Homework #4

Due June 1, 2018 at 09:00 A.M.:

Academic Honesty

Academic honesty is taken seriously in this class during the semester. For homework problems or programming assignments, you are allowed to discuss the problems or assignments verbally with other class members, but under no circumstances can you look at or copy anyone else's written solutions or code relating to homework problems or programming assignments. All problem solutions and code submitted must be material you have personally written during this semester, except for any library or utility functions to be supplied. Failure to adhere to this guideline can result in a student receiving a failing grade in the class. It is the responsibility of each student to follow the course guideline.

1. [20 points, Problem 9.21, pp. 794 in the text book] In Section 9.1.2, we used the fact that $W_N^{-kN}=1$ to derive a recurrent algorithm for computing a specific DFT value $X(k)$ for a finite-length sequence $x(n)$, $n=0,1,\cdots,N-1$.

(a) [10 points] Using the fact that $W_N^{kN}=W_N^{Nn}=1$, show that $X(N-k)$ can be obtained as the output after N iterations of the difference equation depicted in Fig. P9.21-1. That is, show that $X(N-k)=y_k(N)$.

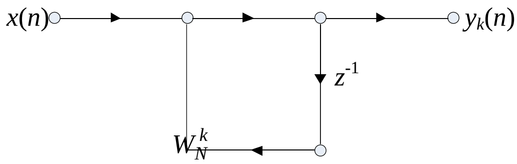

Fig. P9.21-1

(b) [10 points] Show that $X(N-k)$ is also equal to the output after N iterations of the difference equation depicted in Fig. P9.21-2. Note that the system of Figure P.9.21-2 has the same poles as the system in Figure 9.2, but the coefficient required to implement the complex zero in Fig. P.9.21-2 is the complex conjugate of the corresponding coefficient in Figure 9.2; i.e., $W_N^{-k}=\left(W_N^k\right)^*$.

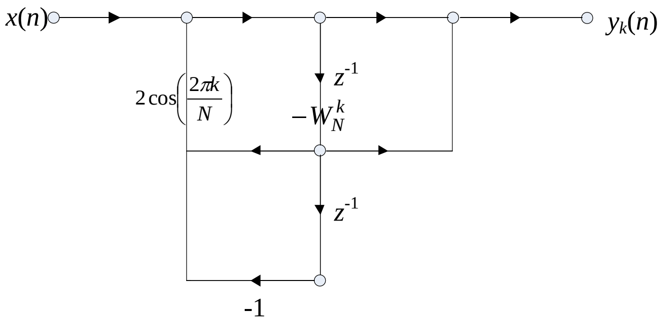

Fig. P9.21-2

2. [20 points] Consider a finite-duration sequence $x(n)$ of length N and its discrete Fourier transform (DFT) $X(k)$. Let $x_i(n)$ be finite-duration sequences of length $2N$ and $X_i(k)$ be their DFTs. Then we have,

$$x_1(n)=\begin{cases} x(n), & 0\le n\le N-1 \\ 0, & N\le n\le 2N-1 \end{cases}$$

$$x_2(n)=\begin{cases} x(n/2), & 0\le n\le 2N-1, n \text{ is even} \\ 0, & elsewhere \end{cases}$$

$$x_3(n)=\begin{cases} x(n), & 0\le n\le N-1 \\ x(n-N), & N\le n\le 2N-1 \end{cases}$$

(a) [5 points] Express $X_1(k)$ in terms of $X(k)$.

(b) [5 points] Express $X_2(k)$ in terms of $X(k)$.

(c) [10 points] Express $X_3(k)$ in terms of $X(k)$.

3. [30 points] Consider two finite-duration sequences $x(n)$ and $y(n)$ of lengths, N and M, respectively. More precisely, $x(n)=0$ for $n<0$ and $n\ge N$, and $y(n)=0$ for $n<0$ and $n\ge M$. Assume that $x(n)\ne y(n)$ for at least one point and $N<M$.

(a) [10 points] Consider the M-point DFTs of $x(n)$ and $y(n)$, which are the samplings of the DTFTs $X(e^{j\omega})$ and $Y(e^{j\omega})$ at M uniformly spaced points such as

$$X_1(k)=\sum_{n=0}^{M-1} x(n)e^{-j\frac{2\pi}{M}nk}, \quad k=0,1,\dots,M-1$$

$$Y_1(k)=\sum_{n=0}^{M-1} y(n)e^{-j\frac{2\pi}{M}nk}, \quad k=0,1,\dots,M-1$$

Can we claim that $X_1(k)=Y_1(k), \quad k=0,1,\dots,M-1$? Justify your answer.

(b) [10 points] Let us sample the DTFTs $X(e^{j\omega})$ and $Y(e^{j\omega})$ of $x(n)$ and $y(n)$ at N uniformly spaced points such as

$$X_2(k)=\sum_{n=0}^{M-1} x(n)e^{-j\frac{2\pi}{N}nk}, \quad k=0,1,\dots,N-1$$

$$Y_2(k)=\sum_{n=0}^{M-1} y(n)e^{-j\frac{2\pi}{N}nk}, \quad k=0,1,\dots,N-1$$

Can we claim that $X_2(k)=Y_2(k), \quad k=0,1,\dots,N-1$? Justify your answer.

(c) [10 points] Assume that $M=2N$, and that

$$y(n)=\begin{cases} x(n), & n=0,1,\dots,N-1 \\ x(n-N), & n=N,N+1,\dots,M-1 \end{cases}$$

and we have the sequence $z(n)=y(n)-x(n)$. Now, we want to find the smallest number of samples of the DTFT of $z(n)$ that would be sufficient to reconstruct $z(n)$. That is,

$$Z(k)=Z\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/Q}, \qquad k=0,1,\ldots,Q-1$$

and we want to ensure that $z(n)$ is recoverable from the samples $Z(k),\ k=0,1,\ldots,Q-1$. Then find the smallest Q that would be sufficient. Also write down the explicit formula of how we can reconstruct $z(n)$, given $Z(k),\ k=0,1,\ldots,Q-1$, for this smallest value of Q.

4. [15 points, Problem 9.25 at pp. 796 in the textbook] The DFT is a sampled version of the DTFT of a finite-length sequence; i.e.,

$$\begin{aligned} X(k) &= X\left(e^{j(2\pi/N)k}\right)=X\left(e^{j\omega_k}\right)\Big|_{\omega_k=(2\pi/N)k} \\ &= \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}, \qquad k=0,1,\cdots,N-1 \end{aligned}$$ --- Eq. (P9.25-1)

Furthermore, an FFT algorithm is an efficient way to compute the values $X(k)$. Now consider a finite-length sequence of length N. We want to evaluate $X(z)$, the z-transform of the finite-length sequence, at the following points in the z-plane

$$z_k=re^{j(2\pi/N)k} \qquad k=0,1,\cdots,N-1$$

where r is a positive number. We have an available FFT algorithm.

(a) [5 points] Plot the points z_k in the z-plane for the case $N=8$ and $r=0.9$.

(b) [5 points] Write an equation [similar to Eq. (P9.25-1) above] for $X(z_k)$ that shows that $X(z_k)$ is the DFT of a modified sequence $\tilde{x}(n)$. What is $\tilde{x}(n)$?

(c) [5 points] Describe an algorithm for computing $X(z_k)$ using the given FFT function. (Direct evaluation is not an option) You may describe your algorithm using a combination of English text and equations, but you may give a step-by-step procedure that starts with the sequence $x(n)$ and ends with $X(z_k)$.

5 [15 points, Problem 9.30 at pp. 798 in the textbook] The system in Fig. P9.30 compute an N-point DFT $X(k)$ of an N-point sequence $x(n)$ by decomposing into N/2-point sequences $g_1(n)$ and $g_2(n)$, computing the N/2-point DFT's $G_1(k)$ and $G_2(k)$, and then combining these to form $X(k)$.

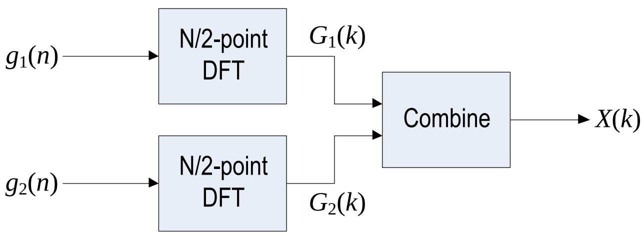

Fig. P9.30

If $g_1(n)$ is the even-indexed values of $x(n)$ and $g_2(n)$ is the odd-indexed values of $x(n)$, i.e., $g_1(n)=x(2n)$ and $g_1(n)=x(2n+1)$, then $X(k)$ will be the DFT of $x(n)$. deterministic autocorrelation of this sequence is the inverse DTFT of

In using the system in Figure P.30, an error is made in forming $g_1(n)$ and $g_2(n)$, such that $g_1(n)$ is incorrectly chosen as the odd-indexed values and $g_2(n)$ as the even-indexed values but $G_1(k)$ and $G_2(k)$ are still combined as in Figure P.30 and the incorrect sequence $\hat{X}(k)$ results. Express $\hat{X}(k)$ in terms of $X(k)$.