13: $\lim\limits_{n\to\infty}\left(\dfrac{a^{1/n}+b^{1/n}}{2}\right)^{n}$ con $a,b>0$

Sea $L=\lim\limits_{n\to\infty}\left(\dfrac{a^{1/n}+b^{1/n}}{2}\right)^{n}$:

$$\log(L)=\log\left[\lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^{n}\right]=\lim_{n\to\infty}\left[n\log\left(\frac{a^{1/n}+b^{1/n}}{2}\right)\right]=\lim_{n\to\infty}\left[\frac{a^{1/n}+b^{1/n}-2}{2}\cdot n\cdot\frac{\log\left(\frac{a^{1/n}+b^{1/n}}{2}+2-2\right)}{\frac{a^{1/n}+b^{1/n}-2}{2}}\right]=$$

$$=\lim_{n\to\infty}\left[\frac{a^{1/n}+b^{1/n}-2}{2}\cdot n\cdot\underbrace{\frac{\log\left(1+\frac{a^{1/n}+b^{1/n}-2}{2}\right)}{\frac{a^{1/n}+b^{1/n}-2}{2}}}_{①}\right]\overset{*}{=}\lim_{n\to\infty}\left[n\cdot\frac{a^{1/n}+b^{1/n}-2}{2}\cdot 1\right]=\frac{1}{2}\lim_{n\to\infty}n\left(a^{1/n}+b^{1/n}-1-1\right)=$$

$$=\frac{1}{2}\left[\underbrace{\lim_{n\to\infty}n\left(a^{1/n}-1\right)}_{②}+\underbrace{\lim_{n\to\infty}n\left(b^{1/n}-1\right)}_{②}\right]\overset{*}{=}\frac{1}{2}\left(\log a+\log b\right)=\frac{1}{2}\log(ab)=\log(ab)^{1/2}=\log\sqrt{ab}\implies\boxed{L=\sqrt{ab}}\ \square$$

↳ $\forall a,b>0$

① $$\lim_{n\to\infty}\frac{\ln(1+n)}{n}=\lim_{n\to\infty}\log(1+n)^{1/n}=\lim_{n\to\infty}\log\left(1+\frac{1}{1/n}\right)^{1/n}\overset{\bullet}{=}\lim_{n\to\infty}\log(e)=1 \longrightarrow 25.4$$

② $$\lim_{n\to\infty}n\left(a^{1/n}-1\right)=\lim_{x\to0}\frac{a^{x}-1}{x}=\lim_{x\to0}\frac{e^{\log a^{x}}-1}{x}=\log a\lim_{x\to0}\frac{e^{x\log a}-1}{x\ln a}\overset{\bullet}{=}\log a\log e=\log a$$ $(\forall a>0)$

$x=\frac{1}{n}$ Cuando $n\to\infty$, $x\to0$

$a=e^{\ln a}$

definición del logaritmo neperiano: • $\lim\limits_{x\to0}\dfrac{a^{x}-1}{x}=\log a$ $\forall x\in\mathbb{R}$

definición del número de Euler: • $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

↳ por definición $\log(e)=1$