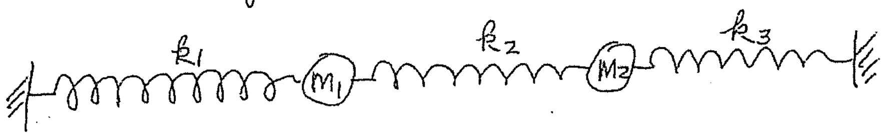
Physics 251 - 11th Class - Tues, Feb. 13, 2024

A. A Systematic Method For Finding Normal Modes

(K-Text, sect 6.14)

Consider the "asymmetric" system (re longitudinal oscillations)



with $m_1 \neq m_2$ and $k_1 \neq k_2 \neq k_3$. The differential equations are

$$m_1 \ddot{\psi}_a = -k_1 \psi_a + k_2(\psi_b - \psi_a)$$

$$m_2 \ddot{\psi}_b = -k_3 \psi_b - k_2(\psi_b - \psi_a)$$

If we attempt to apply the method of finding normal coordinates to these equations we find that it is not so easy. If we simply try adding the two equations, we obtain

$$(m_1 \psi_a + m_2 \psi_b)^{\cdot\cdot} = -k_1 \psi_a - k_3 \psi_b$$

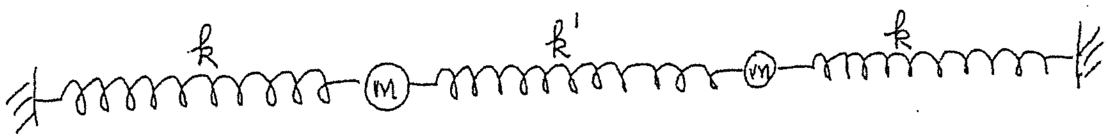
which is not of normal form. Subtracting them also does not lead to anything obviously useful; certainly we do not obtain an equation $\ddot{q} = -\omega^2 q$ for any clearly identifiable "coordinate" $q(t)$.

[These difficulties are not surprising considering that the system lacks left-right symmetry, which means that the mode shapes will not involve equal-magnitude displacements of the masses or displacements that are easy-to-guess multiples of each other]

As well, its difficult in this situation to use the method of physical meaning of ω^2.* However, many situations in "real life" are like this.

For this reason, I will now show you a systematic method that, at least in principle, always finds normal modes if there are any. The point will not be that we have an overiding interest in very asymmetric systems, but rather, that this new method generalizes well to situations involving wave normal modes.

We begin by illustrating this new method on a system that <u>can</u> be solved by the previous two methods. (Indeed, you've done that for homework). The equilibrium configuration of this system is shown below: We consider longitudinal oscillations.



Note that the middle spring has a different spring constant than the other two. The governing differential equations are

$$m\ddot{\psi}_a = -k\psi_a + k'(\psi_b - \psi_a)$$

(1) $$m\ddot{\psi}_b = -k\psi_b - k'(\psi_b - \psi_a)$$

where ψ_a and ψ_b respectively represent the displacements of the left and right masses from equilibrium.

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* Since that method requires that we already have a good guess of the mode configurations

For convenience, we rewrite these as

$$(2)\quad \begin{aligned} \ddot{\psi}_a + \frac{k}{m}\psi_a - \frac{k'}{m}(\psi_b-\psi_a) &= 0 \\ \ddot{\psi}_b + \frac{k}{m}\psi_b + \frac{k'}{m}(\psi_b-\psi_a) &= 0 \end{aligned}$$

The next step in the systematic method is to see if there are any normal mode solutions to these equations by plugging in the <u>assumption</u> of a normal mode. (Of course, this may or may not work, depending on the case at hand. Our attitude is – we will just see what happens). So, we <u>try</u>

$$(3)\quad \begin{aligned} \psi_a(t) &= A\cos(\omega t+\phi) \\ \psi_b(t) &= B\cos(\omega t+\phi) \end{aligned} \quad \left\{\begin{matrix}\text{TRIAL ASSUMPTION} \\ \text{OF A NORMAL MODE}\end{matrix}\right\}^*$$

Here, ω, A, B and ϕ are open parameters – we don't yet know what they should be. Putting (3) into (2) and cancelling the common factor $\cos(\omega t+\phi)$, we wind up with the algebraic equations

$$(4)\quad \begin{aligned} \left(\frac{k+k'}{m}-\omega^2\right)A + \left(-\frac{k'}{m}\right)B &= 0 \\ -\frac{k'}{m}A + \left(\frac{k+k'}{m}-\omega^2\right)B &= 0 \end{aligned}$$

These are simultaneous equations in the unknowns A and B.

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* Recall that, by definition, in a normal mode, all moving parts oscillate simultaneously in s.h.m. at a common frequency. The amplitudes of the moving parts in a normal mode may be different; the individual parts oscillate in phase or 180° out of phase

From a "college algebra" course, these equations will have a nonzero solution if and only if the determinant of the coefficients of A and B are zero - i.e, either $A=0$ and $B=0$, or

$$\begin{vmatrix} \frac{k+k'}{m} - \omega^2 & -\frac{k'}{m} \\ -\frac{k'}{m} & \frac{k+k'}{m} - \omega^2 \end{vmatrix} = 0,$$

which is to say,

$$\left(\frac{k+k'}{m} - \omega^2\right)^2 - \left(\frac{k'}{m}\right)^2 = 0, \quad \text{or}$$

$$\frac{k+k'}{m} - \omega^2 = \pm \frac{k'}{m}, \quad \text{or}$$

(5) $$\omega^2 = \frac{k+k'}{m} \mp \frac{k'}{m}$$

This is very interesting - note that the mathematics is saying that there are <u>two</u> possibilities for ω^2 (and <u>only</u> two).

Thus, our **trial** assumption of normal modes is vindicated - the math is saying that there are two and only two. From (5) they are

(6) $$\omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{k+2k'}{m}$$

exercise: By using the "method of the physical meaning of ω^2" along with a guess for the mode shapes, convince yourself that these values for ω_1 and ω_2 make sense.

Note that, if $k' = k$, these agree with our previous result.

Now we proceed to find the "mode shapes", or "**mode configurations**" (the ratio of the amplitudes of the two masses in each mode).

To do this, we plug these values for ω^2 successively into (4).

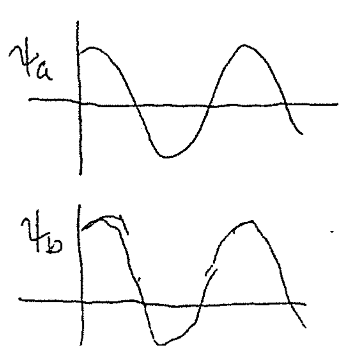
For $\omega^2 = \omega_1^2 = \frac{k}{m}$, (4a) becomes

MODE 1

$$\left(\frac{k+k'}{m} - \frac{k}{m}\right) A - \frac{k'}{m} B = 0 \quad , \text{ or}$$

$$\frac{k'}{m} A - \frac{k'}{m} B = 0, \quad \text{or} \quad \boxed{A = B}$$

In mode 1 for this problem, the amplitudes are equal. Thus, plotting ψ_a and ψ_b in mode 1, they oscillate in phase.



Plugging $\omega^2 = \omega_2^2 = \frac{k+2k'}{m}$ into (4a) we find

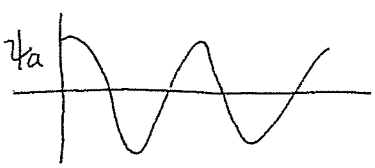
MODE 2

$$\left(\frac{2k'}{m} - \frac{k+2k'}{m}\right) A - \frac{k'}{m} B = 0 \, , \quad \text{or}$$

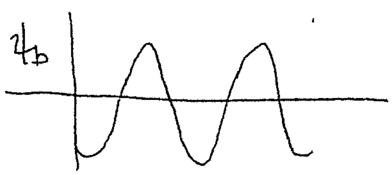
$$-\frac{k'}{m} A = \frac{k'}{m} B \quad \Rightarrow \quad \boxed{B = -A}$$

In this normal mode, the masses oscillate 180° out of phase.

ψ_a



ψ_b



Is this the case for all mode problems (that for mode 1 the amplitudes are equal, and for mode 2 the amplitudes are equal and opposite)?

No. As we already anticipated from an intuitive point of view, and as you will show via example using the "systematic method" developed here, if the masses are not equal,* $|A| \neq |B|$ in (here) either mode.

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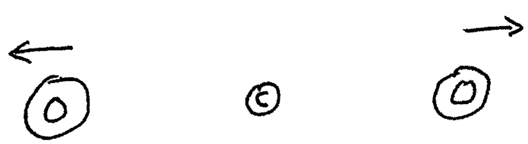
* or, more generally, with left-right asymmetry.

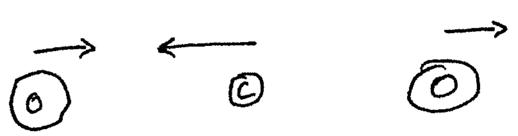
Example: Longitudinal Modes of CO_2, Revisited

As a further example, we look again at the longitudinal normal modes of the CO_2 molecule. We already deduced the mode frequencies using a bit of intuition about the expected mode shapes and then applying the "method of physical meaning of ω^2." We recall that we found:



$$\omega_{s.s.} = \sqrt{\frac{k}{m}}$$

symmetric stretch



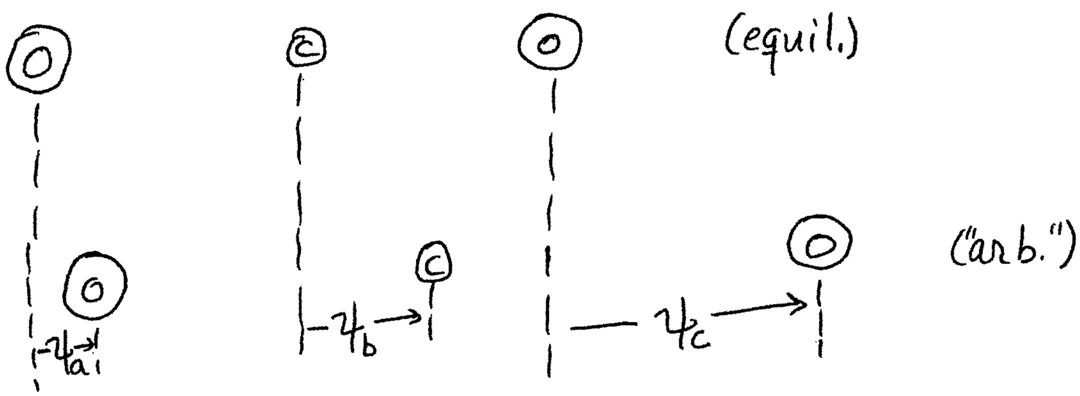
$$\omega_{asy.st.} = \sqrt{\frac{k}{m_o} + \frac{2k}{m_c}}$$

$$\left(\text{i.e., } \omega^2 = \frac{k}{m_o}\left(1 + \frac{2m_o}{m_c}\right)\right).$$

For practice, we now attack the same problem with the systematic (determinant) method. [Since this method is purely "machine-like", it will be longer than our previous intuition/meth. of ω^2 method.]

First we need the governing differential equations.

For those, we must first draw the general (arbitrary) configuration of the system:



("springs" not shown).

Note that since the arbitrary configuration shown includes nonzero motion of the C.M. of the molecule, one of our results is expected to involve overall translational motion.

Anyway, proceding, we ask that you show that the three governing d.e.'s are:

$$m_1 \equiv m_O = m_3$$
$$m_2 = m_C$$

$$m_1 \ddot{\psi}_a = k(\psi_b - \psi_a)$$

$$m_2 \ddot{\psi}_b = k(\psi_c - \psi_b) - k(\psi_b - \psi_a)$$

$$m_3 \ddot{\psi}_c = -k(\psi_c - \psi_b)$$

We ask you to show that this leads to the secular equation

$$\begin{vmatrix} \frac{k}{m_1} - \omega^2 & -\frac{k}{m_1} & 0 \\ -\frac{k}{m_2} & \frac{2k}{m_2} - \omega^2 & -\frac{k}{m_2} \\ 0 & -\frac{k}{m_1} & \frac{k}{m_1} - \omega^2 \end{vmatrix} = 0$$

This 3x3 determinant can be expanded* to give

$$\left(\frac{k}{m_1} - \omega^2\right) \begin{vmatrix} \frac{2k}{m_2} - \omega^2 & -\frac{k}{m_2} \\ -\frac{k}{m_1} & \frac{k}{m_1} - \omega^2 \end{vmatrix} - \left(-\frac{k}{m_1}\right) \begin{vmatrix} -\frac{k}{m_2} & -\frac{k}{m_2} \\ 0 & \frac{k}{m_1} - \omega^2 \end{vmatrix} = 0$$

which is

$$\omega^2 (k - \omega^2 m_1)\left[k(M_2 + 2m_1) - \omega^2 M_2 m_1\right] = 0$$

This means we have $\underline{3}$ possibilities for ω^2:

$$\omega_1^2 = 0$$

$$\left.\begin{aligned} \omega_2^2 &= \frac{k}{m_1} \\ \omega_3^2 &= \frac{k}{m_1} + \frac{2k}{M_2} \end{aligned}\right\} \text{both as we found earlier by other methods.}$$

Since there are 3 longitudinal degrees of freedom, we expected (and found) 3 modes.

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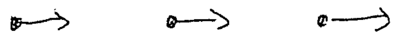
* "Expansion of a determinant "by minors."

The first "mode", with zero frequency, represents translational motion of the entire molecule (technically the math counts this as a "mode")



Modes 2 and 3 are, respectively, the symmetric and asymmetric stretch modes we are already familiar with. Other examples are explored in your text.

For bent molecules such as H_2O the calculations, while the same in principle, are usually more involved, and since we have other things to move on to, we won't do them in this course. Molecular physicists and physical chemists are often very involved with normal modes of molecules. However, we need to stay on the path to Waves...

B. Mode Superposition With Closely Spaced Frequencies; Beats Between Modes

Reading For This:
Chap. 6, Sect. 6.16

Suppose we have a system with two normal modes.

Then each moving part moves in two simultaneous simple harmonic motions - i.e, (for a general

$$\psi_a(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$

(1)
$$\psi_b(t) = B_1 \cos(\omega_1 t + \phi_1) + B_2 \cos(\omega_2 t + \phi_2)$$

Let's consider the case with both ϕ_1 and $\phi_2 = 0$ and with $A_1 = A_2$. (We'll see how to arrange this later). Then, e.g.,

(2)
$$\psi_a(t) = A \cos \omega_1 t + A \cos \omega_2 t$$

Now suppose that ω_1 and ω_2 happen to be close to each other. ($\omega_2 \approx \omega_1$). In this case, something interesting happens.

To see this, let's define

$$\omega_{AV} \equiv \frac{\omega_1 + \omega_2}{2} \qquad \text{("average frequency")}$$

$$\omega_{mod} \equiv \frac{\omega_2 - \omega_1}{2} \qquad \text{("modulation frequency")}$$

Rewriting eqn. (2) in terms of these,

$$\psi(t) = A\cos(\omega_{AV}t - \omega_{mod}t) + A\cos(\omega_{AV}t + \omega_{mod}t)$$

We expand this out:

$$\psi(t) = A\Big[\cos(\omega_{AV}t)\cos(\omega_{mod}t) + \cancel{\sin(\omega_{AV}t)\sin(\omega_{mod}t)}$$
$$+ \cos(\omega_{AV}t)\cos(\omega_{mod}t) - \cancel{\sin(\omega_{AV}t)\sin(\omega_{mod}t)}\Big]$$

or

(3) $$\underline{\psi(t) = 2A\cos(\omega_{mod}t)\cos(\omega_{AV}t)}$$

<u>What does this mean</u>?

<u>Let's think</u>:

We have $\omega_2 \gtrsim \omega_1 \Rightarrow \underline{\omega_{mod} \ll \omega_{AV}}$

Now look at form (3) with this in mind:

$$\psi(t) = \underbrace{(2A\cos\omega_{mod}t)}\cdot\underbrace{(\cos\omega_{AV}t)}$$

Slowly sinusoidally varying "amplitude function"

$A(t) = 2A\cos(\omega_{mod}t)$

Simple harmonic oscillation at the "fast" frequency ω_{AV}

Now - this is all theory. Can we make a practical prediction from it?

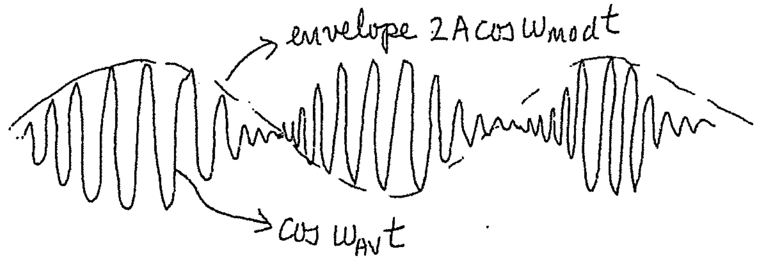
- Consider the eardrum! Suppose two sinusoidal sound waves strike it simultaneously.

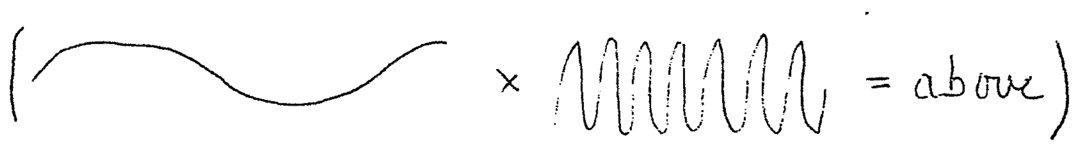
Q: Do we hear it as two separate tones ("form of eqn. (2)"), or as one tone with a "now louder, now lower" amplitude (i.e, "form of eqn. (3)", or "beats form") ?

We experiment (tuning forks
oscillators).

Result: For my ear-brain system, if ω_1 and ω_2 are within a few % of each other, my brain hears it as beats.
If > a few % dif, my brain hears it as two separate tones!

Thus, the multiplication of the two pieces looks like this:





But, now we have <u>two</u> <u>different</u> <u>expressions</u> [(2) and (3)] for the <u>same</u> <u>thing</u>!

<u>How are they related physically</u>?

To find out, <u>see figure next page</u>:

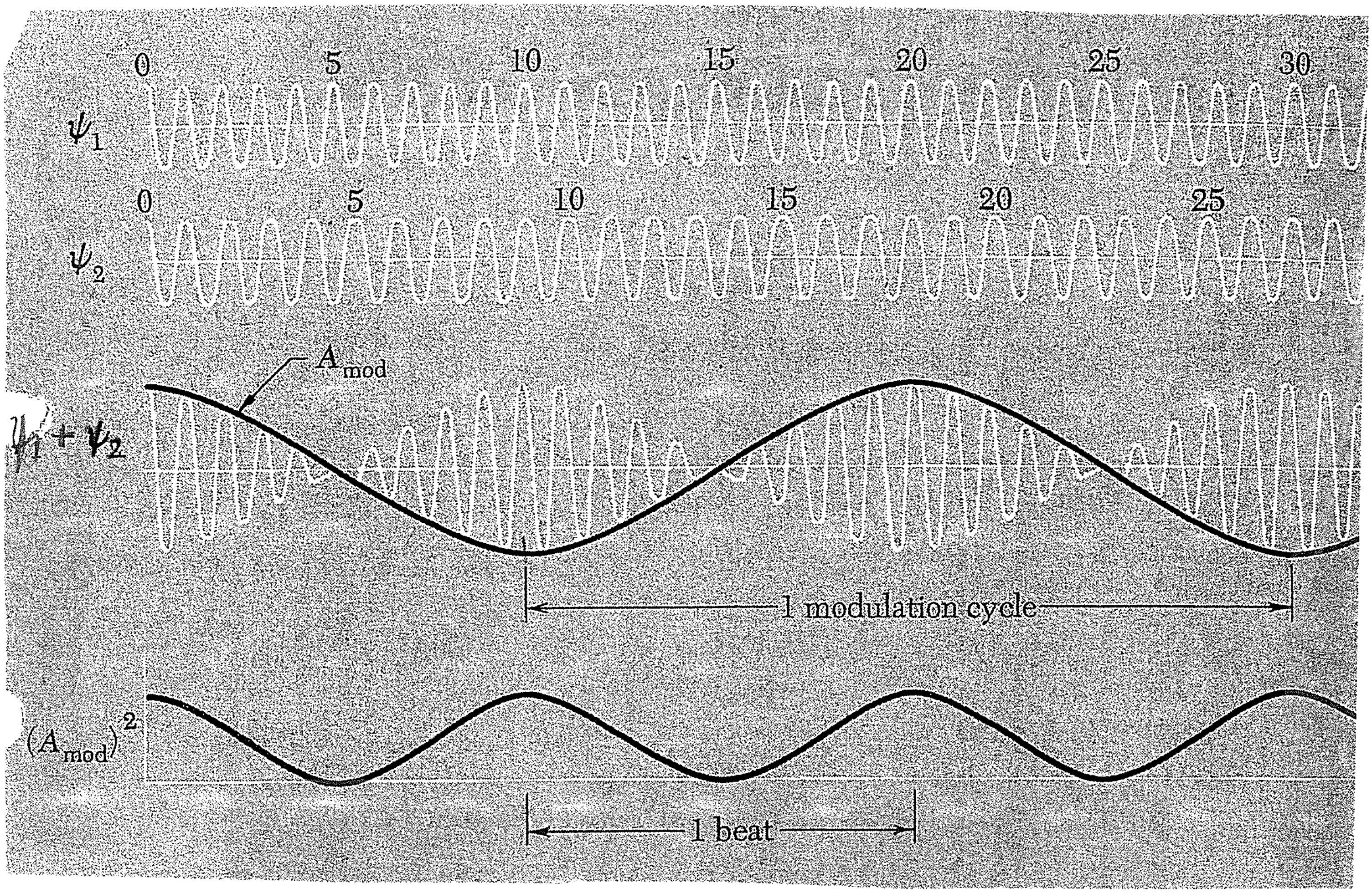


Figure ↑ from Waves, Berkeley Physics Series Vol. 3 by F.S. Crawford (McGraw-Hill), chapter 1

Figure → from Quantum Mechanics A Paradigms Approach by D.H. McIntyre (Pearson).
Figure shows superposition of three different (slightly) frequency sinusoidal sounds.

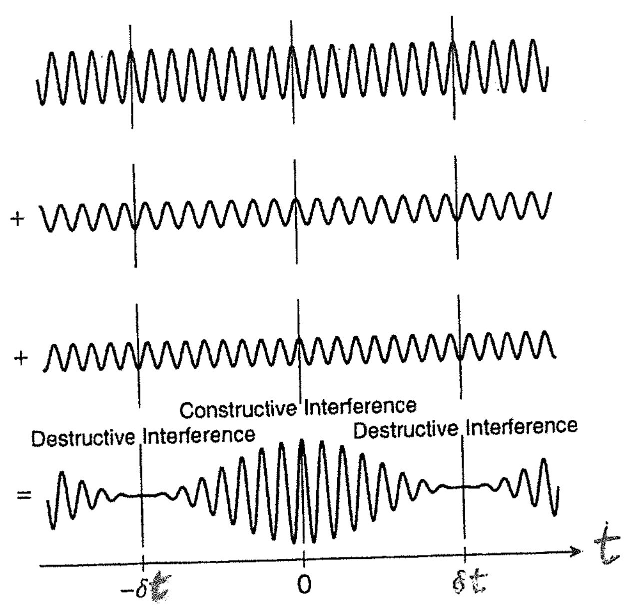


FIGURE 6.4 Discrete wave packet with three components.

Some pictures might help to further clarify this. In Fig. 6.22a I've plotted the sum of two simple harmonic oscillations that differ by 10% in frequency; visually the result appears as a single oscillation at the average ("fast") frequency modulated by a sinusoidally time varying amplitude. You can see the physical origin of this effect by looking at the two cosines in the sum plotted individually, as in Fig. 6.23. When the two components are in phase, $A(t)$ is large, when they are out of phase, $A(t)$ is smaller.

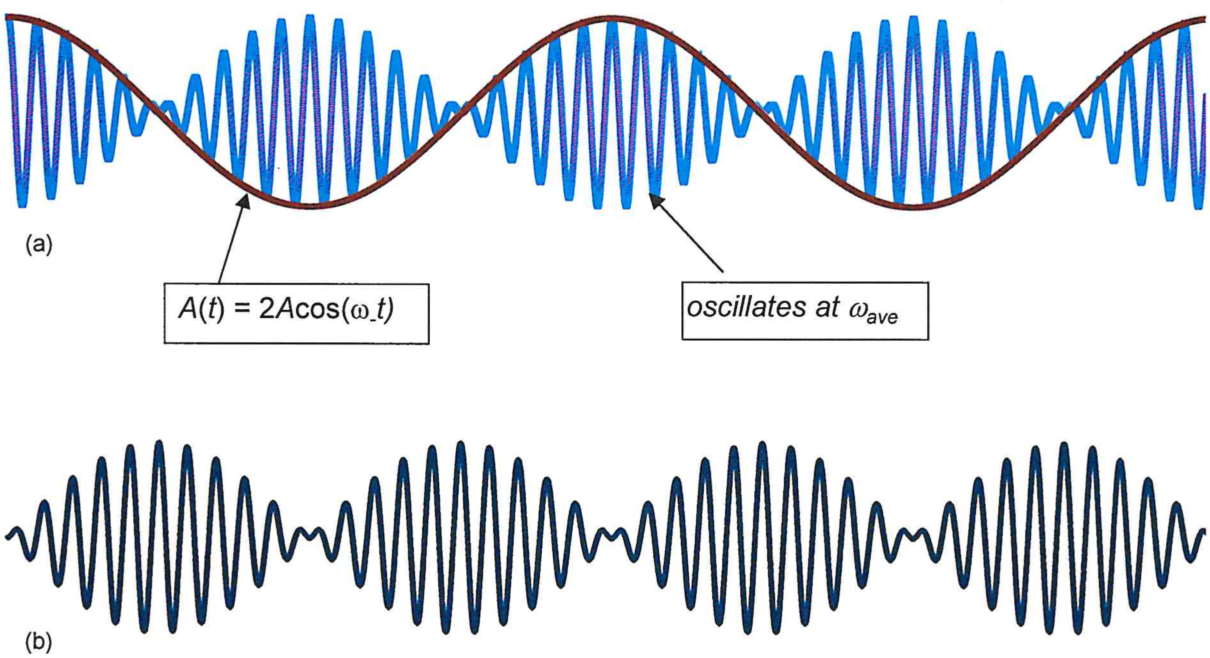


Fig. 6.22. Sum of two simple harmonic oscillations differing by 10% in frequency.
(a) $\cos(\omega_1 t)+\cos(\omega_2 t)$, (b) $\cos(\omega_1 t)-\cos(\omega_2 t)$. $\omega_2 = 1.1\omega_1$. Notice that the nodes in (b) occur at the same times as the maxima do in (a).

It's fun to hear this effect for yourself – get two tuning forks that differ in frequency by about

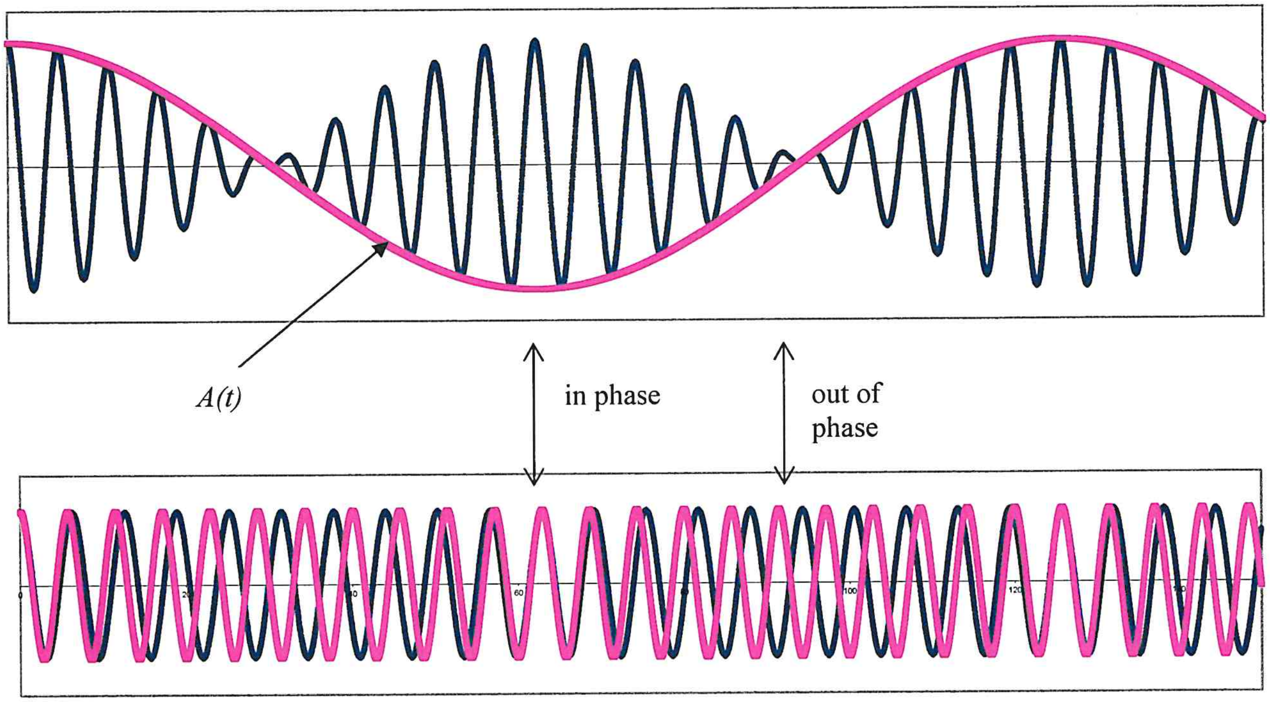
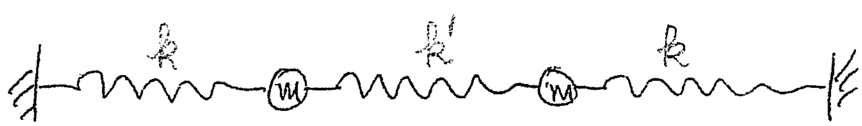


Fig. 6.23. (a) sum of two sinusoidal oscillations – section of Fig. 6.26.
(b) the two sinusoidal components of the sum.

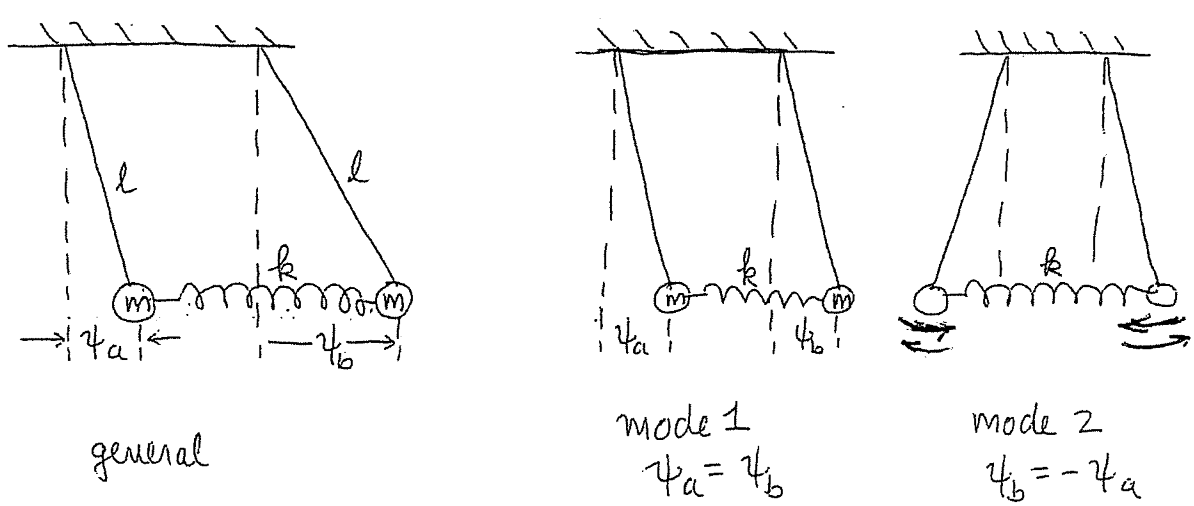
Can we get this in a single system?

Yes! As you know, a single system with two degrees of freedom can have two normal modes. Then, if both modes are excited, either moving part oscillates in a simultaneous superposition of oscillations with two different frequencies. Therefore, if we can arrange for these frequencies to be close, the motion of either moving part should exhibit beats.

<u>Example</u>: How would you arrange this with a system of two masses and three springs? k' weak.



<u>Example</u>: Consider again our coupled pendula:



The two mode frequencies, (as we saw) are $\omega_1^2 = \frac{g}{\ell}$, $\omega_2^2 = \frac{g}{\ell} + \frac{2k}{m}$.

Thus, in general,

$$\psi_a(t) = A_1 \cos\left(\sqrt{\tfrac{g}{\ell}}\, t + \phi_1\right) + A_2 \cos\left(\sqrt{\tfrac{g}{\ell} + \tfrac{2k}{m}}\, t + \phi_2\right)$$

$$\psi_b(t) = A_1 \cos\left(\sqrt{\tfrac{g}{\ell}}\, t + \phi_1\right) - A_2 \cos\left(\sqrt{\tfrac{g}{\ell} + \tfrac{2k}{m}}\, t + \phi_2\right)$$

We see that we can get ω_2 close to ω_1 by choosing k to be weak ("weak coupling") and/or by choosing m large.

Guided by our previous work, for maximal beats we want

$A_1 = A_2$ (= "A"), $\phi_1 = \phi_2 = 0$.

How do find the right initial conditions for this?

We have

$$\psi_a(0) = A_1 \cos\phi_1 + A_2 \cos\phi_2 \,, \quad \psi_b(0) = A_1 \cos\phi_1 - A_2 \cos\phi_2$$

$$\dot{\psi}_a(t) = -\omega_1 A_1 \sin(\omega_1 t + \phi_1) - \omega_2 A_2 \sin(\omega_2 t + \phi_2)$$

$$\dot{\psi}_b(t) = -\omega_1 A_1 \sin(\omega_1 t + \phi_1) + \omega_2 A_2 \sin(\omega_2 t + \phi_2)$$

$$\Rightarrow \quad \dot{\psi}_a(0) = -\omega_1 A_1 \sin\phi_1 - \omega_2 A_2 \sin\phi_2$$

$$\dot{\psi}_b(0) = -\omega_1 A_1 \sin\phi_1 + \omega_2 A_2 \sin\phi_2$$

If we choose $\phi_1 = \phi_2 = 0$, this tells us that $\dot{\psi}_a(0) = \dot{\psi}_b(0) = 0$.

Then

$$\psi_a(0) = A_1 + A_2 \quad , \quad \psi_b(0) = A_1 - A_2$$

If we choose $A_1 = A_2 = A$, then

$$\psi_a(0) = 2A \quad , \quad \psi_b(0) = 0.$$

It is now obvious how to set this up, and we can now enjoy the classroom demonstration of it.