Answer to Assignment 5

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Question 1

(a) $$u_1 = 1v_1 - 3v_2 + 4v_3 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \\ 21 \end{bmatrix}$$

$$u_2 = 2v_1 + 5v_2 + 6v_3 = 2\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + 5\begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} + 6\begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -86 \\ 41 \\ 46 \end{bmatrix}$$

$$u_3 = -1v_1 + 0v_2 + 1v_3 = -\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$$

The basis $\{u_1, u_2, u_3\} = \{ \begin{bmatrix} -6 \\ -5 \\ 21 \end{bmatrix}, \begin{bmatrix} -86 \\ 41 \\ 46 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \}$

(b) $$w_1 = P \times v_1 = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 5 & 0 \\ 4 & 6 & 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 16 \\ 7 \end{bmatrix}$$

$$w_2 = P \times v_2 = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 5 & 0 \\ 4 & 6 & 1 \end{bmatrix} \times \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 49 \\ 0 \end{bmatrix}$$

$$w_3 = P \times v_3 = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 5 & 0 \\ 4 & 6 & 1 \end{bmatrix} \times \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -9 \\ 31 \\ -10 \end{bmatrix}$$

Question 2

(a) From β to C

$$\left[\begin{array}{cc:cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & 1 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & -3 & 0.5 \\ 0 & 1 & -5 & 1.75 \end{array}\right]$$

So the change of coordinates matrix from β to C is $\begin{bmatrix} -3 & 0.5 \\ -5 & 1.75 \end{bmatrix}$

From C to β

$$\left[\begin{array}{cc:cc} 7 & -3 & 1 & -2 \\ 5 & 1 & -5 & -2 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & -0.6364 & 0.1818 \\ 0 & 1 & -1.8182 & 1.0909 \end{array}\right]$$

So the change of coordinates matrix from C to β is $\begin{bmatrix} -0.6364 & 0.1818 \\ -1.8182 & 1.0909 \end{bmatrix}$

(b) From β to C

$$\left[\begin{array}{cc:cc} 1 & 1 & -1 & -1 \\ 4 & 1 & 8 & -5 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & 3 & -1.3333 \\ 0 & 1 & -4 & 0.3333 \end{array}\right]$$

So the change of coordinates matrix from β to C is $\begin{bmatrix} 3 & -1.3333 \\ -4 & 0.3333 \end{bmatrix}$

From C to β

$$\left[\begin{array}{cc:cc} -1 & -1 & 1 & 1 \\ 8 & -5 & 4 & 1 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & -0.0769 & -0.3077 \\ 0 & 1 & -0.9231 & -0.6923 \end{array}\right]$$

So the change of coordinates matrix from C to β is $\begin{bmatrix} -0.0769 & -0.3077 \\ -0.9231 & -0.6923 \end{bmatrix}$

(c) From β to C

$$\left[\begin{array}{cc:cc} 2 & 6 & -6 & 2 \\ -1 & -2 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So the change of coordinates matrix from β to C is $\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$

From C to β

$$\left[\begin{array}{cc:cc} -6 & 2 & 2 & 6 \\ 1 & 0 & -1 & -2 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \end{array}\right]$$

So the change of coordinates matrix from C to β is $\begin{bmatrix} -1 & -2 \\ -2 & -3 \end{bmatrix}$

Question 3

The change of coordinates matrix from β to C is $\begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

The β coordinate vector for $-1+2t$ is $\begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 7 \end{bmatrix}$

Question 4

The rref of A is $\begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.0105 \\ 0 & 1 & 0 & 2.0526 \\ 0 & 0 & 1 & 2.8105 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the column space of A is

$span\{\begin{bmatrix} 7 \\ -5 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ -11 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7 \\ 7 \end{bmatrix}\}$. Since $\begin{bmatrix} 7 & 6 & -4 & \vdots & 1 \\ -5 & -1 & 0 & \vdots & 1 \\ 9 & -11 & 7 & \vdots & -1 \\ 19 & -9 & 7 & \vdots & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0.0105 \\ 0 & 1 & 0 & -1.0526 \\ 0 & 0 & 1 & -1.8105 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is consistent, the

$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ is in the column space of A

In order to find the null space of A, the rref of $\begin{bmatrix} 7 & 6 & -4 & 1 & \vdots & 0 \\ -5 & -1 & 0 & -2 & \vdots & 0 \\ 9 & -11 & 7 & -3 & \vdots & 0 \\ 19 & -9 & 7 & 1 & \vdots & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & -0.0105 & 0 \\ 0 & 1 & 0 & 2.0526 & 0 \\ 0 & 0 & 1 & 2.8105 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

So the null space of A is $span\{\begin{bmatrix} 0.0105 \\ 2.0526 \\ 2.8105 \\ 1 \end{bmatrix}\}$

In order for $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ to be in the null space of A, $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ must be in the $span\{\begin{bmatrix} 0.0105 \\ 2.0526 \\ 2.8105 \\ 1 \end{bmatrix}\}$, but it does not.

As a result, $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ is not in the null space of A.

Final Answer: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ is in the column space but not in the null space of A.

Question 5

In order to form a basis for P_3, the coordinate matrix must span P_3.

(a) $\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so it spans only P_2 not P_3

(b) $\begin{bmatrix} 5 & 9 & 6 & 0 \\ -3 & 1 & -2 & 0 \\ 4 & 8 & 5 & 0 \\ 2 & -6 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.75 & 0 \\ 0 & 1 & 0.25 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so it spans only P_2 not P_3

Question 6

Definition: $\beta = \{b_1 \dots b_n\}$ is a basis of the vector space V if β is a linearly independent set and $V = Span\{b_1 \dots b_n\}$

1) That every x has a unique linear combination of S implies that $\begin{bmatrix} b_1 & \dots & b_n & \vdots & x \end{bmatrix}$ has exactly one answer for each x, which means that $\begin{bmatrix} b_1 & \dots & b_n & \vdots & x \end{bmatrix}$ must be linearly independent.

2) As given, every x in V can be represented as a linear combination of S; in the other words, S spans the vector space V.

According to the two statements above, we can conclude that S is a basis of V

Question 7

In order to find the coordinates of vectors $b_1, b_2, ..., b_n$ with respect to the basis β, we need to construct the augmented matrix $\begin{bmatrix} b_1 & b_2 & \dots & b_n & \vdots & b_1 & b_2 & \dots & b_n \end{bmatrix}$ and reduce the left side of the augmented matrix to the rref form, which will change the right side to the coordinate matrix. The rref of the augmented matrix is $\begin{bmatrix} I & \vdots & e_1 & e_2 & \dots & e_n \end{bmatrix}$ because $b_1, b_2, ..., b_n$ is the basis, which means they are independent. So the coordinate of b_1 is e_1, b_2 is $_2$ and so on until b_n is e_n.

Question 8

$$det(B^4) = (det(B))^4 = ((1\times1\times1)+(0\times2\times1)+(1\times1\times\ 2)-(1\times1\times1)-(2\times2\times1)-(1\times1\times0))^4 =$$

$$(-2)^4 = 16$$

Question 9

(a) $det(AB) = det(A) \times det(B) = -3 \times 4 = -12$

(b) $det(5A) = 5 \times det(A) = 5 \times -3 = -15$

(c) $det(B^t) = det(B) = 4$

(d) $det(A^{-1}) = \frac{1}{det(A)} = -\frac{1}{3}$

(e) $det(A^3) = (det(A))^3 = (-3)^3 = -27$

Question 10

(a) According to the theorem, the determinant of the given matrix is equal to $3\times\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3\times7 = 21$

(b) According to the theorem, the determinant of the given matrix is equal to $5\times\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5\times7 = 35$

(c) According to the theorem, the determinant of the given matrix is equal to $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

Question 11

(a) Suppose the second column is chosen.

$$det = -4 \times \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 3 \\ 6 & -4 & -1 \end{vmatrix} + 8 \begin{vmatrix} 3 & -3 & 1 \\ 3 & 1 & -3 \\ -6 & -4 & 3 \end{vmatrix} = (-4 \times -84) + (8 \times -60) = 336 - 480 = -144$$

(b) Suppose the forth column is chosen.

$$det = -6 \times \begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 4 & 2 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 11 & 4 & 6 \end{vmatrix} = (-6 \times -40) + (3 \times -78) = 240 - 234 = 6$$

Question 12

By referring to the question, $\begin{bmatrix} A_{11(20x20)} & 0 \\ A_{21(30x20)} & A_{22(30x30)} \end{bmatrix} \times \begin{bmatrix} x_1 \\ \vdots \\ x_{20} \\ \cdots \\ x_{21} \\ x_{22} \\ \vdots \\ x_{50} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{20} \\ \cdots \\ b_{21} \\ b_{22} \\ \vdots \\ b_{50} \end{bmatrix}$. The equation $Ax = b$ can be seperated into

1) $A_{11(20x20)} \times \begin{bmatrix} x_1 \\ \vdots \\ x_{20} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{20} \end{bmatrix}$ and 2) $A_{21(30x20)} \times \begin{bmatrix} x_1 \\ \vdots \\ x_{20} \end{bmatrix} + A_{22(30x30)} \times \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{50} \end{bmatrix} = \begin{bmatrix} b_{21} \\ b_{22} \\ \vdots \\ b_{50} \end{bmatrix}$

Both equation can now be solved in the matrix program that is limited to 32 rows and 32 columns.