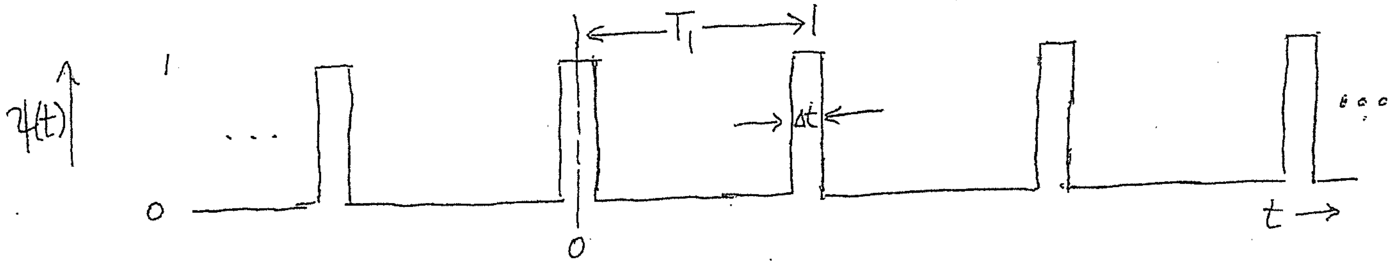
Physics 251 - 19^{th} Class - Tuesday March 19, 2024

Fourier Bandwidth Theorem, continued

A. Recap*: We considered Fourier analyzing the repetitive "flat-top" function



according to (with the symmetric choice of ordinate axis shown)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) \qquad \omega_1 = \frac{2\pi}{T_1}$$

* K-text section 12.1

where $$a_n = \frac{2}{T_1}\int_{t_0}^{t_0+T_1} f(t)\cos(n\omega_1 t)\,dt\,;$$

we found

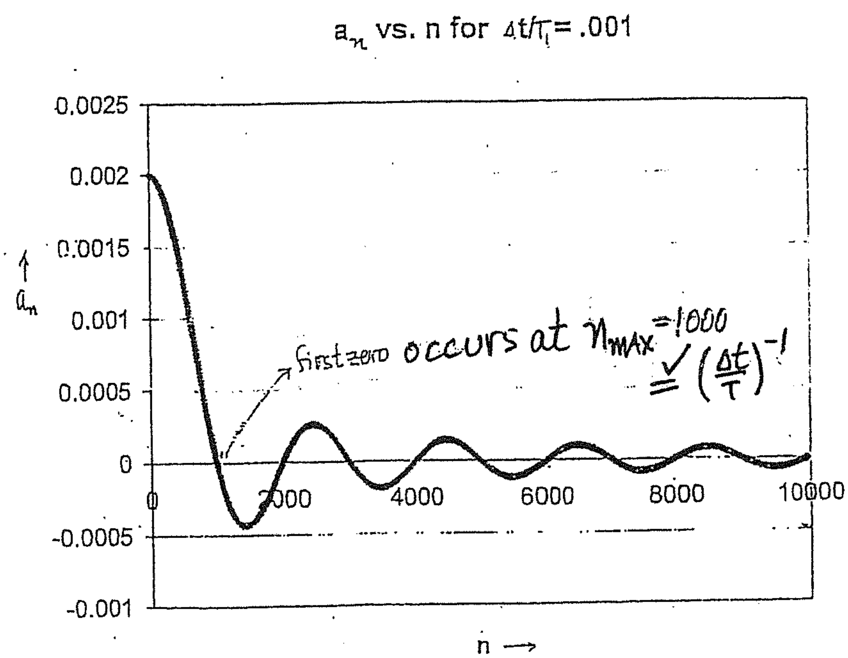
$$a_n = \frac{2}{n\pi}\sin(\pi n f_1 \Delta t)$$

$$= \frac{2}{n\pi}\sin\left(n\pi\frac{\Delta t}{T_1}\right)$$

which has its "first zero" at

(1) $n = \text{"}n_{MAX}\text{"} = \left(\frac{\Delta t}{T}\right)^{-1}$, or at

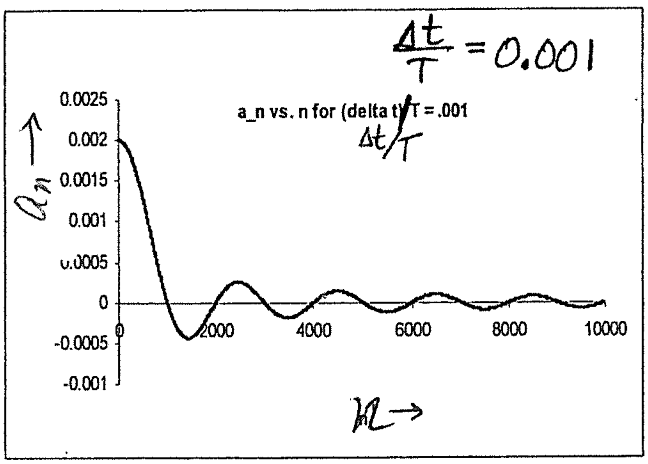
2) $f_{max} = n_{max} f_1 = \frac{1}{\Delta t}$

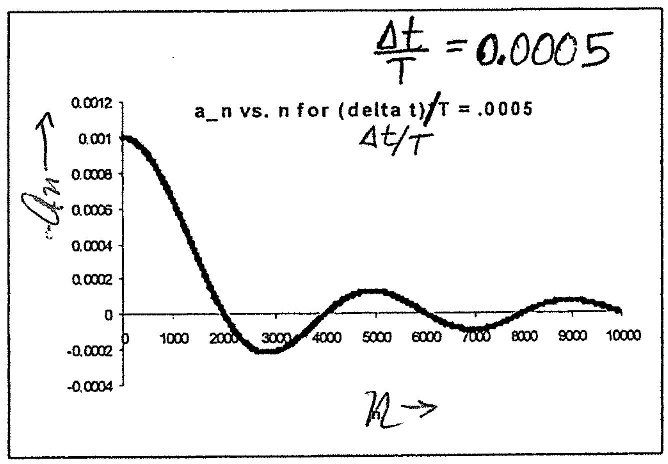


A check:

Suppose we (again) halve the on-fraction $\frac{\Delta t}{T}$. Then, if we plot a_n vs. n, we should see that $n_{MAX} = n_{1ST\ zero}$ doubles.

The following plots show this:





(Do bear in mind that, although the curves look continuous, they actually are just densely packed "dots" since a_n is defined only for integer values of n.)

On the other hand, if I plot a_n vs frequency [i.e, "a_f vs. f"], the first zero of "a_f" occurs at $f = \frac{1}{\Delta t}$, regardless of the repeat period T of the function.

B. Cases of Fractional On-Time Small ($\ll 1$); Example of Fourier Bandwidth Theorem (*)

We now consider a very important class of cases - that for which $\Delta t \ll T_1$. This would be the case if, for example, you clap your hands once per second and $\psi(t)$ is the resulting sound - each clap makes a sound that lasts for about 1 ms or so.

(Cases of narrow pulses that are very separated cf pulse width).

Then the condition for the first zero,

$$n_{MAX} \pi f_1 \Delta t = \pi \Rightarrow n_{MAX} f_1 \Delta t = 1 \Rightarrow n_{MAX} \frac{\Delta t}{T_1} = 1$$

tells us that n_{MAX} is $\gg 1$ (since $\frac{\Delta t}{T_1} \ll 1$).

This means that $f_{MAX} \gg f_1$. Thus, defining the "Frequency bandwidth"

$$\Delta f \equiv f_{MAX} - f_1 \approx f_{MAX}$$

K-text, section 12.3

Then (2) becomes

(3) (3a) $\boxed{\Delta f \approx \frac{1}{\Delta t}}$ or (3b) $\boxed{\Delta f \, \Delta t \approx 1}$

Fourier Bandwidth Theorem* Result, Function of time, Special Case (*sometimes called the "Classical Bandwidth Theorem")

(3) is an example of what is called the "Fourier Bandwidth Theorem". As we will see, as an order of magnitude relation it is quite general, applying to most sharply peaked repetitive func[tions]

From eqn. (3) in the form

$$\Delta f \approx \frac{1}{\Delta t}$$

we see that the frequency bandwidth Δf and the "pulse width" Δt are inversely proportional to each other. That inverse proportionality of the frequency bandwidth with Δt is very general for repetitive peaked functions for which $\Delta t \ll T$ (T = repeat time) - far more general than just the example considered.

Another way of writing eqn. (3) is:

$$\Delta f \, \Delta t \approx 1$$

$$\Rightarrow (2\pi)\Delta f \cdot \Delta t \approx 2\pi$$

(3c) $$\Rightarrow \boxed{\Delta \omega \cdot \Delta t \approx 2\pi}$$

(next page →

B. The Effects of the Approximation of Truncating the Series

[K-text, sect. 12.2]

Suppose that we are asked to sum the truncated series approximation to our repetitive square-topped function f(t) – i.e., suppose we calculated

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{n_{MAX}} a_n \cos(n\,\omega_1 t) \qquad \text{(truncated series approximation)}$$

How (at least semiquantitatively) would g(t) differ from f(t)?

Answer – In two ways:

1. The sudden rises and falls of f(t) [really jumps discontinuities] would appear in g(t) as more gradual, somewhat rounded rises and falls, and

2. The "tops" and the long "bottoms" between peaks would be reproduced with some "wiggle".

We saw both of these effects in considering sums of

(next page →)

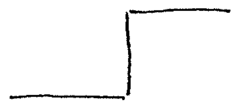
only the first several terms of our very first Fourier example, namely the function  ; which had $\frac{\Delta t}{T} = 0.5$

Figure from Principles of Physical Optics by Charles Bennett (Wiley)

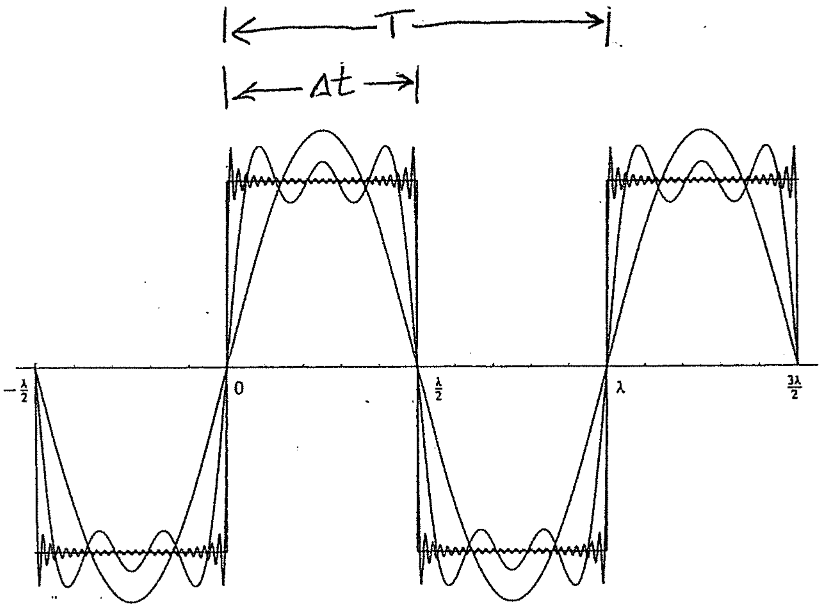


Figure 5.A.2. Fourier expansions of Example 5.15 plotted over two full cycles of the fundamental. In the three expansions, one uses the first nonzero term, another uses three nonzero terms, and the best approximation uses the first 25 nonzero terms in the series.

Now we consider the important class of cases $\Delta t \ll T$.

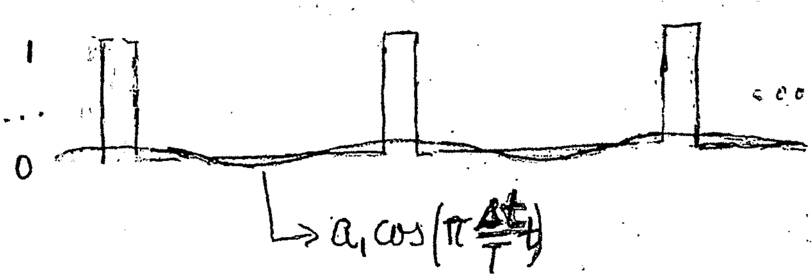
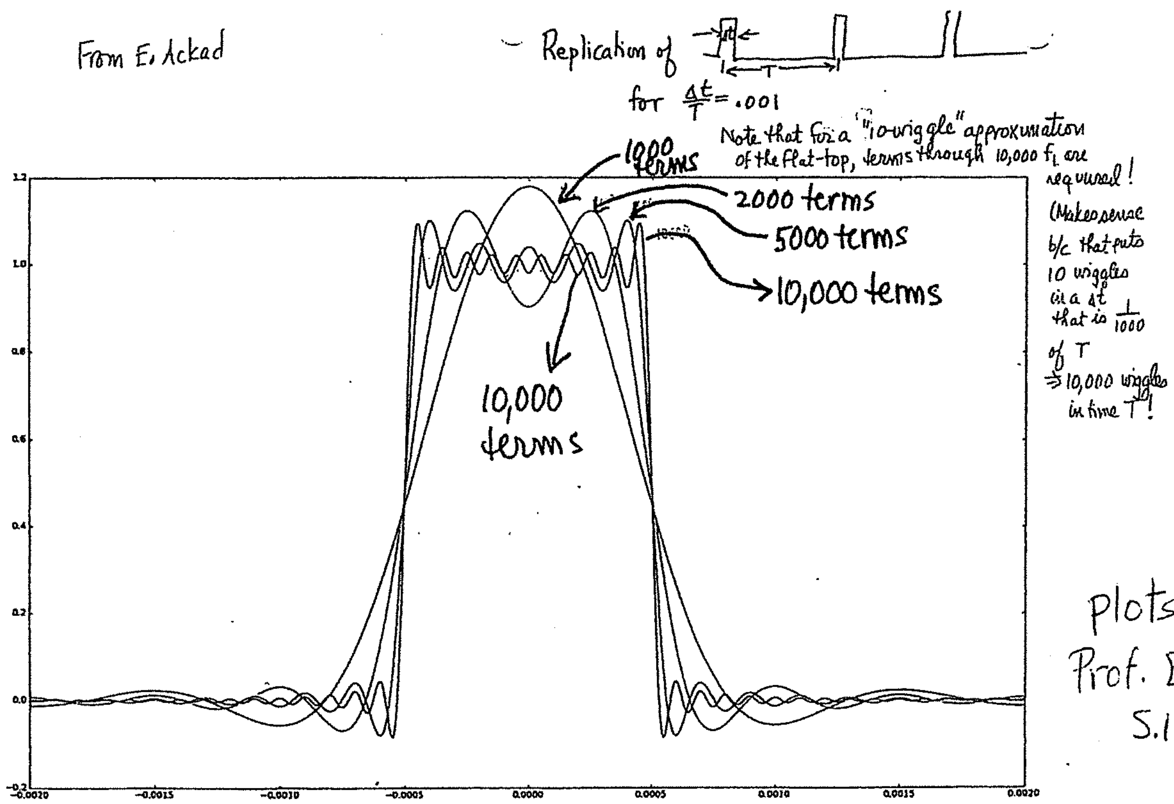


Fig. 11.14. Schematic of repetitive function and lowest frequency Fourier component. The amplitude of the lowest Fourier component is not drawn to scale. [It is greatly exaggerated — a_1 is actually (for case $\frac{\Delta t}{T} = .001$) only $= 0.002$, whereas the height of the peaks of $f(t) = 1$.]

→ The physical reason that the fundamental Fourier component has such small amplitude relative to the amplitude of the peaks of the parent function is that many, many Fourier terms are needed to effect the required relatively lengthy intervals of destructive interference between the peaks – see figure, next page.

To support this conjecture, the following figure (courtesy of S.I.U.E. Prof. E. Ackad) shows ~~several~~ truncated Fourier series



plots courtesy Prof. E. Ackad. S.I.U.E.

approximations to the region of a peak; we see that by including terms only out to the first zero of a_n, → the amplitude of each peak is reproduced, but the flat top and the rise and fall are sacrificed to leave just a smoothly rounded shape with the same average width (Δt); qualitatively we expect this because <u>high frequency Fourier terms are needed to reproduce "sharp fine-structure".</u>

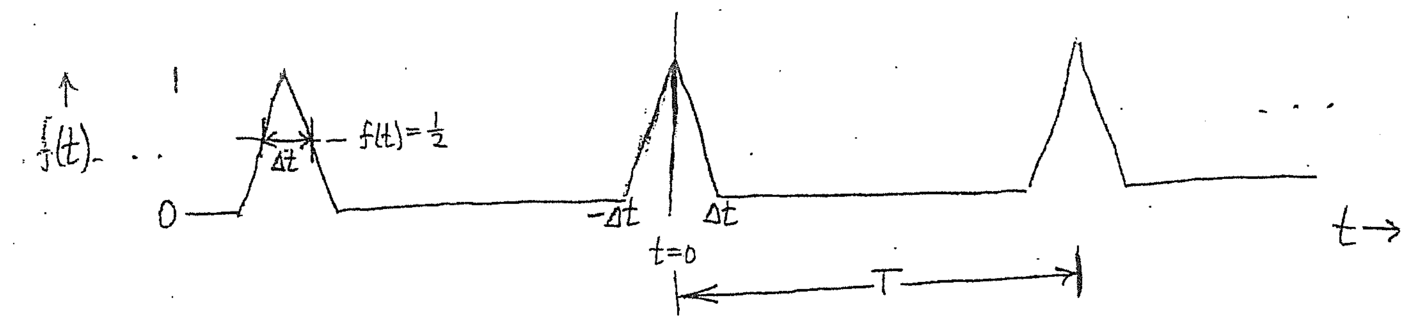
"Take-Away Lesson": The lower Fourier components define the rough overall shape of the function $f(t)$. "Fine-structure" (i.e, changes in $f(t)$ that occur on a short time scale - e.g., sharpness of rises, narrowness of very narrow peaks, etc.) is provided by the higher frequency Fourier terms (Because they "wiggle" rapidly). Thus, truncating the series "rounds" (to at least some extent) fine structure.

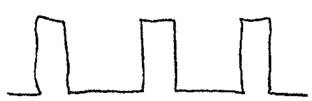
Example: Telephone Transmission:

The useable frequency bandwidth for standard telephone transmission is $\Delta f \approx 2000$ Hz. That is adequate for pure voice transmission but not adequate for good music transmission for two reasons: 1. High frequency Fourier components are cut out, and 2. Sudden changes in sound level (e.g. sudden sharp strike of cymbals, etc.) are "blurred" due to inadequate time resolution since $\Delta t \gtrsim \frac{1}{\Delta f} \gtrsim 0.5$ ms (in fact, it's a few ms).

C. Toward The Generality of the Fourier Bandwidth Theorem*

Let's look at another example – the repeating "triangle function":



If $T \gg \Delta t$, then, on the scale of T, the difference between the  function and the repeating triangle is just a matter of "fine structure".

Of course, this is really not much more than "educated conjecture". A priori, we don't know what the spectral plot for the triangle function even looks like. Perhaps it doesn't have a "main region" of lower frequencies that is clearly defined by the first zero? If that were so, then it might be hard even to usefully define the bandwidth $\Delta\omega$.

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* K-text, section 12.4

Of course, one way to find out is to actually perform the Fourier analysis. Let's try this:

Positioning the ordinate axis as shown, we have a Fourier series for f(t)

$$f(t) = \frac{a_0}{2} + a_1 \cos \omega_1 t + a_2 \cos \omega_2 t + \dots,$$

and our function is

$$f(t) = 0 \qquad -\frac{T}{2} < t < -\Delta t$$

$$f(t) = 1 - \frac{|t|}{\Delta t} \qquad -\Delta t < t < \Delta t$$

$$f(t) = 0 \qquad \Delta t < t < T/2$$

and the periodic repetition of this.

Expanding in a Fourier cosine series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \omega_1 t,$$

one finds

$$\omega_1 = \frac{2\pi}{T} \qquad \text{(as it must)}$$

$$\frac{a_0}{2} = \frac{1}{2}\left(\frac{2\Delta t}{T}\right) = \frac{\Delta t}{T} \quad = \text{average of } f(t)$$

and

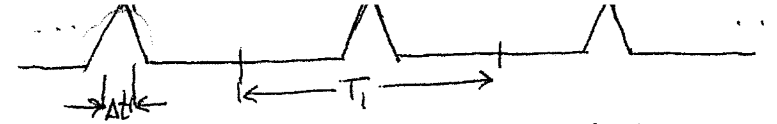
$$a_{n\,(n\neq 1)} \propto \left[\frac{-1}{2\pi^2} \frac{T}{2\Delta t} \frac{1}{n^2} \left[\cos\left(n\pi \frac{2\Delta t}{T}\right) + n\pi \left(\frac{2\Delta t}{T}\right) \sin\left(n\pi \frac{2\Delta t}{T}\right) - 1 \right] + \frac{1}{2\pi} \cdot \frac{1}{n} \sin\left(n\pi \frac{2\Delta t}{T}\right) \right]$$

which is complicated looking, but not difficult to plot in, e.g., Microsoft excel; the result of such a plot (for the case $\frac{\Delta t}{T} = .005$) is shown (see spectral plot, next page).

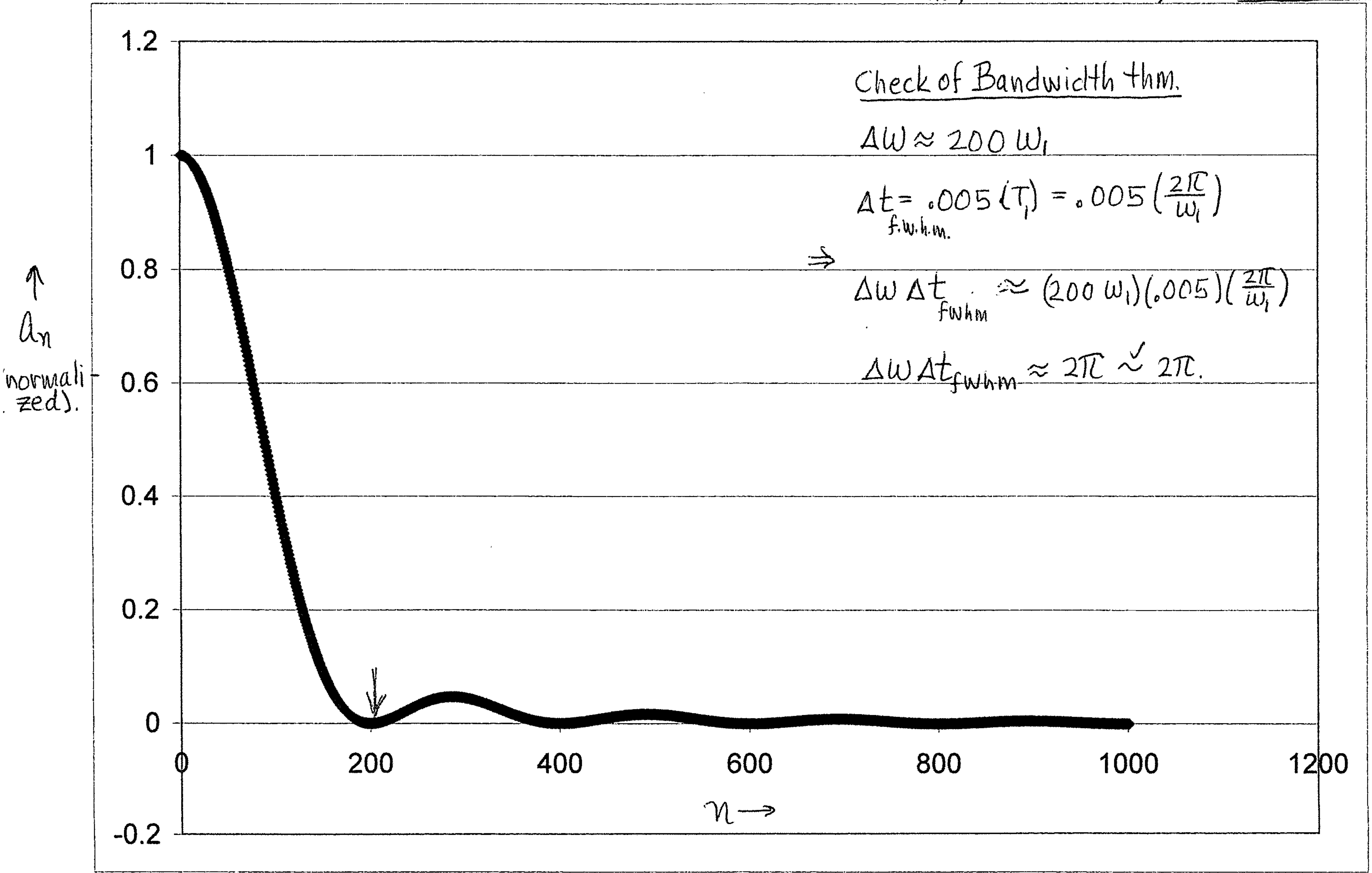
Notice, from that plot, that the "general structure" of the spectral plot is similar to that for the repeating flat-topped function. That is, there is a "main region of lower frequencies" that is clearly deliniated by the first zero in the plot. On the next page, the Bandwidth theorem is explicitly checked for this case.

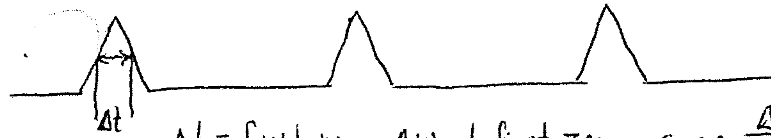
Now suppose we, say, double $\frac{\Delta t}{T}$ to .01. The bandwidth theorem is checked for this case on the following page.

Repeating triangular function

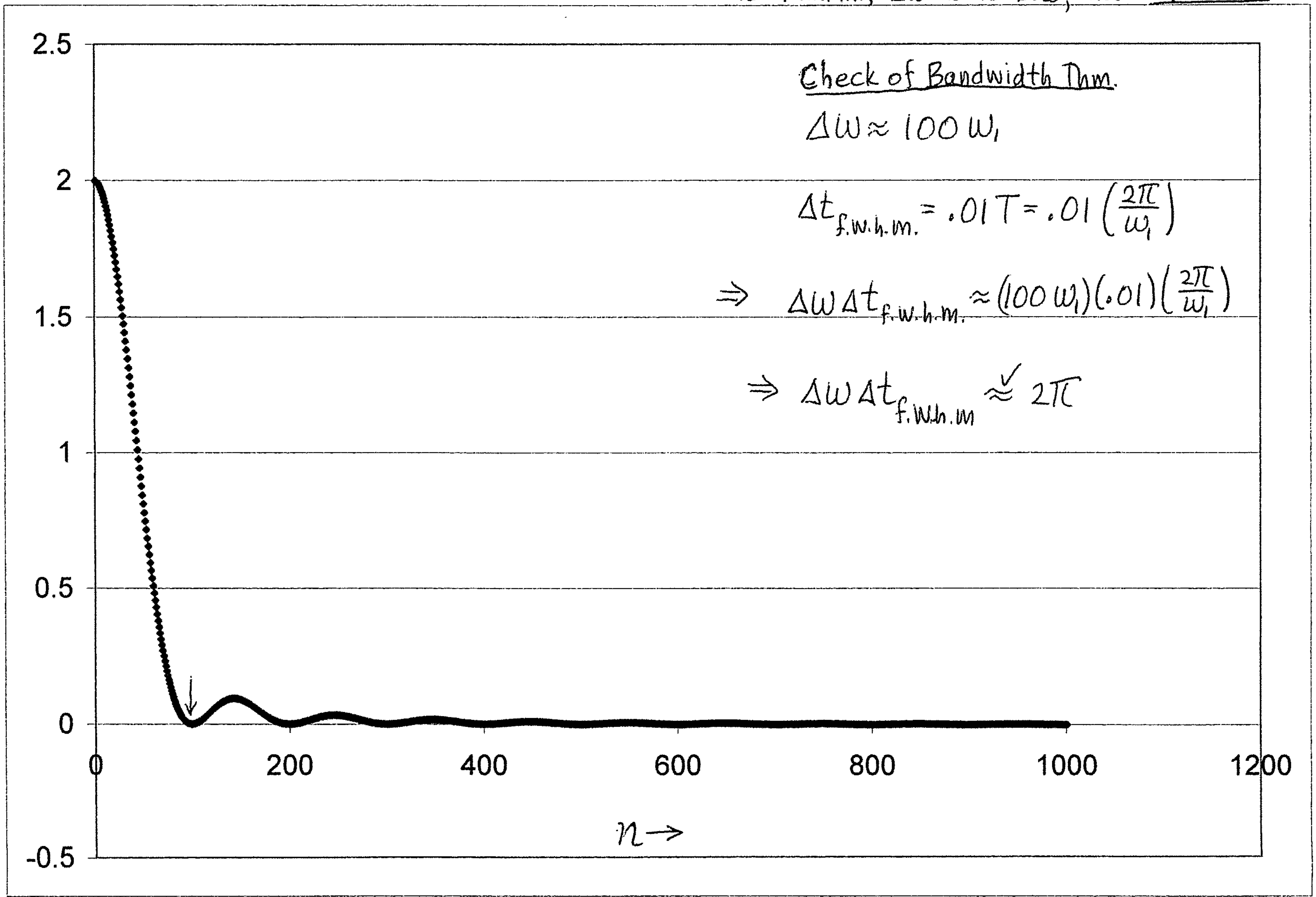


$\Delta t \equiv$ f.w.h.m., $\Delta \omega \equiv$ to first zero, case $\frac{\Delta t}{T} = .005$



Repeating triangular function 

$\Delta t \equiv$ f.w.h.m., $\Delta\omega \equiv$ to first zero, case $\frac{\Delta t}{T} = .01$



Check of Bandwidth Thm.

$\Delta\omega \approx 100\,\omega_1$

$\Delta t_{\text{f.w.h.m.}} = .01\,T = .01\left(\frac{2\pi}{\omega_1}\right)$

$\Rightarrow \quad \Delta\omega\,\Delta t_{\text{f.w.h.m.}} \approx (100\,\omega_1)(.01)\left(\frac{2\pi}{\omega_1}\right)$

$\Rightarrow \quad \Delta\omega\,\Delta t_{\text{f.w.h.m.}} \overset{\checkmark}{\approx} 2\pi$

Question: The spectral plot shown for $\frac{\Delta t}{T} = .01$ is narrower than that shown for $\frac{\Delta t}{T} = .005$. Does that mean that the frequency bandwidth $\Delta\omega$ is less for $\frac{\Delta t}{T} = .01$ than it is for $\frac{\Delta t}{T} = .005$?

Answer: **It depends.** Notice that the plots are a_n vs. $\underline{n}$, <u>not</u> a_ω vs. ω. According to the plots, $n_{max} \propto \frac{1}{(\Delta t/T)}$.

(This makes sense - recall that, for the case, we had $n_{max}\pi \frac{\Delta t}{T} = \pi \Rightarrow n_{max} \propto \frac{1}{(\Delta t/T)}$)

So, the plot of a_n vs. $\underline{n}$ got narrower when $\Delta t/T$ doubled.

However, $\Delta\omega$ depends <u>only</u> on Δt, <u>not</u> on T (as reflected by the Bandwidth Theorem $\Delta\omega\Delta t \approx 2\pi$. (Thus, if we change T, but not Δt, then Δf (and hence also $\Delta\omega$) remain as they were.

But, we could have doubled $\frac{\Delta t}{T}$ by, say, halving Δt and quartering T. Then the theorem predicts $\Delta\omega$ should double. The plots show this also - $\Delta\omega_{new} = n_{max,new}\,\omega_1$, $n_{max,new} = \frac{1}{2} n_{max,old}$, but $\omega_{1,new} = 4\,\omega_{1,old}$, so $\Delta\omega_{new} = \left(\frac{1}{2}\right)(4)\, n_{max,old}\,\omega_{1,old} \stackrel{\checkmark}{=} 2\Delta\omega_{old}$.

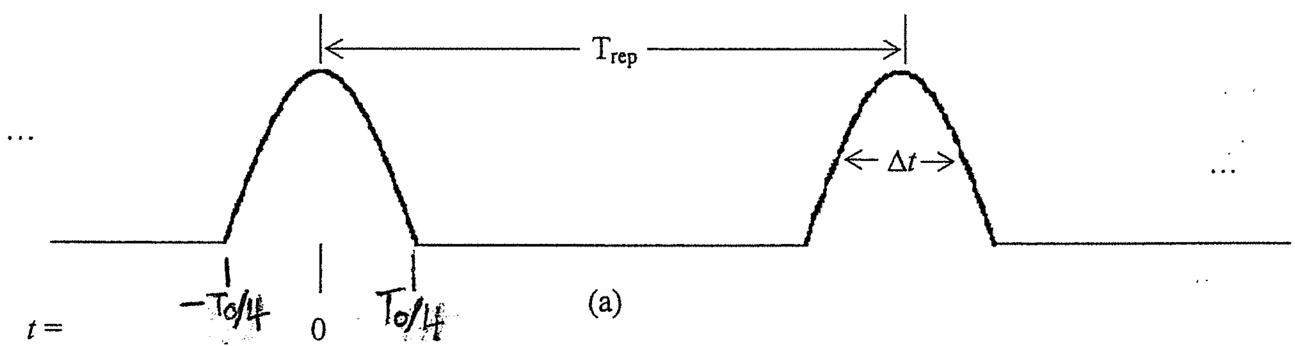
Let's now look at a third example - this time, a rounded repetitive function.

Worked Example 12.2

Investigate the Bandwidth Theorem for the repetitive pulse function

$$\frac{T_0}{4} = \frac{1}{4}\cdot\frac{2\pi}{\omega_0} = \frac{\pi}{2\omega_0}$$

(7.39)
$$f(t) = \begin{cases} 0 & t \le \pi/2\omega_0 \\ \cos(\omega_0 t), & -\pi/2\omega_0 < t < \pi/2\omega_0 \\ 0 & t \ge \pi/2\omega_0 \end{cases}$$
and periodic repetitions (Fig. 7.10 a).



(a)

(A figurative plot of $f(t)$ is shown immediately above.) $\omega_0 = 2\pi f_0$.

Do this by making plots of the Fourier spectra for the cases:

a. $f_0 = 5.5$ Hz, $f_{rep} = 1$ Hz,

b. $f_0 = 16.5$ Hz, $f_{rep} = 1$ Hz

c. $f_0 = 5.5$ Hz, $f_{rep} = 0.33$ Hz

Comment on similarities and differences of your results and check whether or not they are in accord with the predictions of the bandwidth theorem.

Worked Example **continued**

Solution:

Our Fourier series for this function is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \cos(\omega_0 t)\cos(n\omega_1 t)\,dt$$

$$= \frac{2}{T}\int_{-T_0/4}^{T_0/4} \cos(\omega_0 t)\cos(n\omega_1 t)\,dt = \frac{4}{T}\int_0^{T_0/4} \cos(\omega_0 t)\cos(n\omega_1 t)\,dt$$

where $T_0 \equiv 2\pi/\omega_0$.

By a now familiar trig identity, this is

(7.40) $$a_n = \frac{2}{T_1}\cdot\frac{1}{2}\left[\int_{-\pi/\omega_0}^{\pi/\omega_0} \cos(\omega_0 + n\omega_1)t\,dt + \int_{-\pi/\omega_0}^{\pi/\omega_0} \cos(\omega_0 - n\omega_1)t\,dt\right]$$

The integrals are straightforward, and the result is:

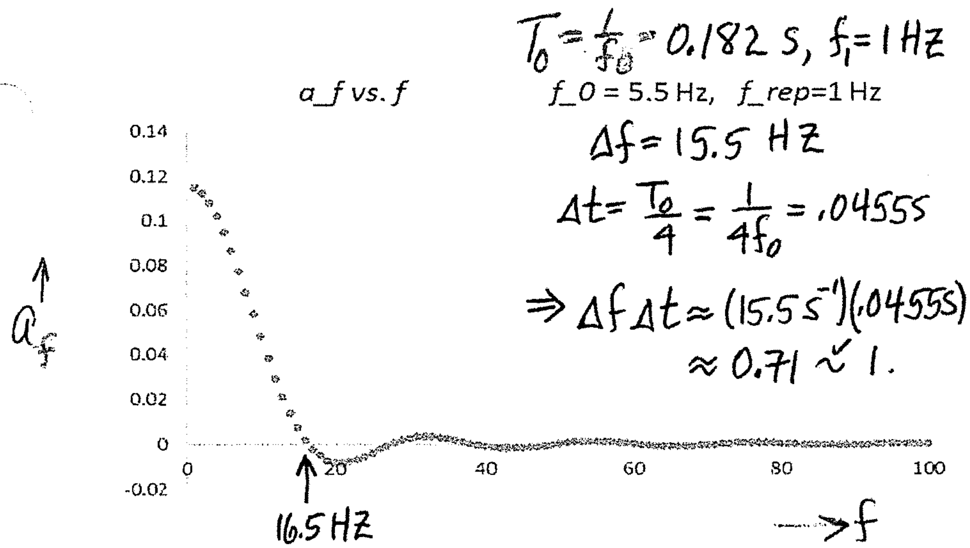
(7.41) $$a_n = \frac{2}{T}\cdot\frac{\pi}{\omega_0}\left\{\frac{\sin[(\omega_0 + n\omega_1)\frac{\pi}{\omega_0}]}{(\omega_0 + n\omega_1)\frac{\pi}{\omega_0}} + \frac{\sin[(\omega_0 - n\omega_1)\frac{\pi}{\omega_0}]}{(\omega_0 - n\omega_1)\frac{\pi}{\omega_0}}\right\}, \qquad n = 1, 2, 3,$$

.... $\frac{a_0}{2}$ = average of ½ cycle of $\cos(\omega_0 t)$ x on-time fraction → $\frac{a_0}{2} = \frac{1}{\sqrt{2}}\cdot\frac{T_0}{T_{rep}}$

Fig. 12.4.2, continued:

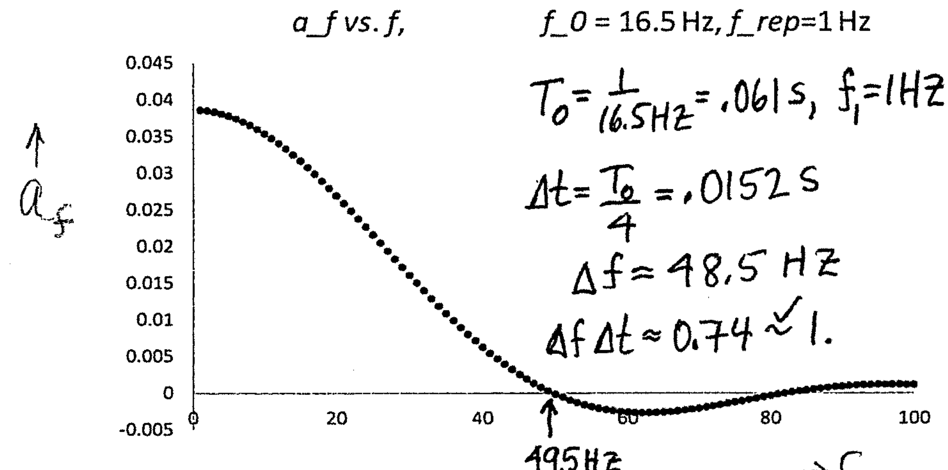
Fourier coefficients a_n vs. $f_n = nf_1$ for repetitive function of eq (12.4.2). The constant coefficient $a_0/2$ is not included.

(a) Shown on p. 11-66. A section of the repeating parent function $f(t)$. T_{rep} is the repetition period.



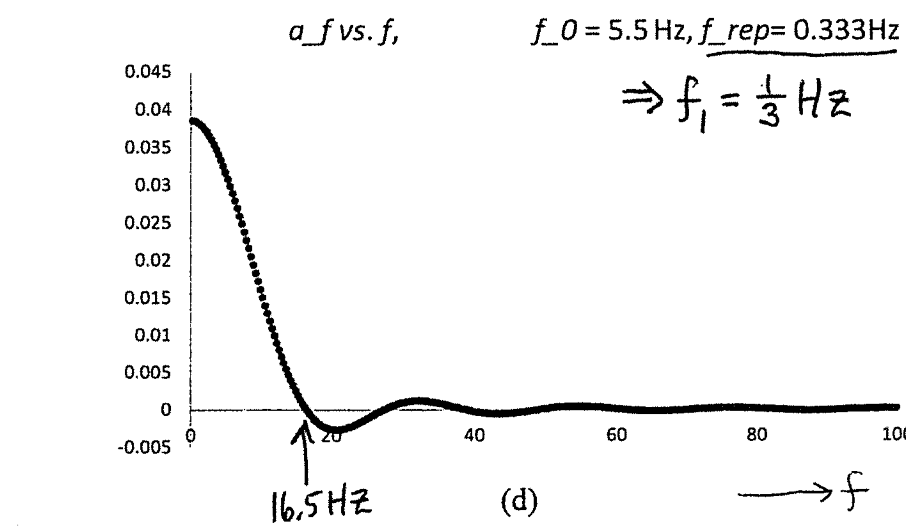
(b)

(b) Case $f_0 = 5.5$ Hz, $f_{rep} = 1/T_{rep} = 1$ Hz. The fundamental f_1 is 1Hz. The first zero of the envelope function for the Fourier coefficients occurs near f=16.5 Hz, so Δf = 15.5Hz. The pulse width $\Delta t = \frac{T_0}{4} = \frac{1}{4f_0} \approx 0.0455$ s, thus $\Delta f \Delta t \approx (15.5 s^{-1})(0.0455 s) \approx 0.71$; agrees with the bandwidth theorem.



(c)

(c) Case $f_0 = 16.5$ Hz, $f_{rep} = 1/T_{rep} = f_1$ = 1 Hz. As in (b), f_1 = 1Hz; however, now $\Delta t = \frac{T_0}{4} = \frac{1}{4(16.5)} \approx 0.0152$ s. Compared to the case of part (b), we have tripled f_0 and thus Δt is 1/3 of what it was. Thus, from the Bandwidth Theorem, we expect Δf to be approximately tripled. Reading from the plot, the first zero occurs at $\approx$3x16.5 = 49.5 Hz, so $\Delta f \approx 48.5$ Hz, confirming this. Thus, $\Delta f \Delta t \approx (48.5 s^{-1})(0.0152 s) \approx 0.74$ in agreement with the BW Theorem.



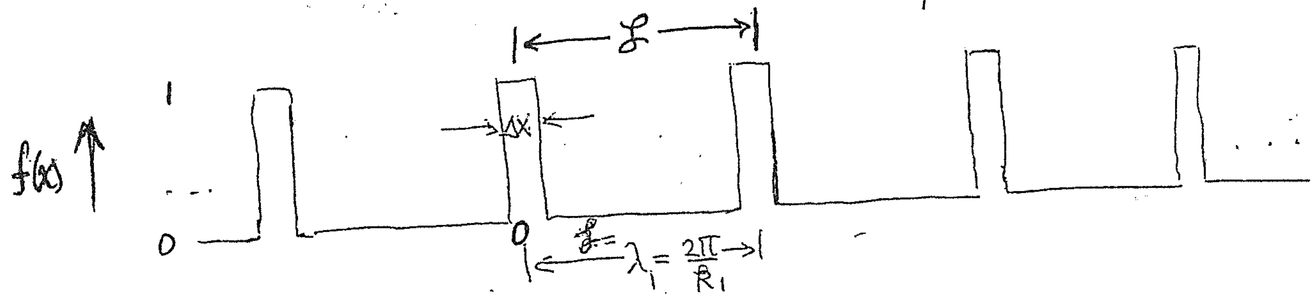
(d)

(d) Case $f_0 = 5.5$ Hz, $f_{rep} = 1/T_{rep} = f_1$ = 1/3 Hz. As in part (b), Δf = 15.5Hz and $\Delta t \approx 0.0455$ s. Since Δt has not changed, we expect the shape of the plot and the frequency of the first zero are the same as in parts (b). However, since Trep has tripled, f_1 is only 1/3 of its value in part (b), i.e., 0.333 Hz. Therefore, we expect the first zero 3 x 16.5 = 49.5f_1; thus the density of points in plot (d) is larger than that of plot (b).

The plots in parts (b), (c) and (d) are all normalized with the same arbitrary normalization constant.

D. Bandwidth Theorem for Repetitive Spatial Functions (in one dimension) [K-text, sect. 12.1]

We've been discussing the Fourier Bandwidth theorem, $\Delta\omega\Delta t \approx 2\pi$, for sharply peaked repetitive functions of time; of course, the same sort of result holds for sharply peaked repetitive functions of x. For example, if we consider a repetitive square-topped function of x, stipulating that $\mathcal{L} \gg \Delta x$,



Then, choosing the origin as shown, f(x) can be expanded in a Fourier cosine series. To help with this, we recall our "dictionary"

$$t \leftrightarrow x, \quad T \leftrightarrow \mathcal{L}, \quad \frac{2\pi}{T_1} = \omega_1 \leftrightarrow \frac{2\pi}{\lambda_1} = \frac{2\pi}{\mathcal{L}} = k_1,$$

i.e, in general, $\omega \leftrightarrow k$, $\Delta t \leftrightarrow \Delta x$, $\Delta\omega \leftrightarrow \Delta k$.

Thus, for the Fourier series,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) \rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nk_1 x)$$

So, ~~the~~ result for the Fourier coefficients for the flat-topped function of time, f(t), namely

$$a_n = \frac{2}{n\pi} \sin\left(n\pi \frac{\Delta t}{T}\right)$$

goes straight over, for the flat-topped function $f(x)$, onto

$$a_n = \frac{2}{n\pi} \sin\left(n\pi \frac{\Delta x}{\mathcal{L}}\right)$$

and the frequency-time Bandwidth theorem result,

$$\underline{\Delta\omega \cdot \Delta t \sim 2\pi} \quad \text{[Bandwidth Thm., Time Domain]}$$

goes straight over onto

$$\boxed{\Delta k \cdot \Delta x \sim 2\pi} \quad \text{[1-d Bandwidth Thm., Spatial Domain]}$$

where $\Delta k = k_{MAX} - k_1 \approx k_{MAX}$

is the "range of important wavenumbers"* in the Fourier series.

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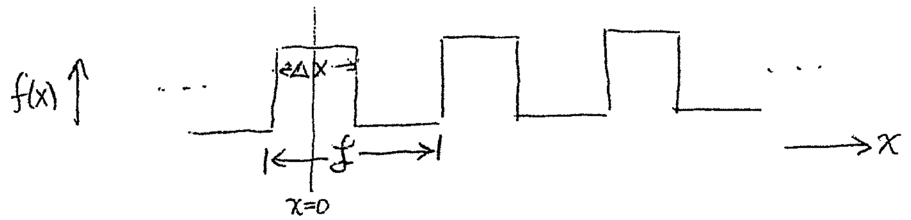
* $n_{MAX} = \left(\frac{\Delta t}{T}\right)^{-1} \longrightarrow n_{MAX} = \left(\frac{\Delta x}{\mathcal{L}}\right)^{-1}$;

$f_{MAX} = n_{MAX} f_1 = \frac{T}{\Delta t} \longrightarrow k_{MAX} = n_{MAX} k_1 = \frac{\mathcal{L}}{\Delta x} k_1$

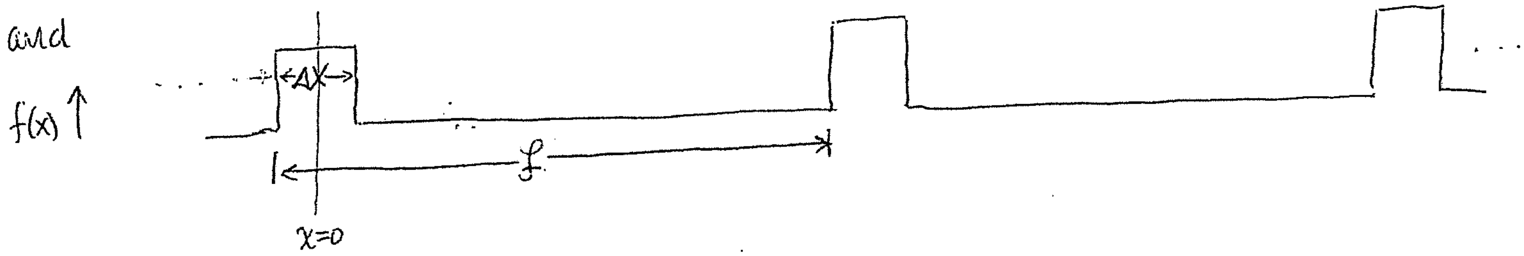
$$\left.\begin{array}{l} \text{Since } n_{MAX} \propto \frac{T}{\Delta t},\ f_1 \propto \frac{1}{T},\ n_{MAX} f_1 = f_{MAX} \propto \frac{1}{\Delta t} \Rightarrow \underline{\Delta f \propto \frac{1}{\Delta t}} \\ \text{Since } n_{MAX} \propto \frac{\mathcal{L}}{\Delta x},\ k_1 \propto \frac{1}{\mathcal{L}},\ n_{MAX} k_1 = k_{MAX} \propto \frac{1}{\Delta x} \Rightarrow \underline{\Delta k \propto \frac{1}{\Delta x}} \end{array}\right\} \text{Bandwidth Thm. Results.}$$

A "Review" Example

Consider a repetitive square f(x) that changes its repetition distance:



and



In both cases Δx is the same, but the repeat distance $\mathcal{L}$ is different. What effect should this have on the Fourier analysis and on the Bandwidth in wavenumber Δk?

Now let us answer the first question. In each case, the Fourier representation is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\underbrace{n k_1}_{k_n} x)$$

where

$$\lambda_1 = \mathcal{L}$$

so $$k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi}{\mathcal{L}}, \quad k_n = n k_1 = n\left(\frac{2\pi}{\mathcal{L}}\right)$$

Thus, the lowest wavenumber is inversely proportional to the repeat distance.

Also, the value of k_n for $n =$ "n of the first zero" (of a_n), call it "k_{MAX}" is the same in the two cases (since, if $\mathcal{L} >> \Delta k$, k_{MAX} is essentially equal to Δk, and the Bandwidth Theorem says $\Delta x \, \Delta k =$ unchanged, and we didn't change Δx).

Note: The numerical value of k_{MAX} is the same. However, the corresponding n-value ("n_{MAX}") is <u>not</u> the same since

$$n_{MAX} \equiv n_{\text{of 1ST zero of } a_n} = \left(\frac{\Delta t}{T}\right)^{-1} \longleftrightarrow \left(\frac{\Delta x}{\mathcal{L}}\right)^{-1}$$ and we've changed $\mathcal{L}$! (In fact, then $n_{MAX} \propto \mathcal{L}$!)

The bandwidth theorem says

$$\Delta x \, \Delta k \approx 2\pi$$

and does not refer to $\mathcal{L}$. Thus, the bandwidth in wavenumber, $\Delta k = \frac{2\pi}{\Delta x}$ $\longrightarrow$ is the same regardless of the repeat distance as long as the pulse width Δx is the same.

Of course, all these statements are exact analogs of statements we already deduced for the case of repetitive function of time. The plots on the next page and the associated captions show, in more detail, what is going on in this example.

$$k_{1ST\ zero} = \frac{2\pi}{\Delta X}, \quad k_1 = \frac{2\pi}{\mathcal{L}}$$

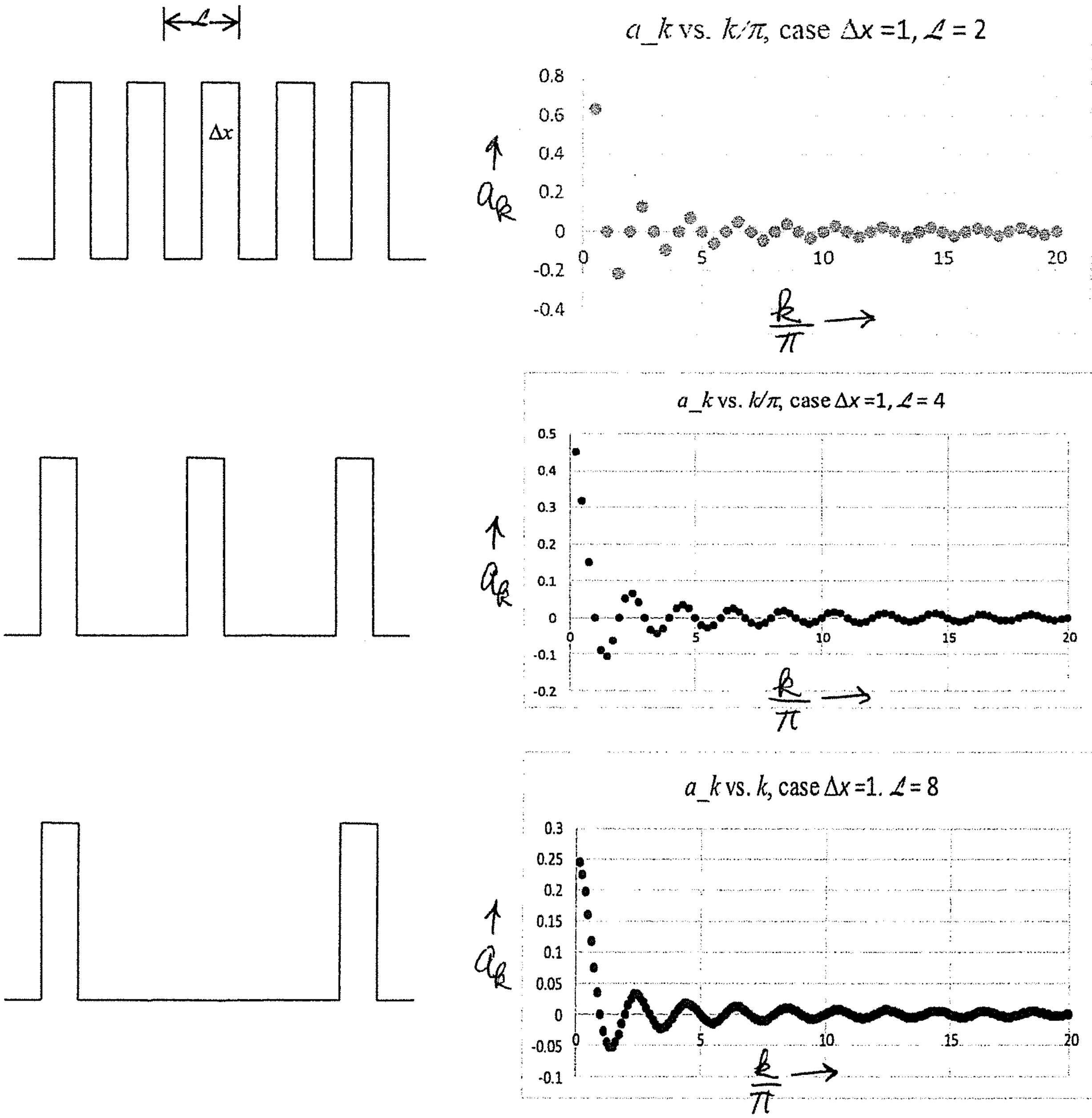


Fig. 11.16 Fourier coefficients a_k vs. k for several cases of infinitely repeating rectangular shapes. In each case, Δx is the "pulse width" and $\mathcal{L}$ is the repeat period. The the a_n's are given by eqn. (11.32) with $\Delta t \rightarrow \Delta x$, $T \rightarrow \mathcal{L}$ and $a_k = a_n$ for $k = nk_1$. Abscissa scales are in units of π. For situations in which the analyzed shape is a function of time and one plots a_ω vs. ω, all three plots would look exactly the same as those shown if one replaces Δx by Δt, $\mathcal{L}$ by T, and a_k by a_ω (again, with abscissa scale in units of 2π.) (a) Case $\Delta x = 1$, $\mathcal{L} = 2$. The first zero occurs at n=2, which is at $k = 2k_1$. Since $k_1 = 2\pi/\mathcal{L} = \pi$, the first zero is at $k = 2\pi$ radians. Thus, $\Delta k\, \Delta x = 2\pi$ for the criteria used in the text. (b) Case $\Delta x = 1$, $\mathcal{L} = 4$. Since Δx has not changed, the frequency of the first zero is still at $k = 2\pi$ rad, but, since k_1 has halved, it now occurs at n=4 and the Fourier components are twice as densely packed as in case a. (c) Case $\Delta x = 1$, $\mathcal{L} = 8$. Again, the frequency bandwidth has not changed, but the density of Fourier components has again doubled, as we expect.