Homework #3

Due May 11, 2018 at 09:00 A.M.:

Academic Honesty

Academic honesty is taken seriously in this class during the semester. For homework problems or programming assignments, you are allowed to discuss the problems or assignments verbally with other class members, but under no circumstances can you look at or copy anyone else's written solutions or code relating to homework problems or programming assignments. All problem solutions and code submitted must be material you have personally written during this semester, except for any library or utility functions to be supplied. Failure to adhere to this guideline can result in a student receiving a failing grade in the class. It is the responsibility of each student to follow the course guideline.

1. [20 points] (Problem 5.4, textbook pp. 370) When the input to an LTI system is

$$x(n)=\left(\frac{1}{2}\right)^n u(n)+2^n u(-n-1),$$

the output is

$$y(n)=6\left(\frac{1}{2}\right)^n u(n)-6\left(\frac{3}{4}\right)^n u(n)$$

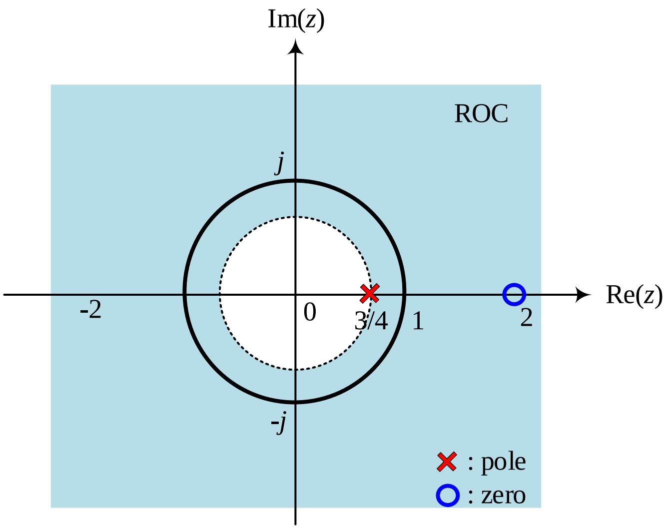
(a) [5 points] Determine the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the ROC.

Solution)

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-2z^{-1}}, \qquad \frac{1}{2}<|z|<2$$

$$Y(z)=\frac{6}{1-\frac{1}{2}z^{-1}}-\frac{6}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4}<|z|$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}\frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}{-\frac{3}{2}z^{-1}}=\frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4}<|z|$$



(b) [5 points] Determine the impulse response $h(n)$ of the system for all values of n.

Solution)

$$H(z)=\frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}=\frac{1}{1-\frac{3}{4}z^{-1}}-\frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4}<|z|$$

$$\therefore h(n)=\left(\frac{3}{4}\right)^{n}u(n)-2\left(\frac{3}{4}\right)^{n-1}u(n-1)$$

(c) [5 points] Write the difference equation that characterizes the system.

Solution)

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4}<|z|$$

$$Y(z)-\frac{3}{4}z^{-1}Y(z)=X(z)-2z^{-1}X(z)$$

$$\therefore y(n)=\frac{3}{4}y(n-1)+x(n)-2x(n-1)$$

(d) [5 points] Is the system stable? Is it causal?

Solution)

Since the pole $p=\frac{3}{4}$ is located inside the unit circle with ROC $\frac{3}{4}<|z|$, the system is stable and causal.

2. [20 points] Consider the following notch filter:

$$H(z)=\frac{b_0\left(1-e^{j\omega_0}z^{-1}\right)\left(1-e^{-j\omega_0}z^{-1}\right)}{\left(1-re^{j\omega_0}z^{-1}\right)\left(1-re^{-j\omega_0}z^{-1}\right)}$$

(a) [5 points] Let $r=0.75$ and $\omega_0=\pi/3$. Find b_0 such that $\left|H\left(e^{j\pi}\right)\right|=\left|H\left(e^{-j\pi}\right)\right|=1$.

Solution) $e^{j\pi}=e^{-j\pi}=-1$

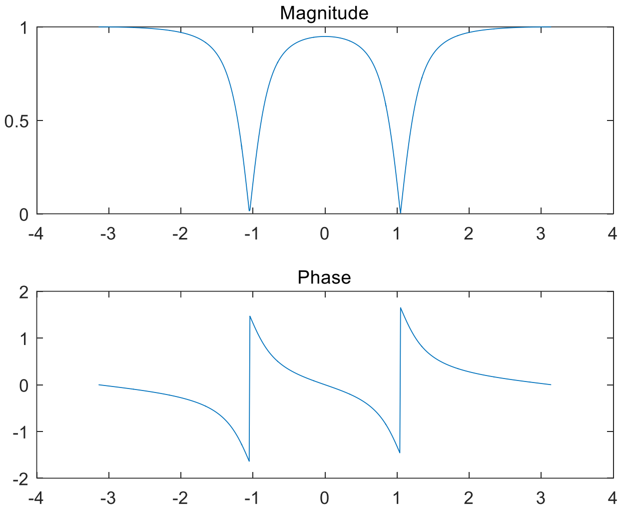
$$H\left(e^{j\pi}\right)=\frac{b_0\left(1-e^{j\omega_0}e^{-j\pi}\right)\left(1-e^{-j\omega_0}e^{-j\pi}\right)}{\left(1-re^{j\omega_0}e^{-j\pi}\right)\left(1-re^{-j\omega_0}e^{-j\pi}\right)}=\frac{b_0\left(1+e^{j\omega_0}\right)\left(1+e^{-j\omega_0}\right)}{\left(1+re^{j\omega_0}\right)\left(1+re^{-j\omega_0}\right)}=\frac{b_0\left(2+2\cos\omega_0\right)}{1+2r\cos\omega_0+r^2}$$

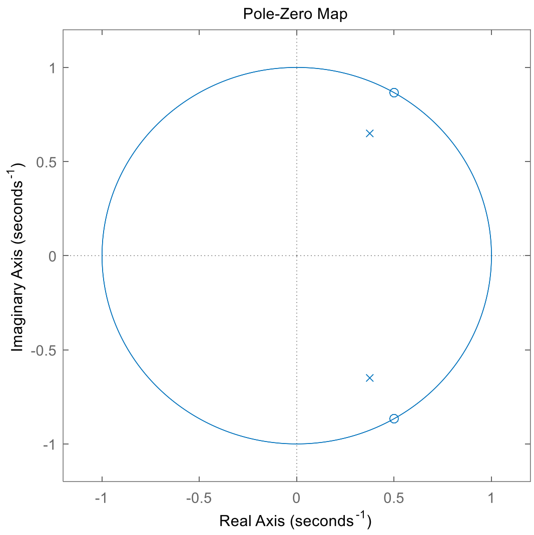
$$=\frac{b_0\left(2+2\cos\frac{\pi}{3}\right)}{1+2(0.75)\cos\frac{\pi}{3}+(0.75)^2}=\frac{3b_0}{1+0.75+(0.75)^2}=\frac{48}{37}b_0$$

$$\therefore b_0=\pm\frac{37}{48}=\pm 0.7708$$

(b) [5 points] Use Matlab to plot the magnitude response $\left|H\left(e^{j\omega}\right)\right|$ and the phase response $\measuredangle H\left(e^{j\omega}\right)$ for the range $-\pi \le \omega \le \pi$. Plot the poles and zeros for $H(z)$ using Matlab.

Solution)





(c) [5 points] Now, let $r = 0.95$ and $\omega_0 = \pi/3$. Find b_0 such that $\left|H\left(e^{j\pi}\right)\right| = \left|H\left(e^{-j\pi}\right)\right| = 1$.

Solution)

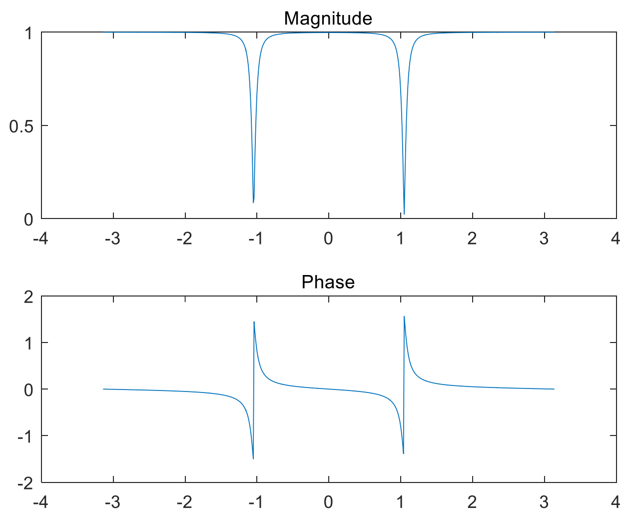
$$H\left(e^{j\pi}\right) = \frac{b_0\left(1-e^{j\omega_0}e^{-j\pi}\right)\left(1-e^{-j\omega_0}e^{-j\pi}\right)}{\left(1-re^{j\omega_0}e^{-j\pi}\right)\left(1-re^{-j\omega_0}e^{-j\pi}\right)} = \frac{b_0\left(1+e^{j\omega_0}\right)\left(1+e^{-j\omega_0}\right)}{\left(1+re^{j\omega_0}\right)\left(1+re^{-j\omega_0}\right)} = \frac{b_0\left(2+2\cos\omega_0\right)}{1+2r\cos\omega_0+r^2}$$

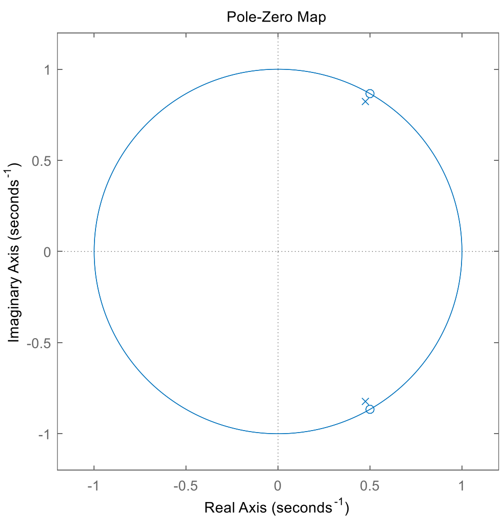
$$= \frac{b_0\left(2+2\cos\frac{\pi}{3}\right)}{1+2(0.95)\cos\frac{\pi}{3}+(0.95)^2} = \frac{3b_0}{1+0.95+(0.95)^2} = \frac{1200}{1141}b_0$$

$$\therefore b_0 = \pm\frac{1141}{1200} = \pm 0.9508$$

(d) [5 points] Use Matlab to plot the magnitude response $\left|H\left(e^{j\omega}\right)\right|$ and the phase response $\measuredangle H\left(e^{j\omega}\right)$ for the range $-\pi \le \omega \le \pi$. Plot the poles and zeros for $H(z)$ using Matlab.

Solution)





3. [30 points] (Similar to Problem 5.23, textbook pp. 377) Consider a causal linear time-invariant system with system function

$$H(z)=\frac{\left(1-a^{-1}z^{-1}\right)\left(1-ja^{-1}z^{-1}\right)\left(1+ja^{-1}z^{-1}\right)}{\left(1-az^{-1}\right)\left(1-jaz^{-1}\right)\left(1+jaz^{-1}\right)}$$

where a is real.

(a) [5 points] Write the difference equation that relates the input and the output of this system.

Solution)

$$H(z)=\frac{\left(1-a^{-1}z^{-1}\right)\left(1-ja^{-1}z^{-1}\right)\left(1+ja^{-1}z^{-1}\right)}{\left(1-az^{-1}\right)\left(1-jaz^{-1}\right)\left(1+jaz^{-1}\right)}=\frac{1-a^{-1}z^{-1}+a^{-2}z^{-2}-a^{-3}z^{-3}}{1-az^{-1}+a^{2}z^{-2}-a^{3}z^{-3}}=\frac{Y(z)}{X(z)}$$

$$Y(z)-az^{-1}Y(z)+a^2z^{-2}Y(z)-a^3z^{-3}Y(z)=X(z)-a^{-1}z^{-1}X(z)+a^{-2}z^{-2}X(z)-a^{-3}z^{-3}X(z)$$

$$\therefore y(n)=ay(n-1)+a^2y(n-2)-a^3y(n-3)+x(n)-a^{-1}x(n-1)+a^{-2}x(n-2)-a^{-3}x(n-3)$$

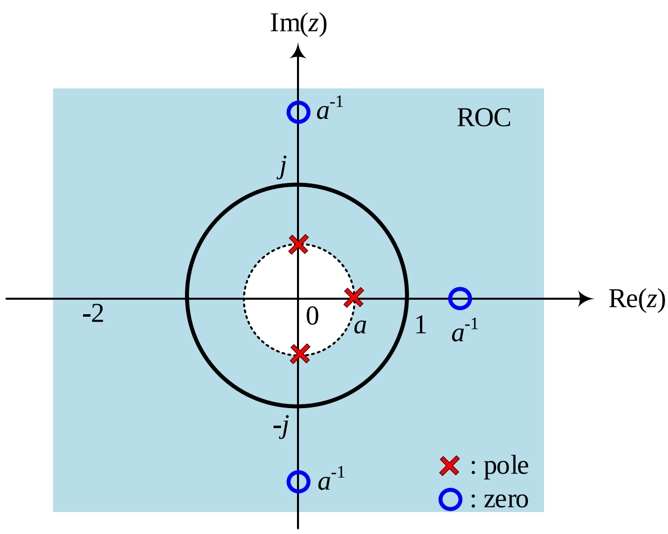
(b) [5 points] For what range of values of a is the system stable?

Solution)

The causal system is stable for $|a|<1$ where the poles are located inside the unit circle in the z-plane.

(c) [5 points] For $a=1/2$, plot the pole-zero diagram using Matlab and shade the region of convergence.

Solution)



(d) [5 points] Find the impulse response $h(n)$ for the system.

Solution)

$$H(z)=\frac{\left(1-a^{-1}z^{-1}\right)\left(1-ja^{-1}z^{-1}\right)\left(1+ja^{-1}z^{-1}\right)}{\left(1-az^{-1}\right)\left(1-jaz^{-1}\right)\left(1+jaz^{-1}\right)}=\frac{A_1}{1-az^{-1}}+\frac{A_2}{1-jaz^{-1}}+\frac{A_3}{1+jaz^{-1}}$$

$$A_1=\left(1-az^{-1}\right)H(z)\Big|_{z=a}=\frac{\left(1-ja^{-2}\right)\left(1+ja^{-2}\right)}{(1-j)(1+j)}=\frac{1+a^{-4}}{2}$$

$$A_2=\left(1-jaz^{-1}\right)H(z)\Big|_{z=ja}=\frac{\left(1-j^{-1}a^{-2}\right)\left(1+a^{-2}\right)}{\left(1-j^{-1}\right)(1+1)}=\left(1+a^{-2}\right)\frac{\left(1+a^{-1}\right)-j\left(1-a^{-1}\right)}{4}$$

$$A_3=\left(1+jaz^{-1}\right)H(z)\Big|_{z=ja}=A_2^*=\left(1+a^{-2}\right)\frac{\left(1+a^{-1}\right)+j\left(1-a^{-1}\right)}{4}$$

We know that $Z^{-1}\left\{\frac{A_k}{1-p_k z^{-1}}+\frac{A_k^*}{1-p_k^* z^{-1}}\right\}=2\left|A_k\right|r_k^n\cos\left(\beta_k n+\alpha_k\right)u(n)$.

$$\left|A_2\right|=\frac{\left(1+a^{-2}\right)}{4}\sqrt{\left(1+a^{-1}\right)^2+\left(1-a^{-1}\right)^2}=\frac{\left(1+a^{-2}\right)}{4}\sqrt{2\left(1+a^{-2}\right)}=\frac{\sqrt{2}}{4}\left(1+a^{-2}\right)^{\frac{3}{2}}$$

$$\measuredangle A_2=-\tan\frac{\left(1-a^{-1}\right)}{\left(1+a^{-1}\right)},\quad p_2=ja=ae^{j\frac{\pi}{2}},\quad p_3=p_2^*=ae^{-j\frac{\pi}{2}}$$

$$h(n)=\left(\frac{1+a^{-4}}{2}\right)a^n u(n)+\frac{\sqrt{2}}{2}\left(1+a^{-2}\right)^{\frac{3}{2}}a^n\cos\left(\frac{\pi}{2}n-\tan^{-1}\frac{\left(1-a^{-1}\right)}{\left(1+a^{-1}\right)}\right)u(n)$$

(e) [5 points] Show that the system is an all-pass system, i.e., that the magnitude of the frequency response is a constant. Also, specify the value of the constant.

Solution)

We plug z = 1 into $H(z)$.

$$H(z)\Big|_{z=1}=\frac{\left(1-a^{-1}z^{-1}\right)\left(1-ja^{-1}z^{-1}\right)\left(1+ja^{-1}z^{-1}\right)}{\left(1-az^{-1}\right)\left(1-jaz^{-1}\right)\left(1+jaz^{-1}\right)}\Bigg|_{z=1}$$

$$=\frac{\left(1-a^{-1}\right)\left(1-ja^{-1}\right)\left(1+ja^{-1}\right)}{(1-a)(1-ja)(1+ja)}=\frac{\left(1-a^{-1}\right)\left(1+a^{-2}\right)}{(1-a)\left(1+a^2\right)}$$

$$=\frac{-a^{-3}(1-a)\left(1+a^2\right)}{(1-a)\left(1+a^2\right)}=-\frac{1}{a^3}$$

(f) [5 points] Plot the magnitude and phase responses of $H\left(e^{j\omega}\right)$ with $a=1/2$ for $-\pi\le\omega\le\pi$ using Matlab.

Solution)

4. [15 points] (Problem 5.40, textbook pp. 383) Consider a stable LTI system given by

$$H(z)=\frac{\left(1-9z^{-2}\right)\left(1+\frac{1}{3}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}$$

(a) [5 points] $H(z)$ can be expressed as a cascade of a minimum-phase system $H_{\min}(z)$ and a unity-gain all-pass system $H_{ap}(z)$. Determine a choice for $H_{\min}(z)$ and $H_{ap}(z)$, and specify whether or not they are unique up to a scale factor.

Solution)

$$H(z)=\frac{\left(1-9z^{-2}\right)\left(1+\frac{1}{3}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}=\frac{\left(1+\frac{1}{3}z^{-1}\right)\left(1-3z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}=\frac{\left(1+\frac{1}{3}z^{-1}\right)^2\left(1-3z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

$$=\underbrace{9\left(1+\frac{1}{3}z^{-1}\right)^2}_{=H_{\min}(z)}\cdot\underbrace{\frac{1}{9}\frac{\left(1-3z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}}_{=H_{ap}(z)}$$

$$\therefore H_{\min}(z)=9\left(1+\frac{1}{3}z^{-1}\right)^2,\quad \text{and}\quad H_{ap}(z)=\frac{1}{9}\frac{\left(1-3z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}.$$

(b) [5 points] Is the minimum-phase system $H_{\min}(z)$ an FIR system? Explain.

Solution)

Yes, it is an FIR filter because $H_{\min}(z)$ has its pole at the origin $z=0$ and has a finite impulse response with length 3 as $h_{\min}(n)=9+6\delta(n-1)+\delta(n-2)$.

(c) [5 points] Is the minimum-phase system $H_{\min}(z)$ a generalized linear-phase system? If not, can $H(z)$ be represented as a cascade of a generalized linear-phase system $H_{lin}(z)$ and an all-pass system $H_{ap2}(z)$? If your answer is yes, determine $H_{lin}(z)$ and $H_{ap2}(z)$. If your answer is no, explain why such representation does not exist.

Solution)

$H_{\min}(z)$ is not a linear phase system because $h_{\min}(n)$ is not symmetric nor anti-symmetric. However, if we formulate $H(z)$ slightly in a different way, we can obtain the product of $H_{\text{lin}}(z)$ and $H_{ap2}(z)$ as follows:

$$H(z)=\frac{\left(1-9z^{-2}\right)\left(1+\frac{1}{3}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}=\frac{\left(1+\frac{1}{3}z^{-1}\right)\left(1-3z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}=\underbrace{3\left(1+\frac{1}{3}z^{-1}\right)\left(1+3z^{-1}\right)}_{=H_{lin}(z)}\cdot\underbrace{\frac{1}{3}\frac{\left(1-3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}}_{=H_{ap2}(z)}$$

5. [15 points] (Problem 5.53, textbook pp. 389) Consider the LTI system whose system function is given by

$$H(z)=\left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{-j0.8\pi}z^{-1}\right)$$

(a) [5 points] Determine all causal system functions that results in the same frequency-response magnitude as $H(z)$ and for which the impulse responses are real valued and of the same length as the impulse response associated with $H(z)$. (There are four different such systems.) Identify which system function is minimum phase and which to within a time shift, is maximum phase.

Solution)

$$H(z)=\left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{-j0.8\pi}z^{-1}\right)$$

$$H_1(z)=(0.9)^2(1.25)^2\left(1-\frac{1}{0.9}e^{j0.6\pi}z^{-1}\right)\left(1-\frac{1}{0.9}e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{1}{1.25}e^{j0.8\pi}z^{-1}\right)\left(1-\frac{1}{1.25}e^{-j0.8\pi}z^{-1}\right)$$

$$H_2(z)=(0.9)^2\left(1-\frac{1}{0.9}e^{j0.6\pi}z^{-1}\right)\left(1-\frac{1}{0.9}e^{-j0.6\pi}z^{-1}\right)\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{-j0.8\pi}z^{-1}\right)$$

$$H_3(z)=(1.25)^2\left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{1}{1.25}e^{j0.8\pi}z^{-1}\right)\left(1-\frac{1}{1.25}e^{-j0.8\pi}z^{-1}\right)$$

(b) [5 points] Determine the impulse responses for the system functions in part (a).

Solution)

$$H(z)=1+2.5788z^{-1}+3.4975z^{-2}+2.5074z^{-3}+1.2656z^{-4}$$

$$\therefore h(n)=\delta(n)+2.5788\delta(n-1)+3.4975\delta(n-2)+2.5074\delta(n-3)+1.2656\delta(n-4)$$

$$H_1(z)=1.2656+2.5074z^{-1}+3.4975z^{-2}+2.5788z^{-3}+z^{-4}$$

$$\therefore h_1(n)=1.2656\delta(n)+2.5074\delta(n-1)+3.4975\delta(n-2)+2.5788\delta(n-3)+\delta(n-4)$$

$$H_2(z)=0.81+2.1945z^{-1}+3.3906z^{-2}+2.8917z^{-3}+1.5625z^{-4}$$

$$\therefore h_2(n)=0.81\delta(n)+2.1945\delta(n-1)+3.3906\delta(n-2)+2.8917\delta(n-3)+1.5625\delta(n-4)$$

$$H_3(z)=1.5625+2.8917z^{-1}+3.3906z^{-2}+2.1945z^{-3}+0.81z^{-4}$$

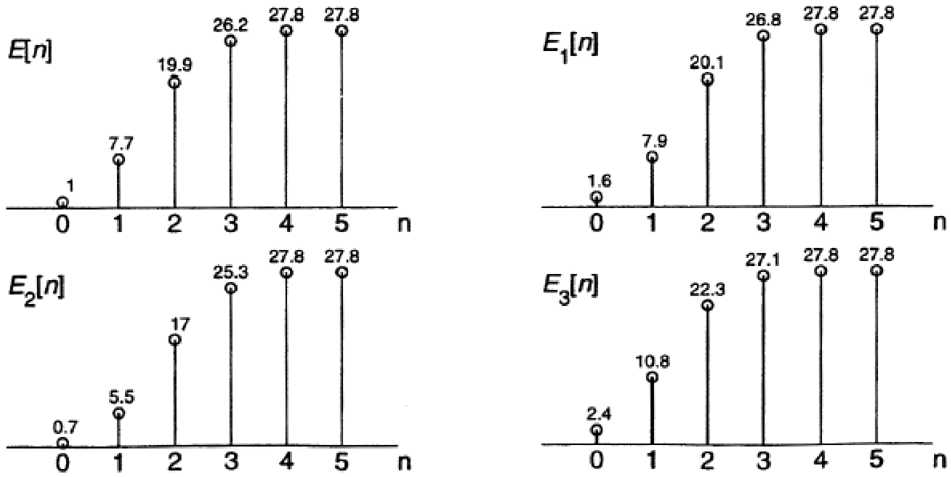
$$\therefore h_3(n)=1.5625\delta(n)+2.8917\delta(n-1)+3.3906\delta(n-2)+2.1945\delta(n-3)+0.81\delta(n-4)$$

(c) [5 points] For each of the sequences in part in (b), compute and plot the quantity $E(n)=\sum_{m=0}^{n} h^2(m)$ for $0 \le n \le 5$. Indicate explicitly which plot corresponds to the minimum-phase system.

Solution)

(c)

n	$E(n)$	$E_1(n)$	$E_2(n)$	$E_3(n)$
0	1.0	1.6	0.7	2.4
1	7.7	7.9	5.5	10.8
2	19.9	20.1	17.0	22.3
3	26.2	26.8	25.3	27.1
4	27.8	27.8	27.8	27.8
5	27.8	27.8	27.8	27.8



The plot of $E_3[n]$ corresponds to the minimum phase sequence.