Quantum Computation and Quantum Information by Michael A. Nielsen and Isaac L. Chuang

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Chapter 2: Introduction to quantum mechanics

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Exercise 2.1: (Linear Dependence Example)

We can observe that:

$$(1,-1) + (1,2) - (2,1) = (0,0) = \mathbf{0}$$

Thus, the set of three provided vectors is linearly dependent.

Exercise 2.2: (Matrix Representations: Example)

V is a vector space with basis $\{|0\rangle, |1\rangle\}$.

$$A|0\rangle = |1\rangle \implies A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \implies \text{ first column of A is } |1\rangle$$

as multiplication of a 2 by 2 matrix by $|0\rangle$ is just extracting the first column. Also

$$A|1\rangle = |0\rangle \implies A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \implies \text{ second column of A is } |0\rangle$$

Thus $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We know, from (2.7), that $|v_1\rangle = |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $|v_2\rangle = |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ span the $\mathbb{C}^2$, they also form a basis for $\mathbb{C}^2$, as they are linearly independent (the only solution for $c_1 |+\rangle + c_2 |-\rangle = 0$ is trivial).

So we can form following linear operators A_i from $\mathbb{C}^2$ to $\mathbb{C}^2$:

1. $A_1 |0\rangle = |+\rangle$ and $A_1 |1\rangle = |-\rangle$
2. $A_2 |0\rangle = |-\rangle$ and $A_2 |1\rangle = |+\rangle$
3. $A_3 |+\rangle = |0\rangle$ and $A_3 |-\rangle = |1\rangle$
4. $A_4 |+\rangle = |1\rangle$ and $A_4 |-\rangle = |0\rangle$

We can compute $A_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. We can trivially see, that $A_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

A case $A_3 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ is a bit more interesting, because we cannot just extract the columns. We can write in matrix form.

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = I$$

We can multiply both sides on the right by a transpose of the second matrix, as it is orthogonal.

$$A_3 = \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Exercise 2.3: (Matrix representation for operator products)

We have the following combination of linear operators and vector spaces $V \xrightarrow{A} W \xrightarrow{B} X$. From (2.12) we can write:

$$A |v_i\rangle = \sum_j A_{ji} |w_j\rangle, \text{ and } B |w_j\rangle = \sum_k B_{kj} |x_k\rangle \tag{1}$$

$$BA |v_j\rangle = B(\sum_j A_{ji} |v_j\rangle) \text{ from (1)}$$

$$\begin{aligned} B(\sum_j A_{ji} |v_j\rangle) &= \sum_j A_{ji} B(|v_j\rangle) \text{ from linearity of inputs in (2.10)} \\ &= \sum_j A_{ji} \sum_k B_{kj} |x_k\rangle \text{ from (1)} \\ &= \sum_k (\sum_j B_{kj} A_{ji}) |x_k\rangle \text{ rearranging the sum order} \end{aligned} \tag{2}$$

If we look from the point of view of a linear operation from V to X, then we need to have some matrix C, that would give us:

$$C\,|v_i\rangle = \sum_k C_{ki}\,|x_k\rangle$$

It is precisely

$$BA_{ki} = \sum_j B_{kj}A_{ji} \text{ from (2)}$$

Exercise 2.4: (Matrix representation for identity)

Exercise 2.5

Exercise 2.6

Here we need to show, that any inner product $(\cdot,\cdot)$ is conjugate-linear in the first argument.

$$\Big(\sum_i \lambda_i\,|w_i\rangle, |v\rangle\Big) = \Big(|v\rangle\,, \sum_i \lambda_i\,|w_i\rangle\Big)^* \text{ from (2.13 (2))}$$

$$= \Big(\sum_i \lambda_i(|v\rangle\,, |w_i\rangle)\Big)^* \text{ from linearity of second inner-product argument (2.13 (1))}$$

$$= \sum_i \left(\lambda_i(|v\rangle\,, |w_i\rangle)\right)^* \text{ as conjugate of a sum is a sum of conjugates}$$

$$= \sum_i \lambda_i^*(|v\rangle\,, |w_i\rangle)^* \text{ as conjugate of a product is a product of conjugates}$$

$$= \sum_i \lambda_i^*(|w_i\rangle\,, |v\rangle) \text{ from (2.13 (2))}$$

Exercise 2.7

$(|v\rangle\,, |w\rangle) = ((1,-1),(1,1)) = \begin{bmatrix}1^* - 1^*\end{bmatrix}\begin{bmatrix}1^*\\1^*\end{bmatrix} = \begin{bmatrix}1 - 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1-1 = 0$. Precisely following (2.14) we establish, that $|w\rangle$ and $|v\rangle$ are orthogonal. Their normalized forms are $\frac{v}{\||v\|\rangle} = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ and $\frac{w}{\||w\|\rangle} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, as both have a norm of $\sqrt{2}$.

Exercise 2.8

Exercise 2.9: (Pauli operators and the outer product)

Consider:

- $|0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
- $|0\rangle\langle 1| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$
- $|1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$
- $|1\rangle\langle 1| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

As if we are just setting a specific bit of the matrix (each outer product only has 1 non-zero value).

- $X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$
- $Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = i\,|0\rangle\langle 1| - i\,|1\rangle\langle 0|$
- $Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

Exercise 2.10

Exercise 2.11: (Eigendecomposition of the Pauli matrices)

1. X $= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \implies$ characteristic equation is $det(\begin{bmatrix}-\lambda & 1\\1 & -\lambda\end{bmatrix}) = \lambda^2 - 1$ and the solution is $\lambda = \pm 1$.
   (a) $\lambda = 1$, Solving $X\,|v\rangle = |v\rangle \implies \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix} \implies \begin{bmatrix}v_2\\v_1\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$. Thus the eigenvector is any scalar multiple of $\begin{bmatrix}1\\1\end{bmatrix}$
   (b) $\lambda = -1$, Solving $X\,|v\rangle = -\,|v\rangle \implies \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}-v_1\\-v_2\end{bmatrix} \implies \begin{bmatrix}v_2\\v_1\end{bmatrix} = \begin{bmatrix}-v_1\\-v_2\end{bmatrix}$. Thus the eigenvector is any scalar multiple of $\begin{bmatrix}1\\-1\end{bmatrix}$

2. Y $= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \implies$ characteristic equation is $det(\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix}) = \lambda^2 - 1$ and the solution is $\lambda = \pm 1$

   (a) $\lambda = 1$, Solving $Y\left|v\right\rangle = i\left|v\right\rangle \implies \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies \begin{bmatrix} -iv_2 \\ iv_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

   (b) $\lambda = -1$, Solving $Y\left|v\right\rangle = -i\left|v\right\rangle \implies \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix} \implies \begin{bmatrix} -iv_2 \\ iv_1 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$

3. Z $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \implies$ characteristic equation is $\det(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}) = \lambda^2 - 1$, and the solution is $\lambda = \pm 1$.

   (a) $\lambda = 1$, Solving $Z\left|v\right\rangle = \left|v\right\rangle \implies \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Thus the eigenvector is any scalar multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   (b) $\lambda = -1$, Solving $Z\left|v\right\rangle = -\left|v\right\rangle \implies \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix} \implies \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$. Thus the eigenvector is any scalar multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Exercise 2.12

Consider $A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} \implies \det(A - \lambda I) = (1-\lambda)^2 = 0 \implies \lambda = 1$ is the only eigenvalue.

Exercise 2.13

$$(\left|w\right\rangle\left\langle v\right|)^\dagger = \left\langle v\right|^\dagger \left|w\right\rangle^\dagger \text{ from the fact, that } (AB)^\dagger = B^\dagger A^\dagger.$$
$$\left\langle v\right|^\dagger \left|w\right\rangle^\dagger = \left|v\right\rangle\left\langle w\right| \text{ by convention}$$

Exercise 2.14: (Anti-linearity of the adjoint)

$$\text{Consider } (\sum_i a_i A_i)^* = \sum_i (a_i A_i)^* = \sum_i a_i^* A_i^* \text{ as conjugation is linear}$$

$$\text{Now consider } ((\sum_i a_i A_i)^*)^T = (\sum_i a_i^* A_i^*)^T = \sum_i a_i^* (A_i^*)^T = \sum_i a_i^* A_i^\dagger \text{ as } a_i \text{ is a scalar}$$

Thus we have established, that adjoint operation is anti-linear, namely: $\sum_i a_i {A_i}^\dagger = \sum_i a_i^* A_i^\dagger$

Exercise 2.15

$(A^\dagger)^\dagger = ((A^\dagger)^T)^* = (((A^T)^*)^T)^* = (((A^T)^*)^*)^T$, as taking a conjugate and transposing can easily be interchanged. Conjugating each element in the matrix twice just yields the same initial value, as $(z^*)^* = z, \forall z \in \mathbb{C}$ So we have: $((A^T)^T) = A$ from the definition of the transpose.

Exercise 2.16

We can write out P^2 explicitly using (2.35) $P^2 = (\sum_{i=1}^{k} |i\rangle \langle i|)(\sum_{j=1}^{k} |j\rangle \langle j|) = \sum_{i=1}^{k} \sum_{j=1}^{k} |i\rangle \langle i| |j\rangle \langle j|$

As we have orthonormality $\langle i| |j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ the above double sum colapses into

a single sum $\sum_{i=1}^{k} |i\rangle \langle i| = P$

Exercise 2.17

!TODO: ASK A QUESTION DID I SPLIT THE IFF CORRECTLY HERE!

$\implies$ Hermitian matrix has real eigenvalues.

$$\text{Consider } (|v\rangle, H|v\rangle) = (H^\dagger |v\rangle, |v\rangle) = (H|v\rangle, |v\rangle) \tag{1}$$

$(|v\rangle, H|v\rangle) = (|v\rangle, \lambda|v\rangle) = \lambda(|v\rangle, |v\rangle)$ from (2.13)

From (1) $(|v\rangle, H|v\rangle) = (\lambda|v\rangle, |v\rangle) = \lambda^*(|v\rangle, |v\rangle)$ from (2.15).

We have $\lambda(|v\rangle, |v\rangle) = (|v\rangle, H|v\rangle) = \lambda^*(|v\rangle, |v\rangle) \implies \lambda^* = \lambda \implies$ eigenvalue is real

As complex conjugate of a complex number is equal to itself and thus imaginary part is zero.

$\impliedby$ A normal matrix that has real eigenvalues is Hermitian

From Theorem 2.1 on page 71 we know, that any normal operator on a vector space V is diagonal with respect to some orthonormal basis for V. Thus it has a diagonal representation. Namely:

$A = \sum_i \lambda_i |i\rangle \langle i|$, where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A.

with corresponding eigenvalues λ_i. Consider $A^\dagger = (\sum_i \lambda_i |i\rangle \langle i|)^\dagger = \sum_i \lambda_i^* \langle i|^\dagger |i\rangle^\dagger$

$$= \sum_i \lambda_i |i\rangle \langle i| \text{ as } \lambda_i \text{ is real.} \implies A^\dagger = A \implies A \text{ is Hermitian}$$

Exercise 2.18

$$\text{For Unitary } U, \text{ following holds for its eigenpairs: } U|v\rangle = \lambda |v\rangle \implies \|U|v\rangle\|_2 = \|\lambda |v\rangle\|_2 \tag{2}$$

$$\text{Say } |\psi\rangle = U|v\rangle \implies \langle\psi| = |\psi\rangle^\dagger = |v\rangle^\dagger U^\dagger = \langle\psi| U^\dagger \tag{3}$$

$$\text{From (2.16) we know, that } \|U|v\rangle\|_2 = \||\psi\rangle\|_2 = \sqrt{\langle\psi||\psi\rangle} = \sqrt{\langle v| U^\dagger U |v\rangle} = \sqrt{\langle v||v\rangle} = \||v\rangle\|_2$$

Thus unitary transformations preserve the inner product and, therefore, the Euclidean norm of vectors.

We can rewrite (1) as $\||v\rangle\|_2 = \|\lambda|v\rangle\|_2 \implies |\lambda| = 1$. It is a complex number on unit circle.

$\lambda = \cos(\theta) + i\sin(\theta) = e^{i\theta}$ from Eulers Formula.

Exercise 2.19: (Pauli matrices: Hermitian and unitary)

Matrix A is Hermitian if $A^\dagger = A$, matrix B is unitary if $B^\dagger B = BB^\dagger = I$

Now let's consider each of the Pauli matrices

1. $\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   $I^\dagger = (I^T)^* = I$ as identity is symmetric, and complex conjugate of a real value is just itself, thus I is Hermitian.

   $I^\dagger I = II = I = II^\dagger \implies I$ is unitary

2. $\sigma_1 = \sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   $X^\dagger = (X^T)^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* = X \implies X$ is Hermitian

   $X^\dagger X = XX = XX^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \implies X$ is unitary

3. $\sigma_2 = \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

   $Y^\dagger = (Y^T)^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y \implies Y$ is Hermitian

   $Y^\dagger Y = YY = YY^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \implies Y$ is unitary

4. $\sigma_3 = \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$Z^\dagger = (Z^T)^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^* = Z \implies Z$ is Hermitian

$Z^\dagger Z = ZZ = ZZ^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^2 \end{bmatrix} = I \implies Z$ is unitary

Exercise 2.20: (Basis changes)

Exercise 2.21

Exercise 2.22

Exercise 2.23

Exercise 2.24: (Hermiticity of positive operators)

$$\forall A, A = \frac{A}{2} + \frac{A}{2} = \frac{A}{2} + \frac{A}{2} + \frac{A^\dagger}{2} - \frac{A^\dagger}{2} = \frac{A + A^\dagger}{2} + \frac{A - A^\dagger}{2} = \frac{A + A^\dagger}{2} + i\frac{A - A^\dagger}{2i}.$$

$$\text{Call } B = \frac{A + A^\dagger}{2}, C + \frac{A - A^\dagger}{2i} \implies A = B + iC.$$

$$\text{Consider } B^\dagger = (\frac{A + A^\dagger}{2})^\dagger = (\frac{A^\dagger + (A^\dagger)^\dagger}{2}) = (\frac{A^\dagger + A}{2}) = B \implies B \text{ is Hermitian}$$

$$A \text{ is positive } \implies (|v\rangle, A|v\rangle) \in \mathbb{R}_{\geq 0}, \forall |v\rangle. \text{ Rewrite as } (|v\rangle, B|v\rangle) + i(|v\rangle, C|v\rangle)$$

as inner product is linear in the second argument from (2.13)

$(|v\rangle, C|v\rangle) = 0$ as for any $|v\rangle$ quantity$(|v\rangle, A|v\rangle)$ is real.

We have $(|v\rangle, A|v\rangle) = (|v\rangle, B|v\rangle) \implies (|v\rangle, (A - B)|v\rangle) = 0$ from (2.13)

Exercise 2.25

Consider inner product $(|v\rangle, A^\dagger A|v\rangle)$. Call $A|v\rangle$ to be $|\psi\rangle$, so we have $(|v\rangle, A^\dagger|\psi\rangle) = (A|v\rangle, |\psi\rangle)$ from the definition of the Hermitian conjugate in (2.32), we get: $(|\psi\rangle, |\psi\rangle) \geq 0$ from the positivity property of inner product described on page 65.

Exercise 2.26

$$|\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle\,, \text{ analogously } |\psi\rangle^{\otimes 3} = |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$$

Explicitly:

$$|\psi\rangle^{\otimes 2} = (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \otimes (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) = \frac{1}{2}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)) \text{ from (2.42)}$$

$$= \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) \text{ from distributive properties (2.43. 2.44)}$$

Using Kronecker product:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, so\,|\psi\rangle^{\otimes 2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Explicitly:

$$|\psi\rangle^{\otimes 3} = (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \otimes (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \otimes (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) = \frac{1}{2\sqrt{2}}(((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)) \otimes (|0\rangle + |1\rangle))$$

$$= \frac{1}{2\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |1\rangle +$$
$$+ |1\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$$

Using Kronecker product:

$$|\psi\rangle^{\otimes 3} = |\psi\rangle^{\otimes 2} \otimes |\psi\rangle = (\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}) \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 2.27

1.

$$X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

2.

$$I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3.

$$X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

As parts 2 and 3 are different, we can conclude, that tensor product is not commutative.

Exercise 2.28

Consider $(A \otimes B)^*$

Exercise 2.29

Explicitly

Consider unitary A, B s.t $AA^\dagger = I, BB^\dagger = I$. Assume A is $m \times n$ matrix, and B is $p \times q$ matrix.

Then $A^\dagger$ is a $n \times m$ matrix, and $B^\dagger$ is a $q \times p$ matrix, as we are transposing, and.

taking a conjugate of each element does not change the dimensions.

$$(A \otimes B)(A \otimes B)^\dagger = (A \otimes B)(A^\dagger \otimes B^\dagger) \text{ from (2.53)}$$

From (2.50) $A \otimes B$ has mp rows and nq columns, and $A^\dagger \otimes B^\dagger$ has qn rows and pm columns.

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \text{ where } a_{ij} \text{ are corresponding elements of } A \tag{4}$$

$$A^\dagger \otimes B^\dagger = \begin{bmatrix} a^\dagger_{11}B^\dagger & a^\dagger_{12}B^\dagger & \dots & a^\dagger_{1m}B^\dagger \\ a_{21}B^\dagger & a_{22}B^\dagger & \dots & a_{2m}B^\dagger \\ \vdots & \vdots & & \vdots \\ a_{n1}B^\dagger & a_{n2}B^\dagger & \dots & a_{nm}B^\dagger \end{bmatrix} \text{ where } a^\dagger_{ij} \text{ are corresponding elements of } A^\dagger \tag{5}$$

As number of block rows in (1) is matching number of block columns in (2), and each block in (1) has the same number of columns as each block has rows in (2) then we can use Block Matrix Multiplication Formula.

$$C_{ij} = \sum_k a_{ik}Ba^\dagger_{kj}B^\dagger = \sum_k a_{ik}a^\dagger_{kj}BB^\dagger = \sum_k a_{ik}a^\dagger_{kj}I_p \text{ as } B \text{ is unitary.} \tag{6}$$

We know, that $AA^\dagger = I_m \implies (AA^\dagger)_{ij} = (I_m)_{ij} \implies \sum_k a_{ik}a^\dagger_{kj} = \delta_{ij}$

$$\text{Thus, } C_{ij} = \begin{cases} I_p & i = j \\ 0 & i \neq j \end{cases} \implies (A\otimes B)(A^\dagger \otimes B^\dagger) = \begin{bmatrix} I_p & 0 & \dots & 0 \\ 0 & I_p & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I_p \end{bmatrix} = I_{mp}$$

One could follow the same line of arguments as above to determine, that $(A^\dagger \otimes B^\dagger)(A\otimes B) = I_{qn}$

Thus tensor product of two unitary operators is unitary.

Mixed-product property

Consider matrices A, B, C, D, where A is $m \times n$, B is $p \times q$, C is $n \times r$, D is $q \times s$. It is important, that number of columns in A, B matches the number of rows in C, D respectively. So when we compute $(A \otimes B)$ we have a matrix of size $mp \times qn$, $(C \otimes D)$ we have a matrix of size $nq \otimes rs$, and when can multiply them, as the dimensions do match, we expect matrix of size $mp \times rs$ as a result.

$$(A\otimes B)(C\otimes D) = \begin{bmatrix} \sum_k a_{1k}Bc_{k1}D & \sum_k a_{1k}Bc_{k2}D & \dots & \sum_k a_{1k}Bc_{kr}D \\ \sum_k a_{2k}Bc_{k1}D & \sum_k a_{2k}Bc_{k2}D & \dots & \sum_k a_{2k}Bc_{kr}D \\ \vdots & \vdots & & \vdots \\ \sum_k a_{mk}Bc_{k1}D & \sum_k a_{mk}Bc_{k2}D & \dots & \sum_k a_{mk}Bc_{kr}D \end{bmatrix} =$$

$$= \begin{bmatrix} (AC)_{11}BD & (AC)_{12}BD & \dots & (AC)_{1r}BD \\ (AC)_{21}BD & (AC)_{22}BD & \dots & (AC)_{2r}BD \\ \vdots & \vdots & & \vdots \\ (AC)_{m1}BD & (AC)_{m2}BD & \dots & (AC)_{mr}BD \end{bmatrix} = (AC) \otimes (BD).$$

Using properties of tensor products

$$(A\otimes B)(A^\dagger \otimes B^\dagger) = (AA^\dagger) \otimes (BB^\dagger) = I_m \otimes I_p = I_{mp}$$

with the number of rows and columns between the original matrix and the transpose matching trivially.

Exercise 2.30

Consider Hermitian A, B s.t $A^\dagger = A, B^\dagger = B$. Now form $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$ from $(2.53) = A \otimes B$

Thus we have established, that the tensor product of two Hermitian matrices is Hermitian, as $(A\otimes B)^\dagger = A\otimes B$

Exercise 2.31

Exercise 2.32

Exercise 2.33

Computing $H^{\otimes 2}$

$$H^{\otimes 2} = H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & 1\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\\ 1\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & -1\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$

Base case n = 1

$$\text{Substituting } |1\rangle \text{ with } \begin{bmatrix}1\\0\end{bmatrix} \text{ and } |0\rangle \text{ with } \begin{bmatrix}0\\1\end{bmatrix}$$

$$\text{(2.54) becomes } H = \frac{1}{\sqrt{2}}[(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix})\begin{bmatrix}1 & 0\end{bmatrix} + (\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix})\begin{bmatrix}0 & 1\end{bmatrix}]$$

$$= \frac{1}{\sqrt{2}}(\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}) = \frac{1}{\sqrt{2}}(\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

Which is indeed the classical matrix representation of the Hadamard operator

Indiction hypothesis n = k

$$\text{Assume for } n = k \text{ the following holds } H^{\otimes k} = \frac{1}{\sqrt{2^k}}\sum_{x,y\in\{0,1\}^k}(-1)^{xy}|x\rangle\langle y|$$

The subscript in the sum directly above just means, that x and y are binary strings of length k

Induction step n = k + 1

$$H^{\otimes(k+1)} = (\frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|]) \otimes \frac{1}{\sqrt{2^k}}\sum_{x,y\in\{0,1\}^k}(-1)^{xy}|x\rangle\langle y|$$

$$= (\frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \otimes H^{\otimes k}$$

$$= \frac{1}{\sqrt{2^1}}\sum_{x,y\in\{0,1\}}(-1)^{xy}|x\rangle\langle y| \otimes \frac{1}{\sqrt{2^k}}\sum_{x,y\in\{0,1\}^k}(-1)^{xy}|x\rangle\langle y| \stackrel{*}{=}$$

We compute xy to determine the sign by performing the binary inner product

xy	x	y
0	0	0
0	0	1
0	1	0
1	1	1

Table 1: $xy \in \{0,1\}$

$xy = x_1y_1 + x_2y_2 \mod 2$	$x = x_1x_2$	$y = y_1y_2$
0	00	00
0	00	01
0	00	10
0	00	11
0	01	00
1	01	01
0	01	10
1	01	11
0	10	00
0	10	01
1	10	10
1	10	11
0	11	00
1	11	01
1	11	10
0	11	11

Table 2: Binary inner product mod 2 for $x, y \in \{0,1\}^2$

$$\text{We can extend (2.43, 2.44) to } \sum_i |a_i\rangle \otimes \sum_i |b_i\rangle = \sum_i \sum_j (|a_i\rangle \otimes |b_i\rangle) \tag{1}$$

$$= \frac{1}{\sqrt{2^{k+1}}} \sum_{x',y' \in \{0,1\}} \sum_{x'',y'' \in \{0,1\}^k} ((-1)^{x'y'} |x'\rangle \langle y'|) \otimes ((-1)^{x''y''} |x''\rangle \langle y''|) \text{ from (1)}$$

$$= \frac{1}{\sqrt{2^{k+1}}} \sum_{x',y' \in \{0,1\}} \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x'y'}(-1)^{x''y''}[(|x'\rangle \langle y'|) \otimes (|x''\rangle \langle y''|)] \text{ from (2.42)}$$

We can explicitly write out all of the combinations, as x', y' can be wither 0 or 1, so we have:

$$\frac{1}{\sqrt{2^{k+1}}}\Big(\sum_{x'',y'' \in \{0,1\}^k} (-1)^{0\cdot 0}(-1)^{x''y''} |0\rangle \langle 0| \otimes |x''\rangle \langle y''| + \sum_{x'',y'' \in \{0,1\}^k} (-1)^{0\cdot 1}(-1)^{x''y''} |0\rangle \langle 1| \otimes |x''\rangle \langle y''| +$$

$$+ \sum_{x'',y'' \in \{0,1\}^k} (-1)^{1\cdot 0}(-1)^{x''y''} |1\rangle \langle 0| \otimes |x''\rangle \langle y''| + \sum_{x'',y'' \in \{0,1\}^k} (-1)^{1\cdot 1}(-1)^{x''y''} |1\rangle \langle 1| \otimes |x''\rangle \langle y''|\Big) =$$

$$\frac{1}{\sqrt{2^{k+1}}}\Big(\sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''} |0\rangle \langle 0| \otimes |x''\rangle \langle y''| + \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''} |0\rangle \langle 1| \otimes |x''\rangle \langle y''| +$$

$$+ \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''} |1\rangle \langle 0| \otimes |x''\rangle \langle y''| + \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''+1} |1\rangle \langle 1| \otimes |x''\rangle \langle y''|\Big) =$$

$$= \frac{1}{\sqrt{2^{k+1}}}\Big(\sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''} \begin{bmatrix} |x''\rangle \langle y''| & 0 \\ 0 & 0 \end{bmatrix} + \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''} \begin{bmatrix} 0 & |x''\rangle \langle y''| \\ 0 & 0 \end{bmatrix} +$$

$$+ \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''} \begin{bmatrix} 0 & 0 \\ |x''\rangle \langle y''| & 0 \end{bmatrix} + \sum_{x'',y'' \in \{0,1\}^k} (-1)^{x''y''+1} \begin{bmatrix} 0 & 0 \\ 0 & |x''\rangle \langle y''| \end{bmatrix}\Big)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} H^{\otimes k} & H^{\otimes k} \\ H^{\otimes k} & -H^{\otimes k} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes H^{\otimes k} = H^{\otimes (k+1)}$$

.

From the assumption for $n = k$ we have established a fact for $n = k + 1$
thus proving by principal of mathematical induction.

Exercise 2.34

Eigenvalues of M are: $\lambda_{1,2} = 4 \pm \sqrt{16-7} = 1, 7$. Corresponding eigenvectors: $|v_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Diagonal representation of A is: } \frac{|v_1\rangle\langle v_1|}{\|v_1\|^2} + \frac{7\,|v_2\rangle\langle v_2|}{\|v_2\|^2}$$

- **Square root**

$$\sqrt{A} \text{ is defined in (2.1.8) to be } \frac{|v_1\rangle\langle v_1|}{\|v_1\|^2} + \frac{\sqrt{7}\,|v_2\rangle\langle v_2|}{\|v_2\|^2} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\sqrt{7}}{2} & \frac{-1+\sqrt{7}}{2} \\ \frac{-1+\sqrt{7}}{2} & \frac{1+\sqrt{7}}{2} \end{bmatrix}$$

- **Logarithm**

$$\text{Equivalently } \log(A) = \frac{\log 1\,|v_1\rangle\langle v_1|}{\|v_1\|^2} + \frac{\log 7\,|v_2\rangle\langle v_2|}{\|v_2\|^2} = \begin{bmatrix} \frac{\log 7}{2} & \frac{\log 7}{2} \\ \frac{\log 7}{2} & \frac{\log 7}{2} \end{bmatrix} \text{ as } \log 1 = 0 \text{ in any base.}$$

Exercise 2.35: (Exponential of the Pauli matrices)

$$\text{Consider } \vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i. \text{ Squaring we get: } \sum_{i=1}^{3}\sum_{j=1}^{3} v_i\sigma_i v_j\sigma_j = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j \sigma_i\sigma_j =$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j (\delta_{ij} I + i\sum_{l=1}^{3} \epsilon_{ijl}\sigma_l) \text{ from (2.78)}$$

$$= \underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3} v_i v_j \delta_{ij} I}_{(1)} + \underbrace{i\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3} \epsilon_{ijl} v_i v_j \sigma_l}_{(2)}$$

$$(1) = \sum_{i=1}^{3} v_i^2 I \text{ as only the terms with the same index last, as } \delta_{ij} = 0 \text{ for } i = j.$$

$$\text{We know, that } i\text{'th component of } \vec{a}\times\vec{b} \text{ is } \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} a_j b_k. \text{ Rewriting (2), get:}$$

$$i\sum_{l=1}^{3}(\sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ijk} v_i v_j)\sigma_l = i\sum_{l=1}(\vec{v}\times\vec{v})_l\sigma_l = 0 \text{ as a cross product of a vector with itself is always0.}$$

$$\text{So } (\vec{v}\cdot\vec{\sigma})^2 = \|v\|_2 I = I \text{ as } \vec{v} \text{ is a unit vector.}$$

Say $A = \vec{v}\cdot\vec{\sigma}$. We know: $AA = I$. Consider eigenvalues$A|v\rangle = \lambda|v\rangle \implies AA|v\rangle = A\lambda|v\rangle \implies$

$$I|v\rangle = \lambda A|v\rangle \implies I|v\rangle = \lambda^2|v\rangle \implies \lambda^2 = 1 \implies \lambda = \pm 1.$$

Say corresponding eigenvectors are $|v_{+1}\rangle, |v_{-1}\rangle$ then we have following spectral decomposition:

$$A = |v_{+1}\rangle\langle v_{+1}| - |v_{-1}\rangle\langle v_{-1}| \text{ and } \exp(i\theta\vec{v}\cdot\vec{\sigma}) = \exp(i\theta A) = e^{i\theta}|v_{+1}\rangle\langle v_{+1}| + e^{-i\theta}|v_{-1}\rangle\langle v_{-1}| \, (3)$$

$$A^2 = |v_{+1}\rangle\langle v_{+1}||v_{+1}\rangle\langle v_{+1}| - |v_{+1}\rangle\langle v_{+1}||v_{-1}\rangle\langle v_{-1}| - |v_{-1}\rangle\langle v_{-1}||v_{+1}\rangle\langle v_{+1}| + |v_{-1}\rangle\langle v_{-1}||v_{-1}\rangle\langle v_{-1}|.$$

$\langle v_{-1}||v_{-1}\rangle = 1, \langle v_{+1}||v_{+1}\rangle = 1, \langle v_{-1}||v_{+1}\rangle = 0$ as $|v_{+1}\rangle$ and $|v_{-1}\rangle$ are orthogonal $\implies$ above is:

$$A^2 = I = |v_{+1}\rangle\langle v_{+1}| - 0 + 0 + |v_{-1}\rangle\langle v_{-1}|$$

We can see, that:

- $|v_{+1}\rangle\langle v_{+1}| = \frac{A+I}{2}$
- $|v_{-1}\rangle\langle v_{-1}| = \frac{I-A}{2}$

So (3) can be rewritten as: $\exp(i\theta\vec{v}\cdot\vec{\sigma}) = e^{i\theta}\dfrac{A+I}{2} + e^{-i\theta}\dfrac{I-A}{2} = A\dfrac{e^{i\theta}-e^{-i\theta}}{2} + I\dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$= Ai\sin(\theta) + I\cos\theta = \cos\theta I + i\sin\theta\vec{v}\cdot\vec{\sigma}$ from Euler's formula.

Note: It is also possible to see, that $AA = I$ can be computed explicitly without "Levi-Civita symbol".

$$A = \vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i = v_1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + v_2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + v_3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}0 & v_1\\ v_1 & 0\end{bmatrix} + \begin{bmatrix}0 & -iv_2\\ iv_2 & 0\end{bmatrix} + \begin{bmatrix}v_3 & 0\\ 0 & -v_3\end{bmatrix} = \begin{bmatrix}v_3 & v_1 - iv_2\\ v_1 + iv_2 & -v_3\end{bmatrix}$$

$$A^2 = \begin{bmatrix}v_3^2 + v_1^2 - iv_1v_2 + iv_1v_2 - iv_2 & v_3v_1 - iv_2v_3 - v_1v_3 + iv_2v_3\\ v_1v_3 + iv_2v_3 - v_3v_1 - iv_2v_3 & v_1^2 + iv_1v_2 - iv_1v_2 - i^2v_2 + v_3\end{bmatrix}$$

$$= \begin{bmatrix}v_1^2 + v_2^2 + v_3^2 & 0\\ 0 & v_1^2 + v_2^2 + v_3^2\end{bmatrix} = I.$$

as $\vec{v}$ is a unit vector.

Exercise 2.36

- $X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} 0 + 0 = 0 \implies \mathrm{tr}(X) = 0.$
- $Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} 0 + 0 = 0 \implies \mathrm{tr}(Y) = 0.$
- $Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} 1 - 1 = 0 \implies \mathrm{tr}(Z) = 0.$

Exercise 2.37: (Cyclic property of the trace)

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} AB_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} \text{ from the definition of matrix multiplication}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ji}A_{ij}, \text{ as } B_{ji} \text{ and } A_{ij} \text{ are elements of the matrix and thus commutative scalars}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji}A_{ij} \textbf{ !!TODO!! ASK A QUESTION}$$

$$= \sum_{j=1}^{n}(BA)_{jj} = \operatorname{tr}(BA)$$

Exercise 2.38: (Linearity of the trace)

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n}(A+B)_{ii} = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = \operatorname{tr}(A) + \operatorname{tr}(B), \text{ as matrix addition is entry-wise.}$$

$$\text{For } z \in \mathbb{C} \text{ consider } \operatorname{tr}(zA) = \sum_{i=1}^{n}(zA)_{ii} = \sum_{i=1}^{n} zA_{ii} = z\sum_{i=1}^{n} A_{ii} = z\operatorname{tr}(A)$$

Exercise 2.39: (The Hilbert–Schmidt inner product on operators)

Exercise 2.40: (Commutation relations for the Pauli matrices)

1. $[X,Y] = XY - YX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\0 & i\end{bmatrix} = \begin{bmatrix}2i & 0\\0 & -2i\end{bmatrix} = 2i\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = 2iZ$
2. $[Y,Z] = YZ - ZY = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}0 & i\\i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\-i & 0\end{bmatrix} = \begin{bmatrix}0 & 2i\\2i & 0\end{bmatrix} = 2i\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = 2iX$
3. $[Z,X] = ZX - XZ = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix} = 2i\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = 2iY$

Note, that ϵ_{jkl} in (2.74) is Levi-Civita symbol.

Exercise 2.41: (Anti-commutation relations for the Pauli matrices)

1. $\{\sigma_1, \sigma_2\} = XY + YX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$
2. $\{\sigma_2, \sigma_3\} = YZ + ZY = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$
3. $\{\sigma_3, \sigma_1\} = ZX + XZ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$
4. $\sigma_1^2 = XX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$
5. $\sigma_2^2 = YY = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$
6. $\sigma_3^2 = ZZ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Exercise 2.42

$$\text{Consider } \frac{[A, B] + \{A, B\}}{2} = \frac{AB - BA + AB + BA}{2} \text{ from (2.66), (2.67)}$$
$$= \frac{2AB}{2} = AB$$

Exercise 2.43

$$\text{Note, that Kronicker delta is defined as } \delta_{jk} = \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k. \end{cases}$$

1. Consider the case $j \neq k$, then from (2.74, 2.75) $[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\} = 2i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l + 0$

   Expanding the commutator and anti-commutator using their definitions we get:

$$\sigma_j\sigma_k - \sigma_k\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_j = 2i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l \implies 2\sigma_j\sigma_k = 2i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l \implies \sigma_j\sigma_k = i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l$$

$$\delta_{jk} = 0 \text{ as } j \neq k \implies \delta_{jk} I = 0 \implies \sigma_j \sigma_k = \delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l$$

2. Now consider the case $j = k \implies \delta_{jk} = 1$ and from (2.76) $\sigma_{jk}\sigma_{jk} = I \implies \sigma_{jk} = \delta_{jk} I$.

$$\text{Consider } \sigma_j \sigma_k = 2i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l = 2i(\epsilon_{jk1}\sigma_1 + \epsilon_{jk2}\sigma_2 + \epsilon_{jk2}\sigma_3 z) = 2i(0) = 0$$

because in all the cases ϵ would have a repeating index, as $j = k$.

$$\text{We have established, that } \sigma_{jk} = \delta_{jk} I + \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l \ \forall j, k$$

Exercise 2.44

$$[A, B] = 0 \implies AB - BA = 0, \{A, B\} = 0 \implies AB + BA = 0 \text{ from definitions (2.66, 2.67)}$$

Then their sum $[A, B] + \{A, B\} = 2AB = 0$ We now multiply both sides on the left by A^{-1}

$$A^{-1} 2AB = A^{-1} 0 \implies 2B = 0 \implies B = 0.$$

Exercise 2.45

Consider Hermitian conjugate of a commutator between two operators A and B

$$[A, B]^\dagger = (AB - BA)^\dagger \text{ from (2.66)}$$

$$= ((AB - BA)^T)^* \text{ from definition of Hermitian conjugate}$$

$$= ((AB)^T - (BA)^T)^* = (B^T A^T - A^T B^T)^* = (B^T)^* (A^T)^* - (A^T)^* (B^T)^*$$

$$= B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$$

Exercise 2.46

Consider $[A, B] = AB - BA = -(-AB + BA) = -(BA - AB) = -[B, A]$

Exercise 2.47

A, B are Hermitian, then each is equal to their conjugate transpose $A = A^\dagger, B = B^\dagger$

$$\text{Consider } (i[A,B])^\dagger = i^*[A,B]^\dagger \text{ as } (cA)^\dagger = c^*A^\dagger \text{ where } c \text{ is a complex scalar.}$$
$$= (-i)[A,B]^\dagger = (-i)[B^\dagger, A^\dagger] \text{ from Exercise 2.45}$$
$$= (-i)[B,A] = -(i)(-[A,B]) \text{ from exercise 2.46} = i[A,B] \implies i[A,B] \text{ is Hermitian}$$

Exercise 2.48

Exercise 2.49

Exercise 2.50

Exercise 2.51

$$\text{Consider } HH^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = I$$
$$\text{Consider } H^\dagger H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = I$$

Thus H is indeed unitary.

Exercise 2.52

$$H^2 = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\times 1 + 1\times 1 & 1\times 1 - 1\times 1\\1\times 1 - 1\times 1 & 1\times 1 + (-1)\times(-1)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = I$$

Exercise 2.53

$$\text{Consider characteristic question } (\frac{1}{\sqrt{2}} - \lambda)(-\frac{1}{\sqrt{2}} - \lambda) - \frac{1}{2} = 0 \implies \lambda^2 = 1 \implies \lambda = \pm 1.$$

Exercise 2.54

Exercise 2.55

Exercise 2.56

Exercise 2.57: (Cascaded measurements are single measurements)

Exercise 2.58

Exercise 2.59

Exercise 2.60

Exercise 2.61

Exercise 2.62

Exercise 2.63

Exercise 2.64

Exercise 2.65

Exercise 2.66

Exercise 2.67

Exercise 2.68

Exercise 2.69

Exercise 2.70

Exercise 2.71: (Criterion to decide if a state is mixed or pure)

Exercise 2.72: (Bloch sphere for mixed states)

Exercise 2.73

Exercise 2.74

Exercise 2.75

Exercise 2.76

Exercise 2.77

Exercise 2.78

Exercise 2.79

Exercise 2.80

Exercise 2.81: (Freedom in purifications)

Exercise 2.82

End of Chapter Exercise 2.1: (Functions of the Pauli matrices)

End of Chapter Exercise 2.2: (Properties of the Schmidt number)

End of Chapter Exercise 2.3: (Tsirelson's inequality)

Chapter 3: Introduction to computer science

1	**2**	**3**	**4**	**5**	**6**	**7**
8	**9**	**10**	**11**	**12**	**13**	**14**
15	**16**	**17**	**18**	**19**	**20**	**21**
22	**23**	**24**	**25**	**26**	**27**	**28**
29	**30**	**31**	**32**	**EOC1**	**EOC2**	**EOC3**
EOC4	**EOC5**	**EOC6**	**EOC7**	**EOC8**	**EOC9**	**EOC10**

Exercise 3.1: Non-computable processes in Nature

Exercise 3.2: Turing numbers

A Turing Machine is just a seven-tuple $(Q, \Sigma, \Gamma, \delta, q_0, a_{accept}, a_{reject})$, where:

- Q is a set of States
- Σ is the input alphabet
- Γ is the tape alphabet, $\square \in \Gamma, \Sigma \subseteq \Gamma$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function
- $q_0 \in Q$ is the accept state
- $a_{accept} \in Q$ is the accept state
- $q_{recect} \in Q$ is the reject state $q_{reject} \neq q_{accept}$

A more detailed explanation of the definition you can find in "Introduction to Theory of Computation" by Michael Sipser, so:

Exercise 3.3: Turing machine to reverse a bit string

Exercise 3.4: Turing machine to add modulo 2

Given two input strings separated by a blank character, we traverse to the last character of the second string (we know this if it is a second blank we encounter, the first one being the separation between the input strings of bits). Based on the value of the last bit, we either transition to a last 1 (q5) state or a last 0 (q4) state. We now have to see what was the last bit in the first input string, so we traverse left, deleting every bit we encounter, not to leave a huge mess after us. Once we encounter the last bit within the first input string, we decide accordingly on what to write on the tape (if we already have 1 from the second input string and encounter 0 now, we write 1, and if we encounter 1, we write 0). We can do addition modulo 2 based on these last two bits.

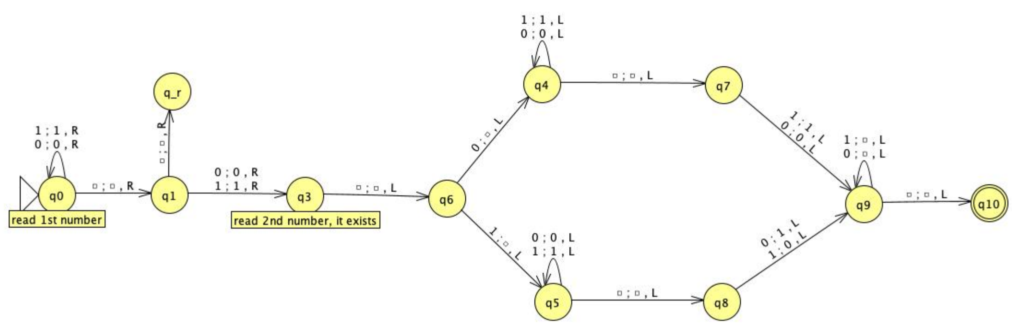

Figure 1: Single-Tape TM

Let's carefully trace what is happening if we want to add 7 and 9 modulo 2. Our tape initially looks like $\ldots \square\square q_0 111\square 1001\square\square \ldots$ with the head on the very first digit if the first input string, the machine is in the position q0.

$$\square\ \square\ q_0\ 1\ 1\ 1\ \square\ 1\ 0\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ q_0\ 1\ 1\ \square\ 1\ 0\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ q_0\ 1\ \square\ 1\ 0\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ q_0\ \square\ 1\ 0\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ q_1\ 1\ 0\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ q_1\ 0\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ 0\ q_1\ 0\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ 0\ 0\ q_1\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ 0\ 0\ 1\ q_1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ 0\ 0\ q_6\ 1\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ 0\ q_5\ 0\ \square\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ 1\ q_5\ 0\ \square\ \square\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ \square\ q_5\ 1\ \square\ \square\ \square\ \square\ \square$$
$$\square\ \square\ 1\ 1\ 1\ q_5\ \square\ \square\ \square\ \square\ \square\ \square\ \square$$
$$\square\ \square\ 1\ 1\ q_8\ 1\ \square\ \square\ \square\ \square\ \square\ \square\ \square$$
$$\square\ \square\ 1\ q_9\ 1\ 0\ \square\ \square\ \square\ \square\ \square\ \square\ \square$$
$$\square\ \square\ q_9\ 1\ \square\ 0\ \square\ \square\ \square\ \square\ \square\ \square\ \square$$
$$\square\ q_9\ \square\ \square\ \square\ 0\ \square\ \square\ \square\ \square\ \square\ \square\ \square$$
$$q_{10}\square\ \ \square\ \square\ \square\ 0\ \square\ \square\ \square\ \square\ \square\ \square\ \square$$

Exercise 3.5: Halting problem with no inputs

Exercise 3.6: Probabilistic halting problem

Exercise 3.7: Halting oracle

Exercise 3.8: Universality of NAND

Note: here I use Sheffer stroke $\uparrow$ notation for NAND

- **AND** $A \wedge B = (A \uparrow B) \uparrow (A \uparrow B)$

A	B	$A \uparrow B$	$A \wedge B$	$(A \uparrow B) \uparrow (A \uparrow B)$
0	0	1	0	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	1

- **NOT** $\neg A = A \uparrow A$

A	$\neg A$	$A \uparrow A$
0	1	1
1	0	0

- **Or** $A \vee B = \neg(\neg A \wedge \neg B) = \neg A \uparrow \neg B = (A \uparrow A) \uparrow (B \uparrow B)$ from the De Morgan's Law.

A	B	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$\neg(\neg A \wedge \neg B)$	$A \uparrow A$	$B \uparrow B$	$(A \uparrow A) \uparrow (B \uparrow B)$	$A \vee B$
0	0	1	1	1	0	1	1	0	0
0	1	1	0	0	1	1	0	1	1
1	0	0	1	0	1	0	1	1	1
1	1	0	0	0	1	0	0	1	1

- $A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$.

Exercise 3.9:

Exercise 3.10:

Exercise 3.11

We need to prove, that $\exists$ constants c, n_0 s.t $\forall n \geq n_0, \log n \leq cn^k, k > 0$.

$$\text{Consider } \lim_{n\to\infty} \frac{\log n}{n^k} \xrightarrow{L'Hopital's rule} \lim_{n\to\infty} \frac{\frac{1}{n}}{kn^{k-1}} = \lim_{n\to\infty} \frac{1}{kn^k} = 0 \implies \frac{\log n}{n^k} \leq 1 \text{ for some large } n.$$

For $c = 1$ and some large $n, \log n \leq n^k,$ as we are considering the non-negative functions.

Thus $\log n$ is $O(n^k)$.

Exercise 3.12: $n^{\log n}$ is super-polynomial

We need to prove, that $\forall k, \exists c, n_0$, s.t $\forall n > n_0, n^k \leq cn^{\log n}$ Consider:

$$\lim_{n\to\infty} \frac{n^k}{n^{\log n}} = \lim_{n\to\infty} \frac{(e^{\ln n})^k}{(e^{\ln n})^{\log n}} = \lim_{n\to\infty} \frac{e^{k \ln n}}{e^{\log n \ln n}} = \lim_{n\to\infty} e^{k \ln n - \log n \ln n} = e^{-\infty} = 0$$

as k is a fixed constant. In conclusion $c = 1$ and some large $n, n^k \leq n^{\log n} \implies n^k$ is $O(n^{\log n})$

Suppose for contradiction $n^{\log n}$ is $O(n^k) \implies \exists c, n_0,$ s.t $n^{\log n} \leq cn^k, \forall n > n_0 \implies \ln n^{\log n} \leq \ln cn^k$

$$\implies \log n \ln n \leq \ln c + k \ln n \implies \ln n(\log n - k) \leq \ln c.$$

But, as $n \to \infty$ the left side tends to infinity, and it cannot be less, than some constant.

Thus we have a contradiction, and out initial statement is false, thus $n^{\log n}$ is never $O(n^k)$.

Exercise 3.13: $n^{\log n}$ is sub-exponential

Exercise 3.14:

Exercise 3.15: Lower bound for compare-and-swap based sorts

Exercise 3.16

Exercise 3.17

Exercise 3.18

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Exercise 3.32

Chapter 4: Quantum circuits

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19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	EOC1	EOC2	EOC3
EOC4	EOC5	EOC6						

Exercise 4.1

Exercise 4.2

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Exercise 4.51

Chapter 5: The quantum Fourier transform and its applications

Chapter 6: Quantum search algorithms

Chapter 7: Quantum computers: physical realization

Chapter 8: Quantum noise and quantum operations

Chapter 9: Distance measures for quantum information

Chapter 10: Quantum error-correction

Chapter 11: Entropy and information

Chapter 12: Quantum information theory