HOW LATE CAN YOU SLEEP IN THE MORNING?

David H. Ahl

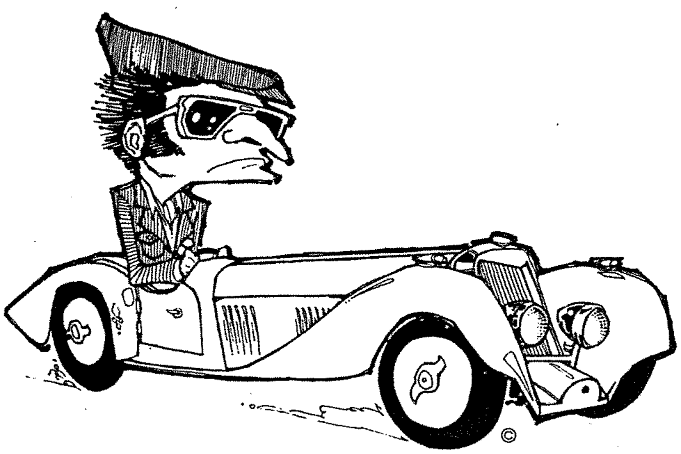
Probabilities and expected values are a vital part to writing almost any game or simulation. Here are two real-life problem situations (I face them practically daily) which can be solved with simple statistics. Be warned: the second is considerably more difficult than the first.

GETTING TO WORK

Driving to work, you can take one of two routes. Route 1 is 5 miles long and has 4 traffic lights. Each light is on a 1-minute cycle but with different intervals. Light 1 is green in your direction for 30 sec., red for 30 sec. Light 2 is 20 sec. green, 40 sec. red. Light 3 is 25 sec. green, 35 sec. red. Light 4 is 40 sec. green, 20 sec. red.

Route 2 is 5.2 miles long with only 1 traffic light which is green your way 20 sec., red 40 sec.

Speed limit in the town is 35 mph.



1. Which is the best route and what is the expected time difference between the two?

2. Route 2 also takes you by a factory loading dock. If a truck is just arriving (occurs 1 day out of 30) you will be held up an average of 3 minutes. Does this change your answer?

AND ONCE THERE

Parking your car, you rush into the 10-story Morristown AT&T Building for your 8:30 am appointment with the vice president whose office is on the 9th floor. As you reach the elevators you glance at your watch and see that it is 8:29. From past experience you know that there is a 40% chance of an elevator stopping at any given floor; a stop takes an average of 10 seconds. The elevator passes from one floor to another in 6 seconds. There are 3 elevators all of which have an indicator on the 1st floor (where you are) that shows the location and direction of that elevator. At the point of your arrival, each elevator is equally likely to be on any of the 10 floors going on either direction.

1. Assuming you take the elevator, what is the probability that you will make your appointment on time? What is the probability that you will be less than 1 minute late? Less than 2 minutes late?

2. Under the conditions stated, exactly when would you expect to arrive on the 9th floor?

3. You also know from past experience that you can run up the stairs to the 2nd floor in 10 seconds. The 2nd to 3rd takes you 10% more time (11 sec.) and 3rd to 4th 10% more time (12.1 sec.) and so on. What position of elevators upon your arrival would cause you to run up all 9 floors? (Many answers are possible — select one "break even" combination).

4. Assuming your office is on the 3rd floor and you are faced with the same situation as above (time 8:29 with 8:30 appointment on the 9th floor), but with no elevator indicators, what is your best strategy to be on time for the meeting or as little late as possible? That is, do you run up or wait for the elevator?

PARTIAL ANSWERS

"Getting to Work." 1. Route 2 is approximately 0.47 sec. faster. 2. The arriving truck has an expected value of -6 sec. per day on Route 2, hence Route 1 now has the edge by 5.53 sec.

"And Once There." Once you get on an elevator, it is a fairly simple matter to determine how long it will take to get to the ninth floor (8 floors x 6 sec. = 48) plus (40% chance of stopping x 10 sec. per stop x 8 floors = 32) equals 80 sec. But figuring when the elevator will arrive on the first floor is something else again. Any elevator can be at any floor going in either direction. Hence, Elevator 1 has a 0.056 probability of being at, say, Floor 3 going up*. How long until it returns to Floor 1? Well, if we know it goes to the 10th floor, that's easy, but it has only a 0.4 chance of going to the 10th floor, 0.4 chance of going to 9 and so on. Multiplied by 18 different possible starting positions and by 3 elevators, this is a nasty problem. In a situation like this you have to ask yourself whether a heuristic, or rule of thumb, or best guess wouldn't provide an adaquate answer. For example, you might want to make the assumption that at least one elevator is at the 4th floor (or below) and heading down.

It is sometimes easier to come up with a solution if you think of the problem in entirely different terms. For example, think of the elevator as a one-way trolley on a circular track — station 1 is Floor 1, station 2 is Floor 2 going up, and so on. Station 18 is Floor 2 coming down and then back to station 1 again. Using this approach *may* make it easier to work the problem.

By the way, it should be apparent that a computer isn't much help in solving this particular problem. However, if this were part of a much larger simulation in which the output of one part provided the input to the next (a very typical situation), a computer would be almost vital to the solution.

If you're still with me and want to read a fun little book on the subject, get "Flaws and Fallacies in Statistical Thinking" by Stephen Campbell published by Prentice-Hall.

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*If there are 10 floors and the elevator has an equal chance of being at any floor going in either direction, why isn't the probability of being at Floor 3 going up 0.1 x 0.5 = 0.05? Simply because Floor 1 and 10 do not have a direction associated with them, hence there are really only 18 locations. Actually, that's over-simplified because the elevator may not even reach Floor 10, or 9 or 8 etc. ■

6/9/95

Elevator Problem

- 4-story building with 1 elevator
- You are on the first floor
- Meeting on 4^{th} floor starts in 30 seconds
- Elevator takes 5 seconds to move one floor
- A stop at a floor takes 10 seconds
- 40% chance that elevator will stop at a floor (includes loading and unloading)
- Elevator automatically returns to the 1^{st} floor when not in use
- Questions:
  1. What is the probability that you arrive on time?
  2. At what time do you expect to arrive?

	1↑	2↑	3↑	4↓	3↓	2↓
1↑	.216	.4	.24	.144	0	0
2↑	.36	0	.4	.24	0	0
3↑	.36	0	0	.4	0	.24
4↓	.36	0	0	0	.4	.24
3↓	.6	0	0	0	0	.4
2↓	1	0	0	0	0	0

← No. We can't tell how many floors were traversed to go from 2↑ to 1↑ with just one stop – we might have stopped at either the 3^{rd} or 4^{th} floor.

17 MARKOV CHAINS

I. Overview

17.1 What is a Stochastic Process?

17.2 What is a Markov Chain?

17.3 n-Step Transition Probabilities

17.4 Classification of States in a Markov Chain

17.5 Steady-State Probabilities and Mean First Passage Times

17.6 Absorbing States

17.1 WHAT IS A STOCHASTIC PROCESS?

I. Notation

(A) A *discrete-time stochastic process* is an indexed collection of random variables $\{X_t\}$, where the time index t runs through a given set T. Sounds complicated, Use Gambler's Problem to illustrate.

(B) States $1, 2, \ldots, s$: mutually exclusive categories for values of X_t.

II. Ex. Gambler's Problem

(A) You have \$1.

(B) You need \$3.

(C) Attempt to win \$3 by placing \$1 bets.

1. Win \$1 with probability 0.4.
2. Lose \$1 with probability 0.6.

(D) Stop gambling when you have either

1. \$0. You do not have any money to bet.
2. \$3. You reached your goal.

(E) Questions

1. What is the probability of going broke?
2. What is the probability of accumulating \$3?

Can we develop formulas to answer these questions?

(F) Let X_t = number of dollars you have after t bets.

(G) Possible states: 0, 1, 2, 3.

III. Ex. Stock Price

(A) Let X_t be the price of a particular stock at the beginning of the tth trading day. Random variable.

17.2 WHAT IS A MARKOV CHAIN?

I. Ex. Gambler's Problem

(A) p = probability of winning a single bet.

(B) States 0, 1, 2, 3 correspond to having \$0, \$1, \$2, \$3.

(C) X_t = state after t bets.

(D) W_t = winnings on tth bet, which equals

1. -1 if lose bet.
2. 0 if no bet is placed. If you already have \$0 or \$3, no bet is placed.
3. 1 if wins bet.

(E) $X_{t+1} = X_t + W_t$.

(F) p_{ij} = probability of moving from state i to state j on single bet
$= P(X_{t+1} = j | X_t = i)$.

(G) Transition Matrix

row must add up to 1.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note: Winston starts states at $s=1$. In this example it is more natural to start at $s=0$.
"Notions, not notation"

(H) $p_{ij}(n)$ = probability of moving from state i to state j in n bets.
$= P(X_{t+n} = j | X_t = i)$. have \$$i$ to start, place n bets (if not reach \$0 or \$3 first), end with \$$j$.

(I) n-step Transition Matrix

$$P^{(n)} = \begin{bmatrix} p_{00}(n) & p_{01}(n) & p_{02}(n) & p_{03}(n) \\ p_{10}(n) & p_{11}(n) & p_{12}(n) & p_{13}(n) \\ \vdots & & & \\ p_{30}(n) & p_{31}(n) & p_{32}(n) & p_{33}(n) \end{bmatrix} = ?$$

II. Notational Answers to Our Questions

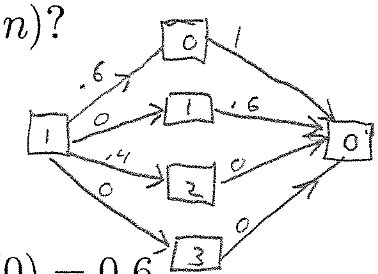
We can answer our question in terms of our notation

(A) Probability of going broke is $\lim_{n\to\infty} p_{10}(n)$, if it exists.

(B) Probability of accumulating \$3 is $\lim_{n\to\infty} p_{13}(n)$, if it exists.

If we ever get to state 0, then we remain there. So $p_{10}(100)$ includes $p_{10}(99)$.

(C) But how do we calculate $p_{ij}(n)$ and $\lim_{n\to\infty} p_{ij}(n)$?



III. Formulas for $p_{ij}(2)$

$$
\begin{aligned}
p_{10}(2) &= p_{10}p_{00} + p_{11}p_{10} + p_{12}p_{20} + p_{13}p_{30} \\
p^{(2)} \qquad &= (.6)(1) + (0)(.6) + (.4)(0) + (0)(0) = 0.6
\end{aligned}
$$

2/14/18 (11)

Test 2/19/18 (12)

P

$$\begin{bmatrix} p_{10}(2) & & & \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} p_{10} & p_{11} & p_{12} & p_{13} \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} p_{00} & & & \\ p_{10} & & & \\ p_{20} & & & \\ p_{30} & & & \end{bmatrix}$$

$$\therefore\ P^{(2)} = P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & .24 & 0 & .16 \\ .36 & 0 & .24 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

IV. Formulas for $p_{ij}(n)$

Notice the parentheses. This is <u>not</u> P raised to the n^{th} power.

$$p_{10}(n) = p_{10}p_{00}(n-1) + p_{11}p_{10}(n-1) + p_{12}p_{20}(n-1) + p_{13}p_{30}(n-1).$$

$$P^{(n)} = P\underbrace{P^{(n-1)} = P}_{\text{by an induction proof.}}P^{n-1} = P^n.$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.60 & 0.24 & 0 & 0.16 \\ 0.36 & 0 & 0.24 & 0.40 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.7440 & 0.0576 & 0 & 0.1984 \\ 0.4464 & 0 & 0.0576 & 0.4960 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.7869 & 0.0033 & 0 & 0.2098 \\ 0.4721 & 0 & 0.0033 & 0.5246 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{16} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.7895 & 0.0000 & 0 & 0.2105 \\ 0.4737 & 0 & 0.0000 & 0.5263 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It appears that $\lim_{n \to \infty} P_{10}(n) = 0.7895$

$$P^{32} = \begin{bmatrix} 1^* & 0 & 0 & 0 \\ 0.7895 & 0.0000 & 0 & 0.2105 \\ 0.4737 & 0 & 0.0000 & 0.5263 \\ 0 & 0 & 0 & 1^* \end{bmatrix}$$

Also provides $\lim_{n \to \infty} P_{20}(n)$, so we know probability of going broke if we start with \$2 instead of \$1.

* Absorbtion states – states 0 and 3.
Transient states – states 1 and 2.
j is <u>reachable</u> from i if $\exists\, n \ni p_{ij}(n) > 0$.
i and j <u>communicate</u> if j is reachable from i and i is reachable from j.

V. Markov Chains

Gambler's Problem

(A) Markovian property

X_t = \$ after bet t.

1. $\{X_t\}$ has the Markovian property if

$(X_{t+1}=2 \mid X_t=1) = .4$

$$P(X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i) = P(X_{t+1} = j \mid X_t = i)$$

regardless how you got to \$1.

for $t = 0, 1, 2, \ldots$ and every $i, j, k_0, k_1, \ldots, k_{t-1}$.

2. Conditional probability depends only on the current state, not the past states. Memoryless property.

(B) Stationary probabilities

Probability of winning a bet remains the same. Contrast with

a. Games of skill, where you might improve over time (shooting baskets at a fair)

b. Fatigue (ringing the bell using a sledge hammer at a fair)

1. $p_{ij} = P(X_{t+1} = j \mid X_t = i) = P(X_1 = j \mid X_0 = i) \quad \forall t.$

$\times$ inserted space

2. $p_{ij}\,(n) = P(X_{t+n} = j \mid X_t = i) = P(X_n = j \mid X_0 = i) \quad \forall t.$

(C) **Finite-state Markov chain** — a stochastic process with the following properties

3, 2, 3

1. Finite number of states.
2. Markovian property.
3. Stationary transition probabilities.
4. A set of initial probabilities $q_i = P(X_0 = i)$.

In our example $P(X_0=i) = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{o.w.} \end{cases}$

(D) Transition matrix

Reminder: states $1, 2, \ldots, s$

Reminder: For Gambler's Problem, we used state 0 for convenience.

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}.$$

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VI. **Ex. Inventory Problem**

(A) Camera store stocks a particular model.

(B) Store can place order for camera on Sat. night and receives order on Mon. morning. Closed on Sunday

(C) D_i = demand for camera in week i.

× (D) D_i are i.i.d. random variables.

Independence ensure Markovian property (dependent sales - high one week, low next week would violate Markovian property) Identical ensures stationary probabilities. (Seasonal sales would violate stationary transition probabilities)

insert space → (E) X_i = # of cameras in stock at end of week i.

(F) Ordering Policy

1. If no cameras are in stock, then order 3.
2. Otherwise, do not order.

(G) Sales are lost when demand exceeds inventory.

$$X_{t+1} = \begin{cases} \max(0, 3-D_t) & \text{if } X_t = 0 \\ \max(0, X_t - D_t) & \text{if } X_t > 0 \end{cases}$$

Again, we use state 0 because it is convenient.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .08 & .184 & .368 & .368 \\ .632 & .368 & 0 & 0 \\ .264 & .368 & .368 & 0 \\ .08 & .184 & .368 & .368 \end{bmatrix} \end{array}$$

Note that rows 0 and 3 must be the same.

No absorbing states.

No transient states. You can always return to a given state.

VII. Ex. Stock Model I

(A) Probability that price of stock increases tomorrow equals

1. 0.7 if price increased today.
2. 0.5 if price decreased today.

State 1: Increased today
2: Decreased today

$$P = \begin{array}{c} \\ +1 \\ -2 \end{array} \begin{array}{c} \begin{array}{cc} +1 & -2 \end{array} \\ \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix} \end{array}$$

VIII. Ex. Stock Model II

(A) Probability that price of stock increases tomorrow equals

1. 0.9 if price increased today and yesterday.
2. 0.6 if price increased today but decreased yesterday.
3. 0.5 if price decreased today but increased yesterday.
4. 0.3 if price decreased today and yesterday.

Might try to formulate as above, but don't have Markov property.

State	
1	++
2	−+
3	+−
4	−−

$$P = \begin{array}{c} \\ ++ \\ -+ \\ +- \\ -- \end{array} \begin{array}{c} \begin{array}{cccc} ++ & -+ & +- & -- \end{array} \\ \begin{bmatrix} .9 & 0 & 0.1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix} \end{array}$$

Note that probabilities depend on two days – need more states to achieve Markovian property.

Do stock prices only depend on price on previous day, or do they depend on a longer history.
This question has been debated.
It appears that for most stocks, price depends only on previous day.

17.3 n-STEP TRANSITION PROBABILITIES CHAPMAN-KOLMOGOROV EQUATIONS

I. 2-Step Equation

We already used this equation for the gambler's proble

(A) $p_{ij}(2) = \sum_{k=1}^{s} p_{ik}p_{kj}$. $p_{i1}p_{1j} + p_{i2}p_{2j} + \cdots + p_{is}p_{sj}$

(B) $P^{(2)} = PP = P^2$.

II. n-Step Equation

X insert space

(A) $p_{ij}(n) = \sum_{k=1}^{s} p_{ik}(v)\, p_{kj}(n-v) \quad \forall i, j, n$ and $0 \le v \le n$.

n-steps = v-steps + $(n-v)$-steps

(B) $p_{ij}(n) = \sum_{k=1}^{s} p_{ik}p_{kj}(n-1)$. Let $V=1$.

(C) $P^{(n)} = PP^{(n-1)} = PP^{n-1} = P^n$.

Stock Model II

$$P^{(2)} = P^2 = \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix} \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix} = \begin{bmatrix} .81 & .05 & .09 & .05 \\ .54 & .20 & .06 & .20 \\ .30 & .15 & .20 & .35 \\ .18 & .21 & .12 & .49 \end{bmatrix}$$

III. Unconditional Probabilities

(A) q_i = probability that process starts in state i.

(B) $P(X_n = j) = q_1p_{1j}(n) + q_2p_{2j}(n) + \cdots + q_sp_{sj}(n)$.

X

Gambler example $q_1 = 1 \quad q_0 = q_2 = q_3 = 0$

$$P(X_n=0) = q_0\, p_{00}(n) + q_1\, p_{10}(n) + q_2\, p_{20}(n) + q_3\, p_{30}(n)$$
$$= 0(\ ) + 1\, p_{10}(n) + 0(\cdot) + 0(\cdot)$$
$$= p_{10}(n)$$

17.4 CLASSIFICATION OF STATES IN A MARKOV CHAIN

I. Reachability

(A) A *path* from state i to j is a sequence of transitions that begins in i and ends in j, such that each transition in the sequence has a positive probability.

(B) State j is *reachable* $(i \to j)$ from i if there is a path from i to j.

1. Equivalently $\exists n \ni p_{ij}(n) > 0$.

(C) States i and j *communicate* $(i \leftrightarrow j)$ if $i \to j$ and $j \to i$.

(D) Properties

1. $i \leftrightarrow i$.
2. $i \leftrightarrow j \Rightarrow j \leftrightarrow i$.
3. $i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$.

(E) States can be partitioned into disjoint classes, where states that communicate belong to the same class.

(F) A Markov chain is *irreducible* if all the states communicate with each other.

(G) A set of states S in a Markov chain is a *closed set* if no state outside of S is reachable from any state in S.

Gambler's Problem

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

0 → 0

1 → 0 1 → 2

2 → 1 2 → 3

3 → 3

{0} and {3} are closed sets.

Classes: {0} {1,2} {3}

Stock Model II

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array}$$

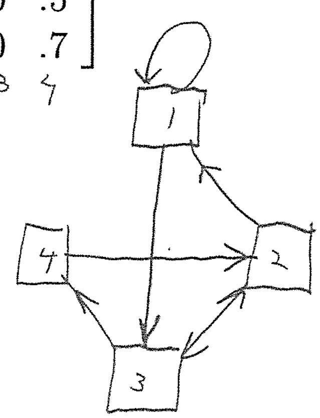
1 → 1 1 → 3

2 → 1 2 → 3

3 → 2 3 → 4

4 → 2 4 → 4

Irreducible



II. Recurrence

(A) f_{ii} = probability that process will ever return to state i given it starts in i.

(B) State i is an *absorbing* state if $p_{ii} = 1$.

(C) State i is *transient* if $f_{ii} < 1$.

1. Equivalently, there is a state j such that $i \to j$ but $j \not\to i$.

(D) State i *recurrent* if $f_{ii} = 1$.

1. Equivalently, i is recurrent if it is not transient.

(E) Properties

1. State i is recurrent $\Rightarrow$ expected number of times that the process is in state i is infinite.
2. State i is transient $\Rightarrow$ expected number of times that the process is in state i is finite.
3. State i is recurrent $\Leftrightarrow \sum_{n=1}^{\infty} p_{ii}(n) = \infty$.

(F) Every state in a class is either recurrent or transient.

(G) All states in an irreducible finite-state Markov chain are recurrent.

Gambler's Problem

$$P = \begin{array}{r} \text{recurrent } 0 \\ \text{Transient } 1 \\ 2 \\ \text{recurrent } 3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \text{-absorbing} \\ p_{10} = 0.6 \Rightarrow f_{11} \le 0.4 \\ p_{23} = .4 \Rightarrow f_{22} \le .6 \\ \text{-absorbing} \end{array}$$

Stock Model II

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \end{array} \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

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3/3/14

How do you tell which states are recurrent for this problem?
Might leave 1 and bounce back and forth between 2 and 3 forever.
Every state communicates with every other state, so all states are recurrent

III. Periodicity

All paths from i to i have a length that is a multiple of t

(A) The *period* of state i is the largest integer $t \ni p_{ii}(n) = 0 \quad \forall$ values of n other than $t, 2t, 3t, \ldots$

Complicated definition but the concept is simple
Can get from i to i only every t^{th} iteration

X (B) A recurrent state is *aperiodic* if it ~~has period 1~~ is not periodic.

Suppose $i \leftrightarrow j$ and shortest path from $i \to j \to i$ has length t. $i \to j \to j \to i$ length is not a multiple of t. ($j \to j$: not a multiple of t)

(C) Every state in a class has the same period.

(D) A chain is *ergodic* if all the states are recurrent (not transient), aperiodic (period 1), and communicate with each other (one class, irreducible).

(E) Ex. Gambler's Problem

$$P^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.600 & 0 & 0.400 & 0 \\ 0 & 0.600 & 0 & 0.400 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.600 & 0.240 & 0 & 0.160 \\ 0.360 & 0 & 0.240 & 0.400 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.744 & 0 & 0.096 & 0.160 \\ 0.360 & 0.144 & 0 & 0.496 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.744 & 0.058 & 0 & 0.198 \\ 0.446 & 0 & 0.058 & 0.496 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.779 & 0 & 0.023 & 0.198 \\ 0.446 & 0.035 & 0 & 0.519 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.779 & 0.014 & 0 & 0.208 \\ 0.467 & 0 & 0.014 & 0.519 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.787 & 0 & 0.006 & 0.208 \\ 0.467 & 0.008 & 0 & 0.525 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.787 & 0.003 & 0 & 0.210 \\ 0.472 & 0 & 0.003 & 0.525 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.789 & 0 & 0.001 & 0.210 \\ 0.472 & 0.002 & 0 & 0.526 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.789 & 0.001 & 0 & 0.210 \\ 0.473 & 0 & 0.001 & 0.526 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$p_{11}(n) = 0$ if n is odd
State 1 has period 2

$p_{22}(n) = 0$ if n is ~~even~~ odd
State 2 has period 2

states 0, 3 are aperiodic

The chain is not ergodic
- transient states
- periodic states
- not all states communicate

(F) Ex. Stock Model II

$$P^1 = \begin{bmatrix} 0.900 & 0 & 0.100 & 0 \\ 0.600 & 0 & 0.400 & 0 \\ 0 & 0.500 & 0 & 0.500 \\ 0.000 & 0.300 & 0 & 0.700 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0.810 & 0.050 & 0.090 & 0.050 \\ 0.540 & 0.200 & 0.060 & 0.200 \\ 0.300 & 0.150 & 0.200 & 0.350 \\ 0.180 & 0.210 & 0.120 & 0.490 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.759 & 0.060 & 0.101 & 0.080 \\ 0.606 & 0.090 & 0.134 & 0.170 \\ 0.360 & 0.205 & 0.090 & 0.345 \\ 0.288 & 0.207 & 0.102 & 0.403 \end{bmatrix} \quad P^4 = \begin{bmatrix} 0.719 & 0.075 & 0.100 & 0.107 \\ 0.599 & 0.118 & 0.097 & 0.186 \\ 0.447 & 0.148 & 0.118 & 0.286 \\ 0.383 & 0.172 & 0.112 & 0.333 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.692 & 0.082 & 0.102 & 0.125 \\ 0.599 & 0.104 & 0.107 & 0.178 \\ 0.491 & 0.145 & 0.104 & 0.260 \\ 0.448 & 0.156 & 0.107 & 0.289 \end{bmatrix} \quad P^6 = \begin{bmatrix} 0.672 & 0.088 & 0.102 & 0.138 \\ 0.612 & 0.107 & 0.103 & 0.179 \\ 0.529 & 0.130 & 0.107 & 0.234 \\ 0.497 & 0.140 & 0.107 & 0.256 \end{bmatrix}$$

$$P^7 = \begin{bmatrix} 0.658 & 0.092 & 0.102 & 0.148 \\ 0.615 & 0.105 & 0.104 & 0.176 \\ 0.554 & 0.124 & 0.105 & 0.217 \\ 0.531 & 0.130 & 0.106 & 0.233 \end{bmatrix} \quad P^8 = \begin{bmatrix} 0.647 & 0.096 & 0.103 & 0.155 \\ 0.616 & 0.105 & 0.103 & 0.175 \\ 0.573 & 0.118 & 0.105 & 0.204 \\ 0.556 & 0.123 & 0.105 & 0.216 \end{bmatrix}$$

$$P^9 = \begin{bmatrix} 0.640 & 0.098 & 0.103 & 0.160 \\ 0.618 & 0.104 & 0.104 & 0.175 \\ 0.586 & 0.114 & 0.104 & 0.196 \\ 0.574 & 0.117 & 0.105 & 0.204 \end{bmatrix} \quad P^{10} = \begin{bmatrix} 0.634 & 0.099 & 0.103 & 0.163 \\ 0.618 & 0.104 & 0.103 & 0.174 \\ 0.596 & 0.111 & 0.104 & 0.189 \\ 0.587 & 0.113 & 0.104 & 0.195 \end{bmatrix}$$

$$P^{11} = \begin{bmatrix} 0.631 & 0.100 & 0.103 & 0.166 \\ 0.619 & 0.104 & 0.103 & 0.174 \\ 0.603 & 0.109 & 0.104 & 0.184 \\ 0.597 & 0.111 & 0.104 & 0.189 \end{bmatrix} \quad P^{12} = \begin{bmatrix} 0.628 & 0.101 & 0.103 & 0.168 \\ 0.620 & 0.104 & 0.103 & 0.173 \\ 0.608 & 0.107 & 0.104 & 0.181 \\ 0.603 & 0.109 & 0.104 & 0.184 \end{bmatrix}$$

$p_{22}(2) = .2 \quad p_{22}(n) > 0 \ \forall n \geq 3 \Rightarrow$ state 2 is aperiodic

$p_{33}(2) = .2 \quad p_{33}(n) > 0 \ \forall n \geq 3 \Rightarrow$ state 3 is aperiodic

$p_{44}(1) = .7 \Rightarrow$ state 4 is aperiodic

Notice that aperiodic does not mean that you can go directly from state i to i in one step. E.g. $p_{22} = 0$

$$P^{32} = \begin{bmatrix} 0.621 & 0.103 & 0.103 & 0.172 \\ 0.621 & 0.103 & 0.103 & 0.172 \\ 0.621 & 0.103 & 0.103 & 0.172 \\ 0.621 & 0.103 & 0.103 & 0.172 \end{bmatrix}$$

All states are recurrent, aperiodic, and one communication class.

$\therefore$ Chain is ergodic.

2/26/18

17.5 STEADY-STATE PROBABILITIES AND MEAN FIRST PASSAGE TIMES

Look at $P^{(32)}$ for Gambler's Problem and Stock Model II.
Note that $P^{(32)}$ has essentially identical rows for Stock Model II, which means probability of being in state 1 after 32 steps is .103 regardless of starting state.

I. Goal: Analyze long-run properties of Markov Chains.

Answer questions < What is the probability of going broke?
" " " " " accumulating \$3?

(A) $\lim_{n\to\infty} p_{ij}(n)$.

(B) $\lim_{n\to\infty} P(X_n = j)$. Independent of starting state.

(C) Expected number of transitions to move from i to j.

all recurrent, aperiodic, one class

II. Steady-State Probabilities for Ergodic Markov Chains

(A) Th: If $\{X_t\}$ is an ergodic Markov chain, then

π may not exist if not ergodic.
Ex. $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $P^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1. $\exists\, \pi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_s \end{bmatrix} \ni$

It is one thing to raise a matrix to a large power and it looks like it converges. It is another thing to provide sufficient conditions and prove convergence.

$$\lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_s \\ \pi_1 & \pi_2 & \cdots & \pi_s \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_s \end{bmatrix}.$$

2. $\lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} P(X_n = j) = \pi_j$.

(B) $\pi_1, \pi_2, \ldots, \pi_s$ found by solving the *steady-state* equations:

Probability of being in state 1 after next step.

$$\rightarrow \pi_1 = \overbrace{\pi_1 p_{11}}^{\text{probability in 1 * probability of moving } 1\to 1} + \pi_2 p_{21} + \cdots + \pi_s p_{s1}$$
$$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \cdots + \pi_s p_{s2}$$
$$\vdots$$
$$\pi_s = \pi_1 p_{1s} + \pi_2 p_{2s} + \cdots + \pi_s p_{ss}$$
$$1 = \pi_1 + \pi_2 + \cdots + \pi_s.$$

- π's represent probabilities, so must sum to 1.
- Need this equation, o.w. $\pi = [0\,0 \cdots 0]$ works

1. In matrix form, $\pi = \pi P$.

3/5/14 (15)

(C) π_j's are called the *steady-state probabilities (distribution).*

(D) π_j is the long-run probability of finding the process in state j.

(E) Ex. Steady-State Probabilities for Stock Model II

$$P = \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix}$$

(1) $\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} + \pi_4 p_{41} = .9\pi_1 + .6\pi_2$

(2) $\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} + \pi_4 p_{42} = .5\pi_3 + .3\pi_4$

(3) $\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} + \pi_4 p_{43} = .1\pi_1 + .4\pi_2$

(4) $\pi_4 = \pi_1 p_{14} + \pi_2 p_{24} + \pi_3 p_{34} + \pi_4 p_{44} = .5\pi_3 + .7\pi_4$

(5) $1 = \pi_1 + \pi_2 + \pi_3 + \pi_4$

(1') $\pi_1 = 6\pi_2$

(4') $\pi_3 = .6\pi_4$

(2') $\pi_2 = .5\pi_3 + .3\pi_4 = .3\pi_4 + .3\pi_4 = .6\pi_4$

$\therefore \pi_2 = \pi_3$

(5') $1 = \pi_1 + \pi_2 + \pi_3 + \pi_4 = 6\pi_2 + \pi_2 + \pi_2 + \frac{\pi_2}{.6}$

$1 = \pi_2 \left(8 + \frac{1}{.6}\right)$

$\pi_1 = .103448276$

$\pi_1 = .620689655$

$\pi_2 = .103448276$

$\pi_3 = .103448276$

$\pi_4 = .172413793$

III. Matrix Approach to Solving Steady-State Probabilities

(A) Method

1. $\pi = \pi P$

$$[0, \cdots 0] = \pi P - \pi$$
$$[0 \cdots\cdots 0] = \pi (P - I)$$

2. Must replace one of the constraints with $1 = \pi_1 + \pi_2 + \cdots + \pi_s$.

   a. Corresponds to replacing a column in $P - I$ with $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

   b. Can replace any column.

   c. For consistency, replace the last column.

   d. Let $\overline{P}$ be the modified matrix.

3. $\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} = \pi \overline{P}$

$$[0 \cdots 0 \ 1]\, \overline{P}^{-1} = \pi \overline{P}\, \overline{P}^{-1} = \pi I = \pi$$
$$\therefore \pi = \text{last row of } \overline{P}^{-1}$$

(B) Ex. Steady-State Probabilities for Stock Model II

$$\pi = \pi P \qquad P = \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix}$$

$$P - I = \begin{bmatrix} -.1 & 0 & .1 & 0 \\ .6 & -1 & .4 & 0 \\ 0 & .5 & -1 & .5 \\ 0 & .3 & 0 & -.3 \end{bmatrix} \qquad \overline{P} = \begin{bmatrix} -.1 & 0 & .1 & 1 \\ .6 & -1 & .4 & 1 \\ 0 & .5 & -1 & 1 \\ 0 & .3 & 0 & 1 \end{bmatrix}$$

$$\overline{P}^{-1} = \begin{bmatrix} -4.2069 & .9655 & -.0345 & 3.2759 \\ -2.0690 & -.3448 & -.3448 & 2.7586 \\ -.4138 & -.0690 & -1.0690 & 1.5517 \\ .6207 & .1034 & .1034 & .1724 \end{bmatrix} \quad \pi$$

IV. Transient Analysis

(A) The behavior of a Markox chain before steady state is reached is the *transient (short-run)* behavior.

(B) There is no general rule for how quickly steady state is reached.

1. Often 10 to 30 steps is sufficient. Ex. $P^{(32)}$ for Gambler's Problem and Stock Model II.

V. Intuitive Interpretation of Steady-State Probabilities

(A) Steady-state equation for state j

$$\pi_j = \pi_1 p_{1j} + \pi_2 p_{2j} + \cdots + \pi_j p_{jj} + \cdots + \pi_s p_{sj}$$

$$\underline{\pi_j (1 - p_{jj})} = \pi_1 p_{1j} + \pi_2 p_{2j} + \cdots + \pi_{j-1} p_{j-1,j} + \pi_{j+1} p_{j+1,j} + \cdots + \pi_s p_{sj}$$

Probability that you leave j given that you are in it.

RHS = probability that you enter j given that you are not in it.

(B) "Flow out" = "Flow in".

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VI. Expectations for Irreducible Finite-State Markov

(A) $\lim_{n\to\infty} p_{ij}(n)$ may not exist for non aperiodic Markov chains.

(B) Ex.

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad P^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p_{11}(n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases} \qquad \therefore \lim_{n\to\infty} p_{11}(n) \text{ does not exist.}$$

Ask students what % of time is spent in state 1.

(C) If X_t is an irreducible finite-state Markov chain, then

$$\frac{\lim_{n\to\infty}\left(\sum_{k=1}^{n} p_{ij}(k)\right)}{n} = \pi_j,$$

where the π_j's satisfy the steady-state equations.

(D) Ex. (cont.)

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = \pi_2 \qquad \pi_1 = \pi_2 = \tfrac{1}{2}$$
$$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} = \pi_1$$
$$1 = \pi_1 + \pi_2$$

(E) π_j = long-run expected fraction of time the system is in state j.

$\pi_j \neq$ probability of being in state j. Use above example. $p_{11}(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

VII. Expected Average Cost Per Unit Time

(A) $C(X_t)$ = cost incurred when the process is in state X_t at time t.

(B) Long-run expected average cost per unit time $= \sum_{j=1}^{s} \pi_j C(j)$.

(C) Ex. Stock Model II Investment Strategy

1. Investment Strategy A

   a. If price increased today, use option to bet price will increase tomorrow.
   $4 if price increases tomorrow.
   $-5 if price decreases tomorrow.

   b. If price decreased today, use option to bet price will decrease tomorrow.
   $-4 if price increases tomorrow.
   $3 if price decreases tomorrow.

   c. What is the long-run expected payoff per day?

$$P = \begin{array}{c} \\ ++ \\ -+ \\ +- \\ -- \end{array} \begin{array}{c} \begin{array}{cccc} ++ & -+ & +- & -- \end{array} \\ \begin{bmatrix} .9 & 0 & .1 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .3 & 0 & .7 \end{bmatrix} \end{array} \qquad \begin{array}{l} C(++)=4 \\ C(-+)=-4 \\ C(+-)=-5 \\ C(--)=3 \end{array} \qquad \begin{array}{l} \pi_1=.621 \\ \pi_2=.103 \\ \pi_3=.103 \\ \pi_4=.172 \end{array}$$

long-run expected payoff per day $= \sum_{j=1}^{4} \pi_j C(j)$

$= (.621)(4)+(.103)(-4)+(.103)(-5)+(.172)(3)$

$= 2.073$

2. Investment Strategy B

   a. What if we only bet when the price increased today?

long-run expected payoff per day $= \pi_1 C(1) + \pi_2 C(2) + \pi_3 C(3) + \pi_4 C(4)$

$= (.621)(4)+(.103)(0)+(.103)(-5)+(.172)(0)$

$= 1.969$

(D) Ex. Simplified Monopoly

1. Board consists of 8 squares.
2. Piece moves clockwise.
   a. One square: probability p.
   b. Two squares: probability $1-p$.
3. Each square is worth a certain amount of \$.
4. What is the long-run expected payoff per turn?

1 \$200	2 \$100	3 -100
8 -100		4 -100
7 300	6 -200	5 100

$$P = \begin{bmatrix} & p & 1-p & & & & & \\ & & p & 1-p & & & & \\ & & & p & 1-p & & & \\ & & & & p & 1-p & & \\ & & & & & p & 1-p & \\ & & & & & & p & 1-p \\ 1-p & & & & & & & p \\ p & 1-p & & & & & & \end{bmatrix}$$

Recurrent states, Aperiodic, Irreducible $\Rightarrow$ ergodic

$\Rightarrow$ steady-state probabilities exist.

$\pi = \pi P$

$$\pi_1 = (1-p)\,\pi_7 + p\pi_8$$
$$\pi_2 = (1-p)\,\pi_8 + p\pi_1$$
$$\pi_3 = (1-p)\,\pi_1 + p\pi_2$$
$$\pi_4 = (1-p)\,\pi_2 + p\pi_3$$
$$\pi_5 = (1-p)\,\pi_3 + p\pi_4$$
$$\pi_6 = (1-p)\,\pi_4 + p\pi_5$$
$$\pi_7 = (1-p)\,\pi_5 + p\pi_6$$
$$\pi_8 = (1-p)\,\pi_6 + p\pi_7$$
$$1 = \pi_1 + \pi_2 + \cdots + \pi_s$$

$\pi_1 = \pi_2 = \cdots = \pi_8 = \frac{1}{8}$

This makes intuitive sense because all squares are alike in their probability of being hit.

long-run expected payoff per turn $= \sum_{j=1}^{8} \pi_j C(j)$

$= \frac{1}{8}(200) + \frac{1}{8}(100) + \frac{1}{8}(-100) + \frac{1}{8}(-100) + \frac{1}{8}(100) + \frac{1}{8}(-200) + \frac{1}{8}(300) + \frac{1}{8}(-100)$

$= \frac{1}{8}(200) = 25$

(Can almost evaluate real Monopoly game after all properties have been purchased and all development has taken place. Exceptions: \$200 for crossing Go, chance cards, jail). Table 3

TABLE 3
Steady-State Probabilities for Monopoly

n	Position	Steady-State Probability
0	Go	.0346
1	Mediterranean Ave.	.0237
2	Community Chest 1	.0218
3	Baltic Ave.	.0241
4	Income tax	.0261
5	Reading RR	.0332
6	Oriental Ave.	.0253
7	Chance 1	.0096
8	Vermont Ave.	.0258
9	Connecticut Ave.	.0237
10	Visiting jail	.0254
11	St. Charles Place	.0304
12	Electric Co.	.0311
13	State Ave.	.0258
14	Virginia Ave.	.0288
15	Pennsylvania RR	.0313
16	St. James Place	.0318
17	Community Chest 2	.0272
18	Tennessee Ave.	.0335
19	New York Ave.	.0334
20	Free parking	.0336
21	Kentucky Ave.	.0310
22	Chance 2	.0125
23	Indiana Ave.	.0305
24	Illinois Ave.	.0355
25	B and O RR	.0344
26	Atlantic Ave.	.0301
27	Ventnor Ave.	.0299
28	Water works	.0315
29	Marvin Gardens	.0289
30	Jail	.1123
31	Pacific Ave.	.0300
32	North Carolina Ave.	.0294
33	Community Chest 3	.0263
34	Pennsylvania Ave.	.0279
35	Short Line RR	.0272
36	Chance 3	.0096
37	Park Place	.0245
38	Luxury tax	.0295
39	Boardwalk	.0295

Source: Reprinted by permission from R. Ash and R. Bishop, "Monopoly as a Markov Process," *Mathematics Magazine* 45(1972):26–29.

VIII. First Passage Times

(A) First passage time = # of steps from state i to j for first time.

(B) Recurrence time = # of steps from state i to i for first time.

(C) $f_{ij}(n)$ = probability that first passage time from i to j is n.

(D) Equations

$$
\begin{aligned}
f_{ij}(1) &= p_{ij} \\
f_{ij}(2) &= p_{ij}(2) - f_{ij}(1)\, p_{jj} \\
&\vdots \\
f_{ij}(n) &= p_{ij}(n) - f_{ij}(1)\, p_{jj}(n-1) - f_{ij}(2)\, p_{jj}(n-2) - \cdots - f_{ij}(n-1)\, p_{jj}
\end{aligned}
$$

IX. Expected First Passage Time

(A) $\sum_{n=1}^{\infty} f_{ij}(n) \leq 1$.

1. $\sum_{n=1}^{\infty} f_{ij}(n) < 1 \Rightarrow$ may never reach j from i.
2. $\sum_{n=1}^{\infty} f_{ij}(n) = 1$, then $f_{ij}(1), f_{ij}(2), \ldots$ is probability distribution for first passage time.

(B) m_{ij} = expected first passage time from i to j.

$$
m_{ij} = \begin{cases} \infty & \text{if } \sum_{n=1}^{\infty} f_{ij}(n) < 1 \\ \sum_{n=1}^{\infty} n f_{ij}(n) & \text{if } \sum_{n=1}^{\infty} f_{ij}(n) = 1 \end{cases}
$$

(C) $m_{1j}, m_{2j}, \ldots, m_{sj}$ found by solving

$$
\begin{aligned}
m_{1j} &= 1 + \sum_{k \neq j} p_{1\cancel{j}k} m_{kj} = p_{1j} \cdot 1 + \sum_{k \neq j} p_{1k}(1 + m_{kj}) \\
m_{2j} &= 1 + \sum_{k \neq j} p_{2\cancel{j}k} m_{kj} \\
&\vdots \\
m_{sj} &= 1 + \sum_{k \neq j} p_{s\cancel{j}k} m_{kj}
\end{aligned}
$$

(D) $m_{ii} = 1/\pi_i$.

Work this problem after Stock Model II

(E) Ex. Passage Times for Gambler's Problem:
$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & .6 & 0 & .4 & 0 \\ 2 & 0 & .6 & 0 & .4 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$\underline{1 \to 0}$

$f_{10}(1) = p_{10} = 0.6$ $\qquad 0.6$

$f_{10}(2) = p_{10}(2) - f_{10}(1)\,p_{00} = .6 - (.6)(1) = 0$ $\qquad 0$

$f_{10}(3) = p_{10}(3) - f_{10}(1)\,p_{00}(2) - f_{10}(2)\,p_{00} = .744 - .6(1) - (0)(1) = .144$ $\qquad .144$

$f_{10}(4) = p_{10}(4) - f_{10}(1)p_{00}(3) - f_{10}(2)p_{00}(2) - f_{10}(3)p_{00} = .744 - .6(1) - (0)(1) - (.144)(1) = 0$ $\qquad 0$

$f_{10}(5) = p_{10}(5) - f_{10}(1)p_{00}(4) - f_{10}(2)p_{00}(3) - f_{10}(3)p_{00}(2) - f_{10}(4)p_{00} = .77856 - .6 - .144 =$ $\qquad .03456$

$f_{10}(6) =$ $\qquad 0$

$f_{10}(7) =$ $\qquad .008294$

$\qquad\qquad\qquad\qquad \overline{.786854}$

$$\sum_{n=1}^{\infty} f_{10}(n) = \lim_{n\to\infty} p_{10}(n) = .789474$$

Expected passage time from 1 to 0 is infinite, since there is a positive probability of never making it.

$$\text{Expected} = \sum_{n=1}^{\infty} n\, f_{10}(n) + \infty\left(\lim_{n\to\infty} p_{13}(n)\right)$$
$$+ \infty\,(.210526)$$

(F) Ex. Passage Times for Stock Model II:
$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & .9 & 0 & .1 & 0 \\ 2 & .6 & 0 & .4 & 0 \\ 3 & 0 & .5 & 0 & .5 \\ 4 & 0 & .3 & 0 & .7 \end{array}$$

$\underline{j \to 1}$

$m_{11} = 1/\pi_1 = \frac{1}{.620690} = 1.611$

$m_{21} = 1 + p_{22}m_{21} + p_{23}m_{31} + p_{24}m_{41} = 1 + .4m_{31}$

$m_{31} = 1 + p_{32}m_{21} + p_{33}m_{31} + p_{34}m_{41} = 1 + .5m_{21} + .5m_{41}$

$m_{41} = 1 + p_{42}m_{21} + p_{43}m_{31} + p_{44}m_{41} = 1 + .3m_{21} + .7m_{41} \to .3m_{41} = 1 + .3m_{21}$

$$\begin{array}{ccc|c} m_{21} & m_{31} & m_{41} & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -.4 & 0 & 1 \\ -.5 & 1 & -.5 & 1 \\ -.3 & 0 & .3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -.4 & 0 & 1 \\ 0 & .8 & -.5 & 3/2 \\ 0 & -.12 & .3 & 1.3 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & -.4 & 0 & 1 \\ 0 & .8 & -.5 & 3/2 \\ 0 & 0 & .225 & 1.525 \end{array}\right]$$

$m_{21} = 3.444$ $\qquad$ Could also find $m_{12}, m_{22}, m_{32}, m_{42}$

$m_{31} = 6.111$ $\qquad\qquad\qquad\qquad m_{13}, m_{23}, m_{33}, m_{43}$

$m_{41} = 6.778$ $\qquad\qquad\qquad\qquad m_{14}, m_{24}, m_{34}, m_{44}$

3/12/18 16
Sick 3/14/18 17
17
3/19/14

17.6 ABSORBING CHAINS

I. Suppose that a Markov chain has absorbtion states and that eventually one of the absorbing states will be reached.

(A) Let k be an absorbing state.

(B) Let f_{ik} = probability of being absorbed into k starting from i.

(C) $f_{kk} = 1$, since $p_{kk} = 1$.

(D) $f_{ik} = 0$, if i is a recurrent state.

(E) $f_{1k}, f_{2k}, \ldots, f_{sk}$ found by solving

$$\begin{aligned} f_{1k} &= p_{11} f_{1k} + p_{12} f_{2k} + \cdots + p_{1s} f_{sk} \\ f_{2k} &= p_{21} f_{1k} + p_{22} f_{2k} + \cdots + p_{2s} f_{sk} \\ &\vdots \\ f_{sk} &= p_{s1} f_{1k} + p_{s2} f_{2k} + \cdots + p_{ss} f_{sk}. \end{aligned}$$

$f = Pf$

II. **Ex. Gambler's Problem:** $P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

$f_{00} = 1$

$f_{10} = p_{10} f_{00} + p_{11} f_{10} + p_{12} f_{20} + p_{13} f_{30} = .6(1) + 0 f_{10} + .4 f_{20} + (0)(0) = .6 + .4 f_{20}$

$f_{20} = p_{20} f_{00} + p_{21} f_{10} + p_{22} f_{20} + p_{23} f_{30} = 0(1) + .6 f_{10} + 0 f_{30} + .4(0) = .6 f_{10}$

$f_{30} = 0$

$f_{20} = .6 f_{10}$ $\quad f_{10} = .6 + .4 f_{20} = .6 + .4(.6 f_{10}) \longrightarrow f_{10} = .789474$

$f_{20} = .473684$

III. Ex. State College

(A) Student's progress can be modeled as a Markov chain.

(B) Student's class status is observed at beginning of fall semester.

(C) When students quit, they never re-enroll.

(D) Probabilities given in the transition matrix below.

(E) Questions:

1. Probability that a freshman graduates?
2. Probability that a freshman graduates in 4 years? $p_{16}(4)$
3. Expected number of years for an entering freshman to graduate? m_{16}

		1 F.	2 So.	3 J.	4 Sen.	5 Q.	6 G.
1	Freshman	.10	.80	0	0	.10	0
2	Sophomore	0	.10	.85	0	.05	0
3	Junior	0	0	.15	.80	.05	0
4	Senior	0	0	0	.10	.05	.85
5	Quits	0	0	0	0	1	0
6	Graduates	0	0	0	0	0	1

Probability that a freshman graduates

$f_{66} = 1$

$f_{56} = 0$

$f_{16} = .1 f_{16} + .8 f_{26} + .1 \cancel{f_{56}}^{0} \longrightarrow .9 f_{16} = .8 f_{26} \longrightarrow f_{16} = .746228$

$f_{26} = .1 f_{26} + .85 f_{36} + .05 \cancel{f_{56}}^{0} \longrightarrow .9 f_{26} = .85 f_{36} \longrightarrow f_{26} = .839506$ 3/19/18 (18)

$f_{36} = .15 f_{36} + .8 f_{46} + .05 \cancel{f_{56}}^{0} \longrightarrow .85 f_{36} = .8 f_{46} \rightarrow f_{36} = .8888$

$f_{46} = .1 f_{46} + .05 \cancel{f_{56}}^{0} + .85 f_{66} = .1 f_{46} + .85 \longrightarrow .9 f_{46} = .85 \quad f_{46} = \frac{.85}{.9} = .9444$

$f_{56} = 0$

$f_{66} = 1$ Probability that a freshman graduates $= .746228$

(18)
3/24/14

Summary of Notation and Definitions

- A *stochastic process* is an indexed collection of random variables $\{X_t\}$, where the index t runs through a given set T
- States $1, 2, \ldots, s$ — mutually exclusive categories for values of X_t
- $p_{ij} = P(X_{t+1} = j | X_t = i)$
- $p_{ij}(n) = P(X_{t+n} = j | X_t = i)$
- $P =$ transition matrix
- $P^{(n)} = n$-step transition matrix
- Chapman-Kolmogorov equations: $p_{ij}(n) = \sum_{k=0}^{s} p_{ik}(v) p_{kj}(n-v) \quad \forall i, j, n$ and $0 \leq v \leq n$
- $q_i = P(X_0 = i)$
- State j is *reachable* $(i \to j)$ from i if there is a path from i to j.
- States i and j *communicate* $(i \leftrightarrow j)$ if $i \to j$ and $j \to i$.
- A Markov chain is *irreducible* if all the states communicate with each other
- A set of states S in a Markov chain is a *closed set* if no state outside of S is reachable from any state in S.
- $f_{ii} =$ probability that process will ever return to state i given it starts in i
- State i is an *absorbing* state if $p_{ii} = 1$.
- State i is *transient* if there is a state j such that $i \to j$ but $j \nrightarrow i$.
- State i is *recurrent* if it is not transient.
- The *period* of state i is the largest integer $t \ni p_{ii}(n) = 0 \quad \forall$ values of n other than $t, 2t, 3t, \ldots$
- A recurrent state is *aperiodic* if it is not periodic.
- Every state in a class has the same period.
- A chain is *ergodic* if all the states are recurrent, aperiodic, and communicate with each other.
- A recurrent state i is *positive recurrent* if, starting in i, the expected time for the process to reenter i is finite. Otherwise state i is *null recurrent.*
- In a finite-state Markov chain all recurrent states are positive recurrent.
- A positive recurrent state that is aperiodic is *ergodic*
- First passage time = # of steps in going from state i to j for the first time
- Recurrence time = # of steps in going from state i to i for the first time
- $f_{ij}(n) =$ probability that first passage time from i to j is n
- $m_{ij} =$ expected first passage time from i to j
- $\pi_j =$ the steady-state probabilities of the Markov chain

Section 17.2

1. State 1: Sunny today
   State 2: Cloudy today

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \end{array}$$

2. State = inventory level at beginning of period
   If $i \leq 1$, order $4-i$ units.

D	$P[D=d]$
0	$\frac{1}{3}$
1	$\frac{1}{3}$
2	$\frac{1}{3}$
3	0

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

3. State = # of machines working at beginning of day
   Each machine has a $\frac{1}{3}$ chance of breaking down during a day.
   Assume machines break down independently.
   Machines return to service two days later.

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix} \end{array}$$

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Section 17.2

4. State 1: Sunny, Sunny
   2: Sunny, Cloudy
   3: Cloudy, Sunny
   4: Cloudy, Cloudy

(Yesterday, Today)

$$P = \begin{array}{c} \\ SS \\ SC \\ CS \\ CC \end{array} \begin{array}{cccc} SS & SC & CS & CC \\ \hline \end{array}$$

$$P = \begin{array}{c|cccc} & SS & SC & CS & CC \\ SS & .95 & .05 & 0 & 0 \\ SC & 0 & 0 & 0.40 & .60 \\ CS & .70 & .30 & 0 & 0 \\ CC & 0 & 0 & .20 & .80 \end{array}$$

6. Continuation of 3.

Machine returns to service three days later.

Each machine has $\frac{1}{3}$ chance of breaking down.

Let a state be described by $[W\ b_0\ b_1]$,

where W = # of machines working at beginning of the day

b_0 = # of broken machines that have spent 0 days being repaired

b_1 = " " " " " " " 1 day " "

$$\begin{array}{c} \\ [2\ 0\ 0] \\ [1\ 0\ 1] \\ [1\ 1\ 0] \\ [0\ 2\ 0] \\ [0\ 1\ 1] \\ [0\ 0\ 2] \end{array} \begin{array}{c} \begin{array}{cccccc} [2\,0\,0] & [1\,0\,1] & [1\,1\,0] & [0\,2\,0] & [0\,1\,1] & [0\,0\,2] \end{array} \\ \begin{bmatrix} \frac{4}{9} & 0 & \frac{4}{9} & \frac{1}{9} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3/3/14

Section 17.3

1 State 1: U = urban
2: S = suburban
3: R = rural

$$P = \begin{array}{c|ccc} & U & S & R \\ \hline U & .80 & .15 & .05 \\ S & .06 & .90 & .04 \\ R & .04 & .06 & .9 \end{array}$$

a.
$$P^{(2)} = P^2 = \begin{bmatrix} .80 & .15 & .05 \\ .06 & .90 & .04 \\ .04 & .06 & .9 \end{bmatrix} \begin{bmatrix} .80 & .15 & .05 \\ .06 & .90 & .04 \\ .04 & .06 & .9 \end{bmatrix} = \begin{bmatrix} .651 & .258 & 0.0910 \\ .1036 & .8214 & .0750 \\ .0716 & .1140 & .8144 \end{bmatrix}$$

$p_{11}(2) = .651 \quad p_{12}(2) = .258 \quad p_{13}(2) = .0910$

b. $q = [.4 \quad .35 \quad .25]$

$$q P^{(2)} = [.4 \quad .35 \quad .25] \begin{bmatrix} .651 & .258 & .0910 \\ .1036 & .8214 & .0750 \\ .0716 & .1140 & .8144 \end{bmatrix} = [.3146 \quad .4192 \quad .2663]$$

$\therefore$ 31.46% of the population of families will live in urban areas.

C. Migration patterns change over time. Therefore, this is a non-stationary stochastic process. Patterns change slowly over time, so the model may be useful for up to 5 or even 10 years.

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Section 17.3

2.
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a.
$$P^{(2)} = P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & p(1-p) & p^2 & 0 & 0 \\ (1-p)^2 & p(1-p) & 2p(1-p) & & \\ & & & & \\ & & & & \end{bmatrix}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & p(1-p) & p^2 & 0 & 0 \\ (1-p)^2 & 0 & 2p(1-p) & 0 & p^2 \\ 0 & (1-p)^2 & 0 & p(1-p) & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$p_{23}(2) = 0$ $\qquad\qquad$ $p_{22}(2) = 2p(1-p)$

b. $$p_{22}(3) = \underbrace{\begin{bmatrix} (1-p)^2 & 0 & p^2 & 0 & p \end{bmatrix}}_{\text{row 2 of } P^2} \underbrace{\begin{bmatrix} 0 \\ p \\ 0 \\ 1-p \\ 0 \end{bmatrix}}_{\text{column 2 of } P}$$

$= 0 + 0 + 0 + 0 + 0$

$= 0$

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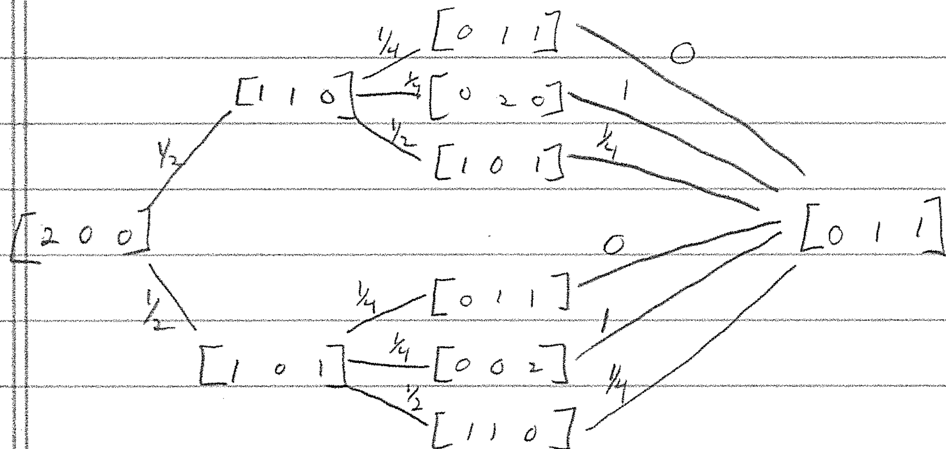
Section 17.3

3. $$P = \begin{array}{c} \\ [0\,1\,1] \\ [0\,2\,0] \\ [0\,0\,2] \\ [2\,0\,0] \\ [1\,1\,0] \\ [1\,0\,1] \end{array} \begin{array}{c} \begin{array}{cccccc} [0\,1\,1] & [0\,2\,0] & [0\,0\,2] & [2\,0\,0] & [1\,1\,0] & [1\,0\,1] \end{array} \\ \begin{bmatrix} & \frac{1}{2} & \frac{1}{2} & & & \\ 1 & & & & & \\ 1 & & & & & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & & & & \frac{1}{2} \\ \frac{1}{4} & & \frac{1}{4} & & \frac{1}{2} & \end{bmatrix} \end{array}$$

a. You can answer this question by constructing $P^{(2)} = P^2$ or by noticing that the only path from [2 0 0] to [0 2 0] in 2 steps is through [1 1 0]

$\therefore \; P_{[200][110]} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}$

b. You can answer this question by constructing $P^{(3)} = P^3$ or by using the diagram below



$$P_{[200][011]}(3) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(0) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(1) + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(0) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(1) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)$$
$$= 3/8$$

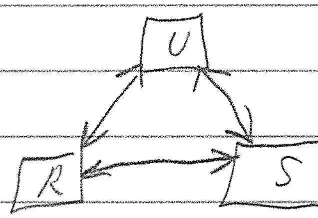
3/3/14

Section 17.4

1.
$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1-p & 0 & p & 0 & 0 \\ 2 & 0 & 1-p & 0 & p & 0 \\ 3 & 0 & 0 & 1-p & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

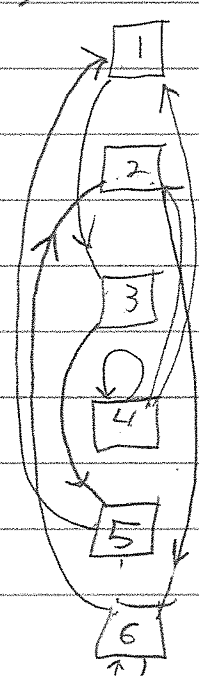
States 1, 2, 3 all have period 2.
Starting from one of these states, it will require an equal number of wins and losses to return. Therefore, it requires an even # of moves.

2.
$$P = \begin{array}{c|ccc} & U & S & R \\ \hline U & .8 & .15 & .05 \\ S & .06 & .9 & .04 \\ R & .04 & .06 & .9 \end{array}$$



All states are recurrent.
All states are aperiodic
There is one communication class
$\Rightarrow$ Chain is ergodic.

3.
$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \end{array}$$



$1 \leftrightarrow 3 \leftrightarrow 5$

$2 \leftrightarrow 6$

4

a. 4 is transient. $4 \to 1$ $1 \not\to 4$

b. 1, 2, 3, 5, 6 are recurrent

c. $\{1,3,5\}$, $\{2,6\}$ are closed sets

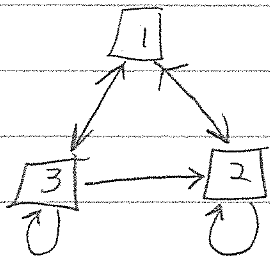
d. The chain is not ergodic.
It contains a transient state.
It has more than one communication class

3/3/14

Section 17.4

4\. a.

$$P_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & .8 & .2 \\ .3 & .7 & 0 \\ .4 & .5 & .1 \end{bmatrix} \end{array}$$

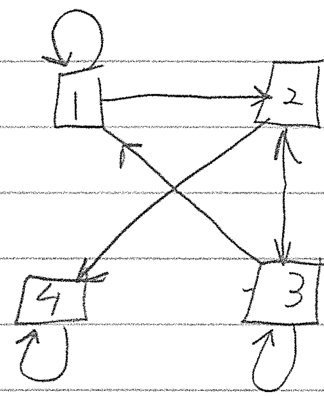


All states recurrent.
All states aperiodic
One communication class

$\Rightarrow$ ergodic

b.

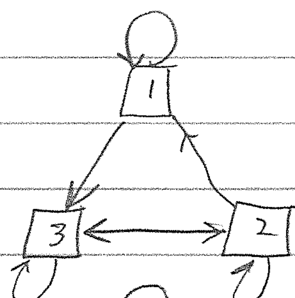
$$P_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} .2 & .8 & 0 & 0 \\ 0 & 0 & .9 & .1 \\ .4 & .5 & .1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



1,2,3 are transient $\Rightarrow$ Chain is not ergodic
4 is recurrent
4 is absorbing

6\. a.

$$P_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .4 & 0 & .6 \\ .3 & .3 & .4 \\ 0 & .5 & .5 \end{bmatrix} \end{array}$$

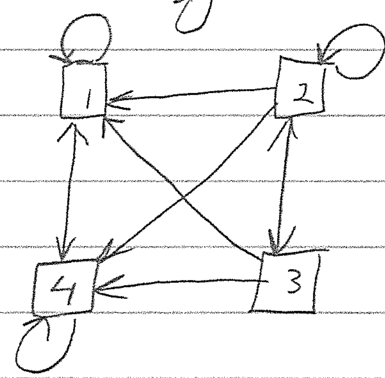


All states recurrent
All states aperiodic
One communication class

$\Rightarrow$ ergodic

b.

$$P_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} .7 & 0 & 0 & .3 \\ .2 & .2 & .4 & .2 \\ .6 & .1 & .1 & .2 \\ .2 & 0 & 0 & .8 \end{bmatrix} \end{array}$$



2,3 are transient $\Rightarrow$ Chain is not ergodic

1, 3, 5, 7, 11, ~~13~~

3/18/14

Section 17.5

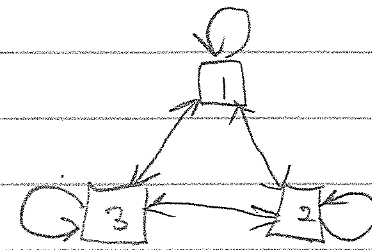
1. $P = \begin{bmatrix} .80 & .15 & .05 \\ .06 & .90 & .04 \\ .04 & .06 & .90 \end{bmatrix}$

$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} = .8\pi_1 + .06\pi_2 + .04\pi_3$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} = .15\pi_1 + .9\pi_2 + .06\pi_3$

$\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} = .05\pi_1 + .04\pi_2 + .9\pi_3$

$1 = \pi_1 + \pi_2 + \pi_3$



$\bar{P} = \begin{bmatrix} -.2 & .15 & 1 \\ .06 & -.10 & 1 \\ .04 & .06 & 1 \end{bmatrix}$ $[0\ 0\ 1] = \pi\bar{P}$

$[0\ 0\ 1]^T = \bar{P}^T \pi^T$

Recurrent, aperiodic, once class.

$\therefore$ Steady-state probabilities exist.

$$\left[\begin{array}{ccc|c} -.2 & .06 & .04 & 0 \\ .15 & -.10 & .06 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ .15 & -.10 & .06 & 0 \\ -.2 & .06 & .04 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -.25 & -.09 & -.15 \\ 0 & .26 & .24 & .20 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -.25 & -.09 & -.15 \\ 0 & 0 & .1464 & .044 \end{bmatrix}$$

$\pi_1 = .2076503$

$\pi_2 = .4918032$

$\pi_3 = .3005464$

3. a. $P = \begin{bmatrix} 2/3 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 2/3\,\pi_1 + \frac{1}{2}\pi_2$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2$

$1 = \pi_1 + \pi_2$ $\pi_2 = 1 - \pi_1$

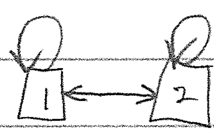
$\pi_1 = 2/3\,\pi_1 + \frac{1}{2}(1 - \pi_1)$

$\pi_1 = 2/3\,\pi_1 + \frac{1}{2} - \frac{1}{2}\pi_1$

$(1 - \frac{2}{3} + \frac{1}{2})\pi_1 = \frac{1}{2}$

$\pi_1 = (\frac{6}{5})(\frac{1}{2}) = 3/5$

$\pi_2 = 2/5$



Recurrent, aperiodic, one class.

$\therefore$ Steady state probabilities exist.

c.

$m_{11} = \frac{1}{\pi_1} = 5/3$

$m_{12} = 1 + p_{12}\, m_{21} = 1 + \frac{1}{3} m_{21} \rightarrow m_{12} = 1 + \frac{1}{3}(1 + \frac{1}{2} m_{12}) \rightarrow m_{12} = \frac{4}{3} + \frac{1}{6} m_{12}$

$m_{21} = 1 + p_{21}\, m_{12} = 1 + \frac{1}{2} m_{12}$

$m_{12} = \frac{6}{5}(\frac{4}{3}) = 8/5$

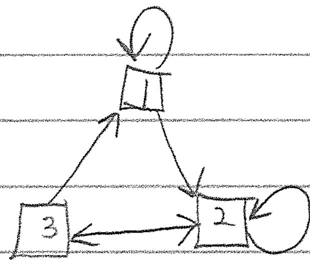
$m_{22} = \frac{1}{\pi_2} = 5/2$

$m_{21} = 1 + \frac{1}{2}(\frac{8}{5}) = 9/5$

3/18/14

Section 17.5

3. b. $p = \begin{bmatrix} .8 & .2 & 0 \\ 0 & .2 & .8 \\ .8 & .2 & 0 \end{bmatrix}$



Recurrent, aperiodic, one class
$\therefore$ Steady-state probabilities exist.

$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} = .8\pi_1 + .8\pi_3$ $\quad .2\pi_1 = .8(.16) \rightarrow \pi_1 = .64$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} = .2\pi_1 + .2\pi_2 + .2\pi_3 \rightarrow \pi_2 = .2(\pi_1 + \pi_2 + \pi_3) = .2$

$\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} = .8\pi_2$ $\quad \rightarrow \pi_3 = (.8)(.2) = .16$

$1 = \pi_1 + \pi_2 + \pi_3$

$\frac{25}{16}$ (1) $m_{11} = \frac{1}{\pi_1} = \frac{25}{16}$

5^* (2) $m_{12} = 1 + p_{11} m_{12} + p_{13} m_{32} = 1 + .8 m_{12} \longrightarrow m_{12} = 5$

6.25 (3) $m_{13} = 1 + p_{11} m_{13} + p_{12} m_{23} = 1 + .8 m_{13} + .2 m_{23} \rightarrow .2 m_{13} = 1 + .2 \overset{5/4}{m_{23}}$ $\quad m_{13} = 6.25$

2.8175 (4) $m_{21} = 1 + p_{22} m_{21} + p_{23} m_{31} = 1 + .2 m_{21} + .8 m_{31}$

5 (5) $m_{22} = \frac{1}{\pi_2} = 5$

$\frac{5}{4}$ (6) $m_{23} = 1 + p_{21} m_{13} + p_{22} m_{23} = 1 + .2 m_{23} \longrightarrow m_{23} = \frac{1}{.8} = \frac{5}{4}$

1.5625 (7) $m_{31} = 1 + p_{32} m_{21} + p_{33} m_{31} = 1 + .2 m_{21} \longrightarrow$

5^* (8) $m_{32} = 1 + p_{31} m_{12} + p_{33} m_{32} = 1 + .8 m_{12} \rightarrow m_{32} = 1 + .8(5) = 5$

$\frac{25}{4}$ (9) $m_{33} = \frac{1}{\pi_3} = \frac{25}{4}$

(4) $.8 m_{21} - .8 m_{31} = 1$

(7) $-.2 m_{21} + m_{31} = 1$

$3.2 m_{31} = 5$

$m_{31} = 1.5625$

$m_{21} = 2.8175$

* Note: m_{12}, m_{32} differ from solution manual.

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Section 17.5

5. A square matrix is *doubly stochastic* if its entries are nonnegative and the entries in each row and column sum to 1.

Th. If P is an ergodic, doubly-stochastic matrix, then all states have same steady-state probabilities.

P is ergodic implies that steady-state probabilities exist.

Proof: We must show that $\pi = \left[\frac{1}{s}\ \frac{1}{s}\ \cdots\ \frac{1}{s}\right]$ is a solution of $\pi = \pi P$.

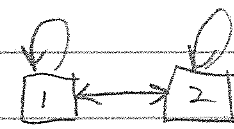
$\frac{1}{s} = \pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \cdots + \pi_s p_{s1} = \frac{1}{s} p_{11} + \frac{1}{s} p_{21} + \cdots + \frac{1}{s} p_{s1} = (p_{11} + p_{21} + \cdots + p_{s1})\frac{1}{s} = \frac{1}{s}$

$\vdots$

$\pi_s = \pi_1 p_{1s} + \pi_2 p_{2s} + \cdots + \pi_s p_{ss} = \frac{1}{s} p_{1s} + \frac{1}{s} p_{2s} + \cdots + \frac{1}{s} p_{ss} = (p_{1s} + p_{2s} + \cdots + p_{ss})\frac{1}{s} = \frac{1}{s}$

$\therefore \pi = \left[\frac{1}{s}\ \cdots\ \frac{1}{s}\right]$ is a solution of $\pi = \pi P$.

7. For stock 1 $P = \begin{array}{c c} & \begin{array}{cc} 10 & 20 \end{array} \\ \begin{array}{c} 10 \\ 20 \end{array} & \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} \end{array}$



Recurrent, aperiodic, one class

$\therefore$ Steady-state probabilities exist.

$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = .8\pi_1 + .1\pi_2 \rightarrow \pi_1 = .8\pi_1 + .1(1-\pi_1) \rightarrow \pi_1 = 1/3$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} = .2\pi_1 + .9\pi_2$ $\qquad \pi_2 = 2/3$

$1 = \pi_1 + \pi_2$

long-run expected price $= \sum_{j=1}^{2} \pi_j C(j) = \frac{1}{3}(10) + \frac{2}{3}(20) = 50/3$

$m_{11} = \frac{1}{\pi_1} = 3$

$m_{12} = 1 + p_{11} m_{12} = 1 + .8 m_{12} \rightarrow .2 m_{12} = 1 \qquad m_{12} = 5$

$m_{21} = 1 + p_{22} m_{21} = 1 + .9 m_{21} \rightarrow .1 m_{21} = 1 \qquad m_{21} = 10$

$m_{22} = \frac{1}{\pi_2} = 3/2$

If the price is \$10 today, then on average it will be 3 days before the price is \$10.

" " " " " " " " " " " " 5 " " " " " 20

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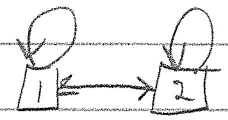
Section 17.5

7. (cont.)

$$P = \begin{array}{cc} 10 & 25 \\ \end{array}$$
$$P = \begin{bmatrix} .9 & .1 \\ .15 & .85 \end{bmatrix}$$



Recurrent, aperiodic, one class
∴ Steady-state probabilities exist.

$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = .9\pi_1 + .15\pi_2 \rightarrow \pi_1 = .9\pi_1 + .15(1-\pi_1) \rightarrow \pi_1 = \frac{.15}{.25} = 3/5$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} = .1\pi_1 + .85\pi_2$ $\qquad \pi_2 = 2/5$

$1 = \pi_1 + \pi_2$

long-run expected price $= \sum_{j=1}^{2} \pi_j C(j) = 3/5(10) + 2/5(25) = \frac{80}{5} = \$16$

$m_{11} = \frac{1}{\pi_1} = 5/3$

$m_{12} = 1 + p_{11} m_{12} = 1 + .9 m_{12} \rightarrow .1 m_{12} = 1 \rightarrow m_{12} = 10$

$m_{21} = 1 + p_{22} m_{21} = 1 + .85 m_{21} \rightarrow .15 m_{21} = 1 \rightarrow m_{21} = 6.67$

$m_{22} = \frac{1}{\pi_2} = 5/2$

If the price is \$10 today, then on average it will be 5/3 days before the price is \$10
" " " " 25 " " " " " " " 6.67 " " " " " \$10

9.

$$P = \begin{array}{c|cccc} & \text{Gray} & \text{Black} & \text{Both} & \text{Neither} \\ \hline \text{Gray} & .7 & .2 & .05 & .05 \\ \text{Black} & .2 & .6 & .1 & .1 \\ \text{Both} & .1 & .1 & .8 & 0 \\ \text{Neither} & .05 & .05 & .1 & .8 \end{array}$$

Solve

$$[0\ 0\ 0\ 1] = [\pi_1\ \pi_2\ \pi_3\ \pi_4] \underbrace{\begin{bmatrix} -.3 & .2 & .05 & 1 \\ .2 & -.4 & .1 & 1 \\ .1 & .1 & -.2 & 1 \\ .05 & .05 & .1 & 1 \end{bmatrix}}_{\bar{P}}$$

$\pi_1 = .285714$
$\pi_2 = .238095$
$\pi_3 = .285714$
$\pi_4 = .190476$

Gray squirrels are present: $\pi_1 + \pi_3 = .285714 + .285714 = .571428$ of the time
Black " " " $\pi_2 + \pi_3 = .238095 + .285714 = .523809$ of the time

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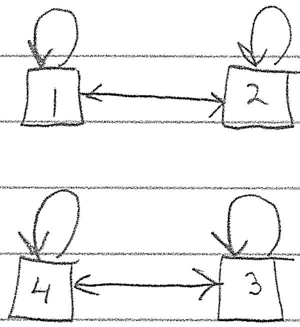
Section 17.5

11. $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$



a. This chain is not ergodic because there are two communication classes.

b. Th. 1 fails because the chain is not ergodic.

$\lim_{n\to\infty} P_{12}(n) > 0$ but $\lim_{n\to\infty} P_{32}(n) = 0$ (since it is impossible to move from 3 to 2).

$\therefore$ Steady-state probabilities do not exist.

c. $\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} + \pi_4 p_{41} = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 \qquad \pi_1 = \pi_2$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} + \pi_4 p_{42} = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$

$\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} + \pi_4 p_{43} = \frac{1}{3}\pi_3 + \frac{2}{3}\pi_4 \qquad \frac{2}{3}\pi_3 = \frac{2}{3}\pi_4 \qquad \pi_3 = \pi_4$

$\pi_4 = \qquad = \frac{2}{3}\pi_3 + \frac{1}{3}\pi_4$

We need to find the stationary probabilities separately for the two communication classes.

Class 1: $\pi_1 = \pi_2 \qquad \pi_1 + \pi_2 = 1 \quad \Rightarrow \quad \pi_1 = \pi_2 = \frac{1}{2}$

Class 2: $\pi_3 = \pi_4 \qquad \pi_3 + \pi_4 = 1 \quad \Rightarrow \quad \pi_3 = \pi_4 = \frac{1}{2}$

$\lim_{n\to\infty} P_{13}(n) = 0 \qquad \lim_{n\to\infty} P_{21}(n) = \frac{1}{2}$

$\lim_{n\to\infty} P_{43}(n) = \frac{1}{2} \qquad \lim_{n\to\infty} P_{41}(n) = 0$

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Section 17.5

13. State 0 = New machine at beginning of the month.

1 = 1 month old " " " " "

2 = 2 " " " " " " "

3 = 3 " " " " " " "

Policy 1

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & .1 & .9 & 0 & 0 \\ 1 & .2 & 0 & .8 & 0 \\ 2 & .5 & 0 & 0 & .5 \\ 3 & .1 & .9 & 0 & 0 \end{array}$$

$\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30} = .1\pi_0 + .2\pi_1 + .5\pi_2 + .1\pi_3$

$\pi_1 = \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} = .9\pi_0 + .9\pi_3$

$\pi_2 = \qquad = .8\pi_1$

$\pi_3 = \qquad = .5\pi_2$

$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4$

Solve to obtain $\pi_0 = .244$ $\pi_1 = .344$ $\pi_2 = .275$ $\pi_3 = .137$

long-run expected cost $= \pi_0(.1)(1500) + \pi_1(.2)(1500) + \pi_2(.5)(1500) + 500\pi_3 + \pi_3(1500)(.1)$

$=$

3(a), 6, 11(a) 3/20/14

Section 17.6

3. a.

$$P = \begin{array}{ll|cccccc} & & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \text{0-5} & .3 & .2 & .4 & .1 & 0 & 0 \\ 2 & >5 & 0 & .3 & 0 & 0 & .2 & .5 \\ 3 & \text{Die} & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & \text{Sold 20} & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & \text{Sold 30} & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & \text{Sold 50} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$f_{13} = .3 f_{13} + .2 f_{23} + .4 \overset{1}{f_{33}} + .1 \cancel{f_{34}}^{0} \rightarrow .7 f_{13} = .2 f_{23} + .4$

$f_{23} = .3 f_{23} + .2 \cancel{f_{53}}^{0} + .5 \cancel{f_{63}}^{0} \rightarrow .7 f_{23} = 0$

$f_{33} = 1$

$.7 f_{13} = .4 \rightarrow f_{13} = 4/7 \approx .571$

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Section 17.6

6. State j = # of dollars currently have

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\left[\begin{array}{ccccccc}
1 & & & & & & \\
.6 & & .4 & & & & \\
.6 & & & & .4 & & \\
.6 & & & & & & .4 \\
 & & .6 & & & & .4 \\
 & & & & .6 & & .4 \\
 & & & & & & 1
\end{array}\right]
\end{array}
$$

$f_{06} = 0$

$f_{16} = .6\,\cancelto{0}{f_{06}} + .4\,f_{26} \rightarrow f_{16} = .4\,f_{26}$

$f_{26} = .6\,\cancelto{0}{f_{06}} + .4\,f_{46} \rightarrow f_{26} = .4\,f_{46}$ $\qquad f_{26} = 4/19$

$f_{36} = .6\,\cancelto{0}{f_{06}} + .4\,\cancelto{1}{f_{66}} \rightarrow f_{36} = .4$ $\qquad f_{36} = .4$

$f_{46} = .6\,f_{26} + .4\,\cancelto{1}{f_{66}} \rightarrow f_{46} = .6\,f_{26} + .4 \rightarrow f_{46} = .6(.4\,f_{46}) + .4 \rightarrow f_{46} = \frac{10}{19}$

$f_{56} = .6\,f_{46} + .4\,\cancelto{1}{f_{66}} \rightarrow f_{56} = .6\,f_{46} + .4$

$f_{66} = 1$

Probability of reaching \$6 starting from \$2 is $f_{26} = 4/19 = .2105$

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Section 17.6

11. a.

$$P = \begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{l} 1 \text{ New} \\ 2 \text{ 1-yr} \\ 3 \text{ 2-yr} \\ 4 \text{ Replaced} \\ 5 \text{ Warr. Exp.} \end{array} & \begin{bmatrix} 0 & .97 & 0 & .03 & 0 \\ 0 & 0 & .95 & .05 & 0 \\ 0 & 0 & 0 & .07 & .93 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$f_{14} = .97 f_{24} + .03 \cancel{f_{44}}^{1} \longrightarrow f_{14} = .97 f_{24} + .03 = .97(.1165) + .03 = .143005$

$f_{24} = .95 f_{34} + .05 \cancel{f_{44}}^{1} \longrightarrow f_{24} = .95 f_{34} + .05 = .95(.07) + .05 = .1165$

$f_{34} = .07 \cancel{f_{44}}^{1} + .93 \cancel{f_{54}}^{0} \qquad f_{34} = .07$

$f_{44} = 1$

$f_{54} = 0$

The fraction of all refrigerators that must be replaced is $f_{14} = .143005$

Homework 3
Due 3/26/14

OR 441 Name________________________________

1. Payoff Insurance Company charges a customer according to his or her accident history during the past 3 years. The amount charged is shown in the table below. Payoff has determined that the probability that a customer will have an accident during the current year is dependent upon the customer's accident history during the past 3 years. These probabilities are shown in the table below. For the sake of simplicity, assume that a customer can have at most one accident during a year.

# of Accidents During Past 3 Years	Annual Premium	Probability of Accident During Current Year
0	$100	0.01
1	200	0.05
2	400	0.10
3	800	0.20

(a) Create a Markov chain to model this problem.

1. Carefully define the states.

2. What is the transition matrix P?

(b) Compute $P^{(8)}$. You are welcome, even encouraged, to use computer software to do this. Please attach some sort of printout to document your work.

(c) Set up the steady-state equations.

(d) Solve the steady-state equations. You are welcome, even encouraged, to use computer software to do this. Please attach some sort of printout to document your work.

(e) Determine the long-run expected average premium paid by a Payoff customer.

Homework 3
Due 3/26/14

OR 441 Name Key

1. Payoff Insurance Company charges a customer according to his or her accident history during the past 3 years. The amount charged is shown in the table below. Payoff has determined that the probability that a customer will have an accident during the current year is dependent upon the customer's accident history during the past 3 years. These probabilities are shown in the table below. For the sake of simplicity, assume that a customer can have at most one accident during a year.

# of Accidents During Past 3 Years	Annual Premium	Probability of Accident During Current Year
0	$100	0.01
1	200	0.05
2	400	0.10
3	800	0.20

(a) Create a Markov chain to model this problem.

1. Carefully define the states.

3 pts.

Let A = accident during a given year.
N = no accident during a given year.

We need 8 states of the form XYZ, where

X = accident record 3 years ago
Y = " " 2 " "
Z = " " last year

2. What is the transition matrix P?

5 pts.

$$
\begin{array}{cc|cccccccc|}
 & & NNN & NNA & NAN & ANN & NAA & ANA & AAN & AAA \\
\hline
1 & NNN & .99 & .01 & & & & & & \\
2 & NNA & & & .95 & & .05 & & & \\
3 & NAN & & & & .95 & & .05 & & \\
4 & ANN & .95 & .05 & & & & & & \\
5 & NAA & & & & & & & .90 & .10 \\
6 & ANA & & & .90 & & .10 & & & \\
7 & AAN & & & & .90 & & .10 & & \\
8 & AAA & & & & & & & .80 & .20 \\
\hline
\end{array}
$$

(b) Compute $P^{(8)}$. You are welcome, even encouraged, to use computer software to do this. Please attach some sort of printout to document your work.

5 pts.

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} NNN & NNA & NAN & ANN & NAA & ANA & AAN & AAA \end{array} \\
\begin{array}{c} NNN \\ NNA \\ NAN \\ ANN \\ NAA \\ ANA \\ AAN \\ AAA \end{array} &
\begin{bmatrix}
.9677 & .0102 & .0102 & .0102 & .0006 & .0006 & .0006 & .0001 \\
.9651 & .0103 & .0105 & .0121 & .0006 & .0007 & .0007 & .0001 \\
.9671 & .0102 & .0104 & .0105 & .0006 & .0006 & .0006 & .0001 \\
.9675 & .0102 & .0102 & .0103 & .0006 & .0006 & .0006 & .001 \\
.9641 & .0104 & .0107 & .0126 & .0006 & .0007 & .0008 & .0001 \\
.9647 & .0103 & .0106 & .0123 & .0006 & .0007 & .0007 & .0001 \\
.9668 & .0103 & .0105 & .0106 & .0006 & .0006 & .0006 & .0001 \\
.9633 & .0104 & .0109 & .0131 & .0006 & .0007 & .0008 & .0001
\end{bmatrix}
\end{array}
$$

(c) Set up the steady-state equations.

5 pts.

$\pi_1 = .99\,\pi_1 + .95\,\pi_4$

$\pi_2 = .01\,\pi_1 + .05\,\pi_4$

$\pi_3 = .95\,\pi_2 + .90\,\pi_6$

$\pi_4 = .95\,\pi_3 + .90\,\pi_7$

$\pi_5 = .05\,\pi_2 + .10\,\pi_6$

$\pi_6 = .05\,\pi_3 + .10\,\pi_7$

$\pi_7 = .90\,\pi_5 + .80\,\pi_8$

$\pi_8 = .90\,\pi_5 + .80\,\pi_8$

$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8$

(d) Solve the steady-state equations. You are welcome, even encouraged, to use computer software to do this. Please attach some sort of printout to document your work.

5 pts

$\pi = [.96767 \;\; .01019 \;\; .01019 \;\; .01019 \;\; .00057 \;\; .00057 \;\; .00057 \;\; .00007]$

(e) Determine the long-run expected average premium paid by a Payoff customer.

5 pts.

Long-run expected average premium $= \sum_{j=1}^{8} \pi_j\, C(j)$

$= 100(.96767) + 200(.01019) + 200(.01019) + 200(.01019) + 400(.00057) + 400(.00057)$

$+ 400(.00057) + 800(.00007)$

$= \$103.61$