PGF 5107 Quantum Field Theory I

Homework 5

Due 10/05/2016

1. **Decay of a Scalar Particle**: Consider the Lagrangian for two scalar particles A and ϕ given by:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A)\partial^\mu A - \frac{1}{2}M^2A^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \mu A\phi\phi$$

with M and m the masses of A and ϕ respectively, and $M > 2m$.

   (a) Derive the Feynman rules of the theory, and compute $\mathcal{M}(A \to \phi\phi)$, the amplitude for this process, at the lowest order in perturbation theory in μ.

   (b) Given the expression for the differential decay width

$$d\Gamma = \frac{1}{2M}\left(\prod_{f=1}^{n}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(A \to \{p_f\})|^2\ (2\pi)^4\ \delta^{(4)}(p_A - \sum p_f)$$

   (with $\mathcal{M}(A \to \{p_f\})$ is the amplitude for the process $A \to f$, and f is a final state), compute the total width Γ for A.

2. **Scattering**: Given the theory in the previous exercise,

   (a) Draw all three allowed distinct Feynman diagrams for the scattering process

$$\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4),$$

   where the p_i's are the initial and final state momenta, at order μ^2 in perturbation theory.

   (b) Using the Feynman rules, compute the scattering amplitude for this process.

   (c) Compute the scattering cross section σ.

   (d) Describe the theory one gets in the limit when $M^2 \gg (p_1+p_2)^2$, etc., i.e. when M is much larger than any external momenta, as well as $M \gg m$.

Homework 5, Q.F.T. I

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P1 Decay of a Scalar Particle.

(a) From $\mathcal{L} = \frac{1}{2}(\partial_\mu A)(\partial^\mu A) - \frac{1}{2}M^2A^2 + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \mu A \phi \phi$,

we immediately see that:

$$\mathcal{H}_{int} = \mu A(x)\, \phi(x)\, \phi(x).$$

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Now consider the three point correlation function:

$$G_3 = \langle 0 | T \{ A(x_1)\, \phi(x_2)\, \phi(x_3)\, \exp\left(-i \int d^4z\, \mu\, A(z)\, \phi(z)\, \phi(z)\right) \} | 0 \rangle,$$

to first order in μ we have:

$$G_3 = \langle 0 | T \{ A(x_1)\, \phi(x_2)\, \phi(x_3)\, \left(\mathbb{1} - i\mu \int d^4z\, A(z)\, \phi(z)\, \phi(z)\right) \} | 0 \rangle$$

With

$$\langle 0 | T\, A(x_1)\, \phi(x_2)\, \phi(x_3)\, A(z)\, \phi(z)\, \phi(z) | 0 \rangle = \langle 0 | T\, A_1\, \phi_2\, \phi_3\, A_z\, \phi_z\, \phi_z | 0 \rangle \times 2$$
$$+ \langle 0 | T\, A_1\, \phi_2\, \phi_3\, A_z\, \phi_z\, \phi_z | 0 \rangle .$$

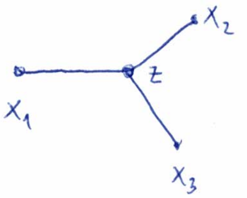
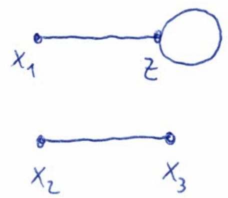
Thus

$$G_3^3 = 2 \times (-i\mu) \int d^4z \, D_F^{(A)}(x_1 - z) \, D_F^{(\phi)}(x_2 - z) \, D_F^{(\phi)}(x_3 - z)$$

$$+ (i\mu) \, D_F^{(\phi)}(x_2 - x_3) \int d^4z \, D_F^{(A)}(x_1 - z) \, D_F^{(\phi)}(z - z).$$

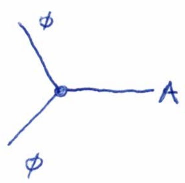
The subscript (ϕ), (A) stands for the correct mass in each propagator.

Diagrammatically we have:

$G_3 = 2 \times$  $+$ 

So the Feynman rules are:

(I) For each propagator  $= D_F^{(?)}(x-y)$. Where $(?)$ depends on if we are treating with ϕ or A.

(II) For each vertex  $= -i\mu \int d^4 z$

(III) For each external point,  $= 1$

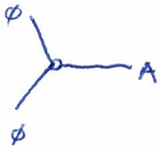
In practice it is more convenient to work in momentum space, by introducing the Fourier expansion of each propagator:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$, and noting that as in $2\phi 4$

theory the momentum is conserved at each vertex.

Thus:

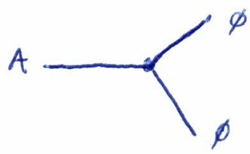
(I) For each propagator  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

(II) For each vertex  $= -i\mu$

(III) For each external point  $= e^{-ip\cdot x}$

(IV) Impose momentum conservation at each vertex

(V) Integrate over each undetermined momentum : $\displaystyle\int \frac{d^4p}{(2\pi)^4}$

Now consider $A \to \phi\phi$: 

Applying the Feynman rules one gets : $\mathcal{M} = -i\mu$, but we must consider the other contribution from the diagram with the ϕ's interchanged. Thus

$$\boxed{\mathcal{M}(A \to \phi\phi) = -2i\mu}$$

(b) For $A \to \phi\phi$ the Golden Rule for decays will give:

$$d\Gamma_A = \frac{|\mathcal{M}|^2}{32\pi^2 M} \frac{\delta^4(P_1 - P_2 - P_3)}{\sqrt{\vec{P}_2^{\,2} + m^2}\sqrt{\vec{P}_3^{\,2} + m^2}} \, d^3\vec{P}_2 \, d^3\vec{P}_3 ,$$

where $P_A = P_1$, $P_{\phi_1} = P_2$, $P_{\phi_2} = P_3$, we will work in the frame of reference where A is at rest : $\vec{P}_A = 0$, $P_A^0 = M$

Thus

$$\delta^4(P_1-P_2-P_3) = \delta(P_1^0-P_2^0-P_3^0)\,\delta^{(3)}(\vec{P}_1-\vec{P}_2-\vec{P}_3)$$

$$= \delta\left(M-\sqrt{\vec{P}_2^{\,2}+m^2}-\sqrt{\vec{P}_3^{\,2}+m^2}\right)\delta^{(3)}(\vec{P}_2+\vec{P}_3)$$

Then

$$\Gamma_A = \frac{|M|^2}{32\pi^2 M}\int \frac{\delta\left(M-\sqrt{\vec{P}_2^{\,2}+m^2}-\sqrt{\vec{P}_3^{\,2}+m^2}\right)\delta^{(3)}(\vec{P}_2+\vec{P}_3)\,d^3\vec{P}_2\,d^3\vec{P}_3}{\sqrt{\vec{P}_2^{\,2}+m^2}\sqrt{\vec{P}_3^{\,2}+m^2}}$$

$$= \frac{|M|^2}{32\pi^2 M}\int \frac{\delta\left(M-2\sqrt{\vec{P}_2^{\,2}+m^2}\right)}{\vec{P}_2^{\,2}+m^2}\,d^3\vec{P}_2$$

Using spherical coordinates : $d^3\vec{P}_2 \rightarrow r^2\,\mathrm{sen}(\theta)\,d\theta\,d\phi\,dr$, $\vec{P}_2^{\,2} = r^2$

$$\Gamma_A = \frac{|M|^2\cdot 4\pi}{32\pi^2 M}\int_0^\infty \frac{\delta\left(M-2\sqrt{r^2+m^2}\right)}{r^2+m^2}\,r^2\,dr$$

Performing the change of variable: $u = 2\sqrt{r^2+m^2} \rightarrow \frac{du}{dr} = \frac{4r}{u}$, we get

$$\Gamma_A = \frac{|M|^2}{16\pi M}\int_{2m}^\infty \delta(M-u)\,\frac{r}{u}\,du$$

this integral sends u to M (remember that $M>2m$) , so that $|\vec{P}_2| = r = \sqrt{\frac{M^2}{4}-m^2}$, and replacing $|M|^2 = 4\mu^2$, we get :

$$\Gamma(A \to \phi\phi) = \frac{\mu^2 |\vec{p}_2|}{4\pi M^2} , \quad \text{with } |\vec{p}_2| = \sqrt{\frac{M^2}{4} - m^2}$$

$\times \frac{1}{2}$ partículas idénticas!

P2) Scattering:

(a) For $\phi(p_1)\,\phi(p_2) \to \phi(p_3)\,\phi(p_4)$

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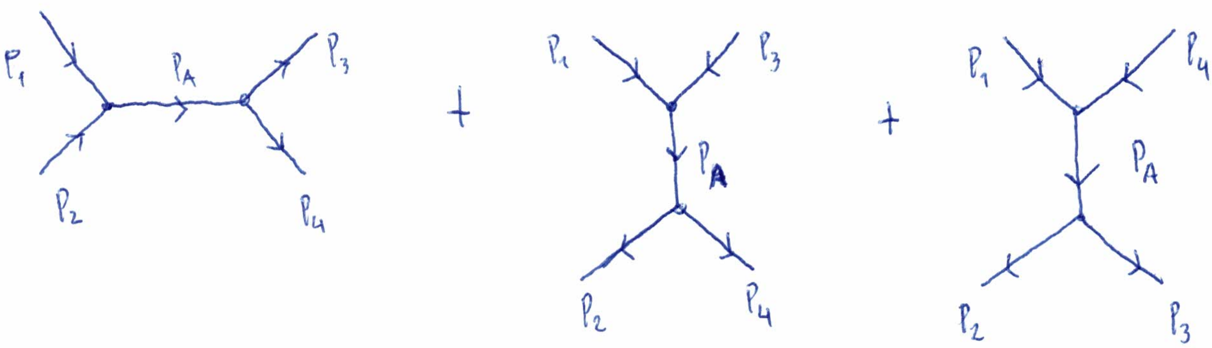
we have the following Feynman diagrams:



(b) Using the Feynman rules we get:

2!!

$$\mathcal{M} = \frac{-i\mu^2}{(p_3+p_4)^2 - M^2} - \frac{i\mu^2}{(p_2+p_4)^2 - M^2} - \frac{i\mu^2}{(p_2+p_3)^2 - M^2}$$

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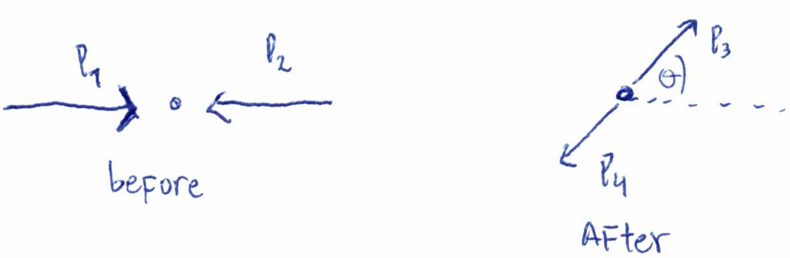
(c) We will compute the scattering cross section in the C.M frame:



In class, we deduced that the expression for the differential cross section for the special case in which all the masses are equal, is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|M|^2}{64\pi^2 E_{CM}^2}$$

where E_{CM} is the total initial energy.

Now notice that as $\vec{P}_1 + \vec{P}_2 = 0$ in C.M. we have also that $\vec{P}_3 + \vec{P}_4 = 0$, thus:

$$\left.\begin{aligned} \vec{P}_1 + \vec{P}_2 = 0 \quad / \cdot \vec{P}_1 \quad &\Rightarrow \quad \vec{P}_1^{\,2} + \vec{P}_1 \cdot \vec{P}_2 = 0 \\ / \cdot \vec{P}_2 \quad &\Rightarrow \quad \vec{P}_1 \cdot \vec{P}_2 + \vec{P}_2^{\,2} = 0 \end{aligned}\right\} \quad |\vec{P}_1| = |\vec{P}_2| \equiv |\vec{P}|$$

Similarly: $|\vec{P}_3| = |\vec{P}_4| \equiv |\tilde{\vec{P}}|$,

besides from the conservation of energy it is straightfoward to notice that $|\vec{P}| = |\tilde{\vec{P}}|$. As $m_1 = m_2 = m_3 = m_4$, we conclude that

$E_1 = E_2 = E_3 = E_4 \equiv E$.

Thus

$$E_{CM} = E_1 + E_2 = 2E$$

$$= 2\sqrt{|\vec{P}|^2 + m^2}.$$

Then

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 \cdot 4(|\vec{p}|^2+m^2)}$$

On the other hand we have

$$|\mathcal{M}|^2 = \mu^4 \left\{ \frac{1}{(P_3+P_4)^2 - M^2} + \frac{1}{(P_2+P_4)^2 - M^2} + \frac{1}{(P_2+P_3)^2 - M^2} \right\}^2$$

with

$$(P_3+P_4)^2 = P_3^2 + P_4^2 + 2P_3P_4 = 2m^2 + 2(E_3E_4 - \vec{P}_3\cdot\vec{P}_4)$$

$$= 2m^2 + 2(E^2 + |\vec{p}|^2)$$

$$= 2m^2 + 2(m^2 + 2|\vec{p}|^2),$$

$$(P_2+P_4)^2 = P_2^2 + P_4^2 + 2P_2\cdot P_4 = 2m^2 + 2(E^2 - \vec{P}_2\cdot\vec{P}_4)$$

$$= 2m^2 + 2(|\vec{p}|^2 + m^2 + \vec{P}_2\cdot\vec{P}_3)$$

$$= 2m^2 + 2(|\vec{p}|^2 + m^2 + |\vec{p}|^2\cos(\theta)),$$

$$(P_2+P_3)^2 = 2m^2 + 2(E^2 - \vec{P}_2\cdot\vec{P}_3)$$

$$= 4m^2 + 2|\vec{p}|^2(1-\cos(\theta))$$

replacing all this, we get:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\mu^4}{256\pi^2(|\vec{p}|^2+m^2)}\left\{\frac{1}{4(m^2+|\vec{p}|^2)-M^2}+\frac{1}{4m^2+2|\vec{p}|^2(1+\cos(\theta))-M^2}\right.$$

$$\left.+\frac{1}{4m^2+2|\vec{p}|^2(1-\cos(\theta))-M^2}\right\}^2$$

which is quite difficult to evaluate !

(d) In such a case we will have that $|\mathcal{M}|^2$ is proporcional to $\frac{\mu^4}{M^4}$ and as it is independent of any external momentum, we will be able to evaluate the total cross section because the angular dependence of $\mathcal{M}$ will vanish.

In this case we will obtain:

$$\sigma = \left(\frac{\mu}{M}\right)^4 \cdot \frac{9}{64\pi(|\vec{p}|^2+m^2)}$$

Teoria vira $\lambda\phi^4$!