PROBLEMA 3.1 Determina dos matrices cuadradas de orden 2, A y B, tales que

$$2A - 5B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \qquad -A + 2B = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}.$$

Sol.: $A = \begin{pmatrix} -12 & -3 \\ -24 & -7 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -1 \\ -10 & -3 \end{pmatrix}$.

Sea $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$2A - 5B = 2\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - 5\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \iff \begin{cases} 2a_{11} - 5b_{11} = 1 \\ 2a_{12} - 5b_{12} = -1 \\ 2a_{21} - 5b_{21} = 2 \\ 2a_{22} - 5b_{22} = 1 \end{cases}$$

$$-A + 2B = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + 2\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} \iff \begin{cases} -a_{11} + 2b_{11} = 2 \Rightarrow a_{11} = 2b_{11} - 2 \\ -a_{12} + 2b_{12} = 1 \Rightarrow a_{12} = 2b_{12} - 1 \\ -a_{21} + 2b_{21} = 4 \Rightarrow a_{21} = 2b_{21} - 4 \\ -a_{22} + 2b_{22} = 1 \Rightarrow a_{22} = 2b_{22} - 1 \end{cases}$$

$$\begin{cases} 4b_{11} - 4 - 5b_{11} = 1 \Rightarrow b_{11} = -5 \\ 4b_{12} - 2 - 5b_{12} = -1 \Rightarrow b_{12} = -1 \\ 4b_{21} - 8 - 5b_{21} = 2 \Rightarrow b_{21} = -10 \\ 4b_{22} - 2 - 5b_{22} = 1 \Rightarrow b_{22} = -3 \end{cases} \Downarrow \qquad \begin{cases} \Rightarrow a_{11} = -12 \\ \Rightarrow a_{12} = -3 \\ \Rightarrow a_{21} = -24 \\ \Rightarrow a_{22} = -7 \end{cases}$$

$$A = \begin{pmatrix} -12 & -3 \\ -24 & -7 \end{pmatrix} \qquad B = \begin{pmatrix} -5 & -1 \\ -10 & -3 \end{pmatrix}$$

PROBLEMA 3.2 Dadas las matrices $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ y $B = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, halla todas las matrices X de orden 2 tales que $AX = X^T B$.

Sol.: $X = \lambda \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

$$AX = X^t B \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$AX = X^t B \iff \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \iff \begin{pmatrix} x_1 & x_2 \\ x_1 + x_3 & x_2 + x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_3 & -x_1 \\ 2x_2 + x_4 & -x_2 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 &= 2x_1 + x_3 \\ x_2 &= -x_1 \\ x_1 + x_3 &= 2x_2 + x_4 \\ x_2 + x_4 &= -x_2 \end{aligned}\right\} \rightarrow \begin{aligned} x_1 &= -x_3 \\ x_1 &= -x_2 \\ x_1 &= \tfrac{1}{2} x_4 \\ x_1 &= \tfrac{1}{2} x_4 \end{aligned} \Rightarrow X = \lambda \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

PROBLEMA 3.3 Prueba que $(A + B)^2 = A^2 + B^2 + 2AB$ si, y sólo si, $AB = BA$.

$$(A+B)^2 = (A+B)(A+B) = \cancel{A^2} + AB + BA + \cancel{B^2} \overset{??}{=} \cancel{A^2} + \cancel{B^2} + 2AB \iff AB + BA = 2AB \iff AB + BA - AB = AB \iff$$
$$\iff BA = AB \ \square$$

PROBLEMA 3.4 Sean A y B dos matrices simétricas de orden n. Prueba que AB es simétrica si, y sólo si, A y B conmutan.

$$¿\underset{①}{AB = (AB)^t} \iff \underset{②}{AB = BA}?$$

$$(AB)^t \overset{②}{=} (BA)^t = A^t B^t = \ldots = AB$$

$\Updownarrow$ por ser simétricas

$$AB \overset{①}{=} (AB)^t = B^t A^t = \ldots = BA$$

$\square$

PROBLEMA 3.5 Dada una matriz cuadrada A, se pide:

1. Probar que $A + A^T$ es simétrica.
2. Probar que $A - A^T$ es antisimétrica.
3. Descomponer A como suma de una matriz simétrica y de una antisimétrica.
4. Demostrar que la descomposición de A, como suma de una matriz simétrica y de una antisimétrica, es única.
5. Si $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$, descomponerla en suma de una matriz simétrica con otra antisimétrica.

1\. $¿A + A^t = (A + A^t)^t?$ $\qquad * \ (A+B)^t = A^t + B^t$

$$A + A^t \overset{*}{=} (A^t)^t + A^t = A + A^t \ \square$$

2\. $¿A - A^t = -(A - A^t)^t?$ $\qquad A - A^t \overset{*}{=} +(A^t)^t - A^t = A - A^t \ \square$

3\.

PROBLEMA 3.6 Sea $A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$. Halla, en cada caso, las matrices cuadradas tales que:

1. $AB = O$. Sol.: $B = \begin{pmatrix} a & b \\ -2a & -2b \end{pmatrix}$, $a, b \in \mathbb{R}$.
2. $BA = O$. Sol.: $B = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{R}$.
3. $AB = BA$. Sol.: $B = \begin{pmatrix} 2a+b & a \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{R}$.

$$① \quad \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow AB = 0$$

$$\begin{cases} 2x_1 + x_3 = 0 \Rightarrow x_3 = -2x_1 \\ 2x_2 + x_4 = 0 \Rightarrow x_4 = -2x_2 \end{cases} \quad \begin{matrix} a = x_1 \\ b = x_2 \end{matrix} \Rightarrow B = \begin{pmatrix} a & b \\ -2a & -2b \end{pmatrix}$$

$$② \quad \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow BA = 0$$

$$\begin{cases} 2x_1 = 0 \\ x_1 = 0 \\ 2x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{matrix} a = x_2 \\ b = x_4 \end{matrix} \Rightarrow B = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

$$③ \quad \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Leftrightarrow AB = BA$$

$$\begin{cases} 2x_1 = 2x_1 + x_3 \\ x_1 = 2x_2 + x_4 \\ 2x_3 = 0 \\ x_3 = 0 \end{cases} \rightarrow \begin{cases} x_3 = 0 \\ x_1 = 2x_2 + x_4 \\ x_3 = 0 \\ x_3 = 0 \end{cases} \quad \begin{matrix} x_2 = a \\ x_4 = b \end{matrix} \Rightarrow B = \begin{pmatrix} 2a+b & a \\ 0 & b \end{pmatrix}$$

PROBLEMA 3.7 Dada la matriz

$$A = \begin{pmatrix} -2 & 4 & 2 & 1 \\ 4 & 2 & 1 & -2 \\ 2 & 1 & -2 & 4 \\ 1 & -2 & 4 & 2 \end{pmatrix},$$

calcula A^2 y A^{-1}.

Sol.: $A^2 = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$, $\quad A^{-1} = \frac{1}{25}\begin{pmatrix} -2 & 4 & 2 & 1 \\ 4 & 2 & 1 & -2 \\ 2 & 1 & -2 & 4 \\ 1 & -2 & 4 & 2 \end{pmatrix}$

$$\left(\begin{array}{cccc|cccc} -2 & 4 & 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 4 & 0 & 0 & 1 & 0 \\ 1 & -2 & 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} F_1 \leftarrow -\frac{1}{2}F_1 \\ F_2 \leftarrow F_2 + (-4F_1) \\ F_3 \leftarrow F_3 + (-2F_1) \\ F_4 \leftarrow F_4 + (-F_1) \end{matrix} \left(\begin{array}{cccc|cccc} 1 & -2 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 & 2 & 1 & 0 & 0 \\ 0 & 5 & 0 & 2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} F_2 \leftarrow \frac{1}{8}F_2 \\ F_3 \leftarrow F_3 + (-5F_2) \end{matrix}$$

$$x = -\frac{568}{525}$$

$$\left(\begin{array}{cccc|cccc} 1 & -2 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 & 1/4 & 1/8 & 0 & 0 \\ 0 & 0 & -25/8 & 2 & -13/4 & -\frac{5}{8} & 1 & 0 \\ 0 & 0 & 5 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} F_3 \leftarrow -\frac{8}{25}F_3 \\ F_4 \leftarrow F_4 + (-5F_3) \end{matrix} \left(\begin{array}{cccc|cccc} 1 & -2 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 & 1/4 & 1/8 & 0 & 0 \\ 0 & 0 & 1 & -\frac{16}{25} & \frac{26}{25} & \frac{1}{5} & -\frac{8}{25} & 0 \\ 0 & 0 & 0 & 21/5 & -\frac{21}{5} & -1 & -5 & 1 \end{array}\right) \begin{matrix} F_4 \leftarrow \frac{5}{21}F_4 \\ F_3 \leftarrow F_3 + \frac{16}{25}F_4 \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & -2 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 5/8 & 0 & 1/4 & 1/8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2/5 & 1/21 & x & \frac{16}{105} \\ 0 & 0 & 0 & 1 & -1 & -\frac{5}{21} & -\frac{25}{21} & \frac{5}{21} \end{array}\right) \begin{matrix} F_2 \leftarrow F_2 + \left(-\frac{5}{8}\right)F_3 \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & -2 & -1 & -\frac{1}{2} & -1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{32} & \frac{2}{21} & \frac{71}{105} & -\frac{2}{21} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & 1/21 & -\frac{568}{525} & \frac{16}{105} \\ 0 & 0 & 0 & 1 & -1 & -\frac{5}{21} & -\frac{25}{21} & \frac{5}{21} \end{array}\right)$$

$$A^{-1} = X \Leftrightarrow IA^{-1} = X \Leftrightarrow I = XA$$

$$A^2 = \begin{pmatrix} -2 & 4 & 2 & 1 \\ 4 & 2 & 1 & -2 \\ 2 & 1 & -2 & 4 \\ 1 & -2 & 4 & 2 \end{pmatrix}\begin{pmatrix} -2 & 4 & 2 & 1 \\ 4 & 2 & 1 & -2 \\ 2 & 1 & -2 & 4 \\ 1 & -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 4+16+4+1 & 0 & 0 & 0 \\ 0 & 4+16+4+1 & 0 & 0 \\ 0 & 0 & 4+16+4+1 & 0 \\ 0 & 0 & 0 & 4+16+4+1 \end{pmatrix} = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$$

PROBLEMA 3.8 Sabiendo que la matriz

$$A = \begin{pmatrix} 0 & 3 & 2 & -1 \\ 7 & 4 & 1 & -6 \\ -9 & -2 & 1 & 7 \\ 2 & 5 & 3 & -3 \end{pmatrix}$$

verifica la igualdad $A^2 = A + I$, calcula A^{-1} y A^4.

Sol.: $A^{-1} = \begin{pmatrix} -1 & 3 & 2 & -1 \\ 7 & 3 & 1 & -6 \\ -9 & -2 & 0 & 7 \\ 2 & 5 & 3 & -4 \end{pmatrix}$, $\quad A^4 = \begin{pmatrix} 2 & 9 & 6 & -3 \\ 21 & 14 & 3 & -18 \\ -27 & -6 & 5 & 21 \\ 6 & 15 & 9 & -7 \end{pmatrix}$.

$$A^2 = A + I \Leftrightarrow A^3 = A^2 + A \Leftrightarrow A^4 = A^3 + A^2 \Leftrightarrow A^4 = 3A + 2I$$

$$\left(\begin{array}{cccc|cccc} 0 & 3 & 2 & -1 & 1 & 0 & 0 & 0 \\ 7 & 4 & 1 & -6 & 0 & 1 & 0 & 0 \\ -9 & -2 & 1 & 7 & 0 & 0 & 1 & 0 \\ 2 & 5 & 3 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} F_4 \leftrightarrow F_1 \\ F_1 = \frac{1}{2}F_1 \\ F_2 = F_2 - 7F_1 \\ F_3 = F_3 + 9F_1 \end{matrix} \left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & \frac{3}{2} & -\frac{3}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -27/2 & -\frac{19}{2} & \frac{9}{2} & 0 & 1 & 0 & -7/2 \\ 0 & 41/2 & \frac{29}{2} & -\frac{13}{2} & 0 & 0 & 1 & 9/2 \\ 0 & 3 & 2 & -1 & 1 & 0 & 0 & 0 \end{array}\right) \begin{matrix} F_2 \leftarrow -\frac{2}{27}F_2 \\ F_3 \leftarrow F_3 - \frac{41}{2}F_2 \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & \frac{3}{2} & -\frac{3}{2} & 0 & 0 & 0 & 1/2 \\ 0 & 1 & \frac{19}{27} & -\frac{1}{3} & 0 & -\frac{2}{27} & 0 & \frac{7}{27} \\ 0 & 0 & 2/27 & \frac{1}{3} & 0 & \frac{41}{27} & 1 & -\frac{22}{27} \\ 0 & 3 & 2 & -1 & 1 & 0 & 0 & 0 \end{array}\right) \begin{matrix} F_3 \leftarrow \frac{27}{2}F_3 \\ F_4 \leftarrow F_4 - 3F_2 \end{matrix} \left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & \frac{3}{2} & -\frac{3}{2} & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 19/27 & -\frac{1}{3} & 0 & -\frac{2}{27} & 0 & 7/27 \\ 0 & 0 & 1 & 9/2 & 0 & \frac{41}{2} & \frac{27}{2} & -11 \\ 0 & 0 & -1/9 & 0 & 1 & \frac{2}{9} & 0 & -7/9 \end{array}\right)$$

$$\begin{matrix} F_4 \leftarrow F_4 + \frac{1}{9}F_3 \\ F_4 \leftarrow 2F_4 \\ F_3 \leftarrow F_3 - \frac{9}{2}F_4 \\ F_2 \leftarrow F_2 - \frac{19}{27}F_3 \end{matrix} \left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & \frac{3}{2} & -\frac{3}{2} & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{19}{3} & \frac{4}{3} & 0 & -\frac{14}{3} \\ 0 & 0 & 1 & 0 & -9 & -2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 & 5 & 3 & -4 \end{array}\right)$$

$$\begin{matrix} F_2 \leftarrow F_2 + \frac{1}{3}F_4 \\ F_1 \leftarrow F_1 + \frac{3}{2}F_4 \end{matrix} \left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & \frac{3}{2} & 0 & 3 & \frac{15}{2} & \frac{9}{2} & -\frac{11}{2} \\ 0 & 1 & 0 & 0 & 7 & 3 & 1 & -6 \\ 0 & 0 & 1 & 0 & -9 & -2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 & 5 & 3 & -4 \end{array}\right) \begin{matrix} F_1 \leftarrow F_1 - \frac{3}{2}F_3 \\ F_1 \leftarrow F_1 - \frac{5}{2}F_2 \end{matrix} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 3 & 2 & -1 \\ 0 & 1 & 0 & 0 & 7 & 3 & 1 & -6 \\ 0 & 0 & 1 & 0 & -9 & -2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 & 5 & 3 & -4 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} -1 & 3 & 2 & -1 \\ 7 & 3 & 1 & -6 \\ -9 & -2 & 0 & 7 \\ 2 & 5 & 3 & -4 \end{pmatrix} //$$

$$A^4 = 3A + 2I \Leftrightarrow A^4 = 3\begin{pmatrix} 0 & 3 & 2 & -1 \\ 7 & 4 & 1 & -6 \\ -9 & -2 & 1 & 7 \\ 2 & 5 & 3 & -3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 9 & 6 & -3 \\ 21 & 13 & 3 & -18 \\ -27 & -6 & 4 & 21 \\ 6 & 15 & 9 & -8 \end{pmatrix} //$$

PROBLEMA 3.9 Calcula los siguientes determinantes usando sus propiedades y efectuando un número reducido de operaciones:

$$a)\ \begin{vmatrix} 2 & 0 & -2 & 4 \\ 0 & 3 & 9 & -6 \\ 1 & 2 & -1 & 0 \\ 0 & 3 & 4 & 2 \end{vmatrix} \qquad b)\ \begin{vmatrix} 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 1 \\ 4 & -1 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix} \qquad c)\ \begin{vmatrix} 1 & 1 & 0 & 1 \\ 2 & x & 0 & 1 \\ 0 & 1 & y & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

$$d)\ \begin{vmatrix} i & 1 & 0 & 0 & 0 \\ 1 & i & 1 & 0 & 0 \\ 0 & 1 & i & 1 & 0 \\ 0 & 0 & 1 & i & 1 \\ 0 & 0 & 0 & 1 & i \end{vmatrix}.$$

Sol.: $a)$ -228. $b)$ 1. $c)$ $(1-x)(1-y)$. $d)$ $8i$.

$$a)\ \underset{F_3 \leftarrow F_3 - F_1}{\overset{F_1 \leftarrow \frac{1}{2}F_1}{\sim}} 2\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 9 & -6 \\ 0 & 2 & 0 & -2 \\ 0 & 3 & 4 & 2 \end{vmatrix} \underset{F_1 \leftarrow F_1 - \frac{1}{5}F_4}{\overset{F_4 \leftarrow F_4 - F_2}{\sim}} 2\begin{vmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 3 & 9 & -6 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -5 & 8 \end{vmatrix} \underset{F_3 \leftrightarrow F_4}{\overset{F_2 \leftrightarrow F_3}{\sim}} 2\begin{vmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -5 & 8 \\ 0 & 3 & 9 & -6 \end{vmatrix} \underset{F_4 \leftarrow F_4 + \frac{9}{5}F_3}{\overset{F_4 \leftarrow F_4 - \frac{3}{2}F_2}{\sim}} 2\begin{vmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -5 & 8 \\ 0 & 0 & 0 & \frac{57}{5} \end{vmatrix}$$

$$|a)| = 2\cdot\left(1\cdot 2\cdot(-5)\cdot\frac{57}{5}\right) = -228 //$$

$$b)\ \overset{F_1 \leftrightarrow F_2}{\sim} \ \begin{matrix} F_2 \leftarrow F_2 - 2F_1 \\ F_3 \leftarrow F_3 - 4F_1 \\ F_4 \leftarrow F_4 - 3F_1 \end{matrix} \ - \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & -1 & 0 & -2 \end{vmatrix} \overset{F_3 \leftarrow F_3 + F_2}{\sim} \ \begin{matrix} F_4 \leftarrow F_4 + F_2 \\ F_4 \leftarrow F_4 - \frac{3}{5}F_3 \end{matrix} \ - \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & -\frac{1}{5} \end{vmatrix} \Rightarrow |b)| = 1 //$$

$$c)\ \overset{F_2 \leftarrow F_2 - 2F_1}{\sim} \ \begin{matrix} F_3 \leftarrow F_3 - \frac{1}{(x-2)}F_2 \\ F_4 \leftarrow F_4 - \frac{1}{(x-2)}F_2 \end{matrix} \ \begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & (x-2) & 0 & -1 \\ 0 & 0 & y & \frac{x-1}{x-2} \\ 0 & 0 & 1 & \frac{x-1}{x-2} \end{vmatrix} \overset{F_4 \leftarrow F_4 - \frac{1}{y}F_3}{\sim} \begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & (x-2) & 0 & -1 \\ 0 & 0 & y & \frac{x-1}{x-2} \\ 0 & 0 & 0 & \left(\frac{x-1}{x-2}\right)\left(\frac{y-1}{y}\right) \end{vmatrix}$$

$$\Rightarrow |c)| = \cancel{(x-2)}\cancel{y}\left(\frac{x-1}{\cancel{x-2}}\right)\left(\frac{y-1}{\cancel{y}}\right) = (x-1)(y-1) //$$

$$d)\ \begin{vmatrix} i & 1 & 0 & 0 & 0 \\ 1 & i & 1 & 0 & 0 \\ 0 & 1 & i & 1 & 0 \\ 0 & 0 & 1 & i & 1 \\ 0 & 0 & 0 & 1 & i \end{vmatrix} \ \begin{matrix} F_2 \leftarrow F_2 + iF_1 \\ F_3 \leftarrow F_3 + \frac{i}{2}F_2 \\ F_4 \leftarrow F_4 + \frac{2i}{3}F_3 \\ F_5 \leftarrow F_5 + \frac{3i}{5}F_4 \end{matrix} \ \begin{vmatrix} i & 1 & 0 & 0 & 0 \\ 0 & 2i & 1 & 0 & 0 \\ 0 & 0 & \frac{3i}{2} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{3}i & 1 \\ 0 & 0 & 0 & 0 & \frac{8}{5}i \end{vmatrix} \Rightarrow |d)| = i\,(2i)\left(\frac{3}{2}i\right)\left(\frac{5}{3}i\right)\left(\frac{8}{5}i\right) = i\left(2\cdot\frac{3}{2}\cdot\frac{5}{3}\cdot\frac{8}{5}\right) = 8i //$$

PROBLEMA 3.11 Resuelve las siguientes ecuaciones en la variable x, aplicando las propiedades de los determinantes:

$$a)\ \begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = 0. \qquad b)\ \begin{vmatrix} a & b & c \\ 2a & x & 2c \\ a^2 & ab & x \end{vmatrix} = 0. \qquad c)\ \begin{vmatrix} a & b & c \\ 2a & x & 2c \\ x & -b & -c \end{vmatrix} = 0.$$

Sol.: $a)$ $x = b$ o $x = c$ si $a \neq 0$. $b)$ $x = 2b$ o $x = ac$ si $a \neq 0$. $c)$ $x = -a$ o $x = 2b$ si $c \neq 0$.

a) si $a=0$ $x\in\mathbb{R}$ es solución
si $a\in\mathbb{R}\setminus\{0\}$ $x_1=b$ y $x_2=c$ son solución.

b) si $a=0$ $x\in\mathbb{R}$ es solución
si $a\in\mathbb{R}\setminus\{0\}$ $x_1=ac$ y $x_2=2b$ son solución

c) si $c=0$ $x\in\mathbb{R}$ es solución
si $c\in\mathbb{R}\setminus\{0\}$ $x_1=2b$ $x_2=-a$ son solución

PROBLEMA 3.12 Sin desarrollar los determinantes, demuestra la identidad:

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ac & b & b^2 \\ ab & c & c^2 \end{vmatrix}.$$

$$\begin{matrix} F_2 \leftarrow F_2 - F_1 \\ F_3 \leftarrow F_3 - F_1 \end{matrix} \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} F_3 \leftarrow F_3 - F_2 \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-b^2 & c^3-b^3 \end{vmatrix} F_3 \leftarrow F_3 - F_2$$