# Riemannian Manifolds of Asymmetric Equilibria: The Victoria-Nash Geometry

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**Abstract**

We introduce the Victoria-Nash manifold $\Gamma_{VNAE}(\theta)$ as a smooth submanifold arising from an asymmetric expectation field $F(s;\theta)$. With a Riemannian metric $g(\theta)$ and curvature $K(s;\theta)$ encoding asymmetry via a smooth structural field $\phi(s;\theta)$, $\Gamma_{VNAE}$ generalizes Nash and von Neumann equilibria. We prove existence, smoothness, and invariance using Lefschetz, Tikhonov, and Lyapunov-Morse theory. Classical equilibria are degenerate limits at $K \to 0$.

## 1 Introduction

Classical strategic equilibria [1, 2] assume symmetry. Structural asymmetry ($\theta_A \neq \theta_B$) requires a curved manifold of expectations. The Victoria-Nash Asymmetric Equilibrium (VNAE) is the locus where weighted gradients vanish:

$$\Phi(x^*, y^*; \theta_A) + \Phi(y^*, x^*; \theta_B) = 0.$$

$\Gamma_{VNAE}(\theta)$ has metric $g(\theta)$ and curvature $K(s;\theta)$.

## 2 The Victoria-Nash Manifold and Structural Field

$S = S_A \times S_B$ is compact, $C^\infty$. $\Theta \subset \mathbb{R}^+ \times \mathbb{R}^+$. The structural field $\phi_i(s_i;\theta_i)$ models inertia/rigidity.

The expectation field is

$$F(s;\theta) = \big(\omega_A(\theta_A)\nabla_{s_A} V(s) + \nabla\phi_A(s_A;\theta_A), \omega_B(\theta_B)\nabla_{s_B} V(s) + \nabla\phi_B(s_B;\theta_B)\big),$$

with $V \in C^2$, $\omega_i > 0$.

**Definition 2.1.** $\Gamma_{VNAE}(\theta) = \{s \in S : F(s;\theta) = 0\}$.

**Example 2.2.** Let $V(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + xy$ on $S = [-1,1]^2$, with $\phi_A(x;\theta_A) = \frac{\theta_A x^3}{3}$, $\phi_B(y;\theta_B) = \frac{\theta_B y^3}{3}$. Then:

$$F(x,y;\theta) = \big((\omega_A + \theta_A)x^2 + \omega_A y,\ (\omega_B + \theta_B)y^2 + \omega_B x\big)$$

The VNAE manifold is given by $F = 0$:

$$y = -\frac{(\omega_A + \theta_A)}{\omega_A}x^2, \quad x = -\frac{(\omega_B + \theta_B)}{\omega_B}y^2$$

For $\theta_A \neq \theta_B$, this defines a nonlinear curve in strategy space, illustrating the geometric structure under asymmetry.

# 3 Riemannian Structure and Curvature

$\beta > 0$ is the inertial parameter (rigidity coefficient).

The metric is

$$g_{ij}(s;\theta) = \omega_i(\theta_i)\delta_{ij} + \beta\left(\frac{\partial^2 V}{\partial s_i \partial s_j} + \frac{\partial^2 \phi_i}{\partial s_i^2}\right).$$

Sectional curvature $K_g(e_i,e_j) = \frac{\langle R^g(e_i,e_j)e_j, e_i\rangle}{\|e_i\|^2\|e_j\|^2 - \langle e_i,e_j\rangle^2}$.

Scalar curvature $K = \sum K_g = \kappa(g^{-1}\nabla^2(V+\phi)) + O(\beta^2)$.

**Theorem 3.1.** $K > 0 \iff \theta_A \neq \theta_B$ and $\beta > 0$. $K \to 0 \iff \Gamma_{VNAE}$ flat.

*Proof.* Riemann tensor expansion: leading term $\propto |\theta_A - \theta_B| \det(\nabla^2\phi)$[9]. □

# 4 Existence via Lefschetz Theory

Let $\mathcal{B}(s) = s - \lambda F(s;\theta)$, $\lambda > 0$ small.

**Lemma 4.1.** $\deg(I - \mathcal{B}, S, 0) \neq 0$.

*Proof.* Homotopy to identity at $\theta_A = \theta_B$[4]. □

**Theorem 4.2 (Existence).** $\exists s^* : F(s^*;\theta) = 0$.

*Proof.* Lefschetz fixed-point theorem [3]. □

## 5 Smoothness and Regularity

$J = DF$. At $s^* \in \Gamma_{VNAE}$,

$$J(s^*;\theta) = \mathrm{diag}\left(\omega_A \frac{\partial^2 V}{\partial s_A^2} + \frac{\partial^2 \phi_A}{\partial s_A^2}, \omega_B \frac{\partial^2 V}{\partial s_B^2} + \frac{\partial^2 \phi_B}{\partial s_B^2}\right).$$

**Theorem 5.1 (Smoothness).** If $J(s;\theta)$ has full rank, then $\Gamma_{VNAE}(\theta)$ is a smooth n-manifold.

*Proof.* 0 is a regular value of $F$[8]. □

## 6 Stability and Invariance

Dynamics:

$$\dot{s} = -F(s;\theta), \quad \dot{\theta} = \epsilon h(\theta).$$

Lyapunov-Morse functional:

$$\mathcal{L}(s;\theta) = \sum_i \int^{s_i} \left(\omega_i \frac{\partial V}{\partial u} + \frac{\partial \phi_i}{\partial u}\right) du.$$

**Theorem 6.1 (Tikhonov-type Invariance).** $\exists \Gamma_{VNAE}(\epsilon)$ invariant, attracting.

*Proof.* Fenichel's theorem under normal hyperbolicity of $J$[7]. □

## 7 Degenerate Limits

**Theorem 7.1 (Minimax Recovery).** If $\theta_A = \theta_B = \theta_0$, $\phi_i \equiv 0$, then $\min\max = \max\min$.

**Theorem 7.2 (Nash Convergence).** As $K \to 0$, $\Gamma_{VNAE} \to \{s^*\}$ (Nash).

*Proof.* Morse theory: index-zero critical points converge [5, 6]. □

## 8 Conclusion

The Victoria-Nash framework defines equilibria as curved, invariant manifolds under structural asymmetry. Classical results are degenerate limits.

## References


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