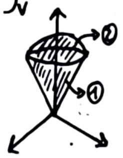
1.- Sea un campo eléctrico de la forma $\bar{E} = (5r^2/4)\hat{r}$ (C/m^2) expresado en coordenadas esféricas, (r, φ, θ) y una esfera de radio 4m, Demostrar que se cumple el Teorema de la Divergencia para el volumen encerrado en la sección cónica de la esfera de $\theta = \pi/4$

T^{ma} Divergencia: $\int_S \bar{A}\,da = \int_V Div(\bar{A})dV = \int_V \bar{\nabla}\cdot\bar{A}\,dV$ $\quad \bar{\nabla}\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{5r^4}{4}\right) = 5r$



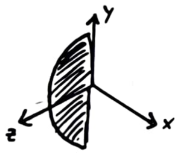
$$\int_V \bar{\nabla}\cdot\bar{A}\,dV = \int_V 5r\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^4 5r^3\sin\theta\,dr\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} 320\sin\theta\,d\theta\,d\varphi = \int_0^{2\pi} 320\,\frac{2-\sqrt{2}}{2}\,d\varphi = 320\cdot\frac{2-\sqrt{2}}{2}\cdot 2\pi = 588'9$$

$\Updownarrow =$

$$\int_S \bar{A}\,da = \phi = \sum_{i=1}^{2}\phi_i = \phi_1 + \phi_2 = \int_{①}\bar{A}\,da^{\,0} + \int_{②}\bar{A}\,da = \underset{② \text{ } \to da = r^2\sin\theta d\theta d\varphi}{} \int_{②} \sin\theta\,\frac{5r^4}{4}\,d\theta\,d\varphi = \frac{5}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} r^4\sin\theta\,d\theta\,d\varphi = \frac{5r^4}{4}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\frac{\pi}{4}} d\varphi = \frac{5\cdot 4^3}{}\int_0^{2\pi} 1 - \frac{\sqrt{2}}{2}\,d\varphi =$$

$$= 5\cdot 4^3\cdot\frac{2-\sqrt{2}}{2}\cdot 2\pi = 588'9$$

2.- Sea un campo vectorial $\bar{F} = -\hat{x} + 2z\hat{y}$, Calcular la integral de superficie a través de un semicírculo de radio $R=2$ perpendicular al eje X, siendo z positiva para todos los puntos del semicírculo. Realizar el cálculo de dos modos distintos, usando cartesianas y cilíndricas.



$$\int_S \bar{F}\,da = \int_S -dy\,dz = \int_{-2}^{2}\int_0^{\sqrt{4-y^2}} -1\,dz\,dy = \int_{-2}^{2} -\sqrt{4-y^2}\,dy = \int_a^b -\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt = \int_a^b -4\cos^2 t\,dt = -4\int_a^b \cos^2 t\,dt =$$

$\to da = \hat{x}\,dy\,dz$

$$= \left[-2(t + \sin(t)\cos(t))\right]_a^b = \left[-2\left(\arcsin\left(\frac{y}{2}\right) + \frac{y}{2}\cos\left(\arcsin\left(\frac{y}{2}\right)\right)\right)\right]_{-2}^{2} = \left[-2\arcsin\left(\frac{y}{2}\right) - y\cos\left(\arcsin\left(\frac{y}{2}\right)\right)\right]_{-2}^{2} =$$

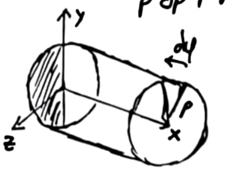
$y = 2\sin t \Leftrightarrow t = \arcsin\left(\frac{y}{2}\right)$
$dy = 2\cos t$

$\sin^2\alpha + \cos^2\alpha = 1$
$\cos^2\alpha = 1 - \sin^2\alpha$

$$= -2\arcsin(1) - 2\cos(\arcsin(1)) + 2\arcsin(-1) + 2\cos(\arcsin(-1)) = -\pi - 2\cos\left(\frac{\pi}{2}\right) - \pi + 2\cos\left(-\frac{\pi}{2}\right) = -2\pi$$

Por el T^{ma} de la Divergencia $\int_S \bar{A}\,da = \int_V Div(\bar{A})dV = \int_V \bar{\nabla}\cdot\bar{A}\,dV$, que en cilíndricas es $\bar{\nabla}\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_x}{\partial x}$

Hacemos coordenadas cilíndricas pero con el cilindro centrado en el eje x:



$$\begin{cases} x = x \\ y = \rho\cos\varphi \\ z = \rho\sin\varphi \end{cases}$$

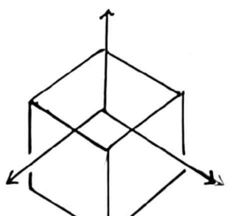
$$\begin{cases} \hat{x} = \hat{x} \\ \hat{\rho} = \cos\varphi\,\hat{y} + \sin\varphi\,\hat{z} \\ \hat{\varphi} = -\sin\varphi\,\hat{y} + \cos\varphi\,\hat{z} \end{cases} \Leftrightarrow \begin{cases} \hat{x} = \hat{x} \\ \hat{y} = \cos\varphi\,\hat{\rho} - \sin\varphi\,\hat{\varphi} \\ \hat{z} = \sin\varphi\,\hat{\rho} + \cos\varphi\,\hat{\varphi} \end{cases}$$

$$\bar{F} = -\hat{x} + 2\rho\sin\varphi(\cos\varphi\,\hat{\rho} - \sin\varphi\,\hat{\varphi})$$
$$= -\hat{x} + 2\rho\sin\varphi\cos\varphi\,\hat{\rho} - 2\rho\sin^2\varphi\,\hat{\varphi}$$

¿ $\int_S \bar{F}\,da = \int_V \bar{\nabla}\cdot\bar{F}\,dV = \int_V 4\sin\varphi\cos\varphi - 4\sin\varphi\cos\varphi + 0 = 0$? $\quad \int_S \bar{F}\,da = \int_S -\rho\,d\rho\,d\varphi = \int_0^{\pi}\int_0^2 -\rho\,d\rho\,d\varphi = \int_0^{\pi} -2\,d\varphi = -2\pi$

$\to da = da_\rho\hat{\rho} + da_\varphi\hat{\varphi} + da_x\hat{x} = da_x\hat{x} = \rho\,d\rho\,d\varphi\,\hat{x}$

3.- Dado el campo escalar $U = x^2 + y^2 + z^2$, calcular su integral de volumen sobre
a) Un cubo de lado L centrado en el origen
b) Una esfera de radio R centrada en el origen

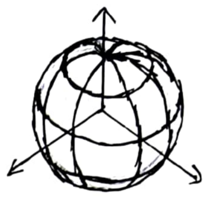


$$\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2+y^2+z^2\,dx\,dy\,dz = \int_V U\,dV = \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{L^3}{12} + L(y^2+z^2)\,dy\,dz = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{L^4}{12} + L\left(\frac{L^3}{12} + Lz^2\right)dz =$$

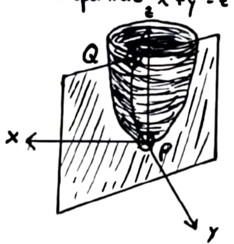
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{L^4}{6} + L^2z^2\,dz = \left[\frac{L^4 z}{6} + \frac{L^2 z^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{L^5}{12} + \frac{L^5}{12} + \frac{L^5}{8\cdot 3} + \frac{L^5}{8\cdot 3} = \frac{L^5}{6} + \frac{L^5}{12} = \frac{2L^5}{12} + \frac{L^5}{12} = \frac{3L^5}{12} = \frac{L^5}{4}$$

$r = (x^2+y^2+z^2)^{1/2} \Leftrightarrow x^2+y^2+z^2 = r^2$
$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$

$$\int_V U\,dV = \int_0^R\int_0^{2\pi}\int_0^{\pi} r^4\sin\theta\,d\theta\,d\varphi\,dr = \int_0^R\int_0^{2\pi} r^4\cdot 2\,d\varphi\,dr = \int_0^R 4\pi r^4\,dr = \frac{4\pi R^5}{5}$$



4.- Sea un campo vectorial $\bar{F} = yz\hat{x} + xz\hat{y} + xy\hat{z}$. Calcular la circulación de dicho campo a lo largo de la curva de intersección de la superficie $x^2+y^2=z$ con la superficie $x=y$ desde el punto $(0,0,0)$ al punto $(2,2,8)$



$$\Upsilon = \int_P^Q \bar{F}\,d\bar{s} = \int_0^2 (2t^3\hat{x} + 2t^3\hat{y} + t^2\hat{z})\cdot(\hat{x} + \hat{y} + 4t\hat{z}) = \int_0^2 8t^3\,dt = \left[2t^4\right]_0^2 = 32$$

Parametrizando

$(x,y,z) = (t, t, 2t^2)$ $\quad x = y = t \Leftrightarrow z = x^2 + y^2 \Leftrightarrow z = 2t^2$

$(\dot{x}, \dot{y}, \dot{z}) = (1, 1, 4t)$