$$f(x) = \begin{cases} \left|\int x\sin\left(\frac{1}{x}\right)dx\right| & \text{si } x \neq 0 \\ \frac{\pi}{4} + x^2 & \text{si } x = 0 \end{cases}$$

$\lim_{x\to 0} \left|\int x\sin\left(\frac{1}{x}\right)dx\right| = \frac{\pi}{4}$ 1) y 2)

$\Rightarrow f$ continua en $\mathbb{R}$

1) Sea $Si(x) \equiv$ seno integral, donde $Si(t) = \int \frac{\sin t}{t}dt$

$$\int x\sin\left(\frac{1}{x}\right)dx \overset{\bullet}{=} -\int x^3\sin(t)\,dt \overset{\bullet}{=} -\int \frac{\sin(t)}{t^3}dt = -\left(-\frac{\sin t}{2t^2} - \int \frac{-\cos t}{2t^2}dt\right) = -\left(-\frac{\sin t}{2t^2} + \frac{1}{2}\int\frac{\cos t}{t^2}\right) \overset{\bullet}{=} -\frac{1}{2}\left[-\frac{\sin t}{t^2} - \frac{\cos t}{t} - \underbrace{\int \frac{\sin t}{t}dt}_{Si(t)}\right] =$$

$t = \frac{1}{x}$

$dt = -\frac{1}{x^2}dx \Leftrightarrow dx = -x^2dt$

$u = \sin(t) \to u' = \cos(t)$

$v' = t^{-3} \to v = -1/2t^2$

$\int uv' = uv - \int vu'$

$u = \cos(t) \to u' = -\sin(t)$

$v' = t^{-2} \to v = -t^{-1}$

$$= \frac{\sin(t) + t\cos(t) + t^2 Si(t)}{2t^2} = \frac{1}{2}\left[x^2\sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right) + Si\left(\frac{1}{x}\right)\right] + C_1 = g(x)$$

2) ¿$\lim_{x\to 0} |g(x)|$?

$$\lim_{x\to 0} |g(x)| = \frac{1}{2}\left[\lim_{x\to 0}\left|x^2\sin\left(\frac{1}{x}\right)\right| + \lim_{x\to 0}\left|x\cos\left(\frac{1}{x}\right)\right| + \lim_{x\to 0}\left|Si\left(\frac{1}{x}\right)\right|\right] + \lim_{x\to 0} x^2 \overset{\bullet\bullet}{=} \frac{1}{2}\lim_{x\to 0}\left|Si\left(\frac{1}{x}\right)\right| = \frac{1}{2}\left|\lim_{x\to 0} Si\left(\frac{1}{x}\right)\right| =$$

$\lim_{x\to 0}\left|x^2\sin\left(\frac{1}{x}\right)\right| \leq \lim_{x\to 0} x^2\cdot 1 = 0 \Leftrightarrow 0 \leq \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \leq 0 \Leftrightarrow \lim_{x\to 0}\left|x^2\sin\left(\frac{1}{x}\right)\right| = 0$ (Para $C_1 = x^2$)

$\lim_{x\to 0}\left|x\cos\left(\frac{1}{x}\right)\right| \leq \lim_{x\to 0} x\cdot 1 = 0 \Leftrightarrow 0 \leq \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) \leq 0 \Leftrightarrow \lim_{x\to 0}\left|x\cos\left(\frac{1}{x}\right)\right| = 0$

$$= \frac{1}{2}\left|Si\left(\lim_{x\to 0}\frac{1}{x}\right)\right| = \frac{1}{2}\left|Si(\infty)\right| \overset{\bullet}{=} \frac{\pi}{4}$$

$Si(+\infty) = +\frac{\pi}{2}$ y $Si(-\infty) = -\frac{\pi}{2} \Rightarrow |Si(\pm\infty)| = \frac{\pi}{2}$

Algunas propiedades de la integral senoidal son:

- Al ser la integral de una función par, es una función impar, esto es, Si(-x) = -Si(x).
- El valor de Si(x) cuando x tiende a infinito es el límite:

$$\lim_{x\to\infty} \mathrm{Si}(x) = \int_0^\infty \frac{\sin(t)}{t}dt = \frac{\pi}{2}$$

Asimismo, el valor de Si(x) cuando x tiende a menos infinito es $-\frac{\pi}{2}$.

(Aplicando el Teorema Fundamental del Cálculo $\Leftrightarrow$ (sea $F(x) = \int f(x)dx \Leftrightarrow F'(x) = f(x)$)

- $$f'(x) = \begin{cases} \left|x\sin\left(\frac{1}{x}\right)\right| & \text{si } x \neq 0 \\ 2x & \text{si } x = 0 \end{cases}$$

$\lim_{x\to 0}\left|x\sin\left(\frac{1}{x}\right)\right| \leq \lim_{x\to 0} x\cdot 1 = 0 \Leftrightarrow 0 \leq \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \leq 0 \Leftrightarrow \lim_{x\to 0}\left|x\sin\left(\frac{1}{x}\right)\right| = 0$

$\Rightarrow f'$ continua en $\mathbb{R}$

- $$f''(x) = \begin{cases} \left|\sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}\right| & \text{si } x \neq 0 \\ 2 & \text{si } x = 0 \end{cases}$$

$\lim_{x\to 0}\left|\sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}\right| = \nexists$

$\Rightarrow f''(x)$ no es continua en 0

51. Sea $f : I \to \mathbb{R}$ y a un punto del intervalo abierto I. Supongamos que f es dos veces derivable en I y que $f''(a) \neq 0$. Si para cada h, $\theta(h)$ representa el punto del intervalo $(0, 1)$ dado por la Fórmula de Lagrange

$$f(a+h) - f(a) = f'(a + \theta(h)h)h,$$

demostrar que

$$\lim_{h\to 0}\theta(h) = \frac{1}{2}.$$

$$f(x) = \begin{cases} \left|\int x\sin\left(\frac{1}{x}\right)dx\right| & x \neq 0 \\ \frac{\pi}{4} + x^2 & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \left|x\sin\left(\frac{1}{x}\right)\right| & x \neq 0 \\ 2x & x = 0 \end{cases}$$

$$f''(x) = \begin{cases} \left|\sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}\right| & x \neq 0 \\ 2 & x = 0 \end{cases}$$

$\lim_{h\to 0}\left|\int x\sin\left(\frac{1}{x}\right)dx\right| = \frac{\pi}{4}$

$\left(f(1) = \frac{\sin(1) + \cos(1) + Si(1)}{2}\right)$

Suponemos que $\lim_{h\to 0}\theta(h) = \frac{1}{2}$ y tratamos de demostrar por reducción al absurdo, para $\boxed{a=1}$

$$\lim_{h\to 0} f(1+h) - f(1) = \lim_{h\to 0} f'(1+\theta(h)h)h \Leftrightarrow \lim_{h\to 0} f(1+h) - \frac{\sin(1)+\cos(1)+Si(1)}{2} = \lim_{h\to 0} f'\left(1+\frac{1}{2}h\right)h \Leftrightarrow$$

$$\lim_{h\to 0} f(h+1) = \frac{1}{2}\left[\lim_{x\to 0}\left|(x+1)^2\sin\left(\frac{1}{x+1}\right)\right| + \lim_{x\to 0}\left|(x+1)\cos\left(\frac{1}{x+1}\right)\right| + \lim_{x\to 0}\left|Si\left(\frac{1}{x+1}\right)\right|\right] + \lim_{x\to 0} x^2 = \frac{\sin(1)+\cos(1)+Si(1)}{2} - \frac{\sin(1)+\cos(1)+Si(1)}{2} = 0 =$$

$$= \lim_{h\to 0} h\sin(1) = 0 \quad \overset{!!}{\cap}$$

$\boxed{a=0}$

$$\lim_{h\to 0} f(h)-f(0) = \lim_{h\to 0} f'(\theta(h)h)h \iff \lim_{h\to 0} f(h) - \frac{\pi}{4} = \lim_{h\to 0} f'(0)h \iff \frac{\pi}{4}-\frac{\pi}{4} = \lim_{h\to 0} 2ah = 0$$

$\boxed{a=2}$

$$\lim_{h\to 0} f(2+h)-f(2) = \lim_{h\to 0} f'(2+\theta(h)h)h \iff \lim_{h\to 0} f(2+h) - \frac{1}{2}\left[4\sin\left(\tfrac{1}{2}\right)+2\cos\left(\tfrac{1}{2}\right)+Si\left(\tfrac{1}{2}\right)\right] = \lim_{h\to 0} f'\left(2+\tfrac{1}{2}h\right)h \iff$$

$$\lim_{h\to 0} f(h+2) = \frac{1}{2}\left[\lim_{x\to 0}\left|(x+2)^2\sin\left(\tfrac{1}{x+2}\right)\right| + \lim_{x\to 0}\left|(x+2)\cos\left(\tfrac{1}{x+2}\right)\right| + \lim_{x\to 0}\left|Si\left(\tfrac{1}{x+2}\right)\right|\right] + \lim_{x\to 0} x^2 = \frac{4\sin\left(\frac{1}{2}\right)+2\cos\left(\frac{1}{2}\right)+Si\left(\frac{1}{2}\right)}{2} - \frac{4\sin\left(\frac{1}{2}\right)+2\cos\left(\frac{1}{2}\right)+Si\left(\frac{1}{2}\right)}{2} = 0 =$$

$$= \lim_{h\to 0} \sin(2)h = 0$$

No consigo encontrar una a tal que no se cumpla