Phys. 251 - 16^{th} class | Thurs. Feb. 29, 2024

More on Fourier Series

We assumed that, for any "reasonable" function $f(x)$ on $[-\pi, \pi]$,

$f(x)$ could be expressed by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \dots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

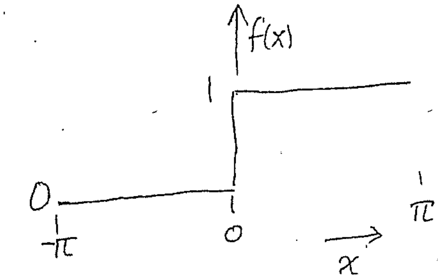
Further, we found that, if this is true, then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \,, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

As an example, for the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$



we found

$$f(x) = \underbrace{\frac{1}{2}}_{\text{bw } \frac{a_0}{2}} + \frac{2}{\pi}\left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

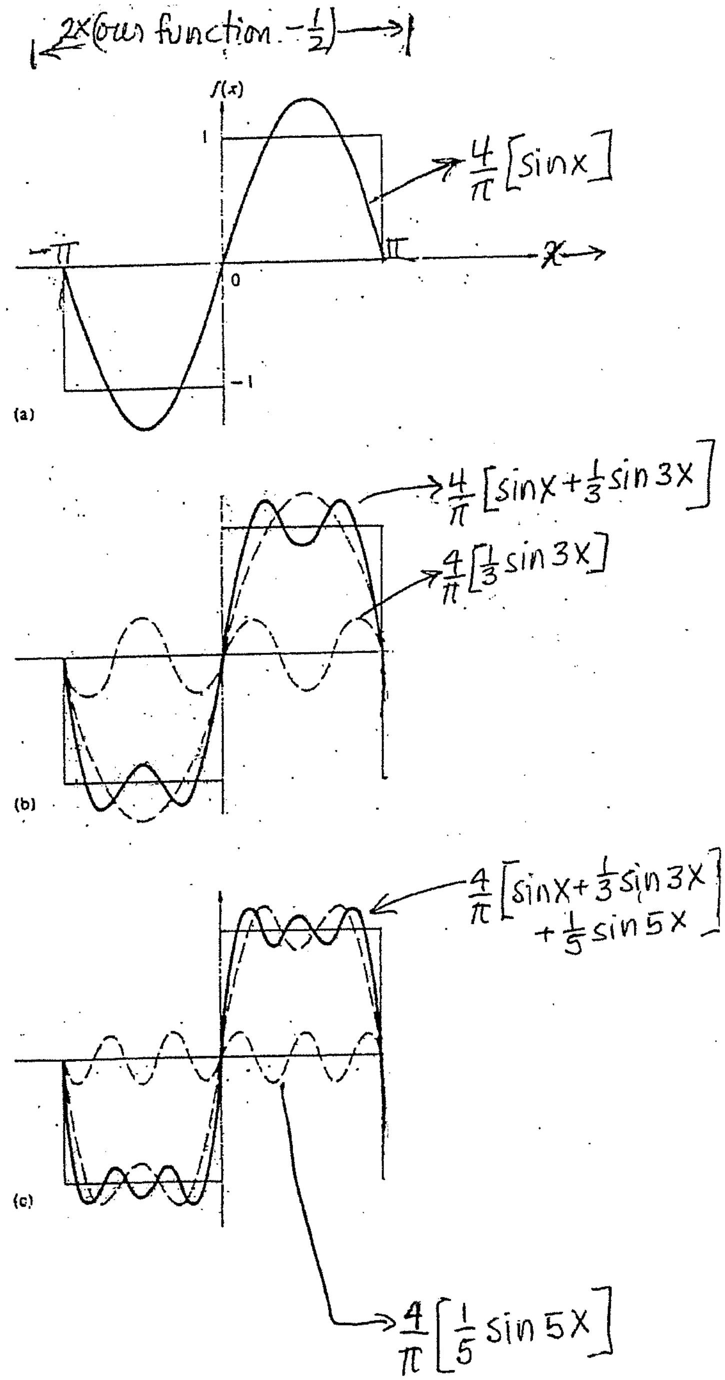
Example:

Fourier term plots for $2\times\left[\text{our function} - \frac{1}{2}\right]$

$\Rightarrow$

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]$$

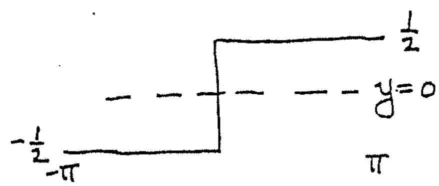
i.e., $\frac{a_0}{2}$ now $= 0$.



Some Observations on Fourier Series   K-text, sect. 11.4

1. The Fourier series for a function* depends on where we put the axes:

   a. Suppose, e.g., we raise the x-axis so that we have



(function called "symstp(x)" in the text.)

- the "shape" is still the same, but now the series works out to be

$$f(x) = \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$$

i.e, a_0 is now zero.

How do I know this so quickly?

Physical Meaning of $\frac{a_0}{2}$ coefficient → $\frac{a_0}{2}$ is the average of the function over the interval.

This is clear from $\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \langle f(x)\rangle_{[-\pi,\pi]}$

(or on $[-L,L]$ from $\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$) $= \langle f(x)\rangle_{[-L,L]}$

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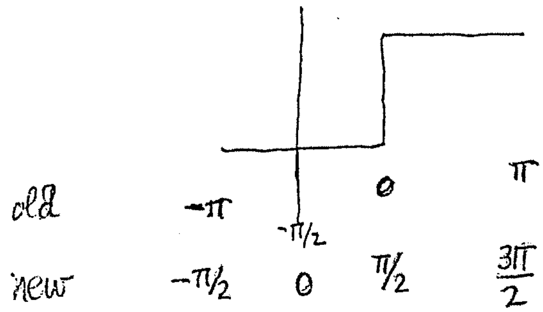
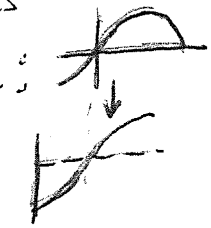
* i.e, a "shape".

b. Now suppose we move the "y-axis", say as follows:
(K-text, sect. 11.5)



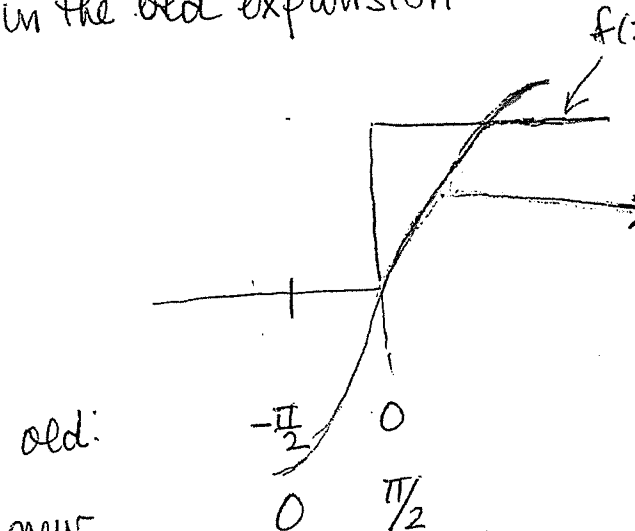
Then, all the sine terms in the expansion change:



$\sin x \to -\cos x$

$\sin 3x \to \cos 3x, \ldots$

To see why, consider, for example, the old "sin x" term in the "old" expansion:

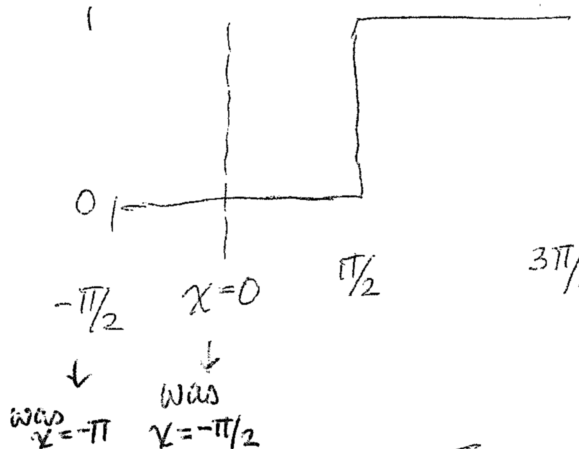


was called "sin x", but now called $\sin(x - \pi/2) = -\cos x$

To generalize: If we move the y-axis "backwards" by ϕ radians, then, e.g., $b_1 \sin x_{old} \longrightarrow b_1 \sin(x_{new} - \phi) = \underbrace{b_1 \cos\phi}_{\text{"}b_{1,new}\text{"}} \sin x_{new} + \underbrace{(-\sin\phi) b_1}_{\text{"}a_{1,new}\text{"}} \cos x_{new}$

Note then what has happened: In the old coordinate system we had only sine functions in the series; in the new coordinate system we have a "different" series with both sine and cosine terms.

So, e.g.



$\Rightarrow \sin x_{old} \to -\cos x_{new}$

$\sin 3x_{old} \to \cos 3x_{new}$

$\sin 5x_{old} \to -\cos 5x_{new}$

$\vdots$

$$\Rightarrow f(x)_{[-\pi/2 \to \frac{3\pi}{2}]} = \frac{2}{\pi}\left[-\cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \dots\right]$$

Has the physics (or here, the geometry) changed? No.
While the names of the function f(x) and the names of the Fourier terms might have changed, the shapes of the function f(x) and of the net "1st order "2nd order terms", etc. have not). [Where by "1st order expansion term" we mean the expansion term with one "wiggle" per 2π interval, by "2nd order term" the one with two "wiggles" per 2π interval, etc.] I.e,
The names may have changed, but the shapes are still the same.

2. a. Change of Interval – "Simple"

Suppose our function is defined and to be expanded, not on $[-\pi, \pi]$, but on $[0, 2\pi]$. This makes only a very minor change – all the $\sin(nx)$ and $\cos(nx)$ still go through n full periods on this interval, so everything is the same except that we change the limits of integration on the coeficients:

on $[0, 2\pi]$: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

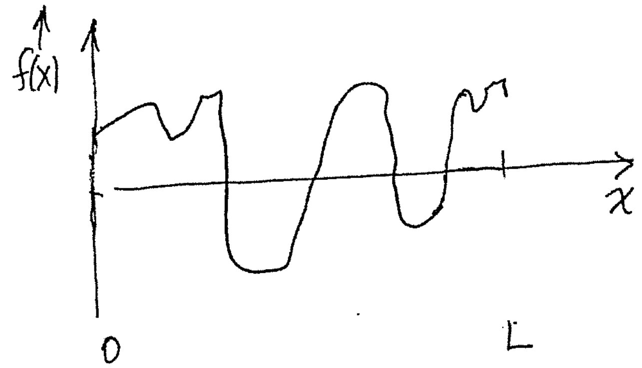
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

(next page →)

b. Expanding a Function on the Arbitrary Finite Interval $[0, L]$.

(K-text, sect 11.6)

Now suppose we want to expand a function $f(x)$ on a more arbitrary interval, say $[0,L]$. How do we do this?



function $f(x)$ defined on $[0,L]$. We seek a Fourier expansion for $f(x)$ that well approximates it or converges to it, on this interval.

Before looking at the more formal math treatment, as usual, let's first ask "what do we expect?"

As a guide, we first note that, for expanding a function on $[0,2\pi]$ a typical term in our expansion is $\propto \sin\left(\frac{2\pi}{\lambda_n} x\right)$.

Why? Typical term was $\sin(nx)$. Now, on $[0,2\pi]$, $\lambda_1 =$ "λ of $\sin x$" $= 2\pi$

$\lambda_2 =$ "λ of $\sin 2x$" $= \frac{2\pi}{2} = \pi$, etc, so $\sin\left(\frac{2\pi}{\lambda_n}x\right) = \sin\left(\frac{2\pi}{2\pi/n}x\right) = \sin(nx)$

$= \sin(nx)$ ✓

$$\left[\lambda_n = \frac{\lambda_1}{n} = \frac{2\pi}{n}\right].$$

Second, we note that $[0,L]$ is just a pure "stretch" (or pure compression, if $L<2\pi$) of $[0,2\pi]$.

Thus, for expansion of a function $f(x)$ on $[0,L]$, we expect "typical term" $\propto \sin\left(\frac{2\pi}{\lambda_n}x\right)$ where $\lambda_n = \frac{L}{n}$ (*), i.e,

"typical term" $\propto \sin\left(n\frac{2\pi}{L}x\right)$, i.e, we expect the series to be

$$f(x)_{[0,L]} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{2\pi}{L}x\right) + b_n \sin\left(n\frac{2\pi}{L}x\right)\right] \quad (\gamma)$$

(*) Note: Here $\lambda_n = \frac{L}{n}$, not $\frac{2L}{n}$! [In case $L=2\pi$, $\frac{L}{n} = \frac{2\pi}{n} \Rightarrow \frac{2\pi}{\lambda_n} = \frac{2\pi}{2\pi/n} \overset{\checkmark}{=} n$]

(γ) Note that in this expansion, the quantities $\left(n\frac{2\pi}{L}\right)$ are generally not integers!

Now let's check this more "formally":

To do this, define the variable $u = \frac{2\pi}{L}x$ $^{(\diamond)}$ Then, as x ranges from 0 to L, u ranges from 0 to 2π. Then

$$f(x) \equiv g(u) \overset{(+)}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nu) + b_n \sin(nu),$$

with $\quad a_n = \frac{1}{\pi}\int_0^{2\pi} g(u)\cos nu\, du, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} g(u)\sin nu\, du$

(+) For example, if $f(x) = e^{-x^2}$, then $u = \frac{2\pi x}{L} \Rightarrow x^2 = \frac{L^2u^2}{4\pi^2}$,

so $g(u) = f(x) = e^{-L^2u^2/4\pi^2}$, $g(u) \neq e^{-u^2}$. (The point by point values of f and g are the same at the same x).

$(\diamond)$ So $\frac{2\pi}{L}$ is the "scale factor" of the change of variable.

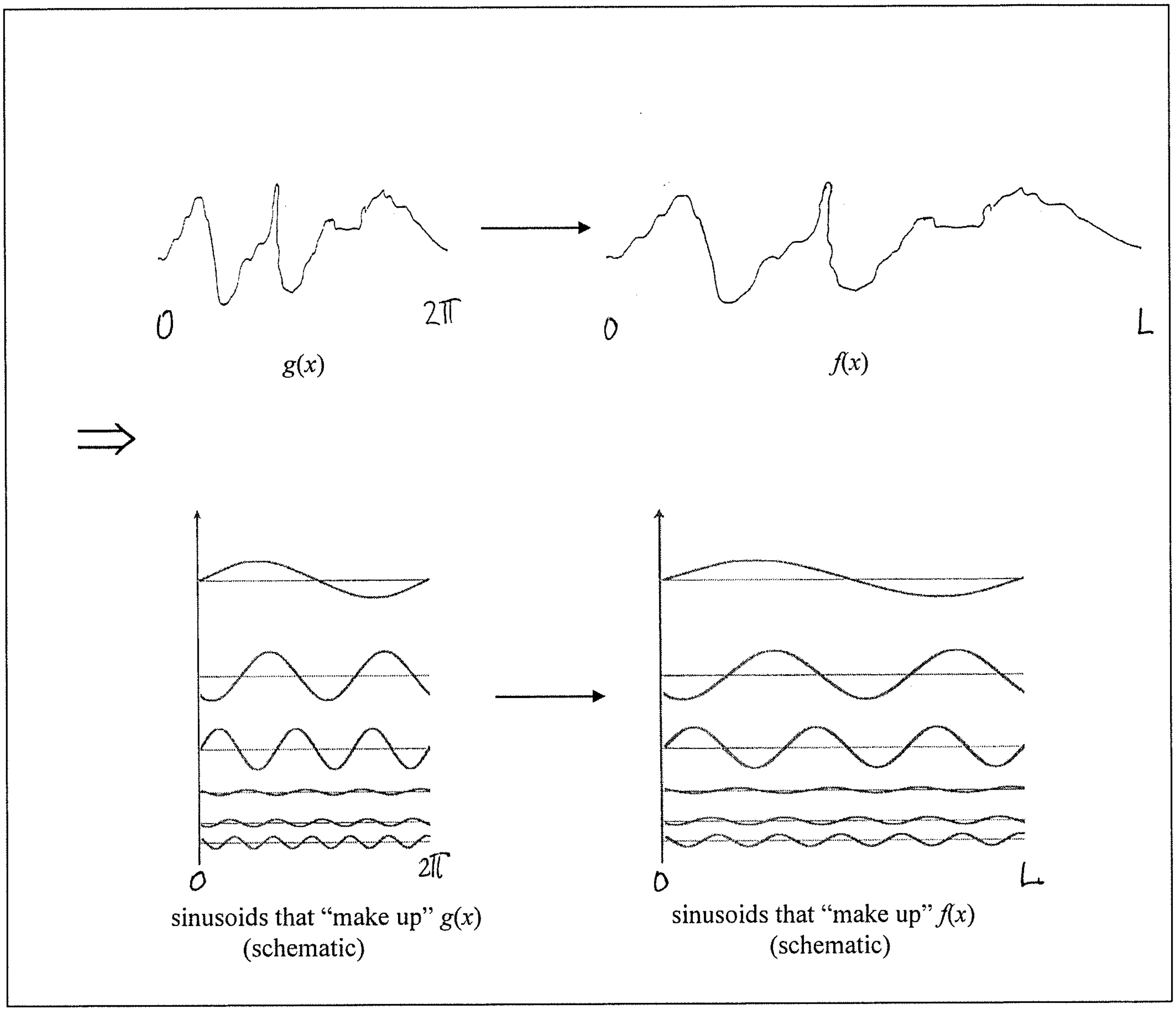


Fig. 11.3. The function $g(x)$ is defined on $(0,2\pi)$. The function $f(x)$ is obtained by stretching $g(x)$ purely horizontally in the figure so that it covers the interval $(0,L)$. The Fourier sinusoids $b_n \sin nx$ and $a_n \cos nx$ that "make up" $g(x)$ must also stretch purely horizontally to make up $f(x)$. Figures are schematic only.

Now that we've thought through what to expect, let's make the same argument mathematically: $f(x)$ is defined and to be expanded on $(0,L)$. Define the "stretched variable"

Expansion of $f(x)$ on $[0,L]$, continued

[recall: $u = \frac{2\pi}{L}x$]

Let us reexpress these in terms of x. We have

(1) $$g(u) = f(x) \underset{[0,L]}{\overset{\checkmark}{=}} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{L}x\right) + b_n \sin\left(n\frac{2\pi}{L}x\right)$$

with
$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(u)\cos nu\, du = \frac{1}{\pi}\int_0^{L} f(x)\cos\left(n\frac{2\pi}{L}x\right)\cdot \underbrace{\frac{2\pi}{L}dx}_{du}, \text{ i.e.,}$$

(2)
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{2\pi}{L}x\right)dx.$$

Similarly,
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{2\pi}{L}x\right)dx.$$

Expand $f(x)$ on $[0,\pi]$.

Example: $L = \pi$, $0 \longmapsto \pi$

then
$$f(x) \underset{[0,\pi]}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(2nx) + b_n\sin(2nx)$$

Notice that only terms are $\cos 2x, \cos 4x, \ldots$ $\sin 2x, \sin 4x, \ldots$

$$= \frac{a_0}{2} + a_1\cos 2x + a_2\cos 4x + \ldots$$
$$+ b_1\sin 2x + b_2\sin 4x + \ldots$$

with
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(2nx)\,dx$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(2nx)\,dx$$

We found, for an arbitrary p.c. function defined on $[0, L]$, where L is any positive real number,

$$(1) \quad f(x)\Big|_{[0,L]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{L} x\right) + b_n \sin\left(n \frac{2\pi}{L} x\right)$$

$\left[\Rightarrow \lambda_1 = L,\ \lambda_n = \frac{L}{n} \text{ cf on } [0, 2\pi]\ \lambda_1 = 2\pi\right]$

with

$$(2) \quad a_n = \frac{2}{L}\int_0^L f(x) \cos\left(n \frac{2\pi}{L} x\right) dx, \quad b_n = \frac{2}{L}\int_0^L \cos\left(n \frac{2\pi}{L} x\right) dx$$

Note that the terms in the above series for $f(x)$ [convergent to $f(x)$ on $[0,L]$] are no longer generally of the form

$\begin{matrix}\cos \\ \sin\end{matrix}(\text{integer} \cdot x)$! [e.g., if $L = 3.7\pi$, cos terms are $\cos\left(n \frac{2}{3.7} x\right) = \cos n(0.541 n)$]

<u>Example</u>: <u>$L = 2\pi$</u>: Should reproduce previously known results. <u>Check that</u>!

<u>Example</u>: <u>$L = \pi$</u>: Then, eqn. (1) $\Rightarrow$

$$f(x)_{[0,\pi]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2nx) + b_n \sin(2nx)$$

$$= \frac{a_0}{2} + a_1 \cos 2x + a_2 \cos 4x + \dots$$
$$+ b_1 \sin 2x + b_2 \sin 4x + \dots$$

– i.e, only <u>even</u> harmonics of the fundamental wavenumber are present! [Since $\lambda_1 = \pi \Rightarrow k_1 = \frac{2\pi}{\lambda_1} = 2$].

B: Fourier Expansion valid for arbitrary interval (a,b): [K-text, sect. 11.6, pp 11-23, →11-24].

a. What Do We Expect?

To get an idea, note that the physics (and the shapes involved) don't care if we call the beginning of the interval $x=0$ or $x=a$; nor does it care whether we call the end of the interval $x=L$ or $x=b$; since L was arbitrary, this is just a shift in the position of the y-axis; → plus, perhaps a stretch or compression. Now, in (1) and in (2), L is just the length of the interval; for (a,b) the length of the interval is $|b-a|$, so we expect, that for expansion of $f(x)$ on (a,b), from (1) and (2) with $L \to |b-a|$,

(3) $$f(x)_{(a,b)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{|b-a|} x\right) + b_n \sin\left(n \frac{2\pi}{|b-a|} x\right) \quad (*)$$

where

(4) $$a_n = \frac{2}{|b-a|} \int_a^b f(x) \cos\left(n \frac{2\pi}{|b-a|} x\right) dx$$

(5) $$b_n = \frac{2}{|b-a|} \int_a^b f(x) \sin\left(n \frac{2\pi}{|b-a|} x\right) dx$$

* Note that eqn. (3) is of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\lambda_1} x\right) + b_n \sin\left(n \frac{2\pi}{\lambda_1} x\right)$ with $\lambda_1 = |b-a|$. Thus $\lambda_n = \frac{\lambda_1}{n} = \frac{|b-a|}{n}$.

But - what about the shift in the ordinate ("y") axis?

As we've seen, in a "shift of the y-axis", $\sin\theta \to \sin(\theta+\phi)$, and $\cos\theta \to \cos(\theta+\delta)$ (phase angles). However, $\sin(\theta+\phi)$ can be written in the form $\alpha\sin\theta + \beta\cos\theta$, sim. for $\cos(\theta+\delta)$. Thus, the form (3) should still be correct.(+) Accepting this, by the logic in our first class on Fourier expansion, (4) and (5) should follow. In fact, (3), (4) and (5) are correct.

(+) Albeit that "some of what used to be called sine is now cosine" and vice versa.

Example:

A common case is that where the interval is symmetric around $x=0$ - i.e, $(-L,L)$. Then, from (4), since $\lambda_1 = 2L \Rightarrow k_1 = \frac{2\pi}{2L} = \frac{\pi}{L}$,

(6) on $(-L,L)$:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi}{L}x\right)$$

with
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

(7)
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

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[K-text, sect. 11.7]

C. Periodicity of Fourier Series

Let us look at our original series for the step  on $[-\pi, \pi]$:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right]$$

Note: The term $\sin x$ is periodic with period 2π

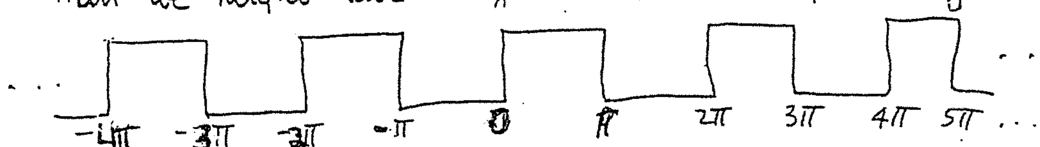
The term $\sin 3x$ is periodic with period $\frac{2\pi}{3}$ and hence also with per. 2π

The term $\sin 5x$ is periodic with period $\frac{2\pi}{5}$ and hence also with per 2π

The term "$\frac{1}{2}$" is periodic with any period.

$\therefore$ - <u>the entire series is periodic with period 2π.</u>

So - we've really found a series representation for something bigger than we might have thought at first - we get the larger entity



This is, of course, true in the general case: If we find on $[-L, L]$ that

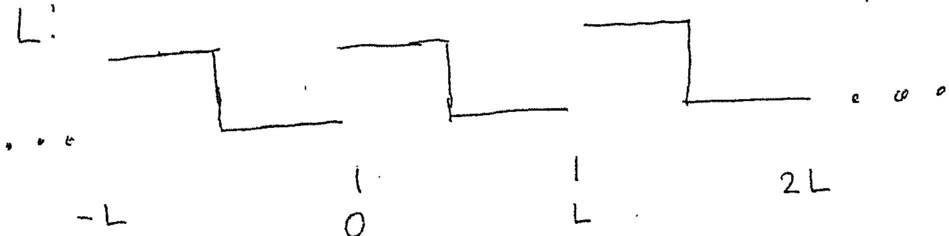
$$\rightarrow \quad f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

the series converges to more:

Likewise consider our expansion of a function given on $[0, L]$, say

with series (*) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)$$

converges to a repetitive version of the figure above with repeat distance L:

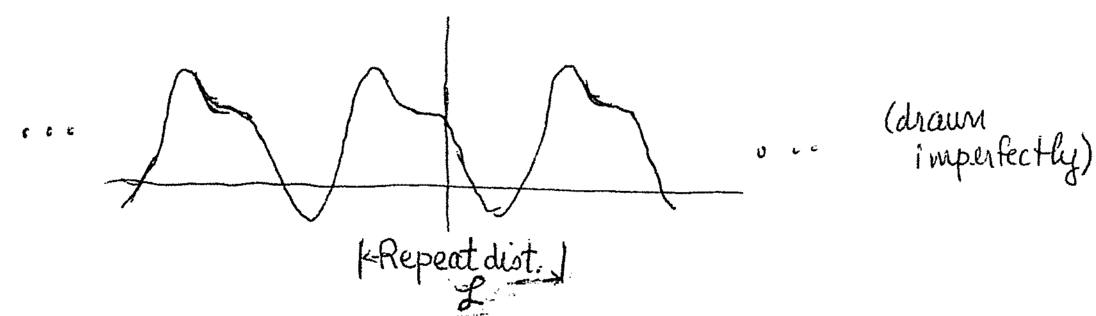


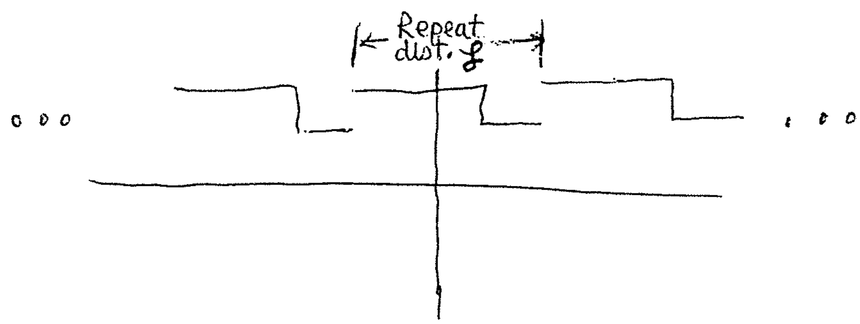
and, in fact, ~~for~~ any function on $[0, L]$ to which the series (*) converges, the series (*) converges to the <u>periodic extended function</u> of $f(x)$ with repeat distance $\lambda_1 = L$.

In each case, the wavelength ~~of~~ the longest-wavelength term (λ_1) is equal to the repeat distance. That must be, since all terms of the series repeat in distance λ_1 as λ_1 is the longest common repeat distance.

D. Fourier Analysis of Repetitive Functions

Frequently the "opposite" sort of situation occurs: we are presented, in the first place, with a repetitive function and we need to "Fourier analyze" it (that means, determine a Fourier series representation in which the repetititive function is viewed as a sum of sinusoidal oscillations, each with its own wavelength). [We will see many examples of the analog of this in the time domain (i.e, function of time t rather than function of space x) when we deal with the physics of musical instruments later.] Two examples of repetitive functions are shown:





Note that the second has a finite number of jump discontinuities on any finite interval. (Mathematicians say that it is "piecewise continuous").

Then, from our previous considerations, we can state the following version of Fourier's Theorem:

Let $f(x)$ be a repetitive function of x with repeat distance $\mathcal{L}$ on the entire x-axis. Further, suppose that both $f(x)$ and $f'(x)$ are piecewise continuous. Then, except at the points of discontinuity,

(8) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) + b_n \sin\left(n \frac{2\pi}{\mathcal{L}} x\right)$$

(9) with $$a_n = \frac{2}{\mathcal{L}} \int_{\text{"}x_0\text{"}}^{\text{"}x_0+\mathcal{L}\text{"}} f(x) \cos\left(n \frac{2\pi}{\mathcal{L}} x\right) dx, \quad b_n = \frac{2}{\mathcal{L}} \int_{x_0}^{x_0+\mathcal{L}} f(x) \sin\left(n \frac{2\pi}{\mathcal{L}} x\right) dx,$$

Where x_0 is <u>any</u> abscissa value,

converges to $f(x)$ at all points where f is continuous.

At a point at which $f(x)$ has a jump discontinuity, the series given converges to the average of the values of f on either side of the jump (i.e, to $\frac{1}{2}[f(x_+) + f(x_-)]$ where $x_+ = \lim_{x \to x_d} x$ from $x > x_d$, and $x_- = \lim_{x \to x_d}$ from $x < x_d$ where x_d is the point of discontinuity.

Comment: Note that, in eqns. (8) and (9), $\mathcal{L}$ is the wavelength of the fundamental (i.e, term with the longest wavelength).

Thus, we can write eqn. (8) as (again)

$$(10) \quad f(x)_{[0,\mathcal{L})} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{\lambda_n} x\right) + b_n \sin\left(\frac{2\pi}{\lambda_n} x\right)$$

where $\lambda_n = \frac{\lambda_1}{n} = \frac{\mathcal{L}}{n}$

Likewise, eqns. (9) are

$$a_n = \frac{2}{\lambda_1} \int_{x_0}^{x_0+\lambda_1} f(x) \cos\left(\frac{2\pi}{\lambda_n} x\right) dx \,,$$

$$b_n = \frac{2}{\lambda_1} \int_{x_0}^{x_0+\lambda_1} f(x) \sin\left(\frac{2\pi}{\lambda_n} x\right) dx.$$

So, alternatively, we could write the series as

$$f(x)_{[0,\mathcal{L}]} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{2\pi}{\lambda_1} x\right) + b_n \sin\left(n \frac{2\pi}{\lambda_1} x\right)$$

where, again, $\lambda_1 = \mathcal{L}$.

2. Further, we could have chosen "$x=0$" anywhere and gotten the same results (e.g., (8) and (9)). Thus, for any point x_0, on $[x_0, x_0+\lambda_1]$ the series (1) [with (2)] converges to $f(x)$.

Since $f(x)$ and all the terms in (1) repeat with repeat distance λ_1 [terms $\cos(n\frac{2\pi}{\lambda_1}x)$ and $\sin(\frac{n2\pi}{\lambda_1}x)$ repeat n times in distance λ_1], the series (1) converges to $f(x)$ everywhere on the x-axis.

Comment: It follows from the above argument that the numerical value of a_n is the same for all choices of the point x_0. You can also see this explictly from

$$a_n = \frac{2}{\lambda_1}\int_{x_0}^{x_0+\lambda_1} f(x)\cos\left(n\frac{2\pi}{\lambda_1}x\right)dx$$

– since both $f(x)$ and $\cos(n\frac{2\pi}{\lambda_1}x)$ both repeat every distance λ_1, then, the entire integrand is repetitive with repeat distance λ_1 – "it doesn't matter where you start the integral, in total, the integral covers the same terrain." (As long as you integrate over one full distance λ_1.)

Of course, the same is true for all b_n.

The last sentence of the theorem, concerning the convergence at a point of discontinuity, is not obvious from our previous considerations, but a mathematical analysis (which we leave to your math course) shows it to be true.

Again: Note that the function repeat distance λ_1 is the same as the wavelength of the $n=1$ terms in the Fourier series. (That's why we called it "λ_1".)

<u>comment:</u>

We can also write the basic series in terms of the basic wavenumber $k_1 \equiv 2\pi/\lambda_1$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nk_1 x) + b_n \sin(nk_1 x)$$

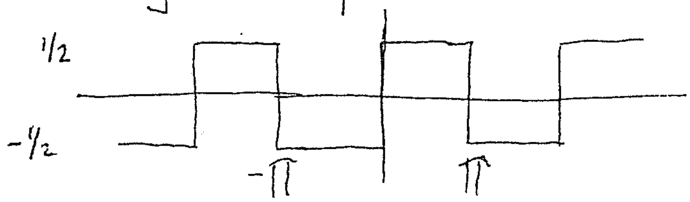
Since $k_n = \frac{2\pi}{\lambda_n} = nk_1$ (show this), this is also

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(k_n x) + b_n \sin(k_n x)$$

All these forms will be useful.

1. Expansion of Even and Odd Functions - [K-text, sect 11.9]

Consider again our "square wave" with axes as shown.



The series is $f(x) = \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$

Note that "it turned out" that there are no cosines in this series.

Actually, this is obvious in advance (i.e, before calculating the coefficients).

Why? Because with the choice of axes as given, $f(x)$ is an odd function (i.e, $f(-x) = -f(x)$). But all cosines are even functions (i.e, $f(-x) = f(x)$). Thus, there cannot be any cosines in the expansion of an odd function. Similarly, there cannot be any sines in the expansion of an even function. [We say that an odd (even) function has odd (even) parity around the origin.]

Fourier Coefficients For Even and Odd Functions

Suppose $f(x)$ is even on $[-L, L]$.

We had (still correct, technically)

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$ But, this is

$$a_n = \frac{1}{L}\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

But, If $f(x)$ is even, the two integrals are equal. Thus,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\ ,\quad a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \textbf{ all } b_n = 0.\ (\underline{\text{Why}}?)$$

Now suppose $f(x)$ is odd on $[-L, L]$. Then, by similar logic,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \textbf{ all } a_n = 0 \text{ (including } a_0!)$$

A function with "no parity" is one that is neither even nor odd.

Summary of Fourier Expansions for functions defined on $[-L, L]$:

a. <u>no parity on $[-L, L]$</u>:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

b. <u>$f(x)$ even on $[-L, L]$</u> (or extended that way from $[0, L]$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\ ,\quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

"NO SINES ALLOWED HERE"!

c. <u>$f(x)$ odd on $[-L, L]$</u> (or extended that way from $[0, L]$):

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\ ,\quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

"NO COSINES ALLOWED HERE"!
