Physics 251 - 22nd Class - Thursday March 28, 2024

Reading for Part I of today's class: K-text, sects 13.11, 13.13 and 13.14

I. Standing and Traveling Wave Synthesis

In the last class we saw that a traveling wave pulse reflects back inverted on encountering a bound end, and reflects back without inversion on encountering a free end.

A. Formation of Standing Wave: Anticipatory Argument

Now suppose that we continually shake one end of a stretched slinky up and down in simple harmonic motion while the far end of the slinky remains bound down. Then, the traveling sinusoidal wave generated from the shaking end and moving (say) right, will, at the bound end, generate a left-moving reflected (and inverted) sinusoidal wave, and if the incident wave "keeps coming", the reflected wave will superpose with it.

Now, the sending end can, at any given instant, be considered a bound end (so it is really a "moving bound end"*), so the

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* especially if the shaking amplitude is small, as we assume. This is because that end is moving in a way that we, rather than the string control.

first reflected wave's arrival back at the shaking end generates a "second reflected wave" heading right, and then we get a "third reflected wave" heading left, and so on. In general the right-moving waves are not in phase with each other; neither are the left moving waves, and after a sufficient number of reflections, at any given point on the string, and at any time, the phasors of all the waves are more or less distributed uniformly "around the clock", and thus we have much destructive interference, and hence, not much of anything resulting.

Let us now ask a question: Is there a condition under which all of the right moving waves will be in phase? The answer is yes. This will occur if the round trip time $\frac{2L}{v}$ is the same as one period (then, e.g., for the incident and second reflected waves, since the two 180° phase shifts cancel, we'd have perfect constructive interference; the same would be true for any pair of right-moving waves).

This condition is

$$T = \frac{2L}{v} = \frac{2L}{f\lambda} = \frac{2LT}{\lambda} \Rightarrow 1 = \frac{2L}{\lambda}$$

$$\Rightarrow \quad \lambda = 2L$$

Then, each round trip causes the new reflected waves to all be just in phase with the incident wave.

As you can easily convince yourself, a similar situation occurs when

$$\frac{2L}{v} = nT \Rightarrow L = \frac{n\lambda}{2} \text{ or } \lambda = \frac{2L}{n},$$

a familiar sounding condition. Also, as you can easily convince yourself, all of the left moving waves will then also add up to a net substantial left-moving wave.

Thus, we would have, if this condition is satisfied, a net result of one sizeable amplitude wave moving right and an equal amplitude net wave moving to the left. (Damping keeps the amplitudes from being infinite)

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B. Basic Properties of Standing Waves As Superposition of Traveling Waves

What might the superposition of these two "net" waves look like?
Before getting involved in the mathematics, let's see if a picture will help:

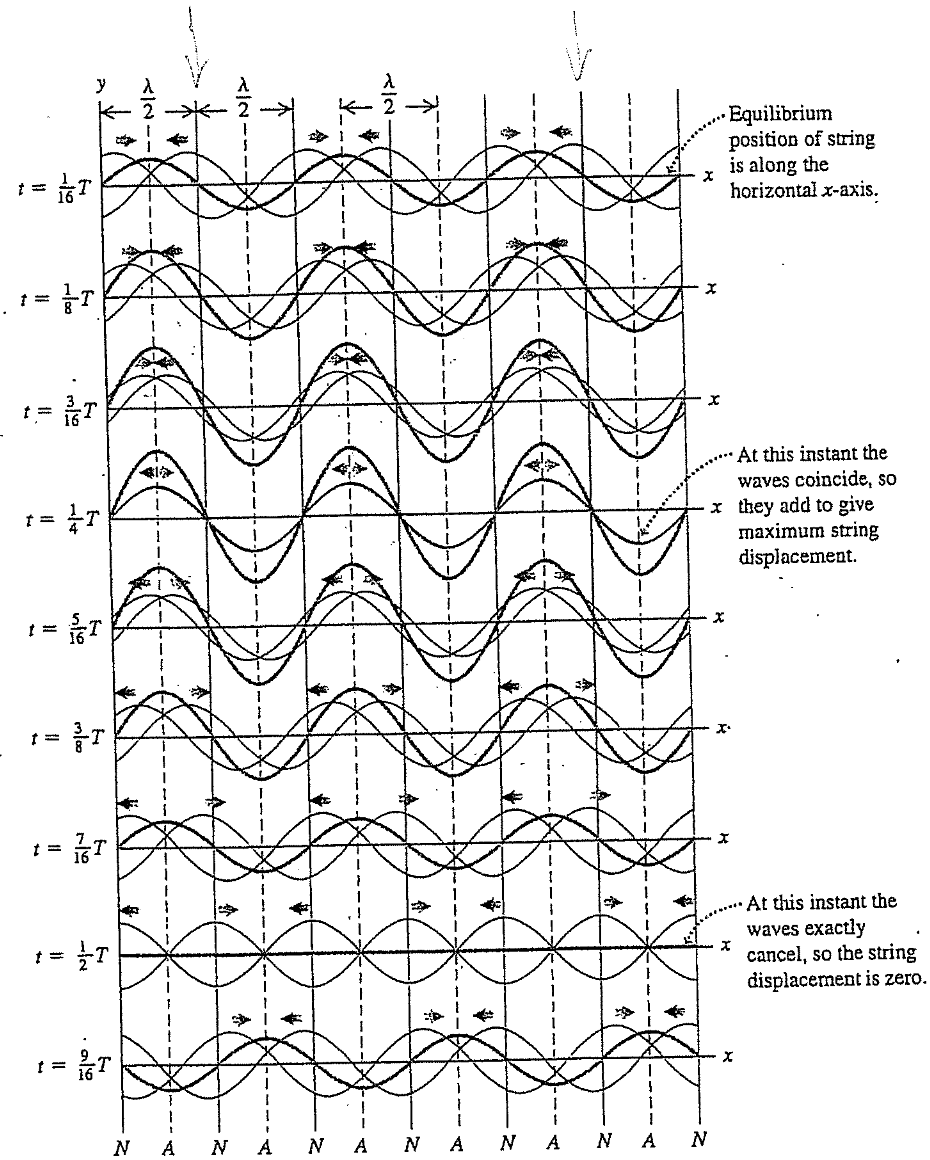


Figure from Young & Freedman, op. cit., chap. 15.

One of our first observations about the resultant that we make is that
it is sinusoidal in shape at any time - so, the sum of two oppositely directed
but same frequency and same amplitude sinusoidal traveling waves is also sinusoidal.
Notice also, however, that at certain points (solid "guide lines") the displacement of
the string is always zero (nodes). Thus, the resultant, while sinusoidal, is not a traveling wave.

To see the net effect as a function of time more clearly, let's look at another figure of this, this one with the vertical "guide lines" removed:

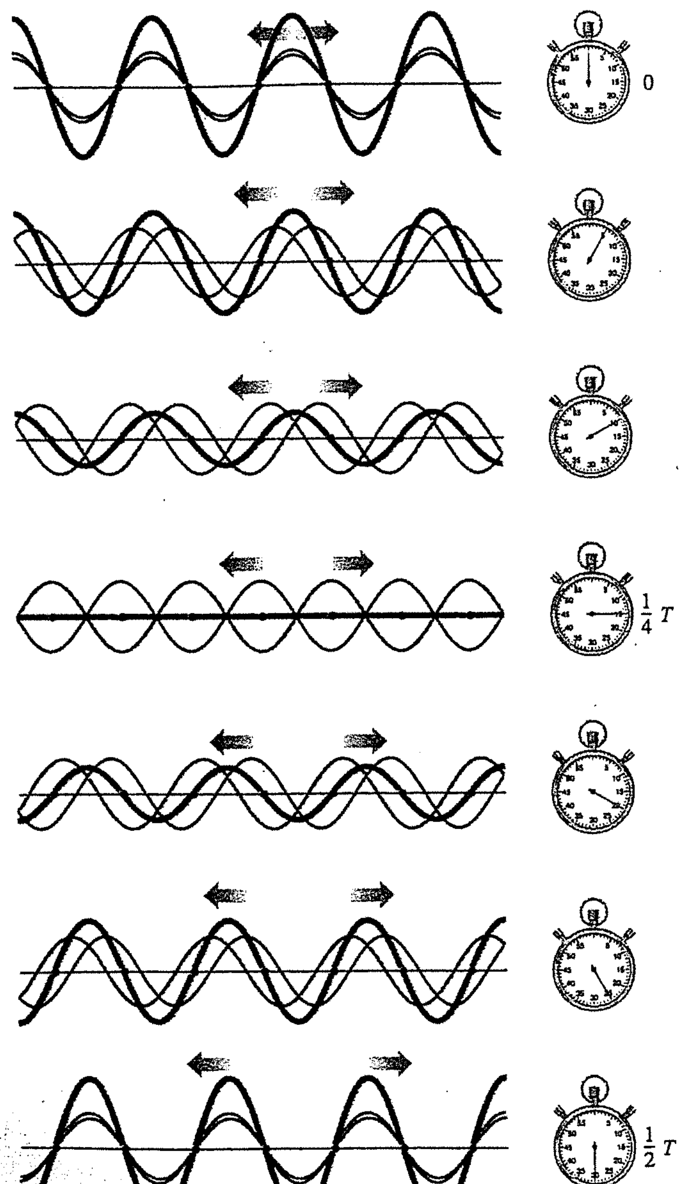


Figure 13.22 The creation of standing waves. Two waves of the same amplitude and wavelength traveling in opposite directions form a stationary disturbance that oscillates in place.

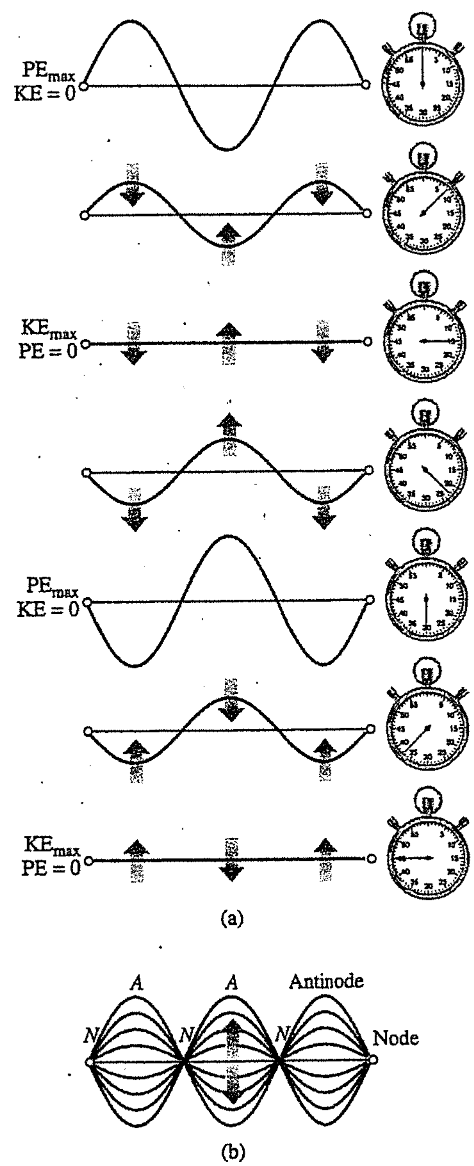


Figure 13.23 (a) A standing wave on a string that has both ends fixed. (b) represents a composite of all the configurations demonstrated in (a).

(Figures from <u>Physics</u> Chap. 13 by Eugene Hecht, op. cit.)

On the right is shown the resultant as a function of time; note that it is a standing wave! Thus, <u>the sum of two</u> oppositely directed equal amplitude traveling waves <u>can</u> form a standing wave.

The sum of two equal amplitude but oppositely directed traveling waves with the same wavelength (and hence, the same frequency) in a medium is a pure standing wave in that medium!

Notice that some points never get any displacement (nodes). Thus, if we put walls at these points and bind the string down at them, we get our familiar standing waves (normal modes) with wavelength condition $\lambda_n = \frac{2L}{n}$:

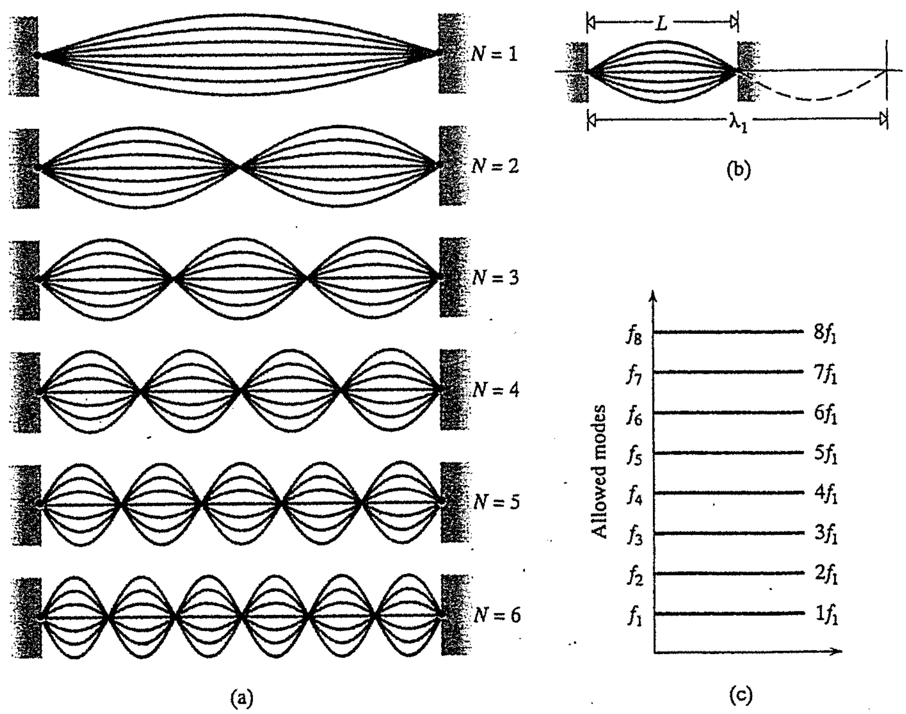


Figure 13.25 (a) Standing-wave modes with a node at both ends. (b) The wavelength of the $N = 1$ fundamental equals $2L$. (c) Allowed modes of oscillation.

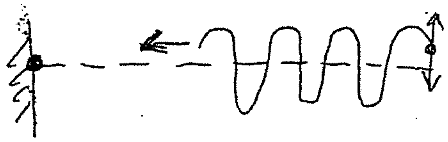
(From E. Hecht, Physics, op. cit.)

What if one or both ends are free? Then, if the wavelengths satisfy the right condition (we leave this to you) we also get standing waves.

C. "Formation" of Standing Waves - Mathematical

1. Simple Case First:

Consider, then, shaking one end of a string with simple harmonic



z=0

motion so as to make a left-traveling wave on a string bound at $z=0$. (I.e., the harmonic wave travels toward $z=0$ from positive z). An example of such a wave would be

$$\Psi_1(z,t) = A\sin(\omega t + kz).$$

Upon encountering the bound end, Ψ_1 generates another traveling wave - an inverted reflection Ψ_2. Thus

$$\Psi_2(z,t) = -A\sin(\omega t - kz)$$

where Ψ_2 has reverse amplitude and reverse direction of travel than does Ψ_1. [Note that this automatically incorporates the B.C. $\Psi=0$ at $z=0$ - not surprising, since our derivation of reflection sign last class assumed this B.C.!]

Now the superposition thm tells us that the net wave on the string is the superposition of Ψ_1 and Ψ_2 - i.e., simultaneously they each travel independently without changing each other.

Thus, after the reflected wave is generated, and as long as the "incident" wave is still being generated,

$$\Psi(z,t) = \Psi_1(z,t) + \Psi_2(z,t) \quad ^{(\dagger)}$$

$$= A \sin(\omega t + kz) - A \sin(\omega t - kz)$$

$$= A\left[\sin\omega t \cos kz + \sin kz \cos\omega t\right]$$

$$+ A\left[-\sin\omega t \cos kz + \sin kz \cos\omega t\right]$$

$$\Psi(z,t) = 2A \sin kz \cos\omega t$$

Note that this is a <u>pure standing wave</u> ("factorized form"). This then shows again that a standing wave can be viewed as the sum of two oppositely directed harmonic traveling waves.

Note also from our previous logic that, if the string is bound at both ends* <u>the period of the lowest normal mode is the same as the "down and back" travel time for a traveling wave of the same frequency in the same medium.</u> For the higher modes, the down and back travel time is $n \cdot$ the period of the mode.

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† We are ignoring the "higher-order" reflected left- and right-traveling waves, which, if the damping is light, to good approximation only change the net

* or, if one end is a "moving bound end"

D. Generalization; Application to Plucked String

Let us now both generalize the preceeding and apply it. Suppose we pluck a string that is bound down at its two ends ($z=0$ and $z=L$). Then, presumably, we generate traveling waves of "all frequencies" (or, at least, many) that travel in both directions; these will reflect off the bound ends multiple times, etc., and there will be much wave interference. From our "anticipatory argument" at the beginning of this class, we expect that, after all the interference, only certain frequencies will survive, and these as standing waves (at the normal mode frequencies).

Let's now see how this works out mathematically. Ignoring damping, we have

$$\Psi(z,t) = \text{“}\sum_{\omega}\text{”} F_\omega \sin(kz - \omega t + \phi_\omega^{(-)}) + G_\omega \sin(kz + \omega t + \phi_\omega^{(+)})$$

we focus on the contibutions from just one frequency:

$$\Psi_\omega(z,t) = F_\omega \sin(kz - \omega t + \phi_\omega^{(-)}) + G_\omega \sin(kz + \omega t + \phi_\omega^{(+)})$$

writing each term in "sine-cosine form",

$$\Psi(z,t) = A\sin(kz-\omega t) + C\cos(kz-\omega t) + B\sin(kz+\omega t) + D\cos(kz-\omega t)$$

Now, here is a crucial point: <u>The application of the boundary conditions automatically ensures that the proper reflections take place.</u> [We commented earlier on that.]

Let us see how this works: We have:

$$0 = \Psi(0,t) = -A\sin\omega t + B\sin\omega t + C\cos\omega t + D\cos\omega t$$

$$= (B-A)\sin\omega t + (C+D)\cos\omega t$$

$$\Rightarrow \quad B = A \quad \text{and} \quad C = -D$$

$$\Rightarrow \quad \Psi(z,t) = A\left[\sin(kz-\omega t) + \sin(kz+\omega t)\right] + C\left[\cos(kz-\omega t) - \cos(kz+\omega t)\right]$$

Applying the other boundary condition,

$$0 = A\left[\sin(kL-\omega t) + \sin(kL+\omega t)\right] + C\left[\cos(kL-\omega t) - \cos(kL+\omega t)\right]$$

As you can show, for this to be true at all times,

$$0 = A\left[\sin kL\cos\omega t - \cancel{\cos kL\sin\omega t} + \sin kL\cos\omega t + \cancel{\cos kL\sin\omega t}\right]$$

$$0 = 2A\sin kL\cos\omega t \Rightarrow \sin kL = 0 \Rightarrow kL = n\pi = k_n = \frac{n\pi}{L} \Rightarrow \omega_n = k_n v_\phi = \frac{n\pi}{L} v_\phi$$

the surviving frequencies must satisfy the

$$\omega_n = n\frac{\pi}{L}v_\phi \qquad \left(v_\phi = \sqrt{\frac{T_0}{\rho_e}}\right)$$

which is exactly the familiar condition for standing waves.

Thus,

$$\Psi(z,t) = \sum_{n=1}^{\infty} A_n\left[\sin(k_n z - \omega_n t) + \sin(k_n z + \omega_n t)\right]$$
$$+ \sum_{n=1}^{\infty} C_n\left[\cos(k_n z - \omega_n t) - \cos(k_n z + \omega_n t)\right]$$

where $k_n = \dfrac{\omega_n}{v_\phi}$

Expanding the terms on the right and following through a few lines of algebra, we find

$$\Psi(z,t) = \sum_n \left[J_n \sin k_n z \cos \omega_n t + K_n \sin k_n z \sin \omega_n t\right] \quad \begin{pmatrix} J_n = 2A_n \\ K_n = 2C_n \end{pmatrix}$$

or

$$\Psi(z,t) = \sum_{n=1}^{\infty} L_n \sin k_n z \cos(\omega_n t + \delta_n)$$

where L_n and δ_n are constants (which you can find); thus, the result of plucking is a superposition of standing waves at the normal mode frequencies; a result we've already encountered.

A Possible Paradox?

Now we have an interesting situation.

We saw that the most general solution of the CWE for a stretched string with its ends at $z=0$ and $z=L$ bound down (and for all points on the string having zero initial velocity) is a mode superposition

$$(1) \quad \Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin k_n z \cos \omega_n t$$

For arbitrary boundary conditions and arbitrary initial velocity profile, Ψ is broader; the general soln. is then

$$(2) \quad \Psi(z,t) = \sum_{n=0}^{\infty} A_n \sin k_n z \cos(\omega_n t + \alpha_n) + B_n \cos k_n z \cos(\omega_n t + \beta_n)$$

The contribution to this at frequency ω_n is

$$(3) \quad \Psi_n(z,t) = A_n \sin k_n z \cos(\omega_n t + \alpha_n) + B_n \cos k_n z \cos(\omega_n t + \beta_n)$$

This is a superposition of two different standing waves at the same frequency (the nodes are displaced by $\frac{1}{4}\lambda$ and the time-phases are different).

This, thus, is the most general solution at frequency ω.

Now, we've seen that a standing wave is a superposition of oppositely directed traveling waves - e.g.,

$$2A \sin kz \cos(\omega t) = A \sin(kz-\omega t) + A \sin(kz+\omega t)$$

But, consider a right traveling sinusoidal wave <u>by itself</u>.

$$\Psi_{\rightarrow}(z,t) = A \cos(kz-\omega t)$$

This does not appear to be of the form (3)!

Yet it is (as you've verified) a solution of the CWE!

<u>Have we gone wrong</u>??

Notice that:

$$\Psi_{\rightarrow}(z,t) = A \cos(kz-\omega t)$$

$$= A \cos kz \cos \omega t - A \sin kz \sin \omega t$$

Thus: While a standing wave can be viewed as made up of two traveling waves,
a traveling wave can also be viewed as made up of two standing waves!

Reading for this part Chap. 15, sects 15.1, 15.2 thru page 15-9 up to heading "A Brief Look Ahead"

II. Energy Carried by Traveling Waves

It comes as no surprise that, in general, traveling waves carry energy - e.g., as we already remarked, if you sit in an inner tube and a water wave passes by you, you are given kinetic energy, and at times during the cycle, perhaps considerable potential energy.

We consider a wave on a stretched string that, by definition, has zero potential energy when in equilibrium. The wave is represented by $\Psi(z,t)$. When it is present, an element of the string dz and mass $dm = \rho_0 dz$, has kinetic energy.

Q: To calculate the kinetic energy in a traveling wave, do we use v_ϕ or $v_{particle}$ or some combination of both?

A: The K.E. must be that of mass elements of the medium. We recall that in traveling wave motion the mass elements do not move at the wave velocity (even though the disturbance pattern does). Thus, the relevant velocity is the particle velocity $\frac{\partial \Psi}{\partial t}$.

So

$$dK = \frac{1}{2}(dm)\left(\frac{\partial \Psi}{\partial t}\right)^2 = \frac{1}{2}\rho \cdot dz\, \dot{\Psi}^2$$

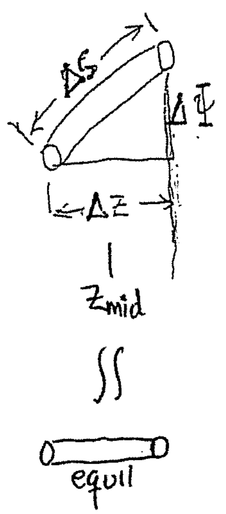
Thus, the kinetic energy per unit length, or "kinetic energy density" is

$$K_1 = \frac{dK}{dz} = \frac{1}{2}\rho_\ell \dot{\Psi}^2$$

and the total kinetic energy on length L (segment $0 \to L$) of the string is

$$K_{0\to L} = \int_0^L \frac{dK}{dz}\, dz = \frac{1}{2}\rho_\ell \int_0^L \dot{\Psi}^2 dz.$$

The element dz also generally has potential energy because the presence of Ψ has stretched it to length $ds > dz$. This P.E. is the work to stretch against the tension. This is hard to calculate unless we assume <u>small amplitude</u> wave motion; then



$$W \approx T_{z=z_{mid}}[\Delta s - \Delta z] \approx T_0(\Delta s - \Delta z) \quad \left[\begin{array}{c}\text{since for}\\ \text{small amps}\\ T \approx T_0.\end{array}\right]$$

From the diagram,

$$(\Delta s)^2 = (\Delta z)^2 + (\Delta \Psi)^2 = \left[1 + \left(\frac{\Delta\Psi}{\Delta z}\right)^2\right](\Delta z)^2$$

so

$$(ds) = \left[1 + \left(\frac{\partial \Psi}{\partial z}\right)^2\right]^{1/2} dz$$

Thus, the amount of stretch of element "dz" from its equilibrium length (dz) is

$$\frac{dW}{T_0} = ds - dz = \left\{\left[1 + \left(\frac{\partial\Psi}{\partial z}\right)^2\right]^{1/2} - 1\right\} dz$$

That's an inconvenient expression. However, we can conveniently approximate i

We assume that $\left(\frac{\partial \Psi}{\partial z}\right)^2 << 1$ (small slope motion), thus we expand the square root and keep only the first two terms:

$$\frac{dW}{T_0} \approx \left\{ 1 + \frac{1}{2}\left(\frac{\partial \Psi}{\partial z}\right)^2 - 1 \right\} dz = \frac{1}{2}\left(\frac{\partial \Psi}{\partial z}\right)^2 dz$$

Thus, the stored potential energy per unit length is

$$V_1 \equiv \frac{dW}{dz} = \frac{1}{2} T_0 \left(\frac{\partial \Psi}{\partial z}\right)^2$$

and the stored potential energy in a finite piece of the string, say the segment extending from $z=0$ to $z=\ell$ is

$$W = \int_0^\ell \frac{dW}{dz}\, dz = \frac{1}{2} T_0 \int_0^\ell \left(\frac{\partial \Psi(z,t)}{\partial z}\right)^2 dz$$

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Thus, the total energy per unit length, or "energy density" in a wave (traveling or standing) is given by (in the small amp. domain)

$$\epsilon = K_1 + V_1 = \frac{1}{2}\rho_0\left(\frac{\partial\Psi}{\partial t}\right)^2 + \frac{1}{2}T_0\left(\frac{\partial\Psi}{\partial z}\right)^2 \equiv \text{"}\frac{dE}{dz}\text{"} \qquad \text{"}\rho_0\text{"} \equiv \rho_\ell$$

<u>Example</u>: Suppose we are dealing with the sinusoidal traveling wave

$$\Psi(z,t) = A\cos(kz - \omega t)$$

Then

$$K_1 = \frac{1}{2}\rho_0\,\omega^2 A^2 \sin^2(kz-\omega t)$$

$$V_1 = \frac{1}{2}T_0\,k^2 A^2 \sin^2(kz-\omega t)$$

For a sinusoidal traveling wave these two expressions happen to be equal, since

$$\rho_0\omega^2 = \rho_0 k^2\frac{\omega^2}{k^2} = \rho_0 k^2 v_\varphi^2 = \rho_0 k^2\frac{T_0}{\rho_0} = T_0 k^2.$$

Thus, <u>for a sinusoidal traveling wave</u>,

(1a) $$\epsilon = \rho_0\left(\frac{\partial\Psi}{\partial t}\right)^2 = T_0\left(\frac{\partial\Psi}{\partial z}\right)^2 = \frac{dE}{dz}$$

(1b) $$= \rho_0\omega^2 A^2\sin^2(kz-\omega t). \qquad \text{(Sinusoidal traveling Wave only)}$$

Note that the energy density is (for a sinusoidal wave on a stretched string)

i. Fluctuating in time,

ii. <u>proportional to the square of the amplitude</u>. ($\varepsilon \propto A^2$)

Thus, e.g., at constant frequency, doubling the amplitude quadruples the energy density at any z, and,

iii <u>proportional to ω^2</u> ($\varepsilon \propto \omega^2$).

[comment: Note that both these proportionalities are also true for a simple harmonic oscillator – there, say for a mass-spring oscillator,

$$E = \frac{1}{2} k A^2 = \frac{1}{2} m \omega^2 A^2 \propto \omega^2 A^2.$$

The same proportionalities turn out to be true for sinusoidal standing waves. This is not surprising since a sinusoidal standing wave oscillates "Like one big extended simple harmonic oscillator."

It's not surprising for a traveling sinusoidal wave either, since in this case, all points on the string move up and down in simple harmonic motion with the same amplitude (albeit, with different phase constants), thus the total energy is that of a sum of simple harmonic oscillators]

Of course, eqn. (1b), since it refers to a specific frequency, applies only to sinusoidal traveling waves on stretched strings.

Let us now try to see how general eqn. (1a) is:

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General traveling Wave Shape On Stretched String

Now suppose we have a general rigidly traveling wave shape

$$\Psi(z,t) = f(z-vt)$$

Then (recall)

$$\frac{\partial\Psi(z,t)}{\partial t} = \frac{\partial f(z-vt)}{\partial t} = \underbrace{\frac{\partial f(z-vt)}{\partial(z-vt)}}_{\equiv \text{"}f'\text{"}} \frac{\partial(z-vt)}{\partial t} = f'(-v)$$

so $$\left(\frac{\partial\Psi}{\partial t}\right)^2 = v^2(f')^2$$

so $$K_1 = \frac{1}{2}\rho_0\left(\frac{\partial\Psi}{\partial t}\right)^2 = \frac{1}{2}\rho_0 v^2(f')^2, \text{ and}$$ $\left(K_1 \equiv \text{"}\frac{dKE}{dz}\text{"}, V_1 \equiv \text{"}\frac{dU}{dz}\text{"}\right.$

$$V_1 = \frac{1}{2}T_0\left(\frac{\partial\Psi}{\partial z}\right)^2 = \frac{1}{2}T_0(f')^2$$ (as you can easily show)

But, again, since $v^2 = \frac{T_0}{\rho_0}$,(*) K_1 and V_1 again turn out to be equal.

Thus, not only for sinusoidal waves, but for any rigidly moving shape,

$$\epsilon = \rho_0\left(\frac{\partial\Psi}{\partial t}\right)^2 = T_0\left(\frac{\partial\Psi}{\partial z}\right)^2$$

Of course, now we cannot say $\epsilon \propto \omega^2$, since we have no one defined frequency for the wave shape.

(*) Recall we showed that the wave velocity v = the phase velocity $v_\phi = \sqrt{\frac{T_0}{\rho_\ell}}$.

Energy Flux

We now ask a new question: Suppose a wave is traveling on a string. Let z be a point on the string. We ask how much energy passes through this point per second. This important quantity is called the "energy flux". On the string it is just the power $P(z,t)$ passing point "z" per second at time t.

We refer to the figure below, which shows the situation for an arbitrarily shaped traveling wave disturbance*.
Consider an interval of time Δt.

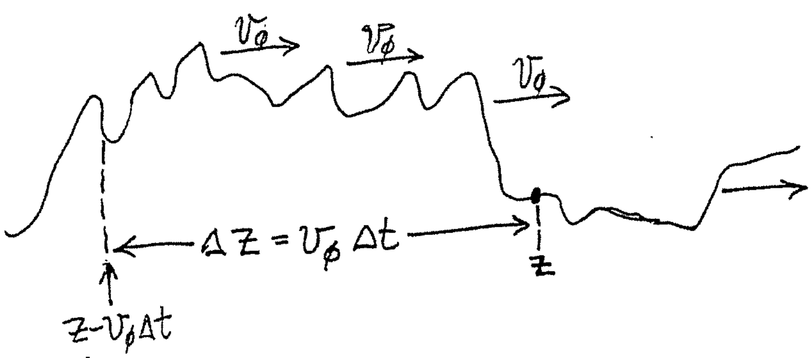

Figure: In time Δt from now all of the energy contained in marked distance $\Delta z = v \Delta t$ will pass thru point z.

In the figure, in the time interval t to $t + \Delta t$ (i.e, in interval duration Δt), all of the energy contained in the marked distance $\Delta z = v_\phi \Delta t$ passes thru point z in time interval Δt and no more energy does.

* Our argument is valid as long as the wave disturbance "fills" Δz; this is not a real restriction since presently we take $\lim \Delta z \to 0$.

Thus, over the time interval Δt, the time averaged power (energy per time) passing through point z is

$$\langle P(z) \rangle_{\Delta t} = \frac{\langle \epsilon \rangle_{\Delta z} \cdot v_\phi \, \Delta t}{\Delta t}$$

or

$$\underline{\overline{P}(z) = \langle \epsilon \rangle_{\Delta z} \cdot v_\phi}$$

where $\langle \epsilon \rangle_{\Delta z}$ is the average of ϵ over the distance Δz at the start of the interval Δt.

Now, if we take Δt very small, then we can treat ϵ as constant, both in time and space over Δt (since if Δt is infinitesimal, so is $\Delta z = v_\varphi \Delta t$). Then

$$P(z,t) = \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \lim_{\Delta t \to 0} \frac{\epsilon \, v_\varphi \, \Delta t}{\Delta t} = \epsilon(z,t) \cdot v_\varphi$$

In the limit $\Delta t \to 0$, then, for any rigidly-moving wave shape that moves with speed $v (= v_\phi)$ the instantaneous power passing point "z" is

$$\boxed{P(z,t) = \epsilon(z,t) \cdot v_\phi}$$

Instantaneous Power Passing through point z at time t.

Combining with our previous results, we have

$$\underline{P(z,t) = \epsilon(z,t) \cdot v_\phi = \rho_0 v_\phi \left(\frac{\partial \Psi}{\partial t}\right)^2 = T_0 v_\phi \left(\frac{\partial \Psi}{\partial z}\right)^2}.$$

Example: Case of Sinusoidal Wave On Stretched String

Say $\Psi(z,t) = A\cos(kz-\omega t)$.

Then
$$P(z,t) = \epsilon(z,t)\, v_\phi = \rho_0\,\omega^2 A^2 v_\phi \sin^2(kz-\omega t).$$

We see that $P(z,t)$ fluctuates in time. Usually*, we just time-average over a cycle

$$\langle P(z,t)\rangle_t \equiv \bar{P}(z) = \rho_0\,\omega^2 A^2 v_\phi \langle \sin^2(kz-\omega t)\rangle = \tfrac{1}{2}\rho_0\,\omega^2 A^2 v_\phi, \text{ i.e.,}$$

$$\boxed{\bar{P} = \tfrac{1}{2}\rho_0\,\omega^2 A^2 v_\phi}$$

which is the same for any z (only true for sinusoidal wave). Again we see the proportionality of $\bar{P}$ to ω^2 and A^2.

As we will see, the equation above for $\bar{P}$ for a sinusoidal traveling wave is both very important and quite useful.

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* Especially if the frequency is high, such as with a light wave. Even with a sound wave ($f \gtrsim 20$ Hz), we are not often interested in the temporal "microstructure". (Again, this is only for single, pure sinusoidal waves. For combinations of different frequency sinusoidal sound waves, often we are — e.g. Compact Audio Disk sampling rate.