Geometric Analysis of VNAE: Metric Tensor g_{ij} and Gaussian Curvature K Calculations

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Paper: "Riemannian Manifolds of Asymmetric Equilibria: The Victoria-Nash Geometry"
Section 3: Riemannian Structure and Curvature
Parameters:

$$\omega_A = 1.0, \quad \theta_A = 0.5$$
$$\omega_B = 1.0, \quad \theta_B = 1.0$$
$$\beta = 0.1 \quad \text{(inertial parameter)}$$
$$|\theta_A - \theta_B| = 0.5$$

Parameter Definitions and Physical Interpretation

Structural Parameters of the VNAE Model:

- $\boldsymbol{\theta_A, \theta_B}$: **Asymmetry coefficients** : Represent the structural differences between players A and B. When $\theta_A \neq \theta_B$, the system exhibits fundamental asymmetry, leading to non-zero Gaussian curvature ($K \neq 0$).
- $\boldsymbol{\beta}$: **Inertial/Rigidity coefficient** : Its value ranges from 0 to 1. Represents the resistance to change in the strategy space. Higher β values indicate greater "stiffness" in the Riemannian metric, amplifying the curvature effects.
- $\boldsymbol{\omega_A, \omega_B}$: **Structural weights** : Their values range from 0 to 1. Determine the relative importance and coupling strength of each player in the system. They control the interaction terms in the expectation field:

  $$F_x = (\omega_A + \theta_A)x^2 + \omega_A y, \quad F_y = (\omega_B + \theta_B)y^2 + \omega_B x$$

  When $\omega_A = \omega_B = 1.0$ (as in this analysis), players have symmetric structural influence.

Physical Interpretation: The VNAE geometry emerges from the interplay between:

- **Asymmetry** ($\theta_A \neq \theta_B$): Creates curved manifold structure
- **Inertia** ($\beta > 0$): Amplifies geometric effects
- **Structural coupling** (ω_A, ω_B): Determines interaction strength

This combination leads to the emergence of non-Euclidean geometry in strategic space, where curvature K serves as a geometric measure of market asymmetry.

Calculation point: $(x, y) = (-0.6057, -0.5503)$ **(Equilibrium point from visualization)**

Step 1: Hessian $H = \nabla^2(V + \phi)$

From visualization code functions:

$$V(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + xy$$
$$\phi_A(x; \theta_A) = \theta_A \cdot \frac{x^3}{3}$$
$$\phi_B(y; \theta_B) = \theta_B \cdot \frac{y^3}{3}$$

Second derivatives:

$$H_{11} = \frac{\partial^2(V + \phi_A)}{\partial x^2} = 2x + 2\theta_A x$$
$$= 2(-0.6057) + 2(0.5)(-0.6057) = -1.2114 - 0.6057 = -1.8171$$
$$H_{12} = \frac{\partial^2 V}{\partial x \partial y} = 1$$
$$H_{22} = \frac{\partial^2(V + \phi_B)}{\partial y^2} = 2y + 2\theta_B y$$
$$= 2(-0.5503) + 2(1.0)(-0.5503) = -1.1006 - 1.1006 = -2.2012$$

Hessian matrix:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} = \begin{bmatrix} -1.8171 & 1 \\ 1 & -2.2012 \end{bmatrix}$$

Determinant:

$$\det(H) = (-1.8171)(-2.2012) - (1)(1) = 3.999 - 1 = 2.999$$

Step 2: Metric Tensor g_{ij}

Formula from paper (Section 3) and visualization code:

$$g_{ij}(s; \theta) = \omega_i(\theta_i)\delta_{ij} + \beta \left[\frac{\partial^2 V}{\partial s_i \partial s_j} + \frac{\partial^2 \phi_i}{\partial s_i^2} \right]$$

Metric elements:

$$g_{11} = \omega_A + \beta \cdot H_{11} = 1.0 + 0.1 \cdot (-1.8171) = 1 - 0.18171 = 0.81829$$
$$g_{12} = \beta \cdot H_{12} = 0.1 \cdot 1 = 0.1$$
$$g_{22} = \omega_B + \beta \cdot H_{22} = 1.0 + 0.1 \cdot (-2.2012) = 1 - 0.22012 = 0.77988$$

Metric matrix:

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = \begin{bmatrix} 0.8183 & 0.1 \\ 0.1 & 0.7799 \end{bmatrix}$$

Determinant:

$$\det(g) = (0.8183)(0.7799) - (0.1)(0.1) = 0.6382 - 0.01 = 0.6282$$

Step 3: Gaussian Curvature K

Formula from Theorem 3.1 and visualization code function:

$$K = \frac{\det(H)}{\det(g)} \cdot \beta \cdot |\theta_A - \theta_B|$$

Substitution:

$$K = \frac{2.999}{0.6282} \cdot 0.1 \cdot 0.5 = 4.774 \cdot 0.1 \cdot 0.5 = 0.2387$$

Step 4: Verification of Theorem 3.1

Theorem 3.1: $K > 0 \iff \theta_A \neq \theta_B$ and $\beta > 0$
Conditions:

- $\theta_A \neq \theta_B$? $0.5 \neq 1.0$
- $\beta > 0$? $0.1 > 0$
- $K > 0$? $0.2387 > 0$

Verification satisfied: Curvature is positive due to structural asymmetry.

Step 5: Geometric Interpretation

- $K = 0.239 > 0$: Positive curvature
- Manifold with local "spherical" geometry
- Equilibrium point shows significant curvature
- Structural asymmetry $\rightarrow$ non-zero curvature $\rightarrow$ curved manifold

Step 6: Visualization Results

The generated plots from the R visualization code confirm:

- **Vector Field Plot:** Shows the expectation field $F(s;\theta)$ converging to the equilibrium point
- **3D Curvature Surface:** Shows maximum curvature near equilibrium regions
- **Phase Portrait:** Equilibrium manifold passes through the calculated point
- **Curvature Analysis:** Confirms $K > 0$ for asymmetric cases

Numerical Results from Visualization Code

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VNAE COMPLETE ANALYSIS
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Structural Parameters (Example 2.2):
  A = 0.5 (Player A structural coefficient)
  B = 1 (Player B structural coefficient)
  A = 1, B = 1 (Structural weights)
   = 0.1 (Inertial parameter)
  |A - B| = 0.5 (Asymmetry measure)

Equilibrium Manifold _VNAE:
  Equilibrium 1: ( 0 0 )
  Equilibrium 2: ( -0.6057 -0.5503 )

Riemannian Geometry at Equilibrium:
  Metric g = [ 0.8183  0.1
               0.1     0.7799 ]
  det(g) = 0.6282
  Gaussian Curvature K = 0.2387

Theorem 3.1 Verification:
   K > 0 confirmed: A  B and  > 0
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Final Summary

Hessian $H = \begin{bmatrix} -1.8171 & 1 \\ 1 & -2.2012 \end{bmatrix}$

Metric $g_{ij} = \begin{bmatrix} 0.8183 & 0.1 \\ 0.1 & 0.7799 \end{bmatrix}$

Gaussian Curvature $K = 0.239$

As a conclusion, we can say that the VNAE is active in a strategy space curved by asymmetry. The equilibrium point shows significant positive curvature ($K = 0.239$), confirming the geometric significance of VNAE equilibria in the Riemannian manifold structure.

This supplementary material provides detailed calculations supporting the geometric framework proposed in the main paper. As an initial exploration of Victoria-Nash Asymmetric Equilibrium geometry, these computations establish foundational metrics while acknowledging that future studies may reveal additional geometric complexities. The current focus on Gaussian curvature K and metric tensor g_{ij} provides a tractable entry point to this rich geometric domain, with more advanced connections (Christoffel symbols, Ricci curvature) reserved for specialized applications where such complexity becomes necessary.