Homework Set 6 — Víctor Mira Ramírez

1. a. Derive the first five nonvanishing terms of a Fourier series that represents the function $f(x) = |x|$ on the interval $(-\pi, \pi)$. Show all steps of the derivation. Note the parity of the function $f(x) = |x|$ around the origin (x=0).

   Write out the first five non-vanishing terms of your Fourier series explicitly (*i.e.*, no Σ symbol). The Fourier coefficients for these terms must be evaluated and must be displayed as part of the series. In displaying the calculations for the Fourier coefficients, distinguish between the variable of integration (call it, say, x', or "u,") and the x that appears as the argument of $f(x)$. Note that x and $|x|$ are not the same function on the half interval $(-\pi, 0)$.

   b Using a computer plotting routine, plot both the function $f(x) = |x|$ and the sum of the first four nonzero series terms on the same plot over all of $(-\pi, \pi)$. Comment on the comparison.

The function $|x|$ has a derivability issue around $x=0$, point from which the function has symmetry, as the absolute value function is an even function.
The Fourier series of a function $f(x)$ on the interval $[-\pi,\pi]$ is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n\cos(nx) + b_n\sin(nx)\right)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad \begin{cases} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{cases}$$

For the function $f(x) = |x|$

- $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \frac{1}{\pi}\int_{-\pi}^{0} -x dx + \frac{1}{\pi}\int_{0}^{\pi} x dx = \frac{-1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi} =$$
$$= \frac{-1}{2\pi}(-\pi^2) + \frac{1}{2\pi}\pi^2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

- $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx \overset{1)}{=}$$
$$\overset{1)}{=} \frac{-1}{\pi}\left[\frac{x}{n}\sin(nx) - \int \frac{1}{n}\sin(nx)dx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) - \int \frac{1}{n}\sin(nx)dx\right]_{0}^{\pi} =$$
$$= -\frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{0}^{\pi} \overset{2)}{=}$$
$$\overset{2)}{=} -\frac{1}{\pi}\left(\frac{1}{n^2} + \frac{1}{n^2}\right) + \frac{1}{\pi}\left(-\frac{1}{n^2} - \frac{1}{n^2}\right) = -\frac{4}{\pi n^2}$$

1) (Notation Abuse) $u = x$ $\quad du = dx$
$v = \frac{1}{n}\sin(nx)$ $\quad dv = \cos(nx)dx$

2) $n \in 2\mathbb{N}-1$
$a_n = 0$ for $n \in 2\mathbb{N}$

$$\bullet\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx \overset{3)}{=}$$

$$\overset{3)}{=} -\frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) - \int -\frac{1}{n}\cos(nx)\,dx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) - \int -\frac{1}{n}\cos(nx)\,dx\right]_{0}^{\pi} =$$

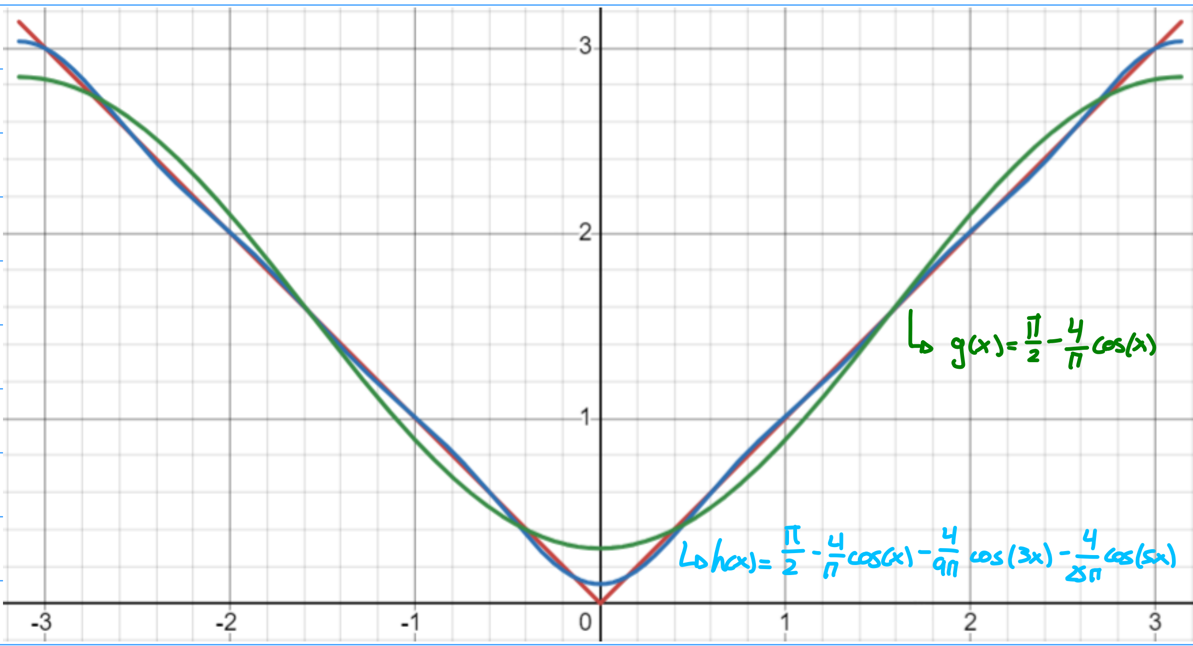
$$= -\frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{0}^{\pi} \overset{4)}{=}$$

$$\overset{4)}{} \left.\begin{cases} = -\frac{1}{\pi}\left(0 - \frac{\pi}{n}\right) + \frac{1}{\pi}\left(\frac{-\pi}{n} - 0\right) = \frac{1}{n} - \frac{1}{n} = 0 & \text{if } n \in 2\mathbb{N} \\ = -\frac{1}{\pi}\left(0 - \left(\frac{-\pi}{n}\right)\right) + \frac{1}{\pi}\left(\frac{\pi}{n} - 0\right) = -\frac{1}{n} + \frac{1}{n} = 0 & \text{if } n \in 2\mathbb{N}-1 \end{cases}\right\} = 0 \,//$$

$$\Rightarrow f(x) = |x| = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + \cancelto{0}{b_n\sin(nx)}\right) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) \simeq$$

$$\simeq a_0 + a_1\cos(x) + a_3\cos(3x) + a_5\cos(5x) + a_7\cos(7x) + a_9\cos(9x) \simeq$$

$$\simeq \boxed{f(x) \simeq \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x) - \frac{4}{49\pi}\cos(7x)}$$



On the plot we can clearly see how the approximation works on the desired interval, being $h(x)$ a closer approximation to $f(x)$ as it contains more terms of the expansion. However, outside of $[-\pi, \pi]$ both of the approximations fall from being close to $|x|$, but that is expected as we would be outside of the period of the trigonometric functions. On $x \in \mathbb{R} \setminus [-\pi, \pi]$ the approximations repeat the plotted pattern meanwhile the function $|x|$ keeps growing to infinity.

3) (Notation Abuse)
4) $u = x \qquad du = dx$
$v = -\frac{1}{n}\cos(nx) \qquad dv = \sin(nx)\,dx$

2. Find a Fourier series for $f(x) = cosx$ that is valid on the interval $(0, \pi/2)$ and that has repeat distance $\pi/2$ on the entire x-axis. To do this:

   a. Sketch a plot of the repetitive function that the asked-for Fourier series is supposed to converge to. Label points on your abscissa axis. Does that repetitive function have a definite parity about the origin?

   b. Now write out the first five terms of the series without including any terms that are "obviously" equal to zero. For this follow the instructions below:

      i. Do this first leaving the Fourier coefficients as "generic" (e.g., like "a_2" or "b_7"). If you concluded that no terms are "obviously" zero, so indicate and justify that conclusion. If you concluded that some terms are "obviously" zero, justify that conclusion, not by simply commenting that "it works out that way," but by citing a general principle or familiar general result.

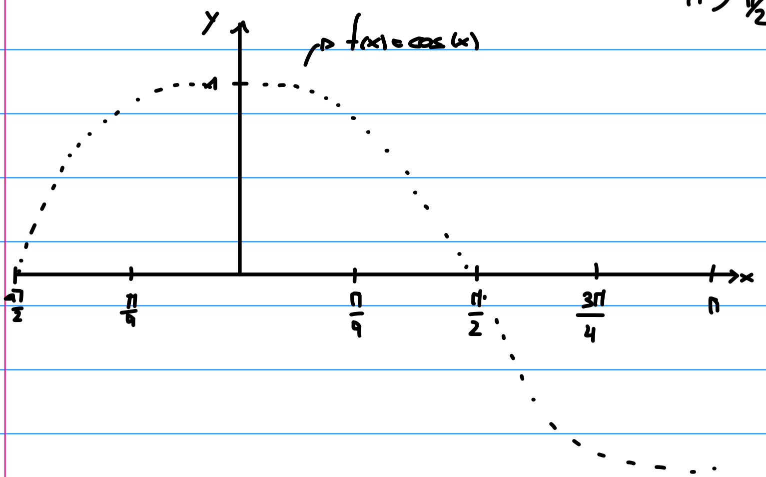
      ii. Now write the five Fourier coefficients as <u>completely specified</u> integrals, but <u>do not evaluate those integrals</u>, just leave them as unevaluated integrals in the answer.

The Fourier series of a function $f(x)$ on the interval $[\frac{\pi}{2},\frac{\pi}{2}]$ is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x)dx \qquad \begin{cases} a_n = \dfrac{2}{\pi}\displaystyle\int_{-\pi/2}^{\pi/2} f(x)\cos(nx)dx \\ b_n = \dfrac{2}{\pi}\displaystyle\int_{\pi/2}^{\pi/2} f(x)\sin(nx)dx \end{cases}$$



There is an obvious symmetry around the OY axis, that tells us that the parity of the function is even. Indeed, the definition of an even function $f(x)=f(-x)$ is satisfied since $\cos(x)=\cos(-x)$.

Because of this parity, all the terms related to sine function vanish (b_n), this is because sine is an odd function, and a composition of odd functions cannot generate an even function and viceversa. When you integrate an odd function over a symmetric interval, the result is zero. This is also why $a_0 \neq 0$.

Now, we have to account for how the a_n will work. If we take a look at the a_n definition on the previous page, we see that we have to analyze how $\cos(x)\cos(nx)$ behaves on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By invoking the orthogonality property of the cosine function:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L}\,dx = \begin{cases} L & \text{if } m=n\ (\neq 0) \\ 0 & \text{if } m \neq n \end{cases} \iff \int_{-\pi/2}^{\pi/2} \cos(2nx)\cos(2mx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$$

And by renaming $2n$ as K we get $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(Kx)\cos(2mx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } K = 2m \\ 0 & \text{if } K \neq 2m \end{cases}$

with $k = 1$, then m clearly has to be an even number, making odd numbers vanish.

Having this in mind: $\boxed{f(x) \simeq a_0 + a_2\cos(2x) + a_4\cos(4x) + a_6\cos(6x) + a_8\cos(8x)}$

We already showed the integral for a_0, explicitly for a_2, a_4, a_6, a_8:

$$\boxed{\begin{array}{ll} a_2 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\cos(2x)\,dx & a_4 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\cos(4x)\,dx \\ a_6 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x)\cos(6x)\,dx & a_8 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x)\cos(8x)\,dx \end{array}}$$

4. <u>Alternate Form of Fourier Series:</u>

Show that the Fourier series for an arbitrary piecewise continuous function defined on $(-L, L)$ can be written as $f(x) = \sum_{n=0}^{\infty} c_n \cos(\frac{n\pi x}{L} + \phi_n)$. Show all steps of your logic/reasoning. Derive expressions for the c_n's and the ϕ_n's in terms of the a_n's and the b_n's we've defined in class.

The expression for a Fourier series of a function f is usually expressed as an infinite sum involving coefficients a_n, b_n as we have previously seen, but it can also be expressed by the coefficients c_n and ϕ_n

Using the trig identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ we can show it

$$\sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L} + \phi_n\right) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)\cos(\phi_n) - c_n \sin\left(\frac{n\pi x}{L}\right)\sin(\phi_n)$$ from where

we can see how $\begin{cases} c_n\cos(\phi_n) = a_n \\ c_n\sin(\phi_n) = b_n \end{cases} \iff a_n \sec\phi_n \sin\phi_n = b_n \iff a_n \tan\phi_n = b_n$

$\iff \boxed{\phi_n = \arctan\left(\frac{b_n}{a_n}\right)}$ If we square an add the eqtns. $\boxed{c_n = \sqrt{a_n^2 + b_n^2}}$

3. Find a Fourier series for the function $f(x) = x$ on interval [-3, 7] with repeat distance 10.

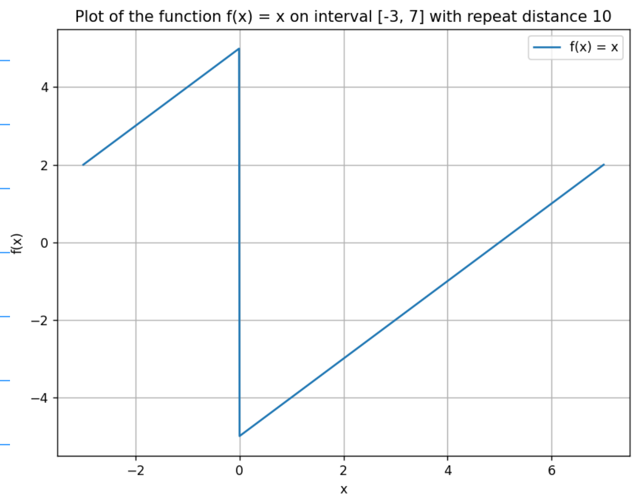
Do this according to the following instructions:

a. Sketch a plot of the repetitive function that the asked-for Fourier series is supposed to converge to. Label points on your abscissa axis. Does that *repetitive* function have a definite parity about the origin?

b. Now write out the first five terms of the series without including any terms that are "obviously" equal to zero. For this follow the instructions below:

i. Do this first leaving the Fourier coefficients as "generic" (e.g., like "a_2" or "b_7"). If you concluded that no terms are "obviously" zero, so indicate and justify that conclusion. If you concluded that some terms are "obviously" zero, justify that conclusion, not by simply commenting that "it works out that way," but by citing a general principle or familiar general result.

ii. Now write the five Fourier coefficients as <u>completely specified</u> integrals, but <u>do not evaluate those integrals</u>, just leave them as unevaluated integrals in the answer.



As we can see from the plot/sketch the function is clearly symmetric, being its parity odd.
Similarly to exercise 2, there will not be any a_n, as an odd function cannot be represented by the composition of even functions, and $\cos(x)$ is even.

There will then be no term a_0. All the terms will involve b_n. As the primitives of $x\cos(nx)$ are not zero regarding the number $n \in \mathbb{N}$, no other terms will vanish. Then,

$$\boxed{f(x) \simeq b_1 \sin\left(\frac{\pi}{10}x\right) + b_3 \sin\left(\frac{3\pi}{10}x\right) + b_5 \sin\left(\frac{\pi}{2}x\right) + b_7 \sin\left(\frac{7}{10}\pi x\right) + b_9 \sin\left(\frac{9}{10}\pi x\right)}$$

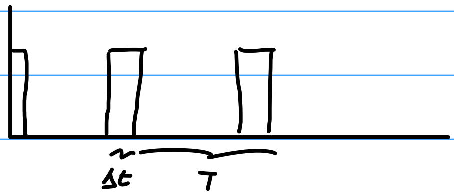
$$b_1 = \frac{1}{5}\int_{-5}^{5} x\sin\left(\frac{\pi}{5}x\right)dx \qquad b_3 = \frac{1}{5}\int_{-5}^{5} x\sin\left(\frac{3}{5}\pi x\right)dx \qquad b_5 = \frac{1}{5}\int_{-5}^{5} x\sin(\pi x)\,dx$$

$$b_7 = \frac{1}{5}\int_{-5}^{5} x\sin\left(\frac{7}{5}\pi x\right)dx \qquad b_9 = \frac{1}{5}\int_{-5}^{5} x\sin\left(\frac{9\pi}{5}x\right)dx$$

5. Laser Pulsing and the Fourier Bandwidth Theorem:

a. An experimenter builds a pulsing circuit for a laser that previously put out continuous, essentially monochromatic green light, say at $\lambda = 530$ nm. If the pulse width is now 1.5 ps (1.5×10^{-12} sec.) and the repetition time is 1 microsecond, can the light that the laser emits still be called green? Why or why not? Explain fully. Be as quantitative as you can without explicitly evaluating a Fourier series expansion. Hint: You may want to take a look at the wavelength ranges for different colors of visible light.

The pulses might look like this:
with $\Delta t = 1'5 ps$ $T = 1 \mu s$



Again, the even parity forces the series to be formed only with cosines, and thus only the a_n exists. From the classnotes, we know that by obtaining the explicit expression for a_n, those only depend on the relation $\frac{\Delta t}{T}$, which in this case it's $1'5 \cdot 10^{-6}$ $\left(a_n = \frac{2}{n\pi} \sin\left(n\pi \frac{\Delta t}{T}\right)\right)$ We can see this as a composition of $\frac{2}{n\pi}$ and the sine function. The first part will make the amplitude decay inversely proportional to n. Moreover, the first part will never be zero, but the sine cancels when the argument is πm with $m \in \mathbb{N} \cup \{0\}$.
So $\sin\left(n\pi \frac{\Delta t}{T}\right) = 0 \iff n\pi \frac{\Delta t}{T} = \pi \iff n = \frac{T}{\Delta t}$ point at which a_n will be equal to zero. For our case, the first zero will apear at $n = 7 \cdot 10^5$
In terms of frequency, we know $\Delta f \Delta t \approx 1 \iff \Delta f \approx \Delta t^{-1}$
and since green light has a frequency of $5'66 \cdot 10^{14}$ hz but $\Delta f = 6'66 \cdot 10^{11}$ hz, this means that the bandwidth will be very small compared to that of green light $5'66 \cdot 10^{14} \pm 3'33 \cdot 10^{11}$ hz So we can say that the window will be smaller than $5'65 \cdot 10^{14}$ hz to $5'67 \cdot 10^{14}$ hz. By a quick search of google we get the image below which tells us that the light will still be well within the expected range for green light.

Color	Wavelength	Frequency	Photon energy
Violet	380–450 nm	668–789 THz	2.75–3.26 eV
Blue	450–495 nm	606–668 THz	2.50–2.75 eV
Green	495–570 nm	526–606 THz	2.17–2.50 eV
Yellow	570–590 nm	508–526 THz	2.10–2.17 eV
Orange	590–620 nm	484–508 THz	2.00–2.10 eV
Red	620–750 nm	400–484 THz	1.65–2.00 eV

b. Consider a pulsed laser with pulse width 1 femtosecond (10^{-15} sec.). If the repetition time is 1 nanosecond and if the color of the unpulsed (i.e., running continuously "on") light output is green, is the pulsed output light still definitely green? Can it be said to be another color? If so, which? If not, why not? Explain fully. Be as quantitative as you can without explicitly evaluating a Fourier series expansion. $\Delta t = 10^{-15} s$

Analogous to the previous exercise, $f < \frac{1}{\Delta t} = 10^{15} hz$ will be most prominent, $\Delta f \Delta t \approx 1 \Leftrightarrow \Delta f = \frac{1}{\Delta t} = 10^{15} hz$ and green light has a frequency of $5'66 \cdot 10^{14} hz$ so we get a range of $5'66 \cdot 10^{14} \pm 5 \cdot 10^{14} hz$
In this case the range is $0'66 \cdot 10^{14} hz$ to $10'66 \cdot 10^{14} hz$ which by looking at the table we used before we can see how this range spans all visible light ranges, and even leaks into infrared/ultraviolet. It is not clear which wavelength will dominate, but I would say that humans would still appreciate it as green light as its amplitude is the greatest and taking in account how eye receptors work (not all frequencies are detected and most colors are interpreted as a superposition of frequencies) I would say that we would still call that light green, although frequency wise the window is way larger.