Physics 251 15th class – Tuesday Feb. 27, 2024

A. Brief Review

We guessed, in our two immediately preceeding classes, that the most general motion of a stretched string bound at both its ends ($z=0$ and $z=L$), is given by a general superposition of all its normal modes going at once, each with its own amplitude and its own phase, i.e.,

$$(1) \quad \Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(n \frac{\pi}{L} z\right) \cos(\omega_n t + \phi_n),$$

which is the general solution to the Classical wave equation (CWE) for a string bound down at $z=0$ and at $z=L$.

The statement that the general motion of a stretched string subject to given boundary conditions at its ends is a general superposition of its normal modes (standing waves) concordant with those boundary conditions is called the "(normal modes) completeness hypothesis."* Last Thursday, we discussed the string IVP briefly. In what follows, we get a bit of additional insight:

---

* We have previously shown that the completeness hypothesis is true for a discrete system of masses and springs. Fourier theory shows it to be true in the continuum limit of a stretched string.

B. Orthogonality of the Standing Wave Eigenfunctions [K-text, sect. 8-9,

As we know, for a single normal mode on a stretched string,

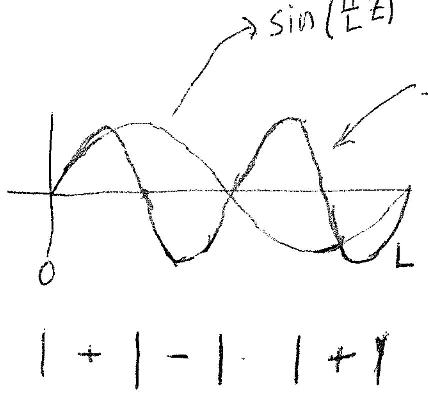
(2) $$\Psi_n(z,t) = A_n(z)\cos(\omega_n t + \phi_n)$$

where $A_n(z)$, the mode profile function (for normal mode # n) is often called the "eigenfunction" of the normal mode. Any normal mode of a system is also often called an "eigenstate" of the system.

A very important property of the eigenfunctions of different frequency normal modes in one dimension is that they are "orthogonal" to each other ["orthogonality property"]. In a one dimensional system, e.g., for a string bound down at $z=0$ and at $z=L$, it is "simply" the fact that

(3) $$\int_{z=0}^{z=L} A_m(z)\,A_n(z)\,dz = \int_{z=0}^{z=L} \sin\left(m\frac{\pi}{L}z\right)\sin\left(n\frac{\pi}{L}z\right)dz = 0 \text{ if } m \neq n$$

Example: $\int_0^L \sin\left(\frac{\pi}{L}z\right)\sin\left(2\frac{\pi}{L}z\right)dz = 0$, as you can verify by tables, or by looking at a figure like the one roughly sketched below:



$$\Rightarrow \int_0^L \sin\left(\frac{\pi}{L} z\right) \sin\left(2\frac{\pi}{L} z\right) dz = 0$$

multiplication results:

The roughly sketched figure above shows that the integral of the product of the two functions $\sin(\frac{\pi}{L} z)$ and $\sin(2\frac{\pi}{L} z)$ is zero because $\sin(2\frac{\pi}{L} z)$ has half the wavelength of $\sin(\frac{\pi}{L} z)$, hence, for half the period of $\sin(\frac{\pi}{L} z)$, the product $\sin(\frac{\pi}{L} z)\sin(2\frac{\pi}{L} z)$ is positive and for the other half of a period it is negative; not surprising, then, that the product integral has value zero.

What the fact that

$$\int_{z=0}^{z=L} \sin\left(m\frac{\pi}{L} z\right) \sin\left(n\frac{\pi}{L} z\right) dz = 0 \text{ if } m \neq n$$ means is

that, taken over the range $z=0$ to $z=L$, neither function has any net positive or negative overlap product with the other:

We can obtain a bit more insight into why that property is called "orthogonality" (over the interval $z=0$ to $z=L$) by considering the following analogy:

Suppose we have an ordinary vector $\vec{A}$ and another ordinary vector $\vec{B}$ in three dimensional space. Then, if $\vec{A}$ and $\vec{B}$ are perpendicular to each other, their "dot" (or "inner") product is zero: $\vec{A} \cdot \vec{B} = 0 = \vec{B} \cdot \vec{A}$. As you know, we can write $\vec{A}$ and $\vec{B}$ as

$$\vec{A} \leftrightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad \text{and} \quad \vec{B} \longleftrightarrow \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

where $A_1 = A_x$, $A_2 = A_y$, $A_3 = A_z$, $B_1 = B_x$, $B_2 = B_y$, $B_3 = B_z$.

Then, if $\vec{A}$ and $\vec{B}$ are perpendicular ("orthogonal") to each other, then

$$(4) \qquad A_1 B_1 + A_2 B_2 + A_3 B_3 = 0.$$

Returning, then, to the eigenfunctions $A_m(z)$ and $A_n(z)$, we can think of each as an infinitely long list list of "components":

$$A_m(z) \doteq \lim_{\substack{N \to \infty \\ \epsilon \to 0}} \begin{pmatrix} A_m(z=0) \\ A_m(z=\epsilon) \\ A_m(z=2\epsilon) \\ \vdots \\ A_m(z=L) \end{pmatrix} \quad \text{and} \quad A_n(z) = \lim_{\substack{N \to \infty \\ \epsilon \to 0}} \begin{pmatrix} A_n(z=0) \\ A_n(z=\epsilon) \\ A_n(z=2\epsilon) \\ \vdots \\ A_n(z=L) \end{pmatrix}$$

Then, their inner product would be given as . .

$$\left(A_m(z), A_n(z)\right) = \lim_{\substack{N \to \infty \\ \epsilon \to 0}} \sum_{i=0}^{\infty} A_m(z_i)\, A_n(z_i) \cdot \epsilon$$

$$\to \int_{z=0}^{z=L} A_m(z)\, A_n(z)\, dz$$

$$= \int_{z=0}^{z=L} \sin\left(m \tfrac{\pi}{L} z\right) \sin\left(n \tfrac{\pi}{L} z\right) dz$$

Thus, the Riemann integral above, which is an "infinite sum of finite products times an infinitesimal (dz)," yields a finite result which is the value of the integral; for the case shown above it is zero unless $m = n$.

We will leave the math at that for now, noting that

$$\int_{z=0}^{z=L} \sin\left(m \tfrac{\pi}{L} z\right) \sin\left(n \tfrac{\pi}{L} z\right) dz$$ is the direct (if a bit abstract) generalization of the dot product of vectors in 3-space to the inner product of two functions (viewed as infinite-dimensional vectors in an infinite dimensional vector space). Since $(A_m(z), A_n(z)) = 0$ if $m \neq n$, the functions $A_m(z)$ and $A_n(z)$ are independent of each other on the interval $[0, L]$.

C. Practice with the Completeness Hypothesis (K-Text, sect. 8.12)

Let's look at a short "Worked Example":

Ex1: A stretched string is bound down at both ends ($z=0$ and $z=L$). At time $t=0$ it is released from rest with profile function $f(z)$ that is coincident with the boundary conditions. Specify as much as you can at this stage about the subsequent profile of the string (i.e, its evolution in time).

Solution: step 1: We know:

IC's and BC's

$$\Psi(z,t=0) = f(z); \quad f(z=0)=0, \quad f(z=L)=0,$$

$$\dot{\Psi}(z,t=0) = v_0(z) = 0 \text{ for all } z.$$

We want to find: $\Psi(z,t)$ for all z and all $t>0$.

Step 2: Whatever $\Psi(z,t)$ is, it is some mode superposition - i.e,

$$\Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} z\right) \cos(\omega_n t + \phi_n) \tag{5}$$

where we don't know $\{A_n\}$ and $\{\phi_n\}$.

Note that the boundary conditions are already incorporated in the mode expansion (5).

step 3: The second initial condition looks easier to apply than the first, so let's apply it first:

From (5),

$$\dot{\Psi}(z,t) = \sum_{n=1}^{\infty}\left[-\omega_n A_n \sin\left(\frac{n\pi}{L}z\right)\sin(\omega_n t + \phi_n)\right]$$

$\Rightarrow$

$$(6)\quad 0 = \dot{\Psi}(z,t=0) = \sum_{n=1}^{\infty}\left[-\omega_n A_n \sin\left(\frac{n\pi}{L}z\right)\sin\phi_n\right]$$

Equation (6) must be true for <u>all z</u> on the string. The only way that can be accomplished is if each term in the sum in (6) vanishes individually. (Convince yourself of that).

Thus, for all n,

$$-\omega_n A_n \sin\left(\frac{n\pi}{L}z\right)\sin\phi_n = 0$$

$\Rightarrow$ All ϕ_n = either 0 or π (2π same as 0, 3π same as π, etc.)

If $\phi_i = \pi$, can call it zero if we change sign of A_i.

So call all $\phi_i = 0$, $i = 1, \ldots \infty$.

So: If $v_0(z) = 0$ for all z, then can take all $\phi_n = 0$!

Thus,

$$(7) \quad \Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} z\right) \cos \omega_n t$$

where

$$(8) \quad \omega_n = 2\pi f_n = 2\pi \frac{v_\phi}{\lambda_n} = \frac{2\pi n}{2L}\sqrt{\frac{T_0}{\rho_0}} = \frac{n\pi}{L}\sqrt{\frac{T_0}{\rho_0}} \qquad \left(\text{from } \lambda_n = \frac{2L}{n}\right)$$

The next step is to determine the A_n's from the first I.C.:

$$(9) \quad \Psi(z,t=0) = f(z) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} z\right)$$

We know $f(z)$. What we want to accomplish is to somehow solve (9) for each A_n in terms of the given $f(z)$. This seems difficult because the sum in (9) has an infinite number of terms.

As we will see, Fourier analysis technique makes this straightforward (in principle)! And we will need it for that!

For <u>now</u>, though, (7) [with (9)] is about as far as we can go in this problem!

D. The Sum of Two Standing Waves of Different Frequency Is Generally Not A Standing Wave (Reading: sect. 8.11)

I've stressed to you the fact that, with a single standing wave, you do not get any left-right motion of the wave profile – all or sections of wave form just oscillate up and down. However, for a *superposition* of standing waves, this is generally not true!* Why is that? Because different modes have different frequencies and hence different time dependences. As an example, in Fig. 6.7 I plotted the superposition "1*mode 1 + 1*mode 2 + (1/3)*mode 3" for a stretched string bound down at both ends, at two different times. (I assumed the initial condition that all three modes pass through their maximum positive displacement from equilibrium at t=0.)

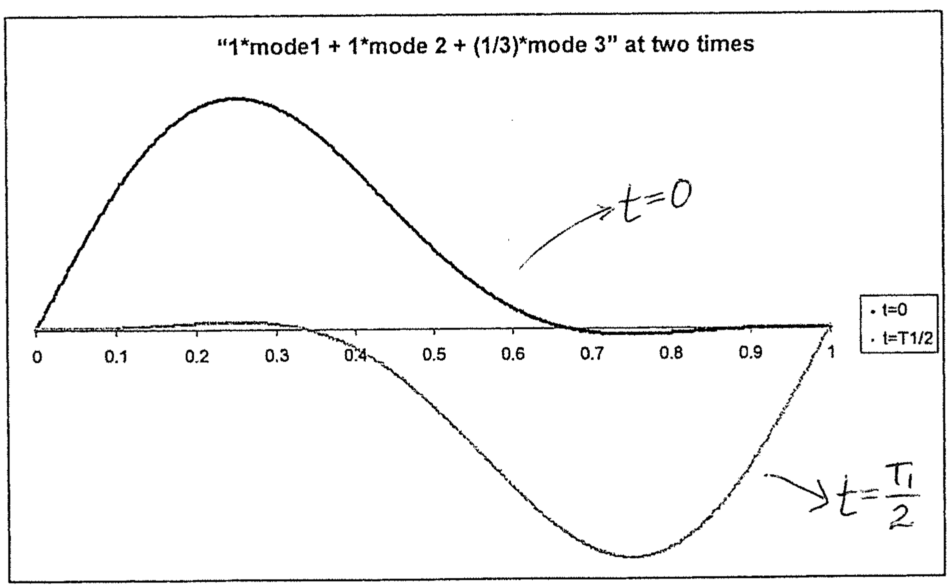


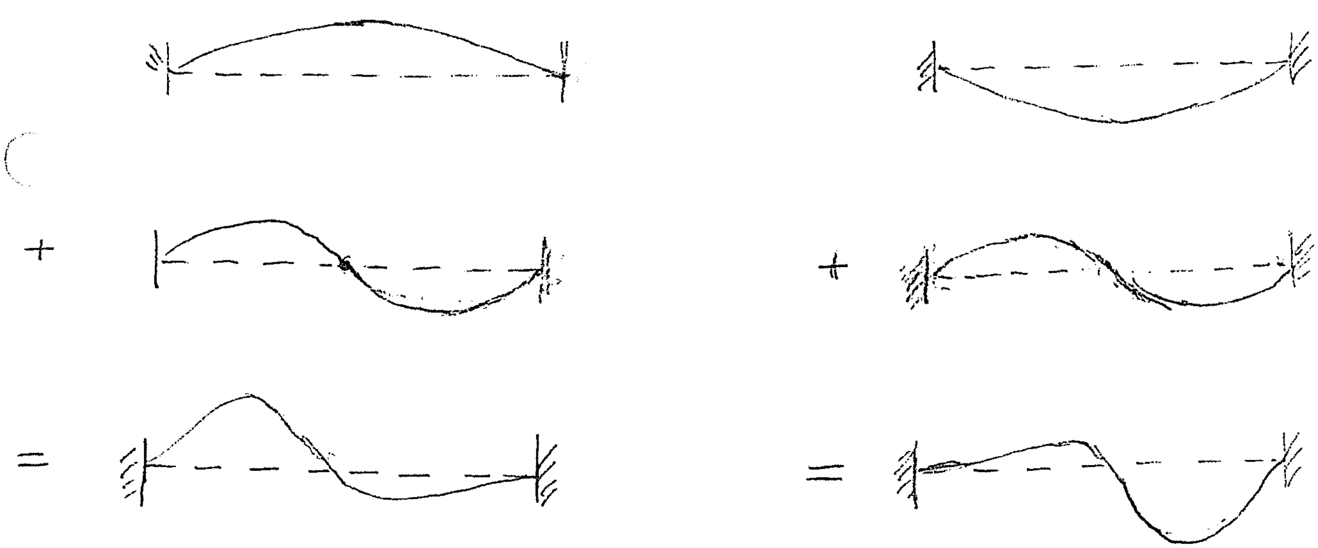
Fig. ~~6.7~~. A mode superposition looked at at two different times. T_1 is the period of the first (lowest frequency) mode.

You can see from the figure that in time $t = T_1/2$, the big "bump" has moved and changed sign!

* Thus, a superposition of standing waves of different frequency is generally not a standing wave.

Let's try to get some insight into this:

Consider an equal amplitude mix of just the two lowest normal modes, each with zero phase constant in the time part for a stretched string that is bound down at each of the two ends. The following rough diagrams crudely illustrate the situation at two different times: $t = 0$ and $t = \frac{1}{2} T_1 = T_2$:



@ $t = 0$ @ $t = \frac{T_1}{2} = T_2$

We see that there has been left-right motion!

You can also see algebraically that you are not going to wind up with a standing wave from this superposition:

We have (taking, for simplicity, $A=1$, $L=\pi$ (so $\pi/L = 1$) and $\omega = 1$),

$\omega = v_\phi \cdot k$

$\omega = v_\phi \cdot \frac{2\pi}{\lambda}$

$\omega_1 = v_\phi \cdot \frac{2\pi}{2}$

so take $v_\phi = \frac{1}{\pi}$.

$$\Psi(z,t) = \sin z \cos t + \sin 2z \cos 2t$$ ← (each mode is factorized form, but sum is not)

$$= \sin z \cos t + (2 \sin z \cos z)(\cos^2 t - \sin^2 t)$$

$$= \sin z \left[\cos t + 2 \cos z (\cos^2 t - \sin^2 t)\right]$$

$$= \sin z \cos t + (2 \sin z \cos z)(2 \cos^2 t - 1)$$

- none of these ways (nor any other way) of writing $\Psi(z,t)$ show the factorized form that is the hallmark of a standing wave - i.e, we find that we cannot write Ψ in factorized form - i.e.,

$$\Psi(z,t) \neq f(z)\, g(t).$$

What's occuring here is somewhat analogous to the phenomenon of "beats" between normal modes of discrete systems.

E. Stretched String - Other Boundary Conditions (Text, sect. 8.1.3)

Bound ends are not the only possible boundary conditions for a stretched string. Another possibility is (one or both) free ends. At a free end, Ψ is free to take on any value. As one means of achieving a free end one might imagine tying the end of the string to a massless "slipring" which can freely move vertically without friction on a vertical pole. Fig. 5.9 shows an arbitrary configuration for a stretched string with two free ends.

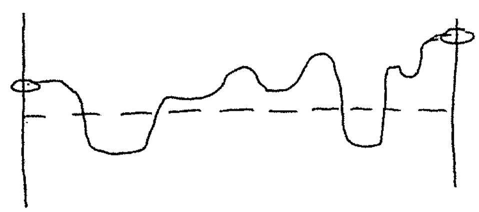


Fig. 5.9. String with both ends free via frictionless "sliprings" constrained to move vertically on frictionless rods. General configuration.

Our goal is to find the standing waves for the case of free ends. Reviewing our previous development, we see that the CWE and the same Helmholtz equation still apply (they are, of course, independent of the boundary conditions). Thus, the general soln. of the Helmholtz eqn. is still the same, namely

$$A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right).$$

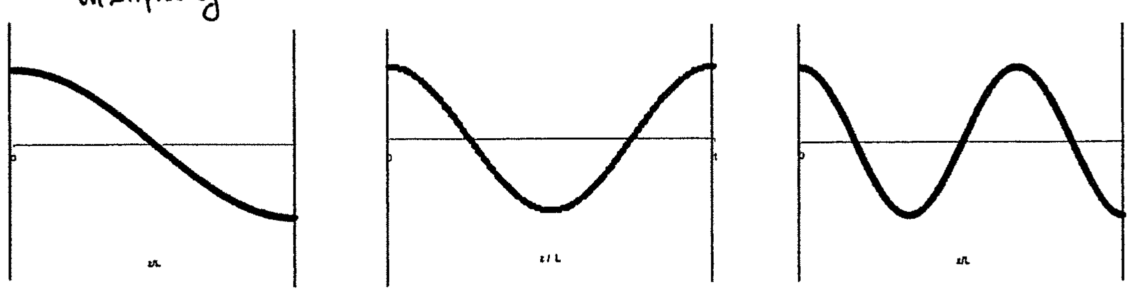
However, the "specific" solution we are after will now not be $A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right)$, since the choice of the B.C.'s does affect the specific solution, as we know.

How, then, do we proceed?

To make progress, we must express the "free-end" boundary condition in mathematical language so that we can input it into the general soln. of the Helmholtz equation to find the appropriate specific class of solutions we need.

How do we mathematically express the boundary conditions for the case of free ends? To see how, consider the right hand end at $z=L$. Because the ring is massless, the net force on it must be zero. (Otherwise, it would have infinite acceleration, which is not physical.) But, any net force on the slipring can only come from two sources – the rod and the last bit of string that connects to it. Now, since there is no friction between the rod and the slipring, the net vertical force of the rod on the slipring is zero. Therefore, the net vertical force of the last bit of string on the ring must also be zero. Consequently, the last bit of string must always be horizontal.* **Thus, the slope of the string is zero at a free end.** Given this and the fact that the modes must be sinusoidal in shape, we can construct pictures of the shapes of the three lowest modes:

(* $F^{\text{vert.}}_{\substack{\text{last bit of string}\\ \text{on slipring}}} = T\sin\theta\Big|_{\substack{\text{at}\\ z=L}} = 0 \Rightarrow \theta\Big|_{z=L} = 0 \Rightarrow$ horizontal)



Mode 1: $\lambda = 2L$. Mode 2: $\lambda = L$ Mode 3: $\lambda = 2L/3$.

Fig. 8.10. Configurations of the three lowest nonzero frequency modes for the case of free ends at $z=0$ and $z = L$. The massless and frictionless sliprings connecting the strings to the rods are not shown.

Half a cycle (for each case) later, each shape will be inverted around the equilibrium line.

It will be very useful for us also to have a mathematical expression of these boundary conditions and mode shapes. Since the slope at any z at time t is given by $\frac{\partial\Psi(z,t)}{\partial z}$, the boundary condition for a free end is

$$\Psi(z=L,t) \text{ is unconstrained.}$$

$$\frac{\partial\Psi(z,t)}{\partial z}\Big|_{z=L} = 0 \quad \text{for all } t.$$

Boundary conditions, free end at, say, $z=L$.

(similar for free end at $z=0$)

If also have a free end at $z=0$, then $\psi(z=0,t)$ is unconstrained & $\frac{\partial\psi}{\partial z}\Big|_{z=0} = 0$

Now let us return to the general solution of the Helmholtz equation

$$A(z) = C\sin(\frac{2\pi}{\lambda}z) + D\cos(\frac{2\pi}{\lambda}z). \quad (8.16)$$

Since, in a mode, $\Psi(z,t) = A(z)\cos(\omega t+\phi)$, $\frac{\partial\Psi(z,t)}{\partial z} = \cos(\omega t+\phi)\frac{dA(z)}{dz}$. Thus, for both ends free, our boundary conditions are

$$\frac{dA(z)}{dz} = 0 \quad \text{at } z=0 \text{ and at } z=L. \quad (8.30)$$ and

I urge you to continue this with exercises 8.15 and 8.16 below.

Exercise 8.15 By using eqn. (8.30) show that the free end boundary at z=0 requires that C = 0 in eqn. (8.16). Then use the boundary condition at z=L to derive the mode wavelengths by a method analogous to that we used in section 8.8. What are the mode frequencies?

Exercise 8.16 Write the most general mathematical expression for the n^{th} lowest frequency transverse normal mode for a string with free ends at z=0 and z=L in terms of T, ρ_l and L.

F. General Solution on the stretched string with both ends free

We guess that the general solution of the CW.E. for a string with both ends (at $z=0$ and at $z=L$) free is the general superposition with all free-ends modes oscillating at once - i.e., in analogy to our reasoning for the case of two bound ends, here, for the case of both ends free, we are led to conjecture that any continuous small amplitude/small slope function that has zero slope at both $z=0$ and $z=L$ can be expanded as [eqn. (8.32)]:

$$(8.32) \quad f(z) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(n \frac{\pi}{L} z\right).$$

As we will see in chapter 11, by relying on Fourier's Theorem, this conjecture turns out to be correct even if the amplitude and slope are not "small".

The question then arises as to how, for a given $f(z)$, we can (and must!) find the B_n's. That, we will learn about in discussing Fourier analysis in the near future.

G. Other Boundary Conditions

It is also possible to have "mixed" boundary conditions – e.g., one end bound and the other end free. We will have you explore that in the homework.

The general solution of the CWE for any set of boundary conditions is a general solution of the normal modes of the system for those boundary conditions. Thus, no matter how the string is initially deformed, its subsequent motion is some superposition of the normal modes consistent with the boundary conditions.

The tool of Fourier analysis will allow us, in most cases, to determine exactly what mode superposition. Once we understand how to effect this, we will be in a very powerful position.

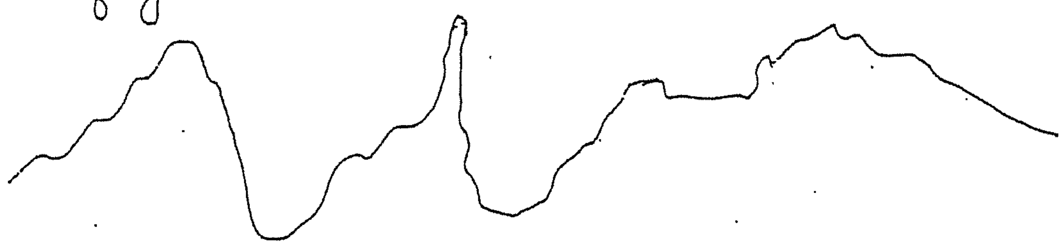
For this, but even more, for many other reasons, we next turn to a general introductory, but fairly thorough (for this level) discussion of Fourier Analysis and Fourier Synthesis . . .

F. Introduction to Fourier Analysis

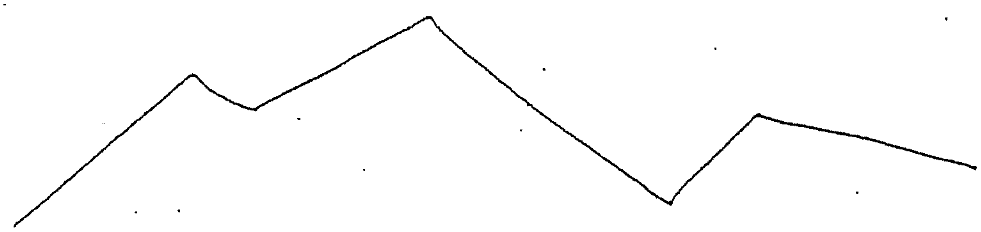
I would now like to tell you what is probably the most important theorem of Mathematical Physics

- by combining different frequency sin and cosine functions you can make up any continuous function at all!

So: if you have



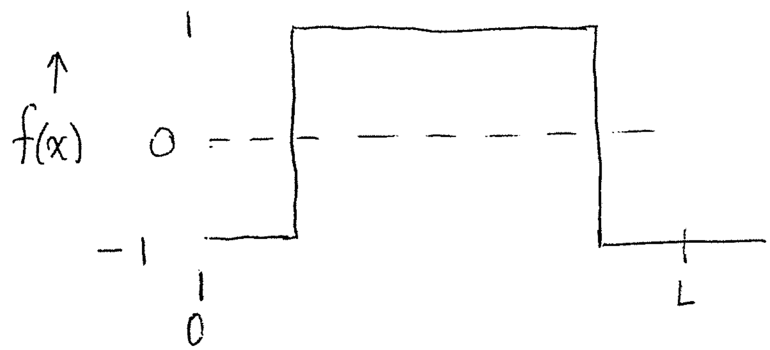
or



or whatever, it is just a combination of different frequency sines and cosines. That is, any continuous function of time can be viewed as being "made up of" a simultaneous superposition of different harmonic oscillations, at different frequencies. Likewise, any continuous function of x can be viewed as being made up of a combination of different sinusoidal oscillations in x, each with its own wavenumber!

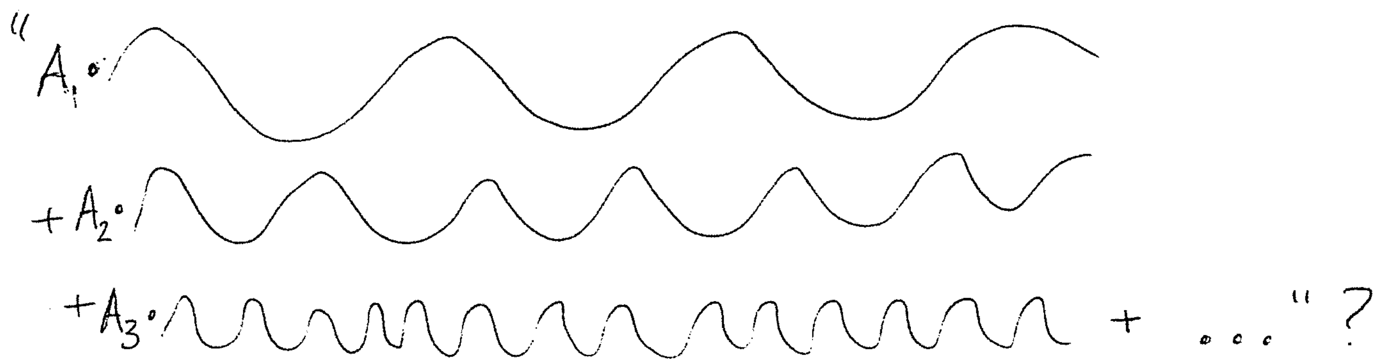
Example: Of What We Are Saying:

Let's pick a seemingly unlikely function for this to work for — one with a discontinuity. For now, we are only concerned with f(x) on



a finite interval, say [−L, L]; later on we generalize to infinite intervals [e.g. (−∞,∞)].

Can it be ~~that this~~ f(x) can be written as $f(x) =$



Before we see "why" this is true, first we'll tell you a version of the theorem in more precise terms, then we'll get practice using it – then we'll tell you why. [It's like learning about cars if you've never seen one - first you take a ride in one, then you drive one, then you worry about understanding the 211 moving parts in an automatic transmission!]

So we state the theorem (actually a restricted case first):

Let $f(x)$ be defined and piecewise continuous on $[-L,L]$. Then, in this interval, for some

$a_0, a_1, a_2, \ldots \; ; \; b_0, b_1, b_2, \ldots$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$$

Having stated that, we consider first only a special case: $L=\pi$.

Then, on $(-\pi,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Which is to say:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

Some obvious questions arise:

1. Is this really true?

2. If it is true, what are the a_n's and b_n's?

To find out we <u>assume</u> the theorem is true and "plug back" in the following way: We <u>assume</u> that on $[-\pi,\pi]$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots$$
$$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots$$

Then

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + a_1\int_{-\pi}^{\pi}\cos x\,dx + a_2\int_{-\pi}^{\pi}\cos 2x\,dx + \dots$$
$$+ b_1\int_{-\pi}^{\pi}\sin x\,dx + b_2\int_{-\pi}^{\pi}\sin 2x\,dx + \dots$$

Now, on the right hand side, all the integrals are zero except the first! So

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\cdot 2\pi \quad , \qquad \text{i.e.,}$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

So far, so good...