Ejercicio 1

Encontrar los puntos críticos de la funciones dadas y determinar su naturaleza :

1. $f(x,y) = \frac{1}{x}e^{x\sin y}$.
2. $f(x,y) = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1+4y^2)}$.
3. $f(x,y) = e^{x^2}(x^4+y^4)$.

1. $f(x,y) = \frac{1}{x}e^{x\sin y}$ $\quad \text{Dom}(f) = \mathbb{R}^2 \setminus \{(0,y) : y\in\mathbb{R}\}$ $\quad$ f de clase $\mathcal{C}_2(\mathbb{R}^2)$ por ser composición de funciones $\mathcal{C}_2(\mathbb{R}^2)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{e^{x\sin y}\sin y}{x} - \frac{e^{x\sin y}}{x^2} \qquad \frac{\partial f}{\partial y}(x,y) = e^{x\sin y}\cos y \Rightarrow \vec{\nabla} f(x,y) = \left(\frac{e^{x\sin y}\sin y}{x} - \frac{e^{x\sin y}}{x^2},\ e^{x\sin y}\cos y\right) = (0,0)$$

$\Rightarrow$ los puntos críticos son aquellos de la forma $(1, \frac{\pi}{2}n)$ y $(-1, \frac{\pi}{2}n)$ con $n\in\mathbb{N}(\{1,3,\dots\})$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\sin^2 y\, e^{x\sin y}}{x} - \frac{2\sin y\, e^{x\sin y}}{x^2} + \frac{2e^{x\sin y}}{x^3} \qquad \frac{\partial^2 f}{\partial y\partial x}(x,y) = \sin y\cos y\, e^{x\sin y} - \frac{\cos y\, e^{x\sin y}}{x} + \frac{\cos y\, e^{x\sin y}}{x} - \sin y\cos y\, e^{x\sin y}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = xe^{x\sin y}\cos^2 y - e^{x\sin y}\sin y \qquad \frac{\partial^2 f}{\partial x\partial y}(x,y) = \sin y\cos y\, e^{x\sin y}$$

$$|Hf(x,y)| = f_{xx}f_{yy} - f_{xy}^2 = \left(\frac{\sin^2 y\, e^{x\sin y}}{x} - \frac{2\sin y\, e^{x\sin y}}{x^2} + \frac{2e^{x\sin y}}{x^3}\right)\left(xe^{x\sin y}\cos^2 y - e^{x\sin y}\sin y\right) - \left(\sin y\cos y\, e^{x\sin y}\right)^2$$

$|Hf(1,\frac{\pi}{2}n)| = (e - 2e + 2e)(0e - e) - 0e = e(-e) = -e^2 < 0 \Rightarrow$ Punto de silla

$|Hf(-1,\frac{\pi}{2}n)| = (-e^{-1} - 2e^{-1} - 2e^{-1})(-e^{-1}) - 0 = -5e^{-1}(-e^{-1}) = 5e^{-2} > 0 \Rightarrow$ Extremo local

$f_{xx}(-1,\frac{\pi}{2}n) = -e^{-1} - 2e^{-1} - 2e^{-1} = -4e^{-1} < 0 \Rightarrow$ máximo local

2. $f(x,y) = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1+4y^2)}$ $\quad \text{Dom}(f) = \mathbb{R}^2$ $\quad$ f de clase $\mathcal{C}_2(\mathbb{R}^2)$ por ser composición de funciones $\mathcal{C}_2(\mathbb{R}^2)$

$x=2$
$x=0$
$x=-1$ $\qquad \neq 0\ \forall x \Rightarrow y=0$

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^3 - x^2 - 2x}{1+4y^2} \qquad \frac{\partial f}{\partial y}(x,y) = \frac{-2y(3x^4 - 4x^3 - 12x^2 + 18)}{3(1+4y^2)^2} \Rightarrow \vec{\nabla} f(x,y) = \left(\frac{x^3 - x^2 - 2x}{1+4y^2}, \frac{-2y(3x^4 - 4x^3 - 12x^2 + 18)}{3(1+4y^2)^2}\right) = (0,0)$$

$\Rightarrow$ puntos críticos $(-1,0), (0,0), (2,0)$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{3x^2 - 2x - 2}{1+4y^2} \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2(3x^4 - 4x^3 - 12x^2 + 18)(12y^2 - 1)}{3(1+4y^2)^3}$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y) = \frac{-8xy(x^3 - x^2 - 2x)}{(1+4y^2)^2} = \frac{\partial^2 f}{\partial x\partial y}(x,y) \qquad |Hf(x,y)| = f_{xx}f_{yy} - f_{xy}^2 = \left(\frac{3x^2 - 2x - 2}{1+4y^2}\right)\left(\frac{(x^3 - x^2 - 2x)(12y^2 - 1)\cdot 2}{3(1+4y^2)^3}\right) - \left(\frac{-8xy(x^3 - x^2 - 2x)}{(1+4y^2)^2}\right)^2$$

$|Hf(0,0)| = 0 \qquad |Hf(2,0)| = 0 \qquad |Hf(-1,0)| = 0$

$f(0,0) = 1'5 \quad f(0,0'5) = \frac{18}{24} \quad f(0,-0'5) = \frac{18}{24} \Rightarrow f(0,0) > f(0,y)$ en un entorno de $(0,0)$
$f(0'5,0) = 1'22 \quad f(-0'5,0) = 1'31 \Rightarrow f(0,0) > f(x,0)$ en un entorno de $(0,0)$ $\Big\} \Rightarrow (0,0)$ máximo local

$f(2,0) = -1'16 \quad f(2,0'1) = -1'12 \quad f(2,-0'1) = -1'12 \Rightarrow f(2,0) < f(2,y)$ en un entorno de $(2,0)$
$f(2'1,0) = -1'13 \quad f(1'9,0) = -1'14 \Rightarrow f(2,0) < f(x,0)$ en un entorno de $(2,0)$ $\Big\} \Rightarrow (2,0)$ mínimo local

$f(-1,0) = 1'08 \quad f(-1,0'1) = 1'04 \quad f(-1,-0'1) = 1'04 \Rightarrow f(-1,0) > f(-1,y)$ en un entorno de $(-1,0)$
$f(-1'1,0) = 1'1 \quad f(-0'9,0) = 1'09 \Rightarrow f(-1,0) < f(x,0)$ en un entorno de $(-1,0)$ $\Big\} \Rightarrow (-1,0)$ punto de silla

tomamos un entorno de cada punto y observamos cómo se comporta la función dentro de dicho entorno

3. $f(x,y) = e^{x^2}(x^4+y^4)$ $\quad \text{Dom}(f) = \mathbb{R}^2$ $\quad$ f clase $\mathcal{C}_2(\mathbb{R}^2)$ por ser composición de funciones $\mathcal{C}_2(\mathbb{R})$

$x=0$
$y=0$

$$\frac{\partial f}{\partial x}(x,y) = 2xe^{x^2}(x^4+y^4) + 4x^3e^{x^2} \qquad \frac{\partial f}{\partial y}(x,y) = 4y^3e^{x^2} \Rightarrow \vec{\nabla} f(x,y) = \left(2xe^{x^2}(x^4+y^4) + 4x^3e^{x^2},\ 4y^3e^{x^2}\right) = (0,0)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 4x^6e^{x^2} + 4x^2y^4e^{x^2} + 8x^4e^{x^2} \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2e^{x^2}$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = 8xy^3e^{x^2} = \frac{\partial^2 f}{\partial y\partial x}(x,y) \qquad |Hf(x,y)| = f_{xx}f_{yy} - f_{xy}^2 = \left(4x^6e^{x^2} + 4x^2y^4e^{x^2} + 8x^4e^{x^2}\right)\left(12y^2e^{x^2}\right) - \left(8xy^3e^{x^2}\right)^2$$

$|Hf(0,0)| = 0 \qquad$ como $f(0,0) = 0$ y $f(x,y) > 0\ \forall (x,y)\neq(0,0) \Leftrightarrow (0,0)$ mínimo global

Ejercicio 5

1. Encontrar y clasificar los puntos críticos de

$$f(x,y) = e^{-(x^2+y^2)}(3x^2+5y^2).$$

2. Mostrar que todos los puntos críticos de $f(x,y) = y + x\sin(y)$ corresponden a puntos silla.

$$\frac{\partial f}{\partial x}(x,y) = -2x\, e^{-(x^2+y^2)}(3x^2+5y^2) + e^{-(x^2+y^2)}\,6x$$

$$\frac{\partial f}{\partial y}(x,y) = -2y\, e^{-(x^2+y^2)}(3x^2+5y^2) + e^{-(x^2+y^2)}\,10y$$

$$\nabla f(x,y) = (0,0) \Leftrightarrow \begin{cases} -6x^3 - 10xy^2 + 6x = 0 \\ -6x^2y - 10y^3 + 10y = 0 \end{cases}$$

si $x=0$: $y=0$, $y=1$, $y=-1$

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$f(x,y) = e^{-(x^2+y^2)}(3x^2+5y^2)$ $\quad$ f continua en $\mathbb{R}^2$ $\quad$ $f\ \mathcal{C}_2(\mathbb{R})$ $\quad \Rightarrow$ puntos críticos $(0,0)$ $(0,1)$ $(0,-1)$ $(1,0)$ $(-1,0)$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -2xe^{-(x^2+y^2)}(-6x^3-10xy^2+6x) + e^{-(x^2+y^2)}(-12x^2-10y^2+6)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -2y\,e^{-(x^2+y^2)}(-6x^2y-10y^3+10y) + e^{-(x^2+y^2)}(-12x^2-30y^2+10)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -2ye^{-(x^2+y^2)}(-6x^3-10xy^2+6x) - 20xye^{-(x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -2xe^{-(x^2+y^2)}(-6x^2y-10y^3+10y) - 12xye^{-(x^2+y^2)}$$

$$|Hf(1,0)| = |Hf(-1,0)| = |Hf(0,1)| = |Hf(0,-1)| = 0 \quad ¡!$$

$$|Hf(0,0)| = 60 > 0 \Rightarrow$$

Ejercicio 6

Encontrar los extremos de f sujetos a las restricciones mencionadas :

1. $f(x,y) = 3x + 2y$, con $2x^2 + 3y^2 = 3$.
2. $f(x,y) = xe^{xy}$, con $x^2 + y = 0$.
3. $f(x,y,z) = xyz$, con $x^2 + y^2 + z^2 - 2x + 2y + 1 = 0$.

2. $f(x,y) = xe^{\lambda y}$ con $x^2+y=0$ $\Leftrightarrow g(x,y)=0$ donde $g(x,y) = x^2+y$ $\quad$ f y g son $\mathcal{C}_1$. T^{ma} Lagrange:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) + \lambda \frac{\partial g}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) + \lambda \frac{\partial g}{\partial y}(x,y) = 0 \end{cases} \Rightarrow \begin{cases} xye^{xy} + 2x\lambda = 0 \\ xye^{xy} + \lambda = 0 \end{cases}$$

para $\lambda \neq 0$ $\quad xe^{xy}(y + 2\lambda) = 0$

$\quad e^{xy}(xy+\lambda) = 0$

$\Rightarrow$ sin puntos críticos

3. $f(x,y,z) = xyz$ $\quad g(x,y,z) = x^2+y^2+z^2-2x+2y+1$ $\quad$ f y g son $\mathcal{C}_1(\mathbb{R}^3)$ T^{ma} Lagrange,

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) + \lambda \frac{\partial g}{\partial x}(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(x,y,z) + \lambda \frac{\partial g}{\partial y}(x,y,z) = 0 \\ \frac{\partial f}{\partial z}(x,y,z) + \lambda \frac{\partial g}{\partial z}(x,y,z) = 0 \end{cases} \Leftrightarrow \begin{cases} yz + \lambda(2x-2) = 0 \\ xz + \lambda(2y+2) = 0 \\ xy + \lambda 2z = 0 \end{cases} \Leftrightarrow \begin{cases} -xyz = x\lambda(2x-2) \\ -xyz = y\lambda(2y+2) \\ -xyz = 2z^2\lambda \end{cases}$$

$\rightarrow x(2x-2) = 2z^2 \Leftrightarrow z^2 = x^2 - x$

$\rightarrow y(2y+2) = x(2x-2) \Leftrightarrow y^2 + y = x^2 - x$

$\rightarrow y(2y+2) = 2z^2 \Leftrightarrow y^2 + y = z^2$

$$2\lambda\sqrt{x^2-x} + xy = 0 \Leftrightarrow x^2 - x = \frac{x^2y^2}{4} \Leftrightarrow 1 - \frac{1}{x} = \frac{y^2}{4} \Leftrightarrow \frac{1}{x} = \frac{4-y^2}{4} \Leftrightarrow x = \frac{4}{4-y^2}$$

$$2\lambda\sqrt{y^2+y} + xy = 0 \Leftrightarrow y^2 + y = \frac{x^2y^2}{y} \Leftrightarrow 1 + \frac{1}{y} = \frac{x^2}{4} \Leftrightarrow x = \sqrt{\frac{4y+1}{y}}$$

$$\frac{4}{4-y^2} = \sqrt{\frac{4y+1}{y}} \Leftrightarrow \frac{16}{(4-y^2)^2} = \frac{4y+1}{y} \Leftrightarrow 16y = (4y+1)(4-y^2)^2 = (4y+1)(16-8y^2+y^4) = 64y - 32y^3 + 4y^5 + 16 - 18y^2 + y^4 \Leftrightarrow$$

$$\Leftrightarrow 4y^5 + y^4 - 32y^3 - 18y^2 + 48y + 16 = 0 \Leftrightarrow$$

c) $y' + y^2 = x^2 - 2x$

$$\left.\begin{aligned} y &= 1-x \\ y' &= -1 \\ y^2 &= (1-x)^2 = x^2-2x+1 \end{aligned}\right\} \Longleftrightarrow -1 + x^2 - 2x + 1 = x^2 - 2x \Leftrightarrow x^2 - 2x = x^2 - 2x$$