# **Modern Classical Mechanics**

**Chapter 12** 

Helliwell & Sahakian

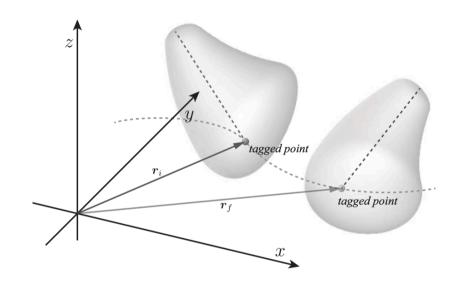
### **Chapter 12**

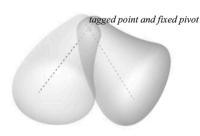
### **Rigid-Body Dynamics**

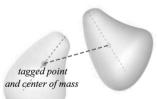
- Euler's theorem
- Lab and body frames
- Euler angles
- Angular momentum
- Torque
- Energy
- Torque-free dynamics
- Gyroscopes

### **Euler's Theorem**

All spatial transformations that leave distances unchanged must be a combination of a translation and a rotation.

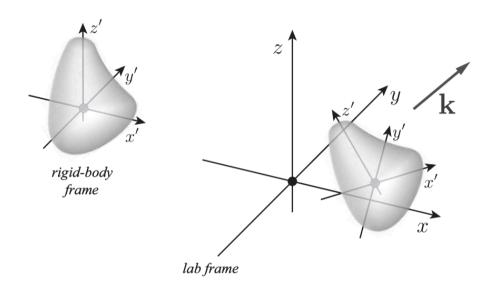






$$\mathbf{R}_{\mathrm{cm}} = \sum_i rac{\Delta m_i \mathbf{r}_i}{M} = rac{1}{M} \int \mathbf{r} dm = rac{1}{M} \int 
ho \mathbf{r} dV$$

### **Rotation Matrices and the Body Frame**



$$\mathbf{k}^{i'}\mathbf{k}^{i'} = \hat{\mathscr{R}}^{i'}_{j}\mathbf{k}^{j}\hat{\mathscr{R}}^{i'}_{k}\mathbf{k}^{k} = \mathbf{k}^{j}\hat{\mathscr{R}}^{i'}_{j}\hat{\mathscr{R}}^{i'}_{k}\mathbf{k}^{k} = \mathbf{k}^{i}\mathbf{k}^{i}$$
 $\hat{\mathscr{R}}^{i'}_{j}\hat{\mathscr{R}}^{i'}_{k} = \delta_{jk} \quad lacksquare$ 
 $\hat{\mathscr{R}}^{i'}_{j}\hat{\mathscr{R}}^{i'}_{k} = \delta_{jk} \quad lacksquare$ 

### **Rotation Matrices and the Body Frame**

$$\hat{\mathscr{R}}_x = egin{pmatrix} 1 & 0 & 0 \ 0 & \coslpha_x & \sinlpha_x \ 0 & -\sinlpha_x & \coslpha_x \end{pmatrix} \qquad \hat{\mathscr{R}}_y = egin{pmatrix} \coslpha_y & 0 & -\sinlpha_y \ 0 & 1 & 0 \ \sinlpha_y & 0 & \coslpha_y \end{pmatrix} \qquad \hat{\mathscr{R}}_z = egin{pmatrix} \coslpha_z & \sinlpha_z & 0 \ -\sinlpha_z & \coslpha_z & 0 \ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{\mathscr{R}}_y = egin{pmatrix} \coslpha_y & 0 & -\sinlpha_y \ 0 & 1 & 0 \ \sinlpha_y & 0 & \coslpha_y \end{pmatrix}$$

$$\hat{\mathscr{R}}_z = egin{pmatrix} \coslpha_z & \sinlpha_z & 0 \ -\sinlpha_z & \coslpha_z & 0 \ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{\mathscr{R}}_1\cdot\hat{\mathscr{R}}_2
eq\hat{\mathscr{R}}_2\cdot\hat{\mathscr{R}}_1$$

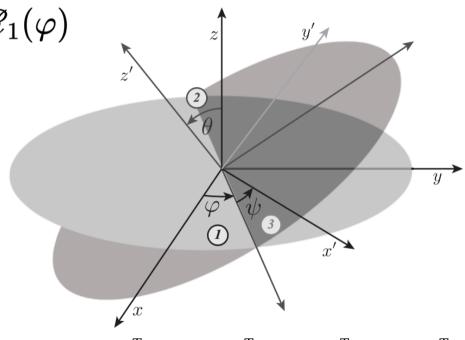
## **The Euler Angles**

$$\hat{\mathscr{R}}(arphi, heta,\psi)=\hat{\mathscr{R}}_3(\psi)\cdot\hat{\mathscr{R}}_2( heta)\cdot\hat{\mathscr{R}}_1(arphi)$$

$$\hat{\mathscr{R}}_1(arphi) = egin{pmatrix} \cosarphi & \sinarphi & 0 \ -\sinarphi & \cosarphi & 0 \ 0 & 0 & 1 \end{pmatrix}$$

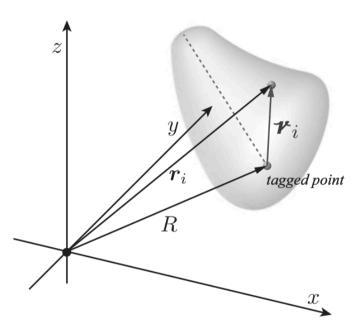
$$\hat{\mathscr{R}}_2( heta) = egin{pmatrix} 1 & 0 & 0 \ 0 & \cos heta & \sin heta \ 0 & -\sin heta & \cos heta \end{pmatrix}$$

$$\hat{\mathscr{R}}_3(\psi) = egin{pmatrix} \cos\psi & \sin\psi & 0 \ -\sin\psi & \cos\psi & 0 \ 0 & 0 & 1 \end{pmatrix}$$



$$egin{aligned} \left[\hat{\mathscr{R}}(arphi, heta,\psi)
ight]^T &= \left[\hat{\mathscr{R}}_1(arphi)
ight]^T \cdot \left[\hat{\mathscr{R}}_2( heta)
ight]^T \cdot \left[\hat{\mathscr{R}}_3(\psi)
ight]^T \ &= \hat{\mathscr{R}}_1(-arphi) \cdot \hat{\mathscr{R}}_2(- heta) \cdot \hat{\mathscr{R}}_3(-\psi) \end{aligned}$$

# **The Euler Angles**



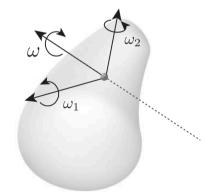
The decomposition of the position vector of a bit of the rigid body in terms of the position of the tagged point  $\mathbf{R}$  and the position of the bit with respect to the tagged point  $\mathbf{r}_a'$ .

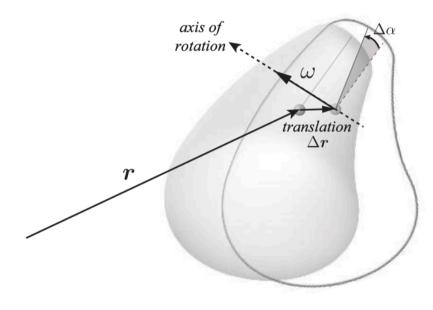
### **Infinitesimal Rotations**

$$\mathbf{r} = \mathbf{R} + \boldsymbol{r}$$

$$\mathbf{v} = rac{dm{r}}{dt} = m{\omega} imes m{r}$$

$$oldsymbol{\omega} = oldsymbol{\omega}_1 + oldsymbol{\omega}_2$$





$$\omega_1 imes oldsymbol{r} + \omega_2 imes oldsymbol{r} = oldsymbol{v}_1 + oldsymbol{v}_2$$

#### **Angular Velocity Transformation**

$$\omega^{(I)} = \dot{\varphi}, \omega^{(II)} = \dot{\theta}, \omega^{(II)} = \dot{\psi}$$

$$\omega = (\omega^{x'}, \omega^{y'}, \omega^{z'})$$

$$= (\dot{\varphi} \sin \theta \sin \psi + \dot{\theta} \cos \psi, \dot{\varphi} \sin \theta \cos \psi - \dot{\theta} \sin \psi, \dot{\varphi} \cos \theta + \dot{\psi})$$

$$\omega = (\omega^{x}, \omega^{y}, \omega^{z})$$

$$= (\dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi_{i} - \dot{\psi} \sin \theta \cos \varphi + \dot{\theta} \sin \varphi, \dot{\psi} \cos \theta + \dot{\varphi})$$

### **Angular Momentum**

$$\mathbf{L}_{\mathrm{tot}} = \sum_{i} oldsymbol{\ell}_{i} = \sum_{i} (\mathbf{r_i} imes \mathbf{p_i})$$

$$\mathbf{p}_i = m_i (\mathbf{V} + oldsymbol{\omega} imes oldsymbol{r}_i)$$

Tagged point is the center of mass of the rigid body

$$\mathbf{L}_{\mathrm{tot}} = \mathbf{R}_{\mathrm{cm}} imes (M \mathbf{V}_{\mathrm{cm}}) + \mathbf{L}$$

Tagged point is a fixed pivot

$$\mathbf{L}_{\mathrm{tot}} = \mathbf{L}$$

$$\mathbf{L} = \sum_i m_i ig[ oldsymbol{\omega}ig(oldsymbol{r}_i^2ig) - (oldsymbol{\omega}\cdotoldsymbol{r}_i)oldsymbol{r}_i ig]$$

$$=\int dmig[oldsymbol{\omega}ig(oldsymbol{r}^2ig)-(oldsymbol{\omega}\cdotoldsymbol{r})oldsymbol{r}ig]$$

## **Angular Momentum**

$$\hat{\mathbf{L}} = \hat{\mathbf{I}} \cdot \hat{oldsymbol{\omega}}$$

$$I_{ab}=\int dmig(r^2\delta_{ab}-r^ar^big)$$

$$egin{align} I_{xx} &= \int dmig(y^2+z^2ig), \quad I_{yy} &= \int dmig(x^2+z^2ig), \quad I_{zz} &= \int dmig(x^2+y^2ig) \ I_{xy} &= \int dm(-xy), \quad I_{xz} &= \int dm(-xz), \quad I_{yz} &= \int dm(-yz) \ \end{array}$$

### **Principal Axes**

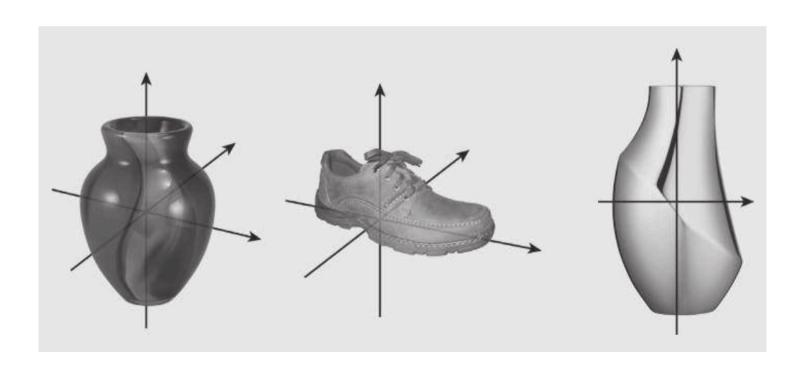
$$\hat{\mathbf{I}} 
ightarrow \hat{\mathscr{A}} \cdot \hat{\mathbf{I}} \cdot \hat{\mathscr{A}}^T$$

$$\int dm \, r^{a'} r^{b'} = 0 \quad ext{ for } a' 
eq b', ext{ in the principal axis frame.}$$

$$\hat{f I} = egin{pmatrix} I_{x'x'} & 0 & 0 \ 0 & I_{y'y'} & 0 \ 0 & 0 & I_{z'z'} \end{pmatrix} = egin{pmatrix} I_1 & 0 & 0 \ 0 & I_2 & 0 \ 0 & 0 & I_3 \end{pmatrix}$$

# **Principal Axes**

$$L_1 = I_1 \omega_1, \;\; L_2 = I_2 \omega_2, \;\; L_3 = I_3 \omega_3$$



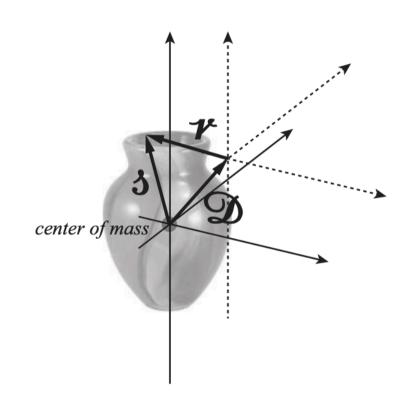
# **Principal Axes**

#### The parallel axis theorem

$$oldsymbol{s} = oldsymbol{r} + \mathscr{D}$$

$$I_{a'a'} = I_{a'}^{
m cm} + M\mathscr{D}^2 - M\mathscr{D}^{a'}\mathscr{D}^a$$

$$I_{a'b'} = -M \mathscr{D}^{a'} \mathscr{D}^{b'} \quad ext{ for } a' 
eq b'$$



### **Torque**

$$\mathbf{N}_{\mathrm{tot}} = \sum_{i} \mathbf{r}_{i} imes \mathbf{F}_{i}^{\mathrm{ext}}$$

$$\mathbf{N}_{\mathrm{tot}} = \sum oldsymbol{r}_i imes \mathbf{F}_i^{\mathbf{ext}} + \mathbf{R} imes \mathbf{F}_i^{\mathbf{ext}} = \mathbf{N} + \mathbf{R} imes \mathbf{F} = rac{d\mathbf{L}_{\mathrm{tot}}}{dt}$$

### **Kinetic Energy**

$$T_{
m tot} = rac{1}{2} \sum_i \Delta m_i \mathbf{v}_i^2 \qquad \qquad \mathbf{v}_i = \mathbf{V} + oldsymbol{\omega} imes oldsymbol{r}_i$$

Tagged point is the center of mass of the rigid body

$$T_{
m tot} = rac{1}{2} M \mathbf{V}_{
m cm}^2 \! + \! M \mathbf{V} \cdot \left(oldsymbol{\omega} imes \mathbf{R}_{
m cm}
ight) \! + \! T$$

Tagged point is a fixed pivot

$$T_{\rm tot} = T$$

$$T = rac{1}{2}oldsymbol{\omega}^T\cdot\hat{\mathbf{I}}\cdotoldsymbol{\omega} 
ightarrow rac{1}{2}oldsymbol{\omega}^T\cdot\hat{\mathscr{A}}^T\cdot\hat{\mathscr{A}}\cdot\hat{\mathbf{I}}\cdot\hat{\mathscr{A}}^T\cdot\hat{\mathscr{A}}\cdotoldsymbol{\omega} = rac{1}{2}I_1\omega_1^2 + rac{1}{2}I_2\omega_2^2 + rac{1}{2}I_3\omega_3^2 = rac{L_1^2}{2I_1} + rac{L_2^2}{2I_2} + rac{L_3^2}{2I_3}$$

$$egin{align} T &= rac{1}{2} \int dm igl( \omega^2 r^2 - (\omega \cdot m{r})^2 igr) \ &= rac{1}{2} \omega^a igg[ \int dm igl( r^2 \delta_{ab} - r^a r^b igr) igg] \omega^b \ &= rac{1}{2} \omega^a I_{ab} \omega^b = rac{1}{2} m{\omega}^T \cdot \hat{\mathbf{I}} \cdot m{\omega} \ \end{aligned}$$

### **Potential Energy**

$$U = \sum_i \Delta m_i g h_i = \sum_i \Delta m_i g \mathbf{r}_i \cdot \hat{\mathbf{z}} = M g \mathbf{R}_{\mathrm{cm}} \cdot \hat{\mathbf{z}} = M g H$$

#### A Hoop Hanging on a Spring

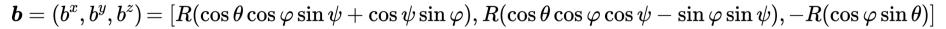
$$T_{
m tot} = rac{1}{2} M \Big( {\dot X}^2 + {\dot Y}^2 + {\dot Z}^2 \Big) + rac{1}{4} M R^2 \Big( {\dot heta}^2 + {\dot arphi}^2 \sin^2 heta \Big) + rac{1}{2} M R^2 ({\dot \psi} + {\dot arphi} \cos heta)^2$$

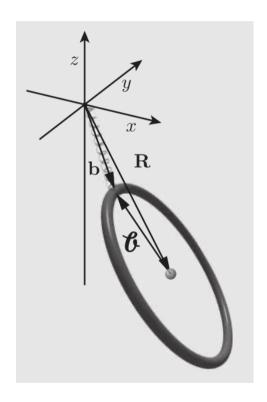
$$U_{
m grav} = MgZ \qquad U_{
m spring} = rac{1}{2}kb^2$$

$$\mathbf{b} = \mathbf{R} + \boldsymbol{b}$$

$$oldsymbol{b} = \left(e^{x'}, heta^{y'}, b^{z'}\right) = (0, R, 0)$$
  $\mathbf{R} = (\mathbf{R}^x, \mathbf{R}^y, \mathbf{R}^z) = (X, Y, Z)$ 

$$\mathbf{b}=(b^x,b^y,b^z)=(X+b^x,y+b^y,Z+b^z)$$





#### A Hoop Hanging on a Spring

$$egin{aligned} U_{ ext{spring}} = &rac{1}{2} k ig( X^2 + Y^2 + Z^2 + R^2 + 2RX (\cos heta \cos arphi \sin \psi + \cos \psi \sin arphi ig) \ &+ 2RY (\cos heta \cos arphi \cos \psi - \sin arphi \sin \psi) - 2RZ \cos arphi \sin heta ig) \end{aligned}$$

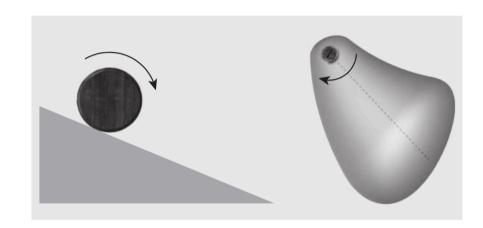
$$\mathscr{L} = T_{
m tot} \, - U_{
m grav} \, - U_{
m spring}$$

#### **Rolling, Fixed-Axis Rotation**

$$L_1=L_2=0,\quad L_3=I_3\omega_3=I_3\dot{\psi}$$

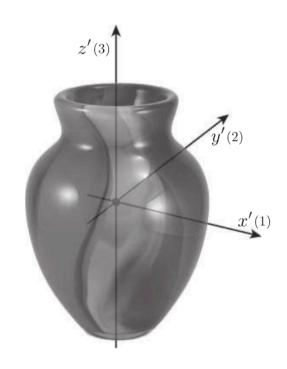
$$I=rac{1}{2}I_3\omega_3^2=rac{1}{2}I_3\dot{\psi}^2=rac{L_3^2}{2/_3}$$

$$rac{dL_3}{dt} = I_3 \dot{\omega}_3 = I_3 lpha \qquad \qquad lpha = \ddot{\psi}$$



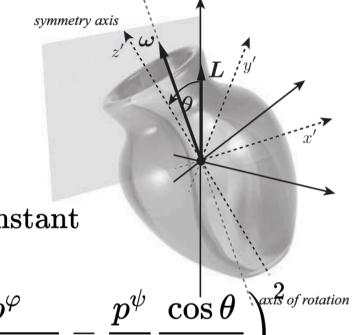
$$egin{align} I_1 &= I_2 \equiv I, \quad I_3 \ \mathscr{L} &= T - U \ T &= rac{1}{2} M ig( \dot{\mathcal{X}}^2 + \dot{\mathcal{Y}}^2 + \dot{\mathcal{Z}}^2 ig) + rac{1}{2} I ig( \dot{ heta}^2 + \dot{arphi}^2 \sin^2 heta ig) + rac{1}{2} I_3 (\dot{\psi} + \dot{arphi} \cos heta)^2 \ U &= MgZ \ p_{\psi} &= rac{\partial \mathscr{L}}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{arphi} \cos heta) = ext{constant} \ \end{aligned}$$

 $p^{arphi} = rac{\partial L}{\partial \dot{arphi}} = I \dot{arphi} \sin^2 heta + p^{\psi} \cos heta = ext{constant}$ 

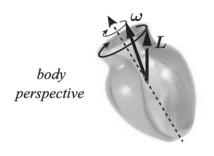


 $I\ddot{ heta} = I\dot{arphi}^2\sin heta\cos heta - p^\psi\dot{arphi}\sin heta$ 

$$H=rac{1}{2}I\dot{ heta}^2+rac{1}{2I_3}ig(p^\psiig)^2+rac{1}{2}I\sin^2 heta\dot{arphi}^2= ext{ constant}$$



#### From body frame perspective



$$\dot{\theta}=0$$

$$oldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = (\dot{arphi} \sin heta \sin \psi, \dot{arphi} \sin heta \cos \psi, \dot{arphi} \cos heta + \dot{\psi})$$

$$I\dot{arphi}\cos heta=p^{\psi}$$

$$p^{arphi}=rac{p^{\psi}}{\cos heta}$$

$$\dot{\psi} = igg(1-rac{I_3}{I}igg)rac{p^\psi}{I_3} = igg(1-rac{I_3}{I}igg)\omega_3$$

#### From lab frame perspective

$$\dot{\theta} = 0$$

$$oldsymbol{\omega} = (\omega^x, \omega^y, \omega^z) = (\dot{\psi} \sin heta \sin arphi, \dot{\psi} \sin heta \cos arphi, \dot{\psi} \cos heta + \dot{arphi})$$

$$egin{align} \dot{arphi} &= rac{1}{I} rac{p^{\psi}}{\cos heta} \ L^2 &= 2HI + ig(p^{\psi}ig)^2igg(1 - rac{I}{I_3}igg) \qquad igstar{} igstar{} igg( L = rac{L}{I} igg) \ \dot{arphi} &= rac{L}{I} igg) \end{aligned}$$

### **Euler's Equations of Motion and Stability**

$$\mathbf{N}_{\mathrm{tot}} = \left(rac{d\mathbf{L}}{dt}
ight)_{\mathrm{lab}} = \left(rac{d\mathbf{L}}{dt}
ight)_{\mathrm{body}} + \omega imes \mathbf{L}$$

$$egin{aligned} N_1 &= I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 \ N_2 &= I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 \ N_3 &= I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 \end{aligned}$$

## **Euler's Equations of Motion and Stability**

no net torque  $\omega_1$  is large and  $\omega_2$  and  $\omega_3$  are both small

$$egin{split} I_1 \dot{\omega}_1 &\simeq 0 \ I_2 \dot{\omega}_2 + [(I_1 - I_3) \omega_1] \omega_3 &\simeq 0 \ I_3 \dot{\omega}_3 + [(I_2 - I_1) \omega_1] \omega_2 &\simeq 0 \end{split}$$

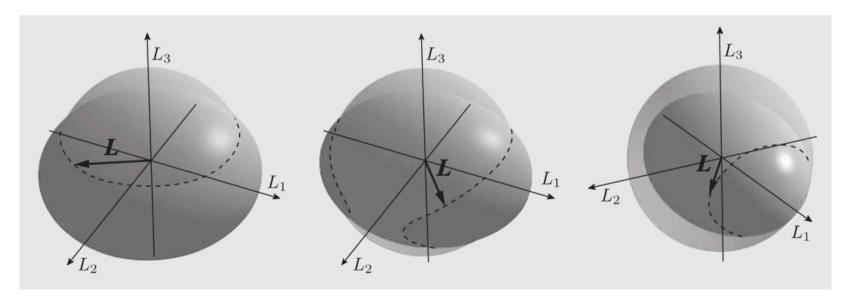
$$\ddot{\omega}_2+\Omega^2\omega_2=0 \quad ext{ and }\quad \ddot{\omega}_3+\Omega^2\omega_3=0 \qquad \Omega^2\equivrac{[(I_1-I_2)(I_1-I_3)]\omega_1^2}{I_2I_3}$$

Stability requires  $I_1$  be the largest or smallest moment of inertia.

### **Euler's Equations of Motion and Stability**

$$L^2 = L_1^2 + L_2^2 + L_3^2 \qquad T = rac{L_1^2}{2I_1} + rac{L_2^2}{2l_2} + rac{L_3^2}{2I_3}$$

$$I_1 \geq I_2 \geq I_3 \quad lacksquare \quad \sqrt{2TI_3} \leq L \leq \sqrt{2TI_1}$$



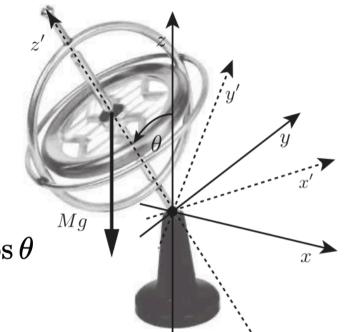
## **Gyroscopes**

$$T=rac{1}{2}Iig(\dot{ heta}^2+\dot{arphi}^2\sin^2 hetaig)+rac{1}{2}I_3(\dot{\psi}+\dot{arphi}\cos heta)^2$$

$$U=MgZ$$
  $Z=-l\cos heta$ 

$$p^{\psi} = I_3(\dot{\psi} + \dot{arphi}\cos heta), \quad p^{arphi} = I\dot{arphi}\sin^2 heta + p^{\psi}\cos heta$$

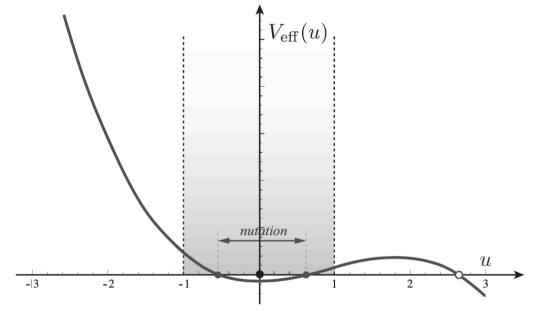
$$H=rac{1}{2}I\dot{ heta}^2+rac{1}{2I_3}ig(p^\psiig)^2+rac{1}{2}I\sin^2 heta\dot{arphi}^2+Mgl\cos heta$$



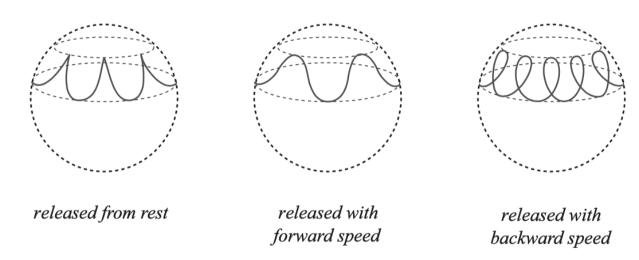
### **Gyroscopes**

$$u = \cos \theta$$

$$rac{1}{2}\dot{u}^2 + rac{1}{2}rac{ig(p^arphi - up^\psiig)^2}{I^2} + rac{1}{2}igg(rac{2HI_3 - ig(p^\psiig)^2}{II_3} - rac{2Mgl}{I}uigg)ig(u^2 - 1ig) = 0$$



### **Gyroscopes**



The nutation pattern of a gyroscope traced out by the gyroscope's z' axis. The spinning about the vertical axis is the familiar tumble, or *precession*. But now we also have a superimposed nutation as gravity tries to pull the z' axis downward.

### **Summary**

- Rigid body dynamics describes the time evolution of the orientation of a rigid body. Euler angles are used to label the orientation.
- Two perspectives for this dynamics are useful: body frame and lab frame.
- Use principal axes whenever possible; and write the Lagrangian in terms of the Euler angles.
- Torque-free dynamics involves a wobble.
- Gyroscope dynamics is characterized by nutation.