Dynamics of second-order systems.

Definition. A second-order system is one whose output $y(t)$ is modeled by a second-order differential equation.

✓ $$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b\, f(t)$$

$$\frac{a_2}{a_0} \frac{d^2y}{dt^2} + \frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} f(t) \qquad a_0 \neq 0.$$

$$\tau^2 \frac{d^2y}{dt^2} + 2\psi\tau \frac{dy}{dt} + y = K_p\, f(t),$$

where,

$$\tau^2 = \frac{a_2}{a_0}, \quad 2\psi\tau = \frac{a_1}{a_0}, \quad K_p = \frac{b}{a_0}.$$

τ = natural period of oscillation

ψ = damping factor

K_p = steady-state / static gain

✓ $$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1}$$

$y, f \Rightarrow$ deviation variables.

- τ determines the speed of response (response time).
- The decrease in size of oscillation (size of two successive peaks) we call "damping". Small value of ψ implies little damping but a large amount of oscillation.

Large damping small oscillation

Small damping oscillation for a long time

Zero damping oscillation with const. amplitude.

$\zeta = 0$	Undamped system	Oscillation with const. amplitude
$\zeta < 0$	Unstable system	Oscillation with increasing ampltd.
$\zeta > 0$	Stable system	Oscillation with decreasing ampltd.

Systems with second- or higher-order dynamics.

1. "Multicapacity processes". Processes that consist of two or more capacities (1st-order systems) in series.
2. "Inherently 2nd-order systems". Examples include the fluid or mechanical solid components of a process that possess inertia and are subjected to acceleration.
3. "System + Controller". It leads to 2nd- or higher-order dynamics.

Dynamic response of Second-order system

✓ $$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1}$$ --- desired before

✓ Determining poles:

$$\tau^2 s^2 + 2\psi\tau s + 1 = 0$$

$$p_1 = \frac{-2\psi\tau + \sqrt{(2\psi\tau)^2 - 4\tau^2}}{2\tau^2} = -\frac{\psi}{\tau} + \frac{\sqrt{\psi^2-1}}{\tau}$$

$$p_2 = \frac{-2\psi\tau - \sqrt{(2\psi\tau)^2 - 4\tau^2}}{2\tau^2} = -\frac{\psi}{\tau} - \frac{\sqrt{\psi^2-1}}{\tau}$$

✓ Thus,

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p/\tau^2}{(s-p_1)(s-p_2)}$$

$y(t)$ will depend strongly on ψ.

Case 1. $\psi > 1$ two distinct and real poles Overdamped system

Case 2. $\psi = 1$ two equal poles $(= -1/\tau)$ Critically damped

Case 3. $0 < \psi < 1$ two complex conjugate poles Underdamped

$$p_1 = -\frac{\psi}{\tau} + i\frac{\sqrt{1-\psi^2}}{\tau} \qquad p_2 = -\frac{\psi}{\tau} - i\frac{\sqrt{1-\psi^2}}{\tau}$$

Step-response time-domain solutions.

✓ For the step input $\bar{f}(s) = \frac{A}{s}$, we have:

$$\bar{y}(s) = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1} \cdot \frac{A}{s}$$

✓ After inverting to the time domain:

Overdamped $(\psi > 1)$

$$y(t) = K_p A \left[1 - e^{-\psi t/\tau} \left\{ \cosh \frac{\sqrt{\psi^2 - 1}}{\tau} t + \frac{\psi}{\sqrt{\psi^2-1}} \sinh \frac{\sqrt{\psi^2-1}}{\tau} t \right\} \right]$$

Critically damped $(\psi = 1)$

$$y(t) = K_p A \left[1 - \left(1 + \frac{t}{\tau}\right) e^{-t/\tau} \right]$$

Underdamped $(0 < \psi < 1)$

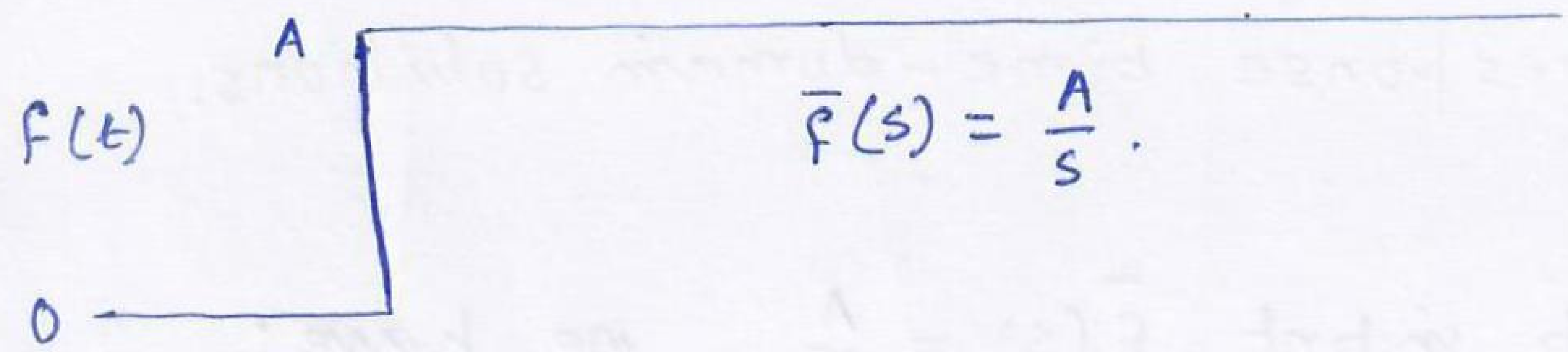
$$y(t) = K_p A \left[1 - e^{-\psi t/\tau} \left\{ \cos \frac{\sqrt{1-\psi^2}}{\tau} t + \frac{\psi}{\sqrt{1-\psi^2}} \sin \frac{\sqrt{1-\psi^2}}{\tau} t \right\} \right]$$

Rearranging,

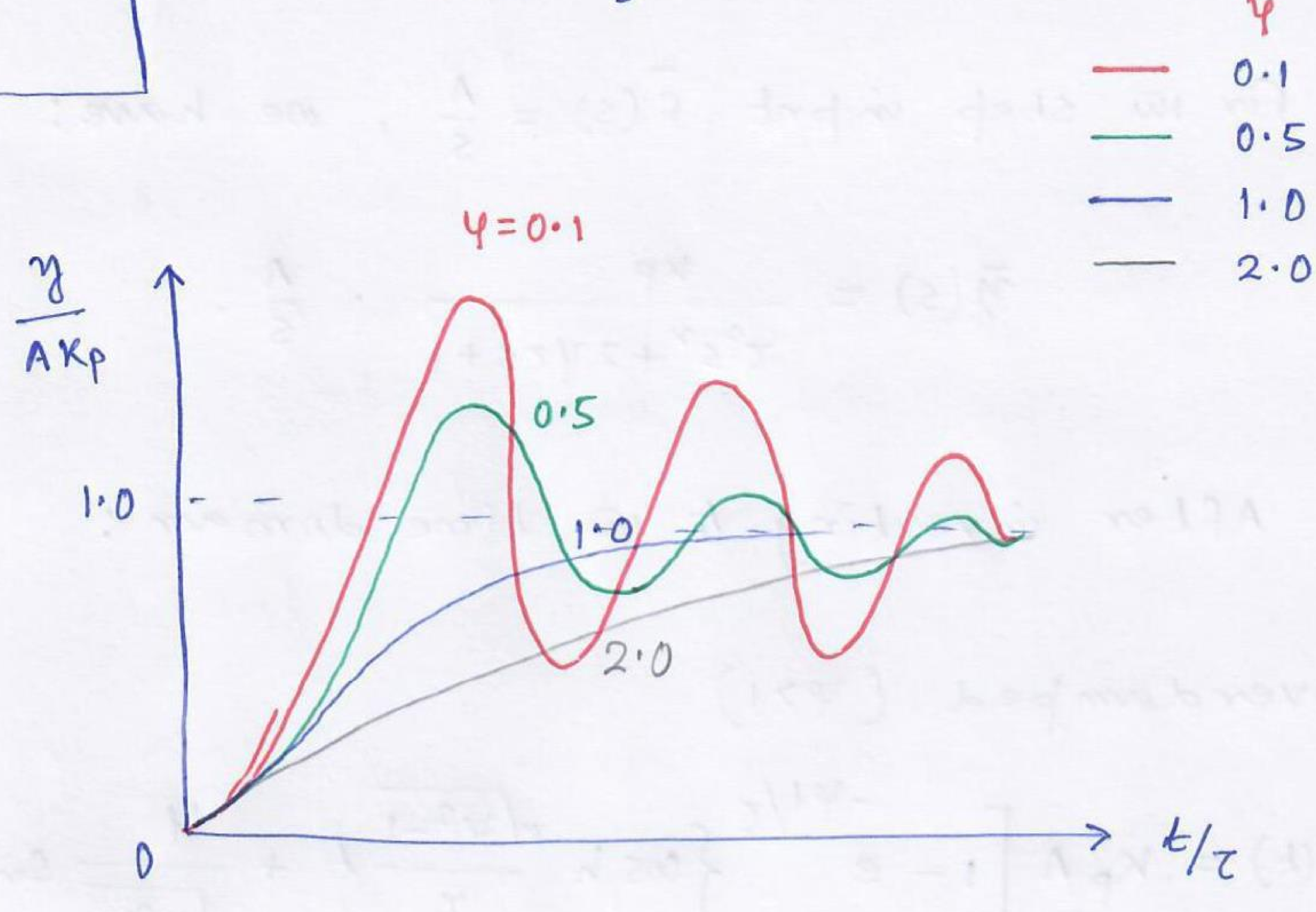
$$y(t) = K_p A \left[1 - \frac{1}{\sqrt{1-\psi^2}} e^{-\psi t/\tau} \sin(\omega t + \phi) \right]$$

Radian frequency (rad/time): $\omega = \frac{\sqrt{1-\psi^2}}{\tau}$

phase angle (rad): $\phi = \tan^{-1}\left(\frac{\sqrt{1-\psi^2}}{\psi} \right)$



$$\bar{f}(s) = \frac{A}{s}.$$

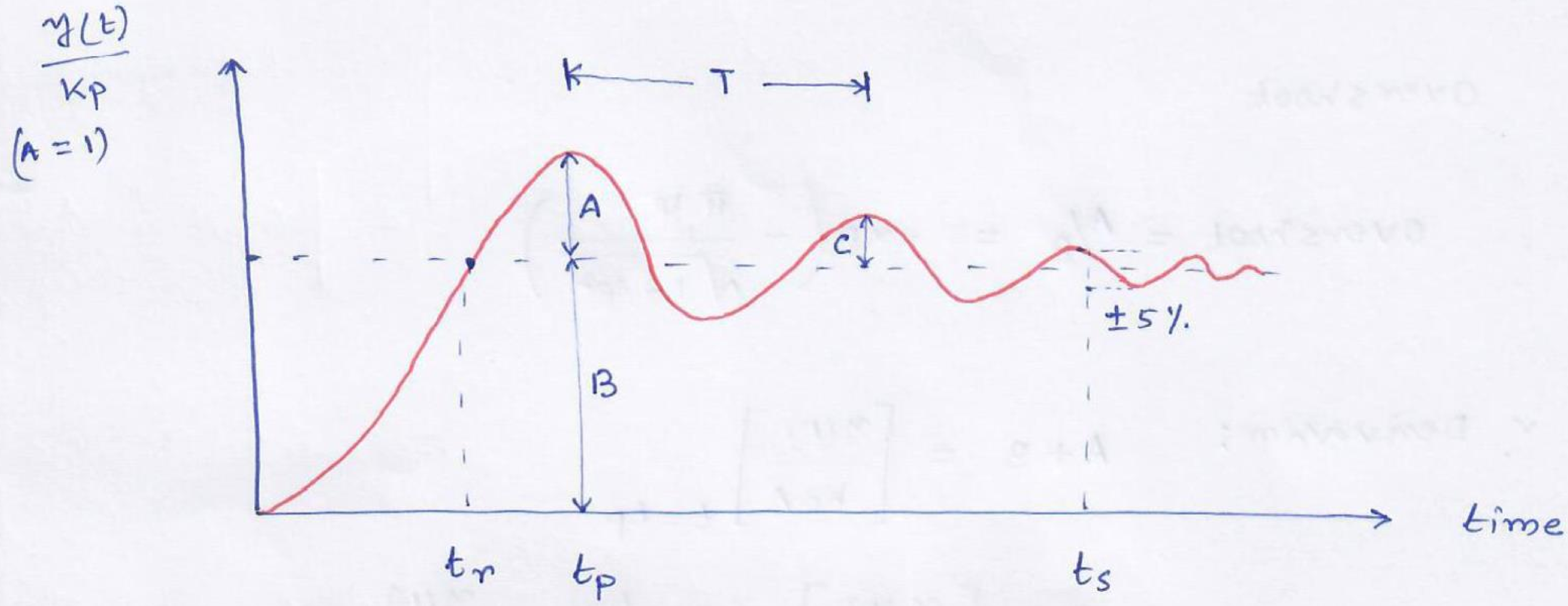


$$\lim_{t \to \infty} \frac{y(t)}{K_p A} = 1.0$$

Underdamped system: time-domain features.

Overdamped response $\equiv$ 1st-order systems connected in series

Underdamped response $\equiv$ system + controller

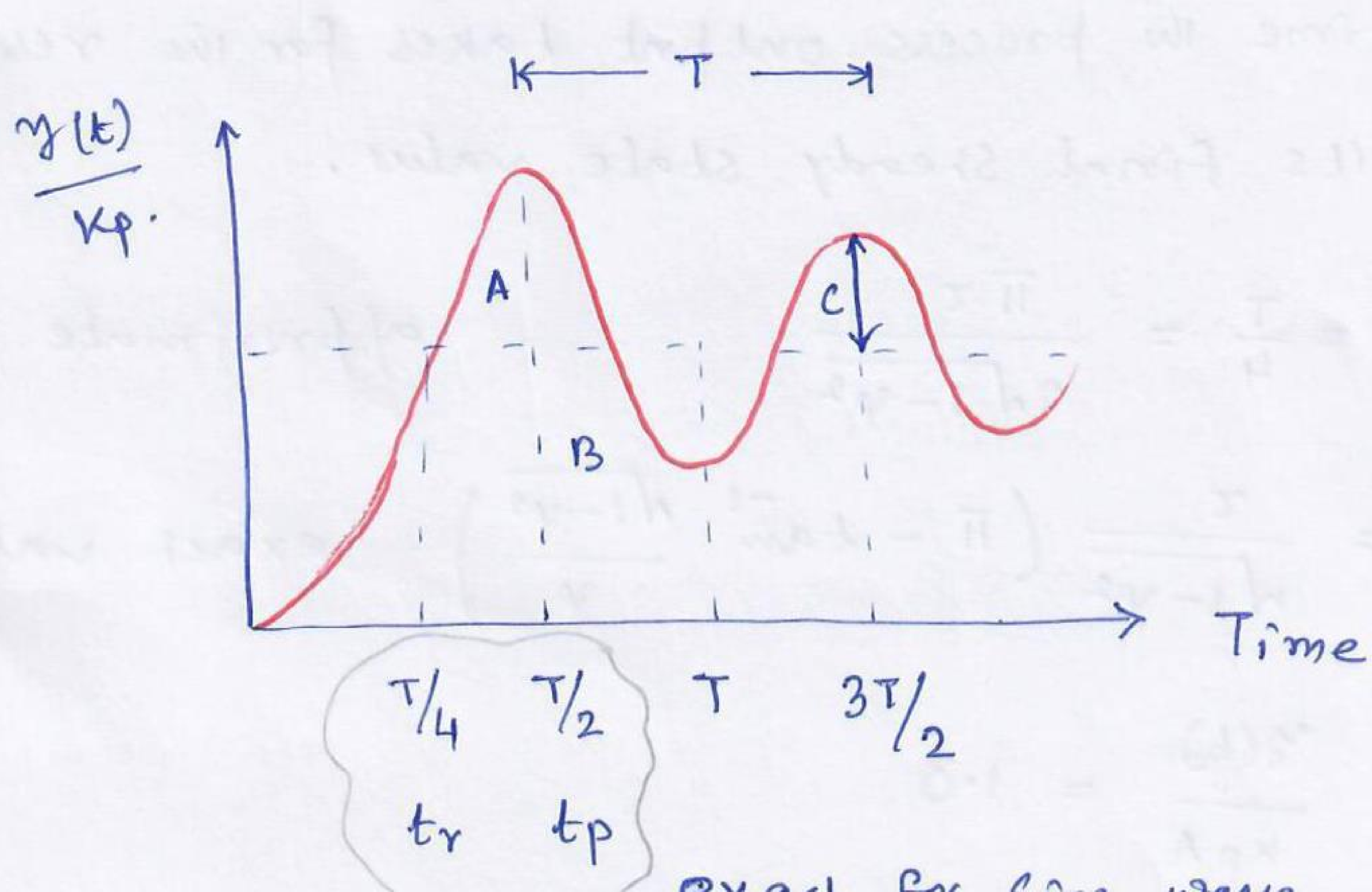


t_r = rise time

t_p = peak time / time to 1st peak

t_s = settling time

T = period.



exact for sine wave

Peak time (t_p).

This is the time required for the output to reach its first max value.

$$t_p = \frac{T}{2} = \frac{\pi \tau}{\sqrt{1-\varphi^2}}$$

$$\omega T = 2\pi$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi\tau}{\sqrt{1-\varphi^2}}$$

✓ Derivation : $\frac{d}{dt}\left[y/K_p A\right] = 0$

Overshoot

$$\text{overshoot} = A/B = \exp\left(-\frac{\pi \varphi}{\sqrt{1-\varphi^2}}\right)$$

✓ Derivation: $A + B = \left[\frac{y(t)}{K_p A}\right]_{t=t_p}$

$$B = \left[\frac{y(t)}{K_p A}\right]_{ss} = \lim_{t\to\infty} \frac{y(t)}{K_p A} = 1.0$$

Rise time (t_r)

This is the time the process output takes for the response to first reach its final steady state value.

$$t_r = \frac{T}{4} = \frac{\pi\tau}{2\sqrt{1-\varphi^2}} \quad \text{---- approximate .}$$

$$t_r = \frac{\tau}{\sqrt{1-\varphi^2}}\left(\pi - \tan^{-1}\frac{\sqrt{1-\varphi^2}}{\varphi}\right) \quad \text{-- exact value}$$

✓ Derivation : $\frac{y(t)}{K_p A} = 1.0$

Decay ratio

$$\text{decay ratio} = \frac{C}{A} = \exp\left(-\frac{2\pi\psi}{\sqrt{1-\psi^2}}\right) = (\text{overshoot})^2$$

✓ Derivation:
$$B + C = \left[\frac{y(t)}{K_p A}\right]_{t = \frac{3T}{2}}$$

$$B = 1.0$$

$t_p = \frac{T}{2}$ (exactly). Another T is there to reach the time corresponding to C. So, $t = T + \frac{T}{2} = \frac{3T}{2}$.

Settling time / Response time (t_s)

This is the time required for the process output to come within some prescribed band of the final steady state value and remain in this band. Typical band limits are ±1%, ±2%, ±3%, ±4%, ±5%..

For band limit of ±1% : $t_s = \frac{5\tau}{\psi}$

±2% : $t_s = \frac{4\tau}{\psi}$

✓ Note: when $t_s = \frac{4\tau}{\psi} \Rightarrow \frac{-\psi t_s}{\tau} = -4 \quad \therefore e^{-4} \approx 0.02 \ (=0.018)$

when $t_s = \frac{5\tau}{\psi} \Rightarrow \quad = -5 \quad \therefore e^{-5} \approx 0.01 \ (=0.0067)$

Ex. A step change of magnitude 10 is introduced into a system having the TF

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{4}{s^2+1.6s+4}$$

(i) comment on the type of response with finding ψ.

(ii) Determine ultimate value of response and overshoot

(iii) Calculate the rise time.

✓ solution: (i)

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{4}{s^2+1.6s+4} = \frac{1}{0.25s^2+0.4s+1} = \frac{K_p}{\tau^2 s^2+2\psi\tau s+1}$$

So, $K_p = 1$, $\tau = 0.5$, $\psi = 0.4$ (<1) Underdamped response

(ii) Since $\bar{f}(s) = \frac{10}{s}$, $\bar{y}(s) = \frac{40}{s(s^2+1.6s+4)}$

ultimate value of response $= B = \lim_{s\to 0}\left[s\,\bar{y}(s)\right] = 10$

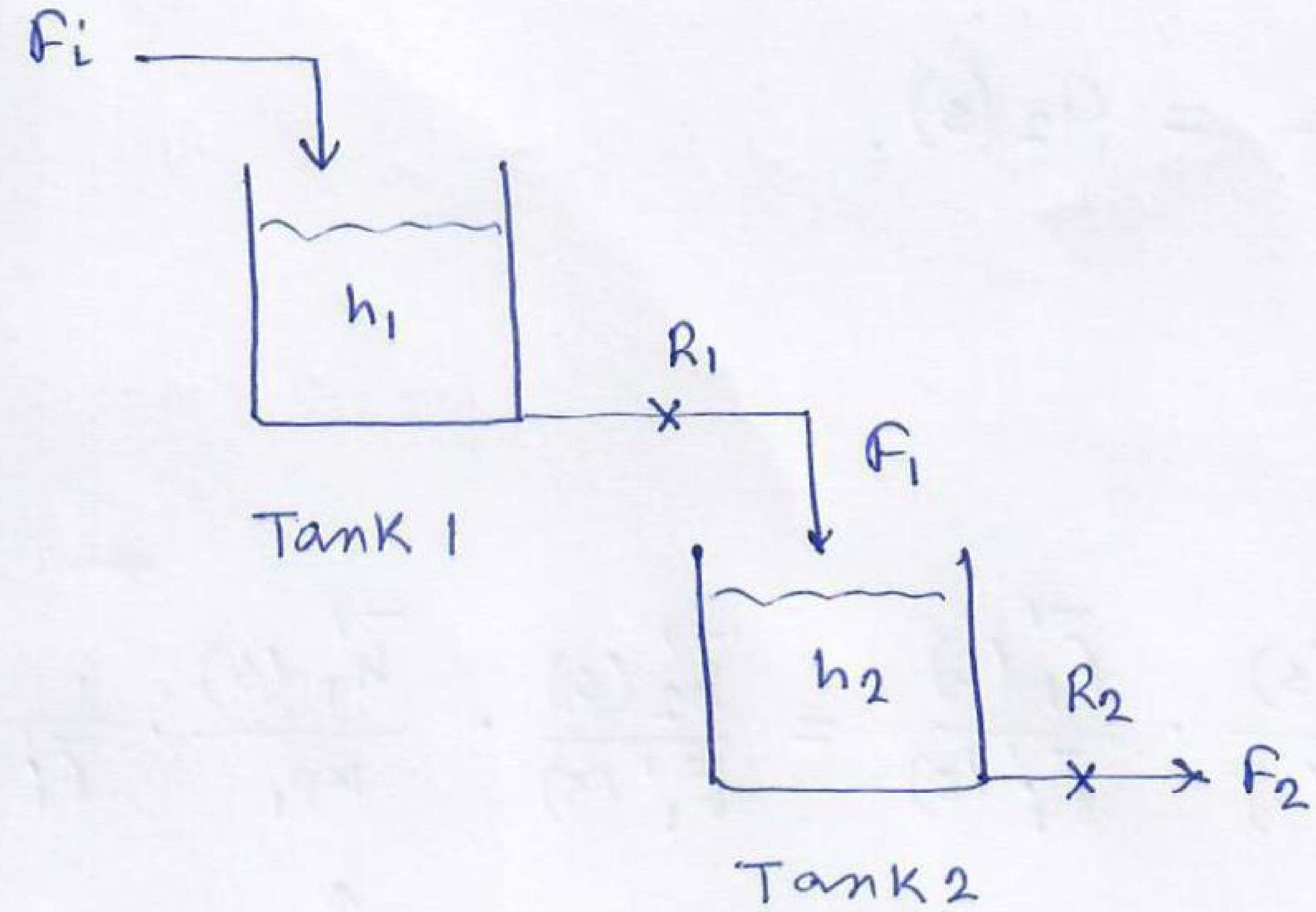
overshoot $= \exp\left(-\frac{\pi\psi}{\sqrt{1-\psi^2}}\right) = 0.254 = A/B.$

(iii) Rise time $t_r = \frac{\tau}{\sqrt{1-\psi^2}}\left(\pi - \tan^{-1}\frac{\sqrt{1-\psi^2}}{\psi}\right) = 1.08$

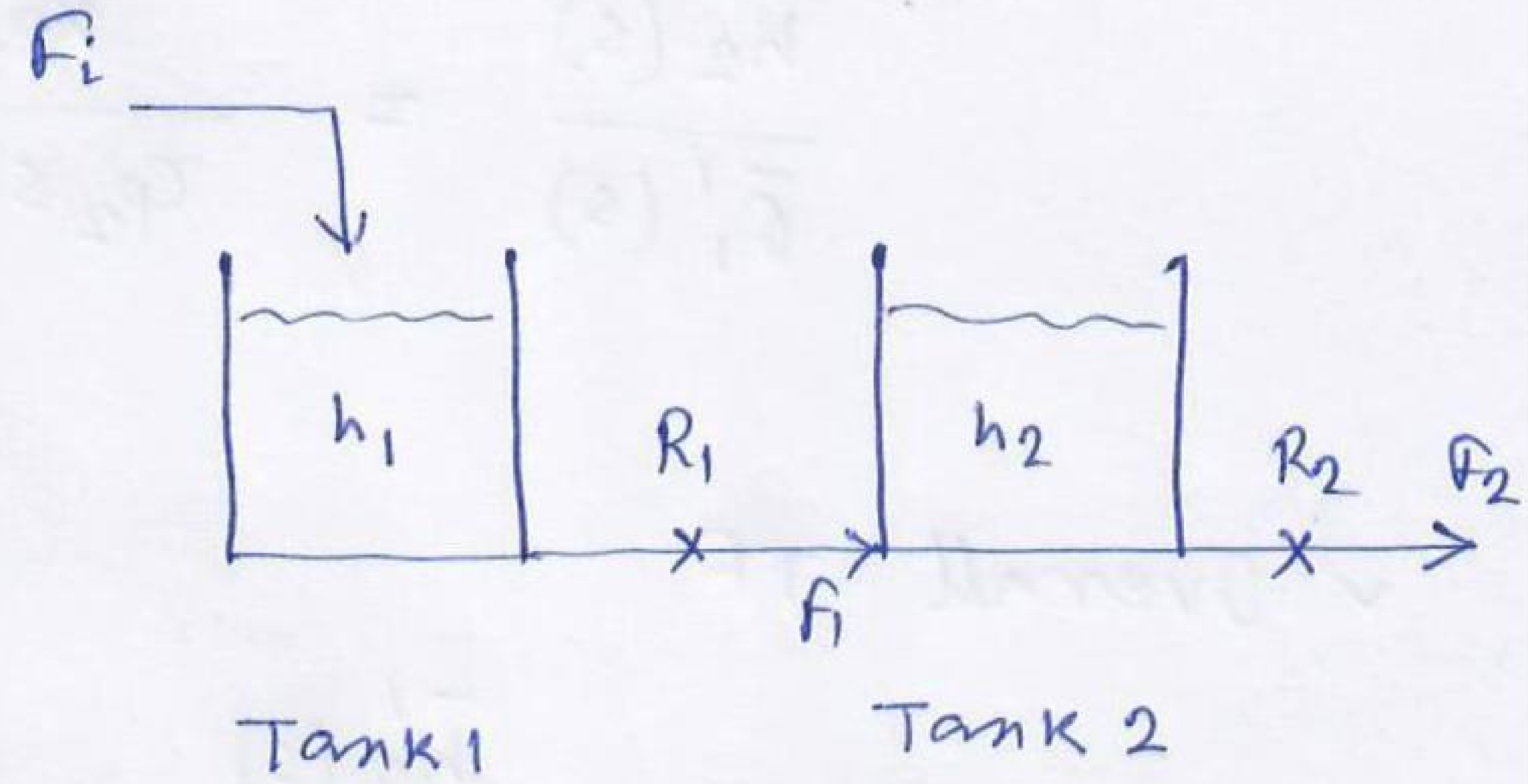
Second - order systems

- → Connecting 2 or more 1st-order systems in series
- → Including a controller with a 1st- or higher-order system

Two 1st-order systems in series



Noninteracting tanks.



Interacting tanks.

Noninteracting tanks.

Tank 1: $A_1 \dfrac{dh_1}{dt} = F_i - F_1 = F_i - \dfrac{h_1}{R_1}$

$$A_1 R_1 \frac{dh_1'}{dt} + h_1' = R_1 F_i'$$

$$\tau_{p_1} \frac{dh_1'}{dt} + h_1' = K_{p_1} F_i' \qquad \tau_{p_1} = A_1 R_1 \qquad K_{p_1} = R_1$$

$$\frac{\bar{h}_1'(s)}{\bar{F}_i'(s)} = \frac{K_{p_1}}{\tau_{p_1} s + 1} = G_1(s).$$

✓ Tank 2 : $A_2 \dfrac{dh_2}{dt} = F_1 - F_2 = F_1 - \dfrac{h_2}{R_2}$

$$A_2 R_2 \frac{dh_2'}{dt} + h_2' = R_2 F_1'$$

$$\tau_{P_2} = A_2 R_2$$

$$\tau_{P_2} \frac{dh_2'}{dt} + h_2' = K_{P_2} F_1'$$

$$K_{P_2} = R_2$$

$$\frac{\bar{h}_2'(s)}{\bar{F}_1'(s)} = \frac{K_{P_2}}{\tau_{P_2} s + 1} = G_2(s).$$

✓ Overall TF

$$G_0(s) = \frac{\bar{h}_2'(s)}{\bar{F}_i'(s)} = \frac{\bar{h}_2'(s)}{\bar{F}_1'(s)} \cdot \frac{\bar{F}_1'(s)}{\bar{F}_i'(s)} = \frac{\bar{h}_2'(s)}{\bar{F}_1'(s)} \cdot \frac{\bar{h}_1'(s)}{K_{P_1}} \cdot \frac{1}{\bar{F}_i'(s)}.$$

$$= \frac{K_{P_2}}{(\tau_{P_1} s+1)(\tau_{P_2} s+1)}$$

$$F_1' = \frac{h_1'}{R_1} = \frac{h_1'}{K_{P_1}}$$

✓ Inverting it,

$$h_2'(t) = K_{P_2}\left[1 + \frac{1}{\tau_{P_2} - \tau_{P_1}}\left(\tau_{P_1} e^{-t/\tau_{P_1}} - \tau_{P_2} e^{-t/\tau_{P_2}}\right)\right]$$

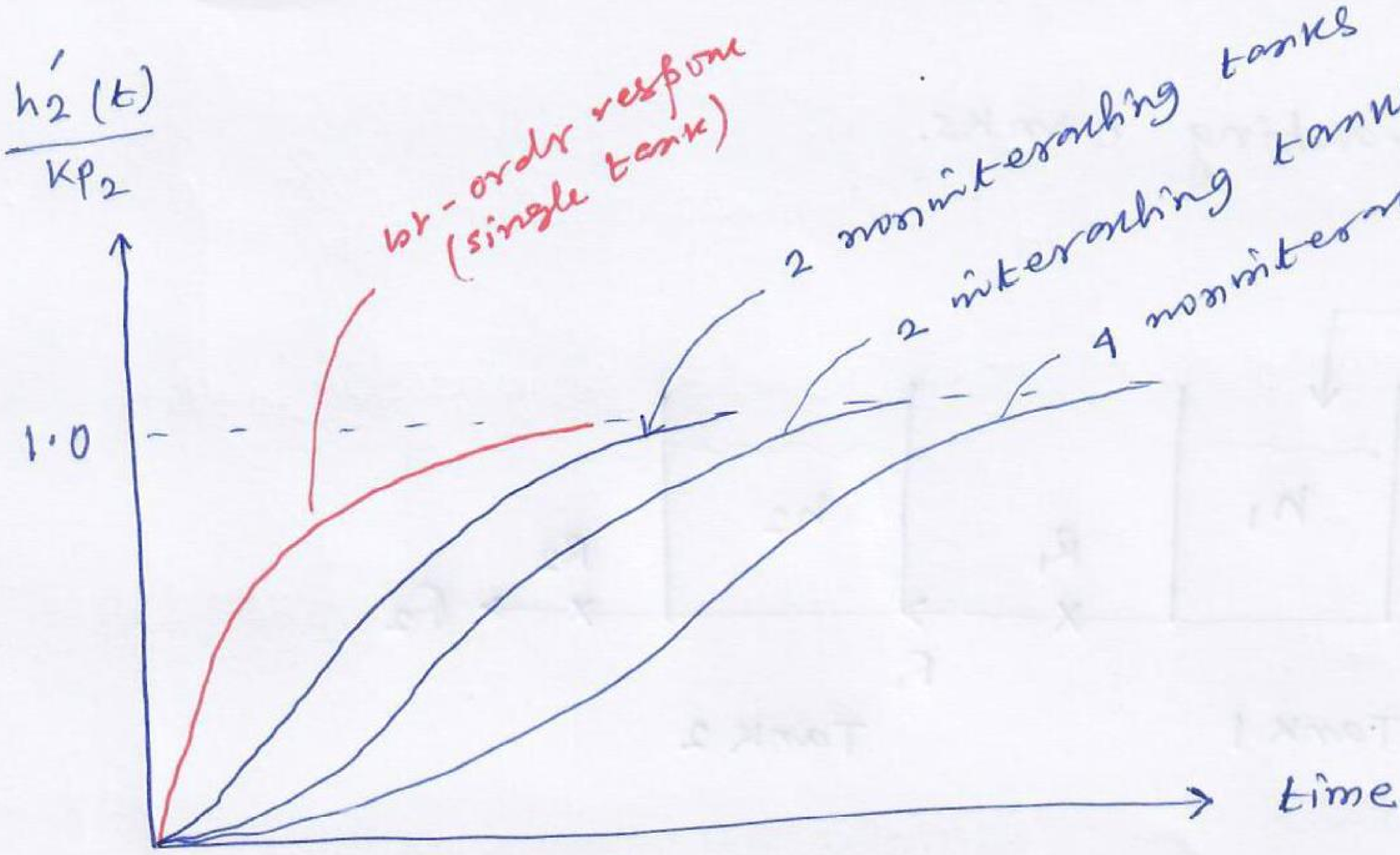
Remarks.

1. Connecting two first-order systems yields 2nd-order system

   see $G_0(s)$.

2. Response of overdamped system is S-shaped @ initially changes slowly and then it picks up speed

   1st-order response — largest rate of change at the beginning.



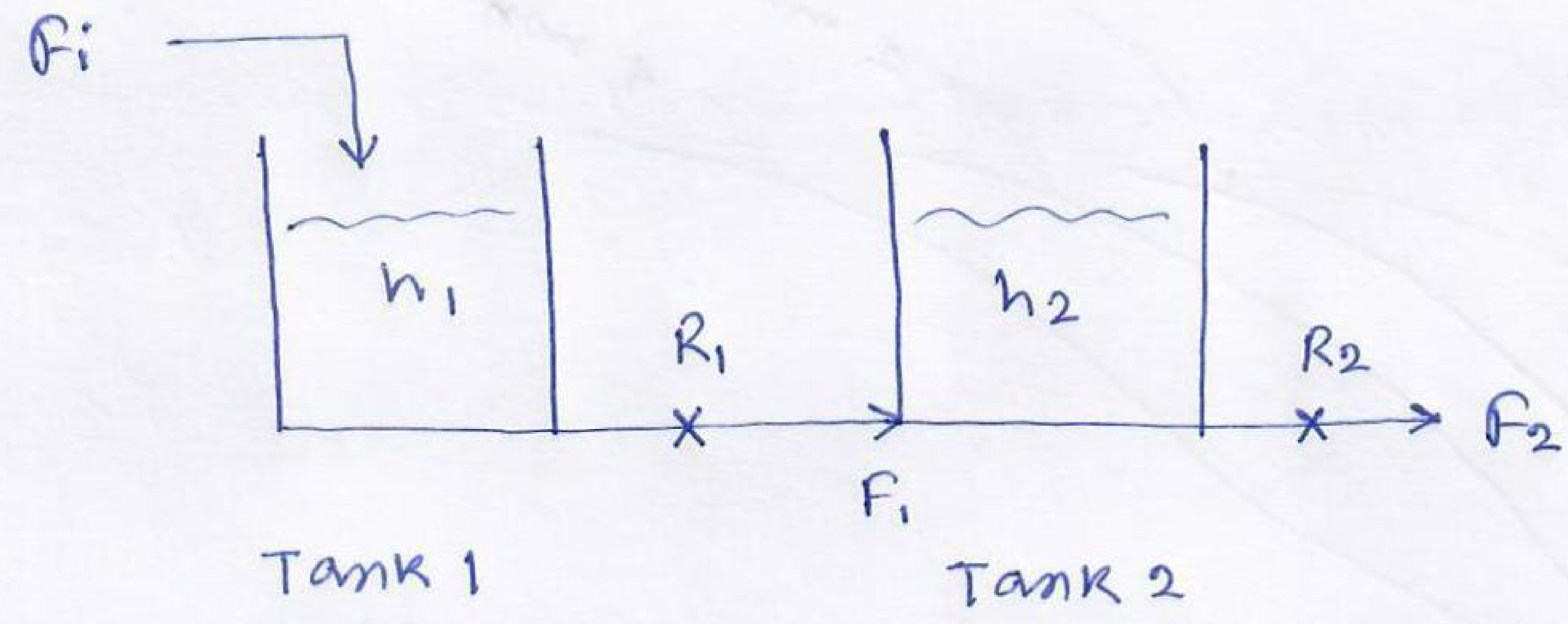
3. As the no. of capacities in series increases, the delay in the initial response becomes more pronounced.

4.

$$\bar{F}_1 \rightarrow \boxed{G_1} \xrightarrow{\bar{y}_1} \boxed{G_2} \xrightarrow{\bar{y}_2} \cdots \xrightarrow{\bar{y}_{N-1}} \boxed{G_N} \rightarrow \bar{y}_N$$

$$G_O(s) = G_1 \cdot G_2 \cdot \cdots G_N = \frac{K_{P_1} \, K_{P_2} \cdots K_{P_N}}{(\tau_{P_1} s+1)(\tau_{P_2} s+1) \cdots (\tau_{P_N} s+1)}$$

Interacting tanks.



Tank 1. $$A_1 \frac{dh_1}{dt} = F_i - F_1 = F_i - \frac{h_1 - h_2}{R_1}$$

$$A_1 R_1 \frac{dh_1'}{dt} + h_1' - h_2' = R_1 F_i'$$

Tank 2. $$A_2 \frac{dh_2}{dt} = F_1 - F_2 = \frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2}$$

$$A_2 R_2 \frac{dh_2'}{dt} + \left(1 + \frac{R_2}{R_1}\right) h_2' - \frac{R_2}{R_1} h_1' = 0$$

Solving,

$$\bar{h}_1'(s) = \frac{\tau_{p_2} R_1 s + (R_1 + R_2)}{\tau_{p_1}\tau_{p_2} s^2 + (\tau_{p_1} + \tau_{p_2} + A_1 R_2) s + 1} \bar{F}_i'(s).$$

$$\bar{h}_2'(s) = \frac{R_2}{\tau_{p_1}\tau_{p_2} s^2 + (\tau_{p_1} + \tau_{p_2} + A_1 R_2) s + 1} \bar{F}_i'(s)$$ .. Interacting

$$\bar{h}_2'(s) = \frac{R_2}{\tau_{p_1}\tau_{p_2} s^2 + (\tau_{p_1} + \tau_{p_2}) s + 1} \bar{F}_i'(s)$$ .. noninteracting

Remarks.

1. Above two equations differ only by the term $A_1 R_2$. This may be thought of as the interaction factor.

2. Connecting two 1st-order systems in series yields 2nd-order system.

3. Poles are:

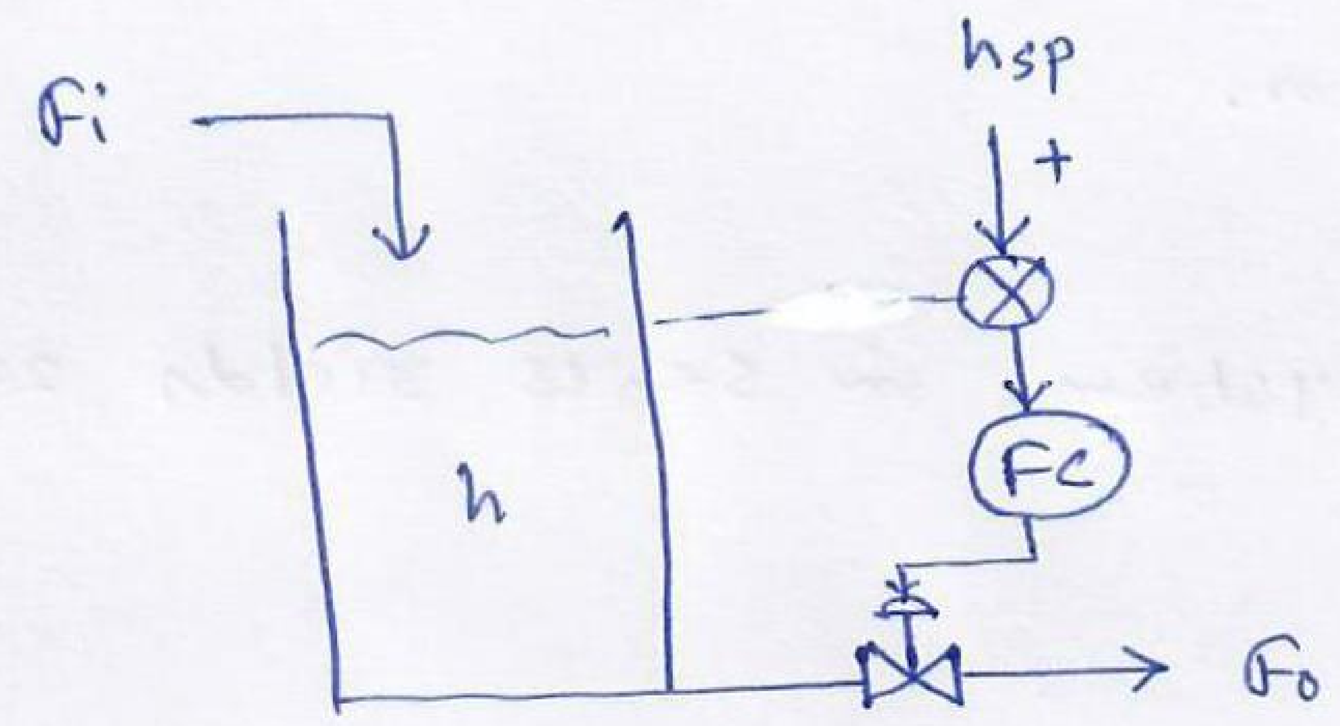
$$P_{1,2} = \frac{-(\tau_{p_1} + \tau_{p_2} + A_1 R_2) \pm \sqrt{(\tau_{p_1} + \tau_{p_2} + A_1 R_2)^2 - 4\tau_{p_1}\tau_{p_2}}}{2\tau_{p_1}\tau_{p_2}}$$

Since

$$(\tau_{p_1} + \tau_{p_2} + A_1 R_2)^2 - 4\tau_{p_1}\tau_{p_2} > 0$$

So P_1 and P_2 are distinct and real poles. Thus, the response of interacting capacities is always overdamped.

1st-order Process + controller ≡ 2nd-order process.



CV	MV
h	Fo

Model : $A \dfrac{dh'}{dt} = F_i' - F_o'$ ... open-loop

Controller : $F_o = F_{os} + K_c h' + \dfrac{K_c}{\tau_i} \int h' \, dt$ ... PI controller

$K_c, \tau_i \longrightarrow$ controller parameters (+ve values).

$h' = h - h_s$

(i) If $h' = 0$ (i.e., $h = h_s$), then $F_o = F_{os}$ no change of valve opening.

(ii) If $h' < 0$ ($h < h_s$, level goes down), then $F_o < F_{os}$ controller reduces F_o and h starts increasing.

(iii) If $h' > 0$ ($h > h_s$, level goes up), then $F_o > F_{os}$ controller increases F_o and h starts decreasing.

Model : $A \dfrac{dh'}{dt} + K_c h' + \dfrac{K_c}{\tau_i} \int h' \, dt = F_i'$ ... closed-loop

Taking L-transform:

$$A s \bar{h}'(s) + K_c \bar{h}'(s) + \frac{K_c}{\tau_i} \cdot \frac{1}{s} \cdot \bar{h}'(s) = \bar{F}_i'(s).$$

$$\frac{\bar{h}'(s)}{\bar{F}_i'(s)} = \frac{\tau_i/K_c \cdot s}{\frac{A\tau_i}{K_c} s^2 + \tau_i s + 1} \equiv \frac{K_p s}{\tau^2 s^2 + 2\psi\tau s + 1}$$

where, $K_p = \dfrac{\tau_i}{K_c}$ $\quad \tau^2 = \dfrac{A\tau_i}{K_c}$ $\quad 2\psi\tau = \tau_i$

Then we get:

$$\tau = \sqrt{\frac{A\tau_i}{K_c}} \qquad \psi = \frac{1}{2}\sqrt{\frac{K_c\tau_i}{A}}$$

Remarks.

1. Liquid level + Controller $\equiv$ 2nd-order process.
(1st-order process)

2.

$\sqrt{\frac{K_c\tau_i}{A}}$	ψ	Type
< 2	< 1	Underdamped
$= 2$	1	Critically damped
> 2	> 1	Overdamped.

Higher-order systems.

Three classes of higher-order systems:

1. N first-order processes in series
2. Processes with dead-time
3. Processes with inverse response.

N capacities in series

Previously we discussed the dynamics of 2 capacities in series when the input is changed by a step. We can extend those conclusions to the systems of N capacities

✓ N noninteracting capacities

(i) Overdamped response (S-shaped and sluggish).

(ii) Increasing the no. of capacities in series increases the sluggishness of the response

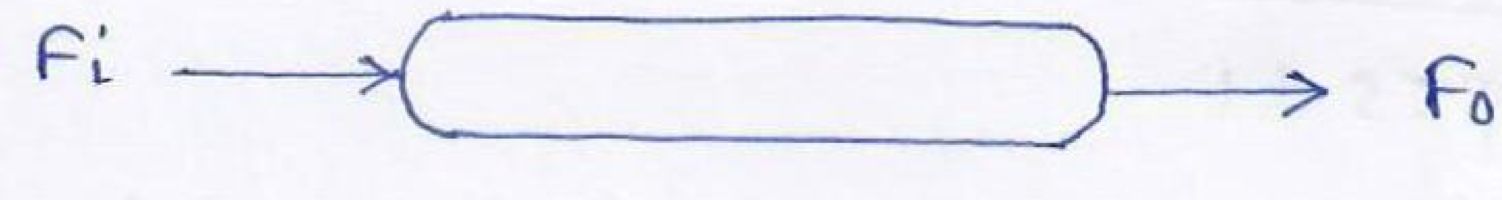
(iii) $G_0(s) = G_1(s)\, G_2(s) \cdots G_N(s) = \dfrac{K_1 \cdot K_2 \cdots K_N}{(\tau_1 s+1)(\tau_2 s+1) \cdots (\tau_N s+1)}$

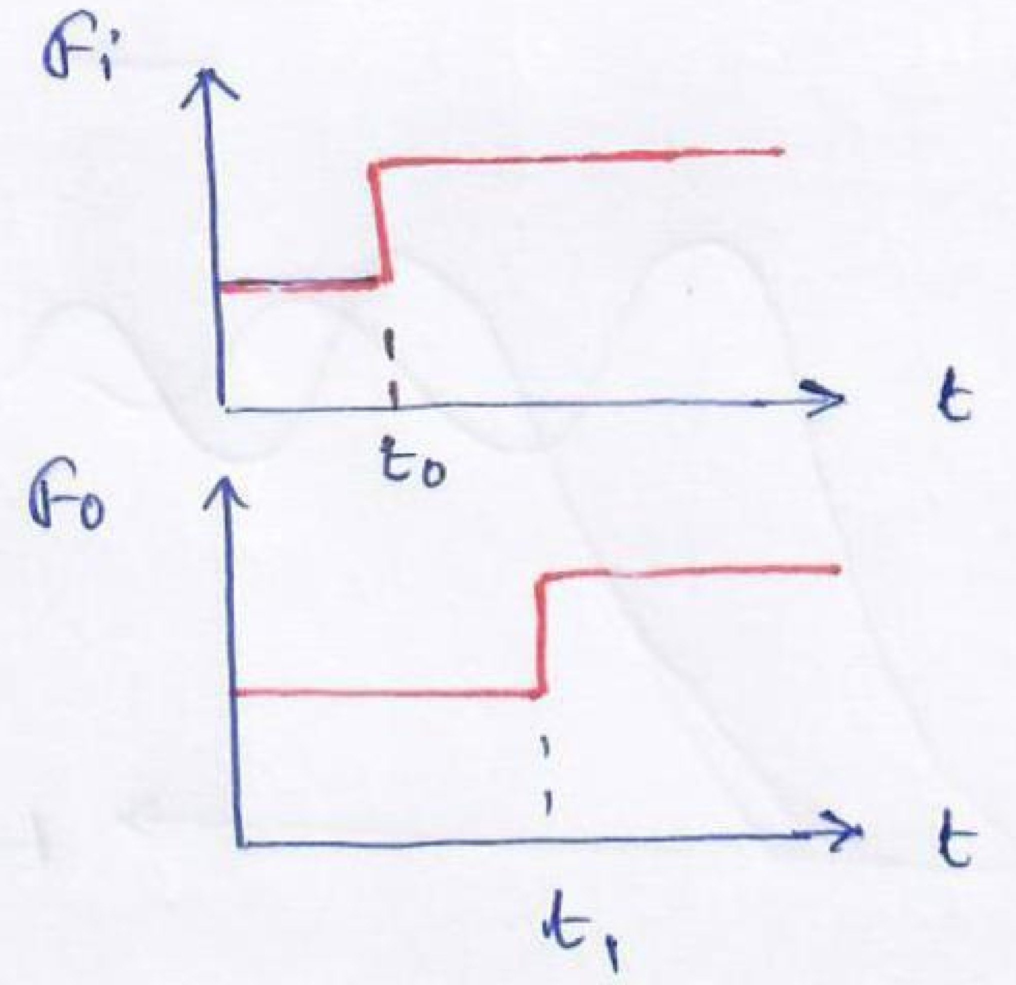
✓ N interacting capacities

(i) Interaction increases the sluggishness of the overall response

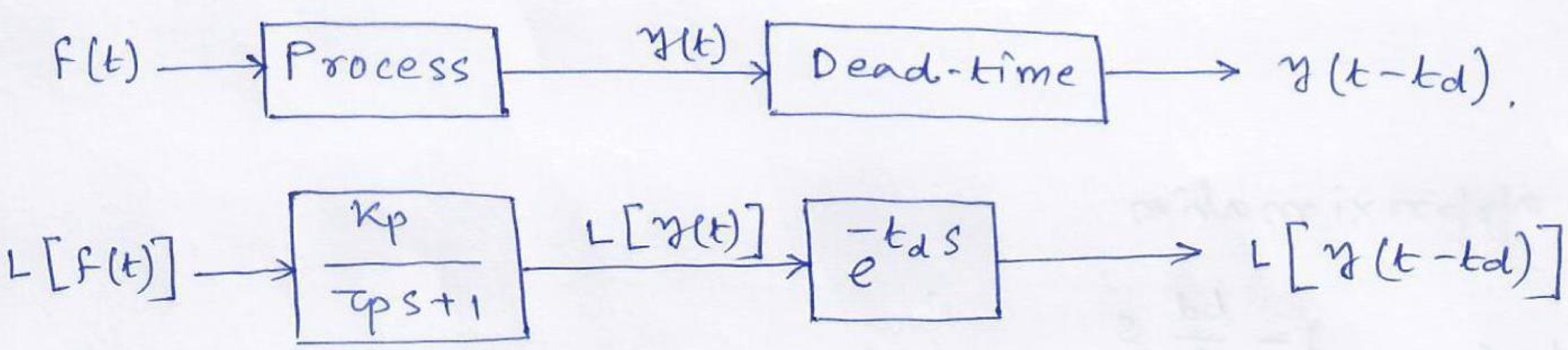
Recall this Fig.

Dynamic systems with dead-time. (t_d)





If we introduce a step change in F_i, after a certain time period it is reflected in F_o. This time lag is called dead-time.



1st-order system

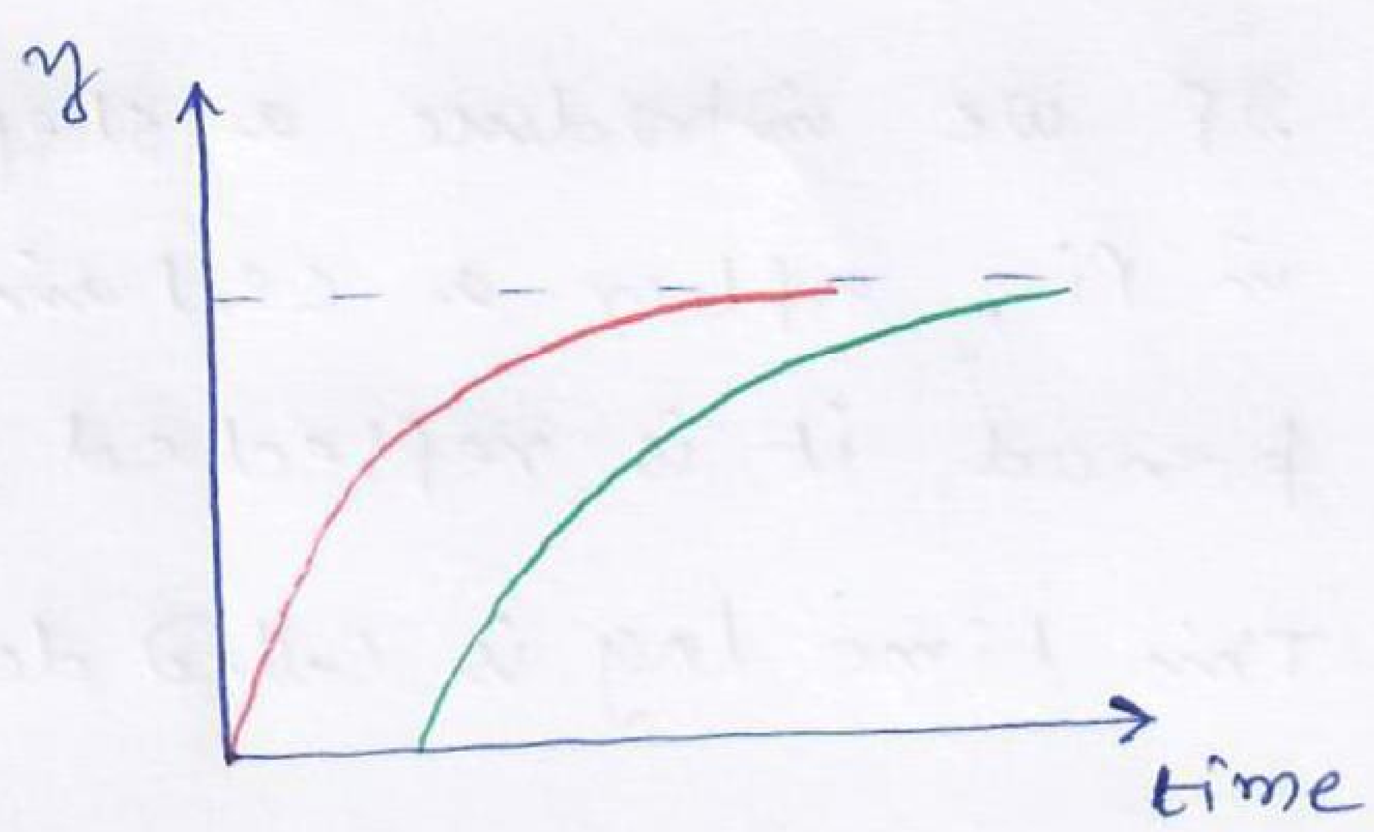
$$\frac{L[y(t)]}{L[f(t)]} = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s+1} \quad \text{--- Process}$$

$$\frac{L[y(t-t_d)]}{L[y(t)]} = e^{-t_d s} \quad \text{--- dead-time element}$$

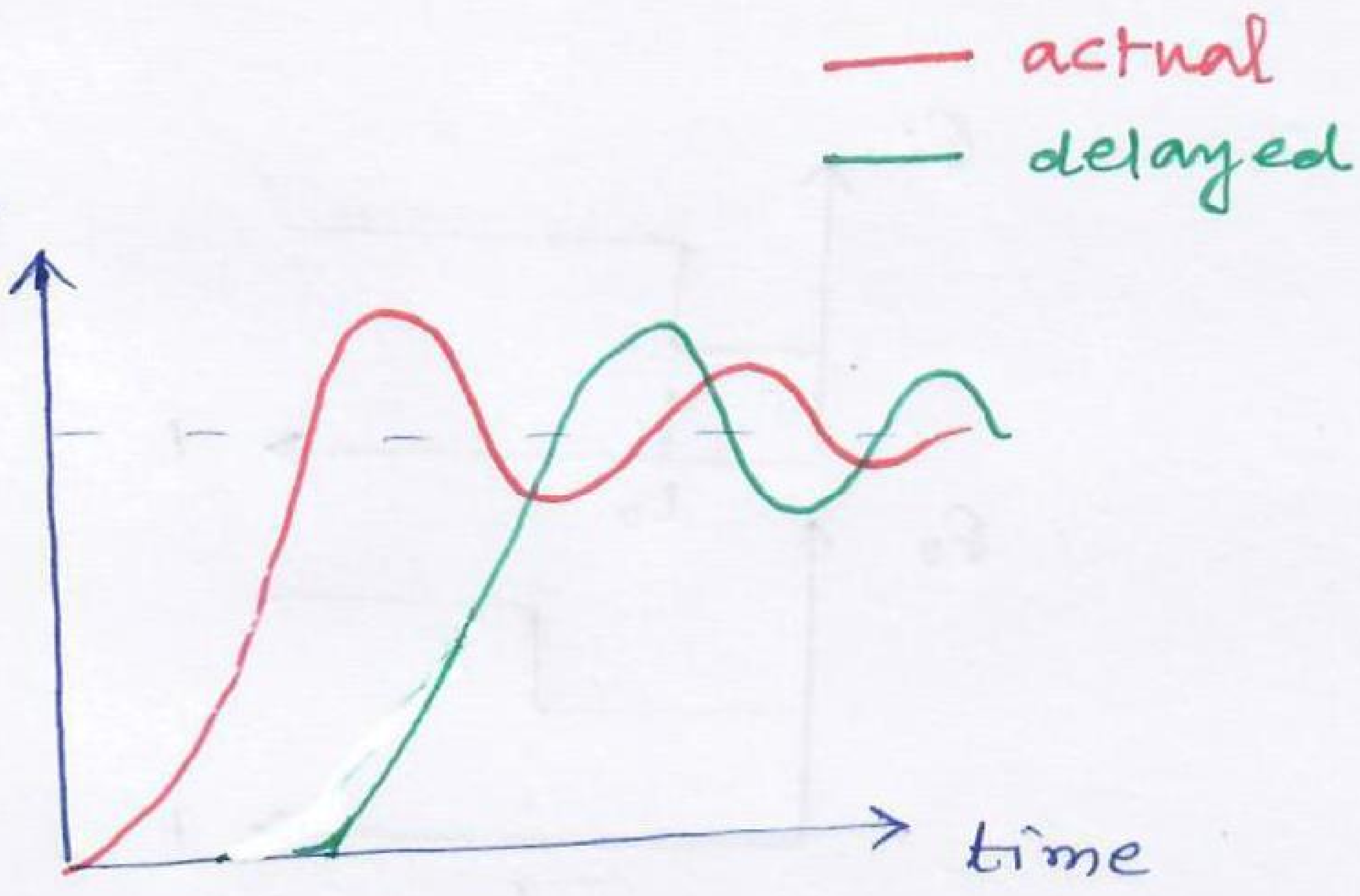
$$G_0(s) = \frac{L[y(t-t_d)]}{L[f(t)]} = \frac{K_p e^{-t_d s}}{\tau_p s+1} \quad \text{--- first-order-plus-dead-time (FOPDT) system}$$

Similarly,

$$\frac{L[y(t-t_d)]}{L[f(t)]} = \frac{K_p e^{-t_d s}}{\tau^2 s^2 + 2\gamma\tau s + 1} \quad \text{... SOPDT system.}$$



first-order system



Second-order system.

Pade approximation

$$e^{-t_d s} \approx \frac{1 - \frac{t_d}{2} s}{1 + \frac{t_d}{2} s} \quad \text{... 1st-order approximation}$$

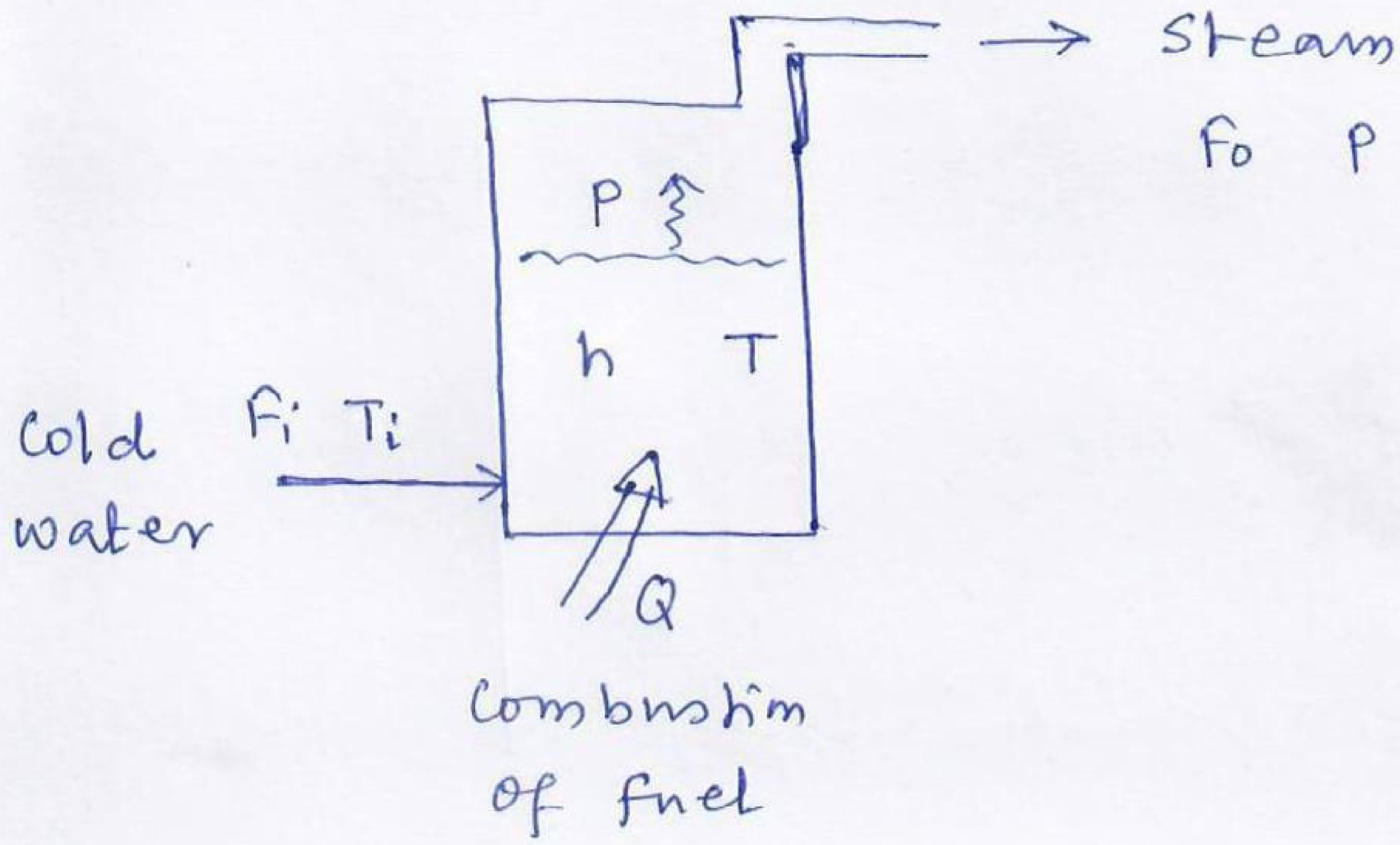
$$e^{-t_d s} = \frac{t_d^2 s^2 - 6 t_d s + 12}{t_d^2 s^2 + 6 t_d s + 12} \quad \text{... 2nd-order approximation}$$

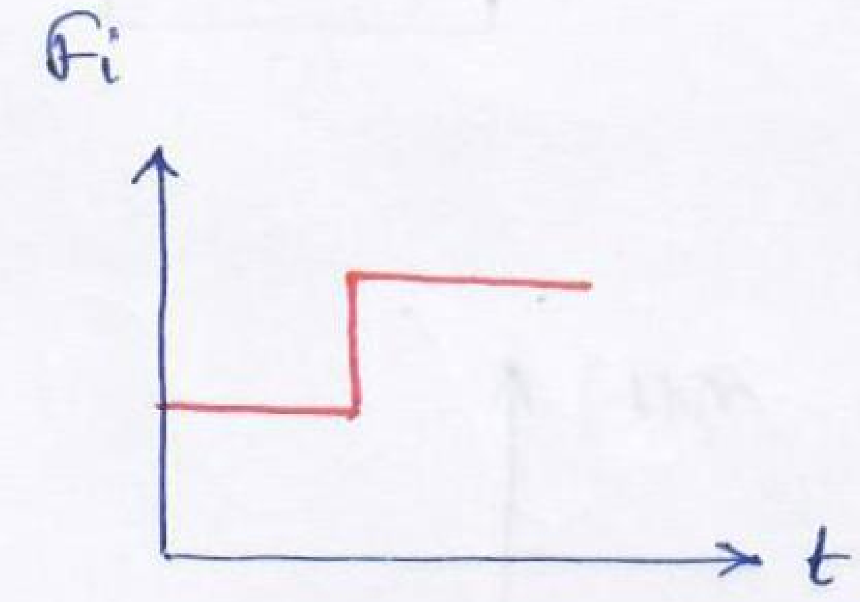
FOPDT system

$$G(s) = \frac{K_p \cdot e^{-t_d s}}{\tau_p s + 1} = \frac{K_p}{\tau_p s + 1} \cdot \frac{1 - \frac{t_d}{2} s}{1 + \frac{t_d}{2} s}.$$

Dynamic systems with inverse response

Boiler system





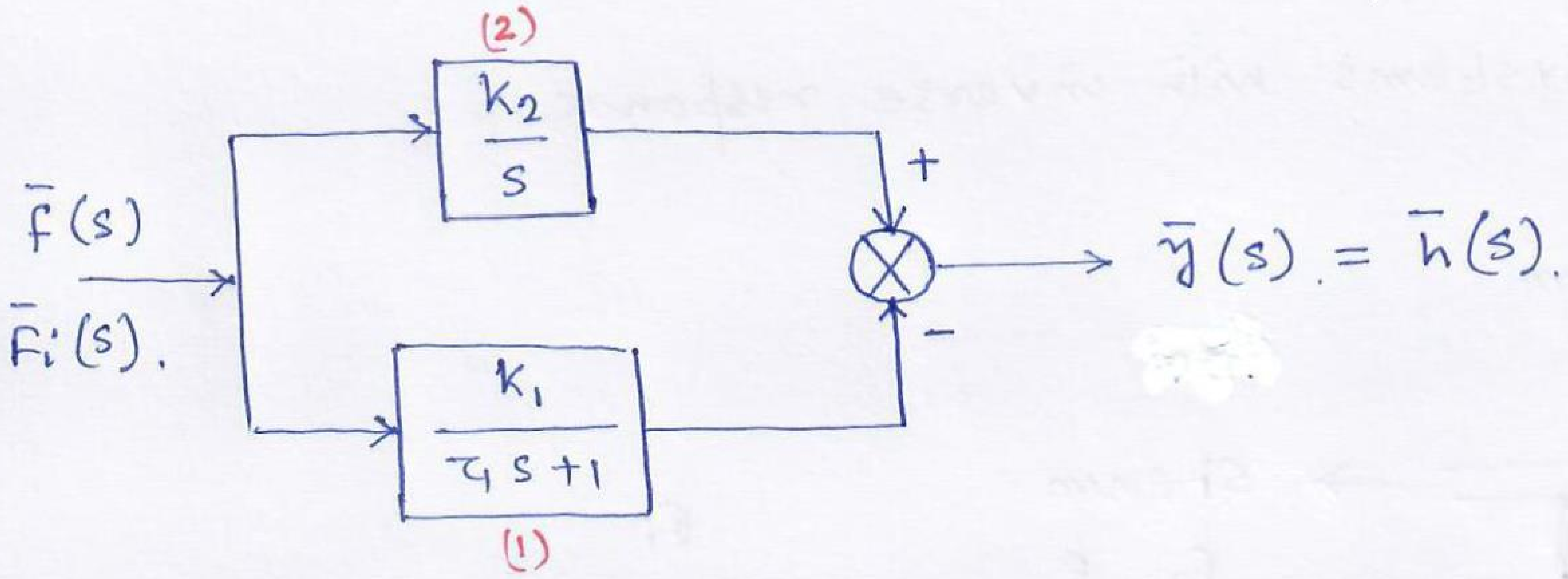
✓ If $F_i \uparrow$ by a step $\Rightarrow$ $h \downarrow$ for a short period & then start increasing.

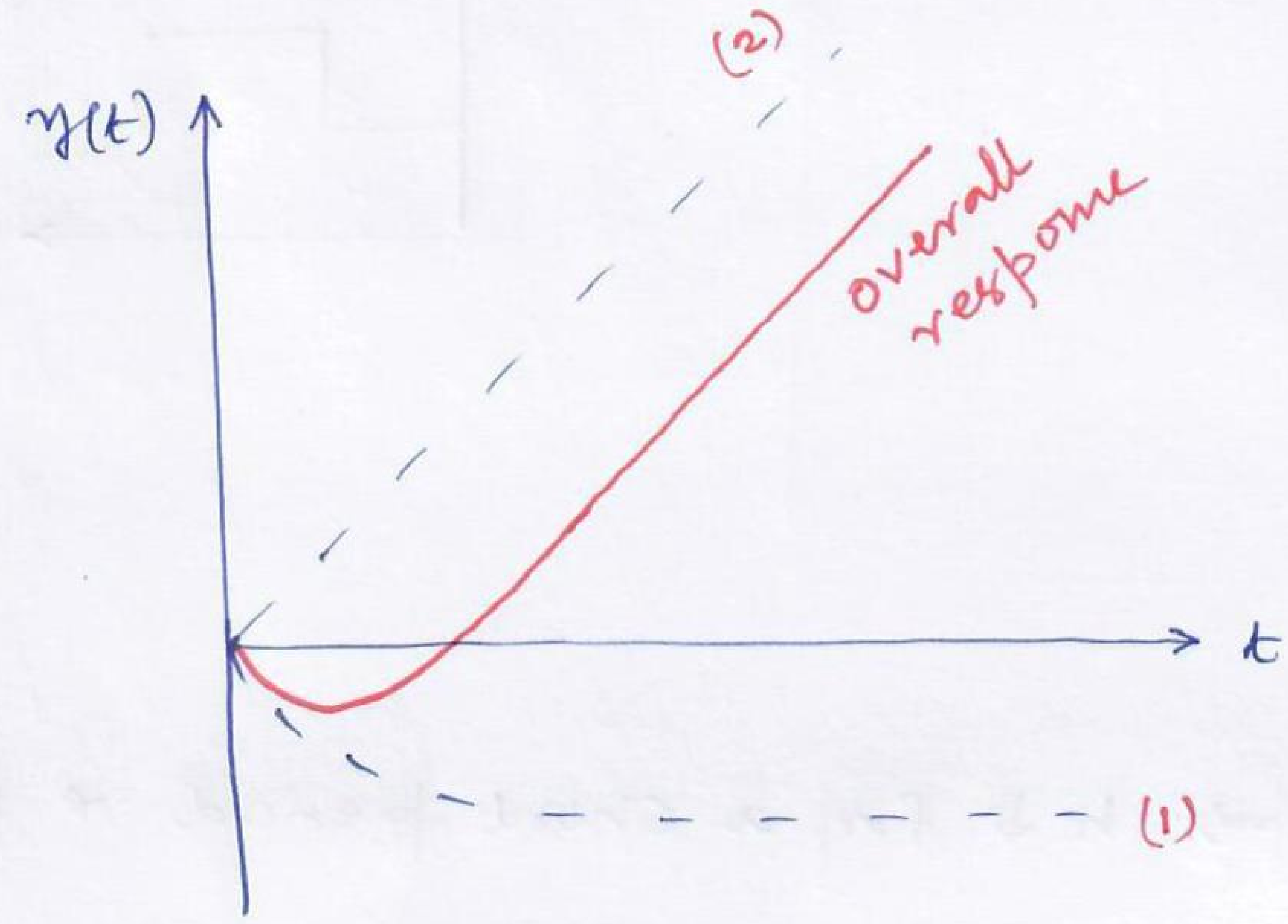
(i) If $F_i \uparrow$, $T \downarrow$ that leads to decrease the vol of the entrained vap bubbles. Thus $h \downarrow$ following first-order behavior :

$$-\frac{K_1}{\tau_1 s+1}$$

(ii) With const. heat supply, the steam production remains const and consequently h starts increasing in an integral form with TF :

$$+\frac{K_2}{s}$$





✓ overall response ≡ result of two opposing effects ≡ inverse response.

✓ $$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_2}{s} - \frac{K_1}{\tau_1 s+1} = \frac{(K_2\tau_1 - K_1)\, s + K_2}{s(\tau_1 s+1)}$$

✓ Condition for inverse response;

$K_1 > K_2 \tau_1$ +ve zero

If this is not satisfied, there is no existence of inverse resp.