La interpretación es: la parte positiva aniquila el vacío:

$$\hat{\varphi}^{+}(x)\,|0\rangle = 0$$

y la parte negativa crea partículas:

$$\hat{\varphi}^{-}(x)\,|0\rangle = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\, e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\, \hat{a}^{+}(\vec{k})\,|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\, e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\, |\vec{k}\rangle$$

Operadores número

Definamos el operador

$$\hat{N}(\vec{k}) = \hat{a}^{+}(\vec{k})\,\hat{a}(\vec{k})$$

$\hookrightarrow$ sus autovalores son los números de ocupación

son enteros: $n(\vec{k}) = 0, 1, 2, \ldots$ que nos informan cuántas partículas con momento $\vec{k}$ hay en cada estado.

Ejemplos:

$\cdot\ |\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle = \hat{a}^+(\vec{k}_1)\hat{a}^+(\vec{k}_2)\dots\hat{a}^+(\vec{k}_n)|0\rangle$

contiene n partículas: una con momento $\vec{k}_1$, otra con $\vec{k}_2$, ...

. Estado con dos partículas con momento $\vec{k}_1$ y una con momento $\vec{k}_2$:

$$|\vec{k}_1, \vec{k}_1, \vec{k}_2\rangle = \frac{\hat{a}^+(\vec{k}_1)\hat{a}^+(\vec{k}_1)}{\sqrt{2}}\hat{a}^+(\vec{k}_2)|0\rangle$$

$$= |\underbrace{n(\vec{k}_1)}_{2}\,\underbrace{n(\vec{k}_2)}_{1}\rangle$$

o bien

$$|n(\vec{k}_1)\, n(\vec{k}_2)\rangle = \frac{\hat{a}^+(\vec{k}_1)^{n(\vec{k}_1)}}{\sqrt{n(k_1)!}}\,\frac{\hat{a}^+(\vec{k}_2)^{n(\vec{k}_2)}}{\sqrt{n(k_2)!}}|0\rangle$$

En general,

$$|n(\vec{k}_1)\, n(\vec{k}_2) \dots n(\vec{k}_m)\rangle = \prod_j \frac{\hat{a}^+(\vec{k}_j)^{n(\vec{k}_j)}}{\sqrt{n(\vec{k}_j)!}} |0\rangle$$

proporciona la densidad de número en realidad

$\hookrightarrow \hat{N}(\vec{k}) = \hat{a}^+(\vec{k})\hat{a}(\vec{k})$

El número total de partículas será

$$N = \int d^3k\, \hat{a}^+(\vec{k})\hat{a}(\vec{k})$$

Ejemplo (sencillo)

$$\hat{N}|\vec{k}'\rangle = ?$$

$$|\vec{k}'\rangle = \hat{a}^+(\vec{k}')|0\rangle$$

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = \hat{a}(\vec{k})\hat{a}^+(\vec{k}') - \hat{a}^+(\vec{k}')\hat{a}(\vec{k}) = \delta(\vec{k}-\vec{k}')$$

$$\hat{a}^+(\vec{k})\hat{a}(\vec{k})|\vec{k}'\rangle = \hat{a}^+(\vec{k})\hat{a}(\vec{k})\hat{a}^+(\vec{k}')|0\rangle$$

$$= \hat{a}^+(\vec{k})\left[\hat{a}^+(\vec{k}')\hat{a}(\vec{k}) + \delta(\vec{k}-\vec{k}')\right]|0\rangle$$

$$= 0$$

$$= \hat{a}^{+}(\vec{k})\, \delta(\vec{k}-\vec{k}')\, |0\rangle$$

$$= \delta(\vec{k}-\vec{k}')\, \hat{a}^{+}(\vec{k})\, |0\rangle$$

$$\hat{N}|\vec{k}'\rangle = \int d^3k\, \hat{a}^{+}(\vec{k})\hat{a}(\vec{k})\, |\vec{k}'\rangle$$

$$= \int d^3k\, \delta(\vec{k}-\vec{k}')\, \hat{a}^{+}(\vec{k})\, |0\rangle$$

$$= \hat{a}^{+}(\vec{k}')\, |0\rangle = |\vec{k}'\rangle$$

$$\Rightarrow n(\vec{k}') = 1.$$

¿Qué sucede con la normalización de estados?

Considerando $\langle 0|0\rangle = 1$, probar que

$$\langle \vec{k}|\vec{k}'\rangle = \delta(\vec{k}-\vec{k}').$$

$$\hat{a}^+(\vec{k})|0\rangle = |\vec{k}\rangle$$

$$\langle 0| = \langle \vec{k}|\hat{a}(k)$$

$$\Rightarrow \quad \langle \vec{a}|\vec{k}'\rangle = \langle 0|\hat{a}(\vec{a})\,\hat{a}^+(\vec{k}')|0\rangle$$

$$= \langle 0|\underbrace{\hat{a}(\vec{k}')\hat{a}(\vec{k})}_{\hat{a}(\vec{k})|0\rangle = 0} + \delta(\vec{k}-\vec{k}')|0\rangle$$

$$= \langle 0|\delta(\vec{k}-\vec{k}')|0\rangle = \delta(\vec{k}-\vec{k}')$$

¿Bosones? En el caso que nos ocupa, estamos tratando con partículas de spin cero.

$$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^+(\vec{k}_1)\hat{a}^+(\vec{k}_2)|0\rangle$$

$$= \hat{a}^+(\vec{k}_2)\hat{a}^+(\vec{k}_1)|0\rangle$$

$$= |\vec{k}_2, \vec{k}_1\rangle$$

Estado simétrico bajo intercambio de partículas $\Rightarrow$ bosones. Si hubiera salido un signo menos serían fermiones.

Ahora sí, vamos a entrar en

Energía y momento:

Comencemos con el desarrollo del campo:

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^{+}(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

su momento conjugado era:

$$\hat{\pi}(x) = -i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \hat{a}^{+}(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

Recordemos que la densidad Hamiltoniana era:

$$\mathcal{H} = \pi(x)\,\dot{\varphi}(x) - \mathcal{L}, \quad \text{con}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2\right), \quad \text{quedando}$$

$$\mathcal{H} = \frac{1}{2}\left[(\partial_0\varphi)^2 + (\nabla\varphi)^2 + m^2\varphi^2\right]$$

$\hookrightarrow \hat{\mathcal{H}}$

Vamos a hacerlo poco a poco

$$\hat{H} = \frac{1}{2}\int (\underbrace{\hat{\pi}^2}_{I_1} + \underbrace{\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi}_{I_2} + \underbrace{m^2\hat{\varphi}}_{I_3})^2 d^3x$$

$$I_1 = \frac{1}{2}\int \hat{\pi}^2 d^3x$$

$$= \frac{1}{2}\int d^3x \left[\frac{i}{(2\pi)^{3/2}}\int d^3k\sqrt{\frac{w_k}{2}}\left[\hat{a}(\vec{k})e^{i(w_k x^0 - \bar{k}\cdot\bar{x})} - \hat{a}^+(\bar{k})e^{-i(w_k x^0 - \bar{k}\cdot\bar{x})}\right]\right.$$

$$\left.\times \frac{i}{(2\pi)^{3/2}}\int d^3k'\sqrt{\frac{w_k'}{2}}\left[\hat{a}(\vec{k}')e^{i(w_k' x^0 - \bar{k}'\bar{x})} - \hat{a}^+(\bar{k}')e^{-i(w_k' x^0 - \bar{k}'\cdot\bar{x})}\right]\right]$$

$$= -\frac{1}{4}\frac{1}{(2\pi)^3}\int d^3x\int d^3k\int d^3k'\sqrt{w_k w_k'}\left(\hat{a}(\vec{k})e^{i(w_k x^0 - \bar{k}\bar{x})} - \hat{a}^+(k)e^{-i(w_k x^0 - \bar{k}\bar{x})}\right)$$

$$\times\left(\hat{a}(\bar{k}')e^{i(w_k' x^0 - \bar{k}'x)} - \hat{a}^+(\bar{k}')e^{-i(w_k' x^0 - \bar{k}'x)}\right)$$

$$= -\frac{1}{4}\frac{1}{(2\pi)^3}\int d^3x\int d^3k\int d^3k'\sqrt{w_k w_{k'}}\left[\hat{a}(\bar{k})\hat{a}(\bar{k}')e^{i[(w_k + w_k')x^0 - (\bar{k}+\bar{k}')\cdot\bar{x}]}\right.$$

$$- \hat{a}(k)\hat{a}^+(\bar{k}')e^{i[(w_k - w_k')x^0 - (\bar{k}-\bar{k}')\cdot\bar{x}]}$$

$$- \hat{a}^+(k)\hat{a}(\bar{k}')e^{-i[(w_k - w_k')x^0 - (\bar{k}-\bar{k}')\cdot\bar{x}]}$$

$$\left. + \hat{a}(\bar{k})\hat{a}^+(\bar{k}')e^{-i[(w_k + w_k')x^0 - (k+k')\cdot\bar{x}]}\right]$$

Utilizamos la representación de Fourier de la delta de Dirac:

$$\frac{1}{(2\pi)^3}\int d^3x\, e^{+i\bar{k}\bar{x}} = \delta^{(3)}(\bar{k}) \quad \text{y además es par.}$$

$$I_1 = -\frac{1}{4}\int d^3k \int d^3k' \sqrt{w_k w_{k'}} \Big[\hat{a}(\bar{k})\hat{a}(\bar{k}')\delta^{(3)}(\bar{k}+\bar{k}')e^{i(w_k+w_{k'})x^0}$$

$$-\hat{a}(\bar{k})\hat{a}^{+}(\bar{k}')\delta^{(3)}(\bar{k}-\bar{k}')e^{i(w_{k}-w_{k'})x^0}$$

$$-\hat{a}^{+}(\bar{k})\hat{a}(\bar{k}')\delta^{(3)}(\bar{k}-\bar{k}')e^{-i(w_k-w_{k'})x^0}$$

$$+\hat{a}^{+}(\bar{k})\hat{a}^{+}(\bar{k}')\delta^{(3)}(\bar{k}+\bar{k}')e^{-i(w_k+w_{k'})x^0}\Big]$$

$$I_1 = -\frac{1}{4}\int d^3k\, w_k \Big[\hat{a}(\bar{k})\hat{a}(-\bar{k})e^{2iw_k x^0}$$

↳ integro en k' $\left(w_{\bar{k}} = w_{-\bar{k}}\ ;\ w_k = +\sqrt{k^2+m^2}\right)$

$$-\hat{a}(\bar{k})\hat{a}^{+}(\bar{k}) - \hat{a}^{+}(\bar{k})\hat{a}(\bar{k})$$

$$+\hat{a}^{+}(\bar{k})\hat{a}^{+}(-\bar{k})e^{-2iw_k x^0}\Big]$$

Vamos a hacer lo mismo con los siguientes términos.

$$\boxed{I_2} = \frac{1}{2}\int d^3x\, \bar{\nabla}\hat{\varphi}\cdot\bar{\nabla}\hat{\varphi}$$

$$= \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x \int \frac{d^3k}{\sqrt{2\omega_k}}\int\frac{d^3k'}{\sqrt{2\omega_{k'}}}(-i\bar{k})\cdot(-i\bar{k}')$$

$$\times\left(\hat{a}(\bar{k})e^{i(\omega_k t^0-\bar{k}\bar{x})} - \hat{a}^+(\bar{k})e^{-i(\omega_k x^0-\bar{k}\bar{x})}\right)$$

$$\times\left(\hat{a}(\bar{k}')e^{i(\omega_{k'}x^0-\bar{k}'\cdot\bar{x})} - \hat{a}^+(\bar{k}')e^{-i(\omega_{k}' x^0-\bar{k}'\cdot\bar{x})}\right)$$

$$= -\frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x\int\frac{d^3k}{\sqrt{2\omega_k}}\int\frac{d^3k'}{\sqrt{2\omega_{k'}}}\,\bar{k}\cdot\bar{k}'\Big[\hat{a}(\bar{k})\hat{a}(\bar{k}')e^{i[(\omega_k+\omega_{k'})x^0-(\bar{k}+\bar{k}')\cdot\bar{x}]}$$

$$-\hat{a}(\bar{k})\hat{a}^+(\bar{k}')e^{i[(\omega_k-\omega_{k'})x^0-(\bar{k}-\bar{k}')\cdot\bar{x}]}$$

$$-\hat{a}^+(\bar{k})\hat{a}(\bar{k}')e^{-i[(\omega_k-\omega_k')x^0-(\bar{k}-\bar{k}')\cdot\bar{x}]} +$$

$$+\hat{a}^+(\bar{k})\hat{a}^+(\bar{k}')e^{-i[(\omega_k+\omega_{k'})x^0-(\bar{k}+\bar{k}')\cdot\bar{x}]}\Big]$$

$$= -\frac{1}{2}\int\frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\,\bar{k}\cdot\bar{k}'\Big[\hat{a}(\bar{k})\hat{a}(\bar{k}')\delta^{(3)}(\bar{k}+\bar{k}')e^{i(\omega_k+\omega_k)x^0}$$

$$-\hat{a}(\bar{k})\hat{a}^+(\bar{k}')\delta^{(3)}(\bar{k}-\bar{k}')e^{i(\omega_k-\omega_k')x^0}$$

$$-\hat{a}^+(\bar{k})\hat{a}(\bar{k}')\delta^{(3)}(\bar{k}-\bar{k}')e^{-i(\omega_k-\omega_k')x^0} +$$

$$+\hat{a}^+(\bar{k})\hat{a}^+(\bar{k}')\delta^{(3)}(\bar{k}+\bar{k}')e^{-i(\omega_k+\omega_k')x^0}\Big]$$

$$= \left[-\frac{1}{4} \int d^3k \, \frac{k^2}{w} \left[-\hat{a}(\bar{k})\hat{a}(-\bar{k}) e^{2i\omega_k x^0} - \hat{a}(\bar{k})\hat{a}^+(\bar{k}) \right.\right.$$

$$\left.\left. - \hat{a}^+(\bar{k})\hat{a}(\bar{k}) - \hat{a}^+(\bar{k})\hat{a}^+(-\bar{k}) e^{-2i\omega_k x^0} \right] = I_2 \right]$$

$$I_3 = \frac{1}{2} \int d^3x \, m^2 \hat{\varphi}^2$$

$$= \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3x \int \frac{d^3k}{\sqrt{2w}} \int \frac{d^3k'}{\sqrt{2w_{k'}}} m^2 \left[\hat{a}(\bar{k}) e^{i(\omega_k x^0 - \bar{k}\bar{x})} + \hat{a}^+(\bar{k}) e^{-i(\omega_k x^0 - \bar{k}\cdot\bar{x})} \right]$$

$$\times \left[\hat{a}(\bar{k}') e^{i(\omega_{k'} x^0 - \bar{k}'\bar{x})} + \hat{a}^+(\bar{k}') e^{-i(\omega_{k'} x^0 - \bar{k}'\bar{x})} \right]$$

$$= \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3x \int \frac{d^3k}{\sqrt{2\omega_k}} \int \frac{d^3k'}{\sqrt{2\omega_{k'}}} m^2 \left[\hat{a}(\bar{k})\hat{a}(\bar{k}') e^{i[(\omega_k + \omega_k')x^0 - (\bar{k}+\bar{k}')\bar{x}]} \right.$$

$$+ \hat{a}(\bar{k})\hat{a}^+(\bar{k}') e^{i[(\omega_k - \omega_{k'})x^0 - (\bar{k}-\bar{k}')\cdot\bar{x}]}$$

$$+ \hat{a}^+(\bar{k})\hat{a}(\bar{k}') e^{-i[(\omega_k - \omega_{k'})x^0 - (\bar{k}-\bar{k}')\bar{x}]} +$$

$$\left. + \hat{a}^+(\bar{k})\hat{a}^+(\bar{k}') e^{-i[(\omega_k + \omega_{k'})x^0 - (\bar{k}+k')\cdot\bar{x}]} \right]$$

$$= \frac{1}{2} \int \frac{d^3k}{\sqrt{2w}} \int \frac{d^3k'}{\sqrt{2w'}} m^2 \left[\hat{a}(\bar{k})\hat{a}(\bar{k}') \delta^{(3)}(\bar{k}+\bar{k}') e^{i(\omega_k + \omega_k')x^0} \right.$$

$$+ \hat{a}(\bar{k})\hat{a}^+(\bar{k}') \delta^{(3)}(\bar{k}-\bar{k}') e^{i(\omega_k - \omega_k')x^0}$$

$$+ \hat{a}^+(\bar{k})\hat{a}(\bar{k}') \delta^{(3)}(\bar{k}-\bar{k}') e^{-i(\omega_k - \omega_k')x^0} + \hat{a}^+(\bar{k})\hat{a}^+(\bar{k}')$$

$$\delta^{(3)}(\bar{k}+\bar{k}') e^{-i(\omega_k + \omega_{k'})x^0}$$

$$\left[= \frac{1}{4}\int d^3k\, \frac{m^2}{\omega_k}\Big[\hat a(\bar k)\hat a(-\bar k)\, e^{2i\omega_k x^0} + \hat a(\bar k)\hat a^\dagger(\bar k) + \right.$$

$$\left. + \hat a^\dagger(\bar k)\hat a(\bar k) + \hat a^\dagger(\bar k)\hat a(-\bar k)\, e^{-2i\omega_k x^0}\Big]\right.$$

$$\left. = I_3 \right]$$

Juntándolo todo (antes de morir) y agrupando

$$\hat H = I_1 + I_2 + I_3$$

$$= -\frac{1}{4}\int d^3k\, \cancelto{0}{\left(\omega_k - \frac{k^2}{\omega_k} - \frac{m^2}{\omega_k}\right)}\, \hat a(\bar k)\hat a(-\bar k)\, e^{2i\omega_k x^0}$$

$$-\frac{1}{4}\int d^3k\, \cancelto{0 \;\to\; \text{no se va}}{\left(-\omega_k - \frac{k^2}{\omega_k} - \frac{m^2}{\omega_k}\right)}\, \hat a(\bar k)\hat a^\dagger(\bar k)$$

$$-\frac{1}{4}\int d^3k\, \left(-\omega_k - \frac{k^2}{\omega_k} - \frac{m^2}{\omega_k}\right)\, \hat a^\dagger(\bar k)\hat a(\bar k)$$

$$-\frac{1}{4}\int d^3k\, \cancelto{0}{\left(\omega_k - \frac{k^2}{\omega_k} - \frac{m^2}{\omega_k}\right)}\, \hat a^\dagger(\bar k)\hat a^\dagger(-\bar k)\, e^{-2i\omega_k x^0}$$

Como ~~$w^2 k m^2$~~ $\omega^2 = k^2 + m^2$, se van algunos

y queda

$$\hat{H} = -\frac{1}{4}\int d^3k\,(-2w_k)\,\hat{a}(\bar{k})\hat{a}^+(\bar{k}) - \frac{1}{4}\int d^3k\,(-2w)\,\hat{a}^+(\bar{k})\hat{a}(\bar{k})$$

$$= \frac{1}{2}\int d^3k\, w_k \left(\hat{a}(\bar{k})\hat{a}^+(\bar{k}) + \hat{a}^+(\bar{k})\hat{a}(\bar{k})\right)$$

$$= \frac{1}{2}\int d^3k\, w_k \left(\hat{a}^+(\bar{k})\hat{a}(\bar{k}) + \hat{a}^+(\bar{k})\hat{a}(\bar{k}) + \delta^{(3)}(\bar{k}-\bar{k})\right)$$

$$= \int d^3k\, w_k \left(\hat{a}^+(\bar{k})\hat{a}(\bar{k}) + \frac{1}{2}\delta^{(3)}(0)\right)$$

Comienzan a aparecer los problemas

Hemos obtenido:

$$\hat{H} = \int d^3k\, w_k \left[\hat{a}^+(\vec{k})\hat{a}(\vec{k}) + \frac{1}{2}\delta^{(3)}(0)\right]$$

La energía del estado fundamental será

$$\hat{H}|0\rangle = \int d^3k\, w_k \frac{1}{2}\delta^{(3)}(0) = E_0|0\rangle$$

Esta divergencia podemos quitarla usando el ordenamiento normal de operadores