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CS5016
Computational Methods and Applications

Ordinary Differential Equations

Scribe 06

ODE

An equation involving one or more derivatives of an unknown function.
If all derivatives are taken wrt a single independent variable, we get an ordinary differential equation (ODE).
If p is the maximum order of differentiation in a differential eqn, we say the differential eqn has order p.

Example

$$\frac{d(x(t))}{dt} = -x(t)$$

Verification that $x(t) = e^{-t}$ satisfies this ODE

$$\frac{d}{dt}x(t) = \frac{d}{dt}e^{-t} = -e^{-t} = -(e^{-t}) = -x(t)$$ verified.

Prey Predator dynamics

The Lotka-Volterra eqns are known as predator-prey equations. They are a pair of 1st order non linear differential eqns. They are used to describe the population dynamics of biological systems in which 2 species interact - one predator, and the other prey. Their populations change through time according to

$$\frac{d(x(t))}{dt} = (\alpha)(x(t)) - (\beta)(x(t)\,y(t))$$

$$\frac{d(y(t))}{dt} = (\delta)(x(t)\,y(t)) - (\gamma)(y(t))$$

- $x(t)$ is the number of preys
- $y(t)$ is the number of predators
- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represents instantaneous growth rate of respective populations
- t represents time
- $\alpha, \beta, \gamma, \delta$ are positive real parameters describing interaction btw them

Lotka-Volterra eqns is an eg of Kolmogorov Model.

Order reduction

** An ODE of order $p > 1$ can always be reduced to a system of p equations of order 1.

Consider this order 3 ODE:

$$\frac{d(x(t))}{dt} + x(t)\frac{d^2(x(t))}{dt^2} + 3x(t)^2\frac{d^3(x(t))}{dt^3} = 4x(t) \quad -(0)$$

This ODE is equivalent to the foll sys of ord 1 ODEs

$$u(t) + x(t)\,v(t) + 3x(t)^2\frac{d(v(t))}{dt} - 4x(t) = 0 \quad -(1)$$

$$u(t) - \frac{d(x(t))}{dt} = 0 \quad -(2)$$

$$v(t) - \frac{d(u(t))}{dt} = 0 \quad -(3)$$

To verify this, we put the values of $v(t)$ in (1) from (3)

$$u(t) + x(t)\frac{d(u(t))}{dt} + 3x(t)^2\frac{d^2u(t)}{dt} - 4x(t) = 0 \quad -(4)$$

Putting val of $u(t)$ in (4) from (2),

$$\frac{d(x(t))}{dt} + x(t)\frac{d^2(x(t))}{dt} + 3x(t)^2\frac{d^3(x(t))}{dt^3} - 4x(t) = 0 \quad -(5)$$

(5) and (0) are clearly equivalent.

Hence verified.

The Cauchy Problem

An ODE in gen admits ∞ no. of sol^{n}s. For eg, $\frac{d(x(t))}{dt} = -x(t)$ admits the sol^n $x(t) = Ce^{-t}$ where C is an arbitrary constant.

If we impose condition that $x(0) = 2$, we get a unique sol^n $x(t) = 2e^{-t}$.

Cauchy Problem:

Find $x : I \to \mathbb{R}$ such that

$$x'(t) = f(t, x(t)) \quad \forall t \in I \quad \text{and} \quad x(t_0) = x_0$$

where I is an interval of $\mathbb{R}$

Cauchy problem has a unique solⁿ if all the following condⁿs are met:

i. $f(t, x(t))$ is continuous in a rectangular region R of (t, x) plane that contains the pt (t_0, x_0).

ii. $f(t, x(t))$ is such that for every compact subset K of R there exists a positive constant L s.t. $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$ $\forall (t_1, x_1)$ and (t_2, x_2) in K. A compact subset is one that is small and closed.

iii. The interval I is such that $(t_0 - h, t_0 + h) \subseteq I$ where $h > 0$ is the max val for which rectangle R is contained in the domain of defⁿ of $f(t, x)$.

Explicit and Implicit Solution

The ODE $\frac{d(x(t))}{dt} = -x(t)$ has an explicit solⁿ $x(t) = Ce^{-t}$. ie, x can be written as fn of t.

Consider

$$\frac{dx(t)}{dt} = \frac{x(t) - t}{x(t) + t} = \frac{x/t - 1}{x/t + 1}$$

Let $x/t = v$

$$\Rightarrow x = vt \Rightarrow dx = v\,dt + t\,dv \Rightarrow \frac{dx}{dt} = v + t\frac{dv}{dt}$$

$$v + t\frac{dv}{dt} = \frac{v-1}{v+1} \Rightarrow t\frac{dv}{dt} = \frac{v-1}{v+1} - v\frac{v+1}{v+1} = \frac{-v^2 - 1}{v+1}$$

solving,

$$-\int \frac{v+1}{v^2+1}\,dv = \int \frac{dt}{t}.$$

$$-\ln t + C_1 = \int \frac{dv}{1+v^2} + \frac{1}{2}\int \frac{2v}{1+v^2}\,dv = \tan^{-1} v + \frac{1}{2}\ln(1+v^2) + C_2$$

Putting back $v = x/t$, we get

$$\ln t + \tan^{-1}\left(\frac{x}{t}\right) + \frac{1}{2}\ln\left(1 + \frac{x^2}{t^2}\right) = C_1 - C_2$$

$$\Rightarrow \cancel{\ln t^2} + 2\tan^{-1}\left(\frac{x}{t}\right) + \ln(t^2 + x^2) - \cancel{\ln t^2} = (C_1 - C_2)(2)$$

$$\Rightarrow \tan^{-1}\left(\frac{x(t)}{t}\right) + \frac{1}{2}\ln(t^2 + x^2) = C$$

Hence verified.

$x(t)$ and t are related according to the above law. However it isn't possible to write $x(t)$ as a function of t. It is clear from the above eqn that $x(t)$ cannot be separated out from ln and $\tan^{-1}$ terms simultaneously.

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Euler methods

Subdivide integration intervals $I = [t_0, T]$ with $T < \infty$ into N_h intervals of length $h = (T - t_0)/N_h$: h is called the <u>discretization step</u>.

At each t_n, $n \in \{0, 1, \dots, N_h - 1\}$ we seek the unknown value x_n that approximates $x(t_n)$. The set of values $\{x_n\}_{n=0}^{N_h-1}$ is our numerical solution.

I Forward Euler method

$$x_{n+1} = x_n + h\, f(t_n, x_n) \quad \forall n \in \{0, 1, \dots, N_h - 1\}$$

II Backward Euler method

$$x_{n+1} = x_n + h\, f(t_{n+1}, x_{n+1}) \quad \forall n \in \{0, 1, \dots, N_h - 1\}$$

Consider ODE

$$\frac{d(x(t))}{dt} = -x(t)^4$$

Forward Euler method gives

$$x_{n+1} = x_n - h \cdot x_n^4 \qquad \text{(an Explicit expression)}$$

Backward Euler method gives

$$x_{n+1} = x_n - h \cdot x_{n+1}^4.$$

i.e., x_{n+1} should be a real root of poly

$$y^4 - \frac{y}{h} - \frac{x_n}{h} = 0 \qquad \text{(an Implicit expression)}$$

Implicit methods enjoy better stability properties than explicit ones. This is because implicit methods usually involve solving an equation involving unknown value of the solution at the next time step, which <u>can help to dampen out numerical oscillations</u> and reduce effort of numerical errors. Explicit methods only involve info of current time step. So, it's less stable

Stability ~~of~~ on unbounded intervals

Consider

$$x'(t) = \lambda x(t) \quad \forall t \in (0, \infty) \qquad \text{and} \qquad x(0) = 1$$

It is clear that $x(t) = e^{\lambda t}$ is the exact solution.

If $\lambda < 0$ then $\lim_{t \to \infty} x(t) = 0$.

$\because$

Forward Euler method with $x_0 = 1$ gives

$$x_{n+1} = x_n(1+\lambda h) = (1+\lambda h)^n \quad \forall n \geq 0.$$

$\lim_{n \to \infty} x_n = 0$ only if $|1+\lambda h| < 1$

$\Rightarrow -1 < 1+\lambda h < 1$

$\Rightarrow -2 < \lambda h < 0$

$\Rightarrow h \in (0, 2/|\lambda|)$

Backward Euler method with $x_0 = 1$ gives

$$x_{n+1} = \frac{x_n}{1-\lambda h} = \frac{1}{(1-\lambda h)^n} \quad \forall n \geq 0$$

If $1-\lambda h > 1$, $(1-\lambda h)^n \xrightarrow[\text{as } n\to\infty]{} \infty \Rightarrow \frac{1}{(1-\lambda h)^n} \xrightarrow[\text{as } n\to\infty]{} 0$

$\Downarrow$

$-\lambda h > 0$

Since $\lambda < 0$, this is true for $h > 0$

$\therefore \lim_{n\to\infty} x_n = 0 \quad \forall h > 0$

System of ODEs

Consider the foll sys of 1st order ODEs with unknowns $x_1(t), \dots, x_m(t)$

$$x_1'(t) = f_1(t, x_1(t), \dots, x_m(t))$$

$$\vdots$$

$$x_m'(t) = f_m(t, x_1(t), \dots, x_m(t))$$

where $t \in [t_0, T]$ with initial conditions $x_{1,0}, \dots, x_{m,0}$

Let us write the above system of ODEs as

$$x'(t) = F(t, x(t))$$

Now we apply any of the methods used to solve the Cauchy problem

Higher Order Methods : Range Kutta (Implicit and Explicit)
Range-Kutta methods are effective ways of solving IVP of diff eqns.
This may be used to construct higher order accurate numerical method by fn's self without needing higher order derivatives.

Scipy module scipy.integrate offers methods to solve ODEs.