15-451/651 Assignment 6

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1: Competing with the Best

We will show that the Move-Half-Way-To-Front (MHWTF) algorithm is 6-competitive. Consider the lists $\{a_1, a_2, ..., a_n\}$ of the MHWTF algorithm and another algorithm B. Consider an element x in the MHWTF list that has k elements in front of it ($k < n$) and the set $P = \{a_{\lceil \frac{k}{2} \rceil}, ..., a_k\}$. Partition P into two sets S and T where S is the set of elements that are in front of x in B and T is the set of elements that are behind x in B. Note that $|P| = |S| + |T| = \lceil \frac{k}{2} \rceil$.

The cost of accessing x in MHWTF is

$$\begin{aligned} Cost_{MHWTF} &= Cost_{access} + Cost_{swap} \\ &= \left(1 + |S| + |T| + \left\lfloor \frac{k}{2} \right\rfloor\right) + (|S| + |T|) \\ &\leq \left(1 + |S| + |T| + \left\lceil \frac{k}{2} \right\rceil\right) + (|S| + |T|) \\ &= 1 + 3(|S| + |T|) \end{aligned}$$

The cost of accessing x in B is

$$Cost_B \geq |S| + 1$$

Modify the potential function from lecture to $\Phi = 3\cdot$(number of different inversions in MHWTF and B). So the change in potential is

$$\Delta\Phi = 3(|S| - |T|)$$

And the amortized cost of MHWTF is

$$AC_{MHWTF} = 1 + 3(|S| + |T|) + 3(|S| - |T|) = 1 + 6|S| \leq 6(1 + |S|) \leq 6 \cdot Cost_B$$

Now let B does a swap, then we get $Cost_{MHWTF} = 0, \Delta\Phi = 3(1), Cost_B \geq 1$ and the amortized cost of MHWTF is

$$AC_{MHWTF} = 0 + 3 \leq 6(1) \leq 6 \cdot Cost_B$$

Summing the amortized cost, we get

$$\text{Total Cost}_{MHWTF} + \Phi_{final} - \Phi_{initial} \leq 6(\text{Total Cost}_B)$$

Since $\Phi_{initial} = 0$ (both algorithms started with the same list) and $\Phi_{final} \geq 0$ (the number of inversions cannot be negative),

$$\text{Total Cost}_{MHWTF} \leq 6(\text{Total Cost}_B)$$

Thus, MHWTF is 6-competitive.

To show that MHWTF is not c'-competitive for any $c' < 6$. Consider a series of access request of the form $[a_n, a_{n-1}, a_{n-2}, ..., a_{\frac{n}{2}}]^t$, (i.e.) asking for the last $\frac{n}{2}$ items t times. An optimal offline algorithm would move the last $\frac{n}{2}$ elements to the front first before allowing access. Then the cost to access these $\frac{nt}{2}$ elements are

$$\begin{aligned}
Cost_{MHWTF} &= Cost_{access} + Cost_{swap} \\
&= \left(n + \frac{n}{2}\right)\left(\frac{nt}{2}\right) = \left(\frac{3n}{2}\right)\left(\frac{nt}{2}\right) \\
Cost_{OPT} &= Cost_{swap} + Cost_{access} \\
&= \left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right) + \left(1 + 2 + ... + \frac{n}{2}\right)t \\
&\leq \left(\frac{n}{2}\right)\left(\frac{n}{2}\right) + \frac{1}{2}\left(\frac{n}{2} + 1\right)\left(\frac{n}{2}\right)t
\end{aligned}$$

And the competitive ratio is

$$\begin{aligned}
\frac{Cost_{MHWTF}}{Cost_{OPT}} &\geq \frac{\left(\frac{3n}{2}\right)\left(\frac{nt}{2}\right)}{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right) + \frac{1}{2}\left(\frac{n}{2} + 1\right)\left(\frac{n}{2}\right)t} \\
&= \frac{3nt}{n + \left(\frac{n}{2} + 1\right)t} \\
&= \frac{6nt}{nt + 2n + 2t}
\end{aligned}$$

It is safe to assume that n is sufficiently large and $t >> n$. Then, for a large number L, $\lim_{t \to L} \frac{6nt}{nt + 2n + 2t} = \frac{6n}{n} = 6$. Thus, the competitive ratio must be at least 6.

2: Algorithms from Another Planet

Let x_t, x_t^* be the position on the real line for our algorithm A and an arbitrary algorithm B, respectively.

Define the potential function as $\Phi_t = 2(x_t^* - x_t)$

First, let B move from $x_{t-1}^* \to x_t^*$. Then we have $Cost_A = 0, Cost_B = |x_t^* - x_{t-1}^*| + f_t(x_t^*)$ and the change in potential $\Delta\Phi = 2(x_t^* - x_t) - 2(x_{t-1}^* - x_t) = 2(x_t^* - x_{t-1}^*)$. Assume the worst case scenario that B is an optimal algorithm and will move toward the center c_t to minimize the cost.

We now consider all of the four possible configurations (the remaining cases are symmetric to the ones mentioned below)

1. c_t - x_t^* - x_t - x_{t-1}^*
2. c_t - x_t - x_t^* - x_{t-1}^*
3. x_t - c_t - x_t^* - x_{t-1}^*
4. x_t - x_{t-1}^* - x_t^* - c_t

The amortized cost is $AC_A = Cost_A + \Delta\Phi = 0 + 2(x_t^* - x_{t-1}^*)$.

Case 1: $AC_A = 0 + (\text{negative } \#) \leq Cost_B \leq 4 \cdot Cost_B$

Case 2: $AC_A = 0 + (\text{negative } \#) \leq Cost_B \leq 4 \cdot Cost_B$

Case 3: $AC_A = 0 + (\text{negative } \#) \leq Cost_B \leq 4 \cdot Cost_B$

Case 4: $AC_A = 0 + (x_t^* - x_{t-1}^*) \leq 2|x_t^* - x_{t-1}^*| + f_t(x_t^*) \leq 2 \cdot Cost_B \leq 4 \cdot Cost_B$

Now fix the position in B and let our algorithm A make a move. Then we have $\Delta\Phi = 2(x_t^* - x_t) - 2(x_t^* - x_{t-1}) = 2(x_{t-1} - x_t)$ and the cost of A is

$$Cost_A = \begin{cases} |x_{t-1} - c_t| + b_t & \text{if } |x_{t-1} - c_t| \leq b_t \\ 2|x_{t-1} - x_t| & \text{else} \end{cases}$$

Note that the algorithm A only walks all the way to the center c_t if the starting position $x_{t-1} \leq b_t$. This is true because there exists an isomorphism between the points on the force field line to the set $\mathbb{R} \setminus [c_t - b_t, c_t + b_t]$ (i.e.) for every point outside the $[c_t - b_t, c_t + b_t]$ neighborhood, we will always end in the second scenario of our algorithm. The idea is for every point outside of the neighborhood on the real line, we can go in the direction toward the center and up $45°$ until hit the force field and then drop back down $90°$ to the real line. This constructs a $45 - 45 - 90$ triangle and satisfies the condition for scenario 2.

Again, we consider all of the four possible configurations

1. c_t - x_t - x_t^* - x_{t-1}
2. c_t - $x_t = x_t^*$ - x_{t-1}
3. c_t - x_t^* - x_t - x_{t-1}
4. c_t - x_t - x_{t-1} - x_t^*

So the amortized cost is

Case 1:

$$AC_A = \begin{cases} (|x_{t-1} - c_t| + b_t) + 2(x_{t-1} - c_t) = 3(x_{t-1} - c_t) + b_t \leq 3b_t + b_t = 4b_t \leq 4 \cdot Cost_B \\ 2|x_{t-1} - x_t| + 2(x_{t-1} - x_t) = 4|x_{t-1} - x_t| = 4f_t(x_t) \leq 4f_t(x_t^*) \leq 4 \cdot Cost_B \end{cases}$$

Case 2:

$$AC_A = \begin{cases} (|x_{t-1} - c_t| + b_t) + 2(x_{t-1} - c_t) = 3(x_{t-1} - x_t^*) + b_t \leq 3b_t + b_t = 4b_t \leq 4 \cdot Cost_B \\ 2|x_{t-1} - x_t| - 2(x_{t-1} - x_t) = 0 \leq 4f_t(x_t^*) \leq 4 \cdot Cost_B \end{cases}$$

Case 3:

$$AC_A = \begin{cases} \text{Does not occur} \\ 2|x_t - x_{t-1}| - 2(x_t - x_{t-1}) = 0 \leq 4 \cdot Cost_B \end{cases}$$

Case 4:

$$AC_A = \begin{cases} (|x_{t-1} - c_t| + b_t) + 2(x_{t-1} - c_t) = 3(x_{t-1} - c_t) + b_t \leq 3b_t + b_t = 4b_t \leq 4 \cdot Cost_B \\ 2|x_{t-1} - x_t| + 2(x_{t-1} - x_t) = 4|x_{t-1} - x_t| \leq 4f_t(x_t^*) \leq 4 \cdot Cost_B \end{cases}$$

Summing over the amortized cost of each case, we get

$$\text{Total Cost}_A + \Phi_{final} - \Phi_{initial} \leq 4(\text{Total Cost}_B)$$

Since both algorithms start at $x_0 = 0$, $\Phi_{initial} = 0$. The potential function can either be 0, negative or positive. In the case that the final potential function is positive, we already saw above that none of them can exceed $4 \cdot Cost_B$ and hence, it cannot exceed the total cost. So we get

$$\text{Total Cost}_A \leq 4(\text{Total Cost}_B)$$

Thus, the algorithm is 4-competitive.

3: Adapt, Adopt, Improve

Let n be the total number of experts, m be the number of mistakes made by the best expert in a contiguous block of trials, M be the number of mistakes of the algorithm, and $w_{i,t}$ be the weight of each expert on day t.

Define the potential function at time t as $\Phi_t = \sum_{i=1}^{n} w_{i,t}$ and Φ_1 as the potential for the first day of the block.

Suppose the algorithm made a mistake. Consider the set X of experts that predicted incorrectly but whose weights did not get halved, (i.e.) each of their weight $w_{i,t} \leq \frac{1}{4}\left(\frac{\sum_{i=1}^{n} w_{i,t}}{n}\right)$. Then it follows that $\frac{\sum_{i\in X} w_{i,t}}{|X|} \leq \frac{1}{4}\left(\frac{\sum_{i=1}^{n} w_{i,t}}{n}\right)$. Since $|X| < n$, we get $\frac{\sum_{i\in X} w_{i,t}}{|X|} \leq \frac{\sum_{i\in X} w_{i,t}}{n} \leq \frac{1}{4}\left(\frac{\sum_{i=1}^{n} w_{i,t}}{n}\right)$. This means $\sum_{i\in X} w_{i,t} \leq \frac{1}{4}\left(\sum_{i=1}^{n} w_{i,t}\right) = \frac{1}{4}\Phi_t$, (i.e.) the total weight of X is less than 1/4 of the total weight on day t.

Since the algorithm made a mistake, that means that more than half of the weights predicted incorrectly. Since all of the experts in X predicted incorrectly and their combined weights is less than $\frac{1}{4}\Phi_t$, then at least 1/4 of the remaining weights came from the experts not in X, and these weights will be halved. So $\Phi_t \leq (1 - \frac{1}{4}\cdot\frac{1}{2})\Phi_{t-1} = \frac{7}{8}\Phi_{t-1}$. After making M mistakes after t days, the potential is

$$\Phi_t \leq \left(\frac{7}{8}\right)^M \Phi_1$$

Let $w_{best,1}$ be the weight of the best expert at the beginning. Since the best expert can make at most m mistakes, its weight must be at least

$$(w_{best,1})\left(\frac{1}{2}\right)^m \geq \left(\frac{\Phi_1}{4n}\right)\left(\frac{1}{2}\right)^m$$

The lower bound is due to the fact that the best expert must have the weight at least 1/4 of the average starting weight. So we have

$$\begin{aligned}
\left(\frac{1}{4n}\right)\left(\frac{1}{2}\right)^m \Phi_1 &\leq \left(\frac{7}{8}\right)^M \Phi_1 \\
\left(\frac{1}{2}\right)^m &\leq (4n)\left(\frac{7}{8}\right)^M \\
-m &\leq \log_2 4 + \log_2 n + M \log_2\left(\frac{7}{8}\right) \\
-m &\leq \log_2 4 + \log_2 n - M \log_2\left(\frac{8}{7}\right) \\
M &\leq \log_2\left(\frac{8}{7}\right)(\log_2 4 + m + \log_2 n) \\
&= O(m + \log n)
\end{aligned}$$