Ejercicio 0

Transformar la operación $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ mediante el cambio de variable: $x+y = u$; $x-y = v$

Aplicando la regla de la cadena: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial z}{\partial v} \cdot \underbrace{\frac{\partial v}{\partial y}}_{=1} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

También tenemos: $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \underbrace{\frac{\partial u}{\partial y}}_{=1} + \frac{\partial z}{\partial v} \cdot \underbrace{\frac{\partial v}{\partial y}}_{=-1} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

Viendo ahora las derivadas parciales segundas:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \left(\frac{\partial^2 z}{\partial u^2} \underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial^2 z}{\partial u \partial v} \cdot \underbrace{\frac{\partial v}{\partial x}}_{=1}\right) + \left(\frac{\partial^2 z}{\partial u \partial v} \underbrace{\frac{\partial u}{\partial x}}_{=1} + \frac{\partial^2 z}{\partial v^2} \underbrace{\frac{\partial v}{\partial x}}_{=1}\right) =$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \ldots\ldots = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Al final, tendremos: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 \iff 4\frac{\partial^2 z}{\partial u \partial v} = 0 \iff \frac{\partial^2 z}{\partial u \partial v} = 0$ //

Ej X Sean las ecuaciones $x^2+y^2 = u$ y $x+y = v$ Razonar cerca de qué puntos se puede despejar x e y en función de u y v (Tma. f. inversa)

Sea $P(x_0, y_0)$ un punto que verifica el sistema de ecuaciones dado.

Para poder aplicar el teorema de la función inversa, debemos justificar que en un entorno de (u_0, v_0)

Jacobiano $\frac{\partial(u,v)}{\partial(x,y)}(P) \neq 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 2(x-y) \iff \frac{\partial(u,v)}{\partial(x,y)}(P) = 2(x_0 - y_0)$$

Para que el Jacobiano sea distinto de cero, $x_0 \neq y_0$.

Ej Y Si $u = x^3 y$ encontrar $\frac{du}{dt}$ si: $\begin{cases} x^5 + y = t \\ x^2 + y^2 = t^2 \end{cases}$

Con la regla de la cadena:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = 3x^2 y \underbrace{\frac{\partial x}{\partial t}}_{x'(t)} + x^3 \underbrace{\frac{\partial y}{\partial t}}_{y'(t)} = 3x^2 y x' + x^3 y'$$

Volviendo a las condiciones iniciales, sean f y g tales que $f(x,y,t) = x^5 + y - t$ y $g(x,y,t) = x^2 + y^2 - t^2$

$$\begin{matrix} A \\ B \end{matrix}\begin{cases} x^5 + y = t \\ x^2 + y^2 = t^2 \end{cases} \iff \begin{cases} x^5 + y - t = 0 \\ x^2 + y^2 - t^2 = 0 \end{cases} \iff \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \quad \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} 5x^4 & 1 \\ 2x & 2y \end{vmatrix} = 2x(5x^3 y - 1)$$

$2x(5x^3y - 1) \neq 0 \iff x \neq 0$ y $x^3 y \neq \frac{1}{5}$ Para poder aplicar el Tma de la función inversa $J \neq 0$

Vamos a derivar en función de t A y B

$$\begin{matrix} A' \\ B' \end{matrix}\begin{cases} 5x^4 x' + y' = 1 \\ 2x x' + 2y y' = 2t \end{cases} \implies \text{Obtenemos } x'(t) = \frac{y-t}{x(5x^3y-1)} \quad y'(t) = \frac{5x^3 t - 1}{5x^3 y - 1}$$

(Sistema con (x', y') de incógnitas)

$$\text{Conclusión} \implies \frac{du}{dt} = \frac{3x^2 y (y-t)}{x(5x^3 y - 1)} + \frac{x^3(5x^3 y - 1)}{5x^3 y - 1} = \frac{3xy(y-t) + x^3(5x^3 t - 1)}{5x^3 y - 1} = F(t)$$ //

Ej 2 Probar que las ecuaciones del sistema
$$\begin{cases} x^2 - y\cos(uv) + z^2 = 0 \\ x^2+y^2 - \sin(uv) + 2z^2 = 2 \\ xy - \sin(u)\cos(v) + z = 0 \end{cases}$$
definen a x,y,z como funciones de u y v en un entorno del punto $(x,y,z,u,v) = (1,1,0,\frac{\pi}{2},0)$ (Calcular $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$ en el punto $(\frac{\pi}{2},0)$)

Sean f_1, f_2 y f_3 tres funciones en $\mathbb{R}^5$ tales que:
$$\begin{cases} f_1(x,y,z,u,v) = x^2 - y\cos(uv) + z^2 = 0 \\ f_2(x,y,z,u,v) = x^2+y^2 - \sin(uv) + 2z^2 - 2 = 0 \\ f_3(x,y,z,u,v) = xy - \sin(u)\cos(v) + z = 0 \end{cases}$$

1. Comprobamos si el punto $P(1,1,0,\frac{\pi}{2},0)$ satisface el sistema. (Pista, sí)

2. Comprobamos si las derivadas parciales son continuas (Pista: sí) (en todas las variables) en un entorno del punto (lo son en $\mathbb{R}^5$) de $f_1 f_2 f_3$
(Estudiamos continuidad en el punto, si lo es, por un teorema, lo es en un entorno de P)

3. Jacobiano $\neq 0$
$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & -\cos(u,v) & 2z \\ 2x & 2y & 4z \\ y & x & 1 \end{vmatrix} \rightarrow$$

evaluado en P $\longrightarrow$
$$\begin{vmatrix} 2 & -1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 6 \neq 0$$

$\Rightarrow$ Podemos aplicar el T^{ma} de la función implícita y concluimos que
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

$$\begin{cases} x^2 - y\cos(uv) + z^2 = 0 \\ x^2+y^2 - \sin(uv) + 2z^2 = 2 \\ xy - \sin(u)\cos(v) + z = 0 \end{cases} \qquad \left|\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}(P)\right| = 6 \neq 0$$

Tenemos entonces, $x = x(u,v)$ $y = y(u,v)$ $z = z(u,v)$

$$\frac{\partial}{\partial u}(x^2 - y\cos(uv) + z^2) = 0 \Leftrightarrow \begin{cases} 2x\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u}\cos(uv) + vy\sin(uv) + 2z\frac{\partial z}{\partial u} = 0 \\ 2x\frac{\partial x}{\partial u} + 2y\frac{\partial y}{\partial u} - v\cos(uv) + 2z\frac{\partial z}{\partial u} = 0 \\ y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} - \cos(u)\cos(v) + \frac{\partial z}{\partial u} = 0 \end{cases}$$

Evaluamos en el punto P, obtenemos el sistema:
$$\begin{cases} 2\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} = 0 \\ 2\frac{\partial x}{\partial u} + 2\frac{\partial y}{\partial u} = 0 \\ \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} + \frac{\partial z}{\partial u} = 0 \end{cases}$$

Se trata de un sistema homogéneo cuya matriz de coeficientes tiene un determinante no nulo así que la única solución es
$$\frac{\partial x}{\partial u}(\tfrac{\pi}{2},0) = 0; \quad \frac{\partial y}{\partial u}(\tfrac{\pi}{2},0) = 0; \quad \frac{\partial z}{\partial u}(\tfrac{\pi}{2},0) = 0$$

Luego, hacemos el cálculo de las derivadas parciales primeras respecto a v y evaluamos en P, lo que nos da:
$$\begin{cases} 2\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} = 0 \\ 2\frac{\partial x}{\partial v} + 2\frac{\partial y}{\partial v} = \frac{\pi}{2} \\ \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} + \frac{\partial z}{\partial v} = 0 \end{cases}$$

Usando la regla de Kramer,
$$\frac{\partial x}{\partial v}(\tfrac{\pi}{2},0) = \frac{\pi}{12} \qquad \frac{\partial y}{\partial v}(\tfrac{\pi}{2},0) = \frac{\pi}{6} \qquad \frac{\partial z}{\partial v}(\tfrac{\pi}{2},0) = -\frac{\pi}{4}$$

$$\frac{\partial x}{\partial v} = \frac{\begin{vmatrix} 0 & -1 & 0 \\ \frac{\pi}{2} & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix}}{\det(M)}$$

Probar que el sistema define implícitamente a y, z como funciones de x en un entorno del punto $(x, y, z) = (2, 1, 1)$

$$\begin{cases} x^2 + y^2 - z^2 - 4 = 0 \\ yz + xz - xy - 1 = 0 \end{cases}$$

Calcular $\frac{dy}{dx}$ y $\frac{dz}{dx}$ en el punto

Rosío

$\star(x_0, y_0)$

Ejercicio 3

Se sabe que para f diferenciable la igualdad $f(x+\frac{z}{y}, y+\frac{z}{x})=0$ define la función implícita $z=h(x,y)$. Calcular $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

Sabemos que $f(x+\frac{z}{y}, y+\frac{z}{x})$ define una función implícita: $z=z(x,y)$ $\quad dz = \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y$ $\star$ $\quad u=(x+\frac{z}{y})$ $\quad v=(y+\frac{z}{x})$

$Df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv$ $\quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = dx + \frac{\frac{\partial z}{\partial y}y - z}{y^2}dy$ $\quad dx = \frac{\partial f}{\partial u}$ $\quad dy = \frac{\partial f}{\partial v}$

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial y}dy = dy + \frac{\frac{\partial z}{\partial x}x - z}{x^2}dx$

Entonces, $$Df = \frac{\partial f}{\partial u}\left(\overbrace{\left(1+\frac{1}{y}\frac{\partial z}{\partial x}\right)}^{A}dx + \frac{\frac{\partial z}{\partial y}y - z}{y^2}dy\right) + \frac{\partial f}{\partial v}\left(\overbrace{\left(1+\frac{1}{x}\frac{\partial z}{\partial y}\right)}^{B}dy + \frac{\frac{\partial z}{\partial x}x - z}{x^2}dx\right) = 0 \iff$$

$$\iff A\frac{\partial f}{\partial x}dx + \frac{1}{y^2}\frac{\partial f}{\partial u}\left(\frac{\partial z}{\partial y} - z\right)dy + \frac{1}{x^2}\frac{\partial f}{\partial v}\left(\frac{\partial z}{\partial x} - z\right)dx + B\frac{\partial f}{\partial v}dy = 0 \iff$$

$$\iff A\frac{\partial f}{\partial u}dx + B\frac{\partial f}{\partial v}dy + \frac{dy}{y^2}\frac{\partial f}{\partial u}\frac{\partial z}{\partial y}y - \frac{dy}{y^2}\frac{\partial f}{\partial u}z + \frac{dx}{x^2}\frac{\partial f}{\partial v}\frac{\partial z}{\partial x}x - \frac{dx}{x^2}\frac{\partial f}{\partial v}z = 0 \iff$$

$$\iff \left[A\frac{\partial f}{\partial u} + \frac{1}{x^2}\frac{\partial f}{\partial v}\underline{\frac{\partial z}{\partial x}x} - \frac{1}{x^2}\frac{\partial f}{\partial v}z\right]dx + \left[B\frac{\partial f}{\partial v} + \frac{1}{y^2}\frac{\partial f}{\partial u}\underline{\frac{\partial z}{\partial y}y} - \frac{1}{y^2}\frac{\partial f}{\partial u}z\right]dy = 0 \iff$$

$$\begin{cases} \frac{\partial z}{\partial x}\cdot x = x^2\,\dfrac{\frac{1}{x^2}\frac{\partial f}{\partial v}z - A\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} \\[2ex] \frac{\partial z}{\partial y}\cdot y = y^2\,\dfrac{\frac{1}{y^2}\frac{\partial f}{\partial u}z - B\frac{\partial f}{\partial v}}{\frac{\partial f}{\partial v}} \end{cases}$$