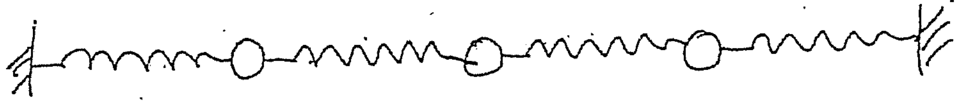
Phys. 251 - 12^{th} Class - Thurs. Feb. 15, 2024

[Reading for this ~~Part:~~ Text, sects. 8.1, 8.2, 8

A. Increasing the Number of Degrees of Freedom*

We have had a fairly detailed discussion of the properties of the normal modes of systems with two degrees of freedom.

Now let us generalize. Suppose we have 3 masses and 4 springs. Then there are three modes in each of three polarizations. Why? Because the three coupled equations of motion lead to a 3×3 determinant which yields a cubic equation in (ω^2) implying 3 independent solutions.(+) These 3 modes again form a complete set for the soln. space.



system of "3 masses and 4 springs"

By this we mean that, within a given polarization, the most general motion of the system is an arbitrary linear combination of the three modes:

$$\begin{pmatrix} \psi_a \\ \psi_b \\ \psi_c \end{pmatrix} = A_1 \begin{bmatrix} \text{unit amplitude} \\ \text{of mode 1} \end{bmatrix} + A_2 \begin{bmatrix} \text{unit} \\ \text{amplitude} \\ \text{of mode 2} \end{bmatrix} + A_3 \begin{bmatrix} \text{unit} \\ \text{amplitude} \\ \text{of mode 3} \end{bmatrix}$$

where A_1, A_2 and A_3 are arbitrary real constants.

The mode shapes for the N-mass systems are shown on the next page.

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* Text Reference: sections 8.1; 8.2, 8.3.

(+) The normal modes of transverse oscillation of three coupled masses are worked out in Chapter 6, Sect. 6.14, Worked Example 6.7 (pp. 6-55 → 6-59).

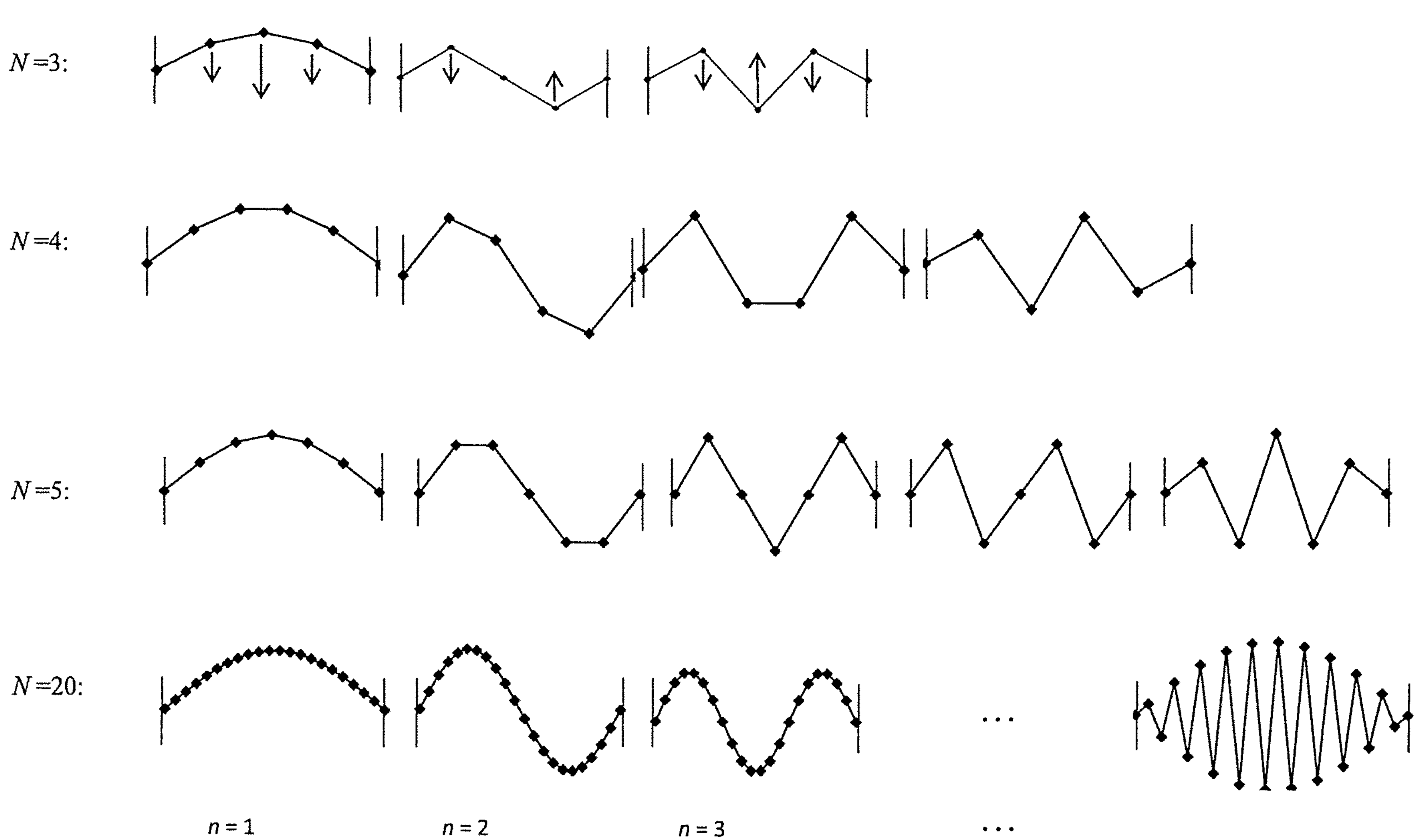

Fig. 8.1. Modes of transverse oscillation for systems of N equal masses coupled by equal strength strings. The "data points" show the position of each mass at a time when the displacements from the equilibrium line are maximized. The first and last "data point" in each case correspond to the motionless walls (shown as vertical lines) anchoring each of the end springs. The springs are shown as straight lines connecting the masses (or connecting a mass to a wall). In each row the mode configurations for the indicated number of masses are ordered in order of increasing mode frequency. For the case N=3, arrows show the direction of motion of the mass points for the next half cycle. For the case N=20, only the lowest frequency three modes (n = 1, 2, 3) and the highest frequency mode (n = 20) configurations are shown. For each value of N, the lowest frequency mode has no connecting lines crossing the equilibrium line, while the highest frequency mode is completely "zigzag" so as to provide the highest return force per unit displacement. In each case, over the course of time masses on the same side (i.e., above or below) the equilibrium line move in phase up and down together, while masses on opposite sides of the equilibrium line move 180^0 out of phase with each other.

Has the author ordered the modes for a given system correctly in order of increasing frequency? The answer is yes. How do we know this? Note that, in his ordering, the springs (shown as straight line segments for simplicity)

* make increasingly large angles with the equilibrium axis as we increase the mode number (taking the displacement of a given bead to be the same). Consequently the return force per unit displacement per unit mass for a given bead in a given system increases when we go from one configuration to the next, and therefore so does the mode frequency.

* From Waves, Berkeley Physics Series Vol. 3 by F.S. Crawford (McGraw Hill)

(next page →

Series, vol. 3, op cit. (F.S. Crawford)

Another thing that is apparent is that our sequence of assumed mode shapes always gives exactly N configurations: the first mode always has zero "nodes" (places where the string crosses the axis, excluding the end points), the second has one node, etc. The highest mode always has the largest possible number of nodes, namely $N - 1$, which is achieved by "zig-zagging" up and down, i.e., crossing the axis once between each two successive masses.

B. *Transverse Modes of Continuous String*

We now consider the case where N is huge, say $N = 1{,}000{,}000$ or so. Then for the lowest modes (say the first few thousand), there are a very large number of "beads" between each node. Thus the displacement varies slowly from one "bead" to the next. We imagine the limit in which there is only one long, flexible "spring" with the mass distributed uniformly along it. Such a one-dimensional continuous flexible system is called a "stretched string" in physics. Under this condition, we can replace a very long list of functions of one variable (t), $\{\psi_a(t), \psi_b(t), \psi_c(t), \psi_d(t), \ldots\}$ by a single continuous function of two variables $\Psi(z,t)$. $\Psi(z,t)$ is the transverse displacement from equilibrium position at time t of "mass particle" having equilibrium coordinate z along the equilibrium line (see figure)

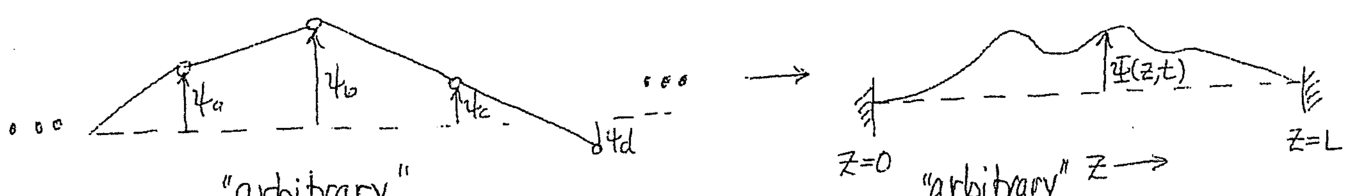

Fig. 8.2. Notations for discrete and continuous systems. In each, the dotted line is the equilibrium line. (a): The discrete notation we used for a system with a relatively small number of moving parts. (b) The notation we use for a stretched string or slinky of length L bound down at both ends z=0 and z=L. Thus, for all t, $\Psi(z=0,t)=0$ and $\Psi(z=L,t)=0$.

As you will see throughout much of the remainder of this course and in your subsequent studies in physics, a solid understanding of features of the possible motions of the stretched string is very important in many areas ranging from the physics of musical instruments (guitar, piano, etc, etc) to the physics of the Earth as an object, to the physics of stars to the physics of the Universe. (As just one example, from contemporary physics, "string theory" was not given that name for no reason.)

As before, we will find that, under ideal conditions, the most general motion of the system is a superposition of the normal modes; partly (but not only) for this reason it is most important to understand the normal modes.

How Do We Mathematically Indicate a Normal Mode on a continuous stretched string?

(see next page →

As you know, in a normal mode, all moving parts oscillate together in simple harmonic motion at the same frequency with a fixed relative phase. Hence, in mathematical language, in a normal mode on a stretched string, the wave function* is

$$\Psi(z,t) = \underbrace{A(z)}_{\text{"Amplitude profile" of the mode ("mode shape function")}} \times \underbrace{\cos(\omega t + \phi)}_{\text{common to all mass points, hence a common factor that multiplies } A(z) \text{ for all } z.}$$

Thus, <u>in a normal mode (only) $\Psi(z,t)$ can be written in "factorized form" as a function only of z multiplied by (another) function only of t.</u>

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* $\Psi(z,t)$ is called the "wave function."

C. Guessing the Shapes of the Lowest Frequency Normal Modes

We are going to apply the systematic method to find the frequencies and shapes of the lowest (frequency) normal modes for our stretched string bound down at both its ends.

However, following our usual approach, before getting involved in a fully mathematical treatment, let us think and guess the features we expect the normal modes to have, so that we will have some expectations in mind.

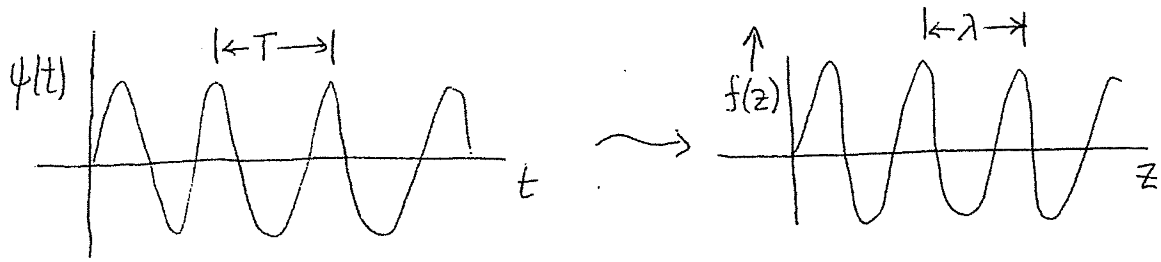
Although we cannot yet say much quantitatively about the frequencies, based on our results and thoughts from earlier, we expect* that the normal mode shapes will be smoothly varying sinusoidal functions of z. (* From the figure on page 2 of this packet of class notes)

How do we write these mode shapes mathematically?

To do this, we first need to understand how to mathematically write a sinusoidal function in <u>space</u>.

We know how to write a sinusoidal function in <u>time</u> - this would be, e.g.,

$$\psi(t) = A \sin \omega t = A \sin\left(\frac{2\pi}{T} t\right)$$



The spatial analog of the period T is the repeat distance, or wavelength, λ.

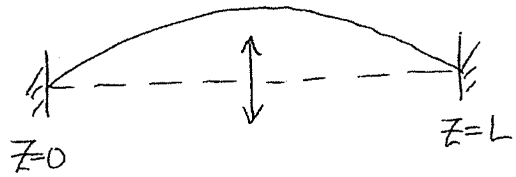
Thus, using the "dictionary"

$$t \leftrightarrow z$$

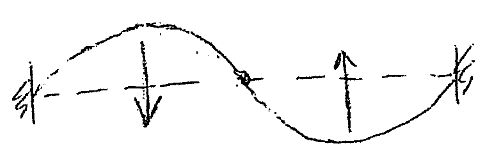
$$T \leftrightarrow \lambda$$

$$\psi(t) = A\sin\left(\frac{2\pi}{T}t\right) \longrightarrow \underline{A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right)}.$$

So far, so good. But this is too general for our purpose — we suppose that the string is <u>bound down</u> ($\Psi = 0$ at all times) at both of its ends ($z=0$ and $z=L$). Then, <u>only certain wavelengths are possible</u> – you can see this pictorially:



<u>mode 1:</u> $\underline{\lambda = 2L}$



<u>mode 2:</u> $\underline{\lambda = L}$

mode 3: $\underline{\lambda = \frac{2L}{3}}$

Thus, we *expect* that, for a uniform continuous string bound down at each of its two ends ($z=0$ and $z=L$), the mode shapes are given by the formula

"mode-shape function", or "amplitude profile" function

$$\left\{ A_n(z) = A_n \sin\left(\frac{2\pi}{\lambda_n} z\right) \right.$$

note: $A_n(z) \neq A_n$

fcn. of z — constant

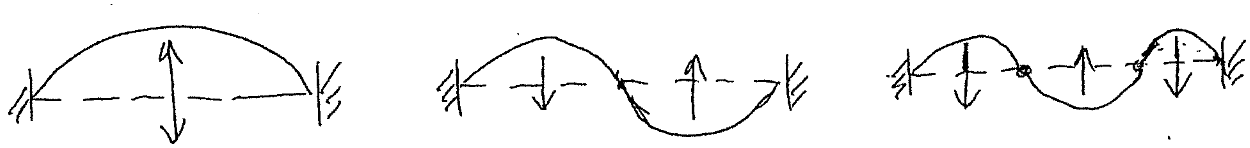
mode number

with $\lambda_n = \dfrac{2L}{n}$, thus $A_n(z) = A_n \sin\left(\dfrac{n\pi}{L} z\right)$

We will confirm this clear expectation soon.

D. Modes of a Bound String With Time Dependence

By definition,
In a normal mode on a stretched string, all parts move up and down together in simple harmonic motion. That is, sections of the string behave like big extended harmonic oscillators.



$n=1$ $n=2$

Thus, in mathematical language, in a normal mode on a stretched string the wave function is given by (as already mentioned)

$$\Psi(z,t) = \underbrace{A(z)}_{\text{Amplitude profile of the mode ("}f(z)\text{" before)}} \times \underbrace{\cos(\omega t + \phi)}_{\text{common to all mass points, hence a common factor that multiplies } A(z) \text{ for every } z.}$$

Thus, for a string bound down at both ends, in mode n, we expect that

(1) $$\Psi_n(z,t) = A_n(z) \cos(\omega_n t + \phi_n)$$

(2) $$= A_n \sin\left(\frac{2\pi}{\lambda_n} z\right) \cos(\omega_n t + \phi_n)$$

or

(2) $$\Psi_n(z,t) = A_n \sin\left(\frac{n\pi}{L} z\right) \cos(\omega_n t + \phi_n) \quad \text{with } A_n = \text{constant}$$

Of course, the mode shape function $A_n(z) = A_n \sin\left(\frac{n\pi}{L} z\right)$ only applies if the stretched string is bound down at $z=0$ and at $z=L$. For other boundary conditions (e.g., free ends) the mode shape function is a different function of z (but still sinusoidal). We will consider other boundary conditions later on.

Let us make sure that we understand equation (2). It says

$$\Psi_1(z,t) = A_1 \sin\left(\frac{\pi z}{L}\right)\cos(\omega_1 t + \phi_1)$$

$$\Psi_2(z,t) = A_2 \sin\left(\frac{2\pi z}{L}\right)\cos(\omega_2 t + \phi_2)$$

$$\Psi_3(z,t) = A_3 \sin\left(\frac{3\pi z}{L}\right)\cos(\omega_3 t + \phi_3)$$

$$\vdots$$

i.e, for normal mode n,

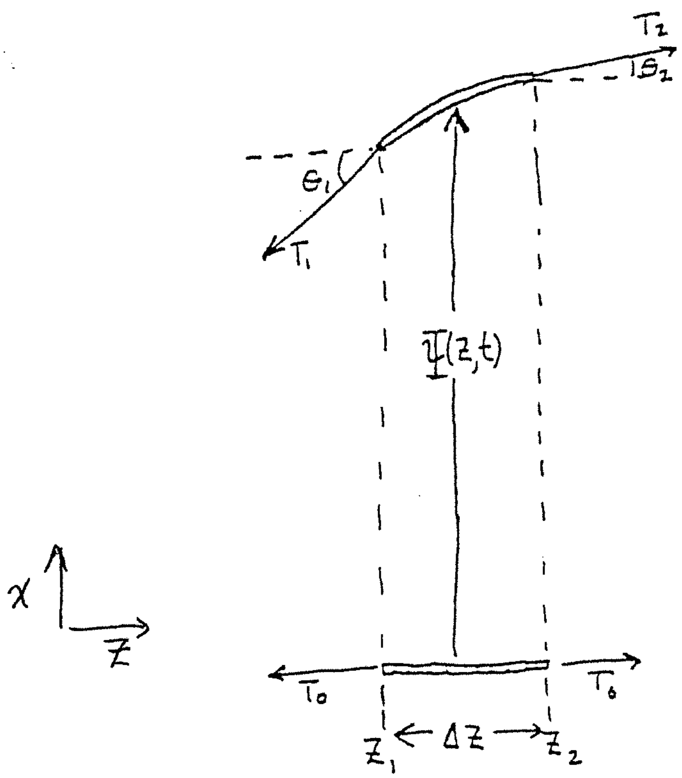
$$\Psi_n(z,t) = A_n \sin\left(\frac{n\pi z}{L}\right)\cos(\omega_n t + \phi_n)$$

Even within the restriction of both ends ($z=0$ and $z=L$) bound down, we must recognize that this → is really still a guess. And, of course, we do not yet know the mode frequencies (ω_n). To confirm or deny it and to find the mode frequencies we will need to employ the systematic method. And that is what we next turn to...

E. Derivation of the Classical Wave Equation

We seek the normal modes of a uniform continuous string. For this we will develop the governing differential equation for the string.* This governing equation, called the "Classical Wave Equation" is really just a transcription of Newton's second law applied to each infinitesimal bit of the string in its general small oscillations configuration. To develop this,

Consider a very small segment of the string that runs from $z_1 = z_1$ to $z = z_2 = z_1 + \Delta z$; the element has mass ΔM given by



$$\Delta M \equiv \rho_0 \Delta z$$

($\rho_0 \equiv$ "linear mass density")
(also called "ρ_ℓ")
$=$ mass per unit length in equilibrium configuration.

[We are taking "upward" as the "positive" direction for x.]

net upward force:

$$F_x(t) = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

* We will proceed via the systematic method, which requires equations of motion.

Now, as we remarked for the case of springs, for small angles (i.e, small oscillations approximation),

$$|T_1| \approx |T_2| \approx |T_0|^{*} \quad \text{(magnitudes same, }\underline{\text{directions different}}\text{)}$$

Further, for small oscillations,

$$\sin\theta \approx \theta \approx \tan\theta$$

and $$\tan\theta = \text{slope} = \frac{\partial \Psi(z,t)}{\partial z}$$

Thus,

$$F_x(t) \approx T_0 \left[\left.\frac{\partial \Psi(z,t)}{\partial z}\right)_{z_2} - \left.\frac{\partial \Psi(z,t)}{\partial z}\right)_{z_1} \right]$$

next page for continuation of main disussion.

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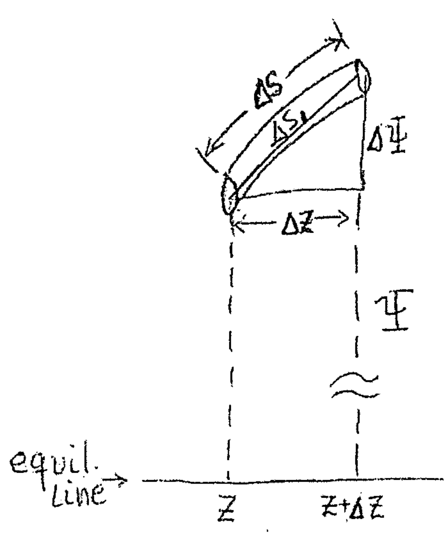
* To investigate in more detail (than we have done hitherto) the approximation quality $|T_1| \approx |T_2| \approx |T_0|$, begin by referring to diagram below.



"Unstretched" (equilibrium) length of segment $\equiv \Delta z$.
"Stretched length" is $\Delta S \approx \Delta S_1$ (small Ψ approximation).
S_0 = relaxed length of segment ($< \Delta z$). Then:

$$T = k(\Delta S - \Delta S_0) \approx k(\Delta S_1 - \Delta S_0) = k\left[\sqrt{(\Delta z)^2 + (\Delta\Psi)^2} - \Delta S_0\right]$$

$$= k\left[\Delta z \sqrt{1 + \left[\frac{(\Delta\Psi)}{(\Delta z)}\right]^2} - \Delta S_0\right]. \text{ Assume } \Delta\Psi \ll \Delta z,$$

$$\approx k\left[\Delta z\left\{1 + \frac{1}{2}\left(\frac{\Delta\Psi}{\Delta z}\right)^2\right\} - \Delta S_0\right]$$

$$= k\left[\Delta z - \Delta S_0\right] + \frac{1}{2} k \Delta z \left(\frac{\Delta\Psi}{\Delta z}\right)^2$$

$$= T_0 + \text{term of 2}^{\text{nd}} \text{ order in } \frac{\Delta\Psi}{\Delta z} \approx T_0.$$

This is unwieldly. To convert it, make first order Taylor expansion for $f(z) \equiv \frac{\partial \Psi(z,t)}{\partial z}$ around $z = z_1$.

Then $$\left.\frac{\partial \Psi(z,t)}{\partial z}\right)_2 \approx \left.\frac{\partial \Psi(z,t)}{\partial z}\right)_1 + (z_2 - z_1)\, \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

so $$F_x(t) \approx T_0\, \Delta z\, \frac{\partial^2 \Psi(z,t)}{\partial z^2} \qquad \Delta z \equiv z_2 - z_1$$

Now the acceleration of the segment is $\frac{\partial^2 \Psi(z,t)}{\partial t^2}$

$\therefore$, $$ma = F \Rightarrow$$

$$\Delta M\, \frac{\partial^2 \Psi(z,t)}{\partial t^2} = T_0\, \Delta z\, \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

but $\Delta M = \rho_\ell\, \Delta z$, so

$$\rho_\ell\, \frac{\partial^2 \Psi}{\partial t^2} = T_0\, \frac{\partial^2 \Psi}{\partial z^2}, \text{ or}$$

$$\boxed{\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_\ell} \cdot \frac{\partial^2 \Psi(z,t)}{\partial z^2}}$$

Classical Wave Equation. (CWE)

This very important result is the Classical Wave Equation (for the stretched string).

We will use it extensively throughout the rest of the course.

Notice what has happened in our derivation: Considering the string as composed of a (very large) number of "mass beads" separated by massless short springs, we would have a very long list of governing d.e.'s, each an ordinary differential equation:

$$\frac{d^2 y_a(t)}{dt^2} = \cdots$$

$$\frac{d^2 y_b(t)}{dt^2} = \cdots$$

$$\vdots$$

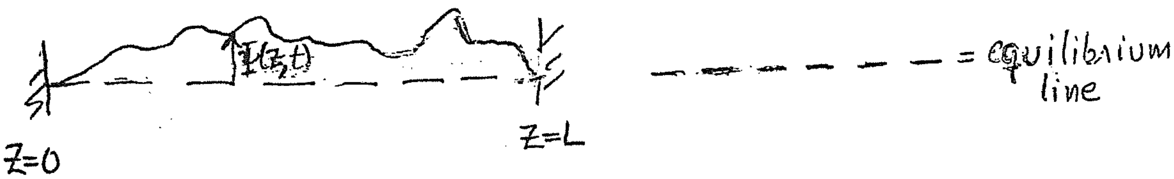
$$\frac{d^2 y_N(t)}{dt^2} = \cdots \qquad \text{where } N \text{ is very large.}$$

By invoking the continuum limit, this entire list has been converted into a <u>single</u> <u>partial differential equation</u> (the CWE).

But, then, any partial differential equation is equivalent to a very long list of ordinary differential equations!

F. Normal Modes On Uniform Stretched String Via Systematic Method

We continue our discussion of normal modes on a uniform stretched string, bound down at each of its two ends ($z=0$ and $z=L$). To corroborate our guesses, and also to find the mode frequencies, we took up the systematic method. For this, from the arbitrary configuration, by applying



Newton's law to a very small string segment, we found that the governing differential equation is a Partial differential equation, the Classical Wave Equation (CWE)

$$\boxed{\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_e}\frac{\partial^2 \Psi(z,t)}{\partial z^2}} \qquad \text{CWE}$$

Where T_0 is the equilibrium tension and ρ_e is the equilibrium linear mass density (mass per unit length).

We now continue to apply the systematic method: step 2:

2. Try this mode assumption in the governing differential equation of motion:

$$\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_\ell} \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

In it, try $\Psi(z,t) = A(z)\cos(\omega t + \phi)$

$$\Rightarrow \quad -\omega^2 A(z)\cos(\omega t+\phi) = \frac{T_0}{\rho} \frac{d^2 A(z)}{dz^2}\cos(\omega t+\phi)$$

$$\Rightarrow \quad \boxed{\frac{d^2 A(z)}{dz^2} = -\omega^2 \frac{\rho_\ell}{T_0} A(z)}$$

Helmholtz, or Mode-Shape Equation

We need to solve the mode shape equation for the profile fcn. $A(z)$

Does it look familiar?

Note that, except for the names of the variables, it is the same as the harmonic oscillator equation!

	Simple Harmonic Oscillator	String Helmholtz
d.e.	$\frac{d^2\psi(t)}{dt^2} = -\frac{k}{m}\psi(t)$	$\frac{d^2 A(z)}{dz^2} = -\omega^2 \frac{\rho_\ell}{T_0} A(z)$
General Soln.	$\Rightarrow \psi(t) = C\sin\left(\frac{2\pi}{T}t\right) + D\cos\left(\frac{2\pi}{T}t\right)$	$\hookrightarrow$ General Soln: $A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)$
	To find T (or ω): plug back, find $T = 2\pi\sqrt{\frac{m}{k}}$	To find λ: plug back, as follows:

To find λ:

$$\frac{d^2A(z)}{dz^2} = -\omega^2 \frac{\rho_\ell}{T_0} A(z)$$

Put in $A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)$

$$\Rightarrow \quad -\left(\frac{2\pi}{\lambda}\right)^2 \left[A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)\right] = -\omega^2\frac{\rho_0}{T_0}\left[A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)\right]$$

$$\Rightarrow \left(\frac{2\pi}{\lambda}\right)^2 = \omega^2\frac{\rho_\ell}{T_0}$$

$$\Rightarrow \frac{2\pi}{\lambda} = \omega\sqrt{\frac{\rho_\ell}{T_0}}$$

$$\Rightarrow \frac{2\pi}{\lambda} = 2\pi f\sqrt{\frac{\rho_\ell}{T_0}}$$

$$\Rightarrow \boxed{\lambda f = \sqrt{\frac{T_0}{\rho_0}}} \qquad \rho_0 \equiv \rho_\ell$$

where f is the frequency of the normal mode under consideration.

"Alternative Method": In analogy to $\frac{d^2\psi(t)}{dt^2} = -\omega^2\psi = -\frac{4\pi^2}{T^2}\psi$,

string Helmholtz eqn. must be of form $\frac{d^2A(z)}{dz^2} = -\frac{4\pi^2}{\lambda^2}A(z)$

$$\Rightarrow \frac{4\pi^2}{\lambda^2} = \omega^2\frac{\rho_\ell}{T_0}$$

$$\Rightarrow \frac{\cancel{4\pi^2}}{\lambda^2} = \cancel{4\pi^2} f^2\frac{\rho_\ell}{T_0}$$

$$\Rightarrow \lambda f = \sqrt{\frac{T_0}{\rho_\ell}}.$$

Thus, there are normal modes (of course we already knew that from our intuitive-pictorial analysis of the last class) and they are of the form

(1) $$\Psi_\omega(z,t) = \left[A\sin\left(\frac{2\pi z}{\lambda}\right) + B\cos\left(\frac{2\pi z}{\lambda}\right)\right]\cos(\omega t+\phi) \quad (*)$$

where ω (or $f \equiv \frac{\omega}{2\pi}$) and λ are related by

$$\lambda f = \text{constant} = \sqrt{\frac{T_0}{\rho}}$$

$$[\lambda f] = \frac{m}{s}$$

$$\left[\sqrt{\frac{T_0}{\rho}}\right] = \sqrt{\frac{N}{kg/m}} = \sqrt{\frac{N\cdot m}{kg}} = \sqrt{\frac{kg\cdot\frac{m}{s^2}\cdot m}{kg}} = \frac{m}{s}$$

Let us understand the meaning of this. We see that the product λf has dimensions m/s - i.e., dimensions of velocity. For this reason, it is called by the name "phase velocity." (v_ϕ)

$$v_\phi = \lambda f = \sqrt{\frac{T_0}{\rho_0}}$$

It is, however, important to understand that, in a transverse normal mode (i.e, the sort we are discussing) nothing is traveling either to the left or to the right. For that reason, the normal modes on a stretched string are often called "standing waves". Indeed, as we saw in our pictorial-intuitive discussion,

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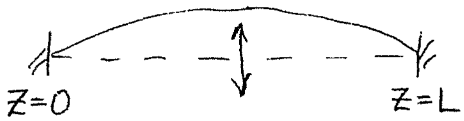
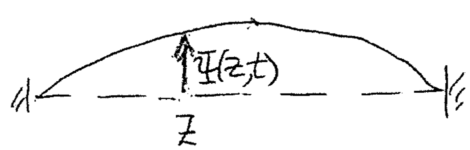
* By $\Psi_\omega(z,t)$ is meant the wave function for the normal mode that oscillates at frequency ω.

in a normal mode, the entire string (or, segments of the string) flap up and down together in s.h.m. at the same frequency



, etc. (*)

Now, of course, there is another velocity in the problem - this is the velocity with which (at any instant t) a particle of the string (or "mass element" dm) moves up and down at frequency $f = \frac{\omega}{2\pi}$. This is the particle velocity; it is $\frac{\partial \Psi(z,t)}{\partial t}$, and



$$\frac{\partial \Psi(z,t)}{\partial t} \neq v_\phi . \quad \text{(Different Concepts).}$$

Of course $\frac{\partial \Psi(z,t)}{\partial t}$ is itself a function of z and t; while v_ϕ is not, since it is a constant $\sqrt{\frac{T_0}{\rho_0}}$ that depends only on equilibrium properties of the medium (the string, or "slinky").

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* These pictures show our expectations of the normal mode shapes for the case where the string is bound down at each end $z=0$ and $z=L$, and for only that case.

A notational point: The quantity $\frac{2\pi}{\lambda}$, which occurs in the mode amplitude profile $A_\omega(z)$, occurs so frequently that it is given its own name and a symbol:

$$\frac{2\pi}{\lambda} \equiv k = \text{the "wavenumber".}$$

This k, the wavenumber, is <u>not a spring constant</u>. (It is unfortunate that the same letter is used for both quantities, but it is always clear which is being referred to by the context.)

Note that

$$[\text{wavenumber}] = \text{meter}^{-1}.$$

Thus,

$$v_\phi = \lambda f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

i.e,

$$\boxed{v_\phi = \lambda f = \frac{\omega}{k} = \sqrt{\frac{T_0}{\rho_0}}}$$

In this, the first two equalities are very general, but the third is true only for the stretched string.

It's now clear what the mode frequencies for an ideal stretched string must be **if** it is bound down at $z=0$ & $z=L$:

<u>mode 1</u>: $\lambda_1 = 2L, \; f\lambda = \sqrt{\frac{T_0}{\rho_0}} \Rightarrow f_1 = \frac{1}{2L}\sqrt{\frac{T_0}{\rho_0}} = \frac{v_\phi}{2L}$

<u>mode 2</u>: $\lambda_2 = L, \; f\lambda = \sqrt{\frac{T_0}{\rho_0}} \Rightarrow f_2 = \frac{1}{L}\sqrt{\frac{T_0}{\rho_0}} = \frac{v_\phi}{L} = 2f_1$

<u>mode 3</u> $\lambda_3 = \frac{2L}{3}, \quad$ " $\quad \Rightarrow f_3 = \frac{3}{2L}\sqrt{\frac{T_0}{\rho_0}} = \frac{3v_\phi}{2L} = 3f_1$

We see that, for an <u>ideal</u> stretched string bound down at both ends, the mode frequencies are all <u>harmonics</u> (integer multiples) of the lowest ("fundamental") mode frequency. As we will see, this is <u>not</u> true for a nonideal string. (More on that, later).