Answer to Assignment 6

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Question 1

First, find the eigenvectors and eigenvalues of A

$(.4-\lambda)(1.2-\lambda)-(.4\times-.3)=0$

$.48-.4\lambda-1.2\lambda+\lambda^2+.12=0$

$\lambda^2-1.6\lambda+.6=0$

$(\lambda-.6)\times(\lambda-1)=0$

$\lambda=.6$ and 1

Second, find the eigenvectors of A

$\lambda=.6$

$$\begin{bmatrix}-.2 & -.3\\ .4 & .6\end{bmatrix}x=0$$

$$\left[\begin{array}{cc:c}1 & 1.5 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$x=span\{\begin{bmatrix}-1.5\\ 1\end{bmatrix}\}$$

$\lambda=1$

$$\left[\begin{array}{cc:c}-.6 & -.3 & 0\\ .4 & .2 & 0\end{array}\right]$$

$$\left[\begin{array}{cc:c}1 & .5 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$x=span\{\begin{bmatrix}-.5\\ 1\end{bmatrix}\}$$

So A can be diagonalized as $A = PDP^{-1}$ where $P = \begin{bmatrix} -1.5 & -.5 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} .6 & 0 \\ 0 & 1 \end{bmatrix}$

$A^k = PD^kP^{-1} = \begin{bmatrix} -1.5 & -.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .6^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} -1 & -.5 \\ 1 & .5 \end{bmatrix}$

As $k \to 0$, $.6^k$ is approaching to 0 and 1^k is approaching to 1.

$$
\begin{aligned}
A^k &= \begin{bmatrix} -1.5 & -.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -.5 \\ 1 & 1.5 \end{bmatrix} \text{ as } k \to 0 \\
&= \begin{bmatrix} 0 & -.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -.5 \\ 1 & 1.5 \end{bmatrix} \\
&= \begin{bmatrix} -.5 & -.75 \\ 1 & 1.5 \end{bmatrix}
\end{aligned}
$$

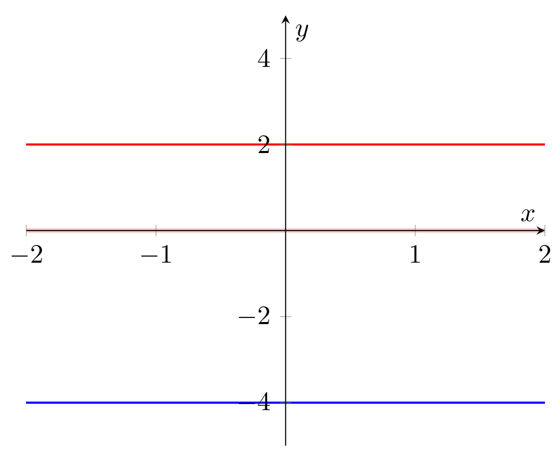
Question 2

For $a = 32$

$$A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & 32 & 25 \end{bmatrix}$$

$$
\begin{aligned}
\text{The characteristic polynomial} &= (-6-\lambda)(-15-\lambda)(25-\lambda) + (28 \times -12 \times -8) + (21 \times 4 \times 32) - \\
&\quad (-8 \times 21)(-15-\lambda) - (32 \times -12)(-6-\lambda) - (4 \times 28)(25-\lambda) \\
&= -\lambda^3 + 4\lambda^2 + 435\lambda + 2250 + 2688 + 2688 - 2520 - 168\lambda - 2304 - 384\lambda - \\
&\quad 2800 + 112\lambda \\
&= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \\
&= -(\lambda - 1)^2(\lambda - 2)
\end{aligned}
$$

The eigenvalues are 1 and 2.



For $a = 31.9$

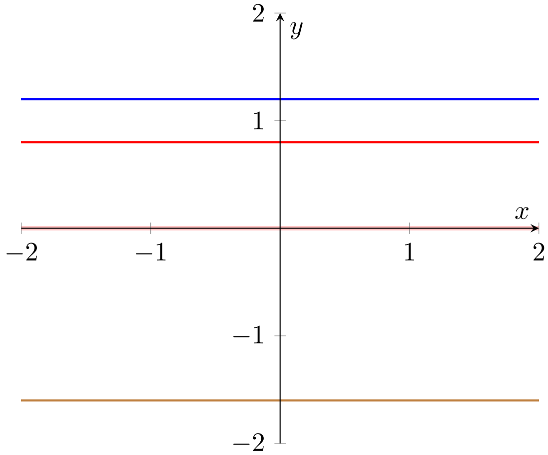
$$A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & 31.9 & 25 \end{bmatrix}$$

The characteristic polynomial $= (-6-\lambda)(-15-\lambda)(25-\lambda) + (28\times -12\times -8) + (21\times 4\times 31.9) - (-8\times 21)(-15-\lambda) - (31.9\times -12)(-6-\lambda) - (4\times 28)(25-\lambda)$

$$= -\lambda^3 + 4\lambda^2 + 435\lambda + 2250 + 2688 + 2679.6 - 2520 - 168\lambda - 2296.8 - 382.8\lambda - 2800 + 112\lambda$$

$$= -\lambda^3 + 4\lambda^2 - 3.8\lambda + .8$$

The eigenvalues are 1, 2.70416, and 0.295841.



For $a = 31.8$

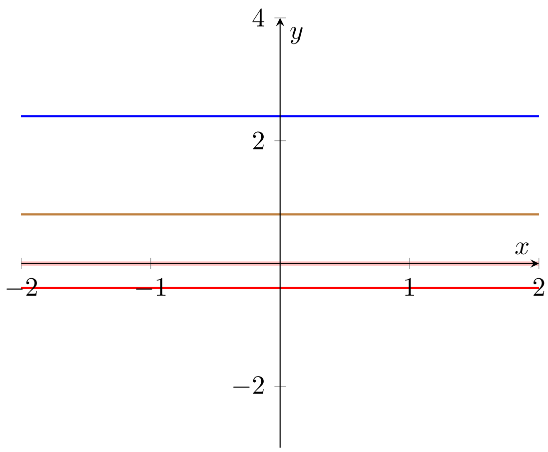
$$A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & 31.8 & 25 \end{bmatrix}$$

The characteristic polynomial $= (-6-\lambda)(-15-\lambda)(25-\lambda) + (28\times -12\times -8) + (21\times 4\times 31.8) - (-8\times 21)(-15-\lambda) - (31.8\times -12)(-6-\lambda) - (4\times 28)(25-\lambda)$

$$= -\lambda^3 + 4\lambda^2 + 435\lambda + 2250 + 2688 + 2671.2 - 2520 - 168\lambda - 2289.6 - 381.6\lambda - 2800 + 112\lambda$$

$$= -\lambda^3 + 4\lambda^2 - 2.6\lambda - 0.4$$

The eigenvalues are 1, -0.127882, and 3.12788.



For $a = 32.1$

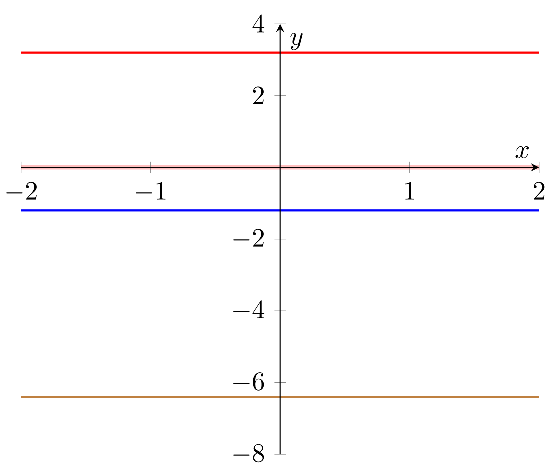
$$A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & 32.1 & 25 \end{bmatrix}$$

The characteristic polynomial $= (-6-\lambda)(-15-\lambda)(25-\lambda) + (28 \times -12 \times -8) + (21 \times 4 \times 32.1) - (-8 \times 21)(-15-\lambda) - (32.1 \times -12)(-6-\lambda) - (4 \times 28)(25-\lambda)$

$= -\lambda^3 + 4\lambda^2 + 435\lambda + 2250 + 2688 + 2696.4 - 2520 - 168\lambda - 2311.2 - 385.2\lambda - 2800 + 112\lambda$

$= -\lambda^3 + 4\lambda^2 - 6.2\lambda + 3.2$

The eigenvalues is 1.



For $a = 32.2$

$$A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & 32.2 & 25 \end{bmatrix}$$

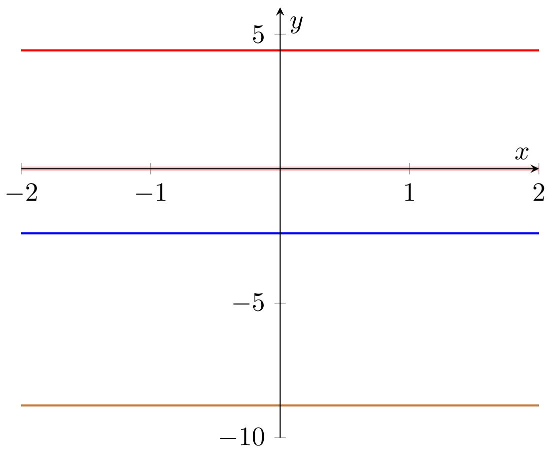
The characteristic polynomial $= (-6-\lambda)(-15-\lambda)(25-\lambda) + (28 \times -12 \times -8) + (21 \times 4 \times 32.2) -$

$$(-8 \times 21)(-15-\lambda) - (32.2 \times -12)(-6-\lambda) - (4 \times 28)(25-\lambda)$$

$$= -\lambda^3 + 4\lambda^2 + 435\lambda + 2250 + 2688 + 2704.8 - 2520 - 168\lambda - 2318.4 - 386.4\lambda - 2800 + 112\lambda$$

$$= -\lambda^3 + 4\lambda^2 - 7.4\lambda + 4.4$$

The eigenvalues is 1.



Question 3

$$T(3b_1 - 4b_2) = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$$

Question 4

$$\text{(a)} = \begin{bmatrix} 5+3(-1) \\ 5+3(0) \\ 5+3(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

(b)

Let $m = ax^2 + bx + c$ and $n = dx^2 + ex + f$

$$T(m+n) = T((a+d)x^2 + (b+e)x + c + f) = \begin{bmatrix} (a+d)-(b+e)+c+f \\ c+f \\ (a+d)+(b+e)+c+f \end{bmatrix} = \begin{bmatrix} a-b+c \\ c \\ a+b+c \end{bmatrix} + \begin{bmatrix} d-e+f \\ f \\ d+e+f \end{bmatrix}$$

$$= T(m) + T(n)$$

So $T(a+b) = T(a) + T(a)$

Let $m = ax^2 + bx + c$ and d is a scalar

$$T(dm) = \begin{bmatrix} da - db + dc \\ dc \\ da + db + c \end{bmatrix} = d \begin{bmatrix} a-b+c \\ c \\ a+b+c \end{bmatrix} = dT(m)$$

So $T(ca) = cT(a)$

As $T(a+b) = T(a) + T(a)$ and $T(ca) = cT(a)$, T is a linear transformation.

(c)

T relative to the basis $\{1, t, t^2\}$

Let $m = ax^2 + bx + c$ $\quad T \times \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} a-b+c \\ c \\ a+b+c \end{bmatrix}$ $\quad T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

T relative to the standard basis

Let $m = \begin{bmatrix} c \\ b \\ a \end{bmatrix}_{IP^2}$ $\quad T \times \begin{bmatrix} c \\ bt \\ at^2 \end{bmatrix} = \begin{bmatrix} a-b+c \\ c \\ a+b+c \end{bmatrix}$ $\quad T = \begin{bmatrix} 1 & -1/t & 1/t^2 \\ 1 & 0 & 0 \\ 1 & 1/t & 1/t^2 \end{bmatrix}$

Question 5

(a) Find the eigenvalues of A

$(-\lambda)(4-\lambda) - (-3 \times 1) = 0 \qquad -4\lambda + \lambda^2 + 3 = 0 \qquad (\lambda - 1)(\lambda - 3) = 0 \qquad \lambda = 1, 3$

According to the theorem, if $A = PDP^{-1}$, the column of P is the basis of β and D is T_β.

Find P

$\lambda = 1$

$\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} x = 0 \qquad x = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$

$\lambda = 3$

$\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} x = 0 \qquad x = span\{\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}\}$

So $P = \begin{bmatrix} 1 & -1/3 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. The basis β is $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}\}$

(b) Follow the same procedure as part (a)

$(5-\lambda)(1-\lambda) - (-7 \times -3) = 0 \qquad 5 - 6\lambda + \lambda^2 - 21 = 0 \qquad \lambda = -2, 8$

Find P

$\lambda = -2$

$\begin{bmatrix} 7 & -3 \\ -7 & 3 \end{bmatrix} x = 0 \qquad x = span\{\begin{bmatrix} 3/7 \\ 1 \end{bmatrix}\}$

$\lambda = 8$

$\begin{bmatrix} -3 & -3 \\ -7 & -7 \end{bmatrix} x = 0 \qquad x = span\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$

So the basis β is $\{\begin{bmatrix} 3/7 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$

Question 6

(a) Find the covariance matrix.

(b) Find the covariance matrix of the matrix *img_in*. It is required since eig() function requires the input to be a square matrix. cov() convert a rectangular matrix into a square matrix. The value of each cell indicates the distance of data from the average.

(c) Find the eigenvalues and eigenvectors.

(d) V is a matrix whose columns are eigenvectors and D is a diagonal matrix of the corresponding eigenvalues.

(e) $V^{-1} = V^T \longrightarrow V^{-1} \times V = V^T \times V \longrightarrow I = V^T \times V$

Because $A = cov(img\_in)$ is a symmetric matrix, A can be decomposed as PDP^{-1} where P is an orthogonal matrix. In this case, V is the same as P. As the property of a orthogonal matrix, $P \times P^T = P^T \times P = I$. As a result, the given equation is true.

(f) The code change the basis from the standard basis to the the basis of the largest 110 eigenvectors and the image get compressed.

(g) 530 to 640 indicates that the *img_in* was compressed by the largest 110 eigenvectors. They are sufficient because they can span most data of the image. The image loses some detail, but the important detail is still preserved.

(h) A image with vertical blue lines. The image loses its important detail because the smallest 400 eigenvectors are much far away from the mean of the data, so they contains unnecessary data.

(i) It changes the basis of *img_in* from the standard basis to the largest 110 eigenvecttors.