Problem 9.30 Confirm that the energy in the TE$_{mn}$ mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle\mathbf{S}\rangle$ and the energy density $\langle u\rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

From Prob. 9.11, $\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)$. Here (Eq. 9.176) $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$, $\tilde{\mathbf{B}}^* = \tilde{\mathbf{B}}_0^* e^{-i(kz-\omega t)}$, and, for the TE$_{mn}$ mode (Eqs. 9.180 and 9.186)

$$\begin{aligned}
B_x^* &= \frac{-ik}{(\omega/c)^2-k^2}\left(\frac{-m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
B_y^* &= \frac{-ik}{(\omega/c)^2-k^2}\left(\frac{-n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
B_z^* &= B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_x &= \frac{i\omega}{(\omega/c)^2-k^2}\left(\frac{-n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
E_y &= \frac{-i\omega}{(\omega/c)^2-k^2}\left(\frac{-m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_z &= 0.
\end{aligned}$$

So

$$\begin{aligned}
\langle\mathbf{S}\rangle &= \frac{\omega k\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{\mathbf{z}}\Big\}.\\
\int\langle\mathbf{S}\rangle\cdot d\mathbf{a} &= \boxed{\frac{1}{8\mu_0}\frac{\omega k\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}ab\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right].}
\end{aligned}$$

[In the last step I used $\int_0^a\sin^2(m\pi x/a)\,dx = \int_0^a\cos^2(m\pi x/a)\,dx = a/2$; $\int_0^b\sin^2(n\pi y/b)\,dy = \int_0^b\cos^2(n\pi y/b)\,dy = b/2$.]

Similarly,

$$\begin{aligned}
\langle u\rangle &= \frac{1}{4}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^*+\frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)\\
&= \frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\
&+\frac{1}{4\mu_0}\Big\{B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\\
&\quad+\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\Big\}.
\end{aligned}$$

$$\int\langle u\rangle\,da = \boxed{\frac{ab}{4}\left\{\frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]+\frac{B_0^2}{4\mu_0}+\frac{1}{4\mu_0}\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]\right\}.}$$

These results can be simplified, using Eq. 9.190 to write $\left[(\omega/c)^2-k^2\right] = (\omega_{mn}/c)^2$, $\epsilon_0\mu_0 = 1/c^2$ to eliminate and Eq. 9.188 to write $\left[(m/a)^2+(n/b)^2\right] = (\omega_{mn}/\pi c)^2$:

$$\int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\omega kabc^2}{8\mu_0\omega_{mn}^2}B_0^2;\qquad \int\langle u\rangle\,da = \frac{\omega^2ab}{8\mu_0\omega_{mn}^2}B_0^2.$$

Evidently

$$\frac{\text{energy per unit time}}{\text{energy per unit length}} = \frac{\int\langle\mathbf{S}\rangle\cdot d\mathbf{a}}{\int\langle u\rangle\,da} = \frac{kc^2}{\omega} = \frac{c}{\omega}\sqrt{\omega^2-\omega_{mn}^2} = v_g \text{ (Eq. 9.192).}\quad\text{qed}$$