■ En el problema 8 del Tema 5 hemos utilizado la igualdad:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r})$$

donde δ es la delta de Dirac y $r = |\vec{r}|$. Ahora se trata de demostrarlo.

Para considerar una situación más general, vamos a demostrar:

$$\boxed{\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}\,')}$$

Para $\vec{r} \neq \vec{r}\,'$

$$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \vec{\nabla}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2} =$$

$$= -\frac{(x-x')\hat{u}_x + (y-y')\hat{u}_y + (z-z')\hat{u}_z}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} = -\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

[Que demuestra además que $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$]

Calculamos ahora:

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \vec{\nabla}\cdot\left(-\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = -\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\right) =$$

$$= -\frac{1}{|\vec{r}-\vec{r}\,'|^3}\vec{\nabla}\cdot(\vec{r}-\vec{r}\,') - (\vec{r}-\vec{r}\,')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|^3}\right) =$$

$$= -\frac{3}{|\vec{r}-\vec{r}\,'|^3} - (\vec{r}-\vec{r}\,')\frac{(-3)(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^5} =$$

$$= -\frac{3}{|\vec{r}-\vec{r}\,'|^3} + \frac{3}{|\vec{r}-\vec{r}\,'|^3} = 0$$

[Ya comenté que $\vec{\nabla}\left(\frac{1}{r^n}\right) = -\frac{n\vec{r}}{r^{n+2}}$]

Para $\vec{r} = \vec{r}\,'$ añadimos una constante ε para evitar la divergencia en el cálculo de la laplaciana:

$$\vec{\nabla}\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2]^{1/2}}\right) =$$

$$= -\frac{(x-x')\hat{u}_x + (y-y')\hat{u}_y + (z-z')\hat{u}_z}{[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2]^{3/2}}$$

y después se hará el límite $\varepsilon \to 0$. La laplaciana queda:

$$\nabla^2\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2]^{1/2}}\right) =$$

$$= \vec{\nabla}\cdot\left(-\frac{(x-x')\hat{u}_x + (y-y')\hat{u}_y + (z-z')\hat{u}_z}{[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2]^{3/2}}\right) =$$

$$= -\frac{\vec{\nabla}\cdot(\vec{r}-\vec{r}\,')}{[(\vec{r}-\vec{r}\,')^2+\varepsilon^2]^{3/2}} + \frac{3(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{[(\vec{r}-\vec{r}\,')^2+\varepsilon^2]^{5/2}} =$$

$$= -\frac{3\varepsilon^2}{[(\vec{r}-\vec{r}\,')^2+\varepsilon^2]^{5/2}}$$

Si ε es suficientemente pequeño, al hacer una integral sólo contribuirá el valor de la función en $\vec{r}$, ya que para $\vec{r}=\vec{r}\,'$ la función varía como $1/\varepsilon^3$, es decir:

$$\left(\begin{array}{l}\vec{r}=\vec{r}\,' \\ f(\vec{r})=f(\vec{r}\,')\end{array}\right)$$

$$\int f(\vec{r}\,')\,\nabla^2\left(\frac{1}{[(\vec{r}-\vec{r}\,')^2+\varepsilon^2]^{1/2}}\right)d^3r' \simeq$$

$$\simeq f(\vec{r})\int\left[-\frac{3\varepsilon^2}{[(\vec{r}-\vec{r}\,')^2+\varepsilon^2]^{5/2}}\right]d^3r' =$$

$$= -f(\vec{r})\int\frac{3\varepsilon^2}{[(\vec{r}-\vec{r}\,')^2+\varepsilon^2]^{5/2}}\,d^3r' = \downarrow$$

Hacemos el cambio de variable:

$$\vec{\rho} = \frac{\vec{r}-\vec{r}\,'}{\varepsilon}$$

y usamos coordenadas esféricas (ρ, θ, φ)

$$\downarrow = -f(\vec{r})\int\frac{\rho^2}{(\rho^2+1)^{5/2}}\,\rho^2 \operatorname{sen}\theta\, d\rho\, d\varphi = \downarrow$$

Hacemos el cambio de variable:

$$\rho = \operatorname{tg} x \longrightarrow d\rho = \frac{1}{\cos^2 x}\,dx = (1+\operatorname{tg}^2 x)\,dx$$

$$\downarrow = -f(\vec{r})\int_0^{2\pi} d\varphi \int_0^{2\pi}\operatorname{sen}\theta\, d\theta \int_0^{\infty}\frac{\rho^2}{(\rho^2+1)^{5/2}}\,d\rho =$$

$$= -12\pi f(\vec{r})\int_0^{\infty}\frac{\rho^2}{(\rho^2+1)^{5/2}}\,d\rho = \downarrow$$

Calculamos:

$$\int_0^\infty \frac{\rho^2}{(\rho^2+1)^{5/2}}\, d\rho = \int_0^{\pi/2} \frac{\operatorname{tg}^2 x}{\cos^2 x} \cos^5 x \, dx =$$

$$= \int_0^{\pi/2} \operatorname{tg}^2 x \cos^3 x \, dx = \int_0^{\pi/2} \operatorname{sen}^2 x \cos x \, dx =$$

$$= \frac{1}{3} \operatorname{sen}^3 x \Big|_0^{\pi/2} = \frac{1}{3}$$

de donde:

$$= -12\pi f(\vec{r})\, \frac{1}{3} = -4\pi f(\vec{r})$$

Como:

$$\int f(\vec{r}\,')\, \delta(\vec{r}-\vec{r}\,')\, d^3r' = f(\vec{r})$$

y hemos obtenido:

$$\int f(\vec{r}\,')\, \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r' = -4\pi f(\vec{r})$$

Entonces:

$$\boxed{\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -4\pi\, \delta(\vec{r}-\vec{r}\,')}$$