Korea Advanced Institute of Science and Technology

School of Electrical Engineering

EE432 Digital Signal Processing Spring 2018

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Homework 3

1. Problem 1

When the input to an LTI system is

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(-n-1)$$

(a) Determine the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the ROC

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}, ROC: \frac{1}{2} < |z| < 2$$

$$y(n) = 6\left(\frac{1}{2}\right)^n u(n) - 6\left(\frac{3}{4}\right)^n u(n)$$

$$\Rightarrow Y(z) = 6\left(\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{3}{4}z^{-1}}\right) = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}, ROC: |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} = \frac{4(z-2)}{4z-3}$$

$H(z)$ has one pole and one zero, $p = \frac{3}{4} = 0.75$ and $z = 2$ with $ROC: |z| > \frac{3}{4}$.

The pole and zero of $H(z)$ are presented in Figure 1.1. The ROC of $H(z)$ is the gray area.

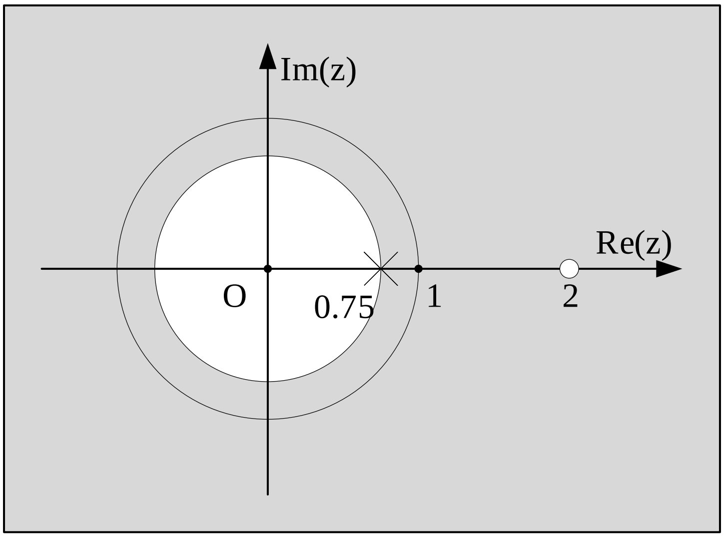

Figure 1.1. Poles and zeros of $H(z)$

(b) Determine the impulse response $h(n)$ of the system for all values of n

$$H(z) = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, ROC: |z| > \frac{3}{4}$$

Inverse z-Transform $H(z)$:

$$h(n) = \left(\frac{3}{4}\right)^{n} u(n) - 2\left(\frac{3}{4}\right)^{n-1} u(n-1)$$

(c) Write the difference equation that characterizes the system

$$\frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} \Leftrightarrow Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

Inverse z-Transform for both sides of the above equation:

$$y(n) - \frac{3}{4}y(n-1) = x(n) - 2x(n-1) \Rightarrow y(n) = \frac{3}{4}y(n-1) + x(n) - 2x(n-1)$$

So, the difference equation, characterizing the system is:

$$y(n) = \frac{3}{4}y(n-1) + x(n) - 2x(n-1)$$

(d) Is the system stable? Is it causal?

The system is stable because the ROC of $H(z)$ includes the unit circle in z-plane.

The system is causal because the output $y(n)$ only depends on the delayed sequences and does not depend on the advanced sequences.

2. Problem 2

(a) Let $r = 0.75$ and $\omega_0 = \frac{\pi}{3}$. Find b_0 such that $\left|H(e^{j\pi})\right| = \left|H(e^{-j\pi})\right| = 1$

$$H(z) = \frac{b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

$$H(e^{j\omega}) = \frac{b_0\left[1 - e^{j(\omega_0-\omega)}\right]\left[1 - e^{-j(\omega_0+\omega)}\right]}{\left[1 - re^{j(\omega_0-\omega)}\right]\left[1 - re^{-j(\omega_0+\omega)}\right]}$$

$$\left|H(e^{j\omega})\right| = |b_0| \frac{\left|1 - e^{j(\omega_0-\omega)}\right|\left|1 - e^{-j(\omega_0+\omega)}\right|}{\left|1 - re^{j(\omega_0-\omega)}\right|\left|1 - re^{-j(\omega_0+\omega)}\right|}$$

$$= |b_0| \frac{\sqrt{[1-\cos(\omega_0-\omega)]^2 + [\sin(\omega_0-\omega)]^2}\sqrt{[1-\cos(\omega_0+\omega)]^2 + [\sin(\omega_0+\omega)]^2}}{\sqrt{[1-r\cos(\omega_0-\omega)]^2 + [r\sin(\omega_0-\omega)]^2}\sqrt{[1-r\cos(\omega_0+\omega)]^2 + [r\sin(\omega_0+\omega)]^2}}$$

$$= |b_0| \frac{\sqrt{2 - 2\cos(\omega_0-\omega)}\,\sqrt{2 - 2\cos(\omega_0+\omega)}}{\sqrt{1 + r^2 - 2r\cos(\omega_0-\omega)}\,\sqrt{1 + r^2 - 2r\cos(\omega_0+\omega)}}$$

For $r = 0.75, \omega_0 = \dfrac{\pi}{3}$ and $\omega = \pi$:

$$\left|H(e^{j\pi})\right| = |b_0| \frac{\sqrt{2 - 2\cos\left(-\frac{2\pi}{3}\right)}\,\sqrt{2 - 2\cos\frac{4\pi}{3}}}{\sqrt{\frac{25}{16} - \frac{3}{2}\cos\left(-\frac{2\pi}{3}\right)}\,\sqrt{\frac{25}{16} - \frac{3}{2}\cos\frac{4\pi}{3}}} = |b_0| \frac{\sqrt{3}\sqrt{3}}{\frac{\sqrt{37}}{4}\frac{\sqrt{37}}{4}} = \frac{48}{37}|b_0|$$

Because $H^*(e^{j\omega}) = H(e^{-j\omega}) \Rightarrow \left|H(e^{j\omega})\right| = \left|H^*(e^{j\omega})\right| = \left|H(e^{-j\omega})\right|$

Let $\left|H(e^{j\pi})\right| = \left|H(e^{-j\pi})\right| = 1 \Leftrightarrow |b_0| = \dfrac{37}{48} \Leftrightarrow b_0 = \pm\dfrac{37}{48}$

(b) Use Matlab to plot the magnitude response $\left|H(e^{j\omega})\right|$ and the phase response $\angle H(e^{j\omega})$ for the range $-\pi \le \omega \le \pi$. Plot the poles and zeros of $H(z)$ using Matlab

❖ Plot the magnitude and phase response of the system

```
clear; clc;
w = (-1000:1:1000)*pi/1000;
b0 = -37/48;
w0 = pi/3;
r = 0.75;
w1 = w0 - w;
w2 = w0 + w;
H = b0*(1-exp(1j*w1)).*(1-exp(-1j*w2))./((1-r*exp(1j*w1)).*(1-r*exp(-
1j*w2)));
magnitudeH = abs(H); phaseH = angle(H);
subplot(2,1,1); plot(w/pi,magnitudeH); grid on;
ylabel('|H(e^j^\omega)|'); title('Magnitude Response');
xlabel('\omega/\pi');
subplot(2,1,2); plot(w/pi,phaseH); grid on;
ylabel('\angleH(e^j^\omega)'); title('Phase Response');
xlabel('\omega/\pi');
```

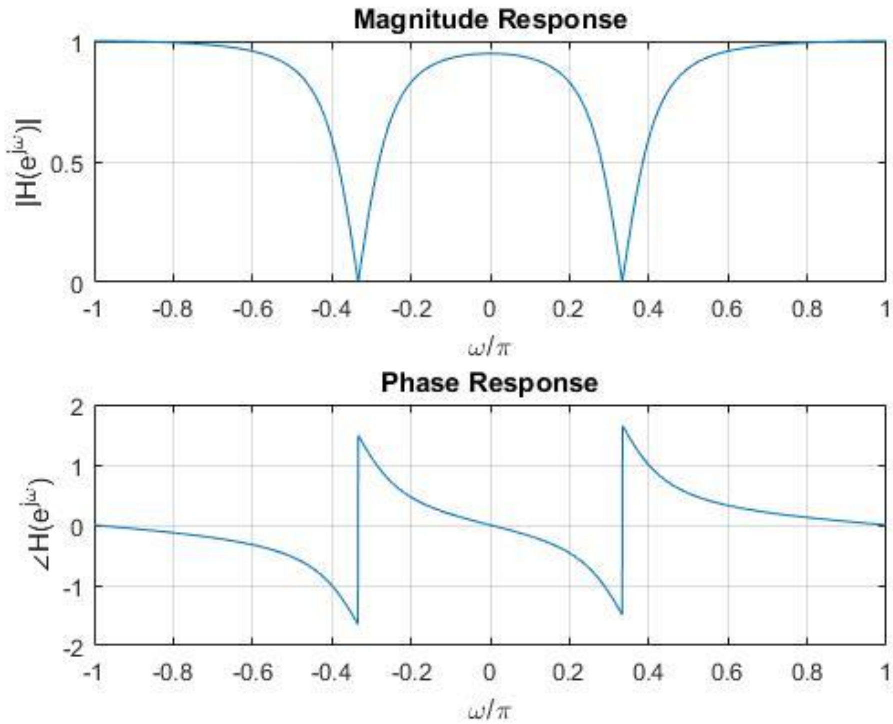

Figure 2.1. The magnitude and phase response of the system with $b_0 = \frac{37}{48}, \omega_0 = \frac{\pi}{3}$ and $r = 0.75$

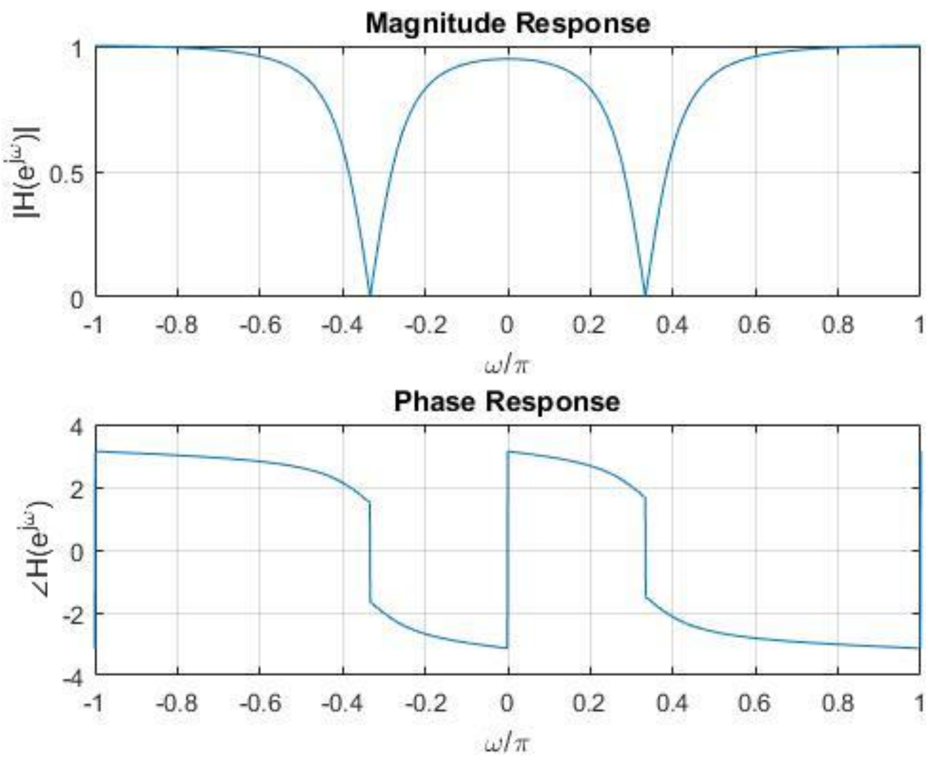

Figure 2.2. The magnitude and phase response of the system with $b_0 = \frac{-37}{48}, \omega_0 = \frac{\pi}{3}$ and $r = 0.75$

❖ Plot the poles and zeros of $H(z)$

$$H(z) = \frac{b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} = b_0\frac{1 - (e^{j\omega_0} + e^{-j\omega_0})z^{-1} + z^{-2}}{1 - r(e^{j\omega_0} + e^{-j\omega_0})z^{-1} + r^2z^{-2}}$$

$$= b_0\frac{1 - 2\cos(\omega_0)\,z^{-1} + z^{-2}}{1 - 2r\cos(\omega_0)\,z^{-1} + r^2z^{-2}}$$

The Figure 2.3 depicts the poles and zeros of $H(z)$ where $b_0 = \pm\frac{37}{48}$.

```
clear; clc;

% Parameter
b0 = 37/48;
w0 = pi/3;
r = 0.75;

% Polynomial parameter
b = b0*[1 -2*cos(w0) 1];
a = [1 -2*r*cos(w0) r^2];

% Calculating pole
[R, p, C] = residuez(b, a);

% Plot zero-pole
[H1, H2, H3] = zplane(b, a);
set(H1, 'markersize', 10, 'color', 'r');
set(H2, 'markersize', 10, 'color', 'b');
title('Pole-Zero Plot');
text(0.85, -0.1, '1.0');
text(0.01, -0.1, '0');
legend('Zero', 'Pole');
grid on;
```

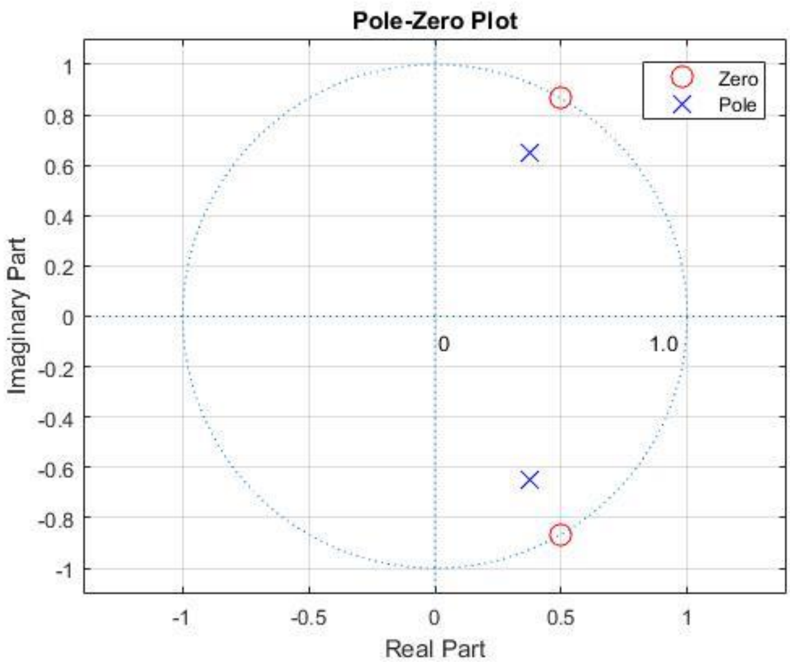

Figure 2.3. The poles and zeros of $H(z)$ for $b_0 = \pm\frac{37}{48}, \omega_0 = \frac{\pi}{3}$ and $r = 0.75$

(c) Now, let $r = 0.95$ and $\omega_0 = \frac{\pi}{3}$. Find b_0 such that $|H(e^{j\pi})| = |H(e^{-j\pi})| = 1$

$$|H(e^{j\omega})| = |b_0| \frac{\sqrt{2 - 2\cos(\omega_0 - \omega)}\sqrt{2 - 2\cos(\omega_0 + \omega)}}{\sqrt{1 + r^2 - 2r\cos(\omega_0 - \omega)}\sqrt{1 + r^2 - 2r\cos(\omega_0 + \omega)}}$$

For $r = 0.95, \omega_0 = \frac{\pi}{3}$ and $\omega = \pi$:

$$|H(e^{j\pi})| = |b_0| \frac{\sqrt{2 - 2\cos\left(-\frac{2\pi}{3}\right)}\sqrt{2 - 2\cos\frac{4\pi}{3}}}{\sqrt{\frac{761}{400} - \frac{19}{10}\cos\left(-\frac{2\pi}{3}\right)}\sqrt{\frac{761}{400} - \frac{19}{10}\cos\frac{4\pi}{3}}} = |b_0| \frac{\sqrt{3}\sqrt{3}}{\frac{\sqrt{1141}}{20}\frac{\sqrt{1141}}{20}}$$
$$= |b_0| \frac{1200}{1141}$$

Let $|H(e^{j\pi})| = |H(e^{-j\pi})| = 1 \Leftrightarrow |b_0| = \frac{1141}{1200} \Leftrightarrow b_0 = \pm\frac{1141}{1200}$

(d) Use Matlab to plot the magnitude response $|H(e^{j\omega})|$ and the phase response $\angle H(e^{j\omega})$ for the range $-\pi \le \omega \le \pi$. Plot the poles and zeros of $H(z)$ using Matlab

❖ Plot the magnitude and phase response of the system

```
clear; clc;
w = (-1000:1:1000)*pi/1000;
b0 = 1141/1200;
w0 = pi/3;
r = 0.95;
w1 = w0 - w;
w2 = w0 + w;
H    =    b0*(1-exp(1j*w1)).*(1-exp(-1j*w2))./((1-r*exp(1j*w1)).*(1-r*exp(-
1j*w2)));
magnitudeH = abs(H);
phaseH = angle(H);
subplot(2,1,1); plot(w/pi,magnitudeH); grid on;
ylabel('|H(e^j^\omega)|'); title('Magnitude Response');
xlabel('\omega/\pi');
subplot(2,1,2); plot(w/pi,phaseH); grid on;
ylabel('\angleH(e^j^\omega)'); title('Phase Response');
xlabel('\omega/\pi');
```

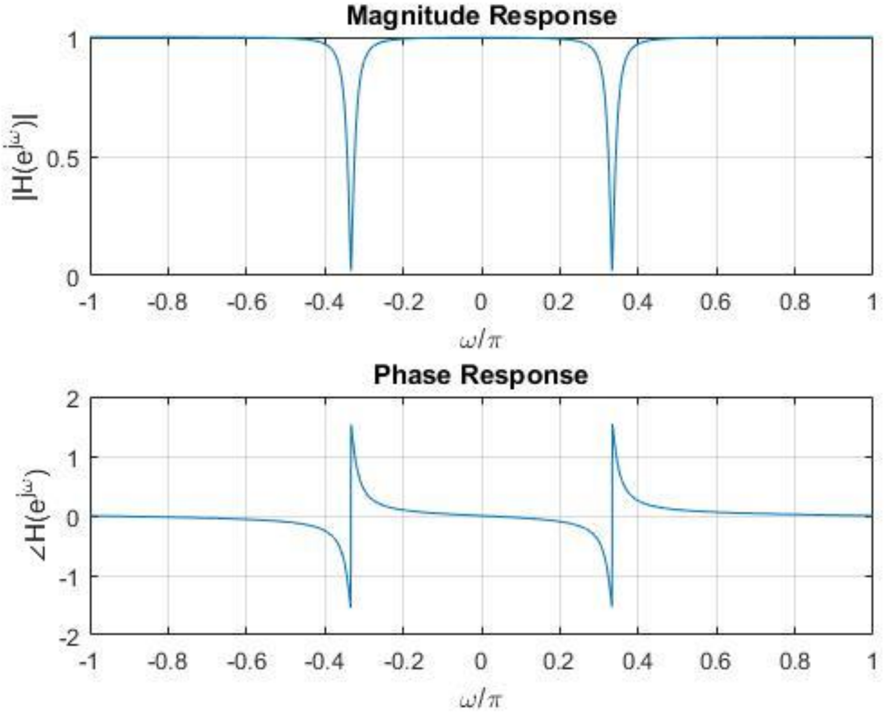

Figure 2.4. The magnitude and phase response of the system with $b_0 = \frac{1141}{1200}, \omega_0 = \frac{\pi}{3}$ and $r = 0.95$

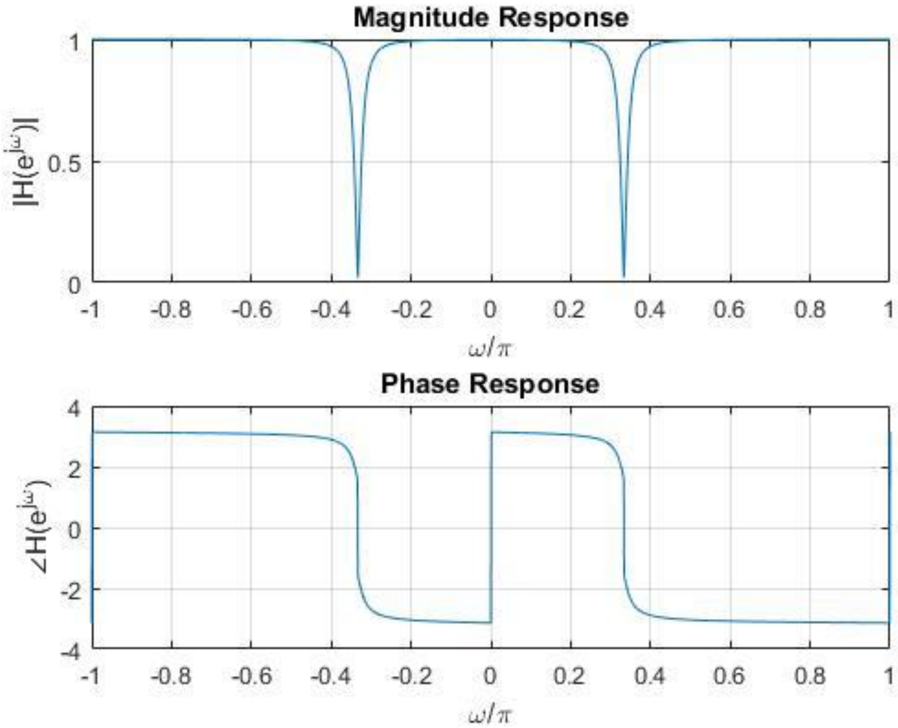

Figure 2.5. The magnitude and phase response of the system with $b_0 = -\frac{1141}{1200}, \omega_0 = \frac{\pi}{3}$ and $r = 0.95$

❖ Plot the poles and zeros of $H(z)$

```
clear; clc;

% Parameter
b0 = 1141/1200;
w0 = pi/3;
r = 0.95;

% Polynomial parameter
b = b0*[1 -2*cos(w0) 1];
a = [1 -2*r*cos(w0) r^2];

% Calculating pole
[R, p, C] = residuez(b, a);

% Plot zero-pole
[H1, H2, H3] = zplane(b, a);
set(H1, 'markersize', 10, 'color', 'r');
set(H2, 'markersize', 10, 'color', 'b');
title('Pole-Zero Plot');
text(0.85, -0.1, '1.0');
text(0.01, -0.1, '0');
legend('Zero', 'Pole');
grid on;
```

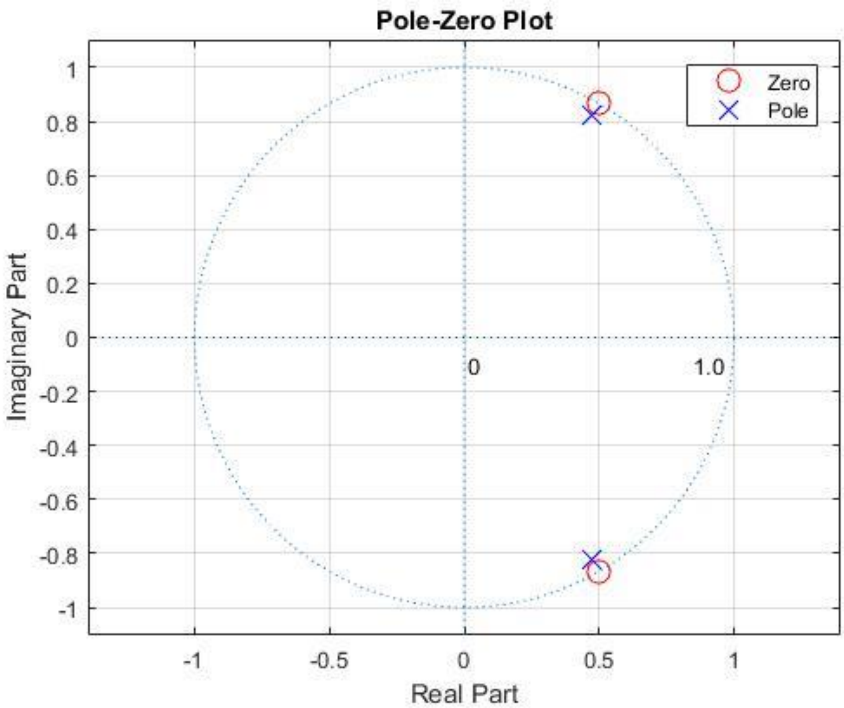

Figure 2.6. The poles and zeros of $H(z)$ for $b_0 = \pm\frac{1141}{1200}, \omega_0 = \frac{\pi}{3}$ and $r = 0.95$

3. Problem 3

Consider a causal linear time-invariant system with system function

$$H(z) = \frac{(1 - a^{-1}z^{-1})(1 - ja^{-1}z^{-1})(1 + ja^{-1}z^{-1})}{(1 - az^{-1})(1 - jaz^{-1})(1 + jaz^{-1})}$$

Where a is real

(a) Write the difference equation that relates the input and the output of this system

$$H(z) = \frac{(1-a^{-1}z^{-1})(1-ja^{-1}z^{-1})(1+ja^{-1}z^{-1})}{(1-az^{-1})(1-jaz^{-1})(1+jaz^{-1})} = \frac{(1-a^{-1}z^{-1})(1+a^{-2}z^{-2})}{(1-az^{-1})(1+a^2z^{-2})}$$

$$= \frac{1-a^{-1}z^{-1}+a^{-2}z^{-2}-a^{-3}z^{-3}}{1-az^{-1}+a^2z^{-2}-a^3z^{-3}} = \frac{Y(z)}{X(z)}$$

$$\Rightarrow (1-az^{-1}+a^2z^{-2}-a^3z^{-3})Y(z) = (1-a^{-1}z^{-1}+a^{-2}z^{-2}-a^{-3}z^{-3})X(z)$$

Inverse z-Transform for both sides of the above equation:

$$y(n) - ay(n-1) + a^2y(n-2) - a^3y(n-3) \\ = x(n) - a^{-1}x(n-1) + a^{-2}x(n-2) - a^{-3}x(n-3)$$

$$\Rightarrow y(n) = ay(n-1) - a^2y(n-2) + a^3y(n-3) + x(n) - a^{-1}x(n-1) \\ + a^{-2}x(n-2) - a^{-3}x(n-3)$$

(b) For what range of values of a is the system stable?

The system always has 3 pole-zero pairs locating in difference sides of the unit circle in z-plane: $\begin{bmatrix} z_1 = a^{-1} \\ p_1 = a \end{bmatrix}, \begin{bmatrix} z_2 = ja^{-1} \\ p_2 = ja \end{bmatrix}, \begin{bmatrix} z_3 = -ja^{-1} \\ p_3 = -ja \end{bmatrix}$

As long as $|a| < 1$, the system is stable.

(c) For $a = \frac{1}{2}$, plot the pole-zero diagram using Matlab and shade the region of convergence

$$H(z) = \frac{(1-2z^{-1})(1-2jz^{-1})(1+2jz^{-1})}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{j}{2}z^{-1}\right)\left(1+\frac{j}{2}z^{-1}\right)} = \frac{(1-2z^{-1})(1+4z^{-2})}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-2}\right)}$$

$z_1 = 2, z_2 = 2j, z_3 = -2j$ and $p_1 = \dfrac{1}{2}, p_2 = \dfrac{j}{2}, p_3 = -\dfrac{j}{2}$

```
clear; clc;

% Parameter
a = 0.5;
r1 = abs(a); r2 = abs(1/a);
r_pole = min(r1, r2);
axis_lim = max(r1, r2) + 0.5;

% Plot the poles
ReP = [a, 0, 0];
ImP = [0, a, -a];
plot(ReP, ImP, 'xr', 'MarkerSize', 7, 'DisplayName', 'Poles');
axis([-axis_lim, axis_lim, -axis_lim, axis_lim]);
axis equal; hold on; grid on;

% Plot the zeros
```

```
ReZ = [1/a, 0, 0];
ImZ = [0, 1/a, -1/a];
plot(ReZ, ImZ, 'ob', 'MarkerSize', 7, 'DisplayName', 'Zeros');
hold on;

% Plot the unit circle
[x_unit, y_unit] = circle(0, 0, 1);
plot(x_unit, y_unit, 'k--', 'LineWidth', 1, 'DisplayName', 'Unit circle');

% Plot the pole circle
[x_pole, y_pole] = circle(0, 0, r_pole);
plot(x_pole, y_pole, 'r', 'LineWidth', 1, 'DisplayName', '|z| = 0.5');

[x, y] = circle(0, 0, 2*axis_lim);
h = patch([x,x_pole], [y,y_pole], [0.5 0.5 0.5], 'LineStyle', 'none',
'FaceAlpha', 0.25, 'DisplayName', 'ROC');

legend('show');
xlabel('Re(z)'); ylabel('Im(z)');
title('Problem 3c');
hold off;

%% Inner function
function [xc, yc] = circle(x, y, r)
    theta = 0:pi/1000:2*pi;
    xc = x + r * cos(theta);
    yc = y + r * sin(theta);
end
```

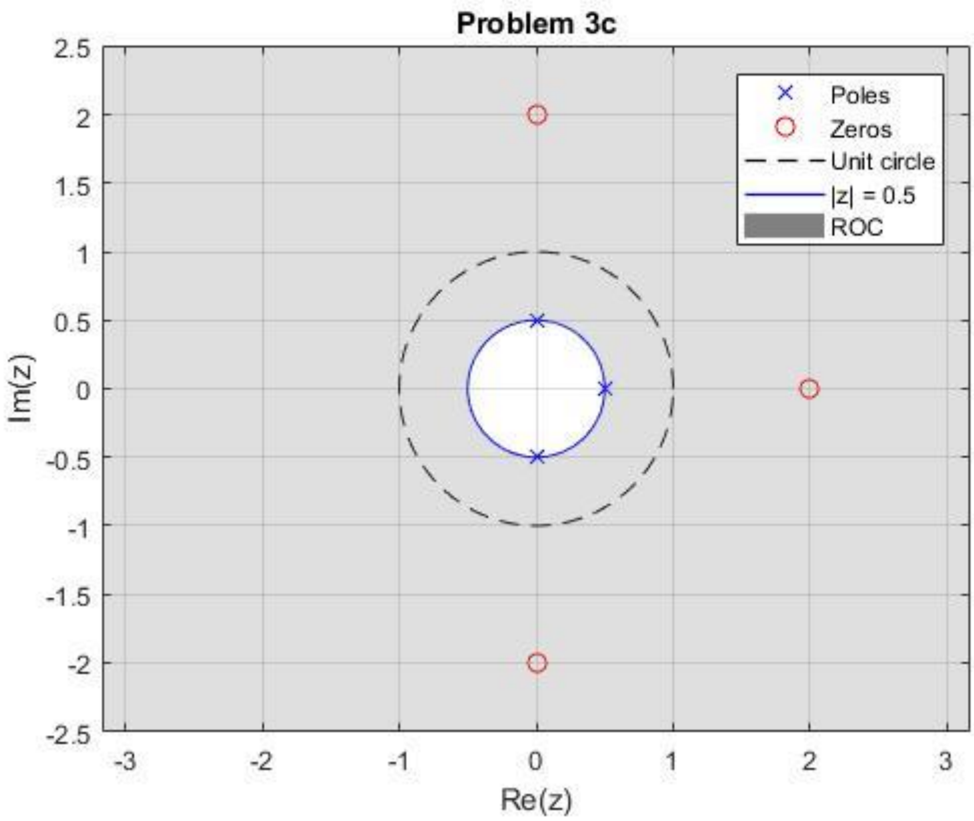

Figure 3.1. The pole-zero and ROC graph

(d) Find the impulse response $h(n)$ for the system

$p_1 = a$

$$p_2 = ja = \begin{cases} ae^{j\frac{\pi}{2}} & if\ a \geq 0 \\ -ae^{-j\frac{\pi}{2}} & if\ a < 0 \end{cases}$$

$p_2 = -ja = p_2^*$

$$H(z) = \frac{1 - a^{-1}z^{-1} + a^{-2}z^{-2} - a^{-3}z^{-3}}{1 - az^{-1} + a^2z^{-2} - a^3z^{-3}}$$

$$= a^{-6} + \frac{1 - a^{-6} + (a^{-5} - a^{-1})z^{-1} + (a^{-2} - a^{-4})z^{-2}}{(1 - az^{-1})(1 - jaz^{-1})(1 + jaz^{-1})}$$

Let $K(z) = \dfrac{1 - a^{-6} + \left(a^{-5} - a^{-1}\right)z^{-1} + \left(a^{-2} - a^{-4}\right)z^{-2}}{(1 - az^{-1})(1 - jaz^{-1})(1 + jaz^{-1})}$

$$\Rightarrow \frac{K(z)}{z} = \frac{(1 - a^{-6})z^2 + (a^{-5} - a^{-1})z + a^{-2} - a^{-4}}{(z - a)(z - ja)(z + ja)} = \frac{A_1}{z - a} + \frac{A_2}{z - ja} + \frac{A_3}{z + ja}$$

$$A_1 = (z - a)\frac{K(z)}{z}\bigg|_{z=a} = \frac{(a^2 - 1)(a^4 + 1)}{2a^6}$$

$$A_2 = (z - ja)\frac{K(z)}{z}\bigg|_{z=ja} = \frac{(1 - a^{-6})(ja)^2 + (a^{-5} - a^{-1})ja + a^{-2} - a^{-4}}{(ja - a)(ja + ja)}$$

$$= \frac{a^4 - 1}{4a^6}[1 + a^2 + j(1 - a^2)] = \frac{a^2 + 1}{4a^6}[a^4 - 1 - j(a^2 - 1)^2]$$

Because $p_3 = p_2^* \Rightarrow A_3 = A_2^*$

$$|A_2| = |A_3| = \left|\frac{a^2 + 1}{4a^6}\right|\sqrt{(a^4 - 1)^2 + (a^2 - 1)^4}$$
$$= \frac{a^2 + 1}{4a^6}|a^2 - 1|\sqrt{(a^2 + 1)^2 + (a^2 - 1)^2}$$

Let $\alpha = \angle A_2 = \text{atan2}\{-(a^2 - 1)^2, a^4 - 1\} \Rightarrow A_2 = |A_2|e^{j\alpha}$

$$K(z) = \frac{A_1}{1 - az^{-1}} + \frac{A_2}{1 - jaz^{-1}} + \frac{A_3}{1 + jaz^{-1}}$$

$$Z^{-1}\{K(z)\} = Z^{-1}\left\{\frac{A_1}{1 - az^{-1}}\right\} + Z^{-1}\left\{\frac{A_2}{1 - jaz^{-1}} + \frac{A_3}{1 + jaz^{-1}}\right\}$$

Because the system is causal, ROC: $|z| > |a| \Rightarrow Z^{-1}\left\{\dfrac{A_1}{1 - az^{-1}}\right\} = A_1 a^n u(n)$

Applying the below inverse z-Transform property:

$$Z^{-1}\left\{\frac{A_k}{1-p_kz^{-1}}+\frac{A_k^*}{1-p_k^*z^{-1}}\right\}=2|A_k|r_k^n\cos(\beta_kn+\alpha_k)\,u(n)$$

Where $A_k=|A_k|e^{j\alpha_k}$ and $p_k=r_ke^{j\beta_k}$

$$Z^{-1}\left\{\frac{A_2}{1-jaz^{-1}}+\frac{A_3}{1+jaz^{-1}}\right\}=Z^{-1}\left\{\frac{A_2}{1-p_2z^{-1}}+\frac{A_2^*}{1-p_2^*z^{-1}}\right\}$$

$$=\begin{cases}2|A_2|a^n\cos\left(\frac{\pi}{2}n+\alpha\right)u(n) & if\ a\geq 0\\ 2|A_2|(-a)^n\cos\left(-\frac{\pi}{2}n+\alpha\right)u(n) & if\ a<0\end{cases}$$

Finally, there are 2 cases:

$$h(n)=\begin{cases}\frac{1}{a^6}\delta(n)+A_1a^nu(n)+2|A_2|a^n\cos\left(\frac{\pi}{2}n+\alpha\right)u(n) & if\ a\geq 0\\ \frac{1}{a^6}\delta(n)+A_1a^nu(n)+2|A_2|(-a)^n\cos\left(-\frac{\pi}{2}n+\alpha\right)u(n) & if\ a<0\end{cases}$$

Where:

$$A_1=\frac{(a^2-1)(a^4+1)}{2a^6}$$

$$|A_2|=\frac{a^2+1}{4a^6}|a^2-1|\sqrt{(a^2+1)^2+(a^2-1)^2}$$

$$\alpha=\operatorname{atan2}\{-(a^2-1)^2,a^4-1\}$$

(e) Show that the system is an all-pass system, i.e., that the magnitude of the frequency response is a constant. Also, specify the value of the constant.

$$H(z)=\frac{(1-a^{-1}z^{-1})(1-ja^{-1}z^{-1})(1+ja^{-1}z^{-1})}{(1-az^{-1})(1-jaz^{-1})(1+jaz^{-1})}$$

$$=-a^{-3}z^{-3}\frac{(1-az)(1-jaz)(1+jaz)}{(1-az^{-1})(1-jaz^{-1})(1+jaz^{-1})}=-a^{-3}z^{-3}\frac{(1-az)(1+a^2z^2)}{(1-az^{-1})(1+a^2z^{-2})}$$

$$H(e^{j\omega})=-a^{-3}e^{-j3\omega}\frac{(1-ae^{j\omega})(1+a^2e^{j2\omega})}{(1-ae^{-j\omega})(1+a^2e^{-j2\omega})}$$

$$\left|H(e^{j\omega})\right|=|-a^{-3}|\times\left|e^{-j3\omega}\right|\times\left|\frac{1-ae^{j\omega}}{1-ae^{-j\omega}}\right|\times\left|\frac{1+a^2e^{j2\omega}}{1+a^2e^{-j2\omega}}\right|$$

$$|-a^{-3}|=\frac{1}{|a^3|}$$

$$\left|e^{-j3\omega}\right|=\sqrt{[\cos(-3\omega)]^2+[\sin(-3\omega)]^2}=1$$

$$\left|\frac{1-ae^{j\omega}}{1-ae^{-j\omega}}\right| = \frac{\sqrt{(1-a\cos\omega)^2+(a\sin\omega)^2}}{\sqrt{(1-a\cos\omega)^2+(a\sin\omega)^2}} = 1$$

$$\left|\frac{1+a^2e^{j2\omega}}{1+a^2e^{-j2\omega}}\right| = \frac{\sqrt{[1+a^2\cos(2\omega)]^2+[a^2\sin(2\omega)]^2}}{\sqrt{[1+a^2\cos(2\omega)]^2+[a^2\sin(2\omega)]^2}} = 1$$

Hence, $\left|H(e^{j\omega})\right| = \dfrac{1}{|a^3|}$

And the system is an all-pass system.

(f) Plot the magnitude and phase responses of $H(e^{j\omega})$ with $a=\frac{1}{2}$ for $-\pi \le \omega \le \pi$ using Matlab

$$H(z) = \frac{(1-a^{-1}z^{-1})(1+a^{-2}z^{-2})}{(1-az^{-1})(1+a^2z^{-2})} \Rightarrow H(e^{j\omega}) = \frac{(1-a^{-1}e^{-j\omega})(1+a^{-2}e^{-j2\omega})}{(1-ae^{-j\omega})(1+a^2e^{-j2\omega})}$$

$$H(e^{j\omega})\Big|_{a=\frac{1}{2}} = \frac{(1-2e^{-j\omega})(1+4e^{-j2\omega})}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{4}e^{-j2\omega}\right)}$$

```
clear; clc;
w = (-1000:1:1000)*pi/1000;
a = 0.5;
H = (1-1/a*exp(-1j*w)).*(1+1/a^2*exp(-2j*w))./((1-a*exp(-
1j*w)).*(1+a^2*exp(-2j*w)));
magnitudeH = abs(H);
phaseH = angle(H);
subplot(2,1,1); plot(w/pi,magnitudeH); grid on;
axis([-1 1 7 9]);
ylabel('|H(e^j^\omega)|'); title('Magnitude Response');
xlabel('\omega/\pi');
subplot(2,1,2); plot(w/pi,phaseH); grid on;
ylabel('\angleH(e^j^\omega)'); title('Phase Response');
xlabel('\omega/\pi');
```

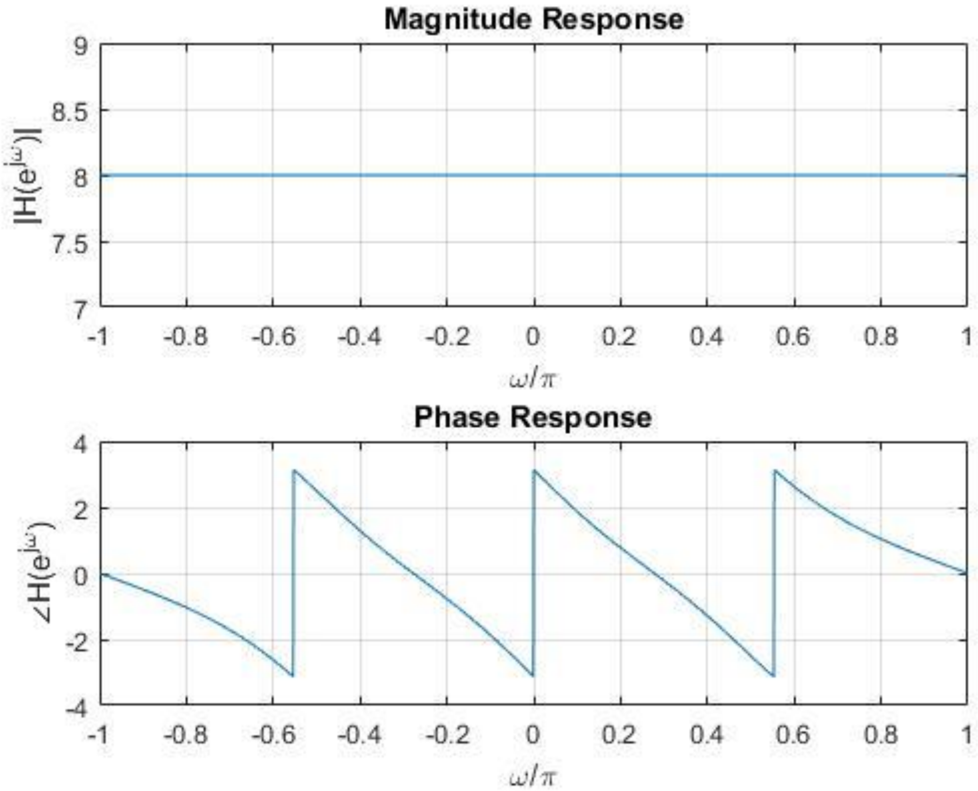

Figure 3.2. The magnitude and phase response of the system where $a = \frac{1}{2}$

4. Problem 4

Consider a stable LTI system given by

$$H(z) = \frac{(1 - 9z^{-2})\left(1 + \frac{1}{3}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}}$$

(a) $H(z)$ can be expressed as a cascade of a minimum-phase system $H_{min}(z)$ and a unity-gain all-pass system $H_{ap}(z)$. Determine a choice for $H_{min}(z)$ and $H_{ap}(z)$, and specify whether or not they are unique up to a scale factor

$$H(z) = \frac{(1 - 9z^{-2})\left(1 + \frac{1}{3}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}} = \frac{(1 - 3z^{-1})(1 + 3z^{-1})\left(1 + \frac{1}{3}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}}$$

$$= \frac{3\left(\frac{1}{3} - z^{-1}\right)(1 + 3z^{-1})\left(1 + \frac{1}{3}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}} = -3(1 + 3z^{-1})\left(1 + \frac{1}{3}z^{-1}\right)\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$$

$$= -9\left(1+\frac{1}{3}z^{-1}\right)^2 \frac{\left(z^{-1}+\frac{1}{3}\right)\left(z^{-1}-\frac{1}{3}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$H_1(z) = H_{min}(z) = -9\left(1+\frac{1}{3}z^{-1}\right)^2$$

$H_1(z)$ is a minimum-phase system because its poles do not exist but its zeros are inside of the unit circle: $z_1 = z_2 = -\frac{1}{3}$

$$H_2(z) = H_{ap}(z) = \frac{\left(z^{-1}+\frac{1}{3}\right)\left(z^{-1}-\frac{1}{3}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$H_2(z)$ is a all-pass system because there are 2 pole-zero pairs, the poles are inside but the zeros are outside of the unit circle.

(b) Is the minimum-phase system $H_{min}(z)$ a FIR system? Explain

$$H_{min}(z) = -9\left(1+\frac{1}{3}z^{-1}\right)^2 = -(3+z^{-1})^2 = -9-6z^{-1}-z^{-2}$$

$$\Rightarrow h_{min}(n) = -9\delta(n) - 6\delta(n-1) - \delta(n-2)$$

Obviously, $h_{min}(n)$ is a FIR.

(c) Is the minimum-phase system $H_{min}(z)$ a generalized linear-phase system? If not, can $H(z)$ be represented as a cascade of a generalized linear-phase system $H_{lin}(z)$ and an all-pass system $H_{ap2}(z)$? If your answer is yes, determine $H_{lin}(z)$ and $H_{ap2}(z)$. If your answer is no, explain why such representation does not exist.

$$H_{min}(z) = -9\left(1+\frac{1}{3}z^{-1}\right)^2$$

$$\begin{aligned}\Rightarrow H_{min}(e^{j\omega}) &= -9\left(1+\frac{1}{3}e^{-j\omega}\right)^2 = -(3+e^{-j\omega})^2 = -9-6e^{-j\omega}-e^{-j2\omega}\\ &= -9-6(\cos\omega - j\sin\omega) - (\cos 2\omega - j\sin 2\omega)\\ &= -9-6\cos\omega-\cos 2\omega + j(6\sin\omega+\sin 2\omega)\end{aligned}$$

$\Rightarrow \angle H_{min}(e^{j\omega}) = \text{atan2}(6\sin\omega+\sin 2\omega\,, -9-6\cos\omega-\cos 2\omega)$ is not linear

Therefore, the minimum-phase system $H_{min}(z)$ is not a generalized linear-phase system.

$H(z)$ can be represented as a cascade of a generalized linear-phase system and an all-pass system as below:

$$H(z) = \frac{(1-3z^{-1})(1+3z^{-1})\left(1+\frac{1}{3}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}$$

$$= \left[(1+3z^{-1})\left(1+\frac{1}{3}z^{-1}\right)\right]\left(\frac{1-3z^{-1}}{1-\frac{1}{3}z^{-1}}\right)$$

$$H_{lin}(z) = (1+3z^{-1})\left(1+\frac{1}{3}z^{-1}\right)$$

$$H_{ap2}(z) = \frac{1-3z^{-1}}{1-\frac{1}{3}z^{-1}}$$

5. Problem 5

Consider the LTI system whose system function is given by

$$H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

(a) Determine all causal system functions that results in the same frequency-response magnitude as $H(z)$ and for which the impulse responses are real value and of the same length as the impulse response associated with $H(z)$. There are four different such systems. Identify which system function is minimum phase and which to within a time shift, is maximum phase.

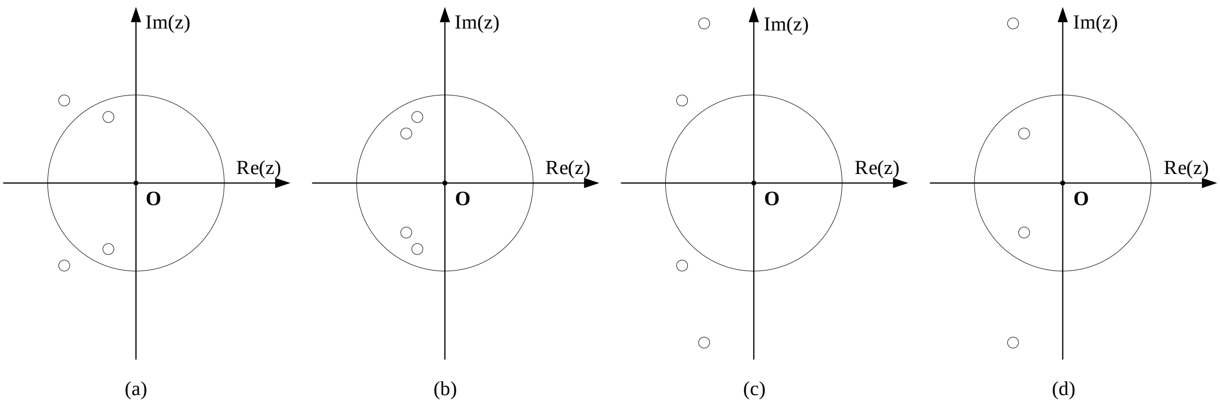

Figure 5.1. The zeros of 4 system (a) $H(z)$, (b) $H_1(z)$, (c) $H_2(z)$ and (d) $H_3(z)$

❖ System 1:

$$H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

$$= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1.25e^{j0.8\pi})(0.8e^{-j0.8\pi} - z^{-1})(1.25e^{-j0.8\pi})(0.8e^{j0.8\pi} - z^{-1})$$

$$= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1.25)^2(z^{-1}-0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})$$

$$= (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})(1 - 0.8e^{j0.8\pi}z^{-1})\frac{(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})}{(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})}$$

$$\Rightarrow H_1(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})(1 - 0.8e^{j0.8\pi}z^{-1})$$

❖ System 2:

$$H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

$$= 0.9e^{j0.6\pi}\left(\frac{10}{9}e^{-j0.6\pi}-z^{-1}\right)0.9e^{-j0.6\pi}\left(\frac{10}{9}e^{j0.6\pi}-z^{-1}\right)(1-1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

$$= (0.9)^2\left(z^{-1}-\frac{10}{9}e^{-j0.6\pi}\right)\left(z^{-1}-\frac{10}{9}e^{j0.6\pi}\right)(1-1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

$$= (0.9)^2(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1 - \frac{10}{9}e^{j0.6\pi}z^{-1}\right)\frac{\left(z^{-1}-\frac{10}{9}e^{-j0.6\pi}\right)\left(z^{-1}-\frac{10}{9}e^{j0.6\pi}\right)}{\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{10}{9}e^{j0.6\pi}z^{-1}\right)}$$

$$\Rightarrow H_2(z) = (0.9)^2(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1 - \frac{10}{9}e^{j0.6\pi}z^{-1}\right)$$

❖ System 3:

$$H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

$$= 0.9e^{j0.6\pi}\left(\frac{10}{9}e^{-j0.6\pi}-z^{-1}\right)0.9e^{-j0.6\pi}\left(\frac{10}{9}e^{j0.6\pi}-z^{-1}\right)(1.25e^{j0.8\pi})(0.8e^{-j0.8\pi} - z^{-1})(1.25e^{-j0.8\pi})(0.8e^{j0.8\pi}-z^{-1})$$

$$= (0.9)^2(1.25)^2\left(z^{-1}-\frac{10}{9}e^{-j0.6\pi}\right)\left(z^{-1}-\frac{10}{9}e^{j0.6\pi}\right)(z^{-1}-0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})$$

$$= (0.9)^2(1.25)^2\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{10}{9}e^{j0.6\pi}z^{-1}\right)(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})\frac{\left(z^{-1}-\frac{10}{9}e^{-j0.6\pi}\right)\left(z^{-1}-\frac{10}{9}e^{j0.6\pi}\right)(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})}{\left(1-\frac{10}{9}e^{-j0.6\pi}\right)\left(1-\frac{10}{9}e^{j0.6\pi}\right)(1-0.8e^{-j0.8\pi})(1-0.8e^{j0.8\pi})}$$

$$\Rightarrow H_3(z) = (0.9)^2(1.25)^2\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{10}{9}e^{j0.6\pi}z^{-1}\right)(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})$$

$H_1(z)$ is a minimum-phase system because all its zeros are inside the unit circle.

$H_2(z)$ is a maximum-phase system because all its zeros are outside the unit circle.

(b) Determine the impulse response for the system function in part (a)

$$H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$$

$$= 1 + 2.5788z^{-1} + 3.4975z^{-2} + 2.5074z^{-3} + 1.2656z^{-4}$$

$$\Rightarrow h(n) = \delta(n) + 2.5788\delta(n-1) + 3.4975\delta(n-2) + 2.5074\delta(n-3) + 1.2656\delta(n-4)$$

$$H_1(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})$$

$$= 1.5625 + 2.8917z^{-1} + 3.3906z^{-2} + 2.1945z^{-3} + 0.81z^{-4}$$

$$\Rightarrow h_1(n) = 1.5625\delta(n) + 2.8917\delta(n-1) + 3.3906\delta(n-2) + 2.1945\delta(n-3) + 0.81\delta(n-4)$$

$$H_2(z) = (0.9)^2(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{10}{9}e^{j0.6\pi}z^{-1}\right)$$

$$= 0.81 + 2.1945z^{-1} + 3.3906z^{-2} + 2.8917z^{-3} + 1.5625z^{-4}$$

$$\Rightarrow h_2(n) = 0.81\delta(n) + 2.1945\delta(n-1) + 3.3906\delta(n-2) + 2.8917\delta(n-3) + 1.5625\delta(n-4)$$

$$H_3(z) = (0.9)^2(1.25)^2\left(1-\frac{10}{9}e^{-j0.6\pi}z^{-1}\right)\left(1-\frac{10}{9}e^{j0.6\pi}z^{-1}\right)(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})$$

$$= 1.2656 + 2.5074z^{-1} + 3.4975z^{-2} + 2.5788z^{-3} + z^{-4}$$

$$\Rightarrow h_3(n) = 1.2656\delta(n) + 2.5074\delta(n-1) + 3.4975\delta(n-2) + 2.5788\delta(n-3) + \delta(n-4)$$

(c) For each of the sequences in part (b), compute and plot the quantity $E(n) = \sum_{m=0}^{n} h^2(m)$ for $0 \leq n \leq 5$. Indicate explicitly which plot corresponds to the minimum-phase system.

Table 5.1. The energy of each sequence

n	$E(n)$	$E_1(n)$	$E_2(n)$	$E_3(n)$
0	1	2.44	0.66	1.6
1	7.65	10.8	5.47	7.89
2	19.88	22.3	16.97	20.12
3	26.17	27.11	25.33	26.77
4	27.77	27.77	27.77	27.77
5	27.77	27.77	27.77	27.77

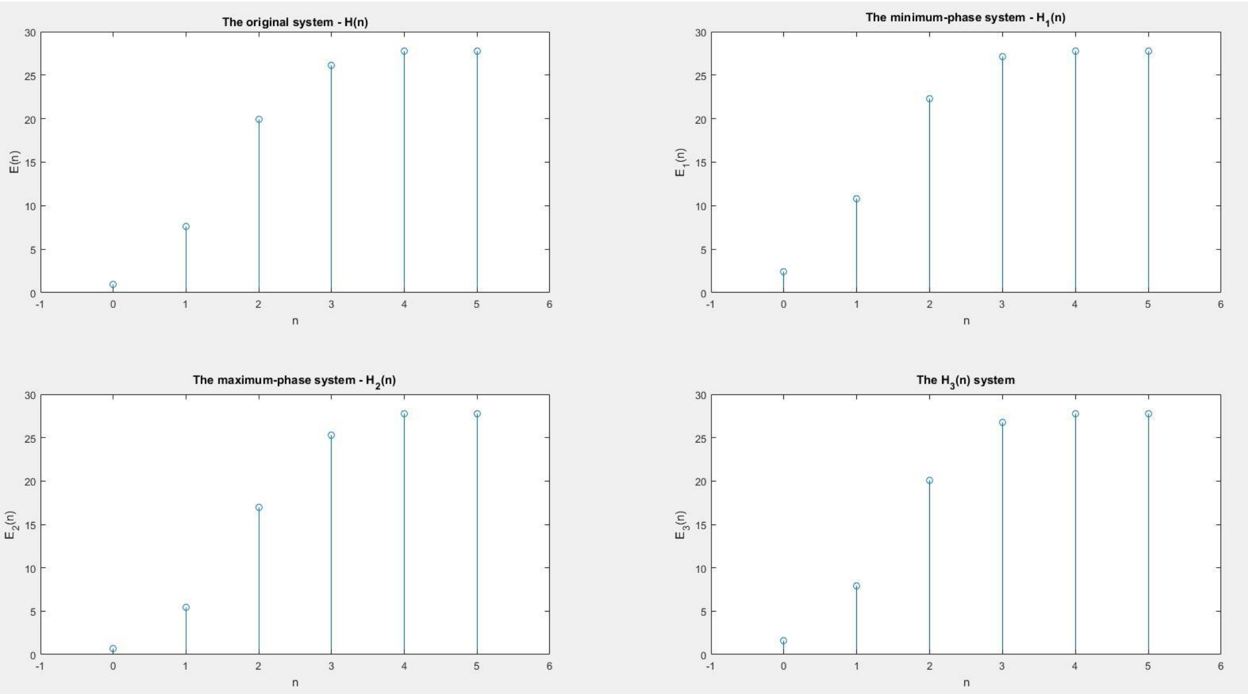

Figure 5.2. The quantity of energy in time domain

The plot of $E_1(n)$ corresponds to the minimum-phase system.