Invariantes del campo electromagnético

$$F^{\mu\nu}F_{\mu\nu} = F^{0\nu}F_{0\nu} + F^{\mu 0}F_{\mu 0} + F^{ij}F_{ij} = F^{0k}F_{0k} + F^{m0}F_{m0} + F^{ij}F_{ij}$$

$$F^{0k}F_{0k} = \left(-\frac{E^k}{c}\right)\left(\frac{E^k}{c}\right) = -\frac{\vec{\mathbf{E}}\cdot\vec{\mathbf{E}}}{c^2} = -\frac{\left|\vec{\mathbf{E}}\right|^2}{c^2} \qquad F^{m0}F_{m0} = \left(\frac{E^m}{c}\right)\left(-\frac{E^m}{c}\right) = -\frac{\vec{\mathbf{E}}\cdot\vec{\mathbf{E}}}{c^2} = -\frac{\left|\vec{\mathbf{E}}\right|^2}{c^2}$$

$$F^{ij}F_{ij} = F^{1j}F_{1j} + F^{2j}F_{2j} + F^{3j}F_{3j}$$

$$F^{1j}F_{1j} = F^{12}F_{12} + F^{13}F_{13} = (-B_z)(-B_z) + (B_y)(B_y) = B_y^2 + B_z^2$$

$$F^{2j}F_{2j} = F^{21}F_{21} + F^{23}F_{23} = (B_z)(B_z) + (-B_x)(-B_x) = B_x^2 + B_z^2$$

$$F^{3j}F_{3j} = F^{31}F_{31} + F^{32}F_{32} = (-B_y)(-B_y) + (B_x)(B_x) = B_x^2 + B_y^2$$

$$F^{ij}F_{ij} = B_y^2 + B_z^2 + B_x^2 + B_z^2 + B_x^2 + B_y^2 = 2(B_x^2 + B_y^2 + B_z^2) = 2\vec{\mathbf{B}}\cdot\vec{\mathbf{B}} = 2\left|\vec{\mathbf{B}}\right|^2$$

$$F^{\mu\nu}F_{\mu\nu} = F^{0k}F_{0k} + F^{m0}F_{m0} + F^{ij}F_{ij} = -\frac{\left|\vec{\mathbf{E}}\right|^2}{c^2} - \frac{\left|\vec{\mathbf{E}}\right|^2}{c^2} + 2\left|\vec{\mathbf{B}}\right|^2 = 2\left(\left|\vec{\mathbf{B}}\right|^2 - \frac{\left|\vec{\mathbf{E}}\right|^2}{c^2}\right)$$

$$F^{\mu\nu}F_{\mu\nu} = 2\left(\left|\vec{\mathbf{B}}\right|^2 - \frac{\left|\vec{\mathbf{E}}\right|^2}{c^2}\right) \quad \Rightarrow \quad \boxed{c^2\left|\vec{\mathbf{B}}\right|^2 - \left|\vec{\mathbf{E}}\right|^2}$$

INVARIANTE

$$\mathcal{F}^{\mu\nu}F_{\mu\nu}=\mathcal{F}^{0\nu}F_{0\nu}+\mathcal{F}^{\mu 0}F_{\mu 0}+\mathcal{F}^{ij}F_{ij}=\mathcal{F}^{0k}F_{0k}+\mathcal{F}^{m0}F_{m0}+\mathcal{F}^{ij}F_{ij}$$

$$\mathcal{F}^{0k}F_{0k}=\left(-B^k\right)\left(\frac{E^k}{c}\right)=-\frac{\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}$$

$$\mathcal{F}^{m0}F_{m0}=\left(B^m\right)\left(-\frac{E^m}{c}\right)=-\frac{\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}$$

$$\mathcal{F}^{ij}F_{ij}=\mathcal{F}^{1j}F_{1j}+\mathcal{F}^{2j}F_{2j}+\mathcal{F}^{3j}F_{3j}$$

$$\mathcal{F}^{1j}F_{1j}=\mathcal{F}^{12}F_{12}+\mathcal{F}^{13}F_{13}=\left(\frac{E_z}{c}\right)\left(-B_z\right)+\left(-\frac{E_y}{c}\right)\left(B_y\right)=-\frac{E_yB_y}{c}-\frac{E_zB_z}{c}$$

$$\mathcal{F}^{2j}F_{2j}=\mathcal{F}^{21}F_{21}+\mathcal{F}^{23}F_{23}=\left(-\frac{E_z}{c}\right)\left(B_z\right)+\left(\frac{E_x}{c}\right)\left(-B_x\right)=-\frac{E_xB_x}{c}-\frac{E_zB_z}{c}$$

$$\mathcal{F}^{3j}F_{3j}=\mathcal{F}^{31}F_{31}+\mathcal{F}^{32}F_{32}=\left(\frac{E_y}{c}\right)\left(-B_y\right)+\left(-\frac{E_x}{c}\right)\left(-B_x\right)=-\frac{E_xB_x}{c}-\frac{E_yB_y}{c}$$

$$\mathcal{F}^{ij}F_{ij}=-\frac{E_yB_y}{c}-\frac{E_zB_z}{c}-\frac{E_xB_x}{c}-\frac{E_zB_z}{c}-\frac{E_xB_x}{c}-\frac{E_yB_y}{c}=-\frac{2\left(E_xB_x+E_yB_yE_zB_z\right)}{c}=-\frac{2\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}$$

$$\mathcal{F}^{\mu\nu}F_{\mu\nu}=\mathcal{F}^{0k}F_{0k}+\mathcal{F}^{m0}F_{m0}+\mathcal{F}^{ij}F_{ij}=-\frac{\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}-\frac{\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}-\frac{2\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}-\frac{2\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}=-\frac{4\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}$$

$$\mathcal{F}^{\mu\nu}F_{\mu\nu}=-\frac{4\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}{c}\quad\Rightarrow\quad\boxed{\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}}$$

INVARIANTE