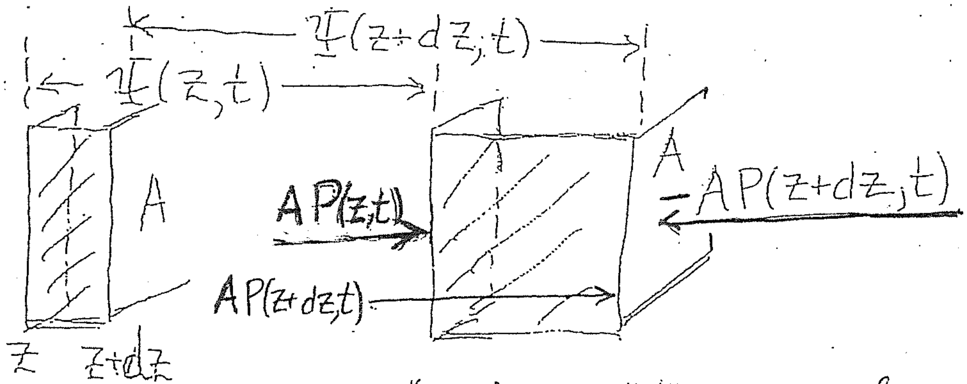
Physics 251 25th class
Tues. April 9, 2024

A. C.W.E. For Sound Waves

To begin here, refer to the figure below, which shows the effects of a plane wave of sound on the medium (air):



"without wave" (equil.)
(a)

"with wave" ("arbitrary configuration")
(b)

Let $AP(z)$ = force, in disturbed configuration (b), that element ~~that was immediately~~ to the left of z exerts, in arb. config on the element that was immediately to the right of z [i.e, force on the displaced slab from the left].

Let $AP(z+dz)$ = force, in disturbed configuration (b), that the mass in the slab exerts on the element to the immediate right of the slab.

Then, by Newton's third law, in the "arbitrary configuration" (i.e, "with wave"),

$$\text{Net force on slab in part (b)} = F_{net} = A\left[P(z) - P(z+dz)\right]$$

Thus, by Newton's Second Law, if dm is the mass within the slab,

$$dm \frac{\partial^2 \Psi(z,t)}{\partial t^2} = A\left[P(z,t) - P(z+dz,t)\right]$$

Now $P(z) - P(z+dz) = [P_0 + P_g(z)] - [P_0 + P_g(z+dz)]$

$$= P_g(z) - P_g(z+dz)$$

$$\approx -\frac{\partial P_g}{\partial z} dz$$

Now recall that

(1) $P_g(z,t) = -B \dfrac{\partial \Psi(z,t)}{\partial z}$, B = Bulk Modulus of the medium.

$\left[\text{where } B = \lim_{\Delta V \to 0}\left(\frac{-\Delta P}{\Delta V/V_0}\right) = -V_0 \left.\frac{dP}{dV}\right)_0\right]$

$$\Rightarrow \quad -\frac{\partial P_g}{\partial z} = B \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

Combining all ~~these~~ we have, using $dm = \rho_0 A\, dz$, [ρ_0 = mass per unit volume]

$$\rho_0 A\, dz \frac{\partial^2 \Psi(z,t)}{\partial t^2} = BA \frac{\partial^2 \Psi(z,t)}{\partial z^2} dz, \text{ or}$$

(2) $$\boxed{\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{B}{\rho_0} \frac{\partial^2 \Psi(z,t)}{\partial z^2}}$$

CWE For Displacement Aspect of Sound

Equation (2) is a Classical Wave Equation! It shows that rigidly standing ~~traveling~~ sound waves* are possible, and that the wave velocity v_ϕ is given by

$$(3) \qquad v_\phi = \sqrt{\frac{B}{\rho_0}}$$

where $B = \lim_{\Delta V \to 0} \left(\frac{-\Delta P}{\Delta V / V} \right)_{V=V_0} = -V_0 \left(\frac{dP}{dV} \right)_0$ (Bulk Modulus)

and where ρ_0 = equilibrium volume mass density (mass/Vol.).

CWE. For Gauge Pressure Wave

Since the "gauge-pressure wave" moves along with the same velocity with the displacement wave, we expect that it satisfies a CWE with the same v_ϕ, i.e.,

$$(4) \qquad \frac{\partial^2 P_g(z,t)}{\partial t^2} = \frac{B}{\rho_0} \frac{\partial^2 P_g(z,t)}{\partial z^2} \qquad \text{(CWE for } P_g\text{)}$$

This is correct.

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* e.g., rigidly traveling sinusoidal sound waves, or any continuous rigidly moving shape.

We can easily verify this as follows: We had

$$(2) \quad \frac{\partial^2 \Psi}{\partial t^2} = \frac{B}{\rho_0} \frac{\partial^2 \Psi}{\partial z^2}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(\frac{\partial \Psi}{\partial z}\right) = \frac{B}{\rho_0} \frac{\partial^2}{\partial z^2}\left(\frac{\partial \Psi}{\partial z}\right)$$ [took $\frac{\partial}{\partial z}$ of both sides of (2)]

But, by (1),

$$\frac{\partial \Psi}{\partial z} = -\frac{1}{B} P_g$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(-\frac{1}{B} P_g\right) = \frac{B}{\rho_0} \frac{\partial^2}{\partial z^2}\left(-\frac{1}{B} P_g\right)$$

Since $\left(-\frac{1}{B}\right)$ is a constant, we can cancel it from both sides, so

$$\frac{\partial^2 P_g(z,t)}{\partial t^2} \stackrel{\checkmark}{=} \frac{B}{\rho_0} \frac{\partial^2 P_g(z,t)}{\partial z^2}$$

From the C.W.E's we derived, it is clear that $V_\phi^{\text{sound}} = \sqrt{\frac{B}{\rho_0}}$.

Of course, we are not done until we know how to <u>evaluate</u> the Bulk Modulus B! For that we will wait until our later discussion of traveling waves of sound.

B. More On The Phase/Wave Velocity of Sound In Air [K-Text, sect. 13.27]

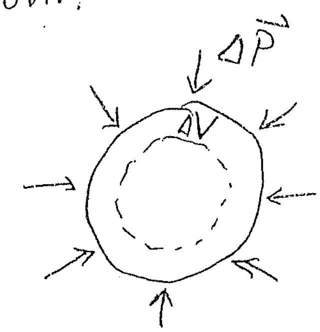
We found CWE's for both $\Psi(z,t)$ and $P_g(z,t)$:

$$\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{B}{\rho_o} \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

$$\frac{\partial^2 P_g(z,t)}{\partial t^2} = \frac{B}{\rho_o} \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

where B is the "Bulk-Modulus" of the medium:

$$B = \lim_{\Delta V \to 0} \left(\frac{-\Delta P}{\Delta V / V_o} \right) = -V_o \frac{dP}{dV_o}$$



From the CWE's above, it is clear that the wave/phase velocity of sound waves is given by

$$v_\phi = \sqrt{\frac{B}{\rho_o}},$$

however, ...

Finding The Bulk Modulus of Air

The question is, then, how do we evaluate B? (Or, at least obtain a formula for it). [Recall $B = -V_0 \left.\frac{dP}{dV}\right)_0$]

To attempt to find out, in 1687 Newton used the newly discovered Boyle's Law, which says that, at constant temperature,

$$P_0 V_0 = PV \qquad \text{(Boyle's Law. Valid for const. } T \text{ only)}.$$

(product of pressure and volume is constant under expansion or compression. Expansion and compression is exactly what happens to a given mass of air molecules as a sound wave goes through).

From Boyle's Law, then, for a mass M of air molecules

$$P = \frac{P_0 V_0}{V} \Rightarrow \frac{dP}{dV} = \frac{-P_0 V_0}{V^2} \Rightarrow \left.\frac{dP}{dV}\right)_0 = \frac{-P_0}{V_0} \Rightarrow -V_0 \left.\frac{dP}{dV}\right)_0 = P_0,$$

so

$$(5) \quad v_\phi = \sqrt{\frac{P_0}{\rho_0}} \quad \text{(wrong!)}$$

Now, for air at STP, using modern numbers,

$$\rho_0 \approx 1.29 \ \mathrm{Kg/m^3}, \ P_0 = 1\,\mathrm{atm.} = 1.01 \times 10^5 \ \mathrm{N/m^2},$$

and putting these into (5) we find

$$v_\phi^{\text{NEWTON}} \approx 280 \ \mathrm{m/s}.$$

$v_{\phi,\text{SOUND}}^{\text{NEWTON}} \simeq 280 \text{ m/s}.$

But, experimentally, at S.T.P.,

$v_\phi^{\text{exp.}} \approx 332 \text{ m/s}.$

- so Newton's theory was off by about 15%!

Correcting Newton's mistake. Now comes the interesting question: How could Newton come so close to the right answer (which shows that something is right with his derivation) and yet miss it by 15% (which shows something is wrong with his derivation)? The trouble came from assuming Boyle's law, which holds only at constant temperature. The temperature in a sound wave does *not* remain constant. The air located (at a given instant) in a region of compression has had work done on it. It is slightly hotter than its equilibrium temperature. The neighboring regions one half-wavelength away are regions of rarefaction. They have cooled slightly in expanding. (Energy is conserved; the excess energy at a compression equals the energy deficit at a rarefaction.) Because of the increase in temperature in a compression, the pressure in the compression is *larger* than predicted by Boyle's law, and the pressure in a rarefaction is *less* than that predicted. This effect produces a larger return force than expected and hence a larger phase velocity.

Excerpted from Waves: Berkeley Physics Series, Vol. 3, pp 167-168. (by F.S. Crawford)

It turns out that instead of Boyle's law (which holds at constant temperature) we should use the *adiabatic gas law,* which gives the relation between p and V when no heat is allowed to flow. (There is not sufficient time for heat to flow from the compressions to the rarefactions so as to equalize the temperature. Before that can happen, a half-cycle has elapsed, and a former region of compression has become a region of rarefaction. Thus the result is the same as if there were "walls" preventing the heat from flowing from one region to another.) This relation can be shown to be given by

$$pV^\gamma = p_0V_0^\gamma, \qquad p = p_0V_0^\gamma V^{-\gamma}, \tag{40}$$

("Adiabatic Gas Law")

where γ is a constant called "the ratio of specific heat at constant pressure to specific heat at constant volume" and has the numerical value

$$\gamma = 1.40 \text{ for air at STP.}$$

so $\quad P = P_0 V_0^{\gamma} V^{-\gamma}$

$$\Rightarrow \frac{dP}{dV} = -\gamma V^{-\gamma-1} P_0 V_0^{\gamma}$$

$$\Rightarrow \quad B = -V_0 \left.\frac{dP}{dV}\right)_0 = \gamma V_0^{-\gamma} P_0 V_0^{\gamma} = \gamma P_0,$$

So, according to the adiabatic gas law, we expect

$$\boxed{v_\phi^{\text{sound}} = \sqrt{\frac{B}{\rho_0}} = \sqrt{\frac{\gamma P_0}{\rho_0}}}$$

which is $\quad v_\phi^{\text{sound}} = \sqrt{1.40}\sqrt{\frac{P_0}{\rho_0}} = 332\ \text{m/s}$

which is correct (for $0°C$, 1 atm (STP))!

This correction was made by Laplace about 120 years after Newton worked on the problem (in 1807)!

The question now is - <u>why is the use of Boyle's law wrong</u>?

This is nicely explained in a famous older text* (pp. 168-169):

Let us examine why the heat does not have time to flow from a compression to a rarefaction and thus to equalize the temperature. In order for the heat flow to keep the temperature everywhere constant, the heat would have to flow a distance of one half-wavelength (from a compression to a rarefaction) in a time which is short compared with one-half of a period of oscillation (after half a period, the compressions and rarefactions will have exchanged places). Thus for the heat flow to be fast enough, one would need

$$v(\text{heat flow}) \gg \frac{\frac{1}{2}\lambda}{\frac{1}{2}T} = v_{\text{sound}}. \qquad (42a)$$

* From Waves, Berkeley Physics Series, Vol. 3 by F.S. Crawford, op. cit.

It turns out that the heat flow is mostly due to conduction, i.e., due to the transfer of translational kinetic energy from one air molecule to another via collisions. For an air molecule of mass M in air at absolute temperature T, the rms thermal velocity (translational velocity due to heat energy) in a given direction z turns out to be

$$v_{\text{rms}} = \langle v_z^2 \rangle^{1/2} = \sqrt{\frac{kT}{M}}, \tag{42b}$$

where k is a constant called Boltzmann's constant. The velocity of sound can be also expressed in terms of T and M. It is given by

$$v_{\text{sound}} = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\frac{\gamma kT}{M}}. \tag{42c}$$

Thus, aside from the constant $\sqrt{\gamma}$, the velocity of sound equals the rms thermal velocity of a molecule along z. Thus *if* the molecules traveled in straight lines for distances of order $\frac{1}{2}\lambda$ before making collisions, they would "just make it" in time to transfer heat. They would not on the average satisfy Eq. (42*a*), but some of the exceptionally fast ones would. There could thus be a significant amount of heat transfer in one half-period. But instead of traveling in straight lines for distances of order $\frac{1}{2}\lambda$, the molecules zigzag their way in a random fashion, only going distances between collisions of the order of 10^{-5} cm (for air at STP). As long as the wavelength is long compared with 10^{-5} cm, the adiabatic law is therefore a very good approximation. (The shortest wavelength for audible sound waves corresponds to $\nu \approx 20{,}000$ cps, so $\lambda = v/\nu \approx 3.32 \times 10^4/2 \times 10^4 = 1.6$ cm.)

Note also from equation (42c) above that we expect that the speed of sound will increase with temperature (like $\sqrt{T}$). This is borne out experimentally. Anyway, with Laplace's correction, our result is pretty good.

B. Standing Waves of Sound

We've looked at standing waves ("normal modes") on a string; it's also possible to set up standing waves of sound in a tube - in fact, that is what happens in musical "wind instruments" such as in clarinets, flutes and organs. The basic cause of this is analogous to the cause of transverse standing waves on a stretched string - it is caused by the superposition of an incident wave with a reflected wave traveling in the opposite direction. Just as the reflection of waves on a string is different depending on whether the reflecting end is bound or free, the reflection of sound waves at the end of a tube depends on whether the end of the tube is closed or open.

For our purposes here, rather than worrying specifically about the sign (i.e, inverted or not) of the reflected wave, we need to understand the character of the resultant wave (i.e, the superposition of the reflected and incident waves) at each of the ends of the tube:

Standing Waves Formed due to reflection at a closed end of a tube

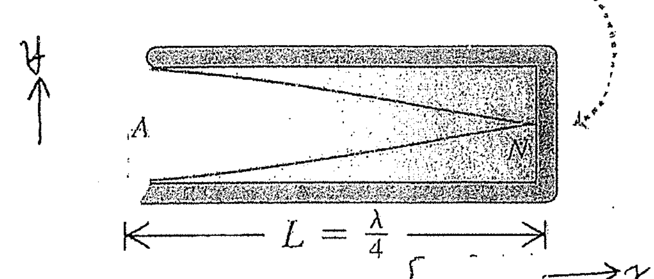
1. At a closed (or "capped") end of a tube, any standing wave ("normal mode") of sound must have a displacement node. That is because, at a closed end the "forward" (longitudinal) displacement of an air parcel or molecule must be zero as the molecule or parcel cannot penetrate the closed end.

2. At an open end of a tube, any standing wave of sound must have a displacement antinode.

   (That is because, at an open end, there is nothing to block the longitudinal "back and forth" motion of air molecules.)

The figure below shows, for the lowest frequency mode (standing wave) in a tube that is open at one end & closed at the other, a plot of the longitudinal (x-direction) displacement of air molecules [plotted in the "vertical" (y-) direction on the plot] of molecules that, with no wave present ("equilibrium") were at position x.

The closed end of this pipe is always a displacement node.



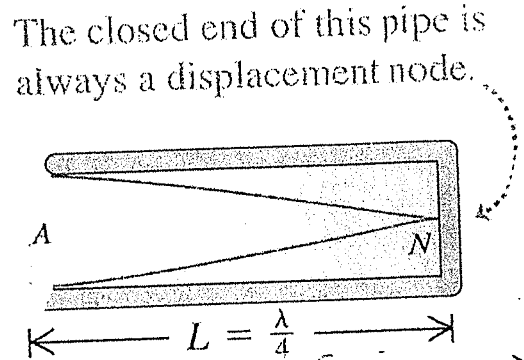
a): Fundamental: $f_1 = \frac{v}{4L}$

[The two curves represent the displacement, plotted vertically, at two instants one-half a period apart in time.]

We see that there is a node (N) at the closed end, and an antinode (A) at the open end.

C. Let's look again at the plot of the longitudinal displacement profile for the lowest frequency mode (or "fundamental mode") of air in a tube that is open at one end and closed at the other (first figure below and to the right):

We see that, for the fundamental mode, in a tube with one end open and one end closed, one-quarter of a wavelength fits into the length of the tube (L). $[L = \frac{1}{4}\lambda_1 \Rightarrow \underline{\lambda_1 = 4L}]$



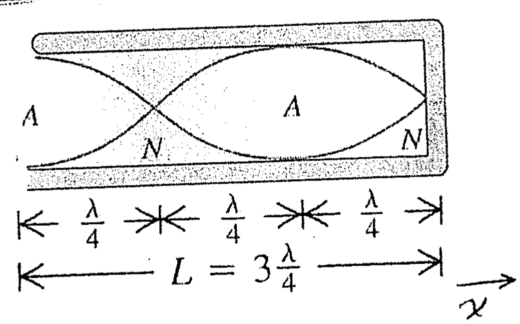
(a): Fundamental mode: $f_1 = \frac{v}{4L}$

Then, from $v = f\lambda$ where v is the speed of sound in air, we have $f = v/\lambda$, so,

fundamental mode: $\lambda_1 = 4L$

$f_1 = \frac{v}{4L}$

[f_1 is the "back & forth" frequency of the longitudinal motion of air parcels]



(b) "Third Harmonic" Mode

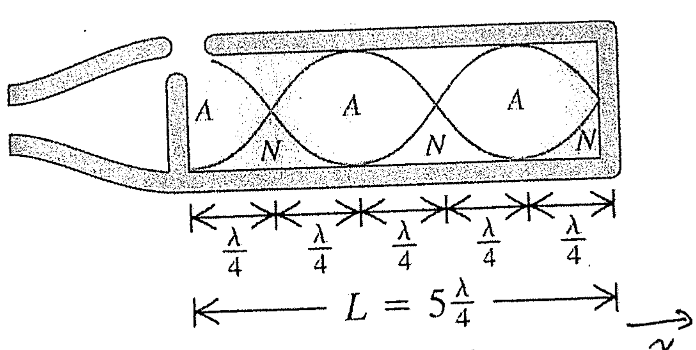
figures from Young, Adams, Chastain, op.cit.

For the next higher frequency mode (middle figure), we have

$L = \frac{3}{4}\lambda \Rightarrow \lambda_3 = \frac{4L}{3}$

$\Rightarrow f_3 = v/\lambda_3 = \frac{v}{\frac{4}{3}L} = 3\frac{v}{4L} = 3f_1$,

so this mode is called the "third harmonic" since its frequency is $3 \times f_1$ ($3f_1$).



(c): Fifth harmonic: $f_5 = 5\frac{v}{4L} = 5f_1$

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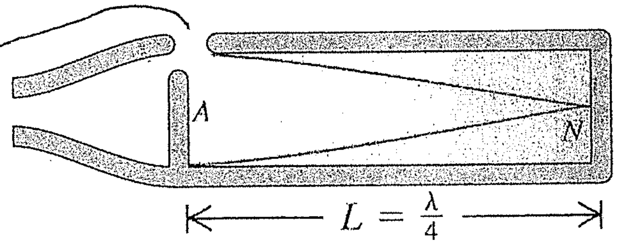
Another look at Standing Waves in a Tube with one End Open and One End Closed

open end is always a displacement antinode

The closed end of this pipe is always a displacement node.

open end



(a): Fundamental: $f_1 = \frac{v}{4L}$

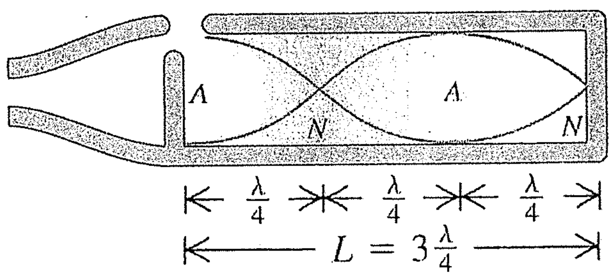
fundamental mode:

$\frac{1}{4}$ wavelength fits between ends

$\Rightarrow L = \frac{\lambda_1}{4} \Rightarrow \lambda_1 = 4L$

$f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$.

Figures from Young, Adams & Chastain, op. cit.



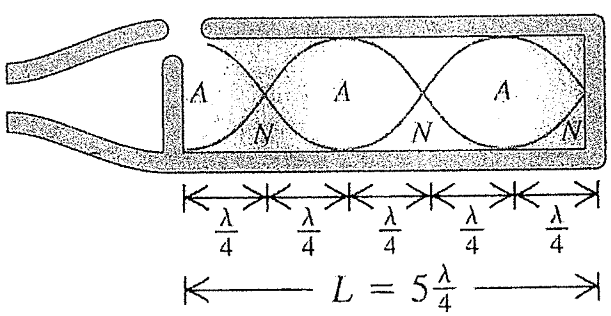
(b): Third harmonic: $f_3 = 3\frac{v}{4L} = 3f_1$

Second Mode: (third harmonic)

$\frac{3}{4}$ of a wavelength fits between ends.

$\Rightarrow L = \frac{3}{4}\lambda_3 \Rightarrow \lambda_3 = \frac{4L}{3}$

$\Rightarrow f_3 = \frac{v}{\lambda_3} = 3\frac{v}{4L} = \underline{3f_1}$



(c): Fifth harmonic: $f_5 = 5\frac{v}{4L} = 5f_1$

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Third Mode: (5^{th} harmonic)

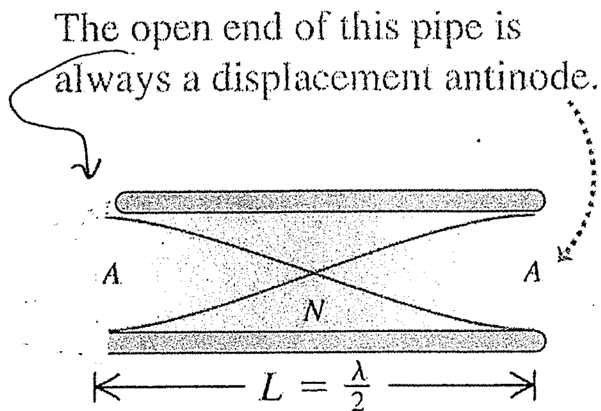
$\frac{5}{4}$ of a wavelength fits between ends.

$\Rightarrow L = \frac{5}{4}\lambda_5 \Rightarrow \lambda_5 = \frac{4L}{5}$

$\Rightarrow f_5 = \frac{v}{\lambda_5} = \frac{5v}{4L} = \underline{5f_1}$

Notice that the mode frequencies for the modes are in the ratio 1 : 3 : 5 : 7 : 9 : .... That is why we denote them by $n=1, n=3, n=5, \ldots$ — then all the overtones are odd harmonics of the fundamental — NO EVEN HARMONICS for a tube with one end open and one end closed.

D. Standing waves (normal modes) of sound waves in a tube open at both ends

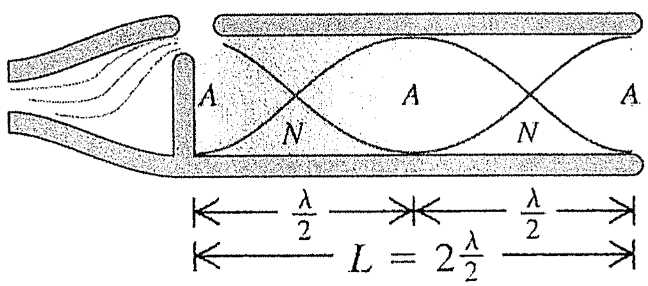


(a): Fundamental: $f_1 = \frac{v}{2L}$

Mode 1: half a wave length fits between ends

$$L = \frac{\lambda_1}{2} \Rightarrow \lambda_1 = 2L$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

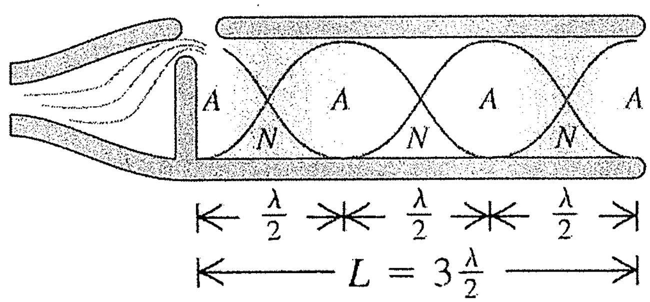


(b): Second harmonic: $f_2 = 2\frac{v}{2L} = 2f_1$

Mode 2: One full wavelength fits between ends

$$\Rightarrow L = \lambda$$

$$\Rightarrow f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1$$



(c): Third harmonic: $f_3 = 3\frac{v}{2L} = 3f_1$

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Mode 3: $1\frac{1}{2}$ wavelengths fit between ends

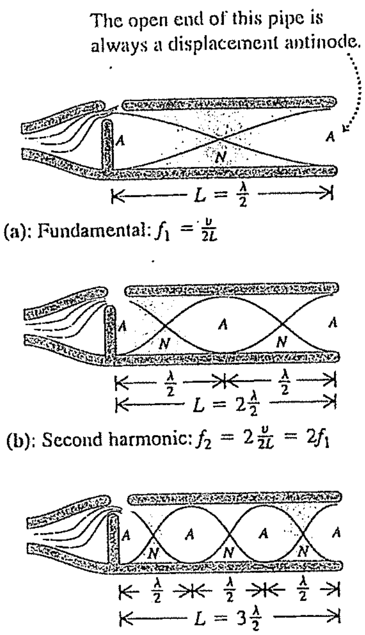
$$\Rightarrow L = \frac{3}{2}\lambda_3 \Rightarrow \lambda_3 = \frac{2L}{3}$$

$$\Rightarrow f_3 = \frac{v}{\lambda_3} = \frac{v}{2L/3}$$

$$\Rightarrow f_3 = 3\frac{v}{2L} = 3f_1$$

We note, from the above (and continuing to higher modes) that for a pipe open at both ends, the mode frequencies are all harmonics of the fundamental, and all harmonics (even and odd n) are possible.

Summarizing our results for the frequencies of the standing waves (normal modes) of sound allowed in pipes:



(a): Fundamental: $f_1 = \frac{v}{2L}$

(b): Second harmonic: $f_2 = 2\frac{v}{2L} = 2f_1$

(c): Third harmonic: $f_3 = 3\frac{v}{2L} = 3f_1$

▲ FIGURE 12.23 A cross section of an open pipe, showing the first three normal modes as well as the displacement nodes and antinodes.

Open-pipe normal-mode frequencies (i.e., both ends open)

$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots) \quad \text{(open pipe)}. \tag{12.10}$$

Units: Hz
Notes:

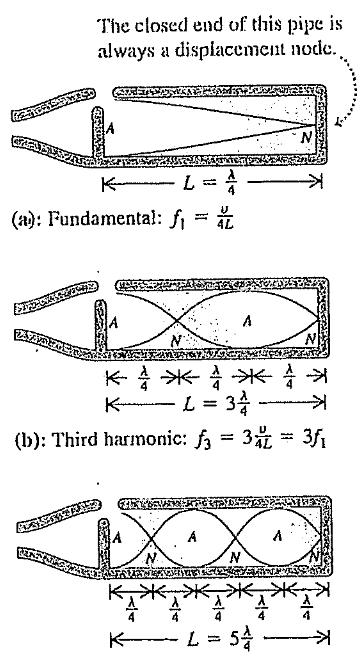
- The value $n = 1$ corresponds to the fundamental frequency, $f_1 = v/2L$.
- v is the speed of sound in air.
- All harmonics of the fundamental allowed (i.e, all integer n $n = 1, 2, 3, 4, 5, \ldots$)

Stopped-pipe normal-mode frequencies (i.e., one end open, one end closed)

$$f_n = n\frac{v}{4L} = nf_1 \quad (n = 1, 3, 5, \ldots) \quad \text{(stopped pipe)}. \tag{12.12}$$

Units: Hz
Notes:

- f_1 is the fundamental frequency given by Equation 12.11.
- Only *odd* values of n are allowed for the stopped pipe.



(a): Fundamental: $f_1 = \frac{v}{4L}$

(b): Third harmonic: $f_3 = 3\frac{v}{4L} = 3f_1$

(c): Fifth harmonic: $f_5 = 5\frac{v}{4L} = 5f_1$

▲ FIGURE 12.24 A cross section of a stopped pipe, showing the first three normal modes as well as the displacement nodes and antinodes. Only odd harmonics

Figures & Text from Young, Adams & Chastain, op. cit.

Note that for both situations (the tube open at both ends or open at one end and closed at the other) the frequency of the fundamental normal mode is inversely proportional to the length of the tube:

$$f_1 \propto \frac{1}{L}$$

Thus, one can achieve low frequency fundamental notes in a musical instrument by using long tubes or pipes. And, as we have seen, if one end is open and one end is closed, the frequency of the fundamental $\left(\frac{v}{4L}\right)$ is only half that if the pipe is open at both ends.

<u>Example</u>: Consider a 4-meter long organ pipe that is closed at the top end. Then, the fundamental freq. f_1 is

$$f_1 = \frac{v}{4L} \approx \frac{344\ \text{m/s}}{4(4\ \text{m})} \approx 22\ \text{HZ}$$

– very close to the lowest frequency that people with completely undamaged hearing can hear.

<u>Example</u>: 12-meter long "open and closed" organ pipe. Then, since $f_1 \propto \frac{1}{L}$, $f_1 = \frac{1}{3}(22\ \text{HZ}) \approx 7\ \text{HZ}$ – too low to hear, but you can <u>feel</u> the vibration from it!

How are the standing waves in a pipe excited?

In some musical instruments (e.g., clarinet, organ pipes) as shown:

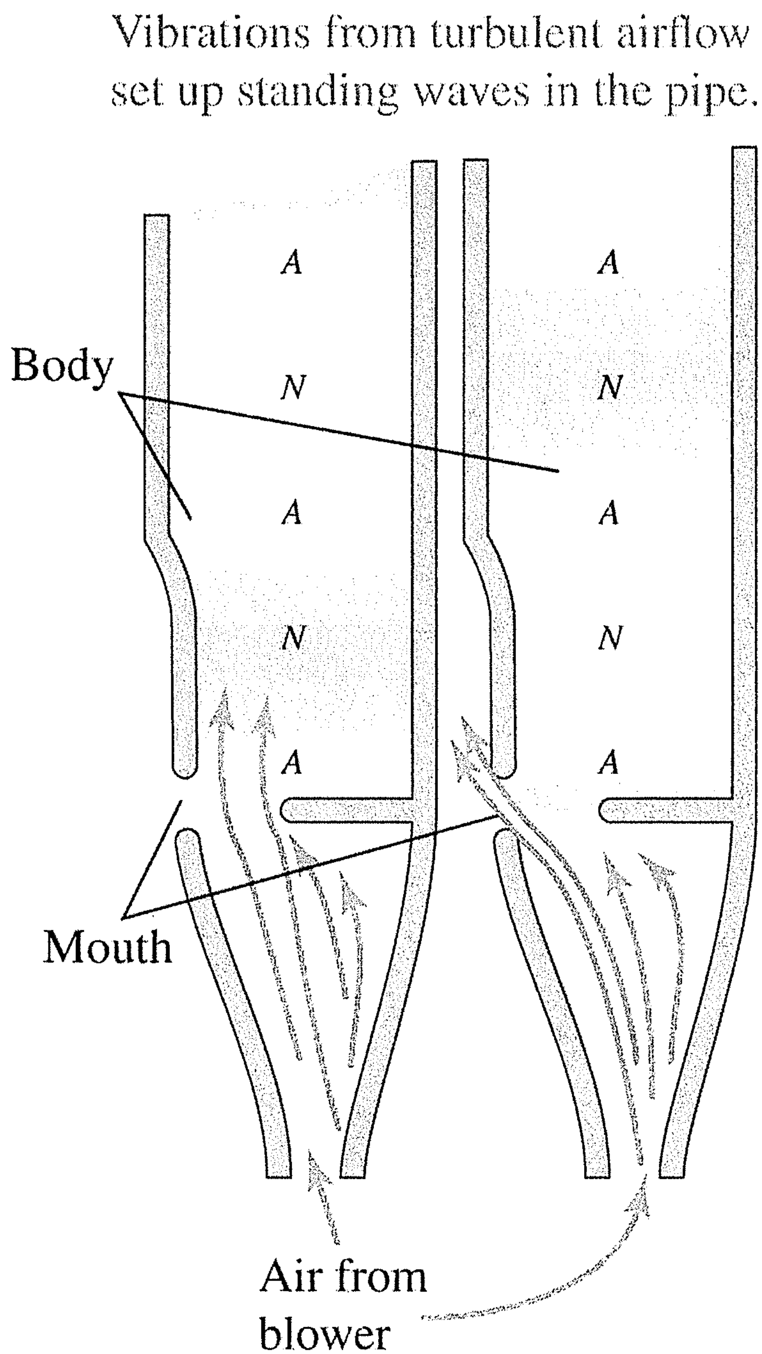


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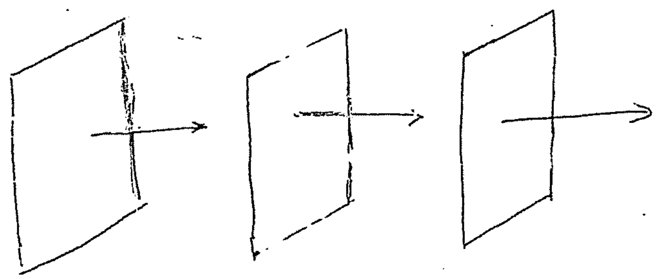
Figure from Young and Freedman, op. cit.

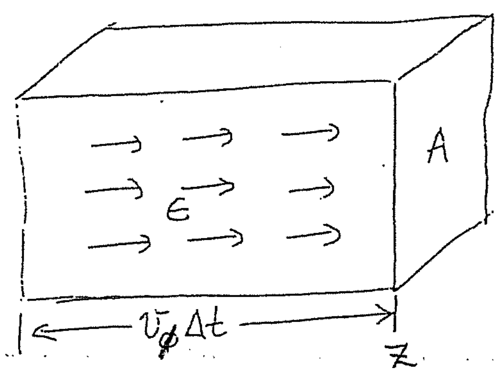
C. Intensity for Traveling Sound Waves

1. Case of Plane Waves in three dimensions – Power

[K-text, sect. 15-6, pp. 23-24]

We now consider the energy per unit area passing a given plane of constant z per second. To do this, we mimick our





For a plane wave propagating in the z-direction, ϵ does not depend on x or y – only on z and t.

analogous argument for the string. The energy is "carried" at speed v_φ. Thus, in time Δt, all the energy in the imaginary box of length $v_\phi \Delta t$ shown above passes through the end plane A.

Let $\epsilon(z,t)$ be the energy per unit volume. We take $\Delta t \to 0$; then $v_\phi \Delta t$ is very small. Take the end of the box at coordinate z, then, the energy per unit time (power) passing thru A at z is

$$\left.\frac{\Delta E}{\Delta t}\right)_z = P(z,t) = \frac{\epsilon(z,t)\cdot(v_\phi \Delta t\, A)}{\Delta t} = \epsilon(z,t)\cdot v_\phi A$$

Thus, the power per unit area ("Energy flux") passing through A is

$$\frac{\text{energy}}{\text{second}\cdot m^2} \equiv \underline{S(z,t) = \epsilon(z,t)\cdot v_\phi} \qquad \text{[Energy Flux]} \tag{1}$$

The time average (over a cycle if the wave is sinusoidal, or otherwise, over a time of your choosing) of the energy flux is called the intensity I;

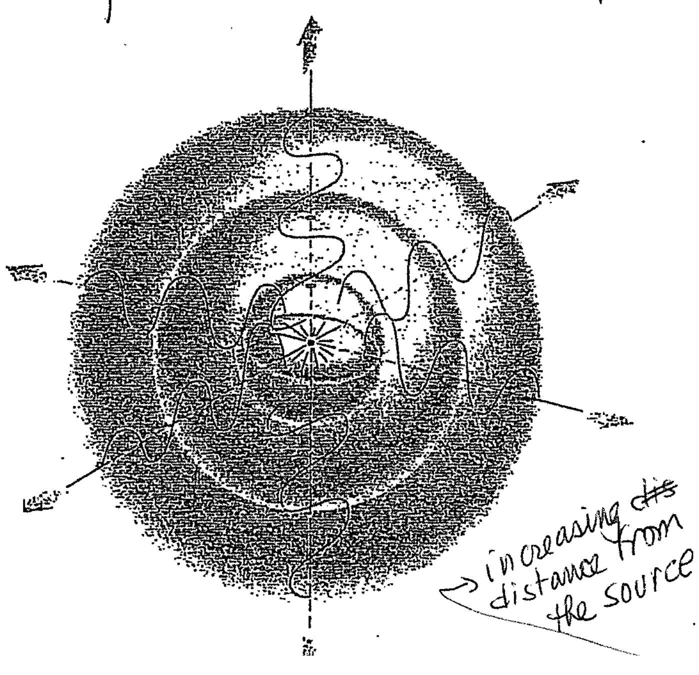
$$I(z,t) = \langle S(z,t)\rangle_t = \langle \epsilon(z,t)\rangle_T \cdot v_\phi = \bar{\epsilon}\, v_\phi \tag{2}$$

2. Case of Curved Wavefronts

As (1) and (2) are local relations, they apply even if the wavefronts are curved.

Ex1:

Suppose that you have a point source of sound radiating at constant power in 3-dimensional space. Then the wave fronts (surfaces of constant phase) are spheres centered on the source. In this situation, if the power is radiated isotropically (same in all directions), then conservation of energy requires that the intensity falls off with



D. What is the Characteristic Impedance of the Air for Sound Waves?

[K-text, chap. 15, sect 15.6
pages 21 through 25]

We already saw that the intensity of a sound wave in 3 dimensions is given by (I assume a wave traveling in the z-direction):

$$(1) \quad I = \langle S(z,t) \rangle_t \equiv \bar{S}(z) = \langle \epsilon(z,t) \rangle_t \cdot v_\phi = \bar{\epsilon}\, v_\phi ,$$

where S is the energy flux, ϵ is the energy density and $v_\phi = \sqrt{\frac{\gamma P_0}{\rho_0}}$.

To proceed, we will need several results, which I summarize here:

For Sound Waves:

$$(2) \quad \text{Kinetic Energy Density} = \text{“}\frac{dK}{dV}\text{”} \equiv \text{“}K_1(z,t)\text{”} = \frac{1}{2}\rho_0 \left(\frac{\partial \Psi(z,t)}{\partial t}\right)^2 .$$

$$(3) \quad \text{Potential Energy Density} = \text{“}\frac{dU}{dV}\text{”} = \frac{1}{2} B \left(\frac{\partial \Psi}{\partial z}\right)^2$$

[Potential Energy Density comes in because the compressions of the air in the sound wave store energy - like a compression in a spring stores potential energy; analogously for the rarefractions.]

That eqn. (2) is true is easy to see: If dm is the mass (of air) that was (in equilibrium) "in dz", then $dm = \rho_0\, dV (= \rho_0 A\, dz)$

$$\text{so} \quad dK = \frac{1}{2}(dm)\left[\frac{\partial \Psi(z,t)}{\partial t}\right]^2 = \frac{1}{2}\rho_0\, dV \left[\frac{\partial \Psi}{\partial t}\right]^2$$

$$\Rightarrow \text{“}\frac{dK}{dV}\text{”} = \frac{\frac{1}{2}\rho_0\, dV \left[\frac{\partial \Psi}{\partial t}\right]^2}{dV} = \frac{1}{2}\rho_0 \left(\frac{\partial \Psi}{\partial t}\right)^2 \checkmark$$

Eqn. (3),

$$(3) \quad \frac{dU}{dV} \equiv \text{"}U_1(z,t) = \frac{1}{2} B \left(\frac{\partial \Psi}{\partial z}\right)^2$$

is a bit more intricate to derive rigorously, so here we simply offer a plausibility argument:

In considering transverse wave on a stretched string, some time ago, we derived

$$(4) \quad U_1(z,t) = \frac{1}{2} T_0 \left[\frac{\partial \Psi(z,t)}{\partial z}\right]^2.$$

On the string, the "resistance to deformation" parameter is the equilibrium tension T_0. (The same is true for longitudinal waves on a long stretched "slinky" (string)).

For sound waves in air, the "resistance to deformation" property is the Bulk Modulus B. Thus, seeing that in $\frac{dK}{dV}$ for sound, ρ_e in $\frac{dK}{dz}$ on a string $\rightarrow \rho_0$, we expect that

<u>string</u>	$\longrightarrow$	<u>sound in air</u>
ρ_e	$\longrightarrow$	ρ_0
T_0	$\longrightarrow$	B

These analogies are corroborated by comparing the expressions for wave velocities in the two systems:

$$v_\phi^{\text{string}} = \sqrt{\frac{T_0}{\rho_e}} \longrightarrow v_\phi^{\text{sound}} = \sqrt{\frac{B}{\rho_0}}$$

if $\rho_e \to \rho_0$ and $T_0 \to B = \gamma P_0$.

Applying these "translations between systems" to eqn. (4)

yields eqn. (3).

(next page →

Now, as we saw some time ago, for rigidly traveling (in $\pm z$-direction) disturbances on a stretched string,

(5) $$\frac{\partial \Psi(z,t)}{\partial z} = \mp \frac{1}{v} \frac{\partial \Psi(z,t)}{\partial t}$$ which led to $K_1(z,t) = U_1(z,t)$.

By analogous logic, for sound waves also

$K_1(z,t) = U_1(z,t)$ (as for waves on a string), and $\therefore$

$\epsilon^{sound}(z,t) = K_1(z,t) + U_1(z,t) \Rightarrow$

(6) $$\epsilon^{sound}(z,t) = \rho_0 \left(\frac{\partial \Psi}{\partial t}\right)^2 = \overset{B}{\gamma P_0} \left(\frac{\partial \Psi}{\partial z}\right)^2$$

$\Rightarrow$ (7) $$I^{sound}(z) = \bar{E}(z)\, v_\phi = \rho_0 v_\phi^{sound} \left\langle \left(\frac{\partial \Psi}{\partial t}\right)^2 \right\rangle_t$$

Now recall that, for waves on a string we defined the characteristic impedance by

(8) $$Z_0^{string} \equiv \rho_e v_\phi$$

so that, on the string

$$\langle P(z,t)\rangle_t = \langle \epsilon(z,t)\rangle_t \cdot v_\phi = \rho_e v_\phi \left\langle \left(\frac{\partial \Psi}{\partial t}\right)^2 \right\rangle_t$$

was given by

$$\bar{P}(z) = \langle P \rangle = Z_0 \left\langle \left(\frac{\partial \Psi}{\partial t}\right)^2 \right\rangle.$$

Analogously, for sound waves in air, with the replacements

$\bar{P} \to I$, $\rho_e \to \rho_0$, $v_\phi = \sqrt{\frac{T_0}{\rho_e}} \to v_\phi^{sound} = \sqrt{\frac{\gamma P_0}{\rho_0}}$, we define

(9) $\underline{Z_{0,\text{sound}} \equiv \rho_0 v_\phi}$ (*) so that $\underline{\underline{I^{\text{sound}} = Z_0 \left\langle \left(\frac{\partial \Psi}{\partial t}\right)^2 \right\rangle}}$

Thus, for sound waves in air

$$Z_{0,\substack{\text{sound}\\ \text{in air}}} = \rho_0 \sqrt{\frac{\gamma P_0}{\rho_0}} \quad \text{or}$$

(10) $$\underline{Z_{0,\substack{\text{sound}\\ \text{in air}}} = \sqrt{\gamma P_0 \rho_0}}\,.$$

Note that (10) is the exact analog of

$$Z_{0,\text{string}} = \sqrt{T_0 \rho_e}$$

with the replacement $T_0 \to B = \gamma P_0$. From (10), $Z_0^{\text{sound in air}}$ is easily evaluated:

(11) $\underline{Z_0^{\text{sound in air}} \approx 428 \frac{\text{N}\cdot\text{s}}{\text{m}^3}}$.