Quantum Field Theory I - Prof. Gustavo Burdman

Homework 1

Due 20/03/20

1. **Relativity Gymnastics**

   (a) Show that the invariance of the interval between two events $(t^2 - |\vec{x}|^2)$ under Lorentz transformations is satisfied if we define the position four-vector $x^\mu \equiv (t;\vec{x})$ such that

   $$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu}$$

   where we defined the Lorentz transformations as Λ^{μ}_{ν}. The inverse Lorentz transformation is defined by $x^{\nu} = \Lambda^{\nu}_{\mu}\, x'^{\mu}$, and it satisfies

   $$\Lambda^{\mu}_{\nu}\, \Lambda^{\nu}_{\rho} = \delta^{\mu}_{\rho}$$

   (b) The Minkowski space metric is given by

   $$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

   i. Show that the interval is given by the position four-vector squared defined by $x.x \equiv x^\mu g_{\mu\nu} x^\nu$.

   ii. A contra-variant four-vector is denoted as $v^\mu = (t;\vec{v})$ Show that if we define a covariant four-vector as $v_\mu \equiv (t;-\vec{v})$, we ca write $v.v = v^\mu v_\mu$.

   iii. Verify that $v_\mu = g_{\mu\nu}\, v^\nu$, and $v^\mu = g^{\mu\nu}\, v_\nu$.

   (c) Show that the relativistic dispersion relation $E^2 - |\vec{p}|^2 = m^2$, where m is the particle's mass, is consistent with defining the momentum four-vector as $P^\mu = (E;\vec{p})$.

(d) Define the differential operators

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu \equiv (\frac{\partial}{\partial t}; \vec{\nabla}) \qquad\qquad \frac{\partial}{\partial x_\mu} \equiv \partial^\mu \equiv (\frac{\partial}{\partial t}; -\vec{\nabla})$$

i. Construct the Klein-Gordon operator using the identifications

$$E \to i\frac{\partial}{\partial t} \qquad\qquad \vec{p} \to -i\vec{\nabla}$$

ii. Show that if we define the four-vector current $J^\mu \equiv (\rho; \vec{j})$, where ρ and $\vec{j}$ the charge density and current respectively, the continuity equation in electrodynamics can be written as the conservation of the four-current as

$$\partial_\mu J^\mu = 0$$

2. **About those Delta Functions**
Using the defining property of the Delta function

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)$$

for some function $f(x)$, and that $\delta(-x) = \delta(x)$

(a) Show that

$$\int_{-\infty}^{+\infty} f(x)\delta(a\,x)\,dx = \frac{1}{|a|}\,f(0)$$

for an arbitrary constant a.

(b) Using all these show that for a function $g(x)$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(g(x))\,dx = \sum_a \frac{1}{|g'(a)|}\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx\ ,$$

where a are the zeroes of $g(x)$ (i.e. $g(a) = 0, \forall a$) while $g'(a) \neq 0$.

(c) Now verify that for $\omega_p = \sqrt{(\vec{p})^2 + m^2} > 0$ we have

$$\int d^4p\,\delta(P^2 - m^2)\,f(P) = \int d^3p \int dp_0\,\delta(p_0^2 - (\vec{p})^2 - m^2)\,f(P) = \frac{1}{2\omega_p}\int d^3p\,f(\omega_p, \vec{p})\ ,$$

where you need to use the fact that since the four-momentum is always time-like, the sign of p_0 is Lorentz invariant. Thus we need only integrate p_0 from 0 to ∞, which in turn means we only need the positive root.

3. **Electromagnetism**:
The covariant form of the Lagrangian density for the electromagnetic field in the absence of sources is

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \ ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor defined in terms of the potential 4-vector A_μ.

(a) Derive the equations of motion for $A_\mu(x)$ assuming it is the dynamical field, using Euler-Lagrange. Show that these are equivalent to Maxwell's equations in the vacuum. (It is useful to remember that $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.)

(b) If we now consider sources in the form of a current j_μ, what is the form of the interaction term between A_μ and j_μ we must add to $\mathcal{L}$ in order to obtain Maxwell's equations in the presence of the source $j_\mu = (\rho, \vec{j})$?

(c) The Lagrangian in the absence of sources is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \ ,$$

with $\alpha(x)$ an arbitrary function. What is the condition that j_μ must fulfill for the Lagrangian to remain gauge invariant in the presence of sources ?

QFT I – Homework 1

Víctor Muñoz

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P1) (a) the invariance of the interval is proved by showing that $s'^2 = x'^2 = x^2$.

$$x'^2 = x'^\mu x'_\mu = \Lambda^\mu{}_\nu x^\nu g_{\mu\alpha} \Lambda^\alpha{}_\beta x^\beta$$

$$= \Lambda^\mu{}_\nu g_{\mu\alpha} \Lambda^\alpha{}_\beta x^\nu x^\beta$$

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but $\Lambda^\mu{}_\nu g_{\mu\alpha} \Lambda^\alpha{}_\beta = g_{\nu\beta}$ $\boxed{(*)}$

$\Rightarrow$ $x'^2 = g_{\nu\beta} x^\nu x^\beta = x^2$ so that $s'^2 = s^2$.

let us verify $\boxed{(*)}$ in the simplest case : $\nu = \beta = 0$, then it should be true that

$$\Lambda^\mu{}_0 g_{\mu\alpha} \Lambda^\alpha{}_0 = 1$$

As the only non null component of $g_{\mu\nu}$ are $g_{\lambda\lambda}$ with $\lambda = 0,1,2,3$ we have:

$$\Lambda^0{}_0 g_{00} \Lambda^0{}_0 + \cdots + \Lambda^3{}_0 g_{33} \Lambda^3{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$$

in the case of a boost in the x direction: $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

with $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\beta = \dfrac{v}{c}$

So we have: $(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 = \gamma^2 - \beta^2\gamma^2$

$$= \gamma^2(1-\beta^2)$$

$$= \frac{(1-\beta^2)}{(1-\beta^2)}$$

$$= 1. \quad \text{As expected.}$$

The other terms are calculated in similar way.

(b) i. $x \cdot x = x^\mu g_{\mu\nu} x^\nu$

$$= x^0 g_{0\nu} x^\nu + x^1 g_{1\nu} x^\nu + x^2 g_{2\nu} x^\nu + x^3 g_{3\nu} x^\nu$$

The only non null components of $g_{\mu\nu}$ are $g_{\lambda\lambda}$, $\lambda = 0,1,2,3$

$$= x^0 g_{00} x^0 + x^1 g_{11} x^\nu + x^2 g_{22} x^2 + x^3 g_{33} x^3$$

$$= x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$$

which is precisely s^2 ✓

ii. $v^\mu v_\mu = v^0 v_0 + v^1 v_1 + v^2 v_2 + v^3 v_3$

$$= t^2 + (v_x)(-v_x) + (v_y)(-v_y) + (v_z)(-v_z)$$

$$= t^2 - \vec{v}^2$$

$$= v \cdot v$$

iii. $v_\mu = g_{\mu\nu} v^\nu$, so

$$v_0 = g_{0\nu} v^\nu = t$$

$$v_1 = g_{1\nu} v^\nu = -v_x$$

$$v_2 = g_{2\nu} v^\nu = -v_y$$

$$v_3 = g_{3\nu} v^\nu = -v_z$$

similarly $v^\mu = g^{\mu\nu} v_\nu$, so

$$v^0 = g^{0\nu} v_\nu = t \qquad v^2 = g^{2\nu} v_\nu = -v_y$$

$$v^1 = g^{1\nu} v_\nu = v_x \qquad v^3 = g^{3\nu} v_\nu = -v_z$$

(c) $p^\mu = (E; \vec{p})$, so $p^\mu p_\mu = P \cdot P = E^2 - \vec{p}^{\,2}$

on the other hand the 4-momentum is defined by $p^\mu = m U^\mu$

where U^μ is the 4-velocity such that $U^2 = 1$.

Then $p^2 = m^2$ and matching these two results we have:

$E^2 - \vec{p}^{\,2} = m^2$ which gives the correct dispersion relation.

(d) From the relativistic dispersion relation: $E^2 - \vec{p}^{\,2} - m^2 = 0$

and using the identifications: $E \to i\partial_t$ and $\vec{p} \to -i\vec{\nabla}$

we have: $(i\partial_t)(i\partial_t) - (-i\vec{\nabla})(-i\vec{\nabla}) - m^2 = -\partial_t^2 + \vec{\nabla}^2 - m^2$

which is the Klein-Gordon operator.

P2 (a) Consider $\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx$, making the change of variable:

$x = ay$, with a some constant.

$dx = a\,dy$

we have: $\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{+\infty} f(y/a)\,\delta(y)\,dy \cdot \frac{1}{|a|}$

$= \frac{1}{|a|} f(0)$

$= \frac{1}{|a|} \int_{-\infty}^{+\infty} f(y)\,\delta(y)\,dy$

Note: the absolute value over "a" stands because if "a" is minor than zero, then the limits of integration are reversed providing a global minus sign in the equation.

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(b) From the previous answer we have:

$\delta(ax) = \frac{1}{|a|}\delta(x)$. We will conclude that this is just a

Porque? $g(x) \cong g(a) + (x-x_0)g'(a) + \cdots$

special case from $\delta(g(x)) = \frac{\delta(x-x_0)}{|g'(x_0)|}$, with x_0 satisfiyin $g(x_0)=0$

For $\int_{-\infty}^{+\infty} f(x)\,\delta(g(x))\,dx$, let $y = g(x)$, $dy = g'(x)\,dx$

$x = g^{-1}(y)$, $a = f(x \to -\infty)$, $b = f(x \to +\infty)$

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)\,\delta(g(x)) = \int_a^b f(g^{-1}(y))\,\delta(y)\,\frac{dy}{|g'(g^{-1}(y))|}$$

$$= \frac{f(g^{-1}(0))}{|g'(g^{-1}(0))|} = \frac{f(x_0)}{|g'(x_0)|}$$

We are assuming that the condition $y=0$ is satisfied with $x = x_0$ for which $g(x_0)=0$.

The last result can be obtained from:

$$\int_{-\infty}^{+\infty} f(x)\,\frac{\delta(x-x_0)}{|g'(x)|}\,dx = \frac{f(x_0)}{|g'(x_0)|}$$

So that by comparison we have: $\delta(g(x)) = \frac{1}{|g'(x_0)|}\delta(x-x_0)$, with $g(x_0)=0$

And the obvious generalization when $g(x)$ possess multiple (simple) zeros is :

$$\delta(g(x)) = \sum_{x_i} \frac{1}{|g'(x_i)|} \delta(x-x_i) \quad , \quad g(x_i)=0$$

so that

$$\int_{-\infty}^{+\infty} f(x)\, \delta(g(x))\, dx = \sum_{x_i} \frac{1}{|g'(x_i)|} \int_{-\infty}^{+\infty} f(x)\, \delta(x-x_i)\, dx$$

(c)

$$\int d^4P\, \delta(P^2-m^2)\, f(P) = \int d^3p \int dP_0\, \delta(P_0^2-\vec{p}^2-m^2)\, f(P)$$

here $g(P_0) = P_0^2 - \vec{p}^2 - m^2 = P_0^2 - \omega_p^2$ here $g(P_0)$ has two zeros at $P_0 = \pm \omega_p$, and $g'(P_0) = 2P_0$. We are interested only in the case where $\omega_p > 0$, so we have:

$$\int d^4P\, \delta(P^2-m^2)\, f(P) = \int d^3p \int dP_0\, \frac{\delta(P_0-\omega_p)}{2\omega_p}\, f(P_0,\vec{p})$$

$$= \int d^3p\, \frac{f(\omega_p,\vec{p})}{2\omega_p}$$

P3] (a) First notice that :

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$$F^{\mu\nu}F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) - \partial_\nu A_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) - \partial_\mu A_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu)$$

$$= 2\, \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

Therefore $\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu)$

The E-L equations are $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right)$

with $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = 0$ and $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}\partial_\mu A_\nu \times$

$$\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

using that $\partial^\mu A^\nu = \partial_\mu A_\nu \eta^{\mu\mu}\eta^{\nu\nu}$ and $\partial^\nu A^\mu = \partial_\mu A_\nu \eta^{\mu\nu}\eta^{\nu\mu}$

we get

$\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = F^{\mu\nu}$, so that

$$\partial_\mu F^{\mu\nu} = 0$$

hence

$$\partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} = 0$$

As $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$ (we are using $c=1$)

we have For $\nu=0$: $\boxed{\vec{\nabla}\cdot\vec{E}=0}$

For $\nu=1$: $\partial_t E_x = \partial_z B_y - \partial_y B_z$

For $\nu=2$: $\partial_t E_y = \partial_x B_z - \partial_z B_x$

For $\nu=3$: $\partial_t E_z = \partial_x B_y - \partial_y B_x$

Adding the last three equations, we get : $\boxed{\vec{\nabla}\times\vec{B} = \partial_t\vec{E}}$

* Note: The other Maxwell's equations are obtained from the dual tensor : $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ whose divergence is null : $\partial_\mu \tilde{F}^{\mu\nu} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B}=0$ and $\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$.

(b) It's easy to see that if we add the therm

$A_\nu J^\nu$ to the previous lagrangian, then we will have

$\frac{\partial\mathcal{L}}{\partial A_\nu} = J^\nu$ and then we will have $\partial_\mu F^{\mu\nu} = J^\nu$

which are the Maxwell's equations in the presence of a source.

(c) The action : $S = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\nu J^\nu\right)$ is gauge invariant if and only if J^ν is a conserved current : $\partial_\nu J^\nu = 0$.

Because for $A_\nu \to A_\nu(x) + \partial_\nu\alpha(x)$ we have :

$$S' = S - \int d^4x \, \partial_\nu \alpha(x) \, J^\nu(x)$$ if $\alpha(x) \to 0$ when $x \to \pm\infty$

then by the Stokes' theorem we have : $\int d^4x \, \partial_\mu (\alpha J^\nu) = 0$

$$\Rightarrow \int d^4x \, \partial_\nu \alpha(x) \, J^\nu = - \int d^4x \, \partial_\nu J^\nu \alpha(x)$$. So that

$$S' = S + \int d^4x \, \alpha(x) \, \partial_\nu J^\nu$$ and then $S' = S \Leftrightarrow \partial_\nu J^\nu = 0$