and, as we just saw, if $\omega_{ave} >> \frac{1}{4}\Delta\omega$, there must be times when $f(t) = 0$ before the time $t = \Delta t = 2\pi/\Delta\omega$. At those times, even though the phasors are not essentially evenly spread all around the clock, for $f(t)$ to be zero the horizontal components of all the phasors must add to zero. Consider, for example, the time t = $\delta t = \dfrac{\pi}{2\omega_{ave}}$ for which $\cos\omega_{ave}t = 0$. The phasor situation at that time is indicated schematically in Fig. 12.8.1 below.

ignore for our present discussion.

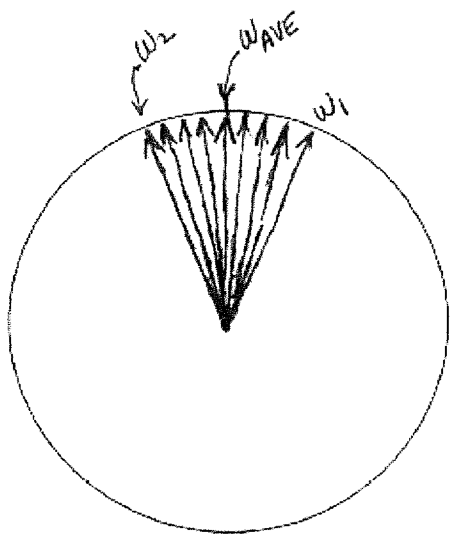

Fig. 12.8.1.

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12.9 The Complex Function Form of Fourier Series [K-text, chap 12, sect. 12.9]

In this and the following few sections I introduce you to the basic idea of the complex function for of Fourier series for real-valued repetitive functions. Writing Fourier series in a form using complex-valued basis functions is often convenient for calculating Fourier coefficients. From a theoretical point of view, it also gives us insight into new complete orthogonal sets of functions – e.g., the sets of complex exponential functions $\{e^{in\frac{2\pi}{\Lambda}x}, n = 1, 2, 3, \ldots\}$, where Λ is the smallest repetition distance of the function being expanded in the Fourier series. It is important that concepts related to the complex function form of Fourier series have direct analogs in the extremely widely used technique of Fourier transforms, which I will discuss with you in chapter 17.

To begin, reconsider our basic Fourier series for repetitive functions with repeat distance 2π:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx . \tag{12.9.1}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \, , \, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \, .$$

We recall that

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \tag{12.9.2a}$$

and

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i} . \tag{12.9.2b}$$

Putting eqns. (12.9.2) into eqn. (12.9.1), we have after collecting terms,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2}\right) e^{-inx} . \tag{12.9.3}$$

We now define a new set of Fourier coefficients as

(12.9.4) $c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')dx'$,

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')(\cos nx' - i\sin nx')\,dx' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \quad (n \geq 1)$$

$$d_n = \frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')(\cos nx' + i\sin nx')\,dx' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \quad (n \geq 1),$$

so our Fourier expansion can be written as

(12.9.5) $$f(x) = c_0 + \sum_{n=1}^{\infty} c_n\, e^{inx} + \sum_{n=1}^{\infty} d_n\, e^{-inx}\ .$$

We can simplify the appearance of this Fourier series with a little trick: We define the set of coefficients $\{c_{-n}\}$ for n = 1. 2. 3. … [i.e., $c_{-1}, c_{-2}, c_{-3}, \ldots$] by

$c_n = \frac{a_n - ib_n}{2}$

(12.9.6) $$c_{-n} \equiv d_n = \frac{a_n + ib_n}{2}\ ,$$

so that

(12.9.7) $$f(x) = \sum_{n=0}^{\infty} c_n\, e^{inx} + \sum_{n=1}^{\infty} c_{-n}\, e^{-inx}\ ,$$

or,

(12.9.8) $$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}}$$

Complex form of Fourier series for a function with repeat distance 2π defined over the entire x-axis

with

(12.9.9)

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx}$$

for all integer values of n: $n = 0, \pm 1, \pm 2, \pm 3, \ \dots$.

So far, we have not invoked the requirement that $f(x)$ be a real-valued function of (real variable) x. In fact, equations (12.9.8) and (12.9.9) are correct even if $f(x)$ is a complex-valued function of (real variable) x. (In that case, the basic Fourier series given by eqn. (12.9.1) is still correct, but the Fourier coefficients in it (the a_n's and the b_n's) are, in general, complex numbers.)

But now, a careful reader interrupts with a question:

Question: At least at first glance, whether $f(x)$ is real-valued or not, it seems that the terms of the complex form Fourier series for $f(x)$ are all complex-valued since they all contain functions of the form e^{-inx}. But then, if $f(x)$ is a real-valued function of x, how can we see, manifestly, that the complex form of the Fourier series for $f(x)$ yields a real-valued result?

Answer: If $f(x)$ is a real-valued function of x, then all of the a_n's and the b_n's in its sine and cosine form Fourier series [i.e., eqn. (12.9.1)] must be real numbers. So, if $f(x)$ is real-valued, for each n, $\underline{c_n \text{ and } c_{-n} = d_n \text{ are complex conjugates of each other}}$ [recall eqn. (12.9.4)],

(12.9.10)

$$c_{-n} = c_n^*$$

True for real $f(x)$ only

Now look again at eqn. (12.9.7):

(12.9.7) $$f(x) = \sum_{n=0}^{\infty} c_n\, e^{inx} + \sum_{n=1}^{\infty} c_{-n}\, e^{-inx}$$

For any given n, the contribution to the Fourier series is

(12.9.11) $$c_n e^{inx} + c_{-n}\, e^{-inx}$$

Now, for real-valued *f(x)* $c_{-n} = c_n^*$ so for real-valued $f(x)$ the second term in eqn. (12.9.11) is the complex conjugate of the first term. [You will recall that the sum of any complex number z and its complex conjugate number is always a real number (equal to twice the real part of z).] <u>Thus, for each n, the sum in eqn. (12.9.11) is a real-valued function</u>. Hence, if $f(x)$ is a real-valued function, the entire right-hand side of eqn. (12.9.7) and hence, also, the entire right-hand side of eqn. (12.9.8) yields a real-valued function of x. On the other hand, if $f(x)$ is a complex-valued function of real variable x (or t), then it is not true that, for each n, $c_{-n} = c_n^*$.[8]

Eqns. (12.9.8) and (12.9.9) provide the basic complex function form that we sought for the Fourier series for a real-valued repetitive function with repeat distance 2π. In the following Worked Example, I illustrate the use of eqns. (12.9.8) and (12.9.9) to find the complex function Fourier series of a familiar real-valued function.

[8] Although I will not prove it here, it turns out that in the series for a complex-valued even function of a real variable (e.g., x or t), $c_{-n} = c_n$ and for a complex-valued odd function of a real variable, $c_{-n} = -c_n$.

Worked Example 12.4

Consider the repetitive square wave function $symstp(x)$ that we considered in chapter 11,

$$f(x) = symstp(x) = \{ \begin{array}{ll} -1/2 & \text{if } -\pi < x \leq 0 \\ +1/2 & \text{if } 0 < x < \pi \} \end{array}$$

and its periodic repetition over the entire x-axis. For that function we found (eqn. 11.2) the Fourier series representation

(12.9.12) and (11.2) $$f(x) = \frac{2}{\pi} [\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + ...],$$

the indicated series converging to $symstp(x)$ except at points of jump discontinuity. Find a Complex Fourier series representation of this function and explicitly verify that the complex series is term-by-term equal to the series we found for this function in chap. 11, eqn. (11.2).

Solution Steps:

1. First, we note that, according to our derivation of the complex-function form of the Fourier series for a real-valued function $f(x)$, the complex valued function form must, term-by-term be equal to the Fourier series that we found back in chapter 11, just rewritten in a different looking form.

2. **Evaluation of Complex-Valued Fourier coefficients**:

   We have $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$, where $f(x) = symstp(x)$ with its periodic

Worked Example 12.4, continued

extension over the entire x-axis, is an odd function. (That is why we wound up with only sine functions in its Fourier expansion in chapter 11). We expand the exponential function in the integral [$e^{-inx} = \cos nx - i \sin nx$] to write c_n as

$$(12.9.13) \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \,.$$

We note that the integrand of the first integral in eqn. (12.9.13) is an odd function since $f(x)$ here is odd and $\cos nx$ is even (true even for n=0). Hence, since the first integral in eqn. (12.9.13) is taken over a symmetric interval, it is equal to zero.

Next, we consider the second integral in eqn. (12.9.13). For $n = 0$ the integrand is zero since sin(0) = 0. Thus, $c_0 = 0$. That makes sense since the average of $= \ symstp(x)$ over one repetition distance is zero. For $n \neq 0$, the integrand of the second integral is an even function, (product of two odd functions) so we can write c_n for $n \neq 0$ in the form

$$(12.9.14) \qquad c_n = -\frac{i}{\pi} \int_0^{\pi} f(x) \sin nx \, dx == -\frac{i}{2\pi} \int_0^{\pi} \sin nx \, dx,$$

since, on the interval (0, π), $f(x) = +\frac{1}{2}$. Integrating, we find

$$(12.9.15) \qquad c_n = -\frac{i}{2n\pi} \left[-\cos nx\right]_0^{\pi} = \frac{i}{2n\pi} \left[\cos n\pi - 1\right] = \begin{cases} \frac{-i}{n\pi} & n \; odd \\ 0 & n \; even, not \; zero \end{cases}$$

Worked Example 12.4, continued

3. **The Resulting Complex Form Fourier series**:

So from eqn. (12.9.15) we have

$$f(x) = \frac{-i}{\pi}\left(e^{ix} + \frac{1}{3}e^{3ix} + \frac{1}{5}e^{5ix} + \ldots\right) + \frac{-i}{\pi}\left(\frac{1}{-1}e^{-ix} + \frac{1}{-3}e^{-3ix} + \frac{1}{-5}e^{-5ix} + \ldots\right),$$

where the first infinite series includes the positive n terms and the second infinite series the negative n terms. We see that, for this situation, all the c_n's are pure imaginary even though $f(x)$ is a real-valued function.

4. **Expressing the result in manifestly real-valued form**:

Combining the two infinite series above we have

$$(12.9.16)\quad f(x) = \frac{-i}{\pi}\left(e^{ix} + \frac{1}{3}e^{3ix} + \frac{1}{5}e^{5ix} + \cdots - e^{-ix} - \frac{1}{3}e^{-3ix} - \frac{1}{5}e^{-5ix} - \ldots\right).$$

I leave it to you to show that the series (12.9.16) is mathematically the same as the series of eqn. (12.9.12) above – that is exercise 12.9.1 below.

Exercise 12.9.1 Show that eqn. (12.9.16) in Worked Example 12.9.4 above can be rewritten in the form of eqn. (12.9.12). Here, do this by starting specifically with eqn. (12.9.16) and manipulating it so that it specifically reproduces eqn. (12.9.12).

Hint: Note that $e^{inx} - e^{-inx} = 2i sin nx$, for all positive integer n.

17.16 The Complex Fourier Transform – An Introduction to the Formulas

In chapter 12, sect. 12.9 we discussed the complex form of Fourier series for piecewise continuous and bounded repetitive functions defined on an interval of length $2L$ [e.g., $(0, 2L)$ and (-L, L)] and saw that an appropriate Fourier series representation of such a function, e.g., on the interval (-L, L) is

(17.16.1) $$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\frac{\pi}{L}x}$$

with Fourier coefficients given by

(17.16.2) $$c_n = \frac{1}{2L}\int_{x_0}^{x_0+2L} f(x)e^{-in\frac{\pi}{L}x}dx.$$

For this expansion, if $f(x)$ is a real-valued function of x, then $c_{-n} = c_n^*$ so that the sum of each pair of contributions $c_n e^{in\frac{\pi}{L}x}$ and $c_{-n}e^{-in\frac{\pi}{L}x}$ is purely real (since then $[c_{-n}e^{-in\frac{\pi}{L}x}]^* = c_n e^{in\frac{\pi}{L}x}$; the sum of a complex number and its complex conjugate is always purely real).

In section 12.10, I showed that the infinite set of complex exponential functions $\{\frac{1}{\sqrt{2L}}e^{in\frac{\pi}{L}x}\}$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, is an "orthonormal set" on the interval (-L, L) and that the orthonormality condition on that interval (-L, L) for that infinite set of complex-valued functions can be written succinctly as

(12.10.3) $$\frac{1}{2L}\int_{-L}^{L} e^{-im\frac{\pi}{L}x}\, e^{in\frac{\pi}{L}x}dx = \delta_{m,n}$$

where $\delta_{m,n} = 1$ if $m \neq n$ and $\delta_{m,n} = 0$ if $m = n$. There I also mentioned (without proof) that the infinite set of complex exponential functions is a complete orthonormal set on the on interval (-*L*, *L*) – i.e., any piecewise continuous and bounded function on (-*L*, *L*) can be expanded in a complex Fourier series in the form of eqn. (17.16.1) on that interval.

Here, we are interested in such a Fourier expansion in the limit that $L \rightarrow \infty$ so that our interval of expansion is the entire *x*-axis $(-\infty, \infty)$. That will allow us to write a Fourier expansion for nonrepetitive functions on $(-\infty, \infty)$. Based on the discussion above in this section and also from our previous discussion in this chapter on Fourier sine and Fourier cosine transforms of nonrepetitive functions on the entire *x*-axis, it is plausible to expect that such a Fourier expansion is possible and would take the form

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk \qquad (17.16.3)$$

where the integral is the continuum limit of the sum and where the continuous function *g*(*k*) is a continuum analog of the c_n's in the complex Fourier series expansion of a function defined on the interval (-*L*, *L*) and repetitive outside of that interval over the entire *x*-axis. It is implicit in the fact that we have an integral on the right-hand side of eqn. (17.16.3) that successive wavenumbers in the resulting Fourier mix differ from each other only infinitesimally. With this in mind, for any particular value of *k* in the mix, the value of the function *g*(*k*) for that value of *k* is the amplitude ("strength") of the Fourier basis function e^{ikx} in the Fourier recipe for the function $f(x)$ that the right-hand side is aiming to synthesize. Since *dk* in the integral is an infinitesimal step in the variable *k*, as is normal in a Riemann integral, the integral is summing an infinite number of infinitesimal quantities to obtain a finite umber of each value of x. Note that, in eqn. (17.16.3) only k is integrated over, not x. To make this more precise, I will next begin by stating, without rigorous proof [12], a theorem that corresponds to the Fourier expansion of a non-repetitive function on $(-\infty, \infty)$. The theorem that I will state is not the most general version possible, but it will apply to every situation that we will consider in this text unless otherwise indicated. After that, I provide you with some examples.

Theorem on Complex Fourier Transforms for Function of x

Let $f(x)$ be a piecewise smooth and bounded function defined on $(-\infty, \infty)$such that $\int_{-\infty}^{\infty}|f(x)|dx$ is finite. Then, at every point of the interval where $f(x)$ is continuous, the integral on the right-hand side of eqn. (17.16.3) below

(17.16.3) $$f(x) = \int_{-\infty}^{\infty} g(k)e^{ikx}\,dk$$

converges to $f(x)$, where the function $g(k)$ is given by

(17.16.4) $$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx\,.$$

At every discontinuous point of the interval, the integral in eqn. (17.16.03) converges to $\frac{1}{2}[f(x_+) + f(x_-)]$, where $f(x_+)$ is the $\lim_{\epsilon\to 0} f(x+\epsilon)$, and where $f(x_-)$ is the $\lim_{\epsilon\to 0} f(x-\epsilon)$.

[12] A heuristic but still non-rigorous "proof" is presented in Appendix ** to this chapter.

In practice, it doesn't matter how the constant factor $\frac{1}{2\pi}$ is distributed between the two functions $f(x)$ and $g(k)$ since $\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$. From this point onward, we will follow the convention of many modern authors in writing the Fourier transform pair in symmetric form as

(17.16.5) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$$

and

(17.16.6) $$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

The functions $f(x)$ and $g(k)$ in eqns. (17.16.5) and (17.16.6) form a "symmetric Fourier transform pair" and are Fourier transforms of each other. Note that infinitely repetitive functions over the entire x-axis are not covered by these formulas since the integral of an infinitely repetitive function that is not always zero is not finite.

If f is a function of time (t), the symmetric formulas subject to the analogous requirements are[13]:

(17.16.7) $$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega$$

and

(17.16.8) $$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

[13] In dealing with functions of time, not infrequently, one uses the exponential factor $e^{-i\omega t}$ in eqn. (17.16.7) and the exponential factor $e^{+i\omega t}$ in eqn. (17.16.8) for defining the Fourier transform pair. As well, some authors use the opposite sign convention in the exponential functions even in eqns. (17.16.5) and (17.16.6) for $f(x)$ and for $g(k)$.

The form of eqn. (17.16.7) shows a complex Fourier integral expansion of a nonrepetitive function $f(t)$ over the complete set of complex exponential functions $\{e^{i\omega t}\}$ for the continuous range of frequencies $0 < \omega < \infty$. The functions $f(t)$ and $g(\omega)$ are Fourier transforms of each other.

Negative Frequencies? No!

In the Fourier transform formula of eqn. (17.16.7), we see the apparent appearance of "negative frequencies" since the limits of the integral are $-\infty \rightarrow \infty$. In actual physical reality, however, there are no "negative frequencies." To understand the root of this possible confusion in the integral, let's remember that the complex exponential function $e^{i\omega t}$ represents a rotating vector in the complex plane. Then, as a convenient *mathematical* idea, we conventionally say that if the sense of rotation of this vector over the course of time is counterclockwise, then the complex exponential function representing this is $e^{i\omega t}$ with "ω positive." We then also say that if the rotation of the vector in the complex plane is clockwise, ω "is negative", but in reality, the exponential function representing this rotating vector is $e^{-i\omega t}$, which is mathematically equivalent to $e^{i(-\omega)t}$, where ω is positive.

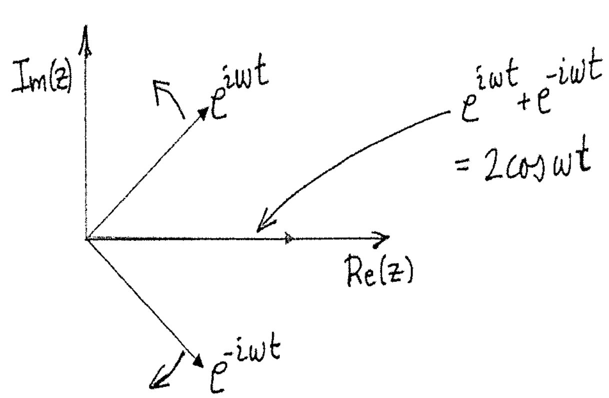


Fig. 17.11 Representation of the quantities $e^{i\omega t}$ and $e^{-i\omega t}$ as oppositely rotating vectors in the complex plane at one instant of time. It is seen that their imaginary parts cancel and that their real parts add so that their sum is purely real and equal to $2\cos(\omega t)$.

If $f(t)$ is a purely real-valued function of t, then for each term $g(\omega)e^{i\omega t}$in the integral of eqn. (17.16.7), we must have a corresponding term $g(-\omega)e^{-i\omega t}$ such that $[g(-\omega)]^* = g(\omega)$ so that then $[g(-\omega)e^{-i\omega t}]^* = [g(-\omega)]^* \cdot [e^{-i\omega t}]^* = g(\omega)e^{i\omega t}$, which enforces the requirement $g(\omega)e^{i\omega t} + g(-\omega)e^{-i\omega t} = 2Re\left[g(\omega)e^{i\omega t}\right]$ which is purely real. Expressing $g(\omega)$ in polar form as $g(\omega) = Re^{i\emptyset}$, where R is real, then $g(\omega)e^{i\omega t} + g(-\omega)e^{-i\omega t} = Re^{i\emptyset}e^{i\omega t} + Re^{-i\emptyset}e^{-i\omega t} = R[e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}] = 2Rcos(\omega t + \phi)$ which is manifestly real. Analogous logic applies to the apparent presence of "negative wavenumbers" in the Fourier transform formulas (17.16.5) and (17.16.6).

It should be noted that the Fourier transform formulas (17.16.5), (17.16.6), (17.16.7) and (17.16.8) are valid whether or not $f(x)$ [or $f(t)$] is purely a real-valued function – i.e., they are also valid if $f(x)$ [or $f(t)$] is complex-valued, as occurs, e.g., and not untypically, in quantum mechanics. If $f(x)$ is complex-valued, then it is *not* true that for each term $g(k)e^{ikx}$in the integral of eqn. (17.16.5), we must have a corresponding term $g(-k)e^{-i\omega t}$ such that $[g(-k)]^* = g(k)$ (and analogously in eqn. (17.16.7) if $f(t)$ is complex-valued).

17.17 Examples of Complex Fourier Transforms

Worked Example 17.7: Complex Fourier Transform of a Box Function

Find the complex Fourier transform of the function

$$f(x) = \begin{cases} A, & -a < x < a \\ 0, & |x| \geq a \end{cases} \text{, where a is a positive real number.}$$

Solution: For this we use eqn. (17.16.6), namely,

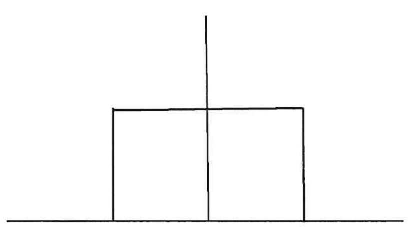
(17.16.6) $$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx,$$

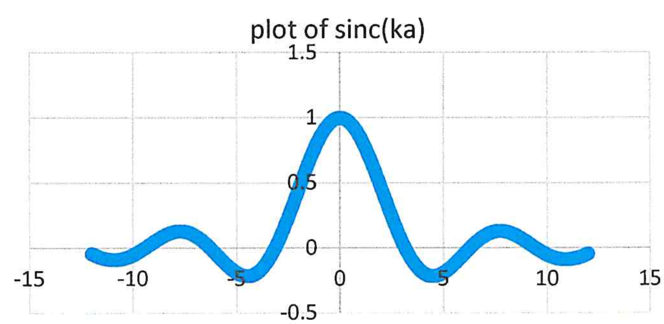
which here is $$g(k) = \frac{A}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx$$

$$= \frac{A}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{-ik}\right]_{-a}^{a} = \frac{A}{\sqrt{2\pi}}\, 2\frac{e^{-ika} - e^{ika}}{-2ik}$$

$$= \sqrt{\frac{2}{\pi}}\, A \left[\frac{\sin ka}{k}\right] = \sqrt{\frac{2}{\pi}}\, Aa\left[\frac{\sin ka}{ka}\right] = \sqrt{\frac{2}{\pi}}\, Aa\, sinc(ka).$$

where $sinc(ka) \equiv \frac{\sin ka}{ka}$. In the figures below, plots proportional to $f(x)$ and $g(k)$ are shown.





Plot of sinc(ka) = $\sqrt{\frac{2}{\pi}}\, Aa\, g(k)$ vs.

From the plots, we take the half-widths at half max as $\Delta x = a$ and $\Delta k \approx 2.0/a$, giving $\Delta x \cdot \Delta k \approx 2.0$, which is consistent with the bandwidth theorem in the form $\Delta x \Delta k \sim 1$. If we use the full widths at half max, then $\Delta x \cdot \Delta k \approx 8.0$, which is consistent with $\Delta x \cdot \Delta k \sim 2\pi$. Either way, obeying the theorem, the bigger Δx, the smaller Δk must be, and vice versa.

Example: Complex Fourier Transform of a Gaussian Function

Find the complex Fourier transform of the Gaussian function $f(x) = Ae^{-bx^2}$, where A and b are both positive real numbers.

Solution: The Fourier formula $g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$, here becomes

$$g(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-bx^2} e^{-ikx}\, dx$$

This integral can be done by a method called "completing the square;" however, for our purposes here we shall simply note that the value of the integral is

$$g(k) = \frac{A}{\sqrt{2b}}\, e^{-\frac{k^2}{4b}} \tag{17.16.9}$$

From eqn. (17.16.9) we see that the Fourier transform of a Gaussian function is another Gaussian function; however, as we saw with Fourier cosine transforms of Gaussian functions, their widths are not equal. Here, in the equation for $f(x)$, the standard deviation parameter σ_x is given by $b = \frac{1}{2\sigma_x^2} \rightarrow \sigma_x^2 = \frac{1}{2b}$, whereas, in $g(k)$, from eqn. (17.16.9) the standard deviation σ_k is given by $4b = 2\sigma_k^2 \rightarrow \sigma_k^2 = 2b$. Thus, for this example, $\sigma_k^2 \cdot \sigma_x^2 = 1 \rightarrow \sigma_k = \frac{1}{\sigma_x}$, which is the same result that we got from the Fourier cosine transform of a Gaussian. Thus, $\sigma_k \sigma_x = 1$, in accord with the bandwidth theorem. Note again that to satisfy the bandwidth theorem, the bigger σ_k is, the smaller σ_x must be, and vice versa,