Teorema de Noether y leyes de conservación

Consideremos un sistema descrito por una densidad lagrangiana:

$$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$$

y cuya acción es:

$$I = \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$$

Supongamos que se lleva a cabo una cierta transformación:

$$\left.\begin{array}{l} x^\mu \longmapsto x'^\mu = x^\mu + \delta x^\mu \\ \varphi(x) \longmapsto \varphi'(x') = \varphi(x) + \delta\varphi(x) \end{array}\right\}$$

Invariancia del sistema significa que no se alteran las ecuaciones de movimiento, lo que debe estar incorporado a través de la invariancia de la acción.

Vamos a calcular δI y hacer que sea cero cuando se tienen en cuenta las transformaciones anteriores para x^μ y φ:

$$\delta I = \delta \int d^4x \mathcal{L} = \int \delta(d^4x \mathcal{L}) = \int \delta(d^4x) \mathcal{L} + \int d^4x \, \delta\mathcal{L}$$

Tenemos que determinar $\delta(d^4x)$ y $\delta\mathcal{L}$. Veamos en primer lugar $\delta(d^4x)$:

$$d^4x' = J d^4x$$

donde J es el jacobiano cuyo valor es:

$$
J = \begin{vmatrix}
\frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^1} & \frac{\partial x'^0}{\partial x^2} & \frac{\partial x'^0}{\partial x^3} \\[6pt]
\frac{\partial x'^1}{\partial x^0} & \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^3}{\partial x^3} \\[6pt]
\frac{\partial x'^2}{\partial x^0} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^2}{\partial x^3} \\[6pt]
\frac{\partial x'^3}{\partial x^0} & \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3}
\end{vmatrix}
$$

Teniendo en cuenta la expresión para x'^μ:

$$x'^\mu = x^\mu + \delta x^\mu$$

Cada elemento del jacobiano es:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu \; \frac{\partial(\delta x^\mu)}{\partial x^\nu}$$

donde δ^μ_ν es la delta de Kronecker. Entonces:

$$
J = \begin{vmatrix}
1 + \frac{\partial(\delta x^0)}{\partial x^0} & \frac{\partial(\delta x^0)}{\partial x^1} & \dots & \frac{\partial(\delta x^0)}{\partial x^3} \\[6pt]
\frac{\partial(\delta x^1)}{\partial x^0} & 1 + \frac{\partial(\delta x^1)}{\partial x^1} & \dots & \frac{\partial(\delta x^1)}{\partial x^3} \\[6pt]
\vdots & \vdots & & \vdots \\[6pt]
\frac{\partial(\delta x^3)}{\partial x^0} & \frac{\partial(\delta x^3)}{\partial x^1} & \dots & 1 + \frac{\partial(\delta x^3)}{\partial x^1}
\end{vmatrix}
$$

por lo que, hasta primer orden, el jacobiano es:

$$J = \det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) = 1 + \frac{\partial(\delta x^{\mu})}{\partial x^{\mu}}$$

luego podemos escribir:

$$d^4x' = J \cdot d^4x \longrightarrow \delta(d^4x) = d^4x' - d^4x$$

es decir:

$$\delta(d^4x) = J\,d^4x - d^4x = (J-1)\,d^4x = \frac{\partial(\delta x^{\mu})}{\partial x^{\mu}}\,d^4x$$

que nos permite obtener:

$$\underline{\underline{\delta(d^4x) = d^4x\,\partial_\mu(\delta x^{\mu})}}$$

Si las transformaciones corresponden a una <u>simetría</u> se tiene $\delta I = 0$, luego:

$$\delta I = \int d^4x\,\left(\delta\mathcal{L} + \mathcal{L}\,\partial_\mu(\delta x^{\mu})\right) = 0$$

Determinamos ahora $\delta\mathcal{L}$:

$$\delta\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \mathcal{L}'(x+\delta x) - \mathcal{L}(x) =$$

$$= \underbrace{\mathcal{L}'(x) - \mathcal{L}(x)}_{=\delta^*\mathcal{L}} + \frac{\partial\mathcal{L}}{\partial x^{\mu}}\,\delta x^{\mu} = \delta^*\mathcal{L} + \frac{\partial\mathcal{L}}{\partial x^{\mu}}\,\delta x^{\mu}$$

es decir:

$$\underline{\underline{\delta\mathcal{L} = \delta^*\mathcal{L} + \frac{\partial\mathcal{L}}{\partial x^{\mu}}\,\delta x^{\mu}}}$$

Veamos ahora lo que es $\delta^*\mathcal{L}$:

$$\delta^*\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi}\delta^*\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta^*(\partial_\mu \varphi) =$$

($\mathcal{L}$ sólo cambia con φ y $\partial_\mu\varphi$ al considerar $\delta^*\mathcal{L}$)

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\right)$$

Ecs. Euler-Lagrange

$$= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta^*\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta^*(\partial_\mu\varphi) =$$

$$= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta^*\varphi\right)$$

y queda finalmente para $\delta\mathcal{L}$:

$$\underline{\underline{\delta\mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta^*\varphi\right) + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu}}$$

Sustituyendo en δI:

$$\delta I = \int d^4x\left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta^*\varphi\right) + \mathcal{L}\,\partial_\mu(\delta x^\mu)\right]$$

Nos falta determinar $\delta^*\varphi$. Sabemos que:

$$\underline{\delta\varphi = \varphi'(x') - \varphi(x)}$$

luego:

$$\delta\varphi = \varphi'(x') - \varphi(x) = \varphi'(x+\delta x) - \varphi(x) =$$

$$= \varphi'(x) + (\partial_\nu\varphi)\delta x^\nu - \varphi(x) =$$

$$= \underbrace{\varphi'(x) - \varphi(x)}_{= \delta^* \varphi} + (\partial_\nu \varphi)\delta x^\nu = \delta^* \varphi + (\partial_\nu \varphi)\delta x^\nu$$

despejando $\delta^* \varphi$ queda:

$$\underline{\underline{\delta^* \varphi = \delta \varphi - (\partial_\nu \varphi)\delta x^\nu}}$$

y δI toma la forma definitiva:

$$\delta I = \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta^* \varphi \right) + \mathcal{L}\, \partial_\mu (\delta x^\mu) \right] =$$

$$= \int d^4x \left\{ \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \left(\delta \varphi - \frac{\partial \varphi}{\partial x^\nu} \delta x^\nu \right) + \mathcal{L}\, \partial_\mu (\delta x^\mu) \right\} =$$

$$= \int d^4x \left\{ \partial_\mu \left[- \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \varphi}{\partial x^\nu} \delta x^\nu + \right. \right.$$

$$\left. \left. + \mathcal{L}\, g^{\mu\nu} \delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta \varphi \right] \right\} = 0$$

Introducimos la <u>corriente de Noether</u> como:

$$\boxed{j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \varphi}{\partial x^\nu} \delta x^\nu - \mathcal{L} g^{\mu\nu} \delta x_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta \varphi}$$

de modo que:

$$\delta I = - \int (\partial_\mu j^\mu)\, \delta^4 x = 0$$

que proporciona la ley de conservación de la corriente de Noether asociada a la simetría:

$$\boxed{\partial_\mu j^\mu = 0}$$

Si hay simetría, entonces:

$$\exists\, j^\mu \,/\, \partial_\mu j^\mu = 0$$

Podemos escribir:

$$\partial_\mu j^\mu = 0 \rightarrow \partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = 0$$

y si aplicamos el teorema de la divergencia:

$$\partial_0 \int_V j^0 \, d^3x = -\int_V \vec{\nabla}\cdot\vec{j}\, d^3x = -\int_S \vec{j}\cdot d\vec{S} = 0$$

(En la superficie del infinito)

luego:

$$\partial_0 \int_V j^0 \, d^3x = 0$$

Si llamamos carga "Q" a la cantidad:

$$\boxed{Q = \int_V j^0 \, d^3x}$$

entonces "Q" es una magnitud que se conserva en el tiempo:

$$\partial_0 Q = 0$$

es decir, tenemos una ley de conservación.

En el caso particular de las tetratraslaciones del grupo de Poincaré:

$$\left.\begin{aligned} x'^{\nu} &= x^{\nu} + \delta x^{\nu} \equiv x^{\nu} + \epsilon^{\nu} \\ \varphi'(x') &= \varphi(x) \rightarrow \delta\varphi = 0 \end{aligned}\right\}$$

por lo que la corriente de Noether se escribe:

$$\boxed{j^{\mu} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)} \partial^{\nu}\varphi - g^{\mu\nu}\mathcal{L} \right) \epsilon_{\nu}}$$

Esta corriente es conservada con independencia de ϵ_{ν} que es arbitraria, por lo que podemos definir un tensor:

$$\boxed{\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)} \partial^{\nu}\varphi - g^{\mu\nu}\mathcal{L}}$$

que es el tensor energía-impulso canónico que cumple:

$$\boxed{\partial_{\mu}\Theta^{\mu\nu} = 0}$$

La invariancia bajo tetratraslaciones da lugar a la conservación del tensor energía-impulso. Esto corresponde a cuatro "cargas" conservadas caracterizadas por el índice ν y que son el tetramomento del campo:

$$P^{\nu} = \int d^3x\, \Theta^{0\nu} = \int d^3x\, \mathcal{P}^{\nu} \rightarrow \partial_0 P^{\nu} = 0$$