$$-u_{30}(1-x) = \sum_{n=1}^{\infty} c_6 \sin n\pi x \qquad t=0$$

$$-u_{30}\int_0^1 (1-x)\sin n\pi x\,dx = \frac{c_6}{2}$$

CYLINDRICAL COORDINATE SYSTEM (r, θ, z)

2D transient

Problem:- (2 dimensional too long θ - symmetry)

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

at $t=0$, $u = u_0$

at $r=0$, u = finite → physical sc.

$r=1$, $u=0$

$$u = T(t)\,R(r)$$

$$R\frac{dT}{dt} = T\frac{1}{r}\left[\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right]$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \text{Const}$$

Case 1 :- Const $=0$

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

$$r\frac{dR}{dr} = c_1$$

$$\frac{dR}{dr} = \frac{c_1}{r}$$

$$\boxed{R = c_1 \ln r + c_2}$$

$\boxed{c_1 = 0}$ → $r=0$, R – finite

$$\boxed{R = c_2}$$

$r=1$, $R=0$, $\boxed{c_2 = 0}$ ⟶ [Trivial soln] → ∴ const $\neq 0$

case 2 :- const = +ve.

$$\frac{1}{R r} \frac{d}{dr}\left(r \frac{dR}{dr}\right) = \lambda^2$$

$$\frac{d}{dr}\left(r \frac{dR}{dr}\right) - \lambda^2 R r = 0$$

$$r \frac{d^2R}{dr^2} + \frac{dR}{dr} - \lambda^2 R r = 0$$

$$r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} - \lambda^2 r^2 R = 0$$

consider, $y = \lambda r$

$$\frac{dR}{dr} = \frac{dR}{dy} \cdot \frac{dy}{dr} = \lambda \frac{dR}{dy}$$

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{dy}\left(\lambda \frac{dR}{dy}\right) \cdot \frac{dy}{dr} = \lambda^2 \frac{d^2R}{dy^2}$$

$$r^2 \lambda^2 \frac{d^2R}{dy^2} + r\lambda \frac{dR}{dy} - \lambda^2 r^2 R = 0.$$

$$y^2 \frac{d^2R}{dy^2} + y\frac{dR}{dy} - y^2 R = 0$$

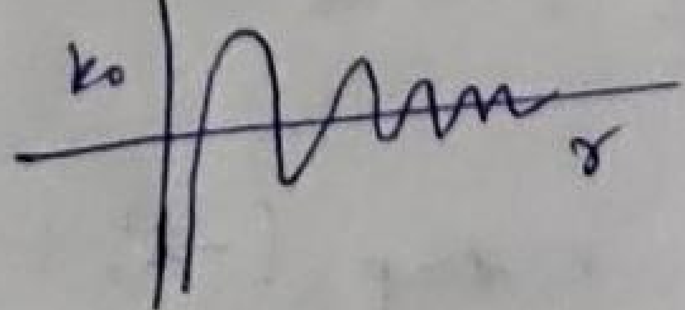
$$\boxed{y \frac{d^2R}{dy^2} + \frac{dR}{dy} - yR = 0.}$$ → Bessel equation of 2nd kind of order '0'.

→ $R = c_1 I_0(y) + c_2 k_0(y)$

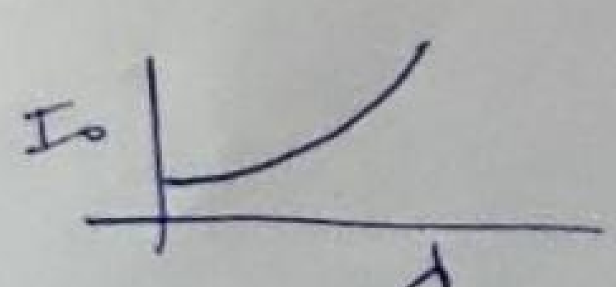
$R(r) = c_1 I_0(\lambda R) + c_2 k_0(\lambda r)$

at $r = 0$, R = finite

$\Rightarrow c_2 = 0$.



$R(r) = c_1 I_0(\lambda r)$



$r = 1, R = 0 \Rightarrow 0 = c_1 \underset{\neq 0}{I_0(\lambda)}$

$\boxed{c_1 = 0}$

Const $= \lambda^2$ → not possible

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Case 3 :- Const $= -ve = -\lambda^2$

$$\frac{1}{R r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 r R = 0$$

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} + \lambda^2 r R = 0.$$

$$y = \lambda r$$

Bessel Eqn of 1st kind of order '0'

$$\boxed{R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)}$$

at $r=0$, $R =$ finite

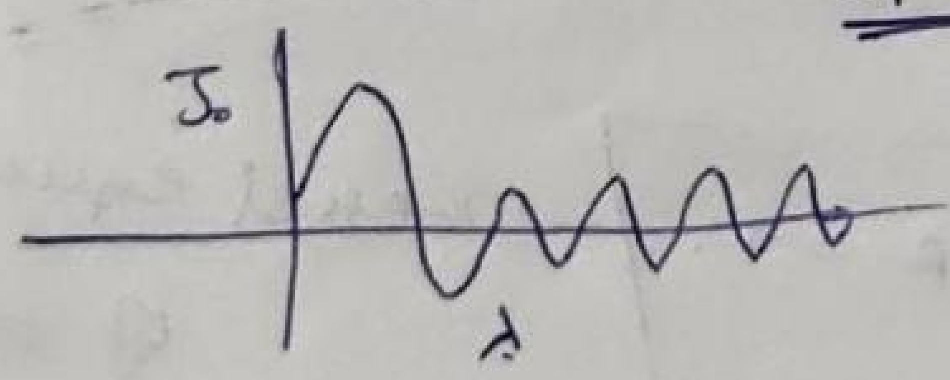
$$c_2 = 0$$

$$R = c_1 J_0(\lambda r)$$

at $r=1$, $R=0$

$$c_1 J_0(\lambda) = 0.$$

$\underline{c_1 \neq 0}$.



$J_0(\lambda_n) = 0$

$$\boxed{R_n = c_1 J_0(\lambda_n r)}$$

[λ_n = eigen values] — Eigen values.

$$\frac{1}{T_n}\frac{dT_n}{dt} = -\lambda_n^2$$

$$T_n = c_2 \exp(-\lambda_n^2 t)$$

→ $$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} c_n \exp(-\lambda_n^2 t)\, J_0(\lambda_n r)$$

at $t=0$, $u = u_0$

$$u_0 = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

Proof:- Bessel functions are orthogonal function w.r.t r.

$$\iiint y_m y_n \underline{r}\, dr = 0 \quad \forall\, m \neq n$$

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 r R = 0$$

λ_m, λ_n are 2 distinct eigenvalues

$\lambda_m \to J_0(\lambda_m r)$, $\lambda_n \to J_0(\lambda_n r)$

① $\frac{d}{dr}\left(r\frac{d}{dr}J_0(\lambda_m r)\right) + \lambda_m^2 r J_0(\lambda_m r) = 0 \quad \times\ r J_0(\lambda_m r)$

② $\frac{d}{dr}\left(r\frac{d}{dr}\left(J_0(\lambda_n r)\right)\right) + \lambda_n^2 r J_0(\lambda_n r) = 0 \quad \times\ r J_0(\lambda_m r)$.

$$J_0(\lambda_m) = J_0(\lambda_n) = 0.$$

$$\int_0^1 J_0(\lambda_n r)\frac{d}{dr}\left(r\frac{d}{dr}J_0(\lambda_m r)\right)dr + \lambda_m^2\int r J_0(\lambda_m r) J_0(\lambda_n r)\,dr -$$

$$\int_0^1 J_0(\lambda_m r)\frac{d}{dr}\left(r\frac{d}{dr}J_0(\lambda_m r)\right)dr - \lambda_m^2\int r J_0(\lambda_m r) J_0(\lambda_n r)\,dr = 0.$$

$$\to \cancel{J_0(\lambda_n r)}\, r\frac{d}{dr}\left(J_0(\lambda_m r)\right)\Big|_0^1 - \int_0^1 \frac{dJ_0(\lambda_n r)}{dr}\, r\, \frac{dJ_0(\lambda_m r)}{dr}\,dr$$

$r=1$

$$-\cancel{J_0(\lambda_m r)}\, r\frac{d}{dr}\left(J_0(\lambda_n r)\right)\Big|_0^1 + \int_0^1 \frac{dJ_0(\lambda_m r)}{dr}\, r\, \frac{dJ_0(\lambda_m r)}{dr}\,dr$$

$r=1$

$$= \underbrace{(\lambda_n^2 - \lambda_m^2)}_{\neq 0}\int_0^1 \underbrace{J_0(\lambda_m r)\, J_0(\lambda_n r)\, r\, dr}_{=0}.$$

$$\boxed{\int_0^1 J_0(\lambda_m r)\, J_0(\lambda_n r)\, \underbrace{r}_{\text{weight funct}}\, dr = 0}$$ → for any order bessel function.

$$\rightarrow \quad u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \exp(-\lambda_n^2 t)$$

at $t=0$, $u = u_0$

$$u_0 = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

$$u_0 \int_0^1 J_0(\lambda_m r)\, r\, dr = \sum_{n=1}^{\infty} \int_0^1 c_n J_0(\lambda_n r)\, J_0(\lambda_m r)\, r\, dr$$

$$u_0 \times \int_0^1 J_0(\lambda_n r)\, r\, dr = c_n \int_0^1 J_0^2(\lambda_n r)\, r\, dr.$$

$$\cancel{c_n = u_0 \int_0^1 J_0(\lambda_n r)\, dr}$$

$$\boxed{c_n = u_0 \frac{\int_0^1 J_0(\lambda_n r)\, r\, dr}{\int_0^1 J_0^2(\lambda_n r)\, dr}}$$

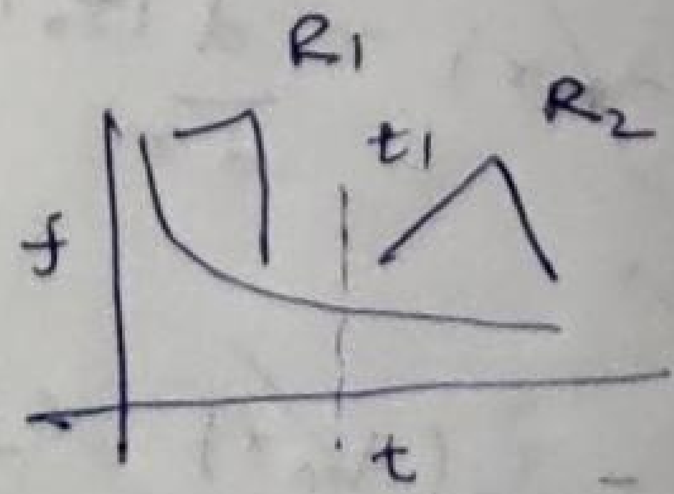
<u>2D transient (3D problem) in cylindrical coordinate</u> (r, θ, t) → no θ symmetry

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

at $t=0$, $u = f(r,\theta) \,/\, f(r) \,/\, f(\theta) \,/$ const

$r = 0$, u = finite → physical BC

$r = 1$, $u = 0$



Periodic BC / Physical BC

$$\left.\begin{aligned} \theta = \pi \rightarrow u|_{\theta=\pi} &= u|_{\theta=-\pi} \\ \left.\frac{\partial u}{\partial \theta}\right|_{\pi} &= +\left.\frac{\partial u}{\partial \theta}\right|_{-\pi} \end{aligned}\right\}$$ Both are homogeneous.

$0 \le t \le t_1 \quad f = g_1(t)$

$t_1 \le t < t_2 \quad f = g_2(t)$

at $t = t_1 \rightarrow g_1 = g_2$

$\frac{dg_1}{dt} = \frac{dg_2}{dt}.$

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$u = T(t)\, R(r) \cdot \Theta(\theta)$

$$R\cancel{\Theta}\frac{dT}{dt} = T\cancel{\Theta}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{RT}{r^2}\frac{d^2\Theta}{d\theta^2}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}$$

$$\frac{r^2}{T}\frac{dT}{dt} = \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$$

$$\frac{r^2}{T}\frac{dT}{dt} - \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = +\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \text{const}$$

<u>Case 1</u> :- const $= 0$

~~$\Theta = c_1\theta$~~ . $\Theta = c_1\theta + c_2$

$\Theta|_{\pi} = \Theta|_{-\pi}$

$c_1\pi + c_2 = -c_1\pi + c_2$

$\boxed{c_1 = 0}$

$\boxed{\Theta = c_2} \rightarrow$ eigen function

$\frac{d\Theta}{d\theta}\Big|_{\pi} = \frac{d\Theta}{d\theta}\Big|_{-\pi} \rightarrow$ always valid

0 is the eigen value | Eigen function = const

<u>Case-2</u> :- const = +ve

$$+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = +\alpha^2$$

$$\Theta = c_1 e^{\alpha\theta} + c_2 e^{-\alpha\theta}$$

$$c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = c_1 e^{-\alpha\pi} + c_2 e^{\alpha\pi}$$

$\boxed{c_1 = c_2}$

$\Theta = c_1(e^{\alpha\theta} + e^{-\alpha\theta})$

$\frac{d\Theta}{d\theta}\Big|_{\pi} = \frac{d\Theta}{d\theta}\Big|_{-\pi}$

$c_1(\alpha e^{\alpha\pi} - \alpha e^{-\alpha\pi}) = c_1(\alpha e^{-\alpha\pi} - \alpha e^{\alpha\pi})$

$\boxed{c_1 = 0}$

$c_1 = 0,\ c_2 = 0.$

[trivial soln]

Case 3: const $= -\alpha^2$

$$\frac{d^2\theta}{d\theta^2} + \alpha^2\theta = 0$$

$$\theta = C_1 \sin\alpha\theta + C_2\cos\alpha\theta$$

$$C_1 \sin\alpha\pi + C_2\cos\alpha\pi = C_1\sin\alpha(-\pi) + C_2\cos\alpha(-\pi)$$

$$2C_1\sin\alpha\pi = 0 \rightarrow \sin\alpha\pi = 0 \quad \alpha\pi = n\pi$$

$$\boxed{\cancel{C_1 = 0}} \qquad \boxed{\alpha = n} \rightarrow n = 1,$$

$$C_1\alpha\cos\alpha\pi - C_2\alpha\sin\alpha\pi = C_1\alpha\cos\alpha(-\pi) - C_2\alpha\sin\alpha(-\pi)$$

$$\boxed{\cancel{C_2 = 0}} \qquad \boxed{C_2\alpha\sin\alpha\pi = 0}$$

$$\alpha\sin\alpha\pi = 0.$$

$$\alpha = 0 \text{ or } \alpha\pi = n\pi$$

$$\alpha = 1, 2, \ldots$$

$$\boxed{\alpha_n = n} \rightarrow n = 1, 2, 3, \ldots$$

$\Rightarrow$ Eigen values for θ soln $= 0, 1, 2, \ldots$ $\rightarrow$ combining Case 1 & 3

Eigen function :- $[\Theta(\theta) = C_3 \sin n\theta + C_4\cos n\theta]$ ✓

for case 1 & case 3 $\rightarrow n = 0, 1, 2, \ldots \infty$

$$\rightarrow \frac{r^2}{T}\frac{dT}{dt} - \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -n^2$$

$$\frac{1}{T}\frac{dT}{dt} - \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{-n^2}{r^2}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2} = \text{const} = -\lambda^2$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2} = -\lambda^2$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R + \lambda^2 r R = 0$$

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \frac{n^2}{r}R + \lambda^2 r R = 0.$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R + \lambda^2 r^2 R = 0.$$

$$\underline{y = \lambda r}$$

$$y^2\frac{d^2R}{dy^2} + y\frac{dR}{dy} - n^2R + y^2R = 0.$$

$$y^2\frac{d^2R}{dy^2} + y\frac{dR}{dy} + R(y^2 - n^2) = 0$$

$\hookrightarrow$ n th order Bessel equation

$$\boxed{R_{mn} = c'J_n(\lambda_{mn})} \rightarrow r = 1 \qquad R_{mn} = c'J_n(\underbrace{y_{mn}}_{\lambda_{mn}r})$$

$$\boxed{J_n(\lambda_{mn}) = 0} \rightarrow r=1,\ u=0. \qquad m \rightarrow 1, 2, \ldots \infty$$

$\hookrightarrow$ eigenvalues $\qquad n = 0, 1, 2, \ldots \infty$

$$\boxed{T_{mn} = C_1' \exp(-\lambda_{mn}^2 t)}$$

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \exp(-\lambda_{mn}^2 t)\, J_n(\lambda_{mn}r)\left[C_{n1}\sin n\theta + C_{n2}\cos n\theta\right].$$

$t = 0,\ u = f(r,\theta)$. ..

$$f(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cancel{\exp(} J_n(\lambda_{mn}r)\left[C_{n1}\sin n\theta + C_{n2}\cos n\theta\right]$$

multiply by $\int J_m(\lambda_{mn}r)\, r\,dr \rightarrow \checkmark$

$$C_{n1} = \frac{\int_0^1\int_{-\pi}^{\pi} f(r,\theta)\, J_n(\lambda_{mn}r)\sin(n\theta)\, r\,dr\,d\theta}{\int_0^1\int_{-\pi}^{\pi} r\, J_n^2(\lambda_{mn}r)\sin^2(n\theta)\,dr\,d\theta}$$

Use Simpsons rule

$$C_{n2} = \frac{\int_0^1\int_{-\pi}^{\pi} r f(r,\theta)\, J_n(\lambda_{mn}r)\cos n\theta\, r\,dr\,d\theta}{\iint r\, J_n^2(\lambda_{mn}r)\cos^2(n\theta)\,dr\,d\theta}$$

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Spherical coordinate system (r, θ, ϕ)

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right) +$$
$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$0 \le r \le 1$

$0 \le \theta \le \pi$

$-\pi \le \phi \le \pi$.

1 D Transient Problem θ, ϕ symmetry

$$\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right].$$

$t = 0, \quad u = u_0$

$r = 0 \quad u = \text{finite}$

$r = 1, \quad u = 0$

$u = \frac{v}{r}$

$$\frac{\partial u}{\partial r} = -\frac{v}{r^2} + \frac{1}{r}\frac{\partial v}{\partial r}$$

$$r^2 \frac{\partial u}{\partial r} = -v + r\frac{\partial v}{\partial r}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = -\frac{\partial v}{\partial r} + \frac{\partial v}{\partial r} + r\frac{\partial^2 v}{\partial r^2}$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial^2 v}{\partial r^2}}$$

$$\frac{1}{r}\frac{\partial v}{\partial t} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2}$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial r^2}$$

$$\left.\begin{aligned} &t = 0, \quad v = r u_0 \\ &r = 0, \quad v = 0 \\ &r = 1, \quad v = 0 \end{aligned}\right\} \checkmark$$

$n\pi \rightarrow$ eigen values

$\sin n\pi \rightarrow$ eigen functions

$$v = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2 t) \sin(n\pi r)$$

$$u_0 r = \sum_{n=1}^{\infty} c_n \sin(n\pi r)$$

$$c_n = 2u_0 \int_0^1 r \sin(n\pi r)\, dr$$

$$\left[u(r,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2 t) \frac{\sin(n\pi r)}{r} \right]$$

2 Dim in space

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\nabla^2 u = 0 \quad . \ \& \ \phi \text{ symmetry}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0$$

$r = 0$, $u =$ finite

$r = 1$, $u = f(\theta)$

$u = R(r)\,T(\theta)$

$$\frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2 \sin\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0.$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0.$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{-1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = \lambda \text{ (+ve const)}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \lambda T \sin\theta = 0 \qquad t = \cos\theta.$$

$$\frac{dT}{d\theta} = \frac{dT}{dt} \times \frac{dt}{d\theta} = -\sin\theta\frac{dT}{dt}$$

$$\sin\theta\frac{dT}{d\theta} = -\sin^2\theta\frac{dT}{dt} = -(1-t^2)\frac{dT}{dt}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = -\frac{d}{d\theta}\left((1-t^2)\frac{dT}{dt}\right)$$

$$= -\frac{d}{dt}\left((1-t^2)\frac{dT}{dt}\right)\frac{dt}{d\theta}$$

$$= \sin\theta\left(\frac{d}{dt}\left((1-t^2)\frac{dT}{dt}\right)\right)$$

$$\sin\theta \frac{d}{dt}\left[(1-t^2)\frac{dT}{dt}\right] + \lambda T \sin\theta = 0 \qquad 0 \le \theta \le \pi$$
$$-1 \le t \le 1.$$

$$(1-t^2)\frac{d^2T}{dt^2} - 2t\frac{dT}{dt} + \lambda T = 0.$$

like $n(n+1)$.

$\lambda_n = n(n+1)$, $n = 0, 1, \dots \infty$

$T_n = c_1 P_n(t) + \cancel{c_2}^{\,0} Q_n(t) \rightarrow Q_n =$ undefined at $t = \pm 1$

$$\boxed{T_n(\theta) = c_1 P_n(\cos\theta)}$$

$P_n =$ legendre polynomial of deg n.

$Q_n =$ legendre function of deg n.

$$\frac{1}{R_n}\frac{d}{dr}\left(r^2\frac{dR_n}{dr}\right) - \lambda_n = 0.$$

$$\frac{d}{dr}\left(r^2\frac{dR_n}{dr}\right) - n(n+1)R_n = 0$$

$$r^2\frac{d^2R_n}{dr} + 2r\frac{dR_n}{dr} - n(n+1)R_n = 0$$

$$R_n \sim r^\alpha$$

$$\alpha(\alpha-1) + 2\alpha - n(n+1) = 0.$$

$$\alpha^2 + \alpha - n(n+1) = 0$$

n^2+n-n^2-n

$$\alpha = \frac{-1 \pm \sqrt{1+n(n+1)\times 4}}{2} = \frac{-1 \pm \sqrt{4n^2+4n+1}}{2}$$

$$\boxed{\alpha = n, -(n+1)}$$

at $r=0$, $R_n =$ finite

$$R_n(r) = c_1 r^n + \frac{\cancel{c_2}^{\,0}}{r^{n+1}} \qquad R_n(r) = c_1 r^n$$

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^n P_n(\cos\theta)$$

at $r=1$, $u = f(\theta)$

$$f(\theta) = \sum_{n=1}^{\infty} C_n \underbrace{P_n(\cos\theta)}$$ legendre polynomials are orthogonal functions w.r.t weight function $\sin\theta$.

$$C_n = \frac{\int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta}{\int_0^{\pi} P_n^2(\cos\theta) \sin\theta \, d\theta}$$

Ex-3

$\nabla^2 u = 0$, & no ϕ symmetry.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

$$u = R(r) \times \Theta(\theta) \times \Phi(\phi)$$

$$\frac{\Theta\Phi}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{dR}{dr}\right) + \frac{R\Phi}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{\Theta R}{r^2 \sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0.$$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

Periodic BCs on Φ :-

Physical BC :- $$\begin{cases} \Phi|_{\pi} = \Phi|_{-\pi} \\ \frac{d\Phi}{d\phi}\Big|_{\pi} = \frac{d\Phi}{d\phi}\Big|_{-\pi} \end{cases}$$

$$\frac{d^2\Phi}{d\phi^2} + \underbrace{\text{const}}_{=m^2}\,\Phi = 0.$$

$m = n \rightarrow n = 0, 1, 2, \dots \infty \rightarrow$ eigen values.

$$\boxed{\Phi_m(\phi) = C_1 \sin(m\phi) + C_2 \cos(m\phi)}$$

$$\frac{m^2\Theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{m^2}{\sin^2\theta}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = \lambda$$

for non-trivial soln

→ 3D in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

at $r=0$, u = finite

$r=1$, $u = f(\theta,\phi)$

$\left.\begin{array}{l}\theta=0\\ \theta=\pi\end{array}\right\}$ u = finite

$$u|_{\phi=\pi} = u|_{-\pi}$$

$$\left.\frac{\partial u}{\partial\phi}\right|_{\pi} = \left.\frac{\partial u}{\partial\phi}\right|_{-\pi}$$

$u = R(r)\,\Theta(\theta)\,\Phi(\phi)$

$$\frac{\Theta\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \text{const}$$

(0 & +ve)

$= m^2$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\Phi|_{\pi} = \Phi|_{-\pi}$$

$$\left.\frac{d\Phi}{d\phi}\right|_{\pi} = \left.\frac{d\Phi}{d\phi}\right|_{-\pi}$$

$\Phi_m = A \sin(m\pi) + B \cos m\pi$

$m = 0, 1, 2, \cdots, \infty$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = +ve = \lambda$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2\Theta}{\sin^2\theta} + \lambda\Theta = 0$$

$t = \cos\theta \quad -1 \le t \le 1$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dt}\frac{dt}{d\theta} = -\sin\theta\frac{d\Theta}{dt}$$

$$\boxed{\sin\theta\frac{d\Theta}{d\theta} = -(1-t^2)\frac{d\Theta}{dt}}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\frac{d}{d\theta}\left[(1-t^2)\frac{d\Theta}{dt}\right]$$

$$= -\frac{d}{dt}\left((1-t^2)\frac{d\Theta}{dt}\right) \times \frac{dt}{d\theta}.$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] = \frac{d}{dt}\left[(1-t^2)\frac{d\Theta}{dt}\right]$$

$$\frac{d}{dt}\left[(1-t^2)\frac{d\Theta}{dt}\right] - \frac{m^2}{1-t^2}\Theta + \lambda\Theta = 0$$

→ associated legendre equation for $\lambda = n(n+1)$

$$\Theta(t) = c_1 P_n^m(t) + c_2 Q_n^m(t)$$

$$\boxed{\Theta(\cos\theta) = c_1 P_n^m(\cos\theta)}$$

$n \to$ degree
$m =$ order.
→ eigen values → $\lambda = n(n+1)$
$n = m, m+1, m+2, \cdots \infty$

→ r-direction

$$\frac{1}{R_{mn}}\frac{d}{dr}\left(r^2\frac{dR_{mn}}{dr}\right) = n(n+1)$$

$$r^2\frac{d^2R_{mn}}{dr^2} + 2r\frac{dR_{mn}}{dr} - n(n+1)R_{mn} = 0 \quad \to \text{euler's equation}$$

$R_{mn}(r) = c_3 r^n$

$$u(r,\theta,\phi) = \sum_{n=m}^{\infty} \sum_{m=0}^{n} r^n P_n^m(\cos\theta)\left[C_{mn}\sin(m\phi) + D_{mn}\cos(m\phi)\right]$$

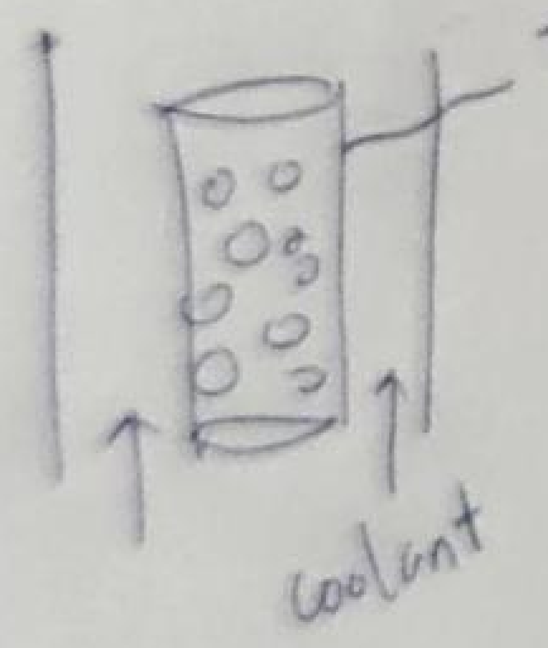
Using orthogonal properties of eigen functions.

at $r=1$, $u = f(r,\theta)$

$$C_{mn} = \frac{\int_{-\pi}^{\pi}\int_{\theta=0}^{\pi} f(r,\theta)\, P_n^m(\cos\theta) \sin(m\phi) \sin\theta \, d\theta\, d\phi}{\int_{-\pi}^{\pi}\int_{\theta=0}^{\pi} P_n^{m\,2}(\cos\theta) \sin^2 m\phi \sin\theta \, d\theta\, d\phi}$$

$$D_{mn} = \frac{\int_{-\pi}^{\pi}\int_{0}^{\pi} f(r,\theta)\, P_n^m(\cos\theta) \cos(m\phi) \sin\theta \, d\phi}{\int_{-\pi}^{\pi}\int_{\theta}^{\pi} P_n^{m\,2}(\cos\theta) \cos^2(m\phi) \sin\theta \, d\theta\, d\phi}$$

Solution of non-homogeneous PDEs



$T_w(t) < T_{mp}$ of MOC of reactor element.

mp → melting pt

$$\rho C_p \frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \pm h(r,t)$$

NH governing eqn

$h(r,t)$ → generation term. because of this term, the equation → NH

+ → source term

− → sink term

→ $\frac{\partial u}{\partial t} = \nabla^2 u + f \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2}$

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

$\boxed{Lu = f}$ $\quad L$ → operator

→ continuous domain → soln $u = L^{-1} f$

Discrete Domain

$Ax = b$

$\boxed{x = A^{-1} b}$

$\rightarrow u = \tilde{L}^{-1} f$

↳ adjoint operator

$L u = f$

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t)$

} → green's function method

at $t = 0$, $u = u_0$

$x = 0$, $u = u_{01}$

$x = 1$, $\frac{\partial u}{\partial x} + \beta u = \alpha$

Particular location (x_0, t_0)

4 sources of non homogeneity. (force all BC too)

~~force~~ force NH in the governing equation to unit impulse.

↳ force $f(x, t)$ to unit impulse. located at (x_0, t_0) → dirac delta function

Definition of green's function (g)

Construction of g (causal green's function)

$$\frac{\partial g}{\partial t}(x, t | x_0, t_0) - \frac{\partial^2 g}{\partial x^2}(x, t | x_0, t_0) = \delta(x - x_0)\delta(t - t_0)$$

at $t = 0$, $g = 0$

$x = 0$, $g = 0$

$x = 1$, $\frac{\partial g}{\partial x} + \beta g = 0$.

Adjoint operator

$L^* = ?$

Solution of NHPDE

$Lu = f$ subject to $Bu = b$ ↳ boundary conditions

$u = L^{-1} f$

$\boxed{u = L^{*} f}$

adjoint operator

Construction of green's function

$Lg(x|x_0) = \delta(x - x_0) \rightarrow 1D$

$Lg(\vec{x}|x_0) = \delta(\vec{x} - x_6) \rightarrow 3D$

$\boxed{Bg(x|x_0) = 0} \rightarrow BC.$

g is dependent variable corresponding to L

g^* is the dependent variable corresponding to L^*

$\left\{ \begin{array}{l} L \text{ is known} \\ L^* \text{ can be derived} \end{array} \right\}$

$\rightarrow \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x^2 t^2$

at $t = 0,\ u = u_0$

$x = 0,\ u = 1$

$x = 1,\ u = 2$

$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

Green's function

$\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} = \underbrace{\delta(x - x_0)\, \delta(t - t_0)}$ ↳ replace NH term by unit impulse

at $t = 0,\ g = 0$

$\left.\begin{array}{l} x = 0 \\ x = 1 \end{array}\right\} g = 0$

$L^* = -\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$

Governing equation of g^* (adjoint green function)

$$L^* g^* = \delta(x-x_1)\,\delta(t-t_1)$$

$$-\frac{\partial g^*}{\partial t} - \frac{\partial^2 g^*}{\partial x^2} = \delta(x-x_1)\,\delta(t-t_1)$$

$Bg^* = ?$ → Bilinear ~~conco~~ concomittant term should be set to 0.

$$g^* = f_2(x,t)$$

$$\langle g^*, Lg \rangle = \langle L^* g^*, g \rangle + \underbrace{J(g,g^*)}_{=0 \text{ to get } g^*}$$

step 1 :- Construction of green's functions

step 2 :- solution of g. → check if $L = L^*$ & $B = B^*$ or not

- if yes: omit 3,4,5 Directly connect u with g
- if no: all steps should be done

step 3 :- Obtain L^*

step 4 :- Construction of adj green's function g^*

step 5 :- solution of $g^*(x,t)$

step 6 :- Connect g^* with u & get soln of $u(x,t)$

→ For laplacian operators (L) → it is self adjoint

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$L = L^*, \quad B = B^*$$

Properties of Dirac delta

(i) $\delta(x-x_0) = 1$ at $x = x_0$

$= 0 \;\forall\; x \neq x_0$

(ii) shifting property

$$\int_{x_0-\epsilon}^{x_0+\epsilon} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

→ Relation between g & g^*

$Lu = f$ subject to: $Bu = h$

Causal green's function :- ① $Lg_{x|x_0} = \delta(x-x_0)$ subject to $Bg = 0$

Adjoint green's function :- ② $L^* g^*_{(x|x_1)} = \delta(x-x_1)$ subject to $Bg^* = 0$

Take inner product of equation ① wrt g^*

Take inner product of equation ② wrt g

and substract

$$< g^*, Lg > - < g, L^* g^* > = < g^*(x|x_1), \delta(x-x_0) > - < g(x|x_0), \delta(x-x_1) >$$

$$< u, Lv > = < u, L^* v > + J(u,v)$$

$$< L^* g^*, g > + \cancel{J(g,g^*)} - < g, L^* g^* > = g^*(x_0|x_1) - g(x_1|x_0)$$

$\because$ all BC are homogeneous.

$$\boxed{g^*(x_0|x_1) = g(x_1|x_0)}$$ → to find g^* → valid for 1D, 2D, 3D...

Example: $g(x|x_0) = x - x_0 + x x_0$

$\downarrow x \to x_1$

$g(x_1|x_0) = x_1 - x_0 + x_1 x_0.$

$= g^*(x_0|x_1)$

$g^*(x_0|x_1) = x_1 - x_0 + x_1 x_0$

$$\boxed{g^*(x|x_1) = x_1 - x + x_1 x}$$

Connection between u & g^*

$Lu = f$, $Bu = h$ → ①

$L^* g^*(x|x_1) = \delta(x - x_1)$ → ②

$Bg^* = 0$

take inner product of ① with g^*

" " " " ② with g

& substract.

$$< g^*, Lu > - < u, L^* g^* > = < f, g^* > - u(x_1)$$

$$< L^* g^*, u > + J(g^*, u) - < u, L^* g^* > = < f g^* > - u(x_1)$$

$$\boxed{u(x_1) = < f, g^* > + J(g^*, u)}$$ → to be used if BC of u are NH

$\underline{Ex}$:- $f = x$

$g^* = x_1 x$

$$\langle f, g^* \rangle = \int_0^1 x\, x_1\, dx = \frac{x_1}{3}.$$

$u(x_1) = p(x_1) \rightarrow$ purely a function of x_1

$x_1 \rightarrow x$.

$$u(x) = \frac{x}{3} + \frac{1}{2} + x^2.$$

→ Q) $\frac{d^2 u}{dx^2} = x$

$$\frac{d^2 g(x|x_0)}{dx^2} = \delta(x - x_0) \qquad 0 \le x \le 1$$

for $x < x_0 \Rightarrow \frac{d^2 g_1}{dx^2} = 0 \rightarrow$ lower half solution

$x > x_0 \rightarrow \frac{d^2 g_2}{dx^2} = 0. \rightarrow$ upper half solution

$$g(x|x_0) = A_1 u_1(x) + B_1 (u_2(x)) \quad \text{for} \quad x < x_0.$$

$$= A_2 u_1(x) + B_2 u_2(x) \quad \text{for} \quad x > x_0.$$

(Consts will be diff for upper & lower half)

→ Relationship between g & $g^* \Rightarrow g(x_1/x_0) = g^*(x_0/x_1)$

solution of g

$$\frac{d^2 g}{dx^2} = \delta(x - x_0)$$

For a typical 2nd order equation:-

$\frac{d^2 g}{dx^2} = 0$ for $x \neq x_0$

$g = c_1 A(x) + c_2 B(x) \quad \forall\ 0 \le x < x_0 \rightarrow$ lower half solution $\rightarrow g_1$

$= c_3 A(x) + c_4 B(x) \quad \forall\ x_0 < x \le 1. \rightarrow$ upper half solution $\rightarrow g_2$

4 constants of integration → C_1, C_2, C_3, C_4

at $\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\}$ $g=0$ $\left\{\begin{matrix} BC_1 \\ BC_2 \end{matrix}\right\}$

$\underline{BC_3}$ → $g_1(x_0-\varepsilon) = g_2(x_0+\varepsilon)$ → continuity of green's function across x_0

$\varepsilon \to 0$

$$\boxed{g_1(x_0) = g_2(x_0)}$$

$\underline{BC_4}$ → Jump discontinuity condition

$$\frac{d^2 g}{dx^2} = \delta(x-x_0)$$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{d^2 g}{dx^2}\, dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)\, dx$$

$$\left[\frac{dg}{dx}\right]_{x_0-\varepsilon}^{x_0+\varepsilon} = 1$$

$$\boxed{\left.\frac{dg}{dx}\right|_{x_0+\varepsilon} - \left.\frac{dg}{dx}\right|_{x_0-\varepsilon} = 1} \rightarrow BC_4$$

$\underline{Ex-1}$ $\frac{d^2 u}{dx^2} = x \longrightarrow (1)$

at $x=0 \longrightarrow u=1$
$x=1 \longrightarrow u=2$

$\underline{\text{Casual green's function}}$:-

$$\frac{d^2 g}{dx^2} = \delta(x-x_0) \rightarrow g=0 \text{ at } x=0,\ x=1.$$

$$\frac{d^2 g}{dx^2} = 0$$

$g(x) = Ax+B \quad \forall \ 0 \le x < x_0$

$= Cx+D \quad \forall \ x_0 < x \le 1$

at $x=0$, $g=0$

$g_1(0)=0$

$\underline{B=0}$

$$\boxed{g_1 = Ax}$$

at $x=1$, $g=0$

$C+D=0$

$$\boxed{D=-C}$$

$$\boxed{g_2 = C(x-1)}$$

$g = Ax \quad \forall\ 0 \le x < x_0$

$ = C(x-1) \quad \forall\ x_0 < x \le 1$

→ Continuity of green's function

$Ax_0 = C(x_0-1)$

→ Jump discontinuity

$C - A = 1$

$$\boxed{A = x_0 - 1} \qquad \boxed{C = x_0}$$

$$\left[\begin{aligned} g &= (x_0-1)x \quad \forall\ 0 \le x < x_0 \\ &= x_0(x-1) \quad \forall\ x_0 < x \le 1 \end{aligned}\right]$$

$\dfrac{d^2 g}{dx^2} = \delta(x-x_0) \longrightarrow ②$ $\qquad \underline{\int ① \times g - \int ② \, u}$

$$\int_0^1 g\frac{d^2u}{dx^2}dx - \int_0^1 u\frac{d^2g}{dx^2} = \int_0^1 ug\,dx - \int_0^1 \delta(x-x_0)\,u\,dx$$

$$g\frac{du}{dx}\Big|_0^1 - u\frac{dg}{dx}\Big|_0^1 = \int xg\,dx - u(x_0)$$

$$\underbrace{g(1)}_{0}u'(1) - \underbrace{g(0)}_{0}u'(0) - u(1)\frac{dg}{dx}\Big|_1 + u(0)\frac{dg}{dx}\Big|_0 = \int xg\,dx - u(x_0)$$

$$-2x_0 + 1\times(x_0-1) = \int xg\,dx - u(x_0)$$

$$\boxed{u(x_0) = \underbrace{\int xg\,dx}_{\text{governing equation NH}} + \underbrace{x_0+1}_{\text{BC}}}$$

↳ 3 sources of NH

∴ 3 terms.

$$u(x_0) = \int_0^{x_0} x(x_0-1)\cancel{x}\, dx + \int_{x_0}^{1} x\, x_0(x-1)\, dx + x_0 + 1$$

$$u(x_0) = \left[(x_0-1)\frac{x^3}{3}\right]_0^{x_0} + x_0\left(\left[\frac{x^3}{3}\right]_{x_0}^{1} - \left[\frac{x^2}{2}\right]_{x_0}^{1}\right) + x_0 + 1$$

$$\cancel{u(x_0) = \frac{x_0^3(x_0-1)}{3} + \frac{x_0}{3} - \frac{x_0^4}{3} - \frac{1}{2} + \frac{x_0^2}{2} + x_0 + 1}$$

$$\boxed{u(x_0) = -\frac{x_0^3}{6} + \cancel{\frac{x_0^2}{2}} + \frac{5x_0}{6} + 1}$$

$x_0 \to x$

$$u(x) = \frac{x^3}{6} + \frac{5}{6}x + 1$$

Ex-2 $\frac{d^2u}{dx^2} = x \rightarrow (1)$

at $x = 0 \longrightarrow du/dx = 1$

$x = 1 \rightarrow u = 2$

(2) $\frac{d^2g}{dx^2} = \delta(x - x_0)$ subject to

$\frac{dg}{dx} = 0$ at $x = 0$

$g = 0$ at $x = 1$

$g = Ax + B$ for $x < x_0$

$= Cx + D$ for $x > x_0$.

$\frac{dg}{dx} = 0 \rightarrow A = 0$ $\boxed{g = B} \rightarrow x < x_0$

$0 = C + D \rightarrow x = 1$

$C = -D$

$\boxed{g = c(x-1)} \rightarrow x > x_0$

$$g = \begin{cases} B & x < x_0 \\ C(x-1) & x > x_0 \end{cases}$$

continuity of green's function

$$B = C(x_0 - 1)$$

Jump discontinuity

$$\left.\frac{dg}{dx}\right|_{x_0^+} - \left.\frac{dg}{dx}\right|_{x_0^-} = 1$$

$$C - 0 = 1$$

$$\boxed{C = 1}$$

$$\boxed{B = (x_0 - 1)}$$

$$g = \begin{cases} x_0 - 1 & x < x_0 \\ x - 1 & x > x_0 \end{cases}$$

$$\int g \frac{d^2u}{dx^2}\,dx - \int u \frac{d^2g}{dx^2}\,dx = \int x g\,dx - u(x_0)$$

By parts

$$-\cancel{g(1)}u'(1) - g(0)u'(0) - u(1)g'(1) - u(0)\cancel{g'(0)} = \int x g\,dx - u(x_0)$$

$$u(x_0) = \underbrace{\int x g\,dx}_{g} + 1 + x_0$$

$$\int_0^{x_0} \quad \int_{x_0}^{1}$$

$$u(x_0) = \int_0^{x_0} x(x_0-1)\,dx + \int_{x_0}^{1} x(x-1)\,dx + x_0 + 1$$

$$u(x_0) = \left(\frac{x^2}{2}(x_0-1)\right)_0^{x_0} + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x_0}^{1} + x_0 + 1$$

$$u(x_0) = \frac{x_0^3}{2} - \frac{x_0^2}{2} + \frac{1}{3} - \frac{1}{2} - \frac{x_0^3}{3} + \frac{x_0^2}{2} + x_0 + 1$$

$$u(x_0) = \frac{x_0^3}{6} + x_0 + 5/6$$

$$u(x) = \frac{x^3}{6} + x + 5/6$$

Ex-3 $\frac{d^2u}{dx^2} = x$

at $x = 0$, $u = 1$

$x = 1$, $\frac{du}{dx} + u = 2$

$$\frac{d^2g}{dx^2} = \delta(x - x_0)$$

at $x = 0$, $g = 0$

$x = 1$, $\frac{dg}{dx} + g = 0$.

$$g(x|x_0) = \begin{cases} (x_0 - 2)\frac{x}{2} & \forall \quad x < x_0 \\ \frac{x_0}{2}(x - 2) & \forall \quad x > x_0 \end{cases}$$ ✓

$u(x) = \frac{x^3}{6} + \frac{x}{6} + 1$ ✓

Ex-4: $\frac{d^2u}{dx^2} = x$

$x = 0$ $\frac{du}{dx} + 2u = 3$

$x = 1$ $\frac{du}{dx} + u = 2$

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Solution of non-homogeneous PDE

Elliptic equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

at $x=0 \rightarrow u=a$ $\quad y=0,\ u=0$

$x=1 \rightarrow u=0 \,/\, \frac{\partial u}{\partial x}=0$ $\quad y=1,\ u=b$

Construction of Causal green's Function:

$$Lg = \delta(x-x_0)\,\delta(y-y_0)$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0)$$

$L\phi = \lambda\phi$ type.

at $\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\}$ $\left.\begin{matrix} y=0 \\ y=1 \end{matrix}\right\}$ → 2 independent eigen value problems

$\underbrace{\qquad}_{g=0\ \ \frac{\partial g}{\partial x}=0}$

$$g(x,y|x_0,y_0) = \sum a_i\,\phi_i(x,y)$$

ϕ_i = eigen function

(full eigen function expansion method)

corresponding eigen value problem.

$$L\phi + \lambda\phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \lambda\phi = 0$$

eigen function is function of both x,y.

at $\left.\begin{matrix} x=0,1 \\ y=0,1 \end{matrix}\right\}$ $\phi = 0 \,/\, \frac{\partial \phi}{\partial x}=0$

$$\phi(x,y) = X(x)\,Y(y)$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \lambda XY = 0$$

→ formulating eigen value problem in both x,y directions

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda = -\alpha^2$$

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0$$

$\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\} X=0$ $\quad \frac{\partial X}{\partial x}=0$

$$\boxed{X_n = C_1 \sin n\pi x}$$

$\alpha_n = n\pi$

$n = 1,2,\ldots\infty$

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$$-\frac{1}{Y}\frac{d^2Y}{dy^2} - \lambda = -\alpha^2$$

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0 \quad \text{at } \left.\begin{matrix} y=0 \\ y=1 \end{matrix}\right\} Y=0$$

$$Y_m = C_2 \sin m\pi y \qquad \lambda_{mn} = (m^2+n^2)\pi$$

$$\beta_m = m\pi, \quad m = 1, 2, \ldots$$

$$\boxed{\phi_{m,n}(x,y) = C_{mn} \sin n\pi x \sin m\pi y}$$

$$g(x,y) = \sum_m \sum_n a_{mn}\, \phi_{mn}(x,y)$$

norm → length

~~$g(x,y) = \sum_m \sum_n a_{mn} \sin$~~

$$a_{mn} = \frac{\langle g, \phi_m \rangle}{\langle \phi_{mn}, \phi_{mn} \rangle} \rightarrow \text{normalization}$$

$\langle \phi_{mn}, \phi_{mn} \rangle = 1 \rightarrow$ [MAKE UR LYF EASY!!!]

$$\iint \phi_{mn}^2 \, dx\, dy = 1$$

$$\int_0^1\int_0^1 C_{mn}^2 \sin^2 n\pi x \sin^2 m\pi y \, dx\, dy = 1$$

$$\boxed{C_{mn} = 2}$$

$$\phi_{mn}(x,y) = 2 \sin n\pi x \sin m\pi y$$

$$a_{mn} = \langle g, \phi_{mn} \rangle$$

Estimation of a_{mn}

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x-x_0)\,\delta(y-y_0) \longrightarrow (1)$$

$$\frac{\partial^2 \phi_{mn}}{\partial x^2} + \frac{\partial^2 \phi_{mn}}{\partial y^2} + \lambda_{mn}\phi_{mn} = 0 \longrightarrow (2)$$

$$\langle 1, \phi_{mn} \rangle - \langle 2, g \rangle$$

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$$\iint \phi_{mn} \frac{\partial^2 g}{\partial x^2} dx\,dy + \iint \phi_{mn} \frac{\partial^2 g}{\partial y^2} dx\,dy - \iint g \frac{\partial^2 \phi_{mn}}{\partial x^2} dx\,dy -$$

$$\iint g \frac{\partial^2 \phi_{mn}}{\partial y^2} dx\,dy - \lambda_{mn} \iint \phi_{mn}\, g\, dx\,dy = \phi_{mn}(x_0, y_0)$$

$$\text{LHS} = \int_y \left[\phi_{mn} \frac{\partial g}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial \phi_m}{\partial x} \frac{\partial g}{\partial x} dx \right] dy + \int_x \left[\phi_{mn} \frac{\partial g}{\partial y}\Big|_0^1 - \int_0^1 \frac{\partial \phi_m}{\partial y} \cancel{\frac{\partial g}{\partial y}} \right] dx$$

0 0 0

cancel

$$- \int_y \left[g \frac{\partial \phi_m}{\partial x}\Big|_0^1 - \int \frac{\partial g}{\partial x} \frac{\partial \phi_{mn}}{\partial x} dx \right] dy$$

0

$$- \int_x \left[g \frac{\partial \phi_{mn}}{\partial y}\Big|_0^1 - \int \frac{\partial g}{\partial y} \cancel{\frac{\partial \phi_{mn}}{\partial y}} dy \right] dx$$

0

$$0 = \lambda_{mn} \langle \phi_{mn}, g \rangle + \phi_{m,n}(x_0, y_0)$$

$$\langle \phi_{m,n}, g \rangle = -\phi_{m,n}(x_0, y_0)$$

$$a_{mn} = -2 \frac{\sin(n\pi x_0)\sin(m\pi y_0)}{\pi^2(m^2+n^2)}$$

$$g(x, y | x_0, y_0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-4) \frac{\sin(n\pi x_0)\sin(m\pi y_0)\sin n\pi x \sin m\pi y}{\pi^2(m^2+n^2)}$$

$\rightarrow L^* = ?$ ~~For L^*~~ ~~Self adjoint~~

$\langle v, Lu \rangle$

$u = g$, $v = g^*$

$$\langle g^*, Lg \rangle = \iint g^* \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) dx\,dy$$

$$= \iint g^* \frac{\partial^2 g}{\partial x^2} dx\,dy + \iint g^* \frac{\partial^2 g}{\partial y^2} dx\,dy$$

$$= \int_y \left[g^* \frac{\partial g}{\partial x}\Big|_0^1 - \int \frac{\partial g^*}{\partial x} \frac{\partial g}{\partial x} dy\,dx \right] + \int_x \left[g^* \frac{\partial g}{\partial y}\Big|_0^1 - \frac{\partial g^*}{\partial y} \frac{\partial g}{\partial y} dy\,dx \right]$$

$$= -\iint \frac{\partial g^*}{\partial x} \frac{\partial g}{\partial x} dx\,dy - \iint \frac{\partial g^*}{\partial y} \frac{\partial g}{\partial y} dy\,dx$$

$$= -\left[\int_y \left[-\frac{\partial g^*}{\partial x} g\right]_0^1 - \int \frac{\partial^2 g^*}{\partial x^2} g \, dx\, dy - \left[\int_x \left[-\frac{\partial g^*}{\partial y} g\right]_0^1 - \int \frac{\partial^2 g^*}{\partial y^2} g \, dx\, dy\right]\right.$$

$$= \iint g \left(\frac{\partial^2 g^*}{\partial x^2} + \frac{\partial^2 g^*}{\partial y^2}\right) dx\, dy = \iint g(L^* g^*)\, dx\, dy$$

$$= \langle g, L^* g^* \rangle$$

$$L^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\underline{L^* = L}$$

$\Rightarrow$ ① $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

② $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \delta(x - x_0)\,\delta(y - y_0)$

$$\iint g \frac{\partial^2 u}{\partial x^2} dx\, dy + \iint g \frac{\partial^2 u}{\partial y^2} dx\, dy = \iint g f \, dx\, dy - u(x_0, y_0)$$

$$- \iint u \frac{\partial^2 g}{\partial x^2} dx\, dy - \iint u \frac{\partial^2 g}{\partial y^2} dy\, dx$$

<u>LHS</u> :- $\int_y \left[g \frac{\partial u}{\partial x}\Big|_0^1 - \int \cancel{\frac{\partial g}{\partial x} \frac{\partial u}{\partial x}} dx\right] dy$

$$+ \int_x \left[g \frac{\partial u}{\partial y}\Big|_0^1 - \int \cancel{\frac{\partial g}{\partial y} \frac{\partial u}{\partial y}} dy\right] dx$$

$$- \int_y \left[u \frac{\partial g}{\partial x}\Big|_0^1 - \int \cancel{\frac{\partial u}{\partial x} \frac{\partial g}{\partial x}} dx\right] dy$$

$$- \int_x \left[u \frac{\partial g}{\partial y}\Big|_0^1 - \int \cancel{\frac{\partial u}{\partial y} \frac{\partial g}{\partial y}} dy\right] dx$$

$$= \int_y \left[g(1) \frac{\partial u}{\partial x}\Big|_{x=1} - g(0) \frac{\partial u}{\partial x}\Big|_{x=0} \right] dy$$

$$+ \int_x \left[g(y=1) \frac{\partial u}{\partial y}\Big|_{y=1} - g(0) \frac{\partial u}{\partial y}\Big|_{y=0} \right] dx.$$

$$- \int_y \left[u(1) \frac{\partial g}{\partial x}\Big|_1 - u(0) \frac{\partial g}{\partial x}\Big|_{x=0} \right] dy$$

$$- \int_x \left[u(1) \frac{\partial g}{\partial y}\Big|_{y=1} - u(0) \frac{\partial g}{\partial y}\Big|_{x=0} \right] dx$$

LHS :- $= a \int \frac{\partial g}{\partial x}\Big|_{x=0} dy - b \int \frac{\partial g}{\partial y}\Big|_{y=1} dx.$

$$a\int_{y=0}^{1} \frac{\partial g}{\partial x}\Big|_{x=0} dy - b\int_{x=0}^{1} \frac{\partial g}{\partial y}\Big|_{y=1} dx = \iint gf \, dx dy - u(x_0, y_0)$$

$$u(x_0, y_0) = \iint gf \, dx\, dy + b\int_x \frac{\partial g}{\partial y}\Big|_{y=1} dx - a\int_y \frac{\partial g}{\partial x}\Big|_{x=0} dy.$$