📈 Linear Algebra

13 Session - 2 SKS

#COMPSCIBINUS🏫1stSEM #math

E-Book

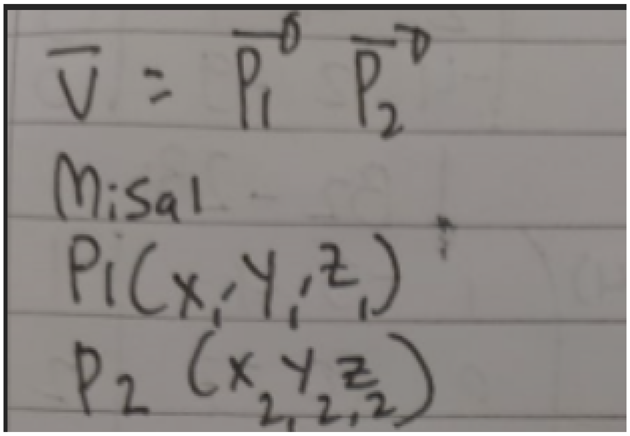
Howard Anton, Anton Kaul - Elementary Linear Algebra (2019, Wiley) - libgen.li.pdf

!!

> Linear Algebra ANKI
>
> TARGET DECK: Linear Algebra
>
> Two vector is called equivalent if they have equal #flashcard
> ***length*** and ***direction***
>
> What is called Vector with no direction and length? #flashcard
> Zero vector
>
> Zero Vector element is (a, b, c), what is a, b, and c? #flashcard
> (0, 0, 0)
>
> Commutative **u** + **v** = #flashcard
> v + u
>
> Associative (**u** + **v**) + **w** = #flashcard
> **u** + (**v** + **w**)
>
> **u** + 0 = #flashcard
> 0 + **u** = **u**
>
> **u** + (-**u**) = #flashcard
> 0
>
> Distributive k(**u** + **v**) = #flashcard
> k**u** + k**v**
>
> Distributive (k + m)**u** = #flashcard
> k**u** + m**u**
>
> Associative k(m**u**) = #flashcard
> (km)**u**
>
> 1**u** = #flashcard
> **u**
>
> 0**v** = #flashcard
> **0**

0v = #flashcard
0

The Definition of parallel vector and colinear vector #flashcard
Parallel Vector = Vektor paralel, gradien/kemiringan sama, tidak akan pernah bertemu
Colinear Vector = Vektor berhimpit, gradien sama, dapat bertemu di suatu titik



||v|| = ? #flashcard
$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

(v1 *n, v2* n, ... vn * n) = #flashcard
n**v**

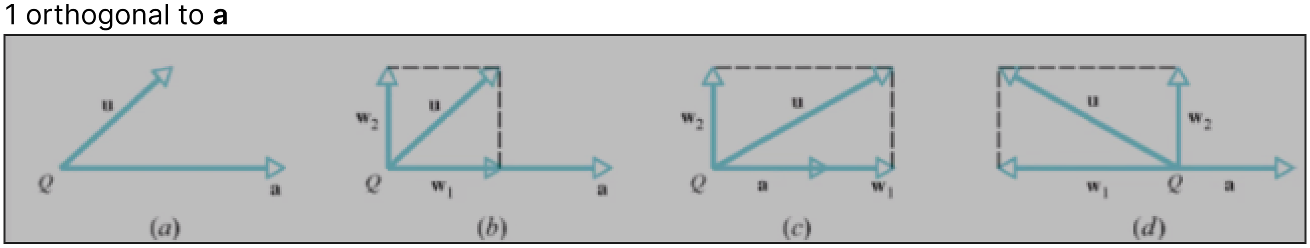
u . v = #flashcard
||u|| *//v// cos θ*
`(u1v1 + u2v2 + ... + unvn)`

if θ acute, obtuse, and orthogonal, **u . v** = #flashcard
if θ acute: **u . v** > 0
if θ obtuse: **u . v** < 0
if θ orthogonal: **u . v** = 0

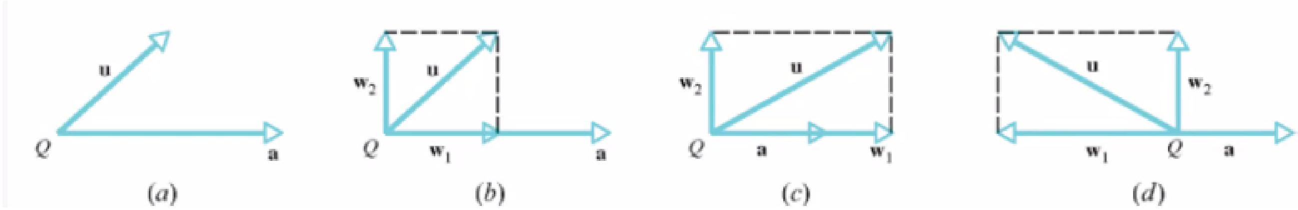
u x **v** = #flashcard
$u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1$

|| **u** x **v** || = #flashcard
||**u**|| ||**v**|| sin θ

What is decomposing a vector to two vector that is
1 scalar multiple of **a** and
1 orthogonal to **a**



#flashcard
Orthogonal Projection



u = ? #flashcard

u = w1 + (u - w1)

u = w1 + w2

proj$_{\mathbf{a}}$**u** = ?

u - proj$_{\mathbf{a}}$**u** = ? #flashcard

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \text{ (}\textit{vector component of } \mathbf{u} \textit{ along } \mathbf{a}\text{)}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \text{ (}\textit{vector component of } \mathbf{u} \textit{ orthogonal to } \mathbf{a}\text{)}$$

If **w** is a vector in R^n then **w** is said to be a linear combination of the vectors v1, v2, ..., v, then **w** = ? #flashcard

$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots. + k_r\mathbf{v}_r$

Determine whether the vectors

$\mathbf{v}_1$ = (1, -2, 3)

$\mathbf{v}_2$ = (5, 6, -1)

$\mathbf{v}_3$ = (3, 2, 1)

are linearly dependent or linearly independent in R^3 #flashcard

Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = 0$$

or, equivalently, of

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

Thus, our problem reduces to determining whether this system has nontrivial solutions. There are various ways to do this; one possibility is to simply solve the system, which yields

$$k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$$

(we omit the details). This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

Determine if $\vec{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is a linear combination of the vectors $\vec{a_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

Linear combination #flashcard

after Gauss jordan of $\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$, it's found that the answer is not found!

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \longrightarrow A^{-1}$$

The Inverse matrix is .. #flashcard

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Matrix A . A^{-1} = ? #flashcard

I_3

Matrix A is invertible if #flashcard

det(A) != 0

Calculating the Inverse of matrix A $= \begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix}$

Calculate the inverse of matrix A #flashcard

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{pmatrix}$$

Properties of matrix $(AB)^{-1}$ = #flashcard

B^{-1} A^{-1}

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

Find the inverse matrix #flashcard

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \longleftarrow$$ We added -2 times the fir row to the second and -1 the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \longleftarrow$$ We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \longleftarrow$$ We multiplied the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \longleftarrow$$ We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \longleftarrow$$ We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example : Finding Minors and Cofactors

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Search for the minorand and cofactors #flashcard

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

$$C_{11} = -1^{1+1} M_{11} = M_{11} = 16$$

Find the determinant of the matrix by cofactor expansion along the first row.

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Find the determinant of the matrix by cofactor expansion along the first row #flashcard

$$det(A) = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = 3\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} - 1\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$$

$$= 3(16) - 10 - 4(3) = -1$$

Using adjoint, find the inverse of this matrix

Example : Calculating the Inverse of matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

#flashcard

Solution :

a. Find the Determinant of A

$$\begin{vmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{vmatrix} = 64$$

b. Find the adjoin of matrix A

$$\mathbf{Adj}\ (\mathrm{A}) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

$$\mathbf{A^{-1}} = \frac{\mathbf{1}}{\mathbf{det(A)}}\ \mathbf{adj\ (A)}$$

$$\mathbf{A^{-1}} = \frac{\mathbf{1}}{\mathbf{64}} \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{12}}{\mathbf{64}} & \frac{\mathbf{4}}{\mathbf{64}} & \frac{\mathbf{12}}{\mathbf{64}} \\ \frac{\mathbf{6}}{\mathbf{64}} & \frac{\mathbf{2}}{\mathbf{64}} & -\frac{\mathbf{10}}{\mathbf{64}} \\ -\frac{\mathbf{16}}{\mathbf{64}} & \frac{\mathbf{16}}{\mathbf{64}} & \frac{\mathbf{16}}{\mathbf{64}} \end{pmatrix}$$

What is the cramer's rule? #flashcard

If $A.\mathbf{x}=\mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the j-th column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Use the cramer's rule to solve

$$\begin{array}{rcrcrcr} x_1 & + & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{array}$$

#flashcard

$$cx_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Find the point at which the line with parametric equations $x = 3 + 4t$, $y = 5 - 2t$, and $z = 4 + 7t$ intersects the plane $2x + 4y - z = 1$.

Find the point at which the line with parametric equations! #flashcard

tbd

Find vector and parametric equations for the line in R^2 that passes through the points P(0,7) and Q(5,0).

Also find vector and parametric equation for P(0, 7) and Q(0, 5). Draw the line in a cartesius!

#flashcard

We will see below that it does not matter which point we take to be x_0 and which we take to be x_1, so let us choose $x_0 = (0,7)$ and

$x_1 = (5,0)$. It follows that $x_1 - x_0 = (5,-7)$ and hence that

$$(x,y) = (0,7) + t\,(5,-7)$$

which we can rewrite in parametric form as

$$x = 5t,\ y = 7-7t$$

Had we reversed our choices and taken $x_0 = (5,0)$ and $x_1 = (0,7)$, then the resulting vector equation would have been

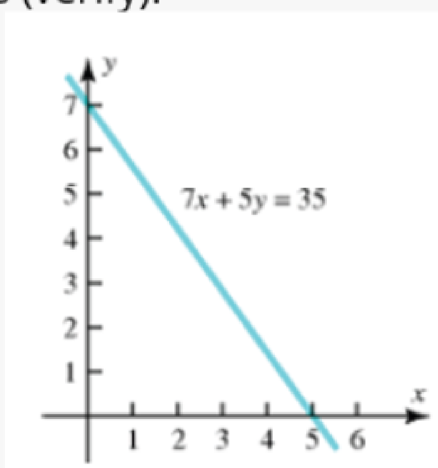
$$(x,y) = (5,0) + t\,(-5,7)$$

and the parametric equations would have been

$$x = 5 - 5t,\ y = 7t$$

$$7x + 5y = 35$$

This can be seen by eliminating the parameter t from the parametric equations (verify).



The plane in R^3 through the point(3, 0, 7) with **n** = (4, 2, -5) equation is... #flashcard

$$4\,(x-3) + 2y - 5(z-7) = 0,$$

or equivalently : $4x + 2y - 5z + 23 = 0$.

Conversely, we can determine the normal vector of a plane from given its equation, as explained in this theorem :

The normal vector of the plane 9x - 8y + 5z + 29 = 0 is? #flashcard

n(9, -8, 5)

> Find vector and parametric equations of the plane x-y+2z=5 #flashcard
> example, solving for *x* in terms of *y* and *z* yields : $x = 5 + y - 2z$, and then using *y* and *z* as parameters t_1 and t_2, respectively, yields the parametric equations
> $$x = 5 + t_1 - 2t_2,\ y = t_1,\ z = t_2$$
> To obtain a vector equation of the plane we rewrite these parametric equations as
> $$(x,y,z) = (5 + t_1 - 2t_2, t_1, t_2)$$
> or, equivalently, as
> $$(x,y,z) = (5,0,0) + t_1 (1,1,0) + t_2 (-2,0,1).$$
>
> If the vector equation of a line is (4t, 8-3t), then the direction vector is? #flashcard
> (4, 3)
> scalar multiplier of parameter t
>
> If the parametric equation of a line is
> x = 6t
> y = 4 + 4t
> z = 10 + 2t
> The direction vector is? #flashcard
> **v**(6, 4, 2)
> scalar multiplier of t
>
> The normal vector of 7x+8y+6z+12 is ? #flashcard
> **n**(7, 8, 6)
>
> The parametric equation of the plane 7x+8y+6z+12 is? #flashcard
> $x = -12 - 8t_1 - 6t_2$
> $y = t_1$
> $z = t_2$
>
> The vector equation of the plane 7x+8y+6z+12 is?
> $(x, y, z) = (-12, 0, 0) + t_1(-8, 1, 0) + t_2(-6, 0, 1)$

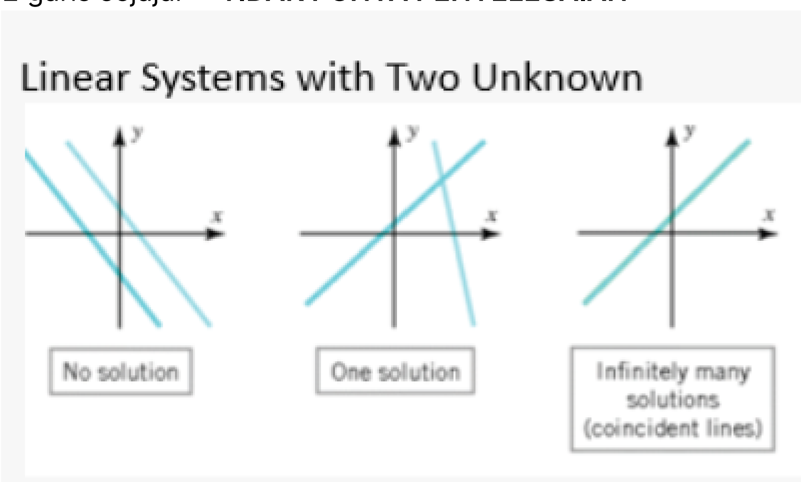
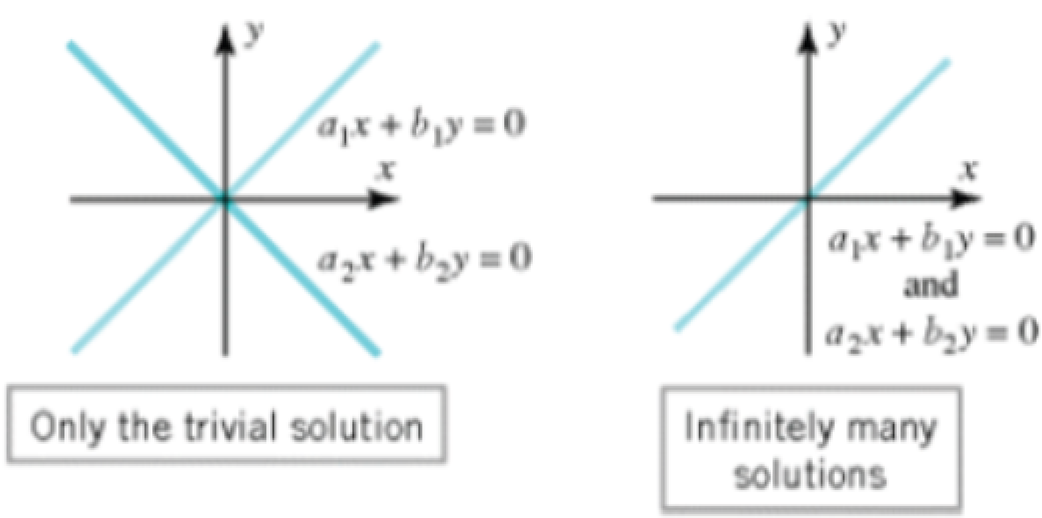
Session 1&2 - System of Linear Equation

!!

> Introduction to Linear Equation
>
> `ax + by = c` is a 2D equation algebra.
> `ax + by + cz = d` is a 3D equation algebra.
> *a, b, c, and d is a constant and cannot be all 0.*
>
> More than one linear equation can produce a result, that is: a point where both of those line equation met.

> Linear Equation Solution

‼

> Linear Equation Solution
>
> Non-Homogen
> Non-homogen adalah ketika konstanta bebasnya ada nilainya
> `ax + by = c`
>
> - Penyelesaian Tunggal
>   2 garis sembarang = pasti ada titik potongnya
> - Penyelesaian Banyak
>   2 garis berhimpit = **PENYELESAIANNYA INFINITE**
> - Tidak punya penyelesaian
>   2 garis sejajar = **TIDAK PUNYA PENYELESAIAN**`
>
>   
>
>
>   berlaku juga untuk 3D dst.
>
> Homogen
> Homogen adalah ketika konstanta yang tidak mengandung variabel bernilai 0
> `ax + by = 0`
>
> 
>

> Penyelesaian *TRIVIAL* dan *NON-TRIVIAL*
> Trivial = the point where all they met is guaranteed to be zero.
> Non-trivial = If there's more unknowns than equation. In this case, the answer will be just relative of other unknown.
>
> Non-trivial example solution is x - $2x_3 + 3x_5 = 0$, so the answer should be:
>
> x_1 = 2s-3t
>
> x_2=0
>
> x_3=s
>
> x_4=0
>
> x_5=t

!!

> Homogenous System of Linear Equation
>
> Homogenous artinya, semua yang ada di kanan (the result of Ax + By + Cz) is all 0.
>
> The answer is either trivial (0) or non trivial(the answer is an equation)

!!

> Gauss-Jordan
>
> https://www.youtube.com/watch?v=eYSASx8_nyg
>
> **Gauss Jordan Using Approximate Method**
>
> Steo
>
> Pitfall
>
> ---
>
> Pitfall 1: Division by 0 error
> Pitfall 2: Large Round off Error
>
> Avoiding Pitfalls
>
> ---
>
> 1. Increase the number of significant digit
>   Decreases round-off error
>   Doesn't solve pitfall 1
> 2. Gaussian Elimination with Partial Pivoting
> 3. UL Decomposition - mingdep
> 4. Metode Saddle - mingdep

Gaussian Elimination.

manipulate the matrix using 3 method.

① For that row, kali semuanya dengan konstanta tertentu.

② Switch that row dengan row diatas / dibawahnya.

③ $R_n = cR_m + R_n$ ↳ contoh = $\left[\begin{array}{ccc:c} 5 & 3 & 7 & 6 \\ 1 & 4 & 2 & 3 \\ 2 & 6 & 5 & 4 \end{array}\right] \xrightarrow{R_3 = (-2)R_2 + R_3} \left[\begin{array}{ccc:c} 5 & 3 & 7 & 6 \\ 1 & 4 & 2 & 3 \\ 0 & -2 & 1 & 4 \end{array}\right]$

All this manipulation is executed to produce: $\left[\begin{array}{ccc:c} 1 & iy, & jz_1 & a \\ 0 & 1 & kz_2 & b \\ 0 & 0 & \boxed{1} & \boxed{c} \end{array}\right]$

From this, z is c

then w/ process of elimination, find i, j, k

to get x, y, z.

Gauss-Jordan.

After you get gauss matrix result, continue to manipulate more till you get:

$\left[\begin{array}{ccc:c} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{array}\right]$ $\left[\begin{array}{ccc:c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$

↓

From this, it is clear that.

x = a, b = y, z = c.

Trivial = The answer is 0

Non-Trivial = The answer is infinite (the solution is an equation with other variable)

Session 3 - Matrices

!!

Intro to Matrix

Matrix is a rectangular array of number

Matrix size(ordo) is n x m
contoh matriks (2×2) (3×5) etc.
where n is a row and m is the collumn

if m = n:
the matrix will be square = `square matrix` dengan ordo n.

If matrix is only consists of 1 collumn, it's called `matrix collumn`
If matrix is only consists of 1 row, it's called `matrix row`

Vector only 1 Dimension, so vector matrix usually is either a matrix collumn or a matrix row
contoh: titik `x = (2,3)`

To select specific element for matrix
A_{ij} = elemen baris ke i, kolom ke j.

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in the matrix below are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

contoh aplikasi:
matriks identitas diagonal utamanya 0
matriks segitiga diagonal utamanya 1

!!

Equality of matrices

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Equality of Matrices

Definition

Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.
The equality of two matrices

$A=[a_{ij}]$ and $B=[b_{ij}]$

of the same size can be expressed by writing $a_{ij}=b_{ij}$, for all values of i and j in those matrices.

Example

Consider the matrices

$$A=\begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix},\quad B=\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix},\quad C=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If x=5 then A=B, b... ces A and B are not equal, since not all of their corresponding entries are equal. There is no value of x for which A=C since A and C have different sizes.

People
Innovation
Excellence

The order and the values inside of it SHOULD be the same in the exact order for two matrix to be equal

!!

Addition and Substraction of Matrices

Matrix Addition and Substraction

Definition

If A and B are matrices of the same size, then the ***sum A+B*** is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference A-B*** is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A=[a_{ij}]$ and $B=[b_{ij}]$ have the same size, then

$$(A+B)_{ij} = (A)_{ij}+(B)_{ij} = a_{ij}+b_{ij}$$

and

$$(A-B)_{ij} = (A)_{ij}-(B)_{ij} = a_{ij}-b_{ij}$$

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To add/substract, the ordo need to be the same. To add/substract, operate addition/substraction of 2 element in the same collumn and row on the other matrix.

Consider the matrices

$$A=\begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A+B=\begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \text{ and } A-B=\begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A+C$, $B+C$, $A-C$, and $B-C$ are undefined.

Always make sure the two matrix is sufficient for addition

!!

Matrix Scalar Multiplication

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Scalar Multiplication

Definition

If A is any matrix and c is any scalar, then the ***product*** $c.A$ is the matrix obtained by multiplying each entry of the matrix A by c. The matrix $c.A$ is said to be a ***scalar multiple*** of A.

In matrix notation, if $A=[a_{ij}]$, then

$$(c.A)_{ij} = c.(A)_{ij} = c.a_{ij}$$

Example

For the matrices

$$A=\begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & 2 & 7\\ -1 & 3 & -5\end{bmatrix},\quad C=\begin{bmatrix}9 & -6 & 3\\ 3 & 0 & 12\end{bmatrix}$$

we have

$$2A=\begin{bmatrix}4 & 6 & 8\\ 2 & 6 & 2\end{bmatrix},\quad (-1)B=\begin{bmatrix}0 & -2 & -7\\ 1 & -3 & 5\end{bmatrix},\quad \tfrac{1}{3}C=\begin{bmatrix}3 & -2 & 1\\ 1 & 0 & 4\end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

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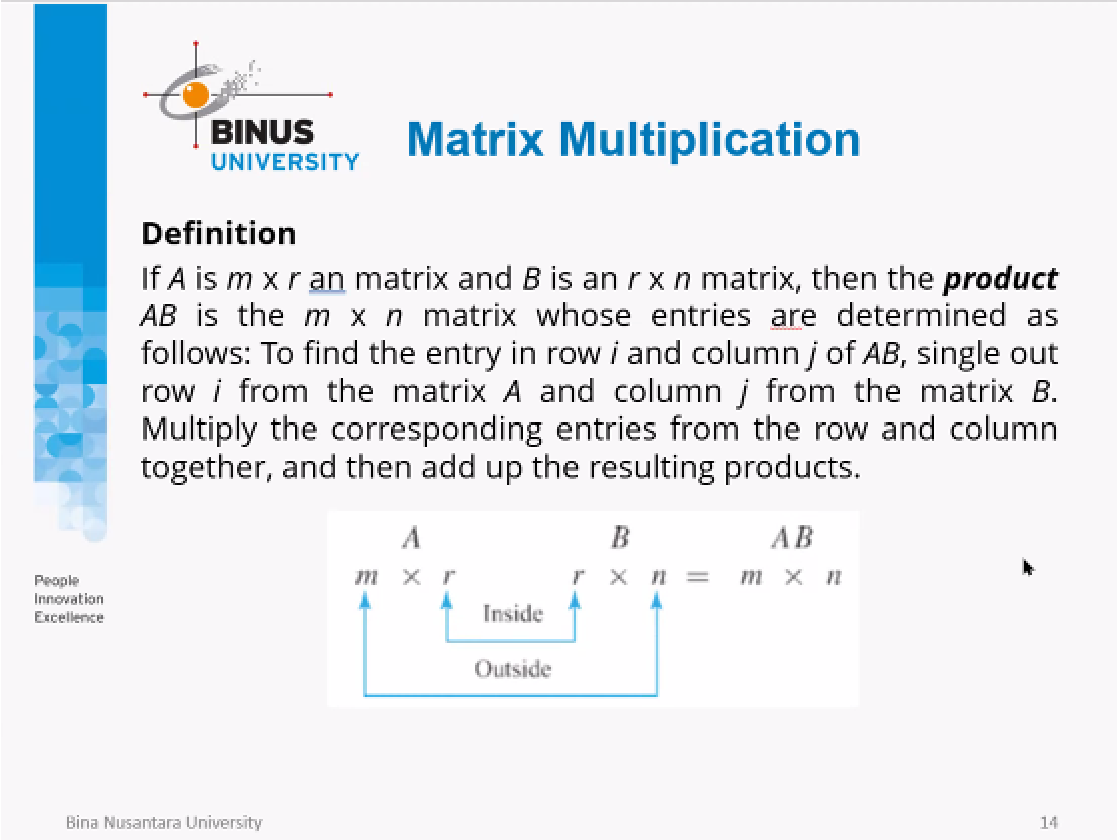
untuk `k(A)` setiap elemen di matriks A dikalikan dengan k.

!!

Matrix Multiplication

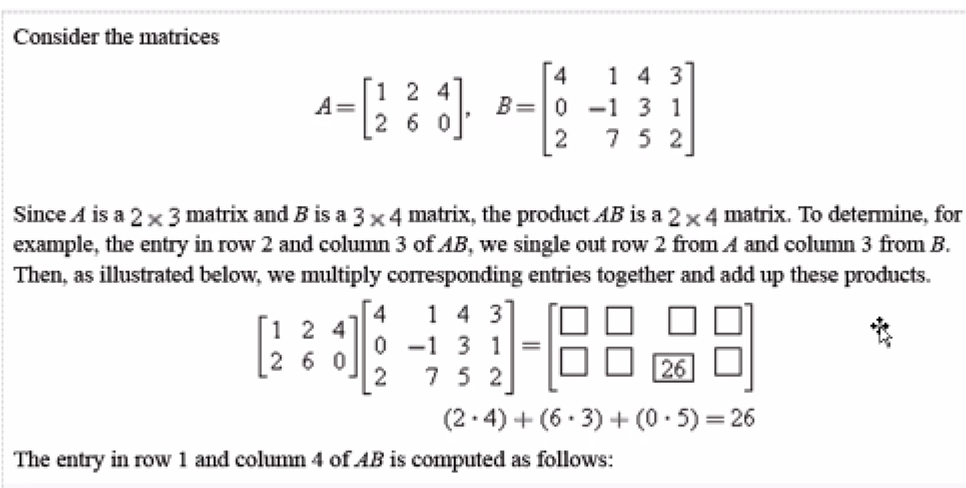


The order of matrix 1 and 2 so that they can be multiplied:

`A(m x n) * B(n x k) = C(m x k)`

The inside should be the same. Order matters!

(The inside means the collumn of the 1st matrix and the row for the 2nd matrix).



Consider the matrices

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix},\quad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}=\begin{bmatrix}\square & \square & \square & \square\\ \square & \square & \boxed{26} & \square\end{bmatrix}$$

$$(2\cdot 4)+(6\cdot 3)+(0\cdot 5)=26$$

The entry in row 1 and column 4 of AB is computed as follows:

The general formula

$(AB)_{ij}$ = $a_{in}b_{nj}$ $+a_{ik}b_{kj}$ + ... sampai selesai

If A=[a_{ij}] is an *m* x *r* matrix and B=[b_{ij}] is an *r* x *n* matrix, then :

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \cdots & \boxed{a_{ir}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & \boxed{b_{1j}} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & \boxed{b_{2j}} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & \boxed{b_{rj}} & \cdots & b_{rn} \end{bmatrix}$$

the entry (AB)ij in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

!!

Matrix Transpose

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Transpose of a Matrix (Cont.)

Observe that not only are the columns of A^T the rows of A, but the rows of A^T are the columns of A. Thus the entry in row i and column j of A^T is the entry in row j and column i of A; that is,

$$(A^T)_{ij} = (A)_{ji}$$

In the special case where A is a square matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal. We see that A^T can also be obtained by "reflecting" A about its main diagonal.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Interchange entries that are symmetrically positioned about the main diagonal

Nusantara University 21

For every element, the collumn number is switched with the row number

if A^T == A:
Matriks simetrik

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Properties of the Transpose

Theorem

If the sizes of the matrices are such that the stated operations can be performed, then:

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(k.A)^T = k.A^T$

(e) $(A.B)T = B^T.A^T$ (*The transpose of a product of any number of matrices is the product of the transposes in the reverse order)*

!!

Trace of Matrix

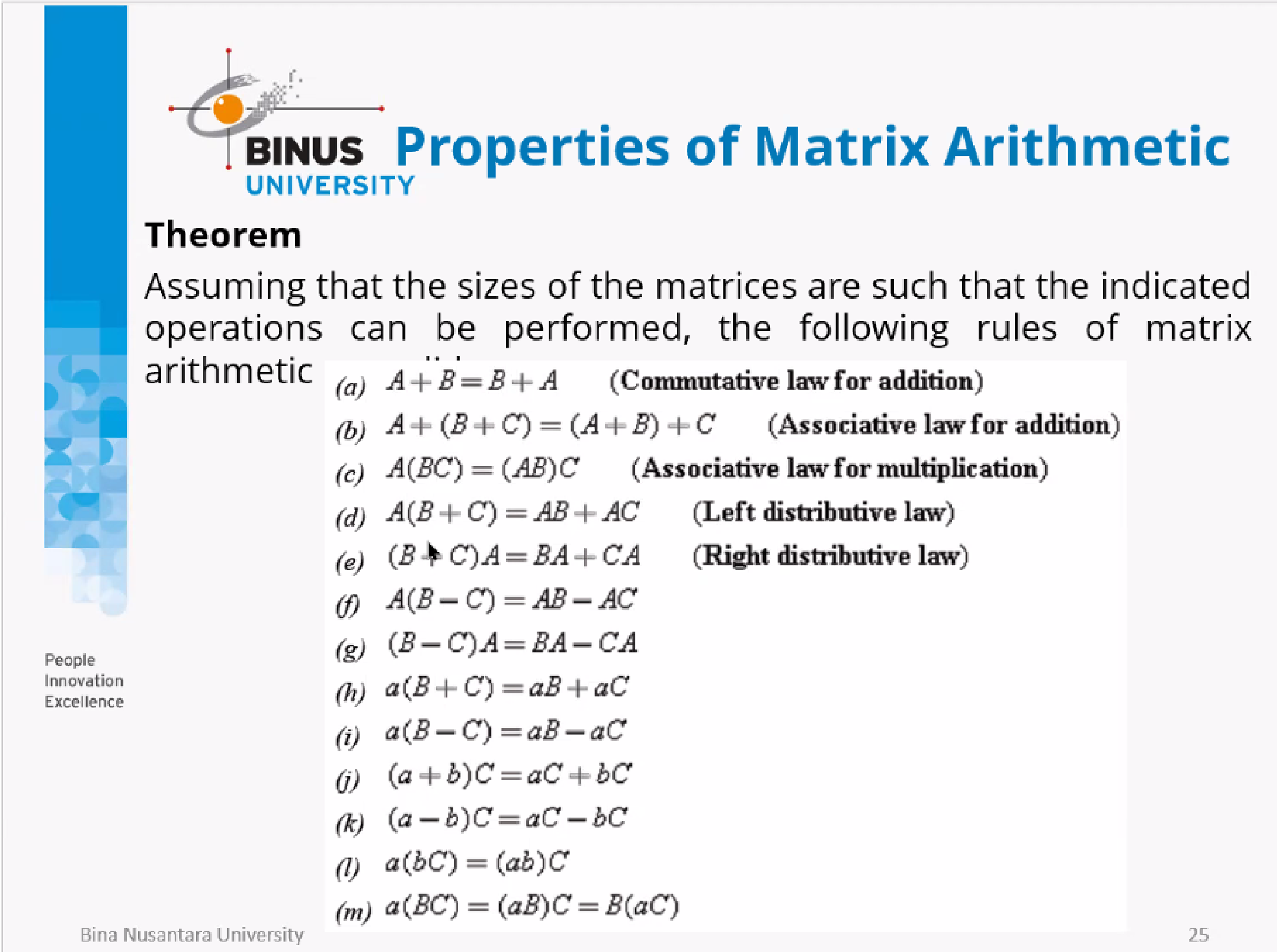
tr(A) is the sum of the main diagonal of matrix A
Look Intro to Matrix: main diagonal

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \quad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

!!![[]]
!!

Properties of Matrix Arithmetics



!!

0 Matrices

All of the element is 0

jika A is a 0 matrix
A + any matrix = The other matrix
A * any matrix = Zero matrix

> adalah alasan kenapa `a * b = a * c, maka b belum tentu = c... a * b`
>
> **EXAMPLE 3 | Failure of the Cancellation Law**
>
> Consider the matrices
>
> $$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$
>
> We leave it for you to confirm that
>
> $$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$
>
> Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication (though there may be particular cases where it is true).
>
> **EXAMPLE 4 | A Zero Product with Nonzero Factors**
>
> Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:
>
> $$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

!!

> Identity Matrices
>
> Spesifically,
> `a square matrix with number 1 on their main diagonal, and 0 elsewhere`
>
> $$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
>
> `Identity matrix * any other matrix = any other matrix itself`

!!

> Diagonal Matrices

`Diagonal Matrix is a matrix with values inside of the main diagonal but 0 elsewhere. (The value can also be 0).`

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

The power of diagonal matrices is equal to if all of the matrix element is powered with the same number.

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$

Also, the multiplication of two diagonal matrices is equal to if the element of the main diagonal in matrix A is multiplied by the element of the main diagonal in matrix B

```
1
  3
    7
  *
2
  4
    6

 =

2
 12
   42
```

!!

Triangular Matrices

`A square matrix with all of the element below the main diagonal 0 = upper triangular matrix.`
`A square matrix with all of the element above the main diagonal 0 = lower triangular matrix`

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4 × 4 upper triangular matrix

A general 4 × 4 lower triangular matrix

Properties of Triangular Matrices

1. The transpose of an upper triangle matrix will result its counterpart in a lower triangle matrix. (and vice versa)
2. The multiplication of two upper triangle matrix will result in an upper triangle matrix. (also applied to lower triangle matrix)

!!

Symmetric Matrices

`A square matrix is a symmetric matrix with mirror images, through the main diagonal.`

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

Properties of Symmetric Matrices

1. The main property is their transpose is itself. A = A^T
   If A is a symmetric matrix and B is another symmetric matrix
2. A + B = C where C is a symmetric matrix
3. A - B = C where C is a symmetric matrix
4. A.B `can` be equal to B.A
   A.B = B.A ↔ A.B = C `(where C is a symmetric matrix)`
   (the multiplication can commute if and only if the result is a symmetric matrix)
   Kalau hasil perkaliannya tidak simetris, maka perkaliannya tidak bisa komutatif
5. k.A = C where C is a symmetric matrix (scalar multiplication)

Session 4 & 5 & 6 & 7- Determinant and Inverses

!!

Determinant

Let A is $matrix$ 2×2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Matrix A is invertible if and only if $ad - bc \neq 0$ and that the expression $ad - bc$ is called determinant of the matrix A. Recall also that this determinant is denoted by writing

$$\det(A) = ad - bc \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

!!

> Minor Cofactor and determinant expansion
>
> Example : Finding Minors and Cofactors
>
> $$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$
>
> Solution
>
> The minorand cofactor of entry a_{ij} is
>
> $$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$
>
> $$C_{11} = -1^{1+1} M_{11} = M_{11} = 16$$
>
> Determinant:
>
> $$A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$
>
> Solution
>
> The minor and cofactor of entry a_{ij} is
>
> $$det(A) = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = 3\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} - 1\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$$
>
> $$= 3(16) - 10 - 4(3) = -1$$

!!

> Row Reduction

Relationship	Operation
$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ $\det(B) = k\det(A)$	The first row of A is multiplied by k.
$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ $\det(B) = -\det(A)$	The first and second rows of A are interchanged.
$$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ $\det(B) = \det(A)$	A multiple of the second row of A is added to the first row.

!!

Inverse Matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$$

$= 1(-1)\begin{vmatrix} 7 & 8 \\ 1 & 1 \end{vmatrix}$

$= -1(7-8)$

$= 1$

but not,

ng Theorem 2.1.2. $\rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

ng the method of Example 4 in Section 1.5.

volves less computation? $[A \,\vdots\, I] \xrightarrow{OBE} [I \,\vdots\, A^{-1}]$

!!

Cramer's Rule Matrix

17.
$$\begin{aligned} 4x + \quad 5y \quad &= 2 \\ 11x + \; y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

18.
$$\begin{aligned} x - 4y + \; z &= 6 \\ 4x - \; y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$

$$A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \Rightarrow \det(A) = 7+6 = 13$$

$$A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \Rightarrow \det(A_1) = 3+10 = 13$$

$$A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow \det(A_2) = 35 - 9 = 26$$

THEOREM 1.6.2

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

$$AX = B \Rightarrow X = A^{-1}B \qquad A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 1 & 2 \\ -3 & 7 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{13}{13} = 1$$

by Cramer's rule, where it applies.

$$\Rightarrow \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad x_2 = \frac{26}{13} = 2$$

$$A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \Rightarrow \det(A) = 7+6 = 13$$

$$A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \Rightarrow \det(A_1) = 3+10 = 13$$

$$A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow \det(A_2) = 35 - 9 = 26$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 1 & 2 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{13} & \frac{2}{13} \\ -\frac{3}{13} & \frac{7}{13} \end{bmatrix}$$

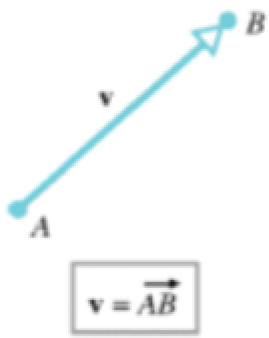
$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{13} & \frac{2}{13} \\ -\frac{3}{13} & \frac{7}{13} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \frac{3}{13} + \frac{10}{13} = 1, \quad \frac{-9}{13} + \frac{35}{13} = 2$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow x_1 = 1,\ x_2 = 2$$

Session 8 & 9 - !!

Vectors

Vector Notation will be bolden.



Zero Vector

0 = (0, 0, 0, ... Vn)

Vector with no direction and length

Two vector is called equivalent if they have equal ***length*** and ***direction***

Vector Basic Operation
Parallel and Colinear Vectors
Vector in Coordinate Systems

Vectors in n-space

Panjang vektor kuadrat adalah jumlah dari kuadrat elemennya

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}$$

Unit Vectors and Standard Unit Vectors

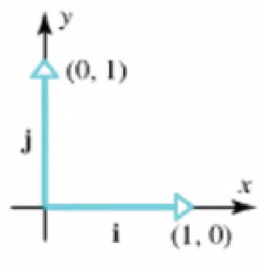
So the vectors **v** can be written as
v = a**i** + b**j** + c**k**

standard unit vectors. In R^2 these vectors are denoted by

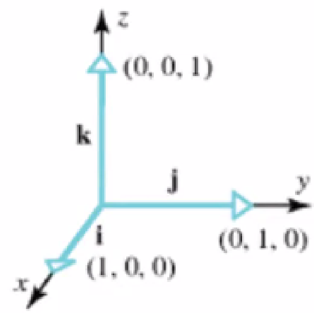
$$\mathbf{i} = (1,0) \qquad \text{and} \qquad \mathbf{j} = (0,1)$$

and in R^3 by

$$\mathbf{i} = (1,0,0), \qquad \mathbf{j} = (0,1,0) \qquad \text{and} \qquad \mathbf{k} = (0,0,1)$$



(a)



(b)

Dot Product
Cross Product

Orthogonal Projection
Linear Combination of Vectors

Session 10 - !!

Line and Plane

t is a parameter
v is a vector direction (a, b, c)

P is a point where the line equation is possible. P parameters are the parametic equation and the Vector equation.

Vector and Parametric Equations of Lines

1. Known:
   - Point P_0
   - Vector Direction **v**

     Vector Equation
     $\mathbf{P} = \mathbf{P_0} + t\mathbf{v}$

     Parametric Equation
     $x = x_0 + ta$

$y = y_0 + tb$

$z = z_0 + tc$

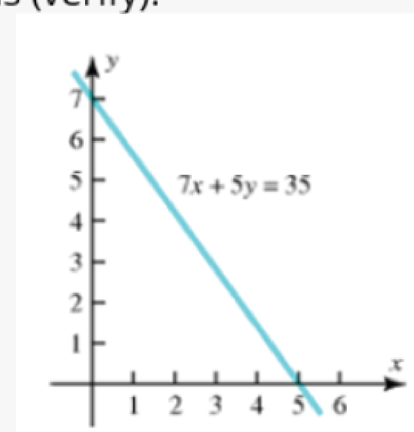
2. Known:
   - Point P_0
   - Point P_1

```
Find the **v** first
**v** = P<sub>1</sub> - P<sub>0</sub>

Vector Equation
**P** = **P<sub>0</sub>** + t(P<sub>1</sub> - P<sub>0</sub>)

Parametric Equation
x = x<sub>0</sub> + t (x<sub>1</sub> - x<sub>0</sub>)
y = y<sub>0</sub> + t (y<sub>1</sub> - y<sub>0</sub>)
z = z<sub>0</sub> + t (z<sub>1</sub> - z<sub>0</sub>)

Jika P<sub>1</sub> dan P<sub>0</sub> dibalik, garisnya akan sama, tetapi arah
vektornya dibalik
```

Although those two results look different, they both represent the line whose equation in rectangular coordinates is

$$7x + 5y = 35$$

This can be seen by eliminating the parameter t from the parametric equations (verify).



Vector and Parametric Equation of Plane

n = normal line that is orthogonal to the plane
n(a, b, c)

1. Search for the point-normal equation of the plane

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

so, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

2. Vector & Parametric Equation
   $x = x_0 + t_1\mathbf{v_1} + t_2\mathbf{v_2}$

   From the point-normal equation, convert y to t_1 and convert z to t_2

   example, solving for x in terms of y and z yields : $x = 5 + y - 2z$, and then using y and z as parameters t_1 and t_2, respectively, yields the parametric equations

   $$x = 5 + t_1 - 2t_2,\ y = t_1,\ z = t_2$$

   To obtain a vector equation of the plane we rewrite these parametric equations as

   $$(x,y,z) = (5 + t_1 - 2t_2, t_1, t_2)$$

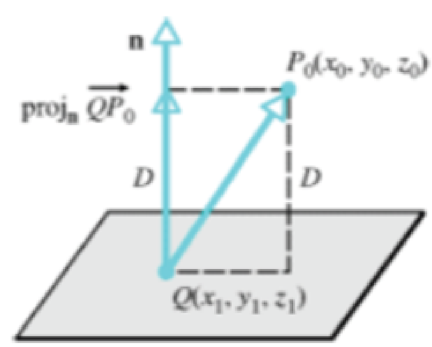
   or, equivalently, as

   $$(x,y,z) = (5,0,0) + t_1 (1,1,0) + t_2 (-2,0,1).$$

Distance between a point and a plane

In R^3 the distance D between the point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$



Distance from P_0 to plane.

Session 11

Session 12