Physics 251 – 29th Class – Tuesday April 23, 2024

A. Brief Review

In our 27th class session we saw that, if we combine two different single-frequency sinusoidal waves traveling in the same direction in the same medium (e.g., on the same stretched string), we obtain, for $\omega_2 \gtrsim \omega_1$, a moving beats pattern like that shown below, for which

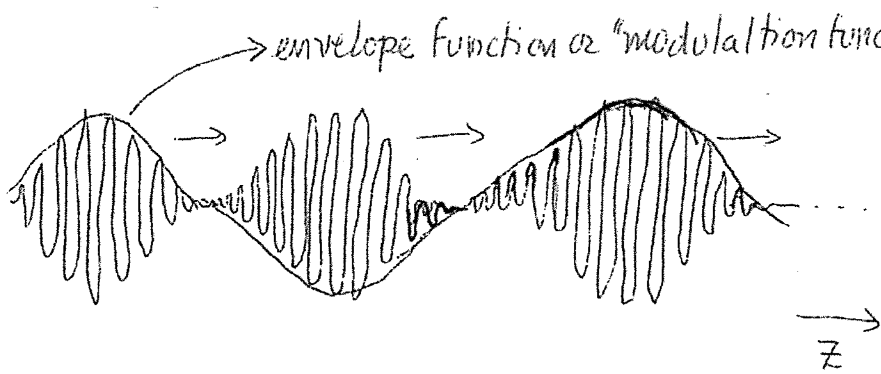


Fig. 17-1 Schematic drawing of moving wave form corresponding to the superposition of two sinusoidal traveling waves of slightly different frequency but of the same amplitude and same phase constant moving in the same direction in the same medium.

$$\Psi(z,t) = A\cos(k_1 z - \omega_1 t) + A\cos(k_2 z - \omega_2 t)$$

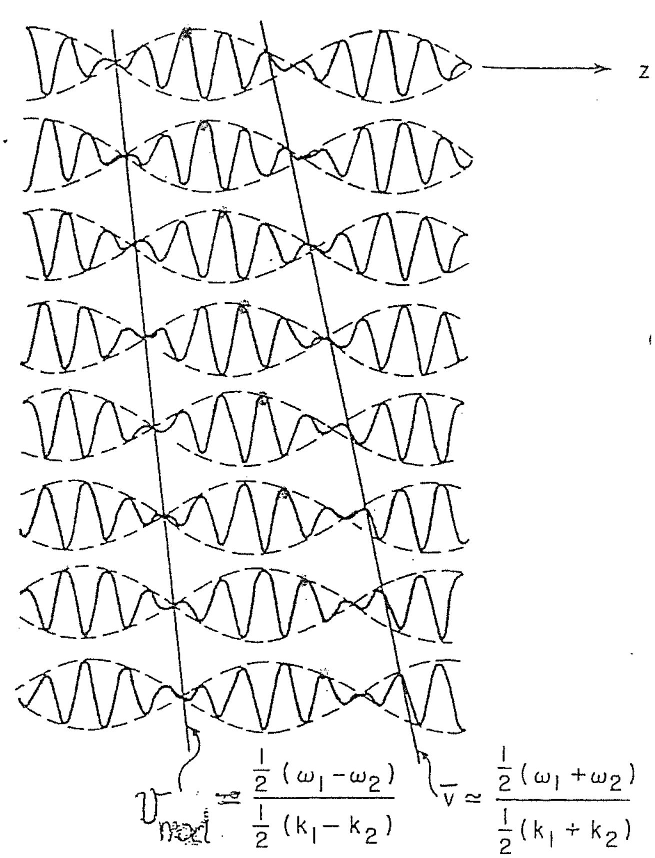
equals, as we showed,

$$\Psi(z,t) = A_{mod}(z,t)\cos\left[k_{AVE}z - \omega_{AVE}t\right]$$

where $A_{mod}(z,t) = 2A\cos\left[k_{mod}z - \omega_{mod}t\right]$ = envelope function, where $k_{mod} = \frac{k_2 - k_1}{2}$, $\omega_{mod} = \frac{\omega_2 - \omega_1}{2}$, $k_{AVE} = \frac{k_1 + k_2}{2}$, $\omega_{AVE} = \frac{\omega_1 + \omega_2}{2}$.

We further saw that, if the medium is <u>dispersive</u> for the wave type in question, [i.e, $v = v_\phi = \frac{\omega}{k}$ depends on k (hence, also depends on ω)], then, in general,

the envelope function $A_{mod}(z,t)$, itself a sinusoidal traveling wave, but one of longer wavelength than that of the "inside" oscillations, moves, with a different velocity, $v_{mod} = \frac{\Delta\omega}{\Delta k}$ than do the inside oscillations (see figure below):



case of $\omega_1 = \frac{9}{7}\omega_2$

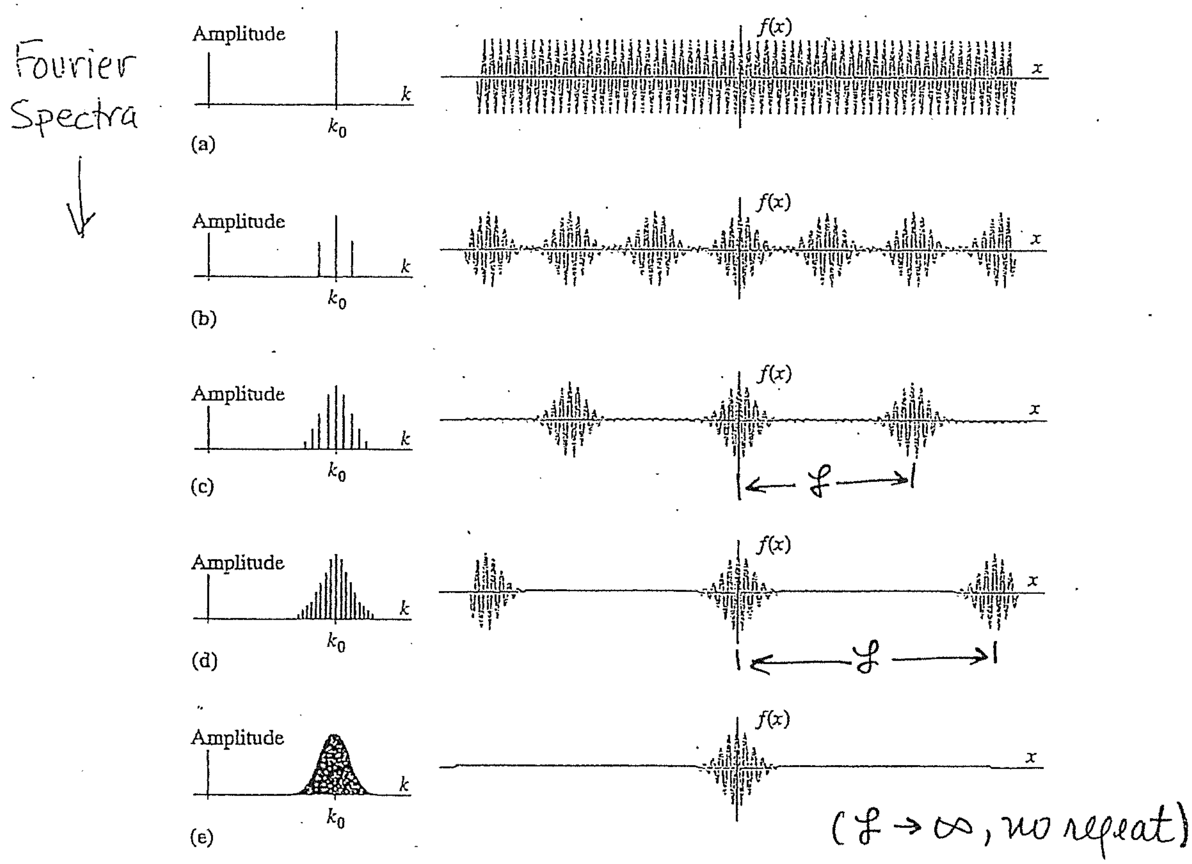
and $k_1 = \frac{5}{3} k_2$

Envelope then moves at half the speed of the "inside waves".

B. Brief Statements on Fourier Analysis of Non-repetitive Functions

Let us look again at the Fourier synthesis of functions of increasing repetition distance:

Figure 4.22 Building a single isolated pulse from pure sine waves requires a continuum of wave numbers.



In the continuum limit of the Fourier k-spectrum ($N \to \infty$, step $\delta k \to 0$) we obtain a single modulated pulse wave shape (shown above).

We next note that we can form a single (i.e., non-repetitive) <u>moving</u> modulated wave form as the limit $\lim_{\substack{N\to\infty \\ \delta k\to 0}}$ (where δk is the Fourier "step size"), for k running from $k_1 \to k_2$,

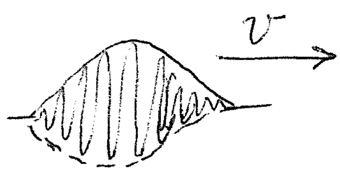
$$\Psi_{\rightarrow}(z,t) = \lim_{\substack{N\to\infty \\ \delta k\to 0}} A(k_1)\cos(k_1 z - \omega_1 t) + A(k_1+\delta k)\cos\left[(k_1+\delta k)z-(\omega_1+\delta\omega)t\right]$$

$$+ A(k_1+2(\delta k))\cos\left[(k_1+2\delta k)z-(\omega_1+2(\delta\omega))t\right]$$

$$+\dots+ A\left[(k_1)+N(\delta k)\right]\cos\left[(k_1+N(\delta k))z-(\omega_1+N(\delta\omega))t\right]$$

which is

$$\Psi_{\rightarrow}(z,t) = \int_{k_1}^{k_2} A(k)\cos(kz-\omega t)\,dk \leftrightarrow$$



where, in the integrand, $A(k)\,dk$ [which is infinitesimal] is the amplitude of the sinusoidal wave of wavenumber k in the Fourier integral superposition. In general, allowing for phase constants,

$$\Psi_{\rightarrow}(z,t) = \int_0^{\infty} A(k)\cos(kz-\omega t)\,dk + \int_0^{\infty} B(k)\sin(kz-\omega t)$$

$$= \int_0^{\infty} A(k)\cos\left[kz-\omega t+\phi_k\right]\,dk$$

C. At What Velocity Does the Envelope of a Non-Sinusoidal Traveling Waveform Move?

For the case of superposing $N=2$ sinusoidal waves traveling in the same direction, we saw that, in a dispersive medium, the envelope moves at the "modulation velocity"

$$v_{mod} = v_{env.} = \frac{\omega_2 - \omega_1}{k_2 - k_1},$$

which, in general, is not equal to

$$v_{AVE} = \frac{\omega_{AVE}}{k_{AVE}} = \frac{\frac{1}{2}(\omega_2+\omega_1)}{\frac{1}{2}(k_2+k_1)} = \frac{\omega_2+\omega_1}{k_2+k_1}.$$

But, we recall, if $v_2 = v_1 = v$, then $\bar{v} = \frac{v_1+v_2}{2} = v$,

and $$v_{env} = v_{mod} = \frac{\omega_2-\omega_1}{k_2-k_1} = \frac{k_2 v - k_1 v}{k_2 - k} = v.$$

However, the quantity $v_{env} = \frac{\Delta\omega}{\Delta k} = \frac{\omega_2-\omega_1}{k_2-k_1}$ can be considered as follows for ω_2 close to ω_1, for then,

$$\omega_2 \approx \omega_1 + \left.\frac{d\omega}{dk}\right)_{k=\bar{k}} (k_2 - k_1)$$

$$\Rightarrow v_{mod} = v_{env} \approx \frac{\omega_2-\omega_1}{k_2-k_1} \approx \frac{\omega_1 + \left.\frac{d\omega}{dk}\right)_{\bar{k}} \cdot (k_2-k_1) - \omega_1}{k_2-k_1} = \left.\frac{d\omega(k)}{dk}\right)_{\bar{k}}$$

The quantity we just found, $\left.\frac{d\omega}{dk}\right)_{k=\bar{k}}$ is called the <u>group velocity</u> (v_g). If the envelope function single wave group has a single well defined peak and the wave packet is symmetric about that peak, then the peak moves at the group velocity.

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Significance of the Group Velocity

For continuous, localized, and non-repetitive wave packets, the group velocity $\frac{d\omega}{dk})_{k=\bar{k}}$ is the velocity that the envelope function of the wave packet moves with.[9] Since the information communicated by a traveling wave packet is carried by its envelope function, the information also travels at the group velocity. And since the energy associated with the wave packet is confined to the region in which the wave amplitude is not zero, the energy as well travels at the group velocity.

Under What Conditions Does the Envelope Function of a Waveform Move Without Changing Shape and Under what Conditions Does it Change Shape as it Moves?

If, for all values of k in the Fourier spectrum of the packet, the group velocity function $\frac{d\omega}{dk}$ does not depend on k, then all narrow bands in wavenumber $[(\delta k)_1, (\delta k)_2, \dots]$ in the Fourier spectrum of sinusoidal waves that make up the packet have the same group velocity, and thus the entire envelope function of the packet then moves without changing shape at that common value of group velocity. But, if the group velocity function $v_g = \frac{d\omega}{dk})_{k=\bar{k}}$ depends on k, then v_g has different values for different narrow bands of wavenumbers that make up the full Fourier spectrum of wavenumbers of the packet (where $\bar{k}$ is the average or dominant wavenumber in a band). In that case, the contributions to the envelope function by those bands with faster-than-average group

[9] In what follows herein I will demonstrate this for nondispersive media and, via examples, for waveforms with a relatively narrow peak in wavenumber distribution that move in media that present a linear dispersion relation for the wave under consideration. If the dispersion relation is nonlinear, the wave packet spreads and the peak amplitude decreases as the packet moves. For that situation the demonstration presented herein applies to a relatively narrow band of wavenumbers around the peak of the wavenumber distribution over which ω varies relatively slowly. If the packet has a relatively broad range of wavenumbers that contribute significantly, the change of shape of the packet occurs rapidly and then the concept of group velocity loses its precise meaning. A more general proof that addresses that situation is beyond the scope of this text.

velocity move to and then define and advance the leading edge of the packet and those bands with slower than average group velocity increasing lag behind at the trailing edge of the packet; thus, the envelope function increasingly spreads in width as the wave packet moves.

So, what determines whether or not $\frac{d\omega}{dk}$ depends on k? That depends on whether or not the second derivative function $\frac{d^2\omega}{dk^2}$ is or is not zero. And, what determines *that*? The dispersion relation! For the simple case of a nondispersive medium [$\omega(k) = ck$, where c is independent of k], then both the phase velocity $v_\phi = \frac{\omega}{k}$ and the group velocity $v_g = \frac{d\omega}{dk}$ are equal to c, so then $\frac{d^2\omega}{dk^2} = 0$ and then the packet and its envelope function move rigidly together without changing shape. But, what if medium is dispersive? For this, we distinguish two different classes of cases:

A. "Linearly Dispersive Medium"

Envelope function moves rigidly at the common group velocity, "inside ripples" also move rigidly, but at a different velocity equal to the average phase velocity.

On the other hand, if the medium is "linearly dispersive" [i.e., of the form $\omega(k) = ck + d$, where c and d are nonzero constants], then the group velocity $v_g = \frac{d\omega}{dk})_{k=\bar{k}} = c$ is independent of wavenumber k and so then all parts of the envelope function move at the same velocity. However, the phase velocity $v_\phi = \frac{\omega}{k} = c + \frac{d}{k}$ then *does* depend on k and does not equal the group velocity. In that situation, if the moving wave packet is a modulated group(s), then depending on the signs of c and d, the group velocity may be greater than the average phase velocity $\bar{v}_\phi$ (in which case, the modulated high frequency carrier wave "ripples" seem to be born in the front of the wave group and move toward the back as the envelope moves), or it may be less than the average phase velocity (then the ripples seem to be born in the rear of the wave group and move forward with respect to the envelope to disappear at the front of the moving envelope – we've seen that (sect. 17.1) in superpositions of two different but closely spaced frequency sinusoidal traveling waves). Fig. 17.9 below shows a schematic representation of the motion of a wave packet in a medium with a linear dispersion relation such that the phase velocity is greater than the group velocity. It can be seen from the figure that in a linearly dispersive medium the wave packet as a whole does not spread or broaden because the "fast oscillations at the average wavenumber" (the "ripples") are contained by the envelope function which does not broaden as the packet moves.

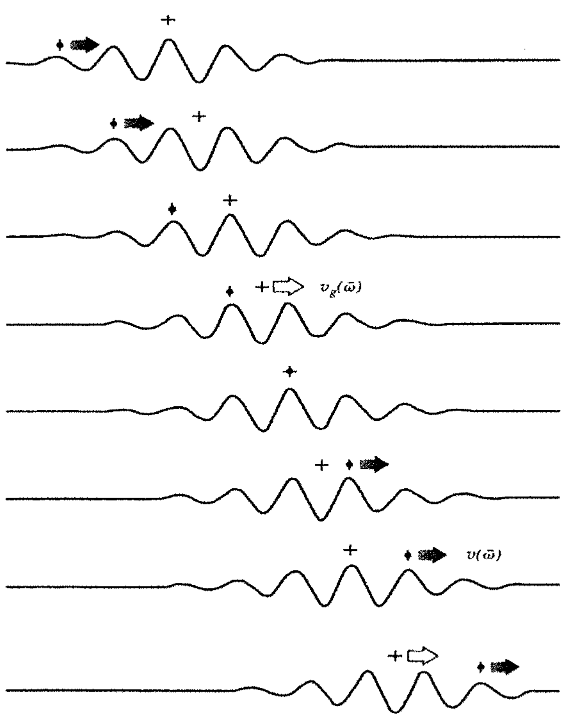

Fig. 17.9 Schematic Illustration of a wave packet moving in a linearly dispersive medium in which the average phase velocity is greater than the group velocity. The average phase velocity is indicated by the progress of the crest marked by the black diamond symbol and the group velocity, indicated by sequential positions of the moving + sign, marks the emotion of the peak amplitude of the envelope function of the packet. From the figure it is seen that as the packet envelope (not shown) moves, the individual "internal wavelets" appear as if born at the left end of the packet and then disappear at the right end. This figure is from Optics, Fourth Edition by Eugene Hecht (Addison Wesley – Pearson Education), page 297. Permission for use to be applied for.

B. Nonlinearly Dispersive Medium – the Wavepacket Broadens as it Moves

What if the medium is "**nonlinearly dispersive**" [e.g, $\omega(k) = ak^2 + b$, where a and b are non-zero constants, or e.g., if $\omega(k) = dk^3 + r$, where d and r are nonzero constants]? Then, in general, the group velocities of different bands of frequencies of the sinusoidal components that synthesize the packet envelope are not equal because then $\frac{d^2\omega}{dk^2} = \frac{d}{dk}(\frac{d\omega}{dk}) \neq 0$. In that case, as mentioned above, the wave packet width spreads as it (the wave packet) moves with concurrent decrease in amplitude so as to conserve energy (see Fig. 17.10 below).

Fig. 17.10 Schematic of a single wave shown at two different times while traveling in a nonlinearly dispersive medium. It is seen that the wave pulse broadens in shape as it travels. Since the energy carried must be conserved, the peak of the pulse decreases in height during the motion.

17.13 An Example – The Traveling Gaussian Wave Packet Revisited

To make some of the concepts of the previous section more concrete, let's consider the motion of the peak a non-modulated Gaussian-shaped moving wave packet of the initial form $\Psi(z, 0) = Ce^{-\frac{z^2}{2\sigma_z^2}}$ that moves in a moderately dispersive medium and which possesses a Fourier wavenumber distribution with relatively narrow standard deviation σ_x. At the position of the peak value of the wave packet the individual sinusoidal components interfere constructively, so that all sinusoidal components at that point have the same (or nearly the same) value of phase function

$\varphi(z_{at\,peak,}t) = \omega t - kz + \phi$, even though, in general, *k*, and ϕ (and *ω*) differ for different wavenumber components. As the peak moves, the phase functions of all Fourier components at its position remain nearly equal so as to maintain that constructive interference. Thus, at the moving position of the peak, the rate of change of the phase function φ of *with respect to Fourier component wavenumber k* (or, equivalently, with respect to *ω*) *does not change to first order* in *k* (or in *ω*). That is, if $z_p(t)$ is the position of the moving peak at time *t*, we require

$$\frac{d}{dk}[\omega t - kz + \phi]_{z=z_p,k=\bar{k}} = 0 \tag{17.13.1}$$

where $\omega = \omega(k)$ is a function of *k* and where $\bar{k}$ is the average wavenumber of the sinusoidal Fourier components in a narrow symmetric band of wavenumbers around the peak of the *wavenumber* distribution. Eqn. (17.13.1) is then $\frac{d\omega}{dk}t - z_p = 0$, or $z_p = \frac{d\omega}{dk}|_{k=\bar{k}}\, t$, showing that the peak moves at the group velocity given by

$$v_g = \frac{d\omega}{dk}|_{k=\bar{k}}\ . \tag{17.13.2}$$

If the medium that the Gaussian wave packet moves in is non-dispersive and non-dissipative, or if it is linearly dispersive, then the shape of the entire Gaussian packet doesn't change as the packet moves at the group velocity given by eqn. (17.13.2). But, if the medium is nonlinearly dispersive, then the packet changes shape as it moves because then $\frac{dv_g}{dk} = \frac{d}{dk}\left(\frac{d\omega}{dk}\right) = \frac{d^2\omega}{dk^2} = \neq 0.$

Logic analogous to that also applies to a symmetric Gaussian-modulated wave packet say, of the form, at time *t*=0,

$$(17.13.3) \qquad \Psi(z,t=0) = D\, e^{-\frac{z^2}{2\sigma_z^2}} \cos\bar{k}z,$$

which results from a Gaussian distribution of wavenumbers where $\bar{k}$ is the average (or dominant) wavenumber[10], namely,

$$(17.**) \qquad A(k) = Ce^{-\frac{(k-k_0)^2}{2\sigma_k^2}},\ B(k) = 0 \text{ for all } k,$$

where $\sigma_z = {}^1/_{\sigma_k}$ and where $D = \frac{1}{\sqrt{2\pi}\sigma_k} C$ [11]. Then, provided that $\Delta k = 2\sigma_k$ is not too large, to first order eqn. (17.13.2) follows, which implies that the position of the peak moves at the group velocity $v_g = \frac{d\omega}{dk}|_{k=\bar{k}}$.

17.14 Examples of Dispersive Media and Comparison of the Group and Average Phase Velocities in These Media

Let's next look at a few examples of dispersive media in which we can easily show that the group and phase velocities are not equal:

Example: **Real metal guitar or piano string**: As I mentioned earlier in this chapter, the dispersion relation for a not perfectly flexible (i.e., a bit stiff) string is given by $\omega \approx ck + dk^3$, where $c =$

[10] See Worked Example 17.6 (and if necessary, Worked Example 17.5), both in sect. 17.8 of this chapter.

[11] Worked Example 17.6, eqns. (17.37) – (17.39); alternately, the Fourier cosine transform of eqn. (17.12.3) is given by eqn. (17.**) with $\frac{C}{\sqrt{2\pi}\sigma_k}$.

$\sqrt{T_0/\rho_l}$ (not the speed of light in vacuum!) and where d is a (small) positive constant. So, a real metal guitar or piano string is a nonlinearly dispersive medium (albeit slightly, is d is small). Thus, $v_\phi(k) = \frac{\omega}{k} \approx \frac{ck+dk^3}{k} = c + dk^2$, and $v_g = \frac{d\omega}{dk})_{k=\bar{k}} = c + 3dk^2 \neq v_\phi$. Note that, in this situation $\frac{dv_g}{dk} = \frac{d^2\omega}{dk^2} = 6dk$ is a function of k, so the shape of the wave packet envelope changes as it moves. Also, for a real metal stretched string that is not perfectly flexible, the added stiffness causes the both the phase velocity and the group velocity to be greater than c, with $v_g > v_\phi$.

Example: Electromagnetic Waves in the ionosphere: The ionosphere (or at least, some layers of it) consists of a dilute plasma of ions and electrons in addition to O_2 and N_2 molecules. The dispersion relation for electromagnetic waves in this medium is

$$\omega^2 = \omega_p^2 + c^2k^2 \tag{17.14.1}$$

where c is the speed of light in vacuum and where ω_p is the plasma frequency we encountered in chapter 2, given by $\omega_p = \sqrt{\frac{ne^2}{m_e\epsilon_0}}$. Thus, for this medium $\omega = \sqrt{\omega_p^2 + c^2k^2}$, which is a nonlinear dispersion relation. Differentiating eqn. (17.14.1) we get $2\omega\frac{d\omega}{dk} = 2c^2k$, which is $\frac{\omega}{k}\left(\frac{d\omega}{dk}\right) = c^2$, or

Note: $\frac{\omega}{k}\left(\frac{d\omega}{dk}\right) = c^2$

$$v_\phi v_g = c^2 . \tag{17.14.2}$$

$\Rightarrow \frac{d\omega}{dk} = \frac{k}{\omega}c^2$

$\neq 0$

$\Rightarrow$ Wave group envelope spreads!

However, it also follows from eqn. (17.14.1) that

$$\omega = \sqrt{c^2 k^2 + \omega_p^2}$$

$$\Rightarrow \frac{\omega}{k} = v_\phi = \sqrt{c^2 + \frac{\omega_p^2}{k^2}}$$

(17.14.3) $$\frac{\omega}{k} = v_\phi = \sqrt{c^2 + \frac{\omega_p^2}{k^2}} \geq c \text{ !!}$$

Thus, in the ionosphere, the phase velocity of electromagnetic waves is general $> c$!

Relativity Violation? No!

It may seem that eqn. (17.14.3) violates the theory of relativity ("nothing can move faster than speed c"), but actually, according to relativity only material objects that are initially moving at speed < c and information cannot move at speed greater than c. A phase velocity in a dispersive medium is the velocity of propagation of a single sinusoidal traveling wave, and as we've already seen, a single purely sinusoidal traveling wave cannot carry information; nor can it carry energy. Note also from eqn. (17.14.3) that, if $v_\phi >$ c then $v_g <$ c. So, there is no conflict with relativity.

Example: **Surface Waves in Deep Water:** In water in which the depth $d \gg \lambda$, for $\lambda \gg 17$ mm, the dispersion relation for surface waves is given to good approximation by

(17.14.4) $$\omega = \sqrt{gk}$$

where $g = 9.81$ m/s^2; such surface water waves are often called "gravity waves" (not to be confused with the gravity waves propagating throughout our Universe in spacetime). The dispersion relation for these waves in deep water is manifestly nonlinear. From it,

(17.14.5) $$v_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$$

and

(17.14.6) $$v_g = \frac{1}{2}\sqrt{\frac{g}{k}},$$

so for gravity waves on a deep body of water for $\lambda \gg 17$ mm, $v_g = \frac{1}{2}\, v_\phi$. From eqn. (17.14.6) we find that $\frac{dv_g}{dk} = \frac{d^2\omega}{dk^2} = -\frac{\sqrt{g}}{4}\, k^{-\frac{3}{2}}$, as the envelope of a wave packet of deep water gravity waves moves, it changes shape.

Exercise 17.** Show that eqns. (17.14.5) and (17.14.6) follow from eqn. (17.14.4).

Example: Nonrelativistic Quantum Mechanical De Broglie Waves: If you have studied basic quantum mechanics, you will recall that the dispersion relation for the De Broglie plane wave associated with a free (i.e., subject to no forces) particle moving at speed nonrelativistic speed is given by

(17.14.7) $$\hbar\omega = \frac{h_{bar}{}^2}{2m}\, k^2,$$

where $\hbar$ (pronounced "*hbar*") is $\frac{h}{2\pi}$, where h is Planck's constant ($\approx$ 6.6 x 10^{-34} Joule-seconds). Eqn. (17.14.7) represents a quadratically nonlinear dispersion relation.

Hence, for a wave packet, as the packet moves, the envelope function of it spreads.

and, as we just saw, if $\omega_{ave} >> \frac{1}{4}\Delta\omega$, there must be times when $f(t) = 0$ before the time $t = \Delta t = 2\pi/\Delta\omega$. At those times, even though the phasors are not essentially evenly spread all around the clock, for $f(t)$ to be zero the horizontal components of all the phasors must add to zero. Consider, for example, the time t = $\delta t = \dfrac{\pi}{2\omega_{ave}}$ for which $\cos\omega_{ave}t = 0$. The phasor situation at that time is indicated schematically in Fig. 12.8.1 below.

ignore for our present discuss-ion.

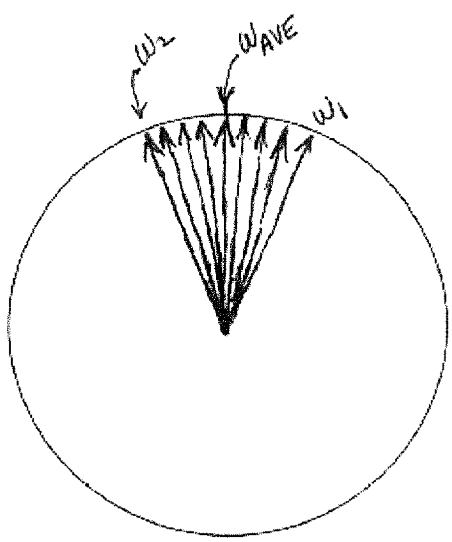

Fig. 12.8.1.

12.9 The Complex Function Form of Fourier Series

[K-text, chap 12, sect. 12.9]

In this and the following few sections I introduce you to the basic idea of the complex function for of Fourier series for real-valued repetitive functions. Writing Fourier series in a form using complex-valued basis functions is often convenient for calculating Fourier coefficients. From a theoretical point of view, it also gives us insight into new complete orthogonal sets of functions – e.g., the sets of complex exponential functions $\{e^{in\frac{2\pi}{\Lambda}x}, n = 1, 2, 3, \ldots\}$, where Λ is the smallest repetition distance of the function being expanded in the Fourier series. It is important that concepts related to the complex function form of Fourier series have direct analogs in the extremely widely used technique of Fourier transforms, which I will discuss with you in chapter 17.

To begin, reconsider our basic Fourier series for repetitive functions with repeat distance 2π:

(12.9.1)
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx .$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx\ ,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx\ .$$

We recall that

(12.9.2a)
$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

and

(12.9.2b)
$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}\ .$$

Putting eqns. (12.9.2) into eqn. (12.9.1), we have after collecting terms,

(12.9.3)
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{inx} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-inx}\ .$$

We now define a new set of Fourier coefficients as

(12.9.4) $c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')dx',$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')(\cos nx' - i\sin nx')\, dx' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \quad (n \geq 1)$$

$$d_n = \frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')(\cos nx' + i\sin nx')\, dx' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \quad (n \geq 1),$$

so our Fourier expansion can be written as

(12.9.5) $$f(x) = c_0 + \sum_{n=1}^{\infty} c_n\, e^{inx} + \sum_{n=1}^{\infty} d_n\, e^{-inx}\, .$$

We can simplify the appearance of this Fourier series with a little trick: We define the set of coefficients $\{c_{-n}\}$ for n = 1. 2. 3. … [i.e., $c_{-1}, c_{-2}, c_{-3}, \ldots$] by

(12.9.6) $$c_{-n} \equiv d_n = \frac{a_n + ib_n}{2}\ ,$$

so that

(12.9.7) $$f(x) = \sum_{n=0}^{\infty} c_n\, e^{inx} + \sum_{n=1}^{\infty} c_{-n}\, e^{-inx}\ ,$$

or,

(12.9.8) $$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}}$$

Complex form of Fourier series for a function with repeat distance 2π defined over the entire x-axis

with

(12.9.9)
$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx}$$

for all integer values of n: $n = 0, \pm 1, \pm 2, \pm 3, \ \ldots$.

So far, we have not invoked the requirement that $f(x)$ be a real-valued function of (real variable) x. In fact, equations (12.9.8) and (12.9.9) are correct even if $f(x)$ is a complex-valued function of (real variable) x. (In that case, the basic Fourier series given by eqn. (12.9.1) is still correct, but the Fourier coefficients in it (the a_n's and the b_n's) are, in general, complex numbers.)

But now, a careful reader interrupts with a question:

Question: At least at first glance, whether $f(x)$ is real-valued or not, it seems that the terms of the complex form Fourier series for $f(x)$ are all complex-valued since they all contain functions of the form e^{-inx}. But then, if $f(x)$ is a real-valued function of x, how can we see, manifestly, that the complex form of the Fourier series for $f(x)$ yields a real-valued result?

Answer: If $f(x)$ is a real-valued function of x, then all of the a_n's and the b_n's in its sine and cosine form Fourier series [i.e., eqn. (12.9.1)] must be real numbers. So, if $f(x)$ is real-valued, for each n, c_n and $c_{-n} = d_n$ are complex conjugates of each other [recall eqn. (12.9.4)],

(12.9.10)
$$c_{-n} = c_n^*$$

True for real $f(x)$ only

Now look again at eqn. (12.9.7):

$$f(x) = \sum_{n=0}^{\infty} c_n\, e^{inx} + \sum_{n=1}^{\infty} c_{-n}\, e^{-inx} \tag{12.9.7}$$

For any given n, the contribution to the Fourier series is

$$c_n e^{inx} + c_{-n}\, e^{-inx} \tag{12.9.11}$$

Now, for real-valued $f(x)$ $c_{-n} = c_n^*$ so for real-valued $f(x)$ the second term in eqn. (12.9.11) is the complex conjugate of the first term. [You will recall that the sum of any complex number z and its complex conjugate number is always a real number (equal to twice the real part of z).] <u>Thus, for each n, the sum in eqn. (12.9.11) is a real-valued function</u>. Hence, if $f(x)$ is a real-valued function, the entire right-hand side of eqn. (12.9.7) and hence, also, the entire right-hand side of eqn. (12.9.8) yields a real-valued function of x. On the other hand, if $f(x)$ is a complex-valued function of real variable x (or t), then it is not true that, for each n, $c_{-n} = c_n^*$.[8]

Eqns. (12.9.8) and (12.9.9) provide the basic complex function form that we sought for the Fourier series for a real-valued repetitive function with repeat distance 2π. In the following Worked Example, I illustrate the use of eqns. (12.9.8) and (12.9.9) to find the complex function Fourier series of a familiar real-valued function.

[8] Although I will not prove it here, it turns out that in the series for a complex-valued even function of a real variable (e.g., x or t), $c_{-n} = c_n$ and for a complex-valued odd function of a real variable, $c_{-n} = -c_n$.

Worked Example 12.4

Consider the repetitive square wave function $symstp(x)$ that we considered in chapter 11,

$$f(x) = symstp(x) = \{ \begin{array}{ll} -1/2 & \text{if } -\pi < x \le 0 \\ +1/2 & \text{if } 0 < x < \pi \end{array} \}$$

and its periodic repetition over the entire x-axis. For that function we found (eqn. 11.2) the Fourier series representation

(12.9.12) and (11.2) $$f(x) = \frac{2}{\pi}\,[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots],$$

the indicated series converging to $symstp(x)$ except at points of jump discontinuity. Find a Complex Fourier series representation of this function and explicitly verify that the complex series is term-by-term equal to the series we found for this function in chap. 11, eqn. (11.2).

Solution Steps:

1. First, we note that, according to our derivation of the complex-function form of the Fourier series for a real-valued function $f(x)$, the complex valued function form must, term-by-term be equal to the Fourier series that we found back in chapter 11, just rewritten in a different looking form.

2. **Evaluation of Complex-Valued Fourier coefficients**:

   We have $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$, where $f(x) = symstp(x)$ with its periodic

Worked Example 12.4, continued

extension over the entire x-axis, is an odd function. (That is why we wound up with only sine functions in its Fourier expansion in chapter 11). We expand the exponential function in the integral [$e^{-inx} = \cos nx - i \sin nx$] to write c_n as

(12.9.13) $$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \, .$$

We note that the integrand of the first integral in eqn. (12.9.13) is an odd function since $f(x)$ here is odd and $\cos nx$ is even (true even for n=0). Hence, since the first integral in eqn. (12.9.13) is taken over a symmetric interval, it is equal to zero.

Next, we consider the second integral in eqn. (12.9.13). For $n = 0$ the integrand is zero since sin(0) = 0. Thus, $c_0 = 0$. That makes sense since the average of = $symstp(x)$ over one repetition distance is zero. For $n \neq 0$, the integrand of the second integral is an even function, (product of two odd functions) so we can write c_n for $n \neq 0$ in the form

(12.9.14) $$c_n = -\frac{i}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = = -\frac{i}{2\pi} \int_0^{\pi} \sin nx \, dx,$$

since, on the interval (0, π), $f(x) = +\frac{1}{2}$. Integrating, we find

(12.9.15) $$c_n = -\frac{i}{2n\pi} [-\cos nx]_0^{\pi} = \frac{i}{2n\pi} [\cos n\pi - 1] = \begin{cases} \frac{-i}{n\pi} & n \; odd \\ 0 & n \; even, not \; zero \end{cases}$$

Worked Example 12.4, continued

3. **The Resulting Complex Form Fourier series**:

So from eqn. (12.9.15) we have

$$f(x) = \frac{-i}{\pi}\ (e^{ix} + \frac{1}{3}e^{3ix} + \frac{1}{5}e^{5ix} + \ldots) + \frac{-i}{\pi}(\frac{1}{-1}e^{-ix} + \frac{1}{-3}e^{-3ix} + \frac{1}{-5}e^{-5ix} + \ldots),$$

where the first infinite series includes the positive n terms and the second infinite series the negative n terms. We see that, for this situation, all the c_n's are pure imaginary even though $f(x)$ is a real-valued function.

4. **Expressing the result in manifestly real-valued form**:

Combining the two infinite series above we have

$$(12.9.16)\quad f(x) = \frac{-i}{\pi}\ (e^{ix} + \frac{1}{3}e^{3ix} + \frac{1}{5}e^{5ix} + \cdots - e^{-ix} - \frac{1}{3}e^{-3ix} - \frac{1}{5}e^{-5ix} - \ldots).$$

I leave it to you to show that the series (12.9.16) is mathematically the same as the series of eqn. (12.9.12) above – that is exercise 12.9.1 below.

Exercise 12.9.1 Show that eqn. (12.9.16) in Worked Example 12.9.4 above can be rewritten in the form of eqn. (12.9.12). Here, do this by starting specifically with eqn. (12.9.16) and manipulating it so that it specifically reproduces eqn. (12.9.12).

Hint: Note that $e^{inx} - e^{-inx} = 2isinnx$, for all positive integer n.