El campo electromagnético:

Consideremos $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$, con

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad y \quad J^\mu = (\rho, j^i) .$$

La acción de Maxwell es

$$S = \int_{\mathcal{M}} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \right)$$

$$\delta S = \int_{\mathcal{M}} d^4x \left(-\frac{1}{4} (\delta F_{\mu\nu}) F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} (\delta F^{\mu\nu}) - J^\mu \delta A_\mu \right)$$

a) $$-\frac{1}{4} (\delta F_{\mu\nu}) F^{\mu\nu} = -\frac{1}{4} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F^{\mu\nu} \overset{*}{=}$$

pero
$$\partial_\nu (\delta A_\mu F^{\mu\nu}) = \partial_\nu \delta A_\mu F^{\mu\nu} + \delta A_\mu \partial_\nu F^{\mu\nu}$$
$$\partial_\mu (\delta A_\nu F^{\mu\nu}) = \partial_\mu \delta A_\nu F^{\mu\nu} + \delta A_\nu \partial_\mu F^{\mu\nu}$$

$$\overset{*}{=} -\frac{1}{4} \Big(\underbrace{\partial_\mu (\delta A_\nu F^{\mu\nu})}_{(1)} - \delta A_\nu \partial_\mu F^{\mu\nu} - \underbrace{\partial_\nu (\delta A_\mu F^{\mu\nu})}_{(2)} + \delta A_\mu \partial_\nu F^{\mu\nu} \Big)$$

(1) y (2) se van a anular al integrar. Son derivadas totales. Stokes. $\delta A_\nu = 0$ en fronteras.

Quedará

$$\overset{(a)}{=} \frac{1}{4} \left(\partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\nu F^{\mu\nu} \delta A_\mu \right)$$

$$= \frac{1}{4} \left(\partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\mu F^{\nu\mu} \delta A_\nu \right) \qquad (F^{\mu\nu} = -F^{\nu\mu})$$

$$= \frac{1}{4} \left(\partial_\mu F^{\mu\nu} \delta A_\nu + \partial_\mu F^{\mu\nu} \delta A_\nu \right)$$

$$= \frac{1}{2} \partial_\mu F^{\mu\nu} \delta A_\nu = -\frac{1}{4} \left(\delta F_{\mu\nu} \right) F^{\mu\nu}$$

b) $-\frac{1}{4} F_{\mu\nu} \delta F^{\mu\nu} =$

$$= -\frac{1}{4} F_{\mu\nu} \delta \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) =$$

$$= -\frac{1}{4} F_{\mu\nu} \left(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu \right)$$

= .... = mismo caso que a) pero hay que bajar índices

$$= \frac{1}{2}\left(\partial_\mu F^{\mu\nu}\right)\delta A_\nu$$

Finalmente, escribimos

$$\delta S = \int d^4x \left(\frac{1}{2}\partial_\mu F^{\mu\nu}\delta A_\nu + \frac{1}{2}\partial_\mu F^{\mu\nu}\delta A_\nu - J^\mu \delta A_\mu \right)$$

$$= \int d^4x \left(\partial_\mu F^{\mu\nu} - J^\nu \right) \underbrace{\delta A_\mu}_{\neq 0 \text{ (es arbitrario)}} = 0$$

$$\Longrightarrow \quad \boxed{\partial_\mu F^{\mu\nu} = J^\nu} \quad (\text{Gauss}^{(E)} + \text{Ampère})$$

Además, $F_{\mu\nu}$ satisface una identidad de Bianchi:

$$\boxed{\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0}$$

$(\text{Gauss}^{(B)} + \text{Faraday})$

Simetrías y leyes de conservación

↳ Cambio en la "visión" de las ecuaciones que las deja invariantes.

- simetrías externas: traslación, tiempo, rotación
- simetrías internas: cambios en los campos que no involucran cambios con respecto al espacio-tiempo.

Consideremos que las coord. espaciotemporales varían según:

$$x^\mu \longrightarrow x^\mu + a^\mu \quad (a^\mu \text{ pequeño y arbitrario})$$

Series de Taylor:

$$\varphi(x) \longrightarrow \varphi(x+a) = \varphi(x) + a^\mu \partial_\mu \varphi$$

Bajo una pequeña perturbación, $\varphi \rightarrow \varphi + \delta\varphi$

luego $\delta\varphi = a^\mu \partial_\mu \varphi$

si $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu\varphi)$, tenemos

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)$$

sabemos que

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right), \text{ luego}$$

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\delta\varphi)$$

si hacemos $u = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}$ $\quad v = \delta\varphi$

$(uv)' = u'v + v'u$, queda

$$\delta\mathcal{L} = (uv)' = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)$$

o bien $\quad \delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi \right) a^\nu$

También podemos escribir

$$\delta\mathcal{L} = \partial_\mu (\mathcal{L}) a^\mu = \delta^\mu_\nu \partial_\mu (\mathcal{L}) a^\nu$$

(considerando cómo varía respecto a $(x^\mu \to x^\mu + a^\mu)$)

Igualando ambas expresiones:

$$\delta\mathcal{L} = \delta^\mu_\nu \partial_\mu (\mathcal{L}) a^\nu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi \right) a^\nu$$

$$\Rightarrow \quad \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \right] a^\nu = 0$$

es arbitrario

$$\Rightarrow \quad \boxed{\partial_\mu T^\mu_\nu = 0}, \quad \text{con}$$

$$\boxed{T^\mu_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}}$$

Tensor energía-momento y su conservación

$$T^0_{\ 0} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} = \mathcal{H} \ (\text{densidad Hamiltoniana})$$

luego $\partial_0 T^0_{\ 0} = 0 \Rightarrow$ conservación energía

$$T^0_{\ i} = p_i \ (\text{densidad de momento})$$

$$P_i = \int d^3x \, p_i = \int d^3x \, T^0_{\ i}$$

Corrientes conservadas:

Hagamos $\varphi \to \varphi + \delta\varphi$

consideremos que, bajo esta variación, $\mathcal{L}$ no va a variar. Esto es,

$$\mathcal{L} \longrightarrow \mathcal{L} + \delta\mathcal{L} \quad \text{con} \quad \delta\mathcal{L} = 0.$$

pero $\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (\delta\varphi)$

como $\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right)$, queda

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (\delta\varphi)$$

$$= \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right]}_{} = 0$$

$$\Rightarrow \quad \boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi}$$ es una

corriente conservada

$$\boxed{\partial_\mu J^\mu = 0}$$

Asociada a cada corriente conservada, hay una carga conservada,

$$Q = \int d^3x \, J^0 .$$

(en. y momento son cargas conservadas debido a invariancia bajo traslaciones espaciotemporales).

Ejercicio: considerad un campo escalar complejo.

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

Deducid y comprobad:

$$(\Box + m^2)\varphi = 0$$

$$(\Box + m^2)\varphi^* = 0$$

$$\pi = \varphi^*_{,t} \qquad \pi^* = \varphi_{,t}$$

$$\mathcal{H} = \pi^* \pi + (\nabla \varphi^*)(\nabla \varphi) + m^2 \varphi^* \varphi$$

$$T_{\mu\nu} = \partial_\mu \varphi^* \partial_\nu \varphi + \partial_\mu \varphi \partial_\nu \varphi^* - g_{\mu\nu} \mathcal{L}$$

$$p^i = \int d^3x \, (\varphi^*_{,t} \partial^i \varphi + \varphi_{,t} \partial^i \varphi^*)$$

$$p^0 = \int d^3x \, \mathcal{H} \quad (\geq 0)$$

$$J^\mu = -i(\varphi \varphi^{*,\mu} - \varphi^* \varphi^{,\mu})$$

$$Q = \int d^3x \, J^0 = i \int (\varphi^* \varphi_{,t} - \varphi \varphi^*_{,t}) \, d^3x$$