Physics 251 - 18th Class - Thursday March 14, 2024

A. Brief Review

[K-text, sect. 11.7]

We recall that we can write the general Fourier expansion for $f(t)$ as

$$(1) \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T}t\right) + b_n \sin\left(n\frac{2\pi}{T}t\right) \quad (*)$$

where, for any t_0,

$$(2) \quad a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\,dt = \text{twice the average of } f(t) \text{ over one period } (T)$$

$$(3) \quad a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(n\frac{2\pi}{T_1}t\right)dt, \quad b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(n\frac{2\pi}{T}t\right)dt$$

Just as it is frequently convenient to write a series for $f(x)$ in terms of $k_1 = 2\pi/\lambda_1$, so, frequently, it's more convenient to write the series in terms of $\omega_1 \equiv \frac{2\pi}{T_1} = \frac{2\pi}{T}$.

Then (1), (2) and (3) become

$$(4) \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t)$$

with

$$(5) \quad a_0 = \frac{\omega_1}{\pi}\int_{t_1}^{t_1+T_1} f(t)\,dt, \quad a_n = \frac{\omega_1}{\pi}\int_{t_1}^{t_1+T_1} f(t)\cos(n\omega_1 t)\,dt, \quad b_n = \frac{\omega_1}{\pi}\int_{t_1}^{t_1+T_1} f(t)\sin(n\omega_1 t)\,dt$$

We could also have gotten (4) and (5) from the "wavenumber form" of the series we developed previously using $k \longleftrightarrow \omega$, but we preferred to work through (1), (2) and (3).

* Thus, the lowest frequency f_1 in the Fourier analysis of $f(t)$ is 1 over the repeat time of $f(t)$.

B. Application to the Physics of Musical Instruments [K-text, sect. 11.11]

A flute does not sound like an oboe when playing the same musical note. Everyone knows this, but why this is so is a good question. That is, why do different instruments playing the same "note" sound different? To get an idea, let's look at some oscilloscope traces:

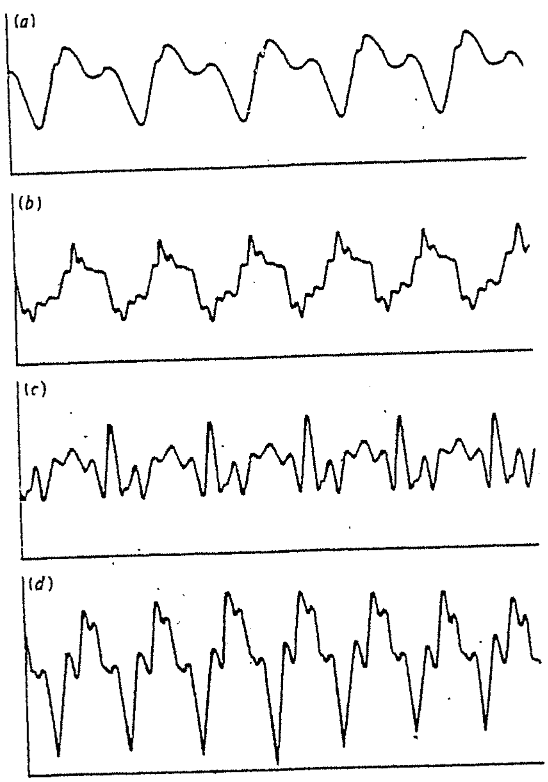


Fig. 13-24 Space-independent wave functions for the same musical note played on (a) a flute, (b) a clarinet, (c) an oboe, and (d) a saxophone. (*After D. C. Miller, Sound Waves, Their Shape and Speed, Macmillan, New York, 1937.*)

We see that the basic repetition periods may be the same, but the details are quite different. Since each is periodic but apparently not pure sinusoidal (!) we can, by Fourier's theorem, view each as composed of a different mix of pure sinusoids.

Thus, we look at the frequency spectra:

A look at the relative amplitudes of the different modes in the mix for a "single note" is very revealing!

Fig. 13-26 Typical Fourier spectra of some common musical instruments. The Fourier spectrum of an instrument varies considerably over its range of pitch and depends to some degree on the loudness with which it is played and other factors. Nevertheless, the characteristic quality of the instrument is still clearly discernible. The reason for this is especially evident for the clarinet, where the odd harmonics ($j = 1, 3, 5, \ldots$) are considerably stronger than the even harmonics ($j = 2, 4, 6, \ldots$).

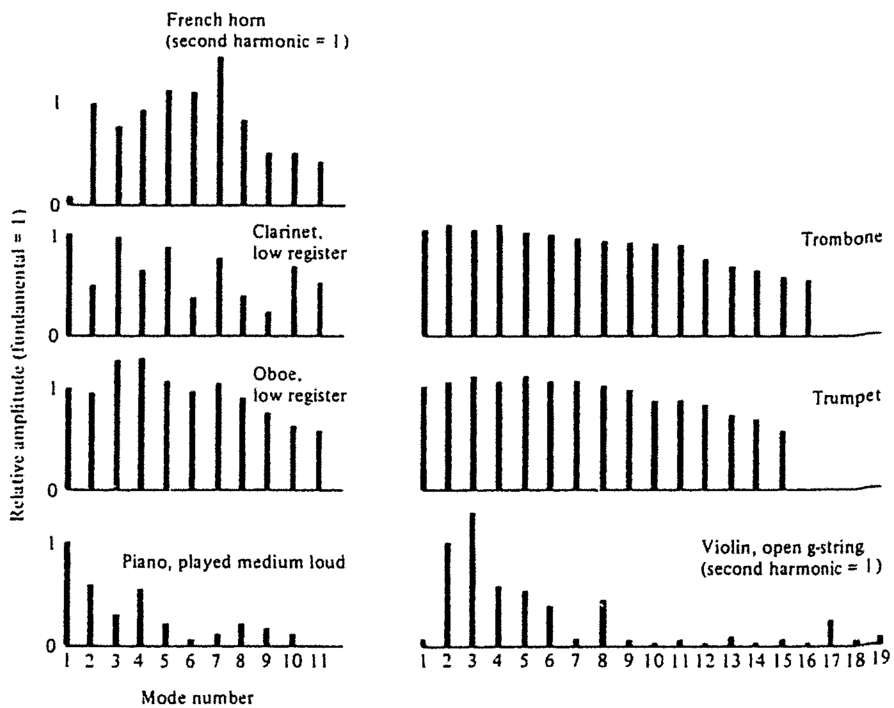


Figure from R. Eisberg & L. Lerner, op. cit., p. 597

On the next two pages we look at more similar cases:

Examples:

The figures are from <u>Physics Foundations and Applications Vol. I</u> by Robert Eisberg and Lawrence Lerner (McGraw Hill), pp. 598-599

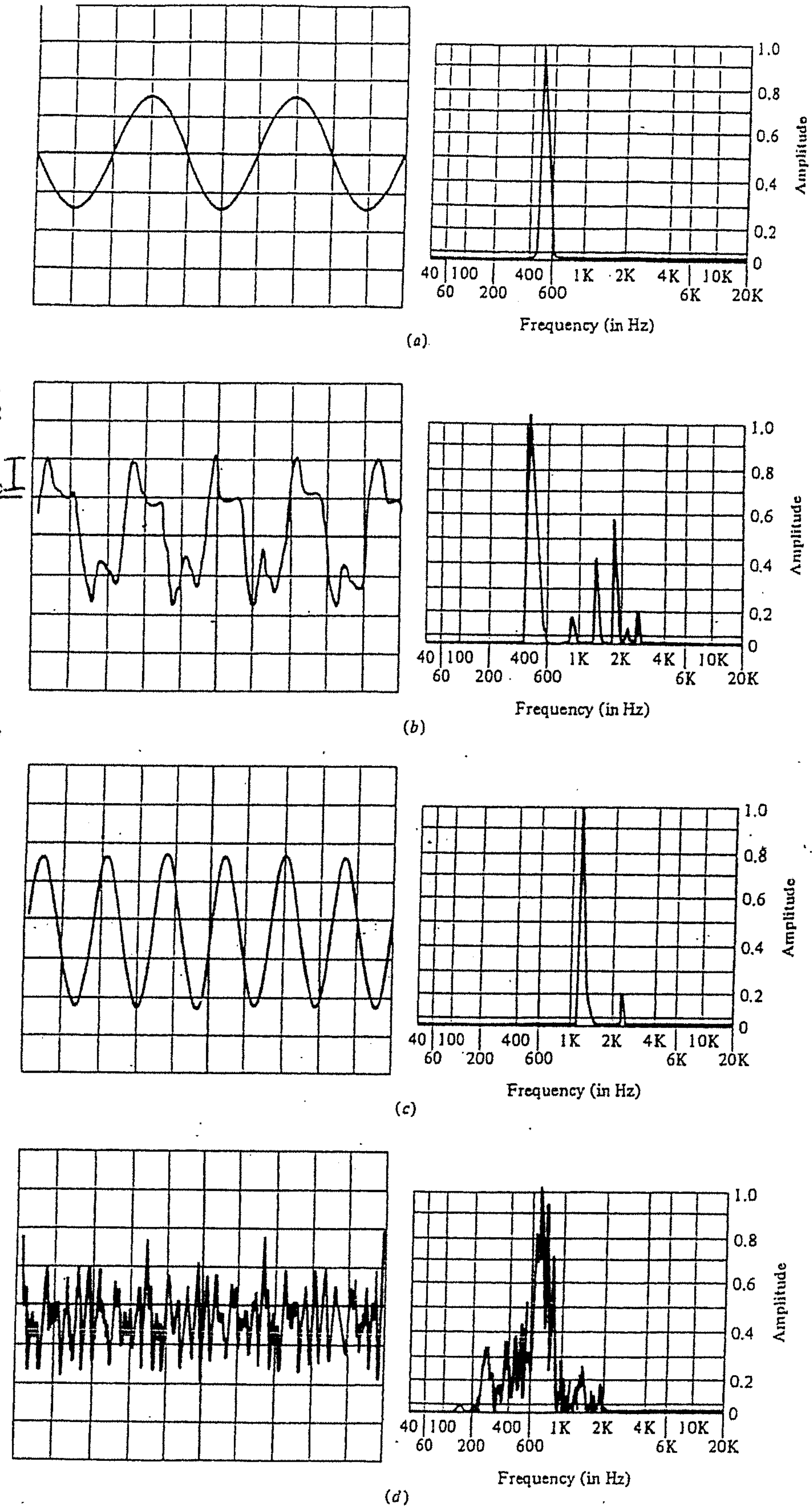
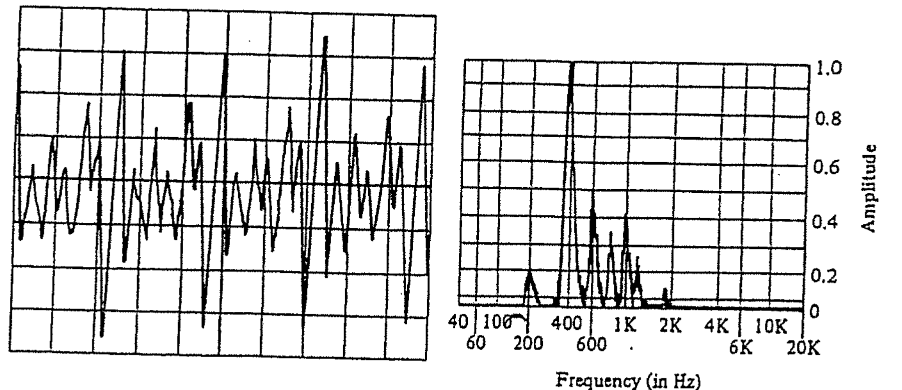
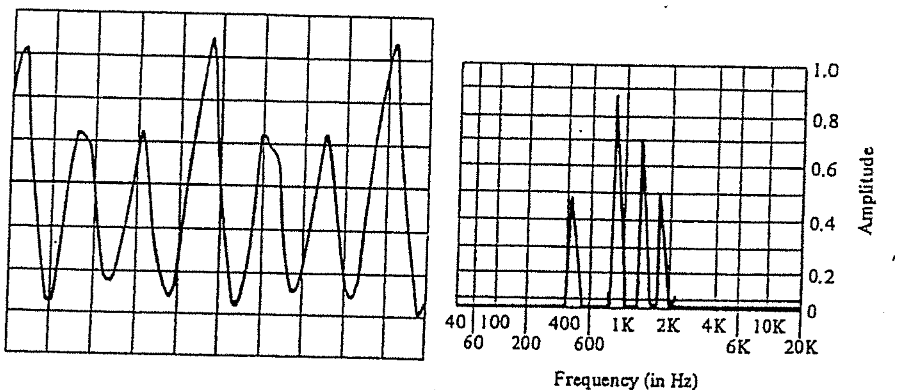


Fig. 13-27 (*a*) Waveform and sound spectrum of a pure (sinusoidal) tone generated electronically (500 Hz). (*b*) Waveform and sound spectrum for a flute playing near the lower end of its range (A = 440 Hz). The fundamental is strongest, but higher frequencies are well represented, too, especially the third and fourth harmonics. (*c*) Flute playing in the middle of its range (D = 1175 Hz). The tone has "cleaned up" to the typically pure quality we associate with this instrument. The second harmonic has only one-fifth the amplitude of the fundamental, and no other harmonics can be seen. The eye can barely distinguish the waveform from a pure sinusoidal. (*d*) Bassoon playing at the very bottom of its range (B-flat = 58 Hz). Note the complete absence of the fundamental and the great weakness of the lower harmonics. The maximum amplitude is in the twelfth harmonic.



(e)



(f)

(e) Bassoon playing in the middle of its range (G = 196 Hz). The fundamental is now visible, but is weaker than five of the harmonics, especially the second. (f) Bassoon near the top of its range (A = 440 Hz). This is the same note being played by the flute in part *b*. The fundamental is now stronger, but still does not dominate the waveform. This sound is much smoother than the lower notes, but still quite characteristic of the bassoon.

Note that the fundamental (pitch of the "note") is often not the strongest Fourier component - indeed, in some cases (e.g., bassoon) it's about absent!

Yet, we perceive the pitch as the fundamental!

How can this be? [Reference: Text, Chap. 3, sect. 3.5]

To some extent the *ear* supplies this!

The ear is far from a perfect **linear** device!

Linear Device:

$$x \propto F$$

displacement response

input force

e.g., $F = (-)kx$

Somewhat Better Model Of Eardrum

$$x = aF + bF^2$$

displacement of eardrum

quadratic nonlinearity.

Example: Suppose that the ear is presented with an equal amplitude sum of two sinusoidal oscillations of different frequencies (these could be, e.g., two of the harmonics in the Fourier decomposition of a "note" played on an instrument). Then

$$F_{in}(t) = \sin \omega_1 t + \sin \omega_2 t \qquad \omega \equiv 2\pi f$$

Then the response of the eardrum is

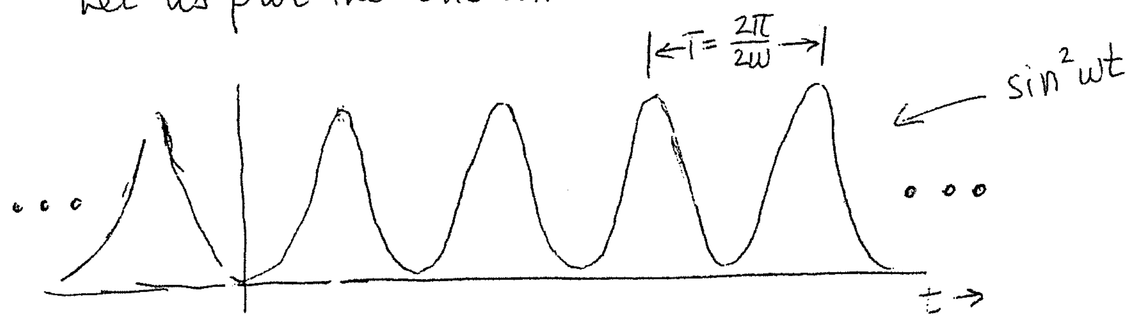
$$x(t) = a(\sin \omega_1 t + \sin \omega_2 t) + b(\sin \omega_1 t + \sin \omega_2 t)^2$$

$$= a \sin \omega_1 t + a \sin \omega_2 t + b \sin^2 \omega_1 t + b \sin^2 \omega_2 t$$
$$+ 2b \sin \omega_1 t \sin \omega_2 t$$

How do we treat the "$\sin^2$" terms?

First, do we expect that a term $\sin^2 \omega t$ to be a pure single frequency harmonic oscillation?

Let us plot the function:



This is technically not a pure sin or cos function* even though it is a repetitive function with defined repeat period π/ω; therefore, it can be Fourier analyzed!

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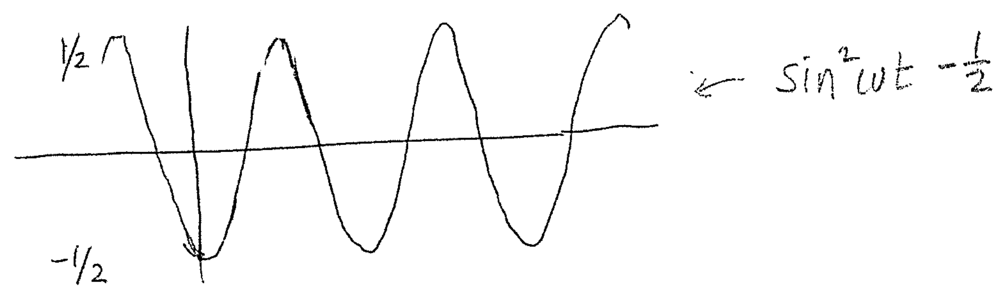
* although, as we will see on the next page, it is a pure sin or cos shape.

In this case, the Fourier analysis is very simple. We note that $\sin^2 \omega t$ oscillates between zero and one symmetrically; hence its average value is $\frac{1}{2}$. Expanding in a cosine series then (since $\sin^2\omega t$ is an even fcn. of t),

$$\sin^2 \omega t = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_1 t$$

where ω_1 is the lowest frequency in the expansion, we have $\frac{a_0}{2} = \frac{1}{2}$.

Subtracting $\frac{1}{2}$,



the function plots exactly as $-\frac{1}{2}\cos(2\omega t)$; hence $\sin^2\omega t - \frac{1}{2} = -\frac{1}{2}\cos(2\omega t)$, or

$$\sin^2 \omega t = \frac{1}{2} - \frac{1}{2}\cos(2\omega t)$$

(Thus, the Fourier cosine series has only two terms. You can confirm that this expansion is correct by trig identity:

$$\cos(2\omega t) = \cos\omega t \cos\omega t - \sin\omega t \sin\omega t$$

$$= \cos^2\omega t - \sin^2\omega t \Rightarrow 1 - \cos(2\omega t) = 1 - \cos^2\omega t + \sin^2\omega t = 2\sin^2\omega t$$

so $\frac{1}{2}[1 - \cos(2\omega t)] \stackrel{\checkmark}{=} \sin^2\omega t.$]

$= \text{constant} + \text{term} \propto \cos(2\omega t).$

Thus, We note the presence in the output of the frequencies $2\omega_1$ and $2\omega_2$, both of which were not present in the input and which are harmonics of the input frequencies ω_1 and ω_2. In an electronic system, this is called <u>harmonic distortion</u> (or, "harmonic generation").

Consider now the output term $2b \sin \omega_1 t \sin \omega_2 t$.

By trig identity this is

$$2b \sin \omega_1 t \sin \omega_2 t = b \cos[(\omega_2 - \omega_1)t] - b \cos[(\omega_1 + \omega_2)t]$$

Note, then, that these response terms are sinusoidal at frequences $f_2 - f_1$ and $f_1 + f_2$ that also are not present in the input. In an electronic system (like a stereo amplifier or speaker) this is called <u>intermodulation (IM) distortion</u>. Unlike harmonic distortion, which is not unpleasant sounding (because it is an octave higher, but the same "note", as the original), IM distortion usually sounds quite unpleasant, unless it is used to provide an implied, but missing in the input, fundamental.

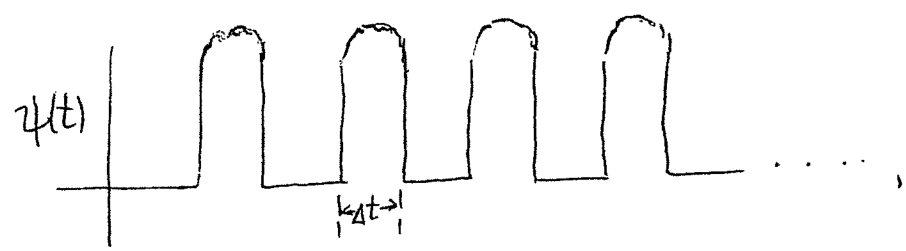
Now let us return to our bassoon playing the "58 Hz note".

We note from the Fourier analysis that the 12^{TH} and 13^{th} harmonics are present in the input to the eardrum; therefore, the quadratic nonlinearity in the eardrum produces the difference frequency $13 f_0 - 12 f_0 = f_0$, which the brain then latches onto as the fundamental.

C. We looked at examples in the physics of music; now we look at some further examples of an important, but different type:

1. Consider a sound that is repeatedly on and off – e.g., someone banging a drum regularly or clapping their hands regularly. [K-text, chapter 12, sect 12.1]

Since the sound is periodic, it can be thought of as being "composed of" many single frequency sinusoids. An idealization of such a function of time (called $\psi(t)$ here) is shown.



The process of Fourier analyzing such a waveform can be referred to as "unbeating the beats". Why do you think this is?

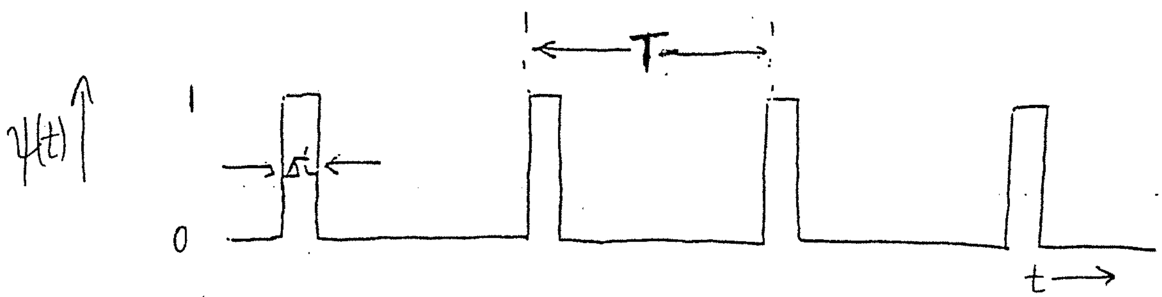
2. Consider the pulsed laser, a common research lab instrument. Such a laser puts out identical bursts of light – say one burst every second. A plot of the output might look (schematically) as shown above. Can the output of such a pulsed laser be truly monochromatic? No. Why not? However, as we will see, the extent of the "nonmonochromaticity" ("bandwidth") depends on the "on-time" for each pulse (Δt in diagram above).

Example: Let us work out an example in detail. The result, it will turn out, is a special case of a very general and important theorem called the Classical Bandwidth theorem (or, by some, the "Fourier Bandwidth Thm.")

Example: Square-Top Repetitive Wave

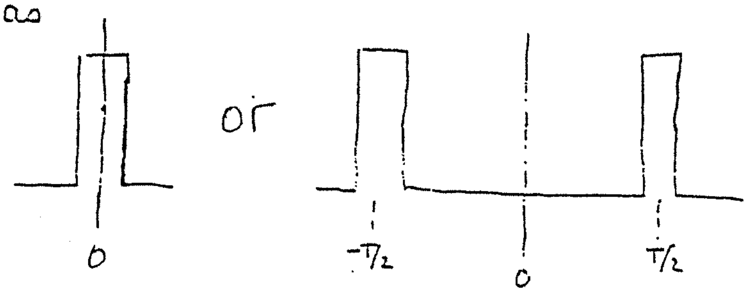
(e.g., sound of someone repetitively clapping their hands).

A plot of the signal as a function of time is shown:



Let us Fourier analyze this signal. We outline the steps:

(a) Choose origin as

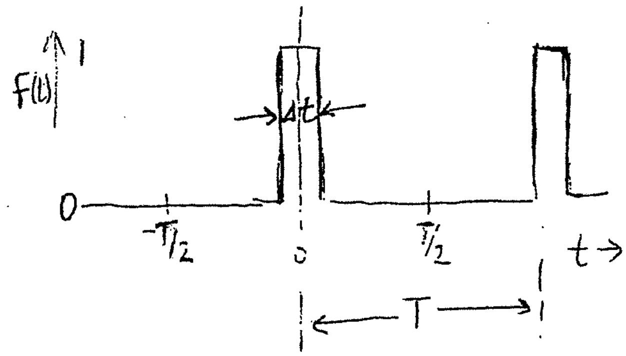


$\Rightarrow$ only cosines appear in the series. (Why?)

Now calculate Fourier coefficients:

$\frac{a_0}{2}$ = average of $F(t)$ over one period

$$= \Delta t / T_1 = \frac{\Delta t}{T}$$



$$a_n = \frac{1}{T/2} \int_{-T/2}^{T/2} F(t) \cos n\omega_1 t \, dt$$

$$= \frac{2}{T/2} \int_{0}^{\Delta t/2} F(t) \cos n\omega_1 t \, dt$$

$$= \frac{4}{T} \int_{0}^{\Delta t/2} \cos(n\omega_1 t) \, dt =$$

$$= \frac{2\cancel{\omega_1}}{\pi} \cdot \frac{1}{n\cancel{\omega_1}} \sin(n\omega_1 t) \Big|_0^{\Delta t/2}$$

$$= \frac{2}{n\pi} \sin\left(n \frac{\cancel{2}\pi}{T_1} \cdot \frac{\Delta t}{\cancel{2}}\right), \quad T_1 = T,$$

or

(1) $$\boxed{a_n = \frac{2}{n\pi} \sin\left(n\pi \frac{\Delta t}{T}\right)}$$

From eqn. (1), note that, other than the running index n, a_n depends only on the quantity $\frac{\Delta t}{T}$, which has physical meaning $\frac{\Delta t}{T}$ = fractional on-time (for this example).

We can also write eqn. (1) in terms of the fundamental frequenc[y] ω_1 as

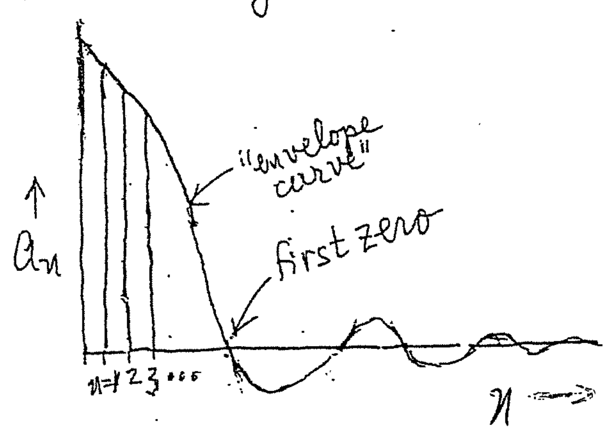
$$(2) \qquad a_n = \frac{2}{n\pi} \sin(n\pi f_1 \Delta t).$$

Now suppose we imagine (for this example) plotting a_n vs. n. Before actually making the plot, let's ask:

<u>What do we expect it to look like?</u>

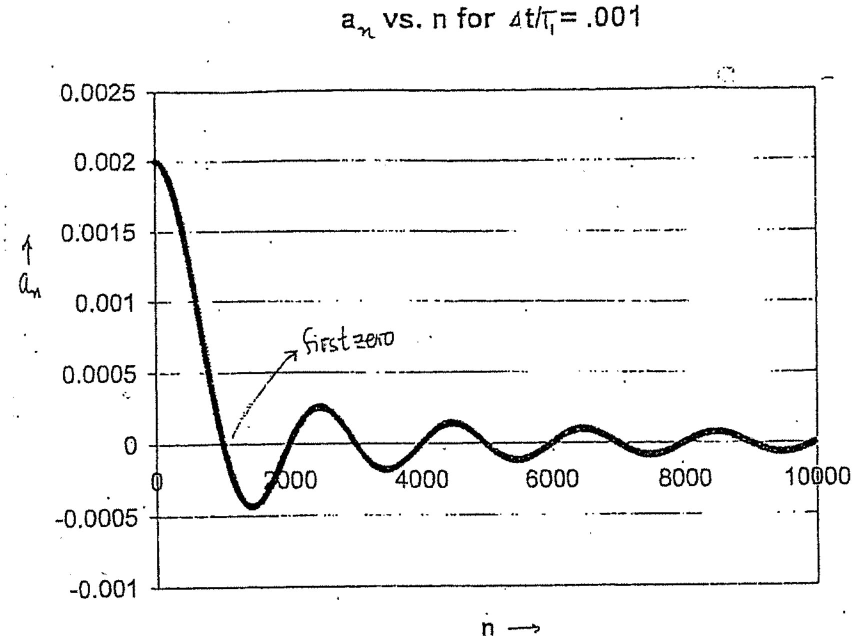
<u>Answer</u>: Since a_n's are discrete ($n = 1, 2, 3, \ldots$), our plot takes the form of a series of "spikes". Since the "$\frac{1}{n}$ piece" in front of [eqn (1) or (2)] decreases the amplitude of the sine function as n increases, we expect something like:

Our plot looks (schematically) like



Next: Before we look at specific plots, mentally consider three cases: $\frac{\Delta t}{T} = 0.5, 0.1, 0.01$. On which plot do you expect the first zero to be at largest n? at smallest n? Why?

Following is the actual plot for the case $\Delta t/T_1 = .001$:



Such a plot (a_n and/or b_n vs. n or, equivalently*, vs. frequency) is called a (Fourier) Spectral Plot; it shows the relative strengths of the Fourier components in the mix.

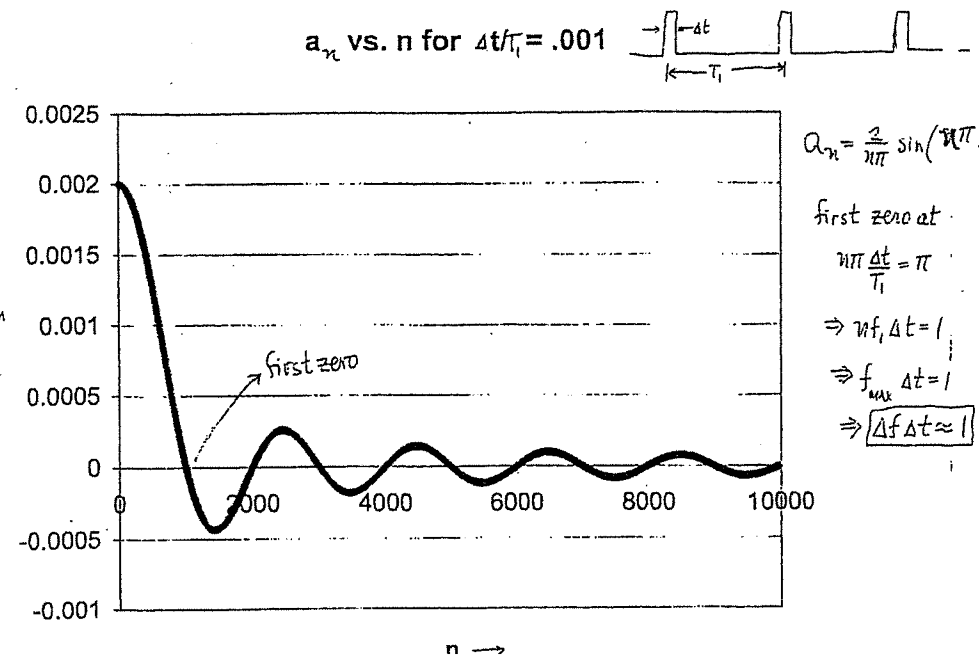
* but see later remarks (e.g. bottom of next page)

Approximation of Fourier Series [K-text, sect. 12.2]

Now each a_n is the coefficient of a frequency nf_1 sinusoid in the composition of $f(t)$. Suppose that we do not wish to calculate the entire series. We can approximate the series in the following way: We can (a bit arbitrarily) say that all terms in the Fourier series out to that that corresponds to (or is closest to) the first zero (see plot above) in a_n vs. n are the "main contributors," and all beyond the first zero are "approximately ignorable."

For this purpose, let us locate the first zero:

We show the spectral plot again:



$a_n = \frac{2}{n\pi}\sin\left(n\pi\frac{\Delta t}{T_1}\right)$

first zero at

$n\pi\frac{\Delta t}{T_1} = \pi$

$\Rightarrow nf_1\Delta t = 1$

$\Rightarrow f_{max}\Delta t = 1$

$\Rightarrow \boxed{\Delta f \Delta t \approx 1}$

(see next page)

As remarked above, if we plot a_n vs. n, the shape of the plot (including the value of n that is closest to the first zero for a_n) depends only on the fractional on-time $\Delta t/T$.

Let's investigate an aspect of this in more detail:

<u>Plot of a_n vs. n, Location of "First zero"</u> (repetitive flat-topped function)

From eqn. (1)*, the envelope function is zero first when

$\sin\left(n\pi \frac{\Delta t}{T}\right) = 0 \Rightarrow n\pi \frac{\Delta t}{T} = \pi$

Call the value of n nearest to where the envelope function has its first zero, "n_{MAX}". Then the eqn. aboves is

(3) $\underline{n_{MAX} = \left(\frac{\Delta t}{T}\right)^{-1}}$ [Exact for repetitive flat-topped function only].

With that result, we can now understand the relative shapes of the plots following:

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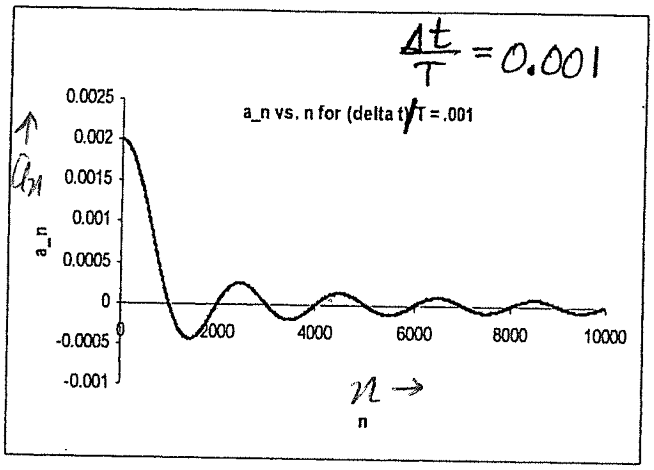
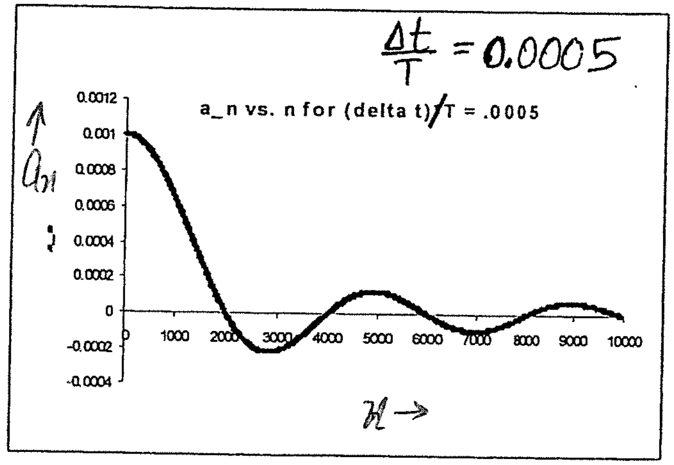
* "Eqn. (1)" is (1) $a_n = \frac{2}{n\pi}\sin\left(n\pi\frac{\Delta t}{T}\right)$, which can be written as

(2) $a_n = \frac{2}{n\pi}\sin(n\pi f_1 \Delta t)$.

Suppose we halve the on-fraction $\frac{\Delta t}{T}$. Then, if we plot a_n vs. n, we should see that $n_{MAX} = n_{1ST\ zero}$ doubles.

The following plots show this:





(Do bear in mind that, although the curves look continuous, they actually are just densely packed "dots" since a_n is defined only for integer values of n.)

On the other hand, if we plot a_n vs. f (frequency), the first zero of "a_f" occurs at $f = \frac{1}{\Delta t}$, regardless of the repeat period T of the function - see next page:

"Cutoff Frequency" ("f_{MAX}") Included in Approximate Truncated Series

For any n, the frequency of the associated term in the Fourier series is $f_n = n f_1$.

Suppose then, that we plot a_n vs. not n, but f (frequency). What value of f is nearest the first-zero of the envelope function on this plot?

Our condition for the first zero of a_n was (by eqn. (3)),

$$(4) \quad n_{MAX} \pi f_1 \Delta t = \pi \qquad \left[(3) \text{ is } n_{MAX} = \left(\frac{\Delta t}{T}\right)^{-1} \right]$$

Now $f_{MAX} = \underbrace{f(n_{MAX})}_{\text{"f of } n_{MAX}\text{"}} = n_{MAX} f_1$

$\Rightarrow$ for first zero, $f_{MAX} \Delta t = 1$, or

$$(5) \qquad \boxed{f_{MAX} = \frac{1}{\Delta t}}$$

(Exact for this repetitive flat-topped function only)

Thus, the position of the first zero on a plot of a_n vs. f does not depend on the repeat time T(!) but only on Δt.

D. Generalization to Class of Cases with $\frac{\Delta t}{T}$ small ($<<1$) [K-text, sect. 12.3]

We now consider a very important class of cases - that for which $\Delta t << T_1$. This would be the case if, for example, you clap your hands once per second and $\psi(t)$ is the resulting sound - each clap makes a sound that lasts for about 1 ms or so.

(Cases of narrow pulses that are very separated cf pulse width).

Then the condition for the first zero,

$$n_{MAX} \pi f_1 \Delta t = \pi \Rightarrow n_{MAX} f_1 \Delta t = 1 \Rightarrow n_{MAX} \frac{\Delta t}{T_1} = 1$$

tells us that n_{MAX} is $>>1$ (since $\frac{\Delta t}{T_1} << 1$).

This means that $f_{MAX} >> f_1$. Thus, defining the "frequency bandwidth"

$$\Delta f \equiv f_{MAX} - f_1 \approx f_{MAX}$$

[(2) is $f_{MAX} = n_{MAX} f_1 = \frac{1}{\Delta t}$]

Then (2) becomes

(3) (3a) $\boxed{\Delta f \approx \frac{1}{\Delta t}}$

or

(3b) $\boxed{\Delta f \, \Delta t \approx 1}$

Fourier Bandwidth Theorem* Result, Function of time, Special Case (*sometimes called the "Classical Bandwidth Theorem")

(3) is an example of the famous "Fourier Bandwidth Theorem". As we will see, as an order of magnitude relation, it is quite general, applying to most sharply peaked repetitive functions.

From eqn. (3) in the form

$$\Delta f \approx \frac{1}{\Delta t}$$

we see that the frequency bandwidth Δf and the "pulse width" Δt are inversely proportional to each other. That inverse proportionality of the frequency bandwidth with Δt is very general for repetitive peaked functions for which $\Delta t \ll T$ (T = repeat time) – far more general than just the example considered.

Another way of writing eqn. (3) is:

$$\Delta f \, \Delta t \approx 1$$

$$\Rightarrow (2\pi) \Delta f \cdot \Delta t \approx 2\pi$$

(3c) $$\Rightarrow \boxed{\Delta \omega \cdot \Delta t \approx 2\pi}$$

(next page →

In words, the classical bandwidth theorem tells us that the "bandwidth in frequency" (i.e, the width in frequency of the range of "important" frequencies) in the Fourier synthesis of a repetitive "pulsing" function of time is inversely proportional to the on-time for a single pulse.

Example: Suppose that you clap your hands repetitively, each clap lasting 1 ms, with claps separated by 1 second. Then:

The lowest (nonzero) frequency in the Fourier synthesis of the sound is $f_1 = \frac{1}{T_1} = 1\,Hz$. As always, f_1 is the inverse of the repetition time t, since $f_1 = 1/T_1 = 1/T$.

The highest "important" frequency in the Fourier synthesis of the sound is (by (3))

$$f_{MAX} \approx \frac{1}{\Delta t} = \frac{1}{10^{-3}s} = 1000\,Hz$$

(approximate since pulses are not exactly flat-topped!). So

$$\Delta f \approx f_{MAX} \approx \frac{1}{\Delta t} = 1000\,Hz.$$