PGF5107 - Quantum Field Theory I – Prof. gustavo Burdman

Homework 2

Due 29/03/2016

1. **Complex Scalar Field**:
Consider a complex scalar field with the action given by:

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right)$$

Given that $\phi(x)$ e $\phi^*(x)$ are independent dynamical degrees of freedom:

a) Derive the equations of motion for $\phi(x)$ e $\phi(x)^*$.

b) Find the conjugate momenta of $\phi(x)$ e $\phi^*(x)$. Show that the Hamiltonian is given by:

$$H = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right)$$

c) Expand the fields and their conjugate momenta in momentum space, in the most general way consistent with the equations of motion in terms of creation and annihilation operators. Impose appropriate commutation rules.

d) Write the Hamiltonian H in terms of creation and annihilation operators. Show that there are two types of particles with the same mass m.

2. **Feynman Propagator**:

(a) Show that the Feynman propagator for a real scalar field $D_F(x-y)$ is a Green function of the Klein-Gordon operator, i.e.

$$(\partial^2 + m^2) D_F(x - y) = -i\delta^{(4)}(x-y) \ ,$$

(b) Derive the form of the momentum space Feynman propagator for the complex scalar field computing $\langle 0 | T\phi(x)\phi^*(y) | 0 \rangle$.

Homework 2, Q.F.T. I

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1. Complex Scalar Field

(a) $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$

From $\dfrac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$ And given that $\phi(x)$ and $\phi^*(x)$ are independent, we have:

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$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^*$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \quad , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi$$

So both fields obeys K-G equation: $(\partial^2 + m^2)\phi^*(x) = 0$

$(\partial^2 + m^2)\phi(x) = 0$

(b) $\mathcal{L} = \dot{\phi}^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi$

And $\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* = \Pi \quad , \quad \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} = \Pi^*$

Thus $H = \displaystyle\int d^3x \left\{ \Pi \dot{\phi} + \Pi^* \dot{\phi}^* - \mathcal{L} \right\}$

$$= \int d^3x \left\{ \Pi^* \Pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right\}$$

(c) As $\phi(x)$ and $\phi^*(x)$ satisfies K-G equation, we have that the most general solution is:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} N_p \left\{ a_p e^{-ip\cdot x} + b_p^+ e^{ip\cdot x} \right\}$$

$$\phi^*(x) = \int \frac{d^3p'}{(2\pi)^3} N_{p'} \left\{ b_{p'} e^{-ip'\cdot x} + a_{p'}^+ e^{ip'\cdot x} \right\}$$

As $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*$ and $\Pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}$, then

$$\Pi = \int \frac{d^3p'}{(2\pi)^3} N_{p'} \left\{ -i\omega_{p'} b_{p'} e^{-ip'\cdot x} + i\omega_{p'} a_{p'}^+ e^{ip'\cdot x} \right\}$$

Keeping in mind that $p\cdot x = \omega_p t - \vec{p}\cdot\vec{x}$ and $p'\cdot x = \omega_{p'} t - \vec{p}'\cdot\vec{x}$ with $\omega_p^2 = \vec{p}^2 + m^2$, $\omega_{p'}^2 = \vec{p}'^2 + m^2$.

Following the scheme of K-G field theory, we impose:

$$[\phi(x), \phi^*(y)] = [\phi(x), \phi(y)] = [\phi^*(x), \phi^*(y)] = 0$$

$$[\Pi(x), \Pi^+(y)] = [\Pi(x), \Pi(y)] = [\Pi^*(x), \Pi^*(y)] = 0$$

$$[\phi(x), \Pi^*(y)] = [\phi^*(x), \Pi(y)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

but

$$[\phi(x), \Pi^*(y)] = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} N_p N_{p'} \Big\{ -i\omega_{p'} e^{-i(p+p')x} [a_p, b_{p'}] + i\omega_{p'} e^{-i(p-p')x} [a_p, a_{p'}^+] - i\omega_{p'} e^{i(p-p')x} [b_p^+, b_{p'}] + i\omega_{p'} e^{i(p+p')x} [b_p^+, a_{p'}^+] \Big\}$$

So now we impose : $[a_p , a_{p'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') = [b_p , b_{p'}^+]$ ②

And $[a_p, b_{p'}] = [b_p^+, a_p^+] = 0$

Thus

$$[\phi(x), \Pi^*(y)] = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} N_p N_{p'} \Big\{ i\omega_{p'} e^{-i(p-p')x} (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') + i\omega_{p'} e^{i(p-p')x} (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \, i \, N_p^2 \, \omega_p \Big\{ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \Big\}$$

we choose $\omega_p N_p^2 = 1/2 \Rightarrow N_p = 1/\sqrt{2\omega_p}$

And we notice that $\delta^{(3)}(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} e^{\pm i\vec{p}\cdot(\vec{x}-\vec{y})}$

So

$[\phi(x), \Pi^*(y)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$, Allways that

$[a_p, a_{p'}^+] = [b_p , b_{p'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')$

(d) We have $H = \int d^3x \left\{ \Pi^*\Pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^*\phi \right\}$

with

$$\int d^3x \, \Pi^*\Pi = \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \sqrt{\frac{\omega_p\omega_{p'}}{4}} \Big\{ -e^{-i(p+p')\cdot x} a_p b_{p'} + e^{-i(p-p')\cdot x} a_p a_{p'}^+ + e^{i(p-p')x} b_p^+ b_{p'} - e^{i(p+p')\cdot x} b_p^+ a_{p'}^+ \Big\}$$

but

$$\int \frac{d^3x}{(2\pi)^3} e^{-i(p+p')\cdot x} = \int \frac{d^3x}{(2\pi)^3} e^{-i(p_0+p_0')x^0} e^{i(\vec{p}+\vec{p}')\cdot\vec{x}}$$

$$= e^{-i(p_0+p_0')x^0} \delta^{(3)}(\vec{p}+\vec{p}')$$

Thus

$$\int d^3x\, \Pi^* \Pi = \int \frac{d^3p}{(2\pi)^3} \int d^3p' \sqrt{\frac{W_p W_{p'}}{4}} \Big\{ -e^{-i(p_0+p_0')x^0} \delta^{(3)}(\vec{p}+\vec{p}')\, a_p b_{p'} +$$

$$e^{-i(p_0-p_0')x^0} \delta^{(3)}(\vec{p}-\vec{p}')\, a_p a_{p'}^+ + e^{i(p_0-p_0')x^0} \delta^{(3)}(\vec{p}-\vec{p}')\, b_p^+ b_{p'} - e^{i(p_0+p_0')x^0} \delta^{(3)}(\vec{p}+\vec{p}') \times$$

$b_p^+ a_{p'}^+ \Big\}$. Now we perform the integral in p', noting that for $\vec{p} = \pm\vec{p}'$, $p_0 = p_0'$:

$$\int d^3x\, \Pi^* \Pi = \int d^3p\, \frac{W_p}{2} \Big\{ -e^{-2ip_0x^0} a_p b_{-p} + a_p a_p^+ + b_p^+ b_p - e^{2ip_0x^0} b_p^+ a_{-p}^+ \Big\}$$

Similarly for $\int \nabla\phi^* \cdot \nabla\phi\, d^3x$, we have

$$\nabla\phi^* = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2W_{p'}}} \Big\{ i\vec{p}' e^{-ip'\cdot x} b_{p'} - i\vec{p}' e^{ip'\cdot x} a_{p'}^+ \Big\}$$

$$\nabla\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2W_p}} \Big\{ i\vec{p} e^{-ip\cdot x} a_p - i\vec{p} e^{ip\cdot x} b_p^+ \Big\}$$

$$\int \nabla\phi^* \cdot \nabla\phi\, d^3x = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int d^3x \frac{1}{\sqrt{4W_pW_{p'}}} \Big\{ -\vec{p}\cdot\vec{p}' e^{-i(p+p')\cdot x} b_{p'} a_p + \vec{p}\cdot\vec{p}' e^{i(p-p')\cdot x} \times$$

$$b_{p'} b_p^+ + \vec{p}\cdot\vec{p}' e^{-i(p-p')\cdot x} a_{p'}^+ a_p - \vec{p}\cdot\vec{p}' e^{i(p+p')\cdot x} a_{p'}^+ b_p^+ \Big\}$$

Again we use

$$\int \frac{d^3x}{(2\pi)^3} e^{\mp i(p\pm p')\cdot x} = e^{\mp i(p_0 \pm p_0')x^0} \delta^{(3)}(\vec{p}\pm\vec{p}')$$

to obtain

$$\int \nabla\phi^* \cdot \nabla\phi \, d^3x = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{\sqrt{4w_p w_{p'}}} (\vec{p}\cdot\vec{p}') \Big\{ -e^{-i(p_0+p_0')x^0} \delta^{(3)}(\vec{p}+\vec{p}') b_{p'} a_p +$$

$$e^{i(p_0-p_0')x^0} \delta^{(3)}(\vec{p}-\vec{p}') b_{p'} b_p^+ + e^{-i(p_0-p_0')x^0} \delta^{(3)}(\vec{p}-\vec{p}') a_{p'}^+ a_p$$

$$- e^{i(p_0+p_0')x^0} \delta^{(3)}(\vec{p}+\vec{p}') a_{p'}^+ b_p^+ \Big\}$$

$$= \int \frac{d^3p}{2w_p} \vec{p}^2 \Big\{ + e^{-2ip_0x^0} b_{-p} a_p + b_p b_p^+ + a_p^+ a_p + e^{2ip_0x^0} a_{-p}^+ b_p^+$$

And Finally

$$\int d^3x \, \phi^* \phi = \int \frac{d^3p}{2w_p} \Big\{ e^{-2ip_0x^0} b_{-p} a_p + b_p b_p^+ + a_p^+ a_p + e^{2ip_0x^0} a_{-p}^+ b_p^+ \Big\}$$

replacing all this in the expression of the hamiltonian :

$$H = \int \frac{d^3p}{2} w_p \Big\{ - e^{-2ip_0x^0} a_p b_{-p} + a_p a_p^+ + b_p^+ b_p - e^{2ip_0x^0} b_p^+ a_{-p}^+$$

$$+ \left(\frac{\vec{p}^2 + m^2}{w_p^2}\right)\left(e^{-2ip_0x^0} b_{-p} a_p + b_p b_p^+ + a_p^+ a_p + e^{2ip_0x^0} a_{-p}^+ b_p^+\right) \Big\}$$

now we use that $[b_{-p}, a_p] = [a_{-p}^+, b_p^+] = 0$

We get:

$$H = \frac{1}{2}\int d^3p\, w_p \left(a_p a_p^\dagger + a_p^\dagger a_p + b_p b_p^\dagger + b_p^\dagger b_p \right)$$

If we now use the commutation rule and the definition of the number operator $N = a^\dagger a$, we get

$$H = \int d^3p\, w_p \left(N_p^{(a)} + N_p^{(b)} \right) + (2\pi)^3 \int d^3p\, \delta^{(3)}(0)$$

We can conclude that there are two kind of particles with the same mass m.

2. Feynman Propagator

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(a) $D_F(x-y) = \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$

$$(\partial^2+m^2)D_F(x-y) = \partial^2\theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + 2\partial_\mu\theta(x^0-y^0)\partial^\mu\langle 0|\phi(x)\phi(y)|0\rangle$$
$$+ \theta(x^0-y^0)(\partial^2+m^2)\langle 0|\phi(x)\phi(y)|0\rangle - \partial^2\theta(x^0-y^0)\langle 0|\phi(y)\phi(x)|0\rangle$$
$$- 2\partial_\mu\theta(x^0-y^0)\partial^\mu\langle 0|\phi(y)\phi(x)|0\rangle + \theta(x^0-y^0)(\partial^2+m^2)\langle 0|\phi(y)\phi(x)|0\rangle$$

As $\partial_0\theta(x^0-y^0) = \delta(x^0-y^0)$, $(\partial^2+m^2)\phi = 0$

And $f(x)\delta'(x) = -f'(x)\delta(x)$, we get

$$(\partial^2+m^2)D_F(x-y) = -\delta(x^0-y^0)\langle 0|\pi(x)\phi(y)|0\rangle + 2\delta(x^0-y^0)\langle 0|\pi(x)\phi(y)|0\rangle$$
$$+\delta(x^0-y^0)\langle 0|\phi(y)\pi(x)|0\rangle - 2\delta(x^0-y^0)\langle 0|\phi(y)\pi(x)|0\rangle$$

$$(\partial^2+m^2)D_F(x-y) = \delta(x^0-y^0)\left\{\langle 0|\pi(x)\,\phi(y)|0\rangle - \langle 0|\phi(y)\,\pi(x)|0\rangle\right\}$$

As $[\pi(x), \phi(y)] = -i\,\delta^{(3)}(\vec{x}-\vec{y})$

we get

$$(\partial^2+m^2)D_F(x-y) = -i\,\delta^{(4)}(x-y)$$

(b) $$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\omega_p 2}}\left\{a_p e^{-ipx} + b_p^+ e^{ipx}\right\}$$

$$\phi^*(y) = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{p'}}}\left\{b_{p'} e^{-ip'y} + a_{p'}^+ e^{ip'y}\right\}$$

As the only non null product of b's and a's are:

$$\langle 0|a_p a_{p'}^+|0\rangle = \langle 0|b_p\, b_{p'}^+|0\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$$

Then $$\langle 0|\phi(x)\phi^*(y)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\langle 0|a_p a_{p'}^+|0\rangle e^{-ipx}e^{ip'y} \times \frac{1}{\sqrt{2\omega_p}}\cdot\frac{1}{\sqrt{2\omega_{p'}}}$$

$$= \int\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2\omega_p}e^{-i(x-y)\cdot p}$$

Similarly we get

$$\langle 0|\phi^*(y)\phi(x)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2\omega_p}e^{-i(y-x)\cdot p}$$

So the Feynman propagator is given by:

$$\langle 0| T\, \phi(x)\, \phi^*(y) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2\omega_p} \left\{ e^{-i(x-y)\cdot p}\, \theta(x^0-y^0) + e^{-i(y-x)\cdot p}\, \theta(y^0-x^0) \right\}$$

write it as

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon}\, e^{-iP\cdot(x-y)}$$

mom. space propagator