

## Practical: Metropolis-Hastings Algorithm

Instructors: Kari Auranen, Elizabeth Halloran and Vladimir Minin

July 18 – July 20, 2022

### Sampling from the standard normal distribution

Suppose our target is a univariate standard normal distribution with density  $f(x) = 1/(\sqrt{2\pi})e^{-x^2/2}$ . Given current state  $x^{(t)}$ , we generate two uniform r.v.s  $U_1 \sim U[-\delta, \delta]$  and  $U_2 \sim U[0, 1]$ . Then set

$$x^{(t+1)} = \begin{cases} x^{(t)} + U_1 & \text{if } U_2 \leq \min \left\{ e^{\left[ \frac{(x^{(t)})^2 - (x^{(t)} + U_1)^2}{2} \right]}, 1 \right\} \\ x^{(t)} & \text{otherwise.} \end{cases}$$

$\delta$  is a tuning parameter. Large  $\delta$  leads to small acceptance rate, small  $\delta$  leads to slow exploration of the state space. The rule of thumb for random walk proposals is to keep acceptance probabilities around 30-40%. If your proposal is close to the target, then higher acceptance rates are favorable.

#### Your task

Implement the above algorithm. Experiment with the tuning parameter  $\delta$  and report empirically estimate acceptance probabilities for different values of this parameter.

### Distribution of the time of infection

Consider a two state continuous-time Markov SIS model, where the disease status  $X_t$  cycles between the two states: 1=susceptible, 2=infected. Let the infection rate be  $\lambda_1$  and clearance rate be  $\lambda_2$ . Suppose that an individual is susceptible at time 0 ( $X_0 = 1$ ) and infected at time  $T$  ( $X_T = 2$ ). We don't know anything else about the disease status of this individual during the interval  $[0, T]$ . If  $T$  is small enough, it is reasonable to assume that the individual was infected only once during this time interval. We would like to obtain the distribution of the time of infection  $I$ , conditional on the information we have.

#### Your task

Implement a Metropolis-Hastings sampler to draw realizations from the distribution

$$\Pr(I \mid X_0 = 1, X_T = 2, N_T = 1) \propto \Pr(0 < t < I : X_t = 1, I < t < T : X_t = 2),$$

where  $N_t$  is the number of infections. Since  $X_t$  is a continuous-time Markov chain, the last probability (it is actually a density) can be written as

$$\Pr(0 < t < I : X_t = 1, I < t < T : X_t = 2) = \underbrace{\lambda_1 e^{-\lambda_1 I}}_{\text{density of waiting time until infection}} \times \underbrace{e^{-\lambda_2(T-I)}}_{\text{prob of staying infected}}.$$

To make things concrete, set  $\lambda_1 = 0.1$ ,  $\lambda_2 = 0.2$  and  $T = 1.0$ . For your proposal distribution, use a uniform random walk with reflective boundaries 0 and  $T$ . In other words, given a current value of the infection time  $t_c$ , generate  $u = \text{Unif}_{[t_c-\delta, t_c+\delta]}$  ( $2\delta < T$ ) and then make a proposal value

$$t_p = \begin{cases} u & \text{if } 0 < u < T, \\ 2T - u & \text{if } u > T, \\ -u & \text{if } u < 0. \end{cases}$$

This is a symmetric proposal, so your M-H ratio will contain only the ratio of target densities:

$$\frac{\lambda_1 e^{-\lambda_1 t_p} e^{-\lambda_2(T-t_p)}}{\lambda_1 e^{-\lambda_1 t_c} e^{-\lambda_2(T-t_c)}} = e^{-\lambda_1(t_p-t_c)-\lambda_2(t_c-t_p)} = e^{(t_p-t_c)(\lambda_2-\lambda_1)}.$$

Plot the histogram of the posterior distribution of the infection time. Try a couple of sets of values for  $\lambda_1$  and  $\lambda_2$  and examine the effect of these changes on the posterior distribution of the infection time.