$\boxed{PDE}$ $\xrightarrow[\text{F.T}]{\text{L.T}}$ ODE $\xrightarrow[\text{soln}]{\text{can}}$

PDE of 2nd order in two independent variables & with constant coefficients.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

a) elliptic if $B^2 - 4AC < 0$

b) hyperbolic if $B^2 - 4AC > 0$

c) parabolic if $B^2 - 4AC = 0$

$u(x, y)$

→ Wave eqn (hyperbolic)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

u – deflection of string

→ Heat eqn (parabolic)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

→ In 2d $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

u – temperature

→ Laplace eqn (elliptic)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(steady state 2D heat eqn in sheet of metal)

★ **Application of Laplace transform in solving PDE**

Criteria for choosing a L.T

(i) At least one of the independent variable should have range from range $(0, \infty)$ in case both have range for either one, but $(0, \infty)$, we can apply L.T. if only has the range $(0, \infty)$ we can apply L.T. only wrt that variable. (convention wrt t)

(ii) Appropriate initial condition must be specified at the lower limit of the variable which has the range from $(0, \infty)$ (Generally at $t=0$) initial condⁿ.

34-35 - Class Test

30/10

Lec. 36

$ If $y(x,t)$ is a fⁿ of x & t and

$$L\{y(x,t)\} = \bar{y}(x,s)$$

(a) $L\left\{\frac{\partial y}{\partial t}\right\} = s\bar{y}(x,s) - y(x,0)$

(b) $L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = s^2\bar{y}(x,s) - s\,y(x,0) - y_t(x,0)$

(c) $L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d\bar{y}}{dx}$

(d) $L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2\bar{y}}{dx^2}$

Proof (a)

$$L\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial y}{\partial t}\,dt$$

$$= \left[e^{-st}y\right]_0^\infty + s\int_0^\infty e^{-st}y(x,t)\,dt$$

$$= s\int_0^\infty e^{-st}y(x,t)\,dt - y(x,0)$$

$$= s\,\bar{y}(x,s) - \underline{y(x,0)}$$ initial condⁿ

(b) Let $v = \frac{\partial y}{\partial t}$

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\} = s L\{v\} - v(x,0)$$

$$= s\left[s L\{y\} - y(x,0)\right] - y_t(x,0)$$

$$= s^2 L\{y\} - s\,y(x,0) - y_t(x,0)$$

$$= s^2 \bar{y}(x,s) - s\,y(x,0) - y_t(x,0)$$ → needs initial cond^n

(c) $$L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial y}{\partial x}\, dt$$

$$= \frac{d}{dx}\left[\int_0^\infty e^{-st} y\, dt\right]$$

$$= \frac{d}{dx}\bar{y}$$

(d) $$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2}{dx^2}\bar{y}$$

e.g. ① Solve the heat eq^n

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x>0,\ t>0$$

can apply for both variables.

subject to (i) $u(0,t) = u_0$ → const. $t>0$

(ii) $u(x,0) = 0$ $x>0$

(iii) $\lim_{x\to\infty} u(x,t) = 0$ $t>0$ → physical condition that temp never goes to ∞ as $x\to\infty$ (on infinite rod temp → 0) not ∞

Scanned by CamScanner

(b) Let $v = \frac{\partial y}{\partial t}$

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\} = s\,L\{v\} - v(x,0)$$

$$= s\left[s\,L\{y\} - y(x,0)\right] - y_t(x,0)$$

$$= s^2 L\{y\} - s\,y(x,0) - y_t(x,0)$$

$$= s^2\,\bar{y}(x,s) - s\,y(x,0) - y_t(x,0)$$ → needs initial cond^n

(c) $$L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^\infty e^{-st}\,\frac{\partial y}{\partial x}\,dt$$

$$= \frac{d}{dx}\left[\int_0^\infty e^{-st}\,y\,dt\right]$$

$$= \frac{d}{dx}\bar{y}$$

(d) $$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2}{dx^2}\bar{y}$$

e.g. (1) Solve the heat eq^n

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x>0,\ t>0$$

can apply for both variables.

subject to (i) $u(0,t) = u_0$ → const. $t>0$

(ii) $u(x,0) = 0$ $\quad x>0$

(iii) $\lim_{x\to\infty} u(x,t) = 0$ $\quad t>0$

→ physical condition that temp never goes to ∞ as $x\to\infty$ (on infinite rod temp → 0 not ∞)

Scanned by CamScanner

Sol^n. Applying L.T. on wrt t

$$s\,\bar{u}(x,s) - \cancel{u(x,0)}^{\to 0} = \frac{d^2}{dx^2}\bar{u}(x,s)$$

$$\Rightarrow \frac{d^2 u}{dx^2} - s\bar{u} = 0$$

soln

$$\boxed{\bar{u}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}}$$

Determine $\bar{u}$, will take in L^{-1}

from (iii) $\lim_{x\to\infty} u(x,t) = 0$ then $\bar{u}(x,s) \to 0$

$\therefore A = 0$

$$\bar{u}(x,s) = Be^{-\sqrt{s}x}$$

Now, $u(0,t) = u_0 \rightarrow$ const

$$\bar{u}(0,s) = \frac{u_0}{s}$$

$$\boxed{B = \frac{u_0}{s}}$$

$$L^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = erf_c$$

$$\therefore \bar{u}(x,s) = \frac{u_0}{s}e^{-\sqrt{s}x}$$

$$u(x,t) = L^{-1}\left\{\frac{u_0}{s}e^{-\sqrt{s}x}\right\}$$

$$u(x,t) = u_0\, erf_c\left[\frac{x}{2\sqrt{t}}\right]$$

Scanned by CamScanner

Sol^n : Applying L.T. on w.r.t t

$$s\,\bar{u}(x,s) - \cancel{u(x,0)}^{\to 0} = \frac{d^2}{dx^2}\bar{u}(x,s)$$

$$\Rightarrow \frac{d^2 \bar{u}}{dx^2} - s\bar{u} = 0$$

sol^n

$$\boxed{\bar{u}(x,s) = A e^{\sqrt{s}x} + B e^{-\sqrt{s}x}}$$

Determine $\bar{u}$, will take in L^{-1}

from (iii) : $\lim_{x\to\infty} u(x,t) = 0$ then $\bar{u}(x,s) \to 0$

$\therefore A = 0$

$$\bar{u}(x,s) = B e^{-\sqrt{s}x}$$

Now, $u(0,t) = u_0 \to$ const

$$\bar{u}(0,s) = \frac{u_0}{s}$$

$$\boxed{B = \frac{u_0}{s}}$$

$$L^{-1}\left\{\frac{e^{-k\sqrt{s}}}{s}\right\} = erf_c$$

$$\therefore \bar{u}(x,s) = \frac{u_0}{s} e^{-\sqrt{s}x}$$

$$u(x,t) = L^{-1}\left\{\frac{u_0}{s} e^{-\sqrt{s}x}\right\}$$

$$u(x,t) = u_0\, erf_c\left[\frac{x}{2\sqrt{t}}\right]$$

② Wave eq^n

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad x>0, \; t>0$$

with $u(x,0) = 0$

$u_t(x,0) = 0$ $\quad x>0$

$u(0,t) = F(t)$; $\lim_{x\to\infty} u(x,t) = 0$ $\quad t>0$

Sol^n

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = c^2 \, L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$s^2 \bar{u}(x,s) - s\,\cancel{u(x,0)}^{\,0} - \cancel{u_t(x,0)}^{\,0} = a^2 \frac{d^2\bar{u}}{dx^2}$$

$$-a^2 \frac{d^2\bar{u}}{dx^2} + s^2 \bar{u}(x,s) = 0$$

$$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{a^2}\bar{u}(x,s) = 0$$

$$\bar{u}(x,s) = A e^{\frac{s}{a}x} + B e^{-\frac{s}{a}x}$$

$u \to 0$ as $x \to \infty$, $\bar{u} = 0$ as $x \to \infty$

so $A = 0$

$$\bar{u}(x,s) = B e^{-s/a\, x}$$

$u(0,t) = F(t)$ — L.T.

$\bar{u}(0,s) = \bar{f}(s)$

$\therefore \bar{u}(x,s) = \bar{f}(s)\, e^{-\frac{s}{a}x}$

$$u(x,t) = L^{-1}\left\{\bar{f}(s)\, e^{-\frac{s}{a}x}\right\}$$

$$= \begin{cases} F\left(t-\frac{x}{a}\right), & t > \frac{x}{a} \\ 0, & t < \frac{x}{a} \end{cases}$$

(2) Wave eqⁿ

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x>0, \; t>0$$

with $u(x,0) = 0$ $\quad x>0$

$u_t(x,0) = 0$

$u(0,t) = F(t)$; $\lim_{x\to\infty} u(x,t) = 0 \quad t>0$

Solⁿ

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = c^2 \, L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$s^2 \bar{u}(x,s) - s\,\cancel{u(x,0)} - \cancel{u_t(x,0)} = a^2 \frac{d^2 \bar{u}}{dx^2}$$

$$-a^2 \frac{d^2\bar{u}}{dx^2} + s^2 \bar{u}(x,s) = 0$$

$$\frac{d^2 \bar{u}}{dx^2} - \frac{s^2}{a^2}\bar{u}(x,s) = 0$$

$$\bar{u}(x,s) = A e^{\frac{s}{a}x} + B e^{-\frac{s}{a}x}$$

$u = 0$ as $x \to \infty$, $\quad \bar{u} = 0$ as $x \to \infty$

so $A = 0$

$$\bar{u}(x,s) = B e^{-s/a\, x}$$

$u(0,t) = F(t)$ — L.T.

$\bar{u}(0,s) = \bar{f}(s)$

$\therefore \bar{u}(x,s) = \bar{f}(s) e^{-\frac{s}{a}x}$

$$u(x,t) = L^{-1}\left\{\bar{f}(s)\, e^{-\frac{s}{a}x}\right\}$$

$$= \begin{cases} F\left(t-\frac{x}{a}\right), & t > \frac{x}{a} \\ 0, & t < \frac{x}{a} \end{cases}$$

$$u(x,t) = F\left(t-\frac{x}{a}\right)\cdot H\left(t-\frac{x}{a}\right)$$

5/11/18 Lec. 37 & 38

Application of $\boxed{\text{F.T.}}$ ~~in solving PDEs~~ -

- Criteria for choosing general F.T.
  (i) One of the independent variables should have the range from $-\infty$ to ∞ & we can apply F.T. wrt that variable (convention is to take F.T. wrt x)
  (ii) Behaviour of u & $\frac{\partial u}{\partial x}$ should be known as $x \to \pm\infty$ [i.e. $|x| \to \infty$] if not specified we take u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$.

- Criteria for choosing sine or cosine transform:

(A) $\int_0^\infty \sin\alpha x \frac{\partial^2 u}{\partial x^2}\,dx$

$$= \left[\sin\alpha x \frac{\partial u}{\partial x}\right]_0^\infty - \alpha\int_0^\infty \cos\alpha x \frac{\partial u}{\partial x}\,dx$$

$$= -\alpha\left[\left[\cos\alpha x\, u\right]_0^\infty + \alpha\int_0^\infty u \sin\alpha x\,dx\right]$$

$\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$$= \alpha\, U\big|_{x=0} - \alpha^2 \hat{U}_s \qquad \because U \to 0 \text{ as } x \to \infty$$

(i) At least one of the independent variables should have the range from 0 to ∞ and we need to apply **FST** wrt that variable.

ii) The value of the unknown for $u(x,t)$ must be known at the lower limit of the variables which has the range from 0 to ∞.

iii) The behaviour of $u(x,t)$ & $\frac{\partial u}{\partial x}$ at $x \to \infty$ should be known. If not specified we take u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$.

(B) $\int_0^\infty \cos \alpha x \frac{\partial^2 u}{\partial x^2} dx$

$$= \left[\frac{\partial u}{\partial x} \cos \alpha x\right]_0^\infty + \alpha \int_0^\infty \sin \alpha x \frac{\partial u}{\partial x} dx$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + \alpha \left[\left[u \sin \alpha x\right]_0^\infty - \alpha^2 \int_0^\infty u \cos \alpha x \, dx\right]$$

if $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - \alpha^2 \hat{u}_c \quad \left[\text{if } u \to 0 \text{ as } x \to \infty\right]$$

(i) same as previous (i)

ii) The value of $\frac{\partial u}{\partial x}$ must be known at the lower limit 0 of the variable selected for exclusion.

iii) same as (iii) of FST

Scanned by CamScanner

(3) Solve $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ $\boxed{\text{Heat Eqn}}$ $0 < x < \infty$, $0 < t < \infty$

subject to condition-

(i) $u = 0$ when $t = 0$, $x > 0$

ii) $\frac{\partial u}{\partial x} = -\mu$. (u is a const) when $x = 0$ & $t > 0$

iii) u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

Solⁿ Taking FCT on both sids wrt x-

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos \alpha x \, dx = K \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx$$

$$\frac{d}{dt} \hat{u}_c = K \sqrt{\frac{2}{\pi}} \left[\cos \alpha x \left[\frac{\partial u}{\partial x}\right]_0^\infty + \alpha \int_0^\infty \frac{\partial u}{\partial x} \sin \alpha x \, dx \right]$$

$$= K \sqrt{\frac{2}{\pi}} \left(\frac{\partial u}{\partial x}\right)_{x=0} + K \alpha \sqrt{\frac{2}{\pi}} \left[u \sin \alpha x\right]_0^\infty - K\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u \cos \alpha x \, dx$$

$\left.\frac{du}{dx}\right)_{x\to\infty} \to 0$; $\left.u\right)_{x\to\infty} \to 0$

$$\frac{d\hat{u}_c}{dt} = \sqrt{\frac{2}{\pi}} K\mu - K\alpha^2 \hat{u}_c$$

$$\frac{d\hat{u}_c}{dt} + K\alpha^2 u_c = \sqrt{\frac{2}{\pi}} K\mu$$

$$IF = e^{\int K\alpha^2 dt} = e^{K\alpha^2 t}$$

$$\hat{u}_c \cdot e^{K\alpha^2 t} = \sqrt{\frac{2}{\pi}} K\mu \int e^{K\alpha^2 t} dt + A$$

$$\hat{u}_c e^{K\alpha^2 t} = \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} e^{K\alpha^2 t} + A$$

At $t=0$, $u>0$

$t=0 \Rightarrow \hat{u}_c = 0$

$$A = -\frac{\mu}{\alpha^2}\sqrt{\frac{2}{\pi}}$$

$$\hat{u}_c = \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}\left[1 - e^{-\kappa\alpha^2 t}\right]$$

$$u = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_c \cos\alpha x \, d\alpha$$

$$u = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\alpha x}{\alpha^2}\left(1 - e^{-\alpha^2 \kappa t}\right) d\alpha$$

(4) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $x>0$, $t>0$

(i) $u=0$ when $x=0$, $t>0$

ii) $u = \begin{cases} 1 & 0<x\le 1 \\ 0 & x\ge 1 \end{cases}$ when $t>0$

iii) u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

Sol^n Applying FST on both sides wrt x

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin\alpha x \, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\alpha x \, dx$$

$$\frac{d}{dt}\hat{u}_s = \sqrt{\frac{2}{\pi}}\left[\sin\alpha x\left[\frac{\partial u}{\partial x}\right]_0^\infty - \alpha\int_0^\infty \frac{\partial u}{\partial x}\cos\alpha x \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - 0 - \alpha\left[\cos\alpha x \cdot u\right]_0^\infty + \alpha^2\int_0^\infty u \sin\alpha x \, dx\right]$$

$$\frac{d\hat{u}_s}{dt} = \sqrt{\frac{2}{\pi}}\,\alpha\, u_{x=0} - \alpha^2\hat{u}_s$$

Scanned by CamScanner

$$\frac{d}{dt}\hat{u}_s + \alpha^2 \hat{u}_s = 0$$

$$IF = e^{\alpha^2 t} \qquad \boxed{\hat{u}_s = A e^{-\alpha^2 t}}$$

$$\cancel{\hat{u}_s e^{\alpha^2 t} = A + \int_0^{\infty} \alpha \sqrt{2}}$$

$t=0$ $$\hat{u}_s(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x,0) \sin\alpha x \, dx$$

$$= \cancel{\sqrt{\frac{2}{\pi}} \left[\int_0^{1} \sin\alpha x \, dx + \sqrt{\frac{2}{\pi}} \int_1^{\infty} \right]}$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{1} \sin\alpha x \, dx + 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos\alpha}{\alpha} \right)$$

$$\hat{u}_s(\alpha, 0) = A = \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos\alpha}{\alpha} \right)$$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos\alpha}{\alpha} \right) e^{-\alpha^2 t}$$

$$u(x,t) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} \hat{u}_s \sin\alpha x \, d\alpha$$

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos\alpha}{\alpha} \right) e^{-\alpha^2 t} \sin\alpha x \, d\alpha$$

(5) 1D Wave eq^n

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}; \quad -\infty < x < \infty, \quad t > 0$$

(i) $u(x,0) = \dfrac{1}{1+x^2}$

(ii) $\dfrac{\partial u}{\partial t}(x,0) = 0$

(iii) u & $\dfrac{\partial u}{\partial x} \to 0$ as $|x| \to \infty$

Scanned by CamScanner

Applies General FT on both side

Sol[4] $\dfrac{d^2 \hat{u}}{dt^2}(\alpha, t) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \dfrac{\partial^2 u}{\partial x^2} dx$

$$\frac{d^2 \hat{u}(\alpha,t)}{dt^2} = \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} e^{-i\alpha x} \right]_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} e^{-i\alpha x} \frac{\partial u}{\partial x} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[i\alpha \left[u e^{-i\alpha x} \right]_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} e^{-i\alpha x} u \, dx \right]$$

$$\frac{d^2}{dt^2} \hat{u}(\alpha, t) = -\alpha^2 \hat{u}(\alpha, t)$$

$$\frac{d^2 \hat{u}}{dt^2} + \alpha^2 \hat{u} = 0$$

f^n of α bcz diff eq^n wrt t.

$$\hat{u}(\alpha, t) = \underline{A(\alpha)} \cos \alpha t + \underline{B(\alpha)} \sin \alpha t$$

$$\boxed{\mathcal{F}\left[\frac{1}{1+x^2} \right] = \sqrt{\frac{\pi}{2}} e^{-|\alpha|}}$$ Imp

* proof

$$\mathcal{F}^{-1}\left[\frac{\pi}{2} e^{-|\alpha|} \right] = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{-|\alpha|} e^{i\alpha x} d\alpha$$

$$= \frac{1}{2} \left[\int_{-\infty}^{0} e^{\alpha} e^{-i\alpha x} d\alpha + \int_{0}^{\infty} e^{-\alpha} e^{-i\alpha x} d\alpha \right]$$

$$= \frac{1}{2} \left[\left. \frac{e^{(1-ix)\alpha}}{1-ix} \right|_{-\infty}^{0} + \left. \frac{e^{-(1+ix)\alpha}}{1+ix} \right|_{0}^{\infty} \right]$$

$$= \frac{1}{2} \left[\frac{1}{1-ix} + \frac{1}{1+ix} \right] = \underline{\underline{\frac{1}{1+x^2}}}$$

Scanned by CamScanner

F.CT of $\frac{1}{1+x^2}$ [alternative lecture 31 & 32]

$$\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty f(x)\cos\alpha x\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx \quad \text{— (1)}$$

$$\hat{f}_c'(\alpha) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin\alpha x}{(1+x^2)}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x^2\sin(\alpha x)}{x(1+x^2)}\,dx = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+x^2-1)\sin\alpha x}{x(1+x^2)}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2} - \int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx\right] \quad \text{— (2)}$$

$$\hat{f}_c''(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx \quad \boxed{= \hat{f}(\alpha)}$$

$$\hat{f}_c(\alpha) = A e^{\alpha} + B e^{-\alpha}$$

$$\hat{f}_c'(\alpha) = A e^{\alpha} - B e^{-\alpha}$$

if $\alpha \to 0$, $\hat{f}_c(0) = \sqrt{\frac{\pi}{2}}$ from eq (1)

$$\Rightarrow A + B = \sqrt{\frac{\pi}{2}}$$

$\hat{f}_c'(0) = -\sqrt{\frac{\pi}{2}}$ — eq (2) $\quad A - B = -\sqrt{\frac{\pi}{2}}$

$$A = 0 \quad B = \sqrt{\frac{\pi}{2}}$$

$$F_c(\alpha) = \sqrt{\frac{\pi}{2}}\, e^{-\alpha}$$

$$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}}\, e^{-|\alpha|}$$

FCT of $\frac{1}{x^2+a^2} = \sqrt{\frac{\pi}{2}}\,\frac{1}{a}\, e^{-a\alpha}$

FT of $\frac{1}{x^2+a^2} = \sqrt{\frac{\pi}{2}}\,\frac{1}{a}\, e^{-a|\alpha|}$

Lec. 31 4:32

Lec - 39

6/11/18

$$\hat{u}(\alpha, t) = A(\alpha)\cos\alpha t + B(\alpha)\sin\alpha t$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

$$\left.\frac{\partial \hat{u}}{dt}\right|_{t=0} = 0$$

$$\left.\frac{d\hat{u}}{dt}\right|_{t=0} = A\alpha\sin(0) + B\alpha(1) = 0$$

$$B\alpha = 0 \quad \boxed{B = 0}$$

$$\hat{u}(\alpha, t) = A(\alpha)\cos\alpha t$$

$$\hat{u}(\alpha, 0) = \sqrt{\frac{\pi}{2}}\, e^{-|\alpha|}$$

$$A(\alpha) = \sqrt{\frac{\pi}{2}}\, e^{-|\alpha|}$$

$$\Rightarrow \boxed{\hat{u}(\alpha, t) = \sqrt{\frac{\pi}{2}}\, e^{-|\alpha|}\cos\alpha t}$$

$$\boxed{u(x, t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\alpha|}\cos\alpha t\; e^{-i\alpha x}\, d\alpha}$$

Scanned by CamScanner

(6) Solve the 1D heat eqⁿ

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad x>0\ ;\ t>0$$

(i) $u(0,t)=0$

(ii) $u(x,0)=e^{-x} \quad x>0$

iii) u & $\frac{\partial u}{\partial x} \to 0$ as $x\to\infty$

Taking $\boxed{FST}$ on both sides wrt x -

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin\alpha x\,dx = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\alpha x\,dx$$

$$\frac{d\hat{u}_s}{dt} = 2\sqrt{\frac{2}{\pi}}\left[\sin\alpha x\frac{\partial u}{\partial x}\Big|_0^\infty - \alpha\int_0^\infty \cos\alpha x\frac{\partial u}{\partial x}\,dx\right]$$

$$= \sqrt{\frac{8}{\pi}}\left[0-\alpha\left[\cos\alpha x\,u\Big|_0^\infty + \alpha\int_0^\infty \sin\alpha x\,u\,dx\right]\right]$$

$$= -\alpha\sqrt{\frac{8}{\pi}}\left[0-u|_{x=0}+\alpha\sqrt{\frac{\pi}{2}}\hat{u}_s\right]$$

$$= -2\alpha^2\hat{u}_s$$

$$\frac{d\hat{u}_s}{dt} + 2\alpha^2\hat{u}_s = 0$$

$$\boxed{\hat{u}_s = Ae^{-2\alpha^2 t}}$$

$t=0$

$u(x,0) = e^{-x}$ ~~$A = e$~~

$$\hat{u}(\alpha,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\alpha x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{\alpha}{1+\alpha^2}$$

$A = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2}$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2} e^{-2\alpha^2 t}$$

$$\boxed{u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\alpha}{1+\alpha^2} e^{-2\alpha^2 t} \sin \alpha x \, d\alpha}$$

(7) Sol 1D heat eqn

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} , \quad x > 0 , \ t > 0$$

(1) $u_x(0,t) = 0$

(2) $u(x,0) = \begin{cases} x & , 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$

(3) u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

Soln Taking FCT on both sides

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx$$

$$\frac{d}{dt} \hat{u}_c = \sqrt{\frac{2}{\pi}} \left[\cos \alpha x \frac{\partial u}{\partial x} \Big|_0^\infty + \alpha \int_0^\infty \sin \alpha x \frac{\partial u}{\partial x} dx \right]$$

$$\frac{d}{dt} \hat{u}_c = \alpha \sqrt{\frac{2}{\pi}} \left[\cos \alpha x \frac{\partial u}{\partial x} - \alpha \int_0^\infty u \cos \alpha x \, dx \right]$$

$$= -\alpha^2 \hat{u}_c$$

$$\frac{d \hat{u}_c}{dt} + \alpha^2 \hat{u}_c = 0$$

$$\boxed{\hat{u}_c = A e^{-\alpha^2 t}}$$

$u(x,0) = \begin{cases} x & , 0 < x \le 1 \\ 0 & x > 1 \end{cases}$

$t=0 \quad \hat{u}_c(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x,0) \cos \alpha x \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^{1} x \cos \alpha x \, dx$

$= \sqrt{\frac{2}{\pi}} \cdot \left[x \frac{\sin \alpha x}{\alpha} + \frac{\cos \alpha x}{\alpha^2} \right]_0^1$

$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha}{\alpha^2} - \frac{1}{\alpha^2} \right]$

$A(\alpha) = \sqrt{\frac{2}{\pi}} \left[\frac{\alpha \sin \alpha + \cos \alpha - 1}{\alpha^2} \right]$

$\hat{u}_c(\alpha, t) = \sqrt{\frac{2}{\pi}} \left[\frac{\cos \alpha + \alpha \sin \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t^2}$

$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\cos \alpha + \alpha \sin \alpha - 1}{\alpha^2} \right) e^{-\alpha^2 t} \cos \alpha x \, d\alpha$