03 - 2.2

Vinicius Prado da Fonseca, Ph.D.
vpradodafons@mun.ca

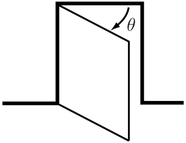
December 19, 2025

Outline

2.2 Degrees of Freedom of a Robot

Consider once again the door example of Figure 2.1(a)

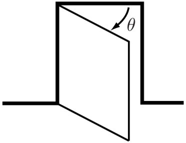
- It consists of a single rigid body connected to a wall by a hinge joint.
- It has only one degree of freedom, conveniently represented by the hinge joint angle θ.
- Without the hinge joint, the door would have six degrees of freedom.
- By connecting the door to the wall via the hinge joint, five independent constraints are imposed, one independent coordinate (θ).
- The door's C-space is represented by some range in the interval $[0, 2\pi)$ over



(a)

- Alternatively, the door can be viewed from above and regarded as a planar body, which has three degrees of freedom.
- The hinge joint then imposes two independent constraints, again leaving only one independent coordinate (θ).

In both cases the joints constrain the motion of the rigid body, thus reducing the overall degrees of freedom. In this section we derive precisely such a formula, called Grübler's formula, for determining the number of degrees of freedom of planar and spatial robots.



(a)

2.2.1 Robot Joints

Figure 2.3 illustrates the basic joints found in typical robots. Every joint connects exactly two links; joints that simultaneously connect three or more links are not allowed.

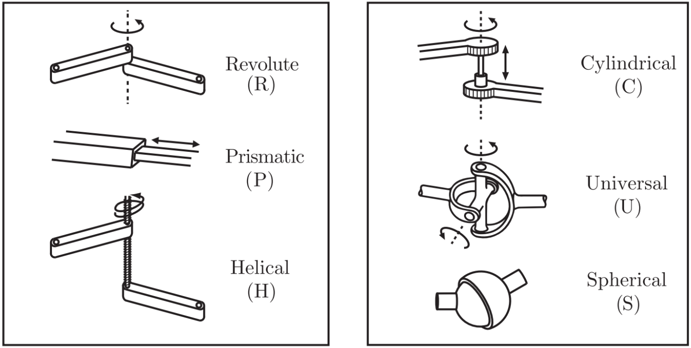
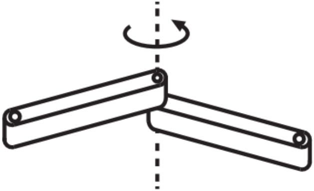


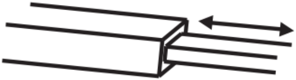
Figure 2.3: Typical robot joints.

The revolute joint (R), also called a hinge joint, allows rotational motion about the joint axis.



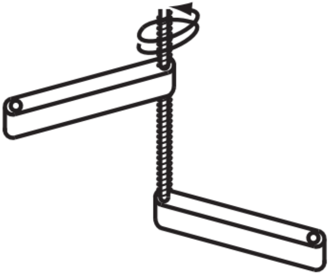
Revolute
(R)

The prismatic joint (P), also called a sliding or linear joint, allows translational (or rectilinear) motion along the direction of the joint axis.



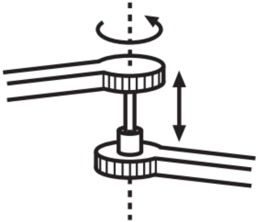
Prismatic
(P)

The helical joint (H), also called a screw joint, allows simultaneous rotation and translation about a screw axis. Revolute, prismatic, and helical joints all have one degree of freedom.



Helical
(H)

Joints can also have multiple degrees of freedom. The cylindrical joint (C) has two degrees of freedom and allows independent translations and rotations about a single fixed joint axis.

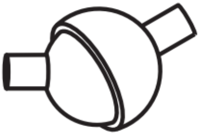


Cylindrical
(C)

The universal joint (U) is another two-degreeof-freedom joint that consists of a pair of revolute joints arranged so that their joint axes are orthogonal.

Universal
(U)

The spherical joint (S), also called a ball-and-socket joint, has three degrees of freedom and functions much like our shoulder joint.



Spherical
(S)

A joint can be viewed as providing freedoms to allow one rigid body to move relative to another. It can also be viewed as providing constraints on the possible motions of the two rigid bodies it connects. Generalizing, the number of degrees of freedom of a rigid body (three for planar bodies and six for spatial bodies) minus the number of constraints provided by a joint must equal the number of freedoms provided by that joint.

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Table 2.1: The number of degrees of freedom f and constraints c provided by common joints.

2.2.2 Grübler's Formula

The number of degrees of freedom of a mechanism with links and joints can be calculated using Grübler's formula, which is an expression of Equation (2.3).

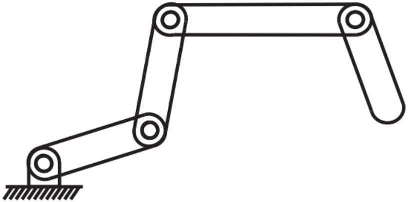
Proposition 2.2. *Consider a mechanism consisting of N links, where ground is also regarded as a link. Let J be the number of joints, m be the number of degrees of freedom of a rigid body ($m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms), f_i be the number of freedoms provided by joint i, and c_i be the number of constraints provided by joint i, where $f_i + c_i = m$ for all i. Then*

The number of degrees of freedom of a mechanism with links and joints can be calculated using Grübler's formula, which is an expression of Equation (2.3).

$$\begin{aligned} \text{dof} &= \underbrace{m(N-1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}} \\ &= m(N-1) - \sum_{i=1}^{J} (m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i. \end{aligned} \tag{2.4}$$

Example 2.4

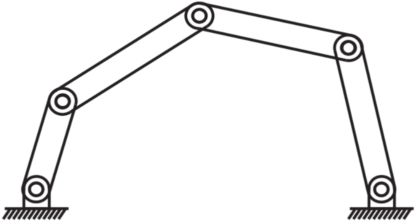
Let us now apply Grübler's formula to several classical planar mechanisms. The k-link planar serial chain of revolute joints in Figure 2.5(a) below.



$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Example 2.4

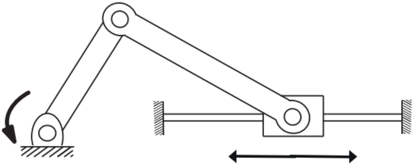
For the planar five-bar linkage of Figure 2.5(b) below.



$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Example 2.3

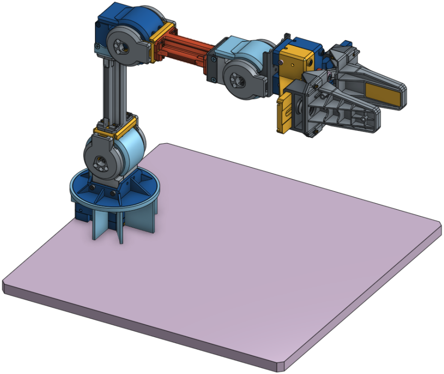
(Four-bar linkage and slider-crank mechanism). The planar fourbar linkage shown in Figure 2.4(a) below consists of four links (one of them ground) arranged in a single closed loop and connected by four revolute joints.



$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Example OMX

OpenManipulator-X



$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

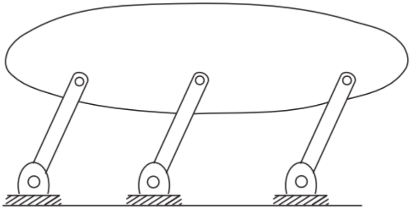
Example Lite6

UFactory Lite6

$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Example 2.6

(Redundant constraints and singularities). For the parallelogram linkage of Figure 2.7(a).



$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

Example 2.6

$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

$N = 5$, $J = 6$, and $f_i = 1$ for each joint. From Grübler's formula, the number of degrees of freedom is $3(5 - 1 - 6) + 6 = 0$. A mechanism with zero degrees of freedom is by definition a rigid structure. It is clear from examining the figure, though, that the mechanism can in fact move with one degree of freedom. Indeed, any one of the three parallel links, with its two joints, has no effect on the motion of the mechanism, so we should have calculated $dof = 3(4 - 1 - 4) + 4 = 1$. In other words, the constraints provided by the joints are not independent, as required by Grübler's formula.

Questions?

Next:

- 04