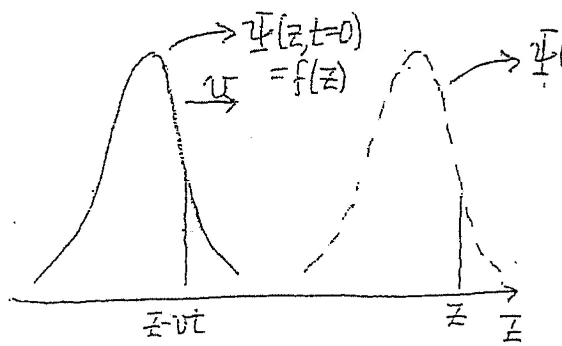
Physics 251 - 20th Class - Thursday March 21, 2024

A. Basic Forms For Rigidly Traveling Waves [K-text, sects. 13.1, 13.2]

We begin by considering pulses traveling on a stretched string or "slinky" with no damping. As we will see soon, ideal media that obey the classical wave equation allow wave shapes to move rigidly (i.e, without changing shape) along the medium at speed equal to the phase velocity $v_\phi = \sqrt{\frac{T_0}{\rho_0}}$ that we found for standing waves. In our initial discussion, however, we keep the "wave velocity" ($v \equiv v_\phi$) as an open parameter and merely discuss some "kinematics" of the motion, most importantly, how to specify traveling wave motion mathematically.

We begin this by assuming* that we have some sort of a shape moving rigidly to the right (positive z) at wave speed v; we suppose that, at time $t=0$ it is denoted mathematically by $f(z)$.



As a concrete example, say

$$\Psi(z,t=0) = f(z) = \frac{3}{10z^4+1}$$

which is peaked at $z=0$.

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* We are not yet addressing the question of whether arbitrary rigidly

If this shape then moves rigidly in the positive-z direction at speed v, what then is its mathematical name at any time t?

Referring back to the diagram, we note that

$$\Psi(z,t) = \Psi(z-vt, t=0) = f(z-vt)$$

is the condition for rigid motion at speed v.

Thus, given $\Psi(z,t=0)$ as a functional form $f(z)$, to form $\Psi(z,t)$ we need merely replace z, wherever it appears in $f(z)$ with $z-vt$.

Thus, for example, if

$$\Psi(z,t=0) = \frac{3}{10z^4+1}$$

and the shape moves rigidly to positive z at speed v, then

$$\Psi(z,t) = \frac{3}{10(z-vt)^4+1}$$

or, if

$\Psi(z,t=0) = Ae^{-z^2}$ and the shape moves rigidly, then

$$\Psi(z,t) = Ae^{-(z-vt)^2}.$$

If the shape moves rigidly in the $-z$-direction, then

$$\Psi(z,t) = \Psi(z+vt, t=0) = g(z+vt)$$

B. Rigidly Traveling Sinusoidal Waves [K-text, sect. 13.3]

Now we look at the most important special case - traveling sinusoidal waves.

Suppose we have an "infinitely long" stretched string (or slinky).
Say it supports a sinusoidal wave traveling rigidly toward positive z with velocity v.

At time "$t=0$" we take a snapshot of this; the waveform is instantaneously frozen in the picture. It's equation at $t=0$ is

$$\Psi(z,t=0) = f(z) = A\cos\left(\frac{2\pi}{\lambda}z + \phi\right)$$

As time progresses, the waveform moves rigidly

What is its name? I.e., what is $\Psi(z,t)$?

Again we can apply our rule

$$\Psi(z,t) = \Psi(z-vt,0), \quad \text{so here}$$

$$\Psi(z,t) = A\cos\left[\frac{2\pi}{\lambda}(z-vt)\right]$$ (where I've ignored the phase constant ϕ).

Now, using our definition

$$k = \frac{2\pi}{\lambda} \quad \text{(wavenumber)}, \text{ this is}$$

(1) $\Psi(z,t) = A\cos(kz - kvt)$

To understand eqn. (1) better, we look at two aspects separately:

i. <u>At fixed $t = t_0$</u> ("snapshot")

Then kvt_0 is a constant; call it α. Then

$$\Psi(z,t_0) = A\cos(kz - \alpha) = A\cos\left(\frac{2\pi}{\lambda}z - \alpha\right)$$

which is a stationary sinusoid in space, as we expect and as shown on the last page

ii. <u>At fixed $z = z_0$</u>:

Then kz_0 is a constant; call it β, then

$$\Psi(z_0,t) = A\cos(\beta - kvt) = A\cos(kvt - \beta)$$

Thus, each point on the slinky oscillates up and down in simple harmonic motion as the wave passes through. The frequency of this up-down motion is

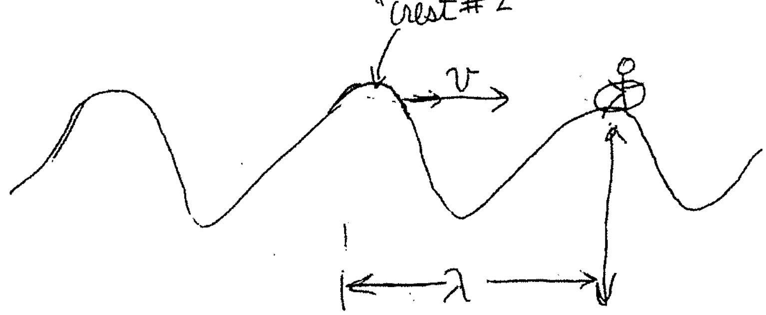
$$\omega = kv \quad \text{or} \quad f = \frac{v}{\lambda} \Rightarrow \boxed{v = f\lambda = \frac{\omega}{k}}$$

$$\Rightarrow 2\pi f = \frac{2\pi}{\lambda} v$$

Note that <u>these are the same relations we had for standing waves</u>! For that reason, $v = v_\phi$ is, in the context of traveling waves, called the "wave velocity" (as well as the "phase velocity")

Here is an "alternate" quick way to think of this last important result:

Suppose you are out on a lake floating in an inner tube at position z, and you are now riding the crest of a sinusoidal wave that is passing under you at speed v.



Your motion is then directly vertically up and down in simple harmonic motion. How long do you have to wait to be at maximum altitude again?

One up-down period T. But, from "another point of view", this is the time for the next crest ("crest # 2") to reach you; since it travels at speed v, and since it is distance one wavelength λ away, this takes a time

$$\Delta t = \frac{\lambda}{v}$$

Hence $T = \frac{1}{f} = \frac{\lambda}{v} \Rightarrow \underline{v = f\lambda = \frac{\omega}{k}}$. Then, since $kv = \omega$, from eqn. (1)

our "right-traveling" (i.e., to positive z) harmonic wave is, e.g.,

(2) $\underline{\Psi_{\rightarrow}(z,t) = A\cos(kz - \omega t) = A\cos(\omega t - kz)}$

and a "left-traveling" (i.e., in the $-z$ direction) harmonic wave is, e.g.,

(3) $$\Psi_{\leftarrow}(z,t) = A\cos(kz+\omega t).$$

Other Forms

Since we can put in an arbitrary fixed phase, the following are all valid harmonic waves

$$\Psi_{\rightarrow}(z,t) = A\cos(kz-\omega t) = A\cos(\omega t-kz)$$

$$\Psi_{\rightarrow}(z,t) = A\sin(kz-\omega t) = B\sin(\omega t-kz), \quad B=-A$$

$$\Psi_{\rightarrow}(z,t) = A\cos(kz-\omega t+\phi)$$

$$\Psi_{\rightarrow}(z,t) = A\sin(kz-\omega t+\phi)$$ (different ϕ than line above)

$$\Psi_{\leftarrow}(z,t) = A\sin(kz+\omega t)$$

$$\Psi_{\leftarrow}(z,t) = A\cos(kz+\omega t)$$

$$\Psi_{\leftarrow}(z,t) = A\cos(kz+\omega t+\phi)$$

etc.

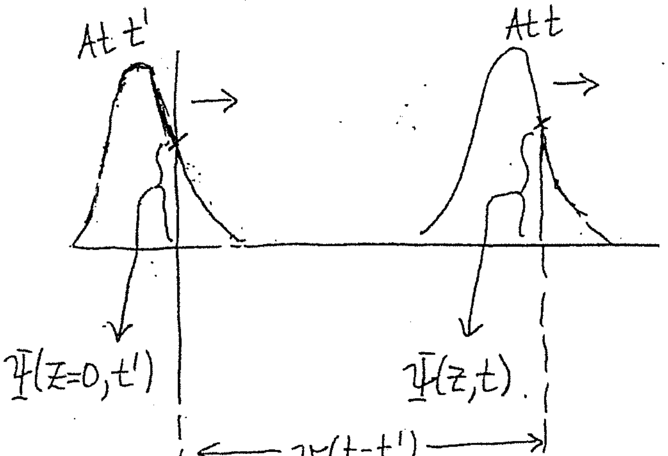
Another example of a "left-moving" wave is

$$\begin{aligned}\Psi_{\leftarrow}(z,t) &= B\sin(-kz-\omega t+\delta)\\ &= -B\sin(kz+\omega t-\delta)\\ &= B\sin[kz+\omega t+(\pi-\delta)]\\ &= B\cos(kz+\omega t+\phi),\end{aligned}$$

the last of which is clearly "left-moving".

C. Alternative Derivation of Mathematical Form of Rigidly Traveling Waveform: "Method of Retarded Time" [K-text, sect 13.3]

Due to the primary importance of understanding why our results are correct, we now look at another, complimentary method of derivation: We note from the figure below that



$\Psi(z,t)$ is the same as Ψ at $z=0$ at the earlier time t', where

$$z = v(t-t')$$

$$\Rightarrow \quad t' = t - z/v.$$

Here, $\frac{z}{v}$ is the travel time for the feature to reach (z,t) from $(0,t')$, so

$$\Psi_{\rightarrow}(z,t) = \Psi_{\rightarrow}(z=0,\ t - z/v)$$

The time $t' = t - \frac{z}{v}$ is sometimes called "the retarded time."

Let us apply this to a right-moving sinusoidal harmonic wave. We can form one by shaking the end of the slinky at $z=0$ up and down in s.h.m according to

$$\Psi(z=0,t) = A\cos\omega t.$$

Thus, here we construct Ψ from a boundary (z=0) condition rather than from an initial (i.e. t=0) condition.

Then

$$\Psi_{\rightarrow}(z,t) = A\cos\left[\omega(t - z/v)\right]$$

$$= A\cos\left[\omega t - \omega z/v\right] = A\cos(\omega t - kz) = A\cos(kz - \omega t)$$

D. <u>Alternative Derivation of Wave Velocity For Traveling Waves</u> - <u>Method of Stationary Phase</u>

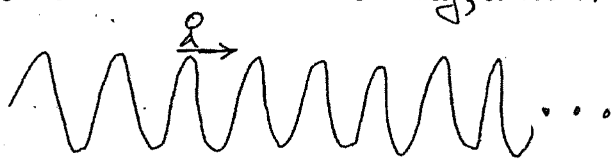
Our "right-traveling" harmonic wave is

$$\Psi(z,t) = A\cos(\omega t - kz)$$

Suppose we want to surf on a given crest of this wave.

We ask - how fast do we move with respect to the "ground?"

This would be the wave velocity, and the same as the "phase velocity."



Riding on the crest, we have a certain value (some integer multiple of 2π) of the phase function $(\omega t - kz) \equiv \varphi(z,t)$, which remains constant as we move along (and as our position z and the time t change accordingly). Thus, as we move along,

$$\begin{aligned} 0 &= d\varphi \\ &= \frac{\partial \varphi(z,t)}{\partial z}\,dz + \frac{\partial \varphi(z,t)}{\partial t}\,dt \\ &= -k\,dz + \omega\,dt \end{aligned}$$

Thus, our position changes at rate

$$v = \frac{dz}{dt} = \frac{\omega}{k},$$

which illustrates the reason for calling v the "phase velocity", since in this method we directly look at the motion of a point of fixed phase on the wave.

E. Complex Function Representation of Rigidly Traveling Sinusoidal Waves [K-text, sect. 13-5]

From Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ $(i = \sqrt{-1})$,

we can write

(1) $\Psi(z,t) = A\cos(kz - \omega t + \phi) = \mathrm{Re}\left[A e^{i(kz-\omega t+\phi)}\right]$

where we are assuming that A, k, ω and ϕ are real. So

(2) $\tilde{\Psi}(z,t) = A e^{i(kz-\omega t+\phi)} = A\cos(kz-\omega t+\phi) + iA\sin(kz-\omega t+\phi)$

We can get some pictorial insight into the function $\tilde{\Psi}(z,t) = A e^{i(kz-\omega t+\phi)}$ from the figure below:

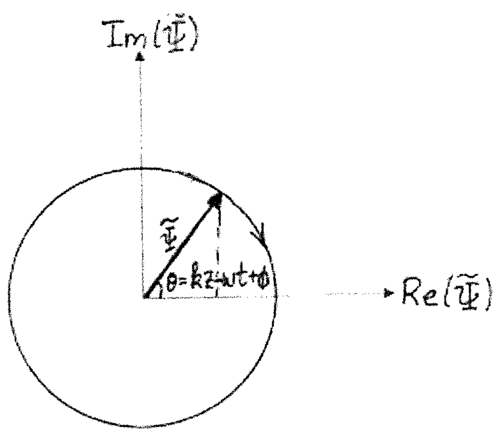


Fig. 13.5.1 Representation of the complex form of the "right-moving" sinusoidal traveling wave function $\tilde{\Psi}(z,t) = Ae^{i(kz-\omega t+\phi)}$ at a fixed point z in space as a rotating vector in the complex Ψ-plane. As time progresses, the tip of the "vector" $\tilde{\Psi}$ traces out a circular path in the clockwise direction in the complex Ψ-plane. If a z-axis (pointing directly straight out of the page) were to be added to the figure, then as time progresses the tip of the vector $\tilde{\Psi}$ would trace out a spiral path with steady progress in the +z direction at the wave velocity $v = \frac{\omega}{k}$.

Similarly, for a sinusoidal wave moving in the $-z$ direction,

(3) $\Psi(z,t) = A\cos(kz + \omega t + \delta) = \mathrm{Re}\left[A e^{i(kz+\omega t+\delta)}\right]$,

for which the complex form $Ae^{i(kz+\omega t+\delta)}$ rotates counterclockwise in the complex-Ψ-plane.

We often leave out the "Re[...]" part as being understood for real-valued wave functions and just write

$$(3) \quad \Psi_{\rightarrow}(z,t) = Ae^{i(kz-\omega t+\phi)}$$

and

$$(4) \quad \Psi_{\leftarrow}(z,t) = Ae^{i(kz+\omega t+\phi)}$$

Eqn. (3) can be also written as

$$(5) \quad \Psi_{\rightarrow}(z,t) = Ae^{i\phi}e^{ikz}e^{-i\omega t}$$

and

$$(6) \quad \Psi_{\leftarrow}(z,t) = Ae^{i\phi}e^{ikz}e^{+i\omega t}$$

Note that the convention we are using has time part $e^{-i\omega t}$ for $\Psi_{\rightarrow}$ and $e^{+i\omega t}$ for $\Psi_{\leftarrow}$.

A nice feature of the complex form of wave functions is that, e.g., if $\Psi(z,t) = Ae^{i(kz-\omega t+\phi)} = Ae^{ikz}e^{-i\omega t}e^{i\phi}$, then

$$\dot{\Psi}(z,t) = -i\omega\Psi(z,t),$$

where the factor $-i$ changes ϕ by $\frac{\pi}{2}$ radians $\left[-i = e^{-i\pi/2} \Rightarrow \Delta\phi = -\frac{\pi}{2}\right]$

F. Do Rigidly Traveling Waves Satisfy the Classical Wave Equation? [K-text, sect. 13.6]

It's nice to talk about rigidly traveling waves, but whether they are allowed in a medium depends on whether they are solutions to the governing differential equation of the medium.

Consider, e.g., the perfectly flexible stretched string. For it, the C.W.E. is

$$(1) \quad \frac{\partial^2 \Psi(z,t)}{\partial z^2} = \frac{1}{v_\phi^2} \frac{\partial^2 \Psi(z,t)}{\partial t^2},$$

where v_ϕ is a number with units m/s.

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It is easy to show (do it, if you have not already) that any of our forms for "right" and "left-traveling" sinusoidal waves with wave velocity parameter v satisfy the CWE above [eqn. (1)] if and only if $v = v_\phi$. Since we already know (from work earlier this semester) that v_ϕ is the phase velocity parameter for standing waves (transverse) on the same stretched string, this thus proves that, the case of a rigidly traveling sinusoidal wave, the velocity of the wave pattern (v) is equal to the phase velocity (v_ϕ) of transverse standing waves [i.e, $v = v_\phi = \sqrt{T_0/\rho_\ell}$].

Now, we would like to answer two questions:

1. Does the ideal stretched string medium allow more general rigidly moving waveform shapes? And,
2. If yes, is the wave velocity v then also equal to v_ϕ?

To answer the first question, we must see if, in general, an arbitrary rigidly moving wave function $f(z-vt)$ satisfies the C.W.E.

So, the question is: Is it true, for "any function" $f(z)$ that

$$\frac{\partial^2 f(z-vt)}{\partial z^2} \stackrel{?}{=} \frac{1}{v_\phi^2} \cdot \frac{\partial^2 f(z-vt)}{\partial t^2} \quad ?$$

To find out, we need some derivatives:

$$\frac{\partial f(z-vt)}{\partial (z)} = \underbrace{\frac{\partial f(z-vt)}{\partial (z-vt)}}_{\text{call this } f'} \cdot \frac{\partial (z-vt)}{\partial z} = f' \cdot 1 = f'$$

similarly,

$$\frac{\partial^2 f(z-vt)}{\partial z^2} = \frac{\partial f'}{\partial z} = \underbrace{\frac{\partial f'}{\partial (z-vt)}}_{f''} \underbrace{\frac{\partial (z-vt)}{\partial z}}_{1} = f''$$

and

$$\frac{\partial f(z-vt)}{\partial t} = \underbrace{\frac{\partial f(z-vt)}{\partial (z-vt)}}_{f'} \cdot \underbrace{\frac{\partial (z-vt)}{\partial t}}_{-v} = -vf'$$

so

$$\frac{\partial^2 f(z-vt)}{\partial t^2} = \frac{\partial}{\partial t}(-vf') = -v \underbrace{\frac{\partial f'}{\partial (z-vt)}}_{f''} \cdot \underbrace{\frac{\partial (z-vt)}{\partial t}}_{-v} = v^2 f'$$

so, indeed

$$\frac{\partial^2 f(z-vt)}{\partial z^2} \stackrel{\checkmark}{=} \frac{1}{v_\phi^2} \frac{\partial^2 f(z-vt)}{\partial t^2} \quad \text{if and only if } v = v_\phi \text{ !}$$

Likewise, any "left moving" shape is also a soln. of the CWE – for any $g(z+vt)$, as you can show;

$$\frac{\partial^2 g(z-vt)}{\partial z^2} \stackrel{\checkmark}{=} \frac{1}{v^2} \frac{\partial^2 g(z-vt)}{\partial t^2}$$

Note: The derivation immediately above is, perhaps, a little abstract and may thus require a careful "re-read" through. In the meantime, however, I suggest that you try the following "concrete example" exercise:

Exercise: Consider the case $\Psi(z,t) = (z - vt)^2$. Explicitly show that this rigidly moving function satisfies the C.W.E.

$$(1) \qquad \frac{\partial^2 \Psi(z,t)}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(z,t)}{\partial t^2}$$

To do this, do not rely on the "general proof" of the previous page. Rather, "do it out" by explicitly taking the appropriate derivatives of $\Psi(z-vt)^2$.

In our derivation on the last page, we showed that, for any medium governed by the C.W.E [eqn. (1) above] for any type of wave, the velocity with which a given waveform moves rigidly is the parameter v in that C.W.E. But, we already know [Chapter 8] that the parameter v in the C.W.E. is equal to the phase velocity for standing waves in that medium. Hence, <u>the wave velocity for a rigidly traveling wave that obeys the C.W.E in a given medium is the same (i.e, numerically equal to, but conceptually different than) the phase velocity for standing waves in that medium.</u>

Example:

Suppose that the medium is an ideal stretched string and that we are considering transverse waves on it.

Then, the relevant C.W.E. is

$$\frac{\partial^2 \Psi}{\partial z^2} = \frac{\rho_e}{T_0} \frac{\partial^2 \Psi}{\partial t^2} ,$$

i.e, $v = \sqrt{\frac{T_0}{\rho_e}}$.

Then, according to our derivation above,

phase velocity for transverse standing waves on ideal string $= \sqrt{\frac{T_0}{\rho_e}}$ = wave velocity v for rigidly moving traveling wave forms (of "arbitrary" shape) on this string.

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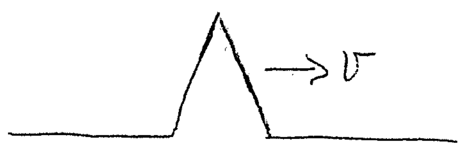
Now, as we've known for quite some time in this course, <u>all</u> motion on the (ideal) stretched string is governed by the CWE and anything that is allowed on the string must satisfy the C.W.E. Conversely, anything that satisfies the CWE is permitted on the string. (To set it up may, of course, require a careful choice of initial conditions.)

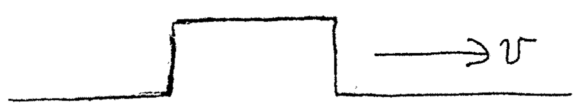
So, allowed on the <u>ideal</u> stretched string are

traveling triangles



traveling bumps

traveling boxes



and so on – as we showed, any function $f(z - vt)$ is a solution. Of course, we could also have any shape rigidly moving to the left, so apparently, a quite general solution

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* where the triangle and "box" shapes are just a bit curved at the bottom so that they are twice differentiable.

of the Classical Wave Equation is

$$(1) \quad \Psi(z,t) = f(z-vt) + g(z+vt) \qquad (*)$$

where $f(z)$ and $g(z)$ are arbitrary ~~twice~~ differentiable functions.

In fact, (1) is a way of writing the most general solution to the CWE!

*[In arriving at this, I used the linearity property (in Ψ) of the CWE to add the general right-moving $[f(z-vt)]$ and left-moving $[g(z-vt)]$ solutions to obtain the general solution [eqn. (1)] to the CWE.]

Note also that, for a string of finite length $[0 \rightarrow L]$ with arbitrary boundary conditions at its ends (e.g., both ends bound, both ends free, etc.), the general solution, as we discussed some time ago, is an "arbitrary general" sum of the normal modes for those boundary conditions. Thus, apparently there are two "different" competitors [eqn (1) above and general superposition of normal modes] for the title of "general solution of the CWE" on a finite stretched string!

The equivalence of these "two forms" of the general solution (back and forth traveling waves superposition and superposition of standing waves (normal modes) can be demonstrated via Fourier analysis; somewhat regretably, that proof is a bit beyond the scope of this course.]

G. More On the Traveling Wave Form of the General Solution of the C.W.E. [K-text, sect. 13.6]

Since the C.W.E.

$$(1) \quad \frac{\partial^2 \Psi(z,t)}{\partial z^2} = \frac{1}{v_\phi^{2}} \frac{\partial^2 \Psi(z,t)}{\partial t^2}$$

is linear and homogeneous, by the superposition theorem, a very general solution to it is

$$(2) \quad \Psi(z,t) = f(z-vt) + g(z+vt)$$

where f and g are arbitrary "reasonable" functions.

Now, since $f(z-vt)$ and $g(z+vt)$ are two linearly independent solutions of the CWE on the entire z-axis, it is eminently plausible that eqn (2) represents a form of the general solution to the CWE on the entire z-axis; it is called "D'Alembert's form of the general solution".

H. Another Interesting Consequence of the Linearity of The CWE and the Superposition Theorem [K-text, sect. 13.10]

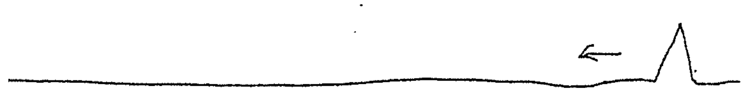
Before getting too involved in mathematical aspects of "the general solution to the CWE in traveling wave form," let's first look to gain some physical understanding of some of the results.

First, consider a solution (to the CWE) $\Psi_1(z,t)$ which is a right-moving "box function"

$\Psi_1(z,t)$

Consider another solution $\Psi_2(z,t)$ which is a "left moving triangle"

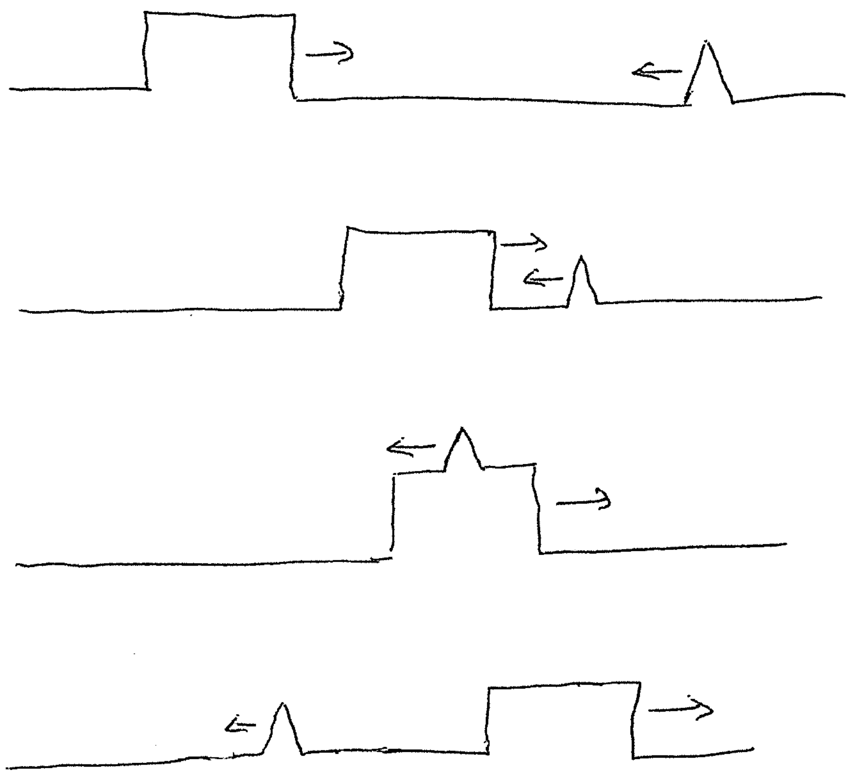
$\Psi_2(z,t)$



Each of these solutions can exist independently on the string. Since the CWE is linear, so can their sum - i.e, a soln. is

$$\Psi_3(z,t) = \Psi_1(z,t) + \Psi_2(z,t).$$

This then implies the following interesting sequence of events can occur:



The shapes "collide", but pass through each other unscathed, each as if the other wasn't there. This is

$$\Psi(z,t) = \Psi_1(z,t) + \Psi_2(z,t) \; ; \qquad \begin{aligned} \Psi_1(z,t) &= \text{moving box} \rightarrow \\ \Psi_2(z,t) &= \text{moving} \leftarrow\Lambda \end{aligned}$$

- a direct consequence of the "<u>Principle of Superposition</u>". <u>It is a direct consequence of having a linear governing differential</u>

equation (e.g., the C.W.E., for media in which it applies).
Thus, the Superposition Principle does not hold for solutions (and, of particular interest to us here, traveling wave solutions) in media in which the governing differential equation(s) is (are) nonlinear ("nonlinear media", we call them).