¿Por qué es necesario el adjunto de Dirac?

## La ecuación adjunta

Partimos de :

$$(\not{p} - m)\psi(x) = 0$$

$$(\gamma^\mu p_\mu - m)\psi(x) = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

Podéis demostrar lo siguiente :

$$\gamma^{i\dagger} = -\gamma^i$$
$$\gamma^{0\dagger} = \gamma^0$$

Entonces : $\left( [i\gamma^\mu \partial_\mu - m]\psi(x) \right)^\dagger =$

$$= \left[ (i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi(x) \right]^\dagger$$

$$= \partial_0\psi^\dagger (i\gamma^0)^\dagger + \partial_i\psi^\dagger (i\gamma^i)^\dagger - m\psi^\dagger$$

$$= -i\partial_0\psi^\dagger\gamma^0 + i\partial_i\psi^\dagger\gamma^i - m\psi^\dagger$$

Problema: la ecuación anterior no la podemos escribir en forma covariante.

• Multipliquemos por $\gamma^0$ por la derecha:

Queda:

$$(-i\partial_0\psi^+\gamma^0 + i\partial_i\psi^+\gamma^i - m\psi^+)\gamma^0 = 0$$

$$(-i\partial_0\bar{\Psi}\gamma^0 + i\partial_i\psi^+\underbrace{\gamma^i\gamma^0}_{-\gamma^0\gamma^i} - m\bar{\Psi}) = 0$$

$$(-i\partial_0\bar{\Psi}\gamma^0 - i\partial_i\bar{\Psi}\gamma^i - m\bar{\Psi}) = 0$$

$$\Rightarrow \boxed{i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0}$$

Ecuación de Dirac adjunta

Tenemos:

Dirac: $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$\overline{\text{Dirac}}$: $(i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi}) = 0$

$$0 = \bar{\Psi}\,(\mathrm{Dirac}) + (\overline{\mathrm{Dirac}})\,\Psi =$$

$$= \bar{\Psi}\,(i\gamma^\mu \partial_\mu \Psi) - \cancel{\bar{\Psi} m \Psi}$$

$$+ (i\partial_\mu \bar{\Psi}\gamma^\mu)\Psi + \cancel{m\bar{\Psi}\Psi}$$

$$= i\left(\bar{\Psi}\gamma^\mu \partial_\mu \Psi + \partial_\mu \bar{\Psi}\gamma^\mu \Psi\right) = 0$$

$$\Rightarrow \quad \partial_\mu (\bar{\Psi}\gamma^\mu \Psi) = 0$$

luego $\boxed{\partial_\mu j^\mu = 0}$ con $\underline{j^\mu = (\rho, j^i)}$

y $\boxed{j^\mu = \bar{\Psi}\gamma^\mu \Psi}$ es la corriente de probabilidad

( es la famosa <u>corriente vectorial</u> ).

¿ Podemos construir una <u>densidad</u>

<u>Lagrangiana</u> para el campo de Dirac?

al contrario que en el caso fermiónico, no podemos tener un análogo clásico del tipo (bosón = masa + muelle). Recordemos que los fermiones no tienen análogo clásico. Aun así, podemos postular una $\mathcal{L}$. Hagámoslo:

$$\boxed{\begin{aligned}\mathcal{L}_D &= \bar{\Psi}(i\not{\partial}-m)\Psi \\ &= \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi\end{aligned}}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = \bar{\Psi}i\gamma^\mu$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\right) = \partial_\mu\bar{\Psi}i\gamma^\mu$$

$$\frac{\partial\mathcal{L}}{\partial\Psi} = -m\bar{\Psi}$$

$$\Longrightarrow \quad i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0 \quad (\overline{\text{Dirac}})$$

De un modo similar, obtened la ecuación de Dirac (ejercicio sencillo). Sigamos avanzando.

• Momento conjugado a $\psi$:

$$\Pi^{\mu}_{\psi} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} = i\bar{\psi}\gamma^{\mu}$$

$$\Pi^{\mu}_{\bar{\psi}} = 0$$

• Densidad Hamiltoniana:

$$\mathcal{H} = \Pi^{0}_{\psi}\partial_0\psi - \mathcal{L}$$

$$= i\bar{\psi}\gamma^{0}\partial_0\psi - \bar{\psi}(i\gamma^{\mu}\partial_\mu - m)\psi$$

$$= i\psi^{+}\partial_0\psi - \psi^{+}\gamma^{0}i\gamma^{\mu}\partial_\mu\psi + m\psi^{+}\gamma^{0}\psi$$

$$= \psi^{+}(i\partial_0 - i\gamma^{0}\gamma^{\mu}\partial_\mu + m\gamma^{0})\psi$$

$$= \psi^{+}(i\partial_0 - i\gamma^{0}\gamma^{0}\partial_0 - i\gamma^{0}\gamma^{i}\partial_i + m\gamma^{0})\psi$$

$$= \psi^{+}(-i\gamma^{0}\vec{\gamma}\cdot\vec{\partial} + m\gamma^{0})\psi$$

Pero sabemos que

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$i\gamma^0 \partial_0 \psi + i\vec{\gamma}\cdot\vec{\nabla}\psi - m\psi = 0$$

$$\Rightarrow \quad (-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi = i\gamma^0 \partial_0 \psi \text{ , luego}$$

$$\mathcal{H} = \psi^\dagger (i\gamma^0 \partial_0 \psi)\gamma^0 = \psi^\dagger i \partial_0 \psi$$

$$\Longrightarrow \quad \boxed{\mathcal{H} = \psi^\dagger i \partial_0 \psi}$$

Prescripción de cuantización

Impondremos relaciones de anticonmutación a tiempos iguales:

$$\{\hat{\psi}_a(t,\vec{x}), \hat{\psi}^\dagger_b(t,\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$$\{\hat{\psi}_a(t,\vec{x}), \hat{\psi}_b(t,\vec{y})\} = \{\hat{\psi}^\dagger_a(t,\vec{x}), \hat{\psi}^\dagger_b(t,\vec{y})\} = 0$$

$(a,b)$ etiquetan las componentes del cuadriespinor $(a,b) = (1,2,3,4)$.