Physics 251 - 8th Class - Thursday Feb. 1, 2024

A. More on Approximation - Approximate SHM and Molecular Radiation

1. Change of Power Series Expansion "reference point" [K-text, sect 3.6]

Consider, e.g., the function $f(x) = x^2$. As stated, it is a one-term "power series" around $x = 0$. But, suppose that we want to expand that function around, say, $x = 3$. Then, the power series expansion is

(1) $\qquad x^2 = 9 + 6(x-3) + \frac{2}{2!}(x-3)^2$

For that expansion, multiplying out the right-hand side, we arrive at

$$x^2 \overset{?}{=} 9 + 6x - 18 + x^2 - 6x + 9 \overset{\checkmark}{=} x^2$$

Thus, eqn. (1) provides a three term power series around the point $x = 3$ that is actually equal to x^2!

On, consider the function $f(x) = \frac{a}{x^9} - \frac{b}{x}$ where a and b are both positive constants and where we only consider values of $x \geq 0$. As written, $f(x) = \frac{a}{x^9} - \frac{b}{x}$ is a two-term

series in inverse powers of x expanded around $x=0$.

As you can show by setting $\frac{df}{dx}$ equal to zero, the function

$\frac{a}{x^9} - \frac{b}{x}$ has a minimum at $x=X$ where $X^8 = 9\frac{a}{b}$

$\Rightarrow x_{min}$ is at $\sqrt[8]{9\frac{a}{b}}$. As we will soon see, in applications

to molecular vibrations of diatomic molecules, "re-expanding"

$f(x) = \frac{a}{x^9} - \frac{b}{x}$ around the point x_{min} (i.e, "re-referencing

the series to $x=X$) can be very helpful.

2. <u>Molecular Vibrational Oscillation</u> (K-text, sect. 3-7)

Context: Molecules were discovered in interstellar space. <u>How?</u>

Each molecular species has its own emitted radiation spectrum.
The details depend on quantum mechanics, but we can use class-
ical mechanics as a rough guide for what frequency range
the radio telescope should look in.

We'll restrict today's discussion to ionically bonded molecules
and to a case in which one atom is much heavier than the other (HCl).

The classical picture is then that the heavy ion can be considered
essentially stationary and the lighter
atom oscillates against it. As the
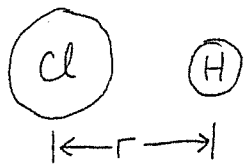lighter ion is charged
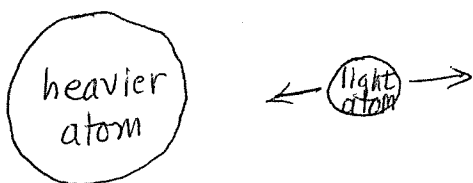
and accelerating, it radiates EM waves at the oscillation frequency.

We wish to estimate the expected frequency. To do this, it is certainly of interest to know if the motion is (at least approximately) simple harmonic. If it is, then you know how to pick off the frequency from the equation of motion.

In order to determine things like this, we need to know something about the force between ions in a molecule. The details of this would come from advanced quantum mechanics, but we can content ourselves here with just asking about some general features.

Actually, it is more convenient for us to discuss the potential energy $U(r)$ [r is the distance between the ion centers].

Cl   H

$|\leftarrow r \rightarrow|$

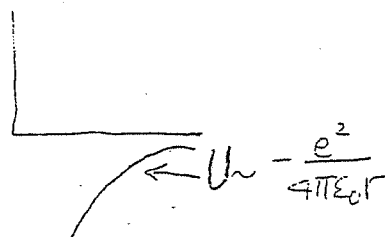Recall that force and potential energy are related by

$$F(r) = -\frac{dU(r)}{dr}$$

What are some general features of $U(r)$?

1. When the ions are far apart, they attract according to Coulomb's law.
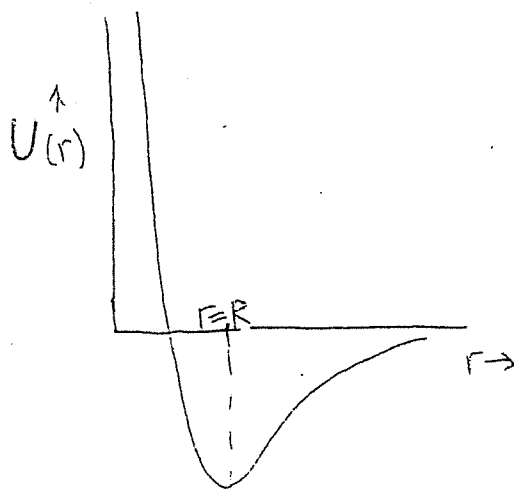
So part of U is

$$U_{far}(r) = \frac{-e^2}{4\pi \varepsilon_0 r}$$

2. When close there is a strong repulsive "force" of quantum mechanical origin. (Otherwise, the molecule would collapse in response to the Coulomb attraction of the oppositely charged ions).

So $U(r)$ must look qualitatively like

To match the two parts of $U(r)$, there must be a minimum. Since the force is zero at the minimum (why?), that value of ionic separation (call it $r = R$) is equilibrium.

Then, expanding $U(R)$ around the minimum at $r = R$, we obtain

$$U(r) = U(R) + \frac{1}{2}\left(\frac{d^2U(r)}{dr^2}\right)_{r=R} \psi^2 + \frac{1}{6}\left(\frac{d^3U(r)}{dr^3}\right)_{r=R} \psi^3 + \ldots$$

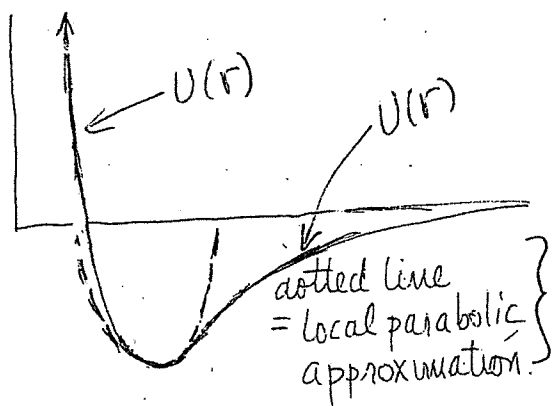where $\psi \equiv r-R$. $\left[\text{Since } \frac{dU(r)}{dr}\Big|_{r=R} = 0\right]$.

Several Lessons from this:

1. If $U(r)$ is a continuous function with a local minimum at $r=R$, if $\frac{d^2U(r)}{dr^2}\Big|_{r=R} > 0$ then $U(r)$ is arbitrarily well represented by a parabolic function $\Big[$ VIZ.

$$U(r) = U(R) + \frac{1}{2}\left(\frac{d^2U(r)}{dr^2}\right)_{r=R}(r-R)^2\Big] \text{ as } r \to R \text{ (as}$$

long as $\frac{d^3U(r)}{dr^3}\Big|_{r=R}$ is finite):

U(r)

U(r)

dotted line
= local parabolic
approximation.

$\to$[Since the curve of $U(r)$ is much shallower to the right of $r=R$ than to the left (Why?) the averaging parabolic approx. must be steeper in slope than $U(r)$ to the right of $r=R$ and shallower to the left.]

2. As you will recall from the Phys. 151 topic "Reading a Potential Energy Plot", if, in the situation under discussion here, if the energy of the oscillation of the lighter ion is somewhat greater than the value of $U(r)$ at the minimum $(r=R)$, then the lighter ion can only oscillate back and forth between the two points ("r_a" and "r_b" in figure below) for which

$$E = U(r) \ [\Rightarrow K.E. = 0 \ \text{at} \ r = r_a, \ r = r_b].$$
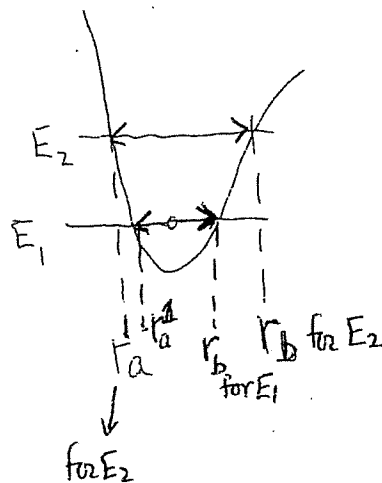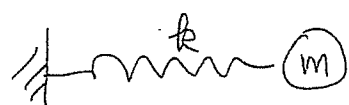
3. If E is only slightly greater than U_{min}, this oscillatory motion is simple harmonic.

This is because simple harmonic motion is motion in a parabolic potential

- recall for, say, a mass on a spring

$$U(x) = \frac{1}{2} k x^2 = \text{parabolic}$$

$$\left[\text{note } U(x) = \frac{1}{2} k x^2 \Rightarrow F(x) = \frac{-dU(x)}{dx} \overset{\checkmark}{=} -kx\right]$$

4. Comparing this to our series expansion (top of last page), we see that for "small oscillations" (E only slightly greater than V_{min}, so r_b not much different than r_a), the "chemical bond" can be conceptually replaced by a simple spring (!) with "effective spring constant"

$$\boxed{k_{eff} = \left(\frac{d^2U(r)}{dr^2}\right)_{r=R}}$$

And the expected angular frequency of radiation will be

$$\omega = \sqrt{\frac{k_{eff}}{\mu}} \qquad \text{denom} = \mu \text{ (reduced mass of the pair)}$$

Now, if we knew U as a function of r, then we could simple calculate $\frac{d^2 U(r)}{dr^2}\Big|_{r=R}$. But, the detailed functional form of $U(r)$ can only come from quantum mechanics. While we aren't in a position to derive $U(r)$ from quantum mechanics in this course, we can at least get some practice with the spirit of how to fit simple approximate forms for $U(r)$ to extract info:
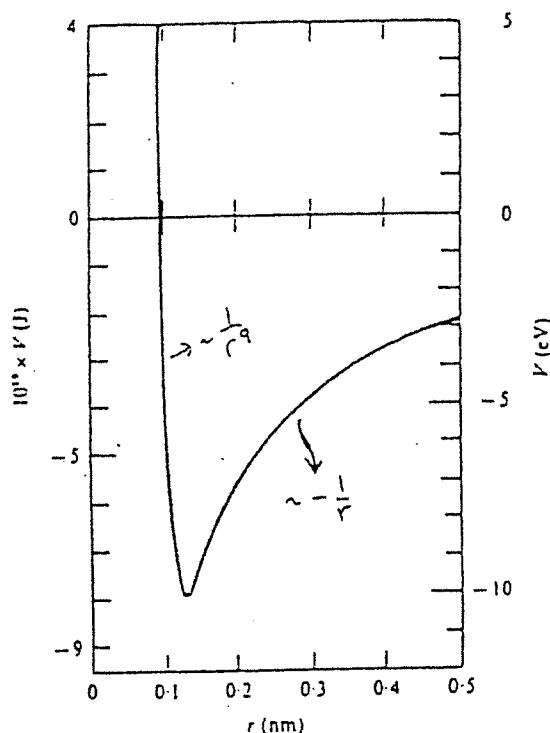
Example: A relatively simple functional form that is sometimes used in this context as a rough approximation is

$$U(r) = \frac{-e^2}{4\pi\varepsilon_0 r} + \frac{B}{r^9}$$

Coulombic attraction

repulsive and very strong at small r.

where B is a constant. This form agrees with our expectations, since it has both a Coulombic attraction (between the ions) piece and also a repulsive piece (+ sign in potential energy of interaction indicates repulsion) that is weak at large r but very strong at very small r, since it is $\propto \frac{1}{r^9}$.

Again, this form is only a simple approximation to the "real $U(r)$", as well, since we wish only an (approximate) frequency prediction, we're really only interested in fitting a parabola for small $|r-R|$.

Let's take this form for $U(r)$ and run with it.

A plot of it (for a particular value of B) is shown below:

$$U(r) = \frac{B}{r^9} - \frac{e^2}{4\pi\varepsilon_0 r}$$

$\left[\begin{array}{l}\text{Recall "B" here} \\ = \text{"C" of Worked Ex!.} \\ \quad \text{text } 3.10 \end{array}\right]$

Fig. 2.6 The potential energy curve for a pair of ions: formula (2.15), with B given by (2.16) and R chosen as 0.13 nm (the measured value for HCl). The zero of potential energy is the value when the ions are infinitely far apart. (It is customary to shift the curve upwards by an amount which allows for the energy required to form the ions from neutral atoms, but this makes no difference to the vibrational behaviour discussed in this book.)

Figure from Vibrations and Waves in Physics, 2nd Ed. by Iain Main (Cambridge).

1. Let's see if we can figure out what B should be.

Letting $r = R$ be the equilibrium separation of the ions, we know that

$$\left.\frac{dU(r)}{dr}\right)_{r=R} = 0 , \quad \text{which is} \quad \frac{e^2}{4\pi\varepsilon_0 R^2} - \frac{9B}{R^{10}} = 0$$

$$\Rightarrow (2) \quad B = \frac{e^2 R^8}{36\pi\varepsilon_0} \quad \left[\begin{array}{l}\text{same as text eqn. (3.27) on page 3-53} \\ \quad \text{in Worked Ex!. 3.10} \\ \quad \text{text}\end{array}\right]$$

2. Now, it would appear that all we've done is trade off one unknown ("B") for another ("R", the equilibrium separation of the ions in the molecule). Let's continue anyway:

To obtain k_{eff} for this molecule, we use

(3) $k_{eff} = \dfrac{d^2U(r)}{dr^2}\Big|_{r=R} = \dfrac{-2e^2}{4\pi\varepsilon_0 R^3} - \dfrac{9(10)B}{R^{11}}$ ($"B"$ here $\equiv "C"$ in text exl.)

(same as Worked Exl 3.10, eqn. (3.28), page 3-53).

Using eqn. (2) above for B in eqn. (3), we find

(4) $k_{eff} = \dfrac{8e^2}{4\pi\varepsilon_0 R^3} \approx \dfrac{1.8\times10^{-27}\, N\cdot m^2}{R^3}$

3. Now we need to input [into (4)] a value for R. Rather than input "the accepted value for R" [as in Worked exl. 3.10], here we use "the method of guessing":

Radius of an atom or ion $\sim 0.5\,\overset{\circ}{A} = 0.5\times10^{-10}$

guess center-to center separation $\sim 3\times$ radius of ion

$\Rightarrow$ guess (rough) $R\sim 1.5\,\overset{\circ}{A}$

Putting that into (4) above $\Rightarrow k_{eff} \approx \dfrac{1.8\times10^{-27}}{3.4\times10^{-30}}\,\dfrac{N}{m}$,

$$k_{eff} \sim 530 \, \frac{N}{m}$$
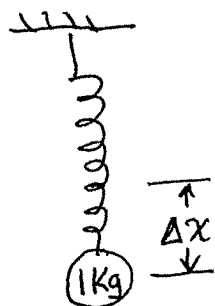
Pause to think:

$k_{eff} \sim 530 \, \frac{N}{m}$: How much is that? Is that "alot"?

To get an idea, consider a spring with spring constant $530 \, \frac{N}{m}$ on which we hang an "everyday" amound of mass — say, 1 Kg. The stretch caused by this is given from

$$k \Delta x = mg$$

$$\Rightarrow \quad \Delta x = \frac{mg}{k} \approx \frac{1(10)}{510} \, \text{meters}$$

$$\approx \frac{1}{50} \, \text{meters} \approx 2 \, \text{cm} !$$

$\Rightarrow$ The "effective spring constant" that models the quantum mechanical chemical bond in HCl in the small-oscillations regime is comparable to that of a spring used in everyday application !!

Anyway — back to our estimate of expected radiation frequency for HCl:

$$\omega = \sqrt{\frac{k_{eff}}{\mu}} \; ;$$

in HCl, Cl^- ion is about 35 times as heavy as H^+ ion

$\Rightarrow$ Cl^- is essentially stationary, $\mu \approx m_{H^+} = $ mass of proton (m_p).

$$m_p \approx 1.67 \times 10^{-27} kg$$

$\Rightarrow$

$$\omega^{pred.} \approx \sqrt{\frac{530 \; N/m}{1.67 \times 10^{-27} \, kg}} \sim 5.6 \times 10^{14} \; rad/sec$$

(for our "rough guess")

$\Rightarrow$

$$f^{\text{"predicted"}} = \frac{1}{2\pi} \omega^{pred} \sim \underline{9.0 \times 10^{13} \; Hz} \quad \leftarrow \text{our rough guess}$$

whereas

$$f^{experimental} \sim \underline{8.9 \times 10^{13} \; Hz}$$

The closeness of our guess to the actual experimental value is mostly fortuitous — we are lucky to have even gotten a result within a factor of 5 of the exptl. value, but the method (an order of magnitude estimate method) has pedagogical value.

B. Superposition Theorem; General Solutions

1. Linear Differential Equations — Homogeneous Case — Superposition

Clearly, differential equations have been important in our work. Now we need to know more: A differential equation is "linear" if it is linear in the _dependent_ variable. We consider linear d.e.'s involving functions $\psi(t)$.

For simplicity we consider second order equations, but our result applies to equations of any order. [Why are we so interested in second order equations?]. The most general 2nd order linear differential equation is

$$a(t)\, \ddot{\psi}(t) + b(t)\, \dot{\psi}(t) + c(t)\, \psi(t) = d(t)$$

Here $a(t)$, $b(t)$, $c(t)$, $d(t)$ are any functions of t (e.g., $6t^2$).

Thus, by "linear" is meant linear in the dependent variable (ψ).

If the right side is zero (i.e, $d(t) = 0$), the equation is said to be "homogeneous." Our first theorem here is:

a. Superposition Principle: Consider the general second order linear, homogeneous d.e.

$$(1) \quad a(t)\,\ddot{\psi}(t) + b(t)\,\dot{\psi}(t) + c(t)\,\psi(t) = 0$$

If $\psi_1(t)$ and $\psi_2(t)$ are any two solutions of (1), then, for any constants c_1 and c_2,

$$(2) \quad \psi(t) = c_1 \psi_1(t) + c_2 \psi_2(t)$$

is also a solution of (1).

Proof: Let $\psi(t) = c_1 \psi_1(t) + c_2 \psi_2(t)$. Then

$$a(t) \, \ddot{\psi}(t) = a(t) \left[c_1 \psi_1(t) + c_2 \psi_2 \right]^{\cdot\cdot} = c_1 \, a(t) \, \ddot{\psi}_1 + c_2 \, a(t) \, \ddot{\psi}_2$$

$$b(t) \, \dot{\psi}(t) = b(t) \left[c_1 \psi_1 + c_2 \psi_2 \right]^{\cdot} = c_1 \, b(t) \, \dot{\psi}_1 + c_2 \, b(t) \, \dot{\psi}_2$$

Also, $c(t)\, \psi = c(t) \left[c_1 \psi_1 + c_2 \psi_2 \right] = c_1 \, c(t) \psi_1 + c_2 \, c(t) \psi_2$

Thus

$$a(t) \, \ddot{\psi} + b(t) \, \dot{\psi}(t) + c(t) \, \psi$$

$$= c_1 \, a(t) \, \ddot{\psi}_1 + c_1 \, b(t) \, \dot{\psi}_1 + c_1 \, \psi_1$$

$$+ c_2 \, a(t) \, \ddot{\psi}_2 + c_2 \, b(t) \, \dot{\psi}_2 + c_2 \, \psi_2$$

$$= c_1 \, \underbrace{\left[a(t) \, \ddot{\psi}_1 + b(t) \, \dot{\psi}_1 + c(t) \, \psi_1 \right]}_{\longrightarrow 0}$$

$$+ c_2 \, \underbrace{\left[a(t) \, \ddot{\psi}_2 + b(t) \, \dot{\psi}_2 + c(t) \, \psi_2 \right]}_{\longrightarrow 0}$$

$$= 0.$$

q.e.d.

Comment: Note that the simple harmonic oscillator equation is

of this form.

Question: What are $a(t)$, $b(t)$, $c(t)$ for the "paradigm" simple

harmonic oscillator?

Here's another very important theorem concerning l.h.d.e.'s:

b. Theorem: Suppose that $\psi_1(t)$ and $\psi_2(t)$ are two linearly independent solutions of (1) on an interval $[a,b]$. Then, on $[a,b]$, the general soln. to the d.e. (1) can be expressed in the form

$$(2) \quad \psi(t) = C_1 \psi_1(t) + C_2 \psi_2(t)$$

for arbitrary constants C_1 and C_2 – i.e, on $[a,b]$ every soln. of eqn. (1) can be expressed in the form (2) for some choice of (C_1, C_2).

We leave the proof of this theorem to your d.e. course; here we only note that it is plausible (as we already discussed earlier)

1. Second order eqn $\Rightarrow$ loosely must integrate given $\ddot{\psi}(t)$ twice to obtain $\psi(t)$ $\Rightarrow$ two constants of integration set by initial conditions; (2) has two different constants multiplying two linearly independent functions.

2. Exl: s.h.o., $\psi_1(t) = \sin \omega t$, $\psi_2(t) = \cos \omega t$,

general soln: $\psi(t) = C \sin \omega t + D \cos \omega t$

2. Now consider a nonlinear homogeneous differential equn:

$$\ddot{\psi}(t) = c\psi^2$$

or $\quad \ddot{\psi}(t) = -\frac{g}{\ell}\sin\psi$

or $\quad \ddot{\psi}(t) = -c\psi + \alpha\psi^2 + b\psi^3$

For any of these (indeed, for any nonlinear differential equation), the superposition theorem is <u>not</u> obeyed — the sum of two solutions is <u>not</u> a solution.

We will have you show that for homework, however, here we offer a plausibility argument:

Linear, homogeneous eqn is expressible as $\mathcal{L}[\psi] = 0$

where $\mathcal{L}$ is a linear operator $\left[e.g., \mathcal{L} = 3t^3\frac{d^2}{dt^2} + 4t\frac{d}{dt} \right]$

As a summary of our proof of the Superposition theorem:
Since $\mathcal{L}$ is linear, $\mathcal{L}(\psi_1 + \psi_2) = \mathcal{L}(\psi_1) + \mathcal{L}(\psi_2) = 0 + 0 \overset{\checkmark}{=} 0$

Now consider nonlinear homogeneous eqn:

$$\eta(\psi) = 0 \quad \text{where } \eta \text{ is a nonlinear operator } [e.g., \eta(\psi) = e^{\psi}]$$

But $\quad \eta(\psi_1 + \psi_2) \neq \eta(\psi_1) + \eta(\psi_2) = 0$

eg:
$$\eta(\psi) = \dot{\psi}^2; \quad (\dot{\psi}_1 + \dot{\psi}_2)^2 = \dot{\psi}_1^2 + \dot{\psi}_2^2 + 2\dot{\psi}_1\dot{\psi}_2 \neq \dot{\psi}_1^2 + \dot{\psi}_2^2 \quad (\eta(\psi) = \dot{\psi}^2)$$

C. <u>Superposition of Solutions for Inhomogeous l.d.e's?</u>

Consider the second order linear inhomogeneous eqn

(4) $a(t)\,\ddot{\psi}(t) + b(t)\,\dot{\psi}(t) + c(t)\,\psi(t) = d(t)$

Then, if $\psi_1(t)$ and $\psi_2(t)$ are linearly independent solns,

then $\psi(t) = \psi_1(t) + \psi_2(t)$ is <u>not</u> a soln of (4).

We leave the proof of that to you.

D. <u>Inhomogeneous Linear Differential Equations</u>, "<u>Superposition of Driving Forces</u>"

Consider the <u>pair</u> of inhomogeneous linear d.e.'s

(5) $a(t)\,\ddot{\psi}(t) + b(t)\,\dot{\psi}(t) + c(t)\,\psi(t) = r(t)$

(6) $a(t)\,\ddot{\psi}(t) + b(t)\,\dot{\psi}(t) + c(t)\,\psi(t) = s(t)$,

For the <u>same functions $a(t)$, $b(t)$, $c(t)$, but different $r(t)$, $s(t)$,</u>

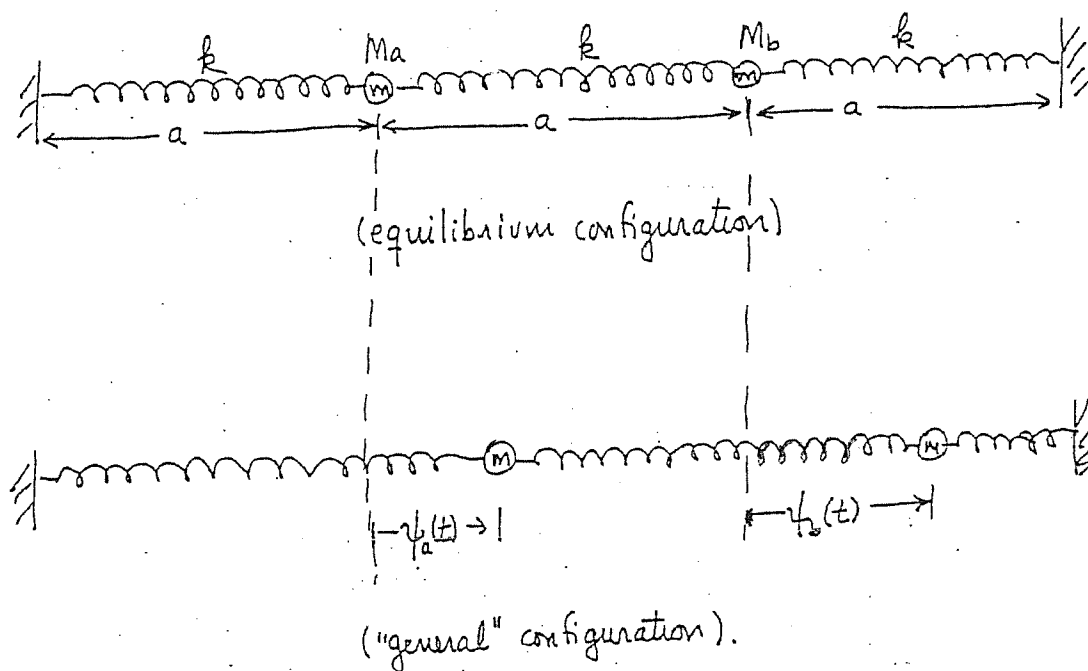Then, if $\psi_1(t)$ and $\psi_2(t)$ are two indep. solns of (5) and (6),

respectively, then $\psi(t) = \psi_1(t) + \psi_2(t)$ is a soln. of

(7) $a(t)\,\ddot{\psi}(t) + b(t)\,\dot{\psi}(t) + c(t)\,\psi(t) = r(t) + s(t)$;
we leave the proof to you.

E.   Introduction to Normal Modes

We come now to something very generally important

in physics - the concept of a "normal mode". To begin looking

at this, we ask you to consider the following system:

(equilibrium configuration)

(general" configuration).

We assume the two masses are equal - $M_a = M_b \equiv M$.

Exercize: Show that the differential equations for this system are

$$m \frac{d^2 \psi_a(t)}{dt^2} = -k \psi_a(t) + k[\psi_b(t) - \psi_a(t)]$$ governing d.e.'s

$$m \frac{d^2 \psi_b(t)}{dt^2} = -k[\psi_b(t) - \psi_a(t)] - k \psi_b(t)$$

These equations are somewhat complicated because the "ψ_a equation"
contains ψ_b and the "ψ_b equation" contains ψ_a - we say

that the equations are <u>coupled</u>. This makes sense because the motion of ψ_a affects that of b, and vice versa, due to the coupling spring.

[This is actually not the first time we've seen coupling between masses — in the "diatomic molecule" example, it is certainly true that the motion of one of the masses affects (via the coupling spring) the motion of the other!]

Also, the general motion of the system (the two masses) is rather complicated* — and, in fact, many different motions are possible. Each "motion of the system" is characterized by a particular pair of functions $\{\psi_a(t), \psi_b(t)\}$. As you might imagine, the particular functions $\psi_a(t)$ and $\psi_b(t)$ are determined by the <u>initial conditions</u> — $\psi_a(0)$, $\psi_b(0)$, $\dot{\psi}_a(0)$, $\dot{\psi}_b(0)$ — i.e., the two initial positions and the two initial velocities.

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* At least from our present naive point of view!

Now, we could try to attack the coupled equations by a purely mathematical method — a method that might proceed by trying to find appropriate linear combinations of ψ_a and ψ_b to use as new coordinates obeying uncoupled equations ("smart coordinates")

We will try that method later. First, however, we will try thinking physically about the system.

We begin this physical approach by asking—"of the many possible pairs of functions describing motions of the system — are there any that are 'simple' (*)?" We ask, in particular, about simple harmonic motion — is there a possible motion in which both masses undergo simple harmonic motion simultaneously? For such a motion of the masses we might have, say,

$$\psi_a(t) = A \cos \omega t$$

$$\psi_b(t) = B \cos \omega t$$

A motion like this, in which both masses undergo simultaneous simple harmonic motion at the same frequency and with fixed relative phase, is called a __Normal Mode__.

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* The word "simple" is a technical term in physics— it means "of that which we already know."

Let us see if a Normal Mode is possible for this system. Suppose we begin by displacing both masses an equal amount to the right from their equilibrium positions.
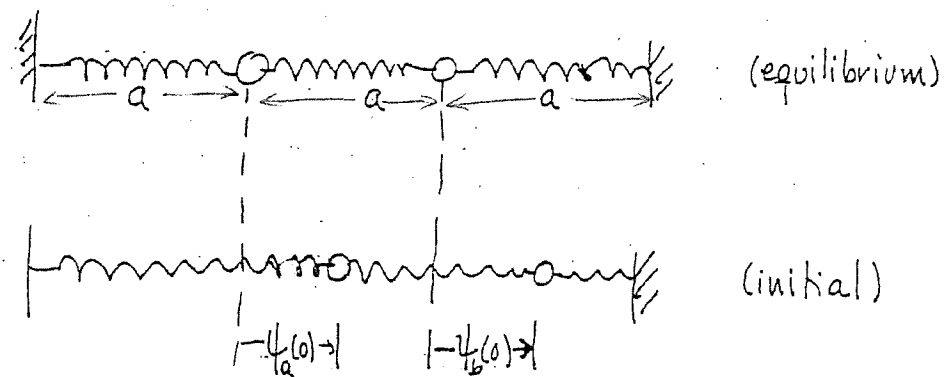
So $\psi_a(t=0) = \psi_b(t=0)$.

Now let us ask about the forces on the masses the instant we let go. We have already learned

that only forces in excess of those used to maintain equilibrium contribute to motion - the forces on each mass used to maintain equilibrium cancel out (that's why there's no motion in equilibrium).

Notice that the extension of the central spring has remained constant at the equilibrium value a. Therefore, at this instant, it contributes no _excess_ force on either mass. Thus, at this instant the only excess force on the left hand mass is that due to the left most spring —

excess force on M_A at $t=0$ = $-k\,\psi_a(t=0)$. Likewise,

excess force on M_B at $t=0$ = $-k\,\psi_b(t=0)$ from right most spring. $\qquad$ (*)

Since $\psi_a(t=0) = \psi_b(t=0)$, these are equal. Thus the masses will start to move with identical accelerations to the left. After a short time Δt, both masses move an equal distance and the center spring remains at extension a. Therefore the masses still have identical forces acting on them, and will move further identically. By continuing this kind of reasoning it becomes clear that the motions of the two masses are identical for all $t>0$ and that (ignoring friction) the motion is stable.

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* As previously, these equations are true regardless of the signs of ψ_a and ψ_b — take a moment and convince yourself of that.

Since the middle spring always remains at constant length, it is easy to see that at any $t > 0$, <u>for motion in this mode</u>

$$m\frac{d^2 \psi_a(t)}{dt^2} = -k\,\psi_a(t)$$

$$m\frac{d^2 \psi_b(t)}{dt^2} = -k\,\psi_b(t)$$

{ not the governing differential eqns. since only valid for present set of physical initial conditions

Thus <u>in this mode</u>, the differential equations have become much simpler - they are uncoupled. (This makes sense, since the coupling device, the middle spring, is not contributing to the motion). The solution to the pair of equations that obeys the prescribed initial conditions is
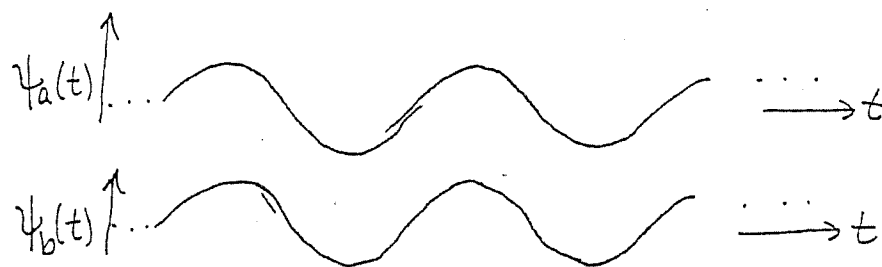
$$\psi_a(t) = C \cos \omega t \qquad \qquad \omega = \sqrt{k/m}$$

$$\psi_b(t) = C \cos \omega t = \psi_a(t) \qquad \omega = \sqrt{k/m}$$

which verifies that this motion is a possible, stable Normal Mode.

In this mode the two masses oscillate in lock-step at the common frequency $\omega = \sqrt{k/m}$.

$\psi_a(t)$ $\longrightarrow t$

$\psi_b(t)$ $\longrightarrow t$

Now let us ask if there are any other normal modes for this system. It turns out that this system does have one other normal mode. Let us try to guess this second mode.

equilibrium

initial condition for mode 2

$\longrightarrow$ $|\psi_a(t)|$ $|\psi_b(t)| = -\psi_a(t)|$
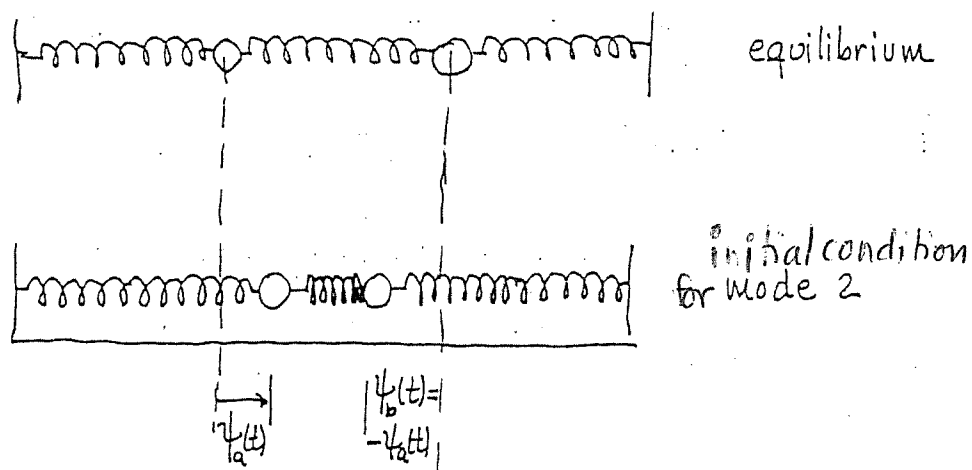
From the symmetry, we guess that if a and b move equally and oppositely we might have a mode. In particular, if at any instant $\psi_b = -\psi_a$, the masses are subjected to equal magnitude (oppositely directed) excess forces.

Consider the left-hand mass a. It is pulled to
the left by the left hand spring with an excess force
$F = -k\psi_a$. But it is also pushed to the left by
the middle spring with an excess force $F = -2k\psi_a$.
(The factor of two comes in because the central spring
is compressed by a total amount $= 2\psi_a$). Thus the
net force on m_a is $-3k\psi_a$. A similar analysis (think
it through) shows that the net force on b is $-3k\psi_b$.
But since, by hypothesis $\psi_b = -\psi_a$, at any instant the
forces on the two masses are the same in magnitude but
oppositely directed. We have, at any $t > 0$, for this motion

$$m\frac{d^2\psi_a}{dt^2} = -3k\psi_a$$

$$m\frac{d^2\psi_b}{dt^2} = -3k\psi_b$$

point: Still true even
if signs of both ψ_a
& ψ_b are simultaneously
reversed (different
IC's)

$$\Rightarrow \quad \psi_a(t) = A\cos\omega t \qquad \omega = \sqrt{\frac{3k}{m}}$$

$$\psi_b(t) = -A\cos\omega t$$

(The relative minus sign comes from the initial conditions).

Thus we have verified that this system has a second

normal mode. In this higher frequency normal mode

the two masses oscillate 180° out of phase with each

other ("against each other").

(equil)

} 2 instants
- second
  normal
  mode.