13 DECISION MAKING UNDER UNCERTAINTY

I. Overview

13.1 Decision Criteria

A. Prototype Example: The News Vendor Problem

B. Decision Rules

13.2 Utility Theory

13.4 Decision Trees

A. Graphical method to solve decision problems.

B. Incorporating New Information

C. Expected Value of Sample Information

D. Expected Value of Perfect Information

E. Sequential Decisions

13.5 Baye's Rule and Decision Trees

13.1 DECISION CRITERIA

I. News Vendor Problem

(A) News vendor buys papers for 10 cents each.

(B) News vendor sells papers for 25 cents each.

(C) Demand is not known in advance.

(D) Probability distribution for demand given below.

(E) Newpapers are worthless at the end of the day.

(F) Decision — how many papers to purchase.

Demand	Probability
0	0.1
1	0.3
2	0.4
3	0.2

II. **Introduction**

(A) Result of your decision depends on nature (or another player).

(B) Nature is indifferent to the outcome.

(C) Problem structure:

1. Decision: You select an action a_i from the set of possible actions $A = \{a_1, a_{2,} \ldots, a_k\}$.
2. A state of nature s_j occurs from a set of possibles states $S = \{s_1, s_2, \ldots, s_n\}$.
3. p_j = probability that state s_j is observed.
4. You receive a reward r_{ij} based on a_i and s_j.

(D) Reward table — return you receive for each possible combination of decisions and states of nature.

Reward Table for the News Vendor Problem

	State of Nature (Demand)			
Decision	0	1	2	3
0	0	0	0	0
1	-10			
2	-20			
3	-30			

III. **Decision Rules**

(A) Dominated Actions

1. Action a_i is *dominated* by $a_{i'}$ if
   $r_{ij} \leq r_{i'j} \quad \forall s_j \in S$ and $r_{ij'} < r_{i'j'}$ for some $s_{j'} \in S$.
2. Ex. News Vendor Problem

Decision	(Demand) 0	1	2	3
0	0	0	0	0
1	-10	15	15	15
2	-20	5	30	30
3	-30	-5	20	45

3. Never choose a dominated action

(B) The Maximin Criterion

1. The *maximin criterion* chooses the action that maximizes the minimum return associated with each action
   a. I.e., choose a_i that maximizes $\min_{j \in S} r_{ij}$
2. Ex. News Vendor Problem

Decision	(Demand) 0	1	2	3	Min. Return
0	0	0	0	0	0
1	-10	15	15	15	-10
2	-20	5	30	30	-20
3	-30	-5	20	45	-30

3. Extremely risk averse.
4. Would you want to apply it to:

Decision	State of Nature 0	1	2	3	4	5	6
1	100	100	2.9	100	100	100	100
2	3	3	3	3	3	3	3

(C) The Maximax Criterion

1. The *maximin criterion* chooses the action that maximizes the maximum return.

   a. I.e., choose a_i that maximizes $\max_{j \in S} r_{ij}$.

2. Ex. News Vendor Problem

	(Demand)				Max.
Decision	0	1	2	3	Return
0	0	0	0	0	0
1	-10	15	15	15	15
2	-20	5	30	30	30
3	-30	-5	20	45	45

3. Extremely optimistic.

4. Would you want to apply it to:

	State of Nature						
Decision	0	1	2	3	4	5	6
1	100	100	100	100	100	100	100
2	3	3	101.1	3	3	3	3

(D) Minimax Regret

1. Regret = extra amount you could have obtained if you had known the state of nature.
2. The *minimax regret criterion* minimizes the maximum regret.
3. Ex. News Vendor Problem

Reward Table for the News Vendor Problem

	State of Nature (Demand)			
Decision	0	1	2	3
0	0	0	0	0
1	-10	15	15	15
2	-20	5	30	30
3	-30	-5	20	45

Regret Table for News Vendor Problem

	(Demand)				Max.
Decision	0	1	2	3	Regret
0	0	15	30	45	45
1	10				30
2	20				20
3	30				30

(E) The above decision rules are most appropriate when estimates of the probabilities p_j are unavailable.

(F) The Expected Value Criterion

1. The *expected value criterion* chooses the action that maximizes the expected return.
2. Expected return if we choose action a_i is:

$$\mathrm{ER}_i = \sum_{j=1}^{n} r_{ij} p_j = r_{i1}p_1 + r_{i2}p_2 + \cdots + r_{in}p_n.$$

3. Ex. News Vendor Problem

	(Demand)				Expected
Decision	0	1	2	3	Return
0	0	0	0	0	0
1	-10	15	15	15	12.5
2	-20	5	30	30	
3	-30	-5	20	45	

(G) Ex. Real Estate Development

1. Developer has 3 options for a piece of land.
   a. Residential proposal.
   b. Commercial proposal 1.
   c. Commercial proposal 2.
2. A shopping center may be built nearby.
   a. No shopping center.
   b. Medium-sized shopping center.
   c. Large shopping center.
3. The shopping center will effect the value of each proposal

Reward Table for Real Estate Developer

Decision	State of Nature (Shopping Center)		
	None	Medium	Large
Residential	4	16	12
Commercial 1	5	6	10
Commercial 2	-1	4	15

13.2 UTILITY THEORY

I. Introductory Example

(A) Urn contains 99 white balls and 1 black ball

(B) Each ball is equally likely to be drawn

(C) If white ball is drawn, you pay \$10,000

(D) If black ball is drawn, you win \$1,000,000

(E) Decision — whether to play or not

Reward Table

Decision	State of Nature		Expected
	White Ball	Black Ball	Return
Play	-10,000	1,000,000	100
Don't Play	0	0	0

II. **Goal — Measure the “attractiveness” of money**

III. **Risk Averse**

(A) Receiving \$1,000,000 is exciting, but receiving \$2,000,000 is not twice as exciting

(B) Utility function is nondecreasing

(C) Utility function is concave, i.e., marginal utility decreases

IV. **Risk Seeking**

(A) An increase of \$100 increases your utility more than a decrease of \$100 decreases it

(B) Utility function is nondecreasing

(C) Utility function is convex, i.e., marginal utility increases

V. **Ex. News Vendor Problem**

(A) Utility Function

(B) Create reward table in terms of utility

(C) Maximize expected utility

Utility Reward Table for Newsboy Problem

Decision	(Demand) 0	1	2	3	Expected Utility
0	0.63	0.63	0.63	0.63	0.63
1	0.50	0.79	0.79	0.79	0.761
2	0.30	0.69	0.90	0.90	0.777
3	0	0.57	0.83	1.00	0.703

VI. **Creating a Utility Function**

(A) Set utility of smallest reward to 0

(B) Set utility of largest reward to 1

(C) For newsboy: $U(-30) = 0$ and $U(45) = 1$

(D) For $U(10)$, select p such that you are indifferent between receiving payment of:

1. \$10 for sure
2. \$45 with prob. p or -\$30 with prob. $1 - p$

(E) Value of p varies from person to person

13.4 DECISION TREES

I. **Decision Trees**

(A) Graphical representation of a decision problem

(B) Useful for a sequence of decisions

(C) A *decision fork* represents a decision point (square nodes)

(D) Lines from squares represent possible decisions

(E) An *event fork* represents uncertain outcomes (circular nodes)

(F) Lines from circles represent possible outcomes

(G) Ex. Decison Tree for News Vendor Problem

(H) Ex. Protrac Problem

1. New line of garden tractors
2. Must decide on marketing and production strategy
   a. **Aggressive (A):**
      New production facility
      Large inventories for prompt delivery
      Nation-wide promotional campaign
      Dealer discounts
   b. **Basic (B):**
      Dedicated production line
      Inventories only for popular items
      Support local or regional advertising
   c. **Cautious (C):**
      Use excess capacity for production
      Production geared to meet demand
      Advertising up to local dealer
3. Market condition categorized as either strong (S) or weak (W)
4. Probability of strong market = 0.45
5. Probability of weak market = 0.55

6. Reward Table for Protrac

Decision	State of Nature		Expected
	Strong (S)	Weak (W)	Return
Aggressive (A)	30	-8	9.10
Basic (B)	20	7	12.85
Cautious (C)	5	15	10.50

7. Decision Tree for Protrac

II. **Solving the Decision Tree**

(A) Calculate expected value for each terminal node

(B) Obtain reduced decision tree

(C) Select alternative with highest expected value

III. **Incorporating New Information**

(A) Board of directors insist on a market study.

(B) Study will report either encouraging (E) or discouraging (D).

(C) History of marketing department indicates:

1. The probability of an encouraging report is $P(E) = 0.435$.
2. The probability of a discouraging report is $P(D) = 0.565$.
3. If the report is encouraging, then the probability that market is strong is $P(S|E) = 0.621$.
4. If the report is discouraging, then the probability that market is strong is $P(S|D) = 0.318$.

(D) Decision Tree with New Information

IV. **Expected Value of Sample Information**

(A) Protrac's Expected Value With Original Information (EVWOI) (i.e., w/o market study) is 12.85.

(B) Protrac's Expected Value With Sample Information (EVWSI) (i.e., with market study):

$$\text{EVWSI} = 15.60(0.435) + 11.81(0.565) = 13.46$$

(C) Protrac's Expected Value of Sample Information (EVSI):

$$\begin{aligned}\text{EVSI} &= \text{EVWSI - EVWOI}\\ &= 13.46 - 12.85 = 0.61.\end{aligned}$$

V. **Expected Value of Perfect Information**

(A) EVPI = largest fee you would pay to know which state will occur.

(B) EVPI provides upper bound on amount to spend gathering information.

(C) r_{bj} = payoff for best decision if state j occurs.

(D) Expected Value With Perfect Information (EVWPI):

$$\text{EVWPI } = \sum_{j=1}^{n} r_{bj} p_j .$$

(E) EVPI = EVWPI - EVWOI.

(F) Ex. News Vendor Problem:

Decision	(Demand) 0	1	2	3
0	0	0	0	0
1	-10	15	15	15
2	-20	5	30	30
3	-30	-5	20	45

Demand	Probability
0	0.1
1	0.3
2	0.4
3	0.2

1. EVWPI =

2. EVPI =

(G) Ex. Protrac

Decision	State of Nature Strong (S)	Weak (W)	Expected Return
Aggressive (A)	30	-8	9.10
Basic (B)	20	7	12.85
Cautious (C)	5	15	10.50

1. EVWPI =

2. EVPI =

3. Conclusion: Market test is not very effective.

VI. **Sequential Decisions**

(A) Ex. Protrac

1. Market test costs 0.5.
2. Should Protrac perform the market test?
3. Formulate as a sequential decision problem.

(B) Ex. Art Dealer

1. Art dealer can sell "Sunplant" painting to client for \$50,000.
2. Dealer can buy the painting on Monday for \$40,000.
3. Dealer can buy it on Tuesday (if still available) for \$30,000.
4. Dealer can buy it on Wednesday (if still available) for \$26,000.
5. Each day there is a 0.6 probability that the painting will be sold.
6. Objective — maximize expected profit.

13.5 BAYES' RULE AND DECISION TREES

I. **Ex. Protrac Revisited**

(A) Past history of marketing department indicates:

1. Results are encouraging 60% of the time when the market is actually strong.
2. Results are discouraging 70% of the time when the market is actually weak.

(B) How should Protrac update probabilities based on report?

(C) How can we set up the decision tree?

1. We do not know $P(E)$ or $P(D)$.
2. We do not know $P(S|E)$ or $P(W|E)$.
3. Instead, we are given $P(E|S)$ and $P(D|W)$.
4. We need to use the laws of probability.

II. **Probability Review**

(A) Illustrative Example

1. A fair die is thrown
   a. If 1 is thrown, draw a ball from urn 1.
   b. If 2 or 3 is thrown, draw a ball from urn 2.
   c. If 4, 5, or 6 is thrown, draw a ball from urn 3.
2. Each ball in an urn is equally likely to be drawn.
   a. Urn 1 contains 28 white and 72 black balls.
   b. Urn 2 contains 40 white and 60 black balls.
   c. Urn 3 contains 92 white and 8 black balls.

(B) Conditional Probabilities

1. $P(W|1)$ means the probability of drawing a white ball given that you are drawing from urn 1.

$$
\begin{aligned}
P(W|1) &= 0.28 \quad \text{and} \quad P(B|1) = 0.72 \\
P(W|2) &= 0.40 \quad \text{and} \quad P(B|2) = 0.60 \\
P(W|3) &= 0.92 \quad \text{and} \quad P(B|3) = 0.08
\end{aligned}
$$

(C) Joint Probabilities

1. We know the probability of drawing from urn 1 is 1/6.
2. Suppose someone tells us a black ball was drawn, but does not tell us which urn was used.
3. We want to know $P(1|B)$.
4. Let I denote any urn.
5. $P(I \text{ and } B) = P(B \text{ and } I) = P(B|I)P(I) = P(I|B)P(B)$.
6. Thus, $P(I|B) = \frac{P(B|I)P(I)}{P(B)}$.
7. $P(B) = P(B|1)P(1) + P(B|2)P(2) + P(B|3)P(3)$.

(D) Bayes' Rule

$$\begin{aligned} P\left(1|B\right) &= \frac{P(B|1)P(1)}{P(B|1)P(1) + P(B|2)P(2) + P(B|3)P(3)} \\ &= \end{aligned}$$

(E) Probability Calculations for Protrac

$$\begin{aligned} P(E) &= P(E|S)P(S) + P(E|W)P(W) \\ &= 0.6(0.45) + 0.3(0.55) = 0.435 \end{aligned}$$

$$P(D) = 1 - P(E) = 0.565$$

$$P(S|E) = \frac{P(E|S)P(S)}{P(E)} = \frac{(0.6)(0.45)}{0.435} = 0.621$$

$$P(W|E) = 1 - P(S|E) = .379$$

$$P(S|D) = \frac{P(D|S)P(S)}{P(D)} = \frac{(0.4)(0.45)}{0.565} = 0.318$$

$$P(W|D) = 1 - P(S|D) = 0.682$$