A. The General Motion of the Stretched String

(K-text, sect. 8.10)

From our previous work, we guess that the general motion must be a general superposition with all the normal modes going at once. That is,

$$\Psi(z,t) = \sum_{n=1}^{\infty} A_n \Psi_n(z,t) \qquad \left[\begin{array}{l}\Psi_n(z,t) = \text{"mode \# } n\text{"} \\ (n=1 = \text{lowest freq., etc.})\end{array}\right]$$

If the string is bound down at the ends $z=0$ and $z=L$, this is

$$\boxed{\Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi z}{L}\right)\cos(\omega_n t + \phi_n)}$$

This is the general solution to the classical wave equation for the case of the string bound down at $z=0$ and at $z=L$.

The statement that the most general motion of a stretched string subject to given boundary conditions is a general superposition of the normal modes of the string concordant with those boundary conditions is called the (normal mode) Completeness Hypothesis*. Fourier analysis shows it to be correct.

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* From our previous work in this class, clearly the hypothesis is true for a discrete system of finite masses and springs. Fourier analysis shows it to also be true in the continuum limit of a stretched string.

B. Initial Value Problem For Stretched String – Introduction

The completeness hypothesis is useful for us in solving an example of a very important general problem in physics:

Stretched String I.V.P.:

Given $\Psi(z, t=0)$ for all z on the string, and given $\left.\dfrac{\partial \Psi(z,t)}{\partial t}\right)_{t=0} \equiv \dot{\Psi}(z, t=0)$ for all z on the string, predict the subsequent $\Psi(z,t)$ for all z on the string and for all $t > 0$.

<u>Comment</u>: Note that this is the "continuous system analog of the IVP problem for the simple mass spring system.

<u>Comment</u>: To have a reasonable hope of solving a string IVP, both $\Psi(z,t)$ and $\dot{\Psi}(z,t) \equiv \dfrac{\partial \Psi(z,t)}{\partial t}$ must be specified (known as functions of z) for all z on the string at a particular time (say, "$t=0$").

<u>This makes sense</u> – the governing differential equation, the CWE

$$\frac{\partial^2 \Psi(z,t)}{\partial z^2} = \frac{1}{v_\phi^2} \frac{\partial^2 \Psi(z,t)}{\partial t^2} \quad , \quad v_\phi = \sqrt{\frac{T_0}{\rho_0}}$$

is second-order in time for each z; as we remarked, it can be viewed as a "list" of second-order-in time o.d.e.'s – one for each z. Now, we saw early on in our course that, for a system governed by a second-order in time o.d.e., specification of two initial conditions is required to uniquely predict the future behavior of the system.

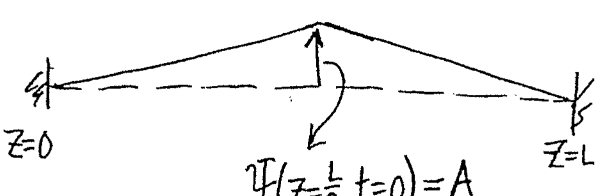
$\hookrightarrow$ i.e., two numbers

[Example: Simple harmonic oscillator, d.e. is $\ddot{\psi}(t) = -\omega^2\psi(t)$,
$\psi(t>0)$ uniquely determined given two numbers: $\psi(t=0)$ and $\dot{\psi}(t=0)$.]

So, we would expect that, if we specify $\Psi(z,t=0)$ and $\dot{\Psi}(z,t=0)$ (two functions of z) for all z on the string, it should, at least in principle, be possible to determine $\Psi(z,t>0)$ uniquely for all $t>0$. As we will see a bit later in our course, this expectation turns out to be correct. Effecting it often involves Fourier analysis.

Example: An IVP for a stretched String:

Suppose that, at time $t=0$, a uniform string is released from rest from triangular deformation with max A. What does the string subsequently do? That is, ~~from~~ just this information, can we predict the subequent $\Psi(z,t)$ for all $t>0$?



We will not study the general string initial value problem now in any great detail; we'll return later to it, but even at this level, it is important to be aware of it. The following example will illustrate one of the main points we are trying to make now:

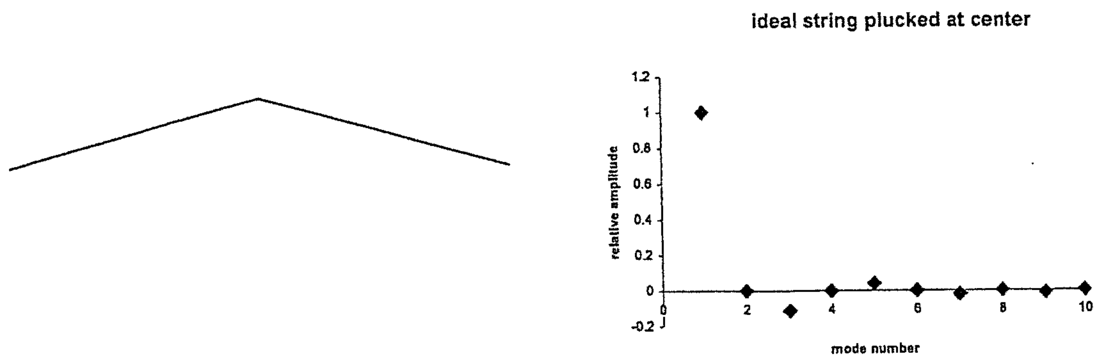
Example: Say you pluck a guitar string. Then you excite some combination of normal modes; that is

$$(12) \quad \Psi(z,t) = \sum_{n=1}^{\text{"}\infty\text{"}} A_n \sin\left(\tfrac{n\pi}{L} z\right) \cos(\omega_n t + \phi_n)$$

That is, you do not get a pure (i.e, "one musical note") tone – many harmonics of the fundamental oscillation are also present. [At this stage, this is plausible, since the initial shape of the string is not exactly that of any one normal mode.].

The relative strengths of the different modes [i.e., the relative sizes of the A_n's] greatly affects the sound and is, in turn, greatly affected by where along the length of the string you pluck (see, e.g., figures on the following page). (That is because, if the pluck point is changed, the initial condition $\Psi(z,t=0)$ is changed, thus changing the mode mix.)

To determine, for a given initial profile pair $\{\Psi_0(z), v_0(z)\}$, the relative strengths of the A_n's in (1) is a problem in Fourier analysis.



(a) (b)

Fig. 6.6 (a) Initial shape of an ideal string "slowly plucked" at its midpoint by a very sharp "pick."* (b) A plot of the relative amplitudes of the first ten normal modes excited under the initial condition shown in part (a). Note that all even numbered modes are absent (have amplitude zero).

* We assume that the initial conditions are:
$\Psi(z, t=0)$ = the function $f(z)$ shown in part (a), and
$\dot{\Psi}(z, t=0) = 0$, both for all z on the string.

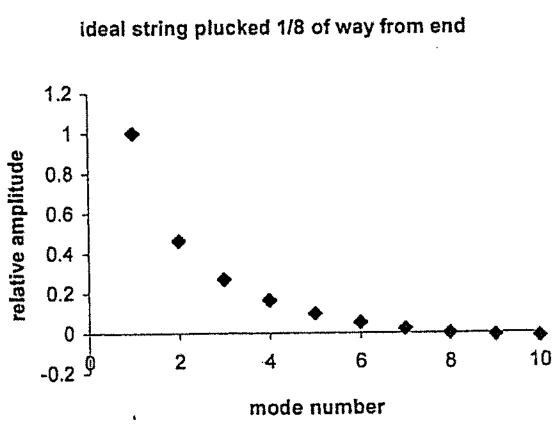


Fig. 6.6. (c) A plot of the relative amplitudes of the first ten normal modes for the case of an ideal string "slowly plucked" at distance 1/8 of the way from one end. Note that mode 8 has zero amplitude.

note: For a real guitar, the mode mix is significantly more complicated as modes of the guitar body and modes in the enclosed air cavity serve to amplify nearby (in frequency) modes excited on the string.

C. Practice with the Completeness Hypothesis (Text, sect. 8.12)

Let's look at a short "Worked Example":

<u>Ex1</u>: A stretched string is bound down at both ends ($z=0$ and $z=L$). At time $t=0$ it is released from rest with profile function $f(z)$ that is coincident with the boundary conditions. Specify as much as you can at this stage about the subsequent profile of the string (i.e, its evolution in time).

<u>Solution</u>: step 1: We know:

IC's and BC's

$$\Psi(z,t=0)=f(z);\quad f(z=0)=0,\ f(z=L)=0,$$

$$\dot{\Psi}(z,t=0)=v_0(z)=0 \text{ for all } z.$$

We want to find: $\Psi(z,t)$ for all z and all $t>0$.

<u>Step 2</u>: Whatever $\Psi(z,t)$ is, it is <u>some</u> mode superposition - i.e,

$$\Psi(z,t)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}z\right)\cos(\omega_n t+\phi_n) \qquad (12)$$

where we don't know $\{A_n\}$ and $\{\phi_n\}$.

Note that the boundary conditions are already incorporated in the mode expansion (12).

step 3: The second initial condition looks easier to apply than the first, so let's apply it first:

From (12),

$$\dot{\Psi}(z,t) = \sum_{n=1}^{\infty} \left[-\omega_n A_n \sin\left(\frac{n\pi}{L} z\right) \sin(\omega_n t + \phi_n) \right]$$

$\Rightarrow$

$$0 = \dot{\Psi}(z,t=0) = \sum_{n=1}^{\infty} \left[-\omega_n A_n \sin\left(\frac{n\pi}{L} z\right) \sin\phi_n \right] \tag{14}$$

Equation (14) must be true for <u>all</u> z on the string. The only way that can be accomplished is if each term in the sum in (14) vanishes individually. (Convince yourself of that).

Thus, for all n,

$$-\omega_n A_n \sin\left(\frac{n\pi}{L} z\right) \sin\phi_n = 0$$

$\Rightarrow$ All ϕ_n = either 0 or π (2π same as 0, 3π same as π, etc.)

If $\phi_i = \pi$, can call it zero if we change sign of A_i.

So call all $\phi_i = 0$, $i = 1, \ldots \infty$.

So: If $v_0(z) = 0$ for all z, then can take all $\phi_n = 0$!

Thus,

$$(15)\quad \Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} z\right) \cos \omega_n t$$

where

$$(16)\quad \omega_n = 2\pi f_n = 2\pi \frac{v_\phi}{\lambda_n} = \frac{2\pi n}{2L}\sqrt{\frac{T_0}{\rho_0}} = \frac{n\pi}{L}\sqrt{\frac{T_0}{\rho_0}}$$

The next step is to determine the A_n's from the first I.C.:

$$(17)\quad \Psi(z,t=0) = f(z) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} z\right)$$

We know $f(z)$. What we want to accomplish is to somehow solve (17) for each A_n in terms of the given $f(z)$. This seems difficult because the sum in () has an infinite number of terms.

As we will see, Fourier analysis technique makes this straight-forward (in principle)! And we will need it for that!

For now, though, (15) [with (16)] is about as far as we can go in this problem!