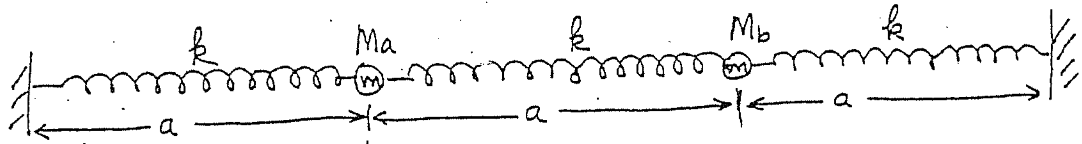
Physics 251 9th Class - Tuesday Feb 6, 2024

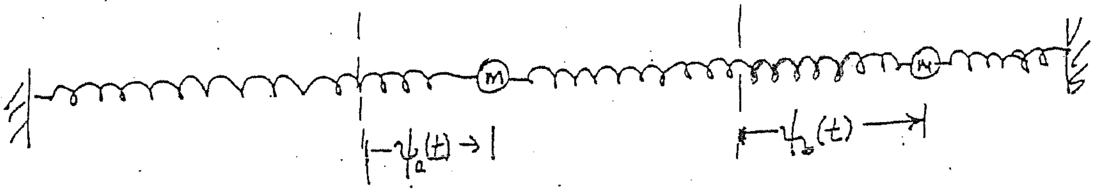
Reading for This: Chap. 6 K-Text, sects. 6.1, 6.2, 6.3.

A. Introduction to Normal Modes

1. We come now to something very generally important in physics - the concept of a "normal mode". To begin looking at this, we ask you to consider the following system:



(equilibrium configuration)



("general" configuration).

We assume the two masses are equal - $M_a = M_b \equiv m$.

Exercize: Show that the gov. differential equations for this system are

(1a) $$m\frac{d^2\psi_a(t)}{dt^2} = -k\psi_a(t) + k[\psi_b(t) - \psi_a(t)]$$ governing d.e.'s.

(1b) $$m\frac{d^2\psi_b(t)}{dt^2} = -k[\psi_b(t) - \psi_a(t)] - k\psi_b(t)$$

These equations are somewhat complicated because the "ψ_a equation" contains ψ_b and the "ψ_b equation" contains ψ_a - we say that the equations are <u>coupled</u>. This makes sense because the motion of ψ_a affects that of b, and vice versa, due to the coupling spring.

[This is actually not the first time we've seen coupling between masses - in the "diatomic molecule" example, it is certainly true that the motion of one of the masses affects (via the "coupling spring") the motion of the other!]

Also, the general motion of the system (the two masses) is rather complicated* - and, in fact, many different motions are possible. Each "motion of the system" is characterized by a particular pair of functions $\{\psi_a(t), \psi_b(t)\}$. As you might imagine, the particular functions $\psi_a(t)$ and $\psi_b(t)$ are determined by the <u>initial</u> <u>conditions</u> - $\psi_a(0)$, $\psi_b(0)$, $\dot{\psi}_a(0)$, $\dot{\psi}_b(0)$ - i.e., the two initial positions and the two initial velocities.

---

* At least from our present naive point of view!

Now, we could try to attack the coupled equations by a purely mathematical method — a method that might proceed by trying to find appropriate linear combinations of ψ_a and ψ_b to use as new coordinates obeying uncoupled equations ("**smart coordinates**")

We will try that method later. First, however, we will try thinking <u>physically</u> about the system.

We begin this physical approach by asking — "of the many possible pairs of functions describing motions of the system — are there any that are 'simple' (*)?" We ask, in particular, about simple harmonic motion — is <u>there a possible motion in which both masses undergo simple harmonic motion simultaneously</u>? For such a motion of the masses we might have, say,

$$\psi_a(t) = A \cos \omega t$$

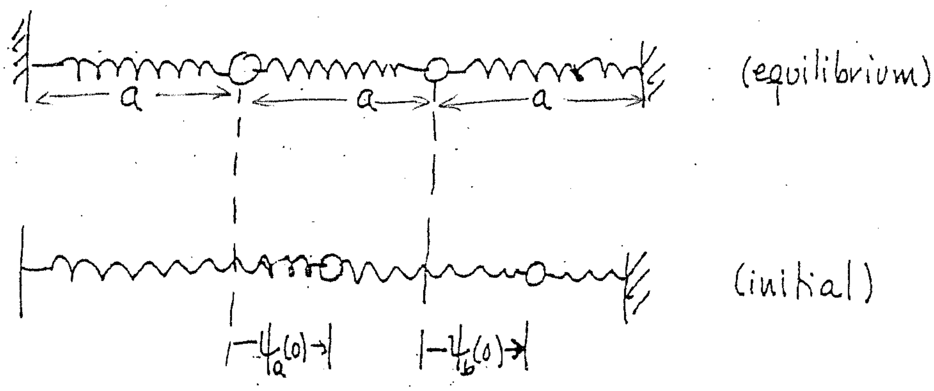
$$\psi_b(t) = B \cos \omega t$$

A <u>motion like this, in which both masses undergo simultaneous simple harmonic motion at the same frequency and with fixed relative phase</u>, is called a <u>Normal</u> <u>Mode</u>.

---

* The word "simple" is a technical term in physics — it means "of that which we already know"

Let us see if a Normal Mode is possible for this system. Suppose we begin by displacing both masses an equal amount to the right from their equilibrium positions.



So $\psi_a(t=0) = \psi_b(t=0)$.

Now let us ask about the forces on the masses the instant we let go. We have already learned that only forces in excess of those used to maintain equilibrium contribute to motion - the forces on each mass used to maintain equilibrium cancel out (that's why there's no motion in equilibrium).

Notice that the extension of the central spring has remained constant at the equilibrium value a. Therefore, at this instant, it contributes no <u>excess</u> force on either mass. Thus, at this instant the only excess force on the left hand mass is that due to the left most spring -

excess force on M_A at $t=0 = -k\psi_a(t=0)$. Likewise,

excess force on M_B at $t=0 = -k\psi_b(t=0)$ from right most spring. (*)

Since $\psi_a(t=0) = \psi_b(t=0)$, these are equal. Thus the masses will start to move with identical accelerations to the left. After a short time Δt, both masses move an equal distance and the center spring remains at extension a. Therefore the masses still have identical forces acting on them, and will move further identically. By continuing this kind of reasoning it becomes clear that the motions of the two masses are identical for all $t>0$ and that (ignoring friction) the motion is stable.

---

* As previously, these equations are true regardless of the signs of ψ_a and ψ_b - take a moment and convince yourself of that.

Since the middle spring always remains at constant length, it is easy to see that at any $t>0$, for motion in this mode

$$m\frac{d^2\psi_a(t)}{dt^2} = -k\,\psi_a(t)$$

$$m\frac{d^2\psi_b(t)}{dt^2} = -k\,\psi_b(t)$$

not the governing differential eqns. since only valid for present set of physical initial conditions

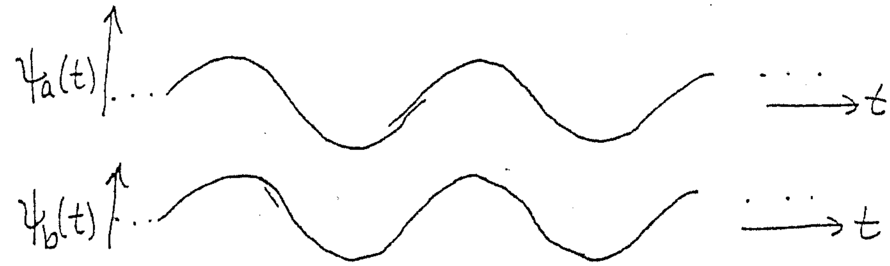
Thus in this mode, the differential equations have become much simpler - they are uncoupled. (This makes sense, since the coupling device, the middle spring, is not contributing to the motion). The solution to the pair of equations that obeys the prescribed initial conditions is

$$\psi_a(t) = C\cos\omega t \qquad \omega = \sqrt{k/m}$$

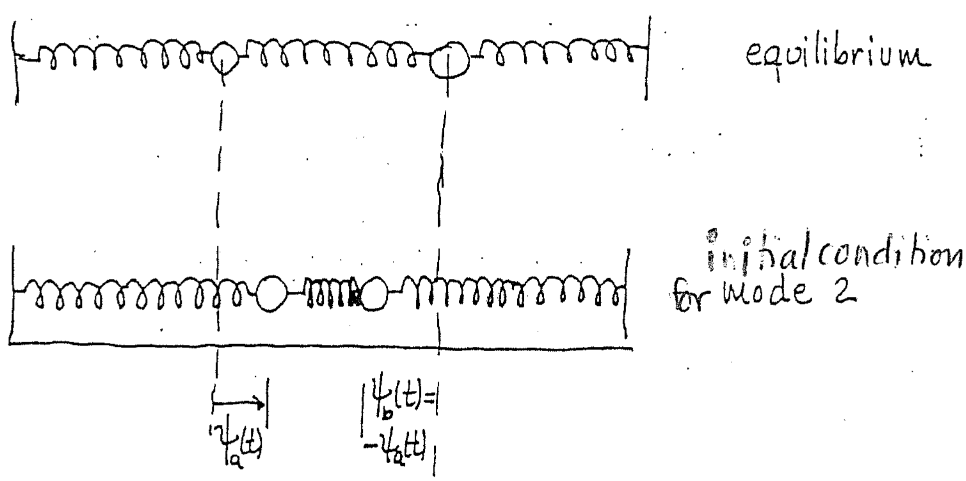
$$\psi_b(t) = C\cos\omega t = \psi_a(t) \qquad \omega = \sqrt{k/m}$$

which verifies that this motion is a possible, stable Normal Mode.

In this mode the two masses oscillate in lock-step at the common frequency $\omega = \sqrt{k/m}$.



Now let us ask if there are any other normal modes for this system. It turns out that this system does have one other longitudinal normal mode. Let us try to guess this second mode.



From the symmetry, we guess that if a and b move equally and oppositely we might have a mode. In particular, if at any instant $\psi_b = -\psi_a$, the masses are subjected to equal magnitude (oppositely directed) <u>excess</u> forces.

Consider the left-hand mass a. It is pulled to the left by the left hand spring with an excess force $F = -k\psi_a$. But it is also pushed to the left by the middle spring with an excess force $F = -2k\psi_a$. (The factor of two comes in because the central spring is compressed by a total amount $= 2\psi_a$). Thus the net force on m_a is $-3k\psi_a$. A similar analysis (think it through) shows that the net force on b is $-3k\psi_b$. But since, by hypothesis $\psi_b = -\psi_a$, at any instant the forces on the two masses are the same in magnitude but oppositely directed. We have, at any $t > 0$, for this motion

$$m\frac{d^2\psi_a}{dt^2} = -3k\psi_a$$

$$m\frac{d^2\psi_b}{dt^2} = -3k\psi_b$$

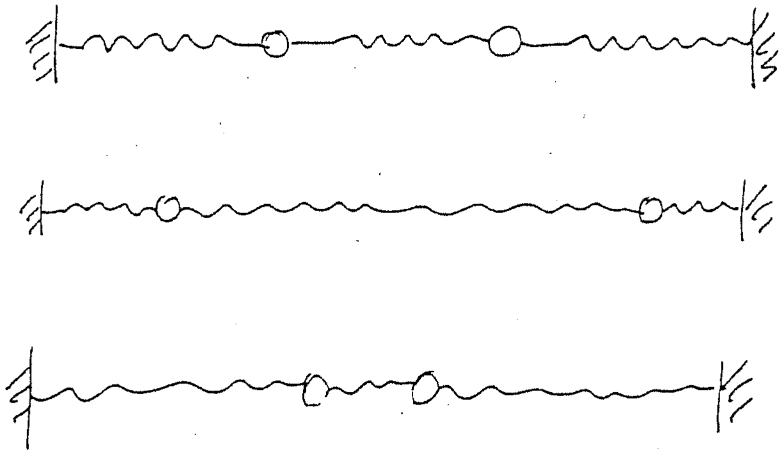
point: Still true even if signs of both ψ_a & ψ_b are simultaneously reversed (different IC's)

$$\Rightarrow \quad \psi_a(t) = A\cos\omega t \qquad \omega = \sqrt{\frac{3k}{m}}$$

$$\psi_b(t) = -A\cos\omega t$$

(The relative minus sign comes from the initial conditions).

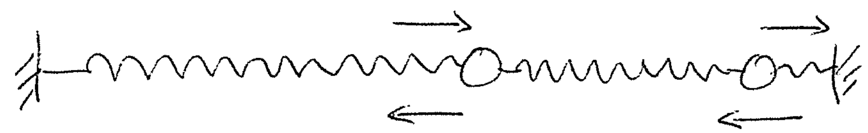
Thus we have verified that this system has a second normal mode. In this higher frequency normal mode the two masses oscillate 180° out of phase with each other ("against each other").



(equil)

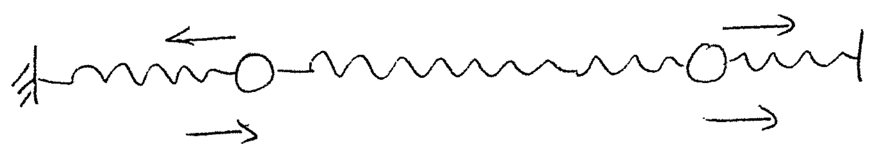
2 instants – second normal mode.

2. We saw, on intuitive grounds, that there are two special "simple" motions of the system - motions in which the two masses undergo simultaneous simple harmonic motion at the same frequency and with a fixed relative phase - the <u>normal modes</u>.



<u>mode 1</u>

$\omega_1^2 = k/m$



mode 2

$\omega_2^2 = 3k/m$

In <u>mode 1</u>,

$$\psi_a(t) = A_1 \cos(\omega_1 t + \phi_1)$$

$$\psi_b(t) = A_1 \cos(\omega_1 t + \phi_1)$$

In <u>mode 2</u>,

$$\psi_a(t) = A_2 \cos(\omega_2 t + \phi_2)$$

$$\psi_b(t) = -A_2 \cos(\omega_2 t + \phi_2)$$

Normal Mode Superpositions [K-Text, sects. 6.4 and 6.5]

Now let us put together what we have.

Since both of the equations of motion are, although coupled with each other, linear and homogeneous, the principle of superposition should apply. Therefore, we can have motions with both modes going at once. That is, it should be possible to have motions corresponding to a simultaneous, arbitrary linear combination of the two normal modes. Writing the pair $\psi_a(t), \psi_b(t)$ as a stacked column, this would be

$$\begin{bmatrix} \psi_a(t) \\ \psi_b(t) \end{bmatrix} = A_1 \times \left(\begin{matrix}\text{unit amplitude of}\\ \text{normal mode 1}\end{matrix}\right) + A_2 \times \left(\begin{matrix}\text{unit amplitude of}\\ \text{normal mode 2}\end{matrix}\right)$$

Putting in what we found for the individual normal modes for this system, this is

$$(2)\quad \begin{matrix} \psi_a(t) = A_1 \\ \psi_b(t) = A_1 \end{matrix} \underbrace{\boxed{\begin{matrix} \cos(\omega_1 t + \phi_1) \\ \cos(\omega_1 t + \phi_1) \end{matrix}}}_{\substack{\text{unit amplitude of}\\ \text{normal mode \#1}}} + \begin{matrix} A_2 \\ A_2 \end{matrix} \underbrace{\boxed{\begin{matrix} \cos(\omega_2 t + \phi_2) \\ (-1)\cos(\omega_2 t + \phi_2) \end{matrix}}}_{\substack{\text{unit amplitude of}\\ \text{normal mode \#2}}}$$

Let us be clear on what eqns. (2) say from the point of view of the motions of the individual masses m_a and m_b. They say that, in general,

$$y_a(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$

(2') $$y_b(t) = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2).$$

That is, in general, mass a moves in a simultaneous superposition of two (different frequency) s.h.m's going on at once. So does m_b, although the sign in the superposition is different for it. [Both normal modes activated].

Now look again at eqns. (2) and (2'). Since there are four arbitrary constants $(A_1, A_2, \phi_1, \phi_2)$, it appears that either of these represents the general solution to the governing d.e.'s eqns. (1). This result is also required by our recent theorem about the general solution of a 2nd order linear homogeneous d.e., since the pair of eqns. (1) represent a linear, homogeneous 2^{nd} order (coupled) system of d.e.'s.

We will explicitly check that eqns. (2) represent the general solution of eqn. (1) for ourselves shortly.

First, however, we note an important consequence of this for our coupled mass-spring system:

There can be no third independent normal mode for that system. (If there were a 3^{rd} mode, it would, having a different frequency "ω_3", be an independent soln. of the d.e.'s, which would violate the superposition theorem.)

Thus, we say that the two normal modes represent a complete set of solutions of the d.e.'s (1).

From K-text, sect. 6.5:

Within the bounds of linearity of the springs, **any internal longitudinal motion of the system, regardless of the specific initial conditions that cause it, must be expressible in the form (2′) for some value of each of the constants A_1, A_2, ϕ_1 and ϕ_2 – *i.e.*, any of these motions is some superposition of the two normal modes.**

Of course, for these assertions to be really believable, we still have to prove those we made about the one-to-one unique correspondences of the sets $\{A_1, A_2, \phi_1, \phi_2\}$ and $\{\psi_a(0), \psi_b(0), \dot{\psi}_a(0), \dot{\psi}_b(0)\}$ to each other. We will do that soon.

So, for the system shown in Fig. 6.1, if eqn. (2′) represents the general solution to the governing differential equations (eqns. 6.1), what would the motion of the masses look like for a given choice of initial conditions? In general, a superposition of two harmonic motions of different frequencies is hard to mentally picture. In Fig. 6.5 I plotted an example for the case $A_1 = A_2 = 1, \phi_1 = 0, \phi_2 = \pi/2$.

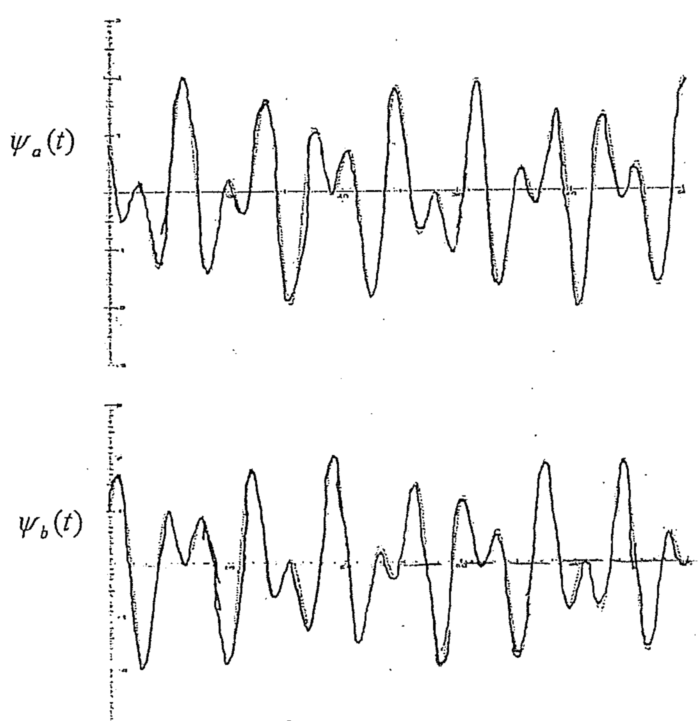


Fig. 6.5: Plots of $\psi_a(t)$ and $\psi_b(t)$ vs. t for the two-mass-three-spring system of Fig. 6.1 for the case $k/m = 1$, $A_1 = A_2 = 1$, $\phi_1 = 0$, $\phi_2 = \pi/2$. Plotted using online freeware *Relplot: A General Equation Plotter* developed by Andrew Myers, Cornell University.

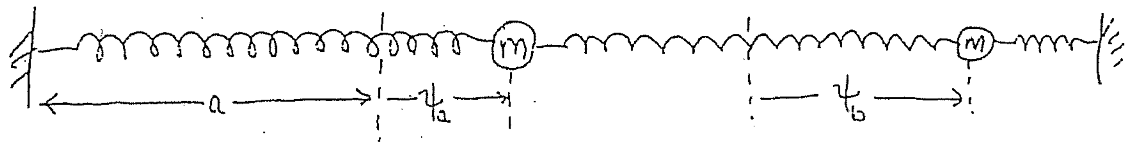
From the figure, $\psi_a(t)$ and $\psi_b(t)$ both look rather complicated. What do you think - will either ever repeat? Why or why not?

Notice an important aspect of what is going on here. Fig. 6.5 illustrates that, with both normal modes excited, the amplitudes of oscillation of the individual masses m_a and m_b are changing in time. Since energies are proportional to squares of amplitudes, this indicates that energy is being exchanged between the masses (and between them and the springs). However, while all that is going on, invisibly, (as long as we can ignore damping) **each normal mode oscillation is continuing on with steady amplitude (determined by the physical initial conditions), independently of the other normal mode**. (That is what eqns. (2') are telling us). As a consequence, the **normal modes do not (again, as long as we can ignore damping) exchange energy.** We will return to further explore this important feature of coupled oscillations later in this chapter.

"Independence of the normal modes"

B. The "Method of Searching For Normal Coordinates"

1. To enhance our insight, we now corroborate these results by another method:



$$(1) \qquad m\frac{d^2\psi_a(t)}{dt^2} = -k\psi_a(t) + k(\psi_b - \psi_a)$$

$$(2) \qquad m\frac{d^2\psi_b(t)}{dt^2} = -k(\psi_b - \psi_a) - k\psi_b$$

Suppose we add these equations:

$$m\frac{d^2[\psi_a(t) + \psi_b(t)]}{dt^2} = -k[\psi_a(t) + \psi_b(t)]$$

and subtract them:

$$m\frac{d^2[\psi_a(t) - \psi_b(t)]}{dt^2} = -3k[\psi_a(t) - \psi_b(t)]$$

Notice now that an almost magical thing has happened - the equations are uncoupled when expressed in terms of the "new coordinates" $\psi_a(t) + \psi_b(t)$ and $\psi_a(t) - \psi_b(t)$!

Let us then define linear combinations (or, "change of variables")

(3) $$q_1(t) \equiv \psi_a(t) + \psi_b(t) \ ; \quad q_2(t) \equiv \psi_a(t) - \psi_b(t)$$

Then we have

(4) $$m \frac{d^2 q_1(t)}{dt^2} = -k\, q_1(t) \ ; \quad m \frac{d^2 q_2}{dt^2} = -3k\, q_2(t)$$

We know the solutions of these - they are

(5) $$q_1(t) = C_1 \cos(\omega_1 t + \phi_1) \ , \quad \omega_1^2 = \frac{k}{m}$$

(6) $$q_2(t) = C_2 \cos(\omega_2 t + \phi_2) \ , \quad \omega_2^2 = \frac{3k}{m}$$

$q_1(t)$ and $q_2(t)$ are called "normal coordinates" since they obey "normal"* (i.e, uncoupled) equations, and for this reason, this method of solving the equations is sometimes called "the method of searching for normal coordinates". In simple cases it can be quite convenient both for insight and as a formal procedure; however, finding the n.c.'s is not always a simple matter of adding and subtracting!

* in this context, in the literature, the phrase "normal equation" is often used for what we have called a "paradigm" equ. - i.e, an equation of the form $\ddot{q} = -\omega^2 q$, $\omega^2 > 0$.

2. Now, we actually want ψ_a and ψ_b.

To get them note that eqns. (3) imply

$$\psi_a(t) = \frac{1}{2}\left[q_1(t) + q_2(t)\right]$$

$$= \frac{1}{2} C_1 \cos(\omega_1 t + \phi_1) + \frac{1}{2} C_2 \cos(\omega_2 t + \phi_2)$$

and

$$\psi_b(t) = \frac{1}{2}\left[q_1(t) - q_2(t)\right]$$

$$= \frac{1}{2} C_1 \cos(\omega_1 t + \phi_1) - \frac{1}{2} C_2 \cos(\omega_2 t + \phi_2)$$

Now define $A_1 \equiv \frac{1}{2} C_1$, $A_2 \equiv \frac{1}{2} C_2$

then this is

$$\underline{\psi_a(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)}$$

$$\underline{\psi_b(t) = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)}$$

Apparently, then, since (5)* and (6)* are the general solutions to eqns. (4)*, <u>the general solutions for ψ_a and ψ_b must be</u>

(7) $\psi_a(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$

(8) $\psi_b(t) = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)$.

---

* (5) $q_1(t) = C_1 \cos(\omega_1 t + \phi_1)$, $\omega_1^2 = k/m$ | (4) $m\ddot{q}_1(t) = -k q_1(t)$

(6) $q_2(t) = C_2 \cos(\omega_2 t + \phi_2)$, $\omega_2^2 = 3k/m$ | $m\ddot{q}_2(t) = -3k q_2(t)$

Let us consider whether this makes sense.

According to (7) and (8), if $A_2 = 0$, i.e., $C_2 = 0 \Rightarrow q_2(t) = 0$ always, we get pure mode 1:

$$(9) \qquad \left.\begin{aligned} \psi_a(t) &= A_1 \cos(\omega_1 t + \phi_1) \\ \psi_b(t) &= A_1 \cos(\omega_1 t + \phi_1) \end{aligned}\right\} \quad \omega_1 = \sqrt{\frac{k}{m}}$$

while, if $A_1 = 0$, we get pure mode 2:

$$(10) \qquad \left.\begin{aligned} \psi_a(t) &= A_2 \cos(\omega_2 t + \phi_2) \\ \psi_b(t) &= -A_2 \cos(\omega_2 t + \phi_2) \end{aligned}\right\} \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

Thus, (7) and (8) are a superposition with both modes going simultaneous

$$\begin{aligned} \psi_a(t) &= A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ \psi_b(t) &= A_1 \underbrace{\cos(\omega_1 t + \varphi_1)}_{\substack{\text{unit amplitude} \\ \text{"mode 1"} \\ \text{(a and b in phase)}}} + A_2 \underbrace{\left[-\cos(\omega_2 t + \varphi_2)\right]}_{\substack{\text{unit amplitude} \\ \text{"mode 2"} \\ \text{(a and b out of phase)}}} \end{aligned}$$

This makes good sense: The differential equations of motion (1) and (2) are <u>linear</u>. Thus, the sum of two solutions is a solution. One solution of (1) and (2) is "mode 1" (eqns. (9)). Another solution is "mode 2" (eqns. (10)). So, the general solution of $\{(1),(2)\}$ is $A_1 \cdot$ mode 1 $+ A_2 \cdot$ mode 2, which is $\{(7),(8)\}$.

C. Completeness of the set of Normal Modes

The pair of equations (7) and (8) represent the general solution to the coupled governing differential eqns. (1a) and (1b) that resulted from applying Newton's law to the longitudinal internal free motions of the system. This means that there is no solution (and hence, no possible internal longitudinal motion) that cannot be expressed in the form of the pair of equations (7) and (8). [By "internal motions" we mean only motions that do not involve, e.g., someone picking up the whole apparatus, "wall anchors" and all, and bodily moving it.] Thus, there is no possible internal free longitudinal motion that cannot be expressed as some linear simultaneous superposition of the two normal modes we found, i.e, any internal longitudinal motion of the system is expressible as

some amplitude ("A_1") × normal mode #1
+ Some amplitude ("A_2") × normal mode #2.

Thus, for example, there cannot be a third independent longitudinal normal mode for this system.

The completeness property of the 2 normal modes is analogous to that of vectors in the "x-y plane". In that plane, any vector $\vec{A}$ can be written as a linear combination of two (and only two) unit-vectors (or, "basis vectors") $\hat{i}$ and $\hat{j}$ as

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

That works because the basis vectors $\hat{i}$ and $\hat{j}$ are independent of each other (neither has a component along the other, as they are perpendicular to each other).

To see the analogy, think of the entity $\begin{pmatrix} \psi_a(t) \\ \psi_b(t) \end{pmatrix}$ as a "vector" in an abstract two-dimensional space ("vector space"). In this abstract space, the "unit vectors" (or "basis vectors") are

$$\left\{ \begin{bmatrix} \text{unit amplitude} \\ \text{of Normal Mode} \\ \#1 \end{bmatrix}, \begin{bmatrix} \text{unit amplitude} \\ \text{of Normal Mode} \\ \#2 \end{bmatrix} \right\}.$$

Eqn. (2) on page 11 herein then shows the expansion of the "vector" $\begin{pmatrix} \psi_a(t) \\ \psi_b(t) \end{pmatrix}$ over these "unit vectors". That expansion works because the two normal modes are independent of each other.

D. Matching To Initial Conditions

Let us now return to the two-mass, three spring system with equal masses. Another way to check whether $\{(2),(2')\}$ is the general solution to the differential equations of motion is to check explicitly whether $\{(2),(2')\}$ covers all possible initial conditions.

To do this, for convenience we reexpress (2) and (2') in "sine + cosine" form:

$$\psi_a(t) = C_1 \sin \omega_1 t + D_1 \cos \omega_1 t + C_2 \sin \omega_2 t + D_2 \cos \omega_2 t$$

(11)

$$\psi_b(t) = C_1 \sin \omega_1 t + D_1 \cos \omega_1 t - C_2 \sin \omega_2 t - D_2 \cos \omega_2 t$$

Of course, either form has four arbitrary constants that are adjustable, and, of course, there are 4 physical initial conditions that must be matched – $\psi_a(0), \psi_b(0), \dot{\psi}_a(0), \dot{\psi}_b(0)$.

At $t=0$ (11) becomes

$$\psi_a(0) = D_1 + D_2 \quad , \quad \psi_b(0) = D_1 - D_2 .$$

Since (D_1+D_2) and (D_1-D_2) are independently arbitrary if D_1 and D_2 are, all possibilities for $\psi_a(0)$ and $\psi_b(0)$ are "covered," and

$$D_1 = \tfrac{1}{2}\left[\psi_a(0) + \psi_b(0)\right], \quad D_2 = \tfrac{1}{2}\left[\psi_a(0) - \psi_b(0)\right]$$

Further, according to (3),

$$\left.\begin{aligned} \dot{\psi}_a(0) &= \omega_1 C_1 + \omega_2 C_2 \\ \dot{\psi}_b(0) &= \omega_1 C_1 - \omega_2 C_2 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= \tfrac{1}{2}\omega_1 [\dot{\psi}_a(0) + \dot{\psi}_b(0)] \\ C_2 &= \tfrac{1}{2}\omega_2 [\dot{\psi}_a - \dot{\psi}_b(0)] \end{aligned}$$

Since C_1 and C_2 are completely arbitrary, the two right-hand sides are independently arbitrary, so all $\dot{\psi}_a(0)$, $\dot{\psi}_b(0)$ are also covered.

Thus eqn. (2) or (2') is, indeed, the general solution of the governing differential equations of motion.

As a corrolary, we see again that, for longitudinal motion of the system that we are considering, there are only two normal modes. The general longitudinal motion of the system is a superposition with both modes going at once - "so much" (A_1) of mode 1 plus "so much" (A_2) of mode 2.

"Regardless of the initial conditions, as long as you restrict to longitudinal internal motion, the general motion is an arbitrary linear superposition of only 2 simpler motions!"