Physics 251 - 17^{th} class - Tues. March 12, 2024

A. Periodicity of Fourier Series [K-text, sect. 11.7]

Let us look at our original series for the step  on $[-\pi, \pi]$:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right]$$

Note: The term $\sin x$ is periodic with period 2π

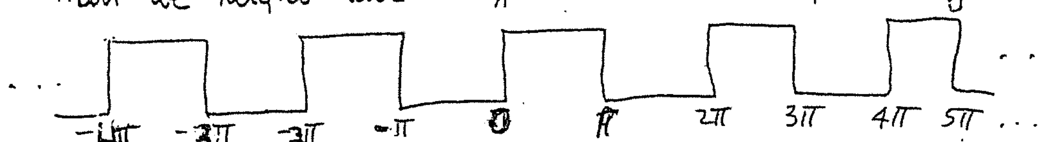
The term $\sin 3x$ is periodic with period $\frac{2\pi}{3}$ and hence also with per. 2π

The term $\sin 5x$ is periodic with period $\frac{2\pi}{5}$ and hence also with per 2π

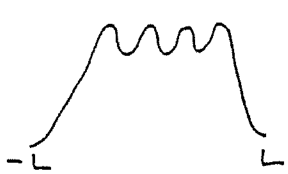
The term "$\frac{1}{2}$" is periodic with any period.

$\therefore$ - the entire series is periodic with period 2π.

So - we've really found a series representation for something bigger than we might have thought at first - we get the larger entity

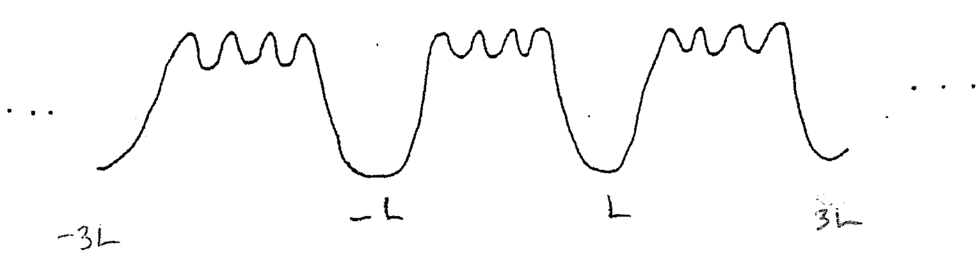


This is, of course, true in the general case: If we find on $[-L, L]$ that

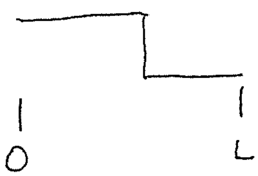


$$\rightarrow \quad f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

the series converges to more:

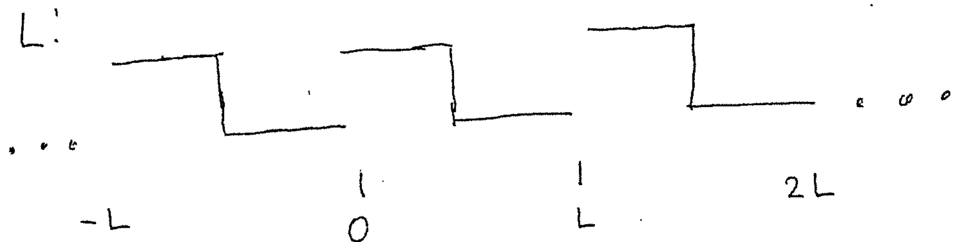


Likewise consider our expansion of a function given on [0, L], say



with series (*) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)$$

converges to a repetitive version of the figure above with repeat distance L:

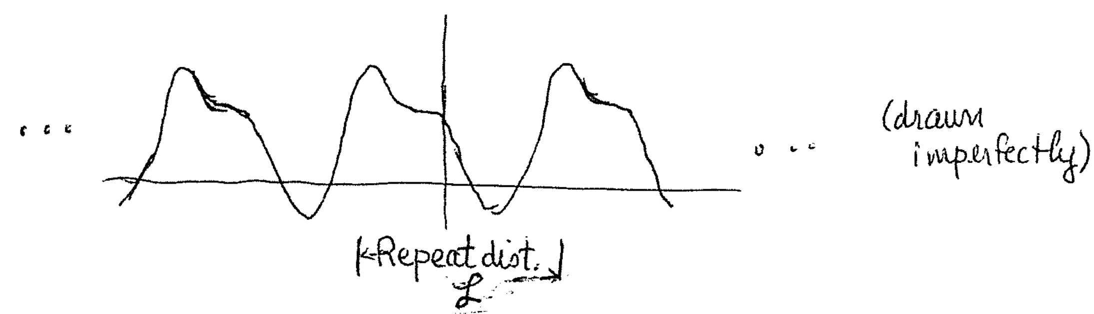


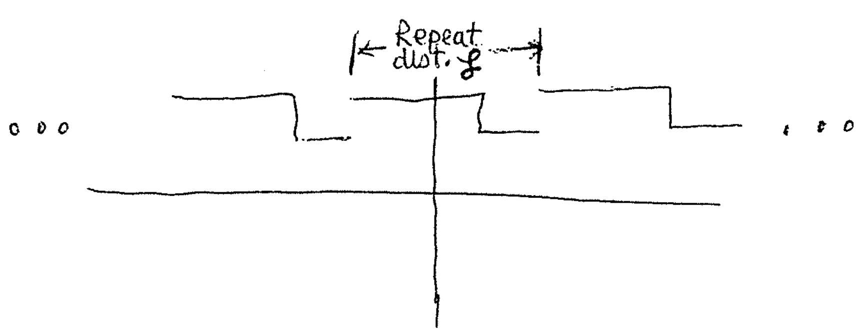
and, in fact, for any function on [0,L] to which the series (*) converges, the series (*) converges to the <u>periodic extended function</u> of f(x) with repeat distance $\lambda_1 = L$.

In each case, the wavelength of the longest-wavelength term (λ_1) is equal to the repeat distance. That must be, since all terms of the series repeat in distance λ_1 as λ_1 is the longest common repeat distance.

B. Fourier Analysis of Repetitive Functions

Frequently the "opposite" sort of situation occurs: we are presented, in the first place, with a repetitive function and we need to "Fourier analyze" it (that means, determine a Fourier series representation in which the repetitive function is viewed as a sum of sinusoidal oscillations, each with its own wavelength). [We will see many examples of the analog of this in the time domain (i.e, function of time t rather than function of space x) when we deal with the physics of musical instruments later.] Two examples of repetitive functions are shown:





Note that the second has a finite number of jump discontinuities on any finite interval. (Mathematicians say that it is "piecewise continuous").

Then, from our previous considerations, we can state the following version of Fourier's Theorem:

Let f(x) be a repetitive function of x with repeat distance $\mathcal{L}$ on the entire x-axis. Further, suppose that both f(x) and f'(x) are piecewise continuous. Then, except at the points of discontinuity,

$$(8)\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) + b_n \sin\left(n \frac{2\pi}{\mathcal{L}} x\right)$$

$$(9)\quad \text{with} \quad a_n = \frac{2}{\mathcal{L}} \int_{\text{"}x_0\text{"}}^{\text{"}x_0+\mathcal{L}\text{"}} f(x) \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) dx, \quad b_n = \frac{2}{\mathcal{L}} \int_{x_0}^{x_0+\mathcal{L}} f(x) \sin\left(n \frac{2\pi}{\mathcal{L}} x\right) dx,$$

Where x_0 is any abscissa value,

converges to f(x) at all points where f is continuous.

At a point at which f(x) has a jump discontinuity, the series given converges to the average of the values of f on either side of the jump (i.e, to $\frac{1}{2}[f(x_+) + f(x_-)]$ where $x_+ = \lim_{x \to x_d} x$ from $x > x_d$, and $x_- = \lim_{x \to x_d}$ from $x < x_d$ where x_d is the point of discontinuity.

<u>Comment</u>: Note that, in eqns. (8) and (9), $\mathcal{L}$ is the wavelength of the fundamental (i.e, term with the longest wavelength).

Thus, we can write eqn. (8) as (again)

$$(10) \quad f(x)_{[0,\mathcal{L})} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{\lambda_n} x\right) + b_n \sin\left(\frac{2\pi}{\lambda_n} x\right)$$

where $\lambda_n = \dfrac{\lambda_1}{n} = \dfrac{\mathcal{L}}{n}$

Likewise, eqns. (9) are

$$a_n = \frac{2}{\lambda_1} \int_{x_0}^{x_0+\lambda_1} f(x) \cos\left(\frac{2\pi}{\lambda_n} x\right) dx \,,$$

$$b_n = \frac{2}{\lambda_1} \int_{x_0}^{x_0+\lambda_1} f(x) \sin\left(\frac{2\pi}{\lambda_n} x\right) dx.$$

So, alternatively, we could write the series as

$$f(x)_{[0,\mathcal{L}]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\lambda_1} x\right) + b_n \sin\left(n \frac{2\pi}{\lambda_1} x\right)$$

where, again, $\lambda_1 = \mathcal{L}$.

2. Further, we could have chosen "$x=0$" anywhere and gotten the same results (e.g., (8) and (9)). Thus, for any point x_0, on $[x_0, x_0+\lambda_1]$ the series (1) [with (2)] converges to $f(x)$.

Since $f(x)$ and all the terms in (1) repeat with repeat distance λ_1 [terms $\cos\left(n\frac{2\pi}{\lambda_1}x\right)$ and $\sin\left(\frac{n2\pi}{\lambda_1}x\right)$ repeat n times in distance λ_1], the series (1) converges to $f(x)$ everywhere on the x-axis.

Comment: It follows from the above argument that the numerical value of a_n is the same for all choices of the point x_0. You can also see this explictly from

$$a_n = \frac{2}{\lambda_1}\int_{x_0}^{x_0+\lambda_1} f(x)\cos\left(n\frac{2\pi}{\lambda_1}x\right)dx$$

– since both $f(x)$ and $\cos\left(n\frac{2\pi}{\lambda_1}x\right)$ both repeat every distance λ_1, then, the entire integrand is repetitive with repeat distance λ_1 – "it doesn't matter where you start the integral, in total, the integral covers the same terrain." (As long as you integrate over one full distance λ_1.)

Of course, the same is true for all b_n.

The last sentence of the theorem, concerning the convergence at a point of discontinuity, is not obvious from our previous considerations, but a mathematical analysis (which we leave to your math course) shows it to be true.

Again: Note that the function repeat distance λ_1 is the same as the wavelength of the $n=1$ terms in the Fourier series. (That's why we called it "λ_1").

Comment:

We can also write the basic series in terms of the basic wavenumber $k_1 \equiv 2\pi/\lambda_1$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n k_1 x) + b_n \sin(n k_1 x)$$

Since $k_n = \frac{2\pi}{\lambda_n} = n k_1$ (show this), this is also

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n x) + b_n \sin(k_n x)$$

All these forms will be useful.

3. What Makes These Fourier Expansions Work? [K-text, page 11-25]

The answer is (as is true on e.g., the interval $[-\pi, \pi]$ for the expansion), orthonormality relations. For example, for interval length (or repeat distance) λ_1 that starts at $x = a$, we have (as you can show)

$$(11a) \quad \int_{x=a}^{x=a+\lambda_1} dx \sin\left(m \frac{2\pi}{\lambda_1} x\right) \sin\left(n \frac{2\pi}{\lambda_1} x\right) = \int_{x=a}^{x=a+\lambda_1} \cos\left(m \frac{2\pi}{\lambda_1} x\right) \cos\left(n \frac{2\pi}{\lambda_1} x\right) dx$$

$$= \begin{cases} 0 \text{ if } m \neq n \text{ (orthogonality)} \\ \lambda/2 \text{ if } m = n \text{ (normalization)} \end{cases}$$

and

$$(11b) \quad \int_{x=a}^{x=a+\lambda} \sin\left(m \frac{2\pi}{\lambda_1} x\right) \cos\left(n \frac{2\pi}{\lambda_1} x\right) dx = 0 \text{ for all integer } m \text{ and } n.$$

(orthogonality)

Example: Expansion interval $[-L, L]$: Then

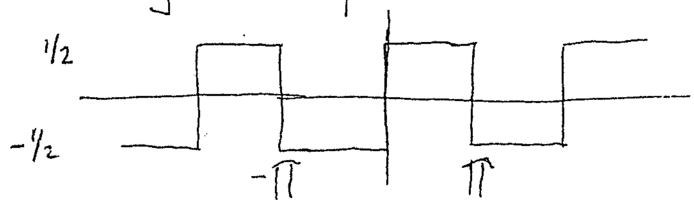
$$(12a) \quad \int_{x=-L}^{x=L} \sin\left(m \frac{\pi}{L} x\right) \sin\left(n \frac{\pi}{L} x\right) dx = \int_{x=-L}^{L} \cos\left(m \frac{\pi}{L} x dx\right) \cos\left(n \frac{\pi}{L} x\right) dx$$

$$= \begin{Bmatrix} 0 \text{ if } m \neq n \\ L \text{ if } m = n \end{Bmatrix}, \text{ and,}$$

$$(12b) \quad \int_{x=-L}^{L} \sin\left(m \frac{\pi}{L} x\right) \cos\left(n \frac{\pi}{L} x\right) dx = 0 \text{ for all integer } m \text{ and } n.$$

C.1. Expansion of Even and Odd Functions - [K-text, Sect 11.9]

Consider again our "square wave" with axes as shown.



The series is $\quad f(x) = \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$

Note that "it turned out" that there are no cosines in this series.

Actually, this is obvious in advance (i.e, before calculating the coefficients).

Why? Because with the choice of axes as given, $f(x)$ is an odd function (i.e, $f(-x) = -f(x)$). But all cosines are even functions (i.e, $f(-x) = f(x)$). Thus, there cannot be any cosines in the expansion of an odd function. Similarly, there cannot be any sines in the expansion of an even function. [We say that an odd (even) function has odd (even) parity around the origin.]

Fourier Coefficients For Even and Odd Functions

Suppose $f(x)$ is even on $[-L, L]$.

We had (still correct, technically)

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \text{ But, this is}$$

$$a_n = \frac{1}{L}\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

But, If $f(x)$ is even, the two integrals are equal. Thus,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\ ,\quad a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \text{ all } b_n = 0. \ (\underline{\text{Why}}?)$$

Now suppose $f(x)$ is odd on $[-L,L]$. Then, by similar logic,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \text{ all } a_n = 0 \text{ (including } a_0!)$$

A function with "no parity" is one that is neither even nor odd.

Summary of Fourier Expansions for functions defined on $[-L,L]$:

a. no parity on $[-L,L]$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

b. $f(x)$ even on $[-L,L]$ (or extended that way from $[0,L]$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right),\quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

"NO SINES ALLOWED HERE"!

c. $f(x)$ odd on $[-L,L]$ (or extended that way from $[0,L]$):

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right),\quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

"NO COSINES ALLOWED HERE"!

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2. Another Way of Looking at It: [From K-text, sect 11.9]

Now let's look at this another way around: Consider an arbitrary function of no parity (i.e., neither even nor odd) defined on, say, $(-L,L)$ that has Fourier series representation given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x) \tag{11.17}$$

Looking at this "full Fourier series," we see that the part $\frac{a_0}{2} + \sum a_n \cos(n\frac{\pi}{L}x)$ corresponds to a Fourier expansion for an even function and that the part $\sum b_n \sin(n\frac{\pi}{L}x)$ corresponds to a Fourier expansion for an odd function. Thus, the implication of eqn. (11.17) is that any function that can be expanded in a Fourier series can be viewed as the sum of an even parity function and an odd parity function. Is this reasonable, or have we gotten into a contradiction?

In fact, it is true that any piecewise continuous function can be written as the sum of an even and an odd function. Here's a way of seeing this: Consider an arbitrary function $f(x)$ of no parity defined on $(-L,L)$. Then the function $g(x) = f(x) + f(-x)$ is even on this interval since, if $x \to -x, f(x) \to f(-x)$ and $f(-x) \to f(x)$. And, the function $h(x) = f(x) - f(-x)$ is odd. But, $f(x) = g(x) + h(x)$! Now notice again how the Fourier series eqn. (11.17) of a function $f(x)$ with no parity reflects this: the sum of the constant term and the cosine terms is even and the sum of the sine terms is odd, and both added together make up $f(x)$ if it has no parity.

D.1. A Question

Let's now raise a question.

We said that, e.g., for a function defined on $[0, L]$, the Fourier expansion is

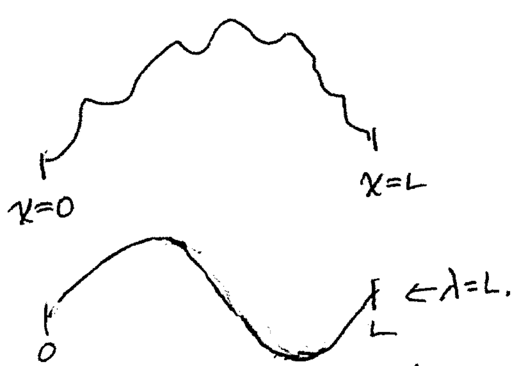
$$(1) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{L} x\right) + b_n \sin\left(n \frac{2\pi}{L} x\right)$$

so that the wavelength of the $n=1$ terms, that is, the longest wavelength of the series is

$$\lambda_1 = L.$$

It may have struck you that there is something "funny" about this. To see this, consider expanding, on $[0, L]$, a shape that looks like that shown below left.

Fig. 1.



Shouldn't the fundamental terms in the expansion for such a function have wavelength $\lambda = 2L$, not $\lambda_1 = L$?

To lead toward resolving this question, we must now take note of another aspect of Fourier expansions on finite intervals – the nonuniqueness of such expansions:

2. Nonuniqueness of Fourier Expansions On Finite Intervals

We can get a clue about this by looking back at

$$(2)\quad f(x)_{[\mathcal{L}]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) + b_n \sin\left(n \frac{2\pi}{\mathcal{L}} x\right)$$

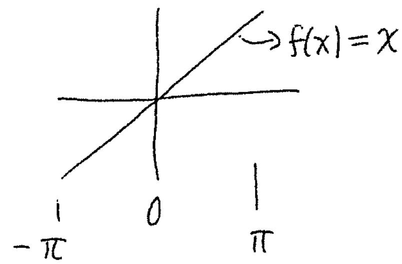
in which $\mathcal{L}$ is the repeat

We ask: Suppose we apply this formula to find a series for the function shown in Fig. 1, but use $\mathcal{L} = 2L$ instead of $\mathcal{L} = L$. To do this, we would have to, e.g., extend the definition of the function $f(x)$ back (arbitrarily) to $[-L, 0]$, so that the repeat period of the extended function is $2L$, rather than L. That might then give a <u>different</u> Fourier series for ~~the entire~~ extended function, and thus, perhaps, a different series on $[0, L]$.

<u>That</u> would imply that, even if the axes are not moved, there is more than one Fourier series representing a function on a finite interval (here $[0, L]$)!

Let's investigate this:

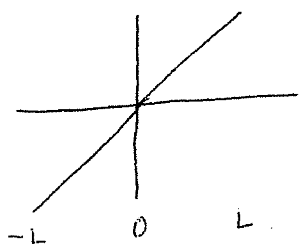
Example: In the text there is provided a Fourier expansion that converged to $f(x) = x$ on $[-\pi, \pi]$.



The series found is

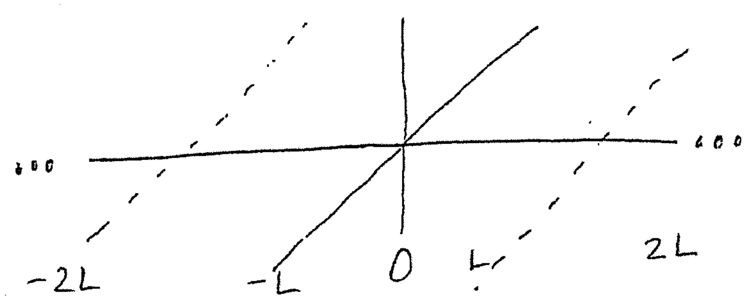
$$(3)\quad x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

You can, of course, generalize this to provide a series expansion for x valid on $[-L, L]$; this is



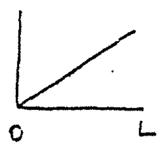
$$(4)\quad x = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Of course, these series converge to the periodic extension of the given $f(x)$, namely, to



Also, the series (4) is valid for x on the restricted interval $[0, L]$; that it contains no cosines is, of course, a result of the fact that the extension of the function back from $[0, L]$ is odd.

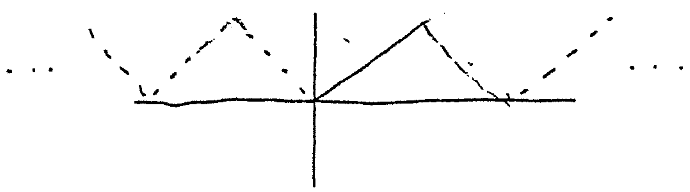
But now a person comes along and says that in their class



they were given the function $f(x) = x$ on only $[0,L]$.

Asked to find an expansion for this $f(x)$,

<u>they</u> completed it periodically as



!

And, in their class they therefore found for <u>their</u> series:

all $b_n = 0$

$$a_n = \frac{2}{L}\int_0^L x \cos\frac{n\pi x}{L}\,dx$$

As you can show, this is $\quad a_n = \frac{-2L}{n^2\pi^2}\left[(-1)^n - 1\right] \quad n \neq 0,$

i.e.,

$$a_n = \begin{cases} 0 & n \text{ even} \\ \frac{-4L}{n^2\pi^2} & n \text{ odd} \end{cases}$$

and for $n=0$: $\quad a_0 = \frac{2}{L}\int_0^L x\,dx = \frac{2}{L}\cdot\frac{L^2}{2} = L$

so - for them,

on $[0,L]$, $\quad x = \frac{L}{2} - \frac{4L}{\pi^2}\left[\cos\frac{\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} + \frac{1}{25}\cos\frac{5\pi x}{L} + \cdots\right]^{*}$

On $[0,L]$ <u>both</u> this and our original sine series (4) converge to the same function x!

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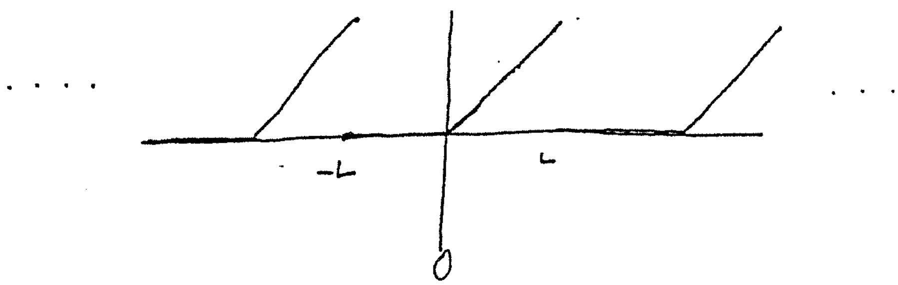
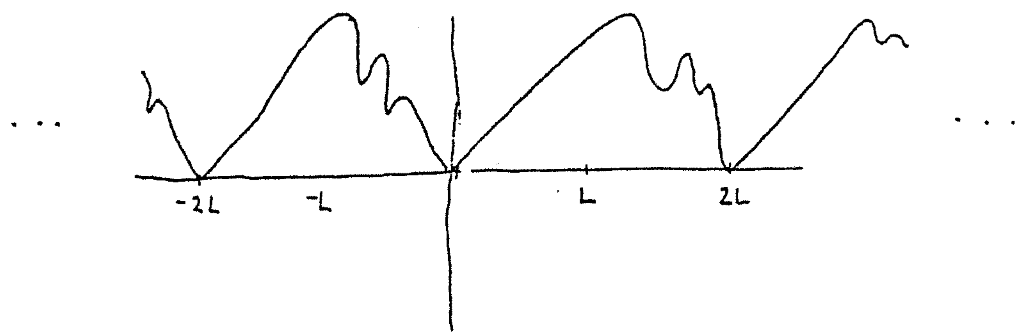
* If $L=\pi$, this $x = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right]$

so - on [0,L] we have two Fourier series for the same function!

In fact - you could have said - lets extend the function periodically as



or like this

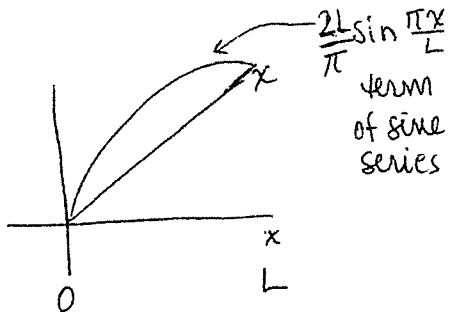


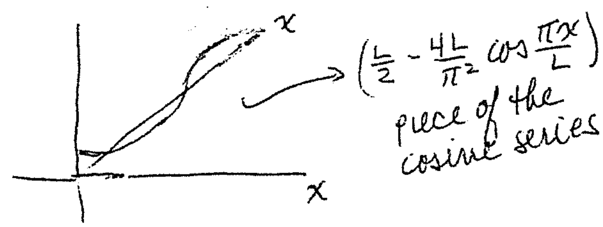
These would give different series - for example, the two immediately above would have both sines and cosines! (since both are not even and both are not odd !)

So - apparently there are an infinite number of Fourier series representations for a function on the finite interval [0,L] ! (even when the x-and y-axes are fixed !)

Note that these really are "different series" - i.e, this is not like the "nonuniqueness" that we generated by moving the y-axis around - there we had a series of the same shapes, only the names of terms changed (e.g., sines → -cosines). Here we actually have different series with different shapes in them!

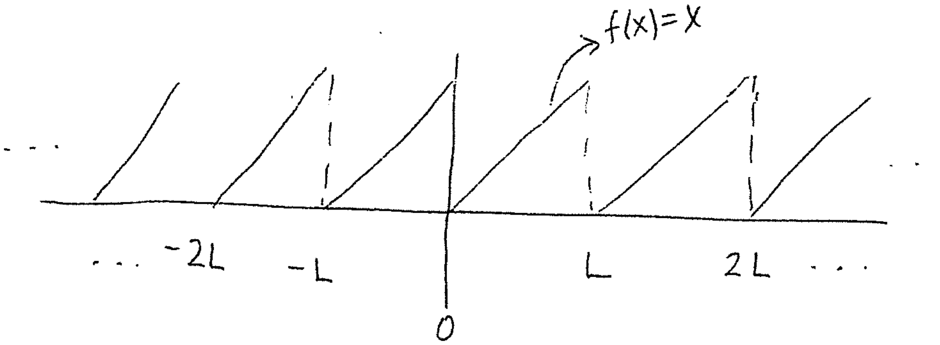
Example: Consider the sine and the cosine series for $f(x) = x$ on $[0, L]$ that we found. Note, e.g., that the qualities of the truncated approximations are different. Consider, e.g., "the first order" approximations in each case:





Also, the two series converge at different rates - see the figures on page next.

Finally (for this) here is yet another series that converges to $f(x) = x$ on $[0, L]$ - extend the function as an exact repeat of what it does on $[0, L]$ (see figure top of next page); then the extended function has repeat distance L, so, as you know,



the Fourier series for this extended function of no parity is

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)$$

[since no parity, series has both sines and cosines]

where you can figure out the a_n's and b_n's; of course, this series also converges to $f(x)=x$ on $[0,L]$. Now only "even harmonics" of $\frac{\pi}{L}$ appear. [Why?

Figures referred to on the previous page. They are from Mathematics For Physicists by Sosan M. Lea, chapter 4 (© Thomson-Brooks/Cole, 2004.)

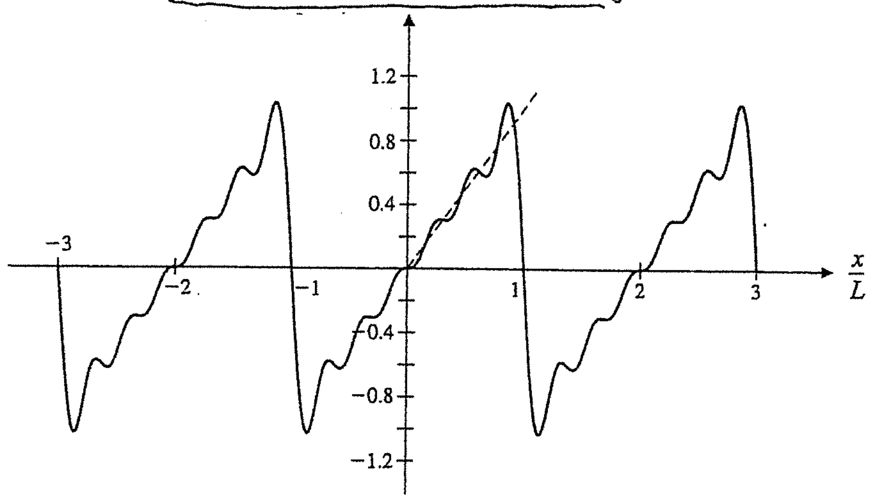


FIGURE 4.5. The first six terms in the sine series of the function $f(x) = x$ defined for $0 < x < L$.

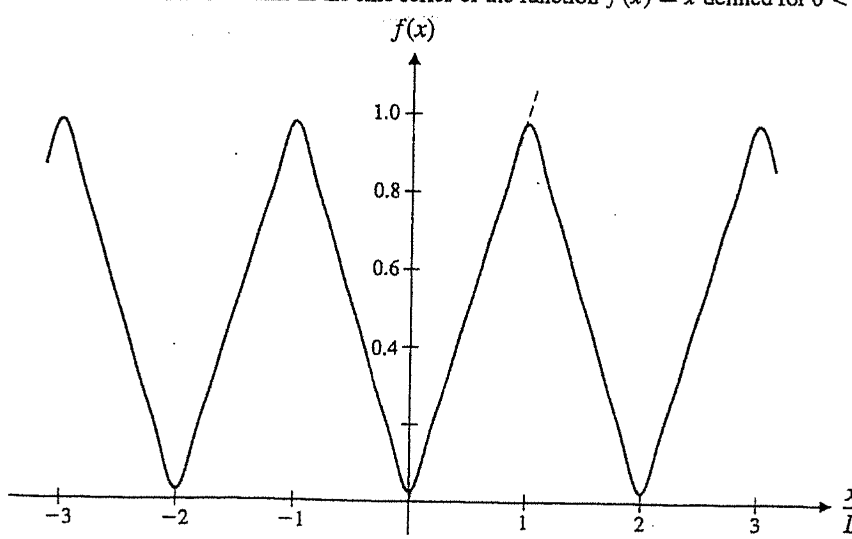
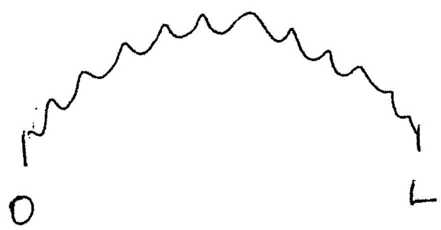


FIGURE 4.6. The first four terms in the cosine series of the function $f(x) = x$ defined for $0 < x < L$. Compare this graph with Figure 4.5. A few terms of the cosine series represent the function more closely than does the sine series.

3. Now let's return briefly to our question of part B.

We were curious about the Fourier expansion of



on $[0, L]$.

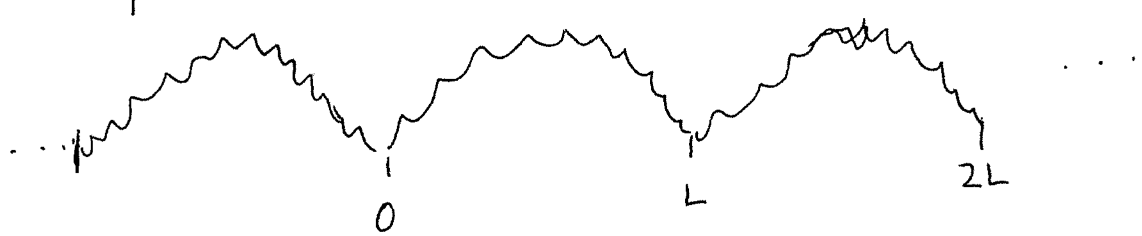
Shouldn't the Fourier $\lambda_1 = 2L$?

As we now see, there are many Fourier expansions that converge to this function on $[0, L]$.

One, of the type we referred to in sect. B above, namely,

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(n \frac{2\pi}{L} x\right) + \cancelto{0}{\sum b_n \sin\left(n \frac{2\pi}{L} x\right)},$$

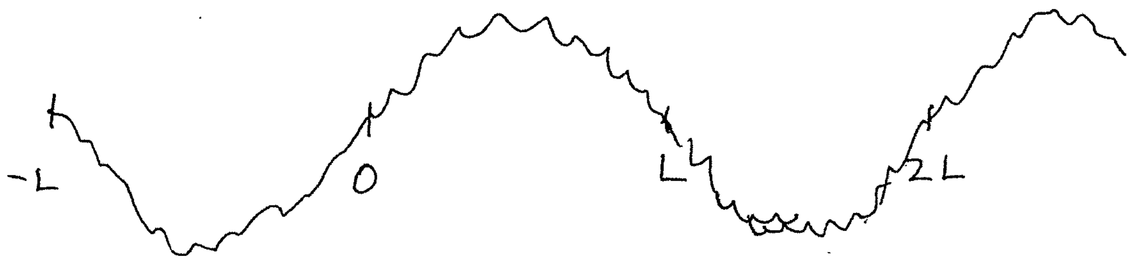
converges to $f(x)$ on $[0, L]$ and to its periodic extension with repeat distance L:



- and its clear why $\lambda_1 = L$ to converge to what's pictured
- all Fourier terms must be periodic with period L and a term with lowest period $2L$ is not!

(next page →)

But another possibility is a series that converges to f(x) on [0,L] is:



The series for that must have repetition distance 2L, so its fundamental wavelength is $\lambda_1 = 2L$! It is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cancelto{0}{a_n \cos\left(n\frac{\pi}{L}x\right)} + \sum b_n \sin\left(n\frac{\pi}{L}x\right).$$

Both series converge to the same function on [0, L], but do not converge to the same function on either of [-L, 0] or [L, 2L], etc.

Of course, this is general. Thus, when we say that

$$f(x)_{[0,L]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{L}x\right) + b_n \sin\left(n\frac{2\pi}{L}x\right),$$

we mean that that is a series that converges to f(x) on [0, L] and has repeat distance L.

Likewise, when we say that

$$f(x)_{[-L,L]} = \frac{a_0}{2} + \sum a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right),$$

we mean that that's a series that converges to f(x) on [-L, L] and has rept. dist. 2L.

Likewise, when we say

$$f(x)\Big|_{[a,b]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{|b-a|}x\right) + b_n \sin\left(n\frac{2\pi}{|b-a|}x\right)$$

we mean a series that converges to $f(x)$ on $[a,b]$ <u>and</u> has repeat distance $|b-a|$.

<u>Question</u>: What would the general form be of a Fourier series that converges to $f(x)$ on $[a,b]$, but which has repeat distance $2|b-a|$?

E. Fourier Analysis Applies to Functions of Time Also.

Suppose we are given a function of time $f(t)$ that we wish to "Fourier analyze." We might be given $f(t)$ as a periodic function with (time) repeat period T, or, we might simply need to find a Fourier series representation convergent to an "arbitrary" function $f(t)$ on a specific interval, say $[-t_0, t_0]$ or $[0, t_0]$.

It should be pretty clear how to do this. To start, we return to the "wavelength form" of our Fourier theorem - if $f(x)$ has repeat distance $\mathcal{L}$, then

(1) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) + b_n \sin\left(n \frac{2\pi}{\mathcal{L}} x\right)$$

(2) where $$a_n = \frac{2}{\mathcal{L}} \int_{x_0}^{x_0+\mathcal{L}} f(x) \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) dx, \text{ etc.,}$$

so that $\lambda_1 = \mathcal{L}$.

Now we use the following "translation dictionary:"

$$x \longrightarrow t$$

$$\mathcal{L} \longrightarrow T$$ where T = (repeat period of parent function $f(t)$) = (longest period of Fourier expansion (T_1))

so $T_1 = T$

Then, the orthonormality relations become [K-Text, page 11-31]:

(3) $$\int_{t=a}^{t=a+T_1} \sin\left(m\frac{2\pi}{T_1}t\right)\sin\left(n\frac{2\pi}{T_1}t\right)dt = \int_{t=a}^{t=a+T_1} \cos\left(m\frac{2\pi}{T_1}t\right)\cos\left(n\frac{2\pi}{T_1}t\right)dt$$

$$= \begin{cases} 0, & m \neq n \text{ (orthogonality)} \\ T_1/2, & m = n \text{ (normalization)} \end{cases}$$

and

(4) $$\int_{t=a}^{t=a+T_1} \sin\left(m\frac{2\pi}{T_1}t\right)\cos\left(n\frac{2\pi}{T_1}t\right)dt = 0$$ for all integer m and n (normalization), so:

Restricted Statement of Fourier's Theorem for function of time, Version III:

Let $f(t)$ be a piecewise continuous repetitive function of t with repeat period T on the entire t-axis. Then, except at the points of discontinuity, the series

(11.32a) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}t\right) + b_n \sin\left(\frac{2n\pi}{T}t\right)$$

with

(11.32b) $$a_n = \frac{2}{T}\int_{T_1} f(t)\cos\frac{2n\pi}{T}t\ dt \quad \text{and} \quad b_n = \frac{2}{T}\int_{T_1} f(t)\sin\frac{2n\pi}{T}t\ dt.$$

converges to $f(t)$. At a point of jump discontinuity, the series above converges to average of the values of $f(t)$ immediately before and after the discontinuity. Since $\omega_1 = 2\pi/T_1$ and T_1 (period of Fourier fundamental) $= T$, another way of writing (~~11.28a~~ 11.32a) is

(11.33) $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t).$$