① Classifications of PDEs

Consider 3 independent variables $\{x_1, x_2, x_3\}$.

Second Order PDE

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_j} = R\left(x_1, x_2, x_3, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)$$

e.g. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

$$\text{LHS} \rightarrow \sum_{i=1}^{3}\left(a_{i1} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_1} + a_{i2} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_2} + a_{i3} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_3}\right)$$

we only look at the second order derivatives for type analysis

$$= \left(a_{11} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_1} + a_{12} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_2} + a_{13} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_3}\right)$$

(substitute i's accordingly)

$$+ \left(a_{i1} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_1} + a_{i2} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_2} + a_{i3} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_3}\right)$$

$$+ \left(a_{i1} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_1} + a_{i2} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_2} + a_{i3} \cdot \frac{\partial^2 u}{\partial x_i \cdot \partial x_3}\right)$$

$$\therefore \; A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let the eigenvalues of this matrix be $\{\lambda_1, \lambda_2, \lambda_3\}$

usually for processes in the transient state

① if atleast 1 e.v. is $= 0$

$\Rightarrow$ "Parabolic PDE"

" steady state

② if all are of the same sign

$\Rightarrow$ "Elliptical PDE".

usually in safety analyses, e.g. vibrations.

③ else

$\Rightarrow$ "Hyperbolic PDE".

② Laplace Example

$$\nabla^2 u = 0$$

laplacian operator.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|A - \lambda \cdot I| = 0$$

$$\Rightarrow \quad (1-\lambda)^3 = 0$$

$$\Rightarrow \quad 1-\lambda = 0$$

$$\Rightarrow \quad \lambda = 1 \rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$$

$\therefore$ elliptical PDE.

(3) Example 2

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$x = x_1$
$y = x_2$
$t = x_3$

not second-order term
$\therefore$ ignore.

$$\therefore \quad A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$\therefore \quad \lambda = 1, 1, 0.$

$\Rightarrow$ Parabolic.

④ Example 3

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

★ 2nd order terms always appear in the mat.

$$\therefore \quad A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$|A - \lambda \cdot I| = 0$$

$$\Rightarrow \quad (1-\lambda)(1-\lambda)\cdot(-1-\lambda) = 0$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = -1$$

$\Rightarrow$ Hyperbolic PDE.

Shortcut method.

⑤ Two independent variables & 1 dependent variable (having 2nd order terms).

Rewrite the PDE as →

$$A \cdot \frac{\partial^2 u}{\partial x^2} + 2 \cdot B \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + C \cdot \frac{\partial^2 u}{\partial y^2}$$

$$= f\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, u\right)$$

$$B^2 - A \cdot C > 0 \Rightarrow \text{Hyperbolic}$$

$$B^2 - A \cdot C = 0 \Rightarrow \text{Parabolic}$$

$$B^2 - A \cdot C < 0 \Rightarrow \text{Elliptical.}$$

e.g. $$\frac{\partial^2 u}{\partial x^2} + x \cdot \frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1 \quad , \quad B = 0 \quad , \quad C = x .$$

$\therefore$

$$-x > 0 \Rightarrow \text{hyper.}$$

$$x = 0 \Rightarrow \text{para.}$$

$$x > 0 \Rightarrow \text{ellip.}$$

Ex 4: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial z^2} = a \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial z} + f(x, y, z)$$

$x = x_1$, $y = x_2$, $z = x_3$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + x \frac{\partial^2 u}{\partial x_1 \partial x_3} - \frac{\partial^2 u}{\partial x_3^2} = R$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{1}{2} \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{2} \frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{x}{2} \frac{\partial^2 u}{\partial x_1 \partial x_3} + \frac{x}{2} \frac{\partial^2 u}{\partial x_3 \partial x_1} - \frac{\partial^2 u}{\partial x_3^2} = R$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{x}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{x}{2} & 0 & -1 \end{bmatrix} \Rightarrow \text{Symmetric Matrix (real valued)}$$

Eigenvalues of "A" $\Rightarrow \lambda = \lambda(x)$

Depending on values of $x \Rightarrow$ classify PDE this indep. var. has no 2nd order term.

⑥ Operator

An operator maps a function to another function or a value.

e.g. $\frac{d}{dx}(f(x))$

e.g. definite integration

- denoted by "L"

e.g. $\frac{d}{dx}(f(x)) = Lf$

e.g. $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$\therefore Lu = 0,$$

where $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

e.g. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Matrices are also operators. $\therefore$ $Lu = 0$,

where $L = \nabla^2$ (Laplacian)

① Linear Operator $\rightsquigarrow$ advantageous $\because$ we can use Linear Superposⁿ. to solve PDEs.

$$L(\alpha \cdot u + \beta \cdot v) = \alpha \cdot L(u) + \beta \cdot L(v)$$

- $$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- $$L = \frac{\partial}{\partial t} - \nabla^2$$

- $$L = \frac{\partial^2}{\partial t^2} - \nabla^2$$

⑧ Resolving N-H using POLC.

For linear operator, one can use Principle of linear superposition.

Principle of linear superposition

$\mathcal{L}u = x$

Consider, $\dfrac{d^2u}{dx^2} = x$ (Non-homogeneous 2nd order ODE)

BC. $\Rightarrow$ $\left.\begin{array}{l} u(x=0)=5 \\ u(x=1)=10 \end{array}\right\}$ D.B.C. / Non-Hom.

Three sources of non-homogeneities.

$$u(x) = \underline{y(x)} + \underline{v(x)} + \underline{w(x)}$$

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Governing Equation of y:

$\dfrac{d^2y}{dx^2} = 0$ Subj to at $x=0$, $y=5$; $x=1$, $y=0$

$\dfrac{dy}{dx} = c_1 \Rightarrow y = c_1 x + c_2$

$5 = c_2$

$y = c_1 x + 5 \Rightarrow 0 = c_1 + 5 \Rightarrow c_1 = -5$

$$\boxed{y(x) = 5 - 5x}$$

Governing Equation of v:

$\dfrac{d^2v}{dx^2} = 0$ Subj. to at $x=0$, $v=0$; at $x=1$, $v=10$

$v(x) = c_1 x + c_2 \Rightarrow c_2 = 0$

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$$\boxed{v(x) = 10x}$$

Governing Equation of $w(x)$:

$\dfrac{d^2W}{dx^2} = x$ Subj to, at $x=0$, $W=0$; at $x=1$, $W=0$

$\Rightarrow \dfrac{dW}{dx} = \dfrac{x^2}{2} + c_1$

$\Rightarrow W(x) = \dfrac{x^3}{6} + c_1 x + c_2$

$0 = c_2$;

$W(x) = \dfrac{x^3}{6} + c_1 x$

$0 = \dfrac{1}{6} + c_1$

$\Rightarrow c_1 = -\dfrac{1}{6}$

$w(x) = \dfrac{x^3}{6} - \dfrac{x}{6}$

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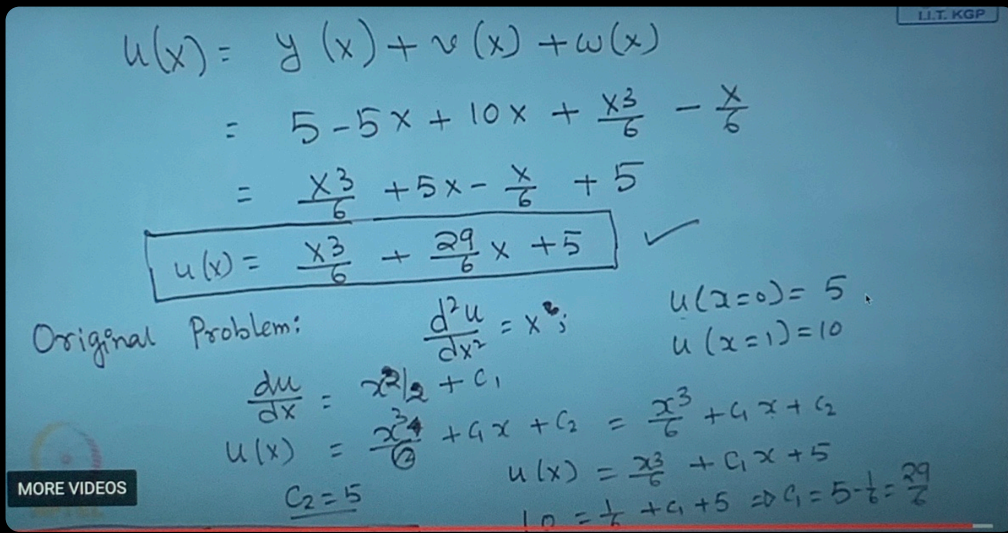

In this case the prob was a simple ODE, but the same logic can be used for PDEs as well.