Tema 5 : teoría Hamilton-Jacobi

1. Ecuación Hamilton-Jacobi

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} \Rightarrow K = H + \frac{\partial F}{\partial t} = 0$$

tomo $F_2 = S$ donde $F_2(q, P)$ y $H(q, P, t)$

ECUACIÓN HAMILTON-JACOBI

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, \dots, q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots, \frac{\partial S}{\partial q_n}, t\right) = 0$$

donde $S = S(q_1, \dots, q_n, P_1, \dots, P_n, t)$

$$P_i = \frac{\partial S}{\partial q_i} = \alpha_i$$

$P_i = cte \quad Q_i = cte$
porque $K = 0$

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S}{\partial \alpha_i} = \beta_i$$

$$S = S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)$$

$$q_i = q_i(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, t)$$

$$P_i = P_i(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t) = P_i(\beta_1, \dots, \beta_n, \alpha_1, \dots, \alpha_n, t)$$

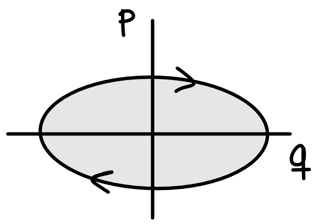
$$S = \int L \, dt$$

(mirar ejemplos )

2 Variables ángulo - acción

sistemas periódicos, conservativos
$H \neq H(t)$

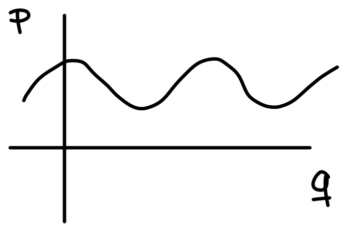
LIBRACIÓN



$$J = \oint p\,dq$$

a lo largo de 1 periodo

ROTACIÓN



$$J = \int p\,dq$$

sistema completamente separable.

$$W(q_1, q_2, \dots, q_n, \alpha_1, \dots, \alpha_n) = \sum W_i(q_i, \alpha_1, \dots, \alpha_n)$$

$$J_i = \oint p_i dq_i \overset{**}{=} \oint \frac{\partial W_i}{\partial q_i} dq_i$$

(integro y solo depende de α_i)

cambio coord.

$$W(q_1, \dots, q_n, \boxed{J_1, \dots, J_n})$$

nuevos momentos transf. canónica

$$\underset{*}{P_i} = J_i = \frac{\partial W}{\partial q_i} \qquad \varphi_i = \frac{\partial W}{\partial J_i}$$

coordenadas angulares

$(q, P) \rightarrow (\varphi, J)$
Transf.
can.

$$\dot{P} = -\frac{\partial H}{\partial Q} \Rightarrow \dot{J} = -\frac{\partial H}{\partial \varphi_i} = 0 \Rightarrow H = H(J_1, \dots, J_n)$$

$J =$ ctes

$$\dot{Q} = \frac{\partial H}{\partial P} \Rightarrow \dot{\varphi}_i = \frac{\partial H}{\partial J_i} = \nu_i \Rightarrow \varphi_i = \nu_i t + \delta_i$$

Son movimientos periódicos, ¿cuál será el periodo?

Cogemos una φ_i, ¿cuánto cambia cuando una de las coordenadas da un ciclo completo? Solo cambio una.

$$\varphi_i = \oint \frac{\partial \boxed{\varphi_i}}{\partial q_j} dq_j =$$ vario q_j y mantengo el resto iguales.

$$\overset{*}{=} \oint \frac{\partial}{\partial q_j} \boxed{\frac{\partial W}{\partial J_i}} dq_j \overset{**}{=} \frac{\partial}{\partial J_i} \oint p_j \, dq_j = \frac{\partial J_j}{\partial J_i} = \delta_{ij}$$

HAMILTON - JACOBI

$$H\left(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}\right) + \frac{\partial S}{\partial t} = 0$$

$$S = S(q, \alpha, t)$$

$$\frac{\partial S}{\partial q} = p \qquad \frac{\partial S}{\partial t} = -H \qquad \frac{\partial S}{\partial \alpha} = \beta$$

OBJETIVO: encontrar ecuaciones de movimiento

PASOS:

1) S?
2) $\frac{\partial S}{\partial \alpha} = \beta \rightarrow q$

ejemplo:

$$H = \frac{1}{2m} p^2 + mgx$$

H-J: $\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + mgx + \frac{\partial S}{\partial t} = 0$

1) $S = f(x) + g(t)$

$$\underbrace{\frac{1}{2m}\left(f'(x)\right)^2 + mgx}_{=\alpha} = \underbrace{-\dot{g}(t)}_{=\alpha}$$

$\rightarrow \dot{g} = -\alpha$

$g = -\alpha t$

$\rightarrow f'(x) = \sqrt{2m(\alpha - mgx)}$

$$f(x) = \sqrt{2m} \frac{(\alpha - mgx)^{3/2}}{\frac{3}{2}} \frac{1}{-mg}$$

2) $\beta = -t + \frac{2}{3} \frac{\sqrt{2m}}{mg} \frac{3}{2} (\alpha - mgx)^{1/2}$

$$-\left((\beta + t) g \sqrt{\frac{m}{2}}\right)^2 = -\alpha + mgx \Rightarrow x = \frac{\alpha}{mg} - \frac{1}{2}(\beta + t)^2 g$$

$$\frac{\partial S}{\partial t} = -E = -\alpha \Rightarrow \alpha = E$$

$$\dot{x} = -g(\beta + t) = \underbrace{-g\beta}_{v_0} - gt \Rightarrow \beta = -\frac{v_0}{g}$$

$$x = \frac{E}{mg} - \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{2}gt^2 + v_0 t$$

Sabemos que $E = \frac{1}{2}mv_0^2 + mgx_0$

$$\frac{E}{mg} = \frac{v_0^2}{2g} + x_0$$

$$x = x_0 + v_0 t - \frac{1}{2}gt^2$$

$$S = -Et - \frac{2\sqrt{2m}}{3mg}(E - mgx)^{3/2}$$

$S(t = t_0, x = x_0) = 0$ (Impongo esto porque $\int_{t_0}^{t_0}$)

$$S(t = t_0, x = x_0) = -Et_0 - \frac{2\sqrt{2m}}{3mg}(E - mgx_0)^{3/2}$$