Espacio de Minkowski

$$x^0 \equiv ct \quad x^1 \equiv x \quad x^2 \equiv y \quad x^3 \equiv z$$

Punto P $\rightarrow x^0, x^1, x^2, x^3$

Espacio de N-dimensiones

Sistemas de referencia de dimensión N $\begin{cases} S & x^\mu \\ S' & x'^\mu \end{cases}$

Fórmulas de transformación de coordenadas

$$x'^\mu = x'^\mu(x^0, x^1, \dots, x^N) \equiv x'^\mu(x^\nu)$$

$$\mu, \nu = 0, 1, \dots, N$$

$$x^\mu = x^\mu(x'^0, x'^1, \dots, x'^N) \equiv x^\mu(x'^\nu)$$

$$\left.\begin{array}{l} x'^\mu = x'^\mu(x^\nu) \rightarrow dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu \\ x^\nu = x^\nu(x'^\sigma) \rightarrow dx^\nu = \dfrac{\partial x^\nu}{\partial x'^\sigma} dx'^\sigma \end{array}\right\} \text{combinándolas}$$

$$\underline{\underline{dx'^\mu}} = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\sigma}} \underline{\underline{dx'^\sigma}} = \delta^\mu_\sigma dx'^\sigma$$

$$\delta^\mu_\sigma = \begin{cases} 1 & \mu = \sigma \\ 0 & \mu \neq \sigma \end{cases} \quad \text{delta de Kronecker}$$

Tensores de orden 0

Escalar $\mapsto$ Permanece invariante bajo un cambio de coordenadas

Tensores de orden 1

N-vector (4-vector en el espacio-tiempo)

Dos tipos $\begin{cases} \text{vectores contravariantes} \\ \text{vectores covariantes} \end{cases}$

Vector contravariante

$$A^{\mu}(x^{\nu})$$

Formula de transformación:

$$\boxed{A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}}$$

Multiplicamos por $\dfrac{\partial x^{\sigma}}{\partial x'^{\mu}}$ y queda:

$$A'^{\mu} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} = \underbrace{\frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\nu}}}_{\delta^{\sigma}_{\nu}} A^{\nu}$$

$$A^{\nu} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} A'^{\mu}$$

Como $dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \Rightarrow dx^{\mu}$ contravariante

Vector covariante

$$A_{\mu}(x^{\nu})$$

Formula de transformación:

$$\boxed{A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}}$$

Multiplicamos por $\frac{\partial x'^{\mu}}{\partial x^{\sigma}}$ y queda:

$$A'_{\mu} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\nu}}{\partial x'^{\mu}}}_{\delta^{\nu}_{\sigma}} A_{\nu}$$

$$A_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} A'_{\mu}$$

Operador derivada: $x'^{\mu} = x'^{\mu}(x^{\nu})$

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}$$

que es covariante $\left(\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}\right)$

Tensor de orden 2

Contravariante:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x'^{\nu}}{\partial x^{\xi}} F^{\sigma\xi}$$

Covariante:

$$F'_{\mu\nu} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x^{\xi}}{\partial x'^{\nu}} F_{\sigma\xi}$$

Mixto:

$$F'^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\xi}}{\partial x'^{\nu}} F^{\sigma}{}_{\xi}$$

Tensores de orden superior

Operaciones con tensores

1) Adición y sustracción (mismo orden y tipo)

$$C^{\lambda\xi}_{\mu} = A^{\lambda\xi}_{\mu} \pm B^{\lambda\xi}_{\mu}$$

2) Multiplicación externa

$$A^{\lambda\xi}_{\mu} B^{\sigma}_{\nu} = C^{\lambda\xi\sigma}_{\mu\nu}$$

Orden = suma de órdenes

3. Contracción: Se igualan un índice covariante y otro contravariante y se suman → Tensor con dos órdenes menos

$$A^{\lambda\nu\sigma}_{\mu\sigma} = B^{\lambda\nu}_{\mu}$$

4. Multiplicación interna = Multiplicación externa seguida de una contracción

$$A^{\lambda\xi}_{\sigma} B^{\sigma}_{\mu\nu} = C^{\lambda\xi}_{\mu\nu}$$

5. Regla del cociente

<u>Tensor métrico</u>

$$x^{\mu} \quad y \quad x^{\mu} + dx^{\mu}$$

$$\boxed{ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}}$$ Cuadrado del diferencial de longitud

$g_{\mu\nu} \mapsto$ Tensor métrico → covariante

Componentes contravariantes (tensor recíproco):

$$g_{\mu\lambda}\, g^{\lambda\nu} = \delta^{\nu}_{\mu}$$

$$\delta^{\nu}_{\mu} \equiv \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

$$g = |g_{\mu\nu}| \text{ determinante de } g_{\mu\nu}$$
$$g \neq 0$$

Definimos

$$g^{\mu\nu} = \frac{\text{adjunto de } g^{\mu\nu}}{g}$$

$g_{\mu\nu}$ es un tensor simétrico pues

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu =$$

$$= g_{\nu\mu}\, dx^\nu dx^\mu = g_{\nu\mu}\, dx^\mu dx^\nu$$

luego:
$$g_{\mu\nu} = g_{\nu\mu}$$

Tensores asociados

Se obtienen mediante el tensor métrico subiendo o bajando índices:

A^μ contravariante $\quad \bar{A}_\mu \equiv g_{\mu\nu} A^\nu$ covariante

A_μ covariante $\quad \bar{A}^\mu \equiv g^{\mu\nu} A_\nu$ contravariante

A^μ y A_μ componentes contravariantes y covariantes del mismo tensor (se suprime la raya de encima)

Teoría de la relatividad especial

4 dimensiones (x^0, x^1, x^2, x^3)

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu \equiv c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu}$$

$$A^\mu \longmapsto A_\mu = g_{\mu\nu} A^\nu$$

$$\boxed{A^\mu \equiv (A^0, \vec{A})}$$

$$A_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$\boxed{A_\mu \equiv (A^0, -\vec{A})}$$

$$x^\mu = (ct, \vec{x}) \qquad x_\mu = (ct, -\vec{x})$$

Producto escalar:

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B}$$

$$\boxed{\text{COVARIANCIA} \longmapsto \text{Invariancia de forma}}$$

Transformación de Lorentz

$$A'^{\mu} = \Lambda^{\mu}_{\cdot\nu} A^{\nu} \qquad \Lambda^{\mu}_{\cdot\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A'_{\mu} = \Lambda^{\cdot\nu}_{\mu} A_{\nu} \qquad \Lambda^{\cdot\nu}_{\mu} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\partial_{\mu} \equiv \left(\frac{\partial}{\partial(ct)}, \vec{\nabla} \right) = \frac{\partial}{\partial x^{\mu}} \quad \text{covariante}$$

$$\partial^{\mu} \equiv \left(\frac{\partial}{\partial(ct)}, -\vec{\nabla} \right) = \frac{\partial}{\partial x_{\mu}} \quad \text{contravariante}$$

$$\partial_{\mu} \partial^{\mu} \equiv g_{\mu\nu} \partial^{\nu} \partial^{\mu} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$$

(Invariante Lorentz – D'Alambertiano)

$$\left.\begin{aligned} A'^{\mu} &= g^{\mu\sigma} A'_{\sigma} \\ A^{\nu} &= g^{\nu\xi} A_{\xi} \end{aligned}\right\}$$

$$g^{\mu\sigma} A'_{\sigma} = \Lambda^{\mu}{}_{\cdot\nu}\, g^{\nu\xi} A_{\xi}$$

$$\underbrace{g_{\mu\lambda}\, g^{\mu\sigma}}_{\delta_{\lambda}^{\sigma}} A'_{\sigma} = g_{\mu\lambda}\, \Lambda^{\mu}{}_{\cdot\nu}\, g^{\nu\xi} A_{\xi}$$

$$A'_{\lambda} = g_{\mu\lambda}\, \Lambda^{\mu}{}_{\cdot\nu}\, g^{\nu\xi} A_{\xi}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda_{\mu}{}^{\cdot\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = -F^{\mu\nu} \quad \text{antisimétrico}$$

Bajamos los índices:

$$F_{\mu\nu} = g_{\mu\sigma} g_{\nu\lambda} F^{\sigma\lambda}$$

$$\left.\begin{matrix}\mu=0\\ \nu=0\end{matrix}\right\} \quad F_{00} = \underbrace{g_{0\sigma}}_{\delta_{0\sigma}} \underbrace{g_{0\lambda}}_{\delta_{0\lambda}} F^{\sigma\lambda} = F^{00} = 0$$

$$\left.\begin{matrix}\mu=i\\ \nu=i\end{matrix}\right\} \quad F_{ii} = \underbrace{g_{i\sigma}}_{-\delta_{i\sigma}} \underbrace{g_{i\lambda}}_{-\delta_{i\lambda}} F^{\sigma\lambda} = F^{ii} = 0$$

$$\left.\begin{matrix}\mu=0\\ \nu=i\end{matrix}\right\} \quad F_{0i} = \underbrace{g_{0\sigma}}_{\delta_{0\sigma}} \underbrace{g_{i\lambda}}_{-\delta_{i\lambda}} F^{\sigma\lambda} = -F^{0i}$$

$$F^{i0} = -F^{0i} = F_{0i}$$

$$\left.\begin{array}{l}\mu = i\\ \nu = j\\ i \neq j\end{array}\right\} F_{ij} = \underbrace{g_{i\sigma}}_{-\delta_{i\sigma}}\underbrace{g_{j\lambda}}_{-\delta_{j\lambda}} F^{\sigma\lambda} = F^{ij}$$

$$F^{ji} = -F^{ij}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -F^{01} & -F^{02} & -F^{03} \\ F_{01} & 0 & F^{12} & F^{13} \\ F_{02} & -F^{21} & 0 & F^{23} \\ F_{03} & -F^{13} & -F^{23} & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} \rightarrow F^{\mu\nu}(\vec{E} \rightarrow -\vec{E})$$