LINEAR REGRESSION

unleash the data scientist in you

Linear Regression-Overview

- Difference between Regression and Classification

- Linear Regression
    - Motivating Example: Predicting Housing Prices
    - Hypothesis function
    - Cost Function
    - Gradient Descent
    - Probabilistic Interpretation

- Applying linear regression for hand digit recognition

Supervised Learning

- Data: $D = \{ d_1, d_2, d_3, \ldots\ldots\ldots, d_n \}$ a set of n examples

  $d_i = < x_i, y_i >$

x_i

- Input vector
- Independent variables
- Explanatory variables
- Features
- Predictors

y_i

- Output scalar
- Dependent variables
- Response
- Outcome

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Supervised Learning

- **Regression:** X discrete or continuous $\longrightarrow$ Y is continuous

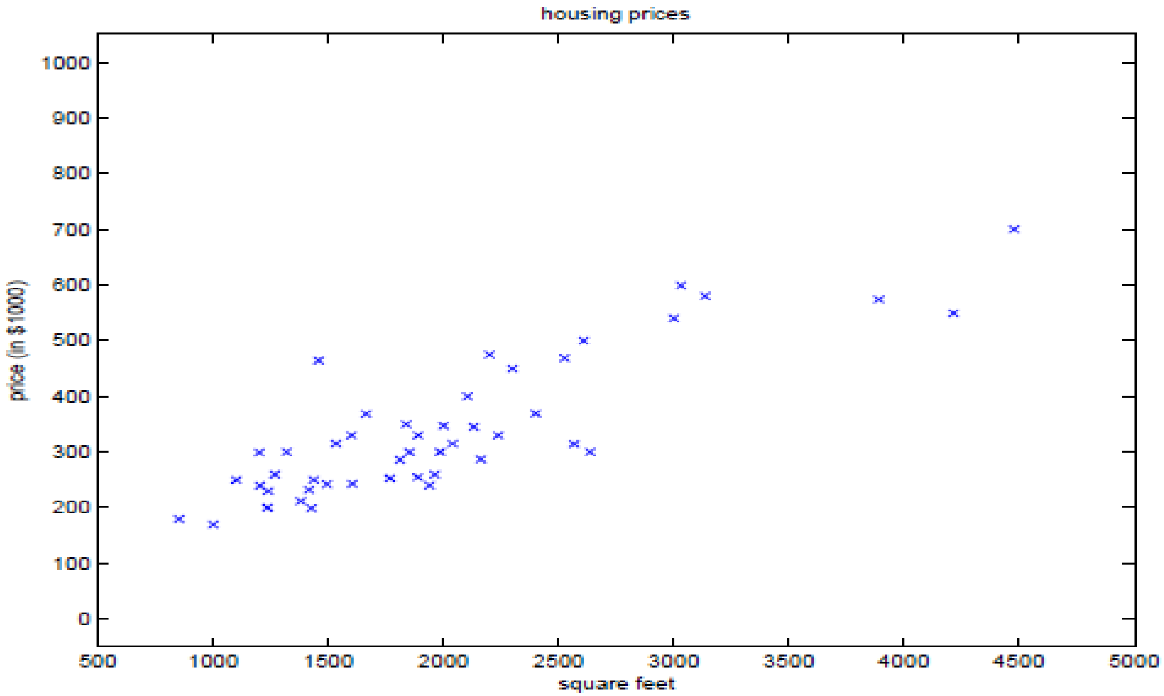
  Eg. Prices, Weight, Height, signal measurement, temperature etc.

- **Classification :** X discrete or continuous $\longrightarrow$ Y is discrete

- **Objective:** learn the mapping of $f : X_i \longrightarrow Y_i$

Predict Housing Prices

Living area ($feet^2$)	Price (1000$s)
2104	400
1600	330
2400	369
1416	232
3000	540
⋮	⋮



Can we learn to predict the price when the price of the house is not in the dataset? **Living Area = 3500 sq. ft.** **Price = ?**

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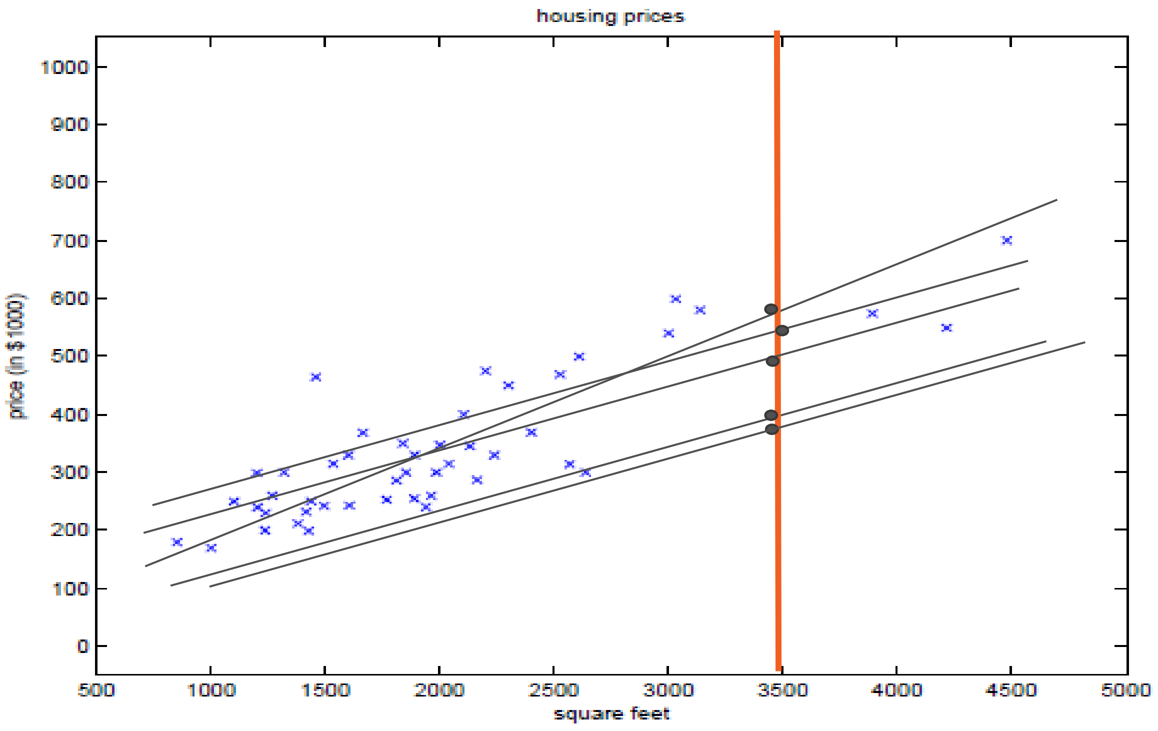
Predict Housing Prices

- Can we learn to predict the price when the price of the house is not in the dataset?
- Living Area = 3500 sq. ft. Price = ?
- Use the line that is somewhere in the middle
- How do we define somewhere in the middle?



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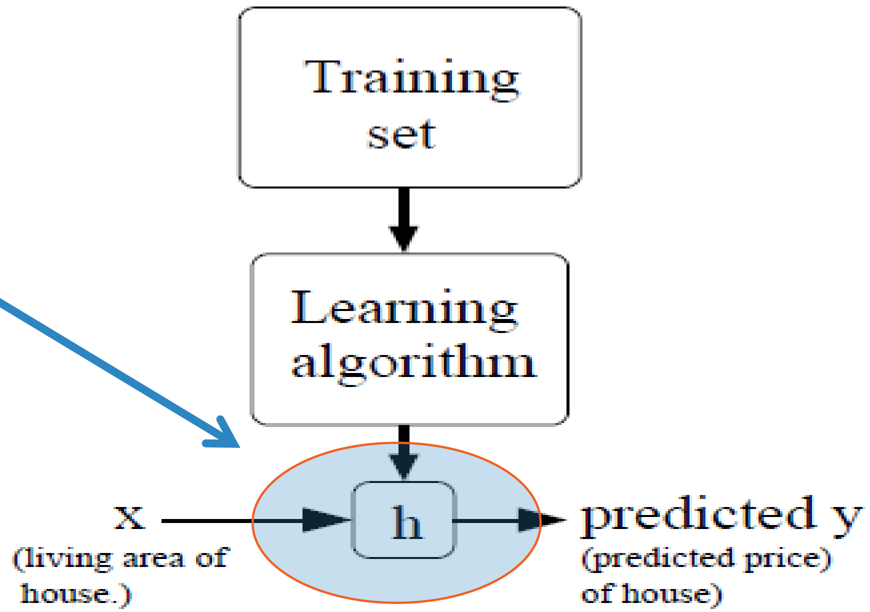
Training Process In Linear Regression

Objective: learn the mapping of $f: X_i \longrightarrow Y_i$

Finding ***h*** is the goal here

h is known as the hypothesis



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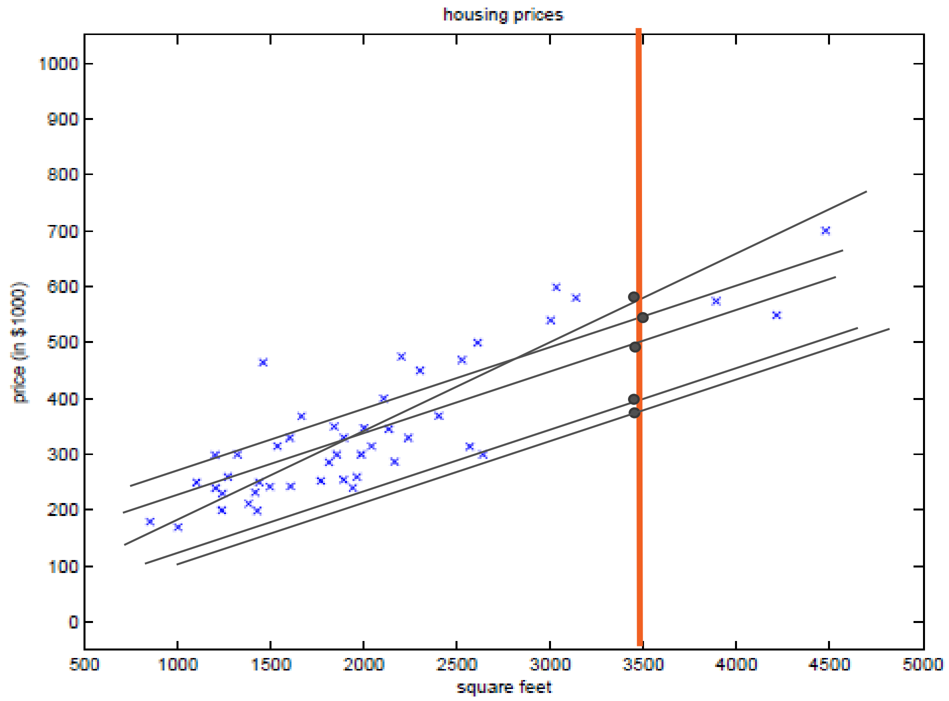
Predict Housing Prices

$$h_\theta(x) = \theta_0 + \theta_1 x$$

	Living area (feet^2)	Price (1000\$s)	
$x^{(1)}$	2104	400	$y^{(1)}$
$x^{(2)}$	1600	330	$y^{(2)}$
$x^{(3)}$	2400	369	$y^{(3)}$
$x^{(4)}$	1416	232	$y^{(4)}$
$x^{(5)}$	3000	540	$y^{(5)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x^{(m)}$			$y^{(m)}$



m = Total number of training examples

θ_j = Parameters

Hypothesis Function-Linear regression

$$h_\theta(x) = \theta_0 + \theta_1 x$$

$h_\theta(x^i)$ is the prediction of the hypothesis h_θ for given input x^i
y^i is the target values (what we would call labels in a classification problem)

We want the prediction $h_\theta(x^i)$ to be as close to the true label y^i as possible.
How do we do that?

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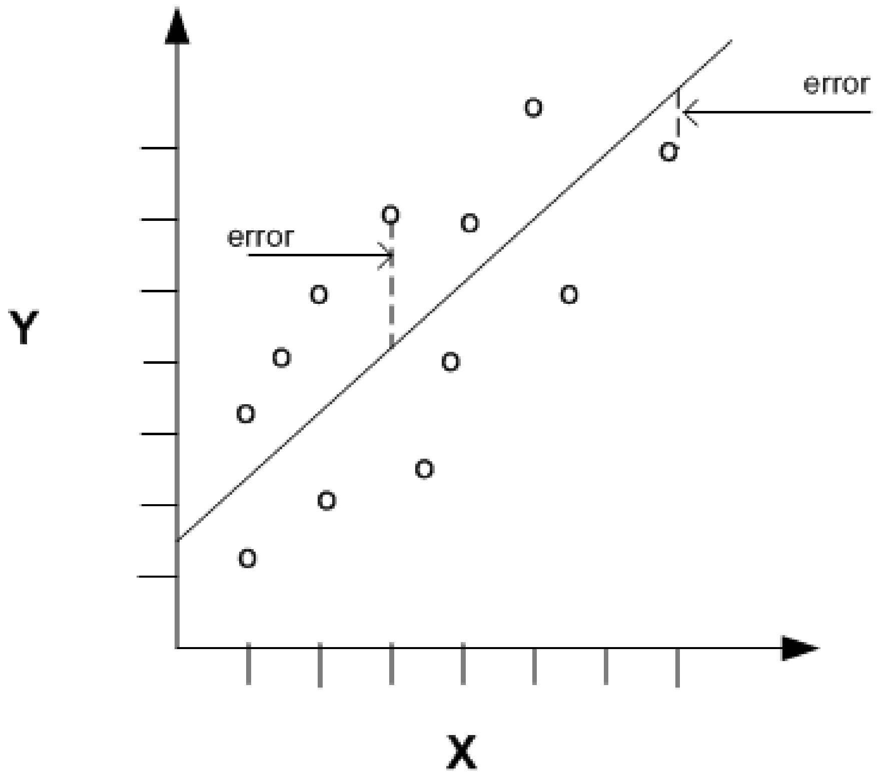
Cost Function

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$



In Plain English

- find a linear line that is as close to the actual outputs as possible.

Mathematically

- Find linear function of X that minimizes the sum of squared residuals from Y.
- a.k.a loss function

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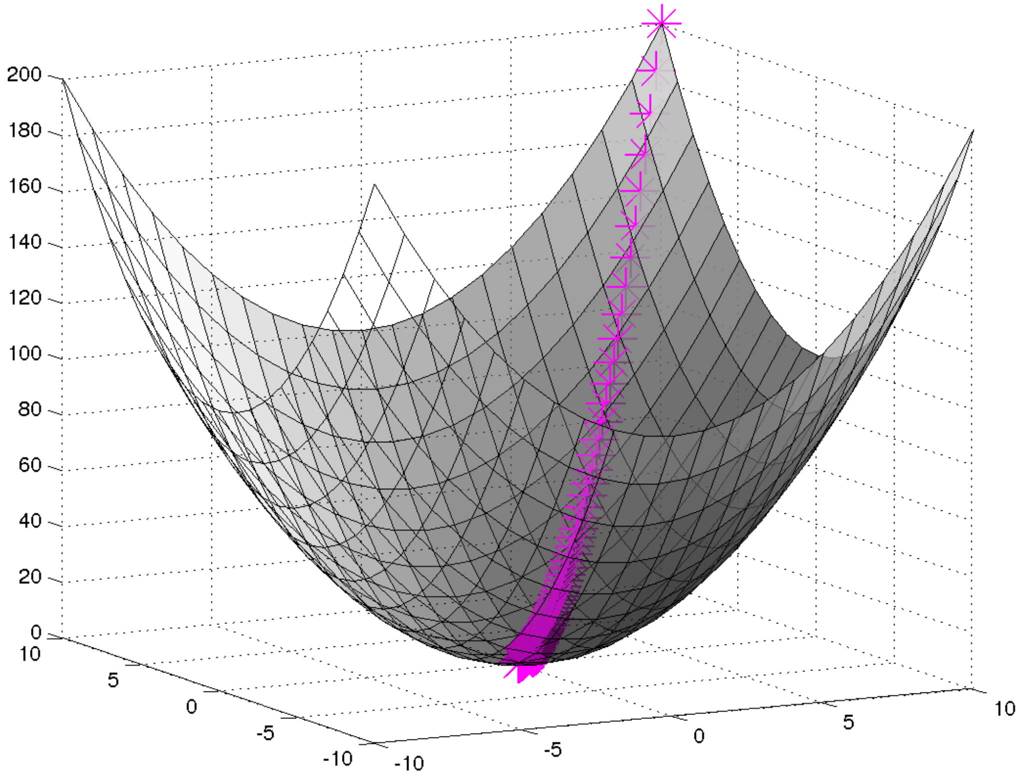
The Cost Function



Hypothesis: $h_\theta(x) = \theta_0 + \theta_1 x$

Cost Function: $J(\theta_0, \theta_1) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$

Parameters: θ_0, θ_1

Goal: $\underset{\theta_0,\theta_1}{\text{minimize}}\ J(\theta_0, \theta_1)$

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Gradient Descent Algorithm

- Goal : $\underset{\theta_0,\theta_1}{\text{minimize}}\, J(\theta_0,\theta_1)$

- Gradient descent starts with some initial θ and then performs an update for each value θ_j

- Repeat until θ converge.

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta).$$

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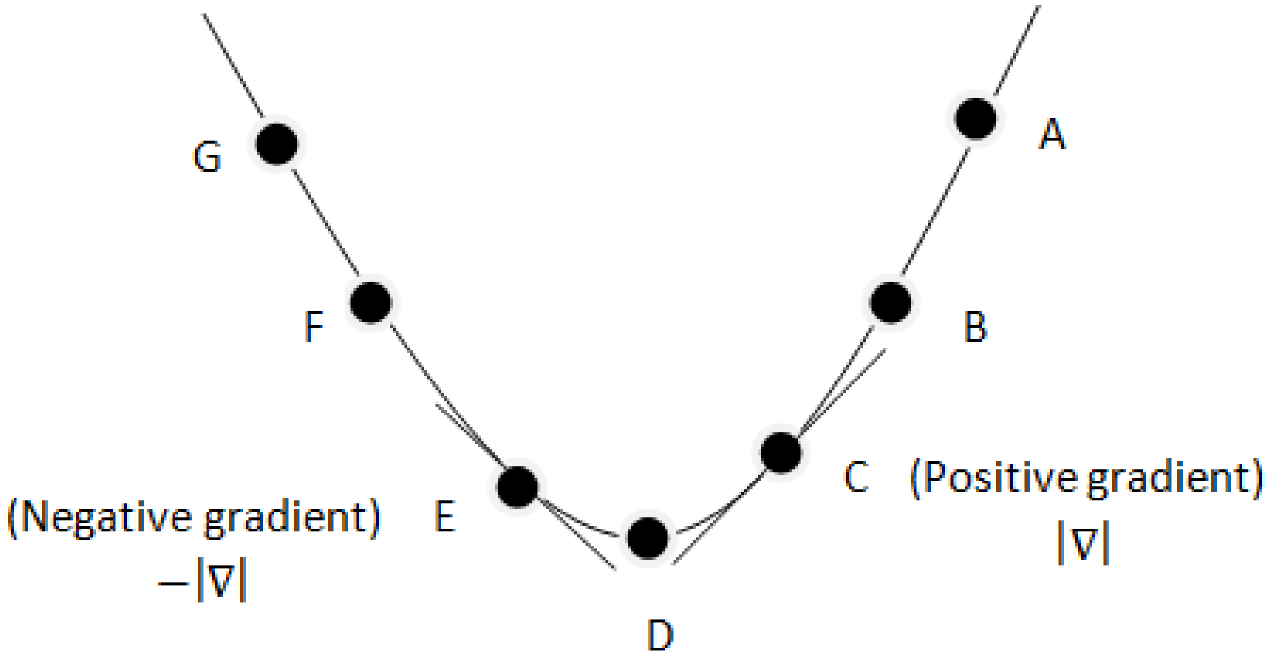
Gradient Descent Algorithm

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta).$$



Gradient Descent Algorithm

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta).$$

For j=0 and j= 1

Repeat until converge {

$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right) \cdot x^{(i)}$$

}

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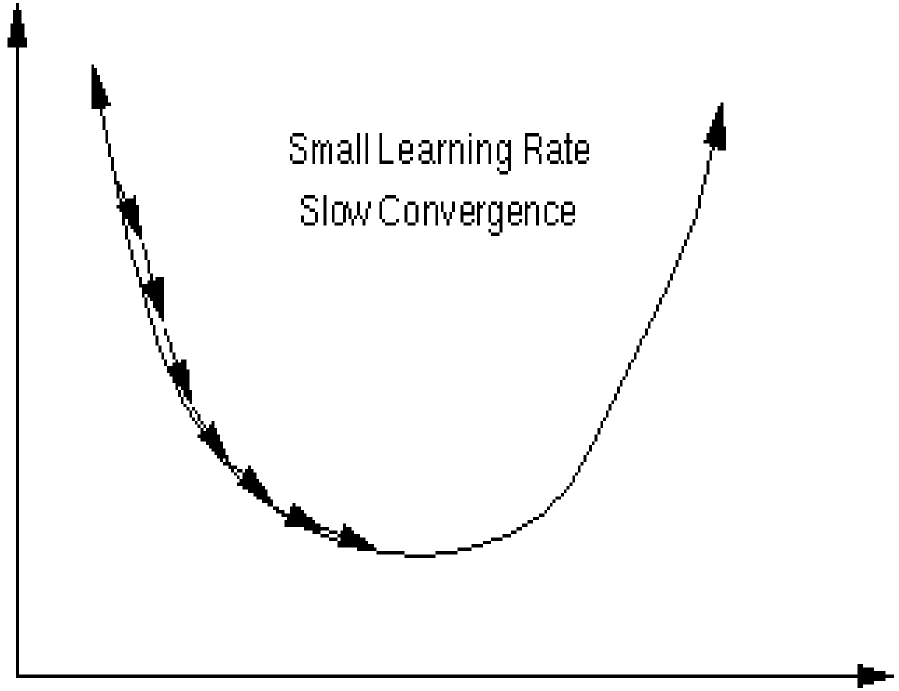
Gradient Descent Algorithm

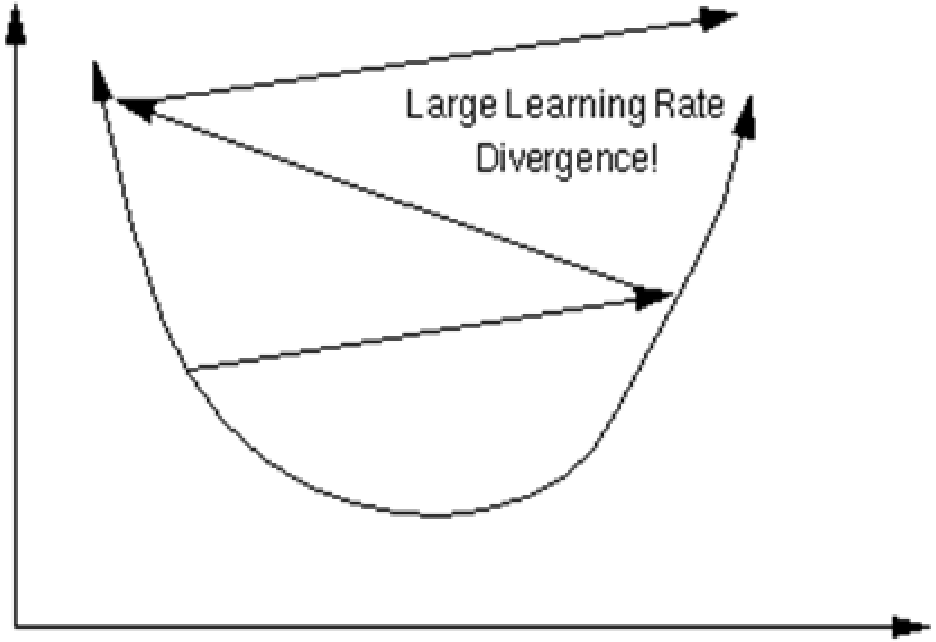
$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta).$$

- α is known as the learning rate
- Each time the algorithm takes a step in the direction of the steepest, $J(\theta)$ decreases.
- α determines how quickly or slowly the algorithm will converge to a solution

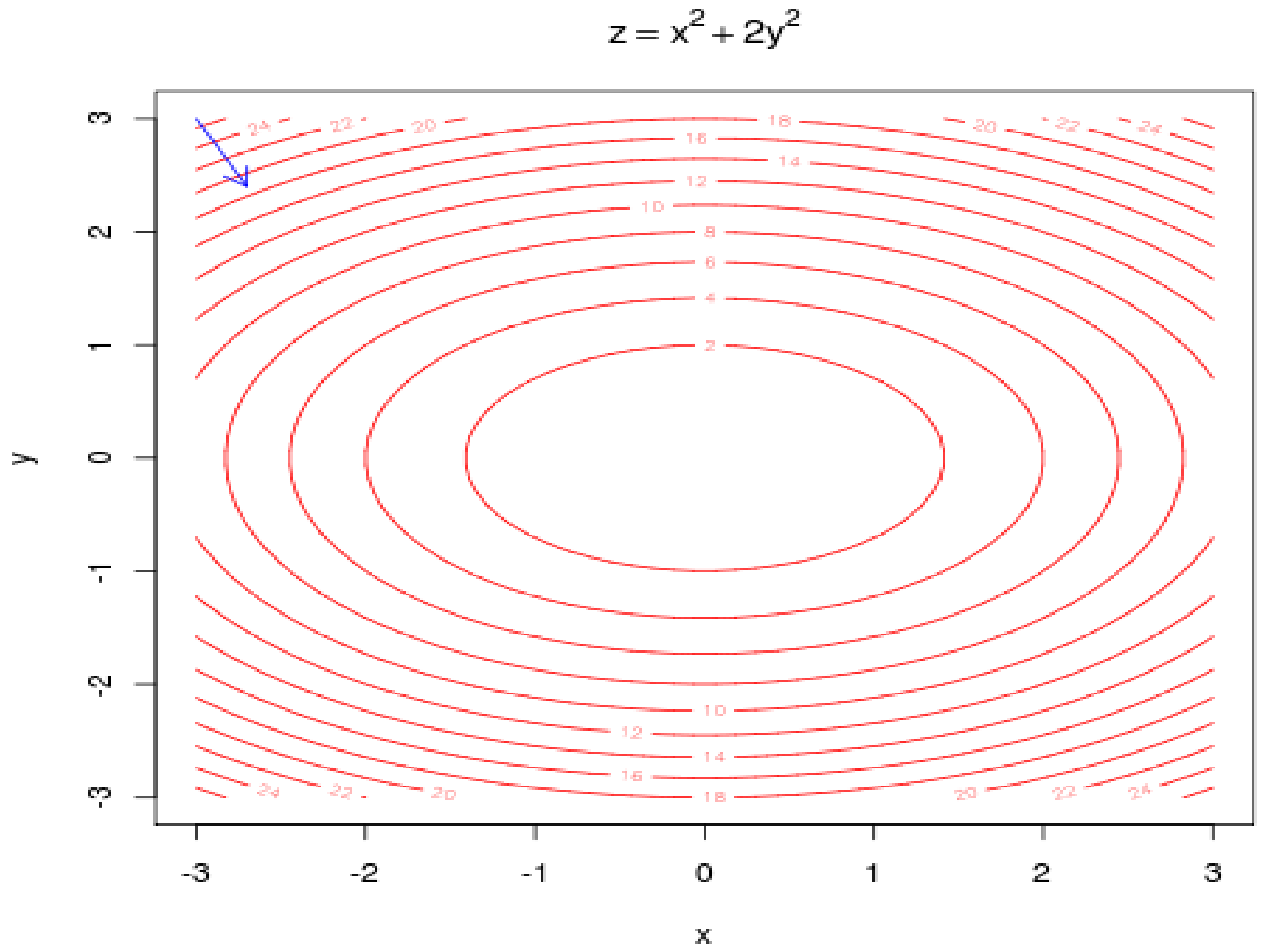
Gradient Descent Algorithm

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta).$$





Digression - Gradient Descent Intuition



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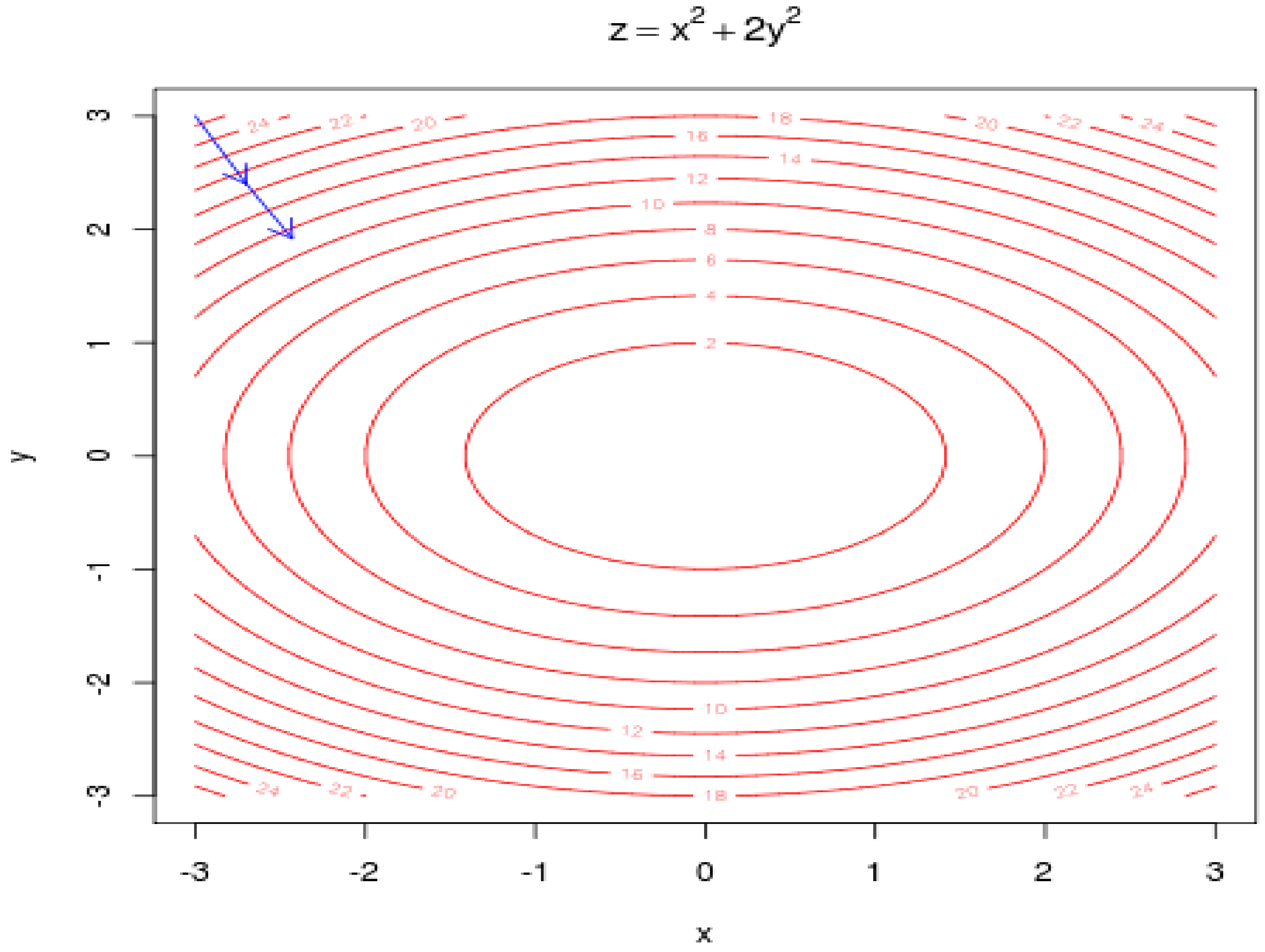

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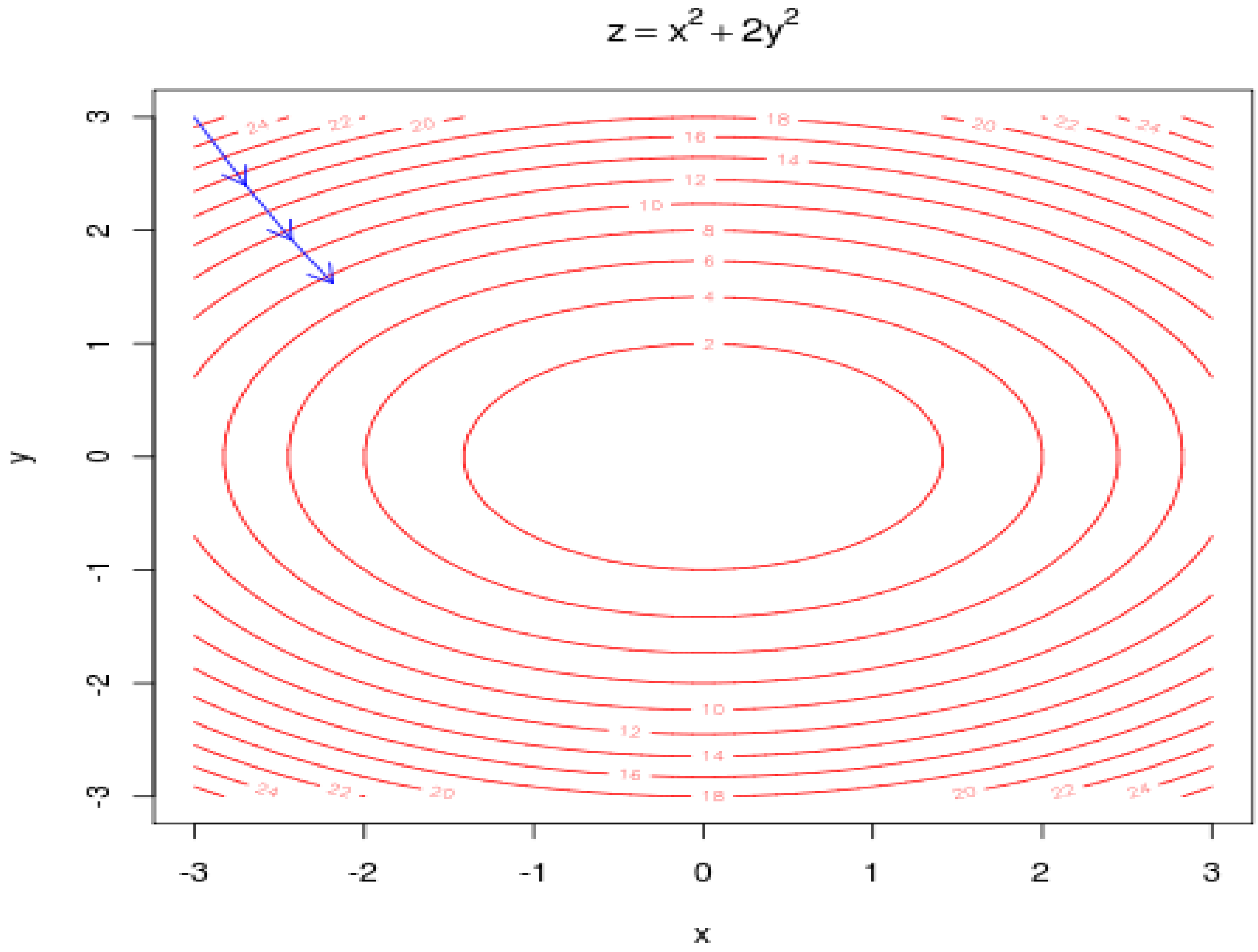

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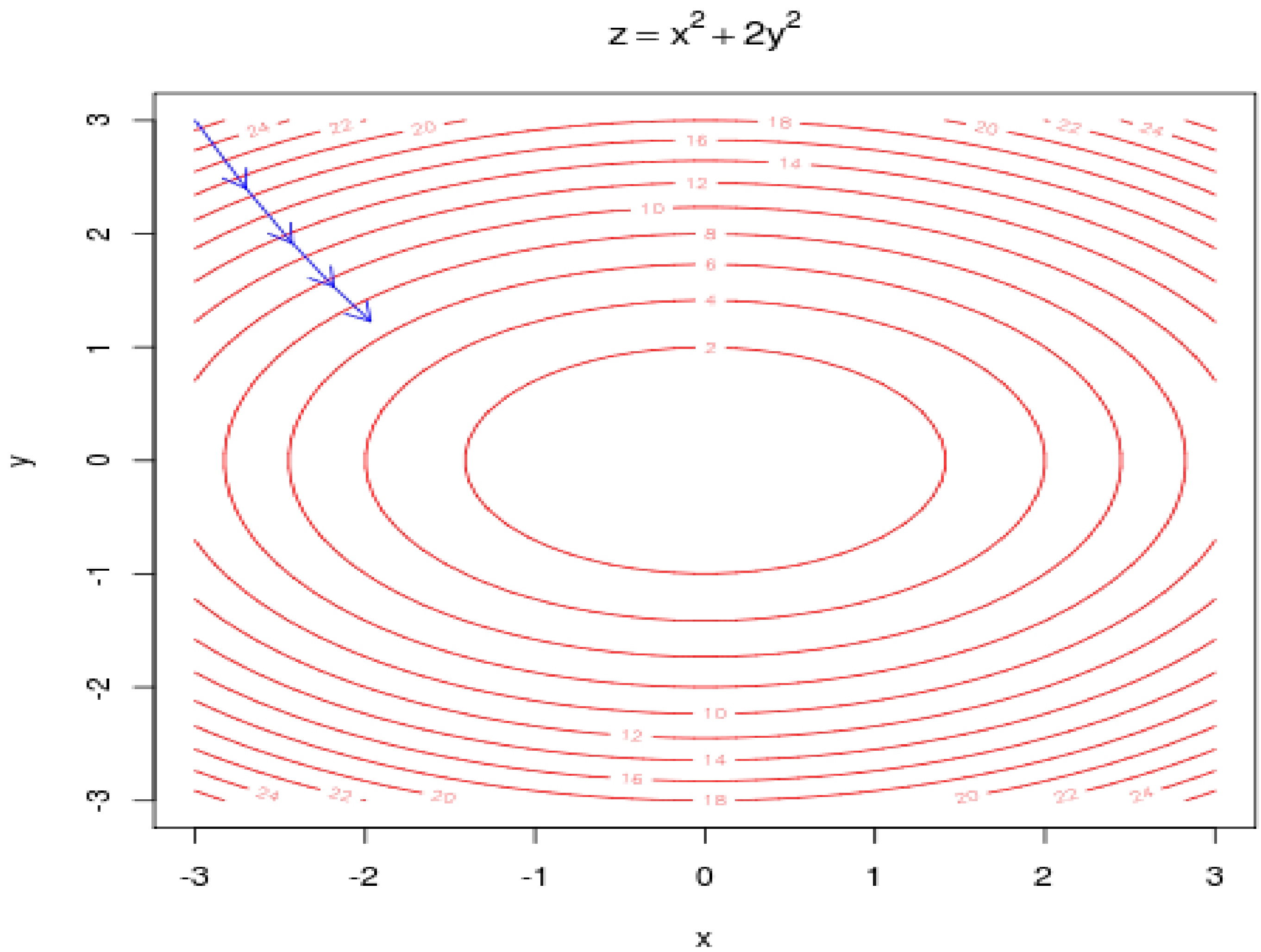

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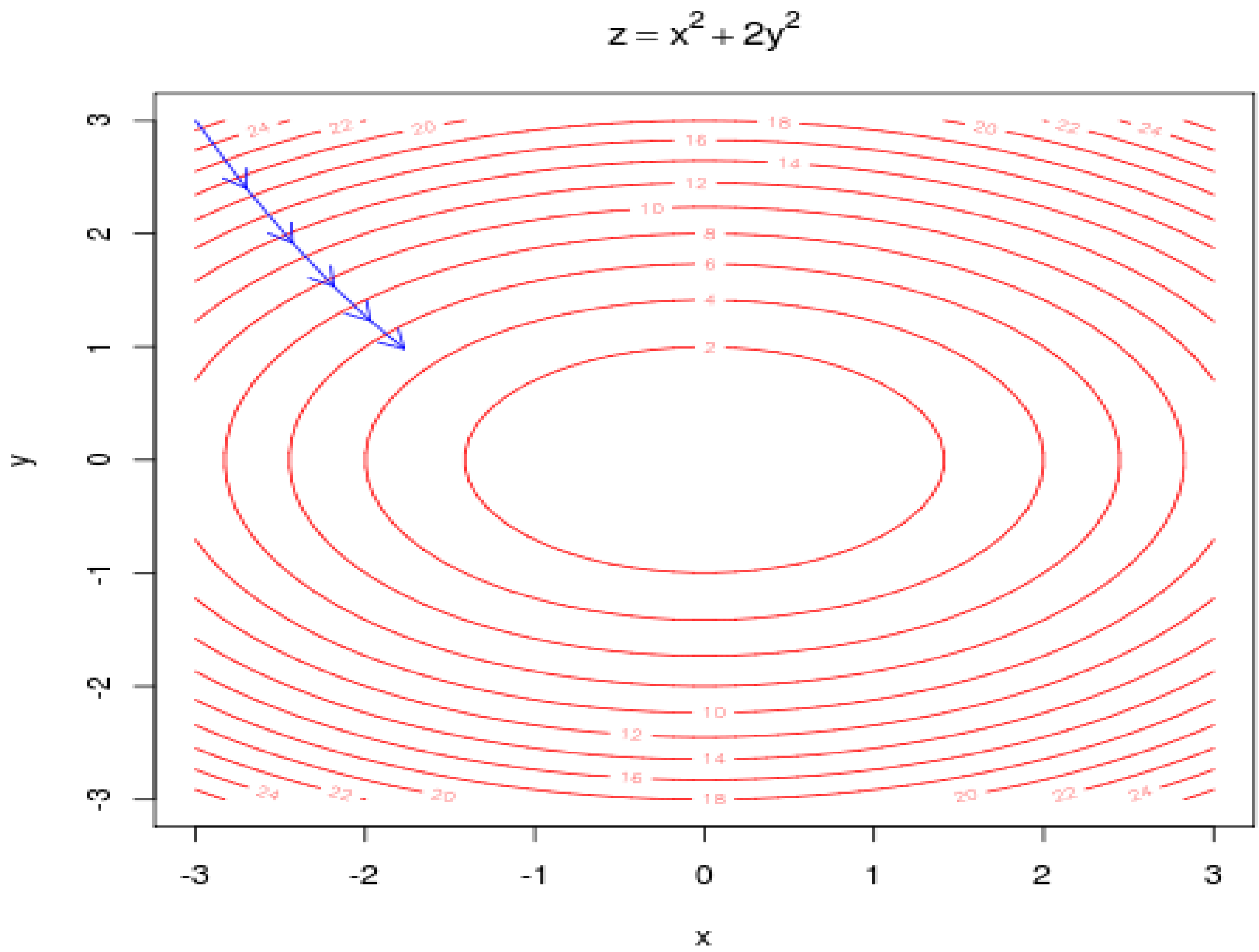

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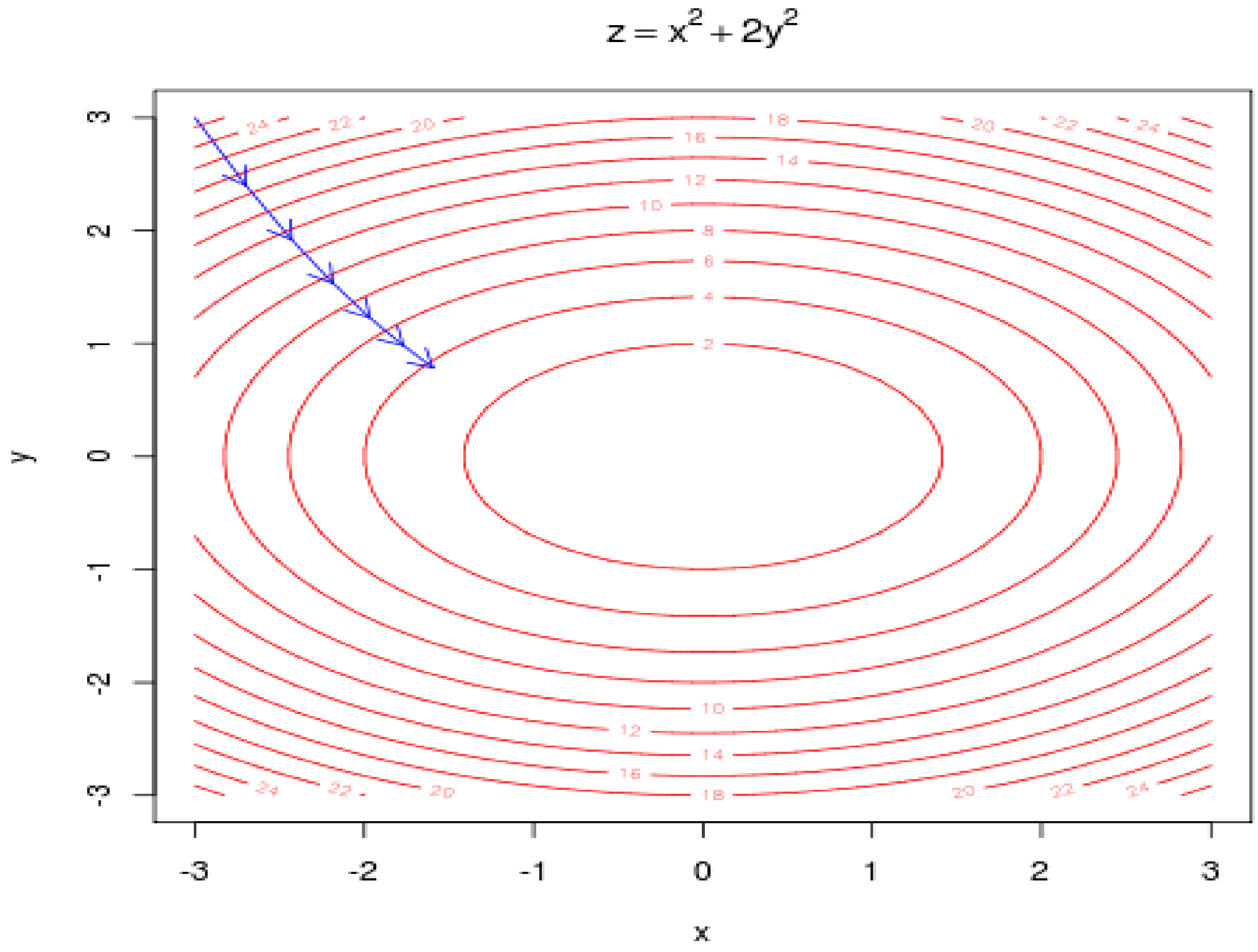

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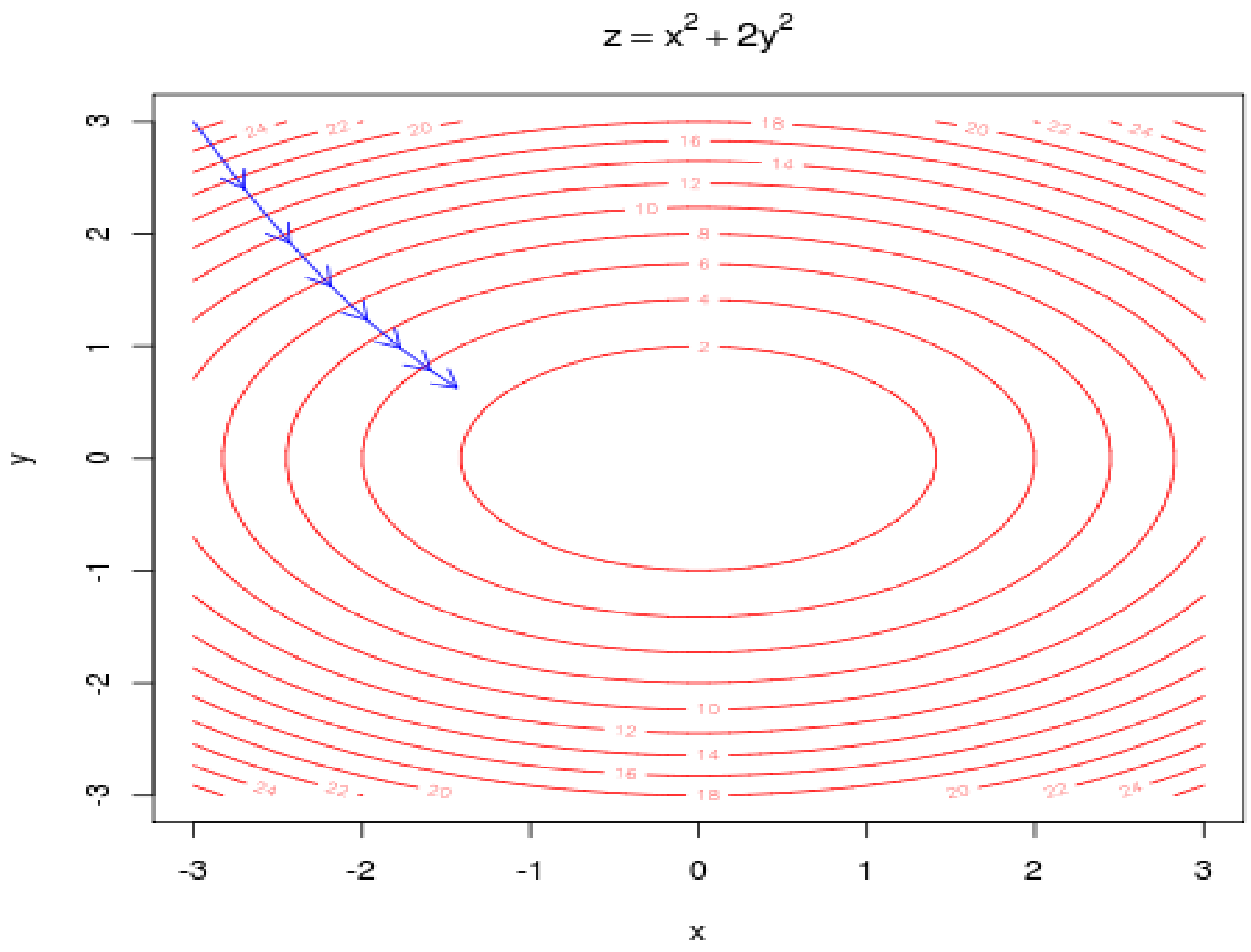

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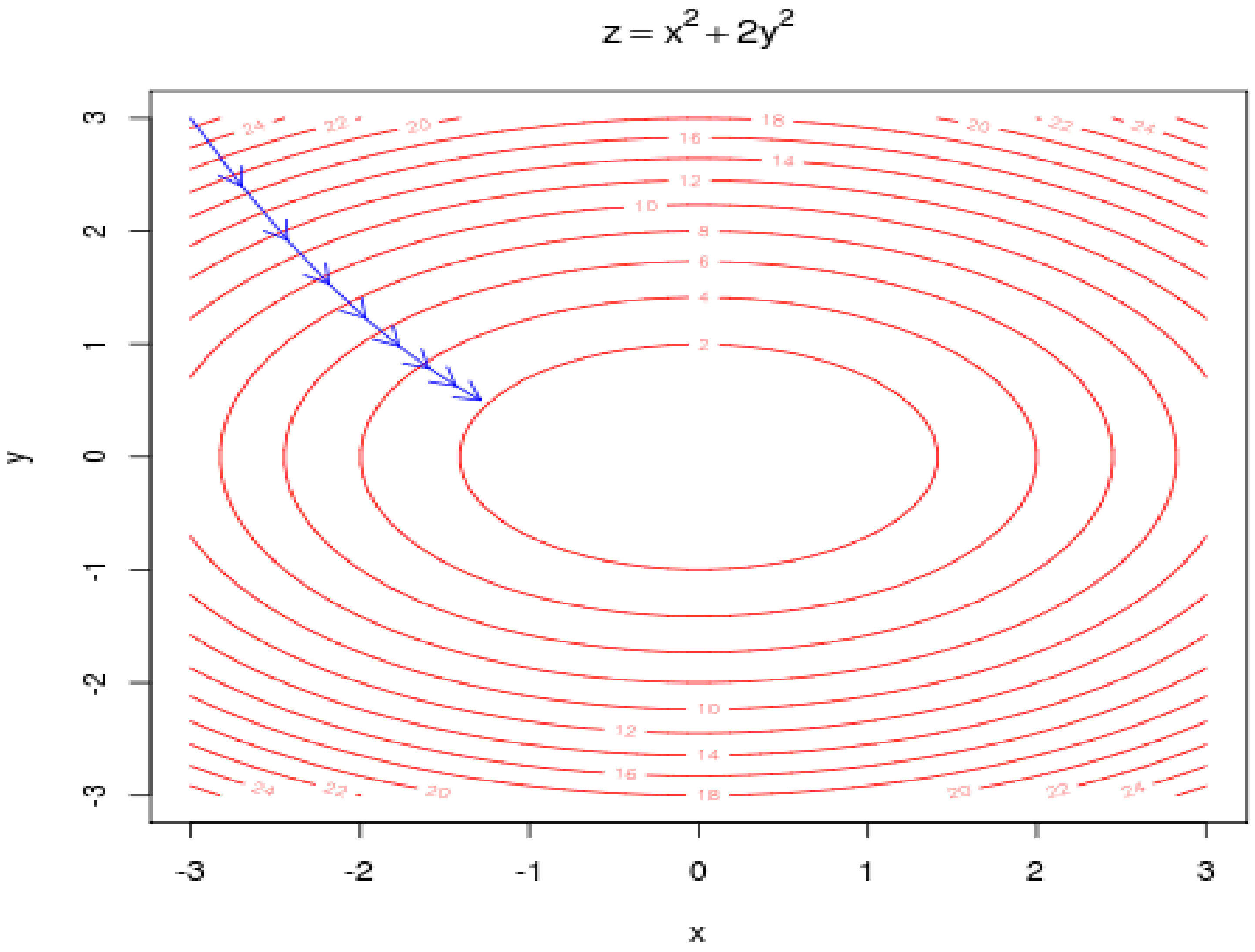

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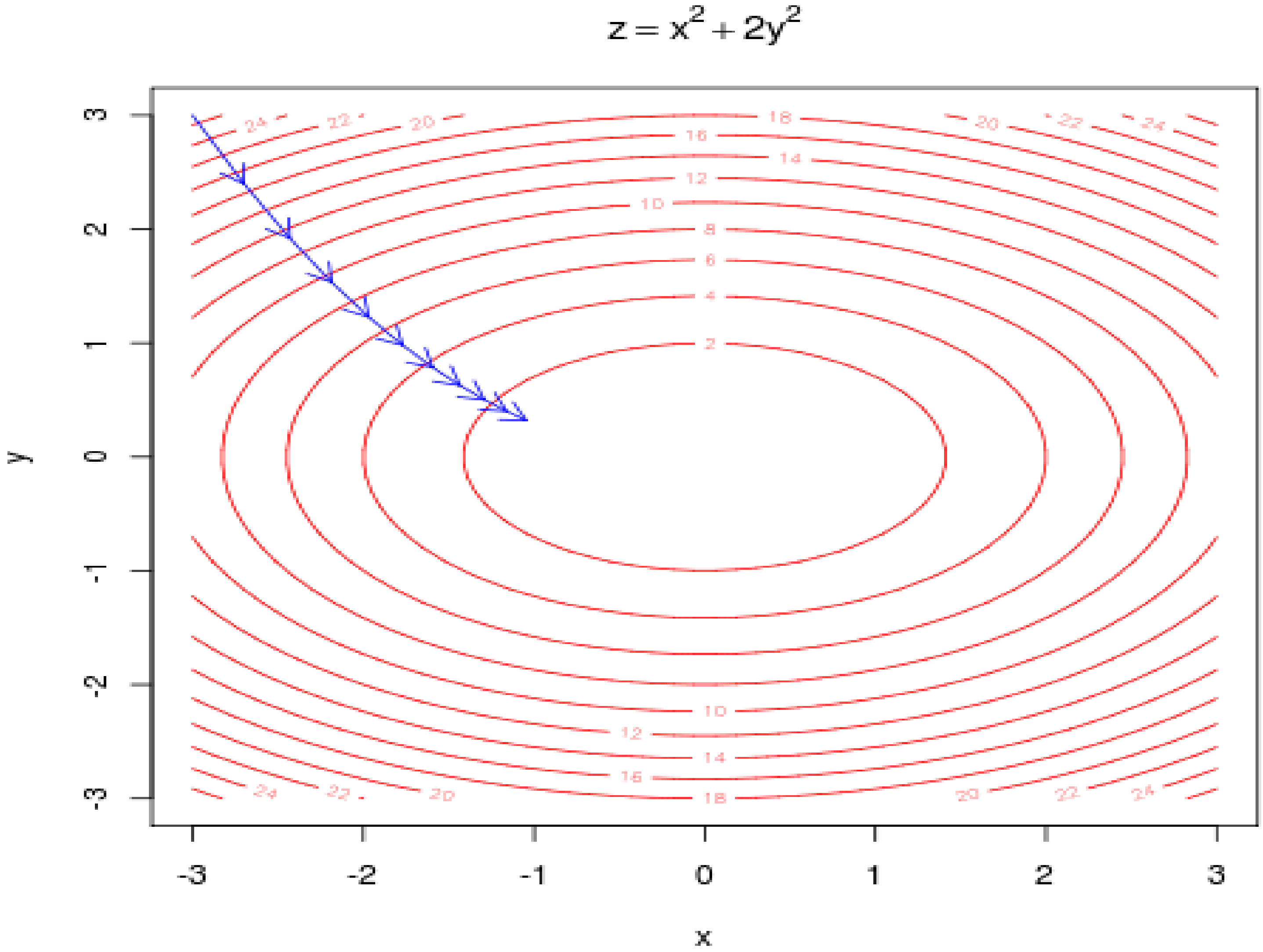

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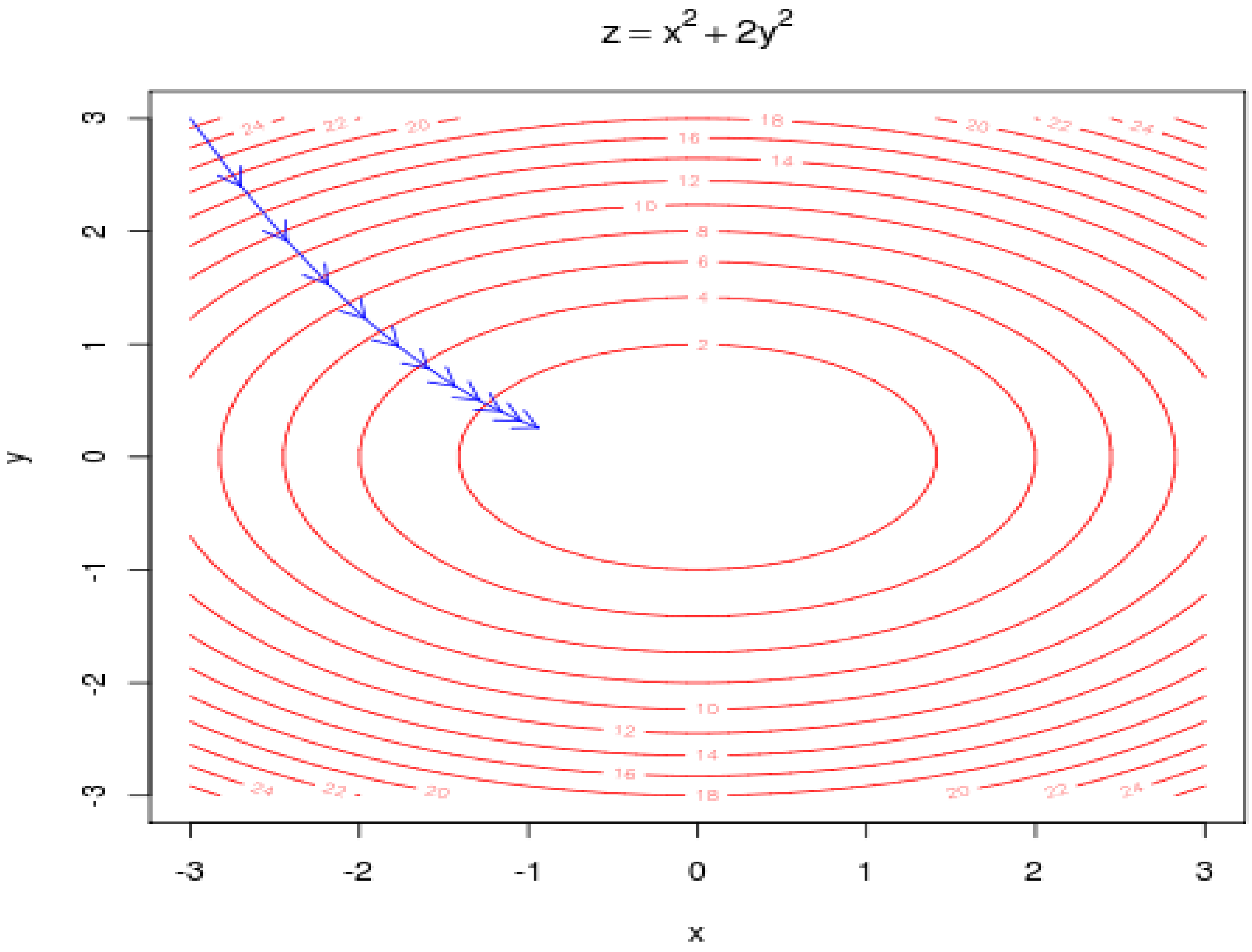

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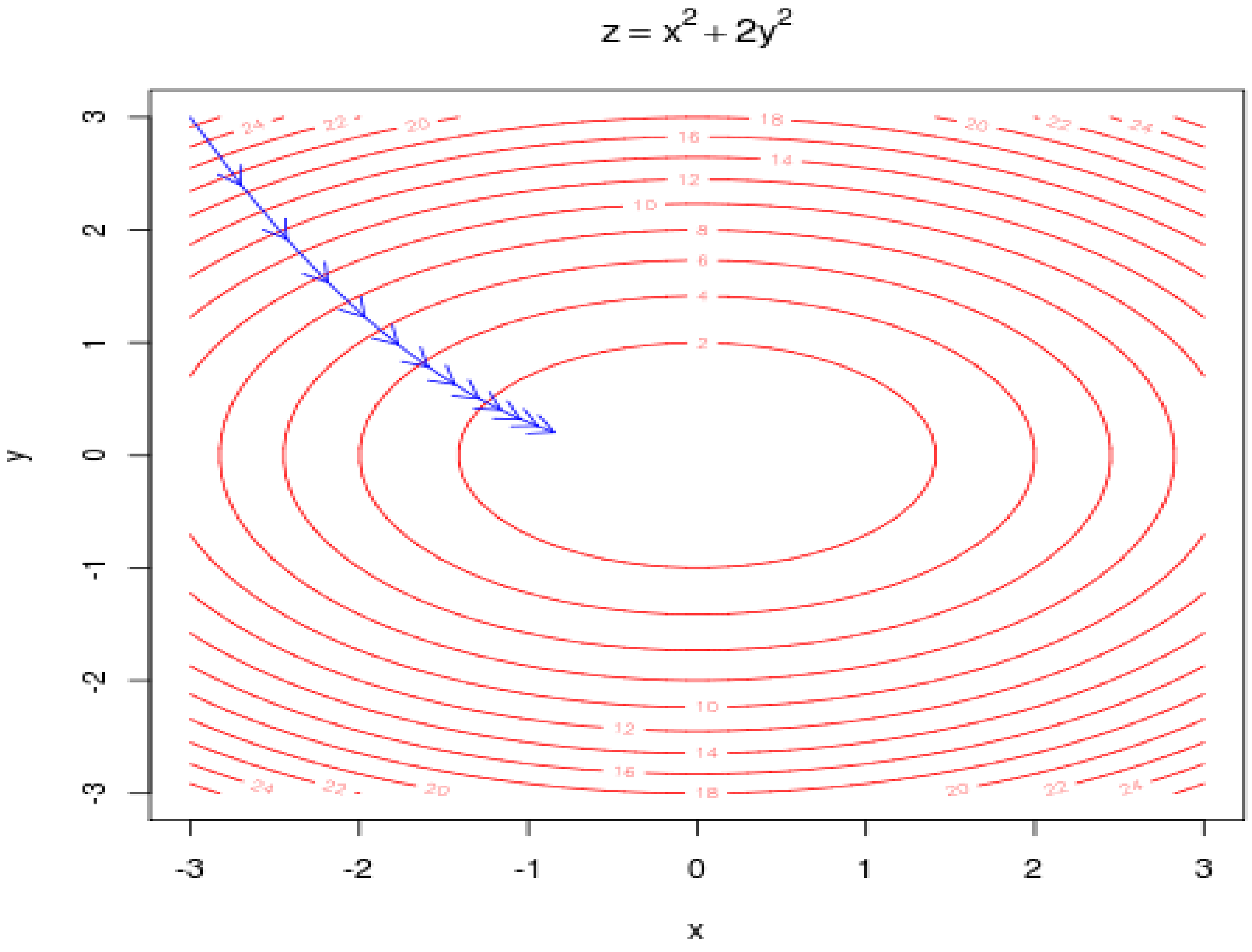

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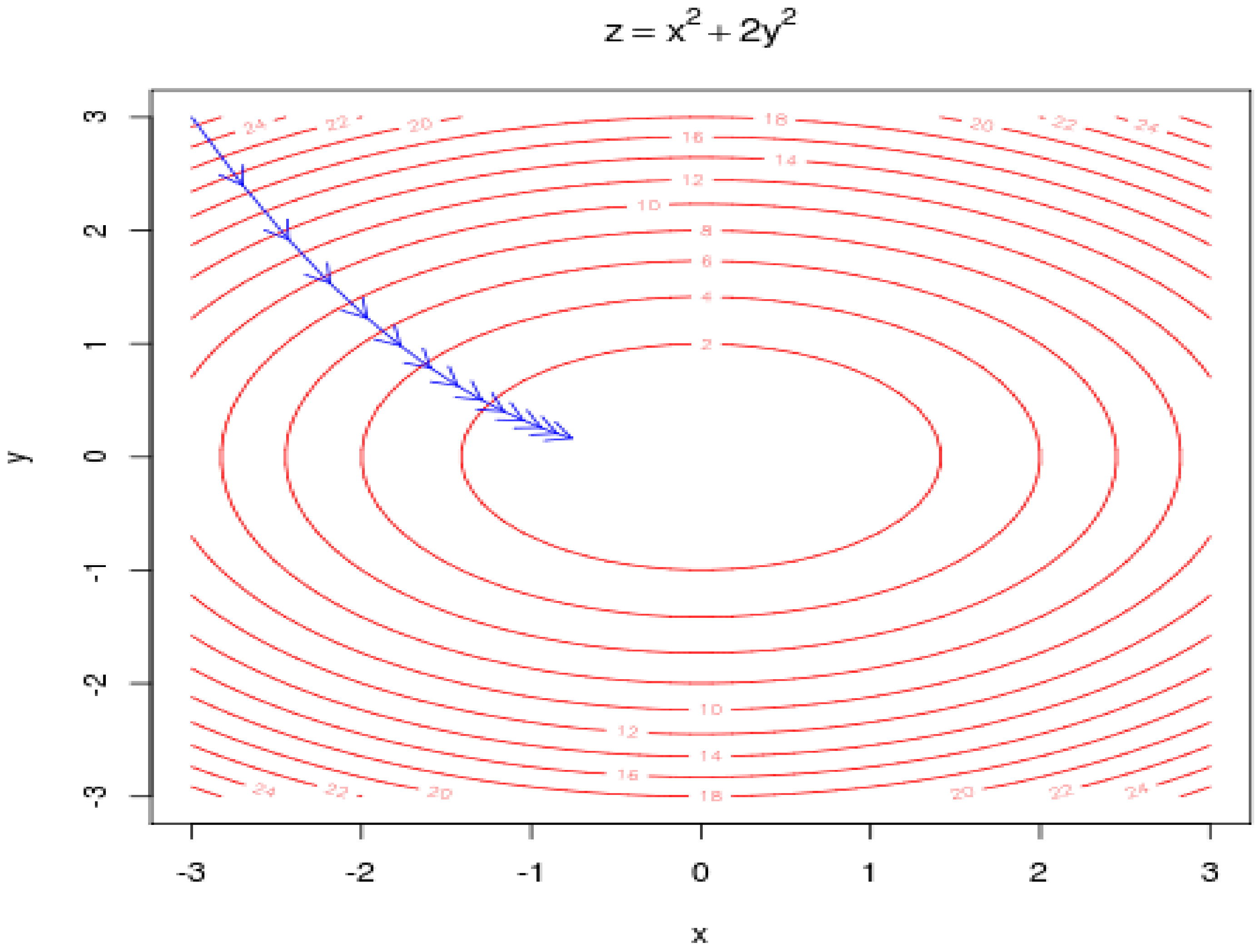

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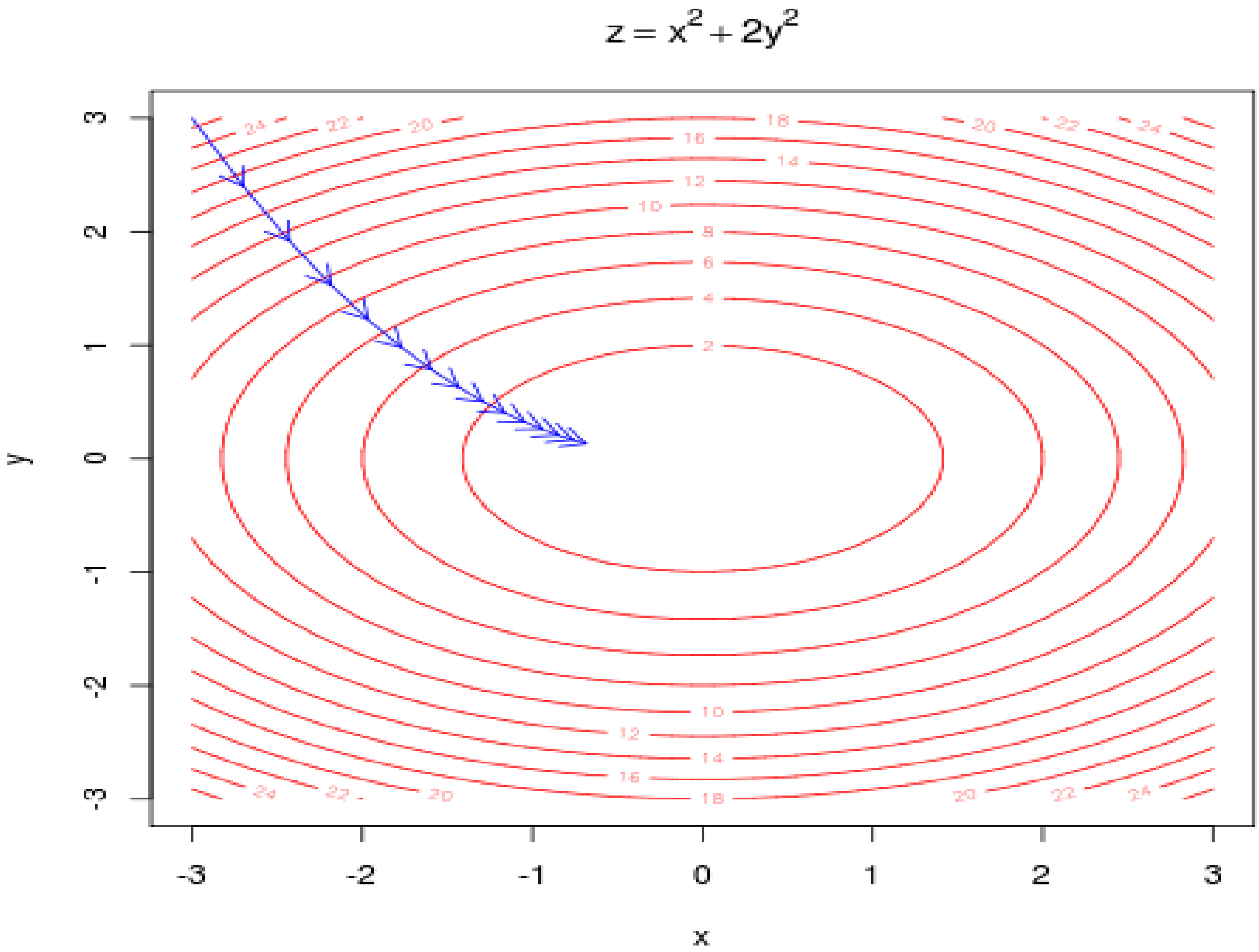

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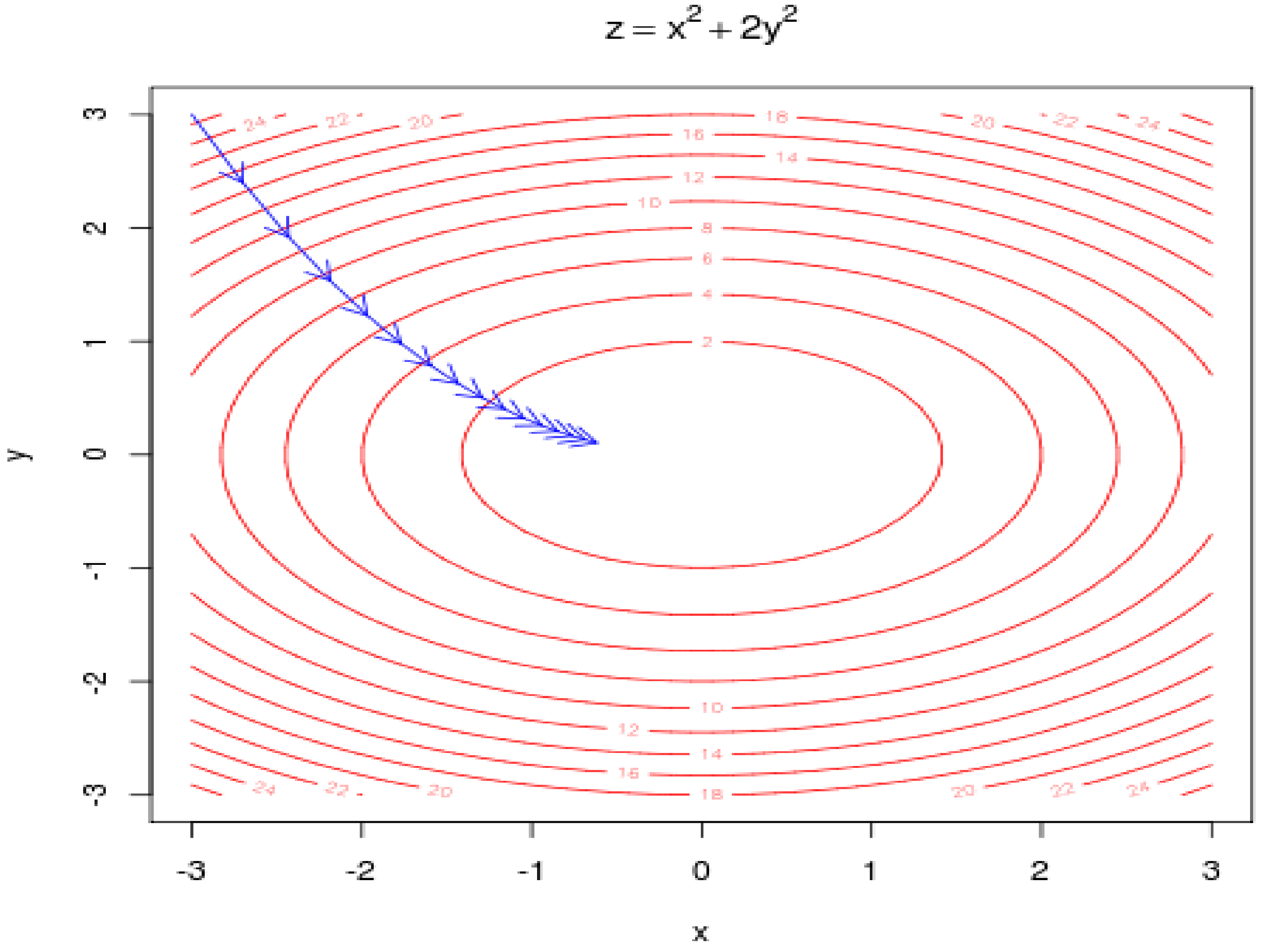

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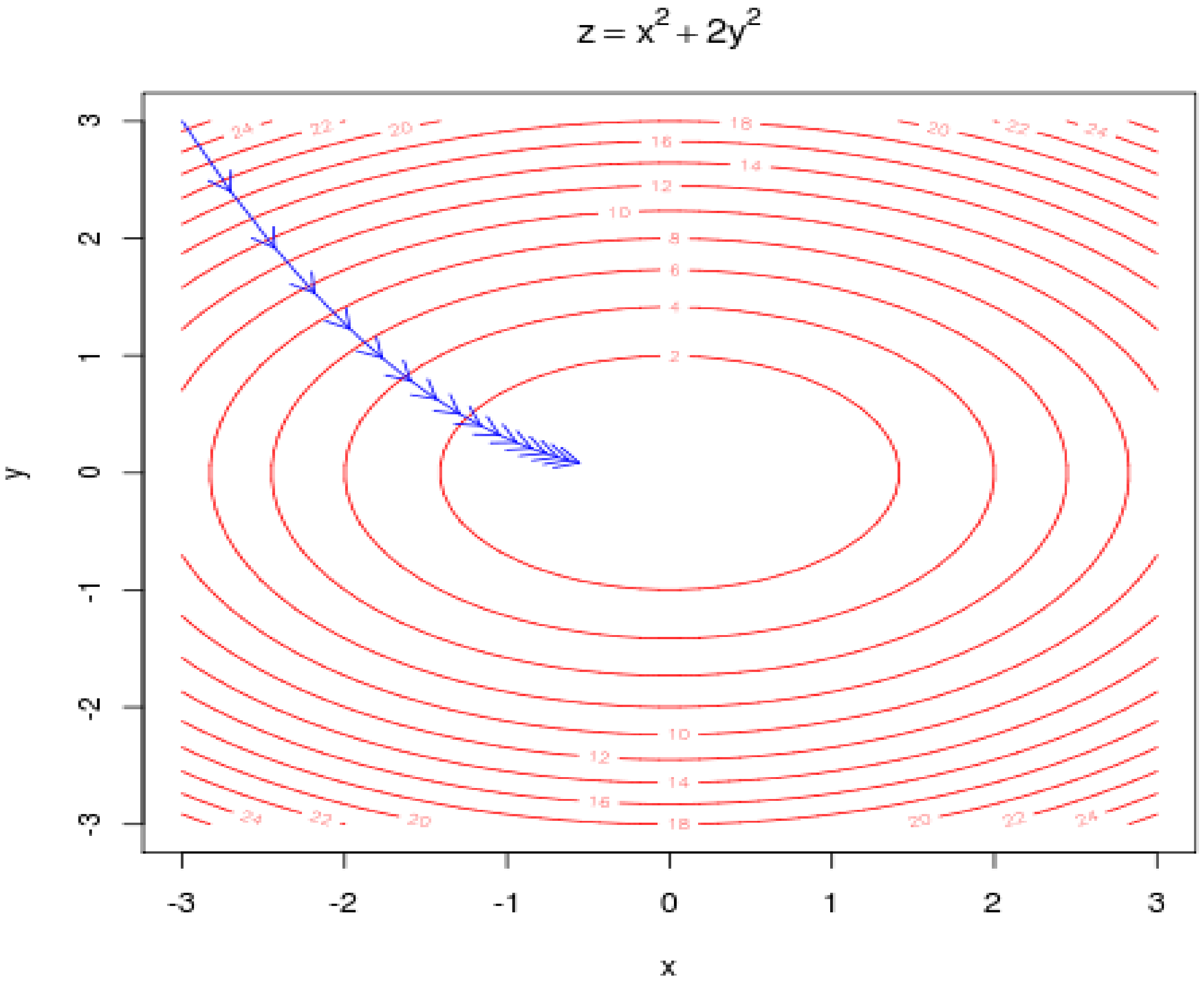

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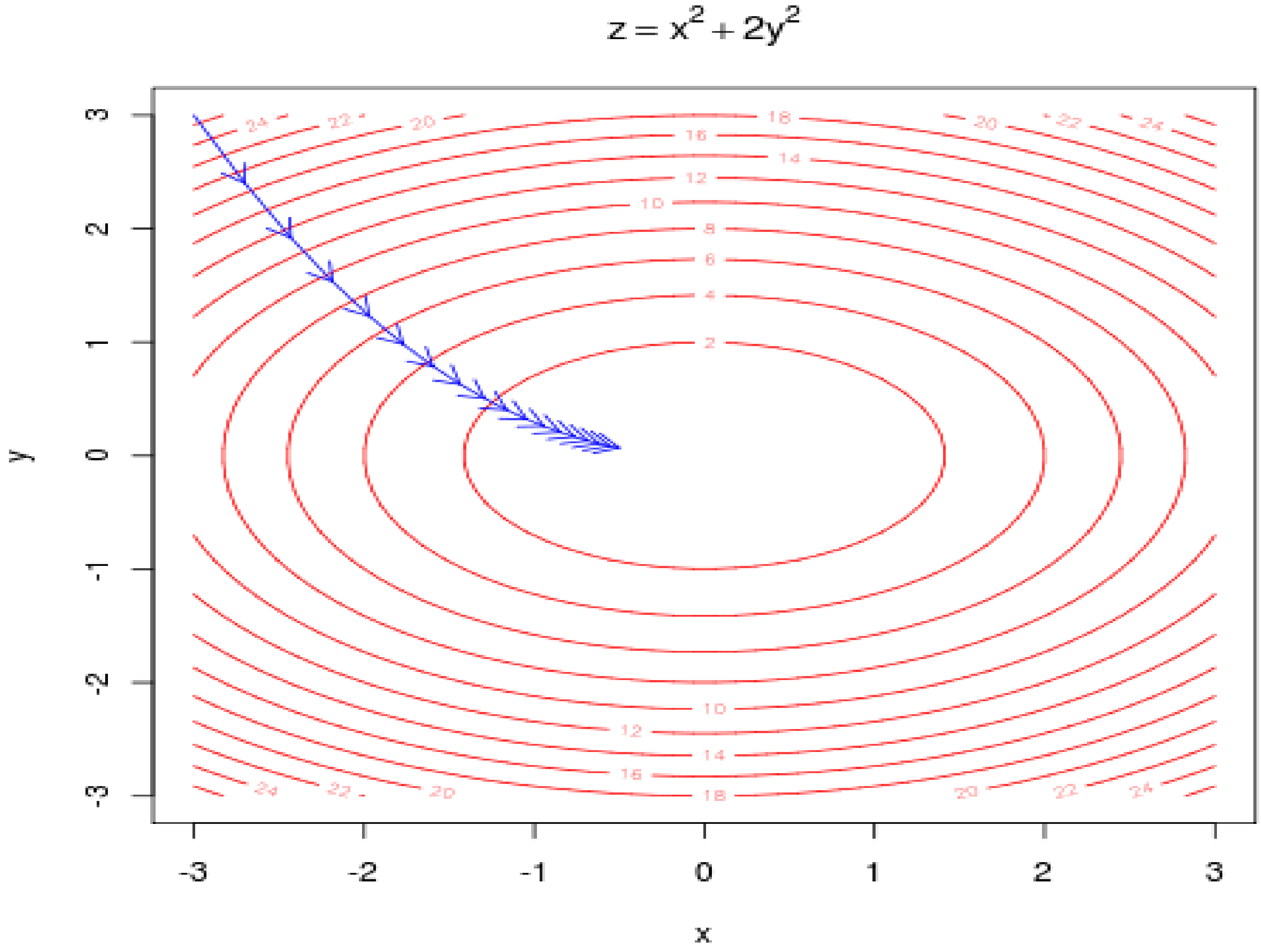

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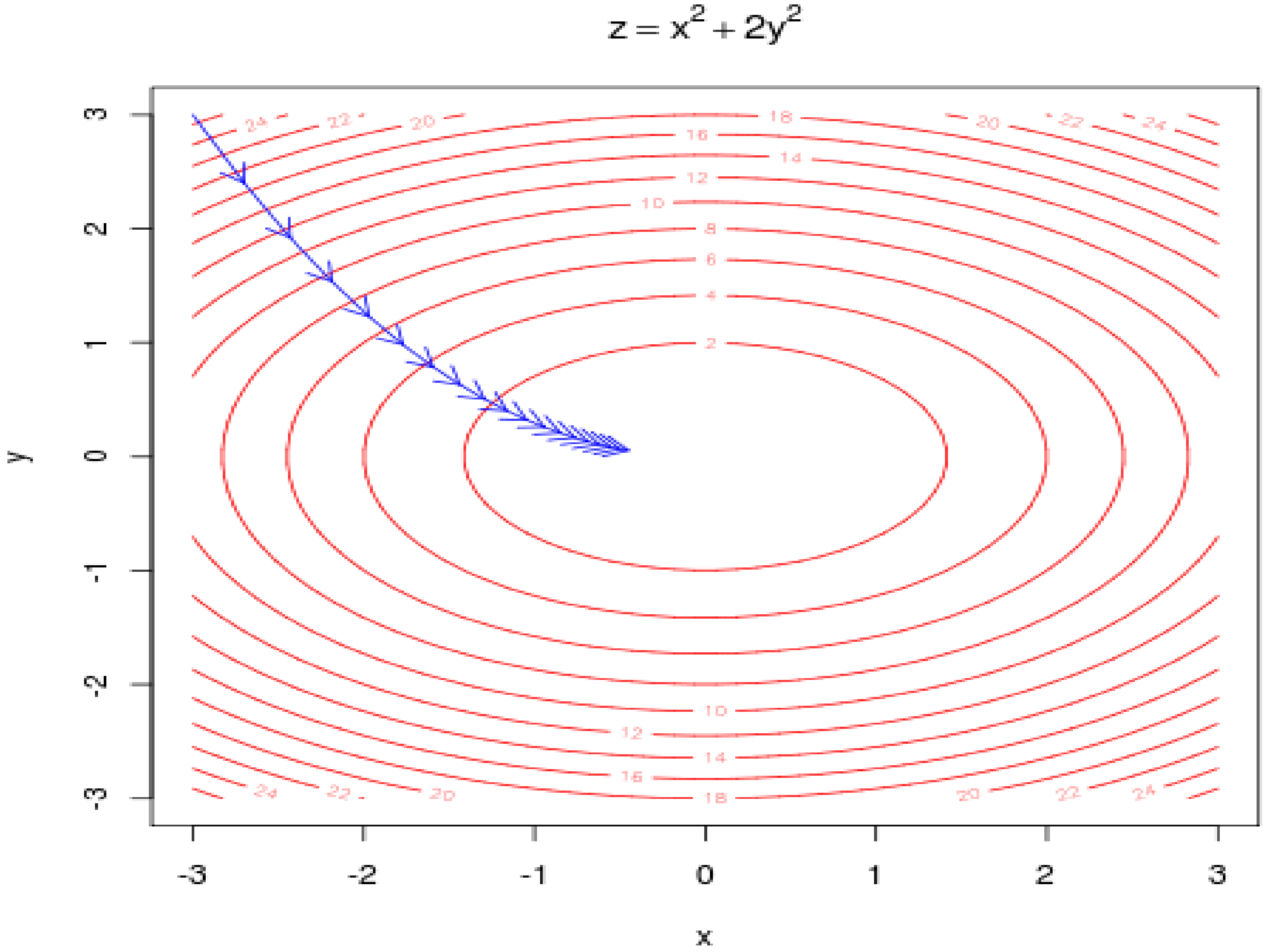

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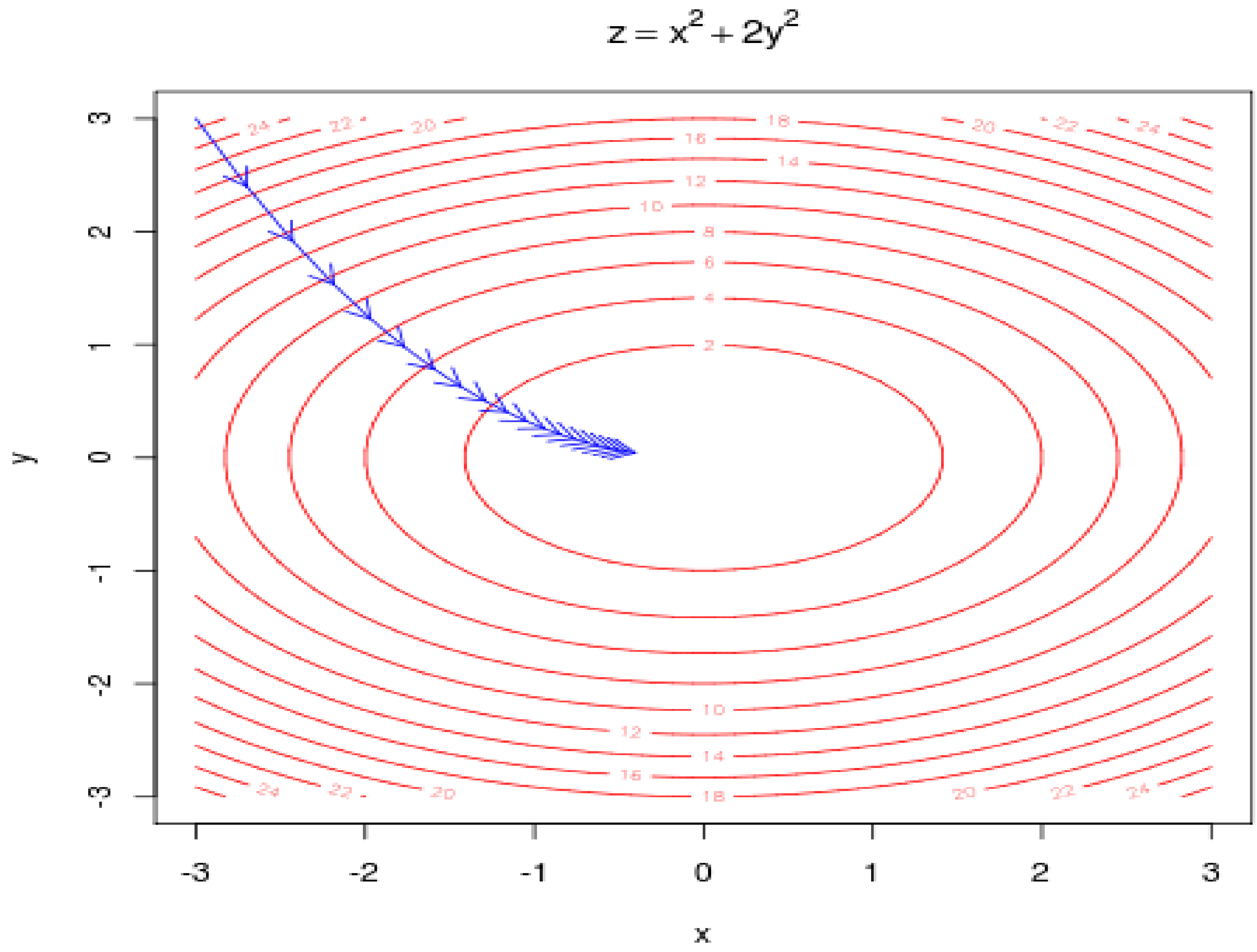

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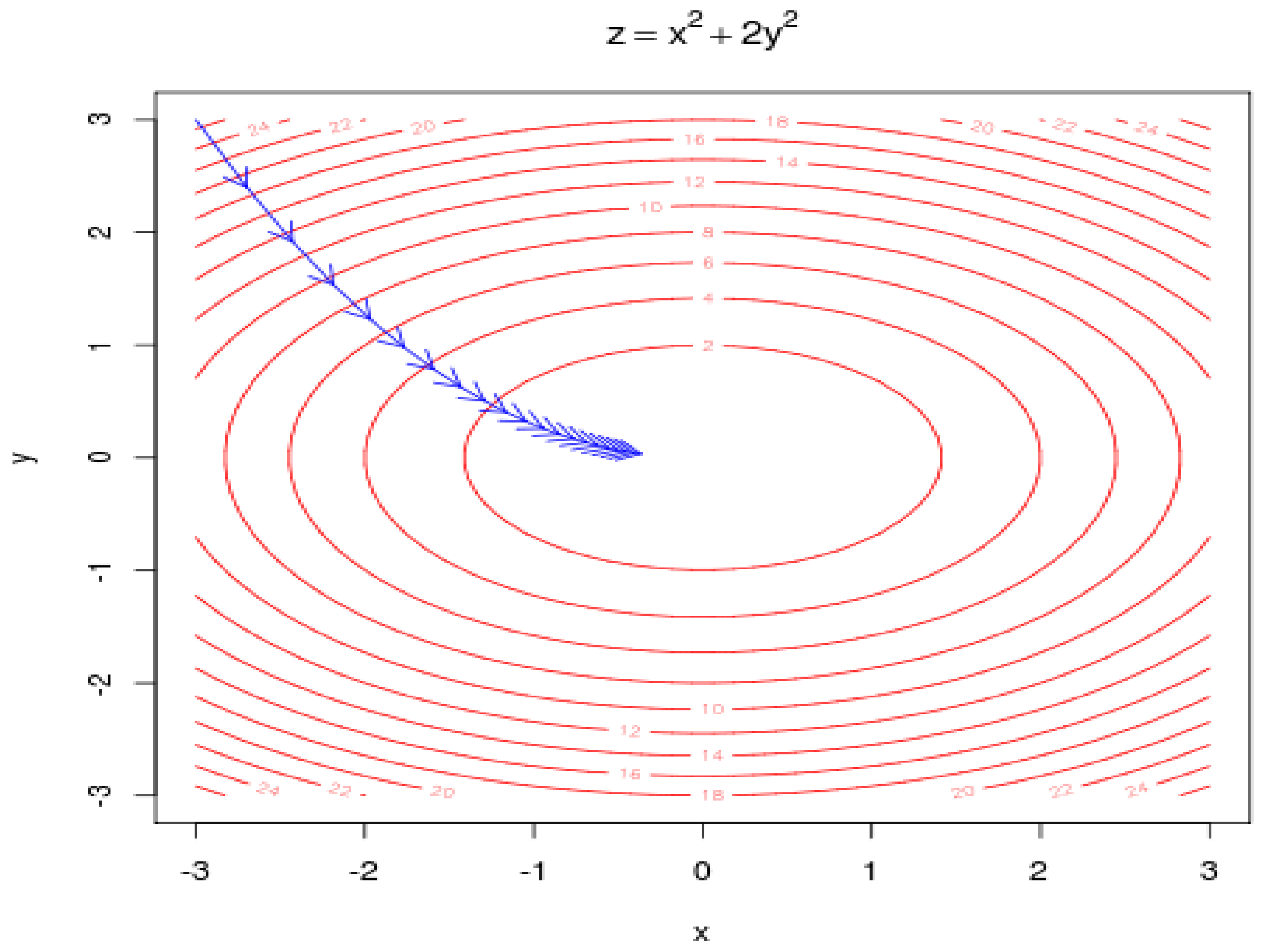

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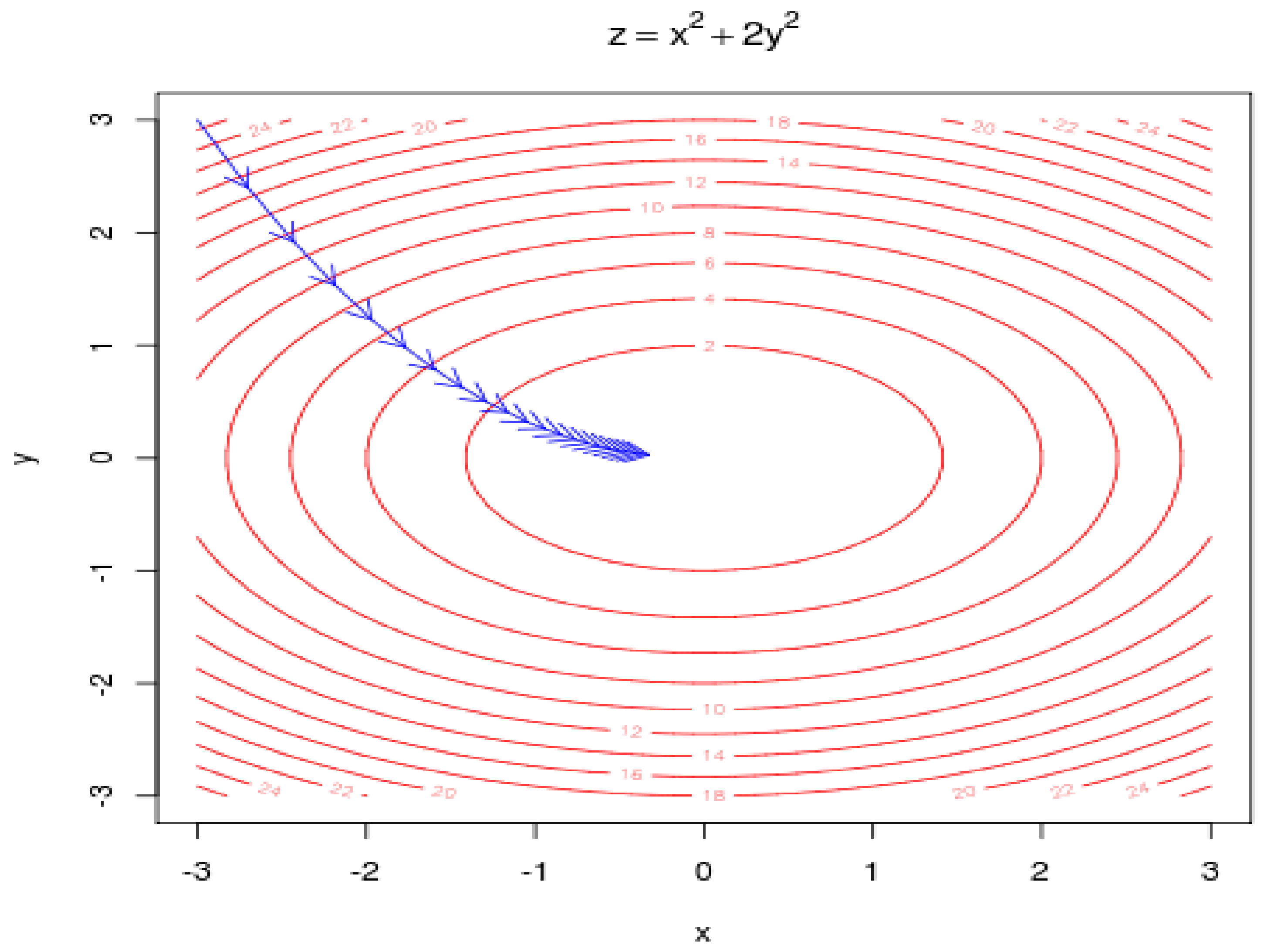

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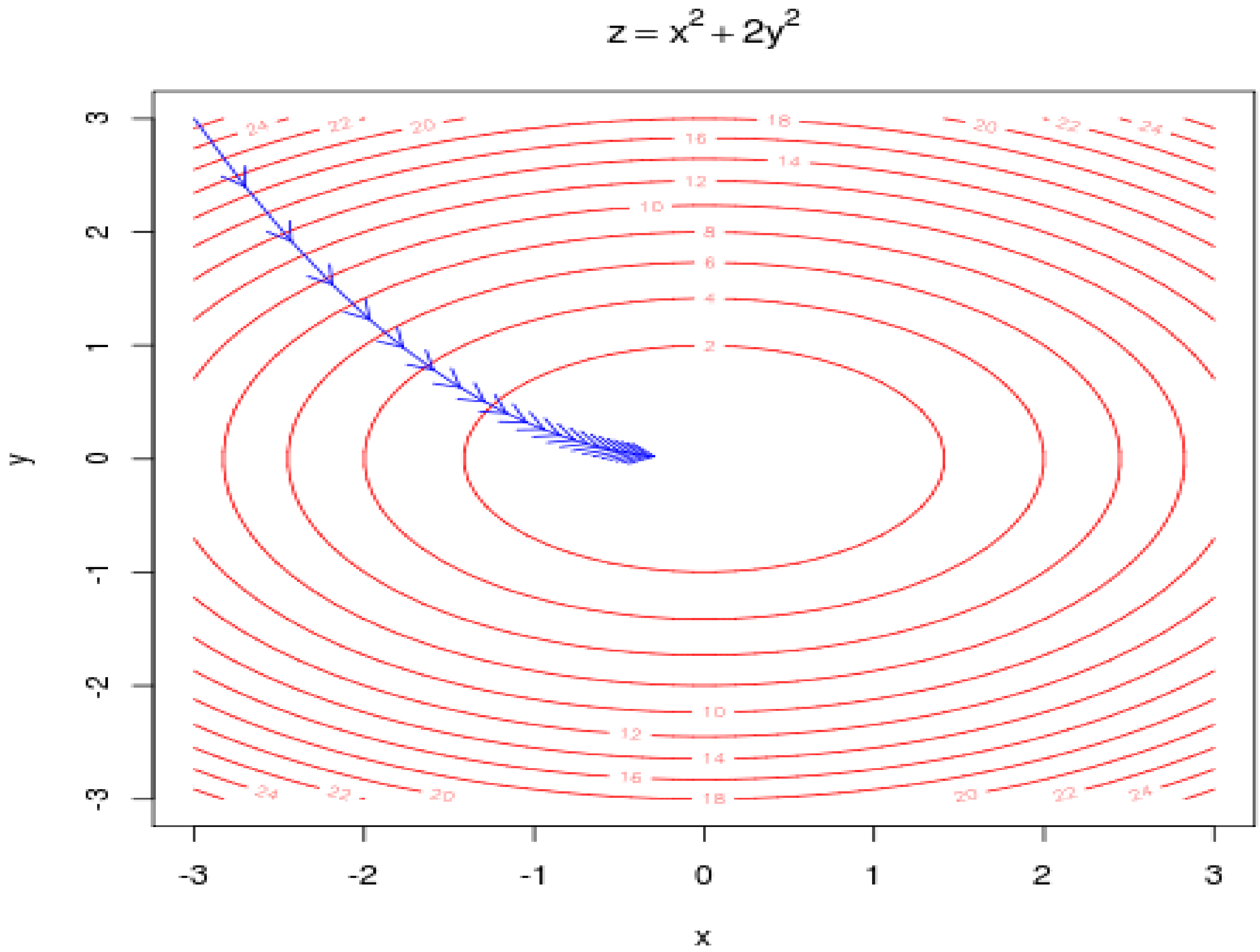

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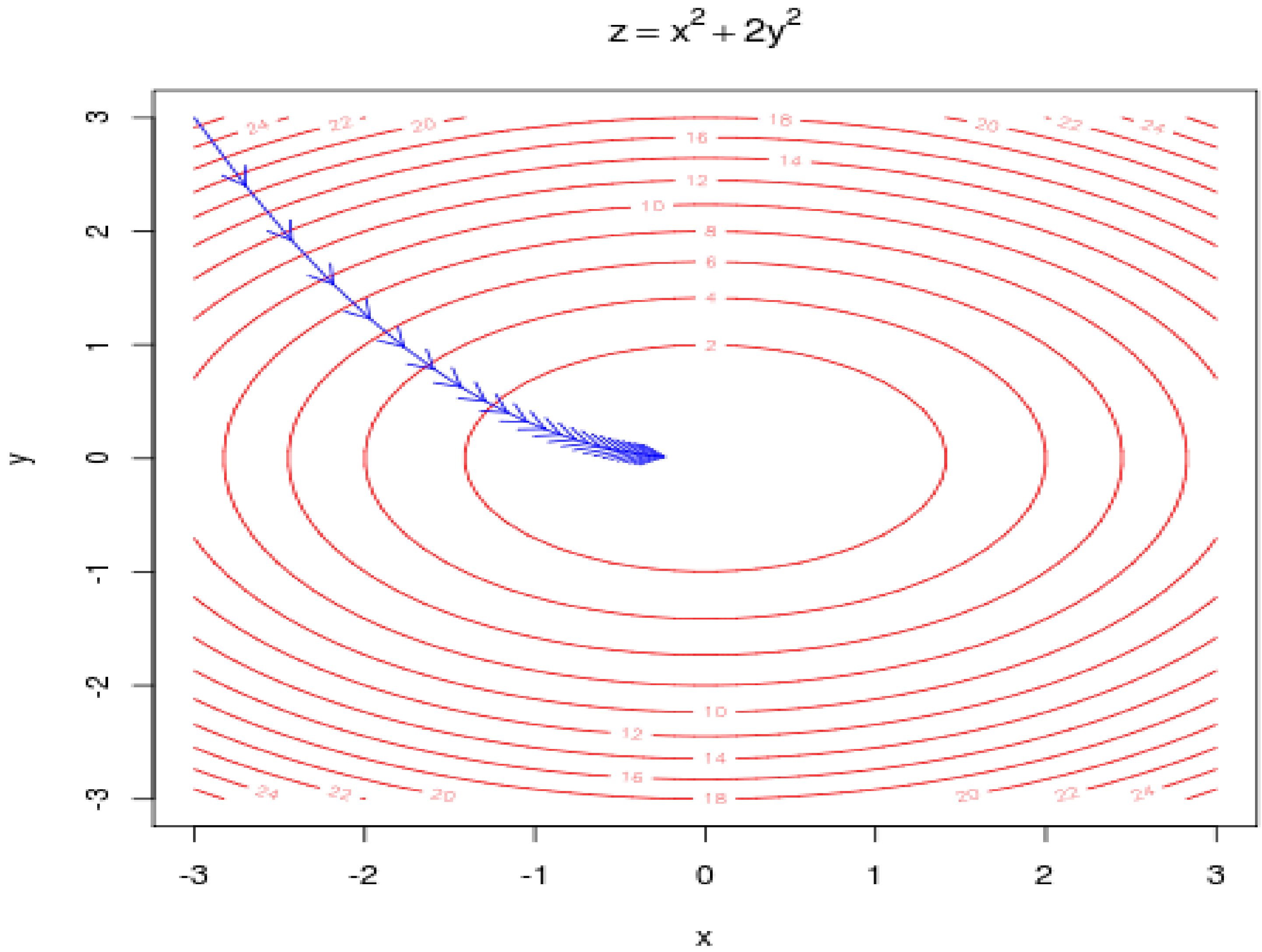

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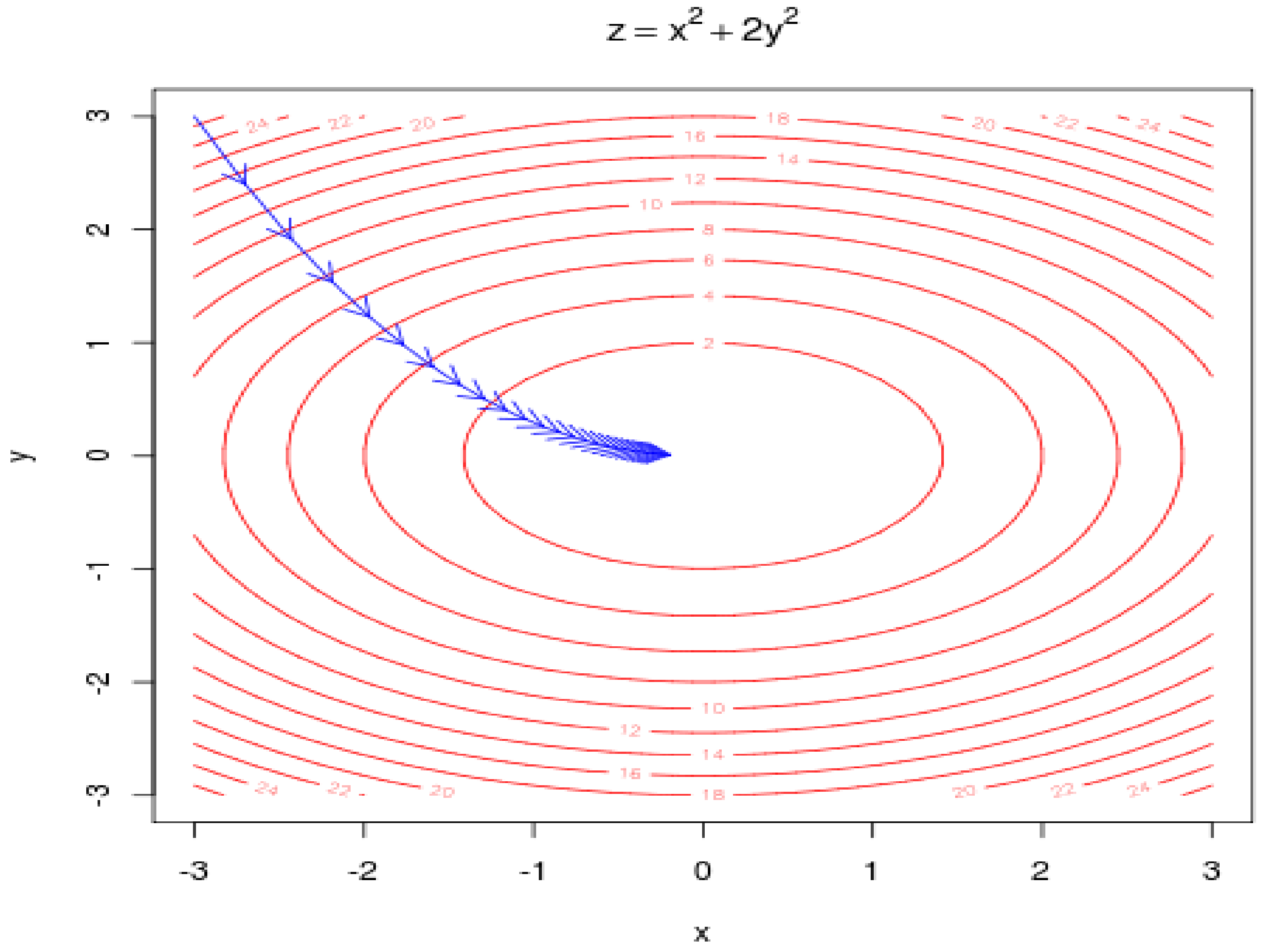

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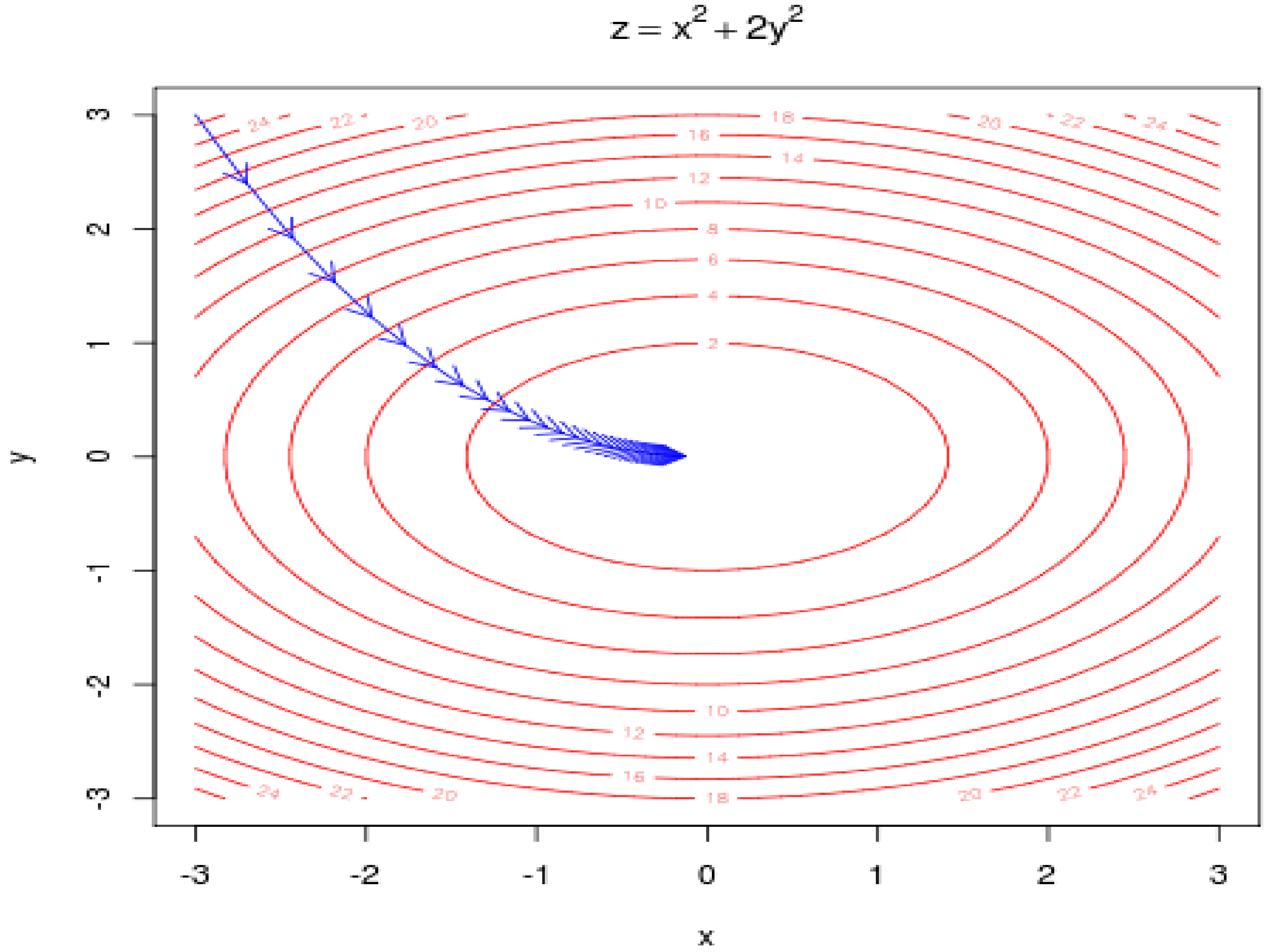

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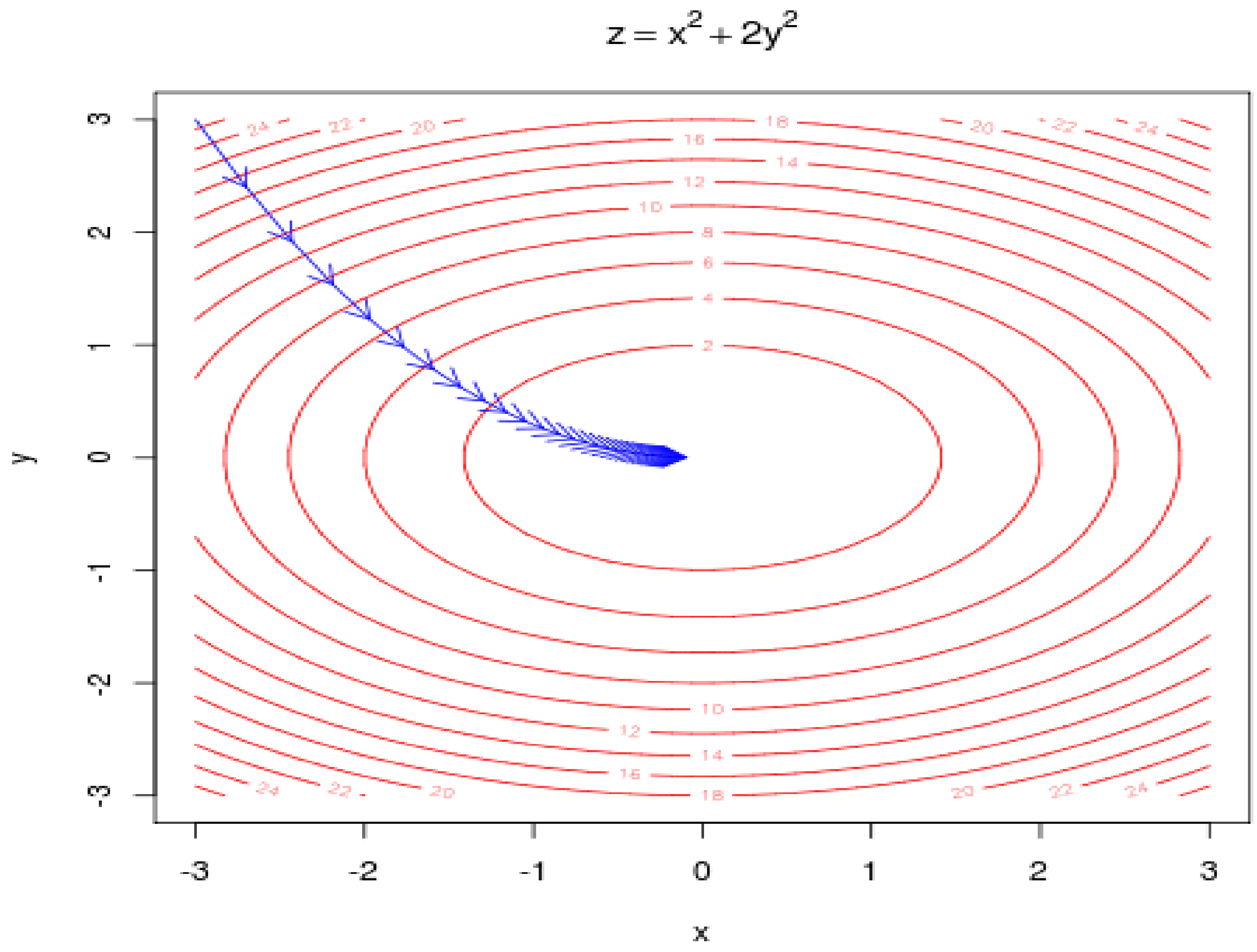

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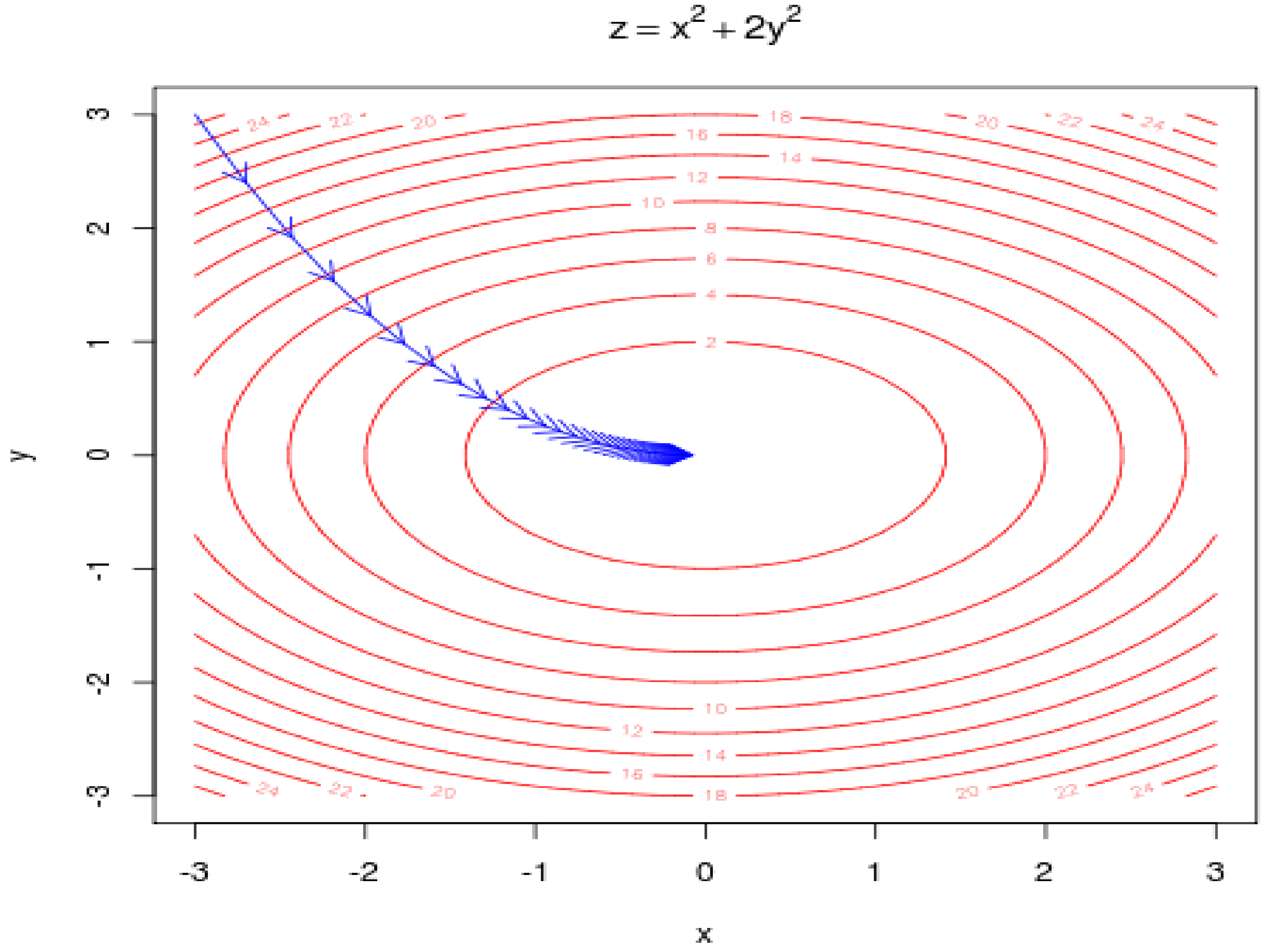

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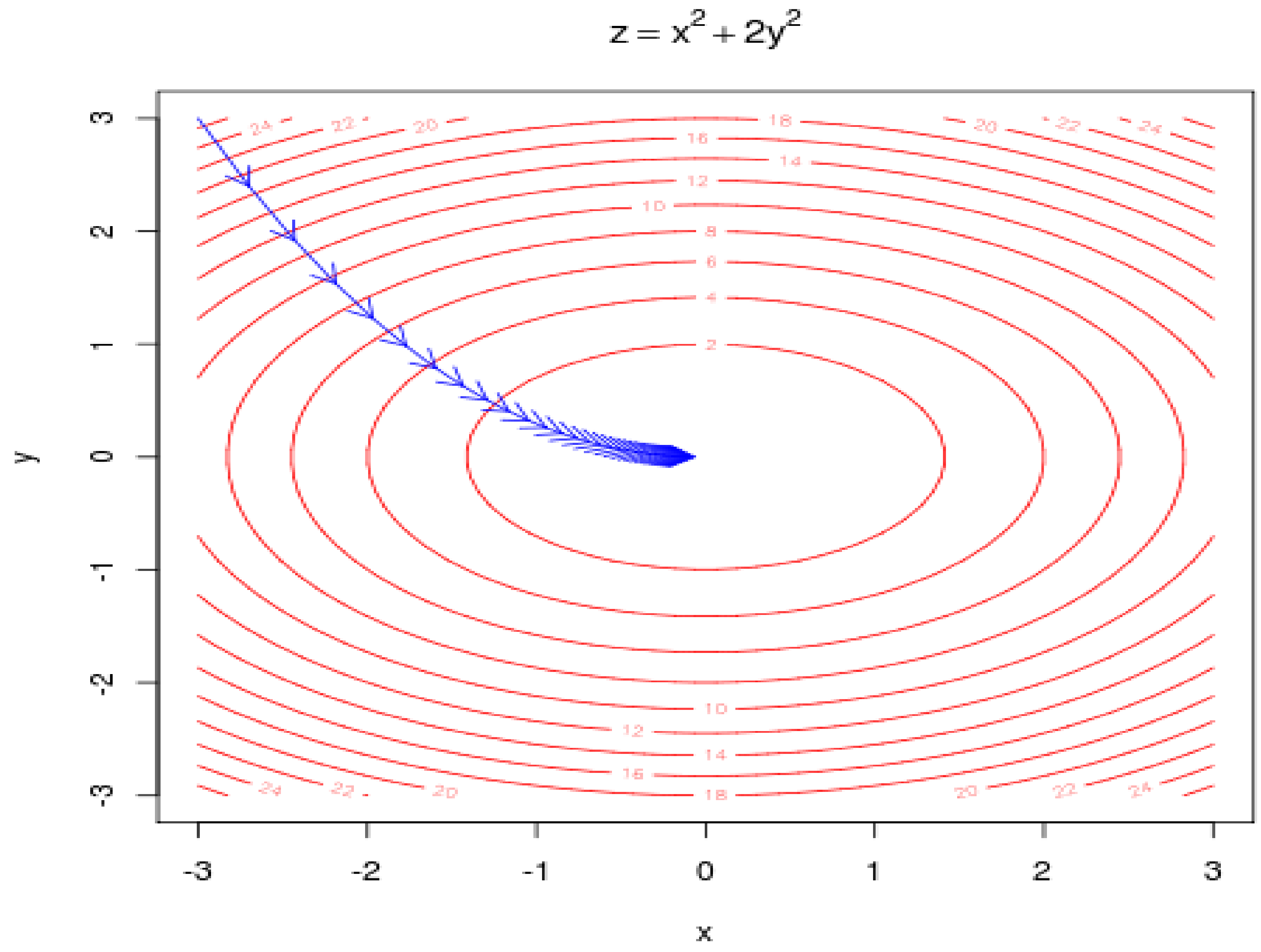

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Multivariate Linear Regression

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Predict Housing Prices – With Two Features

$x^i{}_1$ $x^i{}_2$ y^i

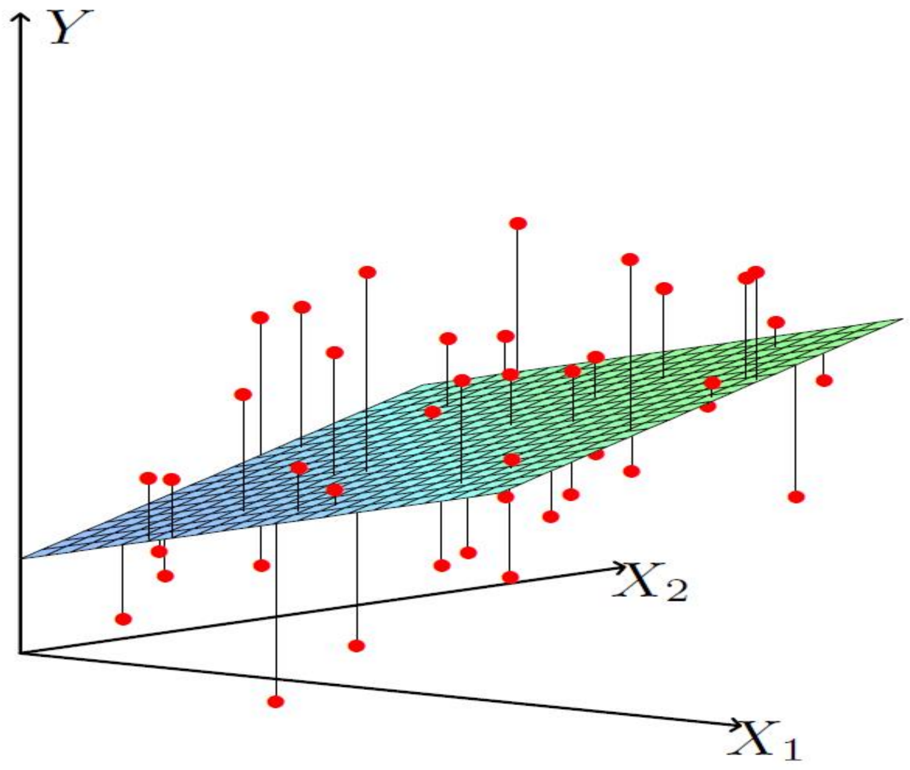
Living area (feet^2)	#bedrooms	Price (1000\$s)
2104	3	400
1600	3	330
2400	3	369
1416	2	232
3000	4	540
⋮	⋮	⋮

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

A linear function is just one of the choices to approximate the target variable.

θ is the space of linear functions mapping the space of input variables to the output/target variables

Predict Housing Prices – With Two Features



- When we have two features, the linearity is in plane instead of line.

Algebraic Notation

- The function approximating the target variable *y* is given by:

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\theta^T = [\theta_0 \; \theta_1 \theta_2]$$

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

as

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$$

$$h(x) = \sum_{i=0}^{n} \theta_i x_i = \theta^T x,$$

$$\theta^T x = [\theta_0 \; \theta_1 \theta_2] \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \sum_{i=0}^{n} \theta_i x_i$$

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Batch Gradient Descent

For ONE training examples, we get the following update rule:

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)}.$$

What do we observe here about the magnitude of the update?

For ALL training examples, we get the following update rule:

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

Stochastic Gradient Descent

- Consider the following algorithm:

Loop {

for i=1 to m, {

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

}

- This algorithm updates the parameters θ_j using each training example instead of all training examples.

- If the training set is big i.e., m is large, this technique converges quicker than batch gradient descent.

- Stochastic gradient descent may oscillate around the minimum of $J(\theta)$ and may not completely converge

Batch vs. Stochastic Gradient Descent

Batch Gradient Descent

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

- To update each parameter value, scan through the whole training data
- Converges to the minimum value slowly
- Preferred for small datasets

Stochastic Gradient Descent

Loop {

for i=1 to m, {

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

}

- Update the parameter values with one training example at a time
- Converges to the 'proximity' of minimum value fast but may keep oscillating near the minimum
- Preferred for large datasets

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Probabilistic Interpretation of Linear Regression

- We will discuss why the least-squares cost function $J(\theta)$ is a reasonable choice.

- Assume that the inputs and the target variable are related by the following equation:

$$y^{(i)} = \theta^T x^{(i)} + \epsilon^{(i)}$$

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Probabilistic Interpretation of Linear Regression

- Where the ϵ^i term captures the un-modeled effects:
  - Random noise
  - Some relevant feature not taken into consideration

- Assumptions:

  - ϵ^i is IID (Independently and Identically Distributed)
  - Normal distribution with mean 0 and variance σ^2
  - variance σ^2 is random

$$\epsilon^i = N(0, \sigma^2)$$

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Probabilistic Interpretation of Linear Regression

The density of ϵ^i is given b $p(\epsilon^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\epsilon^{(i)})^2}{2\sigma^2}\right)$

We can rewrite:

$$p(y^{(i)}|x^{(i)};\theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right)$$

This is the distribution of y^igiven x^i and parameterized by θ

$$y^{(i)} \mid x^{(i)};\theta \sim \mathcal{N}(\theta^T x^{(i)}, \sigma^2)$$

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Probabilistic Interpretation of Linear Regression

- Probability of observing the data as a function of θ is given by:

$$L(\theta) = L(\theta; X, \vec{y}) = p(\vec{y} | X; \theta).$$

- This is known as the likelihood function
- Due to the independence assumption, we can rewrite:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{m} p(y^{(i)} \mid x^{(i)}; \theta) \\ &= \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right). \end{aligned}$$

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Maximizing The Likelihood Function

- Principle of maximum likelihood: Choose θ that makes the observed data as high probability as possible. In other words, choose θ that maximizes $L(\theta)$.

- For mathematical ease, we can maximize the log of the likelihood function $L(\theta)$ instead.

Maximizing The Log Likelihood Function

$$
\begin{aligned}
\ell(\theta) &= \log L(\theta) \\
&= \log \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\
&= \sum_{i=1}^{m} \log \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\
&= m \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{\sigma^2} \cdot \frac{1}{2} \sum_{i=1}^{m} (y^{(i)} - \theta^T x^{(i)})^2.
\end{aligned}
$$

$$
\frac{1}{2} \sum_{i=1}^{m} (y^{(i)} - \theta^T x^{(i)})^2
$$

Maximizing $L(\theta)$ is equivalent to minimizing $J(\theta)$

Linear Regression Using R

- We will learn how to use linear regression for handwritten digit recognition now
- Distinguishing between 2s and 3s
- We will see some R code
  - For processing the inputs
  - Applying linear regression to learn a hypothesis
- You can download R by searching 'R download'

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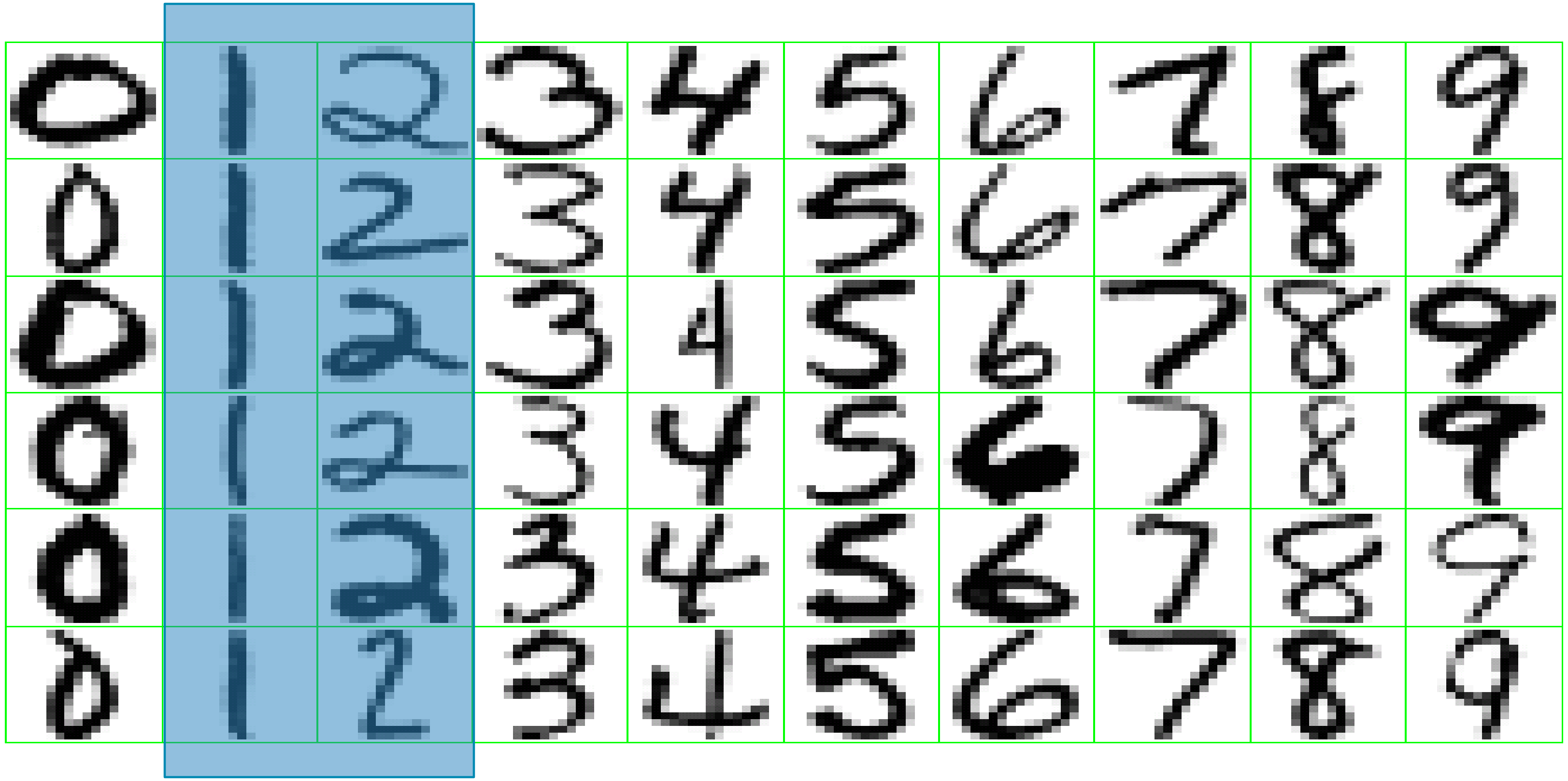
Example: Handwritten Digit Recognition



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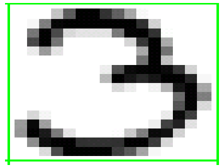
Extracting Features For Learning



$\{x_1, x_2, x_3 \ldots\ldots\ldots\; X_{256}, y$ = 'three'$\}$

- Each x_i corresponds to a feature value in the image
- y is a label of the training data; can be numeric or categorical, '3' or 'three'
- Each image is converted to row vectors and the appropriate learning algorithm is used
- Convention
  - x_i represents the i^{th} feature in a training sample
  - y represents the label for the training sample

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Questions?

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