Modern Classical Mechanics

Chapter 12

Helliwell & Sahakian

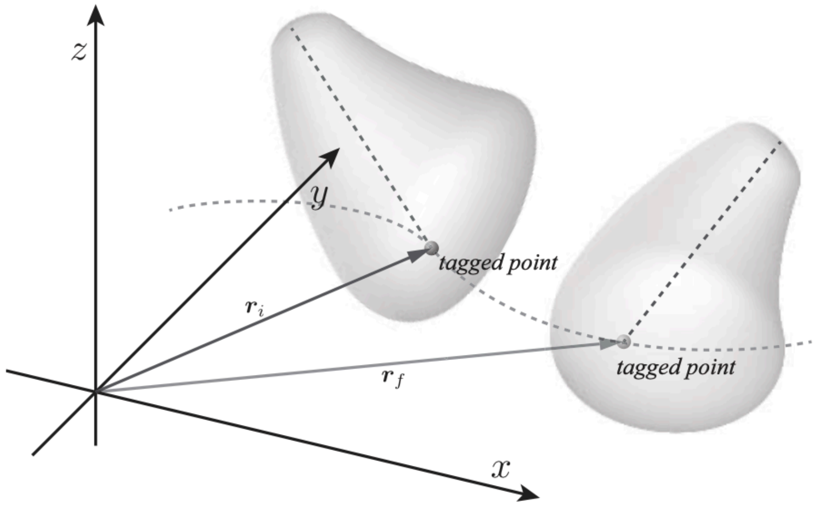
Chapter 12

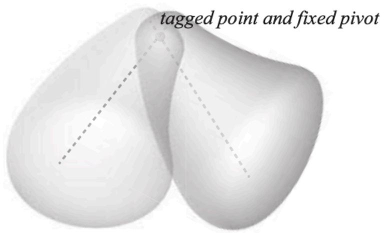
Rigid-Body Dynamics

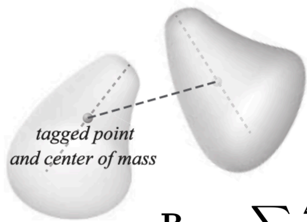
- Euler’s theorem
- Lab and body frames
- Euler angles
- Angular momentum
- Torque
- Energy
- Torque-free dynamics
- Gyroscopes

Euler’s Theorem

All spatial transformations that leave distances unchanged must be a combination of a translation and a rotation.

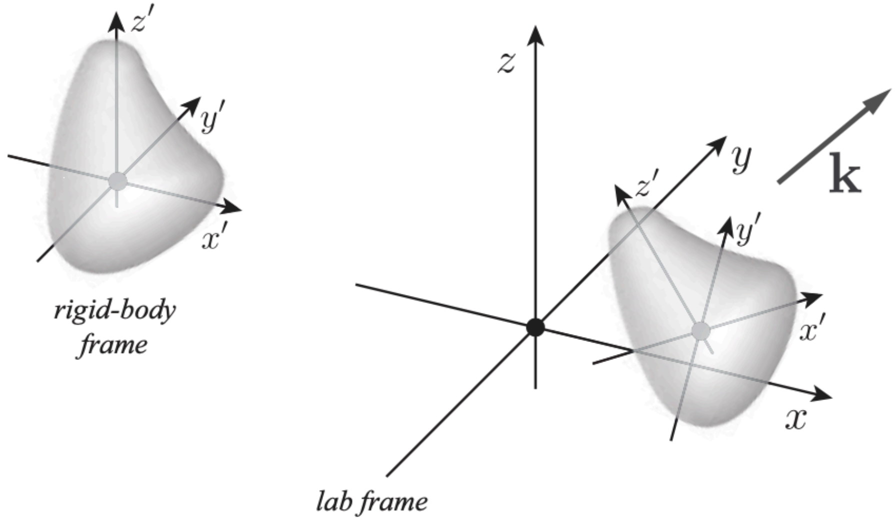






$$\mathbf{R}_{\text{cm}} = \sum_i \frac{\Delta m_i \mathbf{r}_i}{M} = \frac{1}{M}\int \mathbf{r}\,dm = \frac{1}{M}\int \rho\mathbf{r}\,dV$$

Rotation Matrices and the Body Frame



$$\mathrm{k}^{i'}\mathrm{k}^{i'} = \hat{\mathscr{R}}^{i'}_{\;j}\mathrm{k}^{j}\hat{\mathscr{R}}^{i'}_{\;k}\mathrm{k}^{k} = \mathrm{k}^{j}\hat{\mathscr{R}}^{i'}_{\;j}\hat{\mathscr{R}}^{i'}_{\;k}\mathrm{k}^{k} = \mathrm{k}^{i}\mathrm{k}^{i}$$

$$\hat{\mathscr{R}}^{i'}_{\;j}\hat{\mathscr{R}}^{i'}_{\;k} = \delta_{jk} \quad \longrightarrow \quad \hat{\mathscr{R}}^{T} \cdot \hat{\mathscr{R}} = \mathbf{1}$$

Rotation Matrices and the Body Frame

$$\hat{\mathscr{R}}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_x & \sin\alpha_x \\ 0 & -\sin\alpha_x & \cos\alpha_x \end{pmatrix} \qquad \hat{\mathscr{R}}_y = \begin{pmatrix} \cos\alpha_y & 0 & -\sin\alpha_y \\ 0 & 1 & 0 \\ \sin\alpha_y & 0 & \cos\alpha_y \end{pmatrix} \qquad \hat{\mathscr{R}}_z = \begin{pmatrix} \cos\alpha_z & \sin\alpha_z & 0 \\ -\sin\alpha_z & \cos\alpha_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{\mathscr{R}}_1 \cdot \hat{\mathscr{R}}_2 \neq \hat{\mathscr{R}}_2 \cdot \hat{\mathscr{R}}_1$$

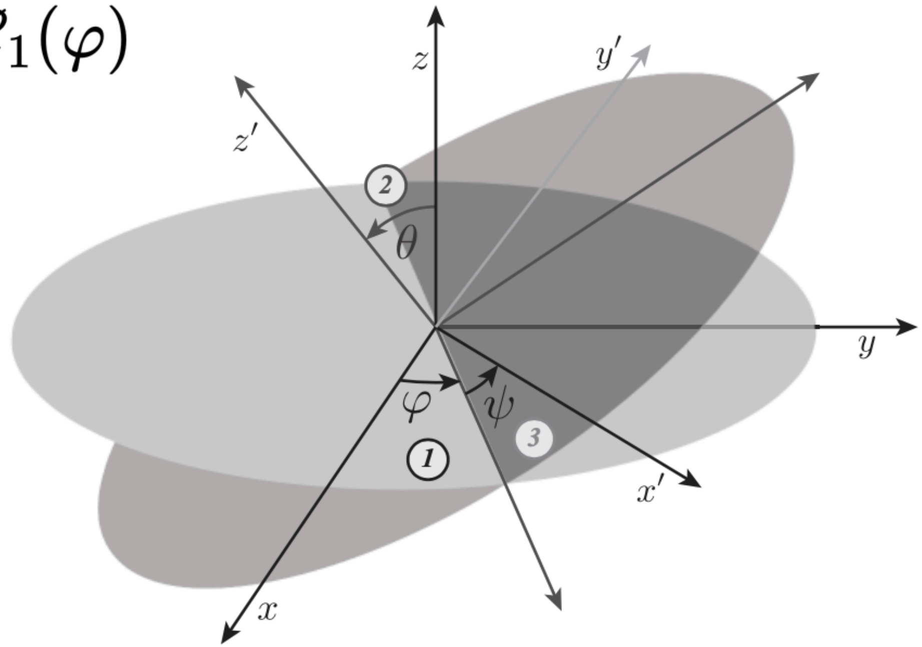
The Euler Angles

$$\hat{\mathscr{R}}(\varphi, \theta, \psi) = \hat{\mathscr{R}}_3(\psi) \cdot \hat{\mathscr{R}}_2(\theta) \cdot \hat{\mathscr{R}}_1(\varphi)$$

$$\hat{\mathscr{R}}_1(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

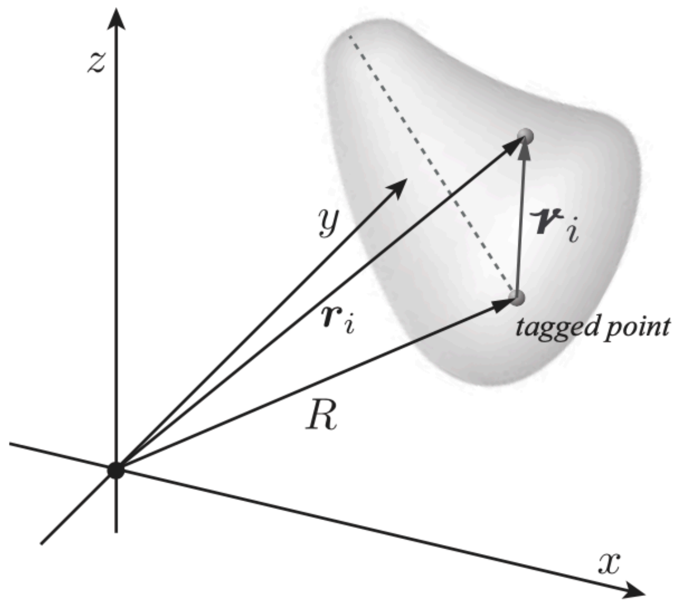
$$\hat{\mathscr{R}}_2(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\hat{\mathscr{R}}_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\left[\hat{\mathscr{R}}(\varphi, \theta, \psi)\right]^T = \left[\hat{\mathscr{R}}_1(\varphi)\right]^T \cdot \left[\hat{\mathscr{R}}_2(\theta)\right]^T \cdot \left[\hat{\mathscr{R}}_3(\psi)\right]^T$$
$$= \hat{\mathscr{R}}_1(-\varphi) \cdot \hat{\mathscr{R}}_2(-\theta) \cdot \hat{\mathscr{R}}_3(-\psi)$$

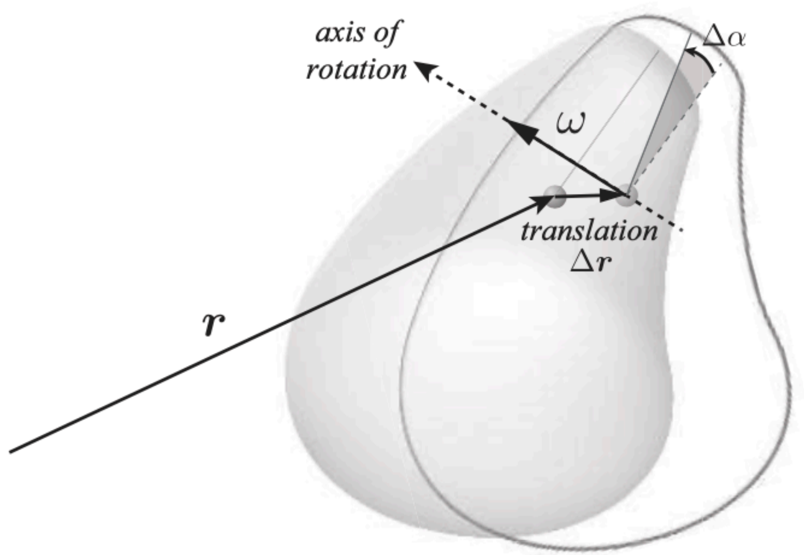
The Euler Angles



The decomposition of the position vector of a bit of the rigid body in terms of the position of the tagged point **R** and the position of the bit with respect to the tagged point $\boldsymbol{r}_a{}'$.

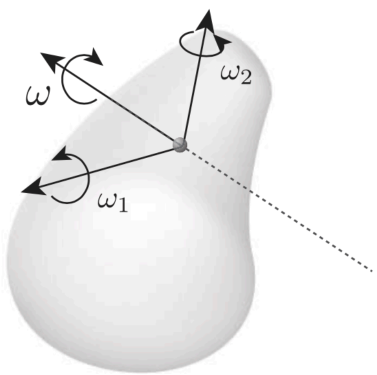
Infinitesimal Rotations

$$\mathbf{r} = \mathbf{R} + \boldsymbol{r}$$



$$\mathbf{v} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$



$$\omega_1 \times \boldsymbol{r} + \omega_2 \times \boldsymbol{r} = \boldsymbol{v}_1 + \boldsymbol{v}_2$$

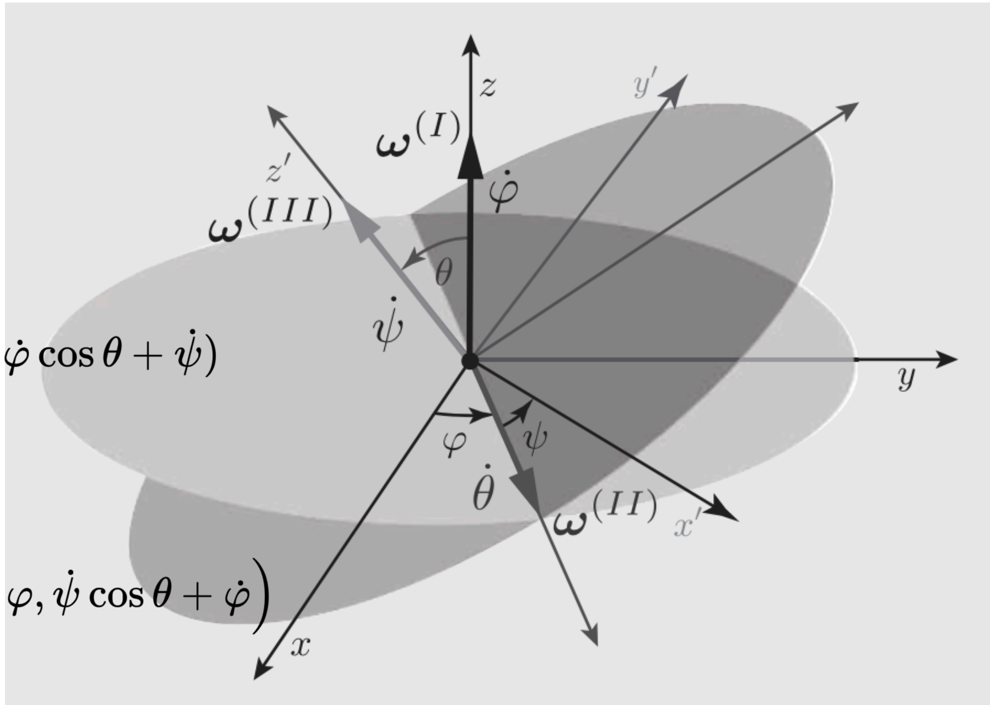
Example 12.2

Angular Velocity Transformation

$$\omega^{(I)} = \dot{\varphi}, \omega^{(II)} = \dot{\theta}, \omega^{(II)} = \dot{\psi}$$

$$\boldsymbol{\omega} = \left(\omega^{x'}, \omega^{y'}, \omega^{z'}\right)$$
$$= (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \dot{\varphi}\cos\theta + \dot{\psi})$$

$$\boldsymbol{\omega} = (\omega^{x}, \omega^{y}, \omega^{z})$$
$$= \left(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi_{\boldsymbol{i}} - \dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi, \dot{\psi}\cos\theta + \dot{\varphi}\right)$$



Angular Momentum

$$\mathbf{L}_{\text{tot}} = \sum_i \boldsymbol{\ell}_i = \sum_i (\mathbf{r_i} \times \mathbf{p_i})$$

$$\mathbf{p}_i = m_i(\mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{r}_i)$$

Tagged point is the center of mass of the rigid body

$$\mathbf{L}_{\text{tot}} = \mathbf{R}_{\text{cm}} \times (M\mathbf{V}_{\text{cm}}) + \mathbf{L}$$

Tagged point is a fixed pivot

$$\mathbf{L}_{\text{tot}} = \mathbf{L}$$

$$\mathbf{L} = \sum_i m_i \left[\boldsymbol{\omega}\left(\boldsymbol{r}_i^2\right) - (\boldsymbol{\omega} \cdot \boldsymbol{r}_i)\boldsymbol{r}_i\right]$$

$$= \int dm \left[\boldsymbol{\omega}\left(\boldsymbol{r}^2\right) - (\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{r}\right]$$

Angular Momentum

$$\hat{\mathbf{L}} = \hat{\mathbf{I}} \cdot \hat{\boldsymbol{\omega}}$$

$$I_{ab} = \int dm \left(r^2 \delta_{ab} - r^a r^b \right)$$

$$I_{xx} = \int dm \left(y^2 + z^2 \right), \quad I_{yy} = \int dm \left(x^2 + z^2 \right), \quad I_{zz} = \int dm \left(x^2 + y^2 \right)$$

$$I_{xy} = \int dm(-xy), \quad I_{xz} = \int dm(-xz), \quad I_{yz} = \int dm(-yz)$$

Principal Axes

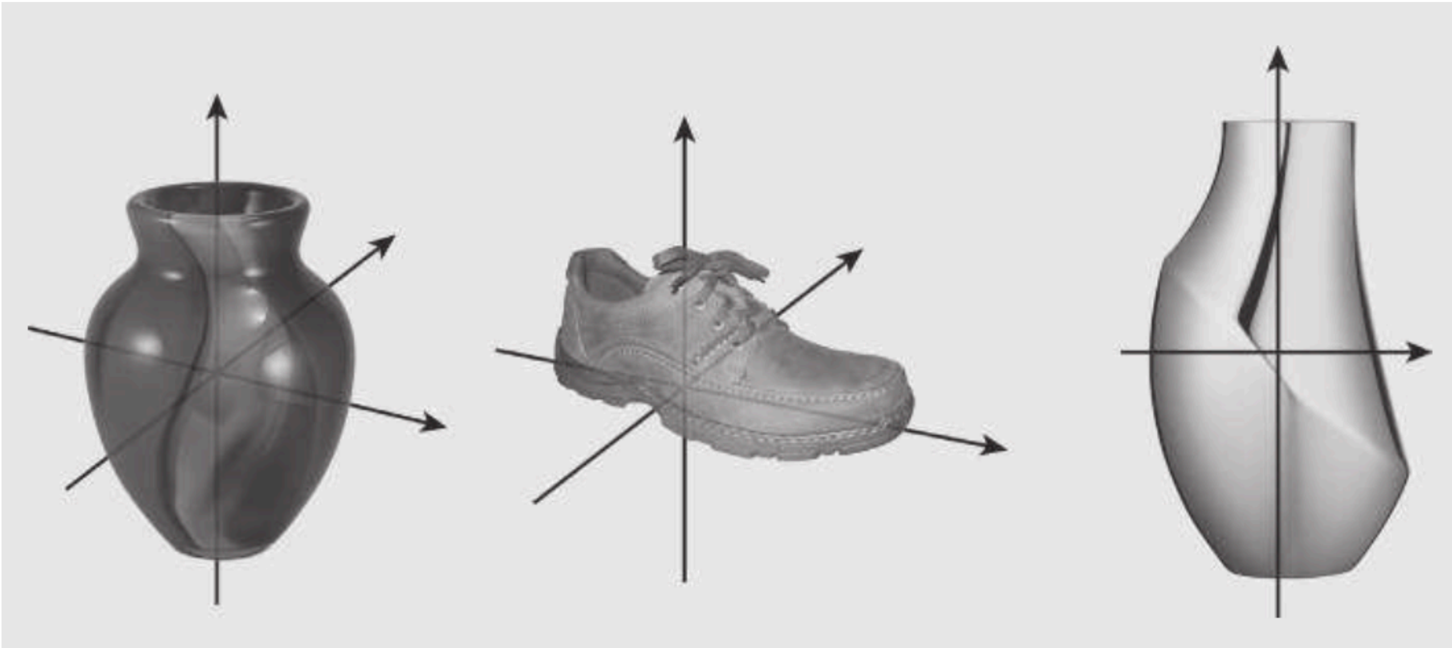
$$\hat{\mathbf{I}} \rightarrow \hat{\mathscr{A}} \cdot \hat{\mathbf{I}} \cdot \hat{\mathscr{A}}^T$$

$$\int dm\, r^{a'} r^{b'} = 0 \quad \text{for } a' \neq b', \text{ in the principal axis frame.}$$

$$\hat{\mathbf{I}} = \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Principal Axes

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3$$



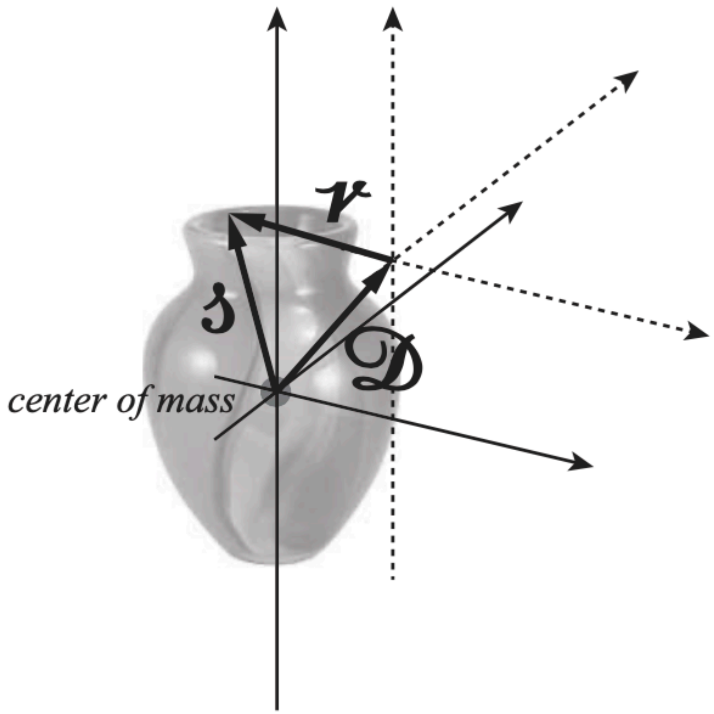
Principal Axes

The parallel axis theorem

$$\boldsymbol{s} = \boldsymbol{r} + \mathscr{D}$$

$$I_{a'a'} = I^{\text{cm}}_{a'} + M\mathscr{D}^2 - M\mathscr{D}^{a'}\mathscr{D}^{a}$$

$$I_{a'b'} = -M\mathscr{D}^{a'}\mathscr{D}^{b'} \qquad \text{for } a' \neq b'$$



Torque

$$\mathbf{N}_{\text{tot}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$$

$$\mathbf{N}_{\text{tot}} = \sum \boldsymbol{r}_i \times \mathbf{F}_i^{\mathbf{ext}} + \mathbf{R} \times \mathbf{F}_i^{\mathbf{ext}} = \mathbf{N} + \mathbf{R} \times \mathbf{F} = \frac{d\mathbf{L}_{\text{tot}}}{dt}$$

Kinetic Energy

$$T_{\text{tot}} = \frac{1}{2}\sum_i \Delta m_i \mathbf{v}_i^2 \qquad\qquad \mathbf{v}_i = \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{r}_i$$

Tagged point is the center of mass of the rigid body

$$T_{\text{tot}} = \frac{1}{2}M\mathbf{V}_{\text{cm}}^2 + M\mathbf{V}\cdot(\boldsymbol{\omega}\times\mathbf{R}_{\text{cm}}) + T$$

Tagged point is a fixed pivot

$$T_{\text{tot}} = T$$

$$\begin{aligned} T &= \frac{1}{2}\int dm\left(\omega^2 r^2 - (\omega\cdot\boldsymbol{r})^2\right) \\ &= \frac{1}{2}\omega^a\left[\int dm\left(r^2\delta_{ab} - r^a r^b\right)\right]\omega^b \\ &= \frac{1}{2}\omega^a I_{ab}\omega^b = \frac{1}{2}\boldsymbol{\omega}^T\cdot\hat{\mathbf{I}}\cdot\boldsymbol{\omega} \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2}\boldsymbol{\omega}^T\cdot\hat{\mathbf{I}}\cdot\boldsymbol{\omega} \to \frac{1}{2}\boldsymbol{\omega}^T\cdot\hat{\mathscr{A}}^T\cdot\hat{\mathscr{A}}\cdot\hat{\mathbf{I}}\cdot\hat{\mathscr{A}}^T\cdot\hat{\mathscr{A}}\cdot\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^T\cdot\hat{\mathbf{I}}\cdot\boldsymbol{\omega} \\ &= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \end{aligned}$$

Potential Energy

$$U = \sum_i \Delta m_i g h_i = \sum_i \Delta m_i g \mathbf{r}_i \cdot \hat{\mathbf{z}} = M g \mathbf{R}_{\mathrm{cm}} \cdot \hat{\mathbf{z}} = M g H$$

Example 12.9

A Hoop Hanging on a Spring

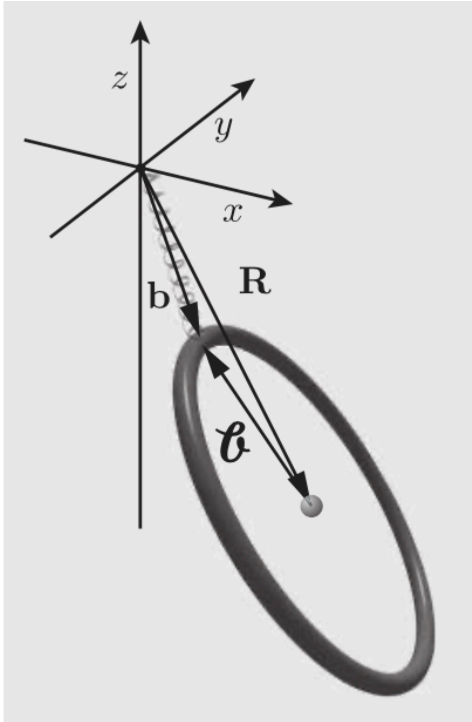
$$T_{\text{tot}} = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{4}MR^2\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) + \frac{1}{2}MR^2(\dot{\psi} + \dot{\varphi}\cos\theta)^2$$

$$U_{\text{grav}} = MgZ \qquad U_{\text{spring}} = \frac{1}{2}kb^2$$

$$\mathbf{b} = \mathbf{R} + \boldsymbol{b}$$

$$\boldsymbol{b} = \left(e^{x'}, \theta^{y'}, b^{z'}\right) = (0, R, 0) \qquad \mathbf{R} = (\mathrm{R}^x, \mathrm{R}^y, \mathrm{R}^z) = (X, Y, Z)$$

$$\mathbf{b} = (b^x, b^y, b^z) = (X + b^x, y + b^y, Z + b^z)$$



$$\boldsymbol{b} = (b^x, b^y, b^z) = [R(\cos\theta\cos\varphi\sin\psi + \cos\psi\sin\varphi), R(\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi), -R(\cos\varphi\sin\theta)]$$

Example 12.9

A Hoop Hanging on a Spring

$$\begin{aligned} U_{\text{spring}} =& \frac{1}{2}k\big(X^2 + Y^2 + Z^2 + R^2 + 2RX(\cos\theta\cos\varphi\sin\psi + \cos\psi\sin\varphi) \\ & + 2RY(\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi) - 2RZ\cos\varphi\sin\theta\big) \end{aligned}$$

$$\mathscr{L} = T_{\text{tot}} - U_{\text{grav}} - U_{\text{spring}}$$

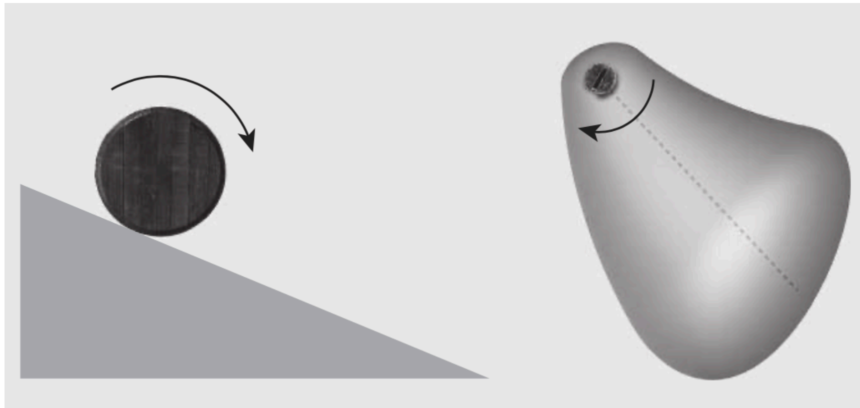
Example 12.11

Rolling, Fixed-Axis Rotation

$$L_1 = L_2 = 0, \qquad L_3 = I_3\omega_3 = I_3\dot{\psi}$$

$$I = \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_3\dot{\psi}^2 = \frac{L_3^2}{2/{}_3}$$

$$\frac{dL_3}{dt} = I_3\dot{\omega}_3 = I_3\alpha \qquad\qquad \alpha = \ddot{\psi}$$



Torque-Free Dynamics

$$I_1 = I_2 \equiv I, \quad I_3$$

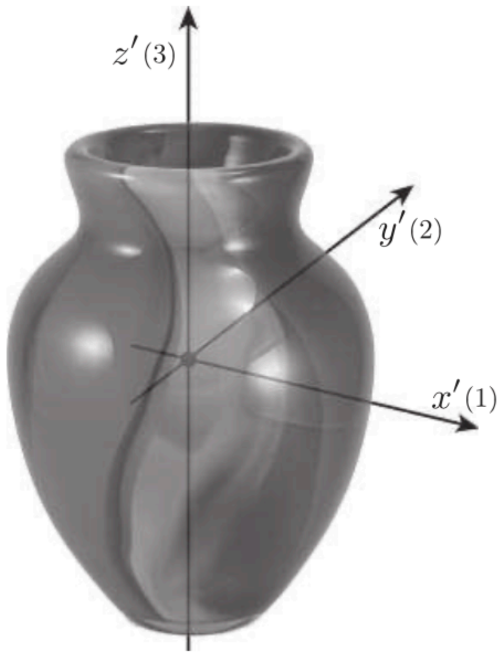
$$\mathscr{L} = T - U$$

$$T = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2$$

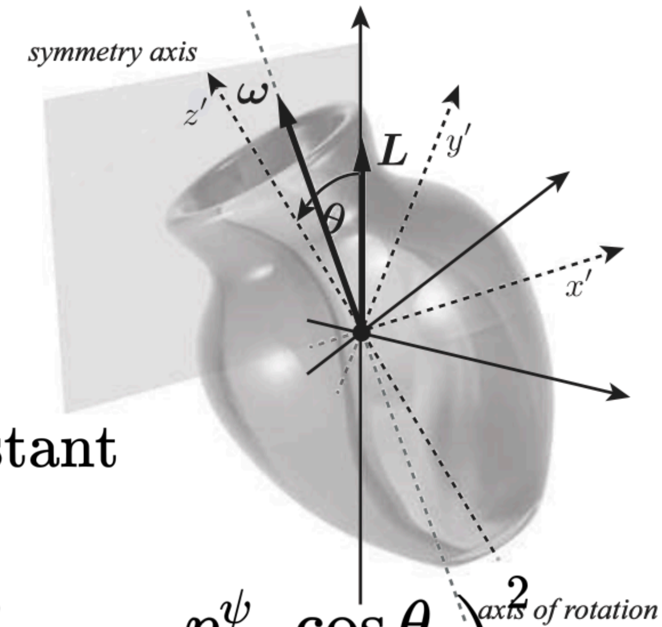
$$U = MgZ$$

$$p_\psi = \frac{\partial\mathscr{L}}{\partial\dot{\psi}} = I_3(\dot{\psi} + \dot{\varphi}\cos\theta) = \text{constant}$$

$$p^\varphi = \frac{\partial L}{\partial\dot{\varphi}} = I\dot{\varphi}\sin^2\theta + p^\psi\cos\theta = \text{constant}$$



Torque-Free Dynamics



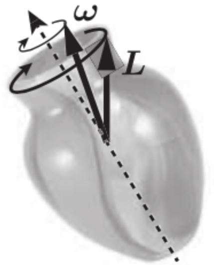
$$I\ddot{\theta} = I\dot{\varphi}^2 \sin\theta\cos\theta - p^{\psi}\dot{\varphi}\sin\theta$$

$$H = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2I_3}\left(p^{\psi}\right)^2 + \frac{1}{2}I\sin^2\theta\dot{\varphi}^2 = \text{ constant}$$

$$H = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2I_3}\left(p^{\psi}\right)^2 + \frac{1}{2}I\sin^2\theta\left(\frac{p^{\varphi}}{I\sin^2\theta} - \frac{p^{\psi}}{I}\frac{\cos\theta}{\sin^2\theta}\right)^2$$

Torque-Free Dynamics

From body frame perspective



$$\dot{\theta} = 0$$

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = (\dot{\varphi}\sin\theta\sin\psi, \dot{\varphi}\sin\theta\cos\psi, \dot{\varphi}\cos\theta + \dot{\psi})$$

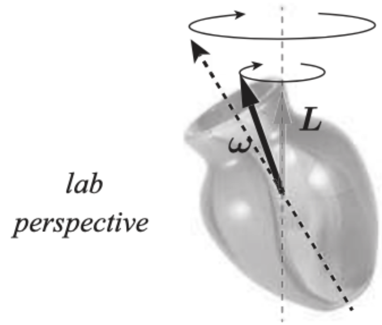
$$I\dot{\varphi}\cos\theta = p^{\psi}$$

$$p^{\varphi} = \frac{p^{\psi}}{\cos\theta}$$

$$\dot{\psi} = \left(1 - \frac{I_3}{I}\right)\frac{p^{\psi}}{I_3} = \left(1 - \frac{I_3}{I}\right)\omega_3$$

Torque-Free Dynamics

From lab frame perspective



$$\dot{\theta} = 0$$

$$\boldsymbol{\omega} = (\omega^x, \omega^y, \omega^z) = (\dot{\psi}\sin\theta\sin\varphi, \dot{\psi}\sin\theta\cos\varphi, \dot{\psi}\cos\theta + \dot{\varphi})$$

$$\dot{\varphi} = \frac{1}{I}\frac{p^\psi}{\cos\theta}$$

$$L^2 = 2HI + \left(p^\psi\right)^2\left(1 - \frac{I}{I_3}\right) \quad \Rightarrow \quad L = \frac{I_3 p^\psi}{\cos\theta}$$

$$\dot{\varphi} = \frac{L}{I}$$

Euler's Equations of Motion and Stability

$$\mathbf{N}_{\text{tot}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{lab}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L}$$

$$N_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$$
$$N_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$$
$$N_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$$

Euler's Equations of Motion and Stability

no net torque
ω_1 is large and ω_2 and ω_3 are both small

$$I_1\dot{\omega}_1 \simeq 0$$
$$I_2\dot{\omega}_2 + [(I_1 - I_3)\omega_1]\omega_3 \simeq 0$$
$$I_3\dot{\omega}_3 + [(I_2 - I_1)\omega_1]\omega_2 \simeq 0$$

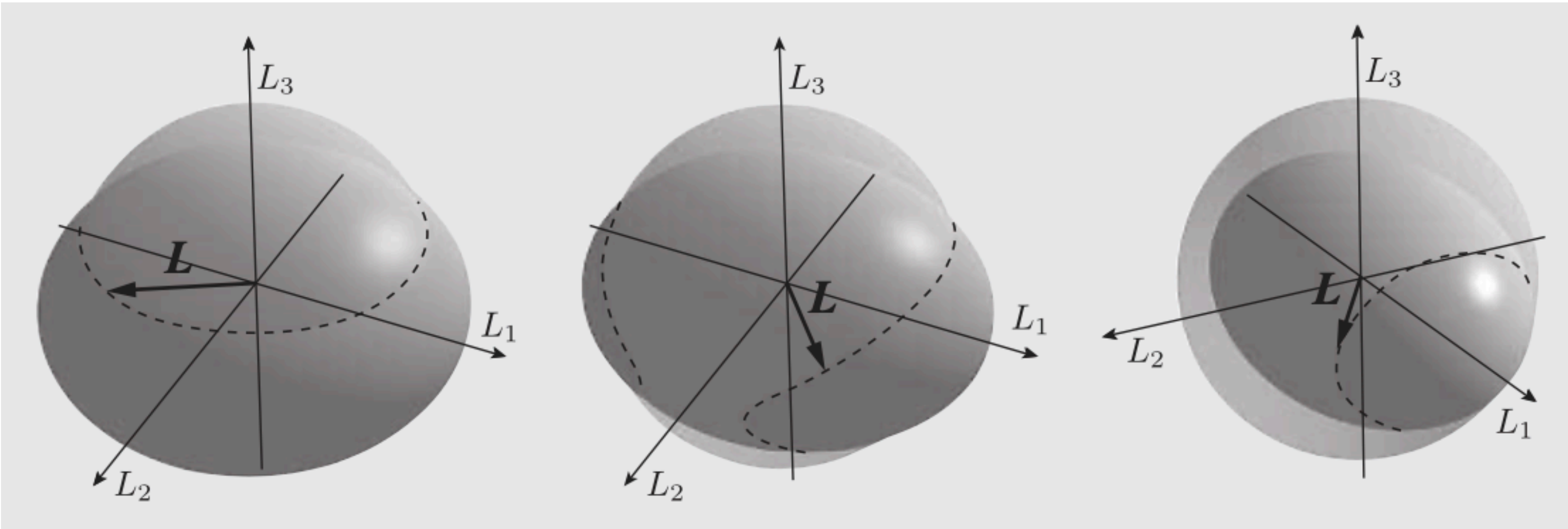
$$\ddot{\omega}_2 + \Omega^2\omega_2 = 0 \quad \text{and} \quad \ddot{\omega}_3 + \Omega^2\omega_3 = 0 \qquad \Omega^2 \equiv \frac{[(I_1 - I_2)(I_1 - I_3)]\omega_1^2}{I_2 I_3}$$

Stability requires I_1 be the largest or smallest moment of inertia.

Euler's Equations of Motion and Stability

$$L^2 = L_1^2 + L_2^2 + L_3^2 \qquad T = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2l_2} + \frac{L_3^2}{2I_3}$$

$$I_1 \geq I_2 \geq I_3 \quad \Rightarrow \quad \sqrt{2TI_3} \leq L \leq \sqrt{2TI_1}$$



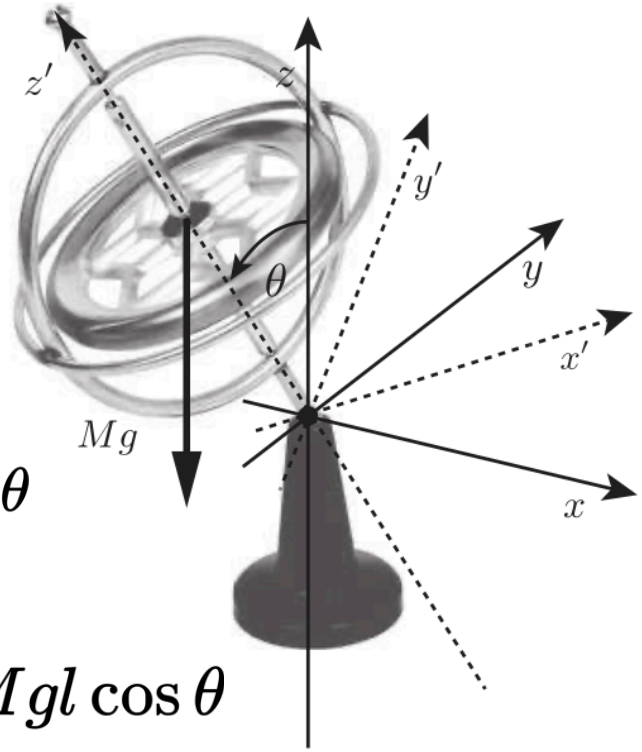
Gyroscopes

$$T = \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2$$

$$U = MgZ \qquad\qquad Z = -l\cos\theta$$

$$p^\psi = I_3(\dot{\psi} + \dot{\varphi}\cos\theta), \quad p^\varphi = I\dot{\varphi}\sin^2\theta + p^\psi\cos\theta$$

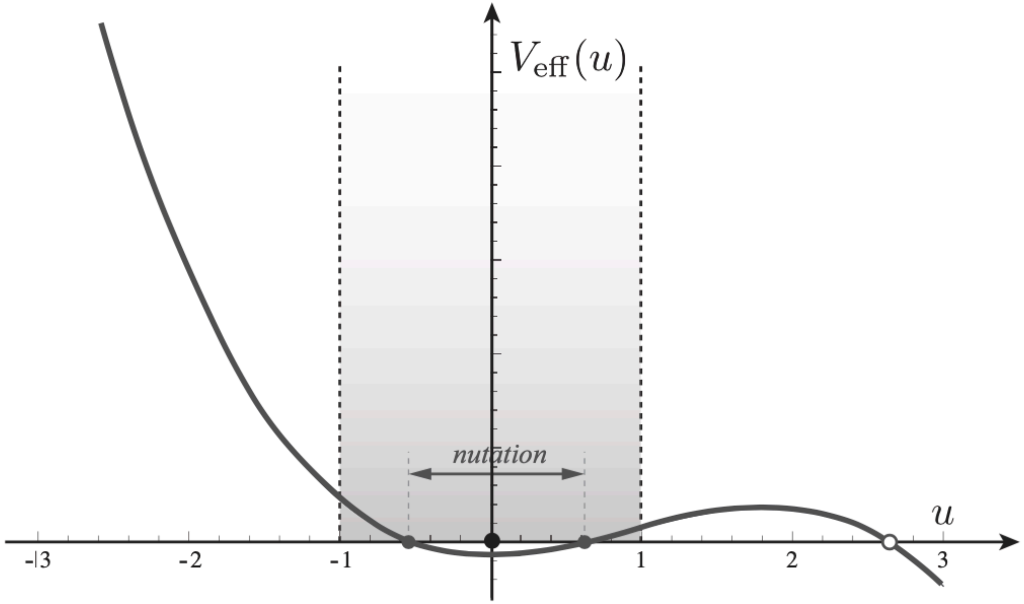
$$H = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2I_3}\left(p^\psi\right)^2 + \frac{1}{2}I\sin^2\theta\dot{\varphi}^2 + Mgl\cos\theta$$



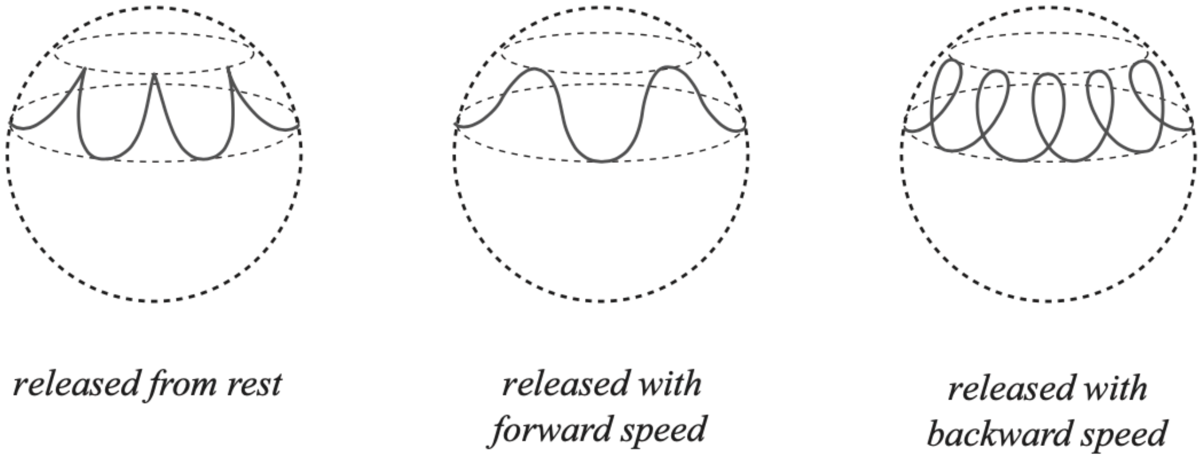
Gyroscopes

$$u = \cos\theta$$

$$\frac{1}{2}\dot{u}^2 + \frac{1}{2}\frac{\left(p^\varphi - up^\psi\right)^2}{I^2} + \frac{1}{2}\left(\frac{2HI_3 - \left(p^\psi\right)^2}{II_3} - \frac{2Mgl}{I}u\right)\left(u^2 - 1\right) = 0$$



Gyroscopes



The nutation pattern of a gyroscope traced out by the gyroscope's z' axis. The spinning about the vertical axis is the familiar tumble, or *precession*. But now we also have a superimposed nutation as gravity tries to pull the z' axis downward.

Summary

- Rigid body dynamics describes the time evolution of the orientation of a rigid body. Euler angles are used to label the orientation.
- Two perspectives for this dynamics are useful: body frame and lab frame.
- Use principal axes whenever possible; and write the Lagrangian in terms of the Euler angles.
- Torque-free dynamics involves a wobble.
- Gyroscope dynamics is characterized by nutation.