TRAZA DEL TENSOR ENERGÍA-IMPULSO SIMÉTRICO

La traza del tensor energía-impulso $\Theta^{\mu\nu}$ no es como la calculé el martes 5-12-23 en clase ni como está calculada en un problema (además erróneamente):

$$\text{Traza}\left(\Theta^{\mu\nu}\right) \neq \Theta^{00} + \Theta^{11} + \Theta^{22} + \Theta^{33}$$

La traza de un tensor se define:

4.3. Contraction of indices

With tensors of at least one covariant and at least one contravariant index we can define a kind of 'internal inner product'. In the simplest case this goes as,

$$t^{\alpha}{}_{\alpha}\,, \tag{4.5}$$

where, as usual, we sum over α. This is the trace of the matrix $t^{\alpha}{}_{\beta}$. Also this construction is invariant under basis transformations. In classical linear algebra this property of the trace of a matrix was also known.

▸ *Exercise 3 of Section C.4.*

One can also perform such a *contraction of indices* with tensors of higher rank, but then some uncontracted indices remain, e.g.

$$t^{\alpha\beta}{}_{\alpha} = v^{\beta}\,, \tag{4.6}$$

In this way we can convert a tensor of type (N, M) into a tensor of type $(N-1, M-1)$. Of course, this contraction procedure causes information to get lost, since after the contraction we have fewer components.

Note that contraction can only take place between one contravariant index and one covariant index. A contraction like $t^{\alpha\alpha}$ is not invariant under coordinate transformation, and we should therefore reject such constructions. In fact, the summations of the summation convention *only* happen over a covariant and a contravariant index, be it a contraction of a tensor (like $t^{\mu\alpha}{}_{\alpha}$) or an inner product between two tensors (like $t^{\mu\alpha}y_{\alpha\nu}$).

es decir,

$$\text{Traza}\left(\Theta^{\mu\nu}\right) = g_{\mu\nu}\,\Theta^{\mu\nu} = \Theta_{\nu}^{\cdot\nu} =$$

$$= \Theta_{0}^{\cdot 0} + \Theta_{1}^{\cdot 1} + \Theta_{2}^{\cdot 2} + \Theta_{3}^{\cdot 3}$$

Calculándola así la traza sí que da cero.

$$\Theta_S^{\mu\nu} = \begin{array}{cc} (00) & (0i) \\ \left(\begin{array}{c|c} \frac{1}{2}\varepsilon_0 \vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2 & \longleftarrow \frac{\vec{E}\times\vec{B}}{\mu_0 c} \longrightarrow \\ \hline (i0)\ \frac{\vec{E}\times\vec{B}}{\mu_0 c} & -\varepsilon_0 E^i E^j - \frac{B^i B^j}{\mu_0} + \frac{1}{2}\delta^{ij}\left(\varepsilon_0\vec{E}^2 + \frac{\vec{B}^2}{\mu_0}\right)\ (ij) \end{array}\right) \end{array}$$

$i0$

$0i$

$$\frac{S^i}{c} = c g^i$$

$\Theta_S^{\mu\nu} = \left(\begin{array}{c\|c} u_{EM} & c\vec{g} \\ \hline c\vec{g} & -T_{MS}^{ij} \end{array}\right)$	$\Theta^S_{\mu\nu} = \left(\begin{array}{c\|c} u_{EM} & -c\vec{g} \\ \hline -c\vec{g} & -T_{MS}^{ij} \end{array}\right)$
fila $\downarrow$ $\Theta^{\mu}_{\cdot\nu}$ $\uparrow$ columna $= \left(\begin{array}{c\|c} u_{EM} & -c\vec{g} \\ \hline c\vec{g} & T_{MS}^{ij} \end{array}\right)$	$\Theta_{\nu\cdot}^{\mu} = \left(\begin{array}{c\|c} u_{EM} & c\vec{g} \\ \hline -c\vec{g} & T_{MS}^{ij} \end{array}\right)$

$$\text{Traza}\left(\Theta_S^{\mu\nu}\right) = \Theta_{\nu}^{\cdot\nu} = \Theta^{\nu}_{\cdot\nu} = u_{EM} + T_{MS}^{ii} =$$

$$= u_{EM} + T_{MS}^{11} + T_{MS}^{22} + T_{MS}^{33}$$

$$\text{Traza}\left(\Theta_s^{\mu\nu}\right) = \Theta^{\cdot\nu}_{\nu} = u_{EM} + \sum_{i=1}^{3} T_{MS}^{ii} =$$

$$= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 + \varepsilon_0 E_x^2 + \frac{1}{2\mu_0} B_x^2 - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) +$$

$$+ \varepsilon_0 E_y^2 + \frac{1}{2\mu_0} B_y^2 - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) +$$

$$+ \varepsilon_0 E_z^2 + \frac{1}{2\mu_0} B_y^2 - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) =$$

$$= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 + \varepsilon_0 (\underbrace{E_x^2 + E_y^2 + E_z^2}_{E^2}) +$$

$$+ \frac{1}{\mu_0}(\underbrace{B_x^2 + B_y^2 + B_z^2}_{B^2}) - \frac{3}{2}\varepsilon_0 E^2 - \frac{3}{2\mu_0} B^2 =$$

$$= \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} + \varepsilon_0 E^2 + \frac{B^2}{\mu_0} - \frac{3}{2}\varepsilon_0 E^2 - \frac{3B^2}{2\mu_0} = 0$$

La traza es nula.