

The risk-free rate in heterogeneous-agent incomplete-insurance economies

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Why has the average real risk-free interest rate been less than one percent? The question is motivated by the failure of a class of calibrated representative-agent economies to explain the average return to equity and risk-free debt. I construct an economy where agents experience uninsurable idiosyncratic endowment shocks and smooth consumption by holding a risk-free asset. I calibrate the economy and characterize equilibria computationally. With a borrowing constraint of one year's income, the resulting risk-free rate is more than one percent below the rate in the comparable representative-agent economy.

1. Introduction

Why has the average real risk-free interest rate been less than one percent? The question is motivated by the work of Mehra and Prescott (1985). They argue that a class of calibrated representative-agent economies does not match the average real return to equity (7%) and risk-free debt (0.8%).¹ The models in this class predict a risk-free rate that is too large and an equity premium that is too small. Subsequent attempts to explain the rate of return observations within the representative-agent structure have been largely unsuccessful [see Weil (1989) for a review]. Mehra and Prescott suggest that we will have to look outside the class of Arrow–Debreu economies for an explanation of the rate of return observations. They also suggest focusing attention on explaining why the risk-free rate has been so low.

I investigate the conjecture that market imperfections are important for determining the risk-free rate. More specifically, I investigate the importance of

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¹ All the data is for the period 1889–1978 and is described in Mehra and Prescott (1985).

idiosyncratic shocks and incomplete insurance. This is operationalized by considering a pure exchange economy where agents experience idiosyncratic endowment shocks and smooth consumption by holding a single asset. Each agent holds a credit balance with a central credit authority. The credit balance must always remain above a fixed credit limit. An agent accumulates credit balances in good times and runs down credit balances in bad times to smooth consumption. The decision problem that an agent faces is similar to the problems studied by Schechtman and Escudero (1977), Mendelson and Amihud (1982), Sotomayor (1984), Clarida (1984), and many others. This paper differs from these papers by examining the general equilibrium implications of economies with decision problems of this type. At this stage a relatively simple explanation is given for why this structure generates a low risk-free rate. With a credit limit, agents are restricted in the level of their indebtedness. However, agents are not restricted from accumulating large credit balances. A low risk-free rate is needed to persuade agents not to accumulate large credit balances so that the credit market can clear. A more rigorous explanation for the result will be provided in section 5. To examine the risk-free rate generated by this structure, I calibrate the economy and characterize equilibria using computational methods.

There has been a considerable amount of work on heterogeneous-agent incomplete-insurance models of asset pricing. In monetary economics, work by Bewley (1980, 1983), Lucas (1980), and Taub (1988) employ a similar structure to that used here. In other areas of economics similar structures have also been used. Imrohoroglu (1989) measures the potential welfare gains from eliminating aggregate fluctuations. Manuelli (1986) and Clarida (1990) study international debt markets. Diaz and Prescott (1989) study movements in the return to money and Treasury bills in response to monetary and fiscal policies. Taub (1991) analyzes the efficiency properties of money and credit in an environment with taste shocks. Aiyagari and Gertler (1991) study the effect of transaction costs on asset returns. Heterogeneous-agent models with different structures that address the equity and debt observations include Mankiw (1986), Kahn (1988), Ketterer and Marcet (1989), Marcet and Singleton (1991), Lucas (1991), and Telmer (1991).

The paper is organized in six sections. Section 2 describes the environment and arrangement in detail. Section 3 describes the equilibrium concept and some theorems that will be used to compute equilibria. Section 4 describes model calibration and computation. Section 5 discusses the results. Section 6 concludes.

2. Environment and arrangement

Consider an exchange economy with a continuum of agents of total mass equal to one. Each period each agent receives an endowment of the one perishable consumption good in the economy. The endowment can either be high (e_h) or low (e_l). The set of possible endowment values is denoted E ,

$E = \{e_h, e_l\}$. Each agent's endowment follows a Markov process with stationary transition probability $\pi(e' | e) = \text{Prob}(e_{t+1} = e' | e_t = e) > 0$ for $e, e' \in E$ that is independent of all other agents' current and past endowments. Each agent has preferences defined over stochastic processes for consumption given by a utility function,

$$E \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad \text{where } \beta \in (0, 1), \quad (1)$$

$$u(c) = \frac{c^{(1-\sigma)}}{(1-\sigma)} \quad \text{where } \sigma > 1. \quad (2)$$

The arrangement investigated here allows each agent to smooth consumption by holding a single asset. The asset can be interpreted as a credit balance with a central credit authority or as a one-period-ahead sure claim on consumption goods. I will use the credit balance interpretation. A credit balance of a units entitles an agent to a goods this period. To obtain a credit balance of a' units next period, an agent must pay $a'q$ goods this period, where q is the price of next-period credit balances. Credit balances must always remain above a credit limit \underline{a} , $\underline{a} < 0$. A period budget constraint for an agent who chooses consumption c and next-period credit balances a' , given credit balance a and endowment e , is

$$c + a'q \leq a + e \quad \text{where } c \geq 0 \quad \text{and} \quad a' \geq \underline{a}.$$

An agent's decision problem will be described at a more technical level after setting down some notation:

* An agent's position at a point in time is described by an individual state vector $x \in X$. $x = (a, e)$ indicates an agent's credit balance a and endowment e . The individual state space is $X = A \times E$, where $A = [\underline{a}, \infty)$, $E = \{e_h, e_l\}$, and $e_h > e_l$.

* Let $q > 0$ be the constant price of credit balances each period.

A functional equation that describes an agent's decision problem is then:

$$v(x; q) = \max_{(c, a') \in \Gamma(x; q)} u(c) + \beta \sum_{e'} v(a', e'; q) \pi(e' | e), \quad (3)$$

where

$$\Gamma(x; q) = \{(c, a'): c + a'q \leq a + e; c \geq 0; a' \geq \underline{a}\}.$$

If a bounded measurable solution v to functional equation (3) exists, then v is the optimal value function [see theorem 9.2 in Stokey and Lucas (1989)]. If v is the optimal value function, then functions $c: X \times R_{++} \rightarrow R_+$ and $a: X \times R_{++} \rightarrow A$ are optimal decision rules provided $c(x; q)$ and $a(x; q)$ are

measurable, feasible, and satisfy

$$v(x; q) = u(c(x; q)) + \beta \sum_{e'} v(a(x; q), e'; q) \pi(e' | e). \quad (4)$$

3. Equilibrium

This section describes the equilibrium concept and some theorems that are used to compute equilibria. First, some background for the equilibrium concept is provided. Since agents will be heterogeneous in their individual state vectors, some way of describing the heterogeneity in the economy at a point in time is needed. A probability measure defined on subsets of the individual state space is a natural way of describing this heterogeneity. So let ψ be a probability measure on (S, β_S) , where $S = [\underline{a}, \bar{a}] \times E$ and β_S is the Borel σ -algebra. Thus, for $B \in \beta_S$, $\psi(B)$ indicates the mass of agents whose individual state vectors lie in B .

In general the price of credit balances, q , will depend on the aggregate state of the economy which is given by ψ . So, as ψ changes over time, the price of credit would be expected to change. For many questions the dynamics caused by changing distributions of individual state vectors are of interest. However, for the question at hand it is best to define a more specialized notion of equilibrium where the probability measure ψ and the price of credit q remain unchanged over time. Since the question at hand concerns the average interest rate, not the dynamic properties of interest rates, this is a useful simplification. An important technical reason for concentrating on stationary equilibria is that general methods for characterizing equilibria of the more general kind do not currently exist. Therefore, this paper adopts the stationary recursive equilibrium structure described in Lucas (1980). To define what it means for a probability measure ψ to be stationary or unchanged over time, a transition function P , $P: S \times \beta_S \rightarrow [0, 1]$, is needed. Intuitively, $P(x, B)$ is the probability that an agent with state x will have an individual state vector lying in B next period. The appendix shows how to construct a transition function from a decision rule $a(x)$ and transition probabilities $\pi(e' | e)$. Equipped with a well-defined transition function P , a probability measure ψ defined on (S, β_S) is stationary provided:

$$\psi(B) = \int_S P(x, B) d\psi \quad \text{for all } B \in \beta_S.$$

Definition. A stationary equilibrium for this economy is $(c(x), a(x), q, \psi)$ satisfying:

- (1) $c(x)$ and $a(x)$ are optimal decision rules, given q .
- (2) Markets clear: (i) $\int_S c(x) d\psi = \int_S e d\psi$, (ii) $\int_S a(x) d\psi = 0$.
- (3) ψ is a stationary probability measure: $\psi(B) = \int_S P(x, B) d\psi$ for all $B \in \beta_S$.

A discussion of the equilibrium concept is in order. The first condition says that agents optimize. The second condition says that consumption and endowment averaged over the population are equal and that credit balances averaged over the population are zero. The third condition says that the distribution of agents over states is unchanging. Note that the measure ψ is defined over subsets of S instead of X . Thus the definition of stationary equilibria requires that individual agents never accumulate credit balances beyond some endogenously determined level \bar{a} . Conditions under which agents optimally decide to do this are given in Theorem 2 below. Fig. 1 shows what is going on. This figure graphs the decision rules $a(a, e_h)$ and $a(a, e_l)$ on a 45° line diagram, where the graph of $a(a, e_h)$ always lies above the graph of $a(a, e_l)$. The credit level a at which $a(a, e_h)$ crosses the 45° line is such an endogenously determined level \bar{a} .

The following theorems will be used to compute equilibria. Theorem 1 states conditions under which for given q there exists a unique solution to (3), provides a method for computing optimal decision rules, and states properties of decision rules. To state Theorem 1, I define a mapping T on $C(X)$, the space of bounded continuous real-valued functions on X , as indicated below:

$$(Tv)(x; q) = \max_{(c, a') \in \Gamma(x; q)} u(c) + \beta \sum_{e'} v(a', e'; q) \pi(e' | e). \quad (5)$$

Use the mapping T to define mappings T^n , where $T^1 v = Tv$, $T^2 v = T(Tv)$, and so on.

Theorem 1. For $q > 0$ and $\underline{a} + e_l - \underline{a}q > 0$, there exists a unique solution $v(x; q) \in C(X)$ to (3) and $T^n v_0$ converges uniformly to v as $n \rightarrow \infty$ from any

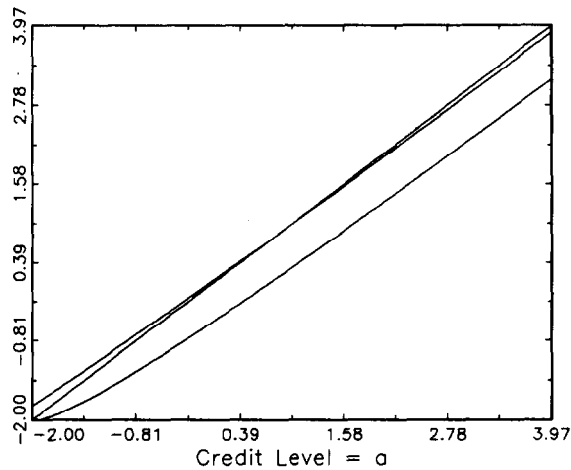


Fig. 1. Optimal decision rule.

$v_0 \in C(X)$. $v(x; q)$ is strictly increasing, strictly concave, and continuously differentiable in a . Furthermore, there exist continuous optimal decision rules $c(x; q)$ and $a(x; q)$. $a(x; q)$ is nondecreasing in a and strictly increasing in a for $(x; q)$ such that $a(x; q) > \underline{a}$.

Theorem 1 is proved in Huggett (1991), and is for the most part a standard result in dynamic programming theory. Theorem 1 requires that if an agent starts out with the smallest credit balance and receives the smallest endowment shock, then the agent can maintain the smallest credit balance and have strictly positive consumption. The one nonstandard step in the proof is to show that if v is a bounded function, then Tv is a bounded function. Because the period utility function is not bounded below, this requires some argument. This is handled by showing that there is some strictly positive level (depending on v) that consumption will never be set below.

Theorem 2 states conditions under which for given q there exists a unique stationary probability measure ψ on (S, β_S) and gives a method for computing excess demand in the credit market. Some additional notation is needed for the statement of the theorem. A transition function P induces a mapping $W: M(S) \rightarrow M(S)$, where $M(S)$ is the space of probability measures on (S, β_S) , defined by

$$(W\psi)(B) = \int_S P(s, B) d\psi \quad \text{for } B \in \beta_S.$$

Use the mapping W to define mappings W^n , where $W^1\psi = W\psi$, $W^2\psi = W(W\psi)$, and so on.

Theorem 2. *If the conditions of Theorem 1 hold, $\beta < q$ and $\pi(e_h | e_h) \geq \pi(e_h | e_l)$, then there exists a unique stationary probability measure ψ (given q) on (S, β_S) and, for any $\psi_0 \in M(S)$, $W^n\psi_0$ converges weakly to ψ as $n \rightarrow \infty$.*

Proof. See the appendix.

Theorem 2 is important for several reasons. First, it states conditions under which for given price q there exists a stationary probability measure $\psi(q)$. Since the definition of equilibrium requires stationary probability measures, this is important. Second, it states that the stationary probability measure is unique. Third, it offers a method to check whether $(a(x; q), c(x; q), q, \psi(q))$ is an equilibrium. Simply pick any initial guess of a probability measure in $M(S)$ and then generate the sequence of probability measures $\psi_n = W^n\psi_0$ described in the theorem. Calculate the sequence of integrals defined below:

$$\int_S a(x; q) d\psi_n.$$

Because ψ_n converges weakly to the unique stationary probability measure $\psi(q)$ and because $a(x; q)$ is bounded and continuous on S , the sequence of integrals converges to

$$\int_S a(x; q) d\psi(q).$$

Since a version of Walras' Law holds in this economy, an easy way to find an equilibrium is to search for prices q that are approximately market-clearing in the credit market.

Theorem 2 is proved by applying theorem 2 in Hopenhayn and Prescott (1987). The proof of Theorem 2 requires two steps. First, I prove that there is a set $S = [\underline{a}, \bar{a}] \times E$ that has the property that if an agent starts out in S , then the agent stays in S . This is accomplished by showing that the decision rule for credit balances has the shape shown in fig. 1. More specifically, it is shown that $a(a, e_l) < a$ for $a > \underline{a}$ and that $a(a, e_h)$ has a fixed point as a function of a . These two facts, together with the fact that $a(a, e)$ is nondecreasing in its first argument, yield the desired result. Schechtman and Escudero (1977) prove a similar result for the case of independent and identically distributed shocks. They assume that the period utility function has an asymptotically bounded coefficient of relative risk aversion and that the interest rate is below the time preference rate (alternatively $\beta < q$). An additional assumption on the transition probabilities is used to prove the result for the Markov shock case considered here. The second step in the proof is to show that the conditions of theorem 2 in Hopenhayn and Prescott (1987) are satisfied.

4. Calibration and computation

I calibrate the economy following the procedures described in Lucas (1981). This involves using microeconomic and macroeconomic observations to set values of the parameters $\{e_h, e_l; \pi(e_h | e_h), \pi(e_h | e_l); \beta; \sigma; \underline{a}\}$ and the period length. I follow Imrohoroglu (1989) in interpreting e_h and e_l as earnings when employed and not employed. With this interpretation, I calibrate the endowment process to roughly match measures of the variability of labor earnings and the time duration in a nonemployed state. As a measure of the variability of labor earnings, consider the data reported in Kydland (1984).² He calculates the standard deviation of annual hours worked for individual prime-age males from 1970–1980. He finds that the standard deviation as a percentage of mean hours

² An earlier version of this paper used a different calibration. The calibration described here uses evidence on hours variability cited in Aiyagari and Gertler (1991). The results obtained with the current and previous calibration are similar.

varies from 16% for the group with the highest education level to 32% for the group with the lowest education level. As a measure of duration, the average duration of unemployment spells for men from 1948–1988 is calculated from data in the Handbook of Labor Statistics. The average duration is 12.3 weeks.

When $e_h = 1.0$, $e_l = 0.1$, $\pi(e_h | e_h) = 0.925$, $\pi(e_h | e_l) = 0.5$, and there are six model periods in one year, the standard deviation of annual earnings as a percentage of mean for an agent is 20% and the average duration of the low endowment shock is two model periods or 17 weeks. The duration of the low endowment shock is higher than the measure cited above. However, Clark and Summers (1979) calculate that in 1974 26% of unemployment spells for men of age 20 and over ended in withdrawal from the labor force. They argue that unemployment duration understates the average time to reemployment.

The discount factor β is set to 0.99322. As there are six model periods in one year the discount factor on an annual basis is 0.96. The microeconomic studies reviewed by Mehra and Prescott (1985) estimate the risk aversion coefficient, σ , to be about 1.5. A range of values for the credit limit are selected, $\underline{a} \in \{-2, -4, -6, -8\}$, to examine the sensitivity of the results to different credit limits. A credit limit of -5.3 is equal to one year's average endowment.

The procedures used to compute equilibria to the calibrated model economies are described next. The computation method consists of three steps:

- (1) Given price q , compute $a(x; q)$ using Theorem 1.
- (2) Given $a(x; q)$, iterate on $\psi_{n+1}(B) = \int_S P(x, B) d\psi_n$ from arbitrary $\psi_0 \in M(S)$. When the sequence of probability measures has approximately converged, use the resulting probability measure in place of ψ to compute $\int_S a(x; q) d\psi$.
- (3) Update q and repeat steps 1 and 2 until market clearing is approximately obtained.

These steps are now discussed in more detail. Step 1 is to iterate on (5) from an arbitrary, bounded, concave, differentiable function v_0 . Each iteration involves solving a concave programming problem. First-order conditions to the concave programming problem implicit in Tv_0 reduce to

$$u'(a + e - a'q)q \geq \beta \sum_{e'} v'_0(a', e') \pi(e' | e), \quad \text{with equality if } a' > \underline{a}. \quad (6)$$

Let $a_1(x; q)$ denote solutions to (6). First-order conditions to the concave programming problem implicit in T^2v_0 also reduce to (6) with $Tv'_0(x) = u'(a + e - a_1(x; q)q)$ substituted in place of $v'_0(x)$. This result follows from Lucas (1978 proposition 2). Values of $a_2(x; q)$ are determined in the same manner. The iterations are repeated until convergence of the decision rule is approximately obtained. To implement this procedure on a computer some

changes need to be made. First, compute $u'(a + e - a'q)$ and $v'_0(a, e)$ on finite grids on $X \times A$ and X , respectively. Between gridpoints let the values of the functions be given by linear interpolation. Next, solve for $a_1(a, e)$ on gridpoints using (6). Iterate until convergence is approximately obtained.³ In summary, the algorithm approximates $a(a, e_h)$ and $a(a, e_l)$, for fixed q , by piecewise linear functions where eq. (6) holds at gridpoints. In practice between 150 and 350 evenly spaced gridpoints are used on the set A . The gridsize is between 0.03 and 0.1 units of credit balances. Sufficiently many gridpoints are used so that $a(a, e)$ clearly crosses the 45° line. See fig. 1 for a typical graph of $a(a, e)$ on a 45° line diagram.

Step 2 involves iterations on $\psi_{n+1}(B) = \int_S P(x, B) d\psi_n$ from arbitrary initial $\psi_0 \in M(S)$ for sets of the form $B = \{x \in S: x_1 \leq a, x_2 = e\}$, where $(a, e) \in S$ and $S = [a, \bar{a}] \times E$. To implement this procedure on a computer, define the function $F_0(a, e) = \psi_0(\{x: x_1 \leq a, x_2 = e\})$ on gridpoints. Between gridpoints let values of the function be given by linear interpolation. Then iterate on

$$F_{t+1}(a', e') = \sum_e \pi(e' | e) F_t(a^{-1}(\cdot, e)(a'), e),$$

on gridpoints (a', e') . Since $a(x)$ may not be invertible in its first argument when \underline{a} is chosen, define $a^{-1}(\cdot, e)(\underline{a})$ as the maximum a such that \underline{a} is chosen when the state is (a, e) . Iterations are continued until the sequence of functions $\{F_t(a, e)\}$

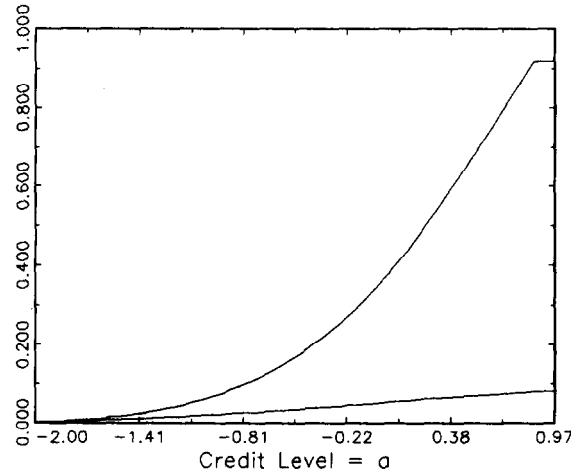


Fig. 2. Stationary distribution.

³ This computation procedure is similar to Coleman's (1988) methods for computing equilibria of representative-agent models.

converges. The decision rule $a(a, e)$ and the converged distribution function $F(a, e)$ are then used to compute the market clearing condition in the credit market. See fig. 2 for a graph of the stationary distribution function. It is interesting to note that in fig. 2 almost a zero mass of agents is near the credit limit in a stationary equilibrium.

The initial value of q is selected to be the midpoint of some interval of candidate q 's. New values are increased if there is an excess demand and decreased if there is an excess supply of credit balances at the previous price. The algorithm is therefore based on the conjecture that the excess demand for credit is decreasing in the price of credit. Although this has not been proven, this appears to be the case for all the economies examined here. The adjustment process is stopped after approximate market clearing is obtained. The criterion for approximate market clearing is that excess demand for credit balances is within 0.0025 units of zero. With this criterion, interest rates that are approximately market clearing vary by less than a tenth of one percent.

5. Results

Tables 1 and 2 present the results. In the tables interest rates (r) are annual rates whereas prices (q) are for model periods. Also note that a credit limit (\underline{a}) of -5.3 is equal to one year's average endowment. The tables illustrate a number of points. First, the experiments listed in the tables show that the risk-free rate decreases as the credit limit increases. The result has an intuitive interpretation.

Table 1
Coefficient of relative risk aversion $\sigma = 1.5$.

Credit limit (\underline{a})	Interest rate (r)	Price (q)
-2	-7.1%	1.0124
-4	2.3%	0.9962
-6	3.4%	0.9944
-8	4.0%	0.9935

Table 2
Coefficient of relative risk aversion $\sigma = 3.0$.

Credit limit (\underline{a})	Interest rate (r)	Price (q)
-2	-23.0%	1.0448
-4	-2.6%	1.0045
-6	1.8%	0.9970
-8	3.7%	0.9940

As the credit limit is increased the interest rate must be lowered to persuade agents from accumulating large credit balances so that the credit market can clear. For a similar result in a different context see Taub (1991). Second, table 2 shows the sensitivity of the interest rates in table 1 to changes in the coefficient of relative risk aversion, σ . The higher risk aversion coefficient reduces the risk-free rate for all credit limits examined. This result is interesting because calibrated representative-agent models typically require high levels of risk aversion to get a large premium on equity, but this also leads to a risk-free rate that is much too large.⁴ For example, Weil (1989) shows that the risk-free rate rises from about 5% to 18% when the coefficient of relative risk aversion rises from 0 to 20. At the same time the equity premium rises from 0% to 6%.

The economy studied here can be compared to a similar representative-agent economy where the representative agent receives the average endowment. In that economy the average endowment is constant, so the risk-free rate for a model period is equal to the time preference rate $[(1 - \beta)/\beta]$. Since there are six model periods in a year and $\beta = 0.99322$, the risk-free rate on an annual basis is 4.2%. So, in all the experiments considered, the heterogeneous-agent incomplete-insurance economy has a lower risk-free rate. I prove in Huggett (1992) that the risk-free rate is strictly less than the time preference rate in a related model. An intuitive reason why the interest rate cannot be greater than or equal to the time preference rate is that each agent's credit balance would diverge almost surely to $+\infty$. See Mendelson and Amihud (1982) or Sotomayor (1984) for an analysis of this point.

To close the section, a caveat is mentioned for interpreting the results. It is not known whether stationary equilibria are unique. If the excess demand for credit balances is continuous and strictly decreasing in the price of credit balances, then the intermediate value theorem would give the existence of a unique equilibria. These properties have been difficult to prove. However, excess demand for credit is strictly decreasing in the price of credit balances for all the cases considered in the tables.

6. Conclusion

The paper investigates why the average real risk-free interest rate has been so low. The main conclusion is that idiosyncratic shocks that cannot be perfectly insured against can generate a risk-free rate well below that of a representative-agent model with the same aggregate fluctuations.

In the future it will be interesting to investigate features that were abstracted from here. For example, how would the introduction of capital change the results? One way to examine this would be to introduce a firm that rents capital

⁴ This result may very well be sensitive to the no-growth abstraction.

and labor in spot markets to produce output. Idiosyncratic uncertainty could be introduced by supposing that an individual's labor endowment is stochastic. Agents would rent labor and capital to the firm and hold capital and credit balances to smooth consumption. I conjecture that in stationary equilibrium the return to capital and credit would be smaller than the time preference rate and the capital stock would be higher than the steady state capital stock in a similar representative-agent model. I would also conjecture that with similar credit limits the risk-free rate would be even closer to the time preference rate as an extra means to smooth consumption has been added without adding extra uncertainty.

Another interesting feature that was abstracted from here is aggregate uncertainty. To investigate the equity premium, one must consider economies with aggregate uncertainty. New techniques for characterizing equilibria will be needed. When economists learn how to analyze those economies, it would be interesting to look at the distribution of asset holdings and to look at how consumption varies with asset returns for agents with different asset holdings. On this point see Mankiw and Zeldes (1990) for an interesting look at some data.

Appendix

A transition function on the state space S is constructed.

Let (S, β_S) be a state space and corresponding Borel σ -algebra. Let z be a random variable defined on the probability measure space (Z, \mathbf{Z}, λ) . Let g be a function mapping $S \times Z$ into S . Define a mapping $P: S \times \beta_S \rightarrow [0, 1]$ by

$$P(s, B) = \lambda(\{z: g(s, z) \in B\}) \quad \text{for } B \in \beta_S. \quad (7)$$

The following lemma gives conditions under which P is a transition function.

Lemma 5 in Hopenhayn and Prescott (1987). If g is measurable in $S \times Z$ (with the product σ -algebra), then P described in (7) is a transition function for a Markov process.

Let (Z, \mathbf{Z}, λ) be Lebesgue measure on the unit interval. Let $g(s, z) = (g_1(s, z), g_2(s, z))$, where

$$g_1(s, z) = a(s)$$

and

$$\begin{aligned} g_2(s, z) &= e_h \quad \text{if } (s_2 = e_h \text{ and } z \in (0, \pi(e_h | e_h)]) \\ &\quad \text{or } (s_2 = e_l \text{ and } z \in (0, \pi(e_h | e_l)]), \\ &= e_l \quad \text{if } (s_2 = e_h \text{ and } z \in (\pi(e_h | e_h), 1]) \\ &\quad \text{or } (s_2 = e_l \text{ and } z \in (\pi(e_h | e_l), 1]). \end{aligned}$$

Note that g is measurable with respect to the product σ -algebra because g_2 is measurable by construction and g_1 is measurable (S, β_S) .⁵

Theorem 2. If the conditions of Theorem 1 hold, $\beta < q$ and $\pi(e_h | e_h) \geq \pi(e_h | e_l)$, then there exists a unique stationary probability measure ψ (given q) on (S, β_S) and, for any $\psi_0 \in M(S)$, $W^n \psi_0$ converges weakly to ψ as $n \rightarrow \infty$.

Proof. Theorem 2 is proved in two steps. The first step is to show that there is a set $S = [\underline{a}, \bar{a}] \times E$ that has the property that if an agent starts out in S , then the agent stays in S . The second step is to show that the conditions of theorem 2 in Hopenhayn and Prescott (1987) are satisfied. The conclusion to Theorem 2 then follows by applying their theorem. Theorem 2 in Hopenhayn and Prescott (1987) is stated below.

Assumption 1. (S, \geq) is an ordered space.

Assumption 2. S is a compact metric space.

Assumption 3. \geq is a closed order.

Assumption 4. (S, β_S) is a measurable space and β_S is the Borel σ -algebra.

Assumption 5. P is a transition function, $P: S \times \beta_S \rightarrow [0, 1]$.

Theorem 2 in Hopenhayn and Prescott (1987). If Assumptions 1–5 hold, P is increasing, S has a greatest (d) and a least (c) element in S and the following condition is satisfied:

Monotone Mixing Condition: There exists $s^* \in S$, $\varepsilon > 0$, and N such that $P^N(d, \{s: s \leq s^*\}) > \varepsilon$ and $P^N(c, \{s: s \geq s^*\}) > \varepsilon$.

Then there exists a unique stationary probability measure ψ and, for any $\psi_0 \in M(S)$, $W^n \psi_0$ converges weakly to ψ as $n \rightarrow \infty$.

For step 1 of the proof consider the following lemmas. Essentially, these lemmas show that the decision rule for credit balances has the shape shown in fig. 1.

Lemma 1. Under the conditions of Theorem 2, $a(a, e_l) < a$ for $a > \underline{a}$.

Proof. Define functions $v_n(x)$ for $n = 0, 1, 2, \dots$ by letting $v_0(x) = 0$ and letting $v_{n+1}(x) = Tv_n(x)$. Define functions $a_{n+1}(x)$ for $n = 0, 1, 2, \dots$ as the optimal setting for credit balances implicit in the mappings Tv_n . Show by induction that

⁵ Conditions under which g maps $S \times Z$ into S are given in Theorem 2. The function g_1 is the optimal decision rule $a(x)$. Theorem 1 states conditions under which $a(x)$ is continuous, hence $a(x)$ defined on S will be measurable (S, β_S) .

$v'_n(a, e_h) \leq v'_n(a, e_l)$, where it is understood that $v'(a, e)$ is the derivative of v with respect to its first component. For $n = 0$ this is obvious. Suppose this property holds up to n . Show that it holds for $n + 1$. Eq. (8) below is a first-order condition for the maximization problem implicit in the mapping Tv_n . The value of a' that solves (8) for fixed $x = (a, e)$ is $a_{n+1}(x)$,

$$u'(a + e - a'q) \geq (\beta/q) \sum_{e'} v'_n(a', e') \pi(e' | e), \text{ with equality if } a' > \underline{a}. \quad (8)$$

Note that the induction assumption and $\pi(e_h | e_h) \geq \pi(e_h | e_l)$ imply that the right-hand side of (8) evaluated at (a', e_h) is always less than or equal to the right-hand side evaluated at (a', e_l) . Similarly, the left-hand side evaluated at (a', e_h) is always less than the left-hand side evaluated at (a', e_l) . It follows that $u'(a + e_h - a_{n+1}(a, e_h)q) \leq u'(a + e_l - a_{n+1}(a, e_l)q)$. This completes the induction since $v'_{n+1}(a, e) = u'(a + e - a_{n+1}(a, e)q)$.

Next, show that $v'_n(a, e)$ converges pointwise to the derivative of the optimal value function $v'(a, e)$. Since $v'_n(a, e) = u'(a + e - a_n(a, e)q)$, $v'(a, e) = u'(a + e - a(a, e)q)$ and u' is continuous, it is sufficient to show that $a_n(a, e)$ converges pointwise to the optimal decision rule $a(a, e)$. It is straight forward to show that the argument in lemma 3.7 in Stokey and Lucas (1989) can be applied to obtain this result. Pointwise convergence of v'_n to v' establishes that $v'(a, e_h) \leq v'(a, e_l)$. The conclusion of Lemma 1 follows because $\beta/q < 1$ and $v'(a, e_h) \leq v'(a, e_l)$ imply that the hypothesis to Lemma 2 below holds for $e = e_l$ and any $a^* > \underline{a}$. ■

Lemma 2. *If $v'(a, e) > (\beta/q) E[v'(a, e') | e]$ for $a \geq a^* > \underline{a}$, then $a(a, e) < a$ for $a \geq a^*$.*

Proof. A first-order condition to an agent's decision problem is

$$u'(a + e - a(a, e)q)q \geq \beta \sum_{e'} v'(a(a, e), e') \pi(e' | e),$$

(9)

with equality if $a(a, e) > \underline{a}$.

For $a \geq \underline{a}^*$, either $a(a, e) = \underline{a}$ or $a(a, e) > \underline{a}$. If the first occurs, then $a(a, e) < a$. If the second occurs, then (9), the hypothesis, $v'(a, e) = u'(a + e - a(a, e)q)$, and v' decreasing in a imply that $a(a, e) < a$. The fact that v is concave and differentiable implies that v' is decreasing in a . ■

Lemma 3. *Under the conditions of Theorem 2, there exists an a such that $a(a, e_h) = a$.*

Proof. Suppose not. Then $a(a, e_h) > a$ for all a . Lemma 1 then implies that $a(a, e_h) \geq a(a, e_l)$ for all a . Three inequalities follow:

$$a + e_l - a(a, e_h)q \leq a + e_l - a(a, e_l)q,$$

$$c(a, e_h) - (e_h - e_l) \leq c(a, e_l),$$

$$c(a, e_l)/c(a, e_h) \geq 1 - (e_h - e_l)/c(a, e_h).$$

Note that v increasing in a and v' decreasing in a imply that $v'(a_1, e) \geq v'(a_2, e) > 0$ for any $a_2 \geq a_1$. The fact that $v'(a, e) = u'(c(a, e))$ then implies that $c(a, e)$ is increasing in a . The fact that v is bounded implies that $c(a, e) \rightarrow \infty$ as $a \rightarrow \infty$. So for all sufficiently large a ,

$$v'(a, e_h)/v'(a, e_l) = (c(a, e_l)/c(a, e_h))^\sigma \geq (1 - (e_h - e_l)/c(a, e_h))^\sigma.$$

Since $\beta/q < 1$, there is an a^* such that $v'(a, e_h)/v'(a, e_l) > \beta/q$ for $a \geq a^*$. This fact and $v'(a, e_h) \leq v'(a, e_l)$ from Lemma 1 imply that the hypothesis of Lemma 2 holds for $e = e_h$. Contradiction. ■

The previous lemmas imply that there is $S = [\underline{a}, \bar{a}] \times E$ such that if an agent starts with state x in S , then the agent stays in S . Lemma 3 shows that $a(a, e_h)$ has a fixed point as a function of a . Choose \bar{a} to be the smallest fixed point. Lemma 1 shows that $a(a, e_l) < a$ for $a > \underline{a}$. Thus, if an agent starts with $x = (a, e_l)$ in S , then next period's state must be in S . Similarly, if an agent starts with $x = (a, e_h)$ in S , then because $a(a, e_h) \leq a(\bar{a}, e_h) = \bar{a}$ next period's state must also be in S .

Now show that the conditions of theorem 2 in Hopenhayn and Prescott (1987) hold. First, define an order \geq on S . For $x, x' \in S$, where $x = (x_1, x_2)$,

$$\begin{aligned} x \geq x' \quad \text{iff} \quad & [(x_1 \geq x'_1 \text{ and } x_2 = x'_2) \quad \text{or} \quad (x' = c = (\underline{a}, e_l)) \\ & \text{or} \quad (x = d = (\bar{a}, e_h))]. \end{aligned}$$

This is a closed order with minimum (c) and maximum (d) elements.

Next, define the transition function P as described earlier in the appendix. To show that P is increasing, Hopenhayn and Prescott (1987) prove that it is sufficient to show:

$$x, x' \in S, \quad x \geq x' \quad \text{imply} \quad \int_S fP(x, dx) \geq \int_S fP(x', dx),$$

where

$$f = \chi_B, \quad B = \{y \in S: y \geq x \text{ for some } x \text{ in } B\} \in \beta_S.$$

Let $B_x = \{z \in Z: g(x, z) \in B\}$ and $B_{x'} = \{z \in Z: g(x', z) \in B\}$, where g was defined earlier in the construction of the transition function. Show $B_{x'} \subseteq B_x$. This is obvious if $g(x, z)$ is monotone in x for fixed z . It is straightforward but tedious to show that this is true. Therefore, $P(x, B) \geq P(x', B)$ as was to be shown.

Lastly, show that the mixing condition holds. Choose $s^* = (a(\bar{a}, e_h) + \bar{a})/2$, e_h). Define a sequence $x_1 = \bar{a}$, $x_2 = a(x_1, e_h)$, $x_3 = a(x_2, e_h)$, \dots and a sequence $y_1 = \bar{a}$, $y_2 = a(y_1, e_l)$, $y_3 = a(y_2, e_l)$, \dots . Note that $\{x_n\} \rightarrow \bar{a}$ monotonically and $\{y_n\} \rightarrow \bar{a}$ monotonically. Therefore, there is an N_1 such that an agent goes from c to $\{x \in S: x \geq s^*\}$ with positive probability in N_1 or greater steps and there is an N_2 such that an agent goes from d to $\{x \in S: x \leq s^*\}$ with positive probability in N_2 or greater steps. Choose $N = \max\{N_1, N_2\}$ in the mixing condition. This completes the argument that the conditions of theorem 2 in Hopenhayn and Prescott (1987) are satisfied. ■

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