$$13.\ \lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n \quad \text{con} \quad a,b>0$$

Sea $L=\lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n$:

$$\log(L)=\log\left[\lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n\right]=\lim_{n\to\infty}\left[n\log\left(\frac{a^{1/n}+b^{1/n}}{2}\right)\right]=\lim_{n\to\infty}\left[\frac{a^{1/n}+b^{1/n}-2}{2}\cdot n\cdot\frac{\log\left(\frac{a^{1/n}+b^{1/n}}{2}+2-2\right)}{\frac{a^{1/n}+b^{1/n}-2}{2}}\right]=$$

$$=\lim_{n\to\infty}\left[\frac{a^{1/n}+b^{1/n}-2}{2}\cdot n\cdot\frac{\log\left(1+\frac{a^{n}+b^{n}-2}{2}\right)}{\frac{a^{1/n}+b^{1/n}-2}{2}}\right]\overset{*}{=}\lim_{n\to\infty}\left[n\cdot\frac{a^{1/n}+b^{1/n}-2}{2}\cdot 1\right]=\frac{1}{2}\lim_{n\to\infty}n\left(a^{1/n}+b^{1/n}-1-1\right)=$$

$$=\frac{1}{2}\left[\lim_{n\to\infty}n\left(a^{1/n}-1\right)+\lim_{n\to\infty}n\left(b^{1/n}-1\right)\right]\overset{*}{=}\frac{1}{2}(\log a+\log b)=\frac{1}{2}\log(ab)=\log(ab)^{1/2}=\log\sqrt{ab}\implies\boxed{L=\sqrt{ab}}\ \square$$

$\forall a,b>0$

$$\lim_{n\to\infty}\frac{\ln(1+n)}{n}=\lim_{n\to\infty}\log(1+n)^{1/n}\overset{*}{=}\lim_{n\to\infty}\log\left(e^{\underbrace{\frac{1}{n}(1+n-1)}_{y_n(x_n-1)}}\right)=\lim_{n\to\infty}\log\left(e^{\frac{n}{n}}\right)=1 \qquad \left(x_n=1+n,\ y_n=\frac{1}{n}\right)$$

$$\lim_{n\to\infty}n\left(a^{1/n}-1\right)=\lim_{n\to\infty}\frac{a^{1/n}-1}{1/n}\underset{\text{R. L'Hôpital}}{=}\lim_{n\to\infty}\frac{-\frac{a^{1/n}\log a}{n^2}}{-\frac{1}{n^2}}=\lim_{n\to\infty}a^{1/n}\log a=\log a\cdot a^{\lim_{n\to\infty}\frac{1}{n}}=\log a \quad (\forall a>0)$$

sin usar derivadas

$$\lim_{n\to\infty}x_n^{y_n}=\lim_{n\to\infty}(x_n+1-1)^{y_n}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{1}{x_n-1}}\right)^{y_n}=\lim_{n\to\infty}\underbrace{\left(1+\frac{1}{\frac{1}{x_n-1}}\right)^{\frac{1}{x_n-1}}}_{e}{}^{y_n\cdot\frac{x_n-1}{1}}=\lim_{n\to\infty}e^{y_n(x_n-1)}$$

$$\lim_{n\to\infty}n\left(a^{1/n}-1\right)\underset{x=\frac{1}{n}}{=}\lim_{x\to 0}\frac{a^x-1}{x}\underset{a=e^{\ln a}}{=}\lim_{x\to 0}\frac{e^{x\log a}-1}{x}=\ln a\lim_{x\to 0}\frac{e^{x\log a}-1}{x\ln a}=\log a\log e=\log a \quad (\forall a>0)$$

definición del logaritmo neperiano: $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$

definición del número de Euler: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \quad \forall x\in\mathbb{R}$

por definición $\log(e)=1$