PGF5107 - Quantum Field Theory I

Homework 4

Due 29/04/2016

1. **Correlation Functions in Perturbation Theory**:

Defining the field in the Heisenberg picture as

$$\tilde{\phi}(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt}$$

and the field in the interaction picture as

$$\phi(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}$$

(a) Show that $\phi(t,\vec{x})$ is a free field. For instance, if this is a real scalar field, show that $\phi(t,\vec{x})$ satisfies the homogenous Klein-Gordon equation.

(b) Show that

$$\tilde{\phi}(x) = \Omega(t)\,\phi(x)\,\Omega^\dagger(t)$$

where we defined

$$\Omega(t) = e^{iHt}\,e^{-iH_0t}$$

(c) Show that for any time order

- $U(t_3,t_1) = U(t_3,t_2)U(t_2,t_1)$,
- $U^\dagger(t_1,t_2) = U(t_2,t_1)$.

with

$$U(t_1,t_2) = \Omega^\dagger(t_1)\,\Omega(t_2)$$

(d) Using the properties of the U operators and the transformation of the $\tilde{\phi}$'s, show that

$$\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_n) =$$
$$U^\dagger(\infty,0)U(\infty,t_1)\phi(x_1)U(t_1,t_2)\phi(x_2)\ldots U(t_{n-1},t_n)\phi(x_n)U(t_n,-\infty)U(-\infty,0)$$

(e) Replacing H_0 by $H_0(1-i\epsilon)$, with $\epsilon > 0$, real and infinitesimal, show that

$$U(-\infty, 0)|\tilde{0}\rangle = \langle 0|\tilde{0}\rangle |0\rangle \ ,$$

and

$$\langle \tilde{0}|U^\dagger(\infty, 0) = \langle 0|\, \langle \tilde{0}|0\rangle \ ,$$

where $|\tilde{0}\rangle$ and $|0\rangle$ are the perturbed and unperturbed vacuum respectively, and in the Heisenberg picture $e^{-iHt}|\tilde{0}\rangle = |\tilde{0}\rangle$. To prove this you want to use the fact that we can expand the perturbed vacuum in terms of eigenstates of H_0, $|n\rangle$, such that $|\tilde{0}\rangle = \sum_n |n\rangle\langle n\,|\tilde{0}\rangle$. As long as $\epsilon > 0$ then the ground state will dominate.

(f) Show that

$$\langle \tilde{0}|T\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_n)|\tilde{0}\rangle =$$
$$\langle 0|U(\infty, t_1)\phi(x_1)U(t_1,t_2)\phi(x_2)\ldots U(t_{n-1},t_n)\phi(x_n)U(t_n,-\infty)|0\rangle \ \times |\langle\tilde{0}|0\rangle|^2$$

(g) Finally,

i. Show

$$\langle \tilde{0}|T\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_n)|\tilde{0}\rangle = \langle 0|T\phi(x_1)\ldots\phi(x_n)\, e^{-i\int_{-\infty}^{+\infty} dt V(t)}|0\rangle \ \times |\langle\tilde{0}|0\rangle|^2$$

where you use that $U(\infty,-\infty) = T e^{-i\int_{-\infty}^{+\infty} dt V(t)}$,

ii. Also show that

$$|\langle\tilde{0}|0\rangle|^2 = \frac{1}{\langle 0|Te^{-i\int_{-\infty}^{+\infty} dt V(t)}|0\rangle}$$

In this way the final answer is

$$\langle \tilde{0}|T\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_n)|\tilde{0}\rangle = \frac{\langle 0|T\phi(x_1)\ldots\phi(x_n)\, e^{-i\int_{-\infty}^{+\infty} dt V(t)}|0\rangle}{\langle 0|Te^{-i\int_{-\infty}^{+\infty} dt V(t)}|0\rangle}$$

$e^{A+B} = e^A e^B e$

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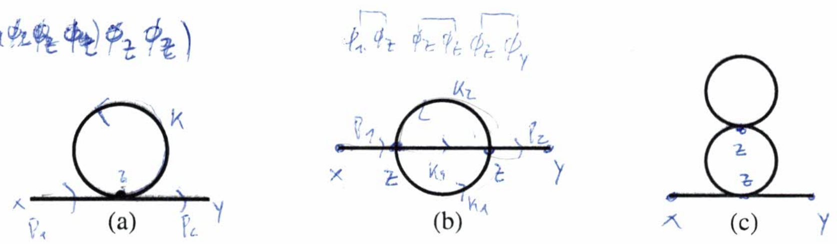
$T(\phi_x \phi_z \phi_z \phi_z \phi_z \phi_y)$



2. **Wick's Theorem and Feynman Diagrams**:

Consider the 3 Feynman diagrams in the figure, for a theory with

$$\mathcal{H}_I = \frac{\lambda}{4!}\phi^4 .$$

(a) Using Wick's Theorem compute the symmetry factors for each, i.e. compute all the possible contractions, etc.

(b) Using the Feynman rules in momentum space, compute the contributions of Figure (a) to the two-point correlation function. Consider the ultra-violet (UV) limit $E \gg m$: how does the answer behave with E ? Is it finite for $E \to \infty$?

(c) Do the same for Figure (b).

4-mom: $P_1 + P_2 = 0$ (a)

(b) $-k_3 + P_1 + k_2 - k_1 = 0, \quad k_3 + k_1 - k_2 - P_2 = 0$

(a) $$-\frac{i\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}$$

(b) $$\frac{(-i\lambda)^2}{6}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}\cdot\frac{i}{k_2^2-m^2+i\varepsilon}\cdot\frac{i}{k_3^2-m^2+i\varepsilon}\cdot\frac{i}{(k_2+P_1-k_3)^2+m^2+i\varepsilon}$$

$$\int d^Dk \frac{1}{(k^2+2p\cdot k-m^2+i\varepsilon)^n} = \frac{i(-1)^n\pi^{D/2}}{\Gamma(n)(m^2+p^2)^{n-D/2}}\Gamma(n-D/2)$$

$$\text{(a)} = \frac{\lambda}{2}(-i\pi^2 m^2)\Gamma(-1)$$

Homework 4. Q.F.T. I

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9.75

1] Correlation Functions in Perturbation Theory:

(a) Let us apply K-G operator to $\phi(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}$

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\phi(t,\vec{x}) = -H_0^2 e^{iH_0t}\phi(0,\vec{x}) + H_0 e^{iH_0t}\phi(0,\vec{x})H_0 e^{-iH_0t}$$
$$+ H_0 e^{iH_0t}\phi(0,\vec{x})H_0 e^{-iH_0t} - e^{iH_0t}\phi(0,\vec{x})H_0^2 e^{-iH_0t}$$
$$+ e^{iH_0t}\{-\vec{\nabla}^2 + m^2\}\phi(0,\vec{x})e^{-iH_0t}$$

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Now notice that $[H_0, e^{\pm iH_0t}] = 0$, and that

from $i\hbar\frac{d}{dt}\phi(x) = [\phi(x), H_0]$, we have that $[\phi(0,\vec{x}), H_0] = 0$.

And obviously for $t=0$, is satisfied $(-\vec{\nabla}^2 + m^2)\phi(0,\vec{x}) = 0$.

Thus

$$(\partial^2 + m^2)\phi(t,\vec{x}) = 0, \quad \text{and } \phi(t,\vec{x}) \text{ is a free field.}$$

(b) $\tilde{\phi}(x) = \Omega(t)\phi(x)\Omega^\dagger(t)$

$$= e^{iHt}e^{-iH_0t}\phi(x)\{e^{iHt}e^{-iH_0t}\}^\dagger$$

$$= e^{iHt}e^{-iH_0t}\phi(x)e^{iH_0t}e^{-iHt}, \quad \text{As } \phi(x) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}$$

$$= e^{iHt}\underbrace{e^{-iH_0t}e^{iH_0t}}_{\mathbb{1}}\phi(0,\vec{x})\underbrace{e^{-iH_0t}e^{iH_0t}}_{\mathbb{1}}e^{-iHt}$$

$$= e^{iHt}\phi(0,\vec{x})e^{-iHt}.$$

(c) As $U(t_3,t_2) = \Omega^\dagger(t_3)\Omega(t_2)$ And $U(t_2,t_1) = \Omega^\dagger(t_2)\Omega(t_1)$

then

$$U(t_3,t_2)\,U(t_2,t_1) = \Omega^\dagger(t_3)\Omega(t_2)\Omega^\dagger(t_2)\Omega(t_1)$$
$$= \Omega^\dagger(t_3)\Omega(t_1)$$
$$= U(t_3,t_1).$$

On the other hand we have:

$$(U(t_1,t_2))^\dagger = (\Omega^\dagger(t_1)\Omega(t_2))^\dagger$$
$$= \Omega^\dagger(t_2)\Omega(t_1)$$
$$= U(t_2,t_1)$$

(d) $$\tilde{\phi}(x_1)\cdots\tilde{\phi}(x_n) = \Omega(t_1)\phi(x_1)\Omega^\dagger(t_1)\Omega(t_2)\phi(x_2)\cdots\Omega(t_n)\phi(x_n)\Omega^\dagger(t_n)$$

Notice: $$U^\dagger(\infty,0)\,U(\infty,t_1) = U(0,\infty)\,U(\infty,t_1)$$
$$= U(0,t_1)$$
$$= \Omega^\dagger(0)\Omega(t_1)$$
$$= \Omega(t_1).$$

Besides $$U(t_n,-\infty)\,U(-\infty,0) = U(t_n,0)$$
$$= \Omega^\dagger(t_n)\Omega(0)$$
$$= \Omega^\dagger(t_n).$$

Using this, we can write:

$$\tilde{\phi}(x_1)\cdots\tilde{\phi}(x_n) = U^\dagger(\infty,0)\,U(\infty,t_1)\,\phi(x_1)\,U(t_1,t_2)\,\phi(x_2)\cdots U(t_{n-1},t_n)\,\phi(x_n)\,U(t_n,-\infty)\,U(-\infty,0)$$

(e) $U(-\infty,0)|\tilde{0}\rangle = \Omega^+(-\infty)\Omega(0)|\tilde{0}\rangle$

$$= \lim_{\Upsilon\to-\infty} e^{iH_0\Upsilon} e^{-iH\Upsilon} |\tilde{0}\rangle$$

$$= \lim_{\Upsilon\to\tilde{\infty}} e^{iH_0\Upsilon} |\tilde{0}\rangle$$

$$= \lim_{\Upsilon\to\tilde{\infty}} \sum_n e^{iH_0\Upsilon} |n\rangle\langle n|\tilde{0}\rangle.$$

As $H_0|n\rangle = E_n|n\rangle$, And taking $H_0 \to H_0(1-i\varepsilon)$, with $\varepsilon>0$:

$$U(-\infty,0)|\tilde{0}\rangle = \lim_{\Upsilon\to-\infty} e^{iE_0\Upsilon} e^{\varepsilon E\Upsilon} |0\rangle\langle 0|\tilde{0}\rangle + \sum_{n\neq 0} e^{iE_n\Upsilon} e^{E_n\varepsilon\Upsilon} |n\rangle\langle n|\tilde{0}\rangle.$$

We can drop the terms with $n\neq 0$ because they die faster than the ground state. Also notice that we can take $E_0=0$, so that $e^{E_0\Upsilon(i+\varepsilon)} \to 0$.

Thus

$$U(-\infty,0)|\tilde{0}\rangle = \langle 0|\tilde{0}\rangle |0\rangle$$

Proceeding in a similar way, we get

$$\langle\tilde{0}|U^+(\infty,0) = \lim_{\Upsilon\to\infty} \langle\tilde{0}|e^{-iH_0\Upsilon}$$

$$= \lim_{\Upsilon\to\infty} \langle\tilde{0}|0\rangle\langle 0|e^{-iE_0\Upsilon}e^{-\varepsilon E_0\Upsilon} + \sum_{n\neq 0}\langle\tilde{0}|n\rangle\langle n|e^{-iE_n\Upsilon}e^{-\varepsilon E_n}$$

$$= \langle 0|\langle\tilde{0}|0\rangle.$$

Where we have used the same argument as before.

(f) using the result that we found in (d), we can write :

$$\langle\tilde{0}| T\, \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) |\tilde{0}\rangle = \langle\tilde{0}| U^{+}(\infty,0)\, U(\infty,0)\, \phi(x_1)\, U(t_1,t_2)\, \phi(x_2) \cdots \phi(x_n)\, U(t_n,-\infty)\, U(-\infty,0)\, |\tilde{0}\rangle$$

but $\langle\tilde{0}| U^{+}(\infty,0) = \langle 0| \langle\tilde{0}|0\rangle$, $U(-\infty,0)|\tilde{0}\rangle = \langle 0|\tilde{0}\rangle |0\rangle$,

$= \langle\tilde{0}|0\rangle^{*} |0\rangle$

thus

$$\langle\tilde{0}| T\, \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) |\tilde{0}\rangle = \langle 0| U(\infty,0)\, \phi(x_1) \cdots \phi(x_n)\, U(t_n,-\infty) |0\rangle\, |\langle\tilde{0}|0\rangle|^2 .$$

(g) (i) Notice that $U(\infty,0)\, \phi(x_1) \cdots U(t_{n-1},t_n)\, \phi(x_n)\, U(t_n,-\infty)$

is already temporary ordered (recall (d)).

So we can write :

$$U(\infty,0)\, \phi(x_1)\, U(t_1,t_2)\, \phi(x_2) \cdots \phi(x_n)\, U(t_n,-\infty) = T\, \phi(x_1)\, \phi(x_2) \cdots \phi(x_n)\, U(\infty,t_1)\, U(t_1,t_2) \cdots U(t_n,-\infty)$$

now we can use that

$$U(t_\alpha,t_\beta) = U(t_\alpha,t_\gamma)\, U(t_\gamma,t_\beta) \qquad (\alpha > \gamma > \beta)$$

repeatedly, so we found:

$$U(\infty,0)\, \phi(x_1) \cdots \phi(x_n)\, U(t_n,-\infty) = T\, \phi(x_1) \cdots \phi(x_n)\, U(+\infty,-\infty)$$

As $U(\infty,-\infty) = T\, e^{-i\int_{-\infty}^{+\infty} dt\, V(t)}$

Then

$$\langle\tilde{0}| T\, \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) |\tilde{0}\rangle = \langle 0| T\, \phi(x_1) \cdots \phi(x_n)\, e^{-i\int_{-\infty}^{+\infty} dt\, V(t)} |0\rangle\, |\langle\tilde{0}|0\rangle|^2$$

(ii.) Consider $U(-\infty,0)|\tilde{0}\rangle = \langle 0|\tilde{0}\rangle |0\rangle$, multipliying by the left by $U(0,-\infty)$:

$$U(0,-\infty)\,U(-\infty,0)\,|\tilde{0}\rangle = \langle 0|\tilde{0}\rangle\, U(0,-\infty)\,|0\rangle$$

$$\Rightarrow \quad |\tilde{0}\rangle = \langle 0|\tilde{0}\rangle\, U(0,-\infty)\,|0\rangle .$$

Now consider $\langle\tilde{0}|\,U^{+}(\infty,0) = \langle 0|\,\langle\tilde{0}|0\rangle$ and multiply this by the right by $U(+\infty,0)$, to get

$$\langle\tilde{0}| = \langle\tilde{0}|0\rangle\,\langle 0|\,U(\infty,0).$$ (we used that $U^{+}(\infty,0) = U(0,\infty)$)

Thus

$$\langle\tilde{0}|\tilde{0}\rangle = |\langle\tilde{0}|0\rangle|^2\,\langle 0|\,U(+\infty,0)\,U(0,-\infty)\,|0\rangle$$

$$= |\langle\tilde{0}|0\rangle|^2\,\langle 0|\,U(+\infty,-\infty)\,|0\rangle$$

$$= |\langle\tilde{0}|0\rangle|^2\,\langle 0|\,T e^{-i\int_{-\infty}^{+\infty} dt\, V(t)}\,|0\rangle$$

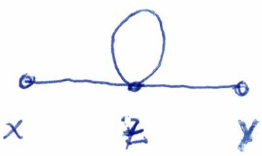
Using that $|\tilde{0}\rangle$ is normalized, we get:

$$|\langle\tilde{0}|0\rangle|^2 = \frac{1}{\langle 0|\,T e^{-i\int_{-\infty}^{+\infty} dt\, V(t)}\,|0\rangle}$$

2| Wick's theorem and Feynman Diagrams: 4.75

(a) From Wick's theorem and the definition of Normal order, we know that the only terms that will contribute in the vacuum expectation value of time-ordered product of Fields, will be those that are totally contracted.

Thus we know that



is related with the contraction:

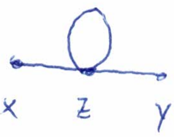
$$\phi(x)\ \phi(z)\ \phi(z)\ \phi(z)\ \phi(z)\ \phi(y)$$

but there is no reason to belive that $\phi(z)$ are distinguishable so it's also true that the contraction:

$$\phi(x)\ \phi(z)\ \phi(z)\ \phi(z)\ \phi(z)\ \phi(y)$$ will work

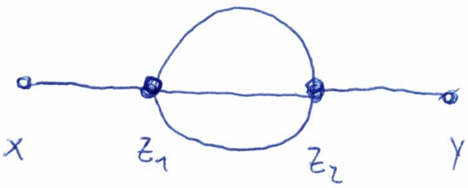
So $\phi(x)$ may be contracted with any of the four $\phi(z)$, and $\phi(y)$ with any of the other three $\phi(z)$ remaining.
We will call $\mathbb{P}$ the number of possible contractions, so that the symmetry factor (S.F.) will be calculated as:

$$(S.F.)^{-1} = \frac{\mathbb{P}}{(4!)^n}$$, where n is the number of vertex of the diagram.

So for 

we have $\mathbb{P} = 4 \cdot 3$

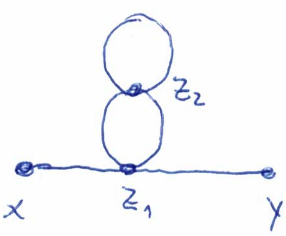
thus $(S.F.)^{-1} = \dfrac{4 \cdot 3}{4!} = \dfrac{1}{2}$

For 

, we have that one of the possible contractions is : $\phi_x \phi_{z_1} \phi_{z_1} \phi_{z_1} \phi_{z_1} \phi_{z_2} \phi_{z_2} \phi_{z_2} \phi_{z_2} \phi_y$

it follows that $\mathbb{P} = 4 \cdot 4 \cdot 3 \cdot 2$,

thus $(S.F.)^{-1} = \dfrac{4 \cdot 4 \cdot 3 \cdot 2}{(4!)^2} = \dfrac{1}{6}$

Finally For 

We have that one of the possible contractions is :

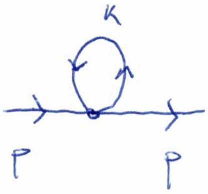
$$\phi_x \phi_{z_1} \phi_{z_1} \phi_{z_1} \phi_{z_1} \phi_{z_2} \phi_{z_2} \phi_{z_2} \phi_{z_2} \phi_y$$

then $\mathbb{P} = 4 \cdot 3 \cdot 12$

thus

$$(S.F.)^{-1} = \frac{4 \cdot 3 \cdot 12}{(4!)^2} = \frac{1}{4}$$

(b)



$$= \frac{-i\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\cdot\frac{i}{k^2-m^2+i\varepsilon} = I$$

Making a continuation to euclidean momentum:

$$I = \frac{-i\lambda}{2}\int \frac{d^4k_E}{(2\pi)^4}\cdot\frac{1}{k_E^2+m^2}$$, using four dimensional spherical coordinates:

$$= \frac{-i\lambda}{2}\int d\Omega_4 \int_0^\infty \frac{dk_E}{(2\pi)^4}\,\frac{k_E^3}{k_E^3+m^2}$$, impose momentum space cutoff Λ:

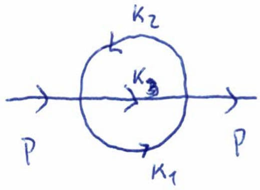
$$= \frac{-i\lambda}{2}\cdot\frac{2\pi^2}{(2\pi)^4}\cdot\frac{1}{2}\left\{k_E^2 - m^2\ln(m^2+k_E^2)\right\}_0^\Lambda$$

$$= \frac{-i\lambda}{32\pi^2}\left\{\Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\right\}$$

for Λ large compared to m:

$$I = \frac{-i\lambda}{32\pi^2}\left\{\Lambda^2 - m^2\ln\left(\frac{\Lambda^2}{m^2}\right)\right\}$$ ✓

We see that this expression diverges for $\Lambda \to \infty$

(c) 

$$= \frac{(-i\lambda)^2}{6} \int \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \cdot \frac{i}{k_2^2 - m^2 + i\varepsilon} \cdot \frac{i}{k_3^2 - m^2 + i\varepsilon} \cdot \frac{i}{(k_2 + p - k_3)^2 + m^2 + i\varepsilon}$$

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