- **Ejercicio 3 (2.25 puntos)** Estudiar la existencia del límite $\lim_{x\to\infty} \frac{(n!)^{1/n}}{n}$ dando, en su caso, su valor.

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n} \quad \text{sea } x_n = \frac{n!}{n^n} \quad \text{criterio de la raíz } \left(\text{si } \exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \lambda\right)$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{(n+1)}}{n!/n^n} = \lim_{n\to\infty} \frac{n^n (n+1)!}{(n+1)^{n+1}\cdot n!} = \lim_{n\to\infty} \frac{n^n(n+1)}{(n+1)^n(n+1)} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^n =$$

$$= \lim_{n\to\infty} \frac{1^n}{(1+\frac{1}{n})^n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e} \;\square$$

$$\underbrace{\lim_{n\to+\infty} \frac{\log(1+2^n)}{n}}_{(1)} > \underbrace{\lim_{n\to+\infty} \frac{3+3^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+3^{\frac{1}{n}}}{n}}_{(2)}$$

Criterio de Stolz $\to \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$

Criterio media aritm $\to \lim_{n\to\infty} \frac{x_1+x_2+\dots+x_n}{n} = x$ si $\lim_{x\to\infty} x_n = x$

$$\lim_{n\to\infty} \frac{\log(1+2^n) - \log(1+2^{n-1})}{n-n-1} = \lim_{n\to\infty} \log(1+2^n) - \log(1+2^{n-1}) = \lim_{n\to\infty} \log\left(\frac{1+2^n}{1+2^{n-1}}\right) = \lim_{n\to\infty} \log\left(\frac{1+2^n}{\frac{1+2^n}{2}}\right) = \log 2$$

$$\lim_{n\to\infty} \frac{3+3^{1/2}+\dots+3^{1/n}}{n} \quad \text{sea } x_n = 3^{1/n}.\; \lim_{n\to\infty} 3^{1/n} = 1 \Rightarrow \lim_{n\to\infty} \frac{3+3^{1/2}+\dots+3^{1/n}}{n} = 1$$

como $\log 2 < 1$ la afirmación es falsa

EJERCICIO 4 [**1,25 puntos**]

Sea $(x_n)_n$ una sucesión de números reales acotada por 2 y que verifica que

$$|x_{n+2} - x_{n+1}| \leq \frac{1}{8}|x_{n+1}^2 - x_n^2|, \quad \forall n \geq 1.$$

Demostrar que $(x_n)_n$ es contractiva y por lo tanto convergente.

$(x_n)_n$ contractiva $\Leftrightarrow |x_{n+2} - x_{n+1}| \leqslant |x_{n+1} - x_n|$

$x_{n+2} \leqslant 2 \quad x_{n+1} < 2 \quad x_n < 2 \dots$

$$|x_{n+2} - x_{n+1}| \leqslant \frac{1}{8}|x_{n+1}^2 - x_n^2| = \frac{1}{8}|x_{n+1}+x_n||x_{n+1}-x_n| \leqslant \frac{1}{8}(2+2)|x_{n+1}-x_n| = \frac{1}{2}|x_{n+1}-x_n| \Rightarrow$$

$$\Rightarrow |x_{n+2} - x_{n+1}| \leqslant |x_{n+1} - x_n| \Rightarrow (x_n)_n \text{ converge} \;\square$$

5. (1'5 puntos). Estudiar la continuidaden el origen, en función del parámetro real α, de la función definida por

$$f(0) = 0 \quad \text{y } f(x) = |x|^\alpha \cos(\frac{\alpha}{x}) \text{ para } x \neq 0.$$

Para que f sea continua en $0 \in \mathbb{R}$, $\lim_{x\to 0} f(x) = 0$

Por tanto calculamos el límite

$$\underline{\lim_{x\to 0} \left||x|^\alpha \cos\left(\frac{\alpha}{x}\right)\right| \leqslant \lim_{x\to 0} |x|^\alpha \cdot 1 = 0 \quad \forall \alpha > 0}, \quad \underline{\text{si } \alpha = 0 \; \lim_{x\to 0} |x|^\alpha = 1}$$

$$-\lim_{x\to 0} |x|^\alpha \cos\left(\frac{\alpha}{x}\right) \leqslant 0 \leqslant \lim_{x\to 0} |x|^\alpha \cos\left(\frac{\alpha}{x}\right) \quad \forall \alpha \in \mathbb{R}\setminus\{0\}$$

$$-\lim_{x\to 0} |x|^\alpha \cos\left(\frac{\alpha}{x}\right) \leqslant 1 \leqslant \lim_{x\to 0} |x|^\alpha \cos\left(\frac{\alpha}{x}\right) \quad \text{con } \alpha = 0$$

$$\Rightarrow \lim_{x\to 0} |x|^\alpha \cos\left(\frac{\alpha}{x}\right) = 0 \quad \forall \alpha \in (0, +\infty)$$

$$\Rightarrow \lim_{x\to 0} |x|^0 \cos\left(\frac{0}{x}\right) = 1$$

$$\forall \alpha \in \mathbb{R} < 0 \quad \lim_{x\to 0} \frac{\cos\left(\frac{\alpha}{x}\right)}{|x|^{-\alpha}} \overset{\text{L'Hôp}}{=}$$

$\Rightarrow f$ es continua $\forall \alpha \in \mathbb{R}\setminus\{0\}$

2. $\lim\limits_{n\to+\infty} \dfrac{\log(n^3+3n^2+2n+1)}{\log(n^2+3n+2)}$.

$$= \lim_{n\to\infty} \frac{(n^2+3n+2)(3n^2+6n+2)}{(n^3+3n^2+2n+1)(2n+3)} = \lim_{n\to\infty} \frac{3n^4}{2n^4} = \frac{3}{2} \quad \square$$

L'Hôp

1. $\lim\limits_{n\to+\infty} \dfrac{n+(-1)^n}{n-(-1)^n}$

$$\left| \lim_{n\to\infty} \frac{n+(-1)^n}{n-(-1)^n} - 1 \right| \leq \left| \lim_{n\to\infty} \frac{n+(-1)^n}{n-(-1)^n} - \frac{n-(-1)^n}{n-(-1)^n} \right| = \left| \lim_{n\to\infty} \frac{2(-1)^n}{n-(-1)^n} \right| =$$

$$= 2\left| \lim_{n\to\infty} \frac{1}{n-(-1)^n} \right| = 0 \implies \lim_{n\to\infty} \frac{n+(-1)^n}{n-(-1)^n} = 1 \quad \square$$

3. (1 punto). Sea $a = \lim_{n\to+\infty} x_n$. Calcular

$$\lim_{n\to+\infty} \frac{x_1+2x_2+\ldots+nx_n}{n^2}.$$

Criterio Stoltz $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n}$

$$\lim_{n\to\infty} \frac{(x_1+\dots+nx_n+(n+1)x_{n+1})-(x_1+\dots+nx_n)}{(n+1)^2-n^2} = \lim_{n\to\infty} \frac{(n+1)x_{n+1}}{2n+1} = \lim_{n\to\infty} \frac{n+1}{2n+1} \lim_{n\to\infty} x_{n+1} = \frac{1}{2}a$$

1.) $\lim\limits_{x\to a} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$

$$= \lim_{x\to a} \left(\frac{x}{x^2-a^2}\right)^{\frac{1}{2}} - \left(\frac{a}{x^2-a^2}\right)^{\frac{1}{2}} + \left(\frac{1}{x+a}\right)^{\frac{1}{2}} = \lim_{x\to a} \frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}} +$$

$$+ \lim_{x\to a} \left(\frac{1}{x+a}\right)^{\frac{1}{2}} = \lim_{x\to\infty} \frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}} + \lim_{x\to a} \frac{1}{(x+a)^{\frac{1}{2}}} = \lim_{x\to\infty} \frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}} + (2a)^{-\frac{1}{2}} =$$

$$= \lim_{x\to a} \frac{\frac{x^{-\frac{1}{2}}}{2}}{\frac{x}{(x^2-a^2)^{\frac{1}{2}}}} + (2a)^{-\frac{1}{2}} = \lim_{x\to a} \frac{(x^2-a^2)^{\frac{1}{2}}}{2x\sqrt{x}} + (2a)^{-\frac{1}{2}} = \frac{1}{\sqrt{2a}} \qquad \text{si } a\in(0,+\infty)$$

Si $a=0$ $\quad \lim\limits_{x\to 0} \dfrac{\sqrt{x}-0+\sqrt{x-0}}{\sqrt{x^2-0^2}} = \lim\limits_{x\to 0} \dfrac{2\sqrt{x}}{x} = +\infty$

2.) $\lim\limits_{x\to 0} \dfrac{(\arctan(\sqrt{x+x^2}))^2}{1-\cos(\sqrt{x^2+2x})}$.

$$= \lim_{x\to 0} \frac{x+x^2}{\frac{x^2+2x}{2}} = 1 \quad \square$$

$1-\cos x \underset{0}{\sim} \dfrac{x^2}{2}$

$\arctan x \underset{0}{\sim} x$