Waves
Homework Set 1

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Question:

Consider the function

$$x(t) = A\cos(bt) \tag{1}$$

Where A and b are both arbitrary positive constants.

By direct substitution, show that, for any constant A, $x(t) = A\cos(bt)$ is a solution which "satisfies" the differential equation:

$$\frac{d^2x(t)}{dt^2} = -\omega^2 x(t) \tag{2}$$

if and only if $b = \sqrt{\frac{k}{m}}$. Show all steps of the logic.

The question is asking us to show that

$$(1) \text{ is a solution of } (2) \iff b = \sqrt{\frac{k}{m}} \qquad \text{assuming that } \omega = \sqrt{\frac{k}{m}}$$

We are essentially going to proof that $\omega = b$, so b will be equal to whatever ω we set for our harmonic oscillator.

($\Longrightarrow$)

If we calculate the derivatives of (1) we get:

$$x(t) = A\cos(bt)$$
$$x'(t) = -Ab\sin(bt)$$
$$x''(t) = -Ab^2\cos(bt)$$

Then, as we are assuming that (1) is a solution of (2), we can substitute the second derivative that we have just calculated into (2), then substitute our solution for $x(t)$ (1) too.

$$-Ab^2\cos(bt) = -\omega^2 x(t) \iff -Ab^2\cos(bt) = -\omega^2 A\cos(bt)$$

If we then divide both sides of our equality by ${}^{-1}/_{A\cos(bt)}$ we get

$$b^2 = \omega^2 \iff b = \omega = \sqrt{\frac{k}{m}}$$

($\Longleftarrow$)

Now, we will assume that $b = \sqrt{\frac{k}{m}}$ and check if (1) is still a solution of (2).
Firstly, we will substitute b into our solution candidate and calculate its derivatives:

$$x(t) = A\cos\left(t\sqrt{k/m}\right)$$
$$x'(t) = -A\sqrt{\frac{k}{m}}\sin\left(t\sqrt{k/m}\right)$$
$$x''(t) = -A\frac{k}{m}\cos\left(t\sqrt{k/m}\right)$$

Finally, we will check if this is a still a solution to the ODE by substituting the derivative and the solution candidate into the ODE:

$$\frac{d^2x(t)}{dt^2} = -\omega^2 x(t) \iff -A\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right) = -A\omega^2\cos\left(\sqrt{\frac{k}{m}}t\right) \iff \omega^2 = \frac{k}{m} \iff \omega = \sqrt{\frac{k}{m}}$$

Which we can see is true for our assumptions.

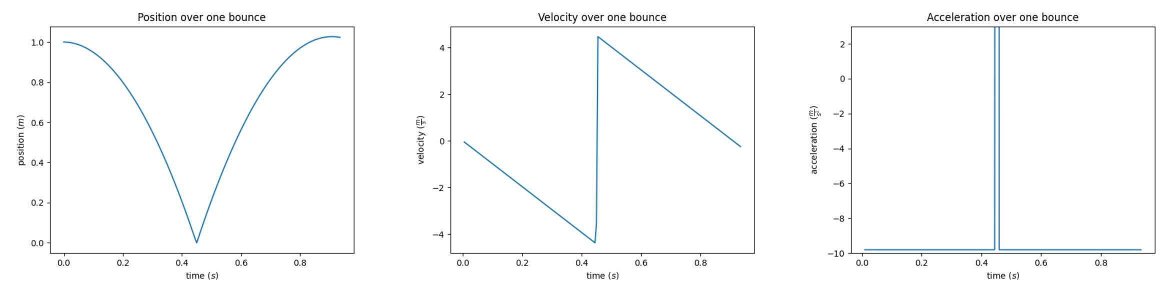
We have proof of both implications, and thus the logic of the exercise has ended.

Question:

Let's return to one of the questions I raised at the beginning of this chapter: Suppose you drop a rubber ball from rest at altitude $1m$ and let it fall vertically straight down. Suppose it bounces perfectly elastically off a smooth hard surface at altitude zero and suppose that we ignore air resistance. Then the ball will repeatedly oscillate in altitude between extremes of zero and $1m$. Is the motion shm? Explain how you know showing all steps of the logic.

No, a bouncing ball would not be a simple harmonic oscillator. Although it is definitelly a periodic movement, the position changes abruptly whenever the ball reaches the floor. This movement can't be represented with a sinusoidal function.

If we plot of the position over the timespan it takes the ball to return to the original point we can see that abrupt change. Velocity has an infinite slope around the bounce time but is constant otherwise. Because of that acceleration is constant besides a delta at the bounce time.



The plots have been calculated using the Uniformly Accelerated Rectilinear Motion equations into a python code that is included as an addenda at the end of the document. An animation was made too, but it couldn't be included in a printed format.

$$\begin{cases} x(t) = x_0 + v_0 t + \frac{1}{2} a t^2 \\ v(t) = v_0 + a t \end{cases} \tag{3}$$

To calculate the time it takes the ball to achieve the soil, we set $x(t) = 0$ and solve for t:

$$x(t) = 0 \Longleftrightarrow 1 - \frac{g}{2} t^2 = 0 \Longleftrightarrow t = \sqrt{\frac{2}{g}} = 0.452 \ s$$

Another way to proof this is to note that $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$ does not satisfy the governing equation (ODE) of Simple Harmonic Oscillators.

Question:

A mass m that carries positive charge Q is released at time $t = 0$, with initial velocity $v_0 = 0$ from position $(x_0, 0, 0)$ in an externally imposed electric field $\vec{E} = -Cx\hat{x}$, where C and x_0 are both positive constants with proper units. The charge Q has no measurable effect on the external field. Ignoring gravity and any effects other than those caused by the applied field.

1. Derive the governing differential equation for the motion of the charge Q. Clearly show all steps of the derivation.
2. Will the position of the charge oscillate? If no, state exactly why not and describe the motion that occurs after release. If yes: Will the oscillation be simple harmonic motion? Exactly why or why not? If yes, what is the angular frequency of the oscillation?
3. Suppose that the initial velocity v0 is in the plus or minus x-direction, not zero, but otherwise arbitrary. Will the position of the charge oscillate? Why or why not? If yes, is the oscillation shm? Exactly why or why not?

Let's start by stating Newton's second law and the electric force exerted on a charge.

$$F = ma \qquad F = QE$$

As we know the value of E_x, we can be sure that:

$$F_x = QE_x = -CQx\hat{x} = ma_x = mx'' \iff \boxed{\frac{d^2x}{dt^2} = -QCx}$$

The position of the charge will oscillate around the origin, as when it goes far from it, the electric force appears and pulls it back to the origin, where it overshoots and starts the cycle again. As this differential equation has the same form as the governing equation for simple harmonic motion, we can state that it will be a SMH. Moreover, it is easy to verify that the ODE has a solution of the form

$$x(t) = A\cos(\omega t + \phi)$$

Which is the one of a simple harmonic oscillator. From the ODE itself we can extract the value of the frequency:

$$\omega^2 = \frac{QC}{m} \iff \boxed{\omega = \sqrt{\frac{QC}{m}}}$$

If we add an initial velocity to the problem, the position of the charge will still describe simple harmonic oscillation, as we are only fixing one initial condition to the governing ODE. In this case this condition is that $v(0) = v_0$ which is the same as saying that $x'(0) = v_0$. The other possible initial condition is the one we are already given: the initial position of the mass, x_0.

Question:

Suppose a particle that can move only along the x-axis is subject to a return force that varies in proportional to the *cube* of the distance from the origin ($x = 0$), *i.e.*, $F(x) = -Cx^3$, where C is a positive constant. The particle is released from rest at a point with coordinate $x_0 > 0$. Will the position of the charge oscillate? How do you know? If yes: Will the oscillation be simple harmonic motion? Exactly why or why not?

This question is analogous to the previous one. If we define the particle's mass as m, then equal Newton's Second Law to the return force given by the question, we get:

$$F = ma = -Cx^3 \iff \boxed{\frac{d^2x}{dt^2} = -\frac{C}{m}x^3}$$

Although the particle will still oscillate, this time the governing equation does not ressemble SHM, and thus the movement will not be SHM. A solution of the form $x(t) = A\cos(\omega t + \phi)$ does not solve the ODE.

$$x'(t) = -A\omega\sin(\omega t + \phi)$$
$$x''(t) = -A\omega^2\cos(\omega t + \phi)$$
$$\boxed{-A\omega^2\cos(\omega t + \phi) \neq \frac{C}{m}A^3\cos^3(\omega t + \phi)}$$

Question:

Hanging a $2kg$ mass from a vertical spring lengthens the spring by $25cm$. While still attached to the spring, the mass is then pulled down an additional $20cm$ and released from rest. How long will it take to reach its maximum vertical height above the starting point?
Note that there's a new feature included in this exercise - the force of gravity. Thus, the governing differential equation would appear to be not of the form $x'' = -\omega^2 x$. (What is the governing differential equation?) Experiments tell us that simple harmonic motion around a new (or "displaced") equilibrium point results, but that the frequency is the same as it is with gravity "turned off." For this exercise, assume that that's correct.

We'll start doing a forces analysis and deriving from there the differential equation.

$$F = -kx - mg = ma \Longleftrightarrow \boxed{x''(t) = -\frac{k}{m}x(t) - g}$$

The solution to this edo is not the one of simple harmonic motion, but one very simillar, as we need to cancel that g dampening with a factor, giving us the solution:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{gm}{k} \qquad x'(t) = -\sqrt{\frac{k}{m}}A\sin\left(\sqrt{\frac{k}{m}}t\right) \qquad x''(t) = -\frac{k}{m}A\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Which satisfies the ode:

$$x''(t) = -\frac{k}{m}x(t) - g \Longleftrightarrow -\frac{k}{m}A\cos\left(\sqrt{\frac{k}{m}}t\right) = -\frac{k}{m}\left(A\cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{gm}{k}\right) - g = -\frac{k}{m}A\cos\left(\sqrt{\frac{k}{m}}t\right)$$

We can obtain the k constant of the spring by using the elongation given to us:

$$F_g = F_k \Longleftrightarrow mg = k\Delta x \Longleftrightarrow k = \frac{mg}{\Delta x} = 8g$$

If we now substitute $A = 0.45m$, $k = 8g$, $m = 2kg$ into the velocity equation ($x'(t)$) and we equal to 0, we will obtain the times at which the position achieves its peak.

$$x'(t) = 0 \Longleftrightarrow -\sqrt{\frac{k}{m}}A\sin\left(\sqrt{\frac{k}{m}}t\right) = 0 \Longleftrightarrow -0.45\sqrt{4g}\sin\left(2\sqrt{g}t\right) = 0 \Longleftrightarrow \sin(2\sqrt{g}t) = 0 \Longleftrightarrow$$

$$\Longleftrightarrow 2\sqrt{g}t = \pi n \Longleftrightarrow \boxed{t = \frac{\pi}{2\sqrt{g}}n, n \in \mathbb{N} \cup 0}$$

As we know that the maximum vertical height will be gained the first time it oscilates upwards, the value of n that interests us is $n = 1$ and thus $t = \frac{\pi}{2\sqrt{g}} = 0.766\ s$

```python
import matplotlib.pyplot as plt
import matplotlib.animation as animation

g = 9.81 # Acceleration due to gravity, m.s-2.
XMAX = 1 # The maximum x-range of ball's trajectory to plot.
YMAX = 1.25
cor = 0.99 # The coefficient of restitution for bounces (-v_up/v_down).
dt = 0.005 # The time step for the animation.

# Initial position and velocity vectors.
x0, y0 = 0, 1
vx0, vy0 = 0, 0

def get_pos(t=0):
    """A generator yielding the ball's position at time t."""
    x, y, vx, vy = x0, y0, vx0, vy0
    while x < XMAX:
        t += dt
        x += vx0 * dt
        y += vy * dt
        vy -= g * dt
        if y < 0:
            y = 0 # bounce!
            vy = -vy * cor
        yield x, y

def init():
    """Initialize the animation figure."""
    ax.set_xlim(-XMAX, XMAX)
    ax.set_ylim(0, YMAX)
    ax.set_xlabel('$x$ /m')
    ax.set_ylabel('$y$ /m')
    line.set_data(xdata, ydata)
    ball.set_center((x0, y0))
    height_text.set_text(f'Height: {y0:.1f} m')
    return line, ball, height_text

def animate(pos):
    """For each frame, advance the animation to the new position, pos."""
    x, y = pos
    xdata.append(x)
    ydata.append(y)
    line.set_data(xdata, ydata)
    ball.set_center((x, y))
    height_text.set_text(f'Height: {y:.1f} m')
    return line, ball, height_text

# Set up a new Figure, with equal aspect ratio so the ball appears round.
fig, ax = plt.subplots()
ax.set_aspect('equal')

# These are the objects we need to keep track of.
line, = ax.plot([], [], lw=2)
ball = plt.Circle((x0, y0), 0.03)
height_text = ax.text(XMAX*0.5, y0*0.8, f'Height: {y0:.1f} m')
ax.add_patch(ball)
xdata, ydata = [], []
```

xdata, ydata = [], []

```python

interval = 1000*dt
ani = animation.FuncAnimation(fig, animate, get_pos, blit=True,
                              interval=interval, repeat=False, init_func=init,save_count=188)
writergif = animation.PillowWriter(fps=30)
ani.save('animation.gif', writer=writergif)

Y=ydata.copy()

for i in range(len(Y)):
    if Y[i]==0:
        bounce = i
        break
for i in range(bounce, len(Y)):
        Y[i] = -Y[i]

Y=np.append(Y,np.negative(Y))
Y=np.append(Y,Y)

plt.figure()
plt.title('"Position" over four bounces')
plt.ylabel('position ($m$)')
plt.xlabel('time ($s$)')
time = np.linspace(0,2*sqrt(2/g)+0.032,len(Y))
plt.plot(time,Y,label='y')
plt.plot(time,np.cos(13.5*time),label='cos(13.5t)',alpha=0.5,linestyle='--')
plt.legend()

dt = time[1]-time[0]
V = []
for i in range(len(Y)-1):
    V.append((Y[i+1]-Y[i])/dt)
A = []
for i in range(len(V)-1):
    A.append((V[i+1]-V[i])/dt)

time = time[::-1]
time = time[:-1]
time = time[::-1]
plt.figure()
plt.title('Velocity over one bounce')
plt.ylabel('velocity ($\\frac{m}{s}$)')
plt.xlabel('time ($s$)')
plt.plot(time,V)

time = time[::-1]
time = time[:-1]
time = time[::-1]
plt.figure()
plt.title('Acceleration over one bounce')
plt.ylabel('acceleration ($\\frac{m}{s^2}$)')
plt.xlabel('time ($s$)')
plt.plot(time,A)

plt.ylim((-10,3))
plt.show()
```