Boolean Algebra

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Boolean Algebra Introduction

- Modern digital systems use 0 and 1 to describe values and boolean equations to describe behaviours.
- Algebra to simplify arithmetic equations
- Boolean algebra to simplify Boolean equations.
- Rules Boolean algebra are much like those of ordinary algebra
- In some cases simpler, because variables have only two possible values: 0 or 1.

Boolean Algebra Introduction - Axioms

- Boolean algebra is based on a set of axioms we assume are correct.
- Axioms are unprovable.
- Set of definitions.
- From these axioms, we prove all the theorems of Boolean algebra.

NOT

$\overline{a}$

a	x
0	1
1	0

AND

$a \cdot b$

a	b	x
0	0	0
0	1	0
1	0	0
1	1	1

OR

$a + b$

a	b	x
0	0	0
0	1	1
1	0	1
1	1	1

Boolean Algebra Introduction - Axioms

- Duality of Boolean Algebra.
- AND and OR are duals of each other.

NOT

$\overline{0} = 1$ and $\overline{1} = 0$

AND

$0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 0 = 0$, and $1 \cdot 1 = 1$,

OR

$0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, and $1 + 1 = 1$,

NOT

$\bar{a}$

a	x
0	1
1	0

AND

$a \cdot b$

a	b	x
0	0	0
0	1	0
1	0	0
1	1	1

OR

$a + b$

a	b	x
0	0	0
0	1	1
1	0	1
1	1	1

Boolean Algebra Introduction - Axioms

- Duality of Boolean Algebra
- AND and OR are duals of each other

	Axiom		Dual	Name
A1	$B = 0$ if $B \neq 1$	A1′	$B = 1$ if $B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$	AND/OR

Single Variable Boolean Algebra Theorems

- **Identity for AND**

  $a \cdot 1 = a$

- Identity for OR

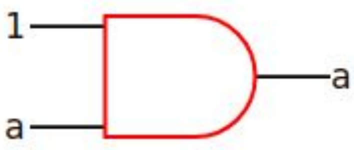
  $a + 0 = a$

- Annihilator for AND

  $a \cdot 0 = 0$

- Annihilator for OR

  $a + 1 = 1$



a	1	a · 1
0	1	0
1	1	1

Single Variable Boolean Algebra Theorems

- Identity for AND

  $a \cdot 1 = a$

- **Identity for OR**

  $\mathbf{a + 0 = a}$

- Annihilator for AND

  $a \cdot 0 = 0$

- Annihilator for OR

  $a + 1 = 1$

a	0	a + 0
0	0	0
1	0	1

Single Variable Boolean Algebra Theorems

- Identity for AND

  $a \cdot 1 = a$

- Identity for OR

  $a + 0 = a$

- **Annihilator for AND**

  $\mathbf{a \cdot 0 = 0}$

- Annihilator for OR

  $a + 1 = 1$

a	0	a · 0
0	0	0
1	0	0

Single Variable Boolean Algebra Theorems

- Identity for AND

  $a \cdot 1 = a$

- Identity for OR

  $a + 0 = a$

- Annihilator for AND

  $a \cdot 0 = 0$

- **Annihilator for OR**

  $a + 1 = 1$

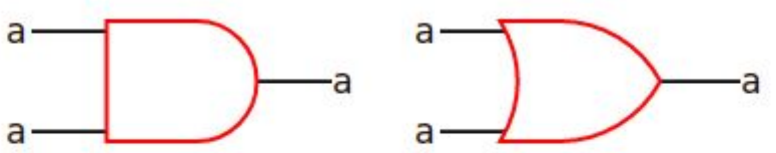
You may want to test all these properties for longer values of input a (e.g., a = 100101)

a	1	a + 1
0	1	1
1	1	1

Single Variable Boolean Algebra Theorems

- Idempotent law (no change) for AND and OR

$a \cdot a = a$ $\qquad$ $a + a = a$



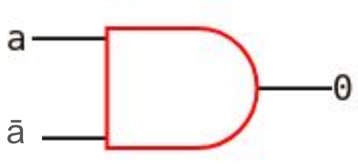
a	a	a · a
0	0	0
1	1	1

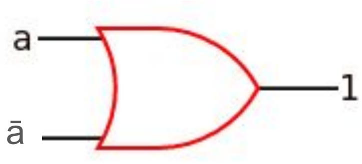
a	a	a + a
0	0	0
1	1	1

Single Variable Boolean Algebra Theorems

- Complement for AND and OR

$a \cdot \bar{a} = 0$ $\qquad$ $a + \bar{a} = 1$





a	ā	a · ā
0	1	0
1	0	0

a	ā	a + ā
0	1	1
1	0	1

Single Variable Boolean Algebra Theorems

- Involution (double NOT)

$$\overline{\overline{a}} = a$$

- "Double negation" cancel each other
  - Not over operators (DeMorgan's):

$$\overline{\overline{A} + \overline{B}} \neq A + B$$

Multi variable Theorems

- Commutativity of AND

  $a \cdot b = b \cdot a$

- Commutativity of OR

  $a + b = b + a$

- Associativity of AND

  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

- Associativity of OR

  $(a + b) + c = a + (b + c)$

Boolean Algebra Theorems

- Distributivity of AND over OR

  a · (b + c) = (a · b) + (a · c)

  The operation inside goes to outside

a	b	c	b+c	a(b+c)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

a	b	c	ab	ac	ab+ac
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

Boolean Algebra Theorems

- Distributivity of OR over AND

  $a + (b \cdot c) = (a + b) \cdot (a + c)$

  The operation inside goes to outside

a	b	c	bc	a+bc
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

a	b	c	a+b	a+c	x
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

More Simplification Boolean Algebra Theorems

Covering (Absorption) of AND and OR (Truth tables)

- $a \cdot (a + b) = a$
  - $0 . (0 + b) = 0$ (annihilator of AND)
  - $1 . (1 + b) = 1$ (annihilator of OR)
- $a + (a \cdot b) = a$ (Same truth table as above, simplification turns this on the other)

Combining of AND and OR

- $(a \cdot b) + (a \cdot \bar{b}) = a . (b + \bar{b}) = a . 1 = a$
- $(a + b) \cdot (a + \bar{b}) = a + (b . \bar{b}) = a + 0 = a$

More Simplification Boolean Algebra Theorems

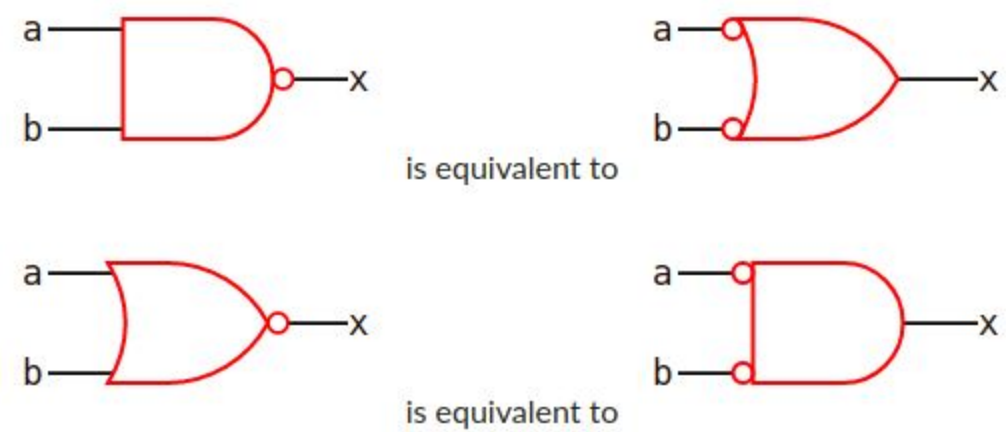
- Consensus (Redundancy)
  - The consensus or resolvent of the terms AB and A'C is BC
  - It is the conjunction of all the unique literals of the terms
  - excluding the literal that appears unnegated in one term and negated in the other.

- $(AB) + (\bar{A}C) + (BC) = AB + \bar{A}C$
- $(A + B) \cdot (\bar{A} + C) \cdot (B + C) = (A + B) \cdot (\bar{A} + C)$

More Simplification Boolean Algebra Theorems

- DeMorgan's Law
- Invert the operations and invert literals

$$\overline{a \cdot b} = \overline{a} + \overline{b}$$
$$\overline{a + b} = \overline{a} \cdot \overline{b}$$



Sum of Products

How to write a Boolean equation for any logic function given its truth table.

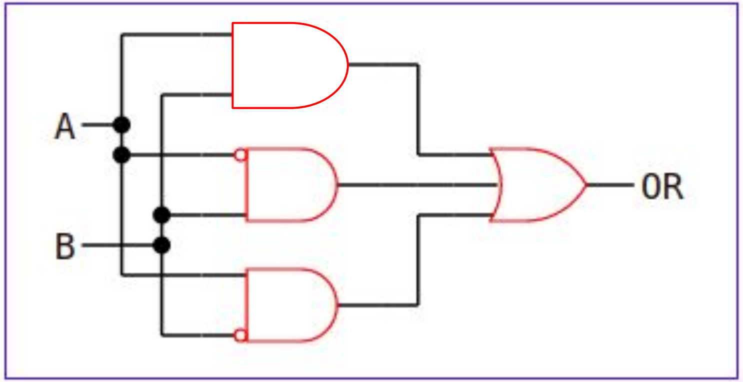
OR

SOP	A	B	x
$\overline{A} \cdot \overline{B}$	0	0	0
$\overline{A} \cdot B$	0	1	1
$A \cdot \overline{B}$	1	0	1
$A \cdot B$	1	1	1

Sum of Products

How to write a Boolean equation for any logic function given its truth table.

$\overline{A} \cdot B + A \cdot \overline{B} + A \cdot B$



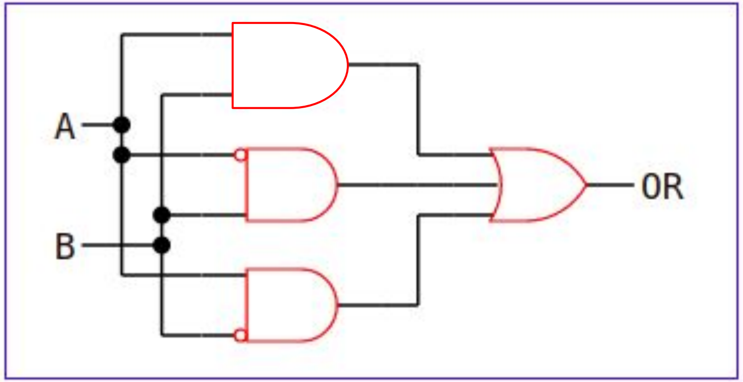
OR

SOP	A	B	x
$\overline{A} \cdot \overline{B}$	0	0	0
$\overline{A} \cdot B$	0	1	1
$A \cdot \overline{B}$	1	0	1
$A \cdot B$	1	1	1

Sum of Products

- Boolean expression contain AND, OR, and NOT operators
- The Canonical form contain all the variables, ANDed. Products are ORed together
- Any Boolean function can be described using a SOP expression
- Non minimal form/canonical form

$$\overline{A}\cdot B + A\cdot\overline{B} + A\cdot B$$



OR

SOP	A	B	x
$\overline{A}\cdot\overline{B}$	0	0	0
$\overline{A}\cdot B$	0	1	1
$A\cdot\overline{B}$	1	0	1
$A\cdot B$	1	1	1

Minterms

- A truth-table can be specified by listing its minterms that are true
- Equivalent to a SOP expression
- Function f has minterms
  - m_3, m_5, m_6, m_7

$\sum(m_3, m_5, m_6, m_7)$ or $\sum(3, 5, 6, 7)$

Minterms	SOP	A	B	C	f
m_0	$\overline{A}\cdot\overline{B}\cdot\overline{C}$	0	0	0	0
m_1	$\overline{A}\cdot\overline{B}\cdot C$	0	0	1	0
m_2	$\overline{A}\cdot B\cdot\overline{C}$	0	1	0	0
m_3	$\overline{A}\cdot B\cdot C$	0	1	1	1
m_4	$A\cdot\overline{B}\cdot\overline{C}$	1	0	0	0
m_5	$A\cdot\overline{B}\cdot C$	1	0	1	1
m_6	$A\cdot B\cdot\overline{C}$	1	1	0	1
m_7	$A\cdot B\cdot C$	1	1	1	1

Product of Sums

- Boolean expression contain AND, OR, and NOT operators
- ANDed (product) a collection of OR terms (sum)

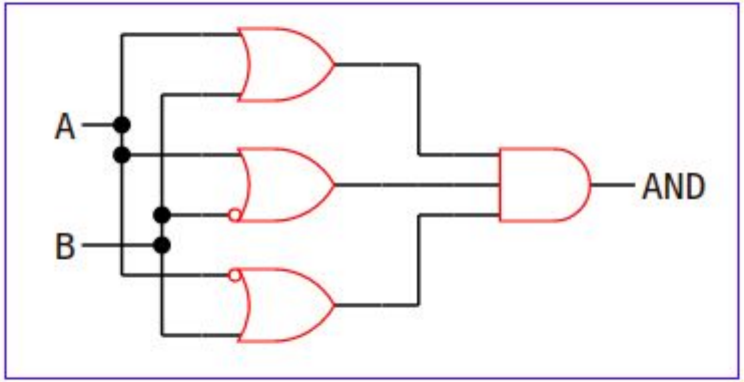
AND

POS	A	B	x
$(A+B)$	0	0	0
$(A+\overline{B})$	0	1	0
$(\overline{A}+B)$	1	0	0
$(\overline{A}+\overline{B})$	1	1	1

Product of Sums

- Boolean expression contain AND, OR, and NOT operators
- ANDed (product) a collection of OR terms (sum)

$$(A+B)\cdot(A+\overline{B})\cdot(\overline{A}+B)$$



AND

POS	A	B	x
$(A+B)$	0	0	0
$(A+\overline{B})$	0	1	0
$(\overline{A}+B)$	1	0	0
$(\overline{A}+\overline{B})$	1	1	1

Maxterms

- A truth-table can be specified by listing its maxterms that are false
- equivalent to a POS expression
- Function f has maxterms
  - M_1, M_2

$\prod(m_1, m_2)$ or $\prod(1, 2)$

Maxterms	POS	A	B	f
M_0	$(A+B)$	0	0	1
M_1	$(A+\overline{B})$	0	1	0
M_2	$(\overline{A}+B)$	1	0	0
M_3	$(\overline{A}+\overline{B})$	1	1	1

Boolean expressions to truth tables

F = ((~b & ~a) | (b & a))

a	b	F
0	0	
0	1	
1	0	
1	1	

Boolean expressions to truth tables

F = ((~b & ~a) | (b & a))

a	b	F
0	0	1
0	1	0
1	0	0
1	1	1

Boolean expressions to truth tables

F = ((~a & c) | (b & a & ~c) | (~b & c) | (~b & ~a))

a	b	c	F
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Boolean expressions to truth tables

F = ((~a & c) | (b & a & ~c) | (~b & c) | (~b & ~a))

a	b	c	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Simplification examples

- Extra video lecture

Questions?

- Next class
- K-Maps
- Simplification of boolean expressions