Cálculo de la componentes del tensor campo electromagnético $F^{\mu\nu}$

$$\boxed{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu}$$

Es evidente que para $\mu = \nu$

$$F^{\mu\mu} = \partial^\mu A^\mu - \partial^\mu A^\mu = 0$$

luego:

$$F^{00} = F^{11} = F^{22} = F^{33} = 0$$

Como $F^{\mu\nu}$ es un tensor antisimétrico:

$$F^{\nu\mu} = -F^{\mu\nu}$$

luego nos basta con calcular las seis componentes F^{10}, F^{20}, F^{30}, F^{21}, F^{31} y F^{32}:

$$F^{10} = \frac{\partial A^0}{\partial x_1} - \frac{\partial A^1}{\partial x_0} = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = -\frac{\partial(\phi/c)}{\partial x} - \frac{\partial A_x}{\partial(ct)} =$$

$$= -\frac{1}{c}\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = \frac{1}{c}\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) = \frac{E_x}{c}$$

$$F^{20} = \frac{\partial A^0}{\partial x_2} - \frac{\partial A^2}{\partial x_0} = -\frac{\partial A^0}{\partial x^2} - \frac{\partial A^2}{\partial x^0} = -\frac{\partial(\phi/c)}{\partial y} - \frac{\partial A_y}{\partial(ct)} =$$

$$= -\frac{1}{c}\frac{\partial \phi}{\partial y} - \frac{1}{c}\frac{\partial A_y}{\partial t} = \frac{1}{c}\left(-\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t}\right) = \frac{E_y}{c}$$

$$F^{30} = \frac{\partial A^0}{\partial x_3} - \frac{\partial A^3}{\partial x_0} = -\frac{\partial A^0}{\partial x^3} - \frac{\partial A^3}{\partial x^0} = -\frac{\partial(\phi/c)}{\partial z} - \frac{\partial A_z}{\partial(ct)} =$$

$$= -\frac{1}{c}\frac{\partial \phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = \frac{1}{c}\left(-\frac{\partial\phi}{\partial z} - \frac{\partial A_z}{\partial t}\right) = \frac{E_z}{c}$$

$$F^{21} = \frac{\partial A^1}{\partial x_2} - \frac{\partial A^2}{\partial x_1} = -\frac{\partial A^1}{\partial x^2} + \frac{\partial A^2}{\partial x^1} = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} =$$

$$= (\vec{\nabla} \times \vec{A})_z = B_z$$

$$F^{31} = \frac{\partial A^1}{\partial x_3} - \frac{\partial A^3}{\partial x_1} = -\frac{\partial A^1}{\partial x^3} + \frac{\partial A^3}{\partial x^1} = -\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x} =$$

$$= -(\vec{\nabla} \times \vec{A})_y = -B_y$$

$$F^{32} = \frac{\partial A^2}{\partial x_3} - \frac{\partial A^3}{\partial x_2} = -\frac{\partial A^2}{\partial x^3} + \frac{\partial A^3}{\partial x^2} = -\frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y} =$$

$$= (\vec{\nabla} \times \vec{A})_x = B_x$$

de donde queda:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

donde se ha tenido en cuenta que:

$F^{01} = -F^{10}$, $F^{02} = -F^{20}$, $F^{03} = -F^{30}$, $F^{12} = -F^{21}$

$F^{13} = -F^{31}$, $F^{23} = -F^{32}$

Para determinar $F_{\mu\nu}$ tenemos en cuenta la relación:

$$F_{\mu\nu} = g_{\mu\lambda}\, g_{\nu\xi}\, F^{\lambda\xi}$$

y que:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

de donde:

$$F_{00} = F_{11} = F_{22} = F_{33} = 0$$

$$F_{01} = g_{0\lambda}\, g_{1\xi}\, F^{\lambda\xi} = g_{00}\, g_{11}\, F^{01} = (1)(-1)\left(-\frac{E_x}{c}\right) = \frac{E_x}{c}$$

$$F_{02} = g_{0\lambda}\, g_{2\xi}\, F^{\lambda\xi} = g_{00}\, g_{22}\, F^{02} = (1)(-1)\left(-\frac{E_y}{c}\right) = \frac{E_y}{c}$$

$$F_{03} = g_{0\lambda}\, g_{3\xi}\, F^{\lambda\xi} = g_{00}\, g_{33}\, F^{03} = (1)(-1)\left(-\frac{E_z}{c}\right) = \frac{E_z}{c}$$

$$F_{12} = g_{1\lambda}\, g_{2\xi}\, F^{\lambda\xi} = g_{11}\, g_{22}\, F^{12} = (-1)(-1)(-B_z) = -B_z$$

$$F_{13} = g_{1\lambda}\, g_{3\xi}\, F^{\lambda\xi} = g_{11}\, g_{33}\, F^{13} = (-1)(-1)(+B_y) = +B_y$$

$$F_{23} = g_{2\lambda}\, g_{3\xi}\, F^{\lambda\xi} = g_{22}\, g_{33}\, F^{23} = (-1)(-1)(-B_x) = -B_x$$

además:

$$F_{10} = -F_{01} \qquad F_{21} = -F_{12}$$

$$F_{20} = -F_{02} \qquad F_{31} = -F_{13}$$

$$F_{30} = -F_{03} \qquad F_{32} = -F_{23}$$

de donde queda:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

es decir:

$$F_{\mu\nu} = F^{\mu\nu}\left[\vec{E} \mapsto -\vec{E}\right]$$