Answer to Assignment 7

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Question 1

(a) $Ax = 0 \quad A^TAx = A^T0 \quad A^TAx = 0$

(b) $A^TAx = 0 \quad x^TA^TAx = x^T0 \quad x^TA^TAx = 0 \quad (Ax)^TAx = 0$ Let $Ax = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$

From $(Ax)^TAx = 0$, we can conclude that ${a_1}^2 + {a_2}^2 + \ldots = 0$. So $a_1 = a_2 = \ldots = 0$, which means $Ax = 0$.

Question 2

(a) According to the fact that A is a linearly independent matrix and $Ax = 0$, there must be a unique solution to x. And from $A^TAx = 0$, A^TA has to be a linearly independent matrix in order to provide a unique solution to x. According to Inverse Matrix Theorem, if columns of a matrix is linearly independent, that matrix is invertable, so A^TA is invertable.

(b) In order to be a linearly independent matrix, a matrix must have pivots in every columns. And to do so, a matrix must have more or equal rows as columns. So A must correspond to the mentioned argument.

(c) Since A is a linearly independent matrix, $Rank(A) = dim(col(A)) = 0$

Question 3

Let $A = \begin{bmatrix} a_1 & a_2 & \ldots \end{bmatrix}$, $|a_1| = |a_2| = \ldots = 1$ and the dot product of the difference columns of A is 0.

According to the equation used to find the least squares solution, $A^TAx = A^Tb$, $A^TA = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & a_1 \cdot a_3 & \ldots \\ a_2 \cdot a_1 & a_2 \cdot a_2 & a_3 \cdot a_2 & \ldots \\ a_3 \cdot a_1 & a_3 \cdot a_2 & a_3 \cdot a_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} =$

$\begin{bmatrix} |a_1|^2 & 0 & 0 & \ldots \\ 0 & |a_2|^2 & 0 & \ldots \\ 0 & 0 & |a_3|^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = I$. So $x = A^Tb$

Question 4

(a) This problem involves 3 unknowns.

According to the equation $A^TAx = A^Tb$ that is used to find the solution to the least square solution, we can construct the augmented matrix $\begin{bmatrix} 4 & 2 & 2 & \vdots & 14 \\ 2 & 2 & 0 & \vdots & 4 \\ 2 & 0 & 2 & \vdots & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & \vdots & 5 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$.

As a result, there are infinite answer to the 3 unknowns: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that corresponds to the first, second, and third column of $A = \begin{bmatrix} 5 - x_3 \\ -3 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$

Code for part (a)

```
A = [1 1 0;1 1 0;1 0 1;1 0 1]
b = [1;3;8;2]
X = rref([A'*A A'*b]) % An augnemted matrix
```

(b) This problem involves 3 unknowns.

According to the equation $A^T Ax = A^T b$ that is used to find the solution to the least square solution, we can construct the augmented matrix $\begin{bmatrix} 6 & 3 & 3 & \vdots & 27 \\ 3 & 3 & 0 & \vdots & 12 \\ 3 & 0 & 3 & \vdots & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & \vdots & 5 \\ 0 & 1 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$.

As a result, there are infinite answer to the 3 unknowns: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that corresponds to the first, second, and third column of $A = \begin{bmatrix} 5 - x_3 \\ -1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$

Code for part (b)

```
A = [1 1 0;1 1 0;1 1 0;1 0 1;1 0 1;1 0 1]
b = [7;2;3;6;5;4]
X = rref([A'*A A'*b]) % An augnemted matrix
```

Question 5

A = $\begin{bmatrix} 1 & x_1 & {x_1}^2 & {x_1}^3 \\ 1 & x_2 & {x_2}^2 & {x_2}^3 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$

So $\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = inv(A^T A)A^T y$

```
x=linspace(0,10,1000)';
y=6+5*x+3*x.^2+3*x.^3+x.^3.*rand(size(x));
plot(x,y,'bo')
hold on;
A = [ones(1000, 1) x x.^2 x.^3]
```

```
ANS = inv(A' * A)*A'*y
plot(x, ANS(1) + ANS(2)*x + ANS(3)*x.^2 + ANS(4)*x.^3, 'r', 'LineWidth', 3)
```

Question 6

```
D = dlmread('Question6matlabData.txt')
A = D(:,1,1)
Y = D(:,2,1)
scatter(A, Y)
hold on;
A = [ones(size(A, 1), 1) A sin(0.5.*pi.*A)]
ANS = inv(A'*A)*A'*Y
plot(A, ANS(1) + ANS(2)*A + ANS(3)*sin(0.5*pi*A), 'r', 'LineWidth', 3)
```