PROBLEMA 3.1 Sean $(V,\langle\,\rangle)$ un espacio euclídeo y $H:\{\bar u+i\bar v \mid \bar u,\bar v\in V\}$. En $H$ se define $(\bar u+i\bar v)*(\bar z+i\bar w)=[\langle\bar u,\bar z\rangle+\langle\bar v,\bar w\rangle]+i[-\langle\bar u,\bar w\rangle+\langle\bar v,\bar z\rangle]$, donde $\bar u+i\bar v,\ \bar z+i\bar w\in H$. Demuestra que $(H,*)$ es un espacio vectorial hermítico.

$\boxed{1}$ ¿$\langle\bar u,\bar v\rangle=\overline{\langle\bar v,\bar u\rangle}$? $\longrightarrow$ Para nuestro caso: ¿$(\bar u+i\bar v)*(\bar z+i\bar w)=\overline{(\bar u+i\bar v)*(\bar z+i\bar w)}$?

$$\cancel{\langle\bar u,\bar v\rangle=(\bar u+i\bar v)*(\bar z+i\bar w)}$$
$$\cancel{\Rightarrow(\bar u+i\bar v)*(\bar z+i\bar w)=(\langle\bar u,\bar z\rangle+\langle\bar v,\bar w\rangle)+i(-\langle\bar u,\bar w\rangle+\langle\bar v,\bar z\rangle)}$$
$$\cancel{\overline{(\bar u+i\bar w)*(\bar z+i\bar w)}=(\langle\bar u,\bar z\rangle+\langle\bar v,\bar w\rangle)-i(-\langle\bar u,\bar w\rangle+\langle\bar v,\bar z\rangle)=\langle\bar u,\bar z\rangle+\langle\bar v,\bar w\rangle+\langle\bar u,\bar w\rangle i-\langle\bar v,\bar z\rangle i}$$
$$\cancel{=(\langle\bar u,\bar z\rangle+\langle\bar v,\bar w\rangle)+i(\langle\bar v,\bar z\rangle+\langle\bar u,\bar w\rangle)}$$

$$\overline{(\bar u+i\bar v)*(\bar z+i\bar w)}=\overline{(\langle\bar u,\bar z\rangle+\langle\bar v,\bar w\rangle)+i(-\langle\bar u,\bar w\rangle+\langle\bar v,\bar z\rangle)}=\overline{\langle\bar u,\bar z\rangle}+\overline{\langle\bar v,\bar w\rangle}-i\,\overline{(-\langle\bar u,\bar w\rangle+\langle\bar v,\bar z\rangle)}$$
$$=\langle\bar z,\bar u\rangle+\langle\bar w,\bar v\rangle-i(\pm\langle\bar w,\bar u\rangle\mp\langle\bar z,\bar v\rangle)=$$
$$=\langle\bar z,\bar u\rangle+\langle\bar w,\bar v\rangle-i(-\langle\bar z,\bar v\rangle+\langle\bar w,\bar u\rangle)=(\bar z+i\bar w)*(\bar u+i\bar v)$$

$\Rightarrow$ la propiedad de conjugación queda mostrada

$\boxed{2}$ Linealidad en la primera componente:

$$[\alpha(\bar u_1+i\bar v_1)+\beta(\bar u_2+i\bar v_2)]*[\bar z+i\bar w]=[(\alpha\bar u_1+\beta\bar u_2)+(\alpha i\bar v_1+\beta i\bar v_2)]*[\bar z+i\bar w]=$$
$$=[\langle\alpha\bar u_1+\beta\bar v_2,\bar z\rangle+\langle\alpha\bar v_1+\beta\bar v_2,\bar w\rangle]+i[-\langle\alpha\bar u_1+\beta\bar u_2,\bar z\rangle+\langle\alpha\bar v_1+\beta\bar v_2,\bar z\rangle]=$$
$$=[\alpha\langle\bar u_1,\bar z\rangle+\beta\langle\bar u_2,\bar z\rangle+\alpha\langle\bar v_1,\bar w\rangle+\beta\langle\bar v_2,\bar w\rangle]+i[-\alpha\langle\bar u_1,\bar z\rangle\mp\beta\langle\bar u_2,\bar z\rangle+\alpha\langle\bar v_1,\bar z\rangle+\beta\langle\bar v_2,\bar z\rangle]$$
$$=\alpha\left([\langle\bar u_1,\bar z\rangle+\langle\bar v_1,\bar w\rangle]+i[-\langle\bar u_1,\bar z\rangle+\langle\bar v_1,\bar z\rangle]\right)+\beta\left([\langle\bar u_2,\bar z\rangle+\langle\bar v_2,\bar w\rangle]+i[-\langle\bar u_2,\bar z\rangle+\langle\bar v_2,\bar z\rangle]\right)=$$
$$=\alpha(\bar u_1+i\bar v_1)*(\bar z+i\bar w)+\beta(\bar u_2+i\bar v_2)*(\bar z+i\bar w)\Rightarrow (H,*)\text{ es lineal en la 1ª componente}$$

$\boxed{3}$ $(\bar u+i\bar v)*(\bar u+i\bar v)\geq 0\ \forall(\bar u+i\bar v)\in H$ y $(\bar u+i\bar v)*(\bar u+i\bar v)=0\Rightarrow(\bar u+i\bar v)=\bar 0$

$$(\bar u+i\bar v)*(\bar u+i\bar v)=[\langle\bar u,\bar u\rangle+\langle\bar v,\bar v\rangle]+i[\langle\bar u,\bar v\rangle+\langle\bar v,\bar u\rangle]=\langle\bar u,\bar u\rangle+\langle\bar v,\bar v\rangle\ \cancel{\geq 0}$$

ya que el producto escalar $\langle,\rangle$ es siempre $\geq 0$ por ser $V$ euclídeo

$$(\bar u+i\bar v)*(\bar u+i\bar v)=0\Longleftrightarrow\langle\bar u,\bar u\rangle+\langle\bar v,\bar v\rangle=0\Longleftrightarrow\begin{cases}\langle\bar u,\bar u\rangle=0\\ \langle\bar v,\bar v\rangle=0\end{cases}$$

y por ser $\langle,\rangle$ el producto escalar de un espacio euclídeo, $\langle\bar u,\bar u\rangle=0\Rightarrow\bar u=\bar 0$ y $\langle\bar v,\bar v\rangle=0\Rightarrow\bar v=\bar 0\Rightarrow(\bar u+i\bar v)*(\bar u+i\bar v)=0\Rightarrow(\bar u+i\bar v)=\bar 0$

$\Rightarrow$ Definida positiva

$\boxed{1},\boxed{2},\boxed{3}\Longrightarrow(H,*)$ define un espacio vectorial hermítico.

**PROBLEMA 3.2** En $\mathbb{C}^3$, dotado del producto ~~escalar~~ hermítico canónico se considera el subespacio vectorial $W = \text{Env}(\{(-i, 1, i)\})$

1. Encuentra el complemento ortogonal de $W$ y una base ortonormal $W^\perp$

$$W^\perp = \text{Env}(\{\vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \ \forall \vec{w} \in W\}) \Rightarrow (x_1 + x_1' i, x_2 + x_2' i, x_3 + x_3' i) \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \overline{\begin{pmatrix} -i \\ 1 \\ i \end{pmatrix}} = 0 \Leftrightarrow$$

$$\Leftrightarrow (x_1 + x_1' i, x_2 + x_2' i, x_3 + x_3' i) \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix} = 0 \ \cancel{\Leftrightarrow x_1 + x_1' i + x_2 + x_2' i - x_3 i + x_3' = 0 \Leftrightarrow \begin{cases} x_1 + x_2 + x_3' = 0 \\ x_1' + x_2' - x_3 = 0 \end{cases} \Leftrightarrow}$$

$$\cancel{\Leftrightarrow \begin{cases} x_1 = -x_2 - x_3' \\ x_1' = x_3 - x_2' \end{cases} \Leftrightarrow W^\perp = \text{Env}(\{(x_1 + x_1' i, x_2 + x_2' i, x_3 + x_3' i) \in \mathbb{C}^3 : x_1 = -x_2 - x_3',\ x_1' = x_2' - x_3\}) \Leftrightarrow}$$

$$\cancel{\Leftrightarrow W^\perp = \text{Env}(\{(-x_2 - x_3' + (x_3 - x_2')i, x_2 + x_2' i, x_3 + x_3' i) \in \mathbb{C}^3\}) = \text{Env}(\{(-1-i, 1+i, 0), (-1+i, 0, 1+i)\})}$$

$$\Leftrightarrow x_1 i - x_1' + x_2 + x_2' i - x_3 i + x_3' \Leftrightarrow \begin{cases} x_1 + x_2' - x_3 = 0 \\ -x_1' + x_2 + x_3' = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_3 - x_2' \\ x_1' = x_2 + x_3' \end{cases} \Leftrightarrow$$

$$\Leftrightarrow W^\perp = \text{Env}(\{(x_1 + x_1' i, x_2 + x_2' i, x_3 + x_3' i) \in \mathbb{C}^3 ; x_1 = x_3 - x_2',\ x_1' = x_2 + x_3'\}) = \text{Env}(\{(x_3 - x_2' + (x_2 + x_3')i, x_2 + x_2' i, x_3 + x_3' i) \in \mathbb{C}^3\})$$

$$= \text{Env}(\{(1+i, 0, 1+i), (-1+i, 1+i, 0)\})$$

$$\|W_1^\perp\| = \left((1+i, 0, 1+i) \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \overline{\begin{pmatrix} 1+i \\ 0 \\ 1+i \end{pmatrix}}\right)^{1/2} = \left((1+i, 0, 1+i) \begin{pmatrix} 1-i \\ 0 \\ 1-i \end{pmatrix}\right)^{1/2} = (4)^{1/2} = \sqrt{4} = 2$$

$$\|W_2^\perp\| = \left((-1+i, 1+i, 0) \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \overline{\begin{pmatrix} -1+i \\ 1+i \\ 0 \end{pmatrix}}\right)^{1/2} = \left((-1+i, 1+i, 0) \begin{pmatrix} -1-i \\ 1-i \\ 0 \end{pmatrix}\right)^{1/2} = (4)^{1/2} = \sqrt{4} = 2$$

$$W^\perp = \text{Env}\left(\left\{\left(\tfrac{1}{2} + \tfrac{1}{2}i, 0, \tfrac{1}{2} + \tfrac{1}{2}i\right), \left(-\tfrac{1}{2} + \tfrac{1}{2}i, \tfrac{1}{2} + \tfrac{1}{2}i, 0\right)\right\}\right)$$

**PROBLEMA 3.3** En el espacio $\mathbb{C}^2$ dotado del producto ~~escalar~~ hermítico canónico, estudia cuales de las siguientes matrices son unitarias

1. $\begin{pmatrix} 3+2i & 5-i \\ 2+i & 1+i \end{pmatrix}$ $\cancel{(a+bi)(c+di) \begin{pmatrix} 3+2i & 5-i \\ 2+i & 1+i \end{pmatrix} \begin{pmatrix} a-bi \\ c-di \end{pmatrix} = (3a+2b+(-3b+2a)i + 5c - d + (-5d + c)i)}$

Si son unitarias respecto al producto hermítico, entonces $A^\dagger A = I$

$$A = \begin{pmatrix} 3+2i & 5-i \\ 2+i & 1+i \end{pmatrix} \quad A^T = \begin{pmatrix} 3+2i & 2+i \\ 5-i & 1+i \end{pmatrix} \quad A^\dagger = \begin{pmatrix} \overline{3+2i} & \overline{2+i} \\ \overline{5-i} & \overline{1+i} \end{pmatrix} = \begin{pmatrix} 3-2i & 2-i \\ 5+i & 1-i \end{pmatrix}$$

$$\begin{pmatrix} 3-2i & 2-i \\ 5+i & 1-i \end{pmatrix} \begin{pmatrix} 3+2i & 5-i \\ 2+i & 1+i \end{pmatrix} = \begin{pmatrix} 18 & 16+12i \\ 16+12i & 28 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \Rightarrow \text{no es unitaria}$$

2. $\begin{pmatrix} \frac{1+\sqrt{2}i}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1-\sqrt{2}i}{2} \end{pmatrix}$ $\quad A^\dagger = \begin{pmatrix} \frac{1-\sqrt{2}i}{2} & -i/2 \\ -\frac{i}{2} & \frac{1+\sqrt{2}i}{2} \end{pmatrix} \quad A^\dagger A = \begin{pmatrix} \frac{1-\sqrt{2}i}{2} & -i/2 \\ -i/2 & \frac{1+\sqrt{2}i}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{2}i}{2} & i/2 \\ i/2 & \frac{1-\sqrt{2}i}{2} \end{pmatrix} \cancel{\neq} I^2 \Rightarrow$ ~~no~~ es unitaria

no

3. $\begin{pmatrix} 1+i & 2+i \\ -1-i & -1 \end{pmatrix}$ $\quad A^\dagger = \begin{pmatrix} 1-i & -1+i \\ 2-i & -1 \end{pmatrix} \quad A^\dagger A = \begin{pmatrix} 1-i & -1+i \\ 2-i & -1 \end{pmatrix} \begin{pmatrix} 1+i & 2+i \\ -1-i & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4-2i \\ 4+2i & 6 \end{pmatrix} \neq I^2$

**PROBLEMA 3.4.** Determina bajo qué condiciones en $a,b \in \mathbb{R}$ la matriz $A=\begin{pmatrix} 0 & a & 0 & ib \\ a & 0 & ib & 0 \\ 0 & ib & 0 & a \\ ib & 0 & a & 0 \end{pmatrix}$ es unitaria

$$A^{\dagger}=\begin{pmatrix} 0 & a & 0 & -ib \\ a & 0 & -ib & 0 \\ 0 & -ib & 0 & a \\ -ib & 0 & a & 0 \end{pmatrix} \qquad A^{\dagger}A=\begin{pmatrix} 0 & a & 0 & -ib \\ a & 0 & -ib & 0 \\ 0 & -ib & 0 & a \\ -ib & 0 & a & 0 \end{pmatrix}\begin{pmatrix} 0 & a & 0 & ib \\ a & 0 & ib & 0 \\ 0 & ib & 0 & a \\ ib & 0 & a & 0 \end{pmatrix}=\begin{pmatrix} a^2+b^2 & 0 & 0 & 0 \\ 0 & a^2+b^2 & 0 & 0 \\ 0 & 0 & a^2+b^2 & 0 \\ 0 & 0 & 0 & a^2+b^2 \end{pmatrix}$$

**PROBLEMA 4.1** Encuentra la base dual de $D=\{\vec{v}_1=(1,2,1), \vec{v}_2=(0,1,1), \vec{v}_3=(1,1,1)\}$ en $\mathbb{R}^3$. Si $\beta(x_1,x_2,x_3)=x_1+3x_2+x_3$, calcula las coordenadas de $\beta$ en la base dual de $D$

$D=\mathrm{Env}(\{(1,2,1),(0,1,1),(1,1,1)\})$ sean $e^1:\mathbb{R}^3\to\mathbb{R}$ $e^2:\mathbb{R}^3\to\mathbb{R}$ $e^3:\mathbb{R}^3\to\mathbb{R}$

$D^*=\mathrm{Env}(\{e^1,e^2,e^3: e^i(v_j)=\delta_{ij}\ \forall v_j\in D\})$

$(x,y,z)=\alpha(1,2,1)+\beta(0,1,1)+\gamma(1,1,1)=(\alpha+\gamma, 2\alpha+\beta+\gamma, \alpha+\beta+\gamma)$

$$\begin{cases} x=\alpha+\gamma \\ y=2\alpha+\beta+\gamma \\ z=\alpha+\beta+\gamma \end{cases} \Longleftrightarrow \begin{cases} \alpha=x-\gamma \\ y=2x-\gamma+z-x \\ \beta=z-x \end{cases} \Longleftrightarrow \begin{cases} \alpha=x-\gamma \\ \gamma=x+z-y \\ \beta=y-x \end{cases} \Longleftrightarrow \begin{cases} \alpha=y-z \\ \beta=z-x \\ \gamma=x-y+z \end{cases}$$

- $e^1(x,y,z)=e^1(\alpha e_1,\beta e_2,\gamma e_3)\overset{\text{linearidad}}{=}\alpha \underbrace{e^1 e_1}_{1}+\beta \underbrace{e^1 e_2}_{0}+\gamma \underbrace{e^1 e_3}_{0}=\alpha=y-z$
- $e^2(x,y,z)=e^2(\alpha e_1,\beta e_2,\gamma e_3)=\alpha \underbrace{e^2 e_1}_{0}+\beta \underbrace{e^2 e_2}_{1}+\gamma \underbrace{e^2 e_3}_{0}=\beta=z-x$
- $e^3(x,y,z)=e^3(\alpha e_1,\beta e_2,\gamma e_3)=\alpha \underbrace{e^3 e_1}_{0}+\beta \underbrace{e^3 e_2}_{0}+\gamma \underbrace{e^3 e_3}_{1}=\gamma=x-y+z$

$\Rightarrow \boxed{D^*=\mathrm{Env}(\{(0,1,-1),(-1,0,1),(1,-1,1)\})}$

$$\beta^*=(P_{V^*}^{V})^T\beta=\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}^T\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$(x_1,x_2,x_3)\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}=2x_1-x_3$ $\qquad \boxed{\beta^*(x_1,x_2,x_3)=2x_1-x_3}$

**PROBLEMA 4.2** Encuentra la base dual de $D=\{1, x+1, x^2-2, x^3-x^2\}$ en el espacio vectorial $\mathbb{R}_3[x]$

$D^*=\mathrm{Env}(\{e^1,e^2,e^3,e^4: e^i(v_j)=\delta_{ij}\ \forall v_j\in D\})$ sean $e^1:\mathbb{R}_3[x]\to\mathbb{R}$ $e^2:\mathbb{R}_3[x]\to\mathbb{R}$

$a+bx+cx^2+dx^3=\alpha\cdot 1+\beta(x+1)+\gamma(x^2-2)+\psi(x^3-x^2)$ $\quad e^3:\mathbb{R}_3[x]\to\mathbb{R}$ $e^4:\mathbb{R}_3[x]\to\mathbb{R}$

$$\begin{cases} a=\alpha+\beta-2\gamma \\ b=\beta \\ c=\gamma-\psi \\ d=\psi \end{cases} \Longleftrightarrow \begin{cases} \beta=b \\ \psi=d \\ c=\gamma-d \\ a=\alpha+b+2d-2c \end{cases} \Longleftrightarrow \begin{cases} \alpha=a-b+2c-2d \\ \beta=b \\ \gamma=c+d \\ \psi=d \end{cases}$$

$e^1(x_1,x_2,x_3,x_4)=e^1(\alpha e_1,\beta e_2,\gamma e_3,\psi e_4)=\alpha \underbrace{e^1 e_1}_{1}+\beta \underbrace{e^1 e_2}_{0}+\gamma \underbrace{e^1 e_3}_{0}+\psi \underbrace{e^1 e_4}_{0}=\alpha=x_1-x_2x+2x_3x^2-2x_4x^3$

$e^2(x_1,x_2,x_3,x_4)=\beta=x_2x$ $\qquad e^3(x_1,x_2,x_3,x_4)=\gamma=x_3x^2+x_4x^3$ $\quad e^3(x_1,x_2,x_3,x_4)=x_4x^3$

$D^*=\mathrm{Env}(\{(1,-1,2,-2),(0,1,0,0),(0,0,1,1),(0,0,0,1)\})$