Lecture 2, September 25th, 2025

1. Bayesian inference: general formulation + examples
2. Monte Carlo algorithms

Bayesian inference (generic setup)

Inputs

- $\underline{y}$ — vector of
- $\underline{\theta}$ — vector of
- $p(\underline{y} \mid \underline{\theta})$ — density/prob defined by
- $p(\underline{\theta})$ — density of the

Output: $p(\underline{\theta} \mid \underline{y}) =$

In practice, we use the following summaries of the posterior distribution:

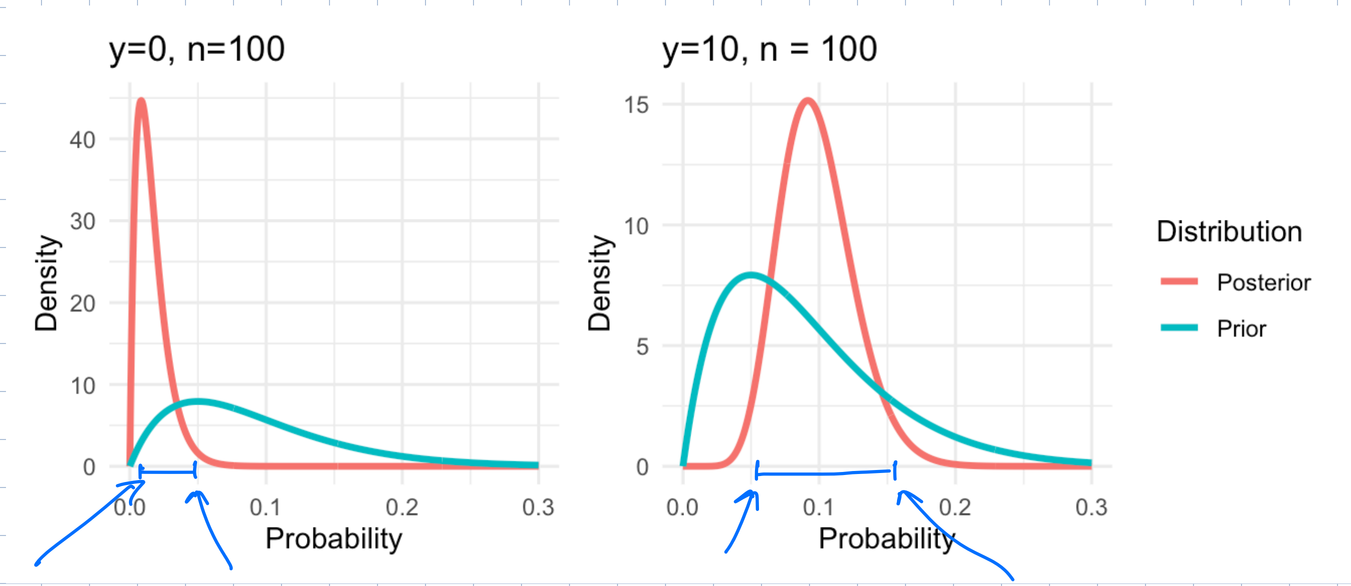
1) point estimation:

For example,

$\hat{\underline{\theta}} =$

More concretely, in our proportion estimation example
$\theta \mid y \sim Beta(\alpha + y, n - y + \beta) \Rightarrow$

$E(\theta \mid y) =$ $E(\theta \mid y) =$



2) uncertainty quantification:

Bayesian

In our proportion estimation example:

$y = 0, \ n = 100, \ \alpha = 2, \ \beta = 20, \ \underbrace{q = 0.05}_{\text{this gives us}}$

$y = 10, \ n = 100, \ \alpha = 2, \ \beta = 2, \ q = 0.05$

There is no formula for quantiles, so these calculations must be done

Example: multiple linear regression

data:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & & \boxed{x_i^T} & \\ 1 & x_{n1} & & x_{np} \end{bmatrix}$$

we condition on covariates/predictors, so we do not model them

only y vector is random

sample size

number of covariates/predictors

Data generating model:

parameters:

Likelihood:

priors: many choices, but let's use something simple:

posterior distribution/density:

Bad news:

Good news:

Monte Carlo Integration

First, let's review the concept of mathematical expectation:

1) for discrete random variable X

$$E[g(X)] = \sum_{k=1}^{n} g(x_k) P(X = x_k)$$

1^{st} moment: $g(x) = x$

2^{nd} moment $g(x) = x^2$

2) for (absolutely) continuous random variable X:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

<u>Example</u>: exponential random variable

$$f(x) = \lambda e^{-\lambda x} \cdot 1_{\{x \geq 0\}}, \text{ where } \lambda > 0 \text{ - rate parameter}$$

$X \sim Exp(\lambda)$ $E(X) =$

Strong Law of Large Numbers (SLLN)

Let $X_1, X_2, \ldots$ be independent and identically distributed (iid) random variables with

$\mu = E(X_1) < \infty$ Then

Monte Carlo Integration

Objective: $E[h(X)] = \int h(x) f(x)\, dx$, where X is a random variable with probability density function $f(x)$ or

$$E(h(X)) = \sum_{k=1}^{\infty} h(X_k)\, p_k$$

, where X is a discrete random variable with prob. mass function $p_1, p_2, \ldots$

Example: second moment of the Beta distribution

Recall that in our estimating proportions example, one setting had:

$$\theta \sim \text{Beta}(2, 20)$$

$$y = 10, \quad n = 100$$

$\theta | y \sim \text{Beta}(12, 110)$

Objective : $E(\theta^2 | y)$ - second moment

In this case we can derive a formula :

$X \sim \text{Beta}(\alpha, \beta) \Rightarrow$

This result tells us that

Let's see if we can get to the same number using Monte Carlo integration.

Our game plan :

1)

2)

computer demo here