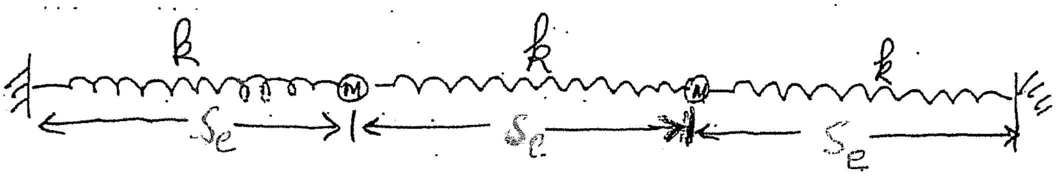
Phys. 251 - 10^{th} Class
Thurs Feb. 8, 2024

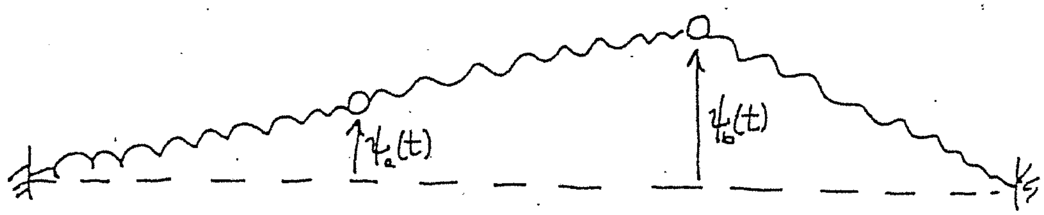
A. Transverse Oscillations of coupled masses-springs system:

Now let us consider the transverse oscillations of the mass-spring system. Are there normal modes of transverse oscillation?

equilibrium



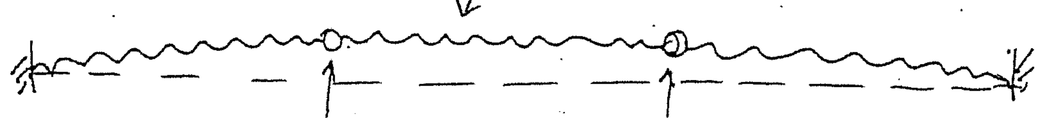
general



Due to the symmetry it's fairly easy to guess the two modes. They are

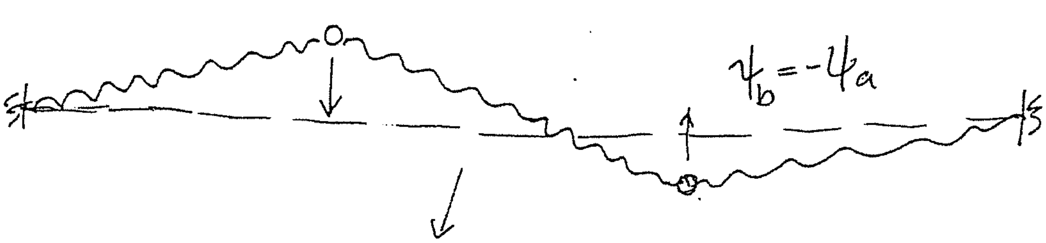
middle spring keeps equil. stretch in normal mode 1. $\psi_a = \psi_b$

mode 1



and

mode 2

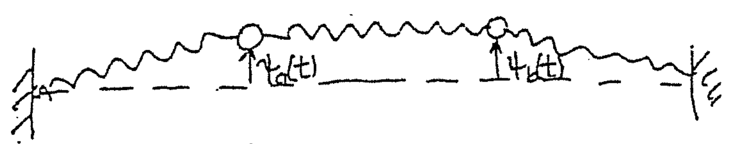


middle spring active in normal mode 2.

Next we find the frequencies of these modes ...

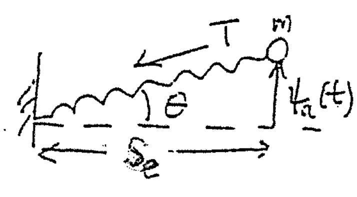
To find the mode frequencies, here we'll use the "quick method" of assuming our guesses on the last page are correct and then using $\omega^2 = $ return force / mass · disp. [Since the motion of either mass in either pure mode is s.h.m, that method ("method of physical meaning of ω^2") must work to provide the mode frequencies; [For homework, you will use the "method of normal coordinates" for the transverse oscillations of this system].

<u>In guessed mode #1,</u> $\psi_a(t) = \psi_b(t)$ at all times and the middle spring always remains at equilibrium stretch (no excess stretch).



Mode 1
$\psi_b = \psi_a$

Thus, for this method, all of the excess force on, say, the left mass comes from the spring that couples it to the wall (and the same is true for the right-hand mass. Thus, in mode 1,



Using $\sin\theta \approx \theta \approx \tan\theta$ <u>in the small oscillations approximation</u>

$$F_{\downarrow} = T\sin\theta \approx T_0\, \psi_a / S_e$$

In small osc. approx., $T \approx T_0$ does not change (hardly) in mag. but of course, direction changes

and, in small osc. approx., $\sin\theta \approx \tan\theta$.

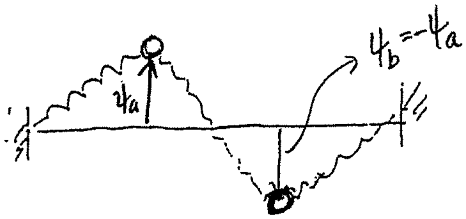
Mode 1
$(\psi_b(t) = \psi_a(t))$

$$\therefore \quad \omega^2 = \frac{\text{return force}}{\text{mass} \cdot \text{disp}} = \frac{T_0\, \psi_a}{S_e\, M \cdot \psi_a} \Rightarrow \boxed{\omega_1^2 = T_0 / M S_e}$$

Normal Mode # 2

Now consider mode 2: on left mass: F_x from left spring $\approx -T_0 \psi_a / s_e$

F_x from ctr. spring $\approx -2T_0 \psi_a / s_e$



$\therefore$ MODE 2: $\boxed{\omega_2^2 = \frac{3T_0}{M s_e}}$, $\psi_b(t) = -\psi_a(t)$

i.e., $\psi_a(t) = A\cos(\omega t + \varphi)$, $\psi_b(t) = -A\cos(\omega t + \varphi)$

Thus, in general, for arbitrary internal transverse motions of the masses within this system, as long as the displacements are "small",

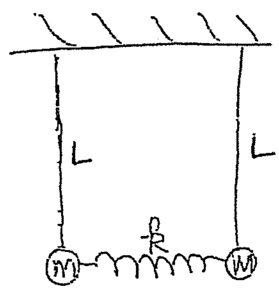
$$\psi_a(t) = A_1 \cos\left(\sqrt{\frac{T_0}{m s_e}}\, t + \phi_1\right) + A_2 \cos\left(\sqrt{\frac{3T_0}{m s_e}}\, t + \phi_2\right)$$

$$\psi_b(t) = A_1 \cos\left(\sqrt{\frac{T_0}{m s_e}}\, t + \phi_1\right) - A_2 \cos\left(\sqrt{\frac{3T_0}{m s_e}}\, t + \phi_2\right)$$

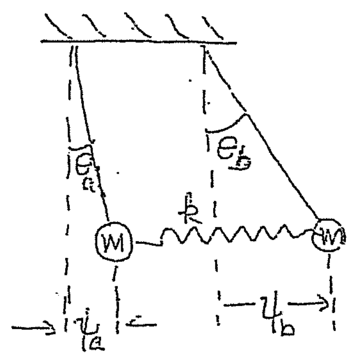
The most general transverse motion (in the "plane of the page") is, then, an arbitrary superposition of both modes going at once with arbitrary amplitudes A_1 and A_2.

B. Further Examples of Normal Modes

1. Coupled Pendula, Equal Masses (Text, sect.



(equilibrium)



What are the equations of motion?

Can work either in θ's or in ψ's. Will arbitrarily choose ψ.

We have you think this through, find, in the small oscillations approx,

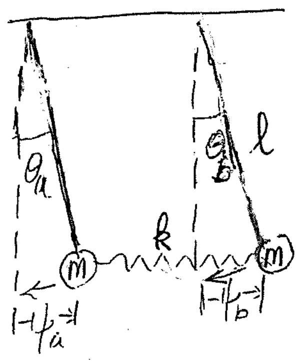
$$(1a)\quad m\ddot{\psi}_a = -m\frac{g}{\ell}\psi_a + k(\psi_b - \psi_a)$$

$$(1b)\quad m\ddot{\psi}_b = -m\frac{g}{\ell}\psi_b - k(\psi_b - \psi_a)$$

(If we decide to "work in the θ's," we get equations of the same form).

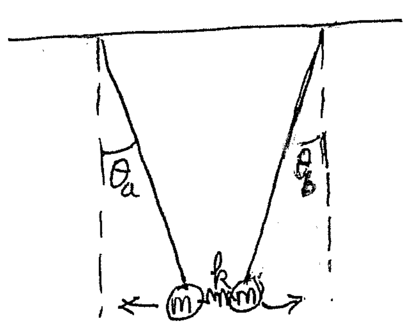
To solve these we can use either the "method of searching for normal coordinates, or we can "guess" by using the "method of physical meaning of ω^2." We will do the latter here:

In the pure first normal mode, both masses swing together in phase at the same frequency; i.e, in this mode $\theta_a(t) = \theta_b(t)$ and $\psi_a(t) = \psi_b(t)$; the frequency is given by



(2a) $\omega_1^2 = \frac{g}{l}$.

In the pure second normal mode, both masses swing $180°$ out of phase with each other (i.e, "against each other"); i.e, in this mode $\theta_b(t) = -\theta_a(t)$ and $\psi_b(t) = -\psi_a(t)$ at all times t; as you can show, the frequency of this mode is given by



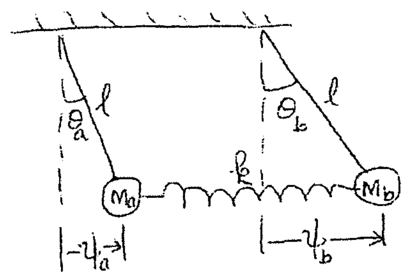
(2b) $\omega_2^2 = \frac{g}{l} + \frac{2k}{m}$.

The most general motion is a general superposition of both modes (each excited with arbitrary amplitude) going at once, i.e,

$$\psi_a(t) = A_1 \cos\left(\sqrt{\tfrac{g}{l}}\, t + \phi_1\right) + A_2 \cos\left(\sqrt{\tfrac{g}{l} + \tfrac{2k}{m}}\, t + \phi_2\right)$$
$$\psi_b(t) = A_1 \cos\left(\sqrt{\tfrac{g}{l}}\, t + \phi_1\right) - A_2 \cos\left(\sqrt{\tfrac{g}{l} + \tfrac{2k}{m}}\, t + \phi_2\right)$$

2. <u>Coupled Pendula, Unequal Masses</u> [Text, Worked Exl. 6.3]

The system is shown (in arbitrary configuration) in the figure:



$m_b \neq m_a$

Fig. 6.10 Coupled pendula of unequal mass

As you can verify, the governing differential eqns. are:

(3a) $$m_a \ddot{\psi}_a = -m_a \frac{g}{\ell} \psi_a + k(\psi_b - \psi_a)$$

(3b) $$m_b \ddot{\psi}_b = -m_b \frac{g}{\ell} \psi_b - k(\psi_b - \psi_a)$$

We will try the method of normal coordinates here.

Adding eqns. (3a) and (3b) we obtain

(4) $$(m_a \psi_a + m_b \psi_b)^{\cdot\cdot} = -\frac{g}{\ell}(m_a \psi_a + m_b \psi_b)$$

which implies that $q_1(t) = m_a \psi_a(t) + m_b \psi_b(t)$

is a normal coordinate. Good.

Now we try to find the second normal coordinate. Suppose we try subtracting eqn. (2b) from eqn. (2a). Then we get

$$(m_a \psi_a - m_b \psi_b)^{\cdot\cdot} = -\frac{g}{\ell}(m_a \psi_a - m_b \psi_b) - 2k(\psi_a - \psi_b),$$

which is not a normal equation!

So, we will have to be a little clever. Suppose we divide eqn. (3a) by m_a, divide eqn. (3b) by m_b, and then subtract. We then get

(5) $$(\psi_a - \psi_b)^{\cdot\cdot} = -\frac{g}{\ell}(\psi_a - \psi_b) - k\left(\frac{1}{m_a} + \frac{1}{m_b}\right)\psi_a - \psi_b)$$

Recalling that the reduced mass μ is defined by

(6) $$\frac{1}{\mu} = \frac{1}{m_a} + \frac{1}{m_b},$$

(6) is

(7) $$(\psi_a - \psi_b)^{\cdot\cdot} = -\left(\frac{g}{\ell} + \frac{k}{\mu}\right)(\psi_a - \psi_b)$$

Eqn. (7) is of normal form, ∴ we can choose our second normal coordinate as $q_2(t) = \psi_a(t) - \psi_b(t)$.

From eqns. (4) and (7) we see that the mode frequencies are

$$(8a) \qquad \omega_1 = \sqrt{\frac{g}{\ell}}$$

$$(8b) \qquad \omega_2 = \sqrt{\frac{g}{\ell} + \frac{k}{\mu}}$$

<u>Question</u>: Suppose that $m_a = m_b$. Then eqn. (8b) should reduce to eqn. (2b). <u>Does it</u>? <u>Justify your answer.</u>

<u>Physical Meanings of The Normal Coordinates</u>

Can we attach a physical meaning to each of the normal coordinates we found in this example? That of $q_2(t) \equiv \psi_a(t) - \psi_b(t)$ is clear; it is just the

of the two masses ("relative coordinate"). That makes sense- in mode 2, the ~~two~~ masses oscillate "against" each other, so the separation distance between them oscillates in s.h.m. So far, so good.

Now, what is the physical meaning of $q_1(t) = m_a \psi_a + m_b \psi_b$?
To answer this question we need to make use of a little theorem:

Theorem: Let $q_\alpha(t)$ be a normal coordinate associated with frequency ω_α $(\alpha = 1, 2.)$. Then, for any constant C, the quantity $C q_\alpha(t)$ is also a normal coordinate associated with frequency ω_α.

Proof: By definition, if $q_\alpha(t)$ is a normal coordinate, it obeys

$$(9) \quad \ddot{q}_\alpha(t) = -\omega_\alpha^2 q_\alpha(t)$$

Multiplying both sides of eqn. (9) by any constant C,

$$(10) \quad [C q_\alpha(t)]^{\cdot\cdot} = -\omega_\alpha^2 [C q_\alpha(t)].$$

Thus, $C q_\alpha$ is also a normal coordinate associated with frequency ω_α.
Since q_α and $C q_\alpha$ are not independent, colloquially

we say that they are both "the same normal coordinate".

Returning, then to $q_1(t)$, we see that we can choose it as any constant times $(m_a \psi_a + m_b \psi_b)$. Let us choose that constant to be $\frac{1}{M}$ where $M \equiv m_a + m_b$. Then

$$(11) \quad q_1(t) = \frac{m_a \psi_a + m_b \psi_b}{m_a + m_b}$$

From eqn. (11) we see that $q_1(t)$ represents the position of the center of mass! (We assume that the spring is massless.) This makes sense - in normal mode #1 the two masses are swinging in lock-step with equal amplitude, with the spring maintaining its equilibrium stretch - the amplitudes are equal in this mode. Therefore, for motion in this mode (only) we can replace the coupling spring with massless rigid rod. Thus, in this mode, the position of the CM indeed oscillates in s.h.m. with frequency $\omega_1 = \sqrt{\frac{g}{\ell}}$!

Thus, we see that, in this situation, the transformation from the (ψ_a, ψ_b) coordinate system to the (q_1, q_2) normal coordinate system is the transformation to center of mass/relative coordinates!

We must now invert to find ψ_a and ψ_b. Doing that, I find

(12) $$\psi_a = q_1 + \frac{m_b}{M} q_2$$ and $$\left[\text{with } q_1 = \frac{m_a \psi_a + m_b \psi_b}{M}, \quad q_2 = \psi_a - \psi_b \right]$$

(13) $$\psi_b = q_1 - \frac{m_a}{M} q_2$$

so that, in general, (with $M \equiv m_a + m_b$)

(14) $$\psi_a(t) = C_1 \cos(\omega_1 t + \phi_1) + \frac{m_b}{M} C_2 \cos(\omega_2 t + \phi_2)$$

(15) $$\psi_b(t) = C_1 \cos(\omega_1 t + \phi_1) - \frac{m_a}{M} C_2 \cos(\omega_2 t + \phi_2)$$

where C_1 and C_2 are arbitrary constants. (<u>You should verify (or negate) this claim.</u> Defining a new constant $G \equiv C_2 \frac{m_b}{M}$, we have

(14') $$\psi_a(t) = C_1 \cos(\omega_1 t + \phi_1) + G \cos(\omega_2 t + \phi_2)$$

(15') $$\psi_b(t) = C_1 \cos(\omega_1 t + \phi_1) - \frac{m_a}{m_b} G \cos(\omega_2 t + \phi_2)$$

We must now check that these results make sense:

1. From eqns. (14) and (15), in <u>mode 1</u> the masses move in phase with equal amplitudes. That makes sense because, in this mode, the coupling spring should maintain its equilibrium length and therefore, the entire assemblage oscillates like one big simple harmonic oscillator at frequency $\omega_1 = \sqrt{\frac{g}{\ell}}$.

2. However, in mode 2, the amplitudes of m_a and m_b are <u>not</u> equal and opposite. Instead, note from eqns. (14') and (15') that, in mode 2,

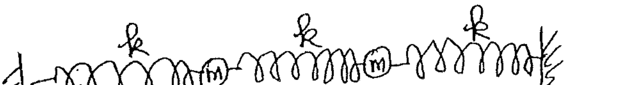
$$\frac{|\mathrm{Amp}_b^{(2)}|}{|\mathrm{amp}_a^{(2)}|} = \frac{m_a}{m_b}$$

– i.e, the motion of each is inversely proportional to its mass. (Oppositely directed.)

That is required to keep the C.M. stationary; hence it "makes sense." That the motions of m_a and m_b are not equal and opposite in mode #2 is also a reflection of the fact that the system is not left-right symmetric about its "mid-line". (Recall discussion in section 6.9 of the text.)

C. Normal Modes In an Electrical Circuit

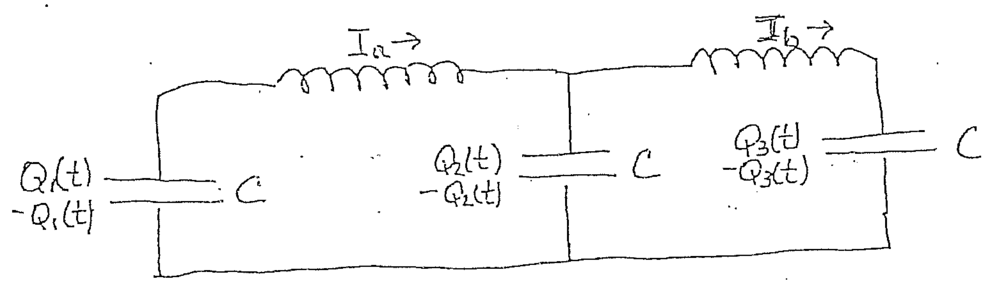
We have considered the normal modes of internal longitudinal oscillation of the coupled mass-spring system 6.1



(equil. configuration).

We now consider the electrical circuit analog of the above system.*

It is [since $m \leftrightarrow L$, $k \leftrightarrow \frac{1}{C}$ (so L's in series, C's in parallel)]:



$$Q_1(t)+Q_2(t)+Q_3(t)=0$$
$$\Rightarrow Q_3(t) = -Q_1(t) - Q_2(t)$$

Are there modes? There should be, since this is an analog of one of the spring-mass systems we considered. We work this out in two ways:

1. Reasoning By Analogy(+). (Serves to tell us what to expect:)

Normal Mode #1: In the mechanical system, in this mode the middle spring always maintains equilibrium length, the two end springs stretch and compress, and the mode frequency is $\omega = \sqrt{k/m}$. Since $C \leftrightarrow \frac{1}{k}$, $L \leftrightarrow m$,

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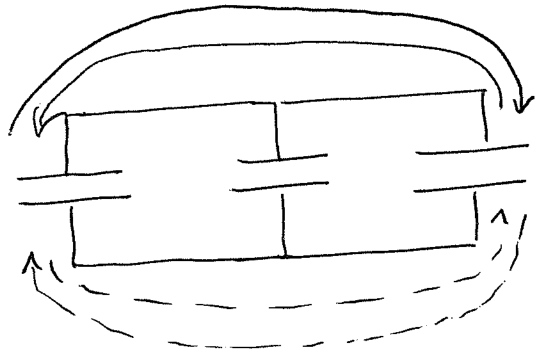
+ Probably the way most physicists would reason on first considering the situation.

* A slight generalization of the electrical analog system we consider here is taken up in Chap. 6 Worked Exs 6.4 and 6.5. You should read that right after our present discussion

in mode #1 in the electrical system we expect no current into or out of the middle capacitor; thus, the charge should oscillate only between the two end capacitors:

MODE 1:



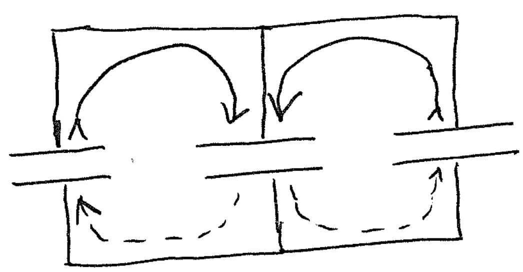
NO CURRENT IN CENTRAL LINE.

Since the mechanical analog system mode 1 frequency is $\omega_1^2 = \sqrt{\frac{k}{m}}$,

the electrical system mode 1 frequency should be $\omega_1^2 = \frac{1}{\sqrt{LC}}$.

Normal Mode #2: In this mode, in the mechanical system, while the end springs both stretch at the same time and compress at the same time, the length of the middle spring also oscillates. Thus, for the electrical system we expect the mode to look like

MODE 2:



, with the arrows reversed half a period later.

Since the mechanical system mode #2 frequency is $\omega_2 = \sqrt{\frac{3k}{m}}$,

for the electrical system we expect that $\omega_2 = \sqrt{\frac{3}{LC}}$.

Thus, the general solution to the governing d.e.'s (eqns. (1) and (2) above) is

(3) $$I_a = C_1 \cos\left(\frac{1}{\sqrt{LC}}t + \phi_1\right) + C_2 \cos\left(\sqrt{\frac{3}{LC}}t + \phi_2\right)$$

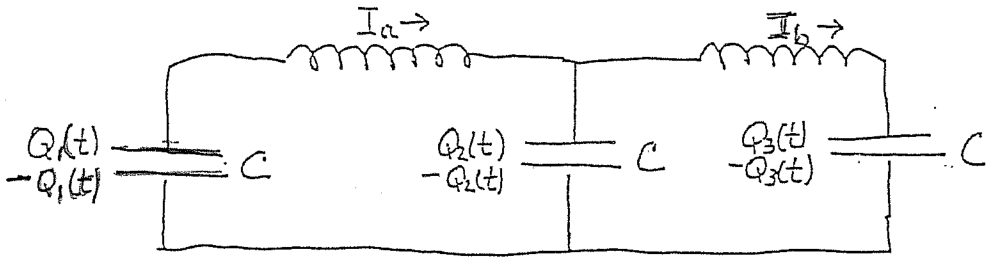
(4) $$I_b(t) = C_1 \cos\left(\frac{1}{\sqrt{LC}}t + \phi_1\right) - C_2 \cos\left(\sqrt{\frac{3}{LC}}t + \phi_2\right)$$

for arbitrary numbers C_1 and C_2.

2. Analysis By Method of Searching for Normal Coordinates

[cf W.E. 6.5, sect. 6.11]

Assuming system is neutral,
$Q_1(t) + Q_2(t) + Q_3(t) = 0$
$\Rightarrow Q_3(t) = -Q_1(t) - Q_2(t)$



We need the equations of motion. Applying the loop theorem to the left loop,

$$\frac{Q_1(t)}{C} - L\frac{dI_a(t)}{dt} - \frac{Q_2(t)}{C} = 0 \quad \text{or} \quad L\frac{dI_a}{dt} = \frac{1}{C}(Q_1 - Q_2)$$

We can express things either in terms of the Q's or the I's. Let's say we use the I language: Recalling $I = \frac{dQ}{dt}$ and taking another derivative,

$$L\frac{d^2I_a}{dt^2} = \frac{1}{C}\left(\frac{dQ_1}{dt} - \frac{dQ_2}{dt}\right)$$

But: $\quad -\frac{dQ_1}{dt} = I_a, \quad \frac{dQ_2}{dt} = I_a - I_b, \quad \therefore$

(5) $$L\frac{d^2I_a}{dt^2} = -\frac{1}{C}I_a + \frac{1}{C}(I_b - I_a)$$, and similarly

(6) $$L\frac{d^2I_b}{dt^2} = -\frac{1}{C}(I_b - I_a) - \frac{1}{C}I_b$$

Eqns. (5) and (6) are the governing differential equations for this system. Note that they are in exact analogy in form to the governing d.e.'s for the mass-spring mechanical system!

Since the physical system is exactly "left-right" symmetric, we should be able to find the normal coords in a straight-forward way. However, first - what do we expect? Since we are working in "I-language" we expect them to be $q_1^{(*)} = I_a + I_b$, $q_2^{(*)} = I_a - I_b$. To confirm this, adding and subtracting eqns. (5) and (6) we find

$$(7) \quad \frac{d^2(I_a + I_b)}{dt^2} = -\frac{1}{LC}(I_a + I_b) \quad , \quad \text{and}$$

$$(8) \quad \frac{d^2(I_a - I_b)}{dt^2} = -\frac{3}{LC}(I_a - I_b).$$

Thus, clearly the normal coordinates are

$$q_{1,2} = C_{1,2}(I_a \pm I_b)$$

where C_1, C_2 are any constants.

And again the most general thing that can happen (in terms of currents internal to the system) is a general superposition of two normal modes - this follows from eqns. (7) and (8) above.

(*) $q_1(t)$ and $q_2(t)$ here are <u>normal coordinates</u>, not charges on cap. a and cap. b.

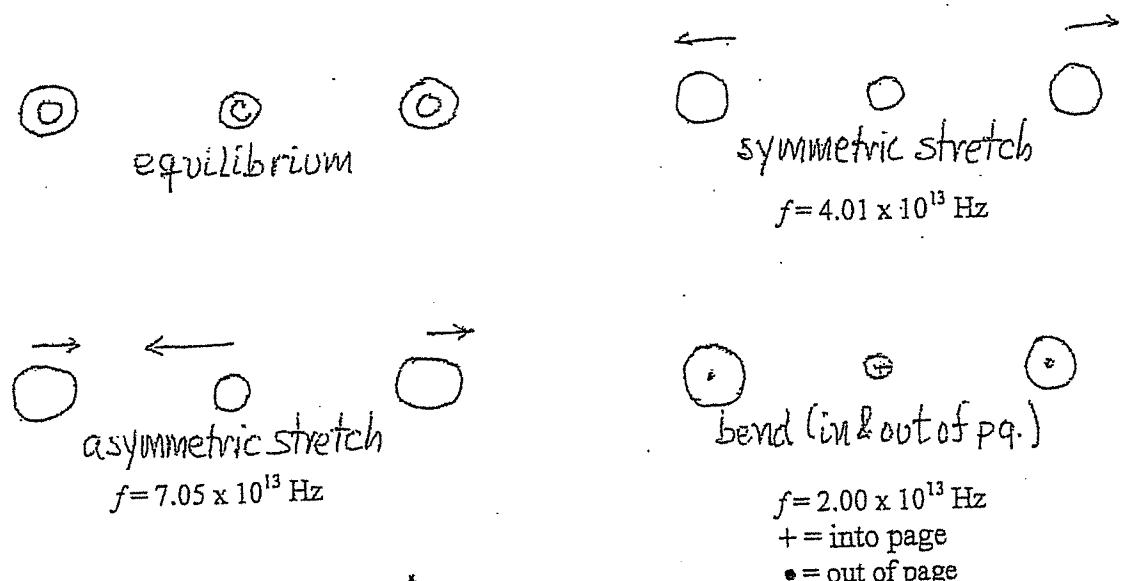
D. Normal Modes and Radiation From Triatomic Molecules; Relation To Planetary Greenhouse Effect.

Reading: Text, sect. 6.13

In the homework, you will treat the case of diatomic molecules with both ions of arbitrary mass by multiple methods.

Here we consider a triatomic case:

Consider the CO_2 molecule. It is a triatomic line-molecule; some of its normal modes are shown below:



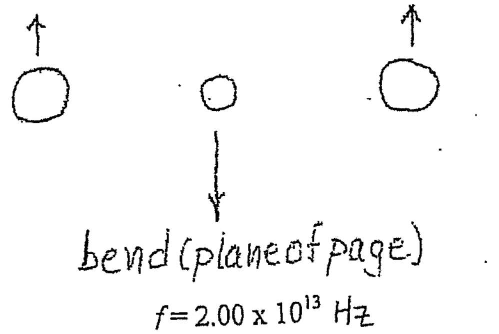


Fig. 6.15 Normal modes of the CO_2 molecule. Not shown is the "normal mode" of frequency zero. All arrows shown are reversed in direction one-half cycle later. The frequencies of electromagnetic radiation measured from these modes are indicated. From these, one can extract the effective spring constant, and from this, information about the C-O chemical bond. By symmetry, the frequencies of the two bend modes shown are equal.

Figure is from your text.

Note from the figure caption that the mode frequencies are in the infrared part of the electromagnetic spectrum. It is interesting that, partly for this reason, the Earth (and also the planet Venus) are warmer than they would otherwise be; this is due to what is called the **greenhouse effect**. The Earth radiates strongly in the infrared, and if there is much CO_2 in the atmosphere, this radiation will be absorbed by the CO_2 molecules and excite their vibrational normal modes. The infrared radiation is then reemitted in all directions, which means that a significant fraction of the heat radiation is sent back down to Earth. On Venus, the effect is very major, since the Venusian atmosphere is almost all CO_2! Another contributor to the greenhouse effect is atmospheric water vapor. As predicted by quantum mechanics, the H_2O molecule cannot be linear; rather the atoms are arranged in a V shape with an opening angle of about 105^0. Three of the normal modes of the water molecule are shown in Fig. 4.8; again, the frequencies are infrared.

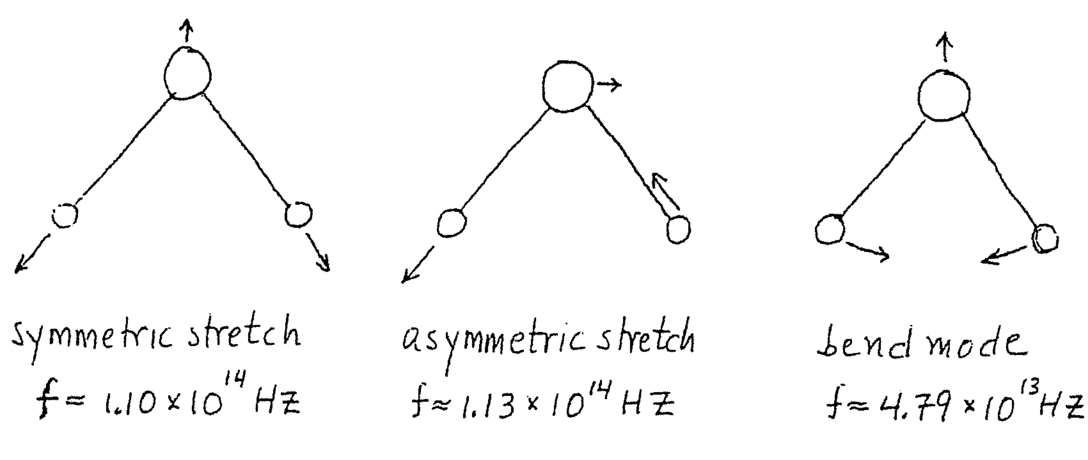
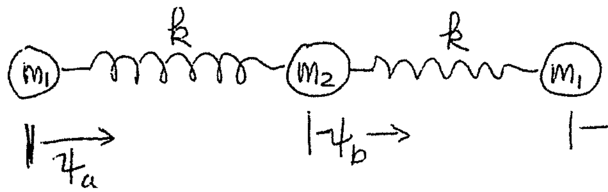


Fig. 6.16 Some of the normal modes of the H_2O molecule and their frequencies. The directions of the arrows are reversed every half cycle.

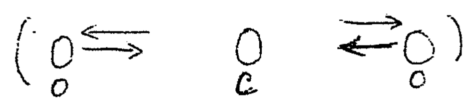
Let us return to CO_2; we are particularly interested in the longitudinal modes. Based on our previous work, we use a spring model for the chemical bonds:



$m_1 = m_{oxygen}$

$m_2 = m_{carbon}$

Within the context of the "spring model," it is not difficult to guess expressions for the frequencies of two of these modes; the symmetric stretch and the asymmetric stretch, both of which occur "in the plane of the page." In each of these modes, the center of mass of the molecule remains stationary[2]. In the symmetric stretch mode (Fig. b), the Carbon in the center remains stationary with the two Oxygen atoms moving toward and away from it with equal amplitudes and 180° out of phase with each other. If each spring modeling the *C-O* bond has spring constant *k*, the mode frequency is $\omega_{ss} = \sqrt{k/m_O}$, since the Carbon acts like a "rigid wall" (in this mode)



In the asymmetric stretch mode, if at any instant t the Oxygens are displaced from equilibrium by amount $\psi_o = \psi_a = \psi_c$, then to keep the CM stationary the Carbon must be displaced in the other

direction by amount $\psi_c(t) = -2\frac{m_o}{m_c}\psi_o(t)$. From this, the total stretch (from equilibrium) of the right-hand spring is

$$\text{total stretch} = \left(1 + \frac{2m_O}{m_C}\right)\psi_0$$

so the force it exerts on the right-hand Oxygen is

$$F_{\text{return on } O} = k\left(1 + \frac{2m_O}{m_C}\right)\psi_0$$

so $$\omega^2 = \frac{F_{\text{return on } O}}{m_O \psi_0} = \frac{k}{m_O}\left(1 + \frac{2m_O}{m_C}\right) = \frac{k}{m_O} + \frac{2k}{m_C}$$

so we really can easily know the mode frequencies in terms of k without having to call on a more general systematic method that we will learn next.

Reading for → Tuesday: Text, sect. 6.14.