The Verification of Cache Coherence Protocols

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Abstract

In this paper we introduce a new verification technique for cache coherence protocols at the behavior level. Protocols are specified by a Finite State Machine (FSM) model. The global state space is the Cartesian product of an arbitrary number of individual cache state spaces and is symbolically expanded. A global FSM characterizing the protocol behavior is built and protocol verification becomes equivalent to finding whether or not the global FSM may enter erroneous states. State expansion only takes a few steps, contrary to current approaches. The verification procedure is applied to the verification of the Illinois protocol.

1.0 Introduction

In a shared-memory multiprocessor, private caches are needed to reduce the effects of memory access latency and contention. Whereas private caches significantly improve system performance, they introduce the *cache coherence* problem. Multiple cached copies of the same memory word must be consistent at any time. A *cache coherence protocol* ensures that changes made to shared memory locations by any processor are visible to all other processors.

A cache coherence protocol is a set of rules coordinating communicating *entities* (usually cache and memory controllers) to enforce consistency among multiple data copies. Many protocols [1,6] have been proposed and implemented; however, they have never been formally validated. The simplicity of these protocols, the lack of verification tools, and the complexity of current formal validation procedures may explain this state of affair. Informal techniques for protocol verification are based on time-consuming, error-prone testing procedures by engineers and require a great deal of ingenuity. As the complexity of protocols grows, it becomes extremely difficult to verify protocols by simply relying on human reasoning.

In a broad sense, the goal of validation is to verify that a protocol satisfies its specification and possesses the required invariant properties. Validation activities, including simulation studies, state reachability analysis and logical reasoning, should exist at all phases of design and implementation. Simulations are conceptually simple but suffer from *incompleteness* since a random test sequence must be run indefinitely to enter all reachable states. It is also very unlikely that validation procedures based on trace-driven simulations can detect most design errors: A protocol passing the test is only shown to be correct for the particular simulation runs.

Approaches based on testing were attempted by Baer and Girault [2] who introduced a Petri Net model of cache protocols. This model comprehensively specifies the underlying hardware structure. The Petri Net model is valuable in capturing the synchronization between communicating hardware entities, and hence, it is an important methodology for mapping protocol designs to actual implementations. The construction of a Petri Net model as shown in [2] is difficult to automate and is very complex even though the protocol is simple. The verification procedure is not specified and is probably very complex.

Reachability analysis is primarily based on exploring exhaustively all the possible interactions between entities interacting in the protocol. The system can be characterized by its *state*. From a given state, the exploration of all possible interactions between entities leads to a number of new states. States in which the protocol fails to preserve expected correctness properties are classified as *erroneous* states; otherwise states are *permissible*. If any erroneous state is reachable, the protocol is incorrect. The major difficulty of this technique is the *"state space explosion"* problem [5, 8]; normally the state exploration complexity quickly blows up with the growing number and complexity of entities involved in the protocol. Reachability analysis

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has been widely adopted for the automated verification of communication protocols [5, 7, 11]. In order to validate a cache coherence protocol, it is not sufficient to track the possible states; the state models must also capture aspects associated with the consistency of data values.

In [15], Rudolf and Segall presented a correctness proof for a snooping protocol by enumerating the various scenarios of reads and writes. Each cache is considered as a finite state automaton and a *product machine* is a collection of n finite state automata. Nanda and Bhuyan [16] presented a similar approach based on the composition of communicating finite state machines and on state enumeration. In an enumeration approach, a large number of redundant states are visited and expanded during the state expansion procedure. Enumerating states for complex protocols faces the state space explosion problem.

Another technique for validating protocols relies on logical proofs [5, 10, 13]. This approach can validate a full range of properties. Ideally, any property which can be formulated in logic can be verified, but proof and formulation of assertions reflecting the desired correctness properties are often error-prone and need considerable ingenuity. In some studies requiring great efforts, correctness conditions are still incorrectly and/or incompletely formulated. More importantly, this approach cannot deal with state-oriented transitions.

An approach combining the advantages of reachability analysis and of logical proof has recently been applied to the verification of communication protocols [3, 4, 5, 10, 14]. Reachability analysis based on state models with augmented variables and processing routines is used to expand major states, while logic properties are formulated and proved over associated context variables. This intermediate approach is suited to the verification of cache coherence protocols: on one hand the coherence activities are mainly reflected by the changes of cache states; on the other hand, the modeling of data aspects are dealt with by augmenting the state description with context variables.

Recent work has focused on the state space explosion problem. Instead of enumerating the state space, a *symbolic* expansion of the state space is advocated in [9]. A complex system often exhibits a great deal of *regularity*. A structure which exploits this regularity is able to compress the representation of a set of states by extracting their common features. For example, the regular expression $\{A,B,C\}^*\{A\}$ represents the set of all strings containing an arbitrary number of replications of $\{A,B,C\}$ and ending with A. This symbolic form provides a concise representation for the enumeration of an infinite set.

This paper introduces a new methodology for validating cache coherence protocols at the early design stage. Our method is based on reachability analysis. The system state is the composition of individual cache states as in [15, 16], but the system state space is symbolically represented and expanded. Relying on the *symmetry* of the system, we derive equivalence relations among states; we then group all caches with equivalent states into classes each separately represented by one symbolic state. The complexity of the state expansion procedure is drastically reduced. A global state graph is reported at the end of the procedure. The global state graph is useful not only to verify data consistency but also to demonstrate the similarities and disparities among protocols.

The first part of this paper provides a protocol model with its fundamental definitions. Equivalence relations leading to the state representation by classes are then introduced, followed by the symbolic construction of the global state graph. Finally, the methodology is applied to the Illinois protocol. The methodology has also been applied to all the protocols described in [1]. The results are presented in [12] but are omitted here for lack of space.

2.0 Protocol Model

2.1 Finite State Automaton

Representing a cache coherence protocol by a finite State Machine (FSM) model is natural from the perspective of protocol designers. Without loss of generality, a formal definition of the protocol model is as follows.

Definition 1 (FSM Model) *A deterministic finite state machine modeling a cache coherence protocol has structure* $\mathcal{M}=(Q, \Sigma, \mathcal{F}, \delta)$ where

Q is a finite set of state symbols,
Σ is the set of operations causing state transitions,
$\mathcal{F}$ is a characteristic function defined for each state and which can be null; and
δ defines the state transition functions $\mathcal{F} \times Q \times \Sigma \rightarrow Q$.

Finally, the finite state machine must be *strongly connected*, that is, starting from any given state there exists at least one path leading to all other states.

Definition 2 (Global or System State) *With respect to a particular memory location, the global or system state is defined as the composition of each individual cache state*[1].

1. Throughout this paper, we track the state of a single block. For simplicity we use the terms *cache state* and *memory state* to signify the state of the block copy in a cache or in memory.

In several protocols, the next state of a cache depends only on its current state and on the type of request issued by the local processor; for these protocols the characteristic functions is null. The characteristic function $\mathcal{F}$ is introduced to deal with protocols whose transitions depend on the global state, and not just on the state of the local cache. For example, in the Illinois protocol, a read miss in a cache may return the desired cache block either in state *Valid-Exclusive* or *Shared*, depending on the presence of the block in other caches. Other protocols exhibiting this property are write-broadcast protocols such as the Dragon and Firefly protocols [1]. We can define a relation and its characteristic (*sharing-detection*) function over all caches' states from the perspective of cache C_i as:

$$f_i(C_1, C_2, \ldots, C_n) = \begin{cases} true & \text{if } \exists C_j \neq C_i \text{ s.t. } state(C_j) \neq invalid \\ false & \text{otherwise} \end{cases}$$

and $\mathcal{F} = (f_1, f_2, \ldots, f_n)$.

We consider that the state *invalid* includes the cases where the block is in cache and has been invalidated or where the block is not present in cache. Note that there is one sharing-detection function per cache. If there is only one valid copy in caches then the function returns 0 for the cache with the valid copy and 1 for all other caches. In fact, the *sharing-detection* function as introduced here provides a more formal framework to model protocols than the framework relying on natural language descriptions given in [1]. Although $\mathcal{F}$ may be generalized to any function of the global state, we limit ourselves to functions used in existing protocols, that is, $\mathcal{F}$ is either null or is the *sharing-detection* function.

The FSM model specifies the protocol behavior in terms of global state transitions and is the support for the protocol verification. To some extent, each cache state carries some semantic interpretation. For example, in the Illinois protocol, a cache block in the *Dirty* state means that the local copy has been modified and that main memory has an obsolete copy, whereas a cache block in the *Shared* state indicates that the local copy is potentially shared with other caches and that all cached copies must be identical with the main memory copy. As a result, if different caches are in states *Dirty* and *Shared* in the same global state then a contradiction in the interpretation of cache states occurs. Similarly several caches in the *Dirty* state signal another contradiction. This suggests a primary verification procedure consisting of searching for all reachable global states and proving that all reached states are *permissible* in the sense that individual cache states are compatible [16]. The problem of searching the global state space is therefore converted into the problem of finding an efficient model for the global FSM.

2.2 Model for Data Consistency

A cache coherence protocol must support correct execution of a program in a multiprocessor system. In general, there are two distinct requirements:

- The ordering of accesses must conform to a well defined consistency model and the parallel code must be written *correctly* for this model through proper use of synchronization primitives (e.g., critical sections).
- For correctly written programs, the cache protocol must always return the latest value on each load.

We formulate this latter condition within the framework of the reachability expansion as follows.

Definition 3 (Data Consistency) *With respect to a particular memory location, the protocol preserves data consistency if and only if the following condition is always true during the reachability analysis: the family of global states originated from G', including G' itself, consistently observe the value written by a STORE transition τ which brings a global state G to G' or the value written by STORE transitions after τ. That is, states reached by expanding G' are not allowed to access the old value defined before τ.*

Global states which violate this condition will be referred to as *erroneous* states. If any erroneous state is reachable, the protocol is incorrect.

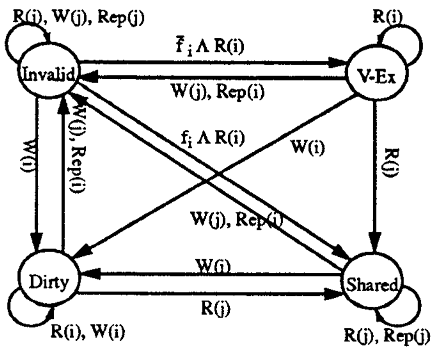


FIGURE 1. The Illinois Transition Diagram from the Perspective of Cache C_i.

2.3 The Illinois Protocol

Figure 1 shows the state transition diagram of the Illinois protocol. There are four states for each cached block: *Invalid*, *Valid-Exclusive* (not modified; only copy in caches), *Shared* (not modified; possibly copies in other caches) and *Dirty* (modified; only copy in caches). On a read miss a block is loaded in states *Valid-Exclusive* or

Shared depending on the value of the *sharing-detection* function *f*. The Illinois protocol can be described as follows:

- State Symbols $Q = \{Invalid, Valid\text{-}Exclusive, Shared, Dirty\}$.
- Operations $\Sigma = \{R, W, Rep\}$, which stand for *read*, *write* and *replacement*.
- Cache algorithm from the perspective of cache C_i:
  1. **Read Hit.** No coherence action need to be taken.
  2. **Read Miss.** If cache C_j has a *Dirty* copy, C_j supplies the missing block and updates main memory at the same time; both C_i and C_j end up in state *Shared*. If there are *Shared* or *Valid-Exclusive* copies in other caches, C_i gets the missing block from one of the caches and all caches with a copy end up in state *Shared*. If there is no cached copy, C_i receives a *Valid-Exclusive* copy from main memory.
  3. **Write Hit.** If the block is *Dirty*, no action is taken. If the block is *Valid-Exclusive*, its state changes to *Dirty*. If the block is *Shared*, all remote copies must be invalidated and C_i's copy turns *Dirty*.
  4. **Write Miss.** Like a read miss, except that all remote copies are invalidated and the block is loaded *Dirty*.
  5. **Replacement.** If the block is in state *Dirty*, it is written back to main memory.

2.4 Augmenting the Global State with Context Variables

To verify the protocol we need to augment the global state with attributes characterizing the value of data.

Definition 4 (Augmented Global State) *In a system with n caches, the global state of a block $G=(q_1, q_2, \ldots, q_n)$, $q_i \in Q$, is augmented by $M=(m_1, m_2, \ldots, m_n)$, where m_i denotes a set of context variables which characterize the block from the point of view of cache i.*

The global state is augmented with auxiliary variables M. Augmented state transition $(G,M) \rightarrow^{\tau} (G',M')$ means that $(G \rightarrow^{\tau} G')$ and that M is changed to M', reflecting the environmental changes, and, in particular, the changes in data aspects such as data transfers and values. The reachability graph is constructed over all reached global states G and associated *Ms* are considered attributes of these states.

The context variables in Definition 4 allow the caches to customize their views of the system in a flexible way. Caches may have different views of their local copies and of the memory copy because of the buffering of stores. For the Illinois protocol, all caches have the same view of the memory block because the effects of a write operation is instantaneous. (We assumed atomic accesses throughout this paper.) Therefore, a single global context variable can represent the state of the memory copy. The more general notation has nonetheless been introduced here so that it will apply to more complex protocols later on.

The formal model of the Illinois protocol associates each cache C_i with an auxiliary variable $cdata_i$ and the memory with auxiliary variable *mdata* to keep track of consistency between memory and cached copies. *cdata* takes values from domain *{nodata, fresh, obsolete}* and *mdata* from domain *{fresh, obsolete}*. Initially, all caches are in the *Invalid* state without data copies ($cdata_i$=*nodata*, for all *i*) and memory has the fresh copy (*mdata=fresh*.) The value assignments to these variables during the state expansion conform to the protocol. A data inconsistency occurs when a processor can access its local copy with value *obsolete*. The data transfer aspects of the Illinois protocol from the perspective of cache C_i can be specified as follows.

1. **Read Miss.**
   if (there exist C_j in *Dirty* state)
   ($mdata = cdata_j$) /* update memory */
   ($cdata_i = cdata_j$)
   else if (there is no cached copy)
   ($cdata_i = mdata$) /* get data from memory */
   else /* arbitrarily choose C_j with a copy */
   ($cdata_i = cdata_j$)
2. **Write Hit.**
   if (C_i has a dirty copy)
   no action is taken
   else
   $\forall j\ cdata_j = nodata$ /* invalidated */
   $cdata_i = fresh$
   $mdata = obsolete$
3. **Write Miss.**
   if (there exist C_j in *Dirty* state)
   $cdata_i = cdata_j$ /* must be a fresh copy */
   else if (there exist C_j with a copy)
   $cdata_i = cdata_j$ /* must observe fresh copy */
   else /* no cached copy */
   $cdata_i = mdata$ /* must be fresh */
   $\forall j\ cdata_j = nodata$ /* invalidated */
   $cdata_i = fresh$
   $mdata = obsolete$
4. **Replacement.**
   if (C_i has a dirty copy)
   $mdata = cdata_i$
   $cdata_i = nodata$

3.0 Reachability Analysis - State Space Expansion

Since the reachability graph is constructed over all global states and since value assignments to auxiliary variables

M are irrelevant to state transitions, we concentrate now on the transitions between global states alone.

3.1 Exhaustive Enumeration of the State Space

To completely verify the protocol at the finite automaton level, all state transitions must be exhaustively simulated. Conventionally, an exhaustive search algorithm as shown in Figure 2 is used to explore the system state space. In this algorithm, a working list of newly produced states and a history list keeping track of visited states are maintained. At each step, all states directly reachable from the current state are generated and inserted in the working list if necessary. Redundant states are pruned by checking the new state against the visited states.

```
Algorithm: exhaustive search.
W : list of working states.
H : list of visited states.
while (W is not empty) do
begin
   get current state A from W and put A in H.
   for all A', A' is a successor state of A.
      if (A' ∉ W∪H)
         then add A' to W.
end.
```

FIGURE 2. Algorithm for Exhaustive Search.

Since the state space is enumerated explicitly, the number of caches must be exactly defined. As a result, the state space must be finite because the number of state symbols and cache events is also finite. Let's have a system with n caches, $|Q| = m$ state symbols, and $|\Sigma| = k$ cache events. The maximum number of states in the system state space is $(m)^n$ states. However, the number of states visited in the expansion process is far more than $(m)^n$ states. For each state in the working list, we must generate all its directly reachable states although some of them may have been visited previously. Without any pruning effort, we need at least approximately $nk(m)^n$ state visits to complete the expansion process. If the *connectivity* information faithfully showing the path leading to a particular state from a given state is stored, the problem of limited memory capacity becomes apparent. The state space grows exponentially with the complexity of the protocol and the number of entities in the validation model. A quantitative analysis of this technique is given in [8].

3.1.1 Pruning the State Space by Counting Equivalence

To keep the state space manageable, pruning of redundant states is necessary. Two system states $(q_1, q_2, \ldots, q_n)$ and $(s_1, s_2, \ldots, s_n)$ are strictly equivalent if and only if $q_i = s_i$, $q_i, s_i \in Q$, for all $1 \le i \le n$. This strict equivalence relation is certainly too conservative. As we mentioned before, the behavior of all cache entities is characterized by a common FSM with deterministic transition functions. All $n!$ permutations of a state $(q_1, q_2, \ldots, q_n)$ are equivalent in the validation process because the order of the tuple is not important. For example, in the validation of a system with three caches, the tuples *(shared, shared, invalid)* and *(shared, invalid, shared)* should represent equivalent states.

This equivalence based on symmetry arguments suggests a concrete way to represent a set of equivalent states by the number of caches in each state. A state for a system with n caches can be represented as $\Pi_{i=1}^{|Q|} q_i^{k_i}$, where k_i denotes the number of caches in state $q_i \in Q$ and $\Sigma_{i=1}^{|Q|} k_i = n$.

Definition 5 (Counting Equivalence) *Two system states* $\Pi_{i=1}^{|Q|} q_i^{k_i}$ *and* $\Pi_{i=1}^{|Q|} q_i^{l_i}$ *are equivalent if* $k_i = l_i$ *for all* i.

The relaxed equivalence of Definition 5 is a first step in the right direction. Equivalence classes can be further broadened.

3.2 Symbolic Expansion

The major difficulties of exhaustive enumeration techniques are the *state space explosion* problem, the large amount of memory required to manipulate the state information, and the inefficiency of testing for the convergence of the state expansion process. Another technical problem is the fact that the validation is done for a fixed number of caches. It is not clear at first that a protocol correct for a system with n caches would also be correct for a system with n' caches, $n \neq n'$.

3.2.1 Composite State

The number of caches in a particular cache state plays an important role in judging whether or not a system state is permissible. For example, several caches in the *Dirty* state signal a data inconsistency. Similarly if a cache is in the *Shared* state, the local copy is clean and *possibly* present in other caches. In theory, an infinite number of caches could have clean copies without affecting the protocol correctness. In all these cases, the actual number of copies is not important. What is critical in all protocols is whether there are 0, 1 or several copies in a given state. These possibilities can be represented by the following set of repetition operators[2].

Definition 6 (Repetition Operator)

1. *The* **Singleton** *(q^1) indicates that there is one and only one cache in state $q \in Q$. This operator can be omitted.*
2. *The* **Positive** *or* **Plus operator** *(q^+) indicates that at least one cache is in state $q \in Q$.*
3. *The* **Star operator** *(q^*) extends the plus operator by including the case of null instance. q^* means that none or some caches are in state $q \in Q$.*

In a *composite* system state, caches in the same state are grouped into a cache state *class* specified by one of the repetition operators. For example, composite state $(q_1^*, q_2^+, q_3, ...)$ indicates that there are one cache in state q_3, at least one cache in state q_2 and none or some caches in state q_1. Formally, the definition for a composite state is as follows.

Definition 7 (Composite State) *A composite state represents the composition of cache states in a system with an arbitrary number of cache entities. It is constructed over cache state classes of the form $(q_1^{r_1}, q_2^{r_2}, ..., q_n^{r_n})$, where $n = |Q|$ and $r_i \in [0, 1, +, *]$*[3].

The representation of a composite state carries all the information needed to verify the protocol. For example, in the Illinois protocol, there are two possible sources of data inconsistencies.The first possibility is that cache(s) in the *Shared* state coexist with a cache in the *Dirty* state and the second possibility is that more than one caches are in the *Dirty* state. It is clear that the first case can be formulated by the cache state classes appearing in the composite state (e.g., the composite state *(Dirty, Shared,...)*). The second case is covered by the repetition operator over state classes with multiple Dirty copies such as $(Dirty^+,...)$, $(Dirty^*,...)$. The definition of composite state expands state equivalence relations beyond the strict counting equivalence of Definition 5.

3.2.2 Information Ordering and Pruning

Repetition operators can be ordered by the possible states they specify. The resulting order is $1 < + < *$, and we also write $q^1 < q^+ < q^*$ where $q \in Q$. q^+ reveals that one cache is in state q (which is always permissible) or that multiple caches are in state q (which may indicate a data inconsistency condition). The star operator adds the possibility of null instance. Therefore, we say that q^{r_1} is *weaker* than q^{r_2} if $r_1 < r_2$, where $q \in Q$ and $r_1, r_2 \in [1, +, *]$. The null instance can be ordered with respect to *, i.e., $0 < *$.

Definition 8 (Structural Covering) *We say that composite state S_2 structurally covers composite state S_1, or $S_1 \le S_2$, if*

$$\forall q^{r_1} \in S_1 \quad \exists q^{r_2} \in S_2 \rightarrow q^{r_1} \le q^{r_2} \text{ i.e. } r_1 \le r_2$$

where r1, r2 are repetition operators.

Definition 9 (Containment) *We say that composite state S_2 contains composite state S_1, or $S_1 \subseteq_{\mathcal{F}} S_2$, if*

$$S_1 \le S_2 \text{ and } \mathcal{F}(S_1) = \mathcal{F}(S_2)$$

where $\mathcal{F}$ is a characteristic function defined in the FSM model of Section 2.1.

The definition of *structural covering* extends the order among repetition operators to composite states. A composite state S_2 *contains* S_1 if S_1 is structurally covered by S_2 and if S_1 and S_2 have the same value of the characteristic function $\mathcal{F}$. An interesting consequence of containment is that if $S_1 \subseteq_{\mathcal{F}} S_2$ and if S_1 is not permissible then S_2 is also not permissible. S_1 could thus be discarded during the verification process provided we keep S_2.

We will show that the expansion process is a *monotonic* operator on the set of composite states S, that is, if $S_1 \subseteq_{\mathcal{F}} S_2$, then $\tau(S_1) \subseteq_{\mathcal{F}} \tau(S_2)$ where τ is an operator representing a cache event. As the expansion process progresses, new composite states are created. A new state is discarded if it is contained in a visited state. Similarly all visited states contained in a new state are discarded. At the end of the expansion process all visited states are *essential* states.

Definition 10 (Essential State) *Composite state S is essential if and only if there does not exist a composite state $\overline{S}$ such that $S \subseteq_{\mathcal{F}} \overline{S}$.*

At the end of the expansion process, the state space is simply decomposed into several families (which may be overlapping) represented by essential composite states.

3.2.3 Rules and Algorithm for the Expansion Process

We need to define the set of operations applicable to composite states in the state generation process. In the following, '/' signifies "or" selection.

2. Note that if a protocol was ever invented in which two dirty copies are permissible, the methodology is still applicable provided that we add this new possibility to the list of repetition operator.

3. The operator 0 means "null instance" and is added for completeness.

1. **Aggregation:** $(q^0, q^r) \equiv q^r$, $(q^*, q^*) \equiv q^*$, $(q, q^{1/+/*}) \equiv q^+$, *and* $(q, q^{1/+/*}) \equiv q^+$. Aggregation rules are equivalence rules between composite states obtained by merging cache states.
2. **Coincident Transition:** $q_1^r \rightarrow^\tau q_2^r$, where $r \in [1, +, *]$ and τ is an observed transition originated by cache $\{C\colon \mathrm{state}(C) \neq q_1\}$. All caches in state q_1 change state coincidentally following a transition originated by another cache not in state q_1.
3. **One-step Transition:** $q_1 \rightarrow^t q_2$ and $q_1^{+/*} \rightarrow^t (q_2, q_3^*)$, where t is a transition originated by cache $\{C\colon \mathrm{state}(C) = q_1\}$. The next state of C is q_2 and all other caches in the state class q_1 change state to q_3. (For example, one cache modifying its local copy in the *Shared* state changes to the *Dirty* state and the copies in all other caches in the same *Shared* state are invalidated.)
4. **N-steps Transitions:** This rule specifies the repetitive application of the same transition N times, where N is an arbitrary positive integer.

   (a) $(Q, q_1^+)_\mathcal{F} \rightarrow^t (Q, q_2^1, q_1^*)_\mathcal{F} \rightarrow^t (Q, q_2^2, q_1^*)_\mathcal{F} \rightarrow^t \ldots \rightarrow^t (Q, q_2^+, q_1^*)_\mathcal{F}$, where t is a transition originated by cache $\{C\colon \mathrm{state}(C) = q_1\}$ and $q_1 \rightarrow^t q_2$. The same transition t can be applied infinitely many times as long as there are caches in state q_1 and every generated state has the same value defined over the characteristic function $\mathcal{F}$. Every application of the transition brings down the number of caches in state q_1 by one and increases the number of caches in state q_2. The transition t has no effect on other caches (denoted as Q in the tuple). One trivial example is a series of consecutive replacements; another example is the case where $|Q| = 0$, q_1 is *Invalid* and q_2 is *Shared* following consecutive read misses.

   (b) $(Q, q_1^+)_\mathcal{F} \rightarrow^t (Q, q_2^1, q_1^+)_\mathcal{F} \rightarrow^t \ldots \rightarrow^t (Q, q_2^+, q_1^+)_\mathcal{F} \rightarrow^t \ldots \rightarrow^t (Q, q_2^+, q_1^{1/0})_\mathcal{F}$. The same interpretation is made as in (a), except that the derivation ends with a composite state with a different value of the characteristic function. This rule is used to model the effect of the *sharing-detection* function introduced in Section 2.1.

During the state expansion process, the next state is produced by stimulating the current state, by exploring all possible cache transactions and by repeatedly applying the above rules. Before formalizing the algorithm for symbolic state expansion and protocol verification, we first prove, as promised, the monotonicity of the expansion process.

Lemma 1 *The immediate successor $\bar{S}_1$ originated from state*

$$S_1 = (q_1^{r_1}, q_2^{r_2}, \ldots, q_{i-1}^{r_{i-1}}, q_i^{i=1}, q_{i+1}^{r_{i+1}}, \ldots, q_n^{r_n})$$

is structurally covered by $\bar{S}_2$ originated from state

$$S_2 = (q_1^{\bar{r}_1}, q_2^{\bar{r}_2}, \ldots, q_{i-1}^{\bar{r}_{i-1}}, q_i^{1/+/*}, q_{i+1}^{\bar{r}_{i+1}}, \ldots, q_n^{\bar{r}_n})$$

if $r_j = \bar{r}_j$ for all $j \neq i$, the same cache event $t \in \Sigma$ is applied to S_1 and S_2, and $\mathcal{F}(S_1)=\mathcal{F}(S_2)$.

Proof: The proof is direct. First of all, conforming to the condition $\mathcal{F}(S1)=\mathcal{F}(S2)$, caches in the same state in S_1 and S_2 will get to the same next cache state under common cache event t. Let's ignore caches in state q_i first, that is, let's consider $\{S_1 - q_i\}$ and $\{S_2 - q_i^r\}$. Since both states are exactly the same, any transition t will bring them to the same state $S1' \equiv S2'$. Next, we augment S_1' and S_2' by adding the effect of applying t on caches in state q_i in S_1 and S_2. The only change is the number of caches in the particular state q_j such that $q_i \rightarrow^t q_j$. It is obvious that $q_j \leq q_j^{1/+/*}$ after the transition. We can thus conclude that $\bar{S}_1 \leq \bar{S}_2$.

Lemma 2 *The claim $\bar{S}_1 \leq \bar{S}_2$ holds if $S_1 \subseteq_\mathcal{F} S_2$, that is, $r_j \leq r_j$ for all j and $\mathcal{F}(S_1)=\mathcal{F}(S_2)$.*

Proof: The result extends the conclusion of lemma 1 and the proof is similar.

The result of lemma 2 indicates that if $S_1 \subseteq_\mathcal{F} S_2$, the immediately reachable state $\bar{S}_1$ of S_1 is structurally covered ($\leq$) by the immediately reachable state $\bar{S}_2$ of S_2. To demonstrate the *monotonicity*, we still need to show that $\bar{S}_1$ and $\bar{S}_2$ have the same characteristic functions, that is, $\mathcal{F}(\bar{S}_1)=\mathcal{F}(\bar{S}_2)$.

Corollary 1 If $\mathcal{F}$ is null and $S_1 \subseteq_{null} S_2$, then for every $\bar{S}_1$ reachable from S_1 there exists $\bar{S}_2$ reachable from S_2 such that $\bar{S}_1 \subseteq_{null} \bar{S}_2$.

Proof: Because $\mathcal{F}$ is null, the transition functions depend only on local cache state and intended operations. By Definition 9, the relation of containment $\subseteq_{null}$ is characterized by the relation of structural covering $\leq$ alone. As a result, the claim is just a recursive induction from lemma 2.

Corollary 1 demonstrates that protocols whose behavior does not depend on any characteristic function exhibits the *monotonic* property of our symbolic expansion process. During the expansion process, S_1 can be discarded because all successors originated from S_1 can be generated by expanding the successors of S_2. Corollary 2 states that protocols depending on a *sharing-detection* function also exhibit the monotonicity of expansion.

Corollary 2 If $\mathcal{F}$ is the *sharing-detection* function and $S_1 \subseteq_\mathcal{F} S_2$, then for every $\bar{S}_1$ reachable from S_1 there exists $\bar{S}_2$ reachable from S_2 such that $\bar{S}_1 \subseteq_\mathcal{F} \bar{S}_2$.

Proof: See the proof in Appendix A.1.

The preceding results suggest a very efficient expansion process shown in Figure 3 to obtain essential states. Two lists keep track of non-expanded and visited states. At each step, a new state is produced by expanding the current state, and then a pruning process justified by the monotonicity property removes contained states. The final output reported in list H is the set of essential states. All possible states are included in the reported essential states, as we now show.

Theorem 1 *The essential composite states generated by the algorithm of Figure 3 are complete. They symbolically characterize all states which can be produced by an exhaustive expansion process such as the algorithm of Figure 2.*

Proof: The number of caches needs to be explicitly specified in the validation model for an enumeration approach, while the symbolic approach employs canonical form to characterize states. Consider states u, v (derived from u by transition τ) in the enumeration approach and composite states s, t (derived from s by transition τ) in the symbolic form such that s symbolically characterize u. t also characterizes v, because the same transition functions are applied and the information is accumulated during the generation of composite states.

Algorithm : essential states generation.

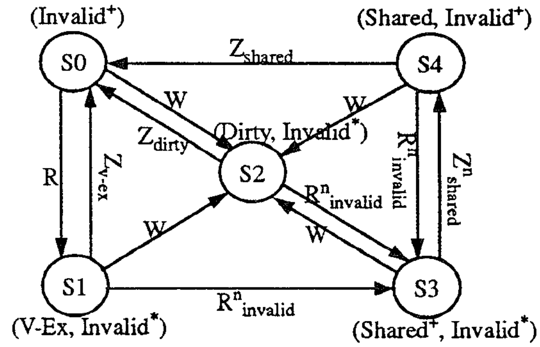
```
W : list of working composite states.
H : list of visited composite states.(output:essential states)

while (W is not empty) do
begin
    get current state A from W.
    for all cache state class v ∈ A
        for all applicable operations τ on v
            A →τ A'.
            for any state P ∈ W and Q ∈ H
                if (A' ⊆𝓕 P or A' ⊆𝓕 Q or A' ⊆𝓕 A)
                    then discard A'.
                else begin
                    remove P from W if P ⊆𝓕 A'.
                    remove Q from H if  Q ⊆𝓕 A'.
                    add A' to W.
                    if (A ⊆𝓕 A') then discard A  and terminate
                      all FOR loops starting a new run.
                    end
            end
        end
    end
    insert A to H if A is fully expanded and is not contained.
end.
```

FIGURE 3. Algorithm for Generating Essential States.

4.0 Verification of the Illinois Protocol

We demonstrate the symbolic expansion by applying the algorithm of Figure 3 to the Illinois protocol. We start the expansion process in an initial state (*Invalid*$^+$), in which no cache has a block copy.

After 22 state visits (listed in Appendix A.2), five essential states, (*Invalid*$^+$), (*V-Ex, Invalid**), (*Dirty, Invalid**), (*Shared*$^+$, *Invalid**) and (*Shared, Invalid*$^+$) are reported and the global transition diagram is shown in Figure 4, where R, W and Z stand for read, write and replacement respectively. An optional subscript identifies the state of the cache originating the transition. A superscript n means that the transition is the result of an N-steps transition. The values of the sharing-detection function and of the auxiliary variables showing the status of the cached data copies and of the memory copy are also listed. It is clear that the data consistency requirement is satisfied because processors always have the most recent data values if a valid copy is in their cache, as shown in the table of Figure 4.



state	sharing($\mathcal{F}$)	cdata	mdata
(Invalid$^+$)	(false)	(nodata)	fresh
(V-Ex, Invalid$^+$)	(false, true)	(fresh, nodata)	fresh
(Dirty, Invalid*)	(false, true)	(fresh, nodata)	obsolete
(Shared$^+$, Invalid*)	(true, true)	(fresh, nodata)	fresh
(Shared, Invalid$^+$)	(false, true)	(fresh, nodata)	fresh

FIGURE 4. The Global Transition Diagram for the Illinois Protocol.

We need to distinguish between states *s3*:(*Shared*$^+$, *Invalid**) and *s4*:(*Shared, Invalid*$^+$) because of different observations of data sharing from the perspective of caches in state class *Shared*. *s4* is structurally covered by *s3* but is not contained in *s3*. By the definition of the *sharing-detection* function in Section 2.1, the boolean values returned are (*true, true*), (*false, true*) for *s3* and *s4* respectively. The state *s0*:(*Invalid*$^+$) is also an essential state and is not contained by another state for the same reason.

The definition of the *plus* (+) operator needs some explanation when the *sharing-detection* function $\mathcal{F}$ is employed. In fact, *Shared*$^+$ in state *s3*: (*Shared*$^+$, *Invalid**) with $\mathcal{F}(s3)$ = (*true, true*) denotes that there are at least two caches in the *Shared* state when *s3* is first constructed. This does not violate the original definition of the *plus* operator and moreover the additional information is carried by the value of $\mathcal{F}$ associated with the composite state. We use the additional information carried by the value of $\mathcal{F}$ to distinguish between *s3* and *s4*. Certainly, we could explicitly express the effect of $\mathcal{F}$ by defining another operator that denotes two or more caches in a particular cache state. However, more expansion rules would be introduced and more essential states would be generated. On the other hand, if we keep the original definition of the *plus* operator, the characteristic function $\mathcal{F}$ shows the subtle difference anyway.

5.0 Conclusion

In this paper, we have introduced a simple method for validating cache coherence protocols at the behavior level. By exploiting equivalence relations among global states, we can symbolically represent and generate the system state space rather than enumerate it. The global transition diagram built upon the symbolic essential states not only facilitates the verification of data consistency but also demonstrates the similarities and disparities among protocols.

The reduction in complexity of the verification procedure over existing approaches is so drastic that we can contemplate efficient verification of much more complex protocols with large numbers of cache states, such as relaxed consistency protocols, protocols with locked states and protocols for hierarchically organized machines. We are currently working at extending the model and the terminology to include more details of the protocol implementation having to do with ordering and consistency models.

Extension of our work beyond more complex protocols could be the definition of a formal specification language capable of describing both the protocol behavior and the processes implementing it. This language should facilitate greater automatization of the verification activities, which would reduce the possibility of errors.

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Appendix A

A.1 Proof of Corollary 2.

The proof is not formal but is based on the enumeration of all possible scenarios. The sketch of the proof is as follows.

Proof: By lemma 1 and lemma 2, we already know that $\bar{S}_2$ must structurally cover $\bar{S}_1$, that is, $\bar{S}_1 \le \bar{S}_2$. It further remains to show that $\mathcal{F}(\bar{S}_1)=\mathcal{F}(\bar{S}_2)$. Consider the value of $\mathcal{F}$ returned by a composite state; there are three possibilities:

1. $\boldsymbol{v1} = (\textit{\textbf{false, false,..., false}})$: no cached copy exists and the only possible composite state is $(Invalid^{+})$.
2. $\boldsymbol{v2} = (\textit{\textbf{true, true,..., true, false}})$: one and only one cached copy exists. The composite state must have the form as $(Invalid^{+/*}, q)$, where $q \in (Q - \{Invalid\})$.
3. $\boldsymbol{v3} = (\textit{\textbf{true, true,..., true}})$: there are several valid copies in the caches. The composite state has the structure $(Invalid^{+/*}, q_1^{r_1}, q_2^{r_2}, \ldots, q_m^{r_m})$, where $q_i \in (Q - \{Invalid\})$ for all i, and there must exist either a q_i with $r_i = +$, or *arbitrary* q_i, q_j with $r_i = r_j = 1$, or any combination of above.

Since $S_1 \subseteq_{\mathcal{F}} S_2$, S_1 and S_2 return the same values of $\mathcal{F}$, either $\boldsymbol{v1}$,$\boldsymbol{v2}$, or $\boldsymbol{v3}$.

Case 1: $\mathcal{F}(S_1) = \mathcal{F}(S_2) = \boldsymbol{v1}$.

In this case, S_1 and S_2 are identical and their successive states $\bar{S}_1$ and $\bar{S}_2$ are also identical, which implies $\mathcal{F}(\bar{S}_1)=\mathcal{F}(\bar{S}_2)$.

Case 2: $\mathcal{F}(S_1) = \mathcal{F}(S_2) = \boldsymbol{v2}$.

Either S_1 and S_2 are identical or $S_1=(Invalid^{+}, q)$ is contained by $S_2 = (Invalid^{*}, q)$. In both cases, the same transition τ results in $\bar{S}_1$ and $\bar{S}_2$ having the same value of $\mathcal{F}$. This could be shown by contradiction. Assume that $\mathcal{F}(\bar{S}_1) = \boldsymbol{v1}$ and $\mathcal{F}(\bar{S}_2) \ne \boldsymbol{v1}$. This can happen only if τ causes $(q \rightarrow^{\tau} Invalid, Invalid \rightarrow^{\tau} Invalid)$ in S_1 and τ has different effects on S_2, which is impossible because $S_1 \subseteq_{\mathcal{F}} S_2$. Similar contradictions could be shown to refute the other two cases, say $\mathcal{F}(\bar{S}_2) \ne \mathcal{F}(\bar{S}_1) = \boldsymbol{v2}$, $\mathcal{F}(\bar{S}_2) \ne \mathcal{F}(\bar{S}_1) = \boldsymbol{v3}$, and thus we could conclude $\mathcal{F}(\bar{S}_1)=\mathcal{F}(\bar{S}_2)$.

Case 3: $\mathcal{F}(S_1) = \mathcal{F}(S_2) = \boldsymbol{v3}$.

Let's consider the values of $\mathcal{F}$ returned by the generated composite state $\bar{S}_1$.

(a). $\mathcal{F}(\bar{S}_1) = \boldsymbol{v1}$. This can happen if the transition τ is repeatedly applied to S_1 on caches in state q with $(q \rightarrow^{\tau} Invalid)$ and S_1 has the structure $(Invalid^{+/*}, q^{+})$. S_2 should have the same structure and result in $\mathcal{F}(\bar{S}_2) = \boldsymbol{v1}$ because the transition τ removes all cached copies in S_1 and it will have the same effect when applied to S_2, and thus $\mathcal{F}(\bar{S}_1) = \boldsymbol{v1}$.

(b). $\mathcal{F}(\bar{S}_1) = \boldsymbol{v2}$. This can happen if a transition τ is applied to S_1 and removes all cached copies except in the cache which originates τ. Because $S_1 \subseteq_{\mathcal{F}} S_2$, τ must have the same effects when applied to S_2, and hence $\mathcal{F}(\bar{S}_2) = \boldsymbol{v2}$.

(c). $\mathcal{F}(\bar{S}_1) = \boldsymbol{v3}$. In this case, $\bar{S}_1$ carries the information that two or more than two cached copies exist. Since we know that $\bar{S}_1 \le \bar{S}_2$, or $\bar{S}_1$ is structurally covered by $\bar{S}_2$, two or more than two cached copies exist in $\bar{S}_2$, that is, $\mathcal{F}(\bar{S}_2) = \boldsymbol{v3}$.

From the above, we can conclude that $\mathcal{F}(\bar{S}_1)=\mathcal{F}(\bar{S}_2)$ in all cases.

A.2 Expansion Steps for The Illinois Protocol.

The intermediate steps for exploring the Illinois state space are listed below.

(Inv^{+})	$\rightarrow$	W_{inv}	$\rightarrow$	$(Dirty, Inv^{*})$
(Inv^{+})	$\rightarrow$	R_{inv}	$\rightarrow$	$(V\text{-}Ex, Inv^{*})$
$(Dirty, Inv^{*})$	$\rightarrow$	Rep_{dirty}	$\rightarrow$	(Inv^{+})
$(Dirty, Inv^{*})$	$\rightarrow$	W_{dirty}	$\rightarrow$	$(Dirty, Inv^{*})$
$(Dirty, Inv^{*})$	$\rightarrow$	R_{dirty}	$\rightarrow$	$(Dirty, Inv^{*})$
$(Dirty, Inv^{*})$	$\rightarrow$	W_{inv}	$\rightarrow$	$(Dirty, Inv^{+})$
$(Dirty, Inv^{*})$	$\rightarrow$	$R^{n}{}_{inv}$	$\rightarrow$	$(Shared^{+}, Inv^{*})$
$(V\text{-}Ex, Inv^{*})$	$\rightarrow$	$Rep_{v\text{-}ex}$	$\rightarrow$	(Inv^{+})
$(V\text{-}Ex, Inv^{*})$	$\rightarrow$	$W_{v\text{-}ex}$	$\rightarrow$	$(Dirty, Inv^{*})$
$(V\text{-}Ex, Inv^{*})$	$\rightarrow$	$R_{v\text{-}ex}$	$\rightarrow$	$(V\text{-}Ex, Inv^{*})$
$(V\text{-}Ex, Inv^{*})$	$\rightarrow$	W_{inv}	$\rightarrow$	$(Dirty, Inv^{+})$
$(V\text{-}Ex, Inv^{*})$	$\rightarrow$	$R^{n}{}_{inv}$	$\rightarrow$	$(Shared^{+}, Inv^{*})$
$(Shared^{+}, Inv^{*})$	$\rightarrow$	$Rep^{n}{}_{shared}$	$\rightarrow$	$(Shared, Inv^{+})$
$(Shared^{+}, Inv^{*})$	$\rightarrow$	W_{shared}	$\rightarrow$	$(Dirty, Inv^{*})$
$(Shared^{+}, Inv^{*})$	$\rightarrow$	R_{shared}	$\rightarrow$	$(Shared^{+}, Inv^{*})$
$(Shared^{+}, Inv^{*})$	$\rightarrow$	W_{inv}	$\rightarrow$	$(Dirty, Inv^{+})$
$(Shared^{+}, Inv^{*})$	$\rightarrow$	$R^{n}{}_{inv}$	$\rightarrow$	$(Shared^{+}, Inv^{*})$
$(Shared, Inv^{+})$	$\rightarrow$	Rep_{shared}	$\rightarrow$	(Inv^{+})
$(Shared, Inv^{+})$	$\rightarrow$	W_{shared}	$\rightarrow$	$(Dirty, Inv^{+})$
$(Shared, Inv^{+})$	$\rightarrow$	R_{shared}	$\rightarrow$	$(Shared, Inv^{+})$
$(Shared, Inv^{+})$	$\rightarrow$	W_{inv}	$\rightarrow$	$(Dirty, Inv^{+})$
$(Shared, Inv^{+})$	$\rightarrow$	$R^{n}{}_{inv}$	$\rightarrow$	$(Shared^{+}, Inv^{*})$