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1 MPI

In this section, we will focus on the optimization and parallelization of Google PageRank problem, more precisely, the calculation of the eigenvectors associated with the highest eigenvalue of a stochastic matrix that has some special attributes.

1.1 Introduction

First of all, we mention briefly the problem. Given a web page P and a measure of its importance $I(P)$, called the page's PageRank. Suppose that page P_i has ℓ_i links. If one of those links is to page P_j, then P_i will pass on $frac1\ell_i$ of its importance to P_j. The importance ranking of P_j is then the sum of all the contributions made by pages linking to it. That is, if we denote the set of pages linking to P_j by B_j,

$$I(P_j) = \sum_{P_i \in B_j} \frac{I(P_i)}{\ell_i}$$

Mathematically, if we define a matrix H where:

$$H_{i,j} = \begin{cases} \frac{1}{\ell_i} & \text{if } P_i \in B_j \\ 0 & \text{otherwise} \end{cases}$$

H is called "hyperlink matrix" and we have $\pi^\top = \pi^\top H$ where $\pi_i = I(P_i)$, a vector contains measure of importance from all pages. In the other words, $(\pi^\top)^\top = (\pi^\top H)^\top \implies \pi = H^\top \pi$ or π is an eigenvector of $H^\top$ whose eigenvalue is 1.

In addition, we could prove that the largest eigenvalue of $H^\top$ less than or equal to 1. Indeed, if λ is an eigenvalue of $H^\top$ and u is its eigenvector (we will choose a vector whose the sum of all of its components is positive), as we can see all entries of H are non negative and the sum of of each row is either 1 or 0 (if that page has no link to other page), we have:

$$\begin{aligned}
& H^\top u = \lambda u \\
\iff & \begin{pmatrix} H_{1,1} & \cdots & H_{n,1} \\ \vdots & \ddots & \vdots \\ H_{1,n} & \cdots & H_{n,n} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \vdots \\ \lambda u_n \end{pmatrix} \\
\implies & \begin{cases} H_{1,1}u_1 + \cdots + H_{n,1}u_n & = \lambda u_1 \ (1) \\ & \vdots \\ H_{1,n}u_1 + \cdots + H_{1,n}u_n & = \lambda u_n \ (n) \end{cases} \\
& \text{By summing up from } (1) \text{ to } (n) \\
\implies & u_1(\sum_{i=1}^{n} H_{1,i}) + \cdots + u_n(\sum_{i=1}^{n} H_{n,i}) = \lambda(\sum_{i=1}^{n} u_i) \\
& \text{As } \sum_{i=1}^{n} H_{j,i} = \text{ either 0 or 1 } \forall j \\
\implies & \sum_{i=1}^{n} u_i \geq \lambda(\sum_{i=1}^{n} u_i) \\
\implies & \lambda \leq 1
\end{aligned}$$

Futhermore, if $\sum_{i=1}^{n} H_{j,i} = 1 \; \forall j$, we have $\lambda = 1$ is the largest eigenvalue of $H^\top$. Therefore, we return to the problem of finding the eigenvector associated with the largest eigenvalue of $H^\top$, which already has a very simple but powerful solution called "Power Iteration" that we will use in next sections. Each component of that vector is a measure of importance of one page and the index of the largest element inside that vector will also be the index of the most meaningful page.

1.2 Power Iteration Method

First, we choose a matrix H as following for testing:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

As proved about, since every sum of a row of H is equal to 1, the largest eigenvalue will be 1 and therefore, we could apply this method to find out the PageRank score of each page.

The method could be described as follow:

Data: A diagonalizable matrix G
Result: Largest eigenvalue and associated eigenvector
$u \leftarrow random$;
$u \leftarrow \frac{u}{\|u\|}$;
while $\|u - v\| > \epsilon$ **do**
  compute $v = Gu$;
  define $\lambda = u^\top v$ and $u = \frac{v}{\|v\|}$;
end

And its implementation in `Python`:

```
def power_iteration(matvec, G, u, v, tol=1e-8, itmax=200):
    t = time.time()
    it = 0
    diff_norm = 2 * tol
    while it < itmax and diff_norm > tol:
        v = matvec(G, u)
        l = dot_product(u, v)
        v = (1 / norm2(v)) * v
        diff_norm = norm2(u - v)
        u = v
        it += 1
    return l, v
n = H.shape[0]
u = np.random.uniform(0, 1, n)
v = np.zeros(n)
l, u = power_iteration(matvec, H.T, u, v)
```

Here, we choose to keep the original power iteration method by calculating $v = Gu$ instead of $v = G^\top u$. It helps us prevent some confusions. Since we want to calculate the eigenvector of $H^\top$, we could pass $G \leftarrow H^\top$. From the output below, the method has successfully converged to 1 and the most meaningful page is the page number 8.

```
## number of iterations = 115
## residual = 8.87e-09
## eigenvalue = 1.000000e+00
## eigenvector = [0.14  0.158 0.07  0.158 0.228 0.473 0.42  0.689]
## highest pagerank = 7
## Computational time = 0.003090381622314453
```

In addition, we check the correctness by calling the function `eigs` from `scipy.sparse.linalg`. As we can see, the two outputs are the similar. Our implementation of power iteration is correct.

```
from scipy.sparse.linalg import eigs
vals, vecs = eigs(H.T)
val, vec = vals[0].real, vecs[:, 0].real
print("Largest eigenvalue ", val)
```

```
## Largest eigenvalue  0.9999999999999996
```

```
print("Associated eigenvector ", vec)
```

```
## Associated eigenvector  [-0.14  -0.158 -0.07  -0.158 -0.228 -0.473 -0.42  -0.689]
```

1.3 Google matrix

In general, there is no guarantee that the algorithm will work. As we can see in the introduction, if one web page does not link to any other page or also called as a dangling node, the sum of that row in the hyperlink matrix will be 0, the largest eigenvalue will be eventually < 1 and therefore we fail. To address this problem, we introduce another type of matrix, so-called Google matrix and use this matrix for calculating the PageRank. It has the following property:

$$G = \alpha(H + \frac{1}{n}de^\top) + (1-\alpha)\frac{1}{n}ee^\top$$

Where

$H :$ a very sparse hyperlink matrix.

$\alpha :$ a scaling parameter between 0 and 1 (generally set at the "magic" value of 0.85).

$d :$ the binary dangling node row vector ($d_i = 1$ if page i is a dangling node and 0 otherwise).

$e^\top :$ the row vector of all entries 1.

We could easily prove that G is stochastic, i.e, sum of each row equals to 1. Indeed,

$$\sum_{i=1}^{n} G_{i,j} = \alpha(\sum_{i=1}^{n} H_{i,j} + \frac{1}{n}\sum_{1}^{n} d_j) + (1-\alpha)$$

$$\implies \sum_{i=1}^{n} G_{i,j} = \alpha((\sum_{i=1}^{n} H_{i,j}) + d_j) + 1 - \alpha$$

Since one of $\sum_{i=1}^{n} H_{i,j}$ or d_j equals to 1 and the other term equals to zero

$$\implies \sum_{i=1}^{n} G_{i,j} = \alpha + 1 - \alpha$$

$$\implies \sum_{i=1}^{n} G_{i,j} = 1$$

We have two approaches for this problem.

1.3.1 Dense

In this approach, we compute in the "traditional" way, i.e, we construct the matrix G and pass it to the power iteration method. The output is shown below.

```
## number of iterations =  54
## residual = 7.99e-09
## eigenvalue = 1.000000e+00
## eigenvector = [0.159 0.233 0.115 0.245 0.277 0.464 0.394 0.632]
## highest pagerank = 7
## Computational time = 0.0007550716400146484
```

1.3.2 Sparse

On the other hand, we could take advantage of two factors: the sparsity of H and the nature of the matrices $de^\top$ and $ee^\top$. First, we note that we want to calculating the eigenvector of

$$G^\top = \alpha(H^\top + \frac{1}{n}ed^\top) + (1-\alpha)\frac{1}{n}ee^\top$$

The matrix multiplication between $G^\top$ and x turns out to be

$$G^\top x = \alpha H^\top x + \alpha\frac{1}{n}ed^\top x + (1-\alpha)\frac{1}{n}ee^\top x$$

Futhermore, give a an arbitrary column vector,

$$\begin{aligned} ea^\top x &= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} a_1 & \dots & a_n \\ \vdots & \ddots & \vdots \\ a_1 & \dots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} a_1x_1 + \dots + a_nx_n \\ \vdots \\ a_1x_1 + \dots + a_nx_n \end{pmatrix} \\ &= \langle a, x\rangle \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

Therefore, instead of calculating $ed^\top x$ and $ee^\top x$, we could just calculate two scalars $\langle d, x\rangle$ and $\langle e, x\rangle$. Thank to the fact that in `Python` world, an operation between an array and a scalar will be resulted in an element-wise operation, we have our desired matrix multiplication. Its implementation in `Python` is:

```
def matvec_sparse(M, x, d, alpha=0.85):
    m, n = np.shape(M)
    e = np.ones(m)
    return (
        alpha * M.dot(x)
        + alpha / n * d.dot(x)
        + (1 - alpha) / n * e.dot(x)
    )
```

The output of this method is shown below.

```
## number of iterations =  54
## residual = 7.99e-09
```

```
## eigenvalue = 1.000000e+00
## eigenvector = [0.159 0.233 0.115 0.245 0.277 0.464 0.394 0.632]
## highest pagerank = 7
## Computational time = 0.001833200454711914
```

1.3.3 Benchmarking

We see the difference between two methods in term of performance is quite small, given that our matrix is too tiny. We varied the initial matrix H and obserbeb that, while the sparse approach performs worse in small cases, it gets better when the size of H increases. That could be explained by the fact that the density of H (number of non-zero elements) decreases when the size of H is bigger (based on the external data). Since 0 is a really special number, a sparsity-aware matrix could take advantage of that fact and perform a lot faster. In our case, the speed-up comes from two factors:

- `M.dot(x)` as $M \leftarrow H$ is a sparse matrix, which means `M` contains only non-zero entries. Therefore while normal algorithm have to do operations with every elements of `M` and `x`, the sparse algorithm will only doing operations between those elements of `M` and `x` (0 multiplies something is 0 anyway) which reduce significanly the number of operations required. if we convert H to a normal matrix beforehand, the performance will be similar to the dense approach.
- When the size of H is too big to be fitted within the RAM, we have an Out Of Memory (OOM) if we try the first approach. The last matrix `ucam2006`, the dense version needs roughly 361,97 GB which is impossible on almost computers. On the opposite side, the second approach only needs a fraction of that amount of memory to store the matrix H since its density is relatively low.

Below is a table that shown the results above where:

- density: the number of non-zero elements of H per mille.
- shape: the number of rows of H (and also of columns because H is a square matrix).
- important: the most meaningful page, we show it here just for asserting the true result.
- memory: the amount of RAM memory in GB that is needed to fit the dense version of that matrix.

Table 1: dense vs sparse approach

algo	time	matrix	shape	important	density	memory
dense	0.0006697	random10	10	4	200.00 ‰	0.00
sparse	0.0013359	random10	10	4	200.00 ‰	
dense	0.0042232	random100	100	90	20.00 ‰	0.00
sparse	0.0023179	random100	100	90	20.00 ‰	
dense	0.0195672	random1000	1000	980	2.00 ‰	0.01
sparse	0.0075064	random1000	1000	980	2.00 ‰	
dense	1.0330197	random10000	10000	6253	0.20 ‰	0.80
sparse	0.0063826	random10000	10000	6253	0.20 ‰	
dense	Inf	ucam2006	212711	-1	0.04 ‰	361.97
sparse	0.3676185	ucam2006	212711	2	0.04 ‰	

1.4 MPI-enabled

Now, since, we has already taken advantage of the sparsity of matrix for optimizing the power iteration method, algorithm-level is now harder to optimize. However, we could try to fully exploit our hardware by parallelization, which will be easier. In this section, we concentrate ourselves in `MPI`, one technology that will allows us to achieve that.

1.4.1 Principal functions

First, we will integrate `MPI` to 4 principal functions: $\|\cdot\|_1$, $\|\cdot\|_2$, $\langle\cdot,\cdot\rangle$ and `matvec` (which is essentially matrix multiplication with the second matrix is a vector).

For the first 3 functions, the principle is simple, we calculate them with a subset of values in each process and reduce them into one at the end. For example, $\langle\cdot,\cdot\rangle$ could be calculated as follow:

$$
\begin{aligned}
\langle x, y\rangle &= \sum_{i=1}^{kn} x_i y_i \\
&= \sum_{i=1}^{k} \sum_{j=(i-1)n+1}^{ni} x_j y_j \\
&= \sum_{i=1}^{k} \underbrace{\langle \begin{pmatrix} x_{(i-1)n+1} \\ \vdots \\ x_{in} \end{pmatrix}, \begin{pmatrix} y_{(i-1)n+1} \\ \vdots \\ y_{in} \end{pmatrix} \rangle}_{\text{calculate in process } i}
\end{aligned}
$$

Our implementation of those 3 functions with `MPI` is shown below. Note that `u` and `v` is only a subset of the real vectors `u` and `v`. However, since we reduce the result of each process in the end and send it to all processes by `Allreduce`, we still get the true value of these operations on vectors `u` and `v` regardless process.

```
def norm1_mpi(u, comm=MPI.COMM_WORLD):
    norm = norm1(u)
    result = np.empty(1)
    comm.Allreduce(norm, result, MPI.SUM)
    return result


def norm2_mpi(u, comm=MPI.COMM_WORLD):
    norm = dot_product(u, u)
    result = np.empty(1)
    comm.Allreduce(norm, result, MPI.SUM)
    return np.sqrt(result)


def dot_product_mpi(u, v, comm=MPI.COMM_WORLD):
    product = dot_product(u, v)
    result = np.empty(1)
    comm.Allreduce(product, result, MPI.SUM)
    return result
```

For `matvec`, everything get more complicated, we note that, given a matrix H and u, if we note $v = Hu$, we have:

$$v_i = \sum_{j=1}^{n} H_{i,j} u_j$$

Therefore, the vector that passed into `matvec` must be a “full” vector.

1.4.2 Power Iteration

With all those principles in mind, we are ready now for a `MPI`-enabled power iteration. The general idea will be:

Data: A diagonalizable matrix G whose size is kn and k processes
Result: Largest eigenvalue and associated eigenvector
In process i

$v_i \leftarrow$ random

gather $u \leftarrow \begin{pmatrix} v_1 & \dots & v_k \end{pmatrix}$ (each process has their own copy of u)

Define

$$G_i = \begin{pmatrix} G_{(i-1)n+1,\cdot} & \dots & G_{in,\cdot} \end{pmatrix}^\top$$

while $\|u_i - v\| > \epsilon$ **do**

define G_i and u_i

compute $v_i = G_i u$ or in other world $v_i = \begin{pmatrix} [Gu]_{(i-1)n+1} & \dots & [Gu]_{in} \end{pmatrix}^\top$

by using MPI-enabled functions above

gather $v \leftarrow \begin{pmatrix} v_1 & \dots & v_k \end{pmatrix}$

define $\lambda = u^\top v$ and $v_i = \frac{v_i}{\|v\|}$

gather $u \leftarrow \begin{pmatrix} v_1 & \dots & v_k \end{pmatrix}$

end

Dense

For the dense approach, the main logic is implemented as follow:

```
        v = matvec(G[start:end], u)
        l = dot_product(u[start:end], v, comm)
        v = (1 / norm2(v, comm)) * v
        diff_norm = norm2(u[start:end] - v, comm)
        u = mpi_all_to_all(u, v, comm=comm)
```

Inside the code above, `matvec` is just a normal matrix multiplication. The step gather `v` above is executed inside `dot_product` and `norm2` functions. `mpi_all_to_all` is the function we used in order to gather `u` at the end with the help of `Allgather` or `Allgatherv` underlying.

Sparse

However, the sparse method requires a slightly more complicated solution. It has one different line from the the first version.

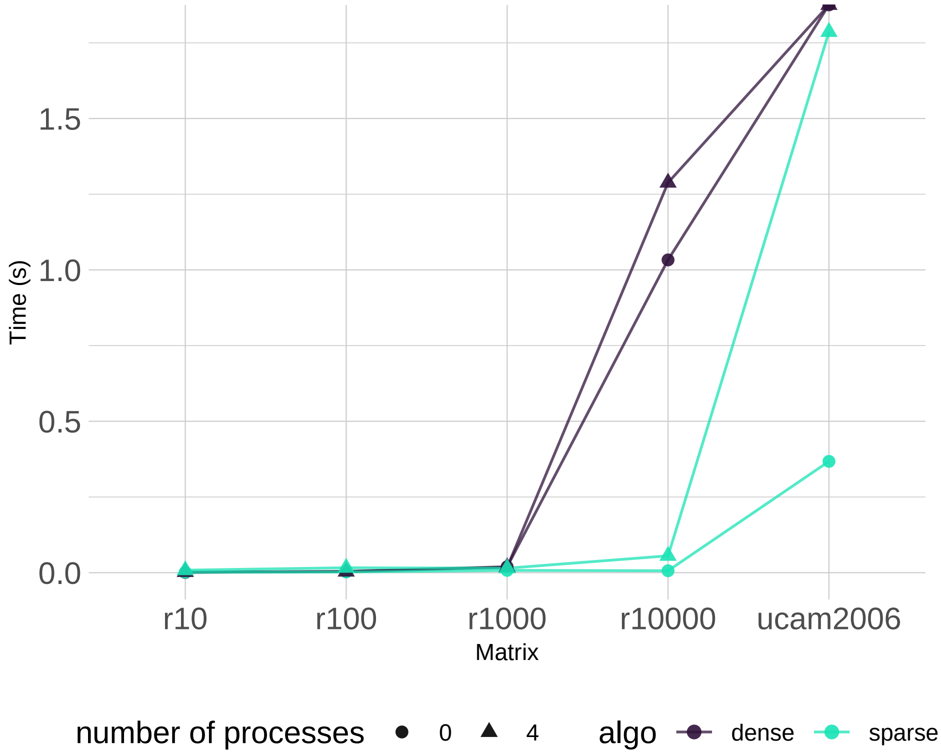
```
        v = matvec(G.T, u, d, start, end)
        l = dot_product(u[start:end], v, comm)
        v = (1 / norm2(v, comm)) * v
        diff_norm = norm2(u[start:end] - v, comm)
        u = mpi_all_to_all(u, v, comm=comm)
```

Where `matvec` is

```
def matvec_sparse(M, x, d, start, end, alpha=0.85, comm=MPI.COMM_WORLD):
    M = M[start:end]
    d = d[start:end]
    xd = x[start:end]
    m, n = np.shape(M)
    e = np.ones(m)
    return (
        alpha * M.dot(x)
        + alpha / n * dot_product(d, xd, comm)
        + (1 - alpha) / n * dot_product(e, xd, comm)
    )
```

Although, `dot_product` produces a global result, it is still true because it is a scalar, not a vector as show above.

1.4.3 Benchmarking



First, both versions (non `MPI` and `MPI`) using the dense algorithm is defeated by the matrix `ucam2006` as explained above. Second, interestingly, when number of processes is 4, it took more time than non `MPI` version. This could be explained by the fact that `MPI` is optimized for a distributed cluster of machines but we are testing on only one computer, hence, the communication overhead overshaddowed any benefit it brought to us.

1.5 Further study: Matrix B

This section, we explore another matrix that has some special constructions and we try to exploit them for a faster computation.

1.5.1 Introduction

We define matrix B size kn as below (with k processes and n is the size of the submatrix that is owned processed in each process):

$$B^\top = \begin{pmatrix} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{kn-1} & & \frac{1}{3} & & & \\ \frac{1}{kn-1} & \frac{1}{2} & & \ddots & & \\ \frac{1}{kn-1} & & \frac{1}{3} & & \frac{1}{3} & \\ \vdots & & & \ddots & & \frac{1}{2} \\ \frac{1}{n-1} & & & & \frac{1}{3} & \end{pmatrix}$$

$B^\top$ has the following non-zeros entries:

- the first column: consits of 0 followed by $n-1$ elements whose value is $\frac{1}{n-1}$.
- the first row: consits of 0 followed by an array that has $\frac{1}{2}$ at its head and tail; $\frac{1}{3}$ at other positions.
- the first superdiagonal: the same array as the first row without the leading 0.

- the first subdiagonal: the same array as the first row except its tail, and the leading 0 is replaced by $\frac{1}{n-1}$.

For example, with k = 3 and n = 3, we have $B^{\top}$:

$$\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{8} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8} & \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{8} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\
\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0
\end{pmatrix}$$

And its corresponding matrix transition:

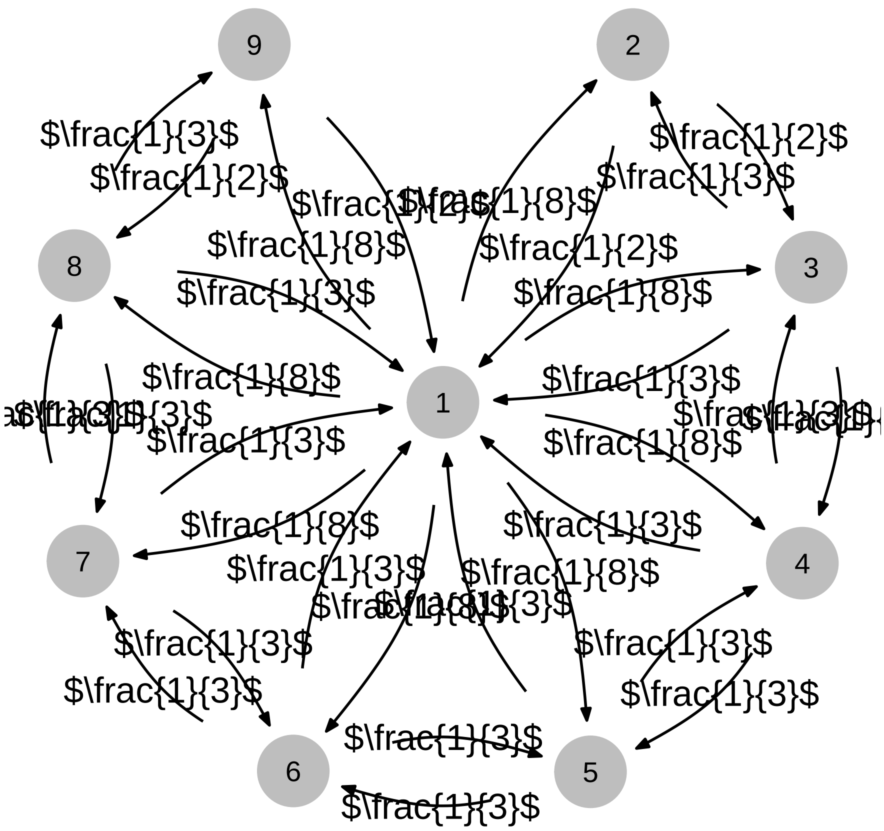


Figure 1: matrix transition of B when kn = 9

We could see easily that:

- Matrix B is non negative.
- Matrix B is stochastic $\forall kn$.
- It doesn't contain any dangling node.
- From 3 observations above, we conclude that largest eigenvalue of $B^{\top}$ is 1.
- Page 1 receives $\frac{1}{2}$ and of the importance of other pages while only share $\frac{1}{8}$ of its. Intuitively, it is safe to guess that page 1 is the most important page.

That assumption is verified by the output followed.

```
## number of iterations =  35
## residual = 9.45e-09
## eigenvalue = 1.000000e+00
## eigenvector = [0.713 0.178 0.267 0.267 0.267 0.267 0.267 0.267 0.178]
## highest pagerank = 0
## Computational time = 0.0010652542114257812
```

1.5.2 MPI-enabled

Same as the Google matrix, we tried to exploit the special properties of $B^\top$. As before, we first try to divide $B^\top$ into n submatrices, each with n rows and kn columns. This is however, the same as the dense version and we lost the sparsity of $B^\top$. But we notice that apart from the top-most submatrix whose the first row contains only 1 0, the other submatrices are quite sparse. Therefore, we have three types of matrix here:

- The first submatrix: nothing could be done here.
- The last submatrix:
  - There are n elements $\frac{1}{kn-1}$.
  - There are n elements of the first subdiagonal which lasts from $(kn-n)^{\text{th}}$ column to $(kn-1)^{\text{th}}$ column.
  - There are $n-1$ elements of the first superdiagonal which lasts from $(kn-n+2)^{\text{th}}$ column to kn^{th} column.
  - From three observations above, we could remove a block from 2^{nd} column to $(kn-n-1)^{\text{th}}$ column which is all zero, which left us a submatrix of size $n \times (n+2)$.
- The i^{th} submatrix:
  - There are n elements $\frac{1}{kn-1}$.
  - There are n elements of the first subdiagonal which lasts from $(in)^{\text{th}}$ column to $(in+n-1)^{\text{th}}$ column.
  - There are n elements of the first superdiagonal which lasts from $(in+2)^{\text{th}}$ column to $(in+n+1)^{\text{th}}$ column.
  - From three observations above, we only take the first column and the block from $(in)^{\text{th}}$ column to $(in+n+1)^{\text{th}}$ column and remove everything else, which left us a submatrix of size $n \times (n+3)$.

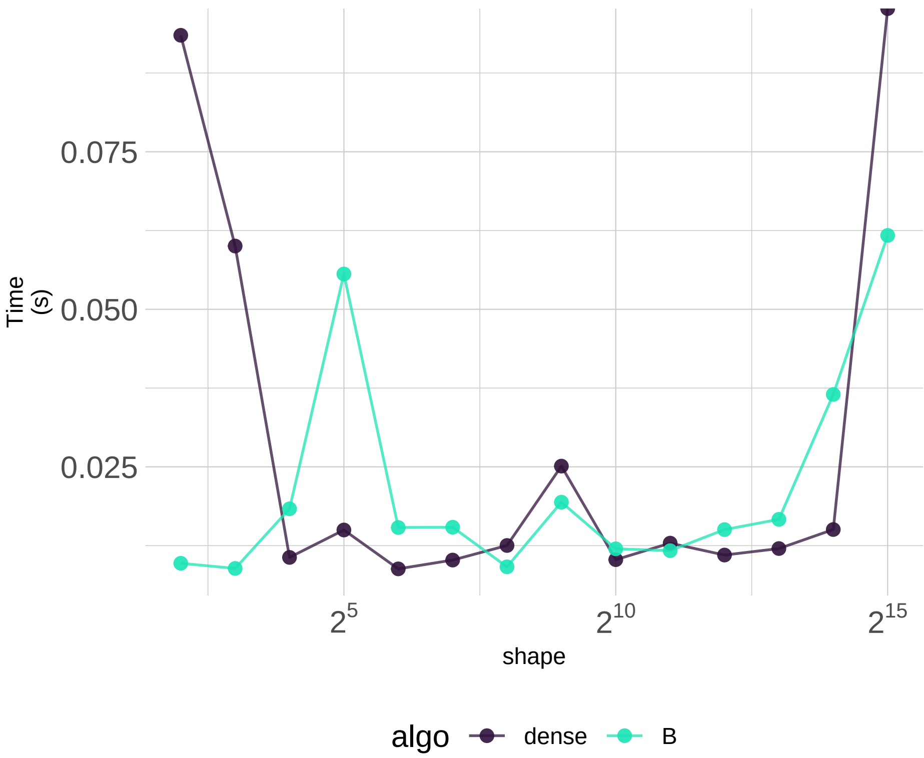
In order to multiply matrix and vector with the modified submatrices, we have to modified u as well because the length of u needs to be the same as the number of columns of each submatrix inside each process (which is constant in the previous methods). It could be done easily by noting that even if we don't remove those blocks, they still contribute nothing to the result, as they are all zeros. Therefore, we could extract the corresponding indices from u, this is how we could take advantage of the sparsity of $B^\top$.

There is one more problem to solve. We needs a modified gathering function. Note that apart from the first submatrix, vector u inside other processes only contain a subset of the real u (which is opposed to the previous method where u is constant in size). Hence, we dicided to send everything to the process that holds the first submatrix in order to assemble from v the real u, broadcast that to every other processes and then let each processed extracting the subset of u they need. The exact code is shown below:

```python
def mpi_all_to_all(x, xd, comm=MPI.COMM_WORLD):
    rank = comm.rank
    size = comm.size
    n = np.shape(xd)[0]
    comm.Gather(xd, x, root=0)
    xbcast = np.empty(n * size) if rank != 0 else x
    comm.Bcast(xbcast, root=0)
    if rank == size - 1:
```

```
        x[0] = xbcast[0]
        x[1 : n + 2] = xbcast[n * rank - 1 : n * rank + n]
    elif rank != 0:
        x[0] = xbcast[0]
        x[1 : n + 3] = xbcast[n * rank - 1 : n * rank + n + 1]
    return x
```

1.5.3 Benchmarking



Compared to the dense version, the $B^\top$-optimized version takes less time to compute in general, which means our algorithm is correct.