dot{q}(r,t) = \frac{q_{sp,i}}{4\pi r_1^2}\delta(r - r_1)\delta(t - t_0) \tag{9.13b}$$

where $q_{sp,i}$ (J) is the strength of the source.

9.3 Slab with Distributed and Plane Heat Sources

As an example, we first consider a slab of thickness L in the x direction, which is initially at a uniform temperature T_i. Internal energy is generated within this slab continuously at a rate of $\dot{q}(x,t)$ per unit volume (volumetric heat source, W/m^3) for times $t \geq 0$, while the surfaces at $x = 0$ and $x = L$ are maintained at the initial temperature T_i. Assuming constant thermophysical properties, the one-dimensional unsteady-state temperature distribution $T(x, t)$ within the slab will satisfy the following initial-and-boundary-value problem for $t > 0$:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}(x,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{9.14a}$$

$$T(x,0) = T_i \tag{9.14b}$$

$$T(0,t) = T(L,t) = T_i \tag{9.14c}$$

where k and α are, respectively, the thermal conductivity and thermal diffusivity of the material of the slab. The solution of this problem can readily be obtained by the methods introduced in the previous chapters (see, for example, Section 7.3) and is given by

$$T(x,t) = T_i + \frac{2}{\rho c L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x \int_0^t e^{\alpha\lambda_n^2 t'} \int_0^L \dot{q}(x',t')\sin\lambda_n x'\,dx'\,dt' \tag{9.15}$$

where ρ and c are, respectively, the density and specific heat of the material of the slab, and

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1,2,3,\ldots \tag{9.16}$$

9.3.1 Instantaneous Volumetric Heat Source

When the volumetric heat source releases heat instantaneously (say, at $t = 0$), a single explosion takes place within the slab and the energy released diffuses throughout the slab. If the instantaneous volumetric heat source strength (i.e., the amount of the energy released per unit volume at $t = 0$) is $\dot{q}_i(x)$ (J/m^3), then it is related to the volumetric heat source strength, $\dot{q}(x,t)$, by

$$\dot{q}(x,t) = \dot{q}_i(x)\delta(t-0) \tag{9.17}$$

After substituting Eq. (9.17) into Eq. (9.15) and noting from Eq. (9.6b) that

$$\int_0^t e^{\alpha\lambda_n^2 t'}\delta(t'-0)\,dt' = 1 \tag{9.18}$$

we obtain

$$T(x,t) = T_i + \frac{2}{\rho c L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x \int_0^L \dot{q}_i(x')\sin\lambda_n x'\,dx' \tag{9.19}$$

which is the transient temperature distribution as response to the instantaneous volumetric heat source within the slab.

If, in particular, $\dot{q}_i(x) = \dot{q}_i = \text{constant}$, then the transient response is given by

$$T(x,t) = T_i \frac{2\dot{q}_i}{\rho c L}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{\lambda_n} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x \tag{9.20}$$

Here it is important to note that

(*a*) as $t \to \infty$, the solution (9.20) yields $T(x, t) = T_i$ as expected, and
(*b*) the total energy removed from the surfaces at $x = 0$ and $x = L$ at all times must be equal to the energy released at $t = 0$ by the source. That is,

$$\int_0^\infty kA\left[\left(\frac{\partial T}{\partial x}\right)_{x=0} - \left(\frac{\partial T}{\partial x}\right)_{x=L}\right]dt = \ldots = \frac{8AL\dot{q}_i}{\pi^2}\sum_{n\ odd}^{\infty}\frac{1}{n^2} \tag{9.21}$$

where A is the surface area of the slab. It can be shown that (see Problem 9.1*a*),

$$\sum_{n\,\text{odd}}\frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \ldots = \frac{1}{8}\pi^2 \tag{9.22}$$

Thus, the energy balance relation (9.21) reduces to

$$\int_0^\infty kA\left[\left(\frac{\partial T}{\partial x}\right)_{x=0} - \left(\frac{\partial T}{\partial x}\right)_{x=L}\right]dt = (LA)\dot{q}_i \tag{9.23}$$

which confirms the validity of the results obtained in this section.

9.3.2 Plane Heat Source

Let us now consider the special case of the problem (9.14) where the heat source is a plane heat source of strength $q''_{pl}(t)$ (W/m^2), which is located at $x = a$ within the slab and releases heat continuously for times $t \geq 0$. In this case, the strength of the plane heat source $q''_{pl}(t)$ is related to the volumetric heat source strength $\dot{q}(x,t)$, as discussed in Section 9.2.1, by

$$\dot{q}(x,t) = q''_{pl}(t)\delta(x-a) \tag{9.24}$$

After substituting Eq. (9.24) into Eq. (9.15) and noting that

$$\int_0^L \delta(x'-a)\sin\lambda_n x'\,dx' = \sin\lambda_n a \tag{9.25}$$

we obtain

$$T(x,t) = T_i + \frac{2}{\rho c L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\sin\lambda_n a\int_0^t e^{\alpha\lambda_n^2 t'}q''_{pl}(t')\,dt' \tag{9.26}$$

If, in particular, $q''_{pl}(t) = q''_{pl}$ = constant, then

$$T(x,t) = T_i + \frac{2q''_{pl}}{kL}\sum_{n=1}^{\infty}\frac{1-e^{-\alpha\lambda_n^2 t}}{\lambda_n^2}\sin\lambda_n x\sin\lambda_n a \tag{9.27}$$

9.3.3 Instantaneous Plane Heat Source

Let the plane heat source considered above release heat instantaneously (say at $t = 0$) at a strength of $q''_{pl,i}$ (J/m^2), which is related to $q''_{pl}(t)$ by

$$q''_{pl}(t) = q''_{pl,i}\delta(t-0) \tag{9.28}$$

Substituting Eq. (9.28) into the solution (9.26) and performing the integral over t' we obtain

$$T(x,t) = T_i + \frac{2q''_{pl,i}}{\rho cL}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\sin\lambda_n a \tag{9.29}$$

Again, it is important to note that

(*a*) as $t \to \infty$, the solution (9.29) yields $T(x, t) = T_i$ as expected, and
(*b*) the total energy removed from the surfaces at $x = 0$ and $x = L$ at all times must be equal to the total energy released by the plane heat source at $t = 0$. That is,

$$\int_0^{\infty} kA\left[\left(\frac{\partial T}{\partial x}\right)_{x=0} - \left(\frac{\partial T}{\partial x}\right)_{x=L}\right]dt = \ldots = \frac{2Aq''_{pl,i}}{L}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\lambda_n}\sin\lambda_n a \tag{9.30}$$

On the other hand, it can be shown that (see Problem 4.11)

$$1 = \frac{2}{L}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\lambda_n}\sin\lambda_n a, \qquad 0 < a < L \tag{9.31}$$

Thus, the energy balance relation (9.30) reduces to

$$\int_0^{\infty} kA\left[\left(\frac{\partial T}{\partial x}\right)_{x=0} - \left(\frac{\partial T}{\partial x}\right)_{x=L}\right]dt = Aq''_{pl,i} \tag{9.32}$$

which confirms the validity of the results we obtained in this section.

9.4 Long Solid Cylinder with Cylindrical Shell and Line Heat Sources

We now consider a long solid cylinder, of radius r_0 and constant thermophysical properties, which is initially at a uniform temperature T_i. Assume that internal energy is generated within the cylinder continuously at a rate of $\dot{q}(r,t)$ per unit volume (W/m^3) for times $t \geq 0$, while the surface at $r = r_0$ is maintained at the initial temperature T_i. The one-dimensional unsteady-state temperature distribution $T(r, t)$ in the cylinder for $t > 0$ is then given by (see Problem 7.19)

$$T(r,t) = T_i + \frac{2}{\rho c r_0^2} \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_o)} \int_0^t e^{\alpha\lambda_n^2 t'} \int_0^{r_0} \dot{q}(r',t') J_0(\lambda_n r') r'\, dr'\, dt' \tag{9.33}$$

where λ_n's are the positive roots of

$$J_0(\lambda r_0) = 0 \tag{9.34}$$

and α, ρ, and c are, respectively, the thermal diffusivity, density, and specific heat of material of the cylinder.

9.4.1 Cylindrical Shell Heat Source

Let us now consider the special case where the heat source is a concentrically situated cylindrical shell heat source of radius r_1 (i.e., $0 < r_1 < r_0$) and of strength $q'_{cy}(t)$ (W/m) per unit length of the cylinder. The cylindrical shell heat source releases heat continuously for times $t \geq 0$. In this case, the strength $q'_{cy}(t)$ of the source is related to the volumetric heat source $\dot{q}(r,t)$ by the relation (9.12*a*). After substituting Eq. (9.12*a*) into Eq. (9.33) and noting that

$$\int_0^{r_0} J_0(\lambda_n r')\delta(r' - r_1) r'\, dr' = r_1 J_0(\lambda_n r_1) \tag{9.35}$$

we obtain

$$T(r,t) = T_i + \frac{1}{\pi \rho c r_0^2} \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t} \frac{J_0(\lambda_n r) J_0(\lambda_n r_1)}{J_1^2(\lambda_n r_0)} \int_0^t e^{\alpha\lambda_n^2 t'} q'_{cy}(t')\, dt' \tag{9.36}$$

If, in particular, $q'_{cy}(t) = q'_{cy} = \text{constant}$, then the transient solution is given by

$$T(r,t) = T_i + \frac{q'_{cy}}{\pi k r_0^2} \sum_{n=1}^{\infty} \frac{1 - e^{-\alpha\lambda_n^2 t}}{\lambda_n^2} \frac{J_0(\lambda_n r) J_0(\lambda_n r_1)}{J_1^2(\lambda_n r_0)} \tag{9.37}$$

where k is the thermal conductivity of the material of the cylinder.

9.4.2 Instantaneous Cylindrical Shell Heat Source

If the cylindrical shell heat source considered above releases heat instantaneously at $t = 0$, then its strength $q'_{cy,i}$ (J/m) is related to $q'_{cy}(t)$ by

$$q'_{cy}(t) = q'_{cy,i}\delta(t-0) \tag{9.38}$$

Substituting Eq. (9.38) into the solution (9.36) and evaluating the integral over t', we obtain

$$T(r,t) = T_i + \frac{q'_{cy,i}}{\pi\rho c r_0^2}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)J_0(\lambda_n r_1)}{J_1^2(\lambda_n r_0)} \tag{9.39}$$

9.4.3 Line Heat Source

Let us now consider the case where the heat source is a *line heat source* of strength $q'_{ln}(t)$ (W/m) situated along the centerline of the cylinder and releasing heat for times $t \geq 0$. In this case, the one-dimensional transient temperature distribution $T(r, t)$ in the cylinder can readily be obtained by letting $r_1 \to 0$ in the solution (9.36),

$$T(r,t) = T_i + \frac{1}{\pi\rho c r_0^2}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0)}\int_0^t e^{\alpha\lambda_n^2 t'} q'_{ln}(t')dt' \tag{9.40}$$

If, in particular, $q'_{ln}(t) = q'_{ln} = \text{constant}$, then

$$T(r,t) = T_i + \frac{q'_{ln}}{\pi k r_0^2}\sum_{n=1}^{\infty}\frac{1-e^{-\alpha\lambda_n^2 t}}{\lambda_n^2}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0)} \tag{9.41}$$

9.4.4 Instantaneous Line Heat Source

If the line heat source releases heat spontaneously at $t = 0$ with a strength of $q'_{ln,i}$ (J/m), the solution can be obtained as a limit by letting $r_1 \to 0$ in the solution (9.39),

$$T(r,t) = T_i + \frac{q'_{ln,i}}{\pi\rho c r_0^2}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_0)} \tag{9.42}$$

Here, we again note that the total energy removed from the cylinder at all times must be equal to the total energy released by the line heat source at $t = 0$; that is,

$$\int_0^{\infty}\left[-2\pi r_0 k\left(\frac{\partial T}{\partial r}\right)_{r=r_0}\right]dt = \ldots = \frac{2q'_{ln,i}}{r_0}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n r_0)} \tag{9.43}$$

On the other hand (see Example 4.5),

$$\frac{2}{r_0}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n r_0)} = 1 \tag{9.44}$$

Thus,

$$\int_0^{\infty}\left[-2\pi r_0 k\left(\frac{\partial T}{\partial r}\right)_{r=r_0}\right]dt = q'_{ln,i} \tag{9.45}$$

which confirms the validity of the results obtained in this section.

9.5 Solid Sphere with Spherical Shell and Point Heat Sources

Consider a solid sphere, of radius r_0 and constant thermophysical properties, which is initially at a uniform temperature T_i. For times $t \geq 0$, internal energy is generated continuously at a rate of $\dot{q}(r,t)$ per unit volume (W/m^3) in this sphere, while the surface at $r = r_0$ is maintained at the initial temperature T_i. The one-dimensional unsteady-state temperature distribution $T(r, t)$ in the sphere for $t > 0$ is given by (see Problem 7.25)

$$T(r,t) = T_i + \frac{2}{\rho c r_0 r}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n r\int_0^t e^{\alpha\lambda_n^2 t'}\int_0^{r_0}\dot{q}(r',t')\sin\lambda_n r' r'\,dr'\,dt' \tag{9.46}$$

where ρ and c are, respectively, the density and specific heat of the material of the sphere, and

$$\lambda_n = \frac{n\pi}{r_0}, \qquad n = 1,2,3,\ldots$$

9.5.1 Spherical Shell Heat Source

We now consider the special case where the heat source is a concentrically situated spherical shell heat source, of radius r_1 (i.e., $0 < r_1 < r_0$) and of strength $q_{sp}(t)$ (W), which releases heat continuously for times $t \geq 0$. For this case, the spherical shell heat source is related to the volumetric heat source by the relation (9.13*a*). Substituting Eq. (9.13*a*) into Eq. (9.46) and performing the integration over r' we get

$$T(r,t) = T_1 + \frac{1}{2\pi\,\rho c r_1 r_0}\frac{1}{r}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n r\sin\lambda_n r_1\int_0^t e^{\alpha\lambda_n^2 t'} q_{sp}(t')\,dt' \tag{9.47}$$

If, in particular, $q_{sp}(t) = q_{sp}$ = constant, then the solution (9.47) reduces to

$$T(r,t) = T_i + \frac{q_{sp}}{2\pi k r_1 r_0} \frac{1}{r} \sum_{n=1}^{\infty} \frac{1 - e^{-\alpha\lambda_n^2 t}}{\lambda_n^2} \sin\lambda_n r \sin\lambda_n r_1 \tag{9.48}$$

where k is the thermal conductivity of the material of the sphere.

9.5.2 Instantaneous Spherical Shell Heat Source

Let the spherical shell heat source considered above release heat spontaneously at $t = 0$ with a strength $q_{sp,i}$ (J). Then, its strength $q_{sp,i}$ is related to $q_{sp}(t)$ by

$$q_{sp}(t) = q_{sp,i}\delta(t-0) \tag{9.49}$$

Substituting Eq. (9.49) into the solution (9.47) and evaluating the integral over t' we obtain

$$T(r,t) = T_i + \frac{q_{sp,i}}{2\pi \rho c r_1 r_0} \frac{1}{r} \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t} \sin\lambda_n r \sin\lambda_n r_1 \tag{9.50}$$

As in the previous cases, as $t \to \infty$, Eq. (9.50) reduces to $T(r, t) = T_i$. Furthermore, the total energy removed from the sphere at all times must be equal to the total energy released $q_{sp,i}$ by the heat source at $t = 0$. That is,

$$\int_0^{\infty} \left[-4\pi r_0^2 k \left(\frac{\partial T}{\partial r} \right)_{r=r_0} \right] dt = \ldots = q_{sp,i} \tag{9.51}$$

where we utilized (see Problem 9.1*b*)

$$\frac{2}{r_1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\lambda_n} \sin\lambda_n r_1 = 1, \qquad 0 < r_1 < r_0 \tag{9.52}$$

9.5.3 Point Heat Source

Let the heat source in the sphere considered above be a *point source* situated at $r = 0$ and releasing heat with a strength $q_{pt}(t)$ (W) for times $t \geq 0$. In this case, the solution can be obtained from (9.47) by letting $r_1 \to 0$. Noting that

$$\lim_{r_1 \to 0} \left\{ \frac{1}{r_1} \sin\lambda_n r_1 \right\} = \lambda_n \tag{9.53}$$

we obtain

$$T(r,t) = T_i + \frac{1}{2\pi \rho c r_0} \frac{1}{r} \sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t} \sin \lambda_n r \int_0^t e^{\alpha\lambda_n^2 t'} q_{pt}(t')\,dt' \tag{9.54}$$

If, in particular, $q_{pt}(t) = q_{pt}$ = constant, then the solution (9.54) reduces to

$$T(r,t) = T_i + \frac{q_{pt}}{2\pi k r_0} \frac{1}{r} \sum_{n=1}^{\infty} \frac{1 - e^{-\alpha\lambda_n^2 t}}{\lambda_n} \sin \lambda_n r \tag{9.55}$$

9.5.4 Instantaneous Point Heat Source

Let the point heat source release heat instantaneously at $t = 0$ with a strength $q_{pt,i}$ (J). Then the solution can be obtained from the solution (9.50) again by letting $r_1 \to 0$, which yields

$$T(r,t) = T_i + \frac{q_{pt,i}}{2\pi \rho c r_0} \frac{1}{r} \sum_{n=1}^{\infty} \lambda_n e^{-\alpha\lambda_n^2 t} \sin \lambda_n r \tag{9.56}$$

Here we note that we cannot evaluate the total heat removed from the sphere at all times by calculating the total heat conducted to the surface at $r = r_0$ by Fourier's law (Why not?). The total heat removed from sphere, however, is also equal to the total heat conducted across any isothermal spherical surface situated at any r $(0 < r < r_0)$. Thus,

$$\int_0^{\infty} \left[-4\pi r^2 k \frac{\partial T}{\partial r} \right] dt = \ldots = q_{pt,i} \tag{9.57}$$

where we have utilized the expansion (see Problem 9.1*c*)

$$\frac{2}{r_0} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} [r\lambda_n \cos \lambda_n r - \sin \lambda_n r] = 1 \tag{9.58}$$

9.6 Infinite Region with Line Heat Source

We now consider heat conduction in a large body, $0 \le r < \infty$, due to a line heat source situated along the z axis in the body at $r = 0$. In the following discussions we will assume that the body is initially at a uniform temperature T_i and its thermophysical properties (ρ, c, k) are constants.

9.6.1 Continuous Heat Release

Let the line heat source be continuously releasing heat at a constant rate q'_{ln} per unit length of the source (W/m) for times $t \geq 0$. The mathematical formulation of the problem for the one-dimensional unsteady temperature distribution $T(r, t)$ can be stated in cylindrical coordinates for $t > 0$, in terms of $\theta(r, t) = T(r, t) - T_i$, as

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{9.59a}$$

$$\theta(r,0) = 0 \tag{9.59b}$$

$$\lim_{r\to 0}\left[-2\pi rk\frac{\partial\theta}{\partial r}\right] = q'_{ln} \quad \text{and} \quad \theta(\infty,t) = 0 \tag{9.59c,d}$$

To solve this problem we will utilize the method of Laplace transforms discussed in Chapter 8. Accordingly, we introduce

$$\bar{\theta}(r,p) = \pounds\{\theta(r,t)\} = \int_0^\infty e^{-pt}\theta(r,t)\,dt \tag{9.60}$$

for the Laplace transforms of $\theta(r, t)$. The transformed differential equation and boundary conditions are then given by

$$\frac{d^2\bar{\theta}}{dr^2} + \frac{1}{r}\frac{d\bar{\theta}}{dr} - \frac{p}{\alpha}\bar{\theta}(r,p) = 0 \tag{9.61a}$$

$$\lim_{r\to 0}\left[r\frac{d\bar{\theta}}{dr}\right] = -\frac{q'_{ln}}{2\pi kp} \quad \text{and} \quad \bar{\theta}(\infty,p) = 0 \tag{9.61b,c}$$

The solution of the differential equation (9.61*a*) which satisfies the condition (9.61*c*) is given by

$$\bar{\theta}(r,p) = AK_0\left(r\sqrt{\frac{p}{a}}\right) \tag{9.62}$$

where A is a non-zero constant and $K_0(r)$ is the zeroth-order modified Bessel function of the second kind. Substituting Eq. (9.62) into the condition (9.6*b*) and making use of the limit as $r \to 0$,

$$\lim_{r\to 0} K_1(r) \cong \frac{1}{r} \tag{9.63}$$

we obtain

$$A = \frac{q'_{ln}}{2\pi kp} \tag{9.64}$$

Thus, the solution for $\bar{\theta}(r,t)$ becomes

$$\bar{\theta}(r,p) = \frac{q'_{ln}}{2\pi kp} K_0\left(r\sqrt{\frac{p}{a}}\right) \tag{9.65}$$

The solution for $\theta(r, t)$ is then obtained as follows:

$$\theta(r,t) = \pounds^{-1}\{\bar{\theta}(r,p)\} = \frac{q'_{ln}}{2\pi k}\pounds^{-1}\left\{\frac{1}{p}K_0\left(r\sqrt{\frac{p}{\alpha}}\right)\right\} \tag{9.66a}$$

which can be rewritten as

$$\theta(r,t) = \frac{q'_{ln}}{2\pi k} = \int_0^t \pounds^{-1}\left\{K_0\left(r\sqrt{\frac{p}{\alpha}}\right)\right\}dt \tag{9.66b}$$

where we have used the relation (8.19). On the other hand [9],

$$\pounds^{-1}\left\{K_0\left(r\sqrt{\frac{p}{\alpha}}\right)\right\} = \frac{1}{2t}\exp\left(-\frac{r^2}{4\alpha t}\right) \tag{9.67}$$

Substituting Eq. (9.67) into Eq. (9.66*b*) yields

$$\theta(r,t) = \frac{q'_{ln}}{4\pi k}\int_0^t \frac{1}{t'}\exp\left(-\frac{r^2}{4\alpha t'}\right)dt' \tag{9.68a}$$

which can also be written as

$$\theta(r,t) = \frac{q'_{ln}}{4\pi k}\int_{\frac{r^2}{4\alpha t}}^{\infty} \frac{1}{\eta}e^{-\eta}\,d\eta \tag{9.68b}$$

or

$$\theta(r,t) = T(r,t) - T_i = \frac{q'_{ln}}{4\pi k}\mathrm{Ei}\left(\frac{r^2}{4\alpha t}\right) \tag{9.68c}$$

where $\mathrm{Ei}(x)$ represents the *exponential integral function*, which is defined as (see Appendix E)

$$\text{Ei}(x) = \int_x^\infty \frac{e^{-\eta}}{\eta} d\eta \tag{9.69}$$

Here we note that, since Ei(0) = ∞, there exists no steady-state solution as $t \to \infty$ for this problem.

9.6.2 Instantaneous Line Heat Source

Let us now consider the case where the line heat source is an instantaneous source, releasing heat at $t = 0$ with a strength $q_{ln,i}$ (J/m). The formulation of the problem for the transient temperature distribution $T(r, t)$ in the body is identical to that discussed above, except the condition (9.60*c*), which is replaced by

$$\int_0^\infty \rho c\theta(r,t)2\pi r dr = q'_{pt,i} \qquad \text{for} \qquad t > 0 \tag{9.70}$$

where $\theta(r, t) = T(r, t) - T_i$. This condition is a statement of the fact that the total energy released by the line heat source at $t = 0$ must be equal to the total increase in the internal energy of the medium at any given time t.

The problem consisting of the differential equation (9.59*a*) and the conditions (9.60*b*,*d*) and (9.71) can also be solved easily by Laplace transforms. Accordingly, the Laplace transforms of the condition (9.70) gives

$$\int_0^\infty \bar{\theta}(r,p) r\, dr = \frac{q'_{ln,i}}{2\pi\rho c p} \tag{9.71}$$

Substituting $\bar{\theta}(r,p)$ from Eq. (9.62) into Eq. (9.71), and evaluating the integral over r, we obtain

$$A = \frac{q'_{ln,i}}{2\pi k} \tag{9.72}$$

In evaluating the integral in Eq. (9.71), we have utilized, in addition to the limit given by Eq. (9.63), the limit as $r \to \infty$,

$$\lim_{r\to\infty} K_1(r) = \sqrt{\frac{\pi}{2r}} e^{-r} \tag{9.73}$$

Thus, the solution for $\bar{\theta}(r,p)$ is given by

$$\bar{\theta}(r,p) = \frac{q'_{ln,i}}{2\pi k} K_0\left(r\sqrt{\frac{p}{a}} \right) \tag{9.74}$$

which, when inverted, yields

$$\theta(r,t) = T(r,t) - T_i = \frac{q'_{ln,i}}{4\pi kt} \exp\left(-\frac{r^2}{4\alpha t}\right) \tag{9.75}$$

where we have used the inverse transform given by Eq. (9.67).

9.7 Infinite Region with Point Heat Source

Let us now consider the case of heat conduction in a large body. $0 \le r < \infty$, due to a point heat source situated at $r = 0$ in the body. Again we assume constant thermophysical properties and that the body is initially at a uniform temperature T_i.

9.7.1 Continuous Heat Release

We first consider the case where the point heat source is releasing heat at a constant rate q_{pt} (W) for times $t \ge 0$. The mathematical formulation of the problem for the one-dimensional unsteady-state temperature distribution $T(r, t)$ can be stated in spherical coordinates for times $t > 0$ as

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{9.76a}$$

$$\theta(r,0) = 0 \tag{9.76b}$$

$$\lim_{r\to 0}\left[-4\pi r^2 k \frac{\partial\theta}{\partial r}\right] = q \quad \text{and} \quad \theta(\infty,t) = 0 \tag{9.76c,d}$$

where $\theta(r, t) = T(r, t) - T_i$.

We will again utilize the method of Laplace transforms to solve the problem (9.76). Accordingly, the Laplace transform of $\theta(r, t)$ will satisfy the following boundary-value problem:

$$\frac{d^2\bar{\theta}}{dr^2} + \frac{2}{r}\frac{d\bar{\theta}}{dr} - \frac{p}{\alpha}\bar{\theta}(r,p) = 0 \tag{9.77a}$$

$$\lim_{r\to 0}\left[r^2 \frac{d\bar{\theta}}{dr}\right] = -\frac{q_{pt}}{4\pi kp} \quad \text{and} \quad \bar{\theta}(\infty,p) = 0 \tag{9.77b,c}$$

The solution of the differential equation (9.77*a*), which satisfies the condition (9.77*c*), is given by

$$\bar{\theta}(r,p) = B\frac{1}{r}e^{-mr}, \qquad m^2 = \frac{p}{\alpha} \tag{9.78}$$

where B is a non-zero constant. Substituting Eq. (9.78) into the condition (9.77*b*) yields

$$B = \frac{q_{pt}}{4\pi kp} \tag{9.79}$$

Thus,

$$\bar{\theta}(r,p) = \frac{q_{pt}}{4\pi krp}\exp\left(-r\sqrt{\frac{p}{\alpha}}\right) \tag{9.80}$$

Noting that, from Appendix D, transform No. 27,

$$\pounds^{-1}\left\{\frac{1}{p}e^{-mr}\right\} = \text{erfc}\left(\frac{r}{2\sqrt{\alpha t}}\right) \tag{9.81}$$

inversion of $\bar{\theta}(r,p)$ yields

$$\theta(r,t) = T(r,t) - T_i = \frac{q_{pt}}{4\pi kr}\text{erfc}\left(\frac{r}{2\sqrt{\alpha t}}\right) \tag{9.82}$$

where erfc(x) is the *complementary error function*.

In this case, we note that, since erfc(0) = 1, the solution (9.82) yields, as $t \to \infty$,

$$T(r) = T_i + \frac{q_{pt}}{4\pi kr} \tag{9.83}$$

for the steady-state temperature distribution in the body.

9.7.2 Instantaneous Point Heat Source

Let the point heat source considered above release heat spontaneously at $t = 0$ with a strength $q_{pt,i}$ (J). The mathematical formulation of the problem for the transient temperature distribution $T(r, t)$ in the body will be given, in terms of $\theta(r, t) = T(r, t) - T_i$, by the same differential equation (9.77*a*), the conditions (9.77*b*,*c*) and

$$\int_0^\infty \rho c\theta(r,t)4\pi r^2\, dr = q_{pt,i} \tag{9.84}$$

Again, this condition expresses the fact that the energy released by the point heat source at $t = 0$ must be equal to the total increase in the internal energy of the body at any given time t.

The Laplace transform of the condition (9.84) gives

$$\int_0^\infty \bar{\theta}(r,p)r^2\, dr = \frac{q_{pt,i}}{4\pi\rho cp} \tag{9.85}$$

Substituting $\bar{\theta}(r,p)$ from Eq. (9.78) into Eq. (9.85) and evaluating the integral over *r*, we get

$$B = \frac{q_{pt,i}}{4\pi k} \tag{9.86}$$

After substituting *B* from Eq. (9.86) into Eq. (9.78) and then inverting the resulting expression for $\bar{\theta}(r,p)$ we obtain

$$\theta(r,t) = T(r,t) - T_i = \frac{q_{pt,i}}{8\rho c(\pi\alpha t)^{3/2}} \exp\left(-\frac{r^2}{4\alpha t}\right) \tag{9.87}$$

In obtaining this result we have utilized (see Appendix D, transform No. 25)

$$\mathcal{L}^{-1}\left\{\exp\left(-r\sqrt{\frac{p}{\alpha}}\right)\right\} = \frac{r}{2(\pi\alpha t^3)^{1/2}} \exp\left(-\frac{r^2}{4\alpha t}\right) \tag{9.88}$$

9.8 Systems with Moving Heat Sources

There are a significant number of engineering applications such as welding, grinding, metal cutting, flame or laser hardening in which the problem of heat transfer can be modelled as heat conduction originating from a moving source of heat. The early approximate theory of such problems has been reviewed by Spraragen and Claussen [4], and the exact analytical theory is due to Rosenthal [5]. An extensive review of the relevant literature can be found in Reference [6].

In this section, following the work of Rosenthal [5], we first present the mathematical modeling of heat conduction with moving sources of heat under the *quasi-steady state* conditions. We then apply the method to solve a representative problem involving a moving source of heat.

9.8.1 Quasi-Steady State Condition

In order to explain the *quasi-steady state* condition, consider one-dimensional heat conduction in a stationary medium in which a plane heat source, of constant strength q''_{pl} (W/m^2) and surface area *A*, is releasing heat continuously while moving with a constant velocity *u* (m/s) in the positive *x* direction as shown in Fig. 9.4*a*.

Let us now define a new coordinate as

$$\xi = x - ut \tag{9.89}$$

Thus, the plane heat source, which is located at *x* at time *t* and moving with the velocity *u* in the positive *x* direction, is located at $\xi = 0$ and is stationary with respect to the new coordinate ξ. The medium, however, moves with the velocity u in the negative ξ direction in the new coordinate system.

Consider a control volume of infinitesimally small thickness $\Delta\xi$ around the plane source as shown in Fig. 9.4*b*. Let the thermophysical properties of the medium (k, ρ, and c) be constants. The net rate of heat conducted into the control volume in the ξ direction is given by

$$\begin{aligned} q_c(\xi) - q_c(\xi + \Delta\xi) &= -\frac{\partial q_c}{\partial \xi}\Delta\xi = -A\frac{\partial q_c''}{\partial \xi}\Delta\xi \\ &= -A\frac{\partial}{\partial \xi}\left(-k\frac{\partial T}{\partial \xi}\right)\Delta\xi = kA\frac{\partial^2 T}{\partial \xi^2}\Delta\xi \end{aligned} \tag{9.90a}$$

where $T = T(\xi, t)$. The net rate of energy carried away from the control volume in the negative ξ direction by the moving medium, on the other hand, is given by

$$\rho cuAT - \rho cuA\left(T + \frac{\partial T}{\partial \xi}\Delta\xi\right) = -\rho cuA\frac{\partial T}{\partial \xi}\Delta\xi \tag{9.90b}$$

The net rate of internal energy generation in the control volume is related to the strength of the plane source by the relation

$$\dot{q}\Delta\xi A = q_{pl}''A \tag{9.90c}$$

A thermal energy balance on the control volume shown in Fig. 9.4*b* then yields

$$kA\frac{\partial^2 T}{\partial \xi^2}\Delta\xi + \rho cuA\frac{\partial T}{\partial \xi}\Delta\xi + q_{pl}''A = \rho cA\Delta\xi\frac{\partial T}{\partial t} \tag{9.91}$$

The term on the right-hand side of Eq. (9.91) represents the rate of increase in the internal energy in the control volume. If, on the other hand, x (= ut) in Fig. 9.4*a* is large, then the

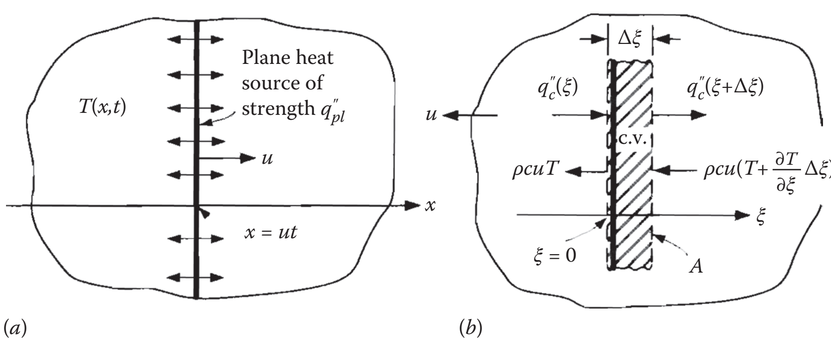

FIGURE 9.4
(*a*) Plane heat source moving with a velocity u in the positive x direction. (*b*) Control volume of infinitesimally small thickness $\Delta\xi$ around the plane source.

temperature distribution around the plane heat source becomes independent of time. In other words, although a stationary observer on the x-axis would notice a change in temperature as the plane source moves in the medium, the same observer would notice no such change in temperature if stationed at a point on the ξ-axis, which is moving with the plane source. This condition has come to be known as the *quasi-steady state* condition. Thus, $\partial T/\partial t = 0$ relative to an observer stationed at $\xi = 0$. The energy balance (9.94) then reduces, as $\Delta\xi \to 0$, to

$$\frac{d^2T}{d\xi^2} + \frac{1}{k} q''_{pl}\delta(\xi - 0) = -\frac{u}{\alpha}\frac{dT}{d\xi} \tag{9.92}$$

where $\alpha = k/\rho c$ is the thermal diffusivity. In obtaining the relation (9.92) we have utilized the representation (see Section 9.2)

$$\lim_{\Delta\xi\to 0} \frac{q''_{pl}}{\Delta\xi} = q''_{pl}\delta(\xi - 0) \tag{9.93}$$

The quasi-steady heat conduction equation (9.92) can also be obtained from the heat conduction equation as follows. For the one-dimensional case considered here the heat conduction equation would be given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{k} q''_{pl}\delta(x - ut) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{9.94}$$

where $T = T(x, t)$. In terms of the transformation (9.89) we get,

$$\frac{\partial T(x,t)}{\partial x} = \frac{\partial T(\xi,t)}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial T(\xi,t)}{\partial \xi} \tag{9.95a}$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial^2 T(\xi,t)}{\partial \xi^2}\frac{\partial \xi}{\partial x} = \frac{\partial^2 T(\xi,t)}{\partial \xi^2} \tag{9.95b}$$

and

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial T(\xi,t)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial T(\xi,t)}{\partial t} = -u\frac{\partial T(\xi,t)}{\partial \xi} + \frac{\partial T(\xi,t)}{\partial t} \tag{9.95c}$$

Substituting Eqs. (9.95*b*) and (9.95*c*) into Eq. (9.94) we obtain

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{k} q''_{pl}\delta(\xi - 0) = \frac{1}{\alpha}\left(-u\frac{\partial T}{\partial \xi} + \frac{\partial T}{\partial t}\right) \tag{9.96}$$

where $T = T(\xi, t)$. Under the quasi-steady condition, we have $\partial T(\xi, t)/\partial t = 0$. Therefore, Eq. (9.96) reduces to

$$\frac{d^2T}{d\xi^2} + \frac{1}{k} q''_{pl} \delta(\xi - 0) = -\frac{u}{\alpha}\frac{dT}{d\xi} \tag{9.97}$$

which the same relation as Eq. (9.92).

9.8.2 Moving Plane Heat Source in an Infinite Solid

As an application, consider a large solid, initially at a uniform temperature T_i and through which a plane heat source of constant strength q''_{pl} (W/m²) moves at a velocity u (m/s) in the positive x direction. With the assumption of the quasi-steady condition, the temperature distribution in the solid will satisfy

$$\frac{d^2T}{d\xi^2} + \frac{1}{k} q''_{pl} \delta(\xi - 0) = -\frac{u}{\alpha}\frac{dT}{d\xi}, \qquad -\infty < \xi < \infty \tag{9.98}$$

Note that, when $\xi \neq 0$, the solution of Eq. (9.98) is given by

$$T(\xi) = Ae^{-(u/\alpha)\xi} + B \tag{9.99}$$

The condition

$$\lim_{\xi \to \infty} T(\xi) = T_i \tag{9.100}$$

yields $B = T_i$. The solution for $\xi > 0$ can then be written as

$$T^+(\xi) = T_i + A^+ e^{-(u/\alpha)\xi} \qquad \text{for} \qquad \xi > 0 \tag{9.101}$$

Similarly, the condition

$$\lim_{\xi \to \infty} T(\xi) = \mathit{finite} \tag{9.102}$$

yields $A = 0$, and the solution for $\xi < 0$ becomes

$$T^-(\xi) = B^- \qquad \text{for} \qquad \xi < 0 \tag{9.103}$$

At $\xi = 0$, we have the conditions

$$T^-(0) = T^+(0) \tag{9.104a}$$

and

$$k\frac{dT^-(0)}{d\xi} - k\frac{dT^+(0)}{d\xi} = q''_{pl} \tag{9.104b}$$

Applying these two conditions at $\xi = 0$, we get

$$A^+ = \frac{q''_{pl}}{\rho c u} \quad \text{and} \quad B^- = T_i + \frac{q''_{pl}}{\rho c u}$$

Substituting A^+ and B^- into Eqs. (9.101) and (9.103) we obtain

$$T^+(\xi) = T_i + \frac{q''_{pl}}{\rho c u} e^{-(u/\alpha)\xi} \quad \text{for} \quad \xi > 0 \tag{9.105a}$$

$$T^-(\xi) = T_i + \frac{q''_{pl}}{\rho c u} \quad \text{for} \quad \xi < 0 \tag{9.105b}$$

From these solutions we see that the temperature remains constant in the region behind the plane source, and this is the maximum temperature in the body.

In this section, we discussed the solution of a representative problem involving a moving plane heat source in a stationary large solid. For further examples involving moving heat sources, see Reference [6].

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Problems

9.1 Show that

$$(a)\quad \sum_{n\,odd} \frac{1}{n^2} = \ldots = \frac{1}{8}\pi^2$$

$$(b)\quad \frac{2}{r_1}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\lambda_n}\sin\lambda_n r_1 = 1, \qquad 0 < r_1 < r_0$$

where $\lambda_n = n\pi / r_0$, $\qquad n = 1, 2, 3, \ldots,$

$$(c)\quad \frac{2}{r_0}\sum_{n=1}^{\infty} \frac{1}{\lambda_n}[r\lambda_n \cos\lambda_n r - \sin\lambda_n r] = 1, \qquad 0 < r < r_0$$

where $\lambda_n = n\pi / r_0$, $\qquad n = 1, 2, 3, \ldots,$

9.2 (*a*) A slab, which extends from $x = 0$ to $x = L$, is initially at a uniform temperature T_∞ at $t = 0$. For times $t \geq 0$, a plane heat source, of strength $q''_{pl}(t)$ (W/m^2) and located normal to the x direction at $x = a$ within the slab, releases heat continuously, while the surface at $x = 0$ is kept perfectly insulated and the surface at $x = L$ dissipates heat by convection with a constant heat transfer coefficient h into a fluid medium maintained at the constant temperature T_∞. Assuming constant thermophysical properties, obtain an expression for the temperature distribution $T(x, t)$ in the slab for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(x, t)$ in the slab for $t > 0$ if the plane heat source releases its heat instantaneously at $t = 0$ at strength $q''_{pl,i}$ (J/m^2).

9.3 (*a*) A solid cylinder of radius r_0 is initially at a uniform temperature T_∞ at $t = 0$. For times $t \geq 0$, a concentrically situated cylindrical shell heat source, of strength $q'_{cy}(t)$ (W/m^2) and of radius r_1 ($0 < r_1 < r_0$), releases heat continuously, while the surface at $r = r_0$ dissipates heat by convection with a constant heat transfer coefficient h into a fluid medium maintained at the constant temperature T_∞. Assuming constant thermophysical properties, obtain an expression for the temperature distribution $T(r, t)$ in the cylinder for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(r, t)$ in the cylinder for $t > 0$ if the source releases its heat instantaneously at $t = 0$ at strength $q'_{cy,i}$ (J/m).

9.4 (*a*) A long solid hollow cylinder, $r_i \leq r \leq r_o$, is initially at a uniform temperature T_i at $t = 0$. For times $t \geq 0$, a concentrically situated cylindrical shell heat source, of strength $q'_{cy}(t)$ (W/m) per unit length and radius r_1 ($r_i < r_1 < r_o$), releases heat continuously, while the surfaces at $r = r_i$ and $r = r_o$ are both kept at the initial temperature T_i. Assuming constant thermophysical properties, obtain an expression for the temperature distribution $T(r, t)$ in the cylinder for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(r, t)$ in the cylinder for $t > 0$ if the source releases its heat instantaneously at $t = 0$ at strength $q'_{cy,i}$ (J/m).

9.5 (*a*) A solid hollow sphere, $r_i \le r \le r_o$, is initially at a uniform temperature T_i at $t = 0$. For times $t \ge 0$, a concentrically situated spherical shell heat source, of strength $q_{sp}(t)$ (W) and of radius r_1 ($r_i < r_1 < r_o$), releases heat continuously, while the surfaces at $r = r_i$ and $r = r_o$ are both kept at the initial temperature T_i. Assuming constant thermophysical properties, obtain an expression for the temperature distribution $T(r, t)$ in the sphere for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for $t > 0$ if the source releases its heat instantaneously at $t = 0$ at strength $q'_{cy,i}$ (J).

9.6 (*a*) Consider a long bar of rectangular cross-section, $0 \le x \le a$ and $0 \le y \le b$, which is initially at a uniform temperature T_i at $t = 0$. For times $t \ge 0$, a line heat source, of strength $q'_{ln}(t)$ (W/m) per unit length of the source and situated along the z axis at (x_1, y_1) inside the bar, releases heat continuously, while the surfaces of the bar are maintained at initial temperature T_i. Assuming constant thermophysical properties, obtain an expression for the temperature distribution $T(x, y, t)$ in the bar for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(x, y, t)$ in the bar for $t > 0$ if the line heat source releases its heat instantaneously at $t = 0$ at strength $q'_{ln,i}$ (J/m).

9.7 (*a*) Consider an infinitely large solid, $0 \le r \le \infty$, which is initially at a uniform temperature T_i at $t = 0$. For times $t \ge 0$, a cylindrical shell heat source, of strength $q''_{cy}(t)$ (W/m) per unit length, radius r_1, and situated with its axis at $r = 0$ along the z direction, releases heat continuously. Obtain an expression for the temperature distribution $T(r, t)$ in the solid for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(r, t)$ in the solid for $t > 0$ if the source releases its heat instantaneously at $t = 0$ at strength $q''_{cy,I}$ (J/m).

(*c*) Discuss the solutions obtained in (*a*) and (*b*) above if $r_1 \to 0$.

9.8 (*a*) Consider an infinitely large solid, $0 \le r < \infty$, which is initially at a uniform temperature T_i at $t = 0$. For times $t \ge 0$, a spherical shell heat source, of strength $q_{sp}(t)$ (W), radius r_1, and centrally situated at $r = 0$, releases heat continuously. Obtain an expression for the temperature distribution $T(r, t)$ in the solid for $t > 0$.

(*b*) Obtain an expression for the temperature distribution $T(r, t)$ in the solid for $t > 0$ if the source releases its heat instantaneously at $t = 0$ at strength $q_{sp,i}$ (J).

(*c*) Discuss the solutions obtained in (*a*) and (*b*) above if $r_1 \to 0$.

9.9 (*a*) Consider a semi-infinite solid. $0 \le x \le \infty$, which is initially at a uniform temperature T_i at $t = 0$. For times $t \ge 0$, a plane heat source, of strength $q''_{pl}(t)$ (W/m^2) and situated perpendicular to the x axis at $x = x_1$ within the solid, releases heat continuously, while the surface at $x = 0$ is maintained at the initial temperature T_i. Obtain an expression for the temperature distribution $T(x, t)$ in the solid for $t > 0$.

(*b*) Obtain the temperature distribution $T(x, t)$ in the solid for $t > 0$ if the source releases its heat instantaneously at $t = 0$ at strength $q''_{pl,i}$ (J/m^2).

9.10 (*a*) Consider a semi-infinite solid, $0 \leq x < \infty$, which is initially at a uniform temperature T_i at $t = 0$. For times $t \geq 0$, a line heat source, of strength $q'_{ln}(t)$ (W/m) per unit length of the source and situated along the z axis at $x = x_1$ and $y = 0$ within the solid, releases heat continuously, while the surface at $x = 0$ is maintained perfectly insulated. Obtain an expression for the unsteady-state temperature distribution in the solid for $t > 0$.

(*b*) Obtain an expression for the unsteady temperature distribution in the solid for $t > 0$ if the source releases its heat instantaneously at f = 0 at strength $q'ln,i$ (J/m).

9.11 Consider a large solid, initially at a uniform temperature T_i. A line heat source, of constant strength q'_{ln} (W/m) per unit length of the source and oriented parallel to the z direction, moves with a constant velocity u (m/s) in the positive x direction. Let the thermophysical properties (k, ρ, c) of the solid be constants. Obtain an expression for the quasi-steady temperature distribution in the solid.

10

Further Analytical Methods of Solution

10.1 Introduction

We have, so far, discussed various analytical methods (i.e., separation of variables, integral transforms and Laplace transforms), and investigated the application of these methods to the solution of several heat conduction problems. In this chapter, we present other analytical methods of solution, including Duhamel's method, the method of similarity transformation, the integral method, and the variational method.

Among the methods introduced in this chapter, the integral method is especially important because it can also be implemented to solve nonlinear problems. That is, nonlinear heat conduction problems need not be linearized because this method is elastic enough to encompass all sorts of nonlinearities associated with such problems.

10.2 Duhamel's Method

Duhamel's method is a superposition technique that can be used to obtain the solution of linear heat conduction problems with time-dependent boundary conditions and/or time-dependent internal energy generation from the solution of the same problem with time-independent boundary conditions and/or time-independent internal energy generation. In this section, we introduce Duhamel's method by applying it to the solution of a representative heat conduction problem. A detailed treatment of the method, as applied to heat conduction problems, can be found in Reference [14].

Consider a semi-infinite solid, $0 \le x < \infty$, initially at a uniform temperature T_i. For times $t \ge 0$, the surface temperature at $x = 0$ is specified as a prescribed function of time as $T(0, t) = f(t)$ (see Fig. 8.5). Assuming constant thermophysical properties, the formulation of the problem for the unsteady-state temperature distribution in terms of $\theta(x, t) = T(x, t) - T_i$ is given by

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{10.1a}$$

$$\theta(x, 0) = 0 \tag{10.1b}$$

$$\theta(0,t) = F(t) \quad \text{and} \quad \lim_{x\to\infty} \theta(x,t) = 0 \tag{10.1c,d}$$

where α is the thermal diffusivity of the solid, and $F(t) = f(t) - T_i$.

Since the problem (10.1) is linear, it is possible to construct an approximate solution for $\theta(x, t)$ by first merely breaking the function $F(t)$ up into a number of constant temperature steps as illustrated in Fig. 10.1. and then superposing the solutions over each step. Accordingly, we now approximate $F(t)$ over a time interval $(0, t)$ as

$$F(t) \cong F(0)H(t) + \sum_{i=1}^{n} \underbrace{[F(\tau_i) - F(\tau_i - 1)]}_{f(\tau_i)-f(\tau_i-1)} H(t-\tau_i) \tag{10.2}$$

where $\tau_0 = 0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n < t < \tau_{n+1}$, and $H(t)$ is the so-called *Heaviside step function*, defined by

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{10.3}$$

The solution to the problem (10.1) at any instant t for the boundary condition $F(t)$ at $x = 0$ as approximated by Eq. (10.2) can now be written in the form

$$\theta(x,t) \cong F(0)\phi(x,t) + \sum_{i=1}^{n} [f(\tau_i) - f(\tau_i - 1)]\phi(x, t-\tau_i) \tag{10.4}$$

where $\phi(x, t)$ is the solution of the following *auxiliary problem*:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{10.5a}$$

$$\phi(x,0) = 0 \tag{10.5b}$$

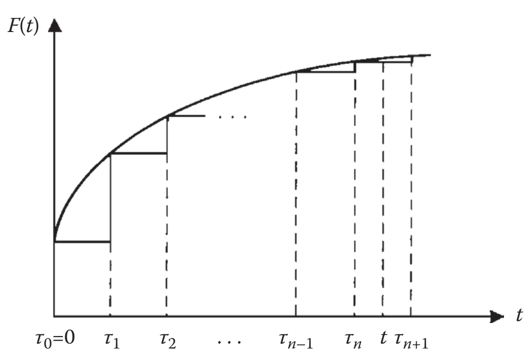

FIGURE 10.1
Stepwise approximation of $F(t)$ over the time interval $(0, t)$.

$$\phi(0,t)=1 \quad \text{and} \quad \lim_{x\to\infty}\phi(x,t)=0 \tag{10.5c,d}$$

The solution of the auxiliary problem (10.5) can readily be obtained from Eq. (8.46):

$$\phi(x,t)=\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{10.6}$$

The approximate solution (10.4) may also be written as

$$\theta(x,t)\cong F(0)\phi(x,t)+\sum_{i=1}^{n}\frac{\Delta f_i}{\Delta\tau_i}\phi(x,t-\tau_i)\Delta\tau_i \tag{10.7}$$

where

$$\Delta f_i = f(\tau_i)-f(\tau_{i-1}) \quad \text{and} \quad \Delta\tau_i=\tau_i-\tau_{i-1}$$

On the other hand, in the limit as $n \to \infty$, Eq. (10.7) takes the form

$$\theta(x,t)=F(0)\phi(x,t)+\int_0^t\frac{df}{d\tau}\phi(x,t-\tau)d\tau \tag{10.8}$$

This is due to the fact that, as $n \to \infty$, the approximation (10.2) becomes an exact representation of $F(t)$, and the summation term in Eq. (10.7), by definition, reduces to the integral term in Eq. (10.8). Equation (10.8) is known as *Duhamel's superposition integral*. After the second term on the right-hand side of Eq. (10.8) is integrated by parts, it reduces to

$$\theta(x,t)=-T_i\phi(x,t)-\int_0^t f(t)\frac{\partial\phi(x,t-\tau)}{\partial\tau}d\tau \tag{10.9}$$

Since

$$\frac{\partial\phi(x,t-\tau)}{\partial\tau}=-\frac{\partial\phi(x,t-\tau)}{\partial t} \tag{10.10}$$

Eq. (10.9) can also be written as

$$\theta(x,t)=-T_i\phi(x,t)-\int_0^t f(t)\frac{\partial\phi(x,t-\tau)}{\partial t}d\tau \tag{10.11}$$

Moreover, from Eq. (10.6) we have

$$\frac{\partial\phi(x,t-\tau)}{\partial t}=\frac{1}{2\sqrt{\pi\alpha}}\frac{x}{(1-\tau)^{3/2}}\exp\left[-\frac{x^2}{4\alpha(t-\tau)}\right] \tag{10.12}$$

Thus, after introducing Eqs. (10.6) and (10.12) into Eq. (10.11), the solution becomes

$$\theta(x,t) = -T_i \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{x}{2\sqrt{\pi\alpha}} \int_0^t \frac{f(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4\alpha(t-\tau)}\right] d\tau \tag{10.13}$$

which can also be written as

$$T(x,t) = T_i - T_i \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{x}{2\sqrt{\pi\alpha}} \int_0^t \frac{f(t-t')}{(t')^{3/2}} \exp\left(-\frac{x^2}{4\alpha t'}\right) dt' \tag{10.14}$$

This result, as expected, is identical to Eq. (8.52).

The example presented here highlights the basics of Duhamel's method. For further applications of the method, see References [1,14].

10.3 The Similarity Method

The use of the *similarity method* dates back to Prandtl [17], who applied this method in 1904 to transform the laminar boundary-layer momentum equation in fluid mechanics into an ordinary differential equation for flows over a flat surface with uniform free-stream velocity. Following the work of Prandtl [17], Blasius in 1908 obtained the first exact solution to the boundary-layer momentum equation [3]. Falkner and Skan [4] in 1931 demonstrated the possibility of the same transformation for a family of problems. Later in 1939, Goldstein [5] investigated in detail the conditions under which such a transform can be carried out. In fact, this method takes advantage of the self-similarity property inherent in certain boundary-layer type problems. It is also an equally attractive method for solving diffusion type problems possessing similarity property, such as certain unsteady-state heat conduction problems in semi-infinite regions. In this section, we introduce the basics of the similarity method and the solution procedure by solving a representative heat conduction problem.

Consider a semi-infinite solid, $0 \le x < \infty$, initially at a uniform temperature T_i (see Fig. 8.4). The surface temperature at $x = 0$ is changed to and kept at a constant temperature T_w for times $t \ge 0$. Assuming constant thermophysical properties, the formulation of the problem for the unsteady-state temperature distribution in terms of $\theta(x, t) = T(x, t) - T_w$ is given by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{10.15a}$$

$$\theta(x,0) = \theta_i \tag{10.15b}$$

$$\theta(0,t) = 0 \quad \text{and} \quad \lim_{x\to\infty}\theta(x,t) = \theta_i \tag{10.15c,d}$$

where $\theta_i = T_i - T_w$, and α is the thermal diffusivity of the solid.

Here we note that the temperature $T(x, t)$ approaches the initial temperature T_i asymptotically as $x \to \infty$. Accordingly, as illustrated in Fig. 10.2*a*, we define, at any instant t, a region of thickness $\delta(t)$ called the *penetration depth* or *thermal layer* such that

$$\frac{\theta(\delta,t)}{\theta_i} = \frac{T(\delta,t)-T_w}{T_i - T_w} = 0.99 \tag{10.16}$$

The problem under consideration has no characteristic length in any direction. Hence, it is reasonable to assume that the temperature profiles are similar at all times as predicted in Fig. 10.2*b* in the sense that

$$\frac{\theta(x,t)}{\theta_i} = \frac{T(x,t)-T_w}{T_i - T_w} = f(\xi) \quad \text{with} \quad \xi = \frac{x}{\delta(t)} \tag{10.17}$$

We now proceed to estimate the thickness of the penetration depth $\delta(t)$. From Eq. (10.17) it follows that

$$\frac{\partial^2\theta}{\partial x^2} = \theta_i \frac{1}{\delta^2}\frac{d^2 f}{d\xi^2} \tag{10.18a}$$

and

$$\frac{\partial\theta}{\partial t} = -\theta_i \frac{\xi}{\delta}\frac{df}{d\xi}\frac{d\delta}{dt} \tag{10.18b}$$

Substitution of Eqs. (10.18*a,b*) into the heat conduction equation (10.15*a*) yields

$$\frac{d^2 f}{d\xi^2} + \xi\frac{df}{d\xi}\frac{\delta}{\alpha}\frac{d\delta}{dt} = 0 \tag{10.19}$$

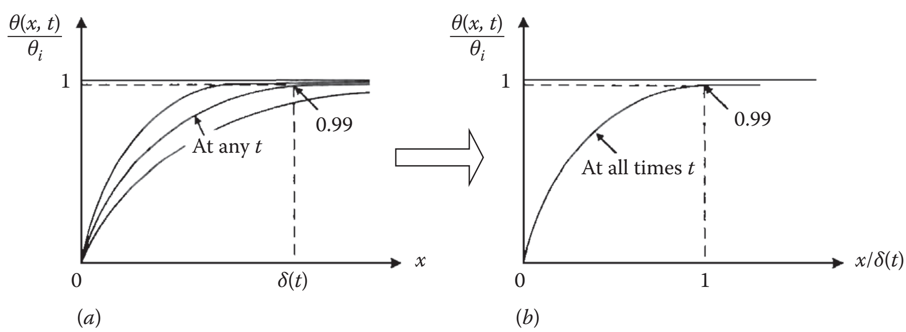

FIGURE 10.2
Penetration depth $\delta(t)$ and self-similar temperature profiles.

Integrating Eq. (10.19) over ξ from $\xi = 0$ to $\xi = 1$, we obtain

$$\delta \frac{d\delta}{dt} = \frac{1}{2} C^2 \alpha \tag{10.20}$$

where the positive constant C^2 is given by

$$C^2 = 2\frac{A}{B}$$

with

$$A = -\int_0^1 \frac{d^2 f}{d\xi^2} d\xi = \frac{df(0)}{d\xi} - \frac{df(1)}{d\xi} > 0$$

and

$$B = \int_0^1 \xi \frac{df}{d\xi} d\xi = \ldots = 0.99 - \int_0^1 f(\xi) d\xi > 0$$

Equation (10.20) is a first-order ordinary differential equation for $\delta(t)$, the solution of which yields

$$\delta^2 = C^2 \alpha t \quad \Rightarrow \quad \delta = C\sqrt{\alpha t} \tag{10.21}$$

where we used the condition that $\delta(0) = 0$. Moreover, substituting Eq. (10.20) into Eq. (10.19), we obtain

$$\frac{d^2 f}{d\xi^2} + \frac{C^2}{2} \xi \frac{df}{d\xi} = 0 \tag{10.22}$$

In order to simplify the algebra that follows, we now introduce a new independent variable η as

$$\eta = \frac{1}{2} C\xi = \frac{x}{2\sqrt{\alpha t}} \tag{10.23}$$

Then, Eq. (10.22) can be rewritten as

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \tag{10.24}$$

Furthermore, realizing that at $x = 0$, $n = 0$ and as $x \to \infty$ (also, at $t = 0$), $\eta \to \infty$, the initial and boundary conditions of the problem, Eqs. (10.15*b,c,d*) reduce to

$$f(0) = 0 \tag{10.25a}$$

$$f(\infty) = 1 \tag{10.25b}$$

The fact that we were successful in transforming the partial differential equation Eq. (10.15*a*) and the initial and the boundary conditions (10.15*b,c,d*) to a second-order ordinary differential equation (10.24) together with the two boundary conditions (10.25*a,b*) indicates that a similarity solution in the form

$$\frac{\theta(x,t)}{\theta_i} = f(\eta) \qquad \text{with} \qquad \eta = \frac{x}{2\sqrt{\alpha t}} \tag{10.26}$$

does indeed exist, where the variable η is named as the *similarity variable*.

If we now introduce

$$g(\eta) = \frac{df}{d\eta} \tag{10.27}$$

then Eq. (10.24) can be rewritten as

$$\frac{dg}{d\eta} + 2\eta g = 0 \tag{10.28a}$$

or

$$\frac{dg}{g} = -2\eta d\eta \tag{10.28b}$$

the solution of which yields

$$g(\eta) = \frac{df}{d\eta} = De^{-\eta^2} \tag{10.29}$$

where D is a constant of integration. One more integration from $\eta = 0$ to any η gives

$$f(\eta) = D\int_0^{n} e^{-(n')^2}\, d\eta' \tag{10.30a}$$

which can also be rewritten as

$$f(\eta) = D\frac{\sqrt{\pi}}{2}\operatorname{erf}(\eta) \tag{10.30b}$$

where erf(η) is the *error function* defined by Eq. (6.93). On the other hand, noting that erf(∞) = 1 (see Appendix C),

$$f(\infty) = 1 \quad \Rightarrow \quad D = \frac{2}{\sqrt{\pi}}$$

the solution (10.30*b*) reduces to

$$f(\eta) = \frac{\theta(x,t)}{\theta_i} = \operatorname{erf}(\eta) \tag{10.31a}$$

which can also be rewritten as

$$\frac{\theta(x,t)}{\theta_i} = erf\left(\frac{x}{2\sqrt{\pi t}}\right) \tag{10.31b}$$

which is, as expected, the same result as Eq. (6.94).

From the values of the error function listed in Appendix C, we see that

$$\operatorname{erf}(\eta) = 0.99 \quad \text{when} \quad \eta = 1.82$$

Accordingly, making use of the definition of the similarity variable η, we obtain

$$\frac{\delta}{2\sqrt{\alpha t}} = 1.82 \quad \Rightarrow \quad \delta = 3.64\sqrt{\alpha t} \tag{10.32}$$

10.4 The Integral Method

The objective of this section is to introduce the basics of the *integral method*, by which approximate analytical solutions to both linear and nonlinear heat conduction problems can be obtained. The integral method was first used by von Karman [19] and Pohlhausen [15] to approximately solve the boundary-layer momentum and energy equations of viscous fluid flow with heat transfer. This method, however, is also equally attractive for solving problems governed by a diffusion type equation, such as unsteady-state heat conduction problems in solids. In 1953, Landahl [12] used the integral method for the first time to solve the diffusion equation in the field of biophysics in connection with the spread of a concentrate. Later on, several investigators implemented this technique to obtain approximate solutions to various linear and nonlinear heat conduction problems. Reviews of the literature on the application of the integral method can be found in References [6,14].

To introduce the basics of the solution procedure by the integral method we now reconsider the unsteady-state heat conduction problem in a semi-infinite solid, solved by a similarity transform in Section 10.3. The formulation of the problem for the unsteady-state temperature distribution in terms of $\theta(x, t) = T(x, t) - T_i$ would be given by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{10.33a}$$

$$\theta(x,0) = \theta_i \tag{10.33b}$$

$$\theta(0,t) = \theta_w \quad \text{and} \quad \lim_{x\to\infty}\theta(x,t) = 0 \tag{10.33c,d}$$

where $\theta_w = T_w - T_i$.

As the first step in the application of the method, we introduce the following approximations:

$$\theta(\delta,t) = 0 \quad \text{and} \quad \frac{\partial\theta(\delta,t)}{\partial x} = 0 \tag{10.34a,b}$$

where $\delta(t)$ is the *penetration depth*. These approximations suggest, that for $x \geq \delta(t)$, the solid is still at the initial temperature and there is no heat transfer beyond this depth. Next, we integrate the heat conduction equation (10.33*a*) over x from $x = 0$ to $x = \delta(t)$:

$$\int_0^{\delta(t)} \frac{\partial^2\theta}{\partial x^2}dx = \frac{1}{\alpha}\int_0^{\delta(t)} \frac{\partial\theta}{\partial t}dx \tag{10.35}$$

The term on the left integrates to

$$\int_0^{\delta(t)} \frac{\partial^2\theta}{\partial x^2}dx = -\left.\frac{\partial\theta}{\partial x}\right|_{x=\delta} - \left.\frac{\partial\theta}{\partial x}\right|_{x=0} = -\left.\frac{\partial\theta}{\partial x}\right|_{x=0} \tag{10.36a}$$

The integral term on the right-hand side of Eq. (10.35) can be rewritten, using *Leibnitz's rule*, as

$$\int_0^{\delta(t)} \frac{\partial\theta}{\partial t}dx = \frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)dx - \underbrace{\theta(\delta,t)}_{=0}\frac{d\delta}{dt} = \frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)dx \tag{10.36b}$$

Thus, Eq. (10.35) becomes

$$\frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)\,dx = -\alpha \left.\frac{\partial \theta}{\partial x}\right|_{x=0} \tag{10.37}$$

which is called the *heat-balance integral* or *energy-integral equation*.

We now assume that the temperature function $\theta(x, t)$ can be represented over $0 \le x \le \delta(t)$ by a second-degree polynomial in the form:

$$\theta(x,t) = a + bx + cx^2 \tag{10.38}$$

where the coefficients a, b, and c may depend on t. Applying Eqs. (10.33*c*) and (10.34*a,b*), these coefficients are found to be

$$a = \theta_w, \qquad b = -2\frac{\theta_w}{\delta}, \qquad \text{and} \qquad c = \frac{\theta_w}{\delta^2}$$

Thus, the assumed profile (10.38) becomes

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = 1 - 2\left(\frac{x}{\delta}\right) + \left(\frac{x}{\delta}\right)^2 \tag{10.39}$$

Substituting this result into the energy integral equation (10.37) and then performing the integration, we obtain

$$\delta \frac{d\delta}{dt} = 6\alpha \tag{10.40}$$

Since $\delta(0) = 0$, the solution of Eq. (10.40) yields

$$\delta = \sqrt{12\alpha t} \tag{10.41}$$

Now that we have obtained $\delta(t)$, Eq. (10.39) yields the temperature distribution $T(x, t)$.

The surface heat flux $q''_w(t)$ at $x = 0$ is a quantity of interest and is given by

$$q''_w(t) = -k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \frac{1}{\sqrt{3}}\frac{k(T_w - T_i)}{\sqrt{\alpha t}} \tag{10.42}$$

The exact solution for the temperature distribution in this problem was obtained in Chapter 8 by Laplace transforms (see Section 8.8), and also in Section 10.2 by a similarity transformation. From Eq. (8.46), for example, we have

$$\frac{T(x,t) - T_i}{T_w - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{(exact)} \tag{10.43}$$

This exact solution gives

$$q''_w(t) = \frac{1}{\sqrt{\pi}} \frac{k(T_w - T_i)}{\sqrt{\alpha t}} \qquad \text{(exact)} \tag{10.44}$$

By comparing Eqs. (10.42) and (10.44), it is seen that the two results for the surface heat flux are of the same form, differing only by a numerical factor. The approximate result (10.42) has an error of about 2.4%.

A different approximate solution can be obtained through a third-degree polynomial for the temperature function as presented in the form below. However, a higher polynomial degree representation does not mean a better approximation, and the question regarding which approximation degree polynomial representation is more accurate can only be answered after each of these solutions is compared with the exact solution of the same problem. A third-degree polynomial can be used for the temperature function in the form

$$\theta(x,t) = a + bx + cx^2 + dx^3 \tag{10.45}$$

where there are four coefficients to be determined. Equations (10.33*c*) and (10.34*a,b*) give only three conditions, which are called the *natural conditions*. A fourth *derived condition* may be obtained, for example, by evaluating the heat conduction equation (10.33*a*) at $x = \delta$, where $\theta(x, t) = 0$. The resulting condition is

$$\left.\frac{\partial^2 \theta}{\partial x^2}\right|_{x=\delta} = 0 \tag{10.46}$$

This derived condition is also referred to as the *smoothing condition*, because it tends to make the assumed profile go smoothly into the undisturbed initial temperature at $x = \delta$. With the natural conditions (10.33*c*) and (10.34*a,b*), and the derived condition (10.46), the third-degree profile (10.45) takes the form

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = \left(1 - \frac{x}{\delta}\right)^3 \tag{10.47}$$

Substituting this result into Eq. (10.37), we are led to a differential equation for $\delta(t)$, the solution of which, with the condition $\delta(0) = 0$, yields

$$\delta = \sqrt{24\alpha t} \tag{10.48}$$

The third-degree polynomial approximation (10.47) then gives the surface heat flux as

$$q''_w(t) = \sqrt{\frac{3}{8}} \frac{k(T_w - T_i)}{\sqrt{\alpha t}} \tag{10.49}$$

which is again of the same form as the exact result (10.44), except for the numerical factor $(3/8)^{1/2}$, and the error in the surface heat flux is about 8.6%.

Evaluating the differential equation (10.33*a*) at $x = 0$, where $\theta = \theta_w$ = constant, another derived condition can also be obtained as

$$\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x=0} = 0 \tag{10.50}$$

If this condition is used together with the natural conditions, the third-degree polynomial (10.45) becomes

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t)-T_i}{T_w - T_i} = 1 - \frac{3}{2}\left(\frac{x}{\delta}\right) + \frac{1}{2}\left(\frac{x}{\delta}\right)^2 \tag{10.51}$$

where

$$\delta = \sqrt{8\alpha t} \tag{10.52}$$

For the surface heat flux, Eq. (10.51) gives

$$q_w''(t) = \frac{3}{4\sqrt{2}}\frac{k(T_w - T_i)}{\sqrt{\alpha t}} \tag{10.53}$$

the error in this case is about 6%.

It is also possible that, with the five conditions (10.33*c*), (10.34*a,b*), (10.46), and (10.50), a fourth-degree polynomial can be assumed for the temperature profile in the form

$$\theta(x,t) = a + bx + cx^2 + dx^3 + ex^4 \tag{10.54}$$

Solving the problem with this profile gives

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t)-T_i}{T_w - T_i} = 1 - 2\left(\frac{x}{\delta}\right) + 2\left(\frac{x}{\delta}\right)^3 - \left(\frac{x}{\delta}\right)^4 \tag{10.55}$$

where

$$\delta = \sqrt{\frac{40}{3}\alpha t} \tag{10.56}$$

The profile (10.54) gives the surface heat flux as

$$q_w''(t) = \sqrt{\frac{3}{10}}\frac{k(T_w - T_i)}{\sqrt{\alpha t}} \tag{10.57}$$

with an error of about 3%.

As illustrated, the choice of the profile is never unique, and the error in the final solution depends, to a large extend, on the selection of the profile. Among the profiles considered above, although the second-degree polynomial (10.38) gives the least error in the surface heat flux, the fourth-degree polynomial (10.54) is the best approximation to the exact temperature distribution (10.43). However, there seems to be no significant improvement in the accuracy of the solution if a polynomial greater than fourth-degree is used [14].

An approximate solution for the unsteady-state temperature distribution in the same semi-infinite solid for an arbitrary time-wise variation of the surface temperature as $T(0, t) = f(t)$ can also be developed from the approximate solutions obtained above by the use of Duhamel's method discussed in Section 10.2. In that case, the use of Duhamel's superposition integral (10.8) gives

$$\theta(x,t) = F(0)\phi(x,t) + \int_0^t \frac{df}{d\tau}\phi(x,t-\tau)\,d\tau \tag{10.58}$$

where, for example, for the fourth-degree polynomial approximation, we would have

$$\theta(x,t) = 1 - 2\left(\frac{x}{\delta}\right) + 2\left(\frac{x}{\delta}\right)^3 - \left(\frac{x}{\delta}\right)^4 \tag{10.59}$$

with δ given by Eq. (10.56).

10.4.1 Problems with Temperature-Dependent Thermal Conductivity

When the thermal conductivity is temperature dependent, the general heat conduction equation is given by

$$\nabla \cdot [k(T)\nabla T] + \dot{q} = \rho c \frac{\partial T}{\partial t} \tag{10.60}$$

Because $k = k(T)$, Eq. (10.60) is a nonlinear differential equation. It can, however, be reduced to a linear form by introducing a new temperature function ($\mathbf{r}$, t) by means of the Kirchhoff transformation introduced in Section 2.4. If the boundary conditions can also be transformed, then the problem may be solved exactly for $\theta(\mathbf{r}, t)$ by the techniques introduced in the previous chapters. But inverting the solution back to $T(\mathbf{r}, t)$ may not be an easy task in most of the cases.

Nonlinear problems involving temperature-dependent thermal conductivity, however, may be solved approximately by the integral method. The following is an example of the use of the integral method to solve such problems.

Consider a semi-infinite solid, $0 \le x < \infty$, initially at a uniform temperature T_i. For times $t \ge 0$, a constant heat flux q''_w is applied to the surface at $x = 0$. Assume that the thermal conductivity k, specific heat c, and density ρ of the solid are all temperature dependent. The formulation of the problem for the temperature distribution $T(x, t)$ is then given by

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \rho c \frac{\partial T}{\partial t} \tag{10.61a}$$

$$T(x,0) = T_i \tag{10.61b}$$

$$-k_w \frac{\partial T(0,t)}{\partial x} = q''_w \quad \text{and} \quad \lim_{x\to\infty} T(x,t) = T_i \tag{10.61c,d}$$

where $k_w = k(T_w)$, with $T_w = T(0, t)$.

Following the work of Goodman [8], we now introduce the following transform:

$$\theta(x,t) = \int_{T_i}^{T(x,t)} \rho c \, dT \tag{10.62}$$

Since

$$\frac{\partial \theta}{\partial t} = \rho c \frac{\partial T}{\partial t} \tag{10.63a}$$

and

$$\frac{\partial \theta}{\partial x} = \rho c \frac{\partial T}{\partial x} \tag{10.63b}$$

the system (10.61) becomes

$$\frac{\partial}{\partial x}\left(\alpha \frac{\partial \theta}{\partial x} \right) = \frac{\partial \theta}{\partial t} \tag{10.64a}$$

$$\theta(x,0) = 0 \tag{10.64b}$$

$$-\alpha_w \frac{\partial \theta(0,t)}{\partial x} = q''_w \quad \text{and} \quad \lim_{x\to\infty} \theta(x,t) = 0 \tag{10.64c,d}$$

where $\alpha = k/\rho c$ and $\alpha_w = \alpha(T_w)$.

We now introduce a penetration depth $\delta(t)$ such that

$$T(\delta,t) = T_i \quad \text{and} \quad \frac{\partial T(\delta,t)}{\partial x} = 0 \tag{10.65a,b}$$

By means of the transformation (10.62) these conditions can also be written as

$$\theta(\delta,t) = 0 \quad \text{and} \quad \frac{\partial \theta(\delta,t)}{\partial x} = 0 \tag{10.66a,b}$$

Upon integrating Eq. (10.64*a*) from $x = 0$ to $x = \delta(t)$, and applying Eqs. (10.64*c*) and (10.66*a,b*), we obtain the following energy integral equation:

$$\frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)\,dx = q''_w \tag{10.67}$$

As an approximation, we adopt a third-degree polynomial representation* for $\theta(x, t)$ as

$$\theta(x,t) = a + bx + cx^2 + dx^3 \tag{10.68}$$

In addition to the three natural conditions, Eqs. (10.64*c*) and (10.66*a,b*), we can obtain an additional derived condition by evaluating the differential equation (10.64*a*) at $x = \delta$ as

$$\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x=\delta} = 0 \tag{10.69}$$

With the natural conditions (10.64*c*) and (10.66*a,b*) and the derived condition (10.69), the assumed profile (10.68) takes the form

$$\theta(x,t) = \frac{q''_w\delta}{3\alpha_w}\left(1 - \frac{x}{\delta}\right)^3 \tag{10.70}$$

Substituting this profile into Eq. (10.67) and performing the indicated operations, we obtain the following differential equation for $\delta(t)$:

$$\delta\frac{d\delta}{dt} = 6\alpha_w \tag{10.71}$$

Notice that only the thermophysical properties at $x = 0$ are involved when the problem is cast in terms of the transformed variable $\theta(x, t)$. The solution of Eq. (10.71), subject to the initial condition $\delta(0) = 0$, is

$$\delta = \sqrt{12\alpha_w t} \tag{10.72}$$

This result cannot yet be used to calculate the penetration depth δ directly, because α_w is the thermal diffusivity to be evaluated at $T(0, t)$, which is also not yet known. On the other hand, by setting $x = 0$ in Eq. (10.70), and eliminating δ between the resulting equation and Eq. (10.72), the following relation is obtained:

$$\sqrt{\alpha_w}\,\theta(0,t) = \sqrt{\frac{4}{3}}q''_w\sqrt{t} \tag{10.73}$$

* The polynomial representation is adequate provided that the thermophysical properties do not vary rapidly with temperature. This characteristic applies to most materials.

Since α_w is a function of $T(0, t)$, or $\theta(0, t)$, Eq. (10.73) is a transcendental equation which enables us to determine $\theta(0, t)$, and therefore α_w, as a function of time. Equation (10.70) then yields the transformed temperature function $\theta(x, t)$. Next, the temperature distribution $T(x, t)$ is obtained by inverting $\theta(x, t)$ by means of the transformation (10.62).

As a special case, suppose that ρc is constant. Hence,

$$\theta(x,t) = \rho c[T(x,t) - T_i] \tag{10.74a}$$

Furthermore, let the thermal conductivity be a linear function of temperature given by $k = k_i\,[1 + \gamma\,(T - T_i)]$, where $k_i = k(T_i)$. Then, Eq. (10.70) becomes

$$T(x,t) - T_i = \frac{q''_w \delta}{3k_w}\left(1 - \frac{x}{\delta}\right)^3 \tag{10.74b}$$

where

$$k_w = k_i[1 + \gamma(T_w - T_i)] \tag{10.75a}$$

By substituting $T_w - T_i$ from Eq. (10.74b) and δ from Eq. (10.72), Eq. (10.75a) can be rewritten as

$$k_w(k_w - k_i)^2 = \frac{4}{3}\frac{\left(k_i \gamma q''_w\right)^2}{\rho c} t \tag{10.75b}$$

from which $k_w(t)$ can be obtained directly. Once $k_w(t)$ is available, Eq. (10.72) will give $\delta(t)$. The temperature distribution $T(x, t)$ is then obtained from Eq. (10.74b).

If k, ρ, and c are all constants, then Eq. (10.74b) reduces to

$$T(x,t) - T_i = \frac{q''_w \delta}{3k}\left(1 - \frac{x}{\delta}\right)^3 \tag{10.76}$$

with

$$\delta = \sqrt{12\alpha t} \tag{10.77}$$

where $\alpha = k/\rho c$. In this case, Eq. (10.76) will give the temperature distribution in the solid directly.

10.4.2 Nonlinear Boundary Conditions

The integral method can also be used to obtain approximate solutions to problems with nonlinear boundary conditions. In the following example we illustrate the use of the method by solving a heat conduction problem in a semi-infinite solid with a nonlinear boundary condition.

Consider a semi-infinite solid, $0 \le x < \infty$, initially at zero temperature. For times $t \ge 0$, the surface at $x = 0$ is subjected to a heat flux q''_w that is a prescribed function of the surface temperature T_w. Assuming constant thermophysical properties, the formulation of the problem for the temperature distribution $T(x, t)$ for times $t > 0$ is then given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{10.78a}$$

$$T(x,0) = 0 \tag{10.78b}$$

$$-k\frac{\partial T(0,t)}{\partial x} = q''_w(T_w) \quad \text{and} \quad \lim_{x\to\infty} T(x,t) = 0 \tag{10.78c,d}$$

where $T_w = T(0, t)$.

Define a penetration depth $\delta(t)$ such that

$$T(\delta,t) = 0 \quad \text{and} \quad \frac{\partial T(\delta,t)}{\partial x} = 0 \tag{10.79a,b}$$

Upon integrating Eq. (10.78*a*) from $x = 0$ to $x = \delta(t)$, and applying Eqs. (10.78*c*) and (10.79*a,b*), we obtain

$$\frac{d}{dx}\int_0^{\delta(t)} T(x,t)dx = \frac{q''_w(T_w)}{\rho c} \tag{10.80}$$

The temperature profile for $T(x, t)$ will be taken to be a third-order polynomial as

$$T(x,t) = a + bx + cx^2 + dx^3 \tag{10.81}$$

With the natural conditions Eqs. (10.78*c*) and (10.79*a,b*), and the derived condition

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=\delta} = 0 \tag{10.82}$$

which is obtained by evaluating the heat conduction equation (10.78*a*) at $x = \delta(t)$, the assumed profile (10.81) takes the form

$$T(x,t) = \frac{q''_w(T_w)}{k}\frac{\delta}{3}\left(1 - \frac{x}{\delta}\right)^3 \tag{10.83}$$

which can also be written as

$$\frac{T(x,t)}{T_w} = \left(1 - \frac{x}{\delta}\right)^3 \tag{10.84}$$

where

$$T_w = \frac{q''_w(T_w)}{k}\frac{\delta(t)}{3} \tag{10.85}$$

Equation (10.85) is a relationship between the surface temperature T_w and the penetration depth $\delta(t)$. As a consequence, there is really only one unknown function of time in Eq. (10.84).

By substituting the profile (10.84) into the energy integral equation (10.80), performing the indicated operations, and then eliminating $\delta(t)$ from the resulting expression by using Eq. (10.85), we obtain

$$\frac{d}{dt}\left(\frac{T_w^3}{q''_w}\right) = \frac{4}{3}\frac{q''_w}{\rho ck} \tag{10.86}$$

which is, in fact, an ordinary differential equation for $T_w(t)$. With the initial condition $T_w(0) = T_i$, Eq. (10.86) can be integrated analytically. The result is

$$\int_0^{T_w}\left[2q''_w(T_w) - T_w\frac{dq''_w}{dT_w}\right]\frac{T_w}{\left[q''_w(T_w)\right]^3}dT_w = \frac{4}{3}\frac{t}{\rho ck} \tag{10.87}$$

Equation (10.87) expresses a relationship between the surface temperature T_w and time t. Once $T_w(t)$ is available, then Eqs. (10.84) and (10.85) yield the temperature distribution $T(x, t)$.

10.4.3 Plane Wall

In the examples considered so far, we applied the integral method to various unsteady-state heat conduction problems in the semi-infinite region. In this section, we apply the method to a plane wall problem. In such a problem the penetration depth $\delta(t)$ loses its significance after a certain time period as it cannot increase indefinitely.

Consider a plane wall of thickness L in the x direction, which is initially at a uniform temperature T_i. For times $t \geq 0$, a constant heat flux q''_w is applied to the surface at $x = 0$, while the surface at $x = L$ is maintained at the initial temperature T_i. Assuming constant thermophysical properties, the formulation of the problem for the temperature distribution $T(x, t)$ in the wall for $t > 0$ can be stated as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{10.88a}$$

$$T(x,0) = T_i \tag{10.88b}$$

$$-k\frac{\partial T(0,t)}{\partial x} = q''_w \quad \text{and} \quad T(L,t) = T_i \tag{10.88c,d}$$

Initially, when the penetration depth $\delta(t)$ is less than the thickness L, the effect of the boundary condition (10.88c) is not felt at $x = L$, and the plane wall behaves as if it were a semi-infinite solid. As soon as $\delta = L$, the concept of penetration depth loses its significance and a different type of analysis is required. Therefore, we separate the analysis into two stages:

Solution for $\delta < L$. As discussed above, when $\delta < L$ the plane wall behaves as if it were a semi-infinite solid. If we adopt a third-degree polynomial representation for the temperature distribution, then the results given by Eqs. (10.76) and (10.77) would be applicable. That is,

$$T(x,t) - T_i = \frac{q''_w \delta}{3k}\left(1 - \frac{x}{\delta}\right)^3 \tag{10.89}$$

with

$$\delta = \sqrt{12\alpha t} \tag{10.90}$$

The solution (10.89) is valid for $\delta \le L$, or, therefore, for $t \le L^2/12\alpha$.

Solution for $t > L^2/12\alpha$. In this case, the penetration depth has no meaning. Again, assume that the temperature profile is given by a third-degree polynomial as

$$T(x,t) = a + bx + cx^2 + dx^3, \qquad t > \frac{L^2}{12\alpha} \tag{10.91}$$

where there are four coefficients to be determined. We, therefore, need four independent conditions to determine these coefficients. Two of these are the two natural conditions (10.88c,d). The third condition can be derived by evaluating the heat conduction equation (10.88a) at $x = L$, yielding

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=L} = 0 \tag{10.92}$$

In terms of these three conditions and the unknown surface temperature $T(0, t)$ the profile (10.91) takes the form

$$T(x,t) - T_i = \frac{1}{2}\left(3\frac{\theta_w}{L} - \frac{q''_w}{k}\right)(L-x) - \frac{1}{2L^2}\left(\frac{\theta_w}{L} - \frac{q''_w}{k}\right)(L-x)^2 \tag{10.93}$$

where $\theta_w(t) = T(0, t) - T_i$, and is an unknown function of time. It can, however, be determined as follows. Integrating the heat conduction equation (10.88a) from $x = 0$ to $x = L$, and making use of the condition (10.88c), we get

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} + \frac{q''_w}{k} = \frac{1}{\alpha}\frac{d}{dt}\int_0^L T(x,t)\,dx \tag{10.94}$$

Substituting the profile (10.93) into Eq. (10.94) yields the following differential equation for the unknown function $\theta_w(t)$:

$$\frac{d\theta_w}{dt} = \frac{12}{5}\frac{\alpha}{L}\left(\frac{q''_w}{k} - \frac{\theta_w}{L}\right) \tag{10.95}$$

The initial condition at $t = L^2/12\alpha$, needed to solve this differential equation, is determined by setting $x = 0$ and $\delta = L$ in Eq. (10.89). That is,

$$\theta_w\left(\frac{L^2}{12\alpha}\right) = \frac{q''_w L}{3k} \tag{10.96}$$

The solution of Eq. (10.95), with the condition (10.96), is given by

$$\theta_w(t) = \frac{q''_w L}{k}\left[1 - 0.814\exp\left(-\frac{12}{5}\frac{\alpha t}{L^2}\right)\right], \qquad t > \frac{L^2}{12\alpha} \tag{10.97}$$

Thus, Eq. (10.93) with $\theta_w(t)$ as given by Eq. (10.97) represents the temperature distribution in the slab for times $t > L^2/12\alpha$.

10.4.4 Problems with Cylindrical and Spherical Symmetry

So far, we have considered some representative problems in the rectangular coordinate system, where the use of polynomial representation for the temperature distribution gives reasonably good results. However, Lardner and Pohle [13] demonstrated that polynomial representations are inappropriate for problems involving cylindrical and spherical symmetry. This is due to the fact that the volume into which heat diffuses does not remain the same for equal increments of r in the cylindrical and spherical coordinate systems. They suggest the following representations:

Cylindrical symmetry:	$T = (\text{polynomial in r}) \cdot \ln r$
Spherical symmetry:	$T = \dfrac{\text{polynomial in } r}{r}$

For further details, see Reference [13].

10.5 Variational Formulation and Solution by the Ritz Method

In this section, we first review the basics of variational calculus and then apply the results to obtain the variational formulation of a heat conduction problem. Next, we introduce the Ritz method as an approximate technique to solve the variational formulation of heat conduction problems.

10.5.1 Basics of Variational Calculus

Let us start with the problem of finding the function $y(x)$, defined over $x_1 \le x \le x_2$ and passing through the given end points $y(x_1) = y_1$ and $y(x_2) = y_2$, such that

$$I = \int_{x_1}^{x_2} f\left(x, y, \frac{dx}{dx}\right) dx = extremum\,(minimum\, or\, maximum) \tag{10.98}$$

A definite integral such as Eq. (10.98) is called a *functional* because its value depends on the choice of $y(x)$.

We can reduce the problem of extremization (i.e., maximization or minimization) of the functional (10.98) to the extremization of a function as follows. Denoting the solution (as yet unknown) as $y(x)$, let us consider a one-parameter family of *"comparison functions"* $\tilde{y}(x)$, defined by

$$\tilde{y}(x) = y(x) + \varepsilon\eta(x) \tag{10.99}$$

where ε is a parameter and $\eta(x)$ is *any* continuously differentiable function satisfying the end conditions

$$\eta(x_1) = \eta(x_2) = 0 \tag{10.100}$$

Note that no matter how $\eta(x)$ is defined, the solution $y(x)$ will certainly be a member of the $\tilde{y}$ family, namely, for $\varepsilon = 0$. Note also that $\tilde{y}(x_1) = y_1$ and $\tilde{y}(x_2) = y_2$; thus, all members of the family satisfy the end conditions of $y(x)$.

Next consider

$$I(\varepsilon) = \int_{x_1}^{x_2} f\left(x, \tilde{y}, \frac{d\tilde{y}}{dx}\right) dx \tag{10.101}$$

which is a function of the parameter ε. Note that

$$\lim_{\varepsilon \to 0} I(\varepsilon) = I \tag{10.102}$$

Thus, the value of $I(\varepsilon)$ is an extremum when $\varepsilon = 0$. On the other hand, for $I(\varepsilon)$ to take on its extremum value when $\varepsilon = 0$, the *necessary* condition is

$$\frac{dI(0)}{d\varepsilon} = 0 \tag{10.103}$$

Since, from Eq. (10.101),

$$\begin{aligned} \frac{dI(\varepsilon)}{d\varepsilon} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial \tilde{y}} \eta + \frac{\partial f}{\partial \tilde{y}'} n' \right) dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial (y + \varepsilon\eta)} \eta + \frac{\partial f}{\partial (y' + \varepsilon\eta')} \eta' \right] dx \end{aligned} \tag{10.104}$$

the condition (10.103) reduces to

$$\frac{dI(0)}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' \right) dx = 0 \tag{10.105a}$$

where primes are used to denote differentiation with respect to x. After integrating the second term under the integral sign by parts, Eq. (10.105*a*) becomes

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \eta(x)\, dx + \frac{\partial f}{\partial y'}\eta(x) \Big|_{x_1}^{x_2} = 0 \tag{10.105b}$$

Since by definition $\eta(x_1) = \eta(x_2) = 0$, Eq. (10.105*b*) reduces to

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \eta(x)\, dx = 0 \tag{10.105c}$$

Equation (10.105*c*) must hold for any continuously differentiable $\eta(x)$ that satisfies the conditions (10.100). We therefore conclude that*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{df}{dy'} \right) = 0 \tag{10.106}$$

Equation (10.106) is the so-called *Euler equation* associated with the functional (10.98). Thus, the condition necessary for $y(x)$ to minimize (or maximize) the functional (10.98) is that $f(x, y, dy/dx)$ must satisfy the corresponding Euler equation. Note that the second term in Eq. (10.106) is

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = \frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial^2 f}{\partial y \partial y'}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y'^2}\frac{d^2 y}{dx^2} \tag{10.107}$$

In general, the Euler equation (10.106) will be a nonlinear second-order differential equation in the unknown $y(x)$.

Equation (10.105*b*) holds for all continuously differentiable $\eta(x)$. If $y(x_1)$ and $y(x_2)$ are prescribed, then $\eta(x_1) = \eta(x_2) = 0$ and Eq. (10.105*b*) reduces to Eq. (10.105*c*). If, on the other hand, $y(x_1)$ and $y(x_1)$ are unspecified, then $\eta(x_1)$ and $\eta(x_2)$ will be arbitrary. However, since Eq. (10.105*b*) must hold for all $\eta(x)$, it must obviously hold for the subclass of $\eta(x)$ that vanish at x_1 and x_2 as well. Then the boundary terms in Eq. (10.105*b*) drop out, and, as before, it again leads to the Euler equation Eq. (10.106).

So, for all permissible values of $\eta(x_1)$ and $\eta(x_2)$, Eq. (10.105*b*) reduces to

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* If $F(x)$ is continuous and $\int_{x_2}^{x_1} F(x)\,\eta(x)dx = 0$ for all continuously differentiable functions $\eta(x)$ for which $\eta(x_1) = \eta(x_2) = 0$, then it must be true that $F(x) = 0$ for all $x_1 \le x \le x_2$ [9].

$$\frac{\partial f}{\partial y'}\eta(x)\bigg|_{x_1}^{x_2} = \frac{\partial f}{\partial y'}\bigg|_{x_2}\eta(x_2) - \frac{\partial f}{\partial y'}\bigg|_{x_1}\eta(x_1) = 0 \tag{10.108}$$

Thus, if $y(x_1)$ and/or $y(x_2)$ are unspecified, then $\eta(x_1)$ and/or $\eta(x_2)$ are completely arbitrary. It therefore follows that their coefficients in Eq. (10.108) must each vanish, yielding the conditions

$$\frac{\partial f}{\partial y'}\bigg|_{x_1} = 0 \quad \text{and/or} \quad \frac{\partial f}{\partial y'}\bigg|_{x_2} = 0, \tag{10.109a,b}$$

which are referred to as *natural boundary conditions*, must be satisfied instead.

10.5.2 Variational Formulation of Heat Conduction Problems

The variational formulation of a heat conduction problem may be obtained by treating the heat conduction equation of the problem as the Euler equation associated with its desired variational formulation and working the steps of the previous section in reverse order. The procedure will now be illustrated below in terms of an example.

Consider a steady-state heat conduction problem formulated as

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{q}(x) = 0 \tag{10.110a}$$

$$T(0) = T_1 \qquad \text{and} \qquad T(L) = T_2 \tag{10.110b,c}$$

We now take the governing differential equation (10.110*a*) of the problem as the Euler equation of the variational formulation to be obtained. A look at the derivation of Eq. (10.106) suggests that we proceed as follows:

$$\int_0^L \left[\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{q}(x)\right]\eta(x)\,dx = 0 \tag{10.111}$$

with $\eta(0) = \eta(L) = 0$ (since T is specified both at $x = 0$ and $x = L$). After integrating the first term in Eq. (10.111) by parts, it reduces to

$$k\frac{dT}{dx}\eta(x)\bigg|_0^L - \int_0^L k\frac{dT}{dx}\frac{d\eta}{dx}dx + \int_0^L \dot{q}(x)\eta(x)\,dx = 0 \tag{10.112}$$

where the first term vanishes because $\eta(0) = \eta(L) = 0$. Thus, Eq. (10.112) becomes

$$\int_0^L \left[\dot{q}(x)\eta(x) - k\frac{dT}{dx}\frac{d\eta}{dx}\right]dx = 0 \tag{10.113}$$

Comparing Eq. (10.113) with Eq. (10.105*a*), we obtain

$$\frac{\partial f}{\partial T} = \dot{q}(x) \quad \text{and} \quad \frac{\partial f}{\partial T'} = -k\frac{dT}{dx} \tag{10.114a,b}$$

where $T' = dT/dx$. Integrating Eqs. (10.114*a,b*), we get

$$f = \dot{q}(x)T(x) - \frac{1}{2}k\left(\frac{dT}{dx}\right)^2 + F(x) \tag{10.115}$$

where $F(x)$ is an arbitrary function of x. Thus, the variational form of the problem is given by

$$I = \int_0^L \left[\dot{q}(x)T(x) - \frac{1}{2}k\left(\frac{dT}{dx}\right)^2\right]dx + C = extremum \tag{10.116}$$

where the constant C, which is given by $C = \int_0^L F(x)dx$, can be eliminated from Eq. (10.116) because the function $T(x)$ that minimizes (or maximizes) the functional I defined by Eq. (10.116) also minimizes (or maximizes) the same functional with $C = 0$. Therefore, the variational form of the problem becomes

$$I = \int_0^L \left[\dot{q}(x)T(x) - \frac{1}{2}k\left(\frac{dT}{dx}\right)^2\right]dx = extremum \tag{10.117}$$

Thus, the problem has been reduced to the determination of a function $T(x)$ such that it will pass through the end points $T(0) = T_1$ and $T(L) = T_2$ and it will make the functional I defined by Eq. (10.117) an extremum.

10.5.3 Approximate Solutions by the Ritz Method

Once the variational form of a heat conduction problem is available, finding the form of the temperature distribution that will make the functional of the formulation an extremum may not be possible. On the other hand, an approximate solution can be obtained by a procedure called the *Ritz method*.

Let us now reconsider the problem of finding the function $y(x)$ passing through the given end points $y(x_1) = y_1$ and $y(x_2) = y_2$ such that it makes the functional (10.98) an extremum (i.e., a minimum or a maximum). The Ritz method is based on approximating the unknown function $y(x)$ in terms of some known linearly independent functions $\phi_n(x)$, $n = 0, 1, 2,\ldots, N$, in the form

$$y(x) = \phi_0(x) + \sum_{n=1}^{N} c_n\phi_n(x) \tag{10.118}$$

where the function $\phi_0(x)$ satisfies the boundary conditions, and the functions $\phi_n(x)$ for $n = 1, 2,\ldots, N$ satisfy the homogeneous part of the boundary conditions. Thus, the expression

(10.118) satisfies the boundary conditions. Inserting Eq. (10.118) into Eq. (10.98), we obtain the functional in the following form:

$$I = I(c_0, c_1, c_2, \ldots, c_N) \tag{10.119}$$

On the other hand, the fact that I = *extremum* requires

$$\frac{\partial I}{\partial c_n} = 0, \qquad n = 0, 1, 2, \ldots, N \tag{10.120}$$

This procedure results in N algebraic equations for the determination of N unknown coefficients c_n. Once c_n are available, then Eq. (10.118) represents an approximate solution to the problem. The accuracy and the convergence of this solution largely depend on the choice of functions $\phi_n(x)$. A discussion of the error estimation in the Ritz method can be found in Reference [11].

Let us now obtain an approximate solution by the Ritz method to the problem formulated as

$$\frac{d^2T}{dx^2} + \frac{x}{k} = 0 \tag{10.121a}$$

$$T(0) = T_w \qquad \text{and} \qquad T(1) = T_w \tag{10.121b,c}$$

This is a special case of the problem considered in the previous section. Thus, the variational form is obtainable from Eq. (10.117) as

$$I = \int_0^1 \left[xT(x) - \frac{1}{2}k\left(\frac{dT}{dx}\right)^2 \right] dx = extremum \tag{10.122}$$

Let the trial solution be

$$T(x) \cong T_w + c_1 x(1 - x) \tag{10.123}$$

Clearly, the trial solution (10.123) satisfies the boundary conditions (10.121*b,c*). Introducing Eq. (10.123) into Eq. (10.122), we obtain

$$I(c_1) = \frac{1}{2}\left(T_w + \frac{c_1}{6}\right) - \frac{1}{6}kc_1^2 \tag{10.124}$$

The coefficient c_1 is determined according to Eq. (10.120) as

$$\frac{dI}{dc_1} = \frac{1}{12} - \frac{1}{3}kc_1 = 0 \quad \Rightarrow \quad c_1 = \frac{1}{4k} \tag{10.125}$$

TABLE 10.1

Comparison of First-Order Approximation with Exact Results

x	0.25	0.5	0.75
Exact	0.0391	0.0625	0.0547
First order approximation	0.0469	0.0625	0.0469
Error	~20%	~0%	~14%

Then, the approximate solution becomes

$$T(x) \cong T_w + \frac{1}{4k} x(1-x) \tag{10.126}$$

which represents a first-order approximation of the temperature distribution.

A second-order approximation can be obtained, for example, by letting

$$T(x) \cong T_w + c_1 x(1-x) + c_2 x^2(1-x) \tag{10.127}$$

Introducing Eq. (10.127) into Eq. (10.122), we obtain $I(c_1, c_2)$. Then, the application of Eq. (10.120) gives two algebraic equations for the determination of the coefficients c_1 and c_2. Solving these equations, we get

$$c_1 = \frac{1}{6k} \quad \text{and} \quad c_2 = \frac{1}{6k} \tag{10.128a,b}$$

Thus, a second-order approximation of the temperature distribution is

$$T(x) \cong T_w + \frac{x}{6k}(1-x^2) \tag{10.129}$$

which is, in fact, the exact solution of the problem. Table 10.1 compares the first-order approximation for $k\,[T(x) - T_w]$ with the exact result for the same term at three different x locations.

In this section, the basics of the variational formulation of heat conduction problems and the Ritz method for approximate solution of the variational formulation have been given. Furthermore, the technique has been demonstrated by means of an illustrative example. For further discussions on this method, see References [1,2,10,18,20].

10.6 Coupled Integral Equations Approach (CIEA)

The coupled integral equations approach (CIEA) [21–23] is a very straightforward reformulation tool that can be employed in the simplification of diffusion problems via averaging processes in one or more of the involved space variables. In this sense, simpler formulations of the original partial differential systems are obtained, through a reduction of the number of independent variables in the multidimensional situations, by integration (averaging) of the full partial differential equations in one or more space variables, but retaining some information in the direction integrated out, provided by the related boundary conditions. The resulting lumped-differential formulation offers substantial enhancement over classical lumping schemes [22] in terms of accuracy, without introducing additional complexity in the corresponding final simplified differential equations to be handled.

Different levels of approximation in such mixed lumped-differential formulations can be used, starting from the plain and classical lumped system analysis, toward improved formulations, obtained through Hermite-type approximations for integrals [21–23]. The Hermite formulae of approximating an integral, based on the values of the integrand and its derivatives at the integration limits, are given in the form [21–23]

$$\int_{x_{i-1}}^{x_i} y(x)\,\mathrm{d}x \cong \sum_{\nu=0}^{\alpha} C_\nu y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} D_\nu y_i^{(\nu)} \tag{10.130a}$$

where $y(x)$ and its derivatives $y^{(n)}(x)$ are defined for all $x \in (x_{i-1}, x_i)$. Furthermore, it is assumed that the numerical values of $y^{(\nu)}(x_{i-1}) \equiv y_{i-1}^{(\nu)}$ for $\nu = 0, 1, 2,\ldots, \alpha$ and $y^{(\nu)}(x_i) \equiv y_i^{(\nu)}$ for $\nu = 0, 1, 2,\ldots, \beta$ are available at the end points of the interval.

In such a manner, the integral of $y(x)$ is expressed as a linear combination of $y(x_{i-1})$, $y(x_i)$, and their derivatives $y^{(\nu)}(x_{i-1})$ up to order $\nu = \alpha$, and $y^{(\nu)}(x_i)$ up to order $\nu = \beta$. This is called the $H_{\alpha,\beta}$ approximation given by [21–23]

$$\int_{x_{i-1}}^{x_i} y(x)\,\mathrm{d}x = \sum_{\nu=0}^{\alpha} C_\nu(\alpha,\beta) h_i^{\nu+1} y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} C_\nu(\beta,\alpha)(-1)^\nu h_i^{\nu+1} y_i^{(\nu)} + O\left(h_i^{\alpha+\beta+3}\right) \tag{10.130b}$$

where

$$h_i = x_i - x_{i-1};\, C_\nu(\alpha,\beta) = \frac{(\alpha+1)!\ (\alpha+\beta+1-\nu)!}{(\nu+1)!\ (\alpha-\nu)!\ (\alpha+\beta+2)!} \tag{10.130c,d}$$

In the present analysis, we consider just the two approximations, $H_{0,0}$ and $H_{1,1}$, given by

$$H_{0,0} \to \int_0^h y(x)\,\mathrm{d}x \cong \frac{h}{2}(y(0)+y(h)) \tag{10.131}$$

$$H_{1,1} \to \int_0^h y(x)\,\mathrm{d}x \cong \frac{h}{2}(y(0)+y(h)) + \frac{h^2}{12}(y'(0)-y'(h)) \tag{10.132}$$

which correspond, respectively, to the well-known trapezoidal and corrected trapezoidal integration rules. The respective expressions for the errors in the approximations $H_{0,0}$ and $H_{1,1}$ are written as

$$E_{0,0} = -\frac{h^3}{12} y''(\eta), \qquad \eta \in (0,h) \tag{10.133a}$$

$$E_{1,1} = +\frac{h^5}{720} y^{iv}(\xi), \qquad \xi \in (0,h) \tag{10.133b}$$

and can be employed to compose the final error expression in the desired averaged potential, which may then be bounded for a priori error analysis.

The fully differential formulations can be markedly simplified through reduction of the number of independent variables involved. Thus, one or more space variables may be integrated out in the original differential formulations, yielding, through the use of expressions such as Eqs. (10.131) and (10.132), approximate formulations that retain local information on the remaining coordinates, and averaged information in the directions eliminated through integration.

A fairly simple example is now considered to illustrate the CIEA approach, related to one-dimensional transient heat conduction in a slab. For constant thermophysical properties, k and α, the transient dimensionless formulation is written as

$$\frac{\partial \theta(X,\tau)}{\partial \tau} = \frac{\partial^2 \theta(X,\tau)}{\partial X^2}, \qquad 0 < X < 1, \tau > 0 \tag{10.134a}$$

$$\theta(X,0) = 1, \quad 0 \le X \le 1 \tag{10.134b}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{X=0} = 0; \qquad \left.\frac{\partial \theta}{\partial X}\right|_{X=1} + \mathrm{Bi}\theta(1,\tau) = 0, \qquad \tau > 0 \tag{10.134c,d}$$

where the dimensionless groups are given by

$$\theta(X,\tau) = \frac{T(x,t) - T_\infty}{T_0 - T_\infty}; X = \frac{x}{L}; \tau = \frac{\alpha t}{L^2}; \mathrm{Bi} = \frac{hL}{k}; \tag{10.135}$$

Thus, the slab has thickness L, is initially at the uniform temperature T_0, is insulated at the boundary $x = 0$, and exchanges heat by convection at $x = L$ with a heat transfer coefficient, h, and a fluid at the constant temperature, T_∞.

The exact solution of Eqs. (10.134) is obtained through separation of variables as

$$\theta(X,\tau) = \sum_{i=1}^{\infty} \frac{1}{N_i} \bar{f}_i \cos(\mu_i X) \exp\left(-\mu_i^2 \tau\right) \tag{10.136a}$$

and

$$\bar{f}_i = \int_0^1 \cos(\mu_i X)\, dX = \frac{\sin(\mu_i)}{\mu_i} \tag{10.136b}$$

where the respective eigenvalues and norms are obtained from

$$\mu_i \tan \mu_i = \text{Bi} \quad \text{and} \quad N_i = \frac{1}{2}\left\{1 + \frac{1}{\text{Bi}} + \left(\frac{\mu_i}{\text{Bi}}\right)^2\right\}\sin^2 \mu_i \tag{10.136c,d}$$

The average dimensionless temperature is determined from

$$\theta_{\text{av}}(\tau) = \int_0^1 \theta(X,\tau)\, dX = \sum_{\text{i}=1}^{\infty} \frac{1}{N_i}\frac{\sin \mu_i}{\mu_i} \bar{f}_i \exp\left(-\mu_{\text{i}}^2 \tau\right) \tag{10.137}$$

The above definition of $\theta_{\text{av}}(\tau)$ is now employed to promote the lumping of the heat conduction equation, Eq. (10.134*a*), operating with $\int_0^1 \_\_\_\_ \, dX$ and recalling the boundary conditions (10.134*c,d*):

$$\frac{d\theta_{\text{av}}}{d\tau} + \text{Bi}\theta(1,\tau) = 0, \; \tau > 0 \tag{10.138a}$$

$$\theta_{\text{av}}(0) = 1 \tag{10.138b}$$

Equations (10.138) are just a formal representation of the lumping of the heat conduction equations, which involves both the average temperature and the local (boundary) temperature. The approximation process starts by assuming a relation between these two potentials. In the classical lumped system analysis, it is assumed that the temperature gradients are sufficiently smooth over the whole space variable domain, so that the boundary temperature can be approximated by the average temperature:

$$\theta(1,\tau) \cong \theta_{\text{av}}(\tau) \tag{10.139}$$

leading to the following lumped formulation:

$$\frac{d\theta_{\text{av}}}{d\tau} + \text{Bi}\theta_{\text{av}}(\tau) = 0, \qquad \tau > 0 \tag{10.140a}$$

$$\theta_{\text{av}}(0) = 1 \tag{10.140b}$$

which is readily solved as

$$\theta_{\text{av}}(\tau) = \exp(-\text{Bi}\; \tau) \tag{10.141}$$

This classical lumped system analysis is, in general, recommended to problems with Bi < 0.1.

Improved lumped-differential formulations can be readily constructed, as discussed above, by considering more informative relations between the boundary and average temperatures, instead of the simplest possible relationship expressed by Eq. (10.139). The idea is to incorporate further information from the problem formulation, Eqs. (10.134), and reach a relation of the form

$$\theta(1,\tau) \cong f[\theta_{\text{av}}(\tau)] \tag{10.142}$$

For this purpose, Hermite formulae for approximating the integrals that define the average temperature and heat flux in the spatial coordinate shall be employed. The integrals that define the spatially averaged temperature and heat flux within the slab are given by

$$\int_0^1 \theta(X,\tau)\,dX = \theta_{\text{av}}(\tau) \tag{10.143a}$$

$$\int_0^1 \frac{\partial\theta(X,\tau)}{\partial X}\,dX = \theta(1,\tau) - \theta(0,\tau) \tag{10.143b}$$

As an illustration, the above integrals are first approximated through the trapezoidal rule, the $H_{0,0}$ formula in Eq. (10.131), to yield

$$\theta_{\text{av}}(\tau) \cong \frac{1}{2}[\theta(0,\tau) + \theta(1,\tau)] \tag{10.144a}$$

$$\theta(1,\tau) - \theta(0,\tau) \cong \frac{1}{2}\left[\left.\frac{\partial\theta}{\partial X}\right|_{X=0} + \left.\frac{\partial\theta}{\partial X}\right|_{X=1}\right] \tag{10.144b}$$

The derivatives in Eq. (10.144b) can be eliminated through the boundary conditions (10.134c,d) to provide a direct relation between the two boundary temperatures as

$$\theta(0,\tau) = \left[1 + \frac{\text{Bi}}{2}\right]\theta(1,\tau) \tag{10.144c}$$

and by eliminating the temperature at $X = 0$ from Eq. (10.144a), one obtains the sought improved relation between the boundary and average temperatures:

$$\theta(1,\tau) = \left[1 + \frac{\text{Bi}}{4}\right]^{-1}\theta_{\text{av}}(\tau) \tag{10.145}$$

Finally, substitution of the improved relation into the lumped equation (10.138a) leads to the improved lumped-differential formulation based on the $H_{0,0}$ Hermite approximation:

$$\frac{d\theta_{\text{av}}}{d\tau} + \text{Bi}^{+}\theta_{\text{av}}(\tau) = 0, \qquad \tau > 0, \tag{10.146a}$$

$$\theta_{\mathrm{av}}(0) = 1 \tag{10.146b}$$

with a modified Biot number given by

$$\mathrm{Bi}^{+} = \frac{\mathrm{Bi}}{1 + \dfrac{\mathrm{Bi}}{4}} \tag{10.146c}$$

which is readily solved as

$$\theta_{\mathrm{av}}(\tau) = \exp(-\mathrm{Bi}^{+}\,\tau) \tag{10.147}$$

This improved lumped system analysis has an extended applicability limit in terms of Biot number, in comparison to the classical lumped system analysis, as shall be examined in what follows.

Higher order approximations can be adopted, such as the corrected trapezoidal rule ($H_{1,1}$ approximation) of Eq. (10.132). For instance, one may adopt Eq. (10.132) in approximating the average temperature integral, as

$$\theta_{\mathrm{av}}(\tau) \cong \frac{1}{2}[\theta(0,\tau) + \theta(1,\tau)] + \frac{1}{12}\left[\left.\frac{\partial\theta}{\partial X}\right|_{X=0} - \left.\frac{\partial\theta}{\partial X}\right|_{X=1}\right] \tag{10.148}$$

and maintaining the $H_{0,0}$ formula, Eq. (10.131), for the average heat flux expression. The resulting improved relation between the boundary and average temperatures then becomes

$$\theta(1,\tau) = \left[1 + \frac{\mathrm{Bi}}{3}\right]^{-1} \theta_{\mathrm{av}}(\tau) \tag{10.149}$$

The resulting $H_{1,1}/H_{0,0}$ improved formulation is similar to the $H_{0,0}/H_{0,0}$ formulation of Eqs. (10.146), and thus with the same solution of Eq. (10.147), but with a redefined modified Biot number, as

$$\mathrm{Bi}^{+} = \frac{\mathrm{Bi}}{1 + \dfrac{\mathrm{Bi}}{3}} \tag{10.150}$$

Again, this improved formulation is expected to allow for extended applicability limits for the lumped approximation.

If accuracy is at a premium, it is still possible to construct even higher order approximations, for instance by considering the corrected trapezoidal rule also for the average heat flux integral, leading to

$$\theta(1,\tau) - \theta(0,\tau) \cong \frac{1}{2}\left[\left.\frac{\partial\theta}{\partial X}\right|_{X=0} + \left.\frac{\partial\theta}{\partial X}\right|_{X=1}\right] + \frac{1}{12}\left[\left.\frac{\partial^2\theta}{\partial X^2}\right|_{X=0} - \left.\frac{\partial^2\theta}{\partial X^2}\right|_{X=1}\right] \tag{10.151}$$

The second derivatives with respect to X that appear in Eq. (10.151) can be readily evaluated from the heat conduction equation itself, Eq. (10.134*a*), and proceeding with the CIEA approach, as detailed in Ref. [22], two coupled ordinary differential equations for the time evolution of the boundary and average temperatures are obtained, after elimination of the boundary temperature, $\theta(0, \tau)$, from these expressions, in the form

$$\frac{d\theta_{\text{av}}(\tau)}{d\tau} = -\text{Bi}\theta(1,\tau), \qquad \tau > 0 \tag{10.152a}$$

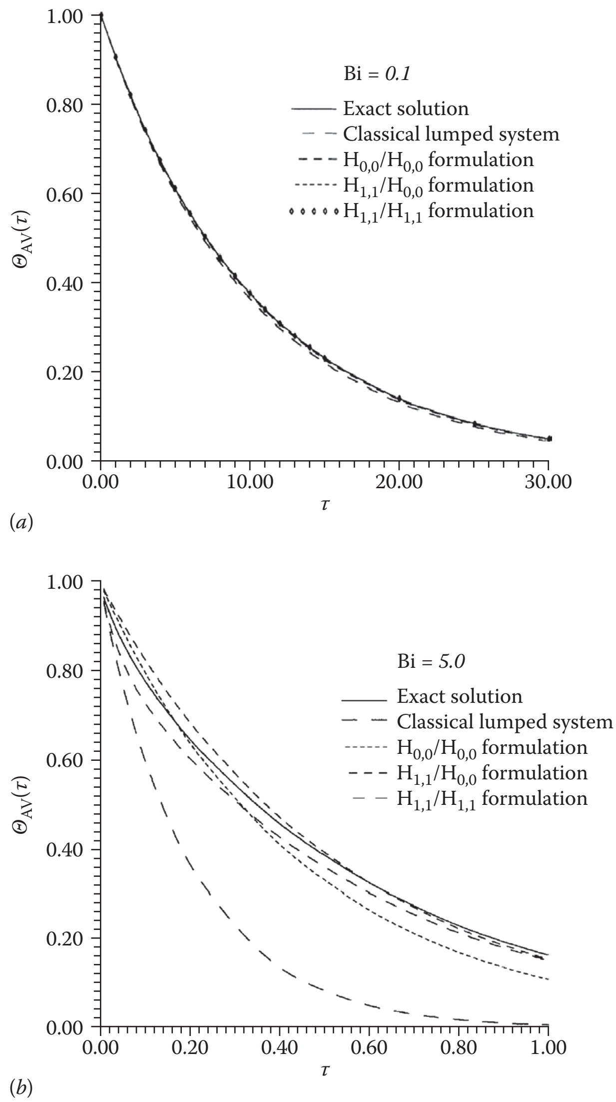

FIGURE 10.3

Comparison of exact fully differential solution and CIEA lumped-differential formulations for transient heat conduction in a slab: (*a*) Bi = 0.1; (*b*) Bi = 5.

$$\frac{d\theta(1,\tau)}{d\tau} = \frac{1}{1+\frac{\mathrm{Bi}}{12}}[-(12+5\mathrm{Bi})\theta\,(1,\tau)+12\theta_{\mathrm{av}}(\tau)]\ \tau > 0 \qquad (10.152b)$$

with the initial conditions

$$\theta_{\mathrm{av}}(0)=1, \quad \theta(1,0)=1 \qquad (10.152c,d)$$

Equations (10.152) can be readily solved in analytical form, but the details are omitted here for conciseness.

Figure (10.3*a*,*b*) present a comparison among the exact fully differential, classical lumped system, and improved lumped formulations ($H_{0,0}/H_{0,0}$, $H_{1,1}/H_{0,0}$, and $H_{1,1}/H_{1,1}$), in terms of the time variation of the dimensionless average temperature for two widespread values of Biot number (Bi = 0.1 and 5.0). In the range of Bi < 0.1, all the approximate formulations show a good agreement with the fully differential heat conduction equation solution, but the classical lumped system solution already shows some deviations from the other lumped formulations. As the Biot number is increased to Bi = 5.0, the improvement offered by the CIEA formulations, as compared to the classical lumped system analysis, becomes quite evident. The $H_{0,0}/H_{0,0}$ formulation is overall the less accurate of the improved formulations, even though with better agreement for the early transient.

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Problems

10.1 A semi-infinite solid, $0 \le x < \infty$, is initially at a uniform temperature T_i. For times $t \ge 0$, a time-dependent heat flux $q''_w(t)$ is applied to the surface at $x = 0$. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution in the solid by Duhamel's method for $t > 0$.

10.2 Re-solve Problem 8.8 by Duhamel's method.

10.3 A long solid cylinder of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. For times $t \ge 0$, the temperature of the peripheral surface at $r = r_0$ is varied according to $T(r_0, t) = f(t)$, where $f(t)$ is a prescribed function of time. Obtain an expression for the unsteady-state temperature distribution in the cylinder by Duhamel's method for $t > 0$.

10.4 A long solid cylinder of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. For times $t \ge 0$, internal energy is generated in the cylinder at a rate $\dot{q}(t)$ per unit volume (W/m^3), while the temperature of the peripheral surface at $r = r_0$ is maintained at the initial temperature T_i. Obtain an expression for the unsteady-state temperature distribution in the cylinder by Duhamel's method for $t > 0$.

10.5 A solid sphere of constant thermophysical properties and radius r_0 is initially at a uniform temperature T_i. For times $t \ge 0$, internal energy is generated in the sphere at a rate $\dot{q}(t)$ per unit volume, while the surface at $r = r_0$ is maintained at the initial T_i. Obtain an expression for the unsteady-state temperature distribution in the sphere by Duhamel's method for $t > 0$.

10.6 Re-solve Problem 7.27 by Duhamel's method.

10.7 Consider an infinite solid medium, $r_0 < r < \infty$, internally bounded by a cylindrical surface at $r = r_0$ as illustrated in Fig. 10.4. Initially, the medium is at a uniform

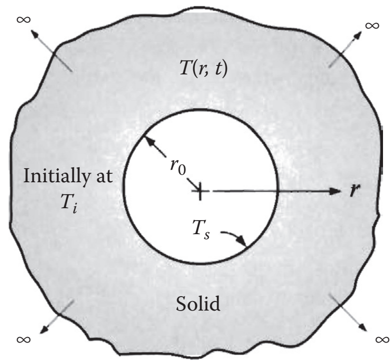

FIGURE 10.4
Figure for Problem 10.7.

temperature T_i. For times $t \geq 0$, the temperature of the surface at $r = r_0$ is kept at a constant temperature T_s different from the initial temperature T_i. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the medium by a similarity transformation for $t > 0$.

10.8 An infinitely large solid, $0 \leq r < \infty$, is initially at a uniform temperature T_i. For times $t \geq 0$, a line heat source situated along the z axis at $r = 0$ releases heat continuously at a constant rate q'_{ln} per unit length (W/m). Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution in the solid by a similarity transformation for $t > 0$.

10.9 An infinitely large solid, $0 \leq r < \infty$, is initially at a uniform temperature T_i. For times $t \geq 0$, a point heat source situated at $r = 0$ releases heat continuously at a rate $q_{pt}(t)$ (W), which varies with time according to

$$q_{pt}(t) = Q_0 t^{1/2}$$

where Q_0 is a prescribed constant. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the solid by a similarity transformation for times $t > 0$.

10.10 A semi-infinite solid, $0 \leq x < \infty$, is initially at a uniform temperature T_i. The surface temperature at $x = 0$ is changed to T_w at $t = 0$ and is maintained constant at this value for times $t > 0$. Assuming that the thermal conductivity k, specific heat c, and density ρ of the solid are all temperature dependent, obtain an expression for the unsteady-state temperature distribution in the solid by integral method for $t > 0$. Assume that the temperature profile can be represented by a third-degree polynomial in x.

10.11 Re-solve Problem 8.6 by the integral method. Approximate the temperature profile by a third-degree polynomial in x.

10.12 Re-solve Problem 8.7 by the integral method. Approximate the temperature profile in the solid by a third-degree polynomial in x.

10.13 A plane wall of thickness L in the x direction is initially at a uniform temperature T_i. For times $t \geq 0$, the surface at $x = 0$ is maintained at a constant temperature T_w and the surface at $x = L$ is kept perfectly insulated. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(x, t)$ in the wall by the integral method for $t > 0$.

10.14 An infinite region, $r \geq r_0$, with a cylindrical hole of radius $r = r_0$ in it is initially at a uniform temperature T_i. For times $t \geq 0$, the cylindrical surface at $r = r_0$ is maintained at a constant temperature T_w. Assuming constant thermophysical properties, obtain an expression for the unsteady-state temperature distribution $T(r, t)$ in the medium by the integral method for $t > 0$ by assuming the following profiles:

(*a*) $T(r, t) = a + br + cr^2$

(*b*) $T(r, t) = (a + br + cr^2) \ln r$

Also, obtain the exact solution of the problem and compare it with the two approximate solutions obtained in (*a*) and (*b*).

10.15 Obtain the variational form of the heat conduction problem formulated as

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{q}(x) = 0$$

$$\left[-k\frac{dT}{dx} + h_1 T(x)\right]_{x=0} = h_1 T_{f_1}$$

$$\left[k\frac{dT}{dx} + h_2 T(x)\right]_{x=L} = h_2 T_{f_2}$$

where h_1, h_2, T_{f_1} and T_{f_2} are constants.

10.16 Obtain an approximate solution by the Ritz method to the problem given by Eqs. (10.121*a*, *b*, *c*) by assuming a trial solution in the form

$$T(x) = T_w + c_1 x(1-x) + c_2 x(1-x)^2$$

10.17 Consider a thin rod of cross-sectional area A, perimeter P, length L and constant thermal conductivity k. The rod is exposed to a fluid maintained at a constant temperature T_∞ with a constant heat transfer coefficient h, while the two ends at $x = 0$ and $x = L$ are both kept at another constant temperature T_0.

(*a*) Obtain an approximate solution by the Ritz method by assuming a trial solution in the form

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = c_0 + c_1\frac{x}{L} + c_2\left(\frac{x}{L}\right)^2$$

(*b*) Obtain an expression for the rate of heat loss from the rod to the surrounding fluid using the approximate profile obtained in (*a*). Also, obtain an exact expression for rate of heat loss and compare the two results.

10.18 Re-solve Problem 10.17 if the end of the rod at $x = 0$ is perfectly insulated, while the end at $x = L$ is kept at the constant temperature T_0.

10.19 Re-solve Problem 10.17 if the end at $x = 0$ is heated by applying a constant heat flux q''_0 (W/m^2), while the end at $x = L$ is kept at the constant temperature T_0.

10.20 Solve the following transient heat conduction problem in a slab with constant heat generation, G, employing the coupled integral equations approach. Compare the exact solution for the dimensionless average temperature against the classical lumped system analysis and the improved lumped formulations obtained by the CIEA ($H_{0,0}/H_{0,0}$, $H_{1,1}/H_{0,0}$, and $H_{1,1}/H_{1,1}$). Consider the Bi values of 0.1 and 5, and vary the dimensionless heat generation term, G, to examine its influence on the accuracy of the improved lumped formulations.

$$\frac{\partial\theta(X,\tau)}{\partial\tau} = \frac{\partial^2\theta(X,\tau)}{\partial X^2} + G, \qquad 0 < X < 1, \tau > 0$$

$$\theta(X,0) = 1, \qquad 0 \le X \le 1$$

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=0} = 0; \qquad \left.\frac{\partial\theta}{\partial X}\right|_{X=1} + \mathrm{Bi}\,\theta(1,\tau) = 0, \qquad \tau > 0$$

10.21 Consider the following two-dimensional transient dimensionless formulation for heat conduction in a solid cylinder:

$$\frac{\partial\theta(R,Z,\tau)}{\partial\tau} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\theta(R,Z,\tau)}{\partial R}\right) + \frac{1}{\mathrm{K}^2}\frac{\partial^2\theta(R,Z,\tau)}{\partial Z^2},$$

$$0 < R < 1, 0 < Z < 1, \tau > 0$$

subjected to the initial and boundary conditions

$$\theta(R,Z,0) = 1 \qquad 0 \le R \le 1, \qquad 0 \le Z \le 1$$

$$\left.\frac{\partial\theta}{\partial R}\right|_{R=0} = 0, \qquad 0 < Z < 1, \qquad \tau > 0$$

$$\left.\frac{\partial\theta}{\partial R}\right|_{R=1} + \mathrm{Bi}\theta(1,Z,\tau) = 0, \qquad 0 < Z < 1, \qquad \tau > 0$$

$$\left.\frac{\partial\theta}{\partial Z}\right|_{Z=0} = 0, \qquad 0 < R < 1, \qquad \tau > 0$$

$$\left.\frac{\partial\theta}{\partial Z}\right|_{Z=1} + \mathrm{Bi}^*\,\theta(R,1,\tau) = 0, \qquad 0 < R < 1, \qquad \tau > 0$$

with the following dimensionless variables:

$$\theta(R,Z,\tau) = \frac{T(r,z,t) - T_\infty}{T_0^* - T_\infty}, \qquad R = \frac{r}{r_w}, \qquad Z = \frac{z}{L/2}, \qquad \tau = \frac{\alpha t}{r_w^2},$$

$$F(R,Z) = \frac{T_0(r,z) - T_\infty}{T_0^* - T_\infty}, \qquad \mathrm{Bi} = \frac{h r_w}{k}, \qquad \mathrm{Bi}^* = \frac{h^* L/2}{k}, \qquad K = \frac{L/2}{r_w}$$

Obtain the exact solution of this problem by separation of variables, and obtain the exact axially averaged temperature distribution along the radial coordinate,

$$\theta_{\mathrm{av}}(R,\tau):$$

$$\theta_{\mathrm{av}}(R,\tau) = \int_0^1 \theta(R,Z,\tau)\,dZ$$

Compare this exact result with the classical lumped system analysis and the CIEA improved formulations ($H_{0,0}/H_{0,0}$ and $H_{1,1}/H_{0,0}$) for different values of the Biot numbers, Bi = Bi* = 0.5 and 5.0, and different aspect ratios, K = 0.1 and 1.

11

Heat Conduction Involving Phase Change

11.1 Introduction

Many important technological applications involve phase change (i.e., freezing and melting), such as the solidification of metals or plastics in casting and coating processes, welding, high-energy laser beam cutting and forming, crystal growth, freezing or thawing of foodstuffs, ice production, aerodynamic ablation, and thermal energy storage using solid–liquid phase-changing materials to name just a few. There are also important medical applications that involve phase change such as cryosurgery or organ preservation.

Heat conduction problems in which phase change occurs are generally referred to as *moving boundary problems*. Such problems are inherently transient because the location of the interface between the two phases changes with time as latent heat is absorbed or released at the interface. The instantaneous location of the interface would not be known *a priori* and, thus, is determined as part of the solution.

In the solidification or melting of pure substances, such as water or pure metals, and eutectic alloys, the phase change takes place at a fixed temperature (*i.e., fusion temperature*), and there is a sharp moving interface between the two phases. On the other hand, in "glassy" materials phase change occurs with a gradual transition of the thermophysical properties, from those of one phase to those of the other. Gradual transition, however, is most distinctly observed in binary mixtures, alloys and impure materials, where phase change takes place over a temperature range and a moving zone of finite extent of a two-phase mixture exists between the solid and liquid phases. Such a two-phase mixture region is called the *mushy zone.* The mushy zone diminishes in the case of *eutectic* freezing.

One of the difficulties in handling phase-change problems is that the moving interface leads to a nonlinear interface condition if the interface is in the form of a sharp interface or two nonlinear interface conditions in the presence of a two-phase mushy zone. Moreover, density changes following phase transformation give rise to some motion of the liquid phase. If the effect of such motion is significant, heat transfer by convection has to be considered in the liquid phase. Convection effects are beyond the scope of this book and are ignored in the treatment of this chapter.

Heat conduction problems involving phase change are mostly associated with Josep Stefan (1835–1893) who appears to have been the first to make an extensive study of them in relation to the melting of polar ice cap [1], even though Gabriel Lamé (1795–1870) and Benoit Clapeyron (1799–1864) had considered such a problem as early as 1831 [2]. In fact, the first important exact analytical solution was obtained by Franz Neumann

(1798–1895), as given in his unpublished lectures delivered in Konigsberg in the early 1860s. However, the first publication of these lectures did not appear until 1912 [3]. Many phase-change problems have appeared in the literature since then. Unfortunately, closed-form exact analytical solutions have been obtained only under the following very restrictive conditions: one-dimensional, semi-infinite geometry, uniform initial temperature, constant imposed surface temperature and constant thermophysical properties in each phase. This is mainly due to the fact that phase-change problems are inherently nonlinear. Accordingly, much research has focused on approximate solution techniques, both analytical and numerical. More detailed discussions on phase-change problems can be found, among others, in the monograph by Alexiades and Solomon [4], in the texts by Carslaw and Jaeger [5], Crank [6], Hill and Dewynne [7], Jiji [8], and Özışık [9], and in the reviews by Fukusako and Seki [10], Lior [11], Muehlbauer and Sunderland [12], Toksoy and Ilken [13], and Yao and Prusa [14].

Two approximate methods most commonly used for the solution of phase-change problems are the *integral method* and the *quasi-steady approximation.* The basic principles of the integral method, which provides a relatively straightforward and simple approach, have already been described in Section 10.4. Goodman [15–18] and subsequently many other investigators [19– 25] have used this method to solve a number of one-dimensional phase-change problems. In Section 11.3.3, we present the application of this method to the solution of a representative problem.

The quasi-steady approximation, which is based on the assumption that the effects of sensible heat are negligible relative to those of latent heat, yields easily computable results for "rough" estimates of phase-change problems. In this approximation, however, the transient interface condition is retained, allowing the prediction of the instantaneous location of the face front. In Section 11.7, we apply this method to obtaining approximate solutions to a number of one-dimensional phase-change problems.

In Section 11.8, we consider the solidification of a binary alloy, as an example to phase-change problems with mushy zone. The presence of a two-phase mushy zone with temperature-dependent concentration and phase proportion complicates the heat transfer analysis in such problems, and requires the simultaneous solution of the heat and mass transfer equations. An introduction to such problems and further references are given by Hayashi and Kunimine [26], and Poulikakos [27].

11.2 Boundary Conditions at a Sharp Moving Interface

Continuity of temperature and the conservation of thermal energy give rise to two conditions at the solid-liquid interface, which are needed for the mathematical formulation of phase-change problems with a sharp moving interface.

Consider a one-dimensional case in which phase change (melting or solidification) takes place at the fusion (melting or solidification) temperature, and therefore the two phases are separated by a sharp interface. Let the thermophysical properties of each phase be constants. Although at least natural convection [28] and sometimes forced convection and radiation may also take place in the liquid phase, assume that heat transfer takes place only by conduction in both phases. Designate the conditions by subscript s in the *solid phase* and by subscript l in the *liquid phase.* Let $x = s(t)$ denote the instantaneous interface location at time t.

11.2.1 Continuity of Temperature at the Interface

If T_f denotes the fusion temperature, then at the solid–liquid interface

$$T_s(x,t)\big|_{x=s(t)} = T_l(x,t)\big|_{x=s(t)} = T_f \tag{11.1}$$

which specifies the continuity of temperature at the interface.

11.2.2 Energy Balance at the Interface

Consider the solidification model depicted in Fig. 11.1, in which a liquid confined to a semi-infinite region, $0 \le x < \infty$, is initially at a temperature higher than the fusion temperature T_f. At $t = 0$, the temperature of the liquid surface at $x = 0$ is suddenly lowered to T_0 ($< T_f$) and maintained at this temperature for times $t > 0$. Consequently, the liquid starts to solidify at $x = 0$ and the solid–liquid interface moves gradually in the positive x direction with a velocity

$$V_i = \frac{ds}{dt} \tag{11.2}$$

Consider a control volume of infinitesimal thickness δ enclosing the interface as shown in Fig. 11.1*b*. Relative to the control volume, solid of constant density ρ_s exits the control surface on the left side with a velocity V_s, and liquid of constant density ρ_l enters the control surface on the right side with a velocity V_l. Since the densities ρ_s and ρ_l are assumed to be constants, independent of temperature, and the solid phase is bounded by the surface at $x = 0$, the velocity V_s of the solid phase relative to the interface (i.e., across the control surface on the left) must be equal to the interface velocity V_i. On the other hand, the law of conservation of mass requires that

$$V_l = \frac{\rho_s}{\rho_l} V_s = \frac{\rho_s}{\rho_l} V_i \tag{11.3}$$

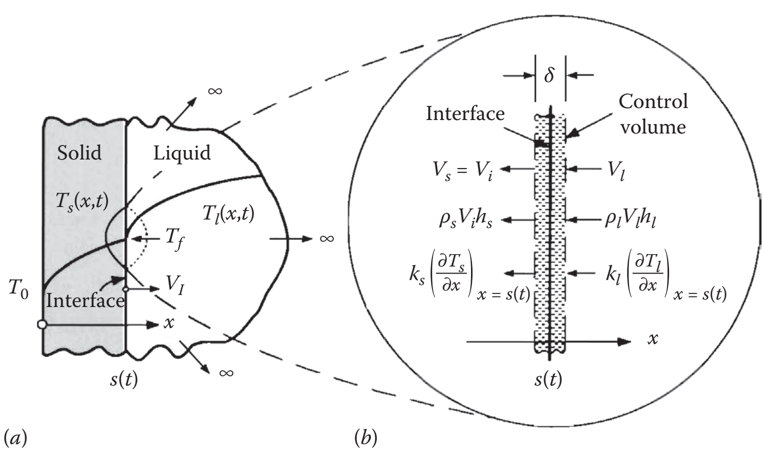

FIGURE 11.1
Solidification of a liquid from a cold plane surface.

Referring to Fig. 11.1*b*, an energy balance on the control volume per unit area gives

$$\rho_l V_l h_l + k_l\left(\frac{\partial T_l}{\partial x}\right)_{x=s(t)} = \rho_s V_i h_s + k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s(t)} \tag{11.4}$$

where k_l and k_s are the thermal conductivities, and h_l and h_s are the specific enthalpies, respectively, of the liquid and solid phases. Making use of the relations (11.2) and (11.3), the interface energy-balance relation (11.4) can be rewritten as

$$k_l\left(\frac{\partial T_l}{\partial x}\right)_{x=s(t)} - k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s(t)} + \rho_s h_{sl}\frac{ds}{dt} = 0 \tag{11.5}$$

where

$$h_{sl} = h_l - h_s \tag{11.6}$$

is the *latent heat of fusion*. Equation (11.5) is known as the *Stefan condition*, which states that the freezing front moves in a such way that its velocity is proportional to the jump in heat flux across the front.

It can be shown that the Stefan condition (11.5) also holds for the melting of a semi-infinite solid as illustrated in Fig. 11.2, except that the density ρ_s is replaced by the density ρ_l (see Problem 11.1).

Effect of Convection in the Liquid Phase. Consider the solidification problem illustrated in Fig. 11.1*a*. If the heat transfer in the liquid phase is controlled by convection, then it may sometimes be more convenient to use Newton's cooling law (i.e., the definition of the heat transfer coefficient) to represent the heat transfer rate from the liquid phase to solid phase at the interface; that is,

$$k_l\left(\frac{\partial T_l}{\partial x}\right)_{x=s(t)} = h(T_b - T_f) \tag{11.7}$$

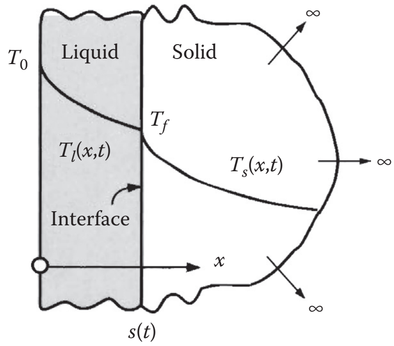

FIGURE 11.2
Melting of a solid from a hot plane surface.

where h is the heat transfer coefficient at the interface on the liquid side and T_b, is the bulk temperature of the liquid phase. Thus, the interface energy-balance relation (11.5) can be rewritten as

$$h(T_b - T_f) - k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s(t)} + \rho_s h_{sl}\frac{ds}{dt} = 0 \qquad (11.8a)$$

For the case of melting illustrated in Fig. 11.2, the energy-balance relation at the interface would be given by

$$h(T_b - T_f) + k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s(t)} - \rho_l h_{sl}\frac{ds}{dt} = 0 \qquad (11.8b)$$

Furthermore, it is to be noted that the relations (11.5) and (11.8*a,b*) also hold for the case of one-dimensional problems in cylindrical or spherical coordinates if x is simply replaced by the radial coordinate r of the cylindrical and spherical systems.

11.3 A Single-Region Phase-Change Problem

Consider the solidification of a semi-infinite liquid region as illustrated in Fig. 11.3. The liquid is initially at the fusion temperature T_f. At $t = 0$, the temperature of the liquid surface at $x = 0$ is suddenly lowered to T_0 and maintained constant at that temperature for times $t > 0$. Consequently, the liquid starts to solidify at $x = 0$ and the solid–liquid interface moves in the positive x direction. Being at the uniform temperature T_f throughout, there is no heat transfer in the liquid phase. We wish to determine the temperature distribution in the solid phase and the location of the interface as a function of time t.

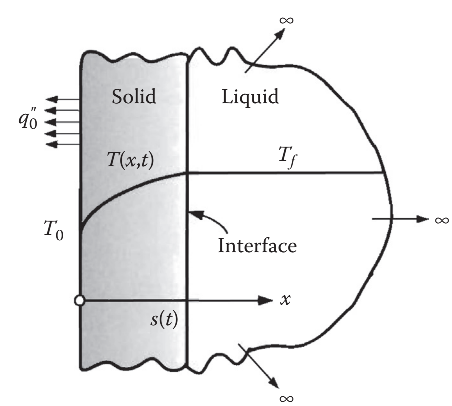

FIGURE 11.3
Solidification of a liquid at fusion temperature.

11.3.1 Formulation

Assuming constant thermophysical properties, the formulation of the problem for the temperature distribution $T(x, t)$ in the solid phase is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{11.9a}$$

$$T(0,t) = T_0 \quad \text{and} \quad T(x,t)\big|_{x=s(t)} = T_f \tag{11.9b,c}$$

Furthermore, at the interface, from Eq. (11.5), we also have

$$-k\frac{\partial T}{\partial x}\bigg|_{x=s(t)} + \rho h_{sl}\frac{ds}{dt} = 0 \tag{11.9d}$$

with the initial condition $s(0) = 0$. In Eqs. (11.9), α, k, and ρ represent the thermal diffusivity, thermal conductivity, and density of the solid phase, respectively. For simplicity, we have dropped the subscript s.

11.3.2 Stefan's Exact Solution

The solidification problem formulated above can be solved by utilizing the similarity method discussed in Section 10.3. Accordingly, it can be shown that this problem accepts a similarity solution in terms of the similarity variable (see Problem 11.2)

$$\eta = \frac{x}{2\sqrt{\alpha t}} \tag{11.10}$$

The heat conduction equation (11.9a) and the conditions (11.9b,c) can then be rewritten for $\theta(\eta) = T(\eta) - T_0$ as

$$\frac{d^2\theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0 \tag{11.11a}$$

$$\theta(0) = 0 \quad \text{and} \quad \theta(\lambda) = T_f - T_0 \tag{11.11b,c}$$

where we have introduced

$$\lambda = \frac{s(t)}{2\sqrt{\alpha t}} \tag{11.12}$$

As shown in Section 10.3, the solution of the ordinary differential equation (11.11a) is given by

$$\theta(\eta) = A + B\,\mathrm{erf}\,(\eta) \tag{11.13}$$

where erf(η) is the *error function*, and A and B are the two constants of integration. Imposing the boundary conditions (11.11*b,c*), we obtain

$$A = 0 \quad \text{and} \quad B = \frac{T_f - T_0}{\text{erf}\,(\lambda)} \tag{11.14a,b}$$

Here we note that, since B is to be a constant for all times, the parameter λ must also be a constant, independent of time t. Now, substituting the constants A and B from Eqs. (11.14*a,b*) into Eq. (11.13), the solution for the temperature distribution in the solid phase is obtained as

$$\frac{\theta(\eta)}{T_f - T_0} = \frac{\text{erf}\,(\eta)}{\text{erf}\,(\lambda)} \tag{11.15a}$$

or

$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\text{erf}\left(\dfrac{x}{2\sqrt{\alpha t}}\right)}{\text{erf}\,(\lambda)} \tag{11.15b}$$

where the parameter λ is yet to be determined. The use of the condition (11.9*d*), on the other hand, yields the relation

$$\lambda e^{\lambda^2}\text{erf}(\lambda) = \frac{1}{\sqrt{\pi}}\text{Ste} \tag{11.16}$$

for the determination of λ. In Eq. (11.16), Ste is the so-called *Stefan number*, which is defined as

$$\text{Ste} = \frac{c(T_f - T_0)}{h_{sl}} \tag{11.17}$$

where c is the specific heat of the solid phase. The Stefan number, as defined by Eq. (11.17), represents the ratio of the *sensible heat* c $(T_f - T_0)$ to the *latent heat* h_{sl}. The sensible heat is the energy added to a unit mass of solid to raise its temperature from T_0 to the fusion temperature T_f. The latent heat is the energy released per unit mass during solidification at the fusion temperature. Thus, one would expect that, for small values of Ste (for example, for most solidifying metals, Ste < 1), the solidification process is very rapid, while for large values of Ste (for example, under laboratory conditions for water Ste ~ 8) the solidification process is much slower.

Equation (11.16) is a nonlinear transcendental algebraic equation and cannot be solved explicitly for the parameter λ. For a given Ste, however, the solution for λ can be obtained numerically by means of a trial-and-error procedure. A plot of λ versus Ste is given in Fig. 11.4. Knowing λ, the instantaneous location of the interface $s(t)$ is obtained from Eq. (11.12) as

$$s(t) = 2\lambda\sqrt{\alpha t} \tag{11.18}$$

and the temperature distribution $T(x, t)$ in the solid phase is given by Eq. (11.15*b*).

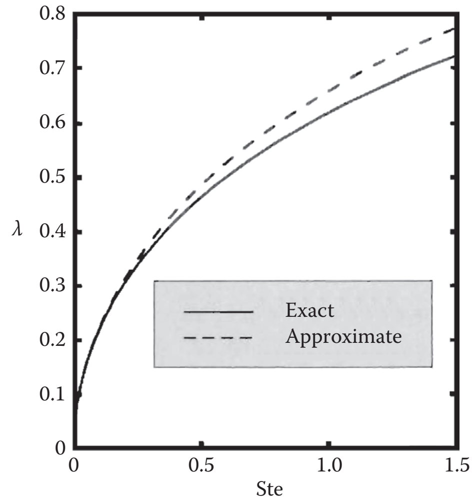

FIGURE 11.4
Exact and approximate values of the parameter λ versus Ste for single-phase Stefan problem.

Once the temperature distribution is known, the heat flux *removed* from the surface at $x = 0$ to maintain its temperature at T_0 is obtained from Fourier's law:

$$q_0'' = k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \frac{k(T_f - T_0)}{\mathrm{erf}\,(\lambda)}\left[\frac{d}{d\eta}\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]_{x=0}$$
$$= \frac{k(T_f - T_0)}{\sqrt{\pi\alpha}\,\mathrm{erf}\,(\lambda)}t^{-1/2} \tag{11.19}$$

Clearly, the surface heat flux decreases as time increases. This is obviously the result of increasing thermal resistance due to the increasing solid phase thickness separating the liquid region from the surface at $x = 0$.

Here we note from Fig. 11.4 that when Ste is small, λ is also small. On the other hand, when λ is small,

$$e^{\lambda^2} = 1 + \lambda^2 + \frac{1}{2!}\lambda^4 + \ldots \cong 1$$

and

$$\mathrm{erf}(\lambda) = \frac{2\lambda}{\sqrt{\pi}}\left(1 - \frac{\lambda^2}{3\times 1!} + \frac{\lambda^4}{5\times 2!} + \ldots\right) \cong \frac{2\lambda}{\sqrt{\pi}}$$

Therefore, from Eq. (11.16), we obtain

$$\lambda \cong \sqrt{\frac{\mathrm{Ste}}{2}} \text{ for small Ste} \tag{11.20}$$

Note that this result is accurate to the first order. Substituting λ from Eq. (11.20) into Eq. (11.18) yields

$$s(t) \cong \sqrt{2\text{Ste}\,\alpha t} \quad \text{for small Ste} \tag{11.21}$$

Thus, this result is consistent with the previous discussion that the case of small Ste corresponds to a slow solidification process. Furthermore, for small values of Ste, the temperature distribution is given by (see Problem 11.3)

$$\frac{T(x,t)-T_0}{T_f - T_0} \cong \frac{x}{s(t)} \quad \text{for small Ste} \tag{11.22}$$

and the heat flux from the surface by

$$q_0'' \cong \frac{k(T_f - T_0)}{s(t)} \quad \text{for small Ste} \tag{11.23}$$

11.3.3 Approximate Solution by the Integral Method

We now obtain an approximate solution by the integral method to the solidification problem solved exactly in the previous section. The formulation of the problem is given by Eqs. (11.9). Upon integrating Eq. (11.9*a*) over x from $x = 0$ to $x = s(t)$ and utilizing Eq. (11.9*d*), we are led to the following energy-integral equation (see Problem 11.5*a*):

$$\frac{d}{dt}\int_0^{s(t)} \left[T(x,t) - T_f\right]dx = \alpha\left[\frac{\rho h_{sl}}{k}\frac{ds}{dt} - \frac{\partial T(0,t)}{\partial x}\right] \tag{11.24}$$

To obtain an approximate solution, we now let $T(x, t)$ be represented by a second-degree polynomial in x in the form

$$T(x,t) = A + B(x-s) + C(x-s)^2 \tag{11.25}$$

where $s = s(t)$. The three conditions needed to determine the coefficients A, B, and C in Eq. (11.25) are given by Eqs. (11.9*b,c,d*). On the other hand, if the condition (11.9*d*) is used in its present form, the resulting temperature profile will involve ds/dt. In turn, the energy-integral equation (11.24) will yield a second-order differential equation for $s(t)$, whereas there is only one initial condition for $s(t)$; that is, $s(0) = 0$. To overcome this difficulty, we form the total derivative of $T(x, t)$ at $x = s(t)$ from the interface condition (11.9*c*) as

$$\left[\frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial t}dt\right]_{x=s(t)} = 0 \tag{11.26a}$$

or

$$\frac{\partial T}{\partial x}\frac{ds}{dt} + \frac{\partial T}{\partial t} = 0 \quad \text{at} \quad x = s(t) \tag{11.26b}$$

Upon eliminating *ds/dt* between Eqs. (11.9*d*) and (11.26*b*), it follows that

$$\left(\frac{\partial T}{\partial x}\right)^2 + \frac{\rho h_{sl}}{k}\frac{\partial T}{\partial t} = 0 \quad \text{at} \quad x = s(t) \tag{11.27}$$

Furthermore, by eliminating $\partial T/\partial t$ between Eqs. (11.9*a*) and (11.27), we get

$$\left(\frac{\partial T}{\partial x}\right)^2 + \frac{h_{sl}}{c}\frac{\partial^2 T}{\partial x^2} = 0 \quad \text{at} \quad x = s(t) \tag{11.28}$$

With this condition, the nonlinearity of the problem is self-evident. The coefficients *A*, *B*, and *C* can now be determined by the use of Eqs. (11.9*b*,*c*) and Eq. (11.28). The results are given by

$$A = T_f \tag{11.29a}$$

$$B = -\frac{h_{sl}}{cs}\left[1 - (1+\kappa)^{1/2}\right] \tag{11.29b}$$

$$C = \frac{Bs - (T_f - T_0)}{s^2} \tag{11.29c}$$

where we have introduced

$$\kappa = \frac{2c(T_f - T_0)}{h_{sl}} = 2\ \text{Ste} \tag{11.30}$$

Substituting the approximate temperature profile (11.25) into the energy-integral equation (11.24) and performing the indicated operations, we finally obtain

$$s\frac{ds}{dt} = 6\alpha\frac{1 - (1+\kappa)^{1/2} + \kappa}{5 + (1+\kappa)^{1/2} + \kappa} \tag{11.31}$$

The solution of Eq. (11.31) with the condition $s(0) = 0$ yields

$$s(t) = 2\lambda\sqrt{\alpha t} \tag{11.32}$$

where

$$\lambda = \left[3\frac{1 - (1+\kappa)^{1/2} + \kappa}{5 + (1+\kappa)^{1/2} + \kappa}\right]^{1/2} \tag{11.33}$$

Here we note that the approximate solution (11.32) for $s(t)$ is of the same form as the exact result (11.18), but the parameter λ is given by Eq. (11.33) for the approximate solution, whereas it is the root of the transcendental equation (11.16) for the exact result.

A graphical comparison of the exact and approximate values of λ as a function of the Stefan number Ste is given in Fig. 11.4. The agreement between the exact and approximate values is reasonably good. In fact, the error is about 7% for Ste = 1.5, the largest value of Ste in Fig. 11.4.

11.4 A Two-Region Phase-Change Problem

We now consider a more general case involving the solidification of a semi-infinite liquid region which is initially at a uniform temperature T_i that is higher than the fusion temperature T_f as indicated in Fig. 11.1*a*. At $t = 0$, the temperature of the liquid surface at $x = 0$ is suddenly lowered to T_0, below T_f, and is maintained at this constant temperature for times $t > 0$. Consequently, the liquid starts to solidify at $x = 0$, and the solid–liquid interface moves in the positive x direction. In this case, we wish to determine the temperature distributions both in the solid and liquid phases and the location of the interface as a function of time. In the following analysis, we assume that the thermophysical properties of each phase remain constant during the solidification process, and that heat transfer takes place only by conduction in both phases.

11.4.1 Formulation

Designating the conditions by subscript s in the solid phase and by l in the liquid phase, the mathematical formulation of this two-phase problem is given by

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}, \quad 0 < x < s(t) \tag{11.34a}$$

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l}\frac{\partial T_l}{\partial t}, \quad s(t) < x < \infty \tag{11.34b}$$

with the conditions at $x = 0$ and as $x \to \infty$,

$$T_s(0,t) = T_0 \quad \text{and} \quad T_l(\infty,t) = T_i \tag{11.34c,d}$$

the interface conditions at $x = s(t)$,

$$T_s(s,t) = T_l(x,t) = T_f \tag{11.34e}$$

$$k_l\left(\frac{\partial T_l}{\partial x}\right)_{x=s} - k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s} + \rho_s h_{sl}\frac{ds}{dt} = 0 \tag{11.34f}$$

and the initial condition $s(0) = 0$, where $s = s(t)$ is the instantaneous location of the interface at time t.

11.4.2 Neumann's Exact Solution

Following the procedure used to solve the single-phase solidification problem exactly in the previous section, we introduce a similarity variable as

$$\eta = \frac{x}{2\sqrt{\alpha_s t}} \tag{11.35}$$

In terms of this similarity variable, Eqs. (11.34*a,b*) reduce to

$$\frac{d^2T_s}{d\eta^2} + 2\eta\frac{dT_s}{d\eta} = 0, \quad 0 < \eta < \lambda \tag{11.36a}$$

$$\frac{d^2T_l}{d\eta^2} + 2\eta\frac{\alpha_s}{\alpha_l}\frac{dT_l}{d\eta} = 0, \quad \lambda < \eta < \infty \tag{11.36b}$$

with the conditions

$$T_s(0) = T_0 \quad \text{and} \quad T_l(\infty) = T_i \tag{11.36c,d}$$

and the interface conditions

$$T_s(\lambda) = T_l(\lambda) = T_f \tag{11.36e}$$

where λ is defined as

$$\lambda = \frac{s(t)}{2\sqrt{\alpha_s t}} \tag{11.37}$$

The solutions of the ordinary differential equations (11.36*a,b*) can be written as

$$T_s(\eta) = A + B\,\text{erf}(\eta) \tag{11.38a}$$

$$T_l(\eta) = C + D\,\text{erfc}\left(\sqrt{\frac{\alpha_s}{\alpha_l}}\eta\right) \tag{11.38b}$$

where erfc(x) is the complementary error function. Application of the conditions (11.36c,d,e) yields

$$A = T_0 \quad \text{and} \quad B = \frac{T_f - T_0}{\text{erf}(\lambda)} \tag{11.39a,b}$$

$$C = T_i \quad \text{and} \quad D = \frac{T_f - T_i}{\operatorname{erfc}\left(\sqrt{\alpha_s/\alpha_l}\,\lambda\right)} \tag{11.39c,d}$$

Here, we note that for B and D to be constants λ must be a constant, independent of time t. Introducing the coefficients A, B, C, and D from Eqs. (11.39) into Eqs. (11.38) we obtain the temperature distributions in the solid and liquid phases as

$$T_s(\eta) = T_0 + (T_f - T_0)\frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)} \tag{11.40a}$$

$$T_l(\eta) = T_i + (T_f - T_i)\frac{\operatorname{erfc}\left(\sqrt{\alpha_s/\alpha_l}\,\eta\right)}{\operatorname{erfc}\left(\sqrt{\alpha_s/\alpha_l}\,\lambda\right)} \tag{11.40b}$$

or, in terms of x and t,

$$\frac{T_s(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left[x/\left(2\sqrt{\alpha_s t}\right)\right]}{\operatorname{erf}(\lambda)} \tag{11.41a}$$

$$\frac{T_l(x,t) - T_i}{T_f - T_i} = \frac{\operatorname{erfc}\left(x/\left(2\sqrt{\alpha_l t}\right)\right)}{\operatorname{erfc}\left(\sqrt{\alpha_s/\alpha_l}\,\lambda\right)} \tag{11.41b}$$

The constant parameter λ appearing in the above results is still unknown. To obtain it, we utilize the condition (11.34f), which yields

$$\frac{\exp(-\lambda^2)}{\operatorname{erf}(\lambda)} + \frac{k_l}{k_s}\sqrt{\frac{\alpha_s}{\alpha_l}}\,\frac{T_f - T_i}{T_f - T_0}\,\frac{\exp[-(\alpha_s/\alpha_l)\lambda^2]}{\operatorname{erfc}\left(\sqrt{\alpha_s/\alpha_l}\lambda\right)} - \sqrt{\pi}\,\frac{\lambda}{\mathrm{Ste}_s} = 0 \tag{11.42}$$

where Ste_s is the Stefan number for the solid region defined as

$$\mathrm{Ste}_s = \frac{c_s(T_f - T_0)}{h_{sl}} \tag{11.43}$$

and c_s is the specific heat of the solid phase. Equation (11.42) is a nonlinear transcendental equation for λ, which can be solved numerically to obtain λ for a given set of values of the temperature T_i and T_0 and properties of the two phases. Knowing λ, the location of the interface $s(t)$ is obtained from Eq. (11.37) as

$$s(t) = 2\lambda\sqrt{\alpha_s t} \tag{11.44}$$

Furthermore, the heat flux *removed* from the surface at $x = 0$ to maintain its temperature at T_0 is obtain from Fourier's law,

$$q_0'' = k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=0} = \frac{k_s (T_f - T_0)}{\sqrt{\pi \alpha_s} \operatorname{erf}(\lambda)} t^{-1/2} \tag{11.45}$$

As in the case of Stefan's solution, an analytical expression for λ can also be obtained for Neumann's solution for small values of λ (i.e., for slow freezing processes) from the transcendental equation (11.42) as (see Problem 11.9)

$$\lambda \cong \left[1 + \frac{k_l}{k_s} \sqrt{\frac{\alpha_s}{\alpha_l}} \frac{T_f - T_i}{T_f - T_0} \right]^{1/2} \sqrt{\frac{\mathrm{Ste}_s}{2}} \tag{11.46}$$

We note that the results for λ given by Eqs. (11.42) and (11.46) reduce to Eqs. (11.16) and (11.20), respectively, for the case of Stefan's solution by setting $T_f = T_i$.

11.5 Solidification Due to a Line Heat Sink in a Large Medium

Consider a large body of liquid initially at a uniform temperature T_i which is higher than the fusion temperature T_f. A line heat sink, of strength per unit length q'_{ln} (W/m) and located at $r = 0$ as shown in Fig. 11.5, is suddenly activated at $t = 0$ to remove heat continuously for times $t > 0$. Consequently, the liquid starts to solidify at $r = 0$ and the cylindrical solid–liquid interface moves in the r direction. As in the cases considered in the previous sections, we again assume constant thermophysical properties in each phase and neglect the effect of any convective motion in the liquid phase. Thus, heat transfer takes place only by conduction in both phases. We also note that the problem has cylindrical symmetry. The mathematical formulation of the two-region problem for the temperature

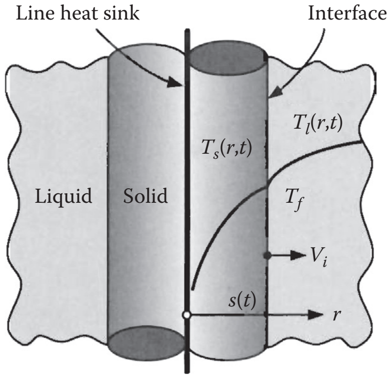

FIGURE 11.5
Solidification due to a line heat sink in a large medium.

distributions $T_s(r, t)$ and $T_l(r, t)$ in the solid and liquid phases, respectively, and the location $s(t)$ of the solid liquid interface as a function of time is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right)=\frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}, \quad 0<r<s(t) \tag{11.47a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_l}{\partial r}\right)=\frac{1}{\alpha_l}\frac{\partial T_l}{\partial t}, \quad s(t)<r<\infty \tag{11.47b}$$

with the conditions at $r = 0$ and as $r \to \infty$,

$$\lim_{r\to 0}\left[2\pi r k_s \frac{\partial T_s}{\partial r}\right]=q'_{ln} \quad \text{and} \quad T_l(\infty,t)=T_i \tag{11.47c,d}$$

the interface conditions at $r = s(t)$,

$$T_s(s,t)=T_l(s,t)=T_f \tag{11.47e}$$

$$k_l\left(\frac{\partial T_l}{\partial x}\right)_{x=s}-k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s}+\rho_s h_{sl}\frac{ds}{dt}=0 \tag{11.47f}$$

and the initial condition $s(0) = 0$.

In terms of a similarity variable defined as

$$\eta=\frac{r^2}{4\alpha_s t} \tag{11.48}$$

Eqs. (11.47*a,b*) can be rewritten as

$$\frac{d^2T_s}{d\eta^2}+\left(1+\frac{1}{\eta}\right)\frac{dT_s}{d\eta}=0 \tag{11.49a}$$

$$\frac{d^2T_l}{d\eta^2}+\left(\frac{\alpha_s}{\alpha_l}+\frac{1}{\eta}\right)\frac{dT_l}{d\eta}=0 \tag{11.49b}$$

with the conditions

$$\lim_{\eta\to 0}\left[4\pi k_s \eta \frac{dT_s}{d\eta}\right]=q'_{ln} \quad \text{and} \quad T_l(\infty)=T_i \tag{11.49c,d}$$

and the interface conditions

$$T_s(\lambda)=T_l(\lambda)=T_f \tag{11.49e}$$

where the parameter λ is defined as

$$\lambda = \frac{s^2(t)}{4\alpha_s t} \tag{11.50}$$

The solutions of ordinary differential equations (11.49*a,b*), which satisfy the interface condition (11.49*e*), can be written as

$$T_s(\eta) = T_f + A\left[\mathrm{Ei}(\lambda) - \mathrm{Ei}(\eta)\right], \quad 0 < \eta < \lambda \tag{11.51a}$$

$$T_l(\eta) = T_f + B\left[\mathrm{Ei}\left(\frac{\alpha_s}{\alpha_l}\lambda\right) - \mathrm{Ei}\left(\frac{\alpha_s}{\alpha_l}\eta\right)\right], \quad \lambda < \eta < \infty \tag{11.51b}$$

where Ei (x) represents the *exponential integral function,* which is defined by (see Appendix E)

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-\eta}}{\eta} d\eta \tag{11.52}$$

Imposing the conditions (11.49*c,d*) we obtain

$$A = \frac{q'_{ln}}{4\pi k_s} \quad \text{and} \quad B = \frac{T_i - T_f}{\mathrm{Ei}\left(\frac{\alpha_s}{\alpha_l}\lambda\right)} \tag{11.53a,b}$$

Here, again, we note that for B to be a constant, λ must also be a constant, independent of time t. Substituting the coefficients A and B into Eqs. (11.51*a,b*) yields

$$T_s(\eta) = T_f + \frac{q'_{ln}}{4\pi k_s}\left[\mathrm{Ei}(\lambda) - \mathrm{Ei}(\eta)\right], \quad 0 < \eta < \lambda \tag{11.54a}$$

$$T_l(\eta) = T_i - (T_i - T_f)\frac{\mathrm{Ei}\left(\frac{\alpha_s}{\alpha_l}\eta\right)}{\mathrm{Ei}\left(\frac{\alpha_s}{\alpha_l}\lambda\right)}, \quad \lambda < \eta < \infty \tag{11.54b}$$

or, in terms of the physical variables r and t,

$$\frac{T_s(x,t) - T_f}{q'_{ln}/(4\pi k_s)} = \mathrm{Ei}(\lambda) - \mathrm{Ei}\left(-\frac{r^2}{4\alpha_s t}\right), \quad 0 < r < s(t) \tag{11.55a}$$

$$\frac{T_l(x,t)-T_i}{T_f-T_i}=\frac{\mathrm{Ei}\left(-\dfrac{r^2}{4\alpha_l t}\right)}{\mathrm{Ei}\left(\dfrac{\alpha_s}{\alpha_l}\lambda\right)},\quad s(t)<r<\infty \tag{11.55b}$$

In these expressions, the parameter λ is still unknown. To obtain it, we substitute Eqs. (11.55*a*,*b*) into the interface energy-balance equation (11.47*f*), which yields

$$\frac{q'_{ln}}{4\pi}e^{-\lambda}-k_l(T_i-T_f)\frac{\exp\left(-\dfrac{\alpha_s}{\alpha_l}\lambda\right)}{\mathrm{Ei}\left(\dfrac{\alpha_s}{\alpha_l}\lambda\right)}-\alpha_s\rho_s h_{sl}\lambda=0 \tag{11.56}$$

The parameter λ is the root of this nonlinear transcendental algebraic equation, which can be solved numerically to yield λ for a given set of q'_{ln}, T_i, T_f and the thermophysical properties of both phases. Once λ is known, Eqs. (11.55) give the temperature distributions in the solid and liquid phases, and the location of the interface is obtained from

$$s(t)=2\sqrt{\lambda\alpha_s t} \tag{11.57}$$

11.6 Solidification Due to a Point Heat Sink an a Large Medium

Consider now solidification due to a point heat sink of strength $q_{pt}(t)$ (W) in a large body of liquid initially at a uniform temperature T_i higher than the fusion temperature T_f. The point heat sink, located at $r = 0$, is suddenly activated at time $t = 0$ to remove heat for times $t > 0$. Consequently, the liquid starts to solidify at $r = 0$ and the spherical solid–liquid interface expands in the r direction as illustrated in Fig. 11.6. Assuming constant thermophysical

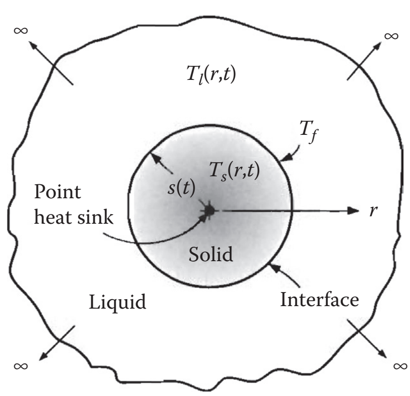

FIGURE 11.6
Solidification due to a point sink in a large medium.

properties in each phase and neglecting convective effects in the liquid phase, the mathematical formulation of the two-region problem for the temperature distributions $T_s(r, t)$ and $T_l(r, t)$ in the solid and liquid phases, respectively, is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_s}{\partial r}\right)=\frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}, \quad 0<r<s(t) \tag{11.58a}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_l}{\partial r}\right)=\frac{1}{\alpha_l}\frac{\partial T_l}{\partial t}, \quad s(t)<r<\infty \tag{11.58b}$$

with the conditions at $r = 0$ and as $r \to \infty$,

$$\lim_{r\to 0}\left[4\pi r^2 k_s\frac{\partial T_s}{\partial r}\right]=q_{pt}(t) \quad \text{and} \quad T_l(\infty,t)=T_i \tag{11.58c,d}$$

the interface conditions at $r = s(t)$,

$$T_s(s,t)=T_l(s,t)=T_f \tag{11.58e}$$

$$k_l\left(\frac{\partial T_l}{\partial r}\right)_{r=s}-k_s\left(\frac{\partial T_s}{\partial x}\right)_{r=s}+\rho_s h_{sl}\frac{ds}{dt}=0 \tag{11.58f}$$

and the initial condition $s(0) = 0$.

11.6.1 Similarity Solution for the Case of $q_{pt}(t) = Q_0 t^{1/2}$

Introduce a similarity variable as

$$\eta=\frac{r^2}{4\alpha_s t} \tag{11.59}$$

In terms of this similarity variable the formulation of the problem is given by

$$\frac{d^2T_s}{d\eta^2}+\left(1+\frac{3}{2\eta}\right)\frac{dT_s}{d\eta}=0 \tag{11.60a}$$

$$\frac{d^2T_l}{d\eta^2}+\left(\frac{\alpha_s}{\alpha_l}+\frac{3}{2\eta}\right)\frac{dT_l}{d\eta}=0 \tag{11.60b}$$

with the conditions at $\eta = 0$ and as $\eta \to \infty$,

$$\lim_{\eta\to 0}\left[16\pi k_s\sqrt{\alpha_s}\,\eta^{3/2}\frac{dT_s}{d\eta}\right]=Q_0 \quad \text{and} \quad T_l(\infty)=T_i \tag{11.60c,d}$$

and interface conditions

$$T_s(\lambda) = T_l(\lambda) = T_f \tag{11.60e}$$

where the parameter λ is defined as

$$\lambda = \frac{s^2(t)}{4\alpha_s t} \tag{11.61}$$

Here we note that the transformed condition at $\eta = 0$, Eq. (11.60*c*), does not explicitly contain the time variable t. That is why the form $Q_0 t^{1/2}$, where Q_0 is a constant, was necessary for the similarity transformation to be possible.

The solutions of the ordinary differential equations (11.60*a,b*), which satisfy the interface condition (11.60*e*), can be written as

$$T_s(\eta) = T_f + \frac{A}{2}\int_\eta^\lambda \frac{e^{-\eta^2}}{\eta^{3/2}}\,d\eta \tag{11.62a}$$

$$T_l(\eta) = T_f + B\int_\lambda^\eta \frac{e^{-(\alpha_s/\alpha_l)\eta}}{\eta^{3/2}}\,d\eta \tag{11.62b}$$

These solutions can also be rewritten as

$$T_s(\eta) = T_f + A\left\{\frac{e^{-\lambda}}{\sqrt{\lambda}} - \frac{e^{-\eta}}{\sqrt{\eta}} + \sqrt{\pi}\left[\operatorname{erf}\left(\sqrt{\lambda}\right) - \operatorname{erf}\left(\sqrt{\eta}\right)\right]\right\} \tag{11.63a}$$

$$T_l(\eta) = T_f + B\left\{\frac{\exp\left(-\dfrac{\alpha_s}{\alpha_l}\lambda\right)}{\sqrt{\lambda\alpha_s/\alpha_l}} - \frac{\exp\left(-\dfrac{\alpha_s}{\alpha_l}\eta\right)}{\sqrt{\eta\alpha_s/\alpha_l}} + \sqrt{\pi}\left[\operatorname{erfc}\sqrt{\frac{\alpha_s}{\alpha_l}\eta} - \operatorname{erfc}\sqrt{\frac{\alpha_s}{\alpha_l}\lambda}\right]\right\} \tag{11.63b}$$

Applying the boundary conditions (11.60*c,d*) yields

$$A = -\frac{Q_0}{8\pi k_s\sqrt{\alpha_s}} \tag{11.64a}$$

and

$$B = (T_i - T_f)\left\{\frac{\exp\left(-\dfrac{\alpha_s}{\alpha_l}\lambda\right)}{\sqrt{\lambda\alpha_s/\alpha_l}} - \sqrt{\pi}\operatorname{erfc}\sqrt{\frac{\alpha_s}{\alpha_l}\lambda}\right\}^{-1} \tag{11.64b}$$

We now note that, for B to be a constant, λ must also be a constant, independent of time t. Finally, substituting A and B into Eqs. (11.63*a*) and (11.63*b*), we get the temperature distributions $T_s(\eta)$ and $T_l(\eta)$. In these expressions, however, the parameter λ is still unknown. It is obtained by substituting the solutions for $T_s(\eta)$ and $T_l(\eta)$ into the interface energy-balance equation (11.58*f*), which yields

$$\frac{k_l(T_i - T_f)}{2\lambda\left[1-\sqrt{\pi\lambda\alpha_s/\alpha_l}\,e^{(\alpha_s/\alpha_l)\lambda}\mathrm{erfc}\sqrt{\lambda\alpha_s/\alpha_l}\right]} - \frac{Q_0}{16\pi\sqrt{\alpha_s}}\frac{e^{-\lambda}}{\lambda^{3/2}} + \alpha_s\rho_s h_{sl} = 0 \tag{11.65}$$

The parameter λ is the root of this nonlinear transcendental algebraic equation, which can be solved numerically to yield λ for a given set of values of the thermophysical properties of both phases, Q_0, T_i, and T_f. Once λ is known, Eqs. (11.63) give the temperature distributions in the solid and liquid phases, and the location of the solid–liquid interface is determined as a function of t from

$$s(t) = 2\sqrt{\lambda\alpha_s t} \tag{11.66}$$

11.7 Solutions by the Quasi-Steady Approximation

The collection of phase-change problems for which exact analytical solutions can be found is extremely small and does not include processes of relevance to most realistic situations. As indicated in the previous sections, this is mainly due to the fact that phase-change problems are inherently nonlinear. Exact solutions mostly exist only for semi-infinite problems with constant thermophysical properties in each phase and constant initial and imposed temperatures. Very few exact analytical solutions are available in cylindrical and spherical geometries and none for finite domains and higher dimensions. Accordingly, for most realistic problems one is forced to seek approximate solutions.

The *quasi-steady approximation* yields easily computable results for the estimates of melting and freezing processes in cases where heat transfer takes place only in one phase. The basic physical assumption underlying the method is that the effects of sensible heat are negligible relative to those of latent heat. Since the Stefan number Ste is defined as the ratio of sensible to latent heat, this amounts to Ste ≈ 0. This is a significant simplification, because the heat conduction equation in the phase where heat transfer takes place becomes independent of time, while allowing the phase-change front to vary in time. Clearly, it is then impossible to meet the initial conditions.

Since the effects of sensible heat are ignored relative to those of latent heat, all of the heat released in solidification or absorbed in melting conditions must be used to derive the phase-change front. It is, therefore, to be expected that this approximation will overestimate the actual interface location. We emphasize, however, that these are just approximations, without full information on the effect of specific problem conditions on the magnitude of the error incurred when using them. Unfortunately, no precise error estimates are available, and thus it is necessary to examine the physical viability of the results when using the quasi-steady approximation.

11.7.1 Melting of a Slab with Prescribed Surface Temperatures

Consider a solid slab, $0 \le x \le L$, initially at the fusion temperature T_f. For times $t \ge 0$, the surface at $x = 0$ is imposed a time-dependent temperature $T_0(t) > T_f$. Consequently, a liquid–solid interface forms at $x = 0$ and moves toward the surface at $x = L$. Assuming constant thermophysical properties, the formulation of the problem for the temperature distribution $T(x, t)$ in the liquid phase is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < s(t) \tag{11.67a}$$

$$T(0,t) = T_0(t) \quad \text{and} \quad T(s,t) = T_f \tag{11.67b,c}$$

Furthermore, at the interface we also have

$$k\left(\frac{\partial T}{\partial x}\right)_{x=s} + \rho\lambda_{sl}\frac{ds}{dt} = 0 \tag{11.67d}$$

with the initial condition $s(0) = 0$, where $s = s(t)$ represents the instantaneous location of the interface at time t. In Eqs. (11.67), α, k, and ρ are the thermal diffusivity, thermal conductivity, and density of the liquid phase, respectively. For simplicity, we have dropped the subscript l.

We note that this single-phase problem admits the Stefan's solution only when T_0 is constant and when $s(t) < L$. In the case under consideration, however, the imposed surface temperature is time dependent. We will, therefore, seek to determine the location of the interface and the temperature distribution in the liquid phase by implementing the quasi-steady approximation.

We now rewrite the formulation of the problem in dimensionless form as follows:

$$\frac{\partial^2 \theta}{\partial \xi^2} = \mathrm{Ste}\frac{\partial \theta}{\partial \tau}, \quad 0 < \xi < \xi_i(\tau) \tag{11.68a}$$

$$\theta(0,\tau) = \theta_0(\tau) \quad \text{and} \quad \theta(\xi_i,\tau) = 0 \tag{11.68b,c}$$

with the interface condition

$$\left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=\xi_i} + \frac{d\xi_i}{d\tau} = 0 \tag{11.68d}$$

and the initial condition $\xi(0) = 0$. Here, we have defined

$$\theta(\xi,\tau) = \frac{T(\xi,\tau) - T_f}{\Delta T}, \quad \theta_0(\tau) = \frac{T_0(\tau) - T_f}{\Delta T} \quad \text{with} \quad \Delta T - max\{T_0(\tau) - T_f\}$$

$$\xi = \frac{x}{\bar{x}}, \quad \xi_i(\tau) = \frac{s(\tau)}{\bar{x}}, \quad \text{and} \quad \tau = \mathrm{Ste}\,\mathrm{Fo}$$

with

$$\text{Ste} = \frac{c\Delta T}{h_{sl}} \quad \text{and} \quad \text{Fo} = \frac{\alpha t}{\bar{x}^2} = \text{Fourier number}$$

and $\bar{x}$ represents an arbitrary characteristic length (the problem under consideration has no natural length scale), and c is the specific heat of the liquid phase.

The quasi-steady approximation (i.e., Ste → 0), replaces the unsteady heat conduction equation (11.68*a*) simply by

$$\frac{\partial^2\theta}{\partial\xi^2} = 0, \quad 0 < \xi < \xi_i(\tau) \tag{11.69}$$

Integrating this equation twice over ξ and then imposing the boundary conditions (11.68*b,c*), we obtain

$$\theta(\xi,\tau) = \theta_0(\tau)\left[1 - \frac{\xi}{\xi_i(\tau)}\right], \quad 0 < \xi < \xi_i(\tau) \tag{11.70}$$

Substituting this result into the interface condition (11.68*d*) yields

$$\frac{d\xi_i}{d\tau} = \frac{\theta_0(\tau)}{\xi_i(\tau)} \quad \text{or} \quad \frac{d\xi_i^2}{d\tau} = 2\theta_0(\tau) \tag{11.71}$$

Integrating Eq. (11.71) from $\tau = 0$ to τ, we get

$$\xi_i(\tau) = \left\{2\int_0^\tau \theta_0(\tau')\,d\tau'\right\}^{1/2} \tag{11.72}$$

which is the quasi-steady solution for the dimensionless interface location. In terms of the original physical variables, the quasi-steady solution for $s(t)$ becomes

$$s(t) = \left\{2\frac{k}{\rho h_{sl}}\int_0^t \left[T_0(t') - T_f\right]dt'\right\}^{1/2} \tag{11.73}$$

and the temperature distribution $T(x, t)$, from Eq. (11.70), is given by

$$T(x,t) = T_0(t) - \left[T_0(t) - T_f\right]\frac{x}{s(t)}, \quad 0 \le x \le s(t), \quad t \ge 0 \tag{11.74}$$

Thus, at any time $t > 0$, the temperature profile $T(x, t)$ is linear across the thickness of the liquid phase.

The heat flux across the liquid phase is obtained from Fourier's law:

$$q''(t) = -k\frac{\partial T}{\partial x} = k\frac{T_0(t) - T_f}{s(t)} \tag{11.75}$$

As a special case, if $T_0(t) = T_0$ = constant, then from Eq. (11.73), we obtain

$$s(t) = \sqrt{\frac{2k(T_0 - T_f)t}{\rho h_{sl}}} = \sqrt{2\,\mathrm{Ste}\,\alpha t} \tag{11.76}$$

Substituting this result into Eqs. (11.74) and (11.75) gives

$$\frac{T(x,t) - T_0}{T_0 - T_f} = \frac{x}{\sqrt{2\,\mathrm{Ste}\,\alpha t}} \tag{11.77}$$

and

$$q'' = k\frac{T_0 - T_f}{\sqrt{2\,\mathrm{Ste}\,\alpha t}} \tag{11.78}$$

These are precisely the same expressions one would obtain from Stefan's exact solution when Ste ≅ 0 (see Section 11.3 for the solidification case).

Furthermore, the time t_{melt} needed for the entire slab to melt is obtained by setting $s(t) = L$ in Eq. (11.76) and solving for t_{melt},

$$t_{melt} = \frac{L^2}{2\alpha\,\mathrm{Ste}} \tag{11.79}$$

Here, we also note that the non-zero root λ of the transcendental equation (11.16) can be shown to be always smaller than $\sqrt{\mathrm{Ste}/2}$ – see Problem (11.11). This, by Eq. (11.76), implies that the quasi-steady approximation for the front $s(t)$ overestimates the exact (Stefan) solution given by Eq. (11.18).

11.7.2 Melting of a Slab with Imposed Surface Heat Flux

Consider the same slab as in the previous section and assume that it is being melted by an imposed heat flux $q_0''(t)$ at $x = 0$. The formulation of the problem for the temperature distribution $T(x, t)$ in the liquid phase is the same as in the previous section, except the boundary condition at $x = 0$ is replaced by

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = q_0''(t) \tag{11.80}$$

There is no exact solution (even with constant q_0'') for this problem. On the other hand, an approximate solution can easily be obtained by the use of the *quasi-steady approximation*, which reduces the unsteady heat conduction equation (11.67*a*) to

$$\frac{\partial^2 T}{\partial x^2} = 0, \quad 0 < x < s(t) \tag{11.81}$$

The solution of Eq. (11.81) is given by

$$T(x,t) = A(t)x + B(t) \tag{11.82}$$

Imposing the boundary conditions (11.67*c*) and (11.80) gives

$$A(t) = -\frac{q_0''(t)}{k} \quad \text{and} \quad B(t) = T_f + \frac{s(t)}{k} q_0''(t) \tag{11.83a,b}$$

Substituting $A(t)$ and $B(t)$ into Eq. (11.82) we get

$$T(x,t) = T_f + \frac{q_0''(t)}{k}[s(t) - x], \quad 0 < x < s(t), \quad t \ge 0 \tag{11.84}$$

where $s(t)$ is determined by substituting this result into the condition (11.67*d*), which yields

$$-q_0''(t) + \rho h_{sl} \frac{ds}{dt} = 0 \tag{11.85}$$

The solution of this equation which, with the use of the initial condition $s(0) = 0$, gives

$$s(t) = \frac{1}{\rho h_{sl}} \int_0^t q_0''(t')dt' \tag{11.86a}$$

As a special case, if $q_0''(t) = q_0''$ = constant, then Eq. (11.85) reduces to

$$s(t) = \frac{q_0''}{\rho h_{sl}} t \tag{11.86b}$$

and Eq. (11.84) becomes

$$T(x,t) = T_f + \frac{q_0''}{k}\left[\frac{q_0''}{\rho h_{sl}} t - x\right] \tag{11.87}$$

which is linear in both x and t.

11.7.3 Melting of a Slab with Convection

We now consider the case where the slab of the previous sections is heated convectively for times $t \geq 0$ by exposing the surface at $x = 0$ to a fluid maintained at temperature $T_\infty(t) > T_f$ with a constant heat transfer coefficient h. In this case, the mathematical formulation for the temperature distribution $T(x, t)$ in the liquid phase is given by Eqs. (11.67), except that the condition (11.67b) is replaced by

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = h[T_\infty(t) - T(0,t)], \quad t \geq 0 \tag{11.88}$$

The quasi-steady approximate solution of the heat conducting equation (11.67*a*) that satisfies the conditions (11.67*c*) and (11.88) is given by

$$\frac{T(x,t) - T_f}{T_\infty(t) - T_f} = \frac{hs(t)}{hs(t) + k} - \frac{hx}{hs(t) + k}, \quad 0 \leq x \leq s(t) \tag{11.89}$$

Substituting $T(x, t)$ from Eq. (11.89) into Eq. (11.67*d*) and integrating the resulting expression from $t = 0$ to t, we obtain

$$s^2(t) + 2\frac{k}{h}s(t) = \frac{2k}{\rho h_{sl}}\int_0^t \left[T_\infty(t') - T_f\right]dt' \tag{11.90}$$

which is a quadric equation in $s(t)$. The physically meaningful root of this equation gives

$$s(t) = -\frac{k}{h} + \left\{\left(\frac{k}{h}\right)^2 + \frac{2k}{\rho h_{sl}}\int_0^t \left[T_\infty(t') - T_f\right]dt'\right\}^{1/2} \tag{11.91}$$

11.7.4 Outward Melting of a Hollow Cylinder

Consider a long hollow solid circular cylinder, $r_i \leq r \leq r_o$, which is initially at its fusion temperature T_f. The inner surface at $r = r_i$ is heated uniformly by maintaining its temperature at $T_0(t) > T_f$ for times $t \geq 0$. Assuming constant thermophysical properties, the formulation of the problem for the temperature distribution $T(r, t)$ in the liquid phase by the quasi-steady approximation is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0, \quad r_i < r < s(t) \tag{11.92a}$$

$$T(r_i, t) = T_0(t) \quad \text{and} \quad T(s,t) = T_f \tag{11.92b,c}$$

with the interface condition

$$k\left(\frac{\partial T}{\partial r}\right)_{r=s} + \rho h_{sl}\frac{ds}{dt} = 0 \tag{11.92d}$$

and the initial condition $s(0) = 0$.

The solution of Eq. (11.92*a*), which satisfies the boundary conditions (11.92*b,c*) is given by

$$T(r,t) = T_f + \left[T_0(t) - T_f\right] \frac{\ln \dfrac{r}{s(t)}}{\ln \dfrac{r_i}{s(t)}}, \quad r_i \le r \le s(t) \tag{11.93}$$

Substituting Eq. (11.93) into Eq. (11.92*d*) and integrating the resulting equation from t = 0 to t, we obtain

$$2s^2 \ln \frac{s}{r_i} = s^2 - r_i^2 + \frac{4k}{\rho h_{sl}} \int_0^t \left[T(t') - T_f\right] dt' \tag{11.94}$$

which is a transcendental equation for the determination of the instantaneous position of the phase front, $s = s(t)$.

The melt-time, t_{melt}, of the entire cylinder is found by setting $s(t) = r_o$ in Eq. (11.94). In particular, for the case $T_0(t) = T_0$ = constant, we get

$$t_{melt} = \frac{\rho h_{sl}}{4k(T_0 - T_f)} \left(r_i^2 - r_o^2 + 2r_o^2 \ln \frac{r_o}{r_i} \right) \tag{11.95}$$

If the melting of the same hollow cylinder is due to an imposed heat flux $q_0''(t)$ at $r = r_i$, then the boundary condition (11.92*b*) is replaced by

$$-k\left(\frac{\partial T}{\partial r}\right)_{r=r_i} = \frac{q_0''(t)}{k}, \quad t > 0 \tag{11.96}$$

The solution of Eq. (11.92*a*), which satisfies the conditions (11.92*c*) and (11.96), is given by

$$T(r,t) = T_f - \frac{r_i q_0''(t)}{k} \ln \frac{r}{s(t)}, \quad r_i < r < s(t) \tag{11.97}$$

where

$$s(t) = \left\{ r_i^2 + 2\frac{r_i}{\rho h_{sl}} \int_0^t q_0''(t')\, dt' \right\}^{1/2}, \quad t > 0 \tag{11.98}$$

which is obtained by substituting Eq. (11.97) into (11.92*d*).

In particular, if $q_0''(t) = q_0''$ = constant, Eq. (11.98) reduces to

$$s(t) = \left\{ r_i^2 + 2\frac{r_i q_0''}{\rho h_{sl}} t \right\}^{1/2} \tag{11.99}$$

and the melt-time is given by

$$t_{melt} = \frac{\rho h_{sl}}{2 r_i q_0''}\left(r_0^2 - r_i^2\right) \tag{11.100}$$

If the melting of the same cylinder is due to convective heating at $r = r_i$ from a fluid at temperature $T_\infty(t) > T_f$ with a constant heat transfer coefficient h, then the condition at $r = r_i$ is given by

$$-k\left(\frac{\partial T}{\partial r}\right)_{r=r_i} = h\left[T_\infty(t) - T(r_i, t)\right] \tag{11.101}$$

The solution for the temperature distribution $T(r, t)$ in the liquid phase then becomes

$$T(r,t) = T_f + \left[T_\infty(t) - T_f\right]\frac{\ln\dfrac{r}{s(t)}}{\ln\dfrac{r_i}{s(r)} - \dfrac{k}{hr_1}}, \quad r_i < r < s(t) \tag{11.102}$$

where $s = s(t)$ is obtained from the solution of the transcendental equation

$$2s^2 \ln\frac{s}{r_i} = \left(1 - \frac{2k}{hr_i}\right)\left(s^2 - r_i^2\right) + \frac{4k}{\rho h_{sl}}\int_0^t \left[T_\infty(t') - T_f\right]dt' \tag{11.103}$$

Here, we note that, as $h \to \infty$, Eqs. (11.102) and (11.103) reduce, as they should, to Eqs. (11.93) and (11.94) with $T_\infty(t)$ replaced by $T_0(t)$.

11.7.5 Inward Melting of a Solid Sphere

Consider a solid sphere, $0 \le r \le r_0$, which is initially at its fusion temperature T_f. The surface of the sphere is heated uniformly by maintaining its temperature at $T_0(t)$ ($> T_f$) for times $t \ge 0$. Consequently, a spherically symmetric phase front, $r = s(t)$, moves inwards from $r = r_0$ with the solid phase in $0 \le r < s(t)$ and the liquid phase in $s(t) < r \le r_0$. Assuming constant thermophysical properties, the formulation of the problem for the temperature distribution $T(r, t)$ in the liquid phase, by the quasi-steady approximation, is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = 0, \quad s(t) < r < r_0 \tag{11.104a}$$

$$T(r_0, t) = T_0(t) \quad \text{and} \quad T(s, t) = T_f \tag{11.104b,c}$$

with the interface condition

$$k\left(\frac{\partial T}{\partial r}\right)_{r=s} + \rho h_{sl}\frac{ds}{dt} = 0 \tag{11.104d}$$

with the initial conditions $s(0) = r_0$.

The solution of the heat conduction equation (11.104*a*), which satisfies the conditions (11.104*b,c*), is given by

$$T(r,t) = T_f + \left[T_0(t) - T_f\right] \frac{1 - \dfrac{s(t)}{r}}{1 - \dfrac{s(t)}{r_0}}, \quad s(t) \le r_0 \tag{11.105}$$

Substituting this result into Eq. (11.104*d*) and integrating the resulting expression from $t = 0$ to t, we obtain

$$2\left(\frac{s}{r_0}\right)^3 - 3\left(\frac{s}{r_0}\right)^2 + 1 = \frac{6k}{\rho h_{sl} r_0^2} \int_0^t \left[T_0(t') - T_f\right] dt' \tag{11.106}$$

for the determination of the position of the phase front $s = s(t)$.

11.8 Solidification of Binary Alloys

So far, we have considered the solidification or melting of pure substances such as water or pure metals, and eutectic alloys. In such cases, the phase change abruptly takes place at the fusion temperature across a sharp moving interface between the two phases. As has been demonstrated by various representative examples in the previous sections, such situations are relatively easier to analyze. In some materials, however, the phase change occurs gradually from one phase to the other, where the transition takes place in a manner that is markedly different from the description above. Gradual phase change is most distinctly observed in binary alloys. The solid and liquid phases are often separated by a moving zone of a two-phase mixture of finite extend, termed the *mushy zone*, which consists of an intricate mixture of liquid and solid phases.

The presence of a two-phase mushy zone obviously complicates the heat transfer analysis. Furthermore, crystals of the solid phase are formed at some preferred locations in the liquid and as freezing progresses the crystals grow in the form of intricately shaped fingers, called *dendrites*. This complicates the geometry significantly and makes mathematical modeling of such processes very difficult. An introduction to such problems and a review of the literature can be found in the article by Hayashi and Kunimine [26] and in the text by Poulikakos [27].

11.8.1 Equilibrium-Phase Diagram

The equilibrium-phase diagram for a binary mixture (or alloy) is of central importance to the phase change process. Consider a typical equilibrium-phase diagram composed of two components, *A* and *B*, shown in Fig. 11.7. In the phase diagram, *C* is the concentration (weight percent) of component *B* in the mixture, *l* denotes the liquid, *s* the solid, S_A a solid with a lattice structure of component *A* in its solid phase but containing some solid molecules of component *B* in that lattice, and S_B a solid with a lattice structure of component

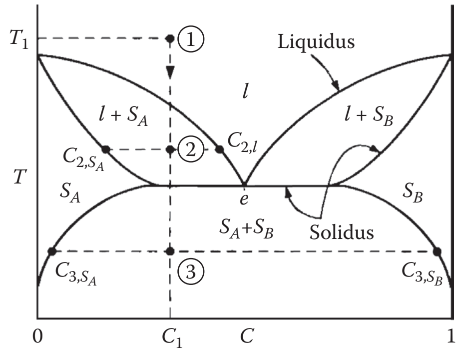

FIGURE 11.7
Equilibrium-phase diagram of a binary mixture.

B in its solid phase but containing some solid molecules of component A in that lattice. The boundary labelled "liquidus" separates the region of the liquid phase from the mushy zone. The "solidus" boundary separates the region of the solid phase from the mushy zone or from another solid phase.

Given a mixture of concentration C and uniform temperature T, the coordinates C and T define a point on the phase diagram. The zone in which this point is located determines the phase present in the system at equilibrium. As an illustration, consider a liquid mixture with concentration C_1 and temperature T_1, which is cooled while maintaining the concentration constant as indicated by the vertical dashed line in Fig. 11.7. As soon as the temperature drops below the liquidus line, solidification starts, creating a mushy zone consisting of the liquid phase and solid S_A. At point 2, for example, the solid phase (S_A) contains a concentration C_{2,S_A} of component B and the liquid phase contains a concentration $C_{2,l}$ of component B. Further cooling to below the solidus line, say to point 3, results in a solid mixture of S_A and S_B, containing concentrations C_{3,S_A} and C_{3,S_B} of component B, respectively.

At the *eutectic point* (e) a unique situation occurs: upon constant-concentration cooling, the liquid forms the solid mixture $S_A + S_B$ without the formation of a mushy zone. The resulting solid mixture (or alloy) is called *eutectic*.

11.8.2 Solidification of a Binary Alloy

We now consider, as an illustration, a large body of a liquid binary alloy of initial concentration C_i, smaller than the eutectic concentration C_e. This alloy, which is contained in a semi-infinite region as shown in Fig. 11.8*a*, is initially at a temperature T_i, higher than the liquidus temperature T_{lq} corresponding to the initial concentration, C_i. A simplified equilibrium-phase diagram of the alloy under consideration is given in Fig. 11.8*b*. At $t = 0$, the temperature of the liquid surface at $x = 0$ is suddenly cooled down to T_w, significantly lower than the eutectic temperature T_e, and is maintained at T_w for times $t > 0$. Consequently, the liquid alloy starts to solidify at $x = 0$, and the solid alloy grows in the positive x direction. The solid and the liquid regions are separated by a mushy zone. We wish to determine the temperature distributions in the solid, liquid and the mixed-phase mushy zone, as well as the locations of the solid–mush and mush–liquid interfaces as functions of t.

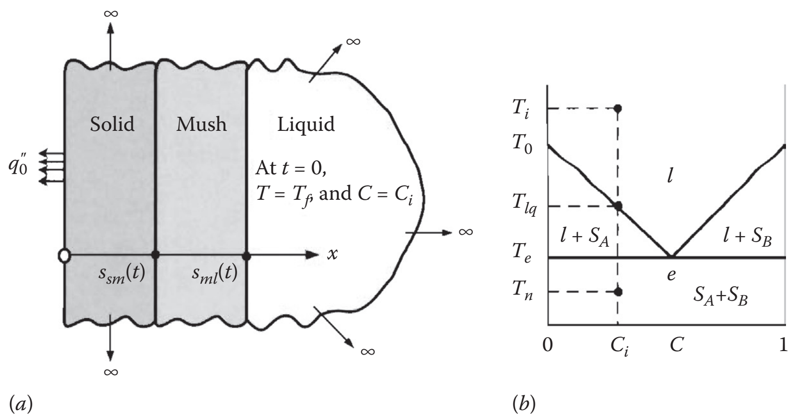

FIGURE 11.8
(*a*) Solidification of a binary alloy, and (*b*) simplified phase-equilibrium diagram.

In the following analysis, we assume constant thermophysical properties in each region with no significant variation in the densities of the liquid and solid phases. In addition, we also neglect the effects of any liquid phase motion and mass diffusion. Designating the conditions by subscripts *s*, *m*, and *l*, respectively, in the solid, mushy, and liquid regions, the formulation of the problem for the temperature distributions in the three regions is given by [27]

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t}, \quad 0 < x < s_{sm}(t) \tag{11.107a}$$

$$\frac{\partial^2 T_m}{\partial x^2} + \frac{\rho h_{sl}}{k_m}\frac{\partial \chi}{\partial t} = \frac{1}{\alpha_m}\frac{\partial T_m}{\partial t}, \quad s_{sm}(t) < x < s_{ml}(t) \tag{11.107b}$$

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l}\frac{\partial T_l}{\partial t}, \quad s_{ml}(t) < x < \infty \tag{11.107c}$$

with the conditions at $x = 0$ and as $x \to \infty$,

$$T_s(0,t) = T_w \quad \text{and} \quad T_l(\infty,t) = T_i \tag{11.107d,e}$$

the interface conditions at $x = s_{sm}(t)$,

$$T_s(s_{sm},t) = T_m(s_{sm},t) = T_e \tag{11.107f}$$

$$k_m\left(\frac{\partial T_m}{\partial x}\right)_{x=s_{sm}} - k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=s_{sm}} + \rho h_{sl}(1-\chi_{sm})\frac{ds_{sm}}{dt} = 0 \tag{11.107g}$$

and the interface conditions at $x = s_{ml}(t)$,

$$T_m(s_{ml}, t) = T_l(s_{ml}, t) = T_{lq} \tag{11.107h}$$

$$k_l\left(\frac{\partial T_l}{\partial x}\right)_{x=s_{ml}} - k_m\left(\frac{\partial T_m}{\partial x}\right)_{x=s_{ml}} + \rho h_{sl}\chi_{ml}\frac{ds_{ml}}{dt} = 0 \tag{11.107i}$$

with the initial conditions $s_{sm}(0) = s_{ml}(0) = 0$, where $s_{sm}(t)$ and $s_{ml}(t)$ denote, at time t, the location of the solid–mush and mush–liquid interfaces, respectively. In addition, the term χ represents the percentage of volume at any point in the mush occupied by the solid.

In Eq. (11.107*b*), the term $\rho h_{sl}(\partial\chi/\partial t)$ accounts for the internal energy generation rate per unit volume by the formation of the solid dendrites or other crystalline structures constituting the solid matrix in the mushy region. This is due to the fact that $\partial\chi/\partial t$ is the time rate of change of the solid volume fraction and when multiplied by ρh_{sl} gives the volumetric internal energy generation rate (W/m^3) caused by the partial solidification inside the mushy region. Furthermore, assuming that the solid fraction χ in the mushy region is a function of temperature only, we get

$$\frac{\partial\chi}{\partial t} = \frac{\partial\chi}{\partial T_m}\frac{\partial T_m}{\partial t} \tag{11.108}$$

Then, combining Eqs. (11.107*b*) and (11.108) we obtain

$$\frac{\partial^2 T_m}{\partial x^2} = \left(\frac{1}{\alpha_m} - \frac{\rho h_{sl}}{k_m}\frac{\partial\chi}{\partial T_m}\right)\frac{\partial T_m}{\partial t}, \quad s_{sm}(t) < x < s_{ml}(t) \tag{11.109}$$

Here, we will further assume, for the purpose of the following analysis, that the term $\partial\chi/\partial T_m$ in Eq. (11.109) is constant at an average value, which can be estimated from the equilibrium-phase diagram for numerical calculations.

The solidification problem as formulated above accepts a similarity solution in terms of the similarity variable

$$\eta = \frac{x}{2\sqrt{\alpha_s t}} \tag{11.110}$$

The formulation of the problem in terms of the dimensionless temperatures

$$\theta_s(\eta) = \frac{T_s(\eta) - T_w}{T_e - T_w}, \quad \theta_m(\eta) = \frac{T_m(\eta) - T_e}{T_0 - T_e} \quad \text{and} \quad \theta_l(\eta) = \frac{T_l(\eta) - T_i}{T_{lq} - T_i}$$

is then given by

$$\frac{d^2\theta_s}{d\eta^2} + 2\eta\frac{d\theta_s}{d\eta} = 0, \quad 0 < \eta < \lambda_{sm} \tag{11.111a}$$

$$\frac{d^2\theta_m}{d\eta^2} + 2A\eta\frac{d\theta_m}{d\eta} = 0, \quad \lambda_{sm} < \eta < \lambda_{ml} \tag{11.111b}$$

$$\frac{d^2\theta_l}{d\eta^2} + 2B\eta\frac{d\theta_l}{d\eta} = 0, \quad \lambda_{ml} < \eta < \infty \tag{11.111c}$$

with the conditions,

$$\theta_s(0) = 0 \quad \text{and} \quad \theta_l(\infty) = 0 \tag{11.111d,e}$$

the interface conditions at $\eta = \lambda_{sm}$,

$$\theta_s(\lambda_{sm}) = 1 \quad \text{and} \quad \theta_m(\lambda_{sm}) = 0 \tag{11.111f,g}$$

$$K_m\left(\frac{d\theta_m}{d\eta}\right)_{\eta=\lambda_{sm}} - \theta_e\left(\frac{d\theta_s}{d\eta}\right)_{\eta=\lambda_{sm}} + 2\lambda_{sm}\frac{1-\chi_{sm}}{\text{Ste}} = 0 \tag{11.111h}$$

and the interface conditions at $\eta = \lambda_{ml}$,

$$\theta_m(\lambda_{ml}) = \theta_{lq} \quad \text{and} \quad \theta_l(\lambda_{ml}) = 1 \tag{11.111i,j}$$

$$K_l\theta_i\left(\frac{d\theta_l}{d\eta}\right)_{\eta=\lambda_{ml}} - K_m\left(\frac{d\theta_m}{d\eta}\right)_{\eta=\lambda_{ml}} + 2\lambda_{ml}\frac{\chi_{ml}}{\text{Ste}} = 0 \tag{11.111k}$$

where λ_{sm} and λ_{ml} are defined as

$$\lambda_{sm} = \frac{s_{sm}(t)}{2\sqrt{\alpha_s t}} \quad \text{and} \quad \lambda_{ml} = \frac{s_{ml}(t)}{2\sqrt{\alpha_s t}} \tag{11.112a,b}$$

Equations (11.111*f,g*) and (11.111*i,j*) indicate that the parameters λ_{sm} and λ_{ml} must be constants. Furthermore, in the above formulation the following dimensionless groups have been introduced:

$$A = \frac{c_m + h_{sl}(\partial\chi/\partial T_m)}{c_s K_m}, \quad B = \frac{c_l}{c_s K_l} \tag{11.113a,b}$$

$$K_m = \frac{k_m}{k_s}, \quad K_l = \frac{k_l}{k_s} \tag{11.113c,d}$$

$$\theta_i = \frac{T_{lq} - T_i}{T_0 - T_e}, \quad \theta_e = \frac{T_e - T_w}{T_0 - T_e}, \quad \theta_{eq} = \frac{T_{lq} - T_e}{T_0 - T_e} \tag{11.113e,f,g}$$

and

$$\text{Ste} = \frac{c_s(T_0 - T_e)}{h_{sl}} \tag{11.113h}$$

is the Stefan number.

As in the pervious cases, the solution to the above formulation, Eqs. (11.112), can be obtained in terms of the error function. Omitting the details, the results for the temperature distributions in the three regions are given by

$$T_s(\eta) = T_w + (T_e - T_w)\frac{\text{erf}(\eta)}{\text{erf}(\lambda_{sm})} \tag{11.114a}$$

$$T_m(\eta) = \frac{T_e \,\text{erf}(A^{1/2}\lambda_{ml}) - T_{lq}\,\text{erf}(A^{1/2}\eta_{sm}) + (T_{lq} - T_e)\text{erf}(A^{1/2}\eta)}{\text{erf}(A^{1/2}\lambda_{ml}) - \text{erf}(A^{1/2}\lambda_{sm})} \tag{11.114b}$$

$$T_l(\eta) = T_i + (T_{lq} - T_i)\frac{\text{erfc}(B^{1/2}\eta)}{\text{erfc}(B^{1/2}\lambda_{ml})} \tag{11.114c}$$

In these expressions, the parameters λ_{sm} and k_{ml} are still unknown. To obtain them we substitute the solutions for $T_s(\eta)$, $T_m(\eta)$, and $T_l(\eta)$ into the interface energy-balance equations (11.111h) and (11.111k) and get

$$\frac{\theta_e e^{-\lambda_{sm}^2}}{\text{erf}(\lambda_{sm})} - \frac{A^{1/2}K_m\theta_{lq}e^{-A\lambda_{sm}^2}}{\text{erf}(A^{1/2}\lambda_{ml}) - \text{erf}(A^{1/2}\lambda_{sm})} - \frac{\sqrt{\pi}\lambda_{sm}(1-\chi_{sm})}{\text{Ste}} = 0 \tag{11.115a}$$

$$\frac{A^{1/2}K_m\theta_{lq}e^{-A\lambda_{ml}^2}}{\text{erf}(A^{1/2}\lambda_{ml}) - erf(A^{1/2}\lambda_{sm})} - \frac{B^{1/2}K_l\theta_i e^{-B\lambda_{ml}^2}}{\text{erf}(B^{1/2}\lambda_{ml})} - \frac{\sqrt{\pi}\lambda_{ml}\chi_{ml}}{\text{Ste}} = 0 \tag{11.115b}$$

The parameters λ_{sm} and λ_{ml} are the roots of these nonlinear transcendental algebraic equations. Once the values of various parameters are prescribed, the roots of these equations can be determined numerically to yield λ_{sm} and λ_{ml}. With these parameters known, Eqs. (11.114) give the temperature distribution at any time in the three regions, and the location of the solid–mush and mush–liquid interfaces are determined as functions of t from

$$s_{sm}(t) = 2\lambda_{sm}\sqrt{\alpha_s t} \quad \text{and} \quad s_{ml}(t) = 2\lambda_{ml}\sqrt{\alpha_s t} \tag{11.116a,b}$$

The heat flux *removed* from the surface at $x = 0$ to maintain its temperature at T_0 is obtained from Fourier's law,

$$q_0'' = k_s\left(\frac{\partial T_s}{\partial x}\right)_{x=0} = \frac{k_s}{2\sqrt{\alpha_s t}}\left(\frac{dT_s}{d\eta}\right)_{n=0} \tag{11.117a}$$

Substituting the temperature distribution $T_s(\eta)$ from Eq. (11.114*a*) into Eq. (11.117*a*) yields

$$q_0'' = \frac{k_s(T_e - T_w)}{\sqrt{\pi \alpha_s t}} \frac{1}{\text{erf}(\lambda_{sm})} \tag{11.117b}$$

The procedure followed in this representative problem can also be extended to conduction-dominated unidirectional binary alloy solidification problems in cylindrical and spherical coordinates.

References

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Problems

11.1 Consider the melting of a semi-infinite solid as illustrated in Fig. 11.2. Obtain an expression for the *Stefan condition* at the interface, $x = s(t)$.

11.2 Show that the problem defined by Eqs. (11.9) accepts a similarity solution in terms of the similarity variable (11.10).

11.3 Show that, for small values of the Ste number, the temperature distribution given by Eq. (11.15*b*) reduces to Eq. (11.22).

11.4 Consider a large solid, $x \geq 0$, initially at the fusion temperature T_f. At $t = 0$, the temperature of the boundary surface at $x = 0$ is raised to T_0 ($> T_f$) and maintained at that constant temperature for times $t > 0$. Assuming constant thermophysical properties for the liquid phase, and neglecting any convective motion in the melt, obtain exact expressions for both the temperature distribution in the liquid phase and the solid–liquid interface location as a function of time.

11.5 Obtain the energy-integral equation (11.24). Show that

(*a*) the coefficients *A*, *B*, and *C* of the second-order polynomial (11.25) are as given by Eqs. (11.29*a*,*b*,*c*),

(*b*) substitution of the approximate temperature profile (11.25) into the energy-integral equation (11.24) results in the ordinary differential equation for $s(t)$ given by Eq. (11.31), and

(*c*) the solution of Eq. (11.31) with the condition $s(0) = 0$ is given by the relation (11.32) with the parameter λ given by Eq. (11.33).

11.6 Re-solve Problem 11.4 by the integral method. Assume a second-degree polynomial for the temperature distribution in the liquid phase, and obtain an expression for the instantaneous location of the solid–liquid interface as a function of time. How does this result compare with the result (11.32) obtained in Section 11.3.3 for the case of solidification?

11.7 A thick slab of solid paraffin wax used for heat storage is initially at the fusion temperature, $T_f = 27.5$°C. Suddenly, the temperature of one side of the slab is raised to 60°C. Neglecting any convective motion in the melt, determine how long it will take for the melt front to reach to a depth of 5 cm. The properties of the liquid paraffin wax can be taken as $\alpha_l = 1 \times 10^{-7}$ m^2/s, $\rho_l = 800$ kg/m^3, $h_{sl} = 250$ kJ/kg, and $c_l = 2.2$ kJ/kg°C.

11.8 A large solid, $x \geq 0$, is initially at the fusion temperature T_f. For times $t \geq 0$, the surface at $x = 0$ is heated by applying a uniform heat flux q''_w. Obtain an expression for the location of the solid–liquid interface as a function of time by the integral method. Assume a second-order polynomial for the temperature distribution in the liquid phase.

11.9 Show that, for small values of λ (i.e., for slow freezing processes), the transcendental equation (11.42) can be approximated by Eq. (11.46).

11.10 A cylinder of radius r_0 and with a line heat source of strength per unit length q'_{ln} (W/m) placed along the centerline at $r = 0$ is filled with a solid material at the fusion temperature T_f. The line heat source is suddenly activated at $t = 0$ to release heat continuously for times $t > 0$. Obtain an expression for the temperature distribution in the liquid phase for times t when $r = s(t) < r_0$, where $s(t)$ is the phase-front. Determine the time needed for the solid to completely melt.

11.11 Show that the non-zero root λ of Eq. (11.16) is always smaller than $\sqrt{\mathrm{Ste}/2}$.

11.12 Consider a long cylindrical rod, $0 \le r \le r_0$, which is initially at the fusion temperature T_f. For times $t \ge 0$, the surface at $r = r_0$ is heated uniformly by maintaining its temperature constant at T_w ($> T_f$). Assuming that the thermophysical properties of the liquid phase are constants with Ste $\ll 1$, and neglecting any convective motion in the melt, estimate the time needed for the solid rod to completely melt.

11.13 Consider two concentric spheres of radii r_1 and r_2. The space between the two spheres is filled with a solid material at the fusion temperature T_f. For times $t \ge 0$, the inner spherical surface at $r = r_1$ is heated uniformly by maintaining its temperature constant at T_w ($> T_f$). Assuming that the thermophysical properties of the liquid phase are constants with Ste $\ll 1$, and neglecting any convective motion in the melt, estimate the time needed for the solid to completely melt.

11.14 Drive the interface conditions (11.107*g*) and (11.107*i*).

11.15 Show that, in terms of the similarity variable (11.110), the system given by Eqs. (11.107) reduces to system given by Eqs. (11.111).

11.16 (*a*) Solve the system given by Eqs. (11.111) to obtain the temperature distributions given by Eqs. (11.114).

(*b*) Substitute the temperature distributions given by Eqs. (11.114) into the interface conditions (11.107*h*) and (11.107*i*) to obtain the transcendental algebraic equations (11.115*a,b*) for the determination of the parameters λ_{sm} and λ_{ml}.

12

Numerical Solutions

12.1 Introduction

In the preceding chapters, various analytical methods were discussed for the solution of heat conduction problems involving relatively simple geometric shapes with certain straightforward boundary conditions in the rectangular, cylindrical, and spherical coordinate systems. The vast majority of problems encountered in practice, however, cannot be solved analytically as they usually involve irregular geometries with mathematically inconvenient mixed boundary conditions. In such cases, numerical, graphical, and/or hybrid numerical–analytical methods (as will be presented in Chapter 13, and for further details, see Ref. [21]) often provide the answer. For example, an exact analytical solution for the temperature distribution in a turbine blade cannot be obtained because the boundary of the blade is not parallel to the coordinate surfaces of an orthogonal system. The geometry, however, can be simplified for an analytical solution, but the results may not be accurate enough for practical applications.

With the development of high-speed computers, numerical techniques have been developed and extended to handle almost any problem of any degree of complexity. Of the numerical methods available, the *finite-difference* method is the most frequently used one. Accordingly, in this chapter we focus on the solution of both steady- and unsteady-state heat conduction problems by the finite-difference method.

The essence of the finite-difference method consists of replacing the pertinent differential equation and boundary conditions by a set of algebraic equations. Our treatment of the method is not intended to be exhaustive in its mathematical rigor, but we present the fundamentals of the method and discuss the solution of finite-difference equations by numerical and graphical means. For more detailed discussion of the applications of the finite-difference and other numerical methods to heat conduction problems, the reader is referred to References [1–7,9,10,13–17,22]. Examples of computer programs for the solution of several heat conduction problems can be found in Reference [12].

The finite-element method is another numerical method of solution. This relatively new method is not as straightforward, conceptually, as the finite-difference method, but it has several advantages over the finite-difference method in solving conduction problems, particularly for problems with complex geometries. Fundamentals of this method are discussed in References [10,20].

12.2 Finite-Difference Approximation of Derivatives

In replacing a differential equation or a boundary condition by a set of algebraic equations, the fundamental operation is to approximate the derivatives by finite differences. Let us consider the function $T(x)$ shown in Fig. 12.1. The definition of the first derivative of $T(x)$ at x_m is

$$\left(\frac{dT}{dx}\right)_{x_m} = \lim_{\Delta x\to 0}\frac{T(x_m+\Delta x)-T(x_m)}{\Delta x} \tag{12.1}$$

As an approximation, we can write

$$\left.\frac{dT}{dx}\right|_{x_m} \cong \frac{T(x_m+\Delta x)-T(x_m)}{\Delta x} = \frac{T_{m+1}-T_m}{\Delta x} \tag{12.2}$$

This approximate expression at x_m is referred to as the *forward-difference form* of the first derivative (12.1). Another approximate expression similar to the forward-difference form (12.2) is written as

$$\left.\frac{dT}{dx}\right|_{x_m} \cong \frac{T_m-T_{m-1}}{\Delta x} \tag{12.3}$$

This approximation is known as the *backward-difference form* of the first derivative at x_m (12.1). An approximate expression more accurate than either the forward-difference form or the backward-difference form can clearly be written as

$$\left.\frac{dT}{dx}\right|_{x_m} \cong \frac{T_{m+1}-T_{m-1}}{2\Delta x} \tag{12.4}$$

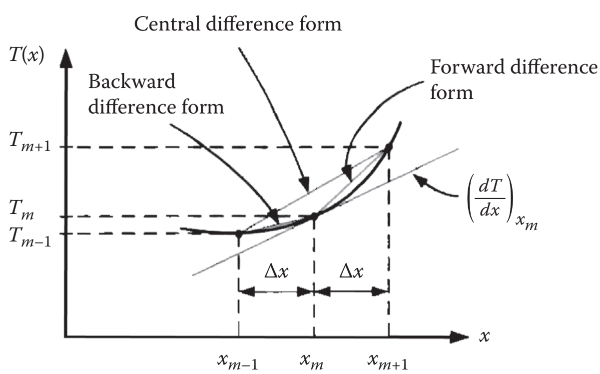

FIGURE 12.1
Finite-difference approximation of derivatives.

which is called the *central-difference form* of the first derivative (12.1).

As we discuss in Section 12.10, both forward-difference and backward-difference forms have a *truncation error* (or *discretization error*) of the order of the magnitude of Δx, whereas the central-difference form has a truncation error of the order of the magnitude of $(\Delta x)^2$.

The second derivative of $T(x)$ at x_m can be approximated in central-difference form as

$$\left.\frac{d^2T}{dx^2}\right|_{x_m} \cong \frac{(dT/dx)\big|_{x_m+\Delta x/2} - (dT/dx)\big|_{x_m-\Delta x/2}}{\Delta x} \tag{12.5a}$$

When the central-difference forms of $(dT/dx)|x_{m+\Delta x/2}$ and $(dT/dx)|x_{m+\Delta x/2}$ are substituted into Eq. (12.5*a*) we get

$$\left.\frac{d^2T}{dx^2}\right|_{x_m} \cong \frac{\left[(T_{m+1} - T_m)/\Delta x\right] - \left[(T_m - T_{m-1})/\Delta x\right]}{\Delta x} \tag{12.5b}$$

which reduces to

$$\left.\frac{d^2T}{dx^2}\right|_{x_m} \cong \frac{T_{m+1} + T_{m-1} - 2T_m}{(\Delta x)^2} \tag{12.5c}$$

The approximation (12.5*c*) for the second derivative also has a truncation error of order $(\Delta x)^2$.

12.3 Finite-Difference Formulation of Steady-State Problems in Rectangular Coordinates

Consider a solid body with a two-dimensional temperature distribution in rectangular coordinates as shown in Fig. 12.2. The results that we shall obtain can readily be extended to three-dimensional problems in rectangular coordinates as well as to problems in other geometries. Let us assume that the thermal conductivity of the solid is constant and that there are no internal energy sources or sinks. Under steady-state conditions the temperature distribution satisfies Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{12.6}$$

everywhere within the solid. In other words, the solution of Eq. (12.6), together with the use of the boundary conditions, gives a continuous temperature distribution within the solid. On the other hand, when the differential equation (12.6) and the boundary conditions are expressed in finite-difference form, the resulting system of equations will be satisfied only at certain points.

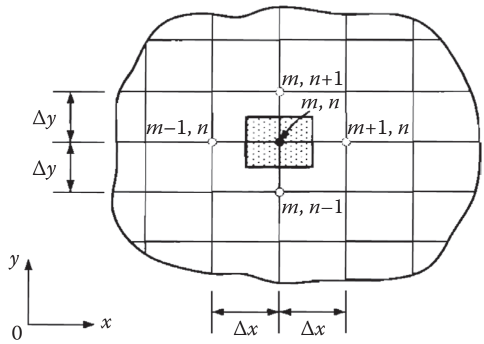

FIGURE 12.2
Network of grid points for the numerical analysis of two-dimensional heat conduction.

Let us divide the solid body both in x and y directions forming the so-called *network of grid* (or *nodal*) points as shown in Fig. 12.2. The grid points are identified by two subscripts, say m and n, m being the number of Δx increments and n the number of Δy increments, respectively. We now approximate Eq. (12.6) at each grid point by replacing the derivatives with their approximate finite-difference equivalents. At the grid point (m, n) we can apply Eq. (12.5*c*) to each second-order derivative, giving the approximation

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} = 0 \tag{12.7}$$

If the step sizes in both x and y directions are taken to be the same, that is, if $\Delta x = \Delta y$, then this approximation reduces to the algebraic equation

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \tag{12.8a}$$

or

$$T_{m,n} = \frac{1}{4}\left(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right) \tag{12.8b}$$

which represents the temperature of the grid point (m, n) in terms of the temperatures of the neighboring grid points (m + 1, n), (m – 1, n), (m, n + 1) and (m, n – 1).

The finite-difference equation (12.8*a*) is applicable to interior points. If the temperature is known over the whole boundary, then the application of Eq. (12.8*a*) to each interior grid point is sufficient to allow the temperatures at these grid points to be found. If there are N interior grid points, this procedure gives N simultaneous algebraic equations, and we know, at least in principle, how to solve such a system of equations.

If small step sizes are used, then the temperature distribution is more closely approximated. On the other hand, the number of grid points and, therefore, the number of finite-difference equations become very large. This is a restriction only if the equations are to be solved by hand. For high-speed and large-capacity computers, however, this is not a restriction.

We have given the derivation of the finite-difference equation (12.8*a*) from a mathematical point of view. That is, we have converted the differential equation (12.6) at the grid point (m, n) into the finite-difference form (12.8*a*) by approximating the derivatives at this grid point. The energy balance concept and the rate equations can also be used directly to arrive at the same finite-difference equation. To explain this alternative method, reconsider the two-dimensional solid shown in Fig. 12.2, and define a system around the grid point (m, n) as shown in Fig. 12.3. The first law of thermodynamics as applied to this system requires that, under steady-state conditions, the net rate of heat transferred to the system be zero. That is,

$$q_{m-1,n} + q_{m,n+1} + q_{m+1.n} + q_{m,n-1} = 0 \tag{12.9}$$

Fourier's law of heat conduction, for unit depth, gives

$$q_{m-1,n} = -k\Delta y\left(\frac{\partial T}{\partial x}\right)_A$$

which can be approximated as

$$q_{m-1,n} \cong k\Delta y\frac{T_{m-1,n} - T_{m,n}}{\Delta x} \tag{12.10a}$$

Similar expressions for $q_{m+1,n}$, $q_{m,n-1}$, and $q_{m,n+1}$ can be written as

$$q_{m+1,n} \cong k\Delta y\frac{T_{m+1,n} - T_{m,n}}{\Delta x} \tag{12.10b}$$

$$q_{m,n-1} \cong k\Delta x\frac{T_{m,n-1} - T_{m,n}}{\Delta y} \tag{12.10c}$$

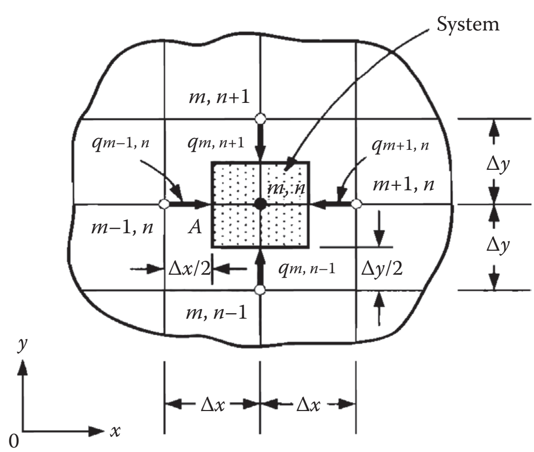

FIGURE 12.3
System defined around the grid point (m, n) for derivation of the finite-difference equation (12.8*a*).

and

$$q_{m,n+1} \cong k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} \tag{12.10d}$$

Substituting Eqs. (12.10*a*–*d*) into Eq. (12.9) yields

$$\begin{gathered} k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta y \frac{T_{m+1.n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} \\ + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} = 0 \end{gathered} \tag{12.11}$$

If the step sizes Δx and Δy are taken to be the same, then Eq. (12.11) reduces

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

which is the same finite-difference relation (12.8*a*) we obtained previously.

12.4 Finite-Difference Approximation of Boundary Conditions

The finite-difference formulation of a heat conduction problem will be complete if the boundary conditions are also written in finite-difference form. When the temperature on the whole boundary is specified, the known value of the boundary temperature enters into the finite-difference equations so that each equation for the grid points adjacent to the boundary would have a prescribed term in it. If the condition on the whole boundary, or a part of it, is other than a temperature boundary condition, then new finite-difference equations need to be derived for the grid points on the boundary. Let us now consider the development of such equations for the grid points on boundaries with different kinds of boundary conditions.

12.4.1 Boundary Exchanging Heat by Convection with a Medium at a Prescribed Temperature

We now consider a grid point (m, n) on a boundary that is exposed to a fluid at temperature T_∞ with a constant heat transfer coefficient h as shown in Fig. 12.4. Application of the first law of thermodynamics to the system shown yields

$$\begin{gathered} k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + h\Delta y (T_\infty - T_{m,n}) + k\frac{\Delta x}{2} \frac{T_{m,n-1} - T_{m,n}}{\Delta y} \\ + k\frac{\Delta x}{2} \frac{T_{m,n+1} - T_{m,n}}{\Delta y} = 0 \end{gathered} \tag{12.12}$$

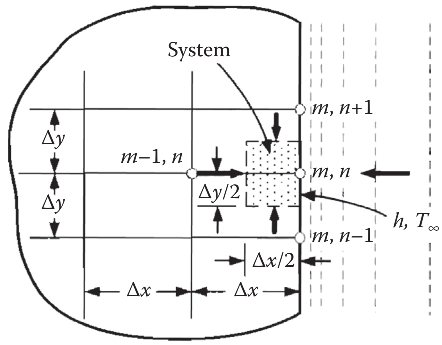

FIGURE 12.4
Grid point on a surface with convection boundary condition.

If the step sizes Δx and Δy are taken to be equal, then Eq. (12.12) reduces to

$$\frac{1}{2}\left(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right) + \frac{h\Delta x}{k}T_\infty - \left(2 + \frac{h\Delta x}{k}\right)T_{m,n} = 0 \tag{12.13}$$

Thus, when a convection boundary condition is present as in Fig. 12.4, the finite-difference equation (12.13) is used for the grid points on the boundary if $\Delta x = \Delta y$, and the finite-difference equation (12.8*a*) is used for the interior grid points. If the grid point happens to be on a corner, such as in Fig. 12.5, then Eq. (12.13) is not applicable. Let us now consider the corner section in Fig. 12.5. Application of the first law of thermodynamics to the system shown, in this case, yields

$$\frac{1}{2}\left(T_{m-1,n} + T_{m,n-1}\right) + \frac{h\Delta x}{k}T_\infty - \left(1 + \frac{h\Delta x}{k}\right)T_{m,n} = 0 \tag{12.14}$$

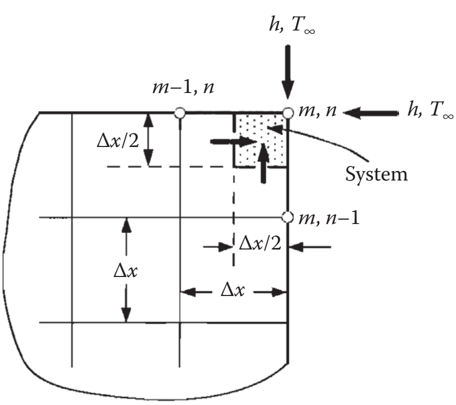

FIGURE 12.5
Grid point on a corner with convection boundary condition.

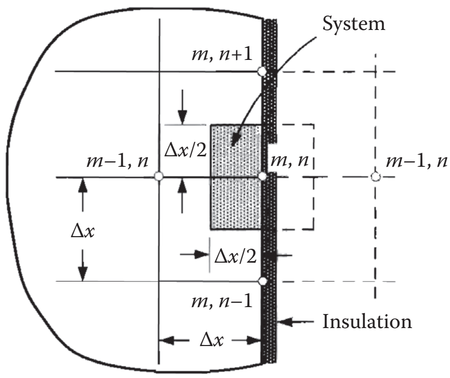

FIGURE 12.6
Grid point on an insulated boundary.

12.4.2 Insulated Boundary

When part of the boundary is insulated as shown in Fig. 12.6, finite-difference equations for the grid points on this insulated section can also be obtained as in the previous case. Thus, applying the first law of thermodynamics to the system shown yields

$$T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n} = 0 \tag{12.15}$$

Since $\partial T/\partial x = 0$ on the boundary, Eq. (12.15) can also be written directly from Eq. (12.8*a*) by setting $T_{m+1,n} = T_{m-1,n}$. (Why?)

12.5 Irregular Boundaries

In problems with simple geometries, one can arrange for certain grid points to lie on the boundaries. However, in cases where the boundary does not fall on regular grid points, the boundary is considered to be irregular. In practical problems, the boundaries are usually irregular and the numerical methods are the only means for solving such problems.

Let us consider the irregular boundary shown in Fig. 12.7 and assume that the temperature of the boundary is known. The finite-difference equation (12.8*a*) cannot be used, for example, for the grid point (m, n) next to the boundary. On the other hand, applying the first law of thermodynamics to the system shown in Fig. 12.7 yields

$$k\frac{(1+b)\Delta y}{2}\frac{T_{m-1,n}-T_{m,n}}{a\Delta x} + k\frac{(1+b)\Delta y}{2}\frac{T_{m+1,n}-T_{m,n}}{\Delta x}$$

$$+k\frac{(1+a)\Delta x}{2}\frac{T_{m,n-1}-T_{m,n}}{b\Delta y} + k\frac{(1+a)\Delta x}{2}\frac{T_{m,n+1}-T_{m,n}}{\Delta y} = 0 \tag{12.16}$$

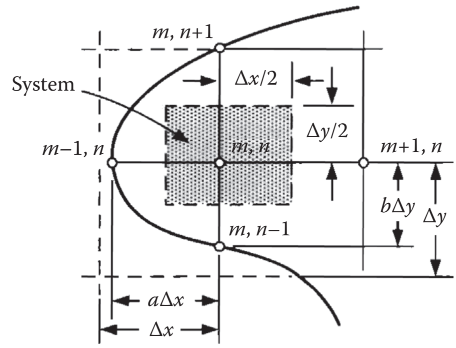

FIGURE 12.7
Grid point adjacent to an irregular boundary.

If the step sizes are taken to be equal, that is, $\Delta x = \Delta y$, then Eq. (12.16) reduces to

$$\begin{aligned} &\frac{1}{1+a}T_{m+1,n} + \frac{1}{a(1+a)}T_{m-1,n} + \frac{1}{1+b}T_{m,n+1} \\ &+\frac{1}{b(1+b)}T_{m,n-1} - \left(\frac{1}{a}+\frac{1}{b}\right)T_{m,n} = 0 \end{aligned} \tag{12.17}$$

where a and b are as defined in Fig. 12.7. We note that the grid point (m, n) is not located at the geometric center of the system defined around (m, n). If $a = b = 1$, then Eq. (12.17) reduces to Eq. (12.8*a*).

12.6 Solution of Finite-Difference Equations

The finite-difference representation of the heat conduction equation together with the boundary conditions of a steady-state problem results in a system of algebraic equations. Except for the problems in which thermal conductivity depends on temperature, these equations will be linear. The resultant system of algebraic equations must be solved to find the temperatures at the various grid points. If the number of grid points and therefore the number of equations are not large, then the system of finite-difference equations can be solved by hand or by the use of a desk calculator using the so-called *relaxation method*, which was first introduced by Southwell [19]. If the number of grid points is large, then the equations can be solved by the use of a large-capacity digital computer,

12.6.1 Relaxation Method

When the temperature of the whole boundary is specified, this method for a two-dimensional steady-state problems is carried out as follows:

a. Set the right-hand side of Eq. (12.8*a*) equal to some residual $R_{m,n}$, as

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = R_{m,n} \tag{12.18}$$

b. Guess values for the temperatures of the interior grid points.

c. Calculate the residual $R_{m,n}$ at each grid point using the assumed values. If the assumed value is the true value of the temperature at each grid point, then all the residuals will, of course, be zero. In general, they will not be zero.

d. Select the largest residual and try to make it, at least approximately, zero by changing the assumed temperature of the corresponding grid point, while holding the temperatures of the other grid points constant.

e. Compute the new residuals. Note that only the residuals of the grid points adjacent to the readjusted grid point are affected. The computation of new residuals may be accomplished by using the relaxation pattern shown in Fig. 12.8. Since $dR_{m,n} = -4dT_{m,n}$ when all other temperatures are held constant, a unit decrease in $T_{m,n}$ results in an increase in $R_{m,n}$ by 4 and a unit decrease in the residuals of the adjacent grid points.

f. Continue to relax the residuals until they are all as close to zero as desired.

The relaxation method will now be demonstrated by an example.

Example 12.1

Let us find the temperature distribution at, and the rate of heat flow across, the surfaces of the two-dimensional solid shown in Fig. 12.9.

SOLUTION

We divide the solid into increments as shown ($\Delta x = \Delta y$). The residual equations for grid points *A*, *B*, *C*, and *D* are given by

$$1000 + T_B - 4T_A = R_A \tag{12.19a}$$

$$600 + T_A + T_C - 4T_B = R_B \tag{12.19b}$$

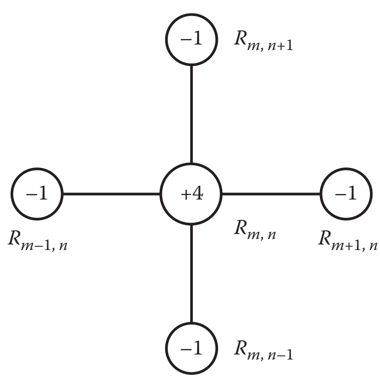

FIGURE 12.8
Two-dimensional relaxation pattern: effect of unit decrease in $T_{m,n}$ on $R_{m,n}$'s.

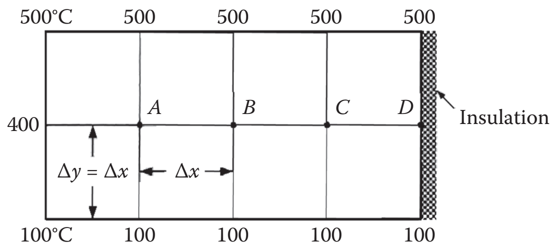

FIGURE 12.9
Figure for Example 12.1.

$$600 + T_B + T_D - 4T_C = R_C \tag{12.19c}$$

$$600 + 2T_C - 4T_D = R_D \tag{12.19d}$$

The relaxation calculations for this problem are given in Table 12.1. The calculation has been stopped while some of the residuals still have nonzero values. However, the accuracy is acceptable because all temperatures are within 1°C of their expected values. Notice that we have overrelaxed or underrelaxed the residuals during the calculations to speed the solution.
The rate of heat loss from the 100°C surface is given by

$$q = kb\left[\frac{1}{4}(400-100)+(327-100)+(307-100)+(302-100)+\frac{1}{2}(300-100)\right] = 886kb$$

where k is the thermal conductivity and b is the depth of the solid. If a sufficiently fine grid network was used, a more accurate value of the heat transfer rate could be obtained. The rate of heat transfer at the 500°C surfaces can be obtained in the same way.

TABLE 12.1
Relaxation Table for Example 12.1[†]

A		B		C		D	
R	**T**	**R**	**T**	**R**	**T**	**R**	**T**
100	300	0	300	0	300	0	300
0	325	25	300	0	300	0	300
6	325	1	306	6	300	0	300
–2	327	3	306	6	300	0	300
–2	327	5	306	–2	302	2	300
–1	327	1	307	–1	302	2	300

[†]Temperatures are given in °C.

12.6.2 Matrix Inversion Method

A system of N finite-difference equations is a set of algebraic equations implicitly involving the unknown temperatures. This system of algebraic equations can be written in matrix notation as

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_N \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_N \end{bmatrix} \tag{12.20a}$$

or

$$\boldsymbol{A} \cdot \boldsymbol{T} = \boldsymbol{C} \tag{12.20b}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{T}$ is the vector of unknown temperatures, and $\mathbf{C}$ is a column vector of known constants. In heat conduction problems, most of the off-diagonal elements of $\mathbf{A}$ would be zero. Multiplication of Eq. (12.20*b*) from the left with the inverse $\mathbf{A}^{-1}$ of the coefficient matrix yields the unknown vector $\mathbf{T}$ as

$$\mathbf{T} = \mathbf{A}^{-1} \cdot \mathbf{C} \tag{12.21}$$

whose elements are the unknown temperatures at the grid points. The problem, then, reduces to inverting the matrix $\mathbf{A}$, but the inverse can be found efficiently by using a digital computer. We now solve an example problem to illustrate the method.

Example 12.2

Let us re-solve the problem in Example 12.1 using the matrix inversion method.

SOLUTION

The finite-difference equations at the grid points A, B, C, and D can be written as

$$4T_A - T_B = 1000 \tag{12.22a}$$

$$-T_A + 4T_B - T_C = 600 \tag{12.22b}$$

$$-T_B + 4T_C - T_D = 600 \tag{12.22c}$$

$$-2T_C + 4T_D = 600 \tag{12.22d}$$

In matrix notation these equations would be

$$\mathbf{A} \cdot \mathbf{T} = \mathbf{C}$$

where

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 2 & -2 & 4 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1000 \\ 600 \\ 600 \\ 600 \end{bmatrix}$$

Now, a digital computer can be employed and standard subroutines can be used to find the inverse $\mathbf{A}^{-1}$. In this special case, $\mathbf{A}^{-1}$ can also be obtained by hand. Here we give the result, and interested readers may refer to books on linear algebra such as [8,11] for matrix inversions. The inverse $\mathbf{A}^{-1}$ is found to be

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.268 & 0.072 & 0.021 & 0.005 \\ 0.072 & 0.289 & 0.083 & 0.021 \\ 0.021 & 0.083 & 0.309 & 0.077 \\ 0.010 & 0.041 & 0.155 & 0.289 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \end{bmatrix} = \begin{bmatrix} 0.268 & 0.072 & 0.021 & 0.005 \\ 0.072 & 0.289 & 0.083 & 0.021 \\ 0.021 & 0.083 & 0.309 & 0.077 \\ 0.010 & 0.041 & 0.155 & 0.289 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 600 \\ 600 \end{bmatrix}$$

which yields

$T_A = 0.268 \times 1000 + (0.072 + 0.021 + 0.005) \times 600 = 326.8°\text{C}$
$T_B = 0.072 \times 1000 + (0.289 + 0.083 + 0.021) \times 600 = 307.8°\text{C}$
$T_C = 0.021 \times 1000 + (0.083 + 0.309 + 0.077) \times 600 = 302.4°\text{C}$
$T_D = 0.010 \times 1000 + (0.041 + 0.155 + 0.289) \times 600 = 301.0°\text{C}$

Clearly, for a large number of grid points, the use of a computer is a necessity.

12.6.3 Gaussian Elimination Method

This is probably one of the most commonly used methods for solving coupled systems of linear algebraic equations. The basic idea behind this method is back substitution. We shall explain the Gaussian elimination method by means of Example 12.3.

Example 12.3

Re-solve the problem in Example 12.1 using the Gaussian elimination method.

SOLUTION

Referring to Eqs. (12.22*a-d*), we divide the first equation by 4 (the coefficient of T_A) to obtain an expression for T_A and use it to eliminate T_A from all other equations. We then divide the second equation by the now modified coefficient of T_B to obtain an expression for T_B, which can then be eliminated in the remaining equations, and so on through the rest of the system. We eventually obtain an equivalent system

$$4T_A - T_B = 1000 \tag{12.23a}$$

$$3.75T_B - T_C = 850 \tag{12.23b}$$

$$3.733T_C - T_D = 826.7 \tag{12.23c}$$

$$3.464T_D = 1042.9 \tag{12.23d}$$

The unknown T_D in the system is now found to be T_D = 301.7°C, and the rest follows by back substitution. Equation (12.23*c*) gives T_C = 302.3°C, Eq. (12.23*b*) gives T_B = 307.3°C, and then Eq. (12.23*a*) gives T_A = 326.8°C.

The last two methods we have seen are examples of *direct methods*. There are also *iterative methods* of solution such as the *Gauss–Seidel method* or the *successive over-relaxation method*. Brief summaries of available numerical methods can be found in [3,4,13,22].

12.7 Finite-Difference Formulation of One-Dimensional, Unsteady-State Problems in Rectangular Coordinates

In one-dimensional, unsteady-state problems in rectangular coordinates, the heat conduction equation that governs the temperature distribution for constant thermophysical properties is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{12.24}$$

which can also be expressed in finite-difference form by approximating the derivatives. Let us form a network of grid points by dividing x and t domains into small intervals of Δx and Δt as shown in Fig. 12.10, so that T_n^p represents the temperature at location $x = n$ Δx at $t = p\ \Delta t$. In Eq. (12.24), the second-order partial derivative with respect to x can be approximated at time t (= $\rho\ \Delta t$) as

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_t \cong \frac{T_{n+1}^p + T_{n-1}^p - 2T_n^p}{(\Delta x)^2} \tag{12.25}$$

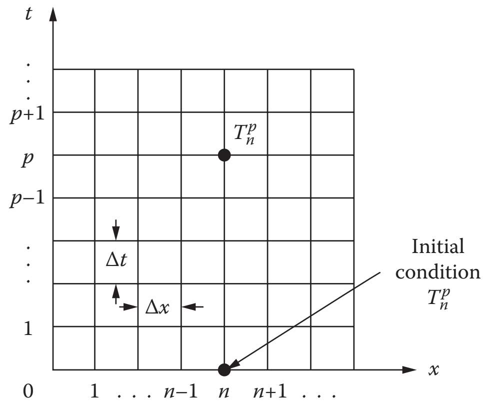

FIGURE 12.10
Network of grid points for one-dimensional, unsteady-state problems.

The time derivative in Eq. (12.24) may be approximated at $x(= n\ \Delta x)$ in terms of central, backward, or forward differences as

$$\left.\frac{\partial T}{\partial t}\right|_x \cong \frac{T^{p+1} - T^{p-1}}{2\Delta t} \tag{12.26a}$$

$$\left.\frac{\partial T}{\partial t}\right|_x \cong \frac{T^{p} - T^{p-1}}{\Delta t} \tag{12.26b}$$

$$\left.\frac{\partial T}{\partial t}\right|_x \cong \frac{T^{p+1} - T^{p}}{\Delta t} \tag{12.26c}$$

These three approximations, as we shall discuss, all have different error levels and also have different stability properties. In other words, each approximation has its own advantages and disadvantages. For example, the finite-difference equations resulting from the forward-difference approximation of the time derivative are uncoupled and therefore easy to solve, but their solutions are not always stable. On the other hand, use of the backward-difference approximation yields finite-difference equations that are coupled and therefore relatively difficult to solve, but their solutions are always stable. We now discuss the representation of Eq. (12.24) in terms of finite-difference equations by different methods.

12.7.1 Explicit Method

If the time derivative in Eq. (12.24) is approximated by the forward-difference approach (12.26*c*), then the heat conduction equation (12.24) can be written in finite-difference form as

$$\frac{T_{n+1}^p + T_{n-1}^p - 2T_n^p}{(\Delta x)^2} = \frac{1}{\alpha}\frac{T_n^{p+1} - T_n^p}{\Delta t} \tag{12.27a}$$

which can be rearranged as

$$T_n^{p+1} = \left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right]T_n^p + \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^p + T_{n-1}^p\right) \tag{12.27b}$$

This formulation is called *explicit* because it is possible to write the temperature T_n^{p+1} (at location n, at time $t + \Delta t$) explicitly in terms of the temperatures at time t and at locations $n - 1$, n, and $n + 1$. Thus, if the temperatures of the grid locations are known at any particular time t, the temperatures after a time increment Δt may be calculated by writing an equation like Eq. (12.27*b*) for each grid location, and obtaining the values of T_n^{p+1}. The calculation proceeds directly from one time increment to the next until the temperature distribution is obtained at the desired time.

The energy balance concept and the rate equations can also be used directly to arrive at the finite-difference equation (12.27*b*). To explain this alternative method, consider a one-dimensional solid and define a system as shown in Fig. 12.11. The first law of thermodynamics as applied to this system gives

$$kA\frac{T_{n-1}^p - T_n^p}{\Delta x} + kA\frac{T_{n+1}^p - T_n^p}{\Delta x} = \rho c A\Delta x\frac{T_n^{p+1} - T_n^p}{\Delta t}$$

which yields Eq. (12.27*a*) when rearranged.

If the space increment Δx and time increment Δt are chosen such that

$$\frac{\alpha\Delta t}{(\Delta x)^2} = \frac{1}{2}$$

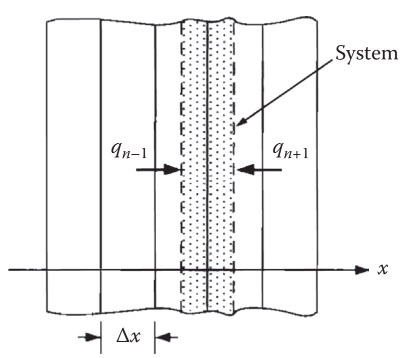

FIGURE 12.11
System defined in a one-dimensional solid for the derivation of the finite-difference equation (11.27*b*).

then Eq. (12.27*b*) reduces to

$$T_n^{p+1} = \frac{1}{2}\left(T_{n+1}^p + T_{n-1}^p\right) \tag{12.28}$$

Therefore, the temperature at location n after one time increment is given by the arithmetic average of the temperatures of the adjacent locations at the beginning of the time increment.

Depending on the values of Δx and Δt, the coefficient of T_n^p in Eq. (12.27*b*) may be negative, zero, or positive. As we shall show in Section 12.11, if this coefficient is negative, a condition is generated such that the second law of thermodynamics is violated and the solution becomes unstable. One might think that small space increments could be used for greater accuracy in combination with large time increments to speed the calculation. This is usually not the case, because it may result in a negative coefficient for T_n^p. To provide some insight into the physical significance of the stability, suppose that the temperatures T_{n+1}^p and T_{n-1}^p, are zero, and the temperature T_n^p is positive. Then, if the coefficient of T_n^{p+1} is negative, the temperature T_n^{p+1} at location n becomes negative. This is impossible because it would violate the second law of thermodynamics as heat cannot flow in the direction of a positive temperature gradient. Therefore, for stable solutions we should have

$$\frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

Once Δx is established, this restriction automatically limits our choice of Δt.

If the surface temperatures are specified, then the finite-difference equation (12.27*b*) is used to determine the temperatures of the internal grid locations as a function of time. If a convection boundary condition prevails on the whole boundary, or part of it, then the above relations are no longer applicable at the boundary. Therefore, such a boundary must be handled separately. For the one-dimensional system shown in Fig. 12.12, the boundary condition at $x = L$ is

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L) - T_\infty] \tag{12.29}$$

The finite-difference approximation of the boundary condition (12.29) can be written as

$$k\frac{T_{m-1}^{p+1} - T_m^{p+1}}{\Delta x} = h\left[T_m^{p+1} - T_\infty\right] \tag{12.30a}$$

or

$$T_m^{p+1} = \frac{1}{1 + h\Delta x / k}\left(T_{m-1}^{p+1} + \frac{h\Delta x}{k}T_\infty\right) \tag{12.30b}$$

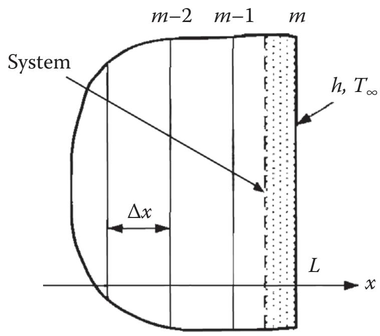

FIGURE 12.12
System defined next to the boundary at $x = L$.

Substitution of T_{m-1}^{p+1} from Eq. (12.27*b*) into Eq. (12.30*b*) gives

$$T_m^{p+1} = \frac{1}{1 + h\Delta x / k}\left\{\left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right]T_{m-1}^p + \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_m^p + T_{m-2}^p\right) + \frac{h\Delta x}{k}T_\infty\right\} \tag{12.31}$$

Equation (12.31) does not include the effect of the heat capacity of the system next to the boundary. If small Δx is used, this approximation works fairly well simply because the heat capacity of the system becomes negligible. A better approximation could be obtained by taking the heat capacity of the system shown into consideration. Application of the first law of thermodynamics then yields

$$k\frac{T_{m-1}^p - T_m^p}{\Delta x} + h\left(T_\infty - T_m^p\right) = \rho c\frac{\Delta x}{2}\frac{T_m^{p+1} - T_m^p}{\Delta t} \tag{12.32a}$$

which can be rearranged as

$$T_m^{p+1} = \frac{\alpha\Delta t}{(\Delta x)^2}\left\{\left[\frac{(\Delta x)^2}{\alpha\Delta t} - 2\frac{h\Delta x}{k} - 2\right]T_m^p + 2T_{m-1}^p + 2\frac{h\Delta x}{k}T_\infty\right\} \tag{12.32b}$$

For the convection boundary condition, the stability of the temperatures of the grid points on the surface must also be ensured and, therefore, the selection of the parameter $(\Delta x)^2/\alpha\Delta t$ is not so simple as it is for interior points. This parameter can be so chosen that the coefficients of T_m^p for both interior and surface grid points become either positive or zero: that is,

$$\frac{(\Delta x)^2}{\alpha\Delta t} \geq 2\left(\frac{h\Delta x}{k} + 1\right)$$

Price and Slack [14] and Schneider [17] discuss the stability of boundary conditions and numerical solutions in some detail. Interested readers may also refer to References [3,6,7] for the stability and convergence of numerical solutions.

Example 12.4

The initial temperature distribution inside a homogeneous flat plate of thickness 60 cm is as shown in Fig. 12.13. The plate is cooled from both sides by a coolant at 100°C. The heat transfer coefficient h on both sides is constant and has a value of 15 W/(m^2·K). Let us find the temperatures at the grid locations as a function of time. The properties of the solid are k = 3 W/(m·K), ρ = 1500 kg/m^3, and c = 0.9 kJ/(kg·K).

SOLUTION

The temperatures of the interior grid locations 1–5 can be calculated from Eq. (12.27*b*) as

$$T_n^{p+1} = \left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right] T_n^p + \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^p + T_{n-1}^p\right),$$
$$n = 1, 2, 3, 4, \text{ and } 5$$

and the surface temperatures can be found using Eq. (12.32*b*).

$$T_0^{p+1} = \frac{\alpha\Delta t}{(\Delta x)^2}\left\{\left[\frac{(\Delta x)^2}{\alpha\Delta t} - 2\frac{h\Delta x}{k} - 2\right] T_0^p + 2T_1^p + 2\frac{h\Delta x}{k} T_\infty\right\}$$

$$T_6^{p+1} = \frac{\alpha\Delta t}{(\Delta x)^2}\left\{\left[\frac{(\Delta x)^2}{\alpha\Delta t} - 2\frac{h\Delta x}{k} - 2\right] T_6^p + 2T_5^p + 2\frac{h\Delta x}{k} T_\infty\right\}$$

Since a convection boundary condition prevails at the boundaries, for a stable solution

$$\frac{(\Delta x)^2}{\alpha\Delta t} \geq 2\left(\frac{h\Delta x}{k} + 1\right)$$

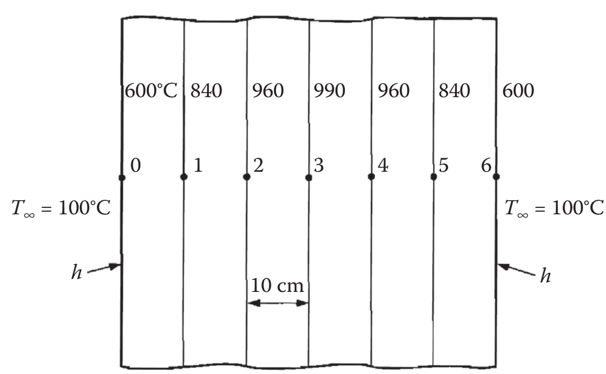

FIGURE 12.13
Figure for Example 12.4.

On the other hand, $h\,\Delta x/k = 0.5$ and, therefore,

$$\frac{(\Delta x)^2}{\alpha\Delta t} \geq 3$$

If $(\Delta x)^2/\alpha\,\Delta t = 3$ is selected, then we can write

$$T_n^{p+1} = \frac{1}{3}\left(T_n^p + T_{n+1}^p + T_{n-1}^p\right), \qquad n = 1, 2, 3, 4, \text{ and } 5$$

and

$$T_0^{p+1} = \frac{1}{3}\left(2T_1^p + T_\infty\right), \qquad T_6^{p+1} = \frac{1}{3}\left(2T_5^p + T_\infty\right)$$

Furthermore

$$\Delta t = \frac{(\Delta x)^2}{3\alpha} = \frac{0.1 \times 1500 \times 0.9}{3 \times 3 \times 10^{-3}} = 1500\,\text{s} = 25\,\text{min}$$

Now, the temperatures of the grid locations can be calculated using the above relations as a function of time with 25-min time intervals as illustrated in Table 12.2.

12.7.2 Implicit Method

In the explicit method discussed in the previous section, we saw that the requirements, such as $\alpha\Delta t/(\Delta x)^2 \leq 1/2$ for interior grid locations, place an undesirable restriction on the time increment Δt. For problems extending over large values of times, this could result in excessive amounts of computation. Furthermore, in the explicit method. T_n^{p+1} depends only on T_n^p, T_{n+1}^p and T_{n-1}^p as depicted in Fig. 12.14. It is natural, however, to expect that T_n^{p+1} should also depend on, for example, T_{n+2}^p.

The implicit method, to be discussed now, overcomes both of these difficulties at the expense of a somewhat more complicated calculational procedure. In this method, $\partial^2 T/\partial x^2$

TABLE 12.2

Results of Example 12.4†

p	t (min)	T_0	T_1	T_2	T_3	T_4	T_5	T_6
0	0	600	840	960	990	960	840	600
1	25	593	800	930	970	930	800	593
2	50	566	774	900	943	900	774	566
3	75	549	747	872	914	872	947	549
4	100	531	723	844	886	844	723	531

†Temperatures are given in °C.

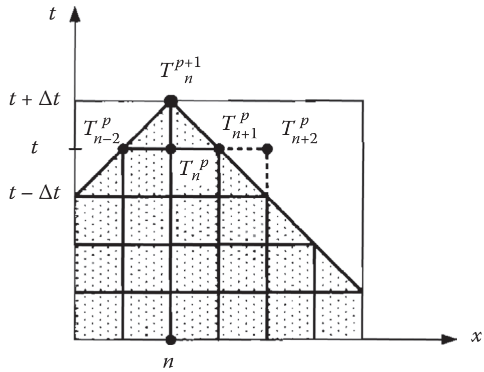

FIGURE 12.14
Limitation of the explicit method.

is replaced by a finite-difference form evaluated at $t + \Delta t$, and $\partial T/\partial t$ by a backward finite-difference form. Then, the differential equation (12.24) becomes

$$\frac{T_{n+1}^{p+1} + T_{n-1}^{p+1} - 2T_n^{p+1}}{(\Delta x)^2} = \frac{1}{\alpha}\frac{T_n^{p+1} - T_n^p}{\Delta t} \tag{12.33a}$$

which can be written as

$$T_n^p = \left[1 + \frac{2\alpha\Delta t}{(\Delta x)^2}\right]T_n^{p+1} - \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^{p+1} + T_{n-1}^{p+1}\right) \tag{12.33b}$$

We note that this formulation does not permit the explicit calculation of T_n^{p+1} in terms of T_n^p. Rather, at any one time level, Eq. (12.33*b*) is written once for each grid point, resulting in a system of algebraic equations that must be solved simultaneously to determine the temperatures T^{p+1}. The solution of such coupled equations, of course, is much more difficult than the solution of the uncoupled ones that result in the explicit method. On the other hand, no restriction is imposed on the step size Δx or the time increment Δt because of the stability conditions. This means that larger time increments can be selected to speed the calculations. The finite-difference equations resulting from the use of the implicit method can be solved using the methods discussed in Section 12.6.

12.7.3 Crank–Nicolson Method

In the Crank–Nicolson method, the arithmetic average of Eqs. (12.27*a*) and (12.33*a*) is taken, yielding

$$\frac{1}{2}\left[\frac{T_{n+1}^p + T_{n-1}^p - 2T_n^p}{(\Delta x)^2} + \frac{T_{n+1}^{p+1} + T_{n-1}^{p+1} - 2T_n^{p+1}}{(\Delta x)^2}\right] = \frac{1}{\alpha}\frac{T_n^{p+1} - T_n^p}{\Delta t} \tag{12.34a}$$

which can also be written as

$$2\left[1+\frac{\alpha\Delta t}{(\Delta x)^2}\right]T_n^{p+1}-\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^{p+1}+T_{n-1}^{p+1}\right)$$
$$=2\left[1-\frac{\alpha\Delta t}{(\Delta x)^2}\right]T_n^{p}+\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^{p}+T_{n-1}^{p}\right) \tag{12.34b}$$

This result is similar to Eq. (12.33*b*) for the implicit method. It can be shown that the Crank–Nicolson method is stable for all values of $\alpha\Delta t/(\Delta x)^2$ and converges, as we discuss in Section 12.10, with the truncation error of the order of the magnitude of $[(\Delta t)^2 + (\Delta x)^2]$. This is a distinct improvement over the previous two methods because both the explicit and the implicit methods lead to a truncation error of the order of the magnitude of $[\Delta t + (\Delta x)^2]$. On the other hand, the finite-difference equations in this case are slightly more involved than the equations of the fully implicit method.

A weighted average of Eq. (12.27*a*) and Eq. (12.33*a*) can also be taken. In the literature, there are other methods for finite differencing obtained in this way. Interested readers can refer to References [5,7] for details of the other methods.

12.8 Finite-Difference Formulation of Two-Dimensional, Unsteady-State Problems in Rectangular Coordinates

Consider a two-dimensional solid as shown in Fig. 12.15. Within this solid, assuming constant thermophysical properties, the temperature distribution satisfies

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{12.35}$$

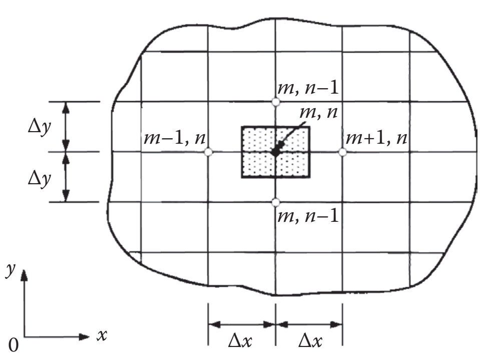

FIGURE 12.15
Nomenclature for the numerical solution of two-dimensional unsteady-state heat conduction in rectangular coordinates.

The initial temperature distribution at $t = 0$ is specified at all points on and within the boundary, and the subsequent conditions on the boundary are usually either of the first, the second, or the third kind. Divide this solid into increments in the x and y directions as we did before.

As in the one-dimensional and unsteady-state case, the finite-difference approximation of Eq. (12.35) can be written in a number of forms. With the notation of Fig. 12.15, the explicit finite-difference representation of Eq. (12.35) is

$$\frac{T^p_{m+1,n} + T^p_{m-1,n} - 2T^p_{m,n}}{(\Delta x)^2} + \frac{T^p_{m,n+1} + T^p_{m,n-1} - 2T^p_{m,n}}{(\Delta y)^2} = \frac{1}{\alpha}\frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta t} \tag{12.36a}$$

Thus, at each grid point, the temperature at any given time is represented in terms of the known values of the previous time step. If the step sizes in the x and y directions are chosen to be the same, that is $\Delta x = \Delta y$, then Eq. (12.36a) can be written as

$$T^{p+1}_{m,n} = \frac{\alpha \Delta t}{(\Delta x)^2}\left(T^p_{m+1,n} + T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1}\right) + \left[1 - 4\frac{\alpha \Delta t}{(\Delta x)^2}\right]T^p_{m,n} \tag{12.36b}$$

As in the case of one-dimensional and unsteady-state problems, this explicit method is stable for certain values of Δx and Δt. For problems with prescribed boundary temperatures for $t \geq 0$, the limit of stability is given by

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{4}$$

The finite-difference equation (12.36b) is useful for determining the temperatures of the interior grid points. If the boundary condition is not of the first kind, then new finite-difference relations have to be developed for the grid points on the boundary as we did in the one-dimensional case. Thus, for a grid point on a boundary where the condition is of the third kind, the finite-difference relation is given by

$$\begin{aligned} T^{p+1}_{m,n} = \frac{\alpha \Delta t}{(\Delta x)^2}\Bigg\{ & 2\frac{h\Delta x}{k}T_\infty + 2T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1} \\ & + \left[\frac{\alpha \Delta t}{(\Delta x)^2} - 2\frac{h \Delta x}{k} - 4\right]T^p_{m,n}\Bigg\} \end{aligned} \tag{12.37}$$

In this case, to ensure the stability of the solution of the finite-difference equations, $(\Delta x)^2/\alpha\Delta t$ should be restricted as

$$\frac{(\Delta x)^2}{\alpha \Delta t} \geq 2\left(\frac{h\Delta x}{k} + 2\right)$$

An implicit method of representing Eq. (12.35) in terms of finite-difference equations can also be given as

$$\frac{T^{p+1}_{m+1,n} + T^{p+1}_{m-1,n} - 2T^{p+1}_{m,n}}{(\Delta x)^2} + \frac{T^{p+1}_{m,n+1} + T^{p+1}_{m,n-1} - 2T^{p+1}_{m,n}}{(\Delta y)^2} = \frac{1}{\alpha}\frac{T^{p+1}_{m,n} - T^{p}_{m,n}}{\Delta t} \tag{12.38a}$$

If Δx is taken to be equal to Δy, then Eq. (12.38) can be rewritten as

$$T^{p}_{m,n} = \left[1 + \frac{4\alpha\Delta t}{(\Delta x)^2}\right]T^{p+1}_{m,n} - \frac{\alpha\Delta t}{(\Delta x)^2}\left(T^{p+1}_{m+1,n} + T^{p+1}_{m-1,n} + T^{p+1}_{m,n+1} + T^{p+1}_{m,n-1}\right) \tag{12.38b}$$

This formulation, although it is stable for all values of Δx and Δy, does not permit the calculation of T^{p+1} explicitly in terms of T^{p}. Rather, it requires the solution of a large number of simultaneous algebraic equations at each time step.

Example 12.5

Consider the solid body, shown in Fig. 12.16, that is initially at a uniform temperature of 100°C. For times $t \geq 0$, the boundary temperatures are maintained at the values given in the figure. Find the temperatures of the grid points *A*, *B*, *C*, and *D* as a function of time.

SOLUTION

If $\alpha\,\Delta t/(\Delta x)^2 = 1/4$ is selected, then the temperatures at the points *A*, *B*, *C*, and *D* can be calculated from

$$T^{p+1}_{A} = \frac{1}{4}\left(1000 + T^{p}_{B}\right), \quad T^{p+1}_{B} = \frac{1}{4}\left(600 + T^{p}_{A} + T^{p}_{C}\right)$$

$$T^{p+1}_{C} = \frac{1}{4}\left(600 + T^{p}_{B} + T^{p}_{D}\right), \quad T^{p+1}_{D} = \frac{1}{4}\left(600 + 2T^{p}_{C}\right)$$

The temperatures of these points as a function of time are tabulated in Table 12.3.

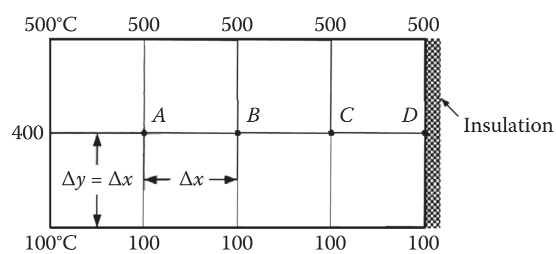

FIGURE 12.16
Figure for Example 12.5.

TABLE 12.3

Results of Example 12.5[†]

p	t	T_A	T_B	T_C	T_D
0	0	100	100	100	100
1	$\Delta t = (\Delta x)^2/4\alpha$	275	200	200	200
2	$2\Delta t$	300	269	250	250
3	$3\Delta t$	317	288	280	275
4	$4\Delta t$	322	299	291	289
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
∞	∞	327	307	302	300

[†]Temperatures are given in °C.

12.9 Finite-Difference Formulation of Problems in Cylindrical Coordinates

We now consider the finite-difference formulation of problems in cylindrical coordinates. The concepts that we introduce here can readily be extended to problems in spherical coordinates. The steady-state temperature distribution in problems in cylindrical coordinates, without internal energy sources and sinks, satisfies

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{12.39}$$

if the thermal conductivity is constant. For problems of the type $T(r, \phi)$, where there is no variation in T along the z axis, Eq. (12.39) reduces to

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0 \tag{12.40}$$

This equation can be represented in finite-difference form at the grid point (m, n) shown in Fig. 12.17 as

$$\begin{aligned} &\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta r)^2} + \frac{1}{r_m}\frac{T_{m+1,n} - T_{m-1,n}}{2\Delta r} \\ &+ \frac{1}{r_m^2}\frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta\phi)^2} = 0 \end{aligned} \tag{12.41}$$

where $r_m = m\Delta r$. For nonzero values, finite difference approximations can be easily employed in Eq. (12.40), but for $r = 0$, the right-hand side of the equation contains singularities. This

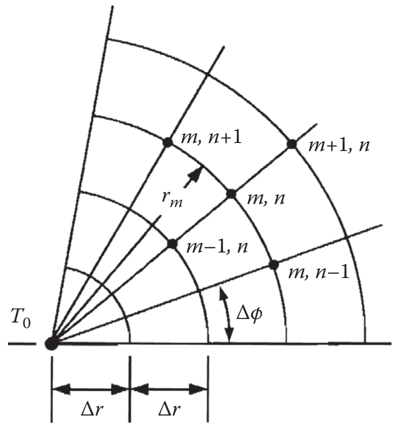

FIGURE 12.17
Nomenclature for finite-difference formulation in cylindrical polar coordinates.

difficulty can be overcome by replacing the polar form of $\nabla^2 T$ by its equivalent in Cartesian coordinates, which transforms Eq. (12.40) into the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

drawing a circle of radius Δr centered at the origin $r = 0$, by tracing the axis xy and denoting the four points of the axes that cross the circle by (T_1, T_2, T_3 and T_4) and placing one more point in the origin (T_0), then $\nabla^2 T$ can be approximated by

$$\nabla^2 T = \frac{T_1 + T_3 - 2T_0}{(\Delta r)^2} + \frac{T_2 + T_4 - 2T_0}{(\Delta r)^2} = \frac{T_1 + T_2 + T_3 + T_4 - 4T_0}{(\Delta r)^2}$$

Rotating the xy axis at different small angles clearly shows a similar equation. Repeating this rotation and summing up all these equations, then $\nabla^2 T$ can be approximated by

$$\nabla^2 T == \frac{4(T_M - T_0)}{(\Delta r)^2}$$

where T_0 is the temperature at $r = 0$ and T_M is the mean temperature of grid points that surround $r = 0$ [18].

When Eq. (12.41) is applied to every grid point, together with the boundary conditions, it results in a system of simultaneous linear algebraic equations similar to those studied previously in rectangular coordinates.

For problems of the type $T(r, z)$ where there is no variation in T along the ϕ direction, Eq. (12.39) reduces to

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{12.42}$$

which can be represented in finite-difference form as

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta r)^2}+\frac{1}{r_m}\frac{T_{m+1,n}-T_{m-1,n}}{2\Delta r}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta z^2)}=0 \tag{12.43}$$

where n represents the number of increments in the z direction, and m the number of increments in the r direction. At $r = 0$, it can be shown that the Laplace equation (12.42) is equivalent to

$$2\frac{\partial^2 T}{\partial r^2}+\frac{\partial^2 T}{\partial z^2}=0 \tag{12.44}$$

which can easily be represented in finite-difference form.

Similar approaches are, of course, necessary for the treatment of *unsteady-state* problems in cylindrical coordinates.

12.10 Errors in Finite-Difference Solutions

Let $T(x)$ and its derivatives be single-valued, finite, and continuous functions of x. The Taylor series of $T(x)$ about x can be written either as

$$T(x+\Delta x)=T(x)+\left.\frac{dT}{dx}\right|_x\Delta x+\frac{1}{2!}\left.\frac{d^2T}{dx^2}\right|_x(\Delta x)^2+\frac{1}{3!}\left.\frac{d^3T}{dx^3}\right|_x(\Delta x)^3+\frac{1}{4!}\left.\frac{d^4T}{dx^4}\right|_x(\Delta x)^4+\cdots \tag{12.45a}$$

or

$$T(x-\Delta x)=T(x)-\left.\frac{dT}{dx}\right|_x\Delta x+\frac{1}{2!}\left.\frac{d^2T}{dx^2}\right|_x(\Delta x)^2-\frac{1}{3!}\left.\frac{d^3T}{dx^3}\right|_x(\Delta x)^3+\frac{1}{4!}\left.\frac{d^4T}{dx^4}\right|_x(\Delta x)^4+\cdots \tag{12.45b}$$

Adding Eqs. (12.45*a*) and (12.45*b*), we obtain,

$$T(x+\Delta x)+T(x-\Delta x)=2T(x)+\left.\frac{d^2T}{dx^2}\right|_x(\Delta x)^2+O[(\Delta x)^4] \tag{12.46}$$

where $O[(\Delta x)^4]$ denotes terms containing fourth and higher powers of Δx. We can rewrite Eq. (12.46) in the form

$$\left.\frac{d^2T}{dx^2}\right|_x = \frac{T(x+\Delta x)+T(x-\Delta x)-2T(x)}{(\Delta x)^2} + O[(\Delta x)^2] \tag{12.47}$$

Comparing Eqs. (12.5*c*) and (12.47), we see that the finite-difference approximation of d^2T/dx^2 has a truncation error of the order of the magnitude of $(\Delta x)^2$. That is, doubling Δx quadruples the truncation error.

Similarly, by subtracting Eq. (12.45*a*) from Eq. (12.45*b*) and rearranging the resulting expression, we get

$$\left.\frac{dT}{dx}\right|_x = \frac{T(x+\Delta x)-T(x-\Delta x)}{2\Delta x} + O[(\Delta x)^2] \tag{12.48}$$

Comparing Eqs. (12.4) and (12.48) shows that the central-difference approximation of the first derivative of $T(x)$ also has a truncation error of the order of magnitude $(\Delta x)^2$.

From Eq. (12.45*a*) we can write

$$\left.\frac{dT}{dx}\right|_x = \frac{T(x+\Delta x)-T(x)}{\Delta x} + O[\Delta x] \tag{12.49}$$

and from Eq. (12.45*b*) we have

$$\left.\frac{dT}{dx}\right|_x = \frac{T(x)-T(x-\Delta x)}{\Delta x} + O[\Delta x] \tag{12.50}$$

Thus, we see that both the forward- and backward-difference representations of the first derivative of $T(x)$ have truncation errors of the order of magnitude of Δx. Therefore, the central-difference form is a more accurate approximation than either the forward- or backward-difference approximation.

When a differential equation is expressed in finite-difference form, the so-called truncation error is introduced into the resulting algebraic equations because of the truncation of the Taylor series expansion of the derivatives. This error is inherent in the method, and it cannot be avoided. It is also independent of the characteristics of the computing equipment. It may be reduced only by selecting a finer grid, that is, smaller sizes for space and time increments.

Finite-difference equations are algebraic equations that are usually solved numerically, and the numerical calculations are carried out only to a finite number of decimal places. In each numerical calculation during the solution, the results are rounded off, and because of this, *round-off* errors are introduced. Since round-off errors can accumulate, they may cause a large cumulative error. It is, actually, difficult to estimate the order of the magnitude of cumulative round-off errors or the cumulative departure of the solution from the true value due to round-off errors. The use of small step sizes increases the accumulation

TABLE 12.4

Various Differencing Schemes and the Truncation Leading Error

Derivative	Grid Points	Finite Difference Form	Error Order
$\dfrac{\partial T(x)}{\partial x}$	2	$\dfrac{T_{m+1}-T_m}{\Delta x}$ (*forward*)	$O(\Delta x)$
$\dfrac{\partial T(x)}{\partial x}$	2	$\dfrac{T_m-T_{m-1}}{\Delta x}$ (*backward*)	$O(\Delta x)$
$\dfrac{\partial T(x)}{\partial x}$	2	$\dfrac{T_{m+1}-T_{m-1}}{\Delta x}$ (*central*)	$O(\Delta x^2)$
$\dfrac{\partial T(x)}{\partial x}$	3	$\dfrac{-3T_m+4T_{m+1}-T_{m+2}}{2\Delta x}$ (*forward*)	$O(\Delta x^2)$
$\dfrac{\partial T(x)}{\partial x}$	3	$\dfrac{T_{m-2}-4T_{m-1}+3T_m}{2\Delta x}$ (*backward*)	$O(\Delta x^2)$
$\dfrac{\partial^2 T(x)}{\partial x^2}$	3	$\dfrac{T_m-2T_{m+1}+T_{m+2}}{\Delta x^2}$ (*forward*)	$O(\Delta x)$
$\dfrac{\partial^2 T(x)}{\partial x^2}$	3	$\dfrac{T_{m-2}-2T_{m-1}+T_m}{\Delta x^2}$ (*backward*)	$O(\Delta x)$
$\dfrac{\partial^2 T(x)}{\partial x^2}$	3	$\dfrac{T_{m-1}-2T_m+T_{m+1}}{\Delta x^2}$ (*central*)	$O(\Delta x^2)$
$\dfrac{\partial^2 T(x)}{\partial x^2}$	4	$\dfrac{(2T_m-5T_{m+1}+4T_{m+2}-T_{m+3})}{\Delta x^2}$ (*forward*)	$O(\Delta x^2)$
$\dfrac{\partial^2 T(x)}{\partial x^2}$	4	$\dfrac{(-T_{m-3}+4T_{m-2}-5T_{m-1}+2T_m)}{\Delta x^2}$ (*backward*)	$O(\Delta x^2)$

of round-off errors, although they are desired for less truncation error, that is, for better approximation. Round-off errors are determined by the characteristics of the machine on which the calculations are carried out.

Often, it may be necessary to represent the first derivative at a node m by using more than two grid points in order to improve the accuracy of approximation; in this sense, a three-point or a four-point formula can be developed through the use of Taylor series expansion. Similarly, the finite-difference approximation for the second derivative given above utilizes three grid points. Approximation utilizing more than three points can also be developed. Table 12.4 lists some such representations.

12.11 Convergence and Stability

The accuracy of the results obtained by solving the finite-difference equations numerically is hard to estimate. It is known, however, that if the numerical method satisfies two criteria termed "convergence" and "stability," the accuracy is determined by the step sizes used, and an increased accuracy may be obtained at the expense of increased labor.

If the approximate numerical solution approaches the exact solution as the grid spacings in time and space approach zero, then the solution is said to be *convergent*. The numerical method is unsatisfactory unless the numerical solution converges to the exact solution in the limit, and a numerical method that converges in the limit to the exact solution is said to fulfill the convergence criterion.

In replacing the differential equation and boundary conditions by finite-difference equations, finite step sizes in both time and space are used, and only a finite number of significant figures can be carried in the calculations. These are practical limitations, and they introduce both truncation and round-off errors into the analysis as discussed in the previous section. These errors may not be serious unless they grow in magnitude as the solution proceeds. If they grow as a solution proceeds and this growth is unbounded, the solution is said to be *unstable*. A numerical method that prevents the growth of errors is said to fulfill the stability criterion. Stability is, in fact, necessary for convergence. Instability also results if errors grow at a rate faster than that at which convergence is approached. Price and Slack [14] discuss the stability and accuracy of the numerical solutions of heat conduction problems. The following is a brief outline of the procedure to determine the stability of finite-difference methods. This procedure was first given by O'Brien et al. [12], and it is also discussed in References [4,13].

To explain the procedure, let us consider the explicit finite-difference form (12.27*b*):

$$T_n^{p+1} = \left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right]T_n^p + \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^p + T_{n-1}^p\right) \tag{12.51}$$

The solution of Eq. (12.24) can be expanded at any time t into a Fourier series in x, and a typical term in the expansion, neglecting the constant coefficient, will be in the form of $\phi(t)\, e^{i\beta x}$, where $i = \sqrt{-1}$. By substituting this term into the finite-difference equation (12.51), the form of $\phi(t)$ can be found, and a criterion can thereby be established as to whether it remains bounded as t becomes large. Substituting $T_n^p = \phi(t)e^{i\beta x}$, we have

$$\phi(t+\Delta t)e^{i\beta x} = \left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right]\phi(t)e^{i\beta x} + \frac{\alpha\Delta t}{(\Delta x)^2}\left[e^{i\beta(x+\Delta x)} + e^{i\beta(x-\Delta x)}\right]\phi(t) \tag{12.52}$$

Rearranging this result we obtain

$$\frac{\phi(t+\Delta t)}{\phi(t)} = 1 - 4\frac{\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{\beta\Delta x}{2} \tag{12.53}$$

For stability, $\phi(t)$ must remain bounded as Δt, and thus Δx, approach zero. Clearly, this requires

$$\max\left|1 - 4\frac{\alpha\Delta t}{(\Delta x)^2}\sin^2\frac{\beta\Delta x}{2}\right| \le 1 \tag{12.54}$$

for all values of β. In fact, components of all frequencies of β may be present; if they are not present in the initial condition, or are not introduced by the boundary conditions, they can be introduced by the round-off errors. To satisfy the condition (12.54), we see that $\alpha\Delta t/(\Delta x)^2$ can be at most 1/2. That is, for the explicit finite-difference form (12.51),

$$\frac{\alpha\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

is a necessary (and sufficient) condition for stability.

12.12 Graphical Solutions

One-dimensional unsteady-state heat conduction problems may also be solved by the graphic method known as the *Schmidt plot*. As we have seen in Section 12.7.1, if Δx and Δt satisfy the condition

$$\frac{\alpha\Delta t}{(\Delta x)^2} = \frac{1}{2}$$

then the temperature at location m and at time $(p + 1)\Delta t$ will be equal to the arithmetic average of the temperatures at locations $(m + 1)$ and $(m - 1)$ at the time $\rho\Delta t$; that is,

$$T_m^{p+1} = \frac{1}{2}\left(T_{m+1}^p + T_{m-1}^p\right)$$

As illustrated in Fig. 12.18, this arithmetic average can easily be constructed graphically.

For a graphical solution of a one-dimensional heat conduction problem, we first draw the initial condition and divide the body into increments of Δx as shown in Fig. 12.19.

The temperature T_m^1 at the location m at time Δt [= ½$(\Delta x)^2/\alpha$] is simply the intersection of the line drawn between T_{m-1}^0 and T_{m+1}^0, with the grid line at location m. This graphic construction, which is known as a Schmidt plot, is repeated until the temperature distribution is obtained at the desired time.

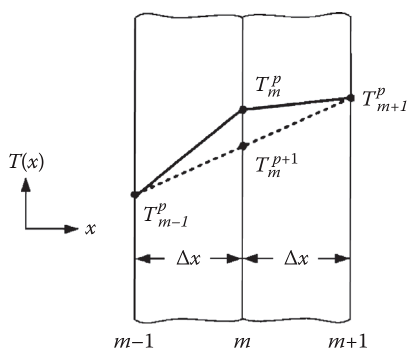

FIGURE 12.18
Graphical determination of T_m^{p+1} when $\alpha\Delta t/(\Delta x)^2 = 1/2$.

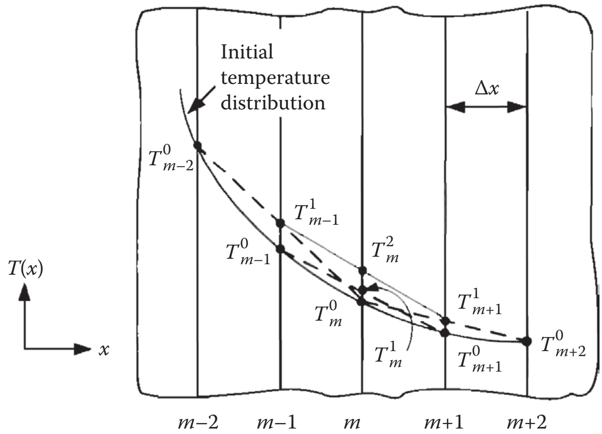

FIGURE 12.19
Schmidt plot technique.

Example 12.6

A plane wall of thickness 60 cm is initially at a uniform temperature of 35°C. At $t = 0$, its surface temperatures are suddenly raised to 150°C and 350°C, and are kept constant at these values for times $t \geq 0$. Find the temperature at a depth of 15 cm from the surface maintained at 150°C after 15 h have passed. The thermal diffusivity of the material of the wall is $\alpha = 0.52 \times 10^{-6}\text{m}^2/\text{s}$.

SOLUTION

To start with the graphical method, let us divide the wall into increments of $\Delta x = 7.5$ cm as shown in Fig. 12.20. The time increment Δt is then given by

$$\Delta t = \frac{(\Delta x)^2}{2\alpha} = \frac{(0.075)^2}{2 \times 0.52 \times 10^{-6} \times 3600} = 1.502\,\text{h}$$

The temperature distributions in the wall at various Δt are constructed approximately by the Schmidt plot, which can easily be followed in Fig. 12.20, where the temperature distributions are depicted after 12 time increments. The temperature at a depth of 15 cm from the surface kept at 150°C, after 15 h, which is approximately equal to 10 time increments, is found to be ~130°C.

The graphical solution with a boundary condition of the third kind, that is, with heat transfer by convection to or from an ambient fluid at a prescribed temperature and with a prescribed heat transfer coefficient may be constructed in the following way: Eq. (12.29) can be written as

$$-\left.\frac{\partial T}{\partial x}\right|_{x=L} = \frac{T_w - T_\infty}{k/h} \tag{12.55}$$

which is the basis for obtaining the surface temperature by Schmidt plot as illustrated in Fig. 12.21. If a line is drawn between T_∞ and T_m^p, the intersection of this line with the surface gives the surface temperature at that particular time. If T_∞ and h are functions of time, the above procedure can still be used to find the surface temperature by simply varying the distances corresponding to T_∞ and k/h.

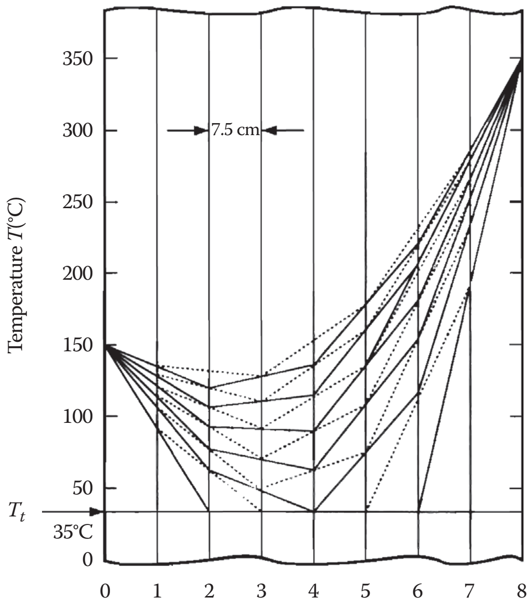

FIGURE 12.20
Schmidt plot solution for Example 12.6.

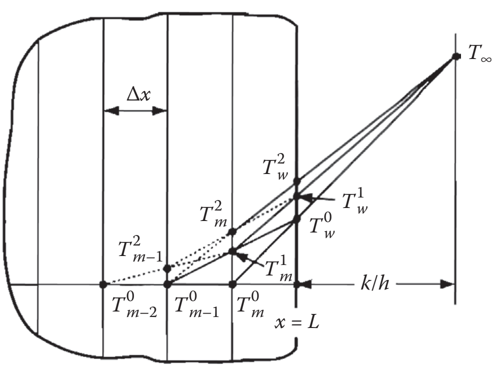

FIGURE 12.21
Schmidt plot with convection boundary condition.

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Problems

12.1 Derive Eq. (12.14).

12.2 Derive Eq. (12.15).

12.3 Obtain an expression in finite-difference form for the temperature of the corner grid point (m, n) of the two-dimensional solid shown in Fig. 12.5 under unsteady-state conditions.

12.4 Consider the one-dimensional fin shown in Fig. 12.22. The base temperature T_b, the surrounding fluid temperature T_∞, and the heat transfer coefficient h are known constants. Divide the fin into six segments as shown, and formulate the problem in terms of finite differences to calculate the temperatures at the six points shown, under steady-state conditions.

12.5 The fin of Problem 12.4 is made of brass, k = 111 W/(m·K), and has a length of 2.5 cm with a 0.1 cm by 30 cm cross section. The base temperature of the fin is 90°C. The surrounding fluid temperature is 20°C, and the heat transfer coefficient is 15 W/(m²·K). Estimate the temperature distribution along the fin and the rate of heat loss to the surrounding fluid.

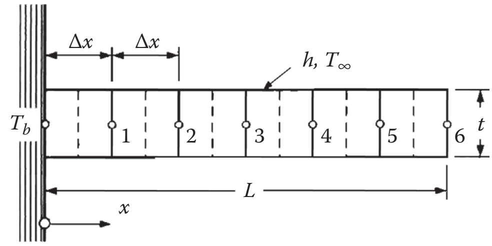

FIGURE 12.22
Figure for Problem 12.4.

12.6 A copper bar of length 15 cm and diameter 0.5 cm is braced to a steel bar with equal length and diameter. This composite bar is placed in an air flow at 0°C. The heat transfer coefficient is 125 W/(m^2·K). The copper-side end temperature of the combined bar is 100°C, and the steel-side end temperature is 50°C. Find the steady-state temperature distribution along the combined bar using the relaxation method.

12.7 Obtain an expression for the residual $R_{m,n}$ at the insulated boundary of Fig. 12.6, if heat is generated within the solid at a rate of $\dot{q}$ per unit volume.

12.8 Figure 12.23 shows the cross section of a long steel bar. The temperatures of the surfaces are as indicated in the figure. Estimate the temperatures at points *A*, *B*, *C*, *D*, *E*, and *F*.

12.9 The cross section of a chimney is shown in Fig. 12.24. The inside and outside surface temperatures are as indicated in the figure. Estimate the temperatures at the points shown.

12.10 The cross section of a long solid bar is shown in Fig. 12.25. The boundaries are insulated except for two, which are kept at constant temperatures as indicated in the figure. Estimate the steady-state temperature distribution in the solid, and the heat transfer rates across the top and bottom surfaces.

12.11 A dry wooden plate is initially at 0°C. It is suddenly exposed to air at 30°C. The plate is 10 cm thick and the other two dimensions are very large. The heat transfer

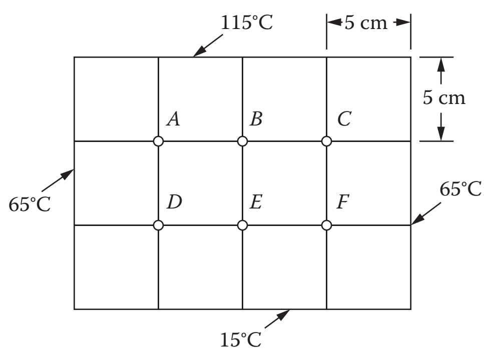

FIGURE 12.23
Figure for Problem 12.8.

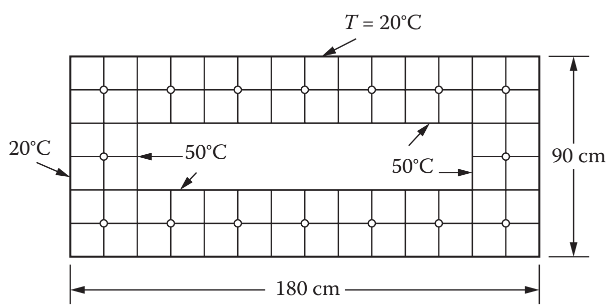

FIGURE 12.24
Figure for Problem 12.9.

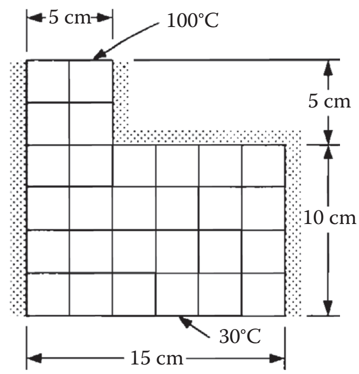

FIGURE 12.25
Figure for Problem 12.10.

coefficient on both sides of the plate is 10 W/(m^2·K), and the properties of the wood are k = 0.109 W/(m·K), ρ = 417 kg/m^3, and c = 2720 J/(kg·K). After 2 h of heating, find the temperatures of the surfaces and the midplane of the plate,

(*a*) by dividing the thickness of the plate into four divisions and applying a finite-difference method, and

(*b*) by the Schmidt plot technique.

12.12 A steel plate of thickness 3.56 cm is initially at 650°C. It is suddenly immersed in cold water. As a result of this, the surface temperatures drop to 94°C and stay at this value for the rest of the cooling process. Estimate the time elapsed for the midplane temperature to decrease to 450°C. For steel, take $\alpha = 1.16 \times 10^{-5}$m^2/s.

12.13 A thin steel plate of thickness 1.6 cm comes out of a mill at 540°C with a velocity of 2.44 m/s. The temperature of the surfaces of decreases linearly at 110°C/m. Assume that the heat flow is in the direction normal to the surfaces. By dividing the thickness of the plate by eight, calculate the temperature distribution across the thickness of the plate and the total surface heat flux at a distance 2.74 cm from the mill. For steel, k = 43.3 W/(m·K) and $\alpha = 0.98 \times 10^{-5}$m^2/s may be taken.

12.14 While the fin given in Problem 12.4 is at a uniform temperature T_i at $t = 0$, it is exposed to an ambient at temperature T_∞ for times $t \geq 0$. Give a formulation of the problem in terms of finite differences.

12.15 Assume that the chimney cross section given in Problem 12.9 is initially at 20°C. The surface temperatures are suddenly changed at $t = 0$ to the values indicated in Fig. 12.24 and are kept at these values for times $t > 0$. Estimate the temperatures at the grid points shown as a function of time.

12.16 The long steel bar given in Problem 12.8 is initially at 65°C. The temperatures of the bottom and top surfaces are then suddenly changed at $t = 0$ to 15°C and 115°C, respectively, and are kept constant at these values for times $t > 0$. Estimate the temperatures at the points shown at various times, and obtain the steady-state temperatures at these points. For steel, take $\alpha = 1.16 \times 10^{-5}$m^2/s.

12.17 Consider steady-state one-dimensional heat conduction in a solid cylinder with internal energy generation. Develop the finite difference solution using a uniform mesh ($\Delta r = 1/4$) with 5 grid points (r_0, r_1, r_2, r_3, and r_4), employing a central differencing scheme with a truncation order of $O(\Delta x^2)$ for all spatial derivatives.

Present the discretized equation for all grid points and construct the resulting algebraic matrix system.

$$\frac{d^2T(r)}{dr^2}+\frac{1}{r}\frac{dT(r)}{dr}+\frac{g(r)}{k}=0 \qquad 0<r<1$$

$$\text{em } r=0: \qquad \left.\frac{dT(r)}{dr}\right|_{r=0}=0$$

$$\text{em } r=1: \qquad T(r=1)=T_\infty$$

12.18 Consider the dimensionless diffusion problem given by

$$\frac{\partial^2\theta(x,t)}{\partial x^2}=\frac{\partial\theta(x,t)}{\partial t}, \qquad 0<x<1, \qquad t>0$$

$$\begin{array}{llll} \theta=0 & em & x=0, & t>0 \\ \theta=0 & em & x=1, & t>0 \\ \theta=Sin(\pi x) & em & t=0, & 0<x<1 \end{array}$$

The exact solution to this problem is given by $\theta(x,t)=e^{-\pi^2 t}Sin(\pi x)$.

Develop the finite difference method solution for this problem using an explicit formulation. Use a uniform mesh with spacing of Δx = 0.1 employing a central differencing scheme with a truncation order of $O(\Delta x^2)$ for all spatial derivatives. Calculate the temperature at x = 0.5 for the case of $\Delta t/\Delta x^2$ = 0.5, and compare the numerical result obtained through the relative error with the exact solution at t = 0.005 and t = 0.01.

12.19 Using the Taylor series expansion, deduce the forward difference approximation for the first derivative, *df/dx*, using 3 points and considering a uniform spacing Δx.

12.20 Consider the problem given by

$$\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t}=\frac{\partial^2 T(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,t)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T(r,t)}{\partial\phi^2}+\frac{1}{k}g(r,\phi,t),$$

$$0\le r\le b, \quad ,0\le\phi\le 2\pi, \quad t>0$$

Employing a central differencing scheme with a truncation order of $O(\Delta x^2)$ for all spatial derivatives, develop the general equation for the internal grid points using the finite differences method applying

(*a*) Explicit formulation

(*b*) Implicit formulation

12.21 Using the Taylor series expansion, deduce the forward difference approximation for the first derivative, *df/dx*, to obtain a truncation order of $O(\Delta x^3)$.

13

Heat Conduction in Heterogeneous Media

13.1 Introduction

Diffusion problems in heterogeneous media are encountered in different contexts of physical sciences and engineering. Specifically on heat conduction in heterogeneous solids, a number of different situations can be identified, including composites with nonuniform nano- or microstructure, multilayer materials, solids with inclusions, nonhomogeneous porous materials, welded or bonded surfaces, etc. The result of the heterogeneity, either in an ordered or random manner, may be to a certain extent expressed by the spatial variation of the associated thermophysical properties and source terms.

Heat conduction problems in heterogeneous media involve spatial variations of thermophysical properties in different forms, depending on the type of heterogeneity involved, such as large-scale variations (as for example in functionally graded materials [FGMs]), abrupt changes in laminated composites, and random variations due to local fluctuations of concentration in dispersed systems, etc. In all these situations, an accurate representation of the heat transfer process requires a detailed local solution of the temperature behavior, since the conception, fabrication, and even self-structuring of the materials in such applications offer uncountable possibilities. Besides, the design of tailor-made materials, in general, requests the solution of computationally intensive optimization problems, requiring advanced techniques for the solution of the associated heat conduction problem and strongly relying on the availability of an accurate and cost-effective solution procedure.

The present chapter provides a systematic derivation of the analytical solution of heat conduction problems in heterogeneous media, introducing a more general perspective of the integral transform method [1–3]. In Chapter 7, the development of integral transform kernels in linear heat conduction, as applied to constant thermophysical properties and different coordinate systems, has been systematically presented, including the analysis of the corresponding specific eigenvalue problems. Here, the generalization of the integral transform method, also known as the generalized integral transform technique (GITT) [2,3], is employed in the solution of linear heat conduction problems with space variable coefficients, for both variable thermophysical properties and source terms, that incorporate all the heterogeneity of the medium under consideration.

Besides the exact analytical solution of diffusion problems with arbitrarily variable coefficients, the material that follows also discusses the concept of a single domain formulation for heterogeneous multidomain problems, originally proposed in the context of conjugated heat transfer problems [4], which upon reformulation are also handled through the same analytical solution with space variable coefficients [5]. Since the application of this analytical solution relies on the accurate solution of the associated eigenvalue problem, the hybrid numerical–analytical solution of the Sturm–Liouville problem with space variable coefficients, through the GITT [2,5], is also described. Finally, the solution approach is described in two different applications, aiming to illustrate the methodology proposed in the treatment of heterogeneities, represented by the spatial variations of the coefficients of the heat conduction equation, with different physical and mathematical natures in each example.

13.2 General Formulation and Formal Solution

The integral transform method described in this chapter comes from the application of the integral transform method in the more general treatment presented in Mikhailov and Ozışık [1], Cotta [2], and Cotta and Mikhailov [3]. The application of this method for the linear diffusion problem analyzed here results in a decoupled transformed system, which is readily solved in analytic form. On the other hand, the eigenvalue problem necessary to construct this exact solution requires the use of the GITT, as applied in the solution of eigenvalue problems, previously proposed for problems with variable coefficients and in irregular domains [2,5,6].

A sufficiently general formulation is considered, for a transient linear diffusion problem for the potential $T(\mathbf{x},t)$, the temperature distribution in the present context, defined in region V with boundary surface S, where $\mathbf{x}$ is the position vector and t is time. The proposed formulation includes the transient term, the diffusion operator, the linear dissipation term, and the source term [1,2], as shown in Eqs. (13.1) to (13.3). The arbitrarily space-variable coefficients $w(\mathbf{x})$, $k(\mathbf{x})$, and $d(\mathbf{x})$ are responsible for the information related to the heterogeneity of the medium, but eventually also include the space variable dependence corresponding to the specific coordinate system. Thus, the diffusion equation together with the initial and boundary conditions are given by

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = \nabla .k(\mathbf{x})\nabla T(\mathbf{x},t) - d(\mathbf{x})T(\mathbf{x},t) + P(\mathbf{x},t), \mathbf{x}\in V, t>0 \tag{13.1}$$

$$T(\mathbf{x},0) = f(\mathbf{x}), \quad \mathbf{x}\in V \tag{13.2}$$

$$\alpha(\mathbf{x})T(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} = g(\mathbf{x},t), \quad \mathbf{x}\in S \tag{13.3}$$

The boundary condition coefficients, $\alpha(\mathbf{x})$and $\beta(\mathbf{x})$, allow for recovering the different boundary condition types (prescribed temperature, prescribed heat flux, and convective condition), while $\mathbf{n}$ denotes the normal vector pointing towards leaving the medium. $P(\mathbf{x},t)$ and $g(\mathbf{x},t)$ correspond to the equation and boundary condition source terms, respectively, while $f(\mathbf{x})$ is the prescribed initial temperature distribution.

The correspondence with Chapter 2, where the general heat conduction equation is written in various coordinate systems, as presented in Table 2.3, can be readily established by taking $w(\mathbf{x}) = \rho c$, $k(\mathbf{x}) = k$, $d(\mathbf{x}) = 0$, and $P(\mathbf{x},t) = \dot{q}$. A convective boundary condition, for instance, as presented in Eq. (2.30b), can be obtained with $\alpha(\mathbf{x}) = h$, $\beta(\mathbf{x}) = 1$, and $g(\mathbf{x},t) = hT_\infty$.

The exact solution to problems (13.1) to (13.3) can be obtained through the integral transform method [1–3], as now described. The approach starts with the proposition of an eigenfunction expansion, with its basis obtained from a chosen eigenvalue problem. The natural choice towards finding the exact solution comes from applying the separation of variables methodology to the homogeneous version of the proposed problem. This homogeneous problem, without the equation and boundary source terms, is written as

$$w(\mathbf{x})\frac{\partial T_h(\mathbf{x},t)}{\partial t} = \nabla.k(\mathbf{x})\nabla T_h(\mathbf{x},t) - d(\mathbf{x})T_h(\mathbf{x},t),\ \mathbf{x}\in V, t>0 \tag{13.4}$$

$$T_h(\mathbf{x},0) = f(\mathbf{x}), \quad \mathbf{x}\in V \tag{13.5}$$

$$\alpha(\mathbf{x})T_h(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T_h(\mathbf{x},t)}{\partial \mathbf{n}} = 0, \quad \mathbf{x}\in S \tag{13.6}$$

where $T_h(\mathbf{x},t)$ is the solution of the homogeneous problem. The basic idea behind separation of variables is to consider the hypothesis that the dependent variable can be expressed as a separable product of the space and time independent variables, in the form

$$T_h(\mathbf{x},t) = \Gamma(t)\phi(\mathbf{x}) \tag{13.7}$$

This proposition is then substituted back into Eq. (13.4):

$$w(\mathbf{x})\phi(\mathbf{x})\frac{d\Gamma(t)}{dt} = \Gamma(t)\{\nabla.k(\mathbf{x})\nabla\phi(\mathbf{x}) - \phi(\mathbf{x})\} \tag{13.8}$$

and after dividing both sides of Eq. (13.8) by the product $w(\mathbf{x})\Gamma(t)\phi(\mathbf{x})$, it results to

$$\frac{1}{\Gamma(t)}\frac{d\Gamma(t)}{dt} = \frac{1}{w(\mathbf{x})\phi(\mathbf{x})}\{\nabla.k(\mathbf{x})\nabla\phi(\mathbf{x}) - \phi(\mathbf{x})\} \tag{13.9}$$

Clearly, the left-hand side of Eq. (13.9) is a function of t only, while the right-hand side is a function of $\mathbf{x}$ only. Thus, the only way that this equality can hold throughout the solution domain is if both sides are equal to the same constant. It may also be advanced that this constant should be negative, so as to take the solution in the time variable to zero as t goes to infinity. Therefore, equating both sides of Eq. (13.9) to the negative constant $-\lambda^2$, two problems are then constructed. First, the ordinary differential equation for the time-dependent function, $\Gamma(t)$,

$$\frac{d\Gamma(t)}{dt} + \lambda^2\Gamma(t) = 0 \tag{13.10}$$

which is readily solved as

$$\Gamma(t) = Ce^{-\lambda^2 t} \tag{13.11}$$

and the partial differential equation for the space variable dependent separable function, $\phi(\mathbf{x})$,

$$\nabla .k(\mathbf{x})\nabla\phi(\mathbf{x})+(\lambda^2 w(\mathbf{x})-d(\mathbf{x}))\phi(\mathbf{x})=0 \tag{13.12}$$

Separation of variables can be similarly applied to the boundary conditions, Eq. (13.2), to obtain the appropriate boundary conditions for Eq. (13.12):

$$\alpha(\mathbf{x})\phi(\mathbf{x})+\beta(\mathbf{x})k(\mathbf{x})\frac{\partial\phi(\mathbf{x})}{\partial\mathbf{n}}=0,\quad \mathbf{x}\in S \tag{13.13}$$

Equations (13.12) and (13.13) form the multidimensional version of the generalized Sturm–Liouville problem, discussed in Section 4.4. For a finite medium, this problem admits an infinite number of nontrivial solutions, for discrete nonnegative values of the parameter λ ($0<\lambda_1<\lambda_2<\lambda_3\ldots$), called the eigenvalues, and the corresponding solutions, $\phi_n(\mathbf{x})$, are called the eigenfunctions. The one-dimensional problem (4.13) is then readily recovered through the correspondence of the coefficients as $w(\mathbf{x})=w(x)$, $k(\mathbf{x})=p(x)$, $d(\mathbf{x})=q(x)$, and $\phi(\mathbf{x})=y(x)$. This eigenvalue problem is itself separable and analytically solvable in terms of special functions, for a few different functional forms of the governing coefficients [1], such as for the particular cases considered in the previous chapters with constant physical properties and for the most usual coordinate systems (Cartesian, cylindrical, and spherical). Otherwise, it can be solved by recalling the GITT, as will be detailed in the next section, and such a solution is assumed to be known at this point.

It can be readily shown, as described for the one-dimensional problem in Chapter 4, Eqs. (4.14) to (4.20), that the eigenfunctions enjoy the following orthogonality property with respect to the weighting function $w(\mathbf{x})$:

$$\int_V w(\mathbf{x})\phi_n(\mathbf{x})\phi_m(\mathbf{x})dv=\begin{cases}0, & \text{for}\quad n\neq m\\ N_n, & \text{for}\quad n=m\end{cases} \tag{13.14}$$

where the normalization integral, N_n, is evaluated from

$$N_n=\int_V w(\mathbf{x})\phi_n^2(\mathbf{x})dv \tag{13.15}$$

So far, the assumed separable product solution for the homogeneous problem resulted in an infinite number of solutions, one for each possible eigenvalue, that satisfy the diffusion equation and the boundary conditions, but not yet the initial condition. Therefore, it is proposed to form the linear combination of these solutions as

$$T_h(\mathbf{x},t)=\sum_{n=1}^{\infty}A_n e^{-\lambda_n^2 t}\phi_n(\mathbf{x}) \tag{13.16}$$

The proposed solution should then satisfy the initial condition for $t = 0$:

$$T_h(\mathbf{x},0) = f(\mathbf{x}) = \sum_{n=1}^{\infty} A_n \phi_n(\mathbf{x}) \tag{13.17}$$

Equation (13.17) is in fact the eigenfunction expansion of the initial condition, and should allow for the determination of the unknown coefficients A_n. Then, the orthogonality property should be recalled, by operating on both sides of Eq. (13.17) with the operator $\int_V w(\mathbf{x})\phi_m(\mathbf{x})\_\_dv$, to find

$$\int_V w(\mathbf{x}) f(\mathbf{x})\phi_m(\mathbf{x})dv = \sum_{n=1}^{\infty} A_n \int_V w(\mathbf{x})\phi_n(\mathbf{x})\phi_m(\mathbf{x})dv \tag{13.18}$$

Clearly, the orthogonality property is recovered on the right-hand side of Eq. (13.19), and therefore all of the terms in the summation go to zero, except one, when the index n assumes the value of m. Naming the left-hand side as $\bar{f}_m$, then the coefficients of the expansion are evaluated as

$$A_m = \frac{1}{\int_V w(\mathbf{x})\phi_m^2(\mathbf{x})dv} \int_V w(\mathbf{x}) f(\mathbf{x})\phi_m(\mathbf{x})dv = \frac{1}{N_m}\bar{f}_m \tag{13.19}$$

The exact solution for the homogeneous problem is then reconstructed as

$$T_h(\mathbf{x},t) = \sum_{n=1}^{\infty} \frac{1}{N_n} \bar{f}_n e^{-\lambda_n^2 t} \phi_n(\mathbf{x}) \tag{13.20}$$

The integral transform method is thus based on constructing the eigenfunction expansion for the nonhomogeneous problem, Eqs. (13.1) to (13.3), based on the above solution of the homogeneous problem. The idea is to propose an eigenfunction expansion with unknown coefficients $A_n(t)$ in the form

$$T(\mathbf{x},t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(\mathbf{x}) \tag{13.21}$$

Operating on both sides of Eq. (13.21) with the operator $\int_V w(\mathbf{x})\phi_m(\mathbf{x})\_\_\_dv$ results to

$$\int_V w(\mathbf{x})\phi_m(\mathbf{x})T(\mathbf{x},t)dv = \sum_{n=1}^{\infty} A_n(t)\int_V w(\mathbf{x})\phi_m(\mathbf{x})\phi_n(\mathbf{x})dv \tag{13.22}$$

Again, the orthogonality property appears on the right-hand side, and naming the left-hand side as $\bar{T}_m(t)$, the expansion coefficients become

$$A_m(t) = \frac{\int_V w(\mathbf{x})T(\mathbf{x},t)\phi_m(\mathbf{x})dv}{\int_V w(\mathbf{x})\phi_m^2(\mathbf{x})dv} \equiv \frac{\bar{T}_m(t)}{N_m} \tag{13.23}$$

Then, the proposed eigenfunction expansion is given by

$$T(\mathbf{x},t) = \sum_{n=1}^{\infty} \frac{1}{N_n} \phi_n(\mathbf{x})\bar{T}_n(t) \tag{13.24}$$

which is called the inversion formula, in terms of the transformed potentials, $\bar{T}_n(t)$, while the so-called transform formula is given by the above definition:

$$\bar{T}_n(t) = \int_V w(\mathbf{x})T(\mathbf{x},t)\phi_n(\mathbf{x})dv \tag{13.25}$$

It is, in general, preferred to express the inversion–transform pair in terms of the normalized eigenfunction, which results in a symmetric kernel of the transformation, in the form

$$T(\mathbf{x},t) = \sum_{n=1}^{\infty} K_n(\mathbf{x})\bar{T}_n(t), \quad \text{inverse} \tag{13.26}$$

$$\bar{T}_n(t) = \int_V w(\mathbf{x})T(\mathbf{x},t)K_n(\mathbf{x})dv, \quad \text{transform} \tag{13.27}$$

$$K_n(\mathbf{x}) = \frac{\phi_n(\mathbf{x})}{\sqrt{N_n}}, \quad \text{normalized eigenfunction (kernel)} \tag{13.28}$$

Here it should be noted that, as previously described in Chapter 7, the kernels $K_n(\mathbf{x})$ are, in fact, the normalized characteristic functions, and the orthogonality property in terms of the kernels is then written as

$$\int_V w(\mathbf{x})K_n(\mathbf{x})K_m(\mathbf{x})dv = \delta_{n,m} \equiv \begin{cases} 0, & \text{for} \quad n \neq m \\ 1, & \text{for} \quad n = m \end{cases} \tag{13.29}$$

The integral transform approach now proceeds towards the integral transformation process itself, which is aimed at transforming the original partial differential equation into an ordinary differential system in time for the transformed potentials, thus eliminating the space variables, by operating on Eq. (13.1) with the operator $\int_V K_n(\mathbf{x})\_\_dv$, which gives

$$\int_V w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}K_n(\mathbf{x})\,dv = \int_V [\nabla.k(\mathbf{x})\nabla T(\mathbf{x},t)]K_n(\mathbf{x})\,dv - \int_V d(\mathbf{x})T(\mathbf{x},t)K_n(\mathbf{x})\,dv$$
$$+\int_V P(\mathbf{x},t)K_n(\mathbf{x})\,dv \tag{13.30}$$

The first integral in Eq. (13.30) can be readily evaluated by substituting the inversion formula, Eq. (13.26), in place of the temperature distribution, to yield, after recalling the definition of the transformed potential,

$$\begin{aligned}\int_V w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}K_n(\mathbf{x})\,dv &= \int_V w(\mathbf{x})\frac{\partial[\sum_{m=1}^{\infty}K_m(\mathbf{x})\bar{T}_m(t)]}{\partial t}K_n(\mathbf{x})\,dv \\ &= \sum_{m=1}^{\infty}\int_V w(\mathbf{x})K_n(\mathbf{x})K_m(\mathbf{x})\,dv\frac{d\bar{T}_m(t)}{dt} \equiv \frac{d\bar{T}_n(t)}{dt}\end{aligned} \tag{13.31}$$

The fourth integral can also be directly evaluated from the known source function:

$$\bar{p}_n^*(t) = \int_V P(\mathbf{x},t)K_n(\mathbf{x})\,dv \tag{13.32}$$

In evaluating the third integral, one first should make use of the eigenvalue problem to rewrite it in the form

$$d(\mathbf{x})K_n(\mathbf{x}) = \nabla.k(\mathbf{x})\nabla K_n(\mathbf{x}) + \lambda_n^2 w(\mathbf{x})K_n(\mathbf{x}) \tag{13.33}$$

which is substituted back into the third integral and combined with the second integral:

$$\int_V \left[\nabla.k(\mathbf{x})\nabla T(\mathbf{x},t)\right]K_n(\mathbf{x})\,dv - \int_V d(\mathbf{x})T(\mathbf{x},t)K_n(\mathbf{x})\,dv$$
$$= \int_V \left[K_n(\mathbf{x})\nabla.k(\mathbf{x})\nabla T(\mathbf{x},t) - T(\mathbf{x},t)\nabla.k(\mathbf{x})\nabla K_n(\mathbf{x})\right]dv - \lambda_n^2\int_V w(\mathbf{x})T(\mathbf{x},t)K_n(\mathbf{x})\,dv \tag{13.34}$$

The second integral on the right-hand side of Eq. (13.34) is directly evaluated by recalling the definition of the transformed potential:

$$-\lambda_n^2\int_V w(\mathbf{x})T(\mathbf{x},t)K_n(\mathbf{x})\,dv = -\lambda_n^2\bar{T}_n(t) \tag{13.35}$$

while the first integral on the right-hand side is transformed into a surface integral by recalling Green's second identity to give

$$\int_V \left[K_n(\mathbf{x})\nabla.k(\mathbf{x})\nabla T(\mathbf{x},t) - T(\mathbf{x},t)\nabla.k(\mathbf{x})\nabla K_n(\mathbf{x})\right]dv$$
$$= \int_S k(\mathbf{x})\left[K_n(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} - T(\mathbf{x},t)\frac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}\right]ds \tag{13.36}$$

This operation is necessary since the original diffusion formulation and the eigenvalue problem do not obey the same boundary conditions due to the nonhomogeneity in the original problem. The right-hand side of Eq. (13.35) would vanish for a homogeneous boundary condition, but it will have a contribution for the more general nonhomogeneous situation. To recover the integrand on the right-hand side of Eq. (13.35), one multiplies the original boundary condition Eq. (13.3) throughout by $K_n(\mathbf{x})$ and the eigenvalue problem boundary condition Eq. (13.13) by $T(\mathbf{x},t)$, and subtract them to find

$$k(\mathbf{x})\left[K_n(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} - T(\mathbf{x},t)\frac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}\right] = g(\mathbf{x},t)\frac{K_n(\mathbf{x})}{\beta(\mathbf{x})} \tag{13.37}$$

Alternatively, one multiplies the original boundary condition Eq. (13.3) throughout by $\partial K_n(\mathbf{x})/\partial \mathbf{n}$, and the eigenvalue problem boundary condition Eq. (13.13) by $\partial T(\mathbf{x},t)/\partial \mathbf{n}$, and subtract them to find

$$k(\mathbf{x})\left[K_n(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} - T(\mathbf{x},t)\frac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}\right] = -g(\mathbf{x},t)\frac{k(\mathbf{x})\dfrac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}}{\alpha(\mathbf{x})} \tag{13.38}$$

One may also combine these two expressions so as to avoid the limiting cases of first kind ($\beta = 0$) or second kind boundary conditions ($\alpha = 0$) in the denominators of Eqs. (13.37) and (13.38), respectively, which yields the following general expression for the surface integral on the right-hand side of Eq. (13.36):

$$\int_S k(\mathbf{x})\left[K_n(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} - T(\mathbf{x},t)\frac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}\right]ds = \int_S g(\mathbf{x},t)\left[\frac{K_n(\mathbf{x}) - k(\mathbf{x})\dfrac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}}{\alpha(\mathbf{x})+\beta(\mathbf{x})}\right]ds \tag{13.39}$$

Substituting back Eqs. (13.39), (13.35), (13.32), and (13.31) into the transformation process, Eq. (13.30), results to the following transformed system:

$$\frac{d\overline{T}_n(t)}{dt} + \lambda_n^2\overline{T}_n(t) = \overline{p}_n(t) \tag{13.40}$$

where the transformed source term that combines the contributions from the equation and boundary condition source terms is given by

$$\overline{p}_n(t) = \int_V P(\mathbf{x},t)K_n(\mathbf{x})\,dv + \int_S g(\mathbf{x},t)\left[\frac{K_n(\mathbf{x}) - k(\mathbf{x})\dfrac{\partial K_n(\mathbf{x})}{\partial \mathbf{n}}}{\alpha(\mathbf{x})+\beta(\mathbf{x})}\right]ds \tag{13.41}$$

To complete the integral transformation process, the initial condition Eq. (13.3) is operated on with $\int_V w(\mathbf{x})K_n(\mathbf{x})\_\_dv$ to yield

$$\bar{T}_n(0) = \bar{f}_n \equiv \int_V w(\mathbf{x})f(\mathbf{x})K_n(\mathbf{x})dv \tag{13.42}$$

The transformed initial value problem formed by Eqs. (13.40) to (13.42) is readily solved through the appropriate integrating factor to obtain

$$\bar{T}_n(t) = \bar{f}_n e^{-\lambda_n^2 t} + \int_0^t \bar{p}_n(t)e^{-\lambda_n^2(t-t')}\,dt' \tag{13.43}$$

and then the inversion formula provides the final solution for the temperature field:

$$T(\mathbf{x},t) = \sum_{n=1}^{\infty} K_n(\mathbf{x})\left(\bar{f}_n e^{-\lambda_n^2 t} + \int_0^t \bar{p}_n(t)e^{-\lambda_n^2(t-t')}\,dt'\right) \tag{13.44}$$

Clearly, for a fully homogeneous problem, the transformed source term, $\bar{p}_n(t)$, would vanish, and then Eq. (13.44) recovers the solution of the homogeneous problem obtained via separation of variables, Eq. (13.20).

Although the above solution for nonhomogeneous diffusion problems is formal and exact, from the computational point of view, it might not be the best possible alternative due to the slower convergence rate on the eigenfunction expansion that can eventually result due to the presence of the source terms. Both the equation and boundary condition source terms deviate the expansion from the ideal spectral convergence governed by the decaying exponential term involving the squared eigenvalue that is achievable by the series corresponding to the homogeneous problem, Eq. (13.20). This aspect becomes clearer when we examine the special case of time-independent source terms, when the transformed source term becomes constant, $\bar{p}_n$. In this case, integrating the second term in Eq. (13.44) results in

$$T(\mathbf{x},t) = \sum_{n=1}^{\infty} K_n(\mathbf{x})\left[\bar{f}_n e^{-\lambda_n^2 t} + \bar{p}_n e^{-\lambda_n^2 t}\int_0^t e^{\lambda_n^2 t'}\,dt'\right] = \sum_{n=1}^{\infty} K_n(\mathbf{x})\left[\bar{f}_n e^{-\lambda_n^2 t} + \frac{\bar{p}_n}{\lambda_n^2}(1-e^{-\lambda_n^2 t})\right] \tag{13.45}$$

It can be noticed that the term $\dfrac{\bar{p}_n}{\lambda_n^2}$ may be responsible for slowing down convergence in comparison to the rates achievable when only the exponential term $e^{-\lambda_n^2 t}$ is present. For this reason, one frequently employed alternative is the proposition of filtering analytical solutions that extract information from the source terms, so as to reduce the importance of the source terms in the final eigenfunction expansion. The solution is written, when accounting for the filter, in the form [3]

$$T(\mathbf{x},t) = T_F(\mathbf{x},t) + T^*(\mathbf{x},t) \tag{13.46}$$

where $T_F(\mathbf{x},t)$ is the filtering solution and $T^*(\mathbf{x},t)$ is the filtered temperature distribution. Once this filtered expression, Eq. (13.46), is substituted back into Eqs. (13.1) to (13.3), the following problem for the filtered temperature is obtained:

$$w(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial t} = \nabla.k(\mathbf{x})\nabla T^*(\mathbf{x},t) - d(\mathbf{x})T^*(\mathbf{x},t) + P^*(\mathbf{x},t), \mathbf{x}\in V, t>0 \tag{13.47}$$

$$T^*(\mathbf{x},0) = f^*(\mathbf{x}), \quad \mathbf{x}\in V \tag{13.48}$$

$$\alpha(\mathbf{x})T^*(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial \mathbf{n}} = g^*(\mathbf{x},t), \quad \mathbf{x}\in S \tag{13.49}$$

where the filtered source terms are evaluated from

$$P^*(\mathbf{x},t) = P(\mathbf{x},t) - w(\mathbf{x})\frac{\partial T_F(\mathbf{x},t)}{\partial t} + \nabla.k(\mathbf{x})\nabla T_F(\mathbf{x},t) - d(\mathbf{x})T_F(\mathbf{x},t) \tag{13.50}$$

$$g^*(\mathbf{x},t) = g(\mathbf{x},t) - \alpha(\mathbf{x})T_F(\mathbf{x},t) - \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T_F(\mathbf{x},t)}{\partial \mathbf{n}} \tag{13.51}$$

and the filtered initial condition is given by

$$f^*(\mathbf{x}) = f(\mathbf{x}) - T_F(\mathbf{x},0) \tag{13.52}$$

It is clear that the filtered problem formulation is similar to the originally proposed problem in Eqs. (13.1) to (13.3), and the general solution developed above can be directly applied. The simplest possible filter would be one that just satisfies the boundary conditions, while more involved filters can be proposed that include information from the original diffusion equation, such as the quasi-steady filter obtained from the following problem formulation:

$$\nabla.k(\mathbf{x})\nabla T_F(\mathbf{x},t) - d(\mathbf{x})T_F(\mathbf{x},t) + P(\mathbf{x},t) = 0, \mathbf{x}\in V, t>0 \tag{13.53}$$

$$\alpha(\mathbf{x})T_F(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T_F(\mathbf{x},t)}{\partial \mathbf{n}} = g(\mathbf{x},t), \quad \mathbf{x}\in S \tag{13.54}$$

In this choice of quasi-steady filter, the filtered source terms of Eqs. (13.50) and (13.51) become

$$P^*(\mathbf{x},t) = -w(\mathbf{x})\frac{\partial T_F(\mathbf{x},t)}{\partial t} \tag{13.55}$$

$$g^*(\mathbf{x},t) = 0 \tag{13.56}$$

The filtering strategy may also be employed recursively, for further improvement in convergence rates, if necessary [3]. For additional discussions on filtering schemes in multidimensional formulations, the reader is referred to Cotta and Mikhailov [3] and Cotta et al. [7].

13.3 Eigenvalue Problem Solution

The formal solution derived above depends on the solution of the corresponding eigenvalue problem, in either one, two, or three dimensions, so as to become a flexible computational approach, including heat conduction problems in heterogeneous media with arbitrarily space variable coefficients. Thus, the GITT is now detailed in solving the eigenvalue problem itself, Eqs. (13.12) and (13.13), by proposing a simpler auxiliary eigenvalue problem, and expanding the unknown eigenfunctions $\phi_n(\mathbf{x})$ in terms of a chosen basis, $\Omega_i(\mathbf{x})$. The solution of problems (13.12) and (13.13) is then proposed as an expansion in terms of the simplified auxiliary eigenvalue problem below:

$$\nabla.k^*(\mathbf{x})\nabla\Omega_i(\mathbf{x}) + (\mu_i^2 w^*(\mathbf{x}) - d^*(\mathbf{x}))\Omega_i(\mathbf{x}) = 0, \quad \mathbf{x} \in V \tag{13.57}$$

with boundary conditions given by

$$\alpha(\mathbf{x})\Omega_i(\mathbf{x}) + \beta(\mathbf{x})k^*(\mathbf{x})\frac{\partial\Omega_i(\mathbf{x})}{\partial\mathbf{n}} = 0, \quad \mathbf{x} \in S \tag{13.58}$$

The coefficients $w^*(\mathbf{x})$, $k^*(\mathbf{x})$, and $d^*(\mathbf{x})$ are simplified forms of the coefficients of the original diffusion equation (13.1), chosen so as to permit analytical solution of the auxiliary problems (13.57) and (13.58), in terms of the eigenfunction, $\Omega_i(\mathbf{x})$, and related eigenvalues, μ_i.

Once the auxiliary eigenfunctions, $\Omega_i(\mathbf{x})$, have been analytically obtained and the auxiliary eigenvalues, μ_i, are computed, the expansion of the original eigenfunction is then proposed as

$$\phi_n(\mathbf{x}) = \sum_{i=1}^{\infty} \tilde{\Omega}_i(\mathbf{x})\bar{\phi}_{n,i}, \quad \text{inverse} \tag{13.59}$$

$$\bar{\phi}_{n,i} = \int_V w^*(\mathbf{x})\phi_n(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})dv, \quad \text{transform} \tag{13.60a}$$

where

$$\tilde{\Omega}_i(\mathbf{x}) = \frac{\Omega_i(\mathbf{x})}{\sqrt{N_i^*}}, \quad \text{normalized eigenfunction} \tag{13.60b}$$

and the orthogonality property in terms of the normalized eigenfunction is then written as

$$\int_V w^*(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})\,dv \equiv \begin{cases} 0, & \text{for} \quad i \neq j \\ 1, & \text{for} \quad i = j \end{cases} \tag{13.60c}$$

$$N_i^* = \int_V w^*(\mathbf{x})\tilde{\Omega}_i^2(\mathbf{x})dv \tag{13.60d}$$

The integral transformation is thus performed by operating Eq. (13.12) on with $\int_V \tilde{\Omega}_i(\mathbf{x})\_\_\_\_dv$ to obtain

$$\int_V \tilde{\Omega}_i(\mathbf{x})(\nabla.k(\mathbf{x})\nabla\phi_n(\mathbf{x}))\,dv + \int_V \tilde{\Omega}_i(\mathbf{x})(\lambda_n^2 w(\mathbf{x}) - d(\mathbf{x}))\phi_n(\mathbf{x})\,dv = 0 \tag{13.61}$$

The first term in Eq. (13.61) is manipulated by Green's second identity, so as to account for the differences in boundary conditions of the two eigenvalue problems, in the following way:

$$\int_V \tilde{\Omega}_i(\mathbf{x})(\nabla.k(\mathbf{x})\nabla\phi_n(\mathbf{x}))dv - \int_V \phi_n(\mathbf{x})(\nabla.k(\mathbf{x})\nabla\tilde{\Omega}_i(\mathbf{x}))dv = \int_S k(\mathbf{x})\left[\phi_n(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}} - \tilde{\Omega}_i(\mathbf{x})\frac{\partial\phi_n(\mathbf{x})}{\partial\mathbf{n}}\right]ds \tag{13.62}$$

Now, by combining boundary conditions (13.13) and (13.58), a more convenient form of the integrand in the surface integral above can be obtained:

$$\int_S k(\mathbf{x})\left[\phi_n(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}} - \tilde{\Omega}_i(\mathbf{x})\frac{\partial\phi_n(\mathbf{x})}{\partial\mathbf{n}}\right]ds = \int_S [k(\mathbf{x}) - k^*(\mathbf{x})]\left[\phi_n(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}}\right]ds \tag{13.63}$$

and Eq. (13.62) can be rewritten as

$$\int_V \phi_n(\mathbf{x})(\nabla.k(\mathbf{x})\nabla\tilde{\Omega}_i(\mathbf{x}))\,dv + \int_S \left(k(\mathbf{x}) - k^*(\mathbf{x}))(\phi_n(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}}\right)ds$$
$$+ \int_V \tilde{\Omega}_i(\mathbf{x})(\lambda_n^2 w(\mathbf{x}) - d(\mathbf{x}))\phi_n(\mathbf{x})dv = 0 \tag{13.64}$$

Substituting the inverse formula, Eq. (13.59), we arrive at the following algebraic eigenvalue problem:

$$\sum_{j=1}^{\infty} \bar{\phi}_{n,j} \left(\begin{aligned} &\int_V \tilde{\Omega}_j(\mathbf{x})(\nabla.k(\mathbf{x})\nabla\tilde{\Omega}_i(\mathbf{x}))\,dv + \int_S k(\mathbf{x})\left(1 - \frac{k^*(\mathbf{x})}{k(\mathbf{x})}\right)\left(\tilde{\Omega}_j(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}}\right)ds \\ &+ \int_V \tilde{\Omega}_i(\mathbf{x})\left(\lambda_n^2 w(\mathbf{x}) - d(\mathbf{x})\right)\tilde{\Omega}_j(\mathbf{x})dv\right) \end{aligned} \right) = 0 \tag{13.65}$$

which in matrix form is concisely given by

$$(\mathbf{C} - \lambda^2 \mathbf{B})\bar{\boldsymbol{\Phi}} = 0, \quad \bar{\boldsymbol{\Phi}} = \{\bar{\phi}_{i,j}\} \tag{13.66}$$

$$\mathbf{B} = \{B_{i,j}\}, \quad B_{i,j} = \int_V w(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})\,dv \tag{13.67}$$

$$\begin{aligned}\mathbf{C} = \{C_{i,j}\}, \quad C_{i,j} = &\int_V \tilde{\Omega}_j(\mathbf{x})\left(\nabla.k(\mathbf{x})\nabla\tilde{\Omega}_i(\mathbf{x})\right)dv + \int_S k(\mathbf{x})\left(1 - \frac{k^*(\mathbf{x})}{k(\mathbf{x})}\right)\tilde{\Omega}_j(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}}\,ds \\ &- \int_V d(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})\,dv\end{aligned} \tag{13.68}$$

From the formulation of the auxiliary problem, the elements of matrix **C** can be rewritten as

$$\begin{aligned}C_{i,j} = &\int_V \tilde{\Omega}_j(\mathbf{x})\left(\nabla.\left(k(\mathbf{x}) - k^*(\mathbf{x})\right)\nabla\tilde{\Omega}_i(\mathbf{x})\right)dv + \int_S \left(k(\mathbf{x}) - k^*(\mathbf{x})\right)\tilde{\Omega}_j(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}})\,ds \\ &- \int_V \left(d(\mathbf{x}) - d^*(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})\,dv + \mu_i^2\delta_{i,j}\end{aligned} \tag{13.69}$$

where $\delta_{i,j}$ is the Kronecker delta. The algebraic problem (13.66) can be numerically solved by providing results for the eigenvalues λ^2 and eigenvectors $\bar{\boldsymbol{\Phi}}$, which combined with the inverse formula, Eq. (13.59), provide the desired original eigenfunction for the reconstruction of the potential given by Eq. (13.45).

For such an eigenvalue problem with space variable coefficients, it is not always possible to employ an auxiliary eigenvalue problem that incorporates even part of this information, since it may result unsolvable in analytic explicit form. Therefore, in many cases, it is required to choose very simple expressions for the auxiliary coefficients, which in some special cases may lead to slowly converging expansions for the original eigenfunctions. This is particularly important when multiple spatial scales and/or very abrupt variations of the coefficients need to be handled. In such cases, an integral balance procedure [8] can be particularly beneficial in accelerating the convergence of such eigenfunction expansions by analytically rewriting the expansion for the eigenfunction itself, while explicitly accounting for the space variable coefficients local variation. The integral balance procedure is a convergence acceleration technique [3], here aimed at obtaining eigenfunction expansions of improved convergence behavior for both the eigenfunction and its derivatives, through integration over the spatial domain, thus benefiting from the better convergence characteristics of the integrals of eigenfunction expansions. The expressions provided by the integral balance approach can then be employed back into the solution of the eigenvalue problem, yielding a modified algebraic eigenvalue that provides the eigenvalues and the eigenvectors.

13.4 Single Domain Formulation

The above diffusion formulation with space variable coefficients defined in an arbitrary domain and the proposed integral transform solution methodology are fairly general and include the situations of complex geometrical configurations and irregular boundary surfaces. For irregular surfaces, with respect to the chosen coordinate system, the GITT has been employed in different applications in heat and fluid flow, either by direct integral transformation of the original partial differential equation [2] or through the integral transform solution of the eigenvalue problem itself [6]. Besides, it also encompasses the analysis of heterogeneous media defined by different subregions, involving different materials, once various formulations are unified in one single region problem, in what has been named the single domain formulation representation [4,9], as described below. This strategy was first proposed aiming at simplifying the treatment of conjugated conduction–convection problems, especially when dealing with multiple channels and with markedly different thermophysical properties among the fluids and substrates, a natural example of a heterogeneous system undergoing convection–diffusion.

Consider the general diffusion problem of Section 13.2, defined in a complex multidimensional configuration that is represented by n_V different subregions with volumes V_p, $p = 1,2,\ldots n_V$, with potential and flux continuity at the interfaces among themselves, as illustrated in Fig. 13.1*a*. The potential in each subregion is to be calculated, $T_p(\mathbf{x},t)$, governed by the general diffusion equation with source terms $P_p(\mathbf{x},t)$ and $g_p(\mathbf{x},t)$. Thus,

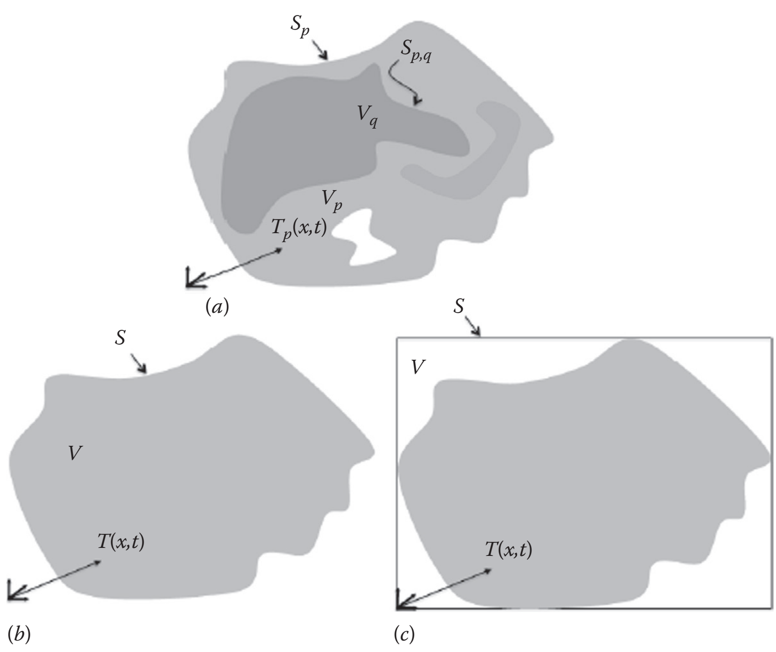

FIGURE 13.1
(*a*) Diffusion in a complex multidimensional configuration with n_V subregions. (*b*) Single domain representation keeping the original overall domain. (*c*) Single domain representation considering a regular overall domain that envelopes the original one.

$$w_p(\mathbf{x})\frac{\partial T_p(\mathbf{x},t)}{\partial t}=\nabla\cdot\left(k_p(\mathbf{x})\nabla T_p(\mathbf{x},t)\right)-d_p(\mathbf{x})T_p(\mathbf{x},t)+P_p(\mathbf{x},t),\ \ \mathbf{x}\in V_p,\ \ t>0,\quad p=1,2,\ldots,n_V \tag{13.70}$$

with initial, interface, and boundary conditions given, respectively, by

$$T_p(\mathbf{x},0)=f_p(\mathbf{x}),\ \ \mathbf{x}\in V_p \tag{13.71}$$

$$T_p(\mathbf{x},t)=T_q(\mathbf{x},t),\ \mathbf{x}\in S_{p,q},\ t>0 \tag{13.72}$$

$$k_p(\mathbf{x})\frac{\partial T_p(\mathbf{x},t)}{\partial \mathbf{n}}=k_q(\mathbf{x})\frac{\partial T_q(\mathbf{x},t)}{\partial \mathbf{n}},\ \ \mathbf{x}\in S_{p,q},\ \ t>0 \tag{13.73}$$

$$\left[\alpha_p(\mathbf{x})+\beta_p(\mathbf{x})k_p(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}\right]T_p(\mathbf{x},t)=g_p(\mathbf{x},t),\ \ \mathbf{x}\in S_p,\ t>0 \tag{13.74}$$

where $\mathbf{n}$ denotes the outward-drawn normal to the interfaces, $S_{p,q}$, and external surfaces, S_p.

One possible approach in solving systems (13.70) to (13.74) would be to promote the integral transformation of the problems in each subregion, individually, which would then be coupled through the interface conditions. However, this could result in a considerable analytical and computational effort, and would most likely require convergence acceleration schemes. Therefore, to alleviate this difficulty, the single domain formulation proposes the representation of the various subdomain transitions in the form of abrupt space variations of the corresponding equation coefficients. Figure 13.1*b* provides the most direct possibility for definition of the single domain, keeping the original external domain surface after unification of the space variable coefficients. As discussed above, irregular regions can be directly integral transformed [2,6]; therefore, there is no drawback in maintaining the original overall irregular domain. However, another possibility provided by the single domain formulation is to consider a regular geometry as an overall domain enveloping the original one, as represented in Fig. 13.1*c*. Eventually, computational advantage may be achieved by enveloping the original irregular domain by a simple regular geometry.

Therefore, as already mentioned and demonstrated in the analysis of conjugated problems [4,9], it is possible to rewrite problems (13.70) to (13.74) as a single domain formulation with space variable coefficients and source terms given by

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}=\nabla\cdot(k(\mathbf{x})\nabla T(\mathbf{x},t))-d(\mathbf{x})T(\mathbf{x},t)+P(\mathbf{x},t),\quad \mathbf{x}\in V,\quad t>0 \tag{13.75}$$

with initial and boundary conditions given, respectively, by

$$T(\mathbf{x},0)=f(\mathbf{x}),\quad \mathbf{x}\in V \tag{13.76}$$

$$\left[\alpha(\mathbf{x})+\beta(\mathbf{x})k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]T(\mathbf{x},t)=g(\mathbf{x},t),\quad \mathbf{x}\in S,\ t>0 \tag{13.77}$$

which coincides with the general formulation proposed in Eqs. (13.1) to (13.3), now with

$$V=\sum_{p=1}^{n_V}V_p,\quad S=\sum_{q=1}^{n_V}S_q \tag{13.78}$$

The space variable coefficients in Eqs. (13.75) to (13.77), besides the new equation and boundary source terms and initial conditions, now without the subscript p for the subregions V_l, incorporate the abrupt transitions among the different subregions and permit the representation of systems (13.70) to (13.74) as a single domain formulation, to be directly handled by integral transforms, as already described in Section 13.2.

13.5 Applications

13.5.1 Functionally Graded Material

The first example of heat conduction in a heterogeneous medium is related to the study of a functionally graded material (FGM), a material designed and manufactured to perform more than one function (for example, structural and thermal), generally in the extreme ranges of the corresponding physical properties. In such cases, the coefficients of the diffusion equation vary by up to a few orders of magnitude along the spatial dimension.

Consider the one-dimensional transient heat conduction problem, in dimensionless form, with initial and boundary conditions given by [5]

$$w(x)\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[k(x)\frac{\partial T(x,t)}{\partial x}\right],\quad 0<x<1,\quad t>0 \tag{13.79}$$

$$T(x,0)=f(x),\quad 0<x<1 \tag{13.80}$$

$$T(0,t)=0,\quad T(1,t)=0,\quad t>0 \tag{13.81}$$

where in this particular example the x-variable thermophysical properties assume the following exponential form [10]:

$$k(x)=k_0e^{2\beta x},\quad w(x)=w_0e^{2\beta x},\quad \alpha_0=\frac{k_0}{w_0}=\text{const.} \tag{13.82}$$

There is no need to repeat the integral transformation procedure described in Section 13.2, since the example is just a special case of the general formulation in Eqs. (13.1) to (13.3).

Instead, one only needs to establish the correspondence between the general and particular formulations. The position vector **x** corresponds to the space coordinate x projected on the unitary vector **i**, then region V in this case corresponds to the interval defined by the slab length, and the surface S corresponds to the boundary surfaces $x = 0$ and $x = 1$. The remaining correspondence of problems (13.79) to (13.81) with the general formulation, Eqs. (13.1) to (13.3), is then given by the following coefficients:

$$\begin{aligned} &w(\mathbf{x}) = w(x), \quad k(\mathbf{x}) = k(x), \quad d(\mathbf{x}) = 0, \qquad P(\mathbf{x},t) = 0, \quad f(\mathbf{x}) = f(x), \\ &\alpha(\mathbf{x}) = 1, \qquad \beta(\mathbf{x}) = 0, \qquad g(\mathbf{x},t) = 0, \quad V = [0,1], \quad \mathbf{x} = x\mathbf{i} \end{aligned} \tag{13.83}$$

Then, the solution of the special case can be directly obtained from the general solution as

$$T(x,t) = \sum_{n=1}^{\infty} K_n(x)\bar{f}_n e^{-\lambda_n^2 t} \tag{13.84}$$

since the transformed source term vanishes due to the absence of sources, and the transformed initial condition is given by

$$\bar{f}_n = \int_0^1 w(x) f(x) K_n(x) dx \tag{13.85}$$

with the normalization integral as

$$N_n = \int_0^1 w(x)\phi_n^2(x) dx \tag{13.86}$$

The eigenvalue problem to be solved for this special case becomes

$$\frac{d}{dx}\left[k(x)\frac{d\phi_n(x)}{dx}\right] + \lambda_n^2 w(x)\phi_n(x) = 0, \quad 0 < x < 1 \tag{13.87}$$

$$\phi_n(0) = 0, \quad \phi_n(1) = 0 \tag{13.88}$$

Thus, to demonstrate the applicability of the present approach, it was considered the simplest form among the several possibilities for the auxiliary problem, based on the choice of constant coefficients $k^*(x) = 1$, $w^*(x) = 1$, and $d^*(x) = 0$, while maintaining the same boundary conditions as given by Eqs. (13.88), resulting in the following solution for the auxiliary eigenvalue problem:

$$\tilde{\Omega}_i(x) = \sqrt{2}\sin(\mu_i x), \quad \text{with} \quad \mu_i = i\pi, \quad i = 1,2,3... \tag{13.89}$$

The algebraic eigenvalue problem of Eq. (13.66) is then formed by the two matrices obtained from Eqs. (13.67) and (13.68), for this particular application, as

$$B_{i,j} = \int_0^1 w(x)\tilde{\Omega}_i(x)\tilde{\Omega}_j(x)\,dx \tag{13.90}$$

$$C_{i,j} = \int_0^1 \tilde{\Omega}_j(x)\frac{dk(x)}{dx}\frac{d\tilde{\Omega}_i(x)}{dx}dx + \mu_i^2\delta_{i,j} \tag{13.91}$$

The resulting algebraic problem is then solved numerically, providing results for the eigenvalues λ^2 and eigenvectors $\bar{\Phi}$, which are combined with the inverse formula, Eq. (13.59), to provide the desired original eigenfunction for the reconstruction of the final solution given by Eq. (13.84).

The particular functional form given by Eqs. (13.82) leads to the formulation of a problem with exact solution via separation of variables, used here as a reference result in the analysis of the proposed solution for this FGM application. Thus, after substituting Eqs. (13.82) into Eq. (13.79), it results to

$$\frac{1}{\alpha_0}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} + 2\beta\frac{\partial T(x,t)}{\partial x}, \quad 0 < x < 1, \quad t > 0 \tag{13.92}$$

A dependent variable transformation can be performed to recover the heat conduction equation form:

$$T(x,t) = u(x,t)e^{-\beta(x+\beta\alpha_0 t)} \tag{13.93}$$

Then, the problem for $u(x,t)$ is rewritten with its initial and boundary conditions in the form

$$\frac{1}{\alpha_0}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \tag{13.94}$$

$$u(x,0) = f^*(x) = f(x)e^{\beta x}, \quad 0 < x < 1 \tag{13.95}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \tag{13.96}$$

The example was solved for different β values, with the initial condition given by

$$f(x) = \frac{1 - e^{2\beta(1-x)}}{1 - e^{2\beta}} \tag{13.97}$$

which corresponds to the steady-state solution for the case of prescribed temperatures $T(0,t) = 1$ and $T(1,t) = 0$.

Figure 13.2 illustrates the effect of the β parameter in the behavior of the thermophysical properties according to Eq. (13.82), associated with the marked variation of the heat diffusion coefficient in the case of a FGM. It is worth mentioning, in the case of $\beta = 3$, the ratio of approximately 400 times between the two values of $k(x)$ in the opposite boundaries [5].

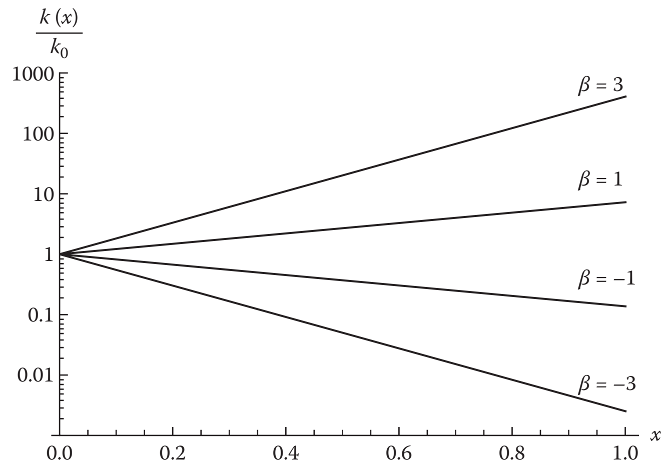

FIGURE 13.2
Behavior of the diffusion coefficient $k(x)$ for heat conduction in FGM, as given by Eq. (13.82) for $\beta = -3, -1, 1$ e 3. (From Naveira-Cotta, C. P. et al., *Int. J. Heat Mass Transfer*, vol. 52, pp. 5029–5039, 2009.)

Numerical results for the eigenvalues and for the temperature distribution in the FGM example are reported next for the numerical values of $\beta = -3$ and 3, and for the values of $w_0 = 10$ and $k_0 = 1$. Table 13.1 shows the excellent convergence of the first 10 eigenvalues, μ_i, associated with the eigenvalue problem, Eqs. (13.87) and (13.88). The different columns correspond to the increase in the truncation order in the expansion of the original eigenfunction in terms of the auxiliary eigenfunction, for NN = 20, 30, 40, and 50. It should be noted that the first 10 eigenvalues are fully converged to five significant digits for the case $\beta = 3$ with 50 terms in the expansion.

Figures 13.3*a*,*b* illustrate the transient behavior of the temperature profiles for three time values, t = 0.01, 0.05, and 0.1, for the two extreme situations considered with $\beta = 3$ and -3, respectively. In Fig. 13.3*a*, corresponding to $\beta = 3$, the thermophysical properties increase

TABLE 13.1
Convergence of Eigenvalues in FGM Heat Conduction Problem, $\beta = 3$

Order n	$NN = 20$	$NN = 30$	$NN = 40$	$NN = 50$
1	1.37371	1.37368	1.37367	1.37367
2	2.2019	2.20182	2.20180	2.20179
3	3.12789	3.12777	3.12774	3.12773
4	4.08578	4.08558	4.08554	4.08552
5	5.05739	5.05716	5.05711	5.05709
6	6.03623	6.03589	6.03582	6.03580
7	7.01911	7.01875	7.01868	7.01865
8	8.00481	8.00426	8.00416	8.00412
9	8.99207	8.99150	8.99139	8.99135
10	9.98090	9.98001	9.97987	9.97982

Source: Naveira-Cotta, C.P. et al., *Int. J. Heat Mass Transfer*, vol. 52, pp. 5029–5039, 2009.

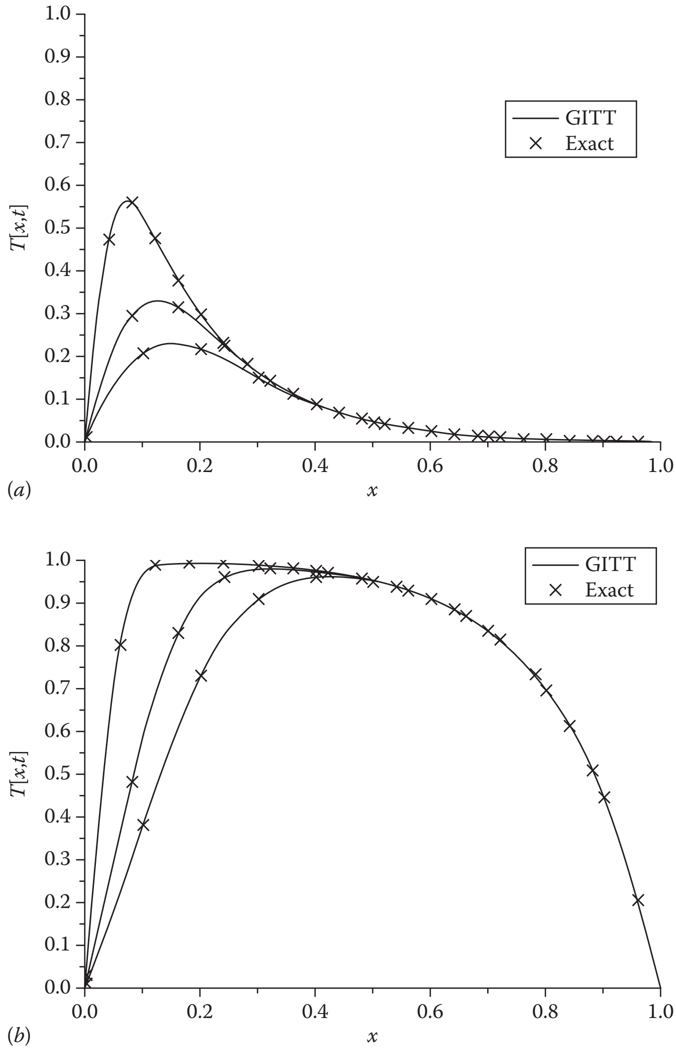

FIGURE 13.3
(*a*) Physical behavior and verification (GITT × Exact) of temperature distributions for the FGM ($\beta = 3$). (*b*) Physical behavior and verification (GITT × Exact) of temperature distributions for the FGM ($\beta = -3$). (From Naveira-Cotta, C. P. et al., *Int. J. Heat Mass Transfer*, vol. 52, pp. 5029–5039, 2009.)

approximately 400 times in the direction of the lowest values of temperature, i.e., the right-hand side of the graph, where both the thermal conductivity and thermal capacity are markedly augmented, and the cooling effect is very effective.

In Fig. 13.3*b*, corresponding to $\beta = -3$, the thermophysical properties are markedly reduced towards the edge $x = 1$, sensibly affecting the cooling of the slab. It should be recalled that the dimensionless thermal diffusivity (α_0) is kept the same in both examples, but the initial conditions are different in light of the variation in the β values. In addition, the exact results are plotted as obtained from the solution of Eqs. (13.93) to (13.96) for verification purposes, for both sets of curves, with a perfect agreement with the GITT results here reported with 50 terms in the eigenfunction expansions.

13.5.2 Variable Thickness Plate: A Benchmark

This example provides a benchmark for spatially variable thermal conductivity and thermal capacity in heterogeneous media [11]. Thin rectangular plates have been manufactured with controlled thickness, varying linearly along the plates' length to simulate one-dimensional spatially variable thermophysical properties. Then, theoretical and experimental analysis was performed, so as to validate the proposed modeling and the associated solution methodology. The heat conduction problem formulation in terms of the thickness averaged temperature distribution $T_{\text{av}}(x,t)$ along the plate length (Fig. 13.4), including the lumped system analysis approximation in the direction of the plate's thickness, is given by [11]

$$w(x)\frac{\partial T_{\text{av}}(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial T_{\text{av}}(x,t)}{\partial x}\right) - d(x)T_{\text{av}}(x,t) + P(x,t), \quad 0 < x < L_x; \quad t > 0 \tag{13.98}$$

$$T_{\text{av}}(x,0) = T_\infty \tag{13.99}$$

$$\left.\frac{\partial T_{\text{av}}(x,t)}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial T_{\text{av}}(x,t)}{\partial x}\right|_{x=L_x} = 0 \tag{13.100}$$

where the coefficients in the formulation are given by

$$w(x) = \hat{w}\varepsilon(x); \quad k(x) = \hat{k}\varepsilon(x); \quad h_{\mathit{eff}}(x) = \hat{h}_{\mathit{eff}}(x)/n_z(x);$$
$$d(x) = \frac{h_{\mathit{eff}}(x)}{L_z}; \quad P(x,t) = \frac{q(x,t)}{L_z} - \frac{h_{\mathit{eff}}(x)}{L_z}T_\infty \tag{13.101}$$

Problems (13.98) to (13.101) model a one-dimensional transient heat conduction problem for a thermally thin plate, including prescribed heat flux at one surface, and convective and radiative heat losses at the opposite plate face, based on a lumped formulation across the sample thickness. Here, $\varepsilon(x)$ is the function that represents the plate's thickness variation,

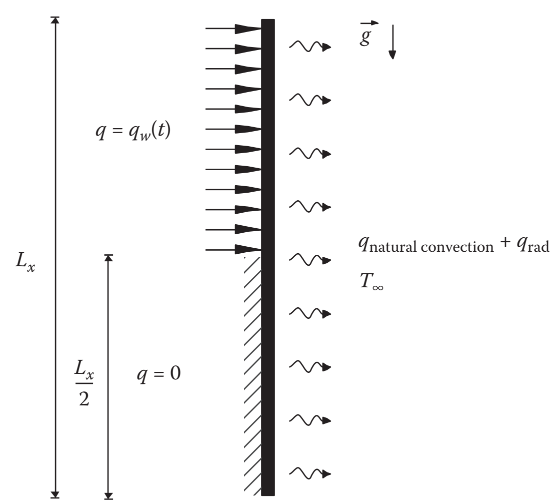

FIGURE 13.4
Physical model for heat conduction in variable thickness plate. (From Knupp, D. C. et al., *Exp. Heat Transfer*, vol. 26, pp. 1–25, 2013.)

while $\hat{w}$, $\hat{k}$, and $\hat{h}_{eff}(x)$ are the actual thermophysical properties and the effective local heat transfer coefficient, respectively; $n_z(x)$ is the direction cosine. The formal exact solution of Problems (13.98) to (13.101) is then obtained with the integral transform method for heterogeneous media, as presented in this chapter.

The correspondence with the general formulation of Eqs. (13.1) to (13.3) is given by

$$w(\mathbf{x}) = w(x),\ k(\mathbf{x}) = k(x),\ d(\mathbf{x}) = d(x),\ P(\mathbf{x},t) = P(x,t),\ f(\mathbf{x}) = T_\infty,$$
$$\alpha(\mathbf{x}) = 0,\ \beta(\mathbf{x}) = 1,\ g(\mathbf{x},t) = 0,\ V = [0,L_x],\ \mathbf{x} = x\mathbf{i},\ T(\mathbf{x},t) = T_m(x,t). \tag{13.102}$$

The eigenvalue problem to be solved for this example is thus given by

$$\frac{d}{dx}\left[k(x)\frac{d\phi_n(x)}{dx}\right] + \left(\lambda_n^2 w(x) - d(x)\right)\phi_n(x) = 0,\ \ x \in [0,L_x] \tag{13.103}$$

with boundary conditions

$$\frac{d\phi_n(x)}{dx} = 0, \quad x = 0, \quad \frac{d\phi_n(x)}{dx} = 0, \quad x = L_x \tag{13.104}$$

Again, the simplest form among the various possibilities for the auxiliary problem has been adopted, basing the choice on constant coefficients equal to $k^*(x) = 1$, $w^*(x) = 1$, and $d^*(x) = 0$, while maintaining the same boundary conditions as given by Eqs. (13.104), resulting in the following solution for the auxiliary eigenvalue problem:

$$\tilde{\Omega}_i(x) = \sqrt{2}\cos(\mu_i x), \quad \text{and} \quad \tilde{\Omega}_0(x) = 1, \quad \text{with} \quad \mu_i = i\pi, \quad i = 0,1,2\ldots \tag{13.105}$$

The algebraic eigenvalue problem of Eq. (13.66) is then formed by the two matrices obtained from Eqs. (13.67) and (13.68) for this particular application as

$$B_{i,j} = \int_0^{L_x} w(x)\tilde{\Omega}_i(x)\tilde{\Omega}_j(x)\,dx \tag{13.106}$$

$$C_{i,j} = \int_0^{L_x} \tilde{\Omega}_j(x)\frac{dk(x)}{dx}\frac{d\tilde{\Omega}_i(x)}{dx}dx - \int_0^{L_x} d(x)\tilde{\Omega}_j(x)\tilde{\Omega}_i(x)\,dx + \mu_i^2\delta_{i,j} \tag{13.107}$$

The resulting algebraic problem is then solved numerically, providing results for the eigenvalues λ^2 and eigenvectors $\bar{\mathbf{\Phi}}$, which are combined with the inverse formula, Eq. (13.59), to provide the desired original eigenfunction. Then, the temperature distribution can be directly obtained from the general solution as

$$T(x,t) = \sum_{i=1}^{\infty} K_n(x)\left(\bar{f}_n e^{-\lambda_n^2 t} + \int_0^t \bar{p}_n(t)e^{-\lambda_n^2(t-t')}\,dt'\right) \tag{13.108}$$

with the transformed source terms obtained from

$$\bar{p}_n(t) = \int_0^{L_x} P(x,t)K_n(x)\,dx \tag{13.109}$$

and the transformed initial condition given by

$$\bar{f}_n = T_\infty \int_0^{L_x} w(x)K_n(x)\,dx \tag{13.110}$$

with the normalization integral as

$$N_n = \int_0^{L_x} w(x)\phi_n^2(x)\,dx \tag{13.111}$$

In order to validate the proposed modeling and solution methodology, an experiment is built and tested [10], which employs samples made of thermally thin plates partially heated with an electrical skin heater on one surface, while the other surface (that faces an infrared thermography system) undergoes combined natural convection and radiation heat transfer with the external environment. The experimental setup employs temperature measurements obtained from the infrared camera FLIR SC660, a high-performance infrared system with 640×480 image resolution, −40°C to 1500°C temperature range.

A nominal voltage of 8 V has been applied through the DC source on the electrical skin heater (38.2 Ω), of 4×4 cm, which allows the definition of the applied heat flux as

$$q(x,t) = \begin{cases} 502\,{}^{W}\!/_{m^2}, & 0 < x < {}^{L_x}\!/_{2} \\ 0, & {}^{L_x}\!/_{2} < x < L_x \end{cases} \tag{13.112}$$

The expression for $\hat{h}_{eff}(x)$ is provided from correlations for natural convection in vertical plates, given by

$$\hat{h}_{eff}(x) = \begin{cases} \dfrac{4.8}{(0.04 - x)^{1/5}}\,{}^{W}\!/_{m^2\,°C} & 0 < x < {}^{L_x}\!/_{2} \\ 0 & {}^{L_x}\!/_{2} < x < L_x \end{cases} \tag{13.113}$$

A polystyrene plate was manufactured with a thickness that varies linearly from 1.101 to 1.880 mm. The variable thickness plate has lateral and vertical dimensions of 40 × 80 mm, i.e., L_x = 80 mm. From thermophysical properties tables, the thermal capacity and thermal conductivity were obtained as $\hat{w} = 1.3 \times 10^6$ J/m³°C and $\hat{k} = 0.116$ W/m°C.

Figures 13.5 and 13.6 show a comparison of experimental and theoretical behaviors in the case where heating was applied on the thicker portion of the plate, respectively, for the time evolution of the surface temperature at x = 2 cm, up to steady state, and for the vertical spatial distribution of the temperature at t = 800 s. In both graphs, an excellent agreement between the experimental data and the simulated temperatures can be observed.

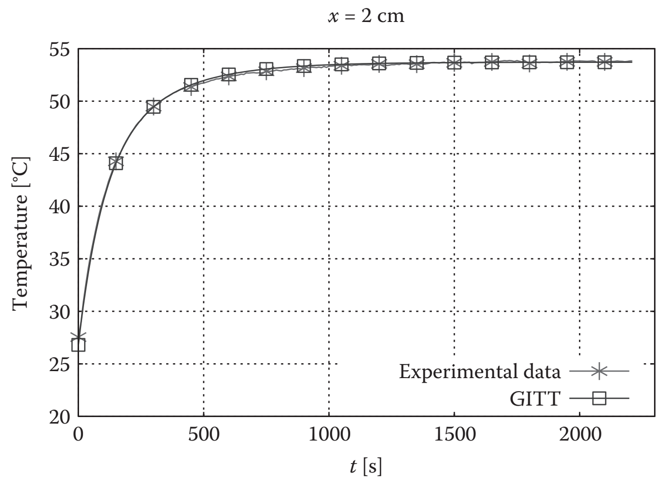

FIGURE 13.5
Time evolution of the plate surface temperature at x = 2 cm for the polystyrene plate with variable thickness. (From Knupp, D. C. et al., *Exp. Heat Transfer*, vol. 26, pp. 1–25, 2013.)

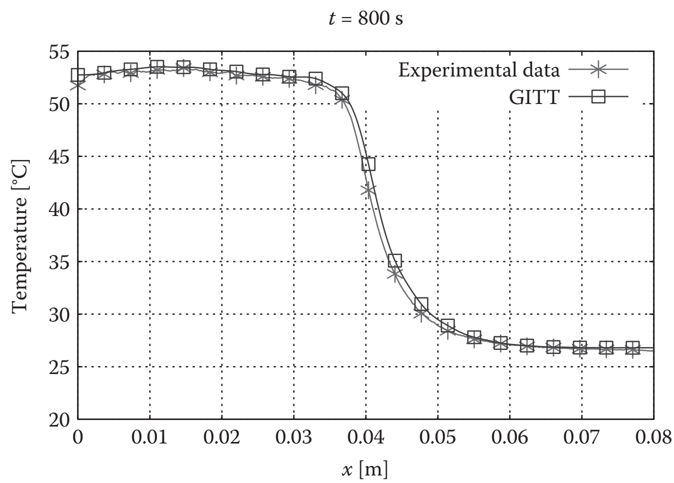

FIGURE 13.6
Vertical spatial distribution of surface temperatures at t = 800 s for the polystyrene plate with variable thickness. (From Knupp, D. C. et al., *Exp. Heat Transfer*, vol. 26, pp. 1–25, 2013.)

The analysis in Ref. [11] also includes the confirmation of the thermophysical properties estimation, through the appropriate inverse problem analysis, employing GITT for the direct problem solution. Another interesting aspect in this inverse problem analysis [11] is the integral transformation of the experimental surface temperatures, as proposed in Ref. [12]. The estimation of the effective heat transfer coefficient from the experimental data, expressed itself as an eigenfunction expansion, also permits an improved comparison of theory and experiments.

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Problems

13.1 Consider the following eigenvalue problem for one-dimensional heat conduction in a solid cylinder:

$$\frac{d}{dx}\left[x\frac{d\psi_i(x)}{dx}\right]+\lambda_i^2 x\psi_i(x)=0,\quad 0<x<1$$

$$\left.\frac{d\psi_i(x)}{dx}\right|_{x=0}=0,\quad \psi_i(x)=0$$

Employ the GITT approach for the solution of this eigenvalue problem, adopting an auxiliary problem with the simplest possible coefficients equal to $k^*(x) = 1$, $w^*(x) = 1$, and $d^*(x) = 0$, while maintaining the same boundary conditions as given above. The resulting algebraic problem should be solved numerically, providing results for the eigenvalues λ^2 and eigenvectors $\bar{\psi}$, which are combined with the inverse formula, Eq. (13.59), to provide the desired original eigenfunction.

Compare these calculations against the exact solution in terms of Bessel functions for the eigenfunctions, eigenvalues, and norms, for the first 10 eigenvalues, for a fully converged GITT solution.

13.2 Employ the above solution of the eigenvalue problem to construct the solution for the heat conduction problem in a solid cylinder, treated as a heterogeneous medium with linearly varying thermophysical properties:

$$x\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[x\frac{\partial T(x,t)}{\partial x}\right],\quad 0<x<1,\quad t>0$$

$$T(x,0)=f(x)\equiv 1-x^2,\quad 0<x<1$$

$$\frac{\partial T(0,t)}{\partial x}=0,\quad T(1,t)=0,\quad t>0$$

Establish the correspondence with the general formulation of Eqs. (13.1) to (13.3) and construct the integral transform solution of the problem without repeating the transformation procedure, just recalling the general solution with the correspondence.

Compare these calculations for the temperature field against the exact solution with the exact eigenfunctions in terms of Bessel functions, with increasing truncation order in the GITT solution.

13.3 Derive the general orthogonality property of Eq. (13.14).

13.4 Consider two-dimensional transient heat conduction in a FGM material, according to the following formulation:

$$w(x)\frac{\partial T(x,y,t)}{\partial t}=\frac{\partial}{\partial x}\left[k(x)\frac{\partial T(x,y,t)}{\partial x}\right]+k(x)\frac{\partial^2 T(x,y,t)}{\partial y^2},\quad 0<x<1,\quad 0<y<H,\quad t>0$$

$$T(x,y,0)=1,\quad 0\le x\le 1,\quad 0\le y\le H$$

$$T(0,y,t)=0,\quad T(1,y,t)=0,\quad t>0$$

$$\frac{\partial T(x,0,t)}{\partial y}=0,\quad k(x)\frac{\partial T(x,H,t)}{\partial y}+hT(x,H,t)=0,\quad t>0$$

where the x-variable thermophysical properties assume the following exponential form:

$$k(x)=k_0e^{2\beta x},\quad w(x)=w_0e^{2\beta x},\quad \alpha_0=\frac{k_0}{w_0}=\text{const.}$$

Establish the correspondence with the general formulation of Eqs. (13.1) to (13.3) and construct the integral transforms solution of both the two-dimensional eigenvalue problem and the temperature distribution, according to the GITT approach presented.

13.5 Consider again two-dimensional transient heat conduction in a FGM material, according to the following formulation:

$$w(x,y)\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\left[k(x,y)\frac{\partial T(x,y,t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y)\frac{\partial T(x,y,t)}{\partial y}\right],$$
$$0 < x < 1, \quad 0 < y < H, \quad t > 0$$

$$T(x,y,0) = 1, \quad 0 \le x \le 1, \quad 0 \le y \le H$$

$$T(0,y,t) = 0, \quad T(1,y,t) = 0, \quad t > 0$$

$$\frac{\partial T(x,0,t)}{\partial y} = 0, \quad k(x,H)\frac{\partial T(x,H,t)}{\partial y} + hT(x,H,t) = 0, \quad t > 0$$

where the x- and y-variable thermophysical properties assume the following:

$$k(x,y) = k_0 e^{2(\beta x + \gamma y)}, \quad w(x,y) = w_0 e^{2(\beta x + \gamma y)}, \quad \alpha_0 = \frac{k_0}{w_0} = \text{const.}$$

Establish the correspondence with the general formulation of Eqs. (13.1) to (13.3) and construct the integral transforms solution of both the two-dimensional eigenvalue problem and the temperature distribution, according to the GITT approach presented.

Appendix A: Thermophysical Properties

TABLE A.1

Thermophysical Properties of Metals

Metal	Temperature Range T, °C	Density ρ, g/cm³	Specific Heat c, kJ/(kg·K)	Thermal Conductivity k, W/(m·K)	Emissivity ε
Aluminum	0–400	2.72	0.895	204–250	0.04–0.06 (polished) 0.07–0.09 (commercial) 0.2–0.3 (oxidized)
Brass (70% Cu, 30% Zn)	100–300	8.52	0.38	104–147	0.03–0.07 (polished) 0.2–0.25 (commercial) 0.45–0.55 (oxidized)
Bronze (75% Cu, 25% Sn)	0–100	8.67	0.34	26	0.03–0.07 (polished) 0.4–0.5 (oxidized)
Constantan (60% Cu, 40% Ni)	0–100	8.92	0.42	22–26	0.03–0.06 (polished) 0.2–0.4 (oxidized)
Copper	0–600	8.95	0.38	385–350	0.02–0.04 (polished) 0.1–0.2 (commercial)
Iron (C ≃ 4%. cast)	0–1000	7.26	0.42	52–35	0.02–0.25 (polished) 0.55–0.65 (oxidized) 0.6–0.8 (rusted)
Iron (C ~ 5%. wrought)	0–1000	7.85	0.46	50–35	0.3–0.35 (polished) 0.9–0.95 (oxidized)
Lead	0–300	11.37	0.13	35–30	0.05–0.08 (polished) 0.3–0.6 (oxidized)
Magnesium	0–300	1.75	1.01	171–157	0.07–0.13 (polished)
Mercury	0–300	13.4	0.125	8–10	0.1–0.12
Molybdenum	0–1000	10.22	0.251	125–99	0.06–0.10 (polished)
Nickel	0–400	8.9	0.45	93–59	0.05–0.07 (polished) 0.35–0.49 (oxidized)
Platinum	0–1000	21.4	0.24	70–75	0.05–0.03 (polished) 0.07–0.11 (oxidized)
Silver	0–400	10.52	0.23	410–360	0.01–0.03 (polished) 0.02–0.04 (oxidized)
Steel (C ≃ 1%)	0–1000	7.80	0.47	43–28	0.07–0.17 (polished)
Steel (Cr ≃ 1%)	0–1000	7.86	0.46	62–33	0.07–0.17 (polished)
Steel (Cr 18%. Ni 8%)	0–1000	7.81	0.46	16–26	0.07–0.17 (polished)
Tin	0–200	7.3	0.23	65–57	0.04–0.06 (polished)
Tungsten	0–1000	19.35	0.13	166–76	0.04–0.08 (polished) 0.1–0.2 (filament)
Zinc	0–400	7.14	0.38	112–93	0.02–0.03 (polished) 0.10–0.11 (oxidized) 0.2–0.3 (galvanized)

TABLE A.2

Thermophysical Properties of Nonmetals

Nonmetal	Temperature Range T, °C	Density ρ, g/cm³	Specific Heat c, kJ/(kg·K)	Thermal Conductivity k, W/(m·K)	Emissivity ε
Asbestos	100–1000	0.47–0.57	0.816	0.15–0.22	0.93–0.97
Brick, rough red	100–1000	1.76	0.84	0.38–0.43	0.90–0.95
Clay	0–200	1.46	0.88	1.3	0.91
Concrete	0–200	2.1	0.88	0.81–1.4	0.94
Glass, window	0–600	2.7	0.84	0.78	0.94–0.66
Glass wool	23	0.024	0.7	0.038	
Ice	0	0.91	1.9	2.2	0.97–0.99
Limestone	100–300	2.5	0.9	1.3	0.95–0.80
Marble	0–100	2.60	0.80	2.07–2.94	0.93–0.95
Plasterboard	0–100	1.25	0.84	0.43	0.92
Rubber (hard)	0–100	1.2	1.42	0.15	0.94
Sandstone	0–300	2.24	0.71	1.83	0.83–0.9
Wood (oak)	0–100	0.6–0.8	2.4	0.17–0.2	0.90

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Appendix B: Bessel Functions

A linear second-order ordinary differential equation with variable coefficients of the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (m^2x^2 - v^2)y = 0 \tag{B.1}$$

is known as *Bessel's differential equation of order v*, where m is a *parameter* and v is any *real* constant. Since only v^2 appears in Eq. (B.1), we may also consider ν to be non-negative without loss of generality. The general solution of Eq. (B.1) may be obtained by using the *method of Frobenius* and the result is [2]

$$y(x) = C_1 J_v(mx) + C_2 Y_v(mx) \tag{B.2}$$

where the functions $J_v(mx)$ and $Y_v(mx)$ are known as the *Bessel functions* of the *first kind* and the *second kind of order v*, respectively. For all values $v \geq 0$, the function $J_v(mx)$ is defined by

$$J_v(mx) = \sum_{k=0}^{\infty} (-1)^k \frac{(mx/2)^{2k+v}}{k!\Gamma(k+v+1)} \tag{B.3}$$

If $v \neq n = 0, 1, 2,...$, the function $Y_v(mx)$ is defined by

$$Y_v(mx) = \frac{J_v(mx)\cos(v\pi) - J_{-v}(mx)}{\sin(v\pi)} \tag{B.4a}$$

and for values of $\nu = 0, 1, 2,...$, it is given by

$$Y_n(mx) = \frac{2}{\pi}\left(\ln\frac{mx}{2} + \gamma\right)J_n(mx) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{mx}{2}\right)^{2k-n}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\phi(k) + \phi(k+n)\right]\frac{(mx/2)^{2k+n}}{k!(n+k)!} \tag{B.4b}$$

The function $J_{-v}(mx)$ in Eq. (B.4*a*) is obtained by replacing v by $-v$ in Eq. (B.3); that is,

$$J_{-v}(mx) = \sum_{k=0}^{\infty} (-1)^k \frac{(mx/2)^{2k-v}}{k!\Gamma(k-v+1)} \tag{B.5}$$

If $v = n = 0, 1, 2,...$, then the functions $J_v(mx)$ and $J_{-v}(mx)$ are linearly dependent, because they are related to each other in the form

$$J_n(mx) = (-1)^n J_{-n}(mx) \tag{B.6}$$

If, however, $\nu \neq n = 0, 1, 2,...$, the functions $J_v(mx)$ and $J_{-v}(mx)$ are linearly independent. Hence, the solution (B.2) can also be written as

$$y(x) = D_1 J_v(mx) + D_2 J_{-v}(mx), \quad v \neq n = 0, 1, 2, \ldots \tag{B.7}$$

The *gamma function* appearing in the above equations is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt, \quad \alpha > 0 \tag{B.8}$$

Integration by parts gives the following important relation:

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \alpha > 0 \tag{B.9}$$

Hence, Eq. (B.9) together with $\Gamma(1) = 1$ yields

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots \tag{B.10}$$

It can be shown that for *fractional* numbers the following relation holds:

$$\Gamma(v)\Gamma(v-1) = \frac{\pi}{\sin(v\pi)} \quad \text{with} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{B.11}$$

Furthermore, the expression $\phi(k)$ in Eq. (B.4*b*) is defined as

$$\phi(k) = \sum_{n=1}^{k} \frac{1}{n} \quad \text{with} \quad \phi(0) = 0 \tag{B.12}$$

and $\gamma = 0.5772156...$ is *Euler's constant*.

It is difficult to predict the behavior of Bessel functions from their series representations. For integer values of v, the general behavior of these functions is depicted in Figs. B.1 and B.2. Note, however, that Bessel functions of the second kind Y_v are unbounded at $x = 0$ for all values of $v \geq 0$. In Table B.1 we present the numerical values of $J_n(x)$ and $Y_v(x)$ for $n = 0$ and 1. In Table B.2 we give the first 40 zeros of $J_0(x)$ and the corresponding values of $J_1(x)$. Furthermore, in Table B.3 we list the first 10 zeros of $J_n(x)$ for $n = 1, 2, 3, 4,$ and 5.

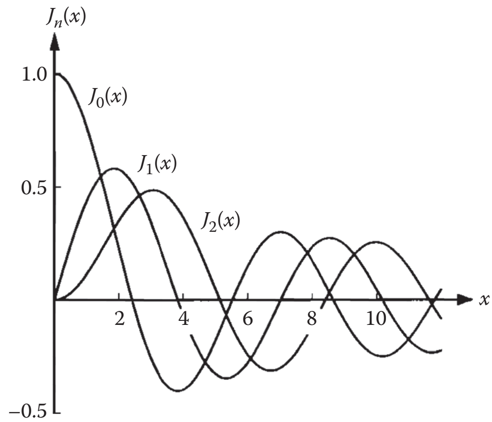

FIGURE B.1
Bessel functions of the first kind.

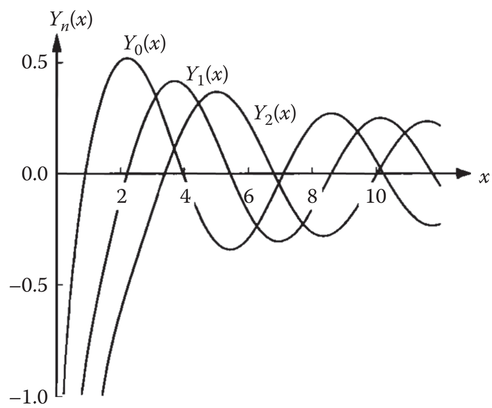

FIGURE B.2
Bessel functions of the second kind.

Modified Bessel Functions

A linear second-order ordinary differential equation with variable coefficients of the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (m^2x^2 + v^2)y = 0 \tag{B.13}$$

is known as the *modified Bessel's differential equation of order v.* The general solution of Eq. (B.13) can be written as

$$y(x) = C_1 J_v(mx) + C_2 K_v(mx) \tag{B.14}$$

TABLE B.1

Numerical Values of $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$

X	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
0.0	1.0000	0.000	$-\infty$	$-\infty$	1.000	0.000	∞	∞
0.1	0.9975	0.0499	–1.5342	–6.4590	1.0025	0.0501	2.4271	9.8538
0.2	0.9900	0.0995	–1.0811	–3.238	1.0100	0.1005	1.7527	4.7760
0.3	0.9776	0.1483	–0.8073	–2.2931	1.0226	0.1517	1.3725	3.0560
0.4	0.9604	0.1960	–0.6060	–1.7809	1.0404	0.2040	1.1145	2.1844
0.5	0.9385	0.2423	–0.4445	–1.4715	1.0635	2.2579	0.9244	1.6564
0.6	0.9120	0.2867	–0.3085	–1.2604	1.0920	0.3137	0.7775	1.3028
0.7	0.8812	0.3290	–0.1907	–1.1032	1.1263	0.3719	0.6605	1.0503
0.8	0.8463	0.3688	–0.0868	–0.9781	1.1665	0.4329	0.5653	0.8618
0.9	0.8075	0.4059	0.0056	0.8731	1.2130	0.4971	0.4867	0.7165
1.0	0.7652	0.4401	0.0883	–0.7812	1.2661	0.5652	0.4210	0.6019
1.1	0.7196	0.4709	0.1622	–0.6981	0.3262	0.6375	0.3656	0.5098
1.2	0.6711	0.4983	0.2281	–0.6211	1.3937	0.7147	0.3185	0.4346
1.3	0.6201	0.5520	0.2865	–0.5485	1.4693	0.7973	0.2782	0.3725
1.4	0.5669	0.5419	0.3379	–0.4791	1.5534	0.8861	0.2437	0.3208
1.5	0.5118	0.5579	0.3824	–0.4123	1.6467	0.9817	0.2138	0.2774
1.6	0.4554	0.5699	0.4204	–0.3476	1.7500	1.0848	0.1880	0.2406
1.7	0.3980	0.5778	0.4520	–0.2847	1.8640	1.1963	0.1655	0.2094
1.8	0.3400	0.5815	0.4774	–0.2237	1.9896	1.3172	0.1459	0.1826
1.9	0.2818	0.5812	0.4968	–0.1644	2.1277	1.4482	0.1288	0.1597
2.0	0.2239	0.5767	0.5104	–0.1070	2.2796	1.5906	0.1139	0.1399
2.1	0.1666	0.5683	0.5183	–0.0517	2.4463	1.7455	0.1008	0.1228
2.2	0.1104	0.5560	0.5208	0.0015	2.6291	1.9141	0.0893	0.1079
2.3	0.0555	0.5399	0.5181	0.0523	2.8296	2.0978	0.0791	0.0950
2.4	0.0025	0.5202	0.5104	0.1005	3.0493	2.2981	0.0702	0.0837
2.5	–0.0484	0.4971	0.4981	0.1459	3.2898	2.5167	0.0624	0.0739
5.6	–0.0968	0.4708	0.4813	0.1884	3.5533	2.7554	0.0554	0.0653
2.7	–0.1424	0.4416	0.4605	0.2276	3.8417	3.0161	0.0493	0.0577
2.8	–0.1850	0.4097	0.4359	0.2635	4.1573	3.3011	0.0438	0.0511
2.9	–0.2243	0.3754	0.4079	0.2959	4.5027	3.6126	0.0390	0.0453
3.0	–0.2601	0.3391	0.3769	0.3247	4.8808	3.9534	0.0347	0.0402
3.1	–0.2921	0.3009	0.3431	0.3496	5.2945	4.3262	0.0310	0.0356
3.2	–0.3202	0.2613	0.3071	0.3707	5.7472	4.7343	0.0276	0.0316
3.3	–0.3443	0.2207	0.2691	0.3879	6.2426	5.1810	0.0246	0.0281
3.4	–0.3643	0.1792	0.2296	0.4010	6.7848	5.6701	0.0220	0.0250
3.5	–0.3801	0.1374	0.1896	0.4102	7.3782	6.2058	0.0196	0.0222
3.6	–0.3918	0.0955	0.1477	0.4154	8.0277	6.7927	0.0175	0.0198
3.7	–0.3992	0.0538	0.1061	0.4167	8.7386	7.4357	0.0156	0.0176
3.8	–0.4226	0.0128	0.0645	0.4141	9.5169	8.1404	0.0140	0.0157
3.9	–0.4018	–0.0272	0.0234	0.4078	10.369	8.9128	0.0125	0.0140
4.0	–0.3971	–0.0660	–0.0169	0.3979	11.302	9.7595	1.1160	1.2484
4.2	–0.3766	–0.1386	–0.0938	0.3680	13.442	11.706	0.8927	0.9938
4.4	–0.3423	–0.2028	–0.1633	0.3260	16.010	14.046	0.7149	0.7923
4.6	–0.2961	–0.2566	–0.2235	0.2737	19.093	16.863	0.5730	0.6325

(*Continued*)

TABLE B.1 (CONTINUED)
Numerical Values of $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$

X	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
4.8	–0.2404	–0.2985	–0.2723	0.2136	22.794	20.253	0.4597	0.5055
5.0	–0.1776	–0.3276	–0.3085	0.1479	27.240	24.336	0.3691	0.4045
5.2	–0.1103	–0.3432	–0.3313	0.0792	32.584	29.254	0.2966	0.3239
5.4	–0.0412	–0.3453	–0.3402	0.0101	39.009	35.182	0.2385	0.2597
5.6	–0.0270	–0.3343	–0.3354	–0.0568	46.738	42.328	0.1918	0.2083
5.8	–0.0917	–0.3110	–0.3177	–0.1192	56.038	50.946	0.1544	0.1673
6.0	0.1506	–0.2767	–0.2882	–0.1750	67.234	61.342	0.1244	0.1344
6.2	0.2017	–0.2329	–0.2483	–0.2223	80.718	73.886	0.1003	0.1081
6.4	0.2433	–0.1816	–0.1999	–0.2596	96.962	89.026	0.0808	0.0869
6.6	0.2740	–0.1250	–0.1452	–0.2857	116.54	107.30	0.0652	0.0700
6.8	0.2931	–0.0652	–0.0864	–0.3002	140.14	129.38	0.0526	0.0564
7.0	0.3001	–0.0047	–0.0259	–0.3027	168.59	156.04	0.0425	0.0454
7.2	0.2951	0.0543	0.0339	–0.2934	202.92	188.25	0.0343	0.0366
7.4	0.2786	0.1096	0.0907	–0.2731	244.34	227.17	0.0277	0.0295
7.6	0.2516	0.1592	0.1424	–0.2428	294.33	274.22	0.0224	0.0238
7.8	0.2154	0.2014	0.1872	–0.2039	354.68	331.10	0.0181	0.0192
8.0	0.1717	0.2346	0.2235	–0.1581	427.56	399.87	0.0146	0.0155
8.2	0.1222	0.2580	0.2501	–0.1072	515.59	483.05	0.0118	0.0126
8.4	0.0692	0.2708	0.2662	–0.0535	621.94	583.66	0.0096	0.0101
8.0	0.0146	0.2728	0.2715	0.0011	750.46	705.38	0.0078	0.0082
8.8	–0.0392	0.2641	0.2659	0.0544	905.80	852.66	0.0063	0.0066
9.0	–0.0903	0.2453	0.2499	0.1043	1093.6	1030.9	0.0051	0.0054
9.2	–0.1367	0.2174	0.2245	0.1491	1320.7	1246.7	0.0041	0.0043
9.4	–0.1768	0.1816	0.1907	0.1871	1595.3	1507.9	0.0033	0.0035
9.6	–0.2090	0.1395	0.1502	0.2171	1927.5	1824.1	0.0027	0.0028
9.8	–0.2323	0.0928	0.1045	0.2379	2329.4	2207.1	0.0022	0.0023
10.0	–0.2459	0.0435	0.0557	0.2490				
10.5	–0.2366	–0.0789	0.0675	0.2337				
11.0	–0.1712	–0.1768	–0.1688	0.1637				
11.5	–0.0677	–0.2284	–0.2252	0.0579				
12.0	0.0477	–0.2234	–0.2252	–0.0571				
12.5	0.1469	–0.1655	–0.1712	–0.1538				
13.0	0.2069	–0.0703	–0.0782	–0.2101				
13.5	0.2150	0.0380	–0.0301	–0.2140				
14.0	0.1711	0.1334	0.1272	–0.1666				
14.5	0.0875	0.1934	0.1903	–0.0810				
15.0	–0.0142	0.2051	0.2055	0.0211				
15.5	–0.1092	0.1672	0.1706	0.1148				

TABLE B.2

Zeros x_k of $J_0(x)$ and the Corresponding Values of $J_1(x)$

k	x_k	$J_1(x_k)$	k	x_k	$J_1(x_k)$
1	2.4048	+0.5191	21	65.1900	+0.09882
2	5.5201	–0.3403	22	68.3315	–0.09652
3	8.6537	+0.2715	23	71.4730	+0.09438
4	11.7915	–0.2325	24	74.6145	–0.09237
5	14.9309	+0.2065	25	77.7560	+0.09049
6	18.0711	–0.1877	26	80.8976	–0.08871
7	21.2116	+0.1733	27	84.0391	+0.08704
8	24.3525	–0.1617	28	87.1806	–0.08545
9	27.4935	+0.1522	29	90.3222	+0.08395
10	30.6346	–0.1442	30	93.4637	–0.08253
11	33.7758	+0.1373	31	96.6053	+0.08118
12	36.9171	–0.1313	32	99.7468	0.07989
13	40.0584	+0.1261	33	102.8884	+0.07866
14	43.1998	–0.1214	34	106.0299	–0.07749
15	46.3412	+0.1172	35	109.1715	+0.07636
16	49.4846	–0.1134	36	112.3131	–0.07529
17	52.6241	+0.1100	37	115.4546	+0.07426
18	55.7655	–0.1068	38	118.5962	–0.07327
19	58.9070	+0.1040	39	121.7377	+0.07232
20	62.0485	–0.1013	40	124.8793	–0.07140

TABLE B.3

Zeros $x_{n,k}$ of $J_n(x)$

k	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	3.832	5.136	6.380	7.588	8.771
2	7.016	8.417	9.761	11.065	12.329
3	10.173	11.620	13.015	14.373	15.700
4	13.324	14.796	16.223	17.616	18.980
5	16.471	17.960	19.409	20.827	22.218
6	19.616	21.117	22.583	24.019	25.430
7	22.760	24.270	25.748	27.199	28.627
8	25.904	27.421	28.908	30.371	31.812
9	29.047	30.569	32.065	33.537	34.989
10	32.190	33.717	35.219	36.699	38.160

where the functions $I_v(mx)$ and $K_v(mx)$ are known as the *modified Bessel functions of the first kind* and *the second kind of order v.* respectively. For all values of $\nu \geq 0$, the function $I_v(mx)$ is given by

$$I_v(mx) = (i)^{-v} J_v(mx), \; i = \sqrt{-1} \tag{B.15}$$

For $\nu \neq n = 0, 1, 2, \ldots$, the function $K_v(mx)$ is defined by

$$K_v(mx) = \frac{\pi}{2} \frac{I_{-v}(mx) - I_v(mx)}{\sin(v\pi)} \tag{B.16}$$

and for the values of $\nu = n = 0, 1, 2,...$, it is given by

$$K_n(mx) = (-1)^{n+1}\left(\ln\frac{mx}{2} + \gamma\right)I_n(mx) + \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!}{k!}\left(\frac{mx}{2}\right)^{2k-n} + \frac{1}{2}(-1)^n\sum_{k=0}^{\infty}[\phi(k)+\phi(k+n)]\frac{(mx/2)^{2k+n}}{k!(n+k)!} \tag{B.17}$$

The function $I_{-v}(mx)$ in Eq. (B.16) is obtained by replacing v and $-v$ in Eq. (B.15). If ν is not an integer or zero, the functions $I_v(mx)$ and $I_{-v}(mx)$ are linearly independent. Hence, the solution (B.14) can also be written as

$$y(x) = D_1 I_v(mx) + D_2 I_{-v}(mx), \qquad v \neq n = 0,1,2,\ldots \tag{B.18}$$

If $v = n = 0, 1, 2,...$, then the functions $I_v(mx)$ and $I_{-v}(mx)$ are related to each other in the form

$$I_n(mx) = I_{-n}(mx) \tag{B.19}$$

The general behavior of these functions for integer values of v is shown in Figs. B.3 and B.4. Note again that the modified Bessel functions of the second kind K_v are unbounded at $x = 0$ for all $v \geq 0$. In Table B.1, we also present the numerical values of $I_n(x)$ and $K_n(x)$ for $n = 0$ and 1.

Asymptotic Formulas for Bessel Functions

For large values of x ($x \to \infty$), the Bessel functions behave as

$$J_v(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right) \tag{B.20a}$$

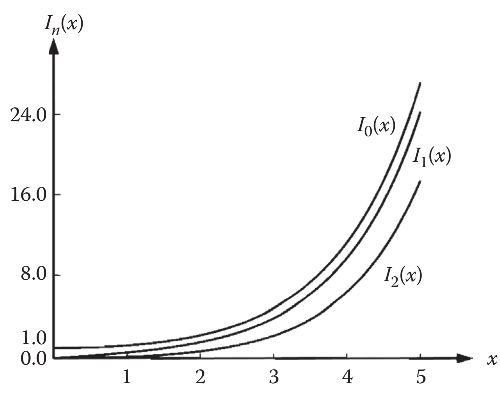

FIGURE B.3
Modified Bessel functions of the first kind.

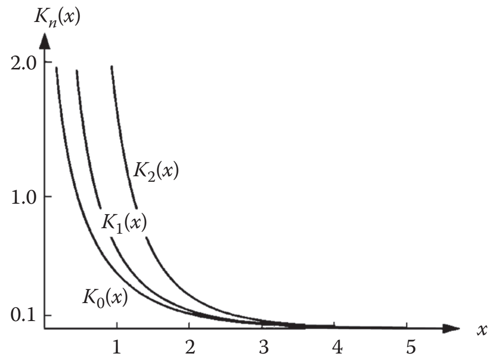

FIGURE B.4
Modified Bessel functions of the second kind.

$$Y_v(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right) \tag{B.20b}$$

$$I_v(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{B.20c}$$

$$K_v(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \tag{B.20d}$$

Note that the asymptotic values $I_v(x)$ and $K_v(x)$ for large x do not depend on v.

Derivatives of Bessel Functions

Some derivatives of Bessel functions are as follows:

$$\frac{d}{dx}\left[W_v(mx)\right] = \begin{cases} mW_{v-1}(mx) - \dfrac{v}{x}W_v(mx), & W = J, Y, I \\ -mW_{v-1}(mx) - \dfrac{v}{x}W_v(mx), & W = K \end{cases} \tag{B.21a}$$

$$\frac{d}{dx}\left[W_v(mx)\right] = \begin{cases} -mW_{v+1}(mx) + \dfrac{v}{x}W_v(mx), & W = J, Y, K \\ mW_{v+1}(mx) + \dfrac{v}{x}W_v(mx), & W = I \end{cases} \tag{B.21b}$$

$$\frac{d}{dx}\left[x^{v}W_v(mx)\right]=\begin{cases} mx^{v}W_{v-1}(mx), & W=J,Y,I \\ -mx^{v}W_{v-1}(mx), & W=K \end{cases} \tag{B.21c}$$

$$\frac{d}{dx}\left[x^{-v}W_v(mx)\right]=\begin{cases} -mx^{v}W_{v+1}(mx), & W=J,Y,K \\ mx^{-v}W_{v+1}(mx), & W=I \end{cases} \tag{B.21d}$$

Equations Transformable into Bessel Differential Equations

The solution of the second-order ordinary differential equation

$$x^2\frac{d^2y}{dx^2}+(1-2k)x\frac{dy}{dx}+(\alpha x^{2r}+\beta^2)y=0 \tag{B.22}$$

where k, α, r, and β are constants, can be written in the form

$$y(x)=x^kZ_v\left(\frac{\sqrt{\alpha}}{r}x^r\right) \tag{B.23}$$

where

$$v=\frac{1}{r}\sqrt{k^2-\beta^2} \tag{B.24}$$

If $\sqrt{\alpha}/r$ is *real*, then Z_v is to be interpreted by Eq. (B.2) or Eq. (B.7), depending on whether v is a positive integer (or zero) or not, respectively. On the other hand, if $\sqrt{\alpha}/r$ is *imaginary*, then Z_v stands for Eq. (B.14) or Eq. (B.18), again depending on v. If $\alpha = 0$, then Eq. (B.22) is a Cauchy–Euler equation and has the solution

$$y(x)=x^k(C_1x^{rv}+C_2x^{-rv}) \tag{B.25}$$

Recurrence Relations

$$W_{v-1}(mx)+W_{v+1}(mx)=\frac{2v}{mx}W_v(mx),\ W=J,Y \tag{B.26a}$$

$$I_{v-1}(mx)-I_{v+1}(mx)=\frac{2v}{mx}I_v(mx) \tag{B.26b}$$

$$K_{v-1}(mx) - K_{v+1}(mx) = \frac{2v}{mx} K_v(mx) \tag{B.26c}$$

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2. Hildebrand, F. B., *Advanced Calculus for Applications*, 2nd ed., Prentice-Hall, 1976.
3. Jahnke, E., Emde, F., and Lösch, F., *Tables of Higher Functions*, McGraw-Hill, 1960.
4. Özışık, M. N., *Heat Conduction*, 2nd ed., John Wiley and Sons, 1993.

Appendix C: Error Function

TABLE C.1

Numerical Values of Error Function

z	erf*z*	*z*	erf*z*	*z*	erf*z*	*z*	erf*z*
0.00	0.00000	0.20	0.22270	0.40	0.42839	0.60	0.60385
0.01	0 01128	0.21	0.23352	0.41	0.43792	0.61	0.61168
0.02	0.02256	0.22	0.24429	0.42	0.44746	0.62	0.61941
0.03	0.03384	0.23	0.25502	0.43	0.45688	0.63	0.62704
0.04	0.04511	0.24	0.26570	0.44	0.46622	0.64	0.63458
0.05	0.05637	0.25	0.27632	0.45	0.47548	0.65	0.64202
0.06	0.06762	0.26	0.28689	0.46	0.48465	0.66	0.64937
0.07	0.07885	0.27	0.29741	0.47	0.49374	0.67	0.65662
0.08	0.09007	0.28	0.30788	0.48	0.50274	0.68	0.66378
0.09	0.10128	0.29	0.31828	0.49	0.51166	0.69	0.67084
0.10	0.11246	0.30	0.32862	0.50	0.52049	0.70	0.67780
0.11	0.12362	0.31	0.33890	0.52	0.52924	0.71	0.68466
0.12	0.13475	0.32	0.34912	0.52	0.53789	0.72	0.69143
0.13	0.14586	0.33	0.35927	0.53	0.54646	0.73	0.69810
0.14	0.15694	0.34	0.36936	0.54	0.55693	0.74	0.70467
0.15	0.16799	0.35	0.37938	0.55	0.56332	0.75	0.71115
0.16	0.17901	0.36	0.38932	0.56	0.57161	0.76	0.71753
0.17	0.18999	0.37	0.39920	0.57	0.57981	0.77	0.72382
0.18	0.20093	0.38	0.40900	0.58	0.58792	0.78	0.73001
0.19	0.21183	0.39	0.41873	0.59	0.59593	0.79	0.73610
0.80	0.74210	1.15	0.89612	1.50	0.96610	1.85	0.99111
0.81	0.74800	1.16	0.89909	1.51	0.96772	1.86	0.99147
0.82	0.77381	1.17	0.90200	1.52	0.96851	1.87	0.99182
0.83	0.75952	1.18	0.90483	1.53	0.96951	1.88	0.99215
0.84	0.76514	1.19	0.90760	1.54	0.97058	1.89	0.99247
0.85	0.77066	1.20	0.91031	1.55	0.97162	1.90	0.99279
0.86	0.77610	1.21	0.91295	1.56	0.97262	1.91	0.99308
0.87	0.78143	1.22	0.91553	1.57	0.97350	1.92	0.99337
0.88	0.78668	1.23	0.91805	1.58	0.97454	1.93	0.99365
0.89	0.79164	1.24	0.92050	1.59	0.97546	1.94	0.99392
0.90	0.79690	1.25	0.92290	1.60	0.97634	1.95	0.99417
0.91	0.80188	1.26	0.92523	1.61	0.97720	1.96	0.99442
0.92	0.80676	1.27	0.92751	1.62	0.97803	1.97	0.99466
0.93	0.81156	1.28	0.92973	1.63	0.97994	1.98	0.99489
0.94	0.81627	1.29	0.93189	1.64	0.97062	1.99	0.99511
0.95	0.82089	1.30	0.93400	1.65	0.98037	2.00	0.99432
0.96	0.82542	1.31	0.93606	1.66	0.98110	2.20	0.99814
0.97	0.82987	1.32	0.93806	1.67	0.98181	2.40	0.99931

(*Continued*)

TABLE C.1 (CONTINUED)

Numerical Values of Error Function

z	erf$z$	z	erf$z$	z	erf$z$	z	erf$z$
0.98	0.83423	1.33	0.94001	1.68	0.98249	2.60	0.99976
0.99	0.83850	1.34	0.94191	1.69	0.98316	2.80	0.99972
1.00	0.84270	1.35	0.94376	1.70	0.98379	3.00	0.999978
1.01	0.84681	1.36	0.94556	1.72	0.98440	3.20	0.999994
1.02	0.85083	1.37	0.94731	1.72	0.98500	3.40	0.999998
1.03	0.85478	1.38	0.94901	1.73	0.98557	3.60	1.000000
1.04	0.85864	1.39	0.95067	1.74	0.98613		
1.05	0.86243	1.40	0.95228	1.75	0.98667		
1.06	0.86614	1.41	0.95385	1.76	0.98719		
1.07	0.86977	1.42	0.95537	1.77	0.98769		
1.08	0.87332	1.43	0.95685	1.78	0.98817		
1.09	0.87680	1.44	0.95829	1.79	0.98875		
1.10	0.88020	1.45	0.95969	1.80	0.98909		
1.11	0.88353	1.46	0.96105	1.81	0.98952		
1.12	0.88678	1.47	0.96237	1.82	0.98994		
1.13	0.88997	1.48	0.96365	1.83	0.99034		
1.14	0.89308	1.49	0.96489	1.84	0.99073		

Appendix D: Laplace Transforms

Table of Laplace Transforms

Transform No.	$\bar{f}(p)$	$f(t)$, $t > 0$
1	$\frac{1}{p}$	1
2	$\frac{1}{p^2}$	t
3	$\frac{1}{p^n}, \quad n = 1,2,3,\ldots$	$\frac{t^{n-1}}{(n-1)!}, \quad 0! = 1$
4	$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{\pi t}}$
5	$\frac{1}{p-a}$	e^{at}
6	$\frac{1}{(p-a)^n}, \quad n = 1,2,3,\ldots$	$\frac{1}{(n-1)!} t^{n-1} e^{at}, \quad 0! = 1$
7	$\frac{a}{p^2 + a^2}$	sin *at*
8	$\frac{a}{p^2 + a^2}$	cos *at*
9	$\frac{a}{p^2 + a^2}$	sinh *at*
10	$\frac{a}{p^2 + a^2}$	cosh *at*
11	$\frac{2ap}{(p^2 + a^2)^2}$	*r* sin *at*
12	$\frac{p^2 - a^2}{(p^2 + a^2)^2}$	*t* cos *at*
13	$\frac{2ap}{(p^2 + a^2)^2}$	*t* sinh *at*
14	$\frac{p^2 + a^2}{p^2 - a^2}$	*t* cosh *at*
15	$\frac{2a^3}{(p^2 + a^2)^2}$	sin *at* – *at* cos *at*
16	$\frac{2a^3}{(p^2 + a^2)^2}$	*at* cosh *at* – sinh *at*
17	$\frac{4a^3}{p^4 + 4a^4}$	sin *at* cosh *at* – cos *at* sinh *at*
18	$\frac{2a^2 p}{p^4 + 4a^4}$	sin *at* sinh *at*

(*Continued*)

Table of Laplace Transforms (CONTINUED)

Transform No.	$\bar{f}(p)$	$f(t)$, $t > 0$
19	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(at)$
20	$\dfrac{1}{\sqrt{p^2+a^2}}$	$I_0(at)$
21	$\dfrac{a}{(p^2+a^2)^{3/2}}$	$t\,J_1(at)$
22	$\dfrac{p}{(p^2+a^2)^{3/2}}$	$t\,J_0(at)$
23	$\dfrac{a}{(p^2-a^2)^{3/2}}$	$t\,I_1(at)$
24	$\dfrac{p}{(p^2-a^2)^{3/2}}$	$t\,I_0(at)$
25	$e^{-x\sqrt{p/a}}$	$\dfrac{x}{2(\pi a t^3)^{1/2}}\,e^{-x^2/4at}$
26	$\dfrac{e^{-x\sqrt{p/a}}}{\sqrt{p/a}}$	$\left(\dfrac{a}{\pi t}\right)^{1/2} e^{-x^2/4at}$
27	$\dfrac{e^{-x\sqrt{p/a}}}{p}$	$\operatorname{erfc}\left[\dfrac{x}{2(at)^{1/2}}\right]$
28	$\dfrac{e^{-x\sqrt{p/a}}}{p\sqrt{p/a}}$	$2\left(\dfrac{at}{\pi}\right)^{1/2} e^{-x^2/4at} - x\operatorname{erfc}\left\{\dfrac{x}{2(at)^{1/2}}\right\}$
29	$\dfrac{e^{-x\sqrt{p/a}}}{p^2}$	$\left(t+\dfrac{x^2}{2a}\right)\operatorname{erfc}\left\{\dfrac{x}{2(at)^{1/2}}\right\} - x\left(\dfrac{t}{\pi a}\right)^{1/2} e^{-x^2/4at}$
30	$\dfrac{e^{-a/p}}{p^{3/2}}$	$\dfrac{\sin 2\sqrt{at}}{\sqrt{\pi a}}$
31	$\dfrac{e^{-a/p}}{\sqrt{p}}$	$\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$
32†	$-\dfrac{(\gamma+\ln p)}{p}$	$\ln t$
33	$\dfrac{\sinh px}{p\sinh pa}$	$\dfrac{x}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\dfrac{n\pi x}{a}\cos\dfrac{n\pi t}{a}$
34	$\dfrac{\sinh px}{p\cosh pa}$	$\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\sin\dfrac{(2n-1)\pi x}{2a}\sin\dfrac{(2n-1)\pi t}{2a}$
35	$\dfrac{\cosh px}{p\sinh pa}$	$\dfrac{t}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\cos\dfrac{n\pi x}{a}\sin\dfrac{n\pi t}{a}$
36	$\dfrac{\cosh px}{p\cosh pa}$	$1+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}\cos\dfrac{(2n-1)\pi x}{2a}\cos\dfrac{(2n-1)\pi t}{2a}$

(Continued)

Table of Laplace Transforms (CONTINUED)

Transform No.	$\bar{f}(p)$	$f(t),\ t > 0$
37	$\dfrac{\sinh x\sqrt{p}}{\sinh a\sqrt{p}}$	$\dfrac{2\pi}{a^2}\sum_{n=1}^{\infty}(-1)^n n e^{-n^2\pi^2 t/a^2}\sin\dfrac{n\pi x}{a}$
38	$\dfrac{\cosh x\sqrt{p}}{\cosh a\sqrt{p}}$	$\dfrac{\pi}{a^2}-\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)e^{-(2n-1)^2\pi^2 t/4a^2}\cos\dfrac{(2n-1)\pi x}{2a}$
39	$\dfrac{\sinh x\sqrt{p}}{p\sinh a\sqrt{p}}$	$\dfrac{x}{a}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}e^{-n^2\pi^2 t/a^2}\sin\dfrac{n\pi x}{a}$
40	$\dfrac{\cosh x\sqrt{p}}{p\cosh a\sqrt{p}}$	$1+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 t/4a^2}\cos\dfrac{(2n-1)\pi x}{2a}$

†γ = Euler's constant = 0.5772156…

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Appendix E: Exponential Integral Functions

The *exponential integral of order n* is defined as

$$E_n(x) = \int_1^\infty e^{-xt} t^{-n} dt = \int_0^1 e^{-x/\mu} \mu^{n-2}\, d\mu,\ n = 0, 1, 2, \ldots \tag{E.1}$$

Differentiating Eq. (E.1) yields

$$\frac{d}{dx} E_n(x) = -E_{n-1}(x),\ n = 1, 2, 3, \ldots \tag{E.2a}$$

where

$$E_0(x) = \int_1^\infty e^{-xt} dt = \frac{1}{x} e^{-x} \tag{E.2b}$$

Moreover, integrating Eq. (E.2*a*) gives

$$\int_x^\infty E_n(x) dx = E_{n+1}(x),\ n = 0, 1, 2, \ldots \tag{E.3}$$

A recurrence relation between the consecutive orders can be obtained by integrating Eq. (E.1) by parts, or

$$E_{n+1}(x) = \frac{1}{n}\left[e^{-x} - xE_n(x)\right],\ n = 1, 2, 3, \ldots \tag{E.4}$$

From Eq. (E.1), for $x = 0$ it follows that

$$E_n(0) = \begin{cases} +\infty, & n = 1 \\ \dfrac{1}{n-1}, & n \geq 2 \end{cases} \tag{E.5}$$

A series expansion for $E_n(x)$ is given by

$$E_n(x) = (-1)^n \frac{x^{n-1}}{(n-1)!}(\ln x + \psi_n) + \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{(-x)^m}{m!(n-1-m)},\ n = 1, 2, 3, \ldots \tag{E.6}$$

where

$$\psi_n = \begin{cases} \gamma, & n = 1 \\ \gamma - \displaystyle\sum_{m=1}^{n-1} \frac{1}{m}, & n \geq 2 \end{cases}$$

and γ = 0.577216... is *Euler's constant*.

For large values of x, the asymptotic expansion for $E_n(x)$ is given by

$$E_n(x) \cong \frac{e^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + \dots\right],\ n = 0, 1, 2, \dots \tag{E.7a}$$

where it is apparent that

$$E_n(x) \cong \frac{e^{-x}}{x},\ \text{as } x \to \infty \tag{E.7b}$$

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