Physics 251 – 13th Class – Tuesday Feb. 20, 2024

Reading for today's class: Text, sect 8.4, 8.7, 8.8

A. Review: Toward The Classical Wave Equation

We are considering the normal modes of a stretched string that is bound down at its two ends ($z=0$ and $z=L$). In a normal mode of the string, between nodes all mass elements move up and down in simultaneous simple harmonic motion at the same frequency; that is, sections of the string behave like big extended harmonic oscillators.*

Based on considerations of imagining the extrapolation to $N \to \infty$ of the transverse modes of N coupled oscillators and based on symmetry considerations, we anticipated that the lowest modes of a homogeneous string look as shown:

$n=1$ $\qquad$ $n=2$ $\qquad$ $n=3$

where n is the mode number in order of increasing mode frequency.

Mathematically, a normal mode on a stretched string is indicated

$$\Psi(z,t) = A(z)\cos(\omega t + \phi)$$

where, for a homogeneous string, we anticipated that

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* In a normal mode on a stretched string, between nodes all mass elements move up and together in phase in shm like an extended harmonic oscillator.

mode shape function", or "amplitude profile" function for mode #n $\left\{ A_n(z) = A_n \sin\left(\frac{2\pi}{\lambda_n} z\right) \right.$ note: $A_n(z) \neq A_n$

A_n: mode number

$A_n(z)$: fcn. of z $\qquad$ A_n: constant

with $\lambda_n = \frac{2L}{n}$; thus $A_n(z) = A_n \sin\left(\frac{n\pi}{L} z\right)$

Thus, for a string bound down at both ends, in mode n,

(1) $$\Psi_n(z,t) = A_n(z) \cos(\omega_n t + \phi_n)$$

(2) $$= A_n \sin\left(\frac{2\pi}{\lambda_n} z\right) \cos(\omega_n t + \phi_n)$$

or

(2) $$\Psi_n(z,t) = A_n \sin\left(\frac{n\pi}{L} z\right) \cos(\omega_n t + \phi_n)$$ with A_n = constant

Of course, the mode shape function $A_n(z) = A_n \sin\left(\frac{n\pi}{L} z\right)$ only applies if the stretched string is bound down at $z=0$ and at $z=L$. For other boundary conditions (e.g., free ends) the mode shape function is a different function of z (but still sinusoidal). We will consider other boundary conditions later on.

Let us make sure that we understand equation (2). It says

$$\Psi_1(z,t) = A_1 \sin\left(\frac{\pi z}{L}\right) \cos(\omega_1 t + \phi_1)$$

$$\Psi_2(z,t) = A_2 \sin\left(\frac{2\pi z}{L}\right) \cos(\omega_2 t + \phi_2)$$

$$\Psi_3(z,t) = A_3 \sin\left(\frac{3\pi z}{L}\right) \cos(\omega_3 t + \phi_3)$$

$$\vdots$$

i.e, for normal mode n,

$$\Psi_n(z,t) = A_n \sin\left(\frac{n\pi z}{L}\right) \cos(\omega_n t + \phi_n) \tag{2}$$

Since, in these normal modes, sections of the string oscillate up and down in unison, but there is no left-right motion of any part of the string or of any pattern, these normal modes are also called "standing waves".

Even within the restriction of both ends ($z=0$ and $z=L$) bound down, we must recognize that equation (2) is really still a guess. And, of course, we do not yet know the mode frequencies (ω_n). To confirm or deny (2) and to find the mode frequencies we will need to employ the systematic method. And that is what we next turn to—a derivation of the governing differential equation for an ideal* stretched string...

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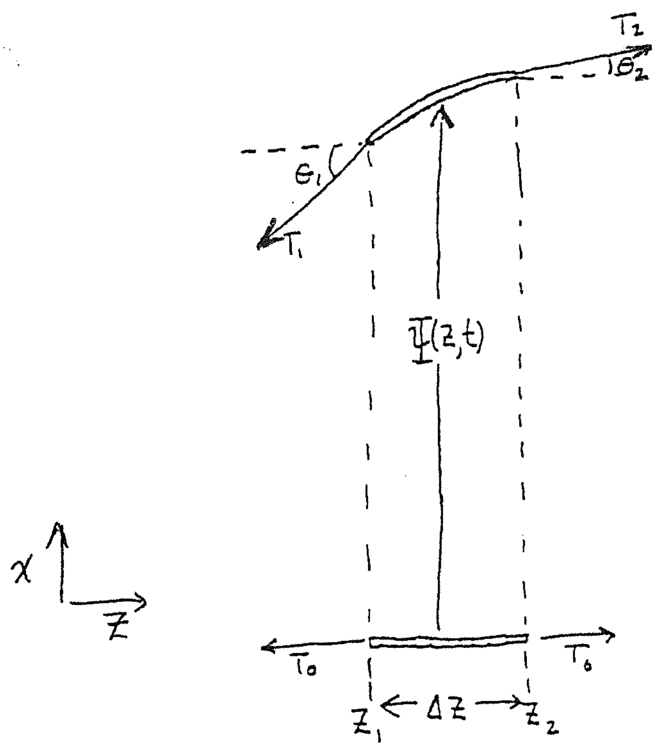
* In physics, an "ideal" stretched string is a stretched string that is "perfectly flexible" (like a long, narrow ideal spring)

B. Derivation of the Classical Wave Equation

We seek the normal modes of a uniform continuous string. For this we will develop the governing differential equation for the string*. This governing equation, called the "Classical Wave Equation" is really just a transcription of Newton's ~~second~~ law applied to each infinitesimal bit of the string in its general small oscillations configuration. To develop this,

Consider a very small segment of the string that runs from $z_1 = z_1$ to $z = z_2 = z_1 + \Delta z$; the element has mass ΔM given by



$$\Delta M \equiv \rho_0 \, \Delta z$$

(ρ_0 = "linear mass density")
(also called "ρ_ℓ")
= mass per unit length in equilibrium configuration.

[We are taking "upward" as the "positive" direction for x.]

net upward force: $\quad F_x(t) = T_2 \sin\theta_2 - T_1 \sin\theta_1$

* We will proceed via the systematic method, which requires equations of motion.

Now, as we remarked for the case of springs, for <u>small angles</u> (i.e, small oscillations approximation),

$$|T_1| \approx |T_2| \approx |T_0| \qquad \text{(magnitudes same, } \underline{\text{directions different}}\text{)}$$

Further, <u>for small oscillations</u>,

$$\sin\theta \approx \theta \approx \tan\theta \quad \left[\begin{matrix} \sin\theta_2 = \cos\theta_2 \tan\theta_2 \approx \tan\theta_2 \\ \sin\theta_1 = \cos\theta_1 \tan\theta_1 \approx \tan\theta_1 \end{matrix} \right\} \begin{matrix}\text{since} \\ \cos\theta_2 \\ \approx \cos\theta_1 \approx 1\end{matrix} \Bigg]^{*}$$

and $$\tan\theta = \text{slope} = \frac{\partial \Psi(z,t)}{\partial z}$$

Thus, $$F_x(t) \approx T_0 \tan\theta_2 - T_0 \tan\theta_1$$

$$\Rightarrow F_x(t) \approx T_0 \left[\left.\frac{\partial \Psi(z,t)}{\partial z}\right)_{z_2} - \left.\frac{\partial \Psi(z,t)}{\partial z}\right)_{z_1} \right]$$

(in the small-oscillations approx.)

next page for continuation of main disussion.

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* For the small angle approximation, $\underset{\approx T_0\cos\theta_2}{F_{z_2}} \approx \underset{\approx T_0\cos\theta_1}{F_{z_1}}$ $\Rightarrow$ no horizontal motion of string segment – motion is purely up or down vertical.
(since $\cos\theta_2 \approx \cos\theta_1$)

This is unwieldly. To convert it, make first order Taylor expansion for $f(z) \equiv \frac{\partial \Psi(z,t)}{\partial z}$ around $z = z_1$.

Then $$\left.\frac{\partial \Psi(z,t)}{\partial z}\right)_2 \approx \left.\frac{\partial \Psi(z,t)}{\partial z}\right)_1 + (z_2 - z_1)\,\frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

so

$$F_x(t) \approx T_0\,\Delta z\,\frac{\partial^2 \Psi(z,t)}{\partial z^2} \qquad \Delta z \equiv z_2 - z_1$$

Now the acceleration of the segment is $\frac{\partial^2 \Psi(z,t)}{\partial t^2}$

$\therefore$, $ma = F \Rightarrow$

$$\Delta M\,\frac{\partial^2 \Psi(z,t)}{\partial t^2} = T_0\,\Delta z\,\frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

but $\Delta M = \rho_\ell\,\Delta z$, so

$$\rho_\ell\,\frac{\partial^2 \Psi}{\partial t^2} = T_0\,\frac{\partial^2 \Psi}{\partial z^2}\text{, or}$$

$$\boxed{\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_\ell}\cdot\frac{\partial^2 \Psi(z,t)}{\partial z^2}}$$

<u>Classical Wave</u> <u>Equation</u>. (CWE)

This very important result is the Classical Wave Equation (for the stretched string).

We will use it extensively throughout much of the rest of the course.

Notice what has happened in our derivation: Considering the string as composed of a (very large) number of "mass beads" separated by massless short springs, we would have a very long list of governing d.e.'s, each an ordinary differential equation:

$$\frac{d^2 y_a(t)}{dt^2} = \dots$$

$$\frac{d^2 y_b(t)}{dt^2} = \dots$$

$$\vdots$$

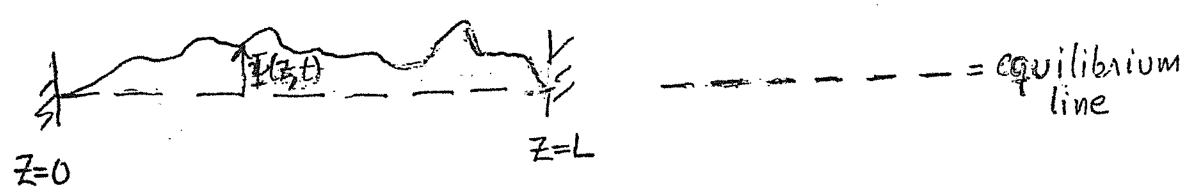
$$\frac{d^2 y_N(t)}{dt^2} = \dots \qquad \text{where } N \text{ is very large.}$$

By invoking the continuum limit, this entire list has been converted into a single partial differential equation (the CWE).

But, then, any partial differential equation is equivalent to a very long list of ordinary differential equations!

C. Normal Modes On Uniform Stretched String Via Systematic Method

We continue our discussion of normal modes on a uniform stretched string, bound down at each of its two ends ($z=0$ and $z=L$). To corroborate our guesses, and also to find the mode frequencies, we took up the systematic method. For this, from the arbitrary configuration, by applying



Newton's law to a very small string segment, we found that the governing differential equation is a Partial differential equation, the <u>Classical Wave Equation</u> (CWE)

$$\boxed{\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_e} \frac{\partial^2 \Psi(z,t)}{\partial z^2}} \qquad \text{CWE}$$

where T_0 is the equilibrium tension and ρ_e is the equilibrium <u>linear</u> mass density (mass per unit length).

We now continue to apply the systematic method (step 2):

2. Try this mode assumption in the governing differential equation of motion:

$$\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_\ell} \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

In it, try $\quad \Psi(z,t) = A(z)\cos(\omega t + \phi)$

$$\Rightarrow \quad -\omega^2 A(z)\cos(\omega t+\phi) = \frac{T_0}{\rho_\ell}\frac{d^2A(z)}{dz^2}\cos(\omega t+\phi)$$

$$\Rightarrow \quad \boxed{\frac{d^2A(z)}{dz^2} = -\omega^2\frac{\rho_\ell}{T_0}A(z)}$$

Helmholtz, or Mode-Shape Equation

We need to solve the mode shape equation for the profile fcn. $A(z)$

Does it look familiar?

Note that, except for the names of the variables, it is the same as the harmonic oscillator equation!

	Simple Harmonic Oscillator	String Helmholtz
d.e.	$\frac{d^2\psi(t)}{dt^2} = -\frac{k}{m}\psi(t)$	$\frac{d^2A(z)}{dz^2} = -\omega^2\frac{\rho_\ell}{T_0}A(z)$
General Soln.	$\Rightarrow \psi(t) = C\sin\left(\frac{2\pi}{T}t\right) + D\cos\left(\frac{2\pi}{T}t\right)$	$\Rightarrow$ General Soln: $A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)$
	To find T (or ω): plug back, find $T = 2\pi\sqrt{\frac{m}{k}}$	To find λ: plug back, as follows:

To find λ:

$$\frac{d^2A(z)}{dz^2} = -\omega^2 \frac{\rho_\ell}{T_0} A(z)$$

Put in $A(z) = A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)$

$$\Rightarrow \quad -\left(\frac{2\pi}{\lambda}\right)^2 \left[A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)\right] = -\omega^2\frac{\rho_0}{T_0}\left[A\sin\left(\frac{2\pi}{\lambda}z\right) + B\cos\left(\frac{2\pi}{\lambda}z\right)\right]$$

$$\Rightarrow \quad \left(\frac{2\pi}{\lambda}\right)^2 = \omega^2\frac{\rho_\ell}{T_0}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda} = \omega\sqrt{\frac{\rho_\ell}{T_0}}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda} = 2\pi f\sqrt{\frac{\rho_\ell}{T_0}}$$

$$\Rightarrow \quad \boxed{\lambda f = \sqrt{\frac{T_0}{\rho_0}}} \qquad \rho_0 \equiv \rho_\ell,$$

where f is the frequency of the normal mode under consideration.

"Alternative Method": In analogy to $\frac{d^2\psi(t)}{dt^2} = -\omega^2\psi = -\frac{4\pi^2}{T^2}\psi$,
string Helmholtz eqn. must be of form $\frac{d^2A(z)}{dz^2} = -\frac{4\pi^2}{\lambda^2}A(z)$

$$\Rightarrow \quad \frac{4\pi^2}{\lambda^2} = \omega^2\frac{\rho_\ell}{T_0}$$

$$\Rightarrow \quad \frac{\cancel{4\pi^2}}{\lambda^2} = \cancel{4\pi^2} f^2\frac{\rho_\ell}{T_0}$$

$$\Rightarrow \quad \lambda f = \sqrt{\frac{T_0}{\rho_\ell}}.$$

Thus, there are normal modes (of course we already knew that from our intuitive-pictorial analysis of the last class) and they are of the form

$$(1) \qquad \Psi_\omega(z,t) = \left[A \sin\left(\frac{2\pi z}{\lambda}\right) + B \cos\left(\frac{2\pi z}{\lambda}\right)\right] \cos(\omega t + \phi) \qquad (*)$$

where ω (or $f \equiv \frac{\omega}{2\pi}$) and λ are related by

$$\lambda f = \text{constant} = \sqrt{\frac{T_0}{\rho}}$$

$$[\lambda f] = \frac{m}{s}$$

$$\left[\sqrt{\frac{T_0}{\rho}}\right] = \sqrt{\frac{N}{kg/m}} = \sqrt{\frac{N \cdot m}{kg}} = \sqrt{\frac{\cancel{kg} \cdot \frac{m}{s^2} \cdot m}{\cancel{kg}}} = \frac{m}{s}$$

Let us understand the meaning of this. We see that the product λf has dimensions m/s – i.e., dimensions of velocity. For this reason, it is called by the name "<u>phase velocity</u>" (v_ϕ)

$$v_\phi = \lambda f = \sqrt{\frac{T_0}{\rho_0}}$$

It is, however, important to understand that, in a transverse normal mode (i.e, the sort we are discussing) <u>nothing is traveling either to the left or to the right</u>. For that reason, the normal modes on a stretched string are often called "<u>standing waves</u>". Indeed, as we saw in our pictorial-intuitive discussion,

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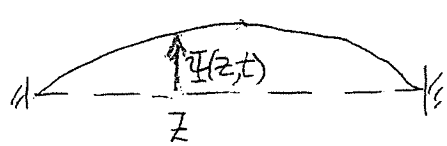
* By $\Psi_\omega(z,t)$ is meant the wave function for the normal mode that oscillates at frequency ω.

in a normal mode, the entire string (or, segments of the string) flap up and down together in s.h.m. at the same frequency

, etc. (*)

Now, of course, there is another velocity in the problem - this is the velocity with which (at any instant t) a particle of the string (or "mass element" dm) moves up and down at frequency $f = \frac{\omega}{2\pi}$. This is the particle velocity; it is $\frac{\partial \Psi(z,t)}{\partial t}$, and



$$\frac{\partial \Psi(z,t)}{\partial t} \neq v_\phi. \quad \text{(Different Concepts).}$$

Of course $\frac{\partial \Psi(z,t)}{\partial t}$ is itself a function of z and t; while v_ϕ is not, since it is a constant $\sqrt{\frac{T_0}{\rho_0}}$ that depends only on equilibrium properties of the medium (the string, or "slinky").

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* These pictures show our expectations of the normal mode shapes for the case where the string is bound down at each end $z=0$ and $z=L$, and for only that case.

A notational point: The quantity $\frac{2\pi}{\lambda}$, which occurs in the mode amplitude profile $A_\omega(z)$, occurs so frequently that it is given its own name and a symbol:

$$\frac{2\pi}{\lambda} \equiv k = \text{the "wavenumber".}$$

This k, the wavenumber, is <u>not a spring constant</u>. (It is a bit unfortunate that the same letter is used for both quantities, but it is always clear which is being referred to by the context.)

Note that

$$[\text{wavenumber}] = \text{meter}^{-1}.$$

Thus,

$$v_\phi = \lambda f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

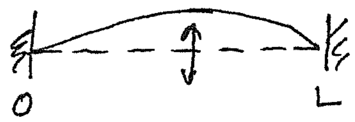
i.e.,

$$\boxed{v_\phi = \lambda f = \frac{\omega}{k} = \sqrt{\frac{T_0}{\rho_0}}}$$

In this, the first two equalities are very general, <u>but the third is true only for the stretched string</u>.

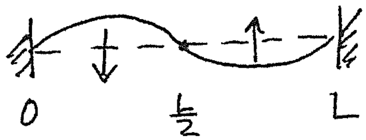
It is now clear what the mode frequencies must be (for an ideal stretched string bound at both ends)

$n=1$ mode:



$\lambda_1 = 2L$, $\underline{f_1 = \frac{v_\phi}{\lambda_1} = \frac{1}{2L}\sqrt{\frac{T_0}{\rho_\ell}}}$.

$n=2$ mode:



$\lambda_2 = L$, $\underline{f_2 = \frac{v_\phi}{\lambda_2} = \frac{1}{L}\sqrt{\frac{T_0}{\rho_\ell}} = 2f_1}$.

$n=3$ mode:

$\lambda_3 = \frac{2L}{3}$, $f_3 = \frac{v_\phi}{\lambda_3} = \underline{\frac{3}{2L}\sqrt{\frac{T_0}{\rho_\ell}} = 3f_1}$.

We see that, for an ideal stretched string bound down at both ends, the mode frequencies are all harmonics (integer multiples) of the lowest ("fundamental") mode frequency. As we will see, this is not true for a non-ideal stretched string. For a real string (e.g., piano string or guitar string), $f\lambda \neq$ constant $\Rightarrow f_n \neq f_1$. [That is because, for example, a real piano string is not perfectly flexible, so the return force for each mode is a bit greater than for an ideal slinky, so the mode frequencies are progressively increasing slightly higher than $f_n = nf_1$.

D. Formal Incorporation of Boundary Conditions

Our mode solution (1) is, in fact, too general for the boundary conditions we have pictured. We note that our string is bound down at each of the two ends, $z=0$ and $z=L$. We need to incorporate these boundary conditions. But, to do this, we need to express the boundary conditions mathematically. These are:

Boundary Conditions

1. $\Psi(z=0, \text{all } t) = 0$

2. $\Psi(z=L, \text{all } t) = 0$

Putting boundary condition 1. into our general solution (5) for a mode

(5) $$\Psi_\omega(z,t) = \left[A \sin \frac{2\pi}{\lambda} z + B \cos \frac{2\pi}{\lambda} z\right] \cos(\omega t + \phi)$$

yields

$0 = \Psi(0,t) = [0 + B] \cos(\omega t + \phi)$, must be true at all times,

$\Rightarrow$ $B = 0$.

so

(8) $$\boxed{\Psi(z,t) = A \sin\left(\frac{2\pi}{\lambda} z\right) \cos(\omega t + \phi)}$$ Bound end at $z=0$.

We see that, with the boundary condition at $z=0$, $\Psi(z,t)$ is simpler than our more general eqn. (1) (which applies for general boundary conditions).

Now we apply boundary condition 2:

(9) $$0 = \Psi(L,t) = A \sin\left(\frac{2\pi}{\lambda} L\right) \cos(\omega t + \phi)$$

Since this must also be true at all t, and since $\cos(\omega t + \phi)$ is not zero at all t, we have two choices: either $A = 0$ (which means $\Psi(z,t) \equiv 0$ (from (8)) which is not what we are interested in, or $\sin\left(\frac{2\pi}{\lambda} L\right) = 0$. So, we take the later choice:

$$\sin\left(\frac{2\pi}{\lambda} L\right) = 0$$

This gives us a number of possibilities - any of

$$\frac{2\pi}{\lambda} L = \pi, 2\pi, 3\pi, \ldots$$

Question:

Why not $\left(\frac{2\pi}{\lambda} L\right) = 0$? What does this say about λ? Is the resulting λ a value we'd be interested in?

Thus $\frac{2\pi}{\lambda_n} L = n\pi \Rightarrow \frac{2L}{\lambda_n} = n$

$$\Rightarrow \lambda_n = \frac{2L}{n}, \qquad \text{(Both ends bound only)}$$

corroborating our "guess", and also corroborating our guess for the mode frequencies

$$f_n = \frac{v_\phi}{\lambda_n} = \frac{n}{2L} v_\phi = \frac{n}{2L}\sqrt{\frac{T_0}{\rho_0}} \qquad \text{(both ends bound only)}$$

So, we have found the frequencies of the normal modes by a rigorous method.

As you know, these normal modes of a stretched string are also often called "standing waves"; however, at the moment I am purposely stressing that they are normal modes (and are the "continuum limit" of the normal modes of discrete mass-sping systems we previously studied.)

E. The General Motion of the Ideal Stretched String

The Classical Wave Equation

$$\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \frac{T_o}{\rho_e} \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

is a linear, homogenous partial differential equation. Therefore, the superposition principle holds for it; thus, any linear combination of normal mode solutions to the CWE should also be a solution, e.g.,

$$(10) \quad \Psi(z,t) = \sum_{n=1}^{N} A_n \Psi_n(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(n \tfrac{\pi}{L} z\right) \cos(\omega_n t + \phi_n)$$

(for both ends of the string fixed ("bound").

Thus, extending our conjecturing here, in principle, in the continuum limit, there would be an infinite number of such normal modes. Thus, we would guess that, as far as the math is concerned, the most general solution to the C.W.E. should be

$$(11) \quad \Psi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(n \tfrac{\pi}{L} z\right) \cos(\omega_n t + \phi_n),$$

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which implies that the most general motion of the stretched string bound at both its ends ($z=0$ and $z=L$) would be given by eqn. (11).

The statement that the most general motion of a stretched string subject to given boundary conditions is a general superposition of the normal modes of the string concordant with those boundary conditions is called the (normal mode) <u>Completeness</u> <u>Hypothesis</u>*. Fourier analysis shows it to be correct.

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* From our previous work in this class, clearly the hypothesis is true for a discrete system of finite masses and springs. Fourier analysis shows it to also be true in the continuum limit of a stretched string.