1) $U = kr^4$

conservación de L $\Rightarrow$ $L = mr^2\dot\theta$ $\Rightarrow$ $r\dot\theta = \dfrac{L}{mr}$

Energía mecánica : $E = \frac{1}{2} m \vec{v}^{\,2} + kr^4 = \frac{1}{2} m (\dot r^2 + r^2\dot\theta^2) + kr^4$

$$= \frac{1}{2} m \dot r^2 + \frac{L^2}{m^2 r^2} \cdot \frac{1}{2} m + kr^4 = \frac{1}{2} m \dot r^2 + \underbrace{\frac{L^2}{2mr^2} + kr^4}_{\text{Pot. efectivo} := V_{eff}}$$

a)

Trayectoria circular $\Rightarrow$ a es el mínimo del potencial efectivo

$$\Rightarrow \left.\frac{\partial V_{eff}}{\partial r}\right)_a = 0 \Rightarrow -\frac{L^2}{ma^3} + 4ka^3 = 0$$

$$\Rightarrow L^2 = 4kma^6 \Rightarrow L = 2a^3 (km)^{1/2}$$

b) $$\dot\theta = \frac{L}{ma^2} = \sqrt{\frac{k}{m}}\, 2a = 2\sqrt{\frac{k}{m}}\, a \Rightarrow \dot\theta T = 2\pi$$

$$T = \frac{2\pi}{\dot\theta} = \frac{\pi}{a}\sqrt{\frac{m}{k}}$$

c)

Desarrollo el potencial efectivo en torno a "a". El término de orden 1 es 0 (la ser mínimo) :

$$V_{eff}(r) = V_{eff}(a) + \frac{1}{2} \left.\frac{d^2 V_{eff}}{dr^2}\right)_a (r-a)^2$$

Es el potencial de un muelle de constante $G = \left.\frac{d^2 V_{eff}}{dr^2}\right)_a$.

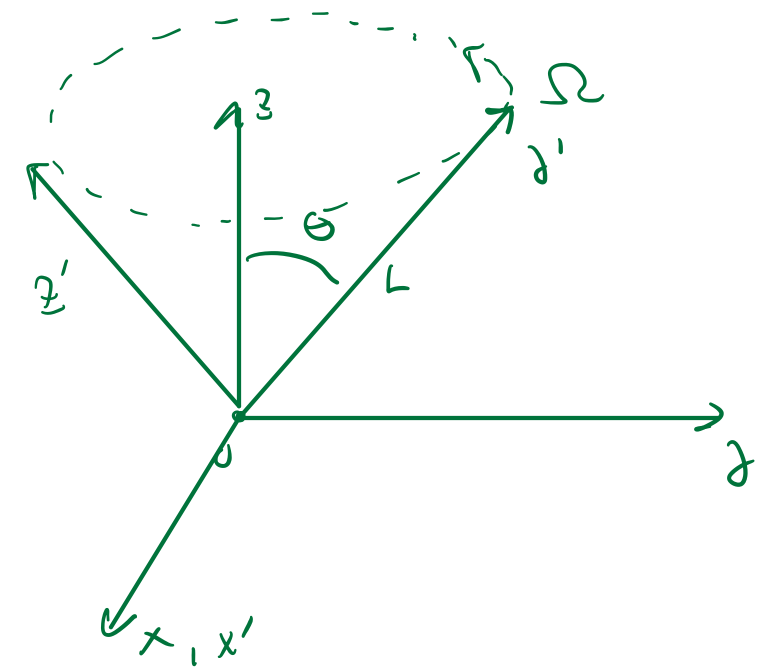
$$G = \frac{3L^2}{m}\frac{1}{a^4} + 12ka^2 = \frac{3\cdot 4k\cancel{m}a^6}{\cancel{m}a^4} + 12ka^2 = 24ka^2$$

$$\Rightarrow \quad T_r = \frac{2\pi}{\omega} \quad y \quad \omega = \sqrt{\frac{G}{m}} = \sqrt{24\frac{k}{m}}\,a \Rightarrow T_r = \frac{1}{\sqrt{6}}\sqrt{\frac{m}{k}}\frac{\pi}{a}$$

$$\frac{T_r}{T_\theta} = \frac{\frac{1}{\sqrt{6}}\cancel{\sqrt{\frac{m}{k}}\frac{\pi}{a}}}{\cancel{\frac{\pi}{a}\sqrt{\frac{m}{k}}}} = \frac{1}{\sqrt{6}}$$

no es racional. La órbita es abierta.

2)



dibujo a t=0

a)

II')

$I'_{yy} = 0$ pues la barra está a lo largo de y'

$I'_{yz} = I'_{yx} = 0$ pues los puntos de la barra tienen $x'=0$ y $z'=0$.

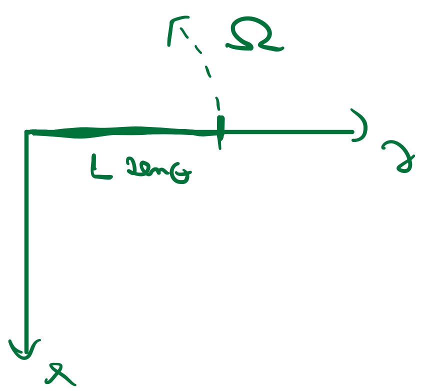
$I'_{zx} = 0$ porque $x' = z' = 0$.

Además la matriz es simétrica.

$$I'_{xx} = \rho \int_0^L dy' \, y'^2 = \frac{M}{L} \left. \frac{y'^3}{3} \right|_0^L = \frac{M}{L}\frac{L^3}{3} = \frac{ML^2}{3} = I'_{zz}.$$

$$\mathbb{I}' = \begin{pmatrix} \frac{ML^2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ML^2}{3} \end{pmatrix}$$

II) Visto desde arriba, a $t=0$.



$y = L \operatorname{sen}\theta \cos \Omega t$; $x = -L \operatorname{sen}\theta \operatorname{sen}\Omega t$; $z = L \cos\theta$

$$I_{xx} = \frac{M}{L}\int_0^L (y^2+z^2)\, dy' = \frac{M}{L}\int_0^L y'^2 \operatorname{sen}^2\theta \cos^2\Omega t + y'^2 \cos^2\theta$$

$$= \frac{M}{L}\left\{ \frac{ML^3}{3} \operatorname{sen}^2\theta \cos^2\Omega t + \frac{ML^3}{3} \cos^2\theta \right\} = \frac{ML^2}{3}\left(\overbrace{\operatorname{sen}^2\theta \cos^2\Omega t + \cos^2\theta}^{H} \right)$$

$$I_{zz} = \frac{M}{L}\int_0^L (y^2+x^2)\, dy' = \frac{ML^2}{3}\left(\underbrace{\operatorname{sen}^2\theta \cos^2\Omega t + \operatorname{sen}^2\theta \operatorname{sen}^2\Omega t}_{G} \right)$$

$$= \frac{ML^3}{3} \operatorname{sen}^2\theta$$

$$I_{yy} = \frac{M}{L}\int_0^L (x^2+z^2)\, dy' = \frac{M}{L}\frac{L^3}{3}\left(\overbrace{\operatorname{sen}^2\theta \operatorname{sen}^2\Omega t + \cos^2\theta}^{F} \right)$$

$$I_{xy} = -\rho\int_0^L x y = +\frac{M}{L}\frac{L^3}{3} \operatorname{sen}^2\theta \operatorname{sen}\Omega t \cos\Omega t = \frac{ML^2}{3} \underbrace{\operatorname{sen}^2\theta \operatorname{sen}\Omega t \cos\Omega t}_{A}$$

$$I_{yz} = -\frac{M}{L}\frac{L^3}{3} \operatorname{sen}^2\theta \cos\Omega t = -\frac{ML^2}{3} \underbrace{\cos^2\theta \cos\Omega t}_{B}$$

$$I_{xz} = \frac{ML^2}{3} \underbrace{\cos^2\theta \operatorname{sen}\Omega t}_{C}$$

b)

$$\mathbb{I}'_{cm} \Big) \qquad \mathbb{I}'_O = \mathbb{I}'_{cm} + \left(M |OG|^2 \mathbb{1} - M \bar{OG} \otimes \bar{OG} \right)$$

$|OG|^2$ en la eje fijo al cuerpo es $\frac{L^2}{4} = |OG|^2$

$$OG = -\frac{L}{2}\hat{y}' \Rightarrow \bar{OG} \otimes \bar{OG} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{L^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}'_0 + M \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{L^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} - M \begin{pmatrix} \frac{L^2}{4} & 0 & 0 \\ 0 & \frac{L^2}{4} & 0 \\ 0 & 0 & \frac{L^2}{4} \end{pmatrix} = \mathbb{I}'_{cm}$$

$$\begin{pmatrix} \frac{ML^2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ML^2}{3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{ML^2}{4} & 0 \\ 0 & 0 & \end{pmatrix} - \begin{pmatrix} \frac{ML^2}{4} & 0 & 0 \\ 0 & \frac{ML^2}{4} & 0 \\ 0 & 0 & \frac{ML^2}{4} \end{pmatrix} = \mathbb{I}'_{cm}$$

$$\begin{pmatrix} \frac{ML^2}{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ML^2}{12} \end{pmatrix} = \mathbb{I}'_{cm}$$

En los ejes fijos del Laboratorio

$$\bar{OG} = \underbrace{\frac{L}{2}\operatorname{sen}\theta\cos\Omega t}_{OG_y}\,\hat{y} - \underbrace{\frac{L}{2}\operatorname{sen}\theta\operatorname{sen}\Omega t}_{OG_x}\,\hat{x} + \underbrace{\frac{L}{2}\cos\theta}_{OG_z}\,\hat{z}$$

$$|OG|^2 = \frac{L^2}{4}\operatorname{sen}^2\theta + \frac{L^2}{4}\cos^2\theta = \frac{L^2}{4}$$

$$\bar{OG}\otimes\bar{OG} = \begin{pmatrix} OG_x^2 & OG_x OG_y & OG_x OG_z \\ OG_y OG_x & OG_y^2 & OG_y OG_z \\ OG_z OG_x & OG_z OG_y & OG_z^2 \end{pmatrix}$$

$$I_{CM} = \frac{ML^2}{3}\begin{pmatrix} H & A & C \\ A & F & B \\ C & B & G \end{pmatrix} - \frac{ML^2}{4}\mathbb{1} +$$

$$+ \frac{ML^2}{4}\begin{pmatrix} \operatorname{sen}^2\theta\operatorname{sen}^2\Omega t & -\operatorname{sen}^2\theta\operatorname{sen}\Omega t\cos\Omega t & -\operatorname{sen}\theta\cos\theta\operatorname{sen}\Omega t \\ -\operatorname{sen}^2\theta\operatorname{sen}\Omega t\cos\Omega t & \operatorname{sen}^2\theta\cos^2\Omega t & \operatorname{sen}\theta\cos\theta\cos\Omega t \\ -\operatorname{sen}\theta\cos\theta\operatorname{sen}\Omega t & \operatorname{sen}\theta\cos\theta\cos\Omega t & \cos^2\theta \end{pmatrix}$$

c)

$$\vec{L} = \mathbb{I}'_0 \vec{\Omega} = \begin{pmatrix} \frac{ML^2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ML^2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ \Omega \cos\theta \\ \Omega \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \Omega \sin\theta \frac{ML^2}{3} \end{pmatrix}$$

$$E_c = \frac{1}{2} \Omega^T \mathbb{I}'_0 \Omega = \frac{1}{2} \left(0, \Omega\cos\theta, \Omega\sin\theta \right) \begin{pmatrix} \frac{ML^2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ML^2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ \Omega\cos\theta \\ \Omega\sin\theta \end{pmatrix}$$

$$= \frac{1}{2} \frac{ML^2}{3} \Omega^2 \sin^2\theta = \frac{1}{6} ML^2 \Omega^2 \sin^2\theta$$

Como $\vec{\Omega}$ es constante, sus componentes en (x', y', z') son constantes. $\Rightarrow$ $\frac{d\Omega_{y'}}{dt} = \frac{d\Omega_{z'}}{dt} = \frac{d\Omega_{x'}}{dt} = 0$.

Entonces $\vec{M} = \frac{d\vec{L}}{dt} = \vec{\Omega} \times \vec{L}$ según las ecuaciones de Euler.

$$\vec{M} = (\Omega \cos\theta\, \hat{y}' + \Omega \sin\theta\, \hat{z}') \times \frac{ML^2}{3}\Omega \sin\theta\, \hat{z}'$$

$$= \boxed{\frac{ML^2}{3}\Omega^2 \sin\theta \cos\theta\, \hat{x}'}$$